MONOMIAL GROUPS

A THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES
OF
MIDDLE EAST TECHNICAL UNIVERSITY

BY

ÖZGE ALMAŞ

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR
THE DEGREE OF MASTER OF SCIENCE
IN
MATHEMATICS

AUGUST 2017
MONOMIAL GROUPS

submitted by ÖZGE ALMAŞ in partial fulfillment of the requirements for the degree of Master of Science in Mathematics Department, Middle East Technical University by,

Prof. Dr. Gülbin Dural Ünver  Dean, Graduate School of Natural and Applied Sciences

Prof. Dr. Mustafa Korkmaz  Head of Department, Mathematics

Prof. Dr. Mahmut Kuzucuoğlu  Supervisor, Mathematics Department, METU

Assoc. Prof. Dr. Ebru Solak  Co-supervisor, Mathematics Department, METU

Examination Committee Members:

Prof. Dr. Ali Erdoğan  Mathematics Department, Hacettepe University

Prof. Dr. Mahmut Kuzucuoğlu  Mathematics Department, METU

Prof. Dr. Yıldray Ozan  Mathematics Department, METU

Assist. Prof. Dr. Mustafa Gökhan Benli  Mathematics Department, METU

Assist. Prof. Dr. Cansu Betin Onur  Mathematics Department, Atılım University

Date: __________________
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Name, Last Name:  ÖZGE ALMAŞ

Signature :
ABSTRACT

MONOMIAL GROUPS

Almaş, Özge
M.S., Department of Mathematics
Supervisor : Prof. Dr. Mahmut Kuzucuoğlu
Co-Supervisor : Assoc. Prof. Dr. Ebru Solak

August 2017, 65 pages

A group $G$ is called a permutation group if it is a subgroup of a symmetric group on a set $\Omega$. $G$ is called a linear group if it is a subgroup of the general linear group $\text{GL}(n, F)$ for a field $F$.

Monomial groups are generalization of permutation groups and restriction of linear groups. In matrix terminology, monomial groups of degree $n$ over a group $H$ are the $n \times n$ invertible matrices in which each row and each column contains only one element of $H$ all the other entries are zero.

Basic properties of finite degree monomial groups are studied by Ore in [2]. Infinite degree monomial groups over an arbitrary group $H$ is studied by Crouch in [1]. This thesis is a survey of the Crouch paper, in particular we will give a complete classification of the structure of centralizers of arbitrary elements in complete monomial groups $\Sigma(H; B, B^+, B^+)$ and conjugacy of the elements in $\Sigma(H; B, B^+, B^+)$. 

Keywords: Monomial groups, Infinite permutation groups, Centralizer of monomial elements, Splitting of monomial groups

Anahtar Kelimeler: Monomial gruplar, Sonsuz permütasyon grupları, Monomial elemanlarının merkezleyenleri, Monomial gruplarda ayrışma
To my parents & to my sister
ACKNOWLEDGEMENTS

I would like to express my gratitude to my supervisor Prof. Dr. Mahmut Kuzucuoğlu for the valuable guidance, enlightening advices, constant support and encouragement. I also appreciate to Assoc. Prof. Dr. Ebru Solak for her encouragement and helps.

I want to thank my friends Sezen Bostan for her helpful comments on writing and for mathematics discussions; Duygu Vargün and Nazmi Oyar for giving me motivation and for their helps. Also, I would like to express my gratitude to Birand Adal for his support, patience and encouragement.

My sincere thanks are due my family for their full support and motivation, especially my sister Ayşê Simge Almaş and my uncle Nuh Naci Kişnişci.
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Mainly there are three kinds of representations of groups; permutation representation, linear representation, and monomial representation.

Permutation representation is a homomorphism from the group into symmetric group, linear representation is a homomorphism from group into the group of invertible linear transformations of a vector space over a field $F$.

Monomial representation is a generalization of permutation representation and restriction of linear representation. If $V$ is a finite dimensional vector space, say $\dim V = n$ over a field $F$, then $GL(V)$ is isomorphic to the general linear group $GL(n,F)$, $n \times n$ invertible matrices over a field $F$.

A monomial matrix is an invertible $n \times n$ matrix where each row and each column contains only one nonzero entry and this entry comes from a fixed group $H$.

If $G$ is a group with a subgroup $H$ of index $n$, then $G$ has a monomial representation of degree $n$ over the subgroup $H$.

Therefore, the study of monomial groups is the study of the structure of groups which has a subgroup of finite index.

In fact, the famous Kalużnin - Krasner Theorem which states that, if a group $G$ has a non-trivial subgroup $H$, then $G$ can be embedded into a monomial group over $H$, but the degree could be infinite. In this respect study of monomial groups is the study of group extensions.
The basic properties of monomial groups of finite degree are studied by Kerber [3], and Ore [2]. In particular, Ore determined conjugacy of the two elements in complete monomial group. Moreover, he finds the structure of a centralizer of an element in complete monomial groups.

The work of Ore is extended to infinite degree monomial groups by R. B. Crouch [1]. Crouch defines monomial groups (symmetries) of arbitrary degree, over an arbitrary group H in the following way.

Let B be an infinite cardinality, and U be a set with cardinality B. We think of U as an ordered set. By $B^+$ we denote the successor cardinal of B.

Let $d$ be the cardinality of natural numbers, i.e., $d = \aleph_0$. A monomial substitution over H is a transformation of the form

$$c = \left( \begin{array}{ccc} \cdots & x_\epsilon & \cdots \\ \cdots & h_\epsilon x_i & \cdots \end{array} \right)$$

(1.1)

where the map $x_\epsilon \mapsto x_{\epsilon_i}$ is a permutation of the set U, and $h_\epsilon \in H$. The product $h_\epsilon x_i$ is a formal product satisfying $h_\epsilon (h_\beta x_i) = (h_\epsilon h_\beta) x_i$, where $x_i \in U$.

The set of all monomial substitutions $\Sigma(H; B, B^+, B^+)$ forms a group with composition of substitutions.

In the first part of this thesis, we find the structure of the centralizer of an arbitrary element in $\Sigma(H; B, B^+, B^+)$. Namely we prove the following theorem:

Let $y$ be conjugate to $y_1$ written in the normal form $y_1 = \prod_i \delta_i$, $\delta_i = \prod_\epsilon \delta^i_\epsilon$, where for a fixed $i$ the $\delta^i_\epsilon$ are the normalized cycles of the same length $n$, and the same determinant class $a$ if $n < d$. Let $\epsilon$ run over a set of cardinal $\mu$ where $0 \leq \mu \leq B$. Then the centralizer $C_{\Sigma(H; B, B^+, B^+)}(y)$ is isomorphic to the strong direct product of symmetries

$$C_{\Sigma(H; B, B^+, B^+)}(y) \cong \prod_i (\Sigma(C_H(a) < \delta >, \mu_i, \mu^+_i, \mu^-_i)) \times \Sigma_{\kappa}(H \times \mathbb{Z}; \kappa, \kappa^+, \kappa^-).$$

The group $C_H(a) < \delta >$ consists of all elements $y_1$ of the form $y_1 = \{k_i\}(c^i_1)^j$ where
k belongs to the centralizer of a in H. The second direct product arises if \( \delta \) is a product of \( \kappa \) infinite cycles where \( \kappa \leq B \).

Let \( G \) be a group and \( N \) be a normal subgroup of \( G \). If there exists a subgroup \( H \leq G \) such that \( G=NH \) and \( N \cap H = 1 \), then we say that \( G \) splits over \( N \) and \( H \) is called complement of \( N \) in \( G \). Clearly if \( H \) is a complement of \( N \), then all conjugates \( H^g, g \in G \) are also complement of \( N \) in \( G \). It is a natural question whether all complements of \( N \) are conjugate in \( G \), i.e., if \( G=NT \) and \( N \cap T = 1 \) does there exist \( x \in G \), such that \( T = H^x \). If all complements of \( N \) are conjugate, then we say that \( G \) splits regularly. Observe that if \( G \) has two complements \( H_1 \) and \( H_2 \) then \( G = NH_1 \) and \( G = NH_2 \), where \( H_1 \cap N = 1 \) and \( H_2 \cap N = 1 \). It follows that \( G/N = H_1N/N = H_1/\text{H}_1 \cap N \cong H_1 \), on the other hand \( G/N = H_2N/N = H_2/H_2 \cap N \cong H_2 \). Hence any two complements of \( N \) are isomorphic. So we are interested in when they are conjugate. Indeed in the following example we have a group \( G \) with a normal subgroup \( N \) such that it has two non-conjugate complements. So the above problem makes sense.

**Example:**

Let \( G = S_6 \). We know that \( A_6 \) is a normal subgroup of \( S_6 \). Let \( T_1 = \langle (1, 2) \rangle \). \( T_1 \) is a subgroup of \( G \) where \( G = A_6T_1 \) and \( A_6 \cap T_1 = 1 \). So \( T_1 \) is a complement of \( A_6 \). Assume \( T_2 = \langle (1, 2), (3, 4), (5, 6) \rangle \). \( T_2 \) is also a subgroup of \( G \). Moreover, \( G = A_6T_2 \) and \( A_6 \cap T_2 = 1 \). But \( T_1 \) and \( T_2 \) are not conjugate. So, \( S_6 \) does not split regularly.

As the following simple observation shows, if a group \( G \) has normal subgroup \( N \), then it may not split. Indeed \( G = Q_8 \) quaternion group of order 8. \( Q_8 = \{1, i, j, k, -1, -i, -j, -k\} \). All subgroups of \( Q_8 \) are normal subgroup. Indeed the subgroups \( \langle i \rangle, \langle j \rangle, \langle k \rangle \) are cyclic subgroups of order 4 and \( \{1, -1\} \) is the center of \( Q_8 \). But the subgroup \( \langle i \rangle \triangleleft Q_8 \) does not split, because there exists no non-trivial subgroup \( H \) such that \( \langle i \rangle \cap H = 1 \). Because all non-trivial subgroups of \( Q_8 \) contain the center \( Z(Q_8) = \{1, -1\} \).

If we come back to monomial groups over \( H \), \( \Sigma(H; B, B^+, C) \) splits over the base
group $V(B, B^+)$ and $S(B, C)$ is a complement of $V(B, B^+)$ in $\Sigma(H; B, B^+, C)$.

In chapter 4, we will discuss the splitting of $\Sigma(H; B, B^+, C)$ and regularity of this splitting. We prove the following:

A necessary and sufficient condition for $\Sigma(H; B, B^+, C)$ where $d^+ \leq C \leq B^+$ to split regularly over the basis group is that $H$ contains no subgroup isomorphic to $S(B, C)$.

An immediate corollary of this result is that $\Sigma(H; B, B^+, C)$ to split regularly over the basis group if and only if $H$ contains no element of order 2.

In the last section we discuss the splitting of alternating groups over the basis group. This discussion is separated into two cases, namely splitting of monomial alternating groups of finite degree $n$, i.e., $\Sigma_{n,A}(H)$ and splitting of monomial alternating groups of infinite degree $B$ where $B$ is an infinite cardinal number. The main difference between the splitting if complete monomial group and monomial alternating group is the following:

In the former one symmetric group $S_n$ is generated by permutations of the form $(1 \ i)$ where $i=2,\ldots,n$ and we study the images of these elements and in the latter one. Alternating groups are generated by permutations of type $(i \ j \ k)$ where $i, j, k$ are pairwise distinct elements of $\{1, 2, \ldots, n\}$. Therefore we study the images of elements of this type.

The examples are given by $\Sigma_{3,A}(H)$ as a special case and in this case the splitting of $\Sigma_{3,A}(H)$ over the basis group is always regular. The second example of the splitting is $\Sigma_{4,A}(H)$ over the basis group is also given. When $n=4$, $A_4$ has a proper normal subgroup is isomorphic to elementary abelian 2-group of order 4. The image of this subgroup into $H$ is studied and for $\Sigma_{4,A}(H)$, we show that there are two types of complement; one comes from the conjugates of $S_n$ and the other complements arises from the homomorphic images of $A_4$ into a cyclic subgroup of order 3 of $H$. In particular, if $H$ has no subgroup of order 3, then all complements of $\Sigma_{4,A}(H)$ over the basis group are conjugate i.e., $\Sigma_{4,A}(H)$ splits regularly over the basis group $V_4(H)$. For the general case we show that, $\Sigma_{n,A}(H)$ splits over the basis group and $\Sigma_{n,A}(H)$ splits regularly over the basis group if and only if $H$ does not contain any
subgroup isomorphic to $A_{n-1}$, for $n \geq 6$.

The splitting of infinite alternating groups is studied with a similar technique and we prove that $\Sigma_A(H; B, B^+, d)$ split regularly over the basis group if and only if $H$ does not have any subgroup isomorphic to $A(B,d)$. 
In this section we define symmetries not only on the finite sets as in the case of Ore [2], but also on the sets arbitrarily large. The group H will be arbitrary.

Let \( d \) be the cardinal of the set of integers, i.e., \( d = \aleph_0 \), \( B \) be any infinite cardinal; \( B^+ \), the successor of \( B \), \( U \) is a set with the cardinal \( B \), and let \( C \) be a cardinal such that

\[ d \leq C \leq B^+. \]

**Definition 2.0.1.** The set of all permutations \( s \) of the set \( U \) onto itself is a group. It is denoted by \( S(B, B^+) \), and is called the infinite symmetric group on the set \( U \).

Let \( s \in S(B, B^+) \). For \( s \), we define support of \( s \). Namely

\[ \text{supp}(s) = \{ x_i \in U | s(x_i) \neq x_i \} \]

By \( |\text{supp}(s)| \) we define the cardinality of the set \( \text{supp}(s) \).

Now, we define the subgroup \( S(B, C) \) of the group \( S(B, B^+) \): \[ S(B, C) = \{ \sigma \in S(B, B^+) : |\text{supp}(\sigma)| < C \} \].

**Lemma 2.0.2.** The set \( S(B, C) \) is a subgroup of \( S(B, B^+) \).

**Proof.** Let \( \sigma_1, \sigma_2 \in S(B, C) \). Then, \( |\text{supp}(\sigma_1)| < C \) and \( |\text{supp}(\sigma_2)| < C \). If

\[ |\text{supp}(\sigma_1)| < C, \text{ since } \text{supp}(\sigma_1) = \text{supp}(\sigma_1^{-1}), |\text{supp}(\sigma_1^{-1})| = |\text{supp}(\sigma_1)| < C. \]
So, $\sigma_1^{-1} \in S(B, C)$.

**Claim:** $\text{supp}(\sigma_1 \sigma_2) \subseteq \text{supp}(\sigma_1) \cup \text{supp}(\sigma_2)$.

Assume $a \in \text{supp}(\sigma_1 \sigma_2)$. Then $a \cdot \sigma_1 \sigma_2 \neq a$. Let $a \notin \text{supp}(\sigma_2)$. If $a \notin \text{supp}(\sigma_2)$, then we show that $a \in \text{supp}(\sigma_1)$.

Since $a \notin \text{supp}(\sigma_2)$, $a \cdot \sigma_2 = a$, and $a \cdot \sigma_1 \sigma_2 \neq a$. It follows that $a \cdot \sigma_1 \neq a$. So, $a \in \text{supp}(\sigma_1)$.

Therefore, $\text{supp}(\sigma_1 \sigma_2) \subseteq \text{supp}(\sigma_1) \cup \text{supp}(\sigma_2)$. Since $C$ is infinite,

$$|\text{supp}(\sigma_1) \cup \text{supp}(\sigma_2)| \leq \{ |\text{supp}(\sigma_1)| + |\text{supp}(\sigma_2)| \} = \max\{|\text{supp}(\sigma_1)|, |\text{supp}(\sigma_2)|\} \leq C$$

This implies $|\text{supp}(\sigma_1 \sigma_2)| \leq C$. So, $\sigma_1 \sigma_2 \in S(B, C)$.

Thus, $S(B, C)$ is a subgroup of $S(B, B^+)$.

**Definition 2.0.3.** If the number of $x$ moved by $s$ is finite, then the group $S(B,d)$ is called finitary symmetric group where $\text{supp}(s) = \{ x_i \in U | s(x_i) \neq x_i \}$. We denote this set as

$$FSym(U) = \{ s \in S(B, B^+) : |\text{supp}(\sigma)| < \infty \}$$

**Definition 2.0.4.** Here we put the constraint that the number of moving elements of $U$ by $s \in A(B, d)$, is less than the cardinality of natural numbers. Since we mention alternating groups we should have evenly many transpositions. So, to define evenly many transpositions, the largest cardinality of $\text{supp}(s)$ should be finite, and so this $s$ comes from $S(B,d)$. The group $A(B, d)$ has elements $s$’s where those $s$’s comes from $S(B,d)$ and each of which have evenly many transpositions. The group $A(B,d)$ is called the infinite alternating group on the set $U$.

In $S(B, B^+)$ every element $s$ determines a set of cycles of the form
\[ c = \begin{pmatrix} x_1 & x_2 & \cdots & x_{n-1} & x_n \\ x_2 & x_3 & \cdots & x_n & x_1 \end{pmatrix} = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix} \]

or

\[ c = \begin{pmatrix} \cdots & x_{-1} & x_0 & x_1 & \cdots \\ \cdots & x_0 & x_1 & x_2 & \cdots \end{pmatrix} = \begin{pmatrix} \cdots & x_{-1} & x_0 & x_1 & \cdots \end{pmatrix}. \]

By well-ordering principle, every set can be well-ordered. Therefore, every permutation or cycle as in the above notation is meaningful.

**Lemma 2.0.5.** Every permutation \( s \in S(B, B^+) \) can be written as a disjoint product of commutative cycles of finite length or infinite cycles.

**Proof.** Let \( U \) be the set as above and \( \sigma \in S(B, B^+) \). So, \( U \) be a set of cardinality \( B \). Define a relation on \( U \). Two elements \( x_i, x_j \in U \) are related \( x_i \sim x_j \) if and only if there exists \( n \in \mathbb{Z} \) such that \( \sigma^n(x_i) = x_j \).

**Claim:** “ \( \sim \)” is an equivalence relation on \( U \).

**(i)** \( x_i \sim x_i \) as \( \sigma^0 = id \) and \( id(x_i) = \sigma^0(x_i) = x_i \).

**(ii)** \( x_i \sim x_j \) implies that there exists \( n \in \mathbb{Z} \) such that \( \sigma^n(x_i) = x_j \). Then \( (\sigma^{-1})^n(x_j) = \sigma^{-n}(x_j) = x_i \). So, \( x_j \sim x_i \) as \( -n \in \mathbb{Z} \).

**(iii)** Assume that \( x_i \sim x_j \) and \( x_j \sim x_k \), \( \{x_i, x_j, x_k\} \subseteq U \). Then there exists \( n \) and \( m \) in \( \mathbb{Z} \) such that \( \sigma^n(x_i) = x_j \), and \( \sigma^m(x_j) = x_k \). It follows that

\[ \sigma^{m+n}(x_i) = \sigma^m(\sigma^n(x_i)) = \sigma^m(x_j) = x_k, n + m \in \mathbb{Z} \]. Hence, \( x_i \sim x_k \).

Consequently, \( \sim \) is an equivalence relation.

The equivalence class containing an element \( x_i \in U \) is of the form \( \{\ldots, \sigma^{-2}(x_i), \sigma^{-1}(x_i), x_i, \sigma(x_i), \sigma^2(x_i), \ldots\} \). If this set is finite, then there exists \( n \in \mathbb{N} \) such that \( \sigma^n(x_i) = x_i \). Then, we have a finite cycle of the form

\( (x_i, \sigma(x_i), \ldots, \sigma^{n-1}(x_i)) \)

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If the equivalence class containing $x_i$ is an infinite set, then we have an infinite cycle of the form

$$(..., \sigma^{-2}(x_i), \sigma^{-1}(x_i), x_i, \sigma(x_i), \sigma^2(x_i), ..., \sigma^{n-1}(x_i), ...)$$

This type of cycles are called infinite cycles.

Since $\sim$ is an equivalence relation, the equivalence classes are disjoint, and union of equivalence classes is the set $U$. Hence, every element of $U$ is contained in a cycle, and one may observe that disjoint cycles commute. Hence, every permutation $\sigma$ can be written in a unique way as a product of disjoint cycles of length finite or infinite up to order.

**Definition 2.0.6.** A cycle with $n$ distinct $x$'s is called an $n$-cycle; $n = 1, 2, \ldots, k$.

**Definition 2.0.7.** A monomial substitution over $H$ is a transformation of the form

$$y = \left( \begin{array}{ccc} \cdots & x_i & \cdots \\ \cdots & h_i x_i & \cdots \end{array} \right) \quad (2.1)$$

where the mapping $x_i \rightarrow x_{i_l}$ is a one to one mapping of $U$ onto itself and $h_l$ belongs to $H$. The $h_l$ will be called factors of $y$.

If $y$ is given by equation (2.1) and $y_1$ is given by

$$y_1 = \left( \begin{array}{ccc} \cdots & x_{i_l} & \cdots \\ \cdots & k_{i_l} x_{j_l} & \cdots \end{array} \right), \quad (2.2)$$

then the product $yy_1$ is defined by

$$yy_1 = \left( \begin{array}{ccc} \cdots & x_{i_l} & \cdots \\ \cdots & h_{i_l} k_{i_l} x_{j_l} & \cdots \end{array} \right). \quad (2.3)$$

The inverse of $y$ is

$$y^{-1} = \left( \begin{array}{ccc} \cdots & x_{i_l} & \cdots \\ \cdots & h_{i_l}^{-1} x_{i_l} & \cdots \end{array} \right). \quad (2.4)$$
The identity element will be

\[ E = \begin{pmatrix} \cdots & x_1 & \cdots \\ \cdots & ex_1 & \cdots \end{pmatrix}. \] (2.5)

**Definition 2.0.8.** By above multiplication, the set of monomial substitution is a group, that will be denoted by \( \Sigma(H; B, B^+, B^+) \), and called the monomial group of \( H \) of degree \( B \) or, more simply, the symmetry of \( H \).

If \( H \) consists only of the identity element, then \( \Sigma(H; B, B^+, B^+) \) is the symmetric group \( S(B, B^+) \).

**Definition 2.0.9.** A permutation in \( \Sigma(H; B, B^+, B^+) \) is a substitution of the form

\[ s = \begin{pmatrix} \cdots & x_\epsilon & \cdots \\ \cdots & ex_{i_\epsilon} & \cdots \end{pmatrix} = \begin{pmatrix} \cdots & \epsilon & \cdots \\ \cdots & i_\epsilon & \cdots \end{pmatrix}. \] (2.6)

**Lemma 2.0.10.** The set of permutations forms a subgroup of \( \Sigma(H; B, B^+, B^+) \) and it is denoted by \( S(B, B^+) \). We call this subgroup as permutation subgroup of \( \Sigma(H; B, B^+, B^+) \).

**Proof.** Let \( \alpha, \beta \in S(B, B^+) \), where

\[ \alpha = \begin{pmatrix} \cdots & x_\epsilon & \cdots \\ \cdots & ex_{i_\epsilon} & \cdots \end{pmatrix} \] (2.7)

\[ \beta = \begin{pmatrix} \cdots & x_\epsilon & \cdots \\ \cdots & ex_{j_\epsilon} & \cdots \end{pmatrix} \text{ and } \beta^{-1} = \begin{pmatrix} \cdots & x_{j_\epsilon} & \cdots \\ \cdots & ex_\epsilon & \cdots \end{pmatrix}, \] (2.8)

\[ \alpha\beta = \begin{pmatrix} \cdots & x_\epsilon & \cdots \\ \cdots & ex_{j_\epsilon} & \cdots \end{pmatrix} \] (2.9)
This product is in $S(B, B^+)$. Therefore, $S(B, B^+)$ is a subgroup of $\Sigma(H; B, B^+, B^+)$. \qed

**Definition 2.0.11.** A multiplication in $\Sigma(H; B, B^+, B^+)$ is a substitution of the form

$$v = \begin{pmatrix} \ldots & x_e & \ldots \\ \ldots & h_e x_e & \ldots \end{pmatrix} = \{\ldots, h_e, \ldots\}. \quad (2.10)$$

**Lemma 2.0.12.** The set of multiplications forms a normal subgroup of $\Sigma(H; B, B^+, B^+)$, denoted by $V(B, B^+)$, and it is called the basis group of $\Sigma(H; B, B^+, B^+)$. 

**Proof.** Let $\kappa, \alpha \in V(B, B^+)$, and $\theta \in \Sigma(H; B, B^+, B^+)$ where

$$\kappa = \begin{pmatrix} \ldots & x_e & \ldots \\ \ldots & h_e x_e & \ldots \end{pmatrix} \quad (2.11)$$

$$\alpha = \begin{pmatrix} \ldots & x_e & \ldots \\ \ldots & l_e x_e & \ldots \end{pmatrix} \quad (2.12)$$

$$\theta = \begin{pmatrix} \ldots & x_e & \ldots \\ \ldots & k_e x_i & \ldots \end{pmatrix}. \quad (2.13)$$

(i) Since

$$\kappa^{-1} = \begin{pmatrix} \ldots & x_e & \ldots \\ \ldots & h_e^{-1} x_e & \ldots \end{pmatrix}, \quad (2.14)$$

clearly $\kappa^{-1} \in V(B, B^+)$. 

(ii) The composition of $\kappa$ and $\alpha$ will be

$$\kappa \alpha = \begin{pmatrix} \ldots & x_e & \ldots \\ \ldots & h_e l_e x_e & \ldots \end{pmatrix} \quad (2.15)$$

So, the composition belongs to the basis group $V(B, B^+)$. 

Thus, $V(B, B^+)$ is a subgroup of $\Sigma(H; B, B^+, B^+)$. 

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Moreover,
\[
\theta^{-1} \kappa \theta = \begin{pmatrix}
\cdots & x_i & \cdots \\
\cdots & k_c^{-1} h_c k_c x_i & \cdots \\
\end{pmatrix}
\]  
(2.16)

So, this product is also in the basis group. Therefore, \( V(B, B^+) \) is a normal subgroup in \( \Sigma(H; B, B^+, B^+) \). \( \square \)

**Lemma 2.0.13.** The basis group is isomorphic to the Cartesian product of \( B \) groups, each of which is isomorphic to \( H \).

**Proof.** Let \( v \in V(B, B^+) \), \( v = \{h_1, h_2, \ldots\} \).

Assume
\[
\theta : V(B, B^+) \longrightarrow \prod H \\
v \longmapsto (h_1, h_2, \ldots)
\]

- \( \theta \) is a homomorphism:

Let \( v_1, v_2 \in V(B, B^+) \), where
\[
v_1 = \{h_1, h_2, \ldots\} \text{ and } v_2 = \{k_1, k_2, \ldots\}, h_i, k_i \in H.
\]
\[
\theta(v_1 v_2) = (h_1 k_1, h_2 k_2, \ldots) = \theta(v_1) \theta(v_2).
\]

So, \( \theta \) is a homomorphism.

- \( \theta \) is one to one:

\[
Ker\theta = \{v \in V(B, B^+) | \theta(v) = id_{\prod H}\}
\]
\[
= \{v \in V(B, B^+) | (h_1, h_2, \ldots) = (e_H, e_H, \ldots)\}
\]

Then \( h_i = e_H \) for \( i = 1, 2, \ldots \)

So, \( \theta \) is one to one.

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• $\theta$ is onto:

For any element $(h_1, h_2, \ldots) \in \prod H$, there exists a $v \in V(B, B^+)$ such that

\[ v = \{h_1, h_2, \ldots\} . \]

So $\theta$ is onto.

Consequently, $\theta$ is an isomorphism, and the basis group is isomorphic to $H \times H \times \ldots \times H \times \ldots$ where the number of $H$ is $B$ many.

\[ \square \]

**Definition 2.0.14.** A scalar in $\Sigma(H; B, B^+, B^+)$ is a multiplication with each factor is the same. Scalars are of the form $\{\ldots, h, h, \ldots\}$ and are denoted by $v = \{h\}$.

**Lemma 2.0.15.** Scalars are the only elements that commute with permutations.

**Proof.** Let $s$ be an arbitrary element of $S(B, B^+)$. It is of the form

\[
s = \begin{pmatrix} x_1 & x_2 & \cdots & x_k & \cdots \\ x_{j_1} & x_{j_2} & \cdots & x_{j_k} & \cdots \end{pmatrix} . \tag{2.17} \]

Let $y \in \Sigma(H; B, B^+, B^+)$ be arbitrary.

\[
y = \begin{pmatrix} x_1 & x_2 & \cdots & x_t & \cdots \\ h_1 x_{i_1} & h_2 x_{i_2} & \cdots & h_t x_{i_t} & \cdots \end{pmatrix} . \tag{2.18} \]

If $ys = sy$, then

\[
ys = \begin{pmatrix} x_1 & x_2 & \cdots & x_t & \cdots \\ h_1 x_{i_1} & h_2 x_{i_2} & \cdots & h_t x_{i_t} & \cdots \end{pmatrix} \begin{pmatrix} x_1 & x_2 & \cdots & x_k & \cdots \\ x_{j_1} & x_{j_2} & \cdots & x_{j_k} & \cdots \end{pmatrix} \tag{2.19}
\]

\[
= \begin{pmatrix} x_1 & x_2 & \cdots & x_t & \cdots \\ h_1 x_{j_1} & h_2 x_{j_2} & \cdots & h_t x_{j_t} & \cdots \end{pmatrix} , \tag{2.20}
\]

and
\[ sy = \begin{pmatrix} x_1 & x_2 & \cdots & x_k & \cdots \\ x_{j_1} & x_{j_2} & \cdots & x_{j_k} & \cdots \end{pmatrix} \begin{pmatrix} x_1 & x_2 & \cdots & x_t & \cdots \\ h_{i_1} & h_{i_2} & \cdots & h_{i_t}x_{i_t} & \cdots \end{pmatrix} \]

\[ = \begin{pmatrix} x_1 & x_2 & \cdots & x_k & \cdots \\ h_{i_1}x_{i_1} & h_{i_2}x_{i_2} & \cdots & h_{i_k}x_{i_k} & \cdots \end{pmatrix} \]  

(2.21)

ys=sy implies that

\[ h_{j_1} = h_1 \]
\[ h_{j_2} = h_2 \]

. . .

\[ h_{j_t} = h_t \]

and

\[ x_{i_{j_1}} = x_{j_{i_1}} \]
\[ x_{i_{j_2}} = x_{j_{i_2}} \]

. . .

\[ x_{i_{j_k}} = x_{j_{i_k}} \]

means that

\[ i_{j_k} = j_{i_k} \implies i_j = j_i \implies j = i. \]

This shows that scalars commute with all permutations.

Now, assume ys=sy for all \( s \in S(B, B^+) \). We should show y is a scalar. Let \( h_a, h_b \) be arbitrary factors of y which occur position a, and position b, respectively.
Consider \( y(x_a, x_b) \) and \( (x_a, x_b)y \). Since \( y \) commutes with all permutations, \( y(x_a, x_b) = (x_a, x_b)y \). If we calculate \( y(x_a, x_b) \) and \( (x_a, x_b)y \), we get

\[
y(x_a, x_b) = \begin{pmatrix}
\cdots & x_a & \cdots & x_b & \cdots \\
\cdots & h_a x_b & \cdots & h_b x_a & \cdots
\end{pmatrix}
\]

(2.23)

\[
(x_a, x_b)y = \begin{pmatrix}
\cdots & x_a & \cdots & x_b & \cdots \\
\cdots & h_b x_b & \cdots & h_a x_a & \cdots
\end{pmatrix}.
\]

(2.24)

As a result, we see that \( h_a = h_b \) for arbitrary \( a \) and \( b \). Thus, in \( y \) all factors are the same.

Now, consider \( y(x_{i\epsilon}, x_a) \) where \( i\epsilon \neq a \). By equation (2.18) we see that with \( y, x_{\epsilon} \) goes to \( x_{i\epsilon} \). Assume also \( i\epsilon \neq \epsilon \). If we calculate \( y(x_{i\epsilon}, x_a) \) and \( (x_{i\epsilon}, x_a)y \) we should get the same result, since \( y \) commutes with all permutation.

\[
y(x_{i\epsilon}, x_a) = \begin{pmatrix}
\cdots & x_{\epsilon} & \cdots \\
\cdots & h_{\epsilon} x_a & \cdots
\end{pmatrix}
\]

(2.25)

\[
(x_{i\epsilon}, x_a)y = \begin{pmatrix}
\cdots & x_{\epsilon} & \cdots \\
\cdots & h_{\epsilon} x_a & \cdots
\end{pmatrix}.
\]

(2.26)

Here since \( x_a = x_{i\epsilon} \) we get a contradiction.

Therefore, \( y \) is a scalar.

Consequently, we get that scalars are the only elements commute with permutations.

\[\square\]

**Lemma 2.0.16.** The center \( Z(\Sigma(H; B, B^+, B^+)) \) of \( \Sigma(H; B, B^+, B^+) \) is the set of all scalars \( v=\{k\} \) where \( k \) belongs to the center of \( H \), and \( Z(\Sigma(H; B, B^+, B^+)) \) is isomorphic to the center of \( H \).

**Proof.** By Lemma 2.0.15 scalars are the only elements that commute with permutations, and permutations are contained in \( \Sigma(H; B, B^+, B^+) \). The elements of
\[
Z(\Sigma(H; B, B^+)) \text{ are contained in scalars.}
\]

Moreover, \( m\{h_1, h_2, \ldots\} = \{h_1, h_2, \ldots\} \) where \( m = \{m, m, \ldots\} \) implies that \( mh_i = h_im \) for all \( h_i \in H \).

Hence, \( m \in Z(H) \).

\[
\varphi : Z(\Sigma(H; B, B^+)) \longrightarrow Z(H)
\]

\[
v = \{k\} \longmapsto k
\]

Let \( v_1 = \{k_1, \ldots\} \), and \( v_2 = \{k_2, \ldots\} \).

- \( \varphi \) is a homomorphism:
  \[
  \varphi(v_1v_2) = k_1k_2 = \varphi(v_1)\varphi(v_2).
  \]
  So \( \varphi \) is a homomorphism.

- \( \varphi \) is a one to one:
  \[
  \text{Ker} \varphi = \{v \in Z(\Sigma(H; B, B^+)) | \varphi(v) = 1_H\} = \{1, 1, \ldots\}.
  \]
  So \( \varphi \) is one to one.

- \( \varphi \) is a onto:
  For any \( k \in Z(H) \), there exists \( v \in Z(\Sigma(H; B, B^+)) \) such that
  \[
  v = \{k, k, \ldots\}.
  \]
  Thus, \( \varphi \) is onto.

Hence \( \varphi \) is an isomorphism.

\[\square\]

**Definition 2.0.17.** A group \( G \) splits over a normal subgroup \( N \) if there exists a subgroup \( M \) of \( G \) such that \( G = \langle M, N \rangle = MN \), \( N \cap M = E \).

The group \( M \) may be replaced by any of its conjugates and the relations will still hold. Indeed, for any element \( g \in G \), we have \( G^g = (NT)^g = N^gT^g \) since \( N \) is normal we obtain \( N^g = N \). Hence, \( T^g \) which is conjugate of \( T \) is also a complement of \( N \) in \( G \).
Therefore, all conjugates of \( T \) will be a complement of \( N \) in \( G \). But for every subgroup \( T \) such that \( G=\langle N, T \rangle \), \( N \cap T=E \) it follows that \( T \) is conjugate to \( M \), then we say that \( G \) splits regularly over \( N \).

**Lemma 2.0.18.** Any substitution \( y \) of \( \Sigma(H; B, B^+, B^+) \) can be written as a multiplication multiplied by a permutation uniquely.

**Proof.** Let
\[
y = \begin{pmatrix}
\cdots & x_{\alpha-1} & x_\alpha & x_{\alpha+1} & \cdots \\
\cdots & h_{\alpha-1}x_{i_{\alpha-1}} & h_\alpha x_i & h_{\alpha+1}x_{i_{\alpha+1}} & \cdots 
\end{pmatrix},
\]
and \( y \in \Sigma(H; B, B^+, B^+) \). Then \( y=vs \) where \( v=\{\ldots, h_{\alpha-1}, h_\alpha, h_{\alpha+1}, \ldots \} \) and
\[
s = \begin{pmatrix}
\cdots & x_{\alpha-1} & x_\alpha & x_{\alpha+1} & \cdots \\
\cdots & x_{i_{\alpha-1}} & x_i & x_{i_{\alpha+1}} & \cdots 
\end{pmatrix},
\]
\[
y = \{\ldots, h_{\alpha-1}, h_\alpha, h_{\alpha+1}, \ldots \} \begin{pmatrix}
\cdots & x_{\alpha-1} & x_\alpha & x_{\alpha+1} & \cdots \\
\cdots & x_{i_{\alpha-1}} & x_i & x_{i_{\alpha+1}} & \cdots 
\end{pmatrix}
\]
So \( y \) can be written as a product of a permutation and a multiplication.

Assume that there exist
\[
s' = \begin{pmatrix}
\cdots & x_{\alpha-1} & x_\alpha & x_{\alpha+1} & \cdots \\
\cdots & x_{j_{\alpha-1}} & x_{j_i} & x_{j_{\alpha+1}} & \cdots 
\end{pmatrix}
\]
a permutation and \( v'=\{\ldots, k_{\alpha-1}, k_\alpha, k_{\alpha+1}, \ldots \} \) a multiplication such that \( y = v's' \).

Then
\[
v's' = \begin{pmatrix}
\cdots & x_{\alpha-1} & x_\alpha & x_{\alpha+1} & \cdots \\
\cdots & k_{\alpha-1}x_{j_{\alpha-1}} & k_\alpha x_{j_i} & k_{\alpha+1}x_{j_{\alpha+1}} & \cdots 
\end{pmatrix}
\]
\[
= \begin{pmatrix}
\cdots & x_{\alpha-1} & x_\alpha & x_{\alpha+1} & \cdots \\
\cdots & h_{\alpha-1}x_{i_{\alpha-1}} & h_\alpha x_i & h_{\alpha+1}x_{i_{\alpha+1}} & \cdots 
\end{pmatrix}
\]
If we look at \( V(B, B^+) \cap S(B, B^+) \);
v=s implies

\[
\begin{pmatrix}
\cdots & x_{\alpha-1} & x_\alpha & x_{\alpha+1} & \cdots \\
\cdots & k_{\alpha-1}x_{\alpha-1} & k_\alpha x_\alpha & k_{\alpha+1}x_{\alpha+1} & \cdots
\end{pmatrix} = \begin{pmatrix}
\cdots & x_{\alpha-1} & x_\alpha & x_{\alpha+1} & \cdots \\
\cdots & x_{j_{\alpha-1}} & x_{j_\alpha} & x_{j_{\alpha+1}} & \cdots
\end{pmatrix}
\]

(2.33)

Then, \(x_{i_1} = x_1\)

.

.

.

\(x_{i_t} = x_t\).

So, \(i_t = t\) for any \(t=1, 2, \ldots\) and \(h_t = e\).

This implies

\[V(B, B^+) \cap S(B, B^+) = E.\]

Then \(vs = v's'\) implies \((v')^{-1}v = s's^{-1} \in V(B, B^+) \cap S(B, B^+) = E.\) Hence, \(v=v'\) and \(s=s'\).

So, this multiplication is unique.

\[\square\]

Thus,

\[\Sigma(H; B, B^+, B^+) = < S(B, B^+), V(B, B^+) > .\]

Let \(B, C, D\) be infinite cardinal such that

\[d \leq C \leq B^+,\]

\[d \leq D \leq B^+.\]

Let \(\Sigma(H; B, C, D)\) be the set of all \(y=vs\) where \(v \in V(B, B^+), s \in S(B, B^+)\) and \(v\) has less than \(C\) non identity factors, \(s\) moves less than \(D\) of the \(x\)'s. Then we get the
Lemma 2.0.19. \( \Sigma(H; B, C, D) \) is a subgroup of \( \Sigma(H; B, B^+, B^+) \).

**Proof.** Let \( y_1, y_2 \in \Sigma(H; B, C, D) \) where \( y_1 = v_1 s_1 \) and \( y_2 = v_2 s_2 \). We know that \( v_1, v_2 \) have less than \( C \) non identity factors; and

\[ |\text{supp}(s_1)|, |\text{supp}(s_2)| < D \]

i) \( y_1^{-1} \in \Sigma(H; B, C, D) \):

\[ y_1^{-1} = s_1^{-1} v_1^{-1} = s_1^{-1} v_1^{-1} s_1 s_1^{-1} (v_1^{-1}) s_1^{-1} \]

Since \( s_1 \) moves only the components of \( v_1 \) according to the action of \( s_1 \) on the set \( U \), we have in the elements \((v_1^{-1}) s_1\) the elements of \( v_1^{-1} \) permuted with respect to the action of \( s_1 \). Hence the cardinality of moved elements will not increase. On the other hand, \( \text{supp}(s_1) = \text{supp}(s_1') \).

Thus, \( y_1^{-1} \in \Sigma(H; B, C, D) \)

ii) \( y_1 y_2 \in \Sigma(H; B, C, D) \):

\[ y_1 y_2 = v_1 s_1 \]

Since \( v_1 \) and \( v_1^{-1} \) has less than \( C \) non identity factors \( v_1 v_1' \) also has less than \( C \) non identity factors.

By Lemma 2.0.2, \( |\text{supp}(s_1 s_2)| < D \).

Hence, \( y_1 y_2 \in \Sigma(H; B, C, D) \), and \( \Sigma(H; B, C, D) \) is a subgroup of \( \Sigma(H; B, B^+, B^+) \). \( \square \)

The set \( \Sigma_A(H; B, C, d) \) of all \( y=vs \) where \( v \) less than \( C \) non identity factors and \( s \) belongs to \( A(B, d) \) forms a subgroup of \( \Sigma(H; B, B^+, B^+) \).

Let \( o(U) = n \) where \( n \) is a finite cardinal. Then the symmetry over \( H \) of \( U \) will be denoted by \( \Sigma(H; n, n + 1, n + 1) = \Sigma_n(H) \). Then \( \Sigma_{n,A}(H) \) where elements of this
group can be written as $y=vs$, and this $s$ belongs to $A_n$, is a subgroup. Here basis group is denoted by $V(n, n+1)=V_n$. 
Let $y$ be an arbitrary element of $\Sigma(H; B, B^+, B^+)$. It has been shown that $y$ has a unique decomposition $y = vs$ where $v$ belongs to $V(B, B^+)$ and $s$ belongs to $S(B, B^+)$. Throughout this section we will mention more about cycles, transformations, and we will give the centralizers of finite and infinite monomial groups, which are written by Ore [2] and Crouch [1].

By Lemma 2.0.5, we know that any permutation $s$ in $S(B, B^+)$ can be written as a disjoint product of commutative cycles. This decomposition induces a decomposition of $v$ such that to each cycle $c_e$ of $s$ there corresponds a multiplication $v_e$ with all factors $e$ in those positions corresponding to $x$ that $s$ does not move and factors the same as in $v$ for the $x$ that $s$ moves. Thus $v_e c_e$ has one of the two forms

$$v_e c_e = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ h_1 x_2 & h_2 x_3 & \cdots & h_n x_1 \end{pmatrix} \text{ when } n < d \quad (3.1)$$

or

$$v_e c_e = \begin{pmatrix} \cdots & x_{-1} & x_0 & x_1 & \cdots \\ \cdots & h_{-1} x_0 & h_0 x_1 & h_1 x_2 & \cdots \end{pmatrix} \text{ when } n = d. \quad (3.2)$$

If $c$ is a cycle of length $n$ and of the form

$$c = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ h_1 x_2 & h_2 x_3 & \cdots & h_n x_1 \end{pmatrix}, \quad (3.3)$$
observe that
\[ c^2 = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ h_1h_2x_3 & h_2h_3x_4 & \cdots & h_nh_1x_2 \end{pmatrix}. \] (3.4)

Then,
\[ c^n = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ h_1h_2\ldots h_nx_1 & h_2h_3\ldots h_nh_1x_2 & \cdots & h_nh_1\ldots h_{n-1}x_n \end{pmatrix} \] (3.5)
\[ = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ \delta_1x_1 & \delta_2x_2 & \cdots & \delta_nx_n \end{pmatrix} \] (3.6)

The factor of \( n^{th} \) power of \( c \) is \{\( \delta_1, \delta_2, \ldots, \delta_n \)\} where \( \delta_1 = h_1 \ldots h_n \),
\[ \delta_2 = h_2 \ldots h_n h_1, \delta_n = h_n h_1 \ldots h_{n-1}. \]

**Definition 3.0.20.** These \( \delta_i \)'s are called the determinants of \( c \).

Note that, \( \delta_i \)'s are conjugate. Indeed,
\[ h_n^{-1} \delta_n h_n = \delta_1. \]
\[ h_2 \delta_3 h_2^{-1} = \delta_2. \]
\[ h_3 \delta_4 h_3^{-1} = \delta_3. \]
\[ \vdots \]
\[ h_n \delta_{n-1} h_n^{-1} = \delta_1 \]

Since \( \delta_i \)'s are conjugate, there exists a unique determinant class for each cycle. Above, we have defined determinant class of a finite cycle.
**Theorem 3.0.21.** Two finite cycles are conjugate if and only if they have the same length and determinant class.

*Proof.* Let,

\[ \kappa = \begin{pmatrix} x_1 & x_2 & \cdots & x_m \\ k_1x_1 & k_2x_2 & \cdots & k_mx_m \end{pmatrix} \]  
(3.7)

\[ \kappa^{-1} = \begin{pmatrix} x_{j_1} & x_{j_2} & \cdots & x_{j_m} \\ k_1^{-1}x_1 & k_2^{-1}x_2 & \cdots & k_m^{-1}x_m \end{pmatrix} \]  
(3.8)

\[ \gamma = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ c_1x_2 & c_2x_3 & \cdots & c_nx_1 \end{pmatrix} \]  
(3.9)

When we consider conjugation of \( \kappa \) with \( \gamma \) there are three cases:

**Case 1:** If \( m=n \),

\[ \kappa^{-1}\gamma\kappa = \begin{pmatrix} x_{j_1} & x_{j_2} & \cdots & x_{j_n} \\ k_1^{-1}c_1k_2x_{j_2} & k_2^{-1}c_2k_3x_{j_3} & \cdots & k_n^{-1}c_nk_1x_{j_1} \end{pmatrix} \]  
(3.10)

**Case 2:** If \( m<n \),

\[ \kappa^{-1}\gamma\kappa = \begin{pmatrix} x_{j_1} & x_{j_2} & \cdots & x_{j_m} & \cdots & x_{j_n} \\ k_1^{-1}c_1k_2x_{j_2} & k_2^{-1}c_2k_3x_{j_3} & \cdots & k_m^{-1}c_mx_{j_{m+1}} & \cdots & c_nk_1x_{j_1} \end{pmatrix} \]  
(3.11)

**Case 3:** If \( m>n \),

\[ \kappa^{-1}\gamma\kappa = \begin{pmatrix} x_{j_1} & x_{j_2} & \cdots & x_{j_n} & x_{j_{n+1}} & \cdots & x_{j_m} \\ k_1^{-1}c_1k_2x_{j_2} & k_2^{-1}c_2k_3x_{j_3} & \cdots & k_n^{-1}c_nk_1x_{j_1} & x_{j_{n+1}} & \cdots & x_{j_m} \end{pmatrix} \]  
(3.12)

Above, it can be seen that \( \kappa^{-1}\gamma\kappa \) has the same form in three cases; they have the same length and the same determinant class. Namely, for case (1) in equation (3.10)
Ore [2] has investigated the result of transforming a finite cycle of an element of monomial group to its normal form. We will state that in the following theorem.

**Theorem 3.0.22.** Any cycle of length \( n \) may be transformed to the normal form

\[
\gamma = \begin{pmatrix}
  x_{i_1} & x_{i_2} & \cdots & x_{i_n} \\
  x_{i_2} & x_{i_3} & \cdots & x_{i_1}
\end{pmatrix} = \{x_{i_2}, \cdots, x_{i_n}, ax_{i_1}\}
\]

(3.13)

where \( a \) is any element in the determinant class of \( \gamma \). Any monomial substitution \( \rho \) is similar to a product of cycles without common variables \( \rho = \gamma_1 \cdots \gamma_r \) where each cycle is in normal form.

**Proof.** Let

\[
\kappa = \begin{pmatrix}
  x_{i_1} & x_{i_2} & \cdots & x_{i_n} \\
  c_1x_{i_2} & c_2x_{i_3} & \cdots & c_nx_{i_1}
\end{pmatrix}
\]

(3.14)

have the same determinant class of the cycle \( \gamma \).

If we can find a \( \beta \) such that \( \beta^{-1} \kappa \beta = \gamma \) then we will get the result.

\( \Delta_\kappa = c_1c_2\cdots c_n \), and \( \Delta_\gamma = a \) are determinants of \( \kappa \) and \( \gamma \). By our assumption, \( \Delta_\kappa \) and \( \Delta_\gamma \) are in the same determinant class, so there exists \( p_1 \) in \( H \) such that \( \Delta_\kappa^{p_1} = \Delta_\gamma \). By Theorem 1 in the paper of Ore [2] there exist \( p_1 \) such that

\[
p_1^{-1}c_1p_2 = 1, \quad p_2^{-1}c_2p_3 = 1, \ldots, \quad p_{n-1}^{-1}c_{n-1}p_n = 1, \quad p_n^{-1}c_np_1 = a.
\]

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Choose
\[
\beta = \begin{pmatrix}
    x_{i1} & x_{i2} & \cdots & x_{in} \\
    p_1 x_{i1} & p_2 x_{i2} & \cdots & p_n x_{in}
\end{pmatrix}.
\] (3.15)

Then, \(\beta^{-1}k\beta = \gamma\). Hence, each cycle may be transformed into normal form.

Since the transformation of \(\gamma\) into normal form may be performed by means of a substitution involving only the same variables, all cycles in \(\rho\) may be transformed into normal form simultaneously.

**Example:** Let \(n=4\), i.e., the set \(U\) has 4 elements, and \(H = S_3\). Let
\[\sigma \in \Sigma(H; 4, 5, 5) = \Sigma_4,\]
where
\[
\sigma = \begin{pmatrix}
    x_1 & x_2 & x_3 & x_4 \\
    (12)x_2 & (123)x_3 & (1)x_4 & (23)x_1
\end{pmatrix}.
\] (3.16)

Then,
\[
\sigma^2 = \begin{pmatrix}
    x_1 & x_2 & x_3 & x_4 \\
    (12)(123)x_3 & (123)(1)x_4 & (1)(23)x_1 & (23)(12)x_2
\end{pmatrix}.
\] (3.17)

\[
= \begin{pmatrix}
    x_1 & x_3 \\
    (13)x_3 & (23)x_1
\end{pmatrix} \begin{pmatrix}
    x_2 & x_4 \\
    (123)x_4 & (123)x_2
\end{pmatrix}.
\] (3.18)

\[
\sigma^3 = \begin{pmatrix}
    x_1 & x_2 & x_3 & x_4 \\
    (13)x_4 & (13)x_1 & (123)x_2 & (132)x_3
\end{pmatrix}.
\] (3.19)

\[
\sigma^4 = \begin{pmatrix}
    x_1 & x_2 & x_3 & x_4 \\
    (123)x_1 & (132)x_2 & (132)x_3 & (132)x_4
\end{pmatrix}.
\] (3.20)

In this case \(\delta_i\)s will be:
\[
\delta_1 = (123)
\]
\[ \delta_2 = (132) \]

\[ \delta_3 = (132) \]

\[ \delta_4 = (132) \]

We know that conjugacy classes of \( S_3 \) are \([(1)], [(1 2)], [(1 2 3)]\) where

\[ [(1)] = \{(1)\} \]

\[ [(1 2)] = \{(1 2), (1 3), (2 3)\} \]

\[ [(1 2 3)] = \{(1 2 3), (1 3 2)\} \]

Since \( \delta_i's \) are in the same conjugacy class, they are conjugate.

Now, we should consider infinite cycles. If a cycle is infinite, then below we will show that any infinite cycle in \( \Sigma(H; B, B^+, B^+) \) is conjugate to an infinite permutation in \( S(B, B^+) \). Indeed, let \( \kappa \) be an arbitrary substitution and \( \gamma \) be an infinite cycle in \( \Sigma(H; B, B^+, B^+) \).

\[
\kappa = \begin{pmatrix}
\cdots & x_{-1} & x_0 & x_1 & \cdots \\
\cdots & k_{-1}x_{j_{-1}} & k_0x_{j_0} & k_1x_{j_1} & \cdots 
\end{pmatrix}
\]

(3.21)

\[
\gamma = \begin{pmatrix}
\cdots & x_{-1} & x_0 & x_1 & \cdots \\
\cdots & h_{-1}x_0 & h_0x_1 & h_1x_2 & \cdots 
\end{pmatrix}
\]

(3.22)

Then,

\[
\kappa^{-1}\gamma\kappa = \begin{pmatrix}
\cdots & x_{j_{-1}} & x_{j_0} & x_{j_1} & \cdots \\
\cdots & k_1^{-1}h_1k_0x_{j_0} & k_0^{-1}h_0k_1x_{j_1} & k_1^{-1}h_1k_2x_{j_2} & \cdots 
\end{pmatrix}
\]

(3.23)

It shows that \( \kappa^{-1}\gamma\kappa \) has an infinite cycle form.

**Theorem 3.0.23.** Two cycles of length \( d \) are conjugate if and only if they leave the same number of \( x \) fixed.
Proof. Let $\gamma$ and $\theta$ be conjugate infinite cycles such that $\gamma$ has $n$ fixed points. Then

$$
\gamma = \begin{pmatrix}
\cdots & x_{i-1} & x_{i0} & x_{i1} & \cdots \\
\cdots & h_{i-1}x_{i0} & h_{i0}x_{i1} & h_{i1}x_{i2} & \cdots \\
\end{pmatrix} (a_1)(a_2)\ldots(a_n) \tag{3.24}
$$

then, for some monomial substitution $\kappa$,

$$
\kappa = \begin{pmatrix}
\cdots & x_{-1} & x_{0} & x_{1} & \cdots \\
\cdots & k_{-1}x_{j-1} & k_{0}x_{j0} & k_{1}x_{j1} & \cdots \\
\end{pmatrix} \tag{3.25}
$$

$$
\theta = \kappa^{-1}\gamma\kappa \tag{3.26}
$$

$$
= \begin{pmatrix}
\cdots & x_{j-1} & x_{j0} & j_{1} & \cdots \\
\cdots & k_{j-1}^{-1}h_{j-1}x_{i0} & h_{i0}x_{i1} & h_{i1}x_{i2} & \cdots \\
\end{pmatrix} (a_1)^\kappa(a_2)^\kappa\ldots(a_n)^\kappa. \tag{3.27}
$$

So, $\theta$ has $n$ fixed points.

Conversely, let

$$
c = \begin{pmatrix}
\cdots & x_{-1} & x_{0} & x_{1} & \cdots \\
\cdots & h_{-1}x_{0} & h_{0}x_{1} & h_{1}x_{2} & \cdots \\
\end{pmatrix}, \tag{3.28}
$$

and

$$
c' = \begin{pmatrix}
\cdots & x_{i-1} & x_{i0} & x_{i1} & \cdots \\
\cdots & r_{-1}x_{i0} & r_{0}x_{i1} & r_{1}x_{i2} & \cdots \\
\end{pmatrix} \tag{3.29}
$$

c and $c'$ leave the same number of $x$ fixed. We should consider if there exists a $y \in \Sigma(H; B, B^+, B^+)$ such that $y^{-1}cy = c'$ where

$$
y = \begin{pmatrix}
\cdots & x_{-1} & x_{0} & x_{1} & \cdots \\
\cdots & k_{-1}x_{j-1} & k_{0}x_{j0} & k_{1}x_{j1} & \cdots \\
\end{pmatrix} \tag{3.30}
$$

$$
y^{-1}cy = \begin{pmatrix}
\cdots & x_{j-1} & x_{j} & x_{j1} & \cdots \\
\cdots & k_{j-1}^{-1}h_{j-1}x_{j} & k_{0}^{-1}h_{0}k_{1}x_{j1} & k_{1}^{-1}h_{1}k_{2}x_{j2} & \cdots \\
\end{pmatrix} \tag{3.31}
$$

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then,

\[ k_{-i}^{-1} h_{-i} k_{-i+1} = r_{-i} \]

\[ \ldots \]

\[ k_{-1}^{-1} h_{-1} k_0 = r_{-1} \]

\[ k_0^{-1} h_0 k_1 = r_0 \]

\[ k_1^{-1} h_1 k_2 = r_1 \]

\[ \ldots \]

\[ k_i^{-1} h_i k_{i+1} = r_i \]

\[ \ldots \]

Let \( k_0 = t \), where \( t \) is arbitrary.

\[ k_{-1}^{-1} = r_{-1} t^{-1} h_{-1}^{-1} \]

\[ k_1 = h_0^{-1} tr_0 \]

\[ k_2 = h_1^{-1} h_0^{-1} tr_0 r_2 \]

Since we can solve these equations, any two cycles of infinite length are conjugate. In particular, if \( c' \) is a permutation still we can solve \( y^{-1} cy = c' \), and \( c' \) is a permutation. Hence, every infinite cycle in \( \Sigma(H; B, B^+, B^+) \) can be made conjugate to an infinite permutation.

\[ \square \]

Now, in the light of the Theorem 3.0.23 and Theorem 3.0.21 we can state the following Theorem.

**Theorem 3.0.24.** Two monomial substitutions \( y \) and \( y_1 \) are conjugate if and only if in their cyclic decomposition the finite cycles can be made to correspond in a one to
one manner such that corresponding cycles have the same lengths and determinant class and cardinality of the set of infinite cycles is the same for both $y$ and $y_1$.

Proof. This Theorem is consequence of Theorem 3.0.21 and Theorem 3.0.23.

3.1 Centralizers of Elements in Monomial Group

Monomial groups appear naturally as centralizer of an element in symmetric groups. The structure of centralizers of elements in finite symmetric groups is well known.

If $\alpha$ is an n-cycle in finite symmetric group $S_n$ on n-letters, then

$$C_{S_n}(\alpha) = \langle \alpha \rangle.$$  

Indeed, $\langle \alpha \rangle \leq C_{S_n}(\alpha)$. Moreover, if $\beta \in C_{S_n}(\alpha)$, then $\alpha^\beta = \alpha$. Since under conjugation cycle type of a permutation is preserved $\alpha^\beta$ must be an n-cycle and conjugation sends

$$(a_1, a_2, \ldots, a_n)^\beta = (a_1^\beta, a_2^\beta, \ldots, a_n^\beta) = (a_1, a_2, \ldots, a_n)$$

implies that if $a_1^\beta = a_j$ for some $j$, then $a_2^\beta = a_{j+1}$, $a_3^\beta = a_{j+2}$, $\cdots$, $a_{n-(j-1)}^\beta = a_n$, $a_{n-j}^\beta = a_1$, $\cdots$, $a_n^\beta = a_{j+n-1} = a_{j-1}$. It shows that

$$y = \begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_{n-(j-1)} & a_{n-j} & \cdots & a_n \\ a_j & a_{j+1} & a_{j+2} & \cdots & a_n & a_1 & \cdots & a_{j-1} \end{pmatrix}$$  \hspace{1cm} (3.32)

So, $\beta = \alpha^{j-1}$. Hence, $\beta \in \langle \alpha \rangle$, i.e., $C_{S_n}(\alpha) = \langle \alpha \rangle$.

Now, if $\alpha$ is in $S_n$ and $\alpha$ is a product of cycles of the same length $k$, then $\alpha$ is of the form

$$\alpha = (a_1, a_2, \ldots, a_k)(a_{k+1}, a_{k+2}, \ldots, a_{2k}) \cdots (a_{(m-1)k+1}, a_{(m-1)k+2}, \ldots, a_{mk}).$$

So, $mk = n$. In this case $C_{S_n}(\alpha) \cong (C_k \times C_k \times \cdots \times C_k) \rtimes S_m$ where $C_k$ is a cyclic group of order $k$, and $S_m$ is the finite symmetric group on m-letters. The elements of $S_m$ permute the cycles in this case.
Let $g \in C_{S_n}(\alpha)$. Then, $\alpha^g = \alpha$. Since conjugation of a permutation by another permutation preserves the cycle type. $\alpha^g$ will be again a permutation of the same type as $\alpha$. Hence $\alpha^g$ is a product of $m$ cycles of length $k$. Moreover, $\alpha^g = \alpha$ implies that the cycles are the same except the order of cycles in $\alpha$, because distinct disjoint cycles commute. Hence we can multiply $g$ with permutation $g_1$ where $g_1 \in S_m$ and $g_1^{-1}g$ fixes each cycle of $g$. Since $g_1^{-1}g$ is again an element of the centralizer and $g_1^{-1}g$ fixes each cycle. By above paragraph we know the centralizer of a $k$-cycle in $S_k$, namely $C_{S_k}(a_1, \ldots, a_k) = \langle (a_1, \ldots, a_k) \rangle$. We multiply by elements $c_{i,k}$ of cyclic group $C_k$ for each cycle of $\alpha$. Hence $c_{i,k}^{-1}c_{2,k}^{-1} \cdots c_{m,k}^{-1}g_1^{-1}g = id$. Hence $g \in C_{S_n}(\alpha) \cong (C_k \times C_k \times \cdots \times C_k) \rtimes S_m \cong \Sigma_m(C_k)$ which is a monomial group of degree $m$ over the cyclic group $C_k$ of order $k$.

If $\alpha$ is a product of cycles of different length then $\alpha$ is of the form $\alpha = (a_{1,1}, \ldots, a_{1,k_1}) \cdots (a_{m,1}, \ldots, a_{m,k_m})$. We need to find $C_{S_n}(\alpha)$. Assume that we have $l$ different lengths of cycles in the cycle decomposition of $\alpha$, and let $Y_m$ be the union of the orbits of the same length $m$, where $m = 1, 2, \ldots, l$. The $Y'_m$'s are a partition of the set with $n$ elements into disjoint sets. Let $x \in C_{S_n}(\alpha)$. It is clear that $Y'_m$'s are $\alpha$ invariant and also $x$ invariant by previous paragraph. Conversely, if we have a permutation $x_m$ of $Y_m$ such that $x_m$ commutes with restriction $\alpha_m$ of an element $\alpha$ to $Y_m$, then $x_m$ is in $C_{S_n}(\alpha)$. Therefore $C_{S_n}(\alpha) = C_{S_{|Y_m|}}(\alpha_1) \times \cdots \times C_{S_{|Y_l|}}(\alpha_l)$ where $S_{|Y_m|}$ is the symmetric group of degree $|Y_m|$. Therefore it is enough to calculate $C_{S_{|Y_k|}}(\alpha)$ where $\alpha$ has a fixed cycle length $k$.

Let $\alpha = (a_{1,1}, \ldots, a_{1,k}) \cdots (a_{m,1}, \ldots, a_{m,mk})$. We want to show that $C_{S_n}(\alpha) \cong (C_k \wr S_m)$.

Define $r'$ by $a_{ij}r' = a_{ij}r$. Then,

$$f : S_m \to C_{S_{mk}}(\alpha)$$

$$r \mapsto r'$$

is a homomorphism.

Let $\theta_l : a_{l,1} \to a_{l,2} \to \cdots \to a_{l,k} \to a_{l,1}$. Certainly $\theta_l$ is in $C_{S_{mk}}(\alpha)$ and $W_i = \langle f(r), \theta_l : l = 1, 2, \ldots, m; r \in S_m \rangle \cong (C_k \wr S_m)$.

Conversely, if $g \in C_{S_{mk}}(\alpha)$, then $g$ permutes the cycles of $\alpha$, so there exists $r \in S_m$
such that \( gr' \) fixes every cycle of \( \alpha \). Since the centralizer in \( S_k \) of a cycle of length \( k \) is a cyclic group of order \( k \).

\[
(gr')^m \prod_{l=1}^{m} \theta_l^{k_l} = 1 \text{ where } 1 \leq k_l \leq k - 1.
\]

Hence \( g \in W_m \), so \( C_{S_m}(\alpha) = W_m \cong C_k \wr S_m \cong \Sigma_m(C_k) \).

In the case \( \alpha \) is a product of cycles of different length since each cycle type will be preserved under conjugation, \( C_{S_n}(\alpha) = \sum_{m_1}(C_{k_1}) \times \cdots \times \sum_{m_l}(C_{k_l}) \) where we have \( l \) different length each length \( k_i \) have \( m_i \) cycles of length \( k_i \).

**Theorem 3.1.1.** Let \( y \) be conjugate to \( y_1 \) written in the normal form \( y_1 = \prod \delta_i \).

\[
\delta_i = \prod_{\epsilon} \delta^\epsilon_i, \text{ where for a fixed } i \text{ the } \delta^\epsilon_i \text{ are the normalized cycles of the same length } n, \text{ and the same determinant class } a \text{ if } n < d. \text{ Let } \epsilon \text{ run over a set of cardinal } \mu \text{ where } 0 \leq \mu \leq B. \text{ Then the centralizer } C_{\Sigma(H;B,B^+,B^+)}(y) \text{ is isomorphic to the strong direct product of symmetries }
\]

\[
C_{\Sigma(H;B,B^+,B^+)}(y) \cong \prod_i (\sum(C_H(a) < \delta >, \mu_i, \mu^+_i, \mu_i^+) \times \Sigma_\kappa(H \times \mathbb{Z}; \kappa, \kappa^+, \kappa^+)).
\]

The group \( C_H(a) < \delta > \) consists of all elements \( y_1 \) of the form \( y_1 = \{k_i\}(c^i_1)^j \) where \( k \) belongs to the centralizer of \( a \) in \( H \). The second direct product arises if \( \delta \) is a product of \( \kappa \) infinite cycles where \( \kappa \leq B \).

**Proof.** Let \( y \) be an arbitrary element of \( \Sigma(H;B, B^+, B^+) \). Then by taking conjugate of \( y \) by elements of \( \Sigma(H;B, B^+, B^+) \), we may assume that \( y \) is a product of cycles in its normal form, say \( y_1 \).

Since \( C_{\Sigma(H;B,B^+,B^+)}(y) \cong C_{\Sigma(H;B,B^+,B^+)}(y_1) \), it is enough to find the structure of centralizer of the elements \( y_1 \), which is a product of symmetries in its normal form.

The element \( y_1 \) may contain finite cycles and infinite cycles. We prove the theorem case by case. In the case of finite cycles we follow the proof of Ore [2] and in the case of infinite cycles we follow Crouch [1].
Since conjugation of an element $y_1 \in \Sigma(H; B, B^+, B^+)$ by an element $g \in \Sigma(H; B, B^+, B^+)$ preserves cycle length and the determinant class, the centralizer of an element $y_1$ will be direct product of centralizers of elements for each cycle length and determinant class. For this reason we will find the structure of centralizers of elements for each cycle length and determinant class.

Step 1: (a) Assume that $y_1$ is just a cycle of length $n$ in $\Sigma_n(H)$, for an arbitrary group $H$, and $y_1$ has determinant class $a$. In this case $y_1$ is of the normal form

$$y_1 = \begin{pmatrix} x_1 & x_2 & x_3 & \cdots & x_n \\ x_2 & x_3 & x_4 & \cdots & ax_1 \end{pmatrix}.$$  \hfill (3.33)

Then for

$$\kappa = \begin{pmatrix} x_1 & x_2 & \cdots & x_m \\ k_1x_{j_1} & k_2x_{j_2} & \cdots & k_mx_{j_m} \end{pmatrix}$$  \hfill (3.34)

and by the calculation

$$\kappa^{-1}y_1\kappa = \begin{pmatrix} x_{j_1} & x_{j_2} & \cdots & x_{j_n} & x_{j_{n+1}} & \cdots & x_{j_m} \\ k_1^{-1}k_2x_{j_2} & k_2^{-1}k_3x_{j_3} & \cdots & k_n^{-1}ak_1x_{j_1} & x_{j_{n+1}} & \cdots & x_{j_m} \end{pmatrix}$$  \hfill (3.35)

$$= \{k_n^{-1}ak_1x_{j_1}, k_1^{-1}k_2x_{j_2}, k_2^{-1}k_3x_{j_3}, \cdots, k_n^{-1}k_nx_{j_n}\}$$  \hfill (3.36)

Since $\kappa$ is in the centralizer of $y_1$ implies that $\kappa^{-1}y_1\kappa = y_1$ we may solve the unknowns $k_1, k_2, \ldots, k_n$ in $H$ and so by Theorem (8) in [2]

$$\kappa = \begin{pmatrix} x_1 & x_2 & \cdots & x_{n-j+1} & x_{n-j+2} & \cdots & x_n \\ rx_j & rx_{j+1} & \cdots & rx_n & rax_1 & \cdots & rax_{j-1} \end{pmatrix} = \{r\}y_1^j \hfill (3.37)

where $r \in C_H(a)$. Clearly, the powers of $y_1, y_1^m \in C_{\Sigma(H; B, B^+, B^+)}(y_1)$. Hence, we obtain $\kappa = \{r\}y_1^j = y_1^j\{r\}$.

Hence, $C_{\Sigma_n(H)}(y_1) \cong C_H(a) < y_1 >= < y_1 > C_H(a)$. $C_{\Sigma_n(H)}(y_1)$ is an extension of $C_H(a)$ by the group $< y_1 >$ of degree $n$.

(b) If $y_1$ is a product of $k$ cycles of length $n$ and each cycle in the normal form has the same determinant class $a$.  

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In this case \( y_1 \) is of the form

\[
y_1 = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ x_2 & x_3 & \cdots & ax_1 \\ & & & \\ x_{(k-1)n+1} & x_{(k-1)n+2} & \cdots & x_{kn} \\ x_{(k-1)n+2} & x_{(k-1)n+3} & \cdots & x_{(k-1)n+1} \end{pmatrix} \cdots (3.38)
\]

any permutation of cycles of type \((x_1, x_{n+1})(x_2, x_{n+2}) \cdots (x_n, x_2)\) and

\[
(x_1, x_{n+1}, x_{2n+1}, \ldots, x_{jn+1})(x_2, x_{n+2}, x_{2n+2}, \ldots, x_{jn+2}) \cdots (x_n, x_{2n}, x_{3n}, \ldots, x_{jn+n})
\]

commute with the given symmetry \( y_1 \) and these type of permutations generate a subgroup isomorphic to \( S_k \), permutations permuting the cycles.

Moreover, for each fixed cycle centralizer be as in then case (a) and these centralizers commute with each other. Hence the centralizer of \( y_1 \) will be isomorphic to

\[
C_H(a) < \delta_1 > \times C_H(a) < \delta_2 > \times \cdots \times C_H(a) < \delta_k > \times S_k 
\]

\[
\cong C_H(a) < \delta_1 > \triangleright S_k \cong \Sigma_k(C_H(a) < \delta_1 >).
\]

Since in our symmetry B might be infinite in the case \( y_1 \) is a product of infinitely many cycles of length \( n \) and determinant class \( a \) and the cardinality of the cycles is say \( \mu \) then

\[
C_{\Sigma(H, B, B^+, B^+)}(y_1) \cong \Sigma(C_H(a) < \delta >; \mu, \mu^+, \mu^+).
\]

Step 2: Now, we find the centralizer of a cycle of infinite length.

First observe by Lemma 3.0.23 that any infinite cycle of the form

\[
\begin{pmatrix} \cdots & x_{-2} & x_{-1} & x_0 & x_1 & x_2 & \cdots \\ \cdots & h_{-1}x_{-2} & h_{-1}x_{-1} & h_0x_0 & h_1x_1 & h_2x_2 & \cdots \end{pmatrix} \cdots (3.40)
\]

by taking its conjugate to a permutation cycle

\[
\begin{pmatrix} \cdots & x_{-2} & x_{-1} & x_0 & x_1 & x_2 & \cdots \\ \cdots & x_{-1} & x_0 & x_1 & x_2 & x_3 & \cdots \end{pmatrix} \cdots (3.41)
\]

Therefore, in the infinite cycle case we may assume that in the normal form \( y_1 \) is a product of say \( \mu_2 \) infinite permutations as above.
First we find the centralizer of an infinite permutation \( \Sigma(H; N_0, N_0^+, N_0^+) \) without fixed point and all elements are moved.

We follow Crouch [1] for find the structure of centralizer of a cycle product of \( \mu_2 \) infinite cycles as above will be the trivial consequence of the same argument.

Now,

\[
\kappa^{-1}_{\kappa c} = \begin{pmatrix}
\cdots & x_{i-1} & x_{i_0} & x_{i_1} & \cdots \\
\cdots & k_i^{-1}k_0x_{i-1_0} & k_0^{-1}k_1x_{i_0} & k_1^{-1}k_2x_{i_1} & \cdots
\end{pmatrix},
\] (3.42)

where \( c = \begin{pmatrix}
\cdots & x_{-1} & x_0 & x_1 & \cdots \\
\cdots & x_{-0} & x_1 & x_2 & \cdots
\end{pmatrix}. \) (3.43)

So, we can solve ..., \( k_{-2}, k_{-1}, k_0, k_1, ... \)

Then we observe that \( \kappa^{-1}_{\kappa c} = c \) implies that

\[
\kappa = \begin{pmatrix}
\cdots & x_{-1} & x_0 & x_1 & \cdots \\
\cdots & kx_{j-1} & x_j & kx_{j+1} & \cdots
\end{pmatrix}.
\] (3.44)

Hence \( \kappa = \{k\}c^j = c^j\kappa \) where \( \kappa \) is not a true scalar on the set \( U \), but \( k \) is a scalar only on the variables which appear in the cycle. \( \{k\} \) will be identity on the elements of \( U \) which \( c \) does not move. It follows that \( C_{\Sigma(H; d, d^+, d^+)}(c) \cong H \times \mathbb{Z} \), where \( \mathbb{Z} \) is an infinite cyclic group which comes from the isomorphism \( <c> \cong \mathbb{Z} \). It is independent of \( c \) as all isomorphic to \( \mathbb{Z} \).

Now, if \( y \) is a product of \( \mu_2 d - \) cycles then

\[
C_{\Sigma(H; B, B^+, B^+)}(y) \cong \Sigma(H \times \mathbb{Z}; \mu_2, \mu_2^+, \mu_2^+). \]

Now, from case 1 and case 2 theorem follows.

\( \square \)
CHAPTER 4

THE SPLITTING OF THE SYMMETRY

**Definition 4.0.2.** Let $G$ be a group, and $N$ be a normal subgroup of $G$. If there exists a subgroup $H \leq G$ such that $G=NH$ and $N \cap H=1$, then we say that $G$ splits over $N$ and $H$ is called as complement of $N$ in $G$.

If $G$ splits over $N$, then for any $g \in G$, $g=nh$ for some $n \in N$, $h \in H$. Moreover, this writing is unique.

Indeed assume that if $g=nh=n_1h_1$ where $n_1, n \in N$, $h_1, h \in H$, then $nh=n_1h_1$ implies $n_1^{-1}n=h_1h^{-1} \in N \cap H=1$. So, $n=n_1$ and $h=h_1$. Hence, the writing $g=nh$ is unique, i.e., $n \in N$, $h \in H$ is unique in the writing $g=nh$.

Observe that for any $g \in G$,

$$G^g=(NH)^g=N^gH^g=NH^g$$

as $N \trianglelefteq G$, $N^g = N$.

Therefore, all conjugates of $H$ are also complement of $N$ in $G$ i.e., $G=NH^g$ and $N \cap H^g = 1$.

**Definition 4.0.3.** A group $G$ splits regularly over $N$, if every complement of $N$ in $G$ is conjugate of $H$.

If $G$ splits regularly, $G=NT$, and $N \cap T=1$, then there exists $g \in G$ such that $T=H^g$.

Let $H$ and $T$ be two complements of $N$ in $G$ such that $H$ is not conjugate to $T$. Then by above, all conjugates of $T$ in $G$ are also complement of $N$ in $G$. 

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Therefore, we may decompose the set of all complements of \( N \) in \( G \).

Let \( \mathcal{C} = \{ T \leq G | TN = G, T \cap N = 1 \} \) be the set of all complements of \( N \) in \( G \). We may define an equivalence relation on \( \mathcal{C} \).

\( T_1 \sim T_2 \) iff \( T_1 \cong T_2^g \) for some \( g \in G \).

\( \sim \) is an equivalence relation:

i) \( T_1 \sim T_1 \), since \( id \in G \) and \( T_1^{id} = T_1 \).

ii) \( T_1 \sim T_2 \) implies that there exists \( g \in G \) such that \( T_1 = T_2^g \) if and only if \( T_1^{g^{-1}} = T_2 \) implies \( T_2 \sim T_1 \).

iii) \( T_1 \sim T_2 \) and \( T_2 \sim T_3 \) implies that there exists \( g_1, g_2 \in G \) such that \( T_1 = T_2^{g_1} \), and \( T_2 = T_3^{g_2} \), and so \( T_3^{g_2 g_1} = (T_3^{g_2})^{g_1} = T_2^{g_1} = T_1 \). \( g_2 g_1 \in G \), so \( \sim \) is an equivalence relation.

The equivalence class containing \( T_1 \),

\[ [T_1] = \{ T | T^g = T_1 \text{ for some } g \in G \} \]

If \( \mathcal{C} \) has only one equivalence class, then \( G \) splits regularly, otherwise \( G \) does not split regularly.

If we go back to \( \Sigma(H; B, B^+, B^+) \) we should notice that \( \Sigma(H; B, B^+, B^+) \) splits over \( V(B, B^+) \) with complement \( S(B, B^+) \).

Now, we will investigate the splitting of \( \Sigma(H; B, B^+, C) \), i.e., the splitting of the subgroup of \( \Sigma(H; B, B^+, B^+) \) such that order of support of permutations is less than \( C \) where \( d \leq C \leq B^+ \).

Clearly, \( \Sigma(H; B, B^+, C) = V(B, B^+)S(B, C) \), and \( V(B, B^+) \cap S(B, C) = 1 \). Hence, \( \Sigma(H; B, B^+, C) \) is a splitting of \( V(B, B^+) \) with the complement \( S(B, C) \).

We are interested in the following question:

Find the necessary and sufficient conditions that \( \mathcal{C} \) has only one equivalence class, i.e., whether all complements of \( V(B, B^+) \) in \( \Sigma(H; B, B^+, C) \) are conjugate or not.

If \( H \) and \( T \) are two complements of \( N \) in \( G \), then \( H \) is isomorphic to \( T \).
Let G=HN=NH and G=NT. Then, G/Ν= HN/Ν= TN/Ν.

HN/Ν≅H/Η∩Ν≅H since Η∩Ν=1.

HN/Ν=TN/Ν≅T/Τ∩Ν≅T since Τ∩Ν=1.

Hence, we have shown that any two complements of Ν in G are isomorphic.

We want to find out when H is conjugate to Τ.

Back to Σ(Η; Β, Β⁺, C). Assume that Τ is a complement of V(Β, Β⁺), Then by the above paragraph Τ≅S(Β, C). Moreover,

Σ(Η; Β, Β⁺, C)=TV(Β, Β⁺)=S(Β, C)V(Β, Β⁺).

Denote θ by the natural isomorphism θ: S(Β, C) → Τ such that θ(s)=vs=t where v∈V(Β, Β⁺), s∈S(Β, C) and v, s are unique satisfying θ(s)=vs=t.

By using the above natural isomorphism, the elements s=(1, α) is a transposition in Σ(Η; Β, Β⁺, C), and

\[ t_α = θ(s) = \{h_{1,α}, h_{2,α}, ..., h_{n,α}, \ldots\}(1, α) \]

Since we can find the elements up to conjugacy of Τ, say T'=vTv⁻¹ where v∈V(Β, Β⁺).

If κ={k₁, k₂, ..., kₙ, ...} where κ∈V(Β, Β⁺), then T' has elements whose first factors are identity. Indeed

\[ t'_α = \{k_1^{-1}, k_2^{-1}, ..., k_n^{-1}, \ldots\}\{h_{1,α}, h_{2,α}, ..., h_{n,α}, \ldots\}(1, α)\{k_1, k_2, ..., k_n, \ldots\} \]

\[ = \{k_1^{-1}h_{1,α}k_α, k_2^{-1}h_{2,α}k_2, k_3^{-1}h_{3,α}k_3, ..., k_n^{-1}h_{α,α}k_1, \ldots\}(1, α) \]

We can choose the first factor k₁⁻¹h₁,αk₁α = 1 since k₁⁻¹h₁,αk₁α = 1 implies k₁ = h₁,αk₁. For each α∈Β, α≠1, we can do this conjugation and choose k₁α = h₁,α⁻¹k₁.

Then simultaneously we can solve this equation and obtain the first component of each t'₁ is identity for all α≠1. k⁻¹₂h₂,αk₂ is a conjugate of h₂,α and other hₖ,α are conjugate except the αᵗʰ component.
Then we have \( t'_\alpha = \{ e, h_{2,\alpha}, \ldots, h_{\epsilon,\alpha}, \ldots \} (1, \alpha) \).

As \( (t'_\alpha)^2 = E \), we have

\[
(t'_\alpha)^2 = \{ h_{\alpha,\alpha}, h_{2,\alpha}^2, \ldots, h_{\alpha-1,\alpha}^2, k_{\alpha}^{-1} h_{1,\alpha} k_{\alpha}, \ldots \} = E
\]

where \( E \) is the identity of \( \Sigma(H; B^+, B^+) \).

So, \( h_{\alpha,\alpha} = e \) and, \( h_{2,\alpha}^2 = e \). Then,

\[
t_{\alpha} = \theta(s) = \{ 1_H, h_{2,\alpha}, \ldots, 1_H, \ldots \} (1, \alpha).
\]

So, we can write

(i) \( t'_\alpha = \{ e, h_{2,\alpha}, \ldots, h_{\epsilon,\alpha}, \ldots \} (1, \alpha) \)

(ii) \( h_{\alpha,\alpha} = e \)

(iii) \( h_{\epsilon,\alpha}^2 = e \) for \( \epsilon \neq 1, \epsilon \neq \alpha \).

Let \( S_1(B, C) \) be the set of all elements of \( S(B, C) \) where

\[
S_1(B, C) = \{ g \in S(B, C) \mid x_1 \cdot g = x_1 \}, \text{i.e., the stabilizer of the point } x_1 \text{ in } U.
\]

Since stabilizer of a point is a subgroup, \( S_1(B, C) \) is a subgroup of \( S(B, C) \).

Observe that, if \( U = \{ 1, 2, \ldots, n \} \), then

\[
S_1(B, C) \cong S_{n-1} \cong S(n-1, n) \leq S(n, n+1).
\]

Moreover if \( U \) is an infinite set, then \( S_1(B, C) \cong S(B, C) \) where \( d \leq C \leq B^+ \).

**Lemma 4.0.4.** If \( s \in S(B, C) \) and \( s \) moves \( x_1 \), then \( s \) can be written uniquely as \( s = (1, \alpha) s_1 \) where \( s_1 \) leaves \( x_1 \) fixed.

**Proof.** Let \( s \in S(B, C) \). Then we may write \( s \) as a product of disjoint cycles. In the writing of \( s \) as a disjoint product of cycles we are interested in only the cycle containing \( 1 \) as the other cycles already fixes \( 1 \) and the product of elements which fixes \( 1 \) is again an element fixing \( 1 \). For this reason we consider the cycle which contains (moves) \( 1 \). If this cycle is finite then we have the following:
Observe first that, if \( s=(1, 2, \ldots, n) \),
\( s=(1, n)(2, 3, \ldots, n) \in S(B, C) \).

We cannot write \( S=(1, k)(\alpha_1, \alpha_2, \ldots, \alpha_l) \) where
\( (\alpha_1, \alpha_2, \ldots, \alpha_l) \in S(B, C) \).

Think of the permutation \( (1, 2, 3, 4)=(1, 3)\beta \) where \( \beta \in S_4 \). There exists no such \( \beta \in S_4 \).

This observation can be generalized for all \( S(B, C) \).

If \( s=(1, n)\alpha_1=(1, n)\alpha_2 \), then \( \alpha_1=\alpha_2 \). So, the writing is unique, i.e.,
\( \alpha_1 \in S_1(B, C) \) is unique.

If \( s \) is an infinite cycle, and \( s \) moves \( x_1 \), then

\[
s = \begin{pmatrix} x_1 & x_2 & \ldots & x_\beta & \cdots \\ x_\alpha & x_\epsilon & \ldots & x_1 & \cdots \end{pmatrix} = (1, \beta) \begin{pmatrix} x_2 & x_3 & \ldots & x_\beta & \cdots \\ x_\epsilon & x_\lambda & \ldots & x_\alpha & \cdots \end{pmatrix}.
\]

On the other hand,

\[
s = \begin{pmatrix} x_2 & x_3 & \ldots & x_\alpha & \cdots & x_\beta & \cdots \\ x_\epsilon & x_\lambda & \ldots & x_\delta & \cdots & x_\alpha & \cdots \end{pmatrix} (1, \alpha)
\]

where \( \begin{pmatrix} x_2 & x_3 & \ldots & x_\alpha & \cdots & x_\beta & \cdots \\ x_\epsilon & x_\lambda & \ldots & x_\delta & \cdots & x_\alpha & \cdots \end{pmatrix} \in S_1(B, C) \).

Assume that there exists \( x_\alpha \in U \) such that \( \alpha \neq 1 \), and \( x_\alpha s_1=x_\alpha \), i.e., \( s_1 \) fixes \( x_\alpha \) for some \( \alpha \neq 1 \). So, \( x_1s_1=x_1 \), and \( x_\alpha s_1=x_\alpha \). Then, consider \( s=(1, \alpha)s_1=s_1(1, \alpha) \Rightarrow s_1=(1, \alpha)s=s(1, \alpha) \)

\[
\theta(1, \alpha) = \{e, h_{2, \alpha}, \ldots, h_{\epsilon, \alpha}, \ldots e, \ldots\}(1, \alpha),
\]

where \( e \) occurs as a factor in the first and \( \alpha^{th} \) positions.

\[
\theta(s_1) = \theta((1, \alpha)s) = \theta(1, \alpha)\theta(s) = \theta(s(1, \alpha)) = \theta(s)\theta(1, \alpha)
\]
If \( s \) belongs to \( S(B,C) \) and moves \( x_1 \), then by Lemma 4.0.4, \( s \) can be written uniquely as \( s = (1, \alpha) s_1 \) where \( s_1 \in S_1(B,C) \).

The image of \((1, \alpha)\) under \( \theta \) has been described above as

\[
\theta(1, \alpha) = \{ e, h_{2,\alpha}, ..., h_{\epsilon,\alpha}, ...e, ... \}(1, \alpha).
\]

To find the image of any element of \( S(B,C) \) it is sufficient to discuss those elements in \( S_1(B,C) \).

Let \( s_1 \in S_1(B,C) \) such that \( x_\alpha s_1 = x_\alpha \) for some \( x_\alpha, \alpha \neq 1 \) i.e., \( s_1 \) fixes \( x_\alpha \) where \( \alpha \neq 1 \) i.e., \( s_1 \in S_1(B,C) \cap S_\alpha(B,C) \).

Let \( s = (1, \alpha)s_1 \). Then \( s_1 = (1, \alpha)s = s(1, \alpha) \) where \( s \) sends \( x_1 \) into \( x_\alpha \), and \( x_\alpha \) into \( x_1 \).

Let, \( \theta(s) = \{k_1, k_2, \ldots, k_\epsilon, \ldots\} \).

So,

\[
\theta(s_1) = \{e, h_{2,\alpha}, ..., h_{\epsilon,\alpha}, ...e, ...\}(1, \alpha)\{k_1, k_2, ..., k_\epsilon, ...\} s
\]

\[
= \begin{pmatrix}
x_1 & x_2 & \ldots & x_\alpha & \cdots \\
k_\alpha x_1 & h_{2,\alpha} k_2 x_\delta & \ldots & k_1 x_\alpha & \cdots
\end{pmatrix}.
\]

\[
\theta(s_1) = \theta(s) \theta(1, \alpha) = \begin{pmatrix}
x_1 & \ldots & x_\alpha & \cdots \\
k_1 x_1 & \ldots & k_\alpha x_\alpha & \cdots
\end{pmatrix} \Rightarrow k_1 = k_\alpha.
\]

This shows that if \( s_1 \) belongs to \( S_1(B, C) \), then the factors of \( v \) where \( \theta(s_1) = v s_1 \) in the positions corresponding to those \( x \) which \( s_1 \) leaves fixed are equal to the first factor of \( v \).

\[\square\]

**Lemma 4.0.5.** Let \( s \) belongs to \( S(B,C) \), and have the following properties: \( s \) moves \( x_1 \), i.e., \( x_1 s \neq x_1 \), and \( x_\alpha s = x_\alpha \) where \( \alpha \neq 1 \), and \( x_\beta s = x_1 \). Then \( s \) has the following form

\[
s = \begin{pmatrix}
x_1 & \ldots & x_\beta & \ldots & x_\alpha & \cdots \\
x_\delta & \ldots & x_1 & \ldots & x_\alpha & \cdots
\end{pmatrix}
\]

(4.4)
where $\delta \neq 1$. Let $\theta(s) = vs$ where $v \in V(B, B^+)$. Then the factors which occur in the first and $\beta^{th}$ positions of $v$ are equal.

**Proof.** Let $\theta(s) = vs = \{c_1, c_2, c_3, \ldots, c_\beta, \ldots, c_\epsilon, \ldots\}$, we need to show $c_1 = c_\beta$.

We may write $s$ in the following form

$$s = (1, \beta) \begin{pmatrix} x_1 & \cdots & x_\beta & \cdots & x_\alpha & \cdots \\ x_1 & \cdots & x_\delta & \cdots & x_\alpha & \cdots \end{pmatrix} = (1, \beta)s_1$$

where $s_1 \in S_1(B, C)$. Also, we may write from right, as

$$s = \begin{pmatrix} x_1 & \cdots & x_\beta & \cdots & x_\alpha & \cdots \\ x_1 & \cdots & x_\delta & \cdots & x_\alpha & \cdots \end{pmatrix} (1, \delta) = s_1(1, \delta).$$

Observe that when we write $s$ as $(1, \beta)s_1$ and $s_1(1, \delta)$, the $s_1$’s are the same. We want to find $\theta(s) = vs$. So we need to understand factors of $v$.

$s = (1, \beta)s_1 = s_1(1, \delta)$, so $\theta(s) = \theta(1, \beta)\theta(s_1)$. As we discussed on page 40,

$\theta(s_1) = \{h_\alpha, \ldots, h_\beta, \ldots, h_\alpha, \ldots, h_\delta, \ldots\} s_1$

$\theta(1, \beta) = \{e, \ldots, e, \ldots, h_{\alpha, \beta}, \ldots, h_{\delta, \beta}, \ldots\} (1, \beta)$

$\theta(1, \delta) = \{e, \ldots, h_{\beta, \delta}, \ldots, h_{\alpha, \delta}, \ldots, e, \ldots\} (1, \delta)$

$\theta(s) = \theta(1, \beta)\theta(s_1) = \begin{pmatrix} x_1 & \cdots & x_\beta & \cdots \\ h_\beta x_\delta & \cdots & h_\alpha x_1 & \cdots \end{pmatrix}$

$\theta(s) = \theta(s_1)\theta(1, \delta) = \begin{pmatrix} x_1 & \cdots & x_\beta & \cdots \\ h_\alpha x_\delta & \cdots & h_\beta x_1 & \cdots \end{pmatrix}$

So, we have $h_\beta = h_\alpha$.

We can do the above computation for any $\alpha$ which is fixed by $s$. So, we may conclude that all the corresponding factors which are fixed by $s$, the factors of $v$ are equal to $h_\beta$ where $x_\beta s = x_1$.

**Claim:** Let $s_1 \in S_1(B, C)$ $x_\alpha s_1 = x_\alpha, \alpha \neq 1$ and $\theta(s_1) = vs_1$. Then $v$ is a scalar.
It is sufficient to show that the factors occupying positions corresponding to $x$ which $s_1$ moves are the same as the first factor of $v$.

We have shown that if $s$ does not move $x_\alpha$, then $h_\alpha = h_1, \forall \alpha$. If we can show that $h_1 = h_\beta$, where $s_1$ moves $x_\beta$, $v$ will be a constant, so that, the factors coming from the $x$’s such that $s_1$ fixes $x$ the factors are equal to $h_1$ and the factors of $x$ which $s_1$ moves $x$ also equal to $h_1$ implies $v$ is a constant.

Let

$$s_1 = \begin{pmatrix} x_1 & \ldots & x_\beta & \ldots & x_\alpha & \ldots \\ x_1 & \ldots & x_\delta & \ldots & x_\alpha & \ldots \end{pmatrix} = \begin{pmatrix} x_1 & \ldots & x_\beta & \ldots & x_\alpha & \ldots \\ x_\delta & \ldots & x_1 & \ldots & x_\alpha & \ldots \end{pmatrix} (1, \delta)$$

where $\delta \neq \beta$ and $\delta \neq 1$.

By the Lemma 4.0.4

$$\theta(s_1) = \{h_1, \ldots, h_\beta, \ldots, h_\alpha, \ldots, h_\delta, \ldots\} s_1$$

where $h_\alpha = h_1$.

Furthermore, $\theta(1, \delta) = \{e, \ldots, h_\beta, \ldots, h_\alpha, \ldots, e, \ldots\} (1, \delta)$.

Using the decomposition of $s_1$ and the fact that $\theta$ is an isomorphism.

$$\theta(s_1) = \theta(s) \theta(1, \delta) = \begin{pmatrix} x_1 & \ldots & x_\beta & \ldots \\ h_\alpha x_1 & \ldots & h_\alpha x_\delta & \ldots \end{pmatrix}$$

Recall that, above if $s_1 \in S_1(B, C)$, and $x_\alpha s_1 = x_\alpha$, then the first factor and $\alpha^{th}$ factor are the same.

So far

$$\theta(s_1) = \{h_1, \ldots, h_\beta, \ldots, h_1, \ldots, h_\delta, \ldots\} s_1$$

By the above calculation, we show $h_1 = h_\alpha = h_\beta$. Hence, under the above condition, $v$ is a constant.

It remains to discuss the case where there exists no $x_\alpha$ such that $x_\alpha s_1 = x_\alpha$, i.e., $s_1$ moves all $x_\beta$ for all $\beta \neq 1$. We need to show under the condition
\( \theta(s_1) = vs_1 \) and \( v \) is a constant. Assume
\[
s_1 = \begin{pmatrix} x_1 & x_2 & \ldots & x_\beta & \ldots \\ x_1 & x_\beta & \ldots & x_\alpha & \ldots \end{pmatrix} = (2, \beta) \begin{pmatrix} x_1 & x_2 & \ldots & x_\beta & \ldots \\ x_1 & x_\alpha & \ldots & x_\beta & \ldots \end{pmatrix} = \bar{s}_1 \bar{s}_1'
\]
where \( \bar{s}_1, \bar{s}_1' \in S_1(B, C) \).

By the above calculation, \( \theta(\bar{s}_1) = \bar{s}_1 \bar{s}_1' \) and \( \theta(\bar{s}_1') = \bar{s}_1' \bar{s}_1' \) where \( \bar{s}_1' \) and \( \bar{s}_1 \) are constant.

\[
\theta(s_1) = \theta(\bar{s}_1 s_1') = \theta(\bar{s}_1) \theta(s_1') = \bar{s}_1 \bar{s}_1' s_1' s_1
\]

Since \( \bar{s}_1' \) and \( \bar{s}_1 \) are constant and constants commute with all permutations we can write
\[
\theta(s) = s_1 s_1' s_1 = \bar{s}_1 \bar{s}_1' s_1, \text{ as } \bar{s}_1 \bar{s}_1' = s_1.
\]

And \( s_1 s_1' s_1 = \bar{s}_1 \bar{s}_1' \) is a constant since product of two constant is constant. So we have shown that if \( s_1 \) does not fix any \( x_\alpha, \alpha \neq 1 \), then \( \theta(s_1) = vs_1 \) where \( v \) is a scalar. Hence, under all conditions for any \( s_1 \in S_1(B, C) \) we have \( \theta(s_1) = vs_1 \) where \( v \) is a constant. \( \square \)

Define a map \( \phi : S_1(B, C) \rightarrow H \) such that \( \phi(s_1) = h_{s_1} \), and
\[
\theta(s_1) = vs_1 \text{ where } v = \{ h_{s_1} \}
\]

\( \theta : S(B, C) \rightarrow T \) such that \( \theta(s) = vs \) where \( vs \in T \).

A computation shows that if
\[
\theta(1, \alpha) = \{ e, \ldots, h_{\beta, \alpha}, \ldots, h_{\alpha, \alpha}, \ldots \}(1, \alpha)
\]
\[
\theta(1, \beta) = \{ e, \ldots, h_{\beta, \beta}, \ldots, h_{\alpha, \beta}, \ldots \}(1, \beta)
\]

where \( h_{\beta, \beta} = e \) and \( h_{\alpha, \alpha} = e \), then
\[
\theta((1, \alpha)(1, \beta)) = \theta(1, \beta)
\]
\[
= \{ e, \ldots, h_{\beta, \alpha, \ldots, e, \ldots}, \alpha \}(1, \alpha) \{ e, \ldots, e, \ldots, h_{\alpha, \beta, \ldots, e, \ldots} \}(1, \beta) \{ e, \ldots, h_{\beta, \alpha, \ldots, e, \ldots} \}(1, \alpha)
\]
\[
= \begin{pmatrix} x_1 & \ldots & x_\alpha & \ldots & x_\beta & \ldots \\ h_{\alpha, \beta} x_1 & \ldots & h_{\beta, \alpha} x_\beta & \ldots & h_{\beta, \alpha} x_\alpha & \ldots \end{pmatrix}
\]

where \( \alpha \neq 1, \beta \neq 1, \alpha \neq \beta \). But as \( \alpha \neq 1, \beta \neq 1 \) we have
\[ \theta(\alpha, \beta) = \{g_{\alpha, \beta}\}(\alpha, \beta). \]

So, \( h_{\alpha, \beta} = h_{\beta, \alpha} = g_{\alpha, \beta} \in H. \)

**Theorem 4.0.6.** The symmetry \( \Sigma(H; B, B^+, C) \) splits over the basis group, \( \Sigma(H; B, B^+, C) = V(B, B^+) \cup T, \Sigma(H; B, B^+, B^+) \cap T = E. \) Any such group \( T \) is the conjugate of some group \( T' \) obtained by the following construction. Let \( G \) be a subgroup of \( H \) that is the homomorphic image of \( S_1(B, C) \) where \( d \leq C \leq B^+. \) Let \( \phi(s) = g_s \) indicate the homomorphism. In particular, \( \phi(\alpha, \beta) = g_{\alpha, \beta}. \) Then the elements of \( T' \) are obtained from the elements of \( S(B, C) \) by the isomorphism defined as follows: Let \( * : S(B, C) \rightarrow T' \) be a map, and \( * s = \{g_s\} \) for \( s \) belonging to \( S_1(B, C), \) \( *(1, \alpha) = \{e, g_{2, \alpha}, ..., g_{\epsilon, \alpha}, ..., e, ...\}(1, \alpha) \) where \( e \) occurs in the first and \( \alpha \)th positions.

In previous pages we have shown that if \( T \) is a complement of \( V(B, B^+), \) then the correspondence gives a homomorphism from \( S(B, C) \) into \( H \) where the above conditions are satisfied. Therefore, we need to prove the converse of the theorem. Namely if there is a correspondence as in the theorem, then it must be an isomorphism.

**Proof.** We have defined a map \( \alpha : S(B, C) \rightarrow T'. \) Now, we want to show that \( * \) is an isomorphism. We know that if an element \( s \in S(B, C), \) then we may write \( s = (1, \alpha)s_1 \) where \( s_1 \in S_1(B, C). \) Indeed if \( s \) is already fixing \( x_1 \) then \( s = s_1, \) and \( \alpha = 1. \) So, we are done. We may assume that \( s \) moves \( x_1. \) Then as \( s \) is a permutation there exists \( \alpha \) such that \( x_\alpha s = x_1. \) Then \( s = (1, \alpha)s_1, \) and where \( j \neq 1 \)

\[
\begin{pmatrix}
  x_1 & \cdots & x_\alpha & \cdots \\
  x_j & \cdots & x_1 & \cdots
\end{pmatrix}
= (1, \alpha)
\begin{pmatrix}
  x_1 & \cdots & x_\alpha & \cdots \\
  x_1 & \cdots & x_j & \cdots
\end{pmatrix}
\]

\[
* (s) = *(1, \alpha)s_1 = *(1, \alpha) * (s_1) = \{e, g_{2, \alpha}, ..., g_{\epsilon, \alpha}, ..., e, ...\}(1, \alpha) * (s_1)
\]

\[
= \{e, g_{2, \alpha}, ..., g_{\epsilon, \alpha}, ..., e, ...\}(1, \alpha) \{h_{s_1}\} s_1
\]

Since \( \{h_{s_1}\} \) is constant, it commutes with \( (1, \alpha). \)

So, \( * (s) = \{h_{s_1}, g_{2, \alpha}h_{s_1}, ..., g_{\epsilon, \alpha}h_{s_1}, ..., h_{s_1}, ...\} s \)

\[ \square \]
Let $\bar{s} = (1, \beta)\bar{s}_1$ be another element of $S(B, C)$ where $x_\beta \bar{s} = x_1$ and $\bar{s}_1 \in S_1(B, C)$.

We want to show that $*$ is a homomorphism, i.e.,

$$*(s\bar{s}) = *(s) * (\bar{s}) = *(1, \alpha) * (s_1) * (1, \beta)(\bar{s}_1).$$

By Ore [2], it is enough to show that $*(s(1, \beta)) = *(s) * (1, \beta)$. This is equivalent to show that

(i) $*(1, \alpha) * (1, \beta) = *((1, \alpha)(1, \beta)$, and

(ii) $*(s_1(1, \beta)) = *(s_1) * (1, \beta)$ for any $s_1 \in S_1(B, C)$.

Indeed

$$\begin{pmatrix} x_1 & \ldots & x_\alpha & \ldots & x_\beta & \ldots \\ g_{\alpha,\beta}x_\alpha & \ldots & x_\beta & \ldots & g_{\beta,\alpha}x_1 & \ldots \end{pmatrix}$$

For $\alpha = \beta$ the case is trivially true. So we assume that $\alpha \neq \beta$. Then we have

$$((1, \alpha)(1, \beta) = (1, \alpha, \beta) = (1, \beta)(1, \alpha)$$

So,

$$*(1, \alpha)(1, \beta) = *[1, \alpha, \beta] = (1, \beta) * (\alpha, \beta) = *(1, \beta)\{g_{\alpha,\beta}\}(\alpha, \beta)$$

$$= \{g_{\alpha,\beta}\}(1, \beta) = \{g_{\alpha,\beta}\}(1, \beta)(\alpha, \beta) = \{g_{\alpha,\beta}\}(1, \beta)(\alpha, \beta)$$

$$= \{g_{\alpha,\beta}\}(1, \alpha)(1, \beta).$$

Now we compute the corresponding factors and obtain $g_{\alpha,\beta} = e$ and $g_{\alpha,\beta}g_{\epsilon,\alpha} = g_{\epsilon,\beta}g_{\alpha,\beta}$ since $(\epsilon, \alpha)(\epsilon, \beta) = (\epsilon, \beta)(\alpha, \beta)$, and where $\phi : S_1(B, C) \rightarrow H$ such that $\phi(\alpha, \beta) = g_{\alpha,\beta}$, $\phi$ is a homomorphism.

Now, we should show that

$$\begin{pmatrix} x_1 & \ldots & x_\alpha & \ldots & x_\beta & \ldots \\ g_{\alpha,\beta}x_\alpha & \ldots & x_\beta & \ldots & g_{\beta,\alpha}x_1 & \ldots \end{pmatrix}$$

For $\alpha = \beta$ the case is trivially true. So we assume that $\alpha \neq \beta$. Then we have

$$((1, \alpha)(1, \beta) = (1, \alpha, \beta) = (1, \beta)(1, \alpha)$$

So,

$$*(1, \alpha)(1, \beta) = *[1, \alpha, \beta] = (1, \beta) * (\alpha, \beta) = *(1, \beta)\{g_{\alpha,\beta}\}(\alpha, \beta)$$

$$= \{g_{\alpha,\beta}\}(1, \beta) = \{g_{\alpha,\beta}\}(1, \beta)(\alpha, \beta) = \{g_{\alpha,\beta}\}(1, \beta)(\alpha, \beta)$$

$$= \{g_{\alpha,\beta}\}(1, \alpha)(1, \beta).$$

Now we compute the corresponding factors and obtain $g_{\alpha,\beta} = e$ and $g_{\alpha,\beta}g_{\epsilon,\alpha} = g_{\epsilon,\beta}g_{\alpha,\beta}$ since $(\epsilon, \alpha)(\epsilon, \beta) = (\epsilon, \beta)(\alpha, \beta)$, and where $\phi : S_1(B, C) \rightarrow H$ such that $\phi(\alpha, \beta) = g_{\alpha,\beta}$, $\phi$ is a homomorphism.
\[ *((s_1(1, \beta))) = *s_1 * (1, \beta) \text{ for all } s_1 \in S_1(B, C). \]

There are two cases in this verification i.e., we will analyze it when \( s_1 \) moves \( x_\beta \) and when \( s_1 \) does not move \( x_\beta \).

Case 1: If \( s_1 \) does not move \( x_\beta \),

we know \( s = (1, \beta)s_1 = s_1(1, \beta) \).

\[ *s = *((1, \beta)) * (s_1) \text{ since } * \text{ is a homomorphism.} \]

Also, \( *((s(1, \beta))) = *s * (1, \beta) = *(1, \beta) * s_1 * (1, \beta) = *s_1 * (1, \beta) * (1, \beta) = *(s_1(1, \beta)) * (1, \beta) = *s * (1, \beta). \]

Case 2: If \( s_1 \) moves \( x_\beta \), then we cannot say anything about \( *(s(1, \beta)) \) with direct computation. But,

\[
\begin{pmatrix}
  x_1 & \cdots & x_\beta & \cdots & x_\delta & \cdots \\
  x_1 & \cdots & x_\alpha & \cdots & x_\beta & \cdots \\
\end{pmatrix}
\begin{pmatrix}
  1, \beta \\
  1, \beta \\
\end{pmatrix}
= (1, \delta)s_1.
\]

(4.5)

Here \( s_1 \) does not move \( x_\beta \) so we can do computation.

\[ *s_1 = \{g_{s_1}\}s_1, \]

\[ *(1, \beta) = \{e, \ldots, e, \ldots, g_{\delta, \beta}, \ldots, g_{e, \beta}, \ldots\}(1, \beta), \]

\[ *(1, \delta) = \{e, \ldots, g_{\beta, \delta}, \ldots, e, \ldots, g_{e, \delta}, \ldots\}(1, \delta) \text{ implies that} \]

\[ *s_1 * (1, \beta) = \{g_{s_1}\}s_1\{e, \ldots, e, \ldots, g_{\delta, \beta}, \ldots, g_{e, \beta}, \ldots\}(1, \beta) \]

\[ = \begin{pmatrix}
  x_1 & \cdots & x_\beta & \cdots & x_\delta & \cdots & x_\varepsilon & \cdots \\
  g_{s_1}x_\beta & \cdots & g_{s_1}g_{\alpha, \beta}x_\alpha & \cdots & g_{s_1}x_1 & \cdots & g_{s_1}g_{e, \beta}x_i & \cdots \\
\end{pmatrix}
\begin{pmatrix}
  1, \beta \\
  1, \beta \\
\end{pmatrix}
= (1, \delta)s_1
\]

(4.6)

since \( * \) is a homomorphism.
Also, $\ast(s_1(1, \beta)) = \ast((1, \delta)s_1) = \ast(1, \delta) \ast (s_1)$

$$= \begin{pmatrix}
    x_1 & \cdots & x_\beta & \cdots & x_\delta & \cdots & x_\epsilon & \cdots \\
    g_{s_1}x_\beta & \cdots & g_{s_1}g_{\beta, \delta}x_\alpha & \cdots & g_{s_1}x_1 & \cdots & g_{s_1}g_{\epsilon, \delta}x_\iota & \cdots 
\end{pmatrix} \quad (4.7)$$

Since $\phi$ is a homomorphism factors of above two computations are the same.

As a consequence, we get that the given correspondence in the theorem preserves the multiplication.

Images of the elements of $S(B, C)$ form a group $T$. This $T$ is isomorphic $S(B, C)$. We can say clearly $V(B, B^+) \cap T = E$. Moreover, $V(B, B^+) \cup T = \Sigma(H; B, B^+, C)$ since if $y \in \Sigma(H; B, B^+, C)$, it can be written $y = vv_1^{-1}v_1s = v_2t$ where $*s = v_1s = t$.

**Theorem 4.0.7.** A necessary and sufficient condition for $\Sigma(H; B, B^+, C)$ where $d^+ \leq C \leq B^+$ to split regularly over the basis group is that $H$ contains no subgroup isomorphic to $S(B, C)$.

**Remark 1:** $S(B, C)\cong S_1(B, C)$ when $B$ is infinite, and this is the case as $d^+ \leq C \leq B^+$.

**Remark 2:** Let $y \in \Sigma(H; B, B^+, C)$, $y=vs$, and $v_1 \in V(B, B^+)$, if we take the conjugate of $y$ by $v_1$, we have $v_1^{-1}yv_1 = v_1^{-1}vs v_1 = v_1^{-1}vs v_1 s^{-1}s = v_1^{-1}v v_2s = v_3s$, where $v_2 = sv_1 s^{-1}$ and $v_3 = v_1^{-1}v v_2 \in V(B, B^+)$. So, $s$ is fixed.

**Remark 3:** $s^{-1}ys \in \Sigma(H; B, B^+, C)$, where $s \in S(B, C)$

**Proof.** Assume that $\Sigma(H; B, B^+, C)$ splits regularly over the basis group. Let $T'$ be another complement of the basis group. Then by assumption there exists $y \in \Sigma(H; B, B^+, C)$ such that

$$(T')^y = y^{-1}(T')y = S(B, C)$$
Every element \( t \in T' \) can be written in the form \( t=vs \) for some \( v \in V(B, B^+) \), and \( s \in S(B, C) \). By remark 2 and 3, \( t^y = (vs)^y \in S(B, C) \), we may take the element \( y \in V(B, B^+) \) because taking conjugate of an element by a permutation only permutes the factors. Therefore, if we want to obtain by taking conjugate we must take conjugation by an element of \( V(B, B^+) \).

Therefore we may assume that \( y \in V(B, B^+) \). Say \( y = \{k_1, k_2, \ldots, k_\alpha, \ldots\} \).

In order to understand the elements of \( T' \) we may consider the elements \( t = \{g_{s_1}\} s_1 \) where \( g_{s_1} \) is a constant element of \( V(B, B^+) \) and \( s_1 \in S_1(B, C) \) i.e. \( s_1 \) fixes the symbol \( x_1 \). \( S_1 \) is the stabilizer of a point \( x_1 \) in \( S(B, C) \).

\[
yty^{-1} = \{k_1, k_2, \ldots\} \{g_{s_1}\} s_1 \{k_1^{-1}, k_2^{-1}, \ldots\} = \begin{pmatrix} x_1 & x_2 & \ldots \\ k_1 g_{s_1} k_1^{-1} x_1 & \ldots \end{pmatrix} \in S_1(B, C)
\]

\( k_1 g_{s_1} k_1^{-1} = e \) implies \( g_{s_1} = e \) \( \forall s_1 \in S_1(B, C) \), and consider

\[\theta : T' \longrightarrow H \text{ such that } \theta(\{g_{s_1}\} s_1) = g_{s_1} = e\]

So, \( t=\{e\} s_1 \) then every element of \( T' \) which is of the form \( \{g_{s_1}s_1\} = \{e\} s_1 \). Hence elements of \( T \) coming from \( S_1(B, C) \) i.e. constant term is actually coming from \( H \) is identity.

Since \( \theta \) sends all \( g_{s_1} \) into identity the above homomorphism sends all elements of the form \( \{g_{s_1}\} \) to identity. Hence \( H \) does not contain a subgroup isomorphic to \( S_1(B, C) \).

Since by remark 1, \( S(B, C) \cong S_1(B, C) \), \( H \) contains no subgroup isomorphic to \( S(B, C) \).

Conversely, assume \( H \) contains no subgroup isomorphic to \( S(B, C) \) and that \( \Sigma(H; B, B^+, C) \) does not split regularly. Then \( H \) contains no subgroup \( G \) which is the homomorphic image of \( S_1(B, C) \). Scott has shown that this implies that \( G \) contains a subgroup isomorphic to \( S(B, C) \), contradicting the hypothesis. Therefore, \( \Sigma(H; B, B^+, C) \) splits regularly.

\[\square\]

**Theorem 4.0.8.** A necessary and sufficient condition for \( \Sigma(H; B, B^+, d) \) to split
regularly over its basis group is that \( H \) contain no element of order 2.

**Proof.** If \( \Sigma(H; B, B^+, d) \) splits regularly, then by Theorem 4.0.6 it contains an isomorphic copy of \( S_1(B, d) \simeq S(B, d) \). Since by assumption \( H \) contains no element of order 2, then the map \( \gamma : T' \rightarrow H \) is the trivial projection i.e.

\[
t_\alpha = \{ h_{1,\alpha}, h_{2,\alpha}, ..., h_{\epsilon,\alpha}, ... \} (1, \alpha),
\]

and \( \gamma : \pi a_\pi \mapsto a_\pi \) where \( a_\pi = 1 \). Then \( T' = S(B, d) \).

By Baer’s Theorem the only normal subgroup of \( S(B, B^+, d) \) are the subgroup \( S(B, C) \) where \( d \leq C \leq B^+, S(B, d) \) and \( \text{Alt}(B, d) \). \( S(B, d) / A(B, d) \simeq Z_2 \) and \( |Z_2| = 2 \).

\( H \) contains no element of order 2. So the map \( \gamma \) is the identity map. \( \square \)

### 4.1 The Splitting of \( \Sigma_{n,A}(H) \)

We first consider the special cases.

**Case 1: Splitting of \( \Sigma_{3,A}(H) \)**

Now we will discuss the splitting for \( n = 3 \) i.e., \( H \) will be an arbitrary group and alternating monomial group of degree 3.

We already know that \( \Sigma_{3,A}(H) = V_3 A_3 \) and \( V_3 \cap A_3 = 1 \). So, \( A_3 \) is a complement of \( V_3 \) in \( \Sigma_{3,A}(H) \).

Recall that \( V_3 = H \times H \times H \). Let \( T \) be an arbitrary complement of \( V_3 \) in \( \Sigma_{3,A}(H) \). Then \( V_3 T = \Sigma_{3,A}(H) \) and \( V_3 \cap T = 1 \). Since arbitrary element of \( \Sigma_{3,A}(H) \) can be written as \( vs \) where \( v \in V_3 \) and \( s \in A_3 \). The elements of \( T \) will be \( \{1, g, g^2\} \). Let \( \theta : A_3 \rightarrow T \) be an isomorphism. Let \( A_3 = \{1, a, a^2\} \). Since \( \theta(1) = 1 \), the image of \( a \) will determine the isomorphism. Let

\[
\theta(a) = \{h_1, h_2, h_3\}(1 2 3).
\]

Since we will find the complement up to conjugacy we may take the conjugate of \( T \) by multiplication

\[
k = \{k_1, k_2, k_3\} \in V_3. \text{ Then } kTk^{-1} \text{ contains the element of the form }
\]

\[
\{k_1, k_2, k_3\}\{h_1, h_2, h_3\}(1 2 3)\{k_1^{-1}, k_2^{-1}, k_3^{-1}\}
\]
$$\left\{ k_1 h_1 k_2^{-1}, k_2 h_2 k_3^{-1}, k_3 h_3 k_1^{-1} \right\}(1\,2\,3).$$

Since $k_1, k_2, k_3$ are arbitrary elements of $H$ we can choose $k_1 = k$ as arbitrary. Then we may choose $k_2 = kh_1$, $k_3 = k_2 h_2 = kh_1 h_2$.

It follows that the third component $k_3 h_3 k_1^{-1} = kh_1 h_2 h_3 k^{-1}$. Then we have $\theta(a) = \left\{ e, e, kh_1 h_2 h_3 k^{-1} \right\}(1\,2\,3) = \left\{ e, e, b \right\}(1\,2\,3)$. Since $a$ has order 3, we have $\theta(a)$ has order 3. Then $1 = \theta(a)^3 = (\left\{ e, e, b \right\}(1\,2\,3))^3 = \left\{ e, e, e \right\}$. It follows that $b=1$. Hence the isomorphism $\theta$ will be the identity automorphism and so $T = A_3$. Hence all complements of $V_3$ will be conjugate to $A_3$ and so $\Sigma_{3,A}(H)$ splits regularly.

**Case 2: Splitting of $\Sigma_{4,A}(H)$**

Recall that the alternating group $A_4$ has order 12 and consists of even permutations of symmetric group on 4 letters. The subgroup $\kappa = \{(1), (1\,2)(3\,4), (1\,3)(2\,4), (1\,4)(2\,3)\}$ will be a normal subgroup of $A_4$ which is isomorphic to elementary Abelian group of order 4 in fact $\kappa \cong Z_2 \times Z_2$. Then $A_4 = \kappa < (1\,2\,3) >$. So $A_4$ is a split extension of $\kappa$ with a cyclic subgroup of order 3. Since $< (1\,2\,3) >$ will be a Sylow 3-subgroup of $A_4$ and by Sylow theorem, all Sylow 3-subgroups are conjugate. We have all complements of $V$ in $A_4$ are conjugate. In our terminology, $A_4$ splits regularly over the normal group $\kappa$.

Since the only nontrivial normal subgroup of $A_4$ is $\kappa$, and any homomorphism $\theta$ from $A_4$ to any other group will be either $\theta(A_4) = 1$ trivial homomorphism or $\theta(A_4) = A_4$ isomorphism or $\theta(A_4) = < d >$ where $< d >$ is a cyclic group of order 3. Therefore, in the above cases $Ker(\theta) = \{1\}$ and $Ker(\theta) = \kappa$.

We will prove the following theorem for this special case.

**Theorem 4.1.1.** The group $\Sigma_{4,A}(H)$ splits over the basis group $V_4 \cong H \times H \times H \times H$ with complement $A_4$. Let $T'$ be another complement of $V_4 \in \Sigma_{4,A}(H)$. Then there exists a homomorphism $\phi : A_4 \rightarrow H$ satisfying $\phi(s) = g_s$ for all $s \in A_4$. Then the isomorphism $\theta$ will be $\theta(s) = \{g_s\}$ for all $s \in A_4$.

**Proof.** Let $T$ be a complement of $V_4$ in $\Sigma_{4,A}(H)$. Then there exists a homomorphism $\theta : A_4 \rightarrow T$. Since by previous pages $A_4$ has an elementary Abelian normal subgroup $\kappa$ isomorphic to $Z_2 \times Z_2$. We will consider the images of $\kappa$ into $T$. Since $\kappa$
is generated by $\sigma_1 = (1 \, 2)(3 \, 4)$, $\sigma_2 = (1 \, 3)(2 \, 4)$ and $\sigma_3 = (1 \, 4)(2 \, 3)$ the group $T$ will have either homomorphic image of $\kappa$ i.e., $\theta(\kappa) \cong \kappa$ or $\theta(\kappa) = \{e\}$.

Let $\theta(\sigma_1) = \{h_{11}, h_{12}, h_{13}, h_{14}\}(1 \, 2)(3 \, 4)$

$\theta(\sigma_2) = \{h_{21}, h_{22}, h_{23}, h_{24}\}(1 \, 3)(2 \, 4)$

$\theta(\sigma_3) = \{h_{31}, h_{32}, h_{33}, h_{34}\}(1 \, 4)(2 \, 3)$

As before, since we want to find the complements up to conjugacy we may take the conjugate of $T$ with a product $\kappa = \{k_1, k_2, k_3, k_4\}$. Then

(i) $\kappa \theta(\sigma_1) \kappa^{-1} = \{k_1, k_2, k_3, k_4\}\{h_{11}, h_{12}, h_{13}, h_{14}\}(1 \, 2)(3 \, 4)\{k_1^{-1}, k_2^{-1}, k_3^{-1}, k_4^{-1}\} = \{k_1 h_{11}k_2^{-1}, k_2 h_{12}k_1^{-1}, k_3 h_{13}k_4^{-1}, k_4 h_{14}k_3^{-1}\}(1 \, 2)(3 \, 4)$

(ii) $\kappa \theta(\sigma_2) \kappa^{-1} = \{k_1, k_2, k_3, k_4\}\{h_{21}, h_{22}, h_{23}, h_{24}\}(1 \, 3)(2 \, 4)\{k_1^{-1}, k_2^{-1}, k_3^{-1}, k_4^{-1}\} = \{k_1 h_{21}k_3^{-1}, k_2 h_{22}k_4^{-1}, k_3 h_{23}k_1^{-1}, k_4 h_{24}k_2^{-1}\}(1 \, 3)(2 \, 4)$

(iii) $\kappa \theta(\sigma_3) \kappa^{-1} = \{k_1, k_2, k_3, k_4\}\{h_{31}, h_{32}, h_{33}, h_{34}\}(1 \, 4)(2 \, 3)\{k_1^{-1}, k_2^{-1}, k_3^{-1}, k_4^{-1}\} = \{k_1 h_{31}k_4^{-1}, k_2 h_{32}k_3^{-1}, k_3 h_{33}k_2^{-1}, k_4 h_{34}k_1^{-1}\}(1 \, 4)(2 \, 3)$.

Then again as $k_1, k_2, k_3, k_4$ are arbitrary elements of $H$, choose $k_1 = k$ fixed, then by (i) and (ii), choose $k_2 = kh_{11}$, then $k = k_2 h_{12}$. $k_4 = k_3 h_{13}$ implies $k_3 = k_4 h_{14}$. $\sigma(i)$ has order 2. So $\theta(\sigma_1)^2 = \{h_{11} h_{12}, h_{12} h_{11}, h_{13} h_{14}, h_{14} h_{13}\} = \{e, e, e, e\}$. Then $h_{12} = h_{11}^{-1}$ and $h_{14} = h_{13}^{-1}$. Hence $\theta(\sigma_1) = \{h_{11}, h_{11}^{-1}, h_{13}, h_{13}^{-1}\}(1 \, 2)(3 \, 4)$.

$\theta(\sigma_2)^2 = \{h_{21} h_{33}, h_{22} h_{24}, h_{23} h_{21}, h_{24} h_{22}\} = \{e, e, e, e\}$. Then $h_{33} = h_{21}^{-1}$ and $h_{24} = h_{22}^{-1}$. Then $\theta(\sigma_2) = \{h_{21}, h_{22}, h_{21}^{-1}, h_{22}^{-1}\}(1 \, 3)(2 \, 4)$.

$\theta(\sigma_3)^2 = \{h_{31} h_{34}, h_{32} h_{33}, h_{33} h_{32}, h_{34} h_{31}\} = \{e, e, e, e\}$. Then $h_{31} = h_{34}^{-1}$ and $h_{32} = h_{33}^{-1}$. Hence $\theta(\sigma_3) = \{h_{31}, h_{32}, h_{32}^{-1}, h_{31}^{-1}\}(1 \, 4)(2 \, 3)$.

Now we use the property that $\kappa$ is an Abelian group. Therefore $\theta(\kappa)$ is an Abelian group.

$\theta(\sigma_1)\theta(\sigma_2) = \theta(\sigma_1 \sigma_2) = \theta(\sigma_2 \sigma_1) = \theta(\sigma_2)\theta(\sigma_1)$ implies that

$53$
\{h_{11}, h_{11}^{-1}, h_{13}, h_{13}^{-1}\}(1\,2)(3\,4)\{h_{21}, h_{22}, h_{21}^{-1}, h_{22}^{-1}\}(1\,3)(2\,4)
= \\
\{h_{11}h_{22}, h_{11}^{-1}h_{21}, h_{13}h_{22}^{-1}, h_{13}^{-1}h_{21}^{-1}\}(1\,4)(2\,3)
= \\
\{h_{21}, h_{22}, h_{21}^{-1}, h_{22}^{-1}\}(1\,3)(2\,4)\{h_{11}, h_{11}^{-1}, h_{13}, h_{13}^{-1}\}(1\,2)(3\,4)\{h_{11}, h_{11}^{-1}, h_{13}, h_{13}^{-1}\}(1\,2)(3\,4)
= \{h_{21}h_{13}, h_{22}h_{13}^{-1}, h_{21}^{-1}h_{11}, h_{22}^{-1}h_{11}^{-1}\}(1\,4)(2\,3). \\

We obtain

(A) \ h_{11}h_{22} = h_{21}h_{13}

(B) \ h_{11}^{-1}h_{21} = h_{22}h_{13}^{-1}

(C) \ h_{13}h_{22}^{-1} = h_{21}^{-1}h_{11}

(D) \ h_{13}^{-1}h_{21}^{-1} = h_{22}^{-1}h_{11}^{-1} implies that A = D^{-1} and B = C^{-1}.

Only the following equation remains.

\ h_{11}h_{22} = h_{21}h_{13}.

\theta(\sigma_2)\theta(\sigma_3) = \theta(\sigma_3)\theta(\sigma_2)

implies that \ \{h_{21}, h_{22}, h_{31}, h_{32}, h_{32}^{-1}, h_{31}^{-1}\}(1\,3)(2\,4)\{h_{31}, h_{32}, h_{32}^{-1}, h_{31}^{-1}\}(1\,4)(2\,3)
= \\
\{h_{21}h_{32}^{-1}, h_{22}h_{31}^{-1}, h_{21}^{-1}h_{31}, h_{22}^{-1}h_{32}\}(1\,2)(3\,4)
= \\
\{h_{31}, h_{32}, h_{32}^{-1}, h_{31}^{-1}\}(1\,4)(2\,3)\{h_{21}, h_{22}, h_{21}^{-1}, h_{22}^{-1}\}(1\,3)(2\,4)
= \\
\{h_{31}h_{22}^{-1}, h_{32}h_{21}^{-1}, h_{32}^{-1}h_{22}, h_{31}^{-1}h_{21}\}(1\,2)(3\,4).

We obtain

(A) \ h_{21}h_{32}^{-1} = h_{31}h_{22}^{-1}

(B) \ h_{22}h_{31}^{-1} = h_{32}h_{21}^{-1}

(C) \ h_{21}^{-1}h_{31} = h_{32}^{-1}h_{22}

(D) \ h_{22}^{-1}h_{32} = h_{31}^{-1}h_{21}

implies that A = B^{-1} and C = D^{-1}.

Only the following equation remain.

\ h_{21}h_{32}^{-1} = h_{31}h_{22}^{-1}
\[ \theta(\sigma_1)\theta(\sigma_3) = \theta(\sigma_3)\theta(\sigma_1) \]

implies that
\[
\begin{align*}
\{h_{11}, h_{11}^{-1}, h_{13}, h_{13}^{-1}\}(1 2)(3 4) & \{h_{31}, h_{32}, h_{32}^{-1}, h_{31}^{-1}\}(1 4)(2 3) = \\
\{h_{11}h_{32}, h_{11}^{-1}h_{31}, h_{13}h_{31}^{-1}, h_{13}^{-1}h_{32}^{-1}\}(1 3)(2 4) = \\
\{h_{31}, h_{32}, h_{32}^{-1}, h_{31}^{-1}\}(1 4)(2 3) & \{h_{11}, h_{11}^{-1}, h_{13}, h_{13}^{-1}\}(1 2)(3 4) = \\
\{h_{31}h_{13}^{-1}, h_{32}h_{13}, h_{32}^{-1}h_{11}^{-1}, h_{31}^{-1}h_{11}\}(1 3)(2 4). \\
\end{align*}
\]

So, we have the following equations

(A) \( h_{11}h_{32} = h_{31}h_{13}^{-1} \)
(B) \( h_{11}^{-1}h_{31} = h_{32}h_{13} \)
(C) \( h_{13}h_{31}^{-1} = h_{32}^{-1}h_{11}^{-1} \)
(D) \( h_{13}^{-1}h_{32}^{-1} = h_{31}^{-1}h_{11} \)

implies that \( A = C^{-1} \) and \( B = C^{-1} \).

We get only \( h_{11}h_{32} = h_{31}h_{13}^{-1} \).

Then we use the property
\[ \theta(\sigma_1\sigma_2) = \theta(\sigma_3), \theta(\sigma_2\sigma_3) = \theta(\sigma_1), \theta(\sigma_1\sigma_3) = \theta(\sigma_2). \]

Then
\[
\begin{align*}
\theta(\sigma_1\sigma_2) & = \{h_{11}h_{22}, h_{11}^{-1}h_{21}, h_{13}h_{22}^{-1}, h_{13}^{-1}h_{21}^{-1}\}(1 4)(2 3) = \\
\{h_{31}, h_{32}, h_{32}^{-1}, h_{31}^{-1}\}(1 4)(2 3) = \theta(\sigma_3) & \\
\end{align*}
\]

implies that
\[
\begin{align*}
h_{31}^{-1} = h_{22}^{-1}h_{11}^{-1} = h_{13}^{-1}h_{21}^{-1}, \\
h_{32} = h_{11}^{-1}h_{21}, \\
h_{32}^{-1} = h_{13}h_{22}^{-1} = h_{21}^{-1}h_{11}.
\end{align*}
\]

\[
\begin{align*}
\theta(\sigma_2\sigma_3) & = \{h_{21}h_{32}^{-1}, h_{22}h_{31}^{-1}, h_{21}^{-1}h_{31}, h_{22}^{-1}h_{32}\}(1 2)(3 4) = \\
\{h_{11}, h_{11}^{-1}, h_{13}, h_{13}^{-1}\}(1 2)(3 4) = \theta(\sigma_1) & \\
\end{align*}
\]
implies that
\[ h_{11}^{-1} = h_{32}^{-1}h_{21}^{-1} = h_{22}h_{31}^{-1}, \]
\[ h_{13} = h_{21}^{-1}h_{31} = h_{32}^{-1}h_{22}. \]

\[ \theta(\sigma_1\sigma_3) = \{ h_{11}h_{32}, h_{11}^{-1}h_{31}, h_{13}h_{31}^{-1}, h_{13}^{-1}h_{32}^{-1} \} (1\ 3)(2\ 4) = \{ h_{21}, h_{22}, h_{21}^{-1}, h_{22}^{-1} \} (1\ 3)(2\ 4) = \theta(\sigma_2) \]

implies that
\[ h_{21} = h_{11}h_{32} = h_{31}h_{13}, \]
\[ h_{22} = h_{11}^{-1}h_{31} = h_{32}h_{13}. \]

Now, we can take conjugate with \( \kappa = \{ k_1, k_2, k_3, k_4 \} \). Then we obtain

(i) \[ \kappa\theta(\sigma_1)^{-1} = \{ k_1, k_2, k_3, k_4 \} \{ h_{11}, h_{11}^{-1}, h_{13}, h_{13}^{-1} \} (1\ 2)(3\ 4) \{ k_1^{-1}, k_2^{-1}, k_3^{-1}, k_4^{-1} \} = \{ k_2h_{11}^{-1}k_1^{-1}, k_1h_{11}h_{21}^{-1}, k_4h_{13}k_3^{-1}, k_3h_{13}k_4^{-1} \} (1\ 2)(3\ 4) \]

(ii) \[ \kappa\theta(\sigma_2)^{-1} = \{ k_1, k_2, k_3, k_4 \} \{ h_{21}, h_{22}, h_{21}^{-1}, h_{22}^{-1} \} (1\ 3)(2\ 4) \{ k_1^{-1}, k_2^{-1}, k_3^{-1}, k_4^{-1} \} = \{ k_3h_{21}^{-1}k_1^{-1}, k_4h_{22}h_{21}^{-1}, k_1h_{21}k_3^{-1}, k_2h_{22}k_4^{-1} \} (1\ 3)(2\ 4) \]

(iii) \[ \kappa\theta(\sigma_3)^{-1} = \{ k_1, k_2, k_3, k_4 \} \{ h_{31}, h_{32}, h_{32}^{-1}, h_{31}^{-1} \} (1\ 4)(2\ 3) \{ k_1^{-1}, k_2^{-1}, k_3^{-1}, k_4^{-1} \} = \{ k_4h_{31}^{-1}k_1^{-1}, k_3h_{32}^{-1}k_2^{-1}, k_2h_{32}k_3^{-1}, k_1h_{31}k_4^{-1} \} (1\ 4)(2\ 3). \]

Since \( \kappa \) is arbitrary, to do first component of \( \kappa\theta(\sigma_i)^{-1}, i=1,2,3 \), is identity we can choose the proper \( k_1, k_2, k_3, k_4 \).

Say \( k_1 = k \) is fixed and
\[ k_1h_{11}^{-1}k_2^{-1} = e, \ k_3h_{21}^{-1}k_1^{-1} = e, \text{ and } k_4h_{31}^{-1}k_1^{-1} = e \]
gives us the following equations.
\[ k_2 = kh_{11}, \]
\[ k_3 = kh_{21}. \]
Using the equations we found up to now, we get

(i) \[
\kappa \theta (\sigma_1) \kappa^{-1} = \{kh_1^{-1}k^{-1}, kh_{11}h_1^{-1}k^{-1}, kh_{31}h_{13}^{-1}h_{21}^{-1}k^{-1}, k^{-1}h_{21}k_{13}h_{31}^{-1}\}(1\ 2)(3\ 4) = \{e, e, e\} (1\ 2)(3\ 4)
\]

(ii) \[
\kappa \theta (\sigma_2) \kappa^{-1} = \{k^{-1}h_{21}h_1^{-1}k^{-1}, kh_{31}h_{22}^{-1}h_{11}^{-1}k^{-1}, kh_{21}h_{21}^{-1}k^{-1}, kh_{11}h_{22}h_{31}^{-1}k^{-1}\}(1\ 3)(2\ 4) = \{e, e, e, e\} (1\ 3)(2\ 4)
\]

(iii) \[
\kappa \theta (\sigma_3) \kappa^{-1} = \{kh_{31}h_{31}^{-1}k^{-1}, k^{-1}h_{21}h_{32}^{-1}h_{11}^{-1}k, kh_{11}h_{32}h_{21}^{-1}k^{-1}, kh_{31}h_{31}^{-1}k^{-1}\}(1\ 4)(2\ 3) = \{e, e, e, e\} (1\ 4)(2\ 3).
\]

As a result, we get that \(t' = \theta (s) = \{e, e, e, e\} s = s\). It follows that where \(\theta (A_4) = T', T' A_4\).

**Case 3: \(\Sigma_{n,A}(H), n \neq 5\)**

**Theorem 4.1.2.** The group \(\Sigma_{n,A}(H)\) splits over the basis group, \(\Sigma_{n,A}(H) = V_n \cup T\), \(V_n \cap T = E\). The group \(T\) is conjugate to some group \(T'\) obtained as follows. Let \(G\) be a subgroup of \(H\) which is the homomorphic image of \(A_{n-1}\). Let \(g_4, ..., g_n\) be generators of \(G\), satisfying the following relations: (i) \(g_i^3 = e, i = 4, ..., n\),

(ii) \((g_i, g_j)^2 = e\) where \(i \neq j\).

Let \(s_i = (1\ i\ 2)\) for \(i = 3, ..., n\) generate the group \(A_n\). Then the elements of \(A_n\) with the aid of the isomorphism \(\theta\) defined by \(\theta (s_3) = t'_3 = \{e, e, e, g_4, ..., g_n\}(1\ i\ 2)\).

\(\theta (s_i) = t'_i = \{e, g_i, g_i^2, g_i^3g_4, ..., g_i^2g_{i-1}, ..., g_i^2g_{i+1}, ..., g_i^2g_n\}(1\ i\ 2)\) for \(i = 4, ..., n\).

**Proof.** The group \(\Sigma_{n,A}(H)\) consists of all symmetries where the permutation part is an element of alternating group \(A_n\). Again the group \(H\) is an arbitrary group as in the case of \(\Sigma_n(H)\) complete monomial group. \(\Sigma_{n,A}(H) = (H \times H \times ... \times H) \times A_n \simeq H \wr A_n\). The action of \(A_n\) on the direct product as before permutes the factors. Let \(V_n = H \times H \times ... \times H\) and \(A_n\) is the alternating group on \(n\) letters. So,
\[ \Sigma_{n,A}(H) = V_n \rtimes A_n \text{ i.e. } V_n \cap A_n = 1 \]

\[ V_n \cdot A_n = \Sigma_{n,A}(H) \text{ so } \Sigma_{n,A}(H) \text{ splits over } V_n. \]

In this section we will consider the splitting problem of \( \Sigma_{n,A}(H) \). Since \( \Sigma_{n,A}(H) = V_n \cdot A_n \) for any element \( g \in \Sigma_{n,A}(H) \), we have \( (V_n \cdot A_n)^g = V_n^g \cdot A_n^g = V_n \cdot A_n^g \), and so when \( A_n \) is a complement of \( V_n \), then any conjugate of \( A_n \) namely \( A_n^g \) is also a complement of \( V_n \). But there are cases that there might be other complements \( T \) of \( V_n \) i.e. \( V_n \cdot T = \Sigma_{n,A}(H) \) and \( V_n \cap T = 1 \) but \( T \) may not be a conjugate of \( A_n \). (It is clear that \( V_n \cdot T = V_n \cdot A_n \) and \( V_n \cdot T/V_n = V_n \cdot A_n/V_n \simeq A_n/A_n \cap V_n \simeq A_n \), and \( V_n \cdot T/V_n \simeq T/V_n \cap T \simeq T \). Hence every complement is isomorphic to alternating group \( A_n \). But we are interested in when \( T \) and \( A_n \) are conjugate If all complements of \( V_n \) are conjugate, then we say that \( \Sigma_{n,A}(H) \) splits regularly.

Assume that \( T \) is a complement of \( V_n \). Then by above, \( T \) is isomorphic to \( A_n \). Moreover, as \( \Sigma_{n,A}(H) = V_n \cdot T \) the isomorphism \( \theta : A_n \rightarrow T \) can be written in the form that \( \theta(a) = v_a a \) where \( v_a \in V_n, a \in A_n \). The natural isomorphism. (Every such isomorphism should be natural isomorphism. )

**Claim:** For \( i \neq j, 1 \neq i, 1 \neq j \) the elements \( (1 \ i \ 2) \) generate the alternating group \( A_n \) where \( i=3, \ldots, n \).

By taking conjugate of \((1 \ i \ 2)\) with \((1 \ j \ 2)\) we have

\[(1 \ i \ 2)^{(1 \ j \ 2)} = (j \ i \ 1) = (1 \ j \ i)\]

So we may obtain all 3-cycles of the form \((1 \ i \ j)\) where \( i \neq j \).

\[(1 \ k \ j)^{(1 \ i \ 2)} = (k \ i \ j) \text{ so we may obtain all 3-cycles of the form } (i \ j \ k) \text{. Hence the group } A_n = \langle (1 \ i \ 2) \mid i = 3, \ldots, n \rangle.\]

Let \( s_i = (1 \ i \ 2) \). Since \( A_n \) is generated by \( s_i \), then \( T \) is generated by \( \theta(s_i) \). Then \( \theta(s_i) = t_i \in T \) and \( t_i = s_i v_i \) where \( t_i = \{h_{i1}, h_{i2}, \ldots, h_{in}\}(1 \ i \ 2) \) where \( i=3, \ldots, n \). So we have \( t_3, t_4, t_5, \ldots, t_n \) i.e. we have \( n-2 \) \( t_i \)'s.
Since we want to find the complement T up to conjugacy we may take conjugate of all $t_i$ with a fixed product $v = \{k_1,k_2,\ldots,k_n\}$.

$t_i' = vt_i v^{-1} = \{k_1,k_2,\ldots,k_n\}\{h_{i1},h_{i2},\ldots,h_{ni}\}(1 \ i \ 2)(k_1^{-1},k_2^{-1},\ldots,k_n^{-1})$

$= \{k_1 h_{i1} k_1^{-1}, k_2 h_{i2} k_2^{-1},\ldots,k_i h_{ii} k_i^{-1},\ldots,k_j h_{jj} k_j^{-1},\ldots\}(1 \ i \ 2)$

Since we want to find complement T of $V_n$ up to conjugacy and $k_i'$s are arbitrary, we may substitute $k_i'$s. So, let $k_1$ be arbitrary fixed element in H. Choose $k_i = k_1 h_{ii}$ for $i=3,\ldots,n$.

Choose $k_2 = k_1 h_{23}$. Then $T'=vTv^{-1}$ contains $t_3' = \{e,e,g_{33},\ldots,g_{nn}\}(132)$ $t_i' = \{e,g_{2i},\ldots,g_{ni}\}(1 \ i \ 2)$ for $i=4,\ldots,n$. So, for $i=3$ and $k_1 = k$ be an arbitrary $k_1 h_{23} k_1^{-1} = e$. We can solve $k_2 = k_1 h_{23}^{-1}$. Hence from the 1st component $k_1 h_{13} k_3^{-1} = e$ then $k_3 = k_1 h_{13}$.

$t_3' = \{e,e,g_{33},\ldots,g_{nn}\}(132)$

Now, for $i \geq 4$ we have,

$t_i' = \{e,g_{2i},\ldots,g_{ni}\}(1 \ i \ 2)$ where $i=4,\ldots,n$, and $g_{ni} = k_n h_{ni} k_n^{-1}$

Since $s_i = (1 \ i \ 2)$ is a 3-cycle, $s_i^3 =1$. Then $t_i = \theta(s_i)^3 = 1$.

Consider $s_i s_j = (1 \ i \ 2)(1 \ j \ 2) = (1 \ i)(2 \ j)$ where $i \neq j$. Then $(s_i s_j)^2 = 1$.

$$(t_i')^3 = \{g_{ii} g_{2i},g_{2i} g_{ii},\ldots,g_{ii} g_{2i} g_{ii},\ldots,g_{ji}^3,\ldots\}$$

$$(t_i't_j')^2 = \{g_{ij} g_{ii} g_{jj},g_{jj} g_{ii} g_{jj},\ldots,g_{ii} g_{2i} g_{ij},\ldots,g_{jj} g_{2j} g_{ij},\ldots,g_{ii} g_{2i} g_{jj},\ldots,(g_{ki} g_{kj})^2\}$$

Recall, we have the isomorphism $*: S_n \rightarrow T'$ such that $*(s_i) = t_i'$. Here, we have a 3-cycle for all $t_i'$ where $i=3,\ldots,n$ for obtaining alternating group $A_n$. So where $*(s_i) = t_i'$, $|s_i| = |t_i'|$ since $*$ is an isomorphism. Therefore, $(t_i')^3 = E$. If we look at the order of $(t_i't_j')$ we should think order of $(s_i s_j)$.

$(s_i s_j) = (1 \ i \ 2)(1 \ j \ 2) = (1 \ i)(2 \ j)$. We see $|s_i s_j| = 2$. So, $|(t_i't_j')| = 2$. We get $(t_i't_j')^2 = E$ where E is the identity of $T'$.

We have from above calculation $(t_i')^3 = \{g_{ii} g_{2i},g_{2i} g_{ii},\ldots,g_{ii} g_{2i} g_{ii},\ldots,g_{ji}^3,\ldots\}$ implies that $g_{ii} g_{2i} = e$ and $g_{ij} g_{ii} g_{2j} = e$ where $i \neq j$ and $i,j \in \{3,4,\ldots,n\}$.
Noting that \( g_{1i} = e = g_{1j} = e = g_{23} = e \), and writing \( g_i \) for \( g_{2i} \) we have \( g_i = g_{2i} = g_{ii}^{-1} \) and \( g_{33} = g_3 = e \) since \( g_{33}^{-1} = g_{23} = e \).

Also, \( g_{ij} = (g_{ii}g_{2j})^{-1} = g_{2j}^{-1}g_{ii}^{-1} = g_{ji}^{-1}g_i = g_{ji}^{-1}g_i \)

\( g_{i3}g_{ii}g_{23} = e \) implies that \( g_{i3} = g_{ii}^{-1} = g_i = g_{3i}^{-1} \) as \( g_{ij} = g_{ji}^{-1} \).

If we use these equations then \( t'_{i} \) will have the form

\[ t'_{i} = \{ e, g_i, g_{ii}^{-1}, g_{3i}, \ldots, g_{ii}^{-1}g_n \} \]

If \( k > 2 \) where \( k \neq j, k \neq i \) in \((t'_{i})^3\) and \((t'_{i}t'_{j})^2\) the \( k^{th} \) factor will satisfy \( g_{ki}^3 = (g_i^{-1}g_k)^3 = e, (g_k, g_{kj})^2 = e \).

For \( k=3 \), we found \( g_3^3 = (g_3g_j)^2 = e \).

So, the elements \( g_i \) where \( i=3, \ldots, n \) generate a homomorphic group to \( A_{n-1} \).

The first, second and the \( i^{th} \) factors of \((t'_{i})^3\) are \( g_3^3, g_1^3, g_3^3 \), respectively, and the first, second, \( i^{th} \) and the \( j^{th} \) factors of \((t'_{i}t'_{j})^2\) are \( g_j^3g_3g_2g_j, g_i^3g_jg_2g_i, g_j^3g_jg_2g_j, g_i^3g_j^2g_i, g_j^3g_j^2g_j \), respectively.

Those above factors are e. If \( k > 2 \), the \( k^{th} \) factors of \((t'_{i})^3\) and \((t'_{i}t'_{j})^2\) are \( (g_i^3g_k)3, (g_i^3g_k)2g_kg_j)2 \) where \( k \neq i \) and \( k \neq j \), respectively. These factors also e.

Therefore there is \( n-2 \) elements in the generating set of the group which is homomorphic image of \( A_n \). Permutation part of \( T' \) is in \( A_n \). So \( T' \) is isomorphic to \( A_n \).

We found that \( T \cong A_n, V \cap T' = E \). Moreover, if \( y \) is in \( \Sigma_{n,A}(H) \), then \( y = vu_1^{-1}v_1s = v_2t \) where \( \theta(s) = v_1s \). Hence, \( \Sigma_{n,A}(H) = V_n \cup T \).

\[ \square \]

**Theorem 4.1.3.** The group \( \Sigma_{n,A}(H) \) splits regularly over the basis group if and only if \( H \) contains no non-trivial subgroup which is homomorphic image of \( A_{n-1} \).

**Proof.** Assume that \( H \) contains no non-trivial subgroup which is isomorphic image of \( A_{n-1} \). Then the complement \( T' \) obtained as in Theorem 4.1.2 is simply \( A_n \). Hence splitting is regular over the basis group.
Conversely, if the group $\Sigma_{n,A}(H)$ splits regularly, then by taking the conjugate of $T'$ with an element $v \in V_n$. Where $v = \{k_1, k_2, \ldots\}$, we obtain $vt_3v^{-1} = \{k_1k_3^{-1}, k_2k_1^{-1}, k_3k_2^{-1}, \ldots, k_ng_nk_n^{-1}\}(132)$. This element is a permutation means that $k_ig_ik_i^{-1} = e$ for all $i=4,\ldots,n$. By multiplying from left by $k_i^{-1}$ and from right by $k_i$ we obtain $g_i = e$ for all $i=4,\ldots,n$. It follows that the image $G = \{e\}$. □

**Corollary 4.1.4.** The group $\Sigma_{n,A}(H)$ for $n=4,5$ splits regularly over the basis group if and only if $H$ contains no element of order 3.

*Proof.* For $n=4$, the group $H$ contains an isomorphic copy of $A_{n-1} = A_3$. But there exists no element of order 3 in $H$ implies that $G = \{e\}$. Then the epimorphism between $A_{n-1}$ and the group $G$ in $H$ will be trivial projection and $T'$ will be $A_3$.

Conversely, if the splitting is regular, then $T'$ will be conjugate of $A_4$. Then the multiplication part of the complement is trivial. Hence $G$ contains no element of order 3.

For $\Sigma_{5,A}(H)$, $A_{n-1} = A_4$ and the epimorphism $\phi : A_4 \rightarrow G$, $G$ is a subgroup of $H$ we can say that all 3-cycles will go to identity. But in the elementary Abelian group $K$ in $A_4$ all elements will be product of two 3 cycles and the homomorphism $\phi$ will send all elements of $A_4$ into identity. Hence $G = \{e\}$ and the splitting will be regular. □

Theorem 4.1.3 implies for $n \geq 6$, the following corollary.

**Corollary 4.1.5.** Let $n \geq 6$. The group $\Sigma_{n,A}(H)$ splits regularly over the basis group if and only if $H$ contains no subgroup isomorphic to $A_{n-1}$.

### 4.2 Splitting of $\Sigma_{A}(H; B, B^+, d)$

Now we go back to infinite case and discuss the splitting of $\Sigma_{A}(H; B, B^+, d)$ over the base group $V(B, B^+)$. 

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In order to be able to talk about infinite alternating groups, each element must have a
finite support and so we can talk the permutation is odd or even. For this reason for
the splitting $\Sigma_A(H; B, B^+, d)$ we lie inside finitary symmetric group and hence even
permutations are in finitary symmetric group.

In the proof of the Theorem 4.1.2 we discuss the isomorphism between $S(B,C)$ and
$S_1(B, C)$ when C is an infinite cardinal.

The same proof will work for infinite alternating groups namely if B is infinite
cardinal then $A(B, d) \cong A_1(B, d)$ where $A_1(B, d)$ is the alternating group on the set
$B \setminus \{1\}$.

By using the similar technique for the infinite case one can prove the following
theorem.

**Theorem 4.2.1.** The complete alternating group $\Sigma_A(H; B, B^+, d)$ splits over the
basis group $V(B, B^+)$.

**Proof.** Two conjugate complements $T$ and $T'$ may be obtained by the following
method.

Let $G$ be a subgroup of $H$ obtained as a homomorphic image of $A(B,d)$. Let $g_4, \ldots, g_\epsilon$
be generators of the group $G$ with the following relations

(a) $g_\epsilon^2 = e$, and

(b) $(g_\epsilon g_5)^2 = e$ for $\epsilon \neq$.

We choose as generators of $A(B,d)$ the three cycles $s_\alpha = (1 \alpha 2)$ where $\alpha=3,4,\ldots$.

Then the elements of the complement $T'$ are obtained by the isomorphism $\theta$ where

$\theta(s_3) = t_3' = \{e, e, e, g_4, \ldots, g_\epsilon, \ldots\}(1 3 2)$

$\theta(s_\alpha) = t_3' = \{e, g_\alpha, g_\alpha^2, g_\alpha g_4, \ldots, g_\epsilon^2, g_\alpha g_\epsilon\}(1 \alpha 2)$

$\Box$

**Theorem 4.2.2.** The group $\Sigma_A(H; B, B^+, d)$ splits regularly over the basis group
$V(B, B^+)$ if and only if the group $H$ does not contain a subgroup isomorphic to the
alternating group $A(B,d)$.  

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Proof. It is well known that infinite alternating groups are simple. See also [4].

By theorem, there exists a homomorphism \( \phi : A(B, d) \rightarrow T \). Since \( A(B, d) \) is simple we have two cases; \( \phi(B, d) \cong A(B, d) \) i.e., \( \text{Ker}\phi = \{1\} \) or \( \text{Ker}\phi = A(B, d) \).

Case 1: If \( \text{Ker}\phi = \{1\} \), then \( \phi(A(B, d)) \) is a subgroup of \( H \) isomorphic to \( A(B, d) \). But by assumption \( H \) does not contain any subgroup isomorphic to \( A(B, d) \). Hence this case is impossible. So, the second case happens.

Case 2: In this case \( \text{Ker}\phi = A(B, d) \), and it follows that \( \phi(A(B, d)) = \{1\} \) i.e., \( \text{Ker}\phi = A(B, d) \). Hence, \( \phi \) is the identity map in this case and so \( \Sigma_A(H; B, B^+, d) \) splits regularly.

Conversely, assume that the complete alternating monomial group splits regularly over the basis group \( V(B, B^+) \). Then by a conjugate of an element of \( \Sigma_A(H; B, B^+, d) \), the complement \( T \) may be transformed to \( A(B, d) \). But as in the case of finite case, this implies that the subgroup \( G \) in \( H \) which is the homomorphic image of \( A(B, d) \) will be the identity group i.e., \( G = \{e\} \). So, \( H \) contains no subgroup isomorphic to \( A(B, d) \).

\[ \square \]

Corollary 4.2.3. For a given group \( H \), there exists a complete monomial alternating group \( \Sigma_A(H; B, B^+, d) \) such that the splitting of the monomial group over the basis group is regular.

Proof. If we choose the cardinal \( B \) such that the order of \( A(B, d) = B \) is strictly greater than the order of \( H \), then by the above Theorem 4.2.2, the isomorphism \( \phi : A(B, d) \rightarrow H \) must be an epimorphism i.e., \( \phi(A(B, d)) = \{1\} \). Because in the other case as \( A(B, d) \) is simple, \( \phi \) must be one to one and hence \( H \) must contain an isomorphic copy of \( A(B, d) \) which is impossible by the order of \( H \), namely \( |H| \leq |B| = |A(B, d)| \). \[ \square \]
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