ADVANCES IN OPTIMAL CONTROL OF MARKOV REGIME-SWITCHING MODELS WITH APPLICATIONS IN FINANCE AND ECONOMICS

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ABSTRACT

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We study stochastic optimal control problems of finance and economics in a Markov regime-switching jump-diffusion market with and without delay component in the dynamics of our model. We formulate portfolio optimization problems as a two player zero-sum and a two player nonzero-sum stochastic differential games. We provide an extension of Dynkin formula to present the Hamilton-Jacobi-Bellman-Isaacs equations in such a more general setting. We illustrate our results for a nonzero-sum stochastic differential game and investigate the impact of regime-switches by comparative statics of a two state Markov regime-switching jump-diffusion model. We prove the existence-uniqueness theorems for a stochastic differential delay equation with jumps and regimes (SDDEJR) and for an anticipated backward stochastic differential equation with jumps and regimes (ABSDEJR). Furthermore, we give the duality between an SDDEJR and an ABSDEJR. We establish necessary and sufficient maximum principles under full and partial information for an SDDEJR. We show that the adjoint equations are represented by an ABSDEJR. We apply our results to a problem of optimal consumption problem from a cash flow with delay and regimes.

Keywords: delayed stochastic differential equations, anticipated backward stochastic differential equations, Markov regime switches, HJBI equations, applications to finance and economics

ÖZ

MARKOV REJİM DEĞİŞİMLİ MODELLERİN FİNANSA VE EKONOMİYE UYGULAMALARIYLA BİRLİKTE OPTİMUM KONTROLÜNDE GELİŞMELER

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Dinamiklerinde zaman ertelemesi öğesi olan ve olmayan Markov rejim değişimli bir sıçrama-difüzyon marketinde finansın ve ekonominin stokastik optimum kontrol problemleri üzerinde çalışılmıştır. Portfolyo optimizasyon problemlerimiz iki oyunculu sıfır toplam ve sıfır olmayan toplam stokastik diferansiyel oyunları olarak formülleştirilmiştir. Böyle daha genel bir düzenlemede, Hamilton-Jacobi-Bellman-Isaacs denklemlerini sunabilmek için Dynkin formülü genelleştirilmiştir. Sonuçlarımız sıfır olmayan toplam stokastik difarensiyel oyunu için örneklendirilmiş ve rejim değişimlerinin etkileri iki durumlu bir Markov rejim değişimli sıçrama-difüzyon modeli için kıyaslamalı statikler yardımıyla araştırılmıştır. Sıçramalı ve rejim değişimli, zaman ertelemesine sahip bir stokastik difarensiyel denklem (SDDEJR) ve sıçramalı ve rejim değişimli bir beklenen geriye doğru stokastik difarensiyel denklem (ABSDEJR) için varlık-teklik teoremleri ispatlanmıştır. Dahası, bir SDDEJR ve bir ABSDEJR arasındaki dualite ispatlanmıştır. Bir SDDEJR için tam ve kısmi bilgi altında gerek ve yeter maksimum prensipleri kurulmulştur. Eşlenik denklemlerinin bir ABSDEJR aracılığıyla temsil edildiği gösterilmiştir. Sonuçlarımız, zaman ertelemeli ve rejimli bir nakit akışı için optimum tüketim problemine uygulanmıştır.

Anahtar Kelimeler : zaman ertelemeli stokastik diferansiyel denklemler, beklenen geriye doğru stokastik diferansiyel denklemler, Markov rejim değişimleri, HJBI denklemleri, finansa ve ekonomiye uygulamaları

To My Family

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LIST OF ABBREVIATIONS

$\mathbb{R}$	Set of Real Numbers
$\mathbb{R}^D$	Set of D-Dimensional Vector of Real Numbers
$\mathbb{R}^+$	Set of Positive Real Numbers
$\mathbb{N}$	Set of Natural Numbers
$\mathcal{C}^1$	Set of Continuously Differentiable Real-Valued Functions
$\mathbb{R}_0$	$\mathbb{R} \setminus \{0\}$
$\mathcal{B}_S$	Sigma field of S
A^T	Transpose of a Matrix A
DPP	Dynamic Programming Principle
MP	Maximum Principle
SDDE	Stochastic Differential Delay Equation
SDDEJR	Stochastic Differential Delay Equation with Jumps and Regimes
BSDE	Backward Stochastic Differential Equation
ABSDE	Anticipated Backward Stochastic Differential Equation
SHSJ	Stochastic Hybrid System with Jumps
HJB	Hamilton-Jacobi-Bellman
HJBI	Hamilton-Jacobi-Bellman-Isaacs
PDE	Partial Differential Equations
ODE	Ordinary Differential Equations
càdlàg	Right Continuous with Left Limits
$a \vee b$	Maximum of a and b
$(c-d)^+$	Maximum of 0 or $(c-d)$
$\langle e, f \rangle$	Inner product of e and f
GE	Good Economy
BE	Bad Economy
GNP	Gross National Product

CHAPTER 1

INTRODUCTION AND MOTIVATION

A Stochastic Hybrid System with Jumps (SHSJ) is a continuous-time process with discrete components. While the discrete variables belong to a countable set, the other dynamics evolve according to a stochastic differential equation with jumps. Hence, an SHSJ can be considered as an interleaving between a finite or countable family of jump-diffusion processes, for closer details, see Chapter 2 and the works [7, 26, 32]. In this thesis, the discrete components are regime-switches represented by a continuous-time Markov chain α with a finite state space S.

A main reason, that regime-switching models received a lot of attention in financial mathematics, is their powerful and efficient nature to capture different modes of the financial market easily, e.g., a shift from a bull to a bear market and vice versa (see [21, 38, 64, 65]). Furthermore, regime-switches can be seen as proxies of the different states of the economy within the framework of macroeconomic instruments such as gross domestic product, purchase management index and sovereign credit rating. On the other hand, a discrete shift from one regime to another may be observed as a result of a change of a macroeconomic indicator, for example a change in economic policy, e.g., a shift in a monetary or an exchange rate policy. Moreover, in some instants, it may be activated by a major event, such as the bankruptcy of Lehman Brothers in September 2008, or the 1973 oil crisis.

In the light of these facts, escaping from large costs encountered because of ignoring the regime-switches leads investors to two main questions. An intuitive and expected one is centered on the existence of an optimal portfolio to hedge against the risk of regime changes. The second question is how to determine the portfolios which should be optimally held in each regime. At this point, the crucial role of stochastic optimal control theory in finance and economics becomes highlighted.

Stochastic optimal control problems for regime-switching models have been studied by many authors for the several fundamental concepts of finance such as option pricing and risk minimization [20, 21], determining optimal selling rules [62] and optimal asset allocation [63]. In this thesis, we present advances for a Markov regime-switching model by the tools of both Dynamic Programming Principle (DPP) and Maximum

Principle (MP). For these methodologies, we refer to the monographs by Øksendal and Sulem [45], Yong and Zhou [61].

A game theoretical approach for a zero-sum and a nonzero-sum stochastic differential games were developed by Mataramvura and Øksendal [41], based on the methods of DPP whose infinitesimal counterpart gives a nonlinear Partial Differential Equation (PDE), called as Hamilton-Jacobi-Bellman-Isaacs (HJBI) equation. Inspired from [41], Elliott and Siu [21] extended a min-max problem to a Markov regime-switching diffusion model and studied on a risk minimization portfolio selection problem. Similarly, Shen and Siu [57] used a DPP approach to select an equivalent martingale measure for the valuation of contingent claims under a Markov regime-switching jump-diffusion model, in which one of the three methods is solving an HJBI equation. Moreover, various versions of a zero-sum stochastic differential game can be found in Browne [10] for optimal investment problems between two investors. Ma, Wu and Lin [39] provided explicit solutions of a nonzero-sum stochastic differential game of optimal portfolio for a regime-switching diffusion model.

First, we follow the methods of DPP under a Markov regime-switching jump-diffusion model. We investigate the solutions of a zero-sum game, i.e., a min-max problem between an investor and the market. This zero-sum game application is an extension of the Theorem 6.3 in Peskir and Shorish [50] and Example 4.1 in Mataramvura and Øksendal [41] by including the states of the Markov chain as the proxies of the different observable macroeconomic indicators. We solve corresponding HJBI equations, whose solutions are compatible with the fundamentals of the arbitrage-free pricing theory. We get best responses of each side, i.e., the solutions of the optimal control problem. Furthermore, we construct a nonzero-sum game between two investors and give explicit solutions for each investor's optimal investment proportions. We establish HJBI equations and utilize Feyman-Kac formula to provide a stochastic representation of the value function in terms of conditional expectation.

On the other hand, it is well-known that the Markov property is a corner stone of the DPP setting. But in the real world, investors tend to look at the historical performance of risky assets. This leads us to consider the time delay in the model, which may represent the memory in the dynamics of the system or the inertia in the financial market. As a consequence of involving memory, we lose Markov property of the state process. However, DPP does not allow to work without Markov property, we do not need to create any Markovian setting for MP. Therefore, in the second step we work on an optimal consumption problem, whose state process is given by a stochastic differential delay equation with jumps and regimes in MP setting.

A comprehensive treatment of the theory of the stochastic differential delay equations (SDDEs) can be found in the monograph by Mohammed [43]. The results of the modern theory of regime-switching models with delay are presented in the monograph by Mao and Yuan [40]. Moreover, optimal control of the SDDEs have already been studied by various authors; see, e.g., Øksendal and Sulem [46], Larssen [37] and Elsanosi, Øksendal and Sulem [22] and references therein.

It is well-known that MP brings together the adjoint equations represented by Backward Stochastic Differential Equations (BSDEs) for the solution of optimal control problems (cf., e.g., [14, 18]). Here, in this new setting, one needs a novel form of BSDEs which is called as Anticipated (Time-advanced) BSDEs and it was first developed by Peng and Yang [49]. A stochastic maximum principle of a forward-backward delayed regime-switching diffusion model has been given by Lv, Tao and Wu [38]. Øksendal, Sulem and Zhang [47] and Tu and Hao [58] extended the existence-uniqueness results of ABSDEs for jump-diffusion models. Our work provides the first extension of the stochastic maximum principle for a Markov regime-switching jump-diffusion model with delay (SDDEJR) and the existence-uniqueness theorem of an ABSDE with jumps and regimes.

This thesis is organized as follows: In Chapter 2, we present a literature overview related to our research and introduce the stochastic dynamics of our model. In Chapter 3, we construct two stochastic differential games with regime-switches, but without delay based on the tools of the DPP and give explicit solutions of the corresponding optimal control problems. In Chapter 4, we develop the main mathematical results, which support and generate the underlying theory of our proposed model with delay. In Chapter 5, we work on an optimal consumption problem whose state process is a Markovian regime switching jump-diffusion model with delay. Chapter 6 is devoted to a conclusion and outlook on future research, as in our financial and economic world of crises and disruptions, further exciting research and applications wait to be done. In Appendix, main remarks are introduced.

CHAPTER 2

A MARKOV REGIME-SWITCHING JUMP-DIFFUSION MODEL

2.1 Literature Review

Regime-switching models first arises in Quandt [51], who derives a method to estimate the parameters of a linear regression system with two different regimes. The original purpose of this work is to determine the position of a single switch in time. Hamilton [30] followed Goldfeld and Quandt's [27] Markov regime switching regression and investigated whether the growth rate of postwar U.S. real GNP depends on these discrete shifts. After this investigation discovered that the business cycle between a recessionary state and a growth state is better denoted by such discrete components, regime-switching models become more popular in financial applications.

Far from econometric approaches, recently, many authors worked on regime-switching models in a diversified field of research, see Chapter 1. In this thesis, we develop new results for stochastic optimal control problems represented by a Markovian regime-switching jump-diffusion model with and without delayed dynamics. We follow both of the theories of DPP without a delayed structure and MP with a time-advanced model. In our model, we illustrate two main risks that an investor faces by the dynamics of our model. The Brownian motion describes the random shock of stock prices and the Poisson random measure interprets larger price fluctuations of the stock as a consequence of sudden changes in the market. On the other hand, the Markov chain represents the uncertainty of the economic modes.

First, the technique of dynamic programing was developed by Bellman [4, 5, 6] in the 1950's by providing several examples of calculus of variations and stochastic optimal control. Moreover, in his work, he presented the intuition behind the theory of DPP, which is now known as Bellman's optimality principle.

Definition 2.1. **(Bellman** [4]) An optimal policy has the property that whatever the initial state and initial decisions are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decisions.

From the perspective of stochastic optimal control, the main problem is to maximize or minimize an objective functional under some technical conditions and find a value function which is the solution of a partial differential equation, called as Hamilton-Jacobi-Bellman (HJB) equation. The Bellman's principle of optimality, i.e., the underlying intuition of DPP, makes the Markov property of the state dynamics and the control processes a key point of this method. Later, this technique was enhanced and its applications were illustrated by several authors. For the comprehensive monographs of this theory, see Øksendal and Sulem [45], Fleming and Soner [23] and Yong and Zhou [61].

Furthermore, the technique of dynamic programming took its place in game theory by providing a dynamic equilibrium as a solution of Hamilton-Jacobi-Bellman-Isaacs (HJBI) equation; see [21, 39, 41, 57]. It is clear that in the financial market, no investor can determine his/her outcome without taking into account other investors' actions, information and expectations. Hence, the analysis of all these strategic interactions lead researchers to the tools of game theory.

John von Neumann and Oscar Morgenstern [59] presented the foundations of game theory in their seminal work, *The Theory of Games and Economic Behavior*. In the last thirty years, game theory has been accepted universally to explain strategic interactions in economics, behavioral and social sciences. The existence of an optimal strategy against others', called as Nash equilibrium, was centered in modern economics as the main solution concept for non-cooperative game theory and has brought the Nobel Prize of Economics to John Nash, John Harsanyi and Reinhard Selten in 1994. Later on, the idea developed and proved by Nash in his PhD thesis refined by Harsanyi [31] and Selten [53]. Furthermore, Isaacs [33] gave the mathematical and theoretical foundations to the differential game theory, due to which counterpart of DPP is also called with Isaacs's name.

Later on, several authors focused on Stochastic Game Theory; see Hamadène and Hassani [29], El-Karoui and Hamadène [16], Karatzas and Li [35] and Shapley [54] and the references therein.

In this thesis, we apply the extensions of the verification theorems of zero-sum and nonzero-sum stochastic differential games for a Markov regime-switching jump diffusion model by the method of dynamic programming. Some of the works with similar approach are Elliott and Siu [21], Bensoussan, Siu, Chi, Yamd and Yang [8], Shen and Siu [57], Ma, Wu and Lin [39].

On the other hand, by paying the price of violating Markov property, more realistic models can be obtained. In fact, this corresponds to the systems of Stochastic Differential Delay Equations (SDDEs). The first existence-uniqueness result of time delayed diffusion processes was driven by Itô and Nisio [34] and Kushner [36] followed them. In his book, Mohammed [43] introduced and provided a very detailed theory of SD-

DEs. At first sight, the difference for the models with and without delay component begins with the method of obtaining a solution of the system. Let us explain it for a diffusion process as in Mohammed [43].

Let $(\Omega, \mathbb{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a probability space satisfying that $(\mathcal{F}_t)_{t\geq 0}$ is a right-continuous filtration and for each $t \geq 0$, $\mathcal{F}_t$ contains all $\mathbb{P}$-null sets in $\mathbb{F}$. Let us define:

$$dX(t) = \sigma X(t-\delta)dW(t), \qquad t \geq 0. \tag{2.1}$$

In Equation (2.1), if $\delta = 0$, then the closed-form solution is:

$$X(t) = X(0)e^{\sigma W(t) - \sigma^2/2}, \qquad t \geq 0.$$

If we assume $\delta > 0$, we need an initial path $\theta(\cdot)$ to solve Equation (2.1) such that

$$X(t) = \theta(t), \qquad -\delta \leq t \leq 0.$$

Then, by recursive Itô integrations over steps of length δ, we observe that there is no closed form solution:

$$\begin{aligned}
&X(t) = \theta(0) + \sigma \int_0^t \theta(u-\delta)dW(u), \qquad 0 \leq t \leq \delta, \\
&X(t) = X(r) + \sigma \int_r^t \left[\theta(0) + \sigma \int_0^{v-\delta} \theta(u-\delta)dW(u)\right] dW(v), \qquad \delta < t \leq 2\delta, \\
&\quad \ldots = \ldots, \qquad 2\delta < t \leq 3\delta, \\
&\quad \vdots
\end{aligned}$$

Here, the solution process $\{X(t) : t \geq -\delta\}$ is still an $\mathcal{F}_t$-martingale but it is not Markovian any more.

Let us define the segment $X_t : [-\delta, 0] \to \mathbb{R}^n$ by

$$X_t(s) = X(t+s) \qquad \text{a.s.} \qquad t \geq 0, \ s \in [-\delta, 0].$$

Then, a general representation for SDDE is as follows:

$$\begin{aligned}
&dX(t) = h(t, X_t)dt + g(t, X_t)dW(t), \qquad t \geq 0, \\
&X_0 = \theta(t), \qquad t \in [-\delta, 0],
\end{aligned}$$

where the initial path $\theta(\cdot) \in C([-\delta, 0], \mathbb{R}^n)$ is an $\mathcal{F}_0$-measurable process.

However, including memory into the dynamics of the system makes us closer to the real-life events, especially from the perspective of finance and economics, the tools of DPP become much more complicated in a delay setting (see [22, 37]). Therefore losing the Markov property leads us to the stochastic MP for which there is not any Markovian assumption. Another main difference between the DPP, introduced by the

American school, and the MP, introduced by the Russian school, is that the DPP depends on the time origin and the state variable at that time origin.

The stochastic MP is a stochastic extension of Pontryagin's maximum principle, which has been a method for the optimal control of a deterministic dynamical system. As in the deterministic case, the stochastic MP introduces adjoint processes called as Backward Stochastic Differential Equations (BSDEs). The first appearance of BSDEs is in Bismut [9], where the author developed the adjoint processes for the stochastic version of the conjugate variable in Pontryagin's maximum principle. Pardoux and Peng [48] provided a systematic work of BSDEs and their connection to financial mathematics was realized very quickly (see El-Karoui et al. [18]).

Let us present the first main result given related to BSDEs, namely the existence-uniqueness result of El-Karoui, Peng and Quenez [18], which may give us a first sight into our works of the following chapter.

Actually, we define a BSDE as follows:

$$-dY(t) = f(t, Y(t), Z(t))dt - Z(t)dW(t), \qquad Y(T) = \xi,$$

or, equivalently,

$$Y(t) = \xi + \int_t^T f(s, Y(s), Z(s))ds - \int_t^T Z_s dW(s),$$

where ξ is an $\mathcal{F}_T$-measurable random variable.

A solution is a pair (Y, Z) such that $\{Y(t) : t \in [0, T]\}$ is a continuous $\mathbb{R}^d$-valued adapted process and $\{Z(t) : t \in [0, T]\}$ is an $\mathbb{R}^{n \times d}$-valued process satisfying the condition $\int_0^T |Z(s)|^2 ds < \infty$, $\mathbb{P}$-a.s.

The theory of BSDEs has attracted several authors by its fruitful nature from both of the perspectives of theory and applications. Here, we refer to Hamadène [28], Crépey and Matoussi [14], Cohen and Elliott [12].

The method of stochastic MP states that an optimal control maximizes a functional, called the *Hamiltonian*, and satisfies the *optimality system*, described by a system of forward-backward stochastic differential equations, i.e., a Necessary Maximum Principle can be established. Moreover, the reverse can also be defined as the Sufficient Maximum Principle. If the non-Markovian nature of this method is taken into account, the counterpart of the PDEs in DPP can be viewed as BSDEs in MP. Cadenillas and Karatzas [11] provided the first use of stochastic MP. Later on, several authors worked on it and gave valuable applications [38, 46, 47, 56, 64].

By the advantage of its non-Markovian structure, we consider a delayed component for our stochastic optimal control problem (see Chapter 5). In this set-up, the corresponding adjoint equations appear in their new forms, called as Anticipated (time-advanced) BSDEs (ABSDEs). This type of equations was developed originally by Peng and Yang [49] in a diffusion setting. In their pioneering study, they also constructed the duality between SDDEs and ABSDEs. Besides, they provided the new existence-uniqueness result and several main theorems related to ABSDE theory, such as the comparison theorem.

Peng and Yang [49] introduced this new form of BSDEs in 2009 as follows:

$$\begin{aligned} -dY(t) &= f(t, Y(t), Z(t), Y(t+\delta_1(t)), Z(t+\delta_2(t)))ds - Z(t)dW(t), \ t \in [0,T], \\ Y(t) &= \xi(t) \text{ and } Z(t) = \psi(t), \qquad t \in [T, T+K], \end{aligned}$$

where $\delta_i(\cdot)$, $i = 1, 2$, be $\mathbb{R}^+$-valued continuous functions on $[0, T]$.

Under some technical conditions, Peng and Yang [49] adopted the existence-uniqueness theorem of a BSDE to this new model.

In the sequel, the existence-uniqueness theorem was extended to a jump-diffusion setting; for closer information see [47, 58]. The duality result of Peng and Yang [49], which was presented by a diffusion model was reorganized by Tu and Hao [58] for a jump-diffusion setting. In our thesis, we extend the duality and the existence-uniqueness theorem of Peng and Yang [49] for an ABSDE with jumps and regimes. These theorems also prepare and support the underlying theory of our optimal control problem which is solved by the tools of MP for a time-delayed state process.

Some recent approaches and applications related to the MP can be found in Meyer-Brandis, Øksendal and Zhou [42], Lv, Tao and Wu [38], Shen and Siu [56], Shen, Meng, Shi [55], Zhang, Elliott and Siu [64].

2.2 Preliminaries for a Markov Regime-Switching Jump-Diffusion Model

Throughout the thesis, we work with a finite time horizon $T > 0$, which represents the maturity time. Let

$$(N(dt, dz) : t \in [0, T], z \in \mathbb{R}_0)$$

be a Poisson random measure on $([0, T] \times \mathbb{R}_0, \mathcal{B}([0, T]) \otimes \mathcal{B}_0)$, where $\mathbb{R}_0 := \mathbb{R} \setminus \{0\}$ and $\mathcal{B}_0$ is the Borel σ-field generated by open subset O of $\mathbb{R}_0$, whose closure does not contain the point 0.

Let $\tilde{N}(dt, dz) := N(dt, dz) - \nu(dz)dt$ be the compensated Poisson random measure, where ν is the Lévy measure of the jump measure $N(\cdot, \cdot)$ such that

$$\nu(\{0\}) = 0 \qquad \text{and} \qquad \int_{\mathbb{R}} (1 \wedge |z|^2)\nu(dz) < \infty.$$

Lévy measure describes the number of the jumps of a certain height in a time interval of length 1.

Furthermore, let $(W(t) : t \in [0, T])$ be a Brownian motion and $(\alpha(t) : t \in [0, T])$ be a continuous-time, finite-state and observable Markov chain. Let $(\Omega, \mathbb{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a complete probability space, where $\mathbb{F} = (\mathcal{F}_t : t \in [0, T])$. Furthermore, $(\mathcal{F}_t)_{t\geq 0}$ is a right-continuous, $\mathbb{P}$-completed filtration generated by the Brownian motion $W(\cdot)$, the Poisson random measure $N(\cdot, \cdot)$ and the Markov chain $\alpha(\cdot)$. We assume that these processes are independent of each other and adapted to $\mathbb{F}$.

The finite-state space of the Markov chain $\alpha(t)$, $S = \{e_1, e_2, ..., e_D\}$, is called a canonical state space, where $D \in \mathbb{N}$, $e_i \in \mathbb{R}^D$ and the jth component of e_i is the Kronecker delta δ_{ij} for each pair of $i, j = 1, 2, ..., D$ (for more details, see Example 2.6.17 by Aggoun and Elliott [1]). We suppose that the chain is homogenous and irreducible. The generator of the chain under $\mathbb{P}$ is defined by $\Lambda := [\lambda_{ij}]_{i,j=1,2,...,D}$. For each $i, j = 1, 2, ..., D$, λ_{ij} is the constant transition intensity of the chain from each state e_i to state e_j at time t. For $i \neq j$, $\lambda_{ij} \geq 0$ and $\sum_{j=1}^{D} \lambda_{ij} = 0$; hence, $\lambda_{ii} \leq 0$. We suppose that for each $i, j = 1, 2, ..., D$, with $i \neq j$, $\lambda_{ij} > 0$ and $\lambda_{ii} < 0$.

Elliott, Aggoun and Moore [19] obtained the following semimartingale representation for the chain α:

$$\alpha(t) = \alpha(0) + \int_0^t \Lambda^T \alpha(u)du + M(t),$$

where $(M(t) : t \in [0, T])$ is an $\mathbb{R}^D$-valued $(\mathbb{F}, \mathbb{P})$-martingale (see Lemma 2.6.18 by Aggoun and Elliott [1]) and Λ^T represents the transpose of the matrix.

Let us introduce a set of Markov jump martingales associated with the chain α.

For each $i, j = 1, 2, ..., D$, with $i \neq j$ and $t \in [0, T]$, let $J^{ij}(t)$ be the number of the jumps from state e_i to state e_j up to time t. Then,

$$J^{ij}(t) := \sum_{0<s\leq t} \langle \alpha(s-), e_i \rangle \langle \alpha(s), e_j \rangle$$

$$
\begin{aligned}
&= \sum_{0<s\leq t} \langle \alpha(s-), e_i \rangle \langle \alpha(s) - \alpha(s-), e_j \rangle \\
&= \int_0^t \langle \alpha(s-), e_i \rangle \langle d\alpha(s), e_j \rangle \\
&= \int_0^t \langle \alpha(s-), e_i \rangle \left\langle \Lambda^T \alpha(s), e_i \right\rangle ds + \int_0^t \langle \alpha(s-), e_i \rangle \langle dM(s), e_j \rangle \\
&= \lambda_{ij} \int_0^t \langle \alpha(s-), e_i \rangle \, ds + m_{ij}(t),
\end{aligned}
$$

where the processes m_{ij} are $(\mathbb{F}, \mathbb{P})$-martingales and called as the basic martingales associated with the chain α. For each fixed $j = 1, 2, ..., D$, let Φ_j be the number of the jumps into state e_j up to time t. Then,

$$
\begin{aligned}
\Phi_j(t) &:= \sum_{i=1, i\neq j}^{D} J^{ij}(t) \\
&= \sum_{i=1, i\neq j}^{D} \lambda_{ij} \int_0^t \langle \alpha(s-), e_i \rangle \, ds + \tilde{\Phi}_j(t).
\end{aligned}
$$

Let us define $\tilde{\Phi}_j(t) := \sum_{i=1, i\neq j}^{D} m_{ij}(t)$ and $\lambda_j(t) := \sum_{i=1, i\neq j}^{D} \lambda_{ij} \int_0^t \langle \alpha(s-), e_i \rangle \, ds$; then for each $j = 1, 2, ..., D$,

$$
\tilde{\Phi}_j(t) = \Phi_j(t) - \lambda_j(t)
$$

is an $(\mathbb{F}, \mathbb{P})$-martingale. Let $\tilde{\Phi}(t) = (\tilde{\Phi}_1(t), \tilde{\Phi}_2(t), ..., \tilde{\Phi}_D(t))^T$ represent an integer-valued random measure on $([0, T] \times S, \mathcal{B}([0, T]) \otimes \mathcal{B}_S)$, where $\mathcal{B}_S$ is a σ-field of S. Let $\mathcal{P}$ be a predictable sigma field on $\Omega \times [0, T]$.

For the rest of this thesis, we utilize this integer-valued random measure Φ which is generated by the Markov chain.

In Chapter 3, we construct a stochastic game and present our results for the following Markov regime-switching jump-diffusion model without delay component:

$$
\begin{cases}
X(t) = b(t, X(t), \alpha(t))dt + \sigma(t, X(t), \alpha(t))dW(t) \\
\qquad + \int_{\mathbb{R}_0} \eta(t, X(t-), \alpha(t-), z) \tilde{N}(dt, dz) + \gamma(t, X(t-), \alpha(t-)) d\tilde{\Phi}(t), \ t \in [0, T], \\
X(0) = x_0 \in \mathbb{R}^N,
\end{cases}
$$

On the other hand, in Chapters 4 and 5, we provide the main mathematical results and the extension of the MP by the following delayed state process:

$$\begin{cases} X(t) = b(t, X(t), Y(t), A(t), \alpha(t))dt \\ \qquad\quad +\sigma(t, X(t), Y(t), A(t), \alpha(t))dW(t) \\ \qquad\quad +\int_{\mathbb{R}_0} \eta(t, X(t-), Y(t-), A(t-), \alpha(t-), z)\tilde{N}(dt, dz) \\ \qquad\quad +\gamma(t, X(t-), Y(t-), A(t-), \alpha(t-))d\tilde{\Phi}(t), \qquad t \in [0, T], \\ X(t) = x_0(t), \qquad t \in [-\delta, 0], \end{cases}$$

where

$$Y(t) = X(t-\delta) \qquad \text{and} \qquad A(t) = \int_{t-\delta}^{t} e^{-\rho(t-r)} X(r)dr, \qquad t \in [0, T].$$

Here, x_0 is a continuous, deterministic function, $\rho \geq 0$ is a constant averaging parameter and $\delta > 0$ is a constant delay.

Let us introduce the following Banach spaces of measurable and integrable random variables and processes:

$L^2(\mathcal{F}_T; \mathbb{R}) = \{\mathbb{R}$-valued, $\mathcal{F}_T$-measurable random variable ϕ such that $E[|\phi|^2] < \infty\}$,

$L^2(\mathcal{B}_0; \mathbb{R}) = \{\mathbb{R}$-valued, $\mathcal{B}_0$-measurable random variable ϕ such that
$\|\phi\|_J^2 = \int_{\mathbb{R}_0} |\phi(z)|^2 \nu(dz) < \infty\}$,

$L^2(\mathcal{B}_S; \mathbb{R}^D) = \{\mathbb{R}^D$-valued, $\mathcal{B}_S$-measurable random variable ϕ such that
$\|\phi\|_S^2 = \sum_{j=1}^{D} |\phi^j|^2 \lambda_j(t) < \infty,\ j = 1, 2, ..., D\}$,

$L^2(\mathcal{F}_T \times \mathcal{B}_0; \mathbb{R}) = \{\mathbb{R}$-valued, $\mathcal{F}_T \times \mathcal{B}_0$-measurable random variable ϕ such that
$E[\int_{\mathbb{R}_0} |\phi(z)|^2 \nu(dz)] < \infty\}$,

$L^2(\mathcal{F}_T \times \mathcal{B}_S; \mathbb{R}^D) = \{\mathbb{R}^D$-valued, $\mathcal{F}_T \times \mathcal{B}_S$-measurable random variable ϕ such that
$E[\sum_{j=1}^{D} |\phi^j|^2 \lambda_j(t)] < \infty,\ j = 1, 2, ..., D\}$,

$L^2_{\mathbb{F}}(0, T; \mathbb{R}) = \{\mathbb{R}$-valued, $\mathcal{F}_t$-adapted stochastic process ϕ such that
$E[\int_0^T |\phi(t)|^2 dt] < \infty\}$,

$S^2_{\mathbb{F}}(0, T; \mathbb{R})$={càdlàg process ϕ in $L^2_{\mathbb{F}}(0, T; \mathbb{R})$ such that $E[\sup_{t \in [0,T]} |\phi(t)|^2] < \infty\}$,

$\mathcal{H}^2_{\mathbb{F}}(0,T;\mathbb{R}) = \{\mathbb{R}$-valued, $\mathcal{P} \otimes \mathcal{B}_0$-measurable stochastic process ϕ such that $\|\phi(t)\|^2_{\mathcal{H}^2} = E[\int_0^T \|\phi(t)\|^2_J \, dt] < \infty\}$,

$\mathcal{M}^2_{\mathbb{F}}(0,T;\mathbb{R}^D) = \{\mathbb{R}^D$-valued, $\mathcal{P} \otimes \mathcal{B}_S-$measurable stochastic process ϕ such that $\|\phi(t)\|^2_{\mathcal{M}^2} = E[\int_0^T \|\phi(t)\|^2_S \, dt] < \infty\}$.

CHAPTER 3

DYNAMIC PROGRAMMING PRINCIPLE APPROACH IN GAME THEORY

In this chapter, we work on zero-sum and nonzero-sum stochastic differential games under Markov regime-switching jump-diffusion model and address their applications to finance. Dynamic programming principle approach is employed for two portfolio games between the market and a trader (zero-sum game) and between two traders (nonzero-sum game) in Sections 3.2 and 3.3, respectively. However, we prefer to follow the methods in Mataramvura and Øksendal [41], in fact, our model is different and more realistic by the additional Markov chain structure not only from mathematical but also from financial perspectives.

3.1 Preliminaries

Let $(\Omega, \mathbb{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a complete probability space, where $\mathbb{F} = (\mathcal{F}_t : t \in [0,T])$. Furthermore, $(\mathcal{F}_t)_{t\geq 0}$ is a right-continuous, $\mathbb{P}$-completed filtration generated by an N-dimensional Brownian motion $W(\cdot)$, an M-dimensional Poisson random measure $N(\cdot,\cdot)$ and a D-dimensional Markov chain $\alpha(\cdot)$. We assume that these processes are independent of each other and adapted to $\mathbb{F}$.

Let us represent our model:

$$\begin{cases} Y(t) = b(t, Y(t), \alpha(t), u_1(t), u_2(t))dt \\ \qquad +\sigma(t, Y(t), \alpha(t), u_1(t), u_2(t))dW(t) \\ \qquad + \displaystyle\int_{\mathbb{R}_0} \eta(t, Y(t-), \alpha(t-), u_1(t-), u_2(t-), z)\tilde{N}(dt, dz) \\ \qquad +\gamma(t, Y(t-), \alpha(t-), u_1(t-), u_2(t-))d\tilde{\Phi}(t), \qquad t \in [0,T], \\ Y(0) = y_0 \in \mathbb{R}^N, \end{cases} \tag{3.1}$$

where $\mathcal{U}_1$ and $\mathcal{U}_2$ are non-empty subsets of $\mathbb{R}^N$. Here,

$$b : [0,T] \times \mathbb{R}^N \times S \times \mathcal{U}_1 \times \mathcal{U}_2 \to \mathbb{R}^N,$$

$$\sigma : [0,T] \times \mathbb{R}^N \times S \times \mathcal{U}_1 \times \mathcal{U}_2 \to \mathbb{R}^{N\times M},$$
$$\eta : [0,T] \times \mathbb{R}^N \times S \times \mathcal{U}_1 \times \mathcal{U}_2 \times \mathbb{R}_0 \to \mathbb{R}^{N\times L},$$
$$\gamma : [0,T] \times \mathbb{R}^N \times S \times \mathcal{U}_1 \times \mathcal{U}_2 \to \mathbb{R}^{N\times D}$$

are given functions such that

$$\int_0^T \left\{ |b(t)| + |\sigma(t)|^2 + \int_{\mathbb{R}_0} |\eta(t,z)|^2 \nu(dz) + \sum_{j=1}^{D} \left|\gamma^j(t)\right|^2 \lambda_j(t) \right\} dt < \infty.$$

Let

$$\tau_G = \inf \{t > 0, Y(t) \notin G\}$$

be the bankruptcy time, where $G \subset \mathbb{R}^N$ is an open set and represents the solvency region.

Furthermore, let $f : [0,T] \times \mathbb{R}^N \times S \times \mathcal{U}_1 \times \mathcal{U}_2 \to \mathbb{R}$ and $g : \mathbb{R}^N \times S \to \mathbb{R}$ be given functions, called as *profit rate* and *terminal gain*, respectively.

Herewith, the performance (objective) functional is defined as follows:

$$J^{u_1,u_2}(t,y,e_i) = E\left[\int_t^{\tau_G} f(s,Y(s),\alpha(s),u_1(s),u_2(s))ds + g(Y(\tau_G),\alpha(\tau_G))\right].$$

We know that under some mild conditions (see Theorem 11.2.3, Øksendal [44]), Markov controls provide as good performance as the more general adapted controls. Hence, we assume that Θ_1 and Θ_2 are given families of admissible Markov control processes $u_1 \in \mathcal{U}_1$ and $u_2 \in \mathcal{U}_2$, respectively.

We say that $\mathcal{U}_1 \times \mathcal{U}_2$-valued, $\mathcal{F}_t$-measurable cádlág control process (u_1, u_2) are admissible, if the following conditions are satisfied:

1. There exists a unique strong solution of the state process $Y(t)$ introduced in Equation (3.1) (see Proposition 7.1 by Crépey [13] or Appendix for an existence-uniqueness theorem of such a system).
2. $E\left[\int_0^{\tau_G} |f(t,Y(t),\alpha(t),u_1(t),u_2(t))|dt + |g(Y(\tau_G),\alpha(\tau_G))|\right] < \infty.$

For any $\phi(\cdot,\cdot,e_i) \in C^{1,2}(G) \cap C(\bar{G})$, let us define the infinitesimal generator $\mathcal{L}^{u_1,u_2}$ for the system (3.1) as in Zhang, Elliott and Siu [64]:

$$\begin{aligned}\mathcal{L}^{u_1,u_2}\left[\phi(t,y,e_i)\right] =& \frac{\partial \phi}{\partial t}(t,y,e_i) + \sum_{k=1}^{N} \frac{\partial \phi}{\partial y_k}(t,y,e_i) b_k(t,y,e_i,u_1,u_2) \\ &+ \frac{1}{2}\sum_{k=1}^{N}\sum_{n=1}^{N} \frac{\partial^2 \phi}{\partial y_k \partial y_n}(t,y,e_i) \sum_{l=1}^{M} \sigma_{kl}\sigma_{nl}(t,y,e_i,u_1,u_2)\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{m=1}^{L} \int_{\mathbb{R}_0} \Bigg[\phi(t, y + \eta^{(m)}(t, y, e_i, u_1, u_2, z), e_i) - \phi(t, y, e_i) \\
&- \sum_{n=1}^{N} \frac{\partial \phi}{\partial y_n}(t, y, e_i) \eta_{nm}(t, y, e_i, u_1, u_2, z) \Bigg] \nu_m(dz) \\
&+ \sum_{j=1}^{D} \lambda_{ij} \Bigg[\phi(t, y + \gamma^{(j)}(t, y, e_i, u_1, u_2), e_j) - \phi(t, y, e_i) \\
&- \sum_{n=1}^{N} \frac{\partial \phi}{\partial y_n}(t, y, e_i) \gamma_{nj}(t, y, e_i, u_1, u_2) \Bigg]. \qquad (3.2)
\end{aligned}
$$

Furthermore, we give the extension of Dynkin formula (see Theorem 1.24, Øksendal and Sulem [45]), by which we establish verification theorems for stochastic differential games in Sections 3.2 and 3.3.

Lemma 3.1. *Let $Y(\cdot) \in \mathbb{R}^N$ be a Markov regime-switching jump-diffusion process, G be an open subset of $\mathbb{R}^N$ and for all $e_i \in S$, $\phi(\cdot, \cdot, e_i) \in C^{1,2}(G) \cap C(\bar{G})$. Let $\tau < \infty$ be a stopping time and $\tau \leq \tau_G := \inf\{t > 0, Y(t) \notin G\}$ and $Y(\tau) \in G$ a.s. for all $e_i \in S$. Moreover,*

$$
\begin{aligned}
&E^{t,y,e_i} \Bigg[|\phi(\tau, Y(\tau), \alpha(\tau))| \\
&+ \int_t^{\tau} \Bigg\{ \mathcal{L}[\phi(s, Y(s-), \alpha(s-))] + \left| \sigma^T(s, Y(s-), \alpha(s-)) \nabla \phi(s, Y(s-), \alpha(s-)) \right|^2 \\
&+ \sum_{k=1}^{L} \int_{\mathbb{R}_0} \Bigg(\phi(s, Y(s-) + \eta^{(k)}(s, Y(s-), \alpha(s-), z), \alpha(s-)) \\
&\qquad - \phi(s, Y(s-), \alpha(s-)) \Bigg)^2 \nu_k(dz_k) \\
&+ \sum_{j=1}^{D} \Bigg(\phi(s, Y(s-) + \gamma^{(j)}(s, Y(s-), \alpha(s-)), e_j) \\
&\qquad - \phi(s, Y(s-), \alpha(s-)) \Bigg)^2 \lambda_j(s) \Bigg\} ds \Bigg] < \infty. \qquad (3.3)
\end{aligned}
$$

Then, we have

$$
E^{t,y,e_i}\left[\phi(\tau, Y(\tau), \alpha(\tau))\right] = \phi(t, y, e_i) + E^{t,y,e_i}\left[\int_t^{\tau} \mathcal{L}\left[\phi(s, Y(s-), \alpha(s-))\right] ds \right]
$$

for each $e_i \in S$.

Proof. Let us apply generalized Itô's differentiation rule on $\phi(s, Y(s), \alpha(s))$ (see Appendix or Theorem 4.1 by Zhang, Elliott and Siu [64]):

$$
\begin{aligned}
\phi(\tau, Y(\tau), \alpha(\tau)) &= \phi(t, y, e_i) + \int_t^\tau \mathcal{L}\left[\phi(s, Y(s-), \alpha(s-))\right] ds \\
&+ \int_t^\tau \sum_{k=1}^{N} \frac{\partial \phi}{\partial y_k}(s, Y(s-), \alpha(s-)) \sum_{n=1}^{M} \sigma_{kn}(s, Y(s-), \alpha(s-)) dW(t) \\
&+ \int_t^\tau \sum_{m=1}^{L} \int_{\mathbb{R}_0} \Big(\phi(s, Y(s-) + \eta^{(m)}(s, Y(s-), \alpha(s-), z), \alpha(s-)) \\
&- \phi(s, Y(s-), \alpha(s-)) \Big) \tilde{N}(ds, dz) \\
&+ \int_t^\tau \sum_{j=1}^{D} \Big(\phi(s, Y(s-) + \gamma^{(j)}(s, Y(s-), \alpha(s-)), e_j) \\
&- \phi(s, Y(s-), \alpha(s-)) \Big) d\tilde{\Phi}_j(s), \qquad (3.4)
\end{aligned}
$$

where $\eta^{(m)}$ and $\gamma^{(j)}$ represents the mth and jth columns of the matrices η and γ, respectively.

Then, by conditioning on Equation (3.1), $Y(t) = y$ and $\alpha(t) = e_i$ for each $e_i \in S$ under $\mathbb{P}$, we obtain

$$
E^{t,y,e_i}\left[\phi(\tau, Y(\tau), \alpha(\tau))\right] = \phi(t, y, e_i) + E^{t,y,e_i}\left[\int_t^\tau \mathcal{L}(\phi(s, Y(s-), \alpha(s-)))ds\right].
$$

Note that by condition (3.3), other stochastic integrals in (3.4) are martingales with null expectation. □

In the following sections, we work under the assumptions of this section.

3.2 A Zero-Sum Stochastic Differential Game and an Application to Finance

In this subsection, first we introduce a zero-sum game within the framework of the dynamic programming principle and give the verification theorem in a general setting. Then, we present an application to finance, herewith demonstrating our results by their way of working.

In a zero-sum stochastic differential game problem, we search the value function $V(t, y, e_i)$ and the optimal control process (the saddle point of the game) $(u_1^*, u_2^*) \in \Theta_1 \times \Theta_2$, if they exists such that

$$
V(t, y, e_i) = \sup_{u_1 \in \Theta_1} \left(\inf_{u_2 \in \Theta_2} J^{u_1,u_2}(t, y, e_i) \right) = J^{u_1^*,u_2^*}(t, y, e_i)
$$

for all $e_i \in S$.

Let $\mathcal{T}$ be the set of all $\mathcal{F}_t$-stopping times $\tau \leq \tau_G$.

Now we can present an HJBI equation for a Markov regime-switching jump-diffusion model, i.e., the following verification theorem for a zero-sum stochastic differential game.

Theorem 3.2. *Suppose that there exists a function $\phi \in C^{1,2}(G) \cap C(\bar{G})$ and a Markov control $(u_1^*, u_2^*) \in \Theta_1 \times \Theta_2$ such that*

i. $\mathcal{L}^{u_1,u_2^*}[\phi(t,y,e_i)] + f(t,y,e_i,u_1,u_2^*) \leq 0$ *for all* $u_1 \in \mathcal{U}_1,\ y \in G$ *and* $e_i \in S$.

ii. $\mathcal{L}^{u_1^*,u_2}[\phi(t,y,e_i)] + f(t,y,e_i,u_1^*,u_2) \geq 0$ *for all* $u_2 \in \mathcal{U}_2,\ y \in G$ *and* $e_i \in S$.

iii. $\mathcal{L}^{u_1^*,u_2^*}[\phi(t,y,e_i)] + f(t,y,e_i,u_1^*,u_2^*) = 0$ *for all* $y \in G$ *and* $e_i \in S$.

iv. $Y^{u_1,u_2}(\tau_G) \in \partial G$ *a.s. on* $\{\tau_G < \infty\}$ *and*
$\lim_{t \to \tau_G^-} \phi(t, Y^{u_1,u_2}(t), \alpha(t)) = g(Y^{u_1,u_2}(\tau_G), \alpha(\tau_G)) 1_{\{\tau_G < \infty\}}$ *a.s.*
for all $(u_1,u_2) \in \Theta_1 \times \Theta_2,\ y \in G$.

v. *The family* $\{\phi(\tau, Y^{u_1,u_2}(\tau), \alpha(\tau))\}_{\{\tau \in \mathcal{T}\}}$ *is uniformly integrable for all* $y \in G$ *and* $(u_1,u_2) \in \Theta_1 \times \Theta_2$.

Then,

$$
\begin{aligned}
\phi(t,y,e_i) =& V(t,y,e_i) = \sup_{u_1 \in \Theta_1} \left(\inf_{u_2 \in \Theta_2} J^{u_1,u_2}(t,y,e_i) \right) \\
=& \inf_{u_2 \in \Theta_2} \left(\sup_{u_1 \in \Theta_1} J^{u_1,u_2}(t,y,e_i) \right) \\
=& \sup_{u_1 \in \Theta_1} J^{u_1,u_2^*}(t,y,e_i) = \inf_{u_2 \in \Theta_2} J^{u_1^*,u_2}(t,y,e_i), \\
=& J^{u_1^*,u_2^*}(t,y,e_i), \qquad y \in G,\ e_i \in S
\end{aligned}
$$

and (u_1^*, u_2^*) is a saddle point (a Markovian optimal control) of the zero-sum stochastic differential game.

Proof. The demonstration of Theorem 3.2 can be obtained similarly as the proof of Theorem 3.2 in Mataramvura and Øksendal [41] by applying Lemma 3.1 above. Therefore, we do not repeat it here. □

Now we can construct our zero-sum game between the market and the representative agent. We assume that there is no transaction cost, but infinite divisible assets are allowed and information is symmetric; in other words, standard assumptions of a financial market hold. In this context, the mean rate of the risky asset $\theta(\cdot)$ is not given a priori, but it is considered that it is a consequence of the portfolio choice $\pi(\cdot)$ of the investor. While the trader tries to maximize his/her expected utility by choosing the optimal portfolio, the market tries to minimize this expected utility by choosing the optimal $\theta(\cdot)$ accordingly.

Let us formulate our problem under the model of Section 3.1 for a 1-dimensional Brownian motion $W(\cdot)$, a 1-dimensional Poisson random measure $N(\cdot,\cdot)$ and a D-dimensional Markov chain, $\alpha(\cdot)$.

The risk-free bond for instantaneous borrowing or lending at the risk-free rate is represented as follows:

$$\begin{aligned} dS_0(t) &= S_0(t) r(t,\alpha(t-))dt, \qquad t \in [0,T], \\ S_0(0) &= s_0 > 0, \end{aligned}$$

where the continuously compounded risk-free rate doesn't depend on the states of the Markov chain, i.e., at time $t \in [0,T]$, $r(t,e_i) := r(t)$ for any $e_i \in S$, $i = 1,2,...,D$.

Let us give the dynamics of risky asset:

$$\begin{aligned} dS(t) =& S(t-)\bigg(\theta(t)dt + \sigma(t,\alpha(t-))dW(t) + \int_{\mathbb{R}_0} \eta(t,\alpha(t-),z)\tilde{N}(dt,dz) \\ & + \gamma(t,\alpha(t-))d\tilde{\Phi}(t)\bigg), \qquad t \in [0,T], \\ S(0) =& s > 0, \end{aligned}$$

where $\sigma(t,\alpha(t-))$, $\eta(t,\alpha(t-),z)$ and $\gamma(t,\alpha(t-))$, $t \in [0,T]$ are deterministic functions. $\pi(t)$ and $1-\pi(t)$ represent the proportion of the trader's wealth invested in the risky asset S and the bond S_0, respectively. Hence the wealth process of the investor is described by:

$$\begin{aligned} dX^{\pi,\theta}(t) &= X(t-)\bigg[\{(1-\pi(t))r(t) + \pi(t)\theta(t)\}dt + \pi(t)\sigma(t,\alpha(t))dW(t) \\ &+ \pi(t-)\int_{\mathbb{R}_0} \eta(t,\alpha(t-),z)\tilde{N}(dt,dz) + \pi(t-)\gamma(t,\alpha(t-))d\tilde{\Phi}(t)\bigg], \ t \in [0,T], \\ X^{\pi,\theta}(0) &= x > 0, \end{aligned}$$

where for $(\theta,\pi) \in \Theta_1 \times \Theta_2$,

$$\begin{aligned} E\bigg[\int_0^T \bigg\{ & |(1-\pi(t))r(t)| + |\pi(t)\theta(t)| + \pi(t)^2\bigg(|\sigma^2(t,e_i)|^2 \\ & + \int_{\mathbb{R}_0} |\eta(t,e_i,z)|^2 \nu(dz) + \sum_{j=1}^{D} |\gamma^j(t,e_i)|^2 \lambda_{ij}\bigg)\bigg\}dt\bigg] < \infty \end{aligned}$$

for all $e_i \in S,\ i = 1, 2, ..., D$.

Note that $\pi(t)$ and $\theta(t)$ are $\mathcal{F}_t$-measurable, cádlág processes.

Our problem is to find the saddle point of this game $(\pi^*, \theta^*) \in \Theta_1 \times \Theta_2$ and the value function $V(t, x, e_i)$ for all $t \in [0, T]$ and $e_i \in S$ such that

$$V(t, x, e_i) = \inf_{\theta \in \Theta_1} \left(\sup_{\pi \in \Theta_2} E^{t,x,e_i} \Big[U(X^{\pi,\theta}(T)) \Big] \right) = E^{t,x,e_i} \Big[U(X^{\pi^*,\theta^*}(T)) \Big].$$

We provide the HJBI equation for the value function V in the form of:

$$V(t, x, e_i) = U(x \exp(\int_t^T r(s)ds)).$$

Then, by applying Equation (3.2), we obtain:

$$\begin{aligned}
&- U'(x \exp(\int_t^T r(s)ds)) x \exp(\int_t^T r(s)ds) r(t) + x((1 - \pi) r(t) + \pi\theta) \\
&\times U'(x \exp(\int_t^T r(s)ds)) \exp(\int_t^T r(s)ds) \\
&+ \frac{1}{2} U''(x \exp(\int_t^T r(s)ds))(\exp(\int_t^T r(s)ds))^2 \pi^2 \sigma^2(t, e_i) x^2 \\
&+ \int_{\mathbb{R}_0} \Big\{ U((x + x\pi\eta(t, e_i, z)) \exp(\int_t^T r(s)ds)) - U(x \exp(\int_t^T r(s)ds)) \\
&- U'(x \exp(\int_t^T r(s)ds)) \exp(\int_t^T r(s)ds)(\pi x \eta(t, e_i, z)) \Big\} \nu(dz) \\
&+ \sum_{j=1}^{D} \lambda_{ij} \Big\{ U((x + x\pi\gamma^j(t, e_i)) \exp(\int_t^T r(s)ds)) - U(x \exp(\int_t^T r(s)ds)) \\
&- U'(x \exp(\int_t^T r(s)ds)) \exp(\int_t^T r(s)ds)(\pi x \gamma^j(t, e_i)) \Big\} = 0. \qquad (3.5)
\end{aligned}$$

Let us apply first order condition to Equation (3.5) with respect to π to receive $\pi^* = \pi^*(\theta)$. This gives:

$$\begin{aligned}
&(\theta - r(t)) x U'(x \exp(\int_t^T r(s)ds)) \exp(\int_t^T r(s)ds) + \frac{1}{2} 2\pi^* \sigma^2(t, e_i) x^2 \\
&\times U''(x \exp(\int_t^T r(s)ds))(\exp(\int_t^T r(s)ds))^2
\end{aligned}$$

$$+ \int_{\mathbb{R}_0} \Big\{ U'((x + x\pi^* \eta(t,e_i,z)) \exp(\int_t^T r(s)ds)) x\eta(t,e_i,z) \exp(\int_t^T r(s)ds)$$

$$- U'(x \exp(\int_t^T r(s)ds)) x\eta(t,e_i,z) \exp(\int_t^T r(s)ds) \Big\} \nu(dz)$$

$$+ \sum_{j=1}^{D} \lambda_{ij} \Big\{ U'((x + x\pi^* \gamma^j(t,e_i)) \exp(\int_t^T r(s)ds)) x\gamma^j(t,e_i,z)$$

$$\times \exp(\int_t^T r(s)ds) - U'(x \exp(\int_t^T r(s)ds)) x\gamma^j(t,e_i) \exp(\int_t^T r(s)ds) \Big\} = 0.$$

If we reorganize the terms above, we get:

$$(\theta - r(t)) U'(x \exp(\int_t^T r(s)ds)) + \pi^* \sigma^2(t,e_i) x U''(x \exp(\int_t^T r(s)ds))$$

$$\times (\exp(\int_t^T r(s)ds)) + \int_{\mathbb{R}_0} \Big\{ U'((x + x\pi^* \eta(t,e_i,z)) \exp(\int_t^T r(s)ds))$$

$$- U'(x \exp(\int_t^T r(s)ds)) \Big\} \eta(t,e_i,z) \nu(dz)$$

$$+ \sum_{j=1}^{D} \lambda_{ij} \gamma^j(t,e_i) \Big\{ U'((x + x\pi^* \gamma^j(t,e_i)) \exp(\int_t^T r(s)ds))$$

$$- U'(x \exp(\int_t^T r(s)ds)) \Big\} = 0. \tag{3.6}$$

Now, let us differentiate the terms in Equation (3.5) with respect to θ, where $\pi^* = \pi^*(\theta)$:

$$(\pi^*)'(\theta) \Big((\theta - r(t)) U'(x \exp(\int_t^T r(s)ds)) + \pi^*(\theta) x \sigma^2(t,e_i)$$

$$\times \exp(\int_t^T r(s)ds) U''(x \exp(\int_t^T r(s)ds)) + \int_{\mathbb{R}_0} \Big\{ U'((x + x\pi^*(\theta) \eta(t,e_i,z))$$

$$\times \exp(\int_t^T r(s)ds)) - U'(x \exp(\int_t^T r(s)ds)) \Big\} \eta(t,e_i,z) \nu(dz)$$

$$+ \sum_{j=1}^{D} \lambda_{ij} \gamma^j(t,e_i) \Big\{ U'((x + x\pi^*(\theta) \gamma^j(t,e_i)) \exp(\int_t^T r(s)ds))$$

$$- U'(x \exp(\int_t^T r(s)ds)) \Big\} \Big) + \pi^*(\theta) U'(x \exp(\int_t^T r(s)ds)) = 0. \tag{3.7}$$

Hence by Equations (3.6) and (3.7), the optimal fraction of the trader's wealth is held in the risky asset $\pi^*(\theta) = 0$ and consequently, by Equation (3.6), $\theta^*(t) = r(t)$. Hence, there is no trade.

Note that for θ^* and $\pi^*(\theta)$, HJBI equation is satisfied. Then,

$$V(t, x, e_i) = U(x \exp(\int_t^T r(s)ds))$$

can be obtained in this form.

Moreover, if we assume $\gamma(t, \alpha(t)) = 0$ for all $t \in [0, T]$, then the continuously compounded risk-free rate can be considered dependent on the Markov chain $\alpha(t)$, for $t \in [0, T]$. In other words, it may be considered that the risk-free rate changes under the effect of macroeconomic conditions. Therefore, the wealth process of the investor becomes as follows:

$$\begin{aligned} dX^{\pi,\theta}(t) &= X(t-)\Bigg[\{(1-\pi(t))r(t,\alpha(t)) + \pi(t)\theta(t)\}\,dt + \pi(t)\sigma(t,\alpha(t))dW(t) \\ &\quad + \pi(t-)\int_{\mathbb{R}_0} \eta(t,\alpha(t-),z)\tilde{N}(dt,dz)\Bigg], \qquad t \in [0,T], \\ X^{\pi,\theta}(0) &= x > 0. \end{aligned}$$

By this setting, one may take the value function in the form of

$$V(t, x, e_i) = U(x \exp(\int_t^T r(s, e_i)ds)),$$

for $e_i \in S,\ i = 1, 2, ..., D$, and follow the same steps. Then, $\pi^*(\theta) = 0$ and $\theta^*(t) = r(t, \alpha(t))$ for all $t \in [0, T]$ are obtained.

This zero-sum game application is an extension of the Theorem 6.3 in Peskir and Shorish [50], which is constructed by a geometric Brownian motion and Example 4.1 in Mataramvura and Øksendal [41], which is presented by a jump-diffusion process. We solve the similar game by including the states of the Markov chain, α with its jump size γ and the compensated random measure Φ as the proxies of the different observable macroeconomic indicators. This solution establishes a "*dynamic equilibrium*" between the market and the investor, i.e., the best action of the investor to the corresponding market force and vice versa. Moreover, this result is compatible with the fundamental equilibrium of risk-neutral asset pricing (see Peskir and Shorish [50] for closer details).

3.3 A Nonzero-Sum Stochastic Differential Game and an Application to Finance

In this section, we give a verification theorem for a nonzero-sum stochastic differential game by the dynamic programming principle approach and an application of a portfolio game between two investors.

Let $u_1 \in \Theta_1$ and $u_2 \in \Theta_2$ be two admissible control processes for Player 1 and Player 2, respectively. Regarding this game, we define two performance functionals (payoff) to Player number k, $k = 1, 2$, respectively:

$$\begin{aligned} J_k^{u_1,u_2}(t, y, e_i) =& E^{t,y,e_i}\left[\int_t^{\tau_G} f_k(s, Y(s), \alpha(s), u_1(s), u_2(s))ds \right. \\ & \left. + g_k(Y(\tau_G), \alpha(\tau_G))\right] \end{aligned} \tag{3.8}$$

for each $e_i \in S$.

If a Nash equilibrium exists for such a game, this means that each player's strategy is optimal or a best response against the other one's move. Furthermore, there can be no unilateral profitable deviation for each player's action. In other words, if one of the players does not move, none of the players change their position. Since a Nash equilibrium is a self-enforcing concept, at the equilibrium, any player knows that moving brings a worse payoff.

Subsequently, we give a mathematical representation for the Nash equilibrium of such a stochastic differential game.

Definition 3.1. Let us assume that for the optimal strategy of Player 2, $u_2^* \in \Theta_2$, the best response of Player 1 satisfies

$$J_1^{u_1,u_2^*}(t, y, e_i) \leq J_1^{u_1^*,u_2^*}(t, y, e_i) \text{ for all } u_1 \in \Theta_1,\ e_i \in S,\ y \in G$$

and for the optimal strategy of Player 1, $u_1^* \in \Theta_1$, the best response of Player 2 satisfies

$$J_2^{u_1^*,u_2}(t, y, e_i) \leq J_2^{u_1^*,u_2^*}(t, y, e_i) \text{ for all } u_2 \in \Theta_2,\ e_i \in S,\ y \in G.$$

Then, the pair of optimal control processes $(u_1^*, u_2^*) \in \Theta_1 \times \Theta_2$ is called a Nash equilibrium for the stochastic differential game of Equations (3.1) and (3.8).

Now we can give the HJBI equations for Nash equilibria, in other words: a verification theorem for a Markov regime-switching jump-diffusion model.

Theorem 3.3. *Suppose that there exists functions $\phi_k \in C^{1,2}(G) \cap C(\bar{G}),\ k = 1, 2$, and a Markov control $(u_1^*, u_2^*) \in \Theta_1 \times \Theta_2$ such that the following conditions are fulfilled:*

i. $\mathcal{L}^{u_1,u_2^*}[\phi(t, y, e_i)] + f(t, y, e_i, u_1, u_2^*) \leq 0$ *for all* $u_1 \in \mathcal{U}_1,\ y \in G$ *and* $e_i \in S$.

ii. $\mathcal{L}^{u_1^*,u_2}[\phi(t, y, e_i)] + f(t, y, e_i, u_1^*, u_2) \leq 0$ *for all* $u_2 \in \mathcal{U}_2,\ y \in G$ *and* $e_i \in S$.

iii. $Y^{u_1,u_2}(\tau_G) \in \partial G$ *a.s. on* $\{\tau_G < \infty\}$ *and*
$\lim_{t \to \tau_G^-} \phi_k(t, Y^{u_1,u_2}(t), \alpha(t)) = g_k(Y^{u_1,u_2}(\tau_G), \alpha(\tau_G))1_{\{\tau_G<\infty\}}$ *a.s.*
for all $(u_1, u_2) \in \Theta_1 \times \Theta_2,\ y \in G,\ k = 1, 2.$

iv. *The family* $\{\phi_k(\tau, Y^{u_1,u_2}(\tau), \alpha(\tau))\}_{\{\tau\in\mathcal{T}\}}$ *is uniformly integrable for all* $y \in G$ *and* $(u_1, u_2) \in \Theta_1 \times \Theta_2,\ k = 1, 2.$

Then, for all $y \in G,\ e_i \in S,\ (u_1^*, u_2^*)$ *is a Nash equilibrium for the game* (3.8) *subject to the goals of the system* (3.1) *such that*

$$\begin{aligned}\phi_1(t, y, e_i) =&V_1(t, y, e_i) = \sup_{u_1\in\Theta_1} J^{u_1,u_2^*}(t, y, e_i) = J^{u_1^*,u_2^*}(t, y, e_i),\\ \phi_2(t, y, e_i) =&V_2(t, y, e_i) = \sup_{u_2\in\Theta_2} J^{u_1^*,u_2}(t, y, e_i) = J^{u_1^*,u_2^*}(t, y, e_i).\end{aligned}$$

Proof. The demonstration of Theorem 3.3 can be obtained similarly as the proof of Theorem 5.2. in Mataramvura and Øksendal [41] by applying Lemma 3.1. Therefore, we do not repeat it here. □

Now let us construct a portfolio game between two investors in a Black-Scholes economy, where the market allows infinitely divisible assets, where no transaction costs exist and information is symmetric. We assume that there are two risky assets with prices $S_1(\cdot)$ for Investor 1 and $S_2(\cdot)$ for Investor 2. Furthermore, there is a risk-free bond (e.g., a bank account) $S_0(\cdot)$. Each investor can invest only one of the risky assets and they are free to invest in the risk-free bond.

Suppose that $r(t)$ represents the continuously compounded risk-free rate.
Let be $r(t) := r(t, \alpha(t)) = \langle r, \alpha(t)\rangle$, where $\langle\cdot,\cdot\rangle$ is the inner product in $\mathbb{R}^D$ and $r := (r^1, r^2, ..., r^D)^T \in \mathbb{R}^D$. Then,

$$\begin{aligned}&dS_0(t) = S_0(t)r(t, \alpha(t-))dt, \qquad t \in [0, T],\\ &S_0(0) = s_0 > 0,\end{aligned}$$

where $\displaystyle\int_0^T |r(t, \alpha(t-))|dt < \infty.$

Let $\mu_m(t) := \mu_m(t, \alpha(t)) = \langle\mu_m, \alpha(t)\rangle,\ \sigma_m(t) := \sigma_m(t, \alpha(t)) = \langle\sigma_m, \alpha(t)\rangle,$ and $\eta_m(t, z) := \eta_m(t, \alpha(t), z) = \langle\eta_m, \alpha(t)\rangle$ denote the appreciation rate, the volatility rate and the jump size of the mth risky asset for $m = 1, 2,\ t \in [0, T]$, where

$$\mu_m := (\mu_m^1, \mu_m^2, ..., \mu_m^D)^T \in \mathbb{R}^D, \qquad \sigma_m := (\sigma_m^1, \sigma_m^2, ..., \sigma_m^D)^T \in \mathbb{R}^D$$

and

$$\eta_m := (z\eta_m^1, z\eta_m^2, ..., z\eta_m^D)^T \in \mathbb{R}^D$$

represent the economy in the ith state. For each $i = 1, 2, ..., D$ and $m = 1, 2,\ \sigma_m^i > 0.$ Please remember that the state space of the Markov chain S is a set of unit basis vectors; hence it is allowed to display the different states of the economy by the inner product of a vector and $\alpha(\cdot)$.

We consider a 1-dimensional standard Brownian motion and a 1-dimensional Poisson random measure. Thus, for each $m = 1, 2$, we represent the risky assets by Markovian regime-switching geometric Lévy processes as follows:

$$
\begin{aligned}
dS_m(t) =& S_m(t-)\Bigg(\mu_m(t,\alpha(t-))dt + \sigma_m(t,\alpha(t-))dW(t) \\
&+ \int_{\mathbb{R}_0} \eta_m(t,\alpha(t-),z)\tilde{N}(dt,dz)\Bigg), \qquad t \in [0,T], \\
S_m(0) =& s_m > 0, \qquad m = 1, 2.
\end{aligned}
$$

Let $\mathcal{U}_m$-valued, $\mathcal{F}_t$-measurable cádlág, Markov control processes $\pi_m,\ m = 1, 2$, be the proportion of the mth trader's wealth invested in the mth risky asset. Then, we can state the dynamics of the wealth processes of each investor as follows:

$$
\begin{aligned}
dX_m(t) &= X_m(t-)\Bigg(\pi_m(t)\mu_m(t,\alpha(t-)) + (1-\pi_m(t))r(t,\alpha(t-))\Bigg)dt + X_m(t-) \\
&\times \pi_m(t)\Bigg(\sigma_m(t,\alpha(t-))dW(t) + \int_{\mathbb{R}_0} \eta_m(t,\alpha(t-),z)\tilde{N}(dt,dz)\Bigg),\ t \in [0,T], \\
X_m(0) &= x_m > 0,\ m = 1, 2, \qquad (3.9)
\end{aligned}
$$

where $\pi_1 \in \mathcal{U}_1$ and $\pi_2 \in \mathcal{U}_2$ are admissible such that Equation (3.9) has a unique strong solution and

$$
\int_0^T |\pi_m(t)|^2 dt < \infty, \qquad m = 1, 2.
$$

The investors act antagonistically to each other by aiming to maximize their own expected terminal gains. Furthermore, we assume that they choose their portfolio strategies $\pi_1 \in \Theta_1$ and $\pi_2 \in \Theta_2$ simultaneously.

We define their performance criterion in such a way that each terminal payoff is proportional to the one of the other investor's:

$$
\begin{aligned}
J_1(t,x_1,x_2,e_i,\pi_1,\pi_2) =& E^{t,x_1,x_2,e_i}\Bigg[\gamma_1 X_1(T)X_2(T)\Bigg], \\
J_2(t,x_1,x_2,e_i,\pi_1,\pi_2) =& E^{t,x_1,x_2,e_i}\Bigg[\gamma_2 X_1(T)X_2(T)\Bigg],
\end{aligned}
$$

where $\gamma_1, \gamma_2 \in \mathbb{R}^+$.

In this way, we can regard one investor's final saving as a factor of *sensitivity* and *marginal gain* for the other investor and vice versa. Thus, maximizing investors' goals means to optimality and jointly direct two investors interests, hence, to collaborate. Then, our problem is to find $(\pi_1^*, \pi_2^*) \in \Theta_1 \times \Theta_2$ and

$$
\begin{aligned}
V_1(t,x_1,x_2,e_i) &= \sup_{\pi_1 \in \Theta_1} J_1(t,x_1,x_2,e_i,\pi_1,\pi_2^*) = J_1^{\pi_1^*,\pi_2^*}(t,x_1,x_2,e_i), \\
V_2(t,x_1,x_2,e_i) &= \sup_{\pi_2 \in \Theta_2} J_2(t,x_1,x_2,e_i,\pi_1^*,\pi_2) = J_2^{\pi_1^*,\pi_2^*}(t,x_1,x_2,e_i).
\end{aligned}
$$

In this concept, $Y(t)$ in Equation (3.1) can be considered as $Y(t) := (X_1(t), X_2(t))$. Now we can define a Markovian regime-switching infinitesimal generator $\mathcal{L}^{\pi_1,\pi_2}$ on ϕ for each investor as follows:

$$\begin{aligned}
\mathcal{L}^{\pi_1,\pi_2^*}[\phi_1(t,x_1,x_2,e_i)] = &\frac{\partial \phi_1}{\partial t}(t,x_1,x_2,e_i) \\
&+ x_1\bigg(\pi_1\mu_1(t,e_1) + (1-\pi_1)r(t,e_i)\bigg)\frac{\partial \phi_1}{\partial x_1}(t,x_1,x_2,e_i) \\
&+ x_2\bigg(\pi_2^*\mu_2(t,e_1) + (1-\pi_2^*)r(t,e_i)\bigg)\frac{\partial \phi_1}{\partial x_2}(t,x_1,x_2,e_i) \\
&+ \frac{1}{2}\pi_1^2 x_1^2\sigma_1^2(t,e_i)\frac{\partial^2 \phi_1}{\partial x_1^2}(t,x_1,x_2,e_i) \\
&+ \frac{1}{2}\pi_2^{*2} x_2^2\sigma_2^2(t,e_i)\frac{\partial^2 \phi_1}{\partial x_2^2}(t,x_1,x_2,e_i) \\
&+ \pi_1\pi_2^* x_1 x_2\sigma_1(t,e_i)\sigma_2(t,e_i)\frac{\partial^2 \phi_1}{\partial x_1 x_2}(t,x_1,x_2,e_i) \\
&+ \int_{\mathbb{R}_0}\bigg[\phi_1(t,x_1+\pi_1x_1\eta_1(t,e_i,z), x_1+\pi_2^*x_2\eta_2(t,e_i,z),e_i) \\
&- \phi_1(t,x_1,x_2,e_i) - \pi_1x_1\eta_1(t,e_i,z)\frac{\partial \phi_1}{\partial x_1}(t,x_1,x_2,e_i) \\
&- \pi_2^*x_2\eta_2(t,e_i,z)\frac{\partial \phi_1}{\partial x_2}(t,x_1,x_2,e_i)\bigg]\nu(dz) \\
&+ \sum_{j=1}^{D}\lambda_{ij}\bigg[\phi_1(t,x_1,x_2,e_j) - \phi_1(t,x_1,x_2,e_i)\bigg]
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{L}^{\pi_1^*,\pi_2}[\phi_2(t,x_1,x_2,e_i)] = &\frac{\partial \phi_2}{\partial t}(t,x_1,x_2,e_i) + x_1\bigg(\pi_1^*\mu_1(t,e_1) \\
&+ (1-\pi_1^*)r(t,e_i)\bigg)\frac{\partial \phi_2}{\partial x_1}(t,x_1,x_2,e_i) \\
&+ x_2\bigg(\pi_2\mu_2(t,e_1) + (1-\pi_2)r(t,e_i)\bigg)\frac{\partial \phi_2}{\partial x_2}(t,x_1,x_2,e_i) \\
&+ \frac{1}{2}\pi_1^{*2} x_1^2\sigma_1^2(t,e_i)\frac{\partial^2 \phi_2}{\partial x_1^2}(t,x_1,x_2,e_i) \\
&+ \frac{1}{2}\pi_2^2 x_2^2\sigma_2^2(t,e_i)\frac{\partial^2 \phi_2}{\partial x_2^2}(t,x_1,x_2,e_i) \\
&+ \pi_1^*\pi_2 x_1 x_2\sigma_1(t,e_i)\sigma_2(t,e_i)\frac{\partial^2 \phi_2}{\partial x_1 x_2}(t,x_1,x_2,e_i)
\end{aligned}$$

$$
\begin{aligned}
&+ \int_{\mathbb{R}_0} \Big[\phi_2(t, x_1 + \pi_1^* x_1 \eta_1(t, e_i, z), x_1 + \pi_2 x_2 \eta_2(t, e_i, z), e_i) \\
&- \phi_2(t, x_1, x_2, e_i) - \pi_1^* x_1 \eta_1(t, e_i, z) \frac{\partial \phi_2}{\partial x_1}(t, x_1, x_2, e_i) \\
&- \pi_2 x_2 \eta_2(t, e_i, z) \frac{\partial \phi_2}{\partial x_2}(t, x_1, x_2, e_i) \Big] \nu(dz) \\
&+ \sum_{j=1}^{D} \lambda_{ij} \Big[\phi_2(t, x_1, x_2, e_j) - \phi_2(t, x_1, x_2, e_i) \Big].
\end{aligned}
$$

Then we can re-state our problem for each investor:

$$
\begin{aligned}
&\sup_{\pi_1 \in \Theta_1} \Big\{ \mathcal{L}^{\pi_1, \pi_2^*}[\phi_1(t, x_1, x_2, e_i)] \Big\} = 0, \\
&\phi_1(T, x_1, x_2, e_i) = \gamma_1 x_1 x_2, \qquad \text{for all } e_i \in S
\end{aligned} \tag{3.10}
$$

and

$$
\begin{aligned}
&\sup_{\pi_2 \in \Theta_2} \Big\{ \mathcal{L}^{\pi_1^*, \pi_2}[\phi_2(t, x_1, x_2, e_i)] \Big\} = 0, \\
&\phi_2(T, x_1, x_2, e_i) = \gamma_2 x_1 x_2, \qquad \text{for all } e_i \in S.
\end{aligned} \tag{3.11}
$$

Let us consider a value function of the form $V_m(t, x_1, x_1, e_i) = k_m(t, e_i) x_1 x_2$ for $m = 1, 2,\ e_i \in S$.

Since the first-order conditions give the optimal portfolio strategies, first let us begin by differentiating the quantity in the HJBI Equation (3.10) with respect to π_1 for fixed π_2^*:

$$
\begin{aligned}
&(\mu_1(t, e_i) - r(t, e_i)) x_1 x_2 k_1(t, e_i) + \pi_2^* \sigma_1(t, e_i) \sigma_2(t, e_i) x_1 x_2 k_1(t, e_i) \\
&\int_{\mathbb{R}_0} \Big((x_2 + \pi_2^* x_2 \eta_2(t, e_i, z)) x_1 \eta_1(t, e_i, z) k_1(t, e_i) - \eta_1(t, e_i, z) x_1 x_2 k_1(t, e_i) \Big) \nu(dz) \\
&= 0.
\end{aligned}
$$

Hence, we obtain

$$
\pi_2^* = \frac{r(t, e_i) - \mu_1(t, e_i)}{\sigma_1(t, e_i) \sigma_2(t, e_i) + \int_{\mathbb{R}_0} \eta_1(t, e_i, z) \eta_2(t, e_i, z) \nu(dz)}, \text{ for all } e_i \in S. \tag{3.12}
$$

Similarly, for fixed π_1^*, the first-order condition for maximizing the quantity in Equation (3.11) with respect to π_2 gives:

$$
\pi_1^* = \frac{r(t, e_i) - \mu_2(t, e_i)}{\sigma_1(t, e_i) \sigma_2(t, e_i) + \int_{\mathbb{R}_0} \eta_1(t, e_i, z) \eta_2(t, e_i, z) \nu(dz)}, \text{ for all } e_i \in S. \tag{3.13}
$$

If we consider π_1^* and π_2^* in Equations (3.10) and (3.11), then for all $e_i \in S$ we obtain D-coupled, linear ordinary differential equations (linear ODEs):

$$
\begin{aligned}
& k_m^{'}(t, e_i) + h(t, e_i) k_m(t, e_i) + \sum_{j=1}^{D} \lambda_{ij} (k_m(t, e_j) - k_m(t, e_i)) = 0 \\
& k_m(T, e_i) = \gamma_m > 0, \qquad \text{for} \quad m = 1, 2,
\end{aligned} \tag{3.14}
$$

where

$$
\begin{aligned}
h(t, e_i) := & \ \pi_1^* \mu_1(t, e_i) + (1 - \pi_1^*) r(t, e_i) + \pi_2^* \mu_2(t, e_i) + (1 - \pi_2^*) r(t, e_i) \\
& + \pi_1^* \pi_2^* \bigg(\sigma_1(t, e_i) \sigma_2(t, e_i) + \int_{\mathbb{R}_0} \eta_1(t, e_i, z) \eta_2(t, e_i, z) \nu(dz) \bigg).
\end{aligned}
$$

By applying the classical procedure of Feynman-Kac representation for the solution of the system of differential equations (3.14), one can obtain the subsequent solution for the value functions of each investor:

$$
k_m(t, \alpha(t)) = \gamma_m E \bigg[\exp \bigg(\int_t^T h(s, \alpha(s)) ds \bigg) | \alpha(t) = e_i \bigg], \qquad \text{for} \quad m = 1, 2.
$$

It is clearly seen that $k_m, \ m = 1, 2$, are nonnegative; consequently $V_m, \ m = 1, 2$, are also nonnegative.

Herewith, we obtain explicit solutions of optimal portfolio processes $\pi_1^*, \ \pi_2^*$ and the value functions $V_1, \ V_2$ of each investor, i.e., the Nash equilibria and the corresponding equilibrium performances of the nonzero-sum game described.

3.3.1 A Special Case

In this subsection, we consider a special case of the game introduced in Section 3.3. We assume that there are just two states, that is $S = \{e_1, e_2\}$, which describe the economy with providing an information rich enough. In this set-up, the states of the economy or the market can be considered as "*good*" and "*bad*", or as "*bear*" and "*bull*", respectively.

In this special case, the rate matrix of the Markov chain α can be represented as follows:

$$
\begin{pmatrix} -\lambda & \lambda \\ \lambda & -\lambda \end{pmatrix}.
$$

By considering the results of the previous section, for the sake of simplicity, we define:

$$\begin{aligned}h(t,e_1) =&d^1 = a^1\mu_1^1 + (1-a^1)r^1 + b^1\mu_2^1 + (1-a^1)r^1 + a^1b^1(\sigma_1^1\sigma_2^1\\&+\int_{\mathbb{R}_0} z^2\eta_1^1\eta_2^1\nu(dz)), \qquad t\in[0,T],\\h(t,e_2) =&d^2 = a^2\mu_1^2 + (1-a^2)r^2 + b^2\mu_2^2 + (1-a^2)r^2 + a^2b^2(\sigma_1^2\sigma_2^2\\&+\int_{\mathbb{R}_0} z^2\eta_1^2\eta_2^2\nu(dz)), \qquad t\in[0,T],\end{aligned}$$

where

$$\pi_1^* = \begin{pmatrix} a^1 \\ a^2 \end{pmatrix} = \begin{pmatrix} \dfrac{r^1-\mu_1^1}{\sigma_1^1\sigma_2^1 + \int_{\mathbb{R}_0} z^2\eta_1^1\eta_2^1\nu(dz)} \\ \dfrac{r^2-\mu_1^2}{\sigma_1^2\sigma_2^2 + \int_{\mathbb{R}_0} z^2\eta_1^2\eta_2^2\nu(dz)} \end{pmatrix}$$

and

$$\pi_2^* = \begin{pmatrix} b^1 \\ b^2 \end{pmatrix} = \begin{pmatrix} \dfrac{r^1-\mu_2^1}{\sigma_1^1\sigma_2^1 + \int_{\mathbb{R}_0} z^2\eta_1^1\eta_2^1\nu(dz)} \\ \dfrac{r^2-\mu_2^2}{\sigma_1^2\sigma_2^2 + \int_{\mathbb{R}_0} z^2\eta_1^2\eta_2^2\nu(dz)} \end{pmatrix}.$$

Thus, by Equation (3.14), we obtain 2-coupled linear ODEs with terminal values:

$$\begin{aligned}&k_1^{'}(t,e_1) + d^1k_1(t,e_1) + \lambda(k_1(t,e_2)-k_1(t,e_1)) = 0, \qquad k_1(T,e_1)=\gamma_1,\\&k_1^{'}(t,e_2) + d^2k_1(t,e_2) + \lambda(k_1(t,e_1)-k_1(t,e_2)) = 0, \qquad k_1(T,e_2)=\gamma_1\end{aligned} \tag{3.15}$$

and, similarly,

$$\begin{aligned}&k_2^{'}(t,e_1) + d^1k_2(t,e_1) + \lambda(k_2(t,e_2)-k_2(t,e_1)) = 0, \qquad k_2(T,e_1)=\gamma_2,\\&k_2^{'}(t,e_2) + d^2k_2(t,e_2) + \lambda(k_2(t,e_1)-k_2(t,e_2)) = 0, \qquad k_2(T,e_2)=\gamma_2.\end{aligned}$$

Then, by writing $k_1(t,e_1)$ in terms of $k_1(t,e_2)$ in Equation (3.15), we get:

$$\begin{aligned}&k_1^{''}(t,e_2) + (d^2+d^1-2\lambda)k_1^{'}(t,e_2) + (d^2d^1-\lambda d^2-\lambda d^1)k_1(t,e_2) = 0,\\&k_1(T,e_2)=\gamma_1.\end{aligned}$$

By the classical methods of solving linear ODEs of second order, solutions of these equations can be received in the following way.

First, let us find Δ:

$$\begin{aligned}\Delta =&(d^2+d^1-2\lambda)^2 - 4(d^2d^1-\lambda d^2-\lambda d^1)\\&= (d^2-d^1)^2 + 4\lambda^2 > 0.\end{aligned}$$

Since Δ is always nonnegative, we consider just real roots. Then,

$$p_{1,2} = \frac{-(d^2 + d^1 - 2\lambda) \pm \sqrt{(d^2 - d^1)^2 + 4\lambda^2}}{2}.$$

Therefore,

$$k_1(T, e_2) = C_1 e^{-p_1(T-t)} + C_2 e^{-p_2(T-t)}, \qquad k_1(T, e_2) = \gamma_1.$$

Similarly,

$$k_1(T, e_1) = C_3 e^{-p_1(T-t)} + C_4 e^{-p_2(T-t)}, \qquad k_1(T, e_1) = \gamma_1.$$

Let us denote:

$$\begin{aligned} C_1 &= -\gamma_1 \frac{(d^2 + p_2)}{p_1 - p_2}, \qquad & C_2 &= \gamma_1 \frac{(d^2 + p_1)}{p_1 - p_2}, \\ C_3 &= -\gamma_1 \frac{(d^1 + p_2)}{p_1 - p_2}, \qquad & C_4 &= \gamma_1 \frac{(d^1 + p_1)}{p_1 - p_2}. \end{aligned}$$

By following similar steps, one can obtain:

$$\begin{aligned} k_2(T, e_2) &= C_5 e^{-p_1(T-t)} + C_6 e^{-p_2(T-t)}, \qquad & k_2(T, e_2) &= \gamma_2, \\ k_2(T, e_1) &= C_7 e^{-p_1(T-t)} + C_8 e^{-p_2(T-t)}, \qquad & k_2(T, e_1) &= \gamma_2, \end{aligned}$$

where

$$\begin{aligned} C_5 &= -\gamma_2 \frac{(d^2 + p_2)}{p_1 - p_2}, \qquad & C_6 &= \gamma_2 \frac{(d^2 + p_1)}{p_1 - p_2}, \\ C_7 &= -\gamma_2 \frac{(d^1 + p_2)}{p_1 - p_2}, \qquad & C_8 &= \gamma_2 \frac{(d^1 + p_1)}{p_1 - p_2}. \end{aligned}$$

Hence, at each state of the economy, we get the optimal investment proportions and value functions for each investor.

In the next subsection, we present simple, heuristic examples to illustrate our results for a two-state case as described in this section. Since it is out of scope of this thesis, we do not follow a statistical approach for the following subsection.

3.3.2 Comparative Statics

We purpose to provide an intuition on how each investor's optimal portfolio strategy varies based on the model parameters. Therefore, we consider some specific annualized hypothetical values for these imaginary parameters. Since it is out of the scope of this thesis, we do not follow a statistical approach using data to determine them for the following simple examples. Actually, such a project will be left to future studies.

We work in a two-state Markov regime-switching financial market with one risk-free bond and two risky assets. We assume that the state space of the Markov chain S represents the states of the economy as *Good Economy (GE)* and *Bad Economy (BE)* based on the relation between expected rates of return of the risky assets, $\mu_k,\ k = 1, 2$, and risk-free rate $r^k,\ k = 1, 2$. The appreciation rates of risky assets are greater than risk-free rate during expansions (GE) and smaller during recessions (BE).

Let $N(dt, dz),\ t \in [0, T]$, be a Poisson random measure and the waiting time between jumps be exponentially distributed with $\lambda_* = 1$. Moreover, the Lévy measure is equal to $\nu(dz) = \lambda_* \times F(dz)$, where $F(dz)$ represents the standard normal distribution, $\mathcal{N}(0, 1)$.

Note that the following graphs can be obtained by Equations (3.12) and (3.13). Furthermore, it is obvious that there is a linear relation between the optimal investment proportions and the appreciation rates of the risky assets. The selected values and Matlab codes can be found in Appendix. Similar graphs can be drawn for π_1^* and π_2^* in view of both of the settings. In these simple examples, we realize how each investor's strategy leads the other one's response.

In Figure 3.1, under BE condition, we present the change of the optimal portfolio strategy of second trader, π_2^*, against the appreciation rate of the first risky asset, μ_1^1; here just the first trader has the right to invest in S_1, for different jump sizes of first risky asset, η_1^1. We observe that when μ_1^1 increases, the optimal proportion invested in S_2 increases for a downward jump value of S_1 and decreases for an upward one. Moreover, it is seen that the second investor should be in short position for $\eta_1^1 < 0$ and in long position for $\eta_1^1 > 0$. Furthermore, there is no trade when $\mu_1^1 = r^1$.

On the other hand, if we focus on the GE state by a similar comparison, in Figure 3.2, we see that when μ_1^1 increases, the optimal proportion invested in S_2 increases for a downward jump value of S_1 and decreases for an upward one, as in BE state. But, the second investor should reverse his/her position, a short position for $\eta_1^1 > 0$ and a long position for $\eta_1^1 < 0$.

Furthermore, in Figure 3.3, we analyze the change of optimal π_2 in BE state against μ_1^1 for two different specific volatility levels. When μ_1^1 increases, the optimal investment strategy for the second trader is affected in a similar direction; however the uncertainty level increases.

We can also focus on the behavior of the optimal portfolio strategies by considering both of the states of the economy, GE-BE, simultaneously. In Figure 3.4, we can view optimal π_1 in GE-BE states against $\mu_2^k,\ k = 1, 2$, for different jump sizes of S_2. The first investor should be in short position for opposite values of $\eta_2^k,\ k = 1, 2$.

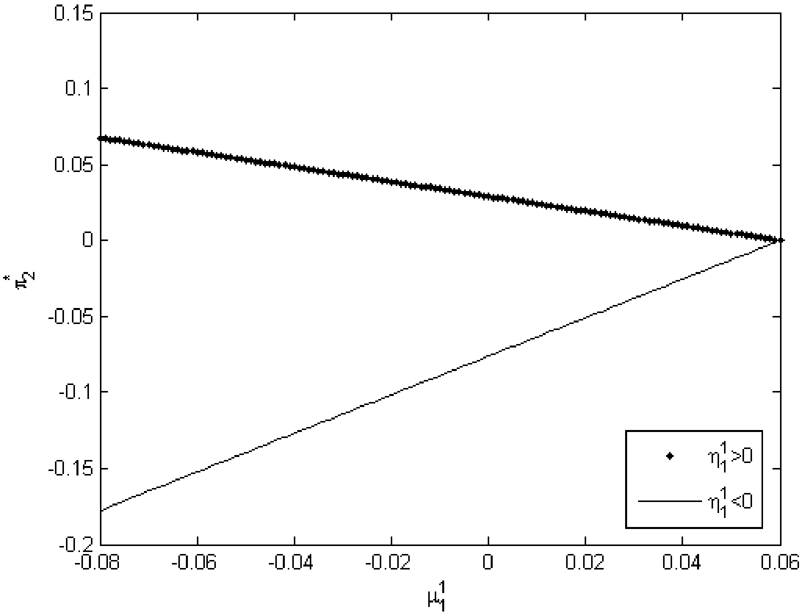

Figure 3.1: Optimal π_2 in BE against μ_1^1 for different η_1^1.

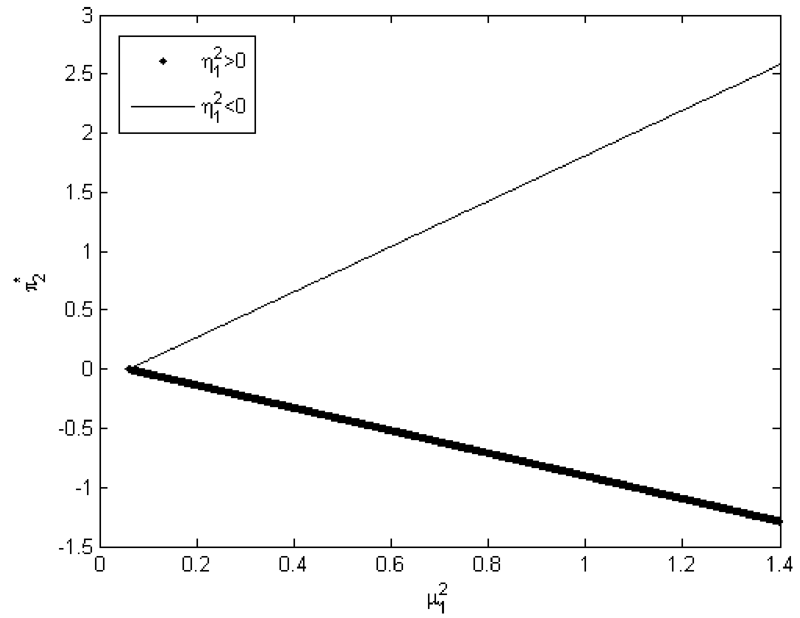

Figure 3.2: Optimal π_2 in GE against μ_1^2 for different η_1^2.

In Figure 3.5, we investigate the increased levels of the risk-free rate. If $\mu_2^k,\ k = 1, 2$, increase, the optimal π_1 decreases. Moreover, the first investor should change his/her position at different levels of the risk-free rate generated at different states of the economy.

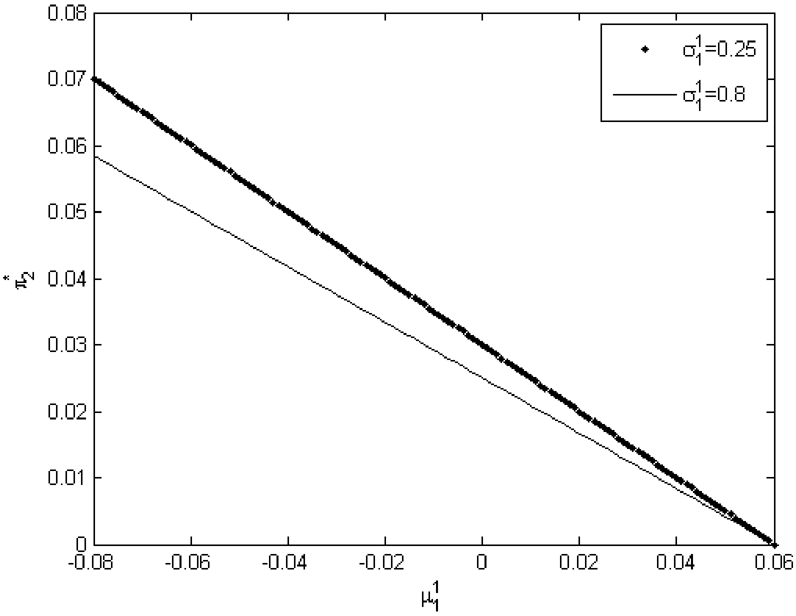

Figure 3.3: Optimal π_2 in BE against μ_1^1 for different σ_1^1.

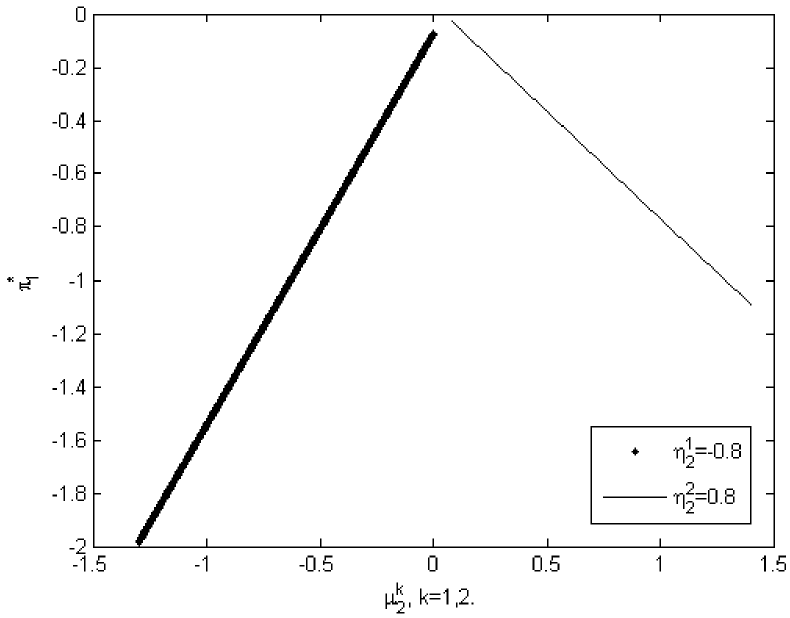

Figure 3.4: Optimal π_1 in GE-BE against $\mu_2^k,\ k = 1, 2$, for different $\eta_2^k,\ k = 1, 2$.

In the last Figure 3.6, we can analyze the optimal π_1 in GE-BE against $\mu_2^k,\ k = 1, 2,$ for different uncertainty levels. Furthermore, we can clearly see the jumps generated by the regime switches from BE to GE in Figure 3.4, Figure 3.5 and Figure 3.6.

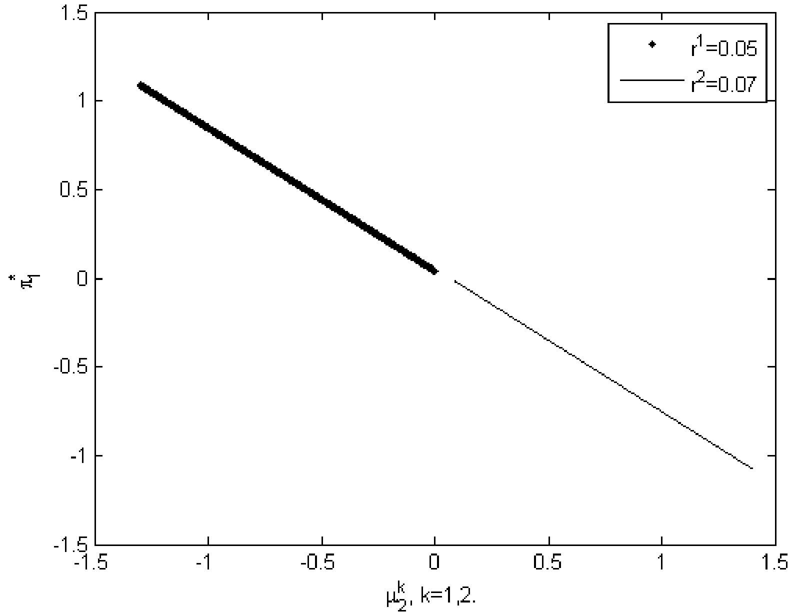

Figure 3.5: Optimal π_1 in GE-BE against $\mu_2^k,\ k=1,2$, for different $r^k,\ k=1,2$.

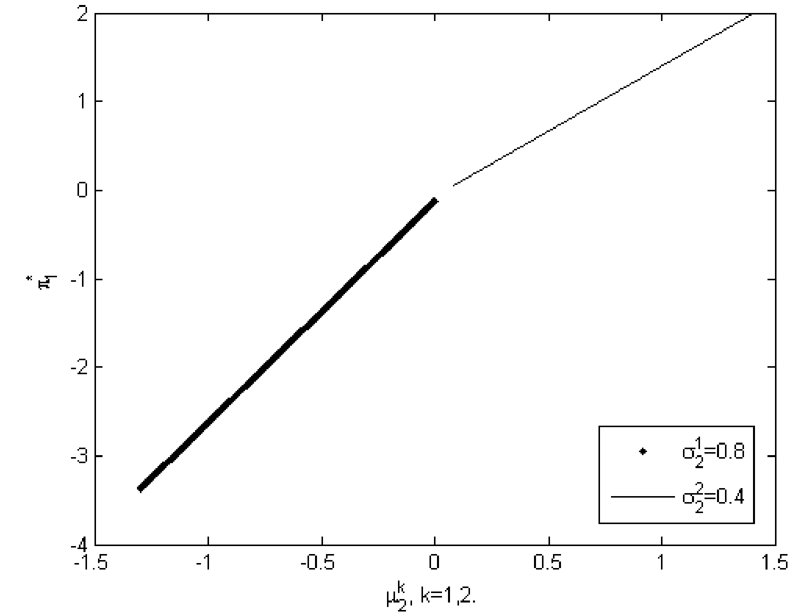

Figure 3.6: Optimal π_1 in GE-BE against $\mu_2^k,\ k=1,2$, for different $\sigma_2^k,\ k=1,2$.

CHAPTER 4

MAIN RESULTS FOR A DELAYED JUMP-DIFFUSION MODEL WITH REGIMES

In this chapter, we present three main theorems for Stochastic Differential Delay Equations (SDDEs) and Anticipated Backward Stochastic Differential Equations (ABSDEs). These results support the results of Chapter 5 and provide an intuition for the relation between SDDEs and ABSDEs. Moreover, this chapter clarifies the existence of admissible control processes in Chapter 5. The techniques applied for the proofs of these underlying theorems are based on the methods of Peng and Yang [49], El-Karoui, Hamadène and Matoussi [17] and Yang, Mao and Yuan [60].

Let $(\Omega, \mathbb{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a complete probability space, where $\mathbb{F} = (\mathcal{F}_t : t \in [0,T])$. Furthermore, $(\mathcal{F}_t)_{t\geq 0}$ is a right-continuous, $\mathbb{P}$-completed filtration generated by a 1-dimensional Brownian motion $W(\cdot)$, a 1-dimensional Poisson random measure $N(\cdot,\cdot)$ and a D-dimensional Markov chain $\alpha(\cdot)$. We assume that these processes are independent of each other and adapted to $\mathbb{F}$.

4.1 Existence-Uniqueness Theorem for an SDDE with Jumps and Regimes

Yang, Mao and Yuan [60] proved an existence-uniqueness theorem for a stochastic differential diffusion process. Then, in the same work, they provided some technical conditions similar to (A1), (A2), (H1) and (H2) (introduced below) and stated that under these assumptions, an existence-uniqueness result can be obtained for a stochastic differential diffusion process with delay; but they skipped the proof. Then, Bao and Yuan [3] presented an extended form of this result for a jump-diffusion process without proving it. In this section, we prove the existence-uniqueness theorem for a system of Markov regime-switching jump-diffusion process with delay. Here, in our setting, the Markov chain α, its jump size γ and the compensated random measure $\tilde{\Phi}$ generated by the Markov chain can be observed as our contribution.

Let us represent our model:

$$
\begin{aligned}
dX(t) =& b(t, X(t), X(t-\delta_1(t)), \alpha(t))dt \\
&+ \sigma(t, X(t), X(t-\delta_2(t)), \alpha(t))dW(t) \\
&+ \int_{\mathbb{R}_0} \eta(t, X(t-), X((t-\delta_3(t))-), \alpha(t-), z)\tilde{N}(dt, dz) \\
&+ \gamma(t, X(t-), X((t-\delta_4(t))-), \alpha(t-))d\tilde{\Phi}(t), \qquad t \in [0,T], \qquad (4.1)\\
X(t) =& x_0(t), \qquad t \in [-\delta, 0],
\end{aligned}
$$

where x_0 is a càdlàg function defined from $[-\delta, 0]$ into $\mathbb{R}$ with the norm

$$\|x_0(t)\| = \sup_{-\delta \le t \le 0} |x_0(t)|.$$

We may call x_0 as pre-history or initial path. Delay components, $\delta_i,\ i = 1, 2, 3, 4$, are nonnegative continuous real-valued functions defined on $[0, T]$ such that:

(A1) There exists a constant $\delta > 0$ such that for each $t \in [0, T]$,

$$-\delta \le t - \delta_i(t) \le t, \qquad i = 1, 2, 3, 4.$$

(A2) There exists a constant $L > 0$ such that for each $t \in [0, T]$ and for each nonnegative, integrable $g(\cdot)$,

$$\int_0^t g(s - \delta_i(s))ds \le L \int_{-\delta}^t g(s)ds, \qquad i = 1, 2, 3, 4.$$

Let us consider the uniform case with $\delta_i(t) = \delta(t)$ for $i = 1, 2, 3, 4$, and assume $X(t - \delta(t)) = Y(t),\ t \in [0, T]$. Now we give further assumptions for Equation (4.1) without losing generality.

Let $b : [0, T] \times \mathbb{R} \times \mathbb{R} \times S \to \mathbb{R},\ \sigma : [0, T] \times \mathbb{R} \times \mathbb{R} \times S \to \mathbb{R}$,
$\eta : [0, T] \times \mathbb{R} \times \mathbb{R} \times S \times \mathbb{R}_0 \to \mathbb{R}$ and $\gamma : [0, T] \times \mathbb{R} \times \mathbb{R} \times S \to \mathbb{R}$ satisfy the following conditions:

(H1) There exists a constant $C > 0$ such that for all $t \in [0, T],\ e_i \in S,\ x_1,\ x_2,\ y_1,\ y_2 \in \mathbb{R}$,

$$
\begin{aligned}
&|b(t, x_1, y_1, e_i) - b(t, x_2, y_2, e_i)| \vee |\sigma(t, x_1, y_1, e_i) - \sigma(t, x_2, y_2, e_i)| \\
&\vee \|\eta(t, x_1, y_1, e_i, z) - \eta(t, x_2, y_2, e_i, z)\|_J \vee \|\gamma(t, x_1, y_1, e_i) - \gamma(t, x_2, y_2, e_i)\|_S \\
&\le C(|x_1 - x_2| + |y_1 - y_2|).
\end{aligned}
$$

(H2) $b(\cdot, 0, 0, e_i) \in L^2_{\mathbb{F}}(0, T; \mathbb{R}),\ \sigma(\cdot, 0, 0, e_i) \in L^2_{\mathbb{F}}(0, T; \mathbb{R}),\ \eta(\cdot, 0, 0, e_i, \cdot) \in \mathcal{H}^2_{\mathbb{F}}(0, T; \mathbb{R})$ and $\gamma(\cdot, 0, 0, e_i) \in \mathcal{M}^2_{\mathbb{F}}(0, T; \mathbb{R}^D)$ for all $e_i \in S$ and $t \in [0, T]$.

Theorem 4.1. *Under the assumptions (A1), (A2), (H1) and (H2), there exists a unique càdlàg adapted solution $X(\cdot) \in L^2_{\mathbb{F}}(0,T;\mathbb{R})$ for Equation* (4.1).

Proof. Let us fix $\beta = 16C^2(1+L)+1$, where C is the Lipschitz constant given in condition (H1) and L is as in assumption (A2). Related to this β, for the sake of convenience, we use a norm in Banach space $L^2_{\mathbb{F}}(0,T;\mathbb{R})$ as follows:

$$\|h(\cdot)\|^2_\beta = E\bigg[\int_0^T e^{-\beta s}\,\|h(s)\|^2\,ds\bigg],$$

which is equivalent to the original norm of $L^2_{\mathbb{F}}(0,T;\mathbb{R})$.

For any given $x(\cdot) \in L^2_{\mathbb{F}}(0,T;\mathbb{R})$ with $x(t) = x_0(t),\ t \in [-\delta, 0]$, we set:

$$\begin{aligned}
X(t) =& b(t,x(t),y(t),\alpha(t))dt + \sigma(t,x(t),y(t),\alpha(t))dW(t) \\
&+ \int_{\mathbb{R}_0} \eta(t,x(t-),y(t-),\alpha(t-),z)\tilde{N}(dt,dz) \\
&+ \gamma(t,x(t-),y(t-),\alpha(t-))d\tilde{\Phi}(t)\bigg], \qquad t \in [0,T], \qquad (4.2)\\
X(t) =& x_0(t), \qquad t \in [-\delta, 0].
\end{aligned}$$

According to the existence-uniqueness results for SDEs with jumps and regimes (see Proposition 7.1 by Crépey [13]), the aforementioned Equation (4.2) has a unique solution. Let us define a mapping,

$$h : L^2_{\mathbb{F}}(0,T;\mathbb{R}) \to L^2_{\mathbb{F}}(0,T;\mathbb{R})$$

such that $h(x)(\cdot) = X(\cdot)$. Note that $y(t) = x(t-\delta(t)),\ t \in [0,T]$ and h is well-defined.

Let us use the following abbreviations:

$$\begin{aligned}
&b_1(s) := b(s,x_1(s),y_1(s),\alpha(s)),\\
&b_2(s) := b(s,x_2(s),y_2(s),\alpha(s)),\ \text{etc.}
\end{aligned}$$

For arbitrary $x_1,\ x_2 \in L^2_{\mathbb{F}}(0,T;\mathbb{R})$, we now apply generalized Itô's formula (see Appendix) to $e^{-\beta t}|h(x_1)(t) - h(x_2)(t)|^2$ and take expectation:

$$E\bigg[e^{-\beta t}(h(x_1)(t) - h(x_2)(t))^2\bigg]$$

$$
\begin{aligned}
&= -\beta E\Big[\int_0^t e^{-\beta s}(h(x_1)(s) - h(x_2)(s))^2 ds\Big] \\
&\quad + 2E\Big[\int_0^t e^{-\beta s}(h(x_1)(s) - h(x_2)(s))\Big\{(b_1(s) - b_2(s))ds \\
&\quad + (\sigma_1(s) - \sigma_2(s))dW(s) + \int_{\mathbb{R}_0}(\eta_1(s,z) - \eta_2(s,z))\tilde{N}(ds,dz) \\
&\quad + (\gamma_1(s) - \gamma_2(s))d\tilde{\Phi}(s)\Big\}\Big] \\
&\quad + E\Big[\int_0^t e^{-\beta s}(\sigma_1(s) - \sigma_2(s))^2 ds\Big] \\
&\quad + E\Big[\int_0^t \int_{\mathbb{R}_0} e^{-\beta s}(\eta_1(s,z) - \eta_2(s,z))^2 \nu(dz)ds\Big] \\
&\quad + E\Big[\int_0^t e^{-\beta s}\sum_{j=1}^{D}(\gamma_1^j(s) - \gamma_2^j(s))^2 \lambda_j(s)ds\Big].
\end{aligned}
$$

Since $a^2 + b^2 \geq 2ab$ and by assumption (A2), we get:

$$
\begin{aligned}
&E\Big[e^{-\beta t}(h(x_1)(t) - h(x_2)(t))^2\Big] \\
&= -\beta E\Big[\int_0^t e^{-\beta s}(h(x_1)(s) - h(x_2)(s))^2 ds\Big] \\
&\quad + E\Big[\int_0^t e^{-\beta s}(h(x_1)(s) - h(x_2)(s))^2 ds\Big] \\
&\quad + E\Big[\int_0^t e^{-\beta s}|b_1(s) - b_2(s)|^2 ds\Big] + E\Big[\int_0^t e^{-\beta s}|\sigma_1(s) - \sigma_2(s)|^2 ds\Big] \\
&\quad + E\Big[\int_0^t e^{-\beta s}\left\|\eta_1(s) - \eta_2(s)\right\|_J^2 ds\Big] + E\Big[\int_0^t e^{-\beta s}\left\|\gamma_1(s) - \gamma_2(s)\right\|_S^2 ds\Big] \\
&\leq (-\beta + 1)E\Big[\int_0^t e^{-\beta s}(h(x_1)(s) - h(x_2)(s))^2 ds\Big] \\
&\quad + 4C^2 E\Big[\int_0^t e^{-\beta s}(|x_1(s) - x_2(s)| + |y_1(s) - y_2(s)|)^2 ds\Big] \\
&\leq (-\beta + 1)E\Big[\int_0^t e^{-\beta s}(h(x_1)(s) - h(x_2)(s))^2 ds\Big] \\
&\quad + 8C^2 E\Big[\int_0^t e^{-\beta s}(|x_1(s) - x_2(s)|^2 + |y_1(s) - y_2(s)|^2 ds\Big] \\
&\leq (-\beta + 1)E\Big[\int_0^t e^{-\beta s}(X_1(s) - X_2(s))^2 ds\Big]
\end{aligned}
$$

$$+ 8C^2 E\left[\int_0^t e^{-\beta s}(|x_1(s) - x_2(s)|^2 ds\right]$$
$$+ 8LC^2 E\left[\int_{-\delta}^t e^{-\beta s}(|x_1(s) - x_2(s)|^2 ds\right].$$

Note that for $s \in [-\delta, 0],\ x_1(s) = x_2(s) = x_0(s)$; then we receive:

$$E\left[e^{-\beta t}(h(x_1)(t) - h(x_2)(t))^2\right] + (\beta - 1)E\left[\int_0^t e^{-\beta s}(X_1(s) - X_2(s))^2 ds\right]$$
$$\leq 8C^2(1+L)E\left[\int_0^t e^{-\beta s}|x_1(s) - x_2(s)|^2 ds\right].$$

Let us also note that $E\left[e^{-\beta t}(h(x_1)(t) - h(x_2)(t))^2\right] > 0.$

Since $\beta = 16C^2(1+L) + 1$, we obtain:

$$E\left[\int_0^t e^{-\beta s}|h(x_1)(s) - h(x_2)(s)|^2 ds\right] \leq \frac{1}{2}E\left[\int_0^t e^{-\beta s}|x_1(s) - x_2(s)|^2 ds\right].$$

It is shown that h is a contraction mapping in Banach space $L^2_{\mathbb{F}}(0,T;\mathbb{R})$. Hence, by Banach Fixed Point Theorem (see Appendix), there exists a unique solution $X(\cdot) \in L^2_{\mathbb{F}}(0,T;\mathbb{R})$ for Equation (4.1). □

4.2 Duality Between SDDEs and ABSDEs with Jumps and Regimes

Peng and Yang [49] constructed the duality between an SDDE and an ABSDE for a diffusion setting. They showed that the solution of an ABSDE can be obtained by the solution of an SDDE. Later, Tu and Hao [58] extended it to a jump-diffusion process. In this section, we establish the relation between the systems of Markovian regime-switching SDDEs and ABSDEs with jumps and regimes similar to Peng and Yang [49] and Tu and Hao [58] for these more general models. Here, in our setting, for both of the processes the Markov chain α and the compensated random measure $\tilde{\Phi}$ generated by the Markov chain can be observed as our contribution.

Let us present the related theorem.

Theorem 4.2. *Suppose $\delta > 0$ is a given constant and $b, \bar{b} \in L^2_{\mathbb{F}}(t-\delta, T+\delta;\mathbb{R})$, $l \in L^2_{\mathbb{F}}(t,T;\mathbb{R})$, $\sigma, \bar{\sigma} \in L^2_{\mathbb{F}}(t-\delta, T+\delta;\mathbb{R})$, $\eta, \bar{\eta} \in H^2_{\mathbb{F}}(t-\delta, T+\delta;\mathbb{R})$, $\gamma, \bar{\gamma} \in M^2_{\mathbb{F}}(t-\delta, T+\delta;\mathbb{R}^D)$ and $b, \bar{b}, \sigma, \bar{\sigma}, \eta, \bar{\eta}, \gamma, \bar{\gamma}$ are uniformly bounded. Then, for all $\xi \in S^2_{\mathbb{F}}(T, T+\delta;\mathbb{R}),\ \psi(t) \in L^2_{\mathbb{F}}(T, T+\delta;\mathbb{R}),\ \zeta \in H^2_{\mathbb{F}}(T, T+\delta;\mathbb{R})$ and*

$\vartheta \in M^2_{\mathbb{F}}(T, T+\delta; \mathbb{R}^D)$, the solution Y of the following ABSDE,

$$
\begin{aligned}
-dY(s) = \Bigg(& b(s,\alpha(s))Y(s) + \bar{b}(s,\alpha(s))E[Y(s+\delta)|\mathcal{F}_s] \\
& + \sigma(s,\alpha(s))Z(s) + \bar{\sigma}(s,\alpha(s))E[Z(s+\delta)|\mathcal{F}_s] \\
& + \int_{\mathbb{R}_0} Q(s,z)\eta(s,\alpha(s-),z)\nu(dz) \\
& + \int_{\mathbb{R}_0} E[Q(s+\delta,z)|\mathcal{F}_s]\bar{\eta}(s,\alpha(s-),z)\nu(dz) \\
& + \sum_{j=1}^{D} V^j(s)\gamma^j(s,\alpha(s-))\lambda_j(s) \\
& + \sum_{j=1}^{D} E[V^j(s+\delta)|\mathcal{F}_s]\bar{\gamma}^j(s,\alpha(s-))\lambda_j(s) + l(s,\alpha(s))\Bigg)dt \\
& - Z(s)dW(s) - \int_{\mathbb{R}_0} Q(s,z)\tilde{N}(ds,dz) - V(s)d\tilde{\Phi}(s),\ s \in [t,T],
\end{aligned}
$$

with terminal values, $Y(s) = \xi(s)$, $Z(s) = \psi(s)$, $Q(s) = \zeta(s)$ and $V(s) = \vartheta(s)$, $s \in [T, T+\delta]$, can be given by the subsequent closed formula:

$$
\begin{aligned}
Y(t) = E\Bigg[& X(T)\xi(T) + \int_t^T X(s)l(s,\alpha(s))ds \\
& + \int_T^{T+\delta} \Big\{ \xi(s)\bar{b}(s-\delta,\alpha(s-\delta))X(s-\delta) \\
& + \psi(s)\bar{\sigma}(s-\delta,\alpha(s-\delta))X(s-\delta) \\
& + \int_{\mathbb{R}_0} \zeta(s,z)\bar{\eta}(s-\delta,\alpha((s-\delta)-),z)X((s-\delta)-)\nu(dz) \\
& + \sum_{j=1}^{D} \vartheta^j(s)\bar{\gamma}^j(s-\delta,\alpha((s-\delta)-))X((s-\delta)-)\lambda_j(s)\Big\} ds \Big| \mathcal{F}_t \Bigg]
\end{aligned}
$$

a.e., a.s., where $X(s)$ is the solution of the following SDDEJR with initial history:

$$
\begin{aligned}
dX(s) = & \Big(b(s,\alpha(s))X(s) + \bar{b}(s-\delta,\alpha(s-\delta))X(s-\delta) \Big) ds \\
& + \Big(X(s)\sigma(s,\alpha(s)) + X(s-\delta)\bar{\sigma}(s-\delta,\alpha(s-\delta)) \Big) dW(s) \\
& + \int_{\mathbb{R}_0} \Big(X(s-)\eta(s,\alpha(s-),z)
\end{aligned}
$$

$$
\begin{aligned}
&\quad + X((s-\delta)-)\bar{\eta}(s-\delta,\alpha((s-\delta)-),z)\Big)\tilde{N}(ds,dz)\\
&\quad + \Big(X(s-)\gamma(s,\alpha(s-)) + X((s-\delta)-)\bar{\gamma}(s-\delta,\alpha(s-\delta)-)\Big)d\tilde{\Phi}(s),\\
&\textit{for } s \in [t,T+\delta], \qquad (4.3)\\
&X(t) = 1,\\
&X(s) = 0, \qquad s \in [t-\delta,t).
\end{aligned}
$$

Proof. First, let us show that Equation (4.3) has a unique solution.

When $s \in [t, t+\delta]$, Equation (4.3) becomes:

$$
\begin{aligned}
dX(s) &= X(s-)\Big\{b(s,\alpha(s))ds + \sigma(s,\alpha(s))dW(s) + \int_{\mathbb{R}_0}\eta(s,\alpha(s-),z)\tilde{N}(ds,dz)\\
&\quad + \gamma(s,\alpha(s-))\Big\}d\tilde{\Phi}(s), \qquad s \in [t,t+\delta], \qquad (4.4)\\
X(t) &= 1.
\end{aligned}
$$

This is an SDE with jumps and regimes without delay and it is known that Equation (4.4) has a unique solution in $L^2_{\mathbb{F}}(0,T;\mathbb{R})$ (cf. [13]). Let $\kappa(\cdot)$ be the solution of Equation (4.4).

For $s \in [t+\delta, T+\delta]$, Equation (4.3) becomes:

$$
\begin{aligned}
dX(s) &= \Big(b(s,\alpha(s))X(s) + \bar{b}(s-\delta,\alpha(s-\delta))X(s-\delta)\Big)ds\\
&\quad + \Big(X(s)\sigma(s,\alpha(s)) + X(s-\delta)\bar{\sigma}(s-\delta,\alpha(s-\delta))\Big)dW(s)\\
&\quad + \int_{\mathbb{R}_0}\Big(X(s-)\eta(s,\alpha(s-),z)\\
&\quad + X((s-\delta)-)\bar{\eta}(s-\delta,\alpha((s-\delta)-),z)\Big)\tilde{N}(dt,dz)\\
&\quad + \Big(X(s-)\gamma(s,\alpha(s-)) + X((s-\delta)-)\bar{\gamma}(s-\delta,\alpha((s-\delta)-))\Big)d\tilde{\Phi}(t),\\
&\quad s \in [t+\delta,T+\delta], \qquad (4.5)\\
X(s) &= \kappa(s), \qquad s \in [t,t+\delta].
\end{aligned}
$$

This is a classical SDDE with jumps and regimes; hence, by Theorem (4.1), it is known that Equation (4.5) has a unique solution.

If we apply product rule to $X(s)Y(s)$ for $s \in [t,T]$ (cf. Lemma 3.2, by Zhang, Elliott and Siu [64] or Appendix), we obtain:

$$
\begin{aligned}
&X(T)Y(T) - X(t)Y(t) \\
&= -\int_t^T X(s-)\bigg\{\bigg(b(s,\alpha(s))Y(s) + \bar{b}(s,\alpha(s))E[Y(s+\delta)|\mathcal{F}_s] \\
&\quad + \sigma(s,\alpha(s))Z(s) + \bar{\sigma}(s,\alpha(s))E[Z(s+\delta)|\mathcal{F}_s] \\
&\quad + \int_{\mathbb{R}_0} (\eta(s,\alpha(s-),z)Q(s,z) + \bar{\eta}(s,\alpha(s-),z)E[Q(s+\delta,z)|\mathcal{F}_s])\nu(dz) \\
&\quad + \sum_{j=1}^{D} (\gamma^j(s,\alpha(s-))V^j(s) + \bar{\gamma}^j(s,\alpha(s-))E[V^j(s+\delta)|\mathcal{F}_s]))\lambda_j(t) \\
&\quad + l(s,\alpha(s))\bigg)ds - Z(s)dW(s) - \int_{\mathbb{R}_0} Q(s,z)\tilde{N}(ds,dz) - V(s)d\tilde{\Phi}(s)\bigg\} \\
&\quad + \int_t^T Y(s)\bigg\{\bigg(b(s,\alpha(s))X(s) + \bar{b}(s-\delta,\alpha(s-\delta))X(s-\delta)\bigg)ds \\
&\quad + \bigg(\sigma(s,\alpha(s))X(s) + \bar{\sigma}(s-\delta,\alpha(s-\delta))X(s-\delta)\bigg)dW(s) \\
&\quad + \int_{\mathbb{R}_0} \bigg(\eta(s,\alpha(s-),z)X(s-) + \bar{\eta}(s-\delta,\alpha((s-\delta)-),z)X((s-\delta)-)\bigg)\tilde{N}(ds,dz) \\
&\quad + \bigg(\gamma(s,\alpha(s-))X(s-) + \bar{\gamma}(s-\delta,\alpha((s-\delta)-)X((s-\delta)-)\bigg)d\Phi(s)\bigg\} \\
&\quad + \int_t^T \bigg\{\bigg(\sigma(s,\alpha(s))X(s) + \bar{\sigma}(s-\delta)X(s-\delta)\bigg)Z(s) + \int_{\mathbb{R}_0}\bigg(\eta(s,\alpha(s-),z) \\
&\quad \times X(s-) + \bar{\eta}(s-\delta,\alpha((s-\delta)-),z)X((s-\delta)-)\bigg)Q(s,z)\nu(dz) \\
&\quad + \sum_{j=1}^{D}\bigg(\gamma^j(s,\alpha(s-))X(s-) + \bar{\gamma}^j(s-\delta,\alpha((s-\delta)-)) \\
&\quad \times X((s-\delta)-)\bigg)V^j(s)\lambda_j(s)\bigg\}ds.
\end{aligned}
$$

Let us arrange the terms and take conditional expectation with respect to $\mathcal{F}_t$. Then, we get:

$$
\begin{aligned}
&E\left[X(T)Y(T) - X(t)Y(t)|\mathcal{F}_t\right] \\
&= E\bigg[\int_t^T \bigg\{\bar{b}(s-\delta,\alpha(s-\delta))Y(s)X(s) - \bar{b}(s,\alpha(s))E[Y(s+\delta)|\mathcal{F}_s]X(s) \\
&\quad + \bar{\sigma}(s-\delta,\alpha(s-\delta))Z(s)X(s-\delta) - \bar{\sigma}(s,\alpha(s))E[Z(s+\delta)|\mathcal{F}_s]X(s) \\
&\quad - l(s,\alpha(s))X(s) - \int_{\mathbb{R}_0} \bar{\eta}(s,\alpha(s-),z)E[Q(s+\delta,z)|\mathcal{F}_s]X(s-)\nu(dz)
\end{aligned}
$$

$$
\begin{aligned}
&- \sum_{j=1}^{D} \bar{\gamma}^j(s,\alpha(s-))E[V^j(s+\delta)|\mathcal{F}_s]X(s-)\lambda_j(s)\\
&+ \int_{\mathbb{R}_0} \bar{\eta}(s-\delta,\alpha((s-\delta)-),z))Q(s,z)X((s-\delta)-)\nu(dz)\\
&+ \sum_{j=1}^{D} \bar{\gamma}^j(s-\delta,\alpha((s-\delta)-))V^j(s)X((s-\delta)-)\Big\}ds|\mathcal{F}_t\Big].
\end{aligned}
$$

We recall that $X(t)=1$ and $X(s)=0$ for $s\in[t-\delta,t)$. By tower property, we obtain:

$$
\begin{aligned}
Y(t) = E\Big[&X(T)Y(T) + \int_t^T X(s)l(s,\alpha(s))ds|\mathcal{F}_t\Big]\\
- E\Big[&\int_t^T X(s-\delta)Y(s)\bar{b}(s-\delta,\alpha(s-\delta))ds\\
&- \int_{t+\delta}^{T+\delta} X(s-\delta)Y(s)\bar{b}(s-\delta,\alpha(s-\delta))ds|\mathcal{F}_t\Big]\\
- E\Big[&\int_t^T X(s-\delta)Z(s)\bar{\sigma}(s-\delta,\alpha(s-\delta))ds\\
&- \int_{t+\delta}^{T+\delta} X(s-\delta)Z(s)\bar{\sigma}(s-\delta,\alpha(s-\delta))ds|\mathcal{F}_t\Big]\\
- E\Big[&\int_t^T \int_{\mathbb{R}_0} X((s-\delta)-)Q(s,z)\bar{\eta}(s-\delta,\alpha((s-\delta)-),z)\nu(dz)ds\\
&- \int_{t+\delta}^{T+\delta} \int_{\mathbb{R}_0} X((s-\delta)-)Q(s,z)\bar{\eta}(s-\delta,\alpha((s-\delta)-),z)\nu(dz)ds|\mathcal{F}_t\Big]\\
- E\Big[&\int_t^T \sum_{j=1}^{D} X((s-\delta)-)V^j(s)\bar{\gamma}^j(s-\delta,\alpha((s-\delta)-))\lambda_j(s)ds\\
&- \int_{t+\delta}^{T+\delta} \sum_{j=1}^{D} X((s-\delta)-)V^j(s)\bar{\gamma}(s-\delta,\alpha((s-\delta)-))\lambda_j(s)ds|\mathcal{F}_t\Big].
\end{aligned}
$$

Hence, we get a closed-form representation for $Y(t)$ as follows:

$$
\begin{aligned}
Y(t) = E\Big[&X(T)\xi(T) + \int_t^T X(s)l(s,\alpha(s))ds\\
&+ \int_T^{T+\delta} \Big\{\xi(s)\bar{b}(s-\delta,\alpha(s-\delta))X(s-\delta)\\
&+ \psi(s)\bar{\sigma}(s-\delta,\alpha(s-\delta))X(s-\delta)
\end{aligned}
$$

$$
\begin{aligned}
&+\int_{\mathbb{R}_0}\zeta(s,z)\bar{\eta}(s-\delta,\alpha((s-\delta)-),z)X((s-\delta)-)\nu(dz)\\
&+\sum_{j=1}^{D}\vartheta^j(s)\bar{\gamma}^j(s-\delta,\alpha((s-\delta)-))X((s-\delta)-)\lambda_j(s)\Big\}ds|\mathcal{F}_t\Big].
\end{aligned}
$$

□

4.3 Existence-Uniqueness Theorem for ABSDEs with Jumps and Regimes

In this final section, we prove our last existence-uniqueness theorem for the following ABSDE with jumps and regimes. Peng and Yang [49] introduced this new form and proved an existence-uniqueness theorem under some technical conditions similar to (a1), (a2), A1.1 and A2.2 (see below). They use the main theorems of BSDEs for their proof. We follow a method as in El-Karoui, Hamadène and Matoussi [17] (see Theorem 1.1) which presents several results related to BSDEs in a diffusion setting. Our contribution (see Theorem 3.1 by Savku and Weber [52]), as previous theorems, is within the framework of Markov regime switches.

Let us introduce a generalized form of BSDEs as follows:

$$
\begin{cases}
-dY(t) = & f(t,Y(t),Z(t),Q(t),V(t),Y(t+\delta_1(t)),Z(t+\delta_2(t)),\\
& Q(t+\delta_3(t)),V(t+\delta_4(t)),\alpha(t))ds - Z(t)dW(t)\\
& -\int_{\mathbb{R}_0} Q(t,z)\tilde{N}(dt,dz) - V(t)d\tilde{\Phi}(t), \qquad t\in[0,T],\\
Y(t) = & \xi(t),\ Z(t)=\psi(t),\ Q(t)=\zeta(t),\ V(t)=\vartheta(t),\ t\in[T,T+K].
\end{cases}
\tag{4.6}
$$

Let $\delta_i(\cdot)$, $i=1,2,3,4$, be $\mathbb{R}^+$-valued continuous functions on $[0,T]$ such that:

(a1) There exists a constant $K\geq 0$ such that for all $t\in[0,T]$ and $i=1,2,3,4$, $t+\delta_i(t)\leq T+K$.

(a2) There exists a constant $L\geq 0$ such that for each $t\in[0,T]$ and for any non-negative integrable function $g(\cdot)$,

$$
\int_t^T g(s+\delta_i(s))ds \leq L\int_t^{T+K} g(s)ds, \qquad \text{for} \qquad i=1,2,3,4.
$$

Assume that for all $t\in[0,T]$ and $e_j\in S$, $f(t,y,z,q,v,\xi,\psi,\zeta,\vartheta,e_j):[0,T]\times\mathbb{R}$ $\times\mathbb{R}\times L^2(\mathcal{B}_0;\mathbb{R})\times L^2(\mathcal{B}_S;\mathbb{R}^D)\times L^2(\mathcal{F}_r;\mathbb{R})\times L^2(\mathcal{F}_{r^*};\mathbb{R})\times L^2(\mathcal{F}_{\hat{r}}\times\mathcal{B}_0;\mathbb{R})$ $\times L^2(\mathcal{F}_{\tilde{r}}\times\mathcal{B}_S;\mathbb{R}^D)\times S\to L^2(\mathcal{F}_t;\mathbb{R})$, where $r,r^*,\hat{r},\tilde{r}\in[t,T+K]$.

Furthermore, f satisfies the following conditions:

A1.1 There exists a constant $C > 0$ such that for all $t \in [0,T],\ e_j \in S,\ y, y, z, z' \in \mathbb{R}$, $q, q' \in L^2(\mathcal{B}_0; \mathbb{R}),\ v, v' \in L^2(\mathcal{B}_S; \mathbb{R}^D),\ \xi, \xi', \psi, \psi' \in L^2_{\mathbb{F}}(t, T+K; \mathbb{R}),\ \zeta, \zeta' \in \mathcal{H}^2_{\mathbb{F}}(t, T+K; \mathbb{R}),\ \vartheta, \vartheta' \in \mathcal{M}^2_{\mathbb{F}}(t, T+K; \mathbb{R}^D)$ and $r, r^*, \hat{r}, \tilde{r} \in [t, T+K]$, we have

$$\begin{aligned}
&|f(t, y, z, q, v, \xi(r), \psi(r^*), \zeta(\hat{r}), \vartheta(\tilde{r}), e_j) - \\
&\qquad f(t, y', z', q', v', \xi'(r), \psi'(r^*), \zeta'(\hat{r}), \vartheta'(\tilde{r}), e_j)| \\
&\le C\Bigg(|y - y'| + |z - z'| + \|q - q'\|_J + \|v - v'\|_S + E\Bigg[|\xi(r) - \xi'(r)| \\
&\qquad + |\psi(r^*) - \psi'(r^*)| + \|\zeta(\hat{r}) - \zeta'(\hat{r})\|_J + \|\vartheta(\tilde{r}) - \vartheta'(\tilde{r})\|_S \,|\mathcal{F}_t \Bigg]\Bigg).
\end{aligned}$$

A2.2 $E\left[\int_0^T |f(t, 0, 0, 0, 0, 0, 0, 0, 0, e_j)|^2\, dt\right] < \infty, \qquad$ for all $e_j \in S$.

Let us give the main result of this section.

Theorem 4.3. *Suppose f fulfills conditions A1.1 and A2.2 and for $i = 1, 2, 3, 4$, δ_i satisfies assumptions (a1) and (a2). Then, for any given terminal variables $\xi(\cdot) \in S^2_{\mathbb{F}}(T, T+K; \mathbb{R})$, $\psi(\cdot) \in L^2_{\mathbb{F}}(T, T+K; \mathbb{R})$, $\zeta(\cdot) \in \mathcal{H}^2_{\mathbb{F}}(T, T+K; \mathbb{R})$ and $\vartheta(\cdot) \in \mathcal{M}^2_{\mathbb{F}}(T, T+K; \mathbb{R}^D)$, the ABSDE* (4.6) *has a unique solution, i.e., there exists a unique 4-tuple of $\mathcal{F}_t$-adapted processes $(Y, Z, Q, V) \in S^2_{\mathbb{F}}(0, T+K; \mathbb{R}) \times L^2_{\mathbb{F}}(0, T+K; \mathbb{R}) \times \mathcal{H}^2_{\mathbb{F}}(0, T+K; \mathbb{R}) \times \mathcal{M}^2_{\mathbb{F}}(0, T+K; \mathbb{R}^D)$ satisfying Equation* (4.6).

Proof. We fix $\beta = 16C^2(L+1)(T+1)$, where C is the Lipschitz constant of f given in condition A1.1 and introduce a norm in the Banach space $S^2_{\mathbb{F}}(0, T+K; \mathbb{R}) \times L^2_{\mathbb{F}}(0, T+K; \mathbb{R}) \times \mathcal{H}^2_{\mathbb{F}}(0, T+K; \mathbb{R}) \times \mathcal{M}^2_{\mathbb{F}}(0, T+K; \mathbb{R}^D)$ as follows:

$$\begin{aligned}
\|(Y(t), Z(t), Q(t), V(t))\|^2_\beta = E\Bigg[\int_0^{T+K} e^{\beta t}\Bigg(&|Y(t)|^2 + |Z(t)|^2 \\
&+ \int_{\mathbb{R}_0} |Q(t,z)|^2 \nu(dz) + \sum_{j=1}^D \left|V^j(t)\right|^2 \lambda_j(t)\Bigg) dt\Bigg].
\end{aligned}$$

It is more convenient to use the equivalent β-norm for applying Banach Fixed Point Theorem. Now we pose the problem,

$$\begin{aligned}
&-dY(t) = \ f(t, y(t), z(t), q(t), v(t), y(t+\delta_1(t)), z(t+\delta_2(t)), q(t+\delta_3(t)), \\
&v(t+\delta_4(t)), \alpha(t))dt - Z(t)dW(t) - \int_{\mathbb{R}_0} Q(t,z)\tilde{N}(dt, dz) - V(t)d\tilde{\Phi}(t),\ t \in [0,T], \\
&Y(t) = \xi(t),\ Z(t) = \psi(t),\ Q(t) = \zeta(t) \text{ and } V(t) = \vartheta(t),\ t \in [T, T+K].
\end{aligned} \tag{4.7}$$

Let us define:

$$\begin{aligned}
h:\ &S^2_{\mathbb{F}}(0, T+K; \mathbb{R}) \times L^2_{\mathbb{F}}(0, T+K; \mathbb{R}) \times \mathcal{H}^2_{\mathbb{F}}(0, T+K; \mathbb{R}) \times \mathcal{M}^2_{\mathbb{F}}(0, T+K; \mathbb{R}^D) \\
&\to S^2_{\mathbb{F}}(0, T+K; \mathbb{R}) \times L^2_{\mathbb{F}}(0, T+K; \mathbb{R}) \times \mathcal{H}^2_{\mathbb{F}}(0, T+K; \mathbb{R}) \times \mathcal{M}^2_{\mathbb{F}}(0, T+K; \mathbb{R}^D).
\end{aligned}$$

For two arbitrary elements $(y(t), z(t), q(t), v(t))$ and $(y'(t), z'(t), q'(t), v'(t))$ in $S^2_{\mathbb{F}}(0, T+K; \mathbb{R}) \times L^2_{\mathbb{F}}(0, T+K; \mathbb{R}) \times \mathcal{H}^2_{\mathbb{F}}(0, T+K; \mathbb{R}) \times \mathcal{M}^2_{\mathbb{F}}(0, T+K; \mathbb{R}^D)$, let us set:

$$h(y(t), z(t), q(t), v(t)) = (Y(t), Z(t), Q(t), V(t)) \qquad \text{and}$$
$$h(y'(t), z'(t), q'(t), v'(t)) = (Y'(t), Z'(t), Q'(t), V'(t)).$$

Furthermore, let us define their differences by

$$(\hat{y}(t), \hat{z}(t), \hat{q}(t), \hat{v}(t)) = (y(t) - y'(t), z(t) - z'(t), q(t) - q'(t), v(t) - v'(t)) \text{ and}$$
$$(\hat{Y}(t), \hat{Z}(t), \hat{Q}(t), \hat{V}(t)) = (Y(t) - Y(t), Z(t) - Z'(t), Q(t) - Q'(t), V(t) - V'(t)).$$

According to the existence-uniqueness results of the BSDEs with jumps and regimes (see Propositions 5.1 and 5.2 by Crépey and Matoussi [14]), the aforementioned equation (4.7) has a unique solution; hence, h is well-defined.

Now we will prove that h is a contraction mapping under the norm $\|\cdot\|_\beta$.

In fact, we apply product rule for regime-switching jump-diffusions (cf. Lemma 3.2 by Zhang, Elliott and Siu [64] or Appendix) and take the expectation:

$$\begin{aligned}
&E[e^{\beta t}(\hat{Y}(t))^2] + E\Big[\int_t^T e^{\beta s}\Big(\Big|\hat{Z}(s)\Big|^2 + \int_{\mathbb{R}}\Big|\hat{Q}(s,z)\Big|^2\nu(dz) + \sum_{j=1}^{D}\Big|\hat{V}^j(s)\Big|^2\lambda_j(s)\Big)ds\Big] \\
&= E\Big[\int_t^T e^{\beta s}\Big(2\hat{Y}(s)\Big(f(s, y(s), z(s), q(s), v(s), y(s+\delta_1(s)), z(s+\delta_2(s)), \\
&\quad q(s+\delta_3(s)), v(s+\delta_4(s)), \alpha(s)) - f(s, y'(s), z'(s), q'(s), v'(s), y'(s+\delta_1(s)), \\
&\quad z'(s+\delta_2(s)), q'(s+\delta_3(s)), v'(s+\delta_4(s)), \alpha(s))\Big)\Big] - \beta(\hat{Y}(s))^2\Big)ds\Big]. \qquad (4.8)
\end{aligned}$$

We note that the terms $2\displaystyle\int_0^t e^{\beta s}\hat{Y}(s)\hat{Z}(s)dW(s)$, $2\displaystyle\int_0^t e^{\beta s}\hat{Y}(s)\hat{Q}(s,z)\tilde{N}(ds,dz)$ and $2\displaystyle\int_0^t e^{\beta s}\hat{Y}(s)\hat{V}(s,z)d\tilde{\Phi}(s)$ are uniformly integrable martingales. Let us show this:

$$\begin{aligned}
&E\Big[\Big(\int_0^T\sum_{j=1}^{D} e^{2\beta t}\Big|\hat{Y}(t)\Big|^2\Big|\hat{V}^j(t)\Big|^2\lambda_j(t)dt\Big)^{\frac{1}{2}}\Big] \\
&\le aE\Big[\sup_{0\le t\le T}\Big|\hat{Y}(t)\Big|\Big(\int_0^T\sum_{j=1}^{D}\Big|\hat{V}^j(t)\Big|^2\lambda_j(t)dt\Big)^{\frac{1}{2}}\Big] \\
&\le \frac{a}{2}E\Big[\sup_{0\le t\le T}\Big|\hat{Y}(t)\Big|^2\Big] + \frac{a}{2}E\Big[\int_0^T\sum_{j=1}^{D}\Big|\hat{V}^j(t)\Big|^2\lambda_j(t)dt\Big];
\end{aligned}$$

since $\hat{Y}(t) \in S^2_{\mathbb{F}}(0, T; \mathbb{R})$ and $\hat{V}(t) \in \mathcal{M}^2_{\mathbb{F}}(0, T; \mathbb{R}^D)$, the associated stochastic integral is a uniformly integrable martingale with null expectation. The others can be obtained similarly.

By Equation (4.8), condition A1.1, assumption (a2) and the inequality $2ab \leq (a^2+b^2)$:

$$\begin{aligned}
&\leq E\bigg[\int_t^T e^{\beta s}\bigg(-\beta\left|\hat{Y}(s)\right|^2 + 2C\left|\hat{Y}(s)\right|\bigg(|\hat{y}(s)| + |\hat{z}(s)| + \|\hat{q}(s)\|_J \\
&\quad + \|\hat{v}(s)\|_S + E\bigg[|\hat{y}(s+\delta_1(s))| + |\hat{z}(s+\delta_2(s))| + \|\hat{q}(s+\delta_3(s))\|_J \\
&\quad + \|\hat{v}(s+\delta_4(s))\|_S\,|\mathcal{F}_s\bigg]\bigg)\bigg)ds\bigg] \\
&\leq E\bigg[\int_t^T e^{\beta s}(-\beta\left|\hat{Y}(s)\right|^2)ds\bigg] + E\bigg[\int_t^T e^{\beta s}2C\left|\hat{Y}(s)\right|\bigg(|\hat{y}(s)| + |\hat{z}(s)| \\
&\quad + \|\hat{q}(s)\|_J + \|\hat{v}(s)\|_S\bigg)ds\bigg] + E\bigg[\int_t^T e^{\beta s}2C\left|\hat{Y}(s)\right|\bigg(|\hat{y}(s+\delta_1(s))| \\
&\quad + |\hat{z}(s+\delta_2(s))| + \|\hat{q}(s+\delta_3(s))\|_J + \|\hat{v}(s+\delta_4(s))\|_S\bigg)ds\bigg] \\
&\leq E\bigg[\int_t^T e^{\beta s}(-\beta\left|\hat{Y}(s)\right|^2)ds\bigg] \\
&\quad + E\bigg[\int_t^T e^{\beta s}\bigg(\frac{\beta}{4}\left|\hat{Y}(s)\right|^2 + \frac{4C^2}{\beta}(|\hat{y}(s)| + |\hat{z}(s)|)^2\bigg)ds\bigg] \\
&\quad + E\bigg[\int_t^T e^{\beta s}\bigg(\frac{\beta}{4}\left|\hat{Y}(s)\right|^2 + \frac{4C^2}{\beta}(\|\hat{q}(s)\|_J + \|\hat{v}(s)\|_S)^2\bigg)ds\bigg] \\
&\quad + E\bigg[\int_t^T e^{\beta s}\bigg(\frac{\beta}{4}\left|\hat{Y}(s)\right|^2 + \frac{4C^2}{\beta}(|\hat{y}(s+\delta_1(s))| + |\hat{z}(s+\delta_2(s))|)^2\bigg)ds\bigg] \\
&\quad + E\bigg[\int_t^T e^{\beta s}\bigg(\frac{\beta}{4}\left|\hat{Y}(s)\right|^2 + \frac{4C^2}{\beta}(\|\hat{q}(s+\delta_3(s))\|_J \\
&\quad + \|\hat{v}(s+\delta_4(s))\|_S)^2\bigg)ds\bigg] \\
&\leq E\bigg[\int_t^T e^{\beta s}\frac{8C^2}{\beta}(|\hat{y}(s)|^2 + |\hat{z}(s)|^2 + \|\hat{q}(s)\|_J^2 + \|\hat{v}(s)\|_S^2)ds\bigg] \\
&\quad + E\bigg[\int_t^T e^{\beta s}\frac{8C^2}{\beta}(|\hat{y}(s+\delta_1(s))|^2 + |\hat{z}(s+\delta_2(s))|^2 + \|\hat{q}(s+\delta_3(s))\|_J^2 \\
&\quad + \|\hat{v}(s+\delta_4(s))\|_S^2)ds\bigg] \\
&\leq \frac{8C^2}{\beta}E\bigg[\int_t^{T+K} e^{\beta s}(|\hat{y}(s)|^2 + |\hat{z}(s)|^2 + \|\hat{q}(s)\|_J^2 + \|\hat{v}(s)\|_S^2)ds\bigg] \\
&\quad + \frac{8C^2L}{\beta}E\bigg[\int_t^{T+K} e^{\beta s}(|\hat{y}(s)|^2 + |\hat{z}(s)|^2 + \|\hat{q}(s)\|_J^2 + \|\hat{v}(s)\|_S^2)ds\bigg]
\end{aligned}$$

$$= \frac{8C^2}{\beta}(L+1)E\left[\int_0^{T+K} e^{\beta s}(|\hat{y}(s)|^2 + |\hat{z}(s)|^2 + \|\hat{q}(s)\|_J^2 + \|\hat{v}(s)\|_S^2)ds\right].$$

In particular,

$$E\left[e^{\beta t}\left|\hat{Y}(t)\right|^2 dt\right] \leq \frac{8C^2}{\beta}(L+1)\left\|(\hat{y}(t), \hat{z}(t), \hat{q}(t), \hat{v}(t))\right\|_\beta^2,$$
$$E\left[\int_0^T e^{\beta t}\left|\hat{Y}(t)\right|^2 dt\right] \leq \frac{8C^2T}{\beta}(L+1)\left\|(\hat{y}(t), \hat{z}(t), \hat{q}(t), \hat{v}(t))\right\|_\beta^2.$$

Hence,

$$E\left[\int_0^{T+K} e^{\beta t}\left(\left|\hat{Y}(t)\right|^2 + \left|\hat{Z}(t)\right|^2 + \int_{\mathbb{R}_0}\left|\hat{Q}(t,z)\right|^2 \nu(dz) + \sum_{j=1}^{D}\left|\hat{V}^j(t)\right|^2 \lambda_j(t)\right)dt\right]$$
$$\leq \frac{8C^2(L+1)(T+1)}{\beta}\left\|(\hat{y}(t), \hat{z}(t), \hat{q}(t), \hat{v}(t))\right\|_\beta^2.$$

Since $\beta = 16C^2(L+1)(T+1)$, we obtain:

$$\left\|(\hat{Y}, \hat{Z}, \hat{Q}, \hat{V})\right\|_\beta \leq \frac{1}{\sqrt{2}}\left\|(\hat{y}, \hat{z}, \hat{q}, \hat{v})\right\|_\beta.$$

Hence, h is a contraction mapping on $S^2_{\mathbb{F}}(0, T+K; \mathbb{R}) \times L^2_{\mathbb{F}}(0, T+K; \mathbb{R}) \times \mathcal{H}^2_{\mathbb{F}}(0, T+K; \mathbb{R}) \times \mathcal{M}^2_{\mathbb{F}}(0, T+K; \mathbb{R}^D)$. Then, by Banach Fixed Point Theorem (see Appendix), Equation (4.6) has a unique solution $(Y, Z, Q, V) \in S^2_{\mathbb{F}}(0, T+K; \mathbb{R}) \times L^2_{\mathbb{F}}(0, T+K; \mathbb{R}) \times \mathcal{H}^2_{\mathbb{F}}(0, T+K; \mathbb{R}) \times \mathcal{M}^2_{\mathbb{F}}(0, T+K; \mathbb{R}^D)$. □

We note that if $\delta_i(\cdot) = \delta \in \mathbb{R}^+$ for all $i = 1, 2, 3, 4$, then one can omit assumption (a2) in the proof and hence, in Theorem 4.3 itself.

In this section, we proved Theorem 4.1 and Theorem 4.3 with a time-dependent delay function, which leads to possible future works in such a setting. In this thesis, we present our applications with a positive constant δ as a delay component.

CHAPTER 5

STOCHASTIC MAXIMUM PRINCIPLE APPROACH

In this chapter, we follow the techniques of Stochastic Maximum Principle and prove the Necessary and Sufficient Maximum Principles under full and partial information for a Markov regime-switching jump-diffusion model with delay. Furthermore, we give an application to finance which shows the optimal consumption rate derived from a cash flow with delay effect. This chapter can be seen as an extension of Øksendal, Sulem and Zhang [47] from a jump-diffusion model with delay to a Markov regime-switching jump-diffusion system with delay. Here, in our setting, the Markov chain α, its jump size γ and the compensated random measure $\tilde{\Phi}$ generated by the Markov chain can be observed as our contribution (cf. Savku and Weber [52]).

5.1 Model Set-up and Control Problem

Let $(\Omega, \mathbb{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a complete probability space, where $\mathbb{F} = (\mathcal{F}_t : t \in [0,T])$. Furthermore, $(\mathcal{F}_t)_{t\geq 0}$ is a right-continuous, $\mathbb{P}$-completed filtration generated by a 1-dimensional Brownian motion $W(\cdot)$, a 1-dimensional Poisson random measure $N(\cdot,\cdot)$ and a D-dimensional Markov chain $\alpha(\cdot)$. We assume that these processes are independent of each other and adapted to $\mathbb{F}$.

Now we represent the controlled Markov regime-switching jump-diffusion with delay:

$$\begin{aligned} X(t) =& b(t, X(t), Y(t), A(t), \alpha(t), u(t))dt \\ &+ \sigma(t, X(t), Y(t), A(t), \alpha(t), u(t))dW(t) \\ &+ \int_{\mathbb{R}_0} \eta(t, X(t-), Y(t-), A(t-), \alpha(t-), u(t-), z)\tilde{N}(dt, dz) \\ &+ \gamma(t, X(t-), Y(t-), A(t-), \alpha(t-), u(t-))d\tilde{\Phi}(t), \quad t \in [0,T], \\ X(t) =& x_0(t), \qquad t \in [-\delta, 0], \end{aligned} \tag{5.1}$$

with delayed and averaged (over the pre-history) states as follows, respectively:

$$Y(t) = X(t-\delta) \qquad \text{and} \qquad A(t) = \int_{t-\delta}^{t} e^{-\rho(t-r)} X(r)dr, \qquad t \in [0,T].$$

Indeed, let x_0 be a continuous, deterministic function, $\rho \geq 0$ be a constant averaging parameter, $\delta > 0$ be a constant delay. Let $\mathcal{U}$ be a non-empty, closed, convex subset of $\mathbb{R}$. We introduce:

$$\begin{aligned}
&b : [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times S \times \mathcal{U} \to \mathbb{R}, \\
&\sigma : [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times S \times \mathcal{U} \to \mathbb{R}, \\
&\eta : [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times S \times \mathcal{U} \times \mathbb{R}_0 \to \mathbb{R}, \\
&\gamma : [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times S \times \mathcal{U} \to \mathbb{R}^D,
\end{aligned}$$

where for all $x, y, a \in \mathbb{R}$, $e_i \in S$, $u \in \mathcal{U}$, $z \in \mathbb{R}_0$ and $t \in [0,T]$, $b(t,x,y,a,e_i,u)$, $\sigma(t,x,y,a,e_i,u)$, $\eta(t,x,y,a,e_i,u,z)$ and $\gamma(t,x,y,a,e_i,u)$ are given $\mathcal{F}_t$-measurable, $\mathcal{C}^1$-functions with respect to x, y, a, u such that for all $x_i = x, y, a, u$, the following condition holds

$$\begin{aligned}
E\bigg[\int_0^T \bigg\{ &\left|\frac{\partial b}{\partial x_i}(t, X(t), Y(t), A(t), \alpha(t), u(t))\right|^2 + \left|\frac{\partial \sigma}{\partial x_i}(t, X(t), Y(t), A(t), \alpha(t), u(t))\right|^2 \\
&+ \int_{\mathbb{R}_0} \left|\frac{\partial \eta}{\partial x_i}(t, X(t-), Y(t-), A(t-), \alpha(t-), u(t-), z)\right|^2 \nu(dz) \\
&+ \sum_{j=1}^D \left|\frac{\partial \gamma}{\partial x_i}(t, X(t-), Y(t-), A(t-), \alpha(t-), u(t-))\right|^2 \lambda_j(t) \bigg\} dt\bigg] < \infty.
\end{aligned}$$

An *admissible control* is a $\mathcal{U}$-valued, $\mathcal{F}_t$-measurable, càdlàg process $u(t)$, $t \in [0,T]$, such that Equation (5.1) has a unique solution and

$$E\bigg[\int_0^T |u(t)|^2 \, dt\bigg] < \infty.$$

We denote the set of all admissible controls by $\mathcal{A}$.

Let us define the performance criterion (objective functional) as follows:

$$J(u) = E\left[\int_0^T f(t, X(t), Y(t), A(t), \alpha(t), u(t))dt + g(X(T), \alpha(T))\right]$$

for all $u \in \mathcal{A}$, where $f : [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times S \times \mathcal{U} \to \mathbb{R}$ and $g : \mathbb{R} \times S \to \mathbb{R}$ are $\mathcal{C}^1$-functions with respect to x, y, a, u, such that for all $x_i = x, y, a, u$,

$$\begin{aligned}
E\bigg[\int_0^T &\left(|f(t, X(t), Y(t), A(t), \alpha(t), u(t))| + \left|\frac{\partial f}{\partial x_i}(t, X(t), Y(t), A(t), \alpha(t), u(t))\right|^2 \right) dt \\
&+ |g(X(T), \alpha(T))| + |g_x(X(T), \alpha(T))|^2 \bigg] < \infty.
\end{aligned}$$

Our problem is to find the optimal control $\hat{u} \in \mathcal{A}$ such that

$$J(\hat{u}) = \sup_{u \in \mathcal{A}} J(u). \tag{5.2}$$

Now, let us define the Hamiltonian as follows:

$$
\begin{aligned}
&H : [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times S \times \mathcal{U} \times \mathbb{R} \times \mathbb{R} \times \mathcal{R} \times \mathbb{R}^D \to \mathbb{R}, \\
&\qquad H(t,x,y,a,e_i,u,p,q,r,w) = f(t,x,y,a,e_i,u) + b(t,x,y,a,e_i,u)p \\
&\qquad\qquad + \sigma(t,x,y,a,e_i,u)q \\
&\qquad\qquad + \int_{\mathbb{R}_0} \eta(t,x,y,a,e_i,u,z) r(t,z)\nu(dz) \\
&\qquad\qquad + \sum_{j=1}^{D} \gamma^j(t,x,y,a,e_i,u) w^j(t)\lambda_{ij},
\end{aligned}
\tag{5.3}
$$

where $\mathcal{R}$ denotes the set of all functions $r : [0,T] \times \mathbb{R}_0 \to \mathbb{R}$, for which the integral in Equation (5.3) converges.

Associated to H, the adjoint, unknown, adapted processes $(p(t) \in \mathbb{R} : t \in [0,T])$, $(q(t) \in \mathbb{R} : t \in [0,T])$, $(r(t,z) \in \mathcal{R} : t \in [0,T], z \in \mathbb{R}_0)$ and $\big(w(t) \in \mathbb{R}^D : t \in [0,T]\big)$ are given by the following ABSDE with jumps and regimes:

$$
\begin{aligned}
dp(t) &= E[\mu(t-)|\mathcal{F}_t]dt + q(t)dW(t) + \int_{\mathbb{R}_0} r(t,z)\tilde{N}(dt,dz) + w(t)d\tilde{\Phi}(t), \qquad (5.4)\\
p(T) &= g_x(X(T),\alpha(T)),
\end{aligned}
$$

where

$$
\begin{aligned}
\mu(t) :=& -\frac{\partial H}{\partial x}(t, X(t), Y(t), A(t), \alpha(t), u(t), p(t), q(t), r(t,\cdot), w(t)) \\
&- \frac{\partial H}{\partial y}(t+\delta, X(t+\delta), Y(t+\delta), A(t+\delta), \alpha(t+\delta), u(t+\delta), p(t+\delta), \\
&q(t+\delta), r(t+\delta,\cdot), w(t+\delta))\mathbf{1}_{[0,T-\delta]}(t) - e^{\rho t}\bigg(\int_t^{t+\delta} \frac{\partial H}{\partial a}(s, X(s), Y(s), \\
&A(s), \alpha(s), u(s), p(s), q(s), r(s,\cdot), w(s)) e^{-\rho s}\mathbf{1}_{[0,T]}(s) ds\bigg).
\end{aligned}
\tag{5.5}
$$

We note that $\mu(t)$ in Equation (5.5) contains future values of the processes $X(s)$, $\alpha(s)$, $u(s),\ p(s),\ q(s),\ r(s,\cdot)$ and $w(s)$ for $s \leq t+\delta$, hence; the BSDE in Equation (5.4) is anticipative (or time advanced). In the following section, we will prove the existence-uniqueness theorem for an ABSDE with jumps and regimes in a general setting and then, we will apply it to a constant delay case, $\delta > 0$, for the rest of the work.

Moreover, the derivatives of $b,\ \sigma,\ \eta,\ \gamma$ with respect to $x,\ y$ and a are bounded. By this assumption, it is easy to check that μ in Equations (5.4)-(5.5) satisfies the Lipschitz condition A1.1 for $p,\ q,\ r,\ w$ and future values of these processes.

Note that by the aforementioned integrability conditions on the derivatives of $b,\ \sigma,\ \eta,\ \gamma$ and $f,$ assumption A2.2 in Theorem 4.3 is satisfied by μ in Equations (5.4)-(5.5).

Furthermore, note that $p(T) = g_x(X(T), \alpha(T))$ (see Equation (5.4)) corresponds to $\xi(\cdot)$ in Theorem 4.3; hence, it has to satisfy $E[|g_x(X(T), \alpha(T))|^2] < \infty$.

5.2 Sufficient Maximum Principle

In this section, we present the sufficient maximum principle and show that under concavity assumptions, maximizing the Hamiltonian provides us the optimal control. For the rest of the work, we will use the following abbreviations and we omit left limit representation for the sake of notational simplicity:

$$\begin{aligned}\frac{\partial \hat{H}}{\partial x}(t) &:= \frac{\partial}{\partial x} H(t, \hat{X}(t), \hat{Y}(t), \hat{A}(t), \alpha(t), \hat{p}(t), \hat{q}(t), \hat{r}(t,z), \hat{w}(t)),\\ \hat{b}(t) &:= b(t, \hat{X}(t), \hat{Y}(t), \hat{A}(t), \alpha(t), \hat{p}(t), \hat{q}(t), \hat{r}(t,z), \hat{w}(t)),\\ b(t) &:= b(t, X(t), Y(t), A(t), \alpha(t), p(t), q(t), r(t,z), w(t)), \text{ etc.}\end{aligned}$$

Theorem 5.1. *Let $\hat{u} \in \mathcal{A}$ with corresponding state processes $\hat{X}(t),\ \hat{Y}(t)$ and $\hat{A}(t)$ and the adjoint processes $\hat{p}(t),\ \hat{q}(t),\ \hat{r}(t,z)$ and $\hat{w}(t)$ assumed to satisfy the SDDEJR in Equation* (5.1) *and the ABSDE with jumps and regimes in Equation* (5.4)*, respectively. Suppose that the following assertions hold:*

1.

$$\begin{aligned}E\bigg[\int_0^T \hat{p}(t)^2 \bigg((\sigma(t) - \hat{\sigma}(t))^2 + \int_{\mathbb{R}_0} (\eta(t,z) - \hat{\eta}(t,z))^2 \nu(dz)\\ + \sum_{j=1}^{D} (\gamma^j(t) - \hat{\gamma}^j(t))^2 \lambda_j(t) \bigg) dt \bigg] < \infty\end{aligned}$$

and

$$\begin{aligned}E\bigg[\int_0^T (X(t) - \hat{X}(t))^2 \bigg\{ \hat{q}^2(t) + \int_{\mathbb{R}_0} \hat{r}^2(t,z)\nu(dz)\\ + \sum_{j=1}^{D} (\hat{w}^j)^2(t)\lambda_j(t) \bigg\} dt < \infty.\end{aligned}$$

2. *For almost all $t \in [0,T]$,*

$$\begin{aligned}&H(t, \hat{X}(t-), \hat{Y}(t-), \hat{A}(t-), \alpha(t-), \hat{u}(t-), \hat{p}(t-), \hat{q}(t), \hat{r}(t,\cdot), \hat{w}(t))\\ &= \max_{u \in \mathcal{U}} H(t, \hat{X}(t-), \hat{Y}(t-), \hat{A}(t-), \alpha(t-), u, \hat{p}(t-), \hat{q}(t), \hat{r}(t,\cdot), \hat{w}(t)).\end{aligned}$$

3. $(x, y, a, u) \mapsto H(t, x, y, a, e_i, u, \hat{p}(t), \hat{q}(t), \hat{r}(t, \cdot), \hat{w}(t))$ *is a concave function for each* $t \in [0, T]$ *almost surely and for each* $e_i \in S$.

4. $g(x, e_i)$ *is a concave function of* x *for each* $e_i \in S$.

Then, $\hat{u}(t)$ *is an optimal control process and* $\hat{X}(t)$, $\hat{Y}(t)$ *and* $\hat{A}(t)$ *are the corresponding controlled state processes.*

Proof. Let $J(u) - J(\hat{u}) = I_1 - I_2$, where

$$I_1 := E\bigg[\int_0^T \bigg\{ f(t, X(t), Y(t), A(t), \alpha(t), u(t)) \\ - f(t, \hat{X}(t), \hat{Y}(t), \hat{A}(t), \alpha(t), \hat{u}(t))\bigg\} dt\bigg]$$

and

$$I_2 := E\bigg[g(X(T), \alpha(T)) - g(\hat{X}(T), \alpha(T))\bigg].$$

Concave functions are bounded from above by their first order Taylor approximation; hence by the concavity of H, we have

$$\begin{aligned}
I_1 = E\bigg[\int_0^T \bigg\{ & H(t, X(t), Y(t), A(t), \alpha(t), u(t), \hat{p}(t), \hat{q}(t), \hat{r}(t, \cdot), \hat{w}(t)) \\
& - H(t, \hat{X}(t), \hat{Y}(t), \hat{A}(t), \alpha(t), \hat{u}(t), \hat{p}(t), \hat{q}(t), \hat{r}(t, \cdot), \hat{w}(t)) \\
& - (b(t) - \hat{b}(t))\hat{p}(t) - (\sigma(t) - \hat{\sigma}(t))\hat{q}(t) \\
& - \int_{\mathbb{R}_0} (\eta(t, z) - \hat{\eta}(t, z))\hat{r}(t, z)\nu(dz) \\
& - \sum_{j=1}^{D} (\gamma^j(t) - \hat{\gamma}^j(t))\hat{w}^j(t)\lambda_j(t) \bigg\}\, dt\bigg] \\
\leq E\bigg[\int_0^T \bigg\{ & \frac{\partial \hat{H}}{\partial x}(t)(X(t) - \hat{X}(t)) + \frac{\partial \hat{H}}{\partial y}(t)(Y(t) - \hat{Y}(t)) \\
& + \frac{\partial \hat{H}}{\partial a}(t)(A(t) - \hat{A}(t)) + \frac{\partial \hat{H}}{\partial u}(t)(u(t) - \hat{u}(t)) - (b(t) - \hat{b}(t))\hat{p}(t) \\
& - (\sigma(t) - \hat{\sigma}(t))\hat{q}(t) - \int_{\mathbb{R}_0} (\eta(t, z) - \hat{\eta}(t, z))\hat{r}(t, z)\nu(dz) \\
& - \sum_{j=1}^{D} (\gamma^j(t) - \hat{\gamma}^j(t))\hat{w}^j(t)\lambda_j(t) \bigg\} dt\bigg].
\end{aligned} \tag{5.6}$$

Via integrating by parts formula and by the concavity of g, we obtain:

$$I_2 \leq E\bigg[\frac{\partial \hat{g}}{\partial x}(T)(X(T) - \hat{X}(T))\bigg]$$

$$
\begin{aligned}
&= E\Big[\hat{p}(T)(X(T)-\hat{X}(T))\Big] \\
&= E\Big[\int_0^T \hat{p}(t)d(X(t)-\hat{X}(t)) + \int_0^T (X(t)-\hat{X}(t))d\hat{p}(t) \\
&\quad + \int_0^T \Big\{ (\sigma(t)-\hat{\sigma}(t))\hat{q}(t) + \int_{\mathbb{R}_0} (\eta(t,z)-\hat{\eta}(t,z))\hat{r}(t,z)\nu(dz) \\
&\quad + \sum_{j=1}^{D} (\gamma^j(t)-\hat{\gamma}^j(t))\hat{w}^j(t)\lambda_j(t)\Big\}dt\Big] \\
&= E\Big[\int_0^T \hat{p}(t)\Big\{ (b(t)-\hat{b}(t))dt + (\sigma(t)-\hat{\sigma}(t))dW(t) \\
&\quad + \int_{\mathbb{R}_0} (\eta(t,z)-\hat{\eta}(t,z))\tilde{N}(dt,dz) + (\gamma(t)-\hat{\gamma}(t))d\tilde{\Phi}(t)\Big\} \\
&\quad + \int_0^T (X(t)-\hat{X}(t))\Big\{E[\hat{\mu}(t)|\mathcal{F}_t]dt + \hat{q}(t)dW(t) + \int_{\mathbb{R}_0} \hat{r}(t,z)\tilde{N}(dt,dz) \\
&\quad + \hat{w}(t)d\tilde{\Phi}(t)\Big\} + \int_0^T \Big\{ (\sigma(t)-\hat{\sigma}(t))\hat{q}(t) + \int_{\mathbb{R}_0} (\eta(t,z)-\hat{\eta}(t,z))\hat{r}(t,z)\nu(dz) \\
&\quad + \sum_{j=1}^{D} (\gamma^j(t)-\hat{\gamma}^j(t))\hat{w}^j(t)\lambda_j(t)\Big\}\, dt\Big]. \qquad (5.7)
\end{aligned}
$$

Note that $X(t) = \hat{X}(t) = x_0(t)$ for all $t \in [-\delta, 0]$. Then, by Equations (5.6)-(5.7), we have

$$
\begin{aligned}
J(u) - J(\hat{u}) &\leq E\Big[\int_0^T \Big\{ \frac{\partial \hat{H}}{\partial x}(t)(X(t)-\hat{X}(t)) + \frac{\partial \hat{H}}{\partial y}(t)(Y(t)-\hat{Y}(t)) \\
&\quad + \frac{\partial \hat{H}}{\partial a}(t)(A(t)-\hat{A}(t)) + \frac{\partial \hat{H}}{\partial u}(t)(u(t)-\hat{u}(t)) \\
&\quad + (X(t)-\hat{X}(t))\hat{\mu}(t)\Big\}\, dt\Big] \\
&= E\Big[\int_\delta^{T+\delta} \Big\{ \frac{\partial \hat{H}}{\partial x}(t-\delta) + \frac{\partial \hat{H}}{\partial y}(t)\mathbf{1}_{[0,T]}(t) + \hat{\mu}(t-\delta)\Big\} \\
&\quad \times (Y(t)-\hat{Y}(t))dt + \int_0^T \frac{\partial \hat{H}}{\partial a}(t)(A(t)-\hat{A}(t))dt \\
&\quad + \int_0^T \frac{\partial \hat{H}}{\partial u}(t)(u(t)-\hat{u}(t))dt\Big]. \qquad (5.8)
\end{aligned}
$$

Substituting $r := t - \delta$, we get

$$
\int_0^T \frac{\partial \hat{H}}{\partial a}(s)(A(t)-\hat{A}(s))ds
$$

$$= \int_0^T \frac{\partial \hat{H}}{\partial a}(s) \int_{s-\delta}^{s} e^{-\rho(s-r)}(X(r) - \hat{X}(r))drds$$
$$= \int_0^T \left(\int_r^{r+\delta} \frac{\partial \hat{H}}{\partial a}(s) e^{-\rho s} \mathbf{1}_{[0,T]}(s)ds \right) e^{\rho r}(X(r) - \hat{X}(r))dr$$
$$= \int_\delta^{T+\delta} \left(\int_{t-\delta}^{t} \frac{\partial \hat{H}}{\partial a}(s) e^{-\rho s} \mathbf{1}_{[0,T]}(s)ds \right) e^{\rho(t-\delta)}(X(t-\delta) - \hat{X}(t-\delta))dt. \tag{5.9}$$

By combining the Equations (5.8)-(5.9), we obtain

$$\begin{aligned} J(u) - J(\hat{u}) \leq E\Bigg[\int_\delta^{T+\delta} \Bigg\{ & \frac{\partial \hat{H}}{\partial x}(t-\delta) + \frac{\partial \hat{H}}{\partial y}(t)\mathbf{1}_{[0,T]}(t) + \hat{\mu}(t-\delta) \\ & + \left(\int_{t-\delta}^{t} \frac{\partial \hat{H}}{\partial a}(s) e^{-\rho s} \mathbf{1}_{[0,T]}(s)ds \right) e^{\rho(t-\delta)} + \Bigg\} (Y(t) - \hat{Y}(t))dt \\ & + \int_0^T \frac{\partial \hat{H}}{\partial u}(t)(u(t) - \hat{u}(t))dt \Bigg] \\ = E\Bigg[\int_0^T & \frac{\partial \hat{H}}{\partial u}(t)(u(t) - \hat{u}(t))dt \Bigg] \leq 0. \end{aligned}$$

Since $\hat{u}(t)$ maximizes $H(t, \hat{X}(t), \hat{Y}(t), \hat{A}(t), \alpha(t), u, \hat{p}(t), \hat{q}(t), \hat{r}(t, \cdot), \hat{w}(t))$, the last inequality holds (see Proposition 2.1 by Ekeland and Temam [15]). Then, we can conclude that $\hat{u}(t)$ is an optimal control for problem (5.2).

□

One of the key facts in this proof is that concave and differentiable functions are bounded from above by their first-order Taylor approximation. Concavity assumptions on H with respect to x, y, a, u and g with respect to x for all $e_i \in S$ have been used in this sense. Furthermore, Proposition 2.1 by Ekeland and Temam [15] works under concavity condition of H with respect to u.

In the next section, we present Necessary Maximum Principle by which one can determine the candidate optimal control processes, but for verification the concavity condition is necessary.

5.3 Necessary Maximum Principle

Let $\hat{u}$ be an optimal control process and β be any other control process, satisfying $\hat{u} + \beta =: v' \in \mathcal{A}$. Since $\mathcal{U}$ is a convex set, for any $v' \in \mathcal{A}$, the perturbed control process $u^s = \hat{u} + s(v' - \hat{u}),\ 0 < s < 1$, is also in $\mathcal{A}$. The directional derivative of the performance criterion $J(\cdot)$ at $\hat{u}$ in the direction of β is given by:

$$\frac{d}{ds} J(\hat{u} + s\beta)|_{s=0} := \lim_{s \to 0^+} \frac{J(\hat{u} + s\beta) - J(\hat{u})}{s}.$$

Since $\hat{u}$ is an optimal control, a necessary condition for optimality is

$$\frac{d}{ds} J(\hat{u} + s\beta)|_{s=0} \leq 0.$$

Let us assume that the derivative process $\xi(t) = \dfrac{d}{ds} X^{u+s\beta}(t)\,|_{s=0}$ for $t \in [0, T]$ exists and it is defined as follows:

$$\begin{aligned} d\xi(t) = &\left\{ \frac{\partial b}{\partial x}(t)\xi(t) + \frac{\partial b}{\partial y}(t)\xi(t-\delta) + \frac{\partial b}{\partial a}(t) \int_{t-\delta}^{t} e^{-\rho(t-r)}\xi(r)dr + \frac{\partial b}{\partial u}(t)\beta(t) \right\} dt \\ &+ \left\{ \frac{\partial \sigma}{\partial x}(t)\xi(t) + \frac{\partial \sigma}{\partial y}(t)\xi(t-\delta) + \frac{\partial \sigma}{\partial a}(t) \int_{t-\delta}^{t} e^{-\rho(t-r)}\xi(r)dr \right. \\ &+ \left. \frac{\partial \sigma}{\partial u}(t)\beta(t) \right\} dW(t) + \int_{\mathbb{R}_0} \left\{ \frac{\partial \eta}{\partial x}(t,z)\xi(t) + \frac{\partial \eta}{\partial y}(t,z)\xi(t-\delta) \right. \\ &+ \left. \frac{\partial \eta}{\partial a}(t,z) \int_{t-\delta}^{t} e^{-\rho(t-r)}\xi(r)dr + \frac{\partial \eta}{\partial u}(t,z)\beta(t) \right\} \tilde{N}(dt,dz) + \left\{ \frac{\partial \gamma}{\partial x}(t)\xi(t) \right. \\ &+ \left. \frac{\partial \gamma}{\partial y}(t)\xi(t-\delta) + \frac{\partial \gamma}{\partial a}(t) \int_{t-\delta}^{t} e^{-\rho(t-r)}\xi(r)dr + \frac{\partial \gamma}{\partial u}(t)\beta(t) \right\} d\tilde{\Phi}(t), \end{aligned} \tag{5.10}$$

where we know that

$$\begin{aligned} \frac{d}{ds} Y^{u+s\beta}(t)|_{s=0} &= \frac{d}{ds} X^{u+s\beta}(t-\delta)|_{s=0} = \xi(t-\delta), \\ \frac{d}{ds} A^{u+s\beta}(t)|_{s=0} &= \frac{d}{ds}\left(\int_{t-\delta}^{t} e^{-\rho(t-r)} X^{u+s\beta}(r)dr \right)|_{s=0} \\ &= \int_{t-\delta}^{t} e^{-\rho(t-r)} \frac{d}{ds} X^{u+s\beta}(r)|_{s=0} dr = \int_{t-\delta}^{t} e^{-\rho(t-r)} \xi(r)dr, \end{aligned}$$

and we have used the following abbreviations:

$$\frac{\partial b}{\partial x}(t) := \frac{\partial b}{\partial x}(t, X(t), Y(t), A(t), \alpha(t), u(t)), \text{ etc.}$$

Note that $\xi(t) = 0$ for all $t \in [-\delta, 0]$.

Theorem 5.2. *Let $\hat{u} \in \mathcal{A}$ be an optimal control of problem in Equation* (5.2) *subject to the controlled system* (5.1) *and let $(\hat{p}(t), \hat{q}(t), \hat{r}(t,z), \hat{w}(t))$ be the unique solution* *of Equation* (5.4). *Moreover, let us assume that*

$$
\begin{aligned}
&E\Big[\int_0^T \hat{p}^2(t)\Big\{\Big(\frac{\partial\hat{\sigma}}{\partial x}\Big)^2(t)\hat{\xi}^2(t)+\Big(\frac{\partial\hat{\sigma}}{\partial y}\Big)^2(t)\hat{\xi}^2(t-\delta)+\Big(\frac{\partial\hat{\sigma}}{\partial a}\Big)^2(t)\\
&\Big(\int_{t-\delta}^t e^{-\rho(t-r)}\hat{\xi}(r)dr\Big)^2+\Big(\frac{\partial\hat{\sigma}}{\partial u}\Big)^2(t)\beta^2(t)+\int_{\mathbb{R}_0}\Big\{\Big(\frac{\partial\hat{\eta}}{\partial x}\Big)^2(t,z)\hat{\xi}^2(t)\\
&+\Big(\frac{\partial\hat{\eta}}{\partial y}\Big)^2(t,z)\hat{\xi}^2(t-\delta)+\Big(\frac{\partial\hat{\eta}}{\partial a}\Big)^2(t,z)\Big(\int_{t-\delta}^t e^{-\rho(t-r)}\hat{\xi}(r)dr\Big)^2\\
&+\Big(\frac{\partial\hat{\eta}}{\partial u}\Big)^2(t,z)\beta^2(t)\Big\}\,\nu(dz)+\sum_{j=1}^D\Big\{\Big(\frac{\partial\hat{\gamma}^j}{\partial x}\Big)^2(t)\hat{\xi}^2(t)\\
&+\Big(\frac{\partial\hat{\gamma}^j}{\partial y}\Big)^2(t)\hat{\xi}^2(t-\delta)+\Big(\frac{\partial\hat{\gamma}^j}{\partial a}\Big)^2(t)\Big(\int_{t-\delta}^t e^{-\rho(t-r)}\hat{\xi}(r)dr\Big)^2\\
&+\Big(\frac{\partial\hat{\gamma}^j}{\partial u}\Big)^2(t)\beta^2(t)\Big\}\,\lambda_j(t)\Big\}\,dt\Big]<\infty
\end{aligned}
$$

and

$$
E\Big[\int_0^T \hat{\xi}^2(t)\Big\{\hat{q}^2(t)+\int_{\mathbb{R}_0}(\hat{r})^2(t,z)\nu(dz)+\sum_{j=1}^D \hat{w}^{j2}(t)\lambda_j(t)\Big\}\,dt\Big]<\infty.
$$

Then, for any $v\in\mathcal{U}$, *we have*

$$
\frac{\partial H}{\partial u}(t,\hat{X}(t),\hat{Y}(t),\hat{A}(t),\alpha(t),\hat{u}(t),\hat{p}(t),\hat{q}(t),\hat{r}(t,\cdot),\hat{w}(t))(v-\hat{u}(t))\le 0,
$$

dt*-a.e,* $\mathbb{P}$*-a.s.*

Proof. For simplicity of notation, let be $\hat{u}=u$, $\hat{X}=X$, $\hat{Y}=Y$, $\hat{p}=p$, $\hat{q}=q$, $\hat{r}=r$ and $\hat{w}=w$. Then,

$$
\begin{aligned}
0&\ge\frac{d}{ds}J(u+s\beta)|_{s=0}\\
&=\frac{d}{ds}E\Big[\int_0^T f(t,X^{u+s\beta}(t),Y^{u+s\beta}(t),A^{u+s\beta}(t),\alpha(t),u(t)+s\beta)dt\\
&\qquad+g(X^{u+s\beta}(T),\alpha(T))\Big]\Big|_{s=0}\\
&=E\Big[\int_0^T\Big\{\frac{\partial f}{\partial x}(t)\xi(t)+\frac{\partial f}{\partial y}(t)\xi(t-\delta)+\frac{\partial f}{\partial a}(t)\int_{t-\delta}^t e^{-\rho(t-r)}\xi(r)dr\\
&\qquad+\frac{\partial f}{\partial u}(t)\beta(t)\Big\}dt+\frac{\partial g}{\partial x}(X(T),\alpha(T))\xi(T)\Big]
\end{aligned}
$$

$$
\begin{aligned}
&= E\Big[\int_0^T \Big\{ \frac{\partial H}{\partial x}(t) - \frac{\partial b}{\partial x}(t)p(t) - \frac{\partial \sigma}{\partial x}(t)q(t) - \int_{\mathbb{R}_0} \frac{\partial \eta}{\partial x}(t,z)r(t,z)\nu(dz) \\
&\quad - \sum_{j=1}^{D} \frac{\partial \gamma^j}{\partial x}(t)w^j(t)\lambda_j(t)\Big\}\,\xi(t)dt + \int_0^T \Big\{ \frac{\partial H}{\partial y}(t) - \frac{\partial b}{\partial y}(t)p(t) - \frac{\partial \sigma}{\partial y}(t)q(t) \\
&\quad - \int_{\mathbb{R}_0} \frac{\partial \eta}{\partial y}(t,z)r(t,z)\nu(dz) - \sum_{j=1}^{D} \frac{\partial \gamma^j}{\partial y}(t)w^j(t)\lambda_j(t)\Big\}\,\xi(t-\delta)dt \\
&\quad + \int_0^T \Big\{ \frac{\partial H}{\partial a}(t) - \frac{\partial b}{\partial a}(t)p(t) - \frac{\partial \sigma}{\partial a}(t)q(t) - \int_{\mathbb{R}_0} \frac{\partial \eta}{\partial a}(t,z)r(t,z)\nu(dz) \\
&\quad - \sum_{j=1}^{D} \frac{\partial \gamma^j}{\partial a}(t)w^j(t)\lambda_j(t)\Big\}\Big(\int_{t-\delta}^{t} e^{-\rho(t-r)}\xi(r)dr\Big)dt + \int_0^T \Big\{ \frac{\partial H}{\partial u}(t) \\
&\quad - \frac{\partial b}{\partial u}(t)p(t) - \frac{\partial \sigma}{\partial u}(t)q(t) - \int_{\mathbb{R}_0} \frac{\partial \eta}{\partial u}(t,z)r(t,z)\nu(dz) \\
&\quad - \sum_{j=1}^{D} \frac{\partial \gamma^j}{\partial u}(t)w^j(t)\lambda_j(t)\Big\}\,\beta(t)dt + \frac{\partial g}{\partial x}(X(T),\alpha(T))\xi(T)\Big]. \qquad (5.11)
\end{aligned}
$$

Via Equation (5.10) and integration by parts, we get:

$$
\begin{aligned}
&E\Big[\frac{\partial g}{\partial x}(X(T),\alpha(T))\xi(T)\Big] = E\Big[p(T)\xi(T)\Big] \\
&= E\Big[\int_0^T p(t)d\xi(t) + \int_0^T \xi(t)dp(t) + \int_0^T q(t)\Big\{ \frac{\partial \sigma}{\partial x}(t)\xi(t) + \frac{\partial \sigma}{\partial y}(t)\xi(t-\delta) \\
&+ \frac{\partial \sigma}{\partial a}(t)\int_{t-\delta}^{t} e^{-\rho(t-r)}\xi(r)dr + \frac{\partial \sigma}{\partial u}(t)\beta(t)\Big\}\,dt + \int_0^T \int_{\mathbb{R}_0} r(t,z)\Big\{\frac{\partial \eta}{\partial x}(t,z)\xi(t) \\
&+ \frac{\partial \eta}{\partial y}(t,z)\xi(t-\delta) + \frac{\partial \eta}{\partial a}(t)\int_{t-\delta}^{t} e^{-\rho(t-r)}\xi(r)dr + \frac{\partial \eta}{\partial u}(t)\beta(t)\Big\}\nu(dz)dt \\
&+ \int_0^T \sum_{j=1}^{D} w^j(t)\Big\{ \frac{\partial \gamma^j}{\partial x}(t)\xi(t) + \frac{\partial \gamma^j}{\partial y}(t)\xi(t-\delta) + \frac{\partial \gamma^j}{\partial a}(t)\int_{t-\delta}^{t} e^{-\rho(t-r)}\xi(r)dr \\
&+ \frac{\partial \gamma^j}{\partial u}(t)\beta(t)\Big\}\,\lambda_j(t)dt\Big]. \qquad (5.12)
\end{aligned}
$$

By Equations (5.11)-(5.12), we obtain:

$$
\begin{aligned}
&0 \geq \frac{d}{ds}J(u+s\beta)|_{s=0} \\
&= E\Big[\int_0^T \Big\{ \frac{\partial H}{\partial x}(t)\xi(t) + \frac{\partial H}{\partial y}(t)\xi(t-\delta) + \frac{\partial H}{\partial a}(t)\int_{t-\delta}^{t} e^{-\rho(t-r)}\xi(r)dr
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{\partial H}{\partial u}(t)\beta(t) + \xi(t)E[\mu(t)|\mathcal{F}_t]\Big\} \, dt\Big] \\
&= E\Big[\int_0^T \xi(t)\Big\{ \frac{\partial H}{\partial x}(t) - \frac{\partial H}{\partial x}(t) - \frac{\partial H}{\partial y}(t+\delta)\mathbf{1}_{[0,T-\delta]}(t) \\
&\quad - e^{\rho t}\Big(\int_t^{t+\delta} \frac{\partial H}{\partial a}(s)e^{-\rho s}\mathbf{1}_{[0,T-\delta]}(s)ds\Big)\Big\} \, dt + \int_0^T \frac{\partial H}{\partial y}(t)\xi(t-\delta)dt \\
&\quad + \int_0^T \Big(\int_{s-\delta}^s e^{-\rho(s-t)}\xi(t)dt\Big)\frac{\partial H}{\partial a}(s)ds + \int_0^T \frac{\partial H}{\partial u}(t)\beta(t)dt\Big] \\
&= E\Big[\int_0^T \xi(t)\Big\{ -\frac{\partial H}{\partial y}(t+\delta)\mathbf{1}_{[0,T-\delta]}(t) \\
&\quad - e^{\rho t}\Big(\int_t^{t+\delta} \frac{\partial H}{\partial a}(s)e^{-\rho s}\mathbf{1}_{[0,T-\delta]}(s)ds\Big)\Big\} \, dt + \int_0^T \frac{\partial H}{\partial y}(t)\xi(t-\delta)dt \\
&\quad + e^{\rho t}\int_0^T \Big(\int_t^{t+\delta} \frac{\partial H}{\partial a}(s)e^{-\rho s}\mathbf{1}_{[0,T-\delta]}(s)ds\Big)\xi(t)dt + \int_0^T \frac{\partial H}{\partial u}(t)\beta(t)dt\Big] \\
&= E\Big[\int_0^T \frac{\partial H}{\partial u}(t)\beta(t)dt\Big].
\end{aligned}
$$

Let $\beta(t) = v'(t) - u(t)$. Since $u(t)$ is optimal, we have

$$
\frac{d}{ds}J(u + s(v' - u))|_{s=0} = \Big[\int_0^T \frac{\partial H}{\partial u}(t)(v'(t) - u(t))dt\Big] \leq 0.
$$

Let us define

$$
v'(t) := \begin{cases} v, & \text{on } B \times (t_0, t_0 + h), \\ u(t), & \text{otherwise,} \end{cases}
$$

for any deterministic element $v \in \mathcal{U}$ and for any element B of $\mathcal{F}_t$. Then,

$$
E\Big[\int_0^T \frac{\partial H}{\partial u}(t)(v'(t) - u(t))dt\Big] = E\Big[\int_{t_0}^{t_0+h} \frac{\partial H}{\partial u}(t)(v - u(t))\mathbf{1}_B dt\Big].
$$

Dividing by h and taking the limit, we get

$$
\begin{aligned}
&\lim_{h\to 0} \frac{1}{h}E\Big[\int_{t_0}^{t_0+h} \frac{\partial H}{\partial u}(t)(v - u(t))\mathbf{1}_B dt\Big] \\
&\qquad = E\Big[\frac{\partial H}{\partial u}(t_0)(v - u(t_0))\mathbf{1}_B\Big] \leq 0 \qquad a.e.
\end{aligned}
$$

for all $B \in \mathcal{F}_{t_0}$; this implies that

$$
E\Big[\frac{\partial H}{\partial u}(t_0)(v - u(t_0))|\mathcal{F}_{t_0}\Big] \leq 0.
$$

Since the quantity inside the conditional expectation is $\mathcal{F}_{t_0}$-measurable, then the inequality in Theorem 5.2 holds $dt - a.e.$, $\mathbb{P} - a.s.$, for all $v \in \mathcal{U}$. $\square$

5.4 Sufficient Maximum Principle under Partial Information

In this section, we establish a maximum principle of sufficient type under partial information. Under the assumptions of Section 5.1, this theorem is the extension of Øksendal, Sulem and Zhang [47] to a Markov regime-switching model.

Let us introduce $\mathcal{E}_t \subseteq \mathcal{F}_t$, $t \in [0,T]$, the subfiltration of $\{\mathcal{F}_t\}_{t\in[0,T]}$ which represents the information available to the controller who decides on the value of $u(t)$ at time t. For example, we may consider $\mathcal{E}_t = \mathcal{F}_{(t-d)^+}$ for some given $d > 0$; this then signifies a retarded flow of information or a retarded learning.

Let $\mathcal{A}_{\mathcal{E}}$ be a given family of admissible control processes $u(t)$, $t \in [0,T]$, included in the set of càdlàg, $\mathcal{E}$-adapted, $\mathcal{U}$-valued processes such that Equation (5.1) has a unique solution.

Theorem 5.3. *Let $\hat{u} \in \mathcal{A}_{\mathcal{E}}$ with corresponding state processes $\hat{X}(t)$, $\hat{Y}(t)$ and $\hat{A}(t)$ and the adjoint processes $\hat{p}(t)$, $\hat{q}(t)$, $\hat{r}(t,z)$ and $\hat{w}(t)$ assumed to satisfy the SDDEJR* (5.1) *and the ABSDE with jumps and regimes as given in Equation* (5.4)*, respectively. Suppose that the following conditions hold:*

1.
$$E\Big[\int_0^T \hat{p}(t)^2\Big((\sigma(t)-\hat{\sigma}(t))^2 + \int_{\mathbb{R}_0}(\eta(t,z)-\hat{\eta}(t,z))^2\nu(dz) + \sum_{j=1}^{D}(\gamma^j(t)-\hat{\gamma}^j(t)\Big)^2\lambda_j(t))dt\Big] < \infty$$

   and
$$E\left[\int_0^T (X(t)-\hat{X}(t))^2\left\{\hat{q}^2(t) + \int_{\mathbb{R}_0}\hat{r}^2(t,z)\nu(dz) + \sum_{j=1}^{D}(\hat{w}^j)^2(t)\lambda_j(t)dt < \infty.\right.\right.$$

2. *For almost all $t \in [0,T]$,*
$$E\left[H(t,\hat{X}(t),\hat{Y}(t),\hat{A}(t),\alpha(t),\hat{u}(t),\hat{p}(t),\hat{q}(t),\hat{r}(t,\cdot),\hat{w}(t))|\mathcal{E}_t\right] = \max_{u\in\mathcal{U}} E\left[H(t,\hat{X}(t),\hat{Y}(t),\hat{A}(t),\alpha(t),u,\hat{p}(t),\hat{q}(t),\hat{r}(t,\cdot),\hat{w}(t))|\mathcal{E}_t\right].$$

3. $(x,y,a,u) \mapsto H(t,x,y,a,e_i,u,\hat{p}(t),\hat{q}(t),\hat{r}(t,\cdot),\hat{w}(t))$ *is a concave function for each $t \in [0,T]$ almost surely and for each $e_i \in S$.*

4. $g(x,e_i)$ *is a concave function of x for each $e_i \in S$.*

Then, $\hat{u}(t)$ is an optimal control process and $\hat{X}(t),\ \hat{Y}(t),\ \hat{A}(t)$ are the corresponding controlled state processes for problem (5.2).

Proof. By the methods of Theorem 5.1, we obtained Equations (5.8)-(5.9). For the sake of completeness, we give the rest of the proof. Then,

$$\begin{aligned} J(u) - J(\hat{u}) &\le E\Big[\int_0^T \frac{\partial \hat{H}}{\partial u}(t)(u(t) - \hat{u}(t))dt\Big] \\ &= E\Big[\int_0^T E\Big[\frac{\partial \hat{H}}{\partial u}(t)(u(t) - \hat{u}(t))|\mathcal{E}_t\Big]dt\Big] \\ &= E\Big[\int_0^T E\Big[\frac{\partial \hat{H}}{\partial u}(t)|\mathcal{E}_t\Big](u(t) - \hat{u}(t))dt\Big] \\ &\le 0. \end{aligned}$$

Since $\hat{u}(t)$ maximizes $E\Big[H(t, \hat{X}(t), \hat{Y}(t), \hat{A}(t), \alpha(t), u, \hat{p}(t), \hat{q}(t), \hat{r}(t,\cdot), \hat{w}(t))|\mathcal{E}_t\Big]$, the last inequality holds. Hence, $\hat{u}(t)$ is an optimal control. □

5.5 Necessary Maximum Principle under Partial Information

In this section, we will provide a necessary maximum principle under partial information which is the extension of the result by Øksendal, Sulem and Zhang [47] to a Markov regime-switching model. Let us represent the technical assumptions as follows:

B1. For all $u \in \mathcal{A}_{\mathcal{E}}$ and all bounded $\beta \in \mathcal{A}_{\mathcal{E}}$, there exists $\epsilon > 0$ such that $u + s\beta \in \mathcal{A}_{\mathcal{E}}$ for all $s \in (-\epsilon, \epsilon)$.

B2. For all $t_0 \in [0,T]$ and all bounded $\mathcal{E}_{t_0}$-measurable random variables v, the control process $\beta(t)$, defined by

$$\beta(t) = v\mathbf{1}_{[t_0,T]}(t), \qquad t \in [0,T], \tag{5.13}$$

belongs to $\mathcal{A}_{\mathcal{E}}$.

B3. For all bounded $\beta \in \mathcal{A}_{\mathcal{E}}$, the derivative process

$$\xi(t) := \frac{d}{ds}X^{u+s\beta}(t)|_{s=0}$$

exists as described by Equation (5.10).

Theorem 5.4. *Let $\hat{u} \in \mathcal{A}_{\mathcal{E}}$ with corresponding solutions $\hat{X}(t),\ \hat{Y}(t)$ and $\hat{A}(t)$ of Equation* (5.1) *and $\hat{p}(t),\ \hat{q}(t),\ \hat{r}(t,z)$ and $\hat{w}(t)$ of Equation* (5.4) *and corresponding derivative process $\hat{\xi}(t)$ given by Equation* (5.10)*. Moreover, we assume that*

$$E\Big[\int_0^T \hat{p}^2(t)\Big\{ \Big(\frac{\partial\hat{\sigma}}{\partial x}\Big)^2(t)\xi^2(t) + \Big(\frac{\partial\hat{\sigma}}{\partial y}\Big)^2(t)\hat{\xi}^2(t-\delta) + \Big(\frac{\partial\hat{\sigma}}{\partial a}\Big)^2(t)$$
$$\Big(\int_{t-\delta}^t e^{-\rho(t-r)}\hat{\xi}(r)dr\Big)^2 + \Big(\frac{\partial\hat{\sigma}}{\partial u}\Big)^2(t)\beta^2(t) + \int_{\mathbb{R}_0}\Big\{ \Big(\frac{\partial\hat{\eta}}{\partial x}\Big)^2(t,z)\hat{\xi}^2(t)$$
$$+ \Big(\frac{\partial\hat{\eta}}{\partial y}\Big)^2(t,z)\hat{\xi}^2(t-\delta) + \Big(\frac{\partial\hat{\eta}}{\partial a}\Big)^2(t,z)\Big(\int_{t-\delta}^t e^{-\rho(t-r)}\hat{\xi}(r)dr\Big)^2$$
$$+ \Big(\frac{\partial\hat{\eta}}{\partial u}\Big)^2(t,z)\beta^2(t)\Big\}\,\nu(dz) + \sum_{j=1}^D\Big\{ \Big(\frac{\partial\hat{\gamma}^j}{\partial x}\Big)^2(t)\hat{\xi}^2(t) + \Big(\frac{\partial\hat{\gamma}^j}{\partial y}\Big)^2(t)\hat{\xi}^2(t-\delta)$$
$$+ \Big(\frac{\partial\hat{\gamma}^j}{\partial a}\Big)^2(t)\Big(\int_{t-\delta}^t e^{-\rho(t-r)}\hat{\xi}(r)dr\Big)^2 + \Big(\frac{\partial\hat{\gamma}^j}{\partial u}\Big)^2(t)\beta^2(t)\Big\}\,\lambda_j(t)\Big\}\,dt\Big] < \infty$$

and

$$E\Big[\int_0^T (\hat{\xi})^2(t)\Big\{ (\hat{q})^2(t) + \int_{\mathbb{R}_0}(\hat{r})^2(t,z)\nu(dz) + \sum_{j=1}^D(\hat{w}^j)^2(t)\lambda_j(t)\Big\}\,dt\Big] < \infty.$$

Then the following equations are equivalent:

(iii) *For all bounded $\beta \in \mathcal{A}_{\mathcal{E}}$,*

$$\frac{d}{ds}J(\hat{u}+s\beta)\,|_{s=0} = 0\ .$$

(iv) *For all $t \in [0,T]$,*

$$E\left[\frac{\partial H}{\partial u}(t,\hat{X}(t),\hat{Y}(t),\hat{A}(t),u,\alpha(t),\hat{p}(t),\hat{q}(t),\hat{r}(t,\cdot),\hat{w}(t))|\mathcal{E}_t\right]_{u=\hat{u}(t)} = 0\ a.s.$$

Proof. By the methods of Theorem 5.2, we obtained Equations (5.11)-(5.12). For the sake of completeness, we give the reminder of the proof. In fact,

$$0 = \frac{d}{ds}J(u+s\beta)|_{s=0} = E\Big[\int_0^T \frac{\partial H}{\partial u}(t)\beta(t)dt\Big].$$

Let us consider $\beta(t) = v(\omega)\mathbf{1}_{[s,T]}(t)$ in Equation (5.13), where $v(\omega)$ is bounded and $\mathcal{E}_{t_0}$-measurable, $s \geq t_0$. Hence, we get

$$E\Big[\int_s^T \frac{\partial}{\partial u}H(t)vdt\Big] = 0.$$

Differentiating with respect to s, we get

$$E\left[\frac{\partial}{\partial u}H(s)v\right] = 0$$

for all $s \geq t_0$ and for all v. Hence, we obtain

$$E\left[\frac{\partial}{\partial u}H(t_0)|\mathcal{E}_{t_0}\right] = 0.$$

This shows that (iii) implies (iv).
Since every bounded $\beta \in \mathcal{A}_{\mathcal{E}}$ can be approximated by linear combinations of controls β of the form (5.13), i.e, by so-called simple processes having the form of step functions, if we reverse the above steps, we show that (iv) implies (iii). □

5.6 An Application to Finance

Let $b(t,\alpha(t)),\ \sigma(t,\alpha(t)),\ \eta(t,\alpha(t),z)$ and $\gamma(t,\alpha(t))$ be given bounded, adapted processes. Let us consider a cash flow $X^0(t)$ with the dynamics:

$$\begin{cases} dX^0(t) = X(t-\delta)\Big[b(t,\alpha(t))dt + \sigma(t,\alpha(t))dW(t) \\ \qquad\qquad + \displaystyle\int_{\mathbb{R}_0} \eta(t,\alpha(t),z)\tilde{N}(dt,dz) + \gamma(t,\alpha(t))d\tilde{\Phi}(t)\Big], \qquad t \in [0,T], \\ X^0(t) = x_0(t), \qquad t \in [-\delta, 0], \end{cases}$$

where $x_0(t)$ is a given $\mathcal{F}_t$-measurable, continuous, non-negative and deterministic function.

A consumption rate $c(t) \geq 0$ is a càdlàg, $\mathcal{F}_t$-adapted process process satisfying

$$E\left[\int_0^t |c(t)|^2\, dt\right] < \infty.$$

Hence the dynamics of the net cash flow $X(t) = X^c(t)$ is given by

$$\begin{cases} dX(t) = (X(t-\delta)b(t,\alpha(t)) - c(t))dt + X(t-\delta)\sigma(t,\alpha(t))dW(t) \\ \qquad\qquad +X(t-\delta)\displaystyle\int_{\mathbb{R}_0} \eta(t,\alpha(t),z)\tilde{N}(dt,dz) \\ \qquad\qquad +X(t-\delta)\gamma(t,\alpha(t))d\tilde{\Phi}(t), \qquad t \in [0,T], \\ X(t) = x_0(t), \qquad t \in [-\delta, 0]. \end{cases} \tag{5.14}$$

Let $U(t,c,e_i,\omega) : [0,T] \times \mathbb{R}^+ \times S \times \Omega \to \mathbb{R}$ be a given stochastic utility function at each $i = 1, 2, ..., D$, so that it is a slightly more general way of modeling here; in fact

U also depends on ω whose notation will be suppressed. Furthermore, U satisfies the following conditions:

$$\begin{aligned}
&t \mapsto U(t,c,e_i) \text{ is } \mathcal{F}_t\text{-adapted for each } c \geq 0 \text{ and } e_i \in S,\\
&c \mapsto U(t,c,e_i) \text{ is } \mathcal{C}^1 \text{ and } \frac{\partial U}{\partial c}(t,c,e_i) > 0 \text{ for each } e_i \in S,\\
&c \mapsto \frac{\partial U}{\partial c}(t,c,e_i) \text{ is strictly decreasing for each } e_i \in S,\\
&\lim_{c\to\infty} \frac{\partial U}{\partial c}(t,c,e_i) = 0 \text{ for all } (t,e_i) \in [0,T] \times S.
\end{aligned}$$

Let $v_0(t,e_i) := \dfrac{\partial U}{\partial c}(t,0,e_i)$ and for preventing from negative consumption values, we define:

$$I(t,v,e_i) := \begin{cases} 0, & \text{if } v \geq v_0(t,e_i),\\ (\dfrac{\partial U}{\partial c}(t,\cdot,e_i))^{-1}(v), & \text{if } 0 \leq v < v_0(t,e_i). \end{cases}$$

We want to find the optimal consumption rate $\hat{c}$ such that

$$\begin{aligned}
J(\hat{c}) &= \sup_{c\in\mathcal{A}} J(c)\\
&= \sup_{c\in\mathcal{A}} E\bigg[\int_0^T U(t,c(t),\alpha(t))dt + g(X(T),\alpha(T))\bigg].
\end{aligned}$$

In this case, the Hamiltonian takes the form

$$\begin{aligned}
H(t,x,y,a,e_i,c,p,q,r(\cdot),w) = U(t,c,e_i) + (b(t,e_i)y - c)p + y\sigma(t,e_i)q\\
+y\int_{\mathbb{R}_0} \eta(t,e_i,z)r(t,z)\nu(dz) + y\sum_{j=1}^{D} \gamma^j(t,e_i)w^j(t)\lambda_{ij}.
\end{aligned} \tag{5.15}$$

Here we observe that maximizing H with respect to c gives

$$\frac{\partial U}{\partial c}(t,\hat{c}(t),\alpha(t)) = p(t).$$

The ABSDE for $p(t),\ q(t),\ r(t,z)$ and $w(t)$ is, by Equations (5.4) and (5.5),

$$\begin{aligned}
dp(t) =& - E[(b(t+\delta,\alpha(t+\delta))p(t+\delta) + \sigma(t+\delta,\alpha(t+\delta))q(t+\delta)\\
&+ \int_{\mathbb{R}_0} \eta(t+\delta,\alpha(t+\delta),z)r(t+\delta,z)\nu(dz)\\
&+ \sum_{j=1}^{D} \gamma^j(t,\alpha(t+\delta))w^j(t+\delta)\lambda_j(t))\mathbf{1}_{[0,T-\delta]}(t)|\mathcal{F}_t]dt\\
&+ q(t)dW(t) + \int_{\mathbb{R}_0} r(t,z)\tilde{N}(dt,dz) + w(t)d\tilde{\Phi}(t),\ t\in[0,T],\\
p(T) =& g_x(X(T),\alpha(T)).
\end{aligned} \tag{5.16}$$

We solve Equation (5.16) inductively in the following way:

Step 1. If $t \in [T-\delta, T]$, the corresponding adjoint equation takes the form:

$$\begin{aligned} dp(t) &= q(t)dW(t) + \int_{\mathbb{R}_0} r(t,z)\tilde{N}(dt,dz) + w(t)d\tilde{\Phi}(t), \qquad t \in [T-\delta, T], \\ p(T) &= g_x(X(T), \alpha(T)), \end{aligned}$$

which has the solution

$$p(t) = E[g_x(X(T), \alpha(T))|\mathcal{F}_t], \qquad t \in [T-\delta, T],$$

with corresponding $q(t),\ r(t,z)$ and $w(t)$ obtained through the martingale representation theorem for regime-switching jump-diffusions, by Crépéy and Matoussi [14].

Step 2. If $t \in [T-2\delta, T-\delta]$ and $T-2\delta > 0$, we get:

$$\begin{aligned} dp(t) = &- E[(b(t+\delta, \alpha(t+\delta))p(t+\delta) + \sigma(t+\delta, \alpha(t+\delta))q(t+\delta) \\ &+ \int_{\mathbb{R}_0} \eta(t+\delta, \alpha(t+\delta), z) r(t+\delta, z)\nu(dz) \\ &+ \sum_{j=1}^{D} \gamma^j(t+\delta, \alpha(t+\delta)) w^j(t+\delta)\lambda_j(t))\mathbf{1}_{[0,T-\delta]}(t)|\mathcal{F}_t]dt \\ &+ q(t)dW(t) + \int_{\mathbb{R}_0} r(t,z)\tilde{N}(dt,dz) + w(t)d\tilde{\Phi}(t),\ t \in [T-2\delta, T-\delta], \end{aligned}$$

with the terminal value $p(T-\delta)$ known from Step 1. When we follow the intervals, it is seen that $p(t+\delta),\ q(t+\delta),\ r(t+\delta, z)$ and $w(t+\delta)$ are known by Step 1. Therefore, this BSDE can be solved for $p(t),\ q(t),\ r(t,z)$ and $w(t)$ in the interval $[T-2\delta, T-\delta]$. We continue in the same way and, by induction, we obtain a solution $p(t) = p_{X(T),\alpha(T)}(t)$ of Equation (5.16).

If $0 \le p(t) \le v_0(t, \alpha(t))$ for all $t \in [0,T]$, then the optimal consumption rate $\hat{c}(t)$ is given by

$$\hat{c}(t) = \hat{c}_{\hat{X}(T),\alpha(T)}(t) := I(t, p(t), \alpha(t)), \qquad t \in [0,T]. \tag{5.17}$$

Hence, we can summarize above findings by the following Proposition.

Proposition 5.5. *Let* $p(t),\ q(t),\ r(t,z),\ w(t)$ *be the solution of Equation* (5.16) *and suppose that* $0 \le p(t) \le v_0(t, \alpha(t))$ *holds for all* $t \in [0,T]$. *Then the corresponding optimal terminal wealth* $X(t)$ *and the optimal consumption rate* $\hat{c}(t)$ *are given implicitly by Equations* (5.14) *and* (5.17)*, respectively.*

To obtain a more explicit solution, let us assume that $b(t, e_i) = b(t)$ is deterministic and $g(x, e_i) = kx,\ k > 0,\ i = 1, 2, ..., D$. We continue our study with the utility function

$U(t,c,e_i) = \phi(t,e_i)\ln(1+c)$ for all $i = 1,2,...,D$, where $\phi(t,e_i)$ is an $\mathbb{R}^+$-valued, càdlàg and $\mathcal{F}_t$-measurable function such that

$$E\left[\int_0^t |\phi(t,e_i)|^2\,dt\right] < \infty.$$

Considering Equation (5.16), since k is deterministic, we can choose $q = r = w = 0$. Hence, the BSDE becomes a deterministic equation:

$$\begin{aligned} dp(t) &= -b(t+\delta)p(t+\delta)\mathbf{1}_{[0,T-\delta]}(t)dt, \qquad t \leq T, \\ p(t) &= k, \qquad t \in [T-\delta, T]. \end{aligned}$$

To solve this, we introduce

$$h(t) = p(T-t), \qquad t \in [0,T].$$

Then, we we arrive at an ordinary delay differential equation:

$$\begin{aligned} dh(t) &= -dp(T-t) = b(T-t+\delta)p(T-t+\delta)dt = b(T-t+\delta)h(t-\delta)dt, \\ &\quad t \in [\delta, T], \\ h(t) &= p(T-t) = k, \qquad t \in [0,\delta]. \end{aligned}$$

Hence, we can determine $h(t)$ inductively on each interval as follows:

If $h(t)$ is known on $[(j-1)\delta, j\delta]$, then

$$h(t) = h(j\delta) + \int_{j\delta}^t h'(s)ds = h(j\delta) + \int_{j\delta}^t b(T-t+\delta)h(s-\delta)ds \tag{5.18}$$

for $t \in [j\delta, (j+1)\delta]$, $j = 1,2,....$

If we substitute the utility function $U(t,c,e_i) = \phi(t,e_i)\ln(1+c)$, $i = 1,2,...,D$, in Equation (5.15), then we have proved the following theorem. Furthermore, since h depends on the constant delay δ and by the coefficient $\phi(t,\alpha(t))$, Theorem 5.6 clarifies the effects of the memory and different states of the economy on the optimal consumption rate.

Theorem 5.6. *The optimal consumption rate $\hat{c}(t)$ under the above construction is explicitly given by*

$$\begin{aligned} \hat{c}(t) &= I(t, h_\delta(T-t), \alpha(t)|_{\alpha(t)=e_i}) \\ &= \begin{cases} 0, & \textit{if} \quad h_\delta(T-t) \geq \phi(t,e_i), \\ \dfrac{\phi(t,e_i)}{h_\delta(T-t)} - 1, & \textit{if} \quad 0 \leq h_\delta(T-t) < \phi(t,e_i), \end{cases} \end{aligned}$$

where $h(\cdot) = h_\delta(\cdot)$ is determined by Equation (5.18).

CHAPTER 6

CONCLUSION AND OUTLOOK

Stochastic hybrid systems have been applied in a diversified field of research up to now. In this thesis, we provide several new results related to a Markov regime-switching jump-diffusion model by the two fundamental techniques of stochastic optimal control theory. We utilized the tools of Dynamic Programming Principle without a delay setting by which we try to show computational challenges for a D-dimensional Markov chain. Furthermore, we provide a clear intuition by reducing the number of the states of the Markov chain in Subsections 3.3.1 and 3.3.2. On the other hand, there are very few works related to stochastic optimal control theory for regime-switching models with a delay setting. At this step, we prefer to follow the methods of Stochastic Maximum Principle. Hence, in this thesis, the models with and without memory are highlighted. We not only contribute to the theory of Financial Mathematics, but also illustrate our results by investment and consumption problems of finance.

In Chapter 3, we give examples of zero-sum and nonzero-sum stochastic differential games by the tools of Dynamic Programming Principle. In other words, we reformulate the general equilibrium problem as a stochastic optimal control problem.

In the first example, we obtain compatible results related to asset pricing with no free-lunch principle. Furthermore, in the last subsection, by establishing the two-state case of the optimal portfolio selection problem between two traders, we show that more complicated differential equations are obtained. This is the effect of the regime switches on the mathematical structure of the game, which is a clear difference between the usual setting and our Markov regime-switching model. For both of the examples, we give explicit solutions of the optimal control processes and closed-form representations for the value functions.

In Chapter 4, we provide three fundamental theorems to present the relation between SDDDEs with jumps and regimes and ABSDEs with jumps and regimes. This relation becomes the main base for the construction of the corresponding adjoint equations of the stochastic optimal control problem studied in Chapter 5. Furthermore, these results open the doors of ABSDE theory in a Markovian regime switching set-up. It is well-known that one of the corner stones of the BSDE theory is the Comparison Theorem which has applications in both of stochastic optimal control and stochastic games. By the help of our results, Comparison Theorem for ABSDEs can be studied and, consequently, the path of several other advances can be lightened from the theoretical and applied perspectives as well. The first work related to ABSDEs was given by Peng and Yang [49] in 2009, and it reached researchers' focus of attention rapidly. The results of BSDEs can be extended to our new and fruitful theory. ABSDEs may be considered in stochastic game theory with carrying future values of the dynamics.

Moreover, we study on a delayed state process of a Markov regime switching jump-diffusion model in Chapter 5. We are concerned with a stochastic optimal control problem by the tools of stochastic maximum principle and prove the necessary and sufficient maximum principles for a delayed SDE with jumps and regimes under full and partial information. We develop the general analytic model setting and methods for the solution of such a model and we apply our results to an optimal consumption problem from a cash flow with delay and regimes. In our setting, under the given conditions, one may prefer any stochastic utility function based on the information about the investor. In this work, we present the optimal consumption rate for a specific stochastic utility function explicitly.

Time-delay can be considered in DPP with a memory challenge (see [22, 37]). To the best of our knowledge, a Markov regime-switching model with delay has not been considered by the tools of DPP.

Stochastic Hybrid Systems are demanding models with their heterogeneous structure for a diversified field of science and technology (see [2, 24, 25]). In this thesis, one of the perspectives that we follow, is to see them as macroeconomic indicators. For example, we may consider currency risk which affects market psychology easily. The country risk is highly sensitive to any abrupt changes in the financial market. Hence, it is expected and effective to consider the Markov regime-switches as macroeconomic factors.

Furthermore, the shifts between the states of a Markov chain may characterize some of the psychological phenomena, e.g., switches between episodes of mania and depression or the periods of recovery and relapses involved in addiction. Human behaviors can be modeled by SHSs at the case of mathematical convenience. When we focus on the necessity of taking into account the investors' preferences, the obvious relation between the psychology and economics is seen.

From the perspective of technology, electric cars can be considered at the top of the list nowadays. For instance, the American company Tesla Motors delivered 70000 electric cars between 2008 and 2015. It is known that electric cars produce about 40 percent less carbon dioxide and ozone than conventional cars, even when we consider the carbon emissions and pollution from the power plants. Hence, increasing demand in electric cars can activate regime switches in a very wide range of life. Especially, this fact may reduce the usage of gasoline as a consequence of decreasing appreciation in gas-powered cars. This is forcing the traditional industry rapidly to switch their techniques, or to leave the market. In this context, we speak of *disruption*. Hence, this development pressure on powerful and rooted companies and whole branches may push the governments to radical economic changes, which means another switch in the financial sector. Moreover, green technology may pep up the dynamics of the nature in a positive direction, which may become generate vital switches for our world. On the other hand, decreasing demand in oil could change the power balances over all over the Middle East and their neighbor regions. This new situation may create the last ring of the chain as a political switch, i.e., a possible revolution for democracy in Middle East might be seen as another regime switch.

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APPENDIX A

Complementary Remarks

Let us present a Markov regime-switching jump-diffusion model as follows:

$$\begin{aligned} Y(t) &= b(t, Y(t), \alpha(t))dt + \sigma(t, Y(t), \alpha(t))dW(t) \\ &\quad + \int_{\mathbb{R}_0} \eta(t, Y(t-), \alpha(t-), z)\tilde{N}(dt, dz) \\ &\quad + \gamma(t, Y(t-), \alpha(t-))d\tilde{\Phi}(t), \qquad t \in [0, T], \\ Y(0) &= y_0 \in \mathbb{R}^N, \end{aligned} \tag{A.1}$$

where

$$\begin{aligned} &b : [0, T] \times \mathbb{R}^N \times S \to \mathbb{R}^N, \\ &\sigma : [0, T] \times \mathbb{R}^N \times S \to \mathbb{R}^{N \times M}, \\ &\eta : [0, T] \times \mathbb{R}^N \times S \times \mathbb{R}_0 \to \mathbb{R}^{N \times L}, \\ &\gamma : [0, T] \times \mathbb{R}^N \times S \times \to \mathbb{R}^{N \times D} \end{aligned}$$

are given functions. By Proposition 7.1 of Crépey [13], the system (A.1) has a unique solution $Y(t) \in L^2_{\mathbb{F}}(0, T; \mathbb{R}^N)$ under the following conditions:

(K1) There exists a constant $K > 0$ such that for all $t \in [0, T],\ e_i \in S,\ x_1,\ x_2 \in \mathbb{R}^N$,

$$\begin{aligned} &\|b(t, x_1, e_i) - b(t, x_2, e_i)\| + \|\sigma(t, x_1, e_i) - \sigma(t, x_2, e_i)\| \\ &+ \|\eta(t, x_1, e_i, z) - \eta(t, x_2, e_i, z)\|_J + \|\gamma(t, x_1, e_i) - \gamma(t, x_2, e_i)\|_S \\ &\leq K \|x_1 - x_2\|. \end{aligned}$$

(K2) $b(\cdot, 0, e_i) \in L^2_{\mathbb{F}}(0, T; \mathbb{R}^N),\ \sigma(\cdot, 0, e_i) \in L^2_{\mathbb{F}}(0, T; \mathbb{R}^{N \times M}),\ \eta(\cdot, 0, e_i, \cdot) \in \mathcal{H}^2_{\mathbb{F}}(0, T; \mathbb{R}^{N \times L})$ and $\gamma(\cdot, 0, e_i) \in \mathcal{M}^2_{\mathbb{F}}(0, T; \mathbb{R}^{N \times D})$ for all $e_i \in S$ and $t \in [0, T]$.

Let us give the extension of Itô's formula as in Zhang, Elliott and Siu [64].

Theorem A.1. *Suppose an N-dimensional process $Y(t),\ t \in [0, T]$, is given as in*

System (A.1) and the function $\phi(\cdot,\cdot,e_i) \in C^{1,2}([0,T]\times\mathbb{R}^N)$ for each $e_j \in S$. Then,

$$
\begin{aligned}
&\phi(T,Y(T),\alpha(T)) - \phi(0,Y(0),\alpha(0))\\
&= \int_0^T \Bigg\{ \bigg(\frac{\partial \phi}{\partial t}(t,Y(t-),\alpha(t-)) + \sum_{k=1}^{N} \frac{\partial \phi}{\partial y_k}(t,Y(t-),\alpha(t-)) b_k(t,Y(t-),\alpha(t-)) \bigg)\\
&\quad + \frac{1}{2}\sum_{k=1}^{N}\sum_{n=1}^{N}\int_0^T \frac{\partial^2 \phi}{\partial y_k \partial y_n}(t,Y(t-),\alpha(t-)) \sum_{l=1}^{M} \sigma_{kl}\sigma_{nl}(t,Y(t-),\alpha(t-))\\
&\quad + \sum_{m=1}^{L}\int_0^T \int_{\mathbb{R}_0} \bigg(\phi(t,Y(t-)+\eta^m(t,Y(t-),\alpha(t-),z),\alpha(t-))\\
&\quad - \phi(t,Y(t-),\alpha(t-))\\
&\quad - \sum_{n=1}^{N} \frac{\partial \phi}{\partial y_n}(t,Y(t-),\alpha(t-)) \eta_{nm}(t,Y(t-),\alpha(t-),z) \bigg) \nu_m(dz)\\
&\quad + \sum_{j=1}^{D} \int_{\mathbb{R}_0} \bigg(\phi(t,Y(t-)+\gamma^{(j)}(t,Y(t-),\alpha(t-)),e_j) - \phi(t,Y(t-),\alpha(t-))\\
&\quad - \sum_{n=1}^{N} \frac{\partial \phi}{\partial y_n}(t,Y(t-),\alpha(t-)) \gamma_{nj}(t,Y(t-),\alpha(t-)) \bigg) \lambda_j(t) \Bigg\} dt\\
&\quad + \int_0^T \sum_{k=1}^{N} \frac{\partial \phi}{\partial y_k}(s,Y(s-),\alpha(s-)) \sum_{n=1}^{M} \sigma_{kn}(s,Y(s-),\alpha(s-)) dW(t)\\
&\quad + \int_0^T \sum_{m=1}^{L} \int_{\mathbb{R}_0} \bigg(\phi(s,Y(s-)+\eta^{(m)}(s,Y(s-),\alpha(s-),z),\alpha(s-))\\
&\quad - \phi(s,Y(s-),\alpha(s-)) \bigg) \tilde{N}(ds,dz)\\
&\quad + \int_0^T \sum_{j=1}^{D} \bigg(\phi(s,Y(s-)+\gamma^{(j)}(s,Y(s-),\alpha(s-)),e_j)\\
&\quad - \phi(s,Y(s-),\alpha(s-)) \bigg) d\tilde{\Phi}_j(s),
\end{aligned}
$$

where $\eta^{(m)}$ and $\gamma^{(j)}$ represents the mth and jth columns of the matrices η and γ, respectively.

Furthermore, we present product rule for Markov regime-switching jump-diffusion models as in Zhang, Elliott and Siu [64], which has been used several times in this thesis.

Lemma A.2. *Suppose that $Y^j(t), j = 1, 2$, are processes defined by the forward SDEs,*

$$\begin{aligned}
Y^j(t) &= b^j(t, Y(t), \alpha(t))dt + \sigma^j(t, Y(t), \alpha(t))dW(t) \\
&\quad + \int_{\mathbb{R}_0} \eta^j(t, Y(t-), \alpha(t-), z)\tilde{N}(dt, dz) \\
&\quad + \gamma^j(t, Y(t-), \alpha(t-))d\tilde{\Phi}(t), \qquad t \in [0, T], \\
Y^j(0) &= y^j \in \mathbb{R}^N, \quad j = 1, 2,
\end{aligned} \tag{A.2}$$

where $b^j(t) \in \mathbb{R}^N$, $\sigma^j(t) \in \mathbb{R}^{N\times M}$, $\eta^j(t) := [\eta^j_{nl}(t)] \in \mathbb{R}^{N\times L}$ and $\gamma^j(t) := [\gamma^j_{nl}(t)]$ $\in \mathbb{R}^{N\times D}$, $t \in [0, T]$, are predictable processes such that the integrals in A.2 exist. Then,

$$\begin{aligned}
&\left\langle Y^1(T), Y^2(T) \right\rangle \\
&= \left\langle y^1, y^2 \right\rangle + \int_0^T \left\langle Y^1(t-), dY^2(t) \right\rangle + \int_0^T \left\langle Y^2(t-), dY^1(t) \right\rangle \\
&\quad + \int_0^T \left[(\sigma^1(t, \alpha(t)))^T \sigma^2(t, \alpha(t)) \right] dt \\
&\quad + \int_0^T \sum_{l=1}^{L} \sum_{n=1}^{N} \eta^1_{nl}(t, \alpha(t-), z)\eta^2_{nl}(t, \alpha(t-), z)\nu^l(dz)dt \\
&\quad + \int_0^T \sum_{l=1}^{D} \sum_{n=1}^{N} \gamma^1_{nl}(t, \alpha(t-))\gamma^2_{nl}(t, \alpha(t-))\lambda_l(t)dt.
\end{aligned}$$

Let us state Banach Fixed Point Theorem which we used in Chapter 4 for the Existence-Uniqueness Theorems of SDDEs and ABSDEs.

Theorem A.3. *Let (X, d) be a complete metric space and $T : X \to X$ be a map such that*

$$d(Tx, Tx') \leq cd(x, x')$$

for some $0 \leq c < 1$ and all $x,\ x' \in X$. Then T has a unique fixed point x^* in X, i.e., $T(x^*) = x^*$.

APPENDIX B

Remarks for Graphs

The following simple codes correspond to each graph in Subsection 3.3.2.

We note that

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\eta_1^k\eta_2^m z^2 e^{-z^2/2}dz = \eta_1^k\eta_2^m,$$

for $k, m = 1, 2$.

For Figure 3.1, $-0.08 \le \mu_1^1 \le 0.06$, $r^1 = 0.06$, $\sigma_1^1 = 0.72$, $\sigma_2^1 = 0.35$, $\eta_1^1 = 1.4$, $\eta_1^1 = -0.8$ $\eta_2^1 = 1.3$.

$N = 1000; h = 1/N; x = -0.08 : h : 0.06;$
$y = (0.06 - x)/(0.72 * 0.35 + 1.4 * 1.3);$
$z = (0.06 - x)/(0.72 * 0.35 + -0.8 * 1.3);$
plot(x,y,'k.',x,z,'k-')
legend($'\eta_1^1 > 0', '\eta_1^1 < 0'$)
xlabel($'\mu_1^1'$), ylabel($'\pi_2^*'$).

For Figure 3.2, $0.06 \le \mu_1^2 \le 1.4$, $r^2 = 0.06$, $\sigma_1^2 = 0.4$, $\sigma_2^2 = 0.2$, $\eta_1^2 = 0.8$, $\eta_1^2 = -0.5$ $\eta_2^1 = 1.2$.

$N = 1000; h = 1/N; x = 0.06 : h : 1.4;$
$y = (0.06 - x)/(0.4 * 0.2 + 1.2 * 0.8);$
$z = (0.06 - x)/(0.4 * 0.2 + 1.2 * -0.5);$
plot(x,y,'k.',x,z,'k-')
legend($'\eta_1^2 > 0', '\eta_1^2 < 0'$)
xlabel($'\mu_1^2'$), ylabel($'\pi_2^*'$).

For Figure 3.3, $-0.08 \le \mu_1^1 \le 0.06$, $r^1 = 0.06$, $\sigma_1^1 = 0.72$, $\sigma_2^1 = 0.25$, $\eta_1^1 = 1.4$, $\eta_2^1 = 1.3$.

$N = 1000; h = 1/N; x = -0.08 : h : 0.06;$
$y = (0.06 - x)/(0.72 * 0.25 + 1.4 * 1.3);$
$z = (0.06 - x)/(0.72 * 0.80 + 1.4 * 1.3);$
plot$(x, y,' k.', x, z,' k-')$
legend$('\sigma_1^1 = 0.25',' \sigma_1^1 = 0.8')$
xlabel$('\mu_1^{1\prime})$, ylabel$('\pi_2^{*\prime})$.

For Figure 3.4, $-1.3 \leq \mu_2^1 \leq 0$, $0.08 \leq \mu_2^2 \leq 1.4$, $r^k = 0.05$, $k = 1, 2$, $\sigma_1^k = 0.7$, $k = 1, 2$, $\sigma_2^k = 0.4$, $k = 1, 2$, $\eta_1^k = 1.2$, $k = 1, 2$, $\eta_2^1 = -0.8$, $\eta_2^2 = 0.8$.

$N = 1000; h = 1/N; x = -1.3 : h : 0; t = 0.08 : h : 1.4;$
$y = (0.05 - x)/(0.7 * 0.4 + 1.2 * -0.8);$
$z = (0.05 - t)/(0.7 * 0.4 + 1.2 * 0.8);$
plot(x,y,'k.',t,z,'k-')
legend$('\eta_2^1 = -0.8',' \eta_2^2 = 0.8')$
xlabel$('\mu_2^k, k = 1, 2.')$, ylabel$('\pi_1^{*\prime})$.

For Figure 3.5, $-1.3 \leq \mu_2^1 \leq 0$, $0.08 \leq \mu_2^2 \leq 1.4$, $r^1 = 0.05$, $r^1 = 0.07$, $\sigma_1^k = 0.7$, $k = 1, 2$, $\sigma_2^k = 0.4$, $k = 1, 2$, $\eta_1^k = 1.2$, $k = 1, 2$, $\eta_2^k = 0.8$, $k = 1, 2$.

$N = 1000; h = 1/N; x = -1.3 : h : 0; t = 0.08 : h : 1.4;$
$y = (0.05 - x)/(0.7 * 0.4 + 1.2 * 0.8);$
$z = (0.07 - t)/(0.7 * 0.4 + 1.2 * 0.8);$
plot(x,y,'k.',t,z,'k-')
legend$('r^1 = 0.05',' r^2 = 0.07')$
xlabel$('\mu_2^k, k = 1, 2.')$, ylabel$('\pi_1^{*\prime})$.

For Figure 3.6, $-1.3 \leq \mu_2^1 \leq 0$, $0.08 \leq \mu_2^2 \leq 1.4$, $r^k = 0.05, k = 1, 2$, $\sigma_1^k = 0.7$, $k = 1, 2$, $\sigma_2^1 = 0.8$, $\sigma_2^2 = 0.4$, $\eta_1^k = 1.2$, $k = 1, 2$, $\eta_2^k = -0.8$, $k = 1, 2$.

$N = 1000; h = 1/N; x = -1.3 : h : 0; t = 0.08 : h : 1.4;$
$y = (0.05 - x)/(0.7 * 0.8 + 1.2 * -0.8);$
$z = (0.05 - t)/(0.7 * 0.4 + 1.2 * -0.8);$
plot(x,y,'k.',t,z,'k-')
legend$('\sigma_2^1 = 0.8',' \sigma_2^2 = 0.4')$
xlabel$('\mu_2^k, k = 1, 2.')$, ylabel$('\pi_1^{*\prime})$.

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PUBLICATIONS

1. E. Savku and G.-W. Weber, Dynamic Programing Principle Approach in Stochastic Differential Games for a Markov Regime-Switching Jump-Diffusion Model with Applications to Finance (in progress, before submission)
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3. **(Book Chapter)** E. Savku, N. Azevedo and G.-W. Weber, Optimal Control of Stochastic Hybrid Models in the Framework of Regime Switches, Springer Volume: Modeling, Dynamics, Optimization and Bioeconomics II, Eds. A. Pinto and D. Zilberman (Accepted 2016, to appear soon)

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2. **(Poster)** E. Savku and G.-W. Weber, Maximum Principle For A Delayed Stochastic Hybrid Model and An Application to Finance, Vienna Congress on Mathematical Finance (VCMF 2016), Austria, September 12-14, 2016
3. E. Savku and G.-W. Weber, Kısmi Bilgi Altında Rejim Değişimine Sahip Sıçrama-Difüzyon Modeli için Stokastik Maksimum Prensibi ve Finansa Bir Uygulaması, 29. Ulusal Matematik Sempozyumu, Mersin, Turkey, August 28-31, 2016
4. E. Savku and G.-W. Weber, Stokastik Hibrit Sistemlerde Optimum Kontrol ve Finansal Uygulamaları, 11. Ankara Matematik Günleri, Ankara, Turkey, May 26-27, 2016
5. E. Savku and G.-W. Weber, A Stochastic Optimal Control Problem in a Regime-Switching Jump-Diffusion Market, Department of Mathematics, University of Aveiro, Portugal, May 16, 2016
6. E. Savku and G.-W. Weber, Optimal Control of Delayed Jump-Diffusions with Regimes and An Application to Finance, 57th EURO Working Group for Commodities and Financial Modeling, University of Coimbra, Portugal, May 12-14, 2016
7. E. Savku and G.-W. Weber, Optimal Control of Stochastic Hybrid Models with An Application to Finance, 27th European Conference on Operational Research, Glasgow, UK, July 12-15, 2015
8. **(Poster)** E. Savku and G.-W. Weber, Optimal Control of Stochastic Hybrid Delayed Models, 2nd Ankara-İstanbul Workshop on Stochastic Processes, İstanbul, Turkey, June 18-19, 2015