NORMALIZERS IN HOMOGENEOUS SYMMETRIC GROUPS

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ABSTRACT

NORMALIZERS IN HOMOGENEOUS SYMMETRIC GROUPS

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We study some properties of locally finite simple groups, which are the direct limit of finite (finitary) symmetric groups of (strictly) diagonal type. The direct limit of the finite (finitary) symmetric groups of strictly diagonal type is called **homogeneous (finitary) symmetric groups**.

In [5], Kegel, Kuzucuoğlu and myself studied the structure of centralizer of finite groups in the homogeneous finitary symmetric groups. Instead of strictly diagonal embeddings, if we have diagonal embeddings, we will have direct limit of finite symmetric groups of diagonal type. We prove the centralizer of a finite subgroup for the symmetric groups of diagonal type is the direct product of homogeneous monomial groups and a symmetric group of diagonal type.

We also study the level preserving automorphisms of the symmetric groups of diagonal type and finitary homogeneous symmetric groups. We prove that the level preserving automorphisms of both groups is isomorphic to the Cartesian product of centralizers of subgroups.

In the last part of the thesis, we study the normalizers of finite subgroups in both homogeneous symmetric groups and homogeneous finitary symmetric groups. In the first class, we find normalizers of finite semi-regular subgroups and in the latter class we find normalizers of finite subgroups, F, satisfying $F_\alpha = F$ or 1. In each class of groups, the quotient of the normalizer of finite subgroup, F, with the centralizer is

isomorphic to the automorphism group of F.

Keywords: Locally finite groups, centralizer, normalizer,...

ÖZ

HOMOJEN SİMETRİK GRUPLARDA NORMALLEYENLER

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Sonlu (sonlumsu) simetrik gruplardan (kati) köşegen tipteki gömmelerle elde edilen lokal sonlu basit direkt limit grupların bazı özellikleri çalışılmıştır. Sonlu (sonlumsu) simetrik gruplardan kati köşegen tipteki gömmelerle elde edilen direkt limit gruplarına homojen (sonlumsu) simetrik gruplar denir.

Homojen sonlumsu simetrik gruplardaki sonlu altgrupların merkezleyenlerinin yapısı, Kegel, Kuzucuoğlu ve benim tarafımdan [5] makalesinde bulunmuştur. Kati köşegen tipteki gömmeler yerine, köşegen tipteki gömmeler kullanırsak sonlu simetrik grupların köşegen tipteki direkt limitlerini bulmuş oluruz. Bu direkt limit grupları içinde sonlu altgrupların merkezleyenlerinin homojen monomial grupların ve köşegen tipteki simetrik grubun direkt çarpımı olduğu kanıtlanmıştır.

Ayrıca, köşegen tipteki simetrik gruplarla, sonlumsu homojen simetrik grupların seviye koruyan otomorfizmaları çalışılmıştır. Seviye koruyan otomorfizmaların, bazı altgrupların merkezleyenlerinin Kartezyen çarpımına izomorf olduğu gösterilmiştir.

Tezin son kısmında, homojen simetrik gruplar ve homojen sonlumsu simetrik gruplardaki sonlu altgrupların normalleyenleri çalışılmıştır. Birinci sınıftaki gruplarda, yarıdüzenli altgruplardaki normalleyenler, diğer sınıfta ise $F_\alpha = F$ ya da 1 koşulunu sağlayan her hangi bir sonlu F altgrubu için normalleyenler bulunmuştur. İki sınıfta da sonlu F altgrubunun merkezleyeniyle normalleyeninin bölüm grubunun, F'nin

otomorfizma grubuna izomorf olduğu gösterilmiştir.

Anahtar Kelimeler: lokal sonlu gruplar, merkezleyenler, normalleyenler,...

To my Mom,
without whom I can not succeed

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CHAPTER 1

INTRODUCTION

In the theory of infinite groups, the class of locally finite groups are of special interest to the group theorists. A group G is called locally finite, if every finite set of elements of G generates a finite group. After the classification of finite simple groups was completed, the interest turned to the classification of infinite locally finite simple groups. However, the work of Kegel-Wehrfritz [7, Corollary 6.12], showed that there are uncountably many non-isomorphic, countable, locally finite simple groups. The authorities accepted that the classification of locally finite simple groups is not easy. However, we know by Meierfrankenfeld that if G is a simple locally finite group, then it must be either finitary group or the group of alternating type or it must accept a Kegel Cover satisfying some certain properties, see [13].

Although the classification may seem harder, the information we get from such groups could help the mathematicians to understand the class of simple locally finite groups better. With this idea, in this thesis, we investigate properties of some certain infinite locally finite, simple groups and give information about the structure of the group itself, subgroups and automorphism group.

There are lots of ways to obtain infinite locally finite simple groups. We are interested in the locally finite (simple) groups, which are the direct limits of finite symmetric (alternating) and finitary symmetric groups, obtained by (strictly) diagonal embeddings.

Definition 1.1. *Let G be a transitive X set and H be a Y set. An embedding d of G into H is called **diagonal**, if the permutation group $(d(G), O)$ is permutationally isomorphic to the group (G, X), for any orbit O of $d(G)$ on Y with length more than* 1.

Definition 1.2. *A diagonal embedding d of G into H is called* ***strictly diagonal****, if all of the orbits of $d(G)$ on Y has length greater than 1.*

In [9], Kroshko and Suschansky studied the direct limits of finite symmetric (alternating) groups with strictly diagonal embedding. Such groups are called **homogeneous symmetric (alternating) groups**. They gave a classification of homogeneous symmetric groups up to isomorphism using the lattice of Steinitz numbers.

Definition 1.3. *A formal product $n = 2^{r_1}3^{r_2}5^{r_3}\ldots$, where $0 \leqslant r_i \leqslant \infty$ for all $i \in \mathbb{N}$ is called a* ***Steinitz number*** *or a* ***supernatural number****.*

After Kroshko and Suschansky's work, in [4] and [5] Kegel, Kuzucuoğlu and myself studied the centralizers of elements and finite groups. We found the structure of the centralizers of elements and finite groups in the homogeneous symmetric groups.

After the classification of homogeneous symmetric groups were completed, the interest turned into the automorphism group of homogeneous symmetric groups. In [12], Lavreniuk and Sushchansky studied automorphism groups of this class and gave a new perspective to understand the homogeneous symmetric groups better, namely, they look at the homogeneous symmetric groups as a subgroup of homeomorphism group of the boundary of a spherically homogeneous rooted tree.

In Chapter 2, first we give the construction of homogeneous symmetric groups and then we give basic definitions and theorems about the trees and the topology that we will use throughout this thesis.

In Chapter 4, we showed that the automorphism group of homogeneous symmetric groups can not act highly transitively on the boundary of the tree whereas we prove that the group of local isometries acts highly transitive on the boundary.

When we change the embeddings in the construction of homogeneous symmetric groups into diagonal embeddings, we get a bigger class of infinite locally finite simple groups. The classification of direct limits of finite symmetric groups of diagonal type is due to Lavreniuk, Nekrashevych and Sushchansky, see [11]. They used the topological approach together with some measure theory.

In this thesis, in Chapter 5, after giving the construction and some theorems given in [11], we prove the following result:

Theorem 1.4. *Let* $\alpha \in S_\chi$, $\chi = \langle(1, k_0), (n_1, k_1), \ldots\rangle$ *and let* $\alpha_0 \in S(\partial T_\chi, l)$ *be the principal beginning of* α *and* $t(\alpha_0) = (r_1, r_2, \ldots, r_k)$ *be the short cycle type of* α_0*, where* r_1 *is the number of fixed vertices other than* $\$\$ \ldots \$$ *in level* l*. Then the centralizer of* α *in* S_χ*;*

$$C_{S_\chi}(\alpha) \cong \underset{i=2}{\overset{k}{Dr}} \Sigma_{\xi_i}(C_i) \times S_{\chi'}$$

where $\xi = (k_0, n_1, n_2, \ldots)$*,* $char(\xi_i) = \frac{char(\xi)}{k_0 n_1 \ldots n_{l-1}} r_i$ *for all* $i \geqslant 2$*,* $\chi' = \langle(1, r_1), (n_l, k_l), \ldots\rangle$ *and* $\Sigma_{\xi_i}(C_i)$ *the homogeneous monomial group over the cyclic group* C_i *of order* i*.*

The structure of centralizers of finite subgroups are also given in the same Chapter 5.

Definition 1.5. *An automorphism of a group acting on the boundary of a tree is called level preserving, if for some* $N = \{n_i |\ i \geqslant 1\}$ *of positive increasing numbers, it preserves all the levels* n_i *of the tree. The group of all level preserving automorphisms for a given* N *will be called* N*-level preserving automorphisms.*

In Chapter 5, we prove the following result:

Theorem 1.6. *The level preserving automorphism group is isomorphic to the Cartesian product of the centralizers of finite subgroups in* S_χ*.*

The idea of homogeneous symmetric groups were extended to the finitary homogeneous symmetric groups, denoted by $FSym(\kappa)(\xi)$, see [5], so that we will have locally finite (simple) groups of any cardinality κ. Instead of taking finite symmetric (alternating) group, we start with finitary symmetric (infinite alternating) groups with a given cardinal κ.

The classification of such groups and the structures of centralizers of elements, as well as the structure of the centralizers of finite subgroups are also given in the same paper.

In this dissertation, we extend the idea of spherically homogeneous rooted tree and in Chapter 6, show that similar to the case of homogeneous symmetric groups, the finitary homogeneous symmetric group has an action on a non-locally finite spherically homogeneous rooted tree.

We get some properties of the automorphism group of finitary homogeneous symmetric groups. We prove the following theorems;

Theorem 1.7. *Any automorphism of finitary homogeneous symmetric group is induced by an element of the homeomorphism group of spherically homogeneous rooted tree.*

We also prove the following;

Theorem 1.8. *For an increasing sequence of natural numbers, $M = \{m_i \mid m_i > 0\}$, the M-level preserving automorphisms are isomorphic to the Cartesian product*

$$Sym(\kappa m_1) \times \prod_{i=2}^{\infty} C_{Sym(\kappa m_i)}(FSym(\kappa)(m_{i-1}))$$

A group, acting on a set Ω is said to be **semi-regular**, if all the point stabilizers are identity. If, in addition, the action is transitive, then we call the group **regular**. If F is a regular subgroup of symmetric group on a set Ω, then, by [2, Corollary 4.2B], the normalizer is isomorphic to the holomorph of F.

Let F be a semi-regular finite subgroup of $Sym(\xi)$. We are interested in normalizers of such groups. Since $Sym(\xi)$ is the union of finite symmetric groups, in Chapter 7, we first find the structure of normalizer of a finite semi-regular group. Notice that we can find semi-regular representation of any finite abstract group, G, just consider regular representation and embed it via strictly diagonal embedding so that the resulting group is semi-regular and isomorphic to G.

We prove the following,

Theorem 1.9. *Let F be a finite semi-regular subgroup of $S(\xi)$, where $\xi =< p_1, p_2, \ldots >$. If F is in S_{n_i} for some $n_i = p_1 p_2 \ldots p_i$, then*

$$N_{S(\xi)}(F)/C_{S(\xi)}(F) \cong Aut(F)$$

Moreover, in Chapter 7, we also showed that the structure of normalizers of finite subgroups of finitary homogeneous symmetric group, satisfying some property, is not so different than the normalizers of semi-regular groups in $S(\xi)$.

Let F be a finite subgroup of $FSym(\kappa)(\xi)$. Hence, F is a subgroup of $FSym(\kappa)(n_i)$ for some n_i. Note that F acts on the set of elements of cardinality κn_i. Let F satisfy the property that the stabilizers of any element α is either F or identity. Then we have the following result.

Theorem 1.10. *Let F be a finite subgroup of $FSym(\kappa)(\xi)$ which satisfies the above property. If $F \in FSym(\kappa n_i)$ for some $n_i = p_1 p_2 \ldots p_i$, then we have*

$$N_{FSym(\kappa)(\xi)}(F)/C_{FSym(\kappa)(\xi)}(F) \cong Aut(F)$$

CHAPTER 2

PRELIMINARIES

In this chapter, we will give basic definitions and results that will be used in the other chapters.

2.1 The homogeneous symmetric group

In this section, the basic definitions and facts about the groups $S(\xi)$ and the strictly diagonal embeddings will be given.

Definition 2.1. *Let G be a transitive permutation group on a set X and H be a permutation group on Y. If we have an embedding d from G into H such that $(d(G), O)$ is permutational isomorphic to (G, X), for any orbit of $d(G)$ on Y of length greater than 1, then d is called* ***diagonal****. On the other hand, if all the orbits have length greater than 1, then the embedding is called* ***strictly diagonal****.*

If $\alpha = \left(\begin{smallmatrix} 1 & 2 & \cdots & n \\ i_1 & i_2 & \cdots & i_n \end{smallmatrix}\right)$, then define a map,

$$d^r : S_n \to S_{nr}$$

and $d^r(\alpha)$ is defined by the rule

$$(kn + i)^{d^r(\alpha)} = kn + i^{\alpha} \quad 0 \leqslant k \leqslant r - 1, \quad 1 \leqslant i \leqslant n.$$

Hence,

$$d^r(\alpha) = \left(\begin{smallmatrix} 1 & 2 & \cdots & n \\ i_1 & i_2 & \cdots & i_n \end{smallmatrix} \middle| \begin{smallmatrix} n+1 & \cdots & 2n \\ n+i_1 & \cdots & n+i_n \end{smallmatrix} \middle| \begin{smallmatrix} \cdots \\ \cdots \end{smallmatrix} \middle| \begin{smallmatrix} (r-1)n+1 & \cdots & (r-1)n+n \\ (r-1)n+i_1 & \cdots & (r-1)n+i_n \end{smallmatrix}\right) \tag{2.1}$$

will become an element of S_{nr}.

Lemma 2.2. *The embedding d^r is a strictly diagonal embedding.*

Proof. It can be easily seen that the image $d^r(S_n)$, acts on the set $\{1, 2, \ldots, rn\}$, by partitioning the set into r pieces of length n. The action of any element $d^r(\alpha) \in d^r(S_n)$ is diagonally same as the action of α on the set $\{1, 2, \ldots n\}$, by Equation 2.1. Hence the orbits of $d^r(S_n)$ in the set $\{1, 2, \ldots, rn\}$ will be of the form

$$\mathscr{O}_k = \{(k-1)n + 1, (k-1)n + 2, \ldots kn\}$$

where $1 \leqslant k \leqslant r$.

Notice that each orbit has length n. For an arbitrary orbit $\mathscr{O}_k$, define the map,

$$\lambda : \{1, 2, \ldots, n\} \to \mathscr{O}_k$$

$$i \mapsto (k-1)n + i$$

Then the map will satisfy the following equation; for any $i \in \{1, 2, \ldots, n\}$, $\alpha \in S_n$,

$$\lambda(i)^{d^r(\alpha)} = ((k-1)n + i)^{d^r(\alpha)} = (k-1)n + i^\alpha = \lambda(i^\alpha)$$

where i^α is used for the action of α on i. Hence, $(d^r(S_n), \mathscr{O}_k)$ is permutational isomorphic to $(S_n, \{1, 2, \ldots, n\})$. $\square$

By using this specific embeddings d^r, one can construct a direct limit group which is locally finite and simple.

Let $\xi = (p_1, p_2, \ldots)$ be an infinite sequence of prime numbers (not necessarily distinct). Consider the embeddings $d^{p_i} : S_{n_i} \to S_{n_{i+1}}$ where $n_i = p_1 p_2 \ldots p_i$. The embeddings will generate a direct limit group and the group will be denoted by $S(\xi)$.

If, in the direct limit group, we denote the image of a permutation α in S_{n_i}, for some n_i by $d(\alpha)$ and the image of S_{n_i} by $S(n_i)$, then the group $S(\xi) := \bigcup\limits_{i=1}^{\infty} S(n_i)$. The groups $S(\xi)$ are called **homogeneous symmetric groups** [9]. Note that $S(\xi)$ is a subgroup of the symmetric group on natural numbers $S(\mathbb{N})$.

Similarly, we can form $A(\xi) = \bigcup\limits_{i=1}^{\infty} A(n_i)$ where $A(n_i)$ is the image of the alternating group A_{n_i} under the embedding d. Moreover $A(\xi) \leqslant S(\xi)$, as for any $\alpha \in A_{n_i}$ and for any r, the image $d^r(\alpha)$ is always an even permutation so $d(\alpha) \in A(\xi)$ and $d(\alpha) \in S(\xi)$.

Definition 2.3. *[9, Page 175] If $\alpha \in S(\xi)$, then there exists a minimal number n_i such that we have $\alpha_0 \in S_{n_i}$ and $d(\alpha_0) = \alpha$. The element α_0 is called **the principal beginning** of α.*

In [9], Kroshko and Suschansky proved the following.

Theorem 2.4. *[9, Page 175, Theorem 1] Let $\xi = (p_1, p_2, \ldots)$ be an infinite sequence consisting of not necessarily distinct primes.*

(1) *If the prime 2 appears infinitely many times in the sequence ξ, then $S(\xi) = A(\xi)$.*
(2) *If in the sequence we have only finitely many 2, then $[S(\xi) : A(\xi)] = 2$.*
(3) *$A(\xi)$ is simple.*

Hence, depending on the sequence ξ, one can determine the simplicity of the group $S(\xi)$. In fact, the classification of the groups $S(\xi)$ also depends on the sequence.

Now in order to give the classification of such groups, we introduce Steinitz numbers.

Definition 2.5. *The formal product $k = 2^{a_1}3^{a_2}5^{a_3}\ldots$ where $0 \leqslant a_i \leqslant \infty$ and a_i is the power of i^{th} prime in the set of all prime numbers is called a **Steinitz number**.*

Lemma 2.6. *The set of all Steinitz numbers is a partially ordered set with respect to division and forms a lattice.*

Proof. Let $\mathscr{S}$ be the set of all Steinitz numbers. Define the division of two Steinitz numbers $k = 2^{a_1}3^{a_2}5^{a_3}\ldots$ and $l = 2^{b_1}3^{b_2}5^{b_3}\ldots$ as follows;
$k|l$ if and only if $a_i \leqslant b_i$ for all $i \in \mathbb{N}$. With the division above, $(\mathscr{S}, |)$ becomes a partially ordered set.
If we define meet and join of arbitrary two Steinitz numbers, $k = 2^{a_1}3^{a_2}5^{a_3}\ldots$ and $l = 2^{b_1}3^{b_2}5^{b_3}\ldots$ as

$$k \vee l = 2^{max\{a_1,b_1\}}3^{max\{a_2,b_2\}}5^{max\{a_3,b_3\}}\ldots$$

$$k \wedge l = 2^{min\{a_1,b_1\}}3^{min\{a_2,b_2\}}5^{min\{a_3,b_3\}}\ldots$$

then obviously, $k \vee l$ and $k \wedge l$ are Steinitz numbers. Hence, with this meet and join operations, the set $\mathscr{S}$ become a lattice. □

Definition 2.7. *Let $\xi = (p_1, p_2, \ldots)$ be an infinite sequence of not necessarily distinct primes. The **characteristic** of the sequence is a Steinitz number,* $\mathbf{char}(\xi) = 2^{r_1}3^{r_2}5^{r_3}\ldots$ *where r_i is the number of i^{th} prime appearing in the sequence ξ. If a prime appears infinitely many times, then set corresponding r_i to be infinity.*

Theorem 2.8. *[9, Lemma 3.3] Let ξ_1 and ξ_2 be two Steinitz numbers. Then $S(\xi_1)$ is a subgroup of $S(\xi_2)$ if and only if $char(\xi_1)$ divides $char(\xi_2)$.*

Therefore, this theorem classifies homogeneous symmetric groups. Moreover, as there are uncountably many Steinitz numbers we will have uncountably many pairwise non-isomorphic homogeneous symmetric groups.

2.2 Trees and The Topology

After the classification of the groups $S(\xi)$ up to isomorphism, natural question arises. What is the structure of automorphism group of homogeneous symmetric groups?

In group theoretic point of view, understanding the structure of automorphism group of a group is generally hard. Although the groups $S(\xi)$ are the union of finite symmetric groups, and the automorphisms of symmetric groups are very well known, to understand the automorphisms of $S(\xi)$, we need some other tools.

We will regard the groups $S(\xi)$ as a subgroup of homeomorphism group of the boundary of a spherically homogeneous tree. For this purpose, in the following subsections, we explain some definitions and facts about trees and the topology they induce. In Chapter 4, the properties about $Aut(S(\xi))$ will be given.

2.2.1 Rooted Trees

Definition 2.9.

- *A **graph** T is a pair, defined by the set of vertices $V(T)$ and the edges $E(T)$ where any $e \in E(T)$ is a two element set $\{v_1, v_2\}$.*
- *Two vertices v_1, v_2 are **adjacent** if there is an edge $e = \{v_1, v_2\} \in E(T)$. In this case we say that edge e connects the vertices v_1 and v_2.*

- *For a vertex v, the number of edges which v belongs to is called the* ***degree*** *of v.*

- *A graph is called* ***locally finite*** *if the degree of every vertex is finite.*

- *A* ***path****, γ, of length $n-1$ is a sequence of pairwise distinct vertices $(v_1, \dots v_n)$ such that for all $1 \leqslant i \leqslant n-1$, $\{v_i, v_{i+1}\}$ forms an edge.*

- *If in the path γ, also $\{v_1, v_n\}$ forms an edge, then the path is called a* ***cycle****.*

- *A graph is* ***connected*** *if we can connect two arbitrary vertices by a path.*

- *A connected graph with no cycles is called* ***tree****.*

Lemma 2.10. *Let T be a graph. For all $v, w \in V(T)$ there exists a unique path connecting them if and only if T is a tree.*

Proof. Let T be a tree. Then by definition of a tree, any two vertices can be connected by a path. If there exist two paths connecting v and w, say $\gamma_1 = (v, v_2, \dots v_{n-1}, w)$ and $\gamma_2 = (v, w_2, \dots w_{m-1}, w)$, then the path $(v, v_2, \dots v_{n-1}, w, w_{m-1}, \dots w_2)$ will become a cycle. But this is a contradiction to the definition of the tree.
Conversely, if for any pair of vertices $v, w \in V(T)$ we have a unique path connecting them, then T is connected. On the other hand, if there exist a cycle $\gamma = (v_1, v_2, \dots v_n)$ so that $\{v_1, v_n\}$ forms an edge, then we have $\gamma_1 = (v_1, v_n)$ a path connecting v_1 and v_n but γ also connects v_1 with v_n so it is a contradiction. Hence there is no cycle in T. Therefore, T is a tree. □

In connection with the above lemma, in a tree for any two vertices u, v, we can define $d(u, v)$ as the length of the path connecting them.

Definition 2.11.

- *A* ***rooted tree*** *(T, v_0) is a tree with a fixed vertex v_0 called the root.*

  In the Figure 2.1, one can see examples of rooted trees. Note that the notion of a rooted tree is just a choice of a vertex that will specify the tree. In the Figure 2.2, we give examples of two trees consisting of same vertex and edge sets but one is rooted and the other one is not.

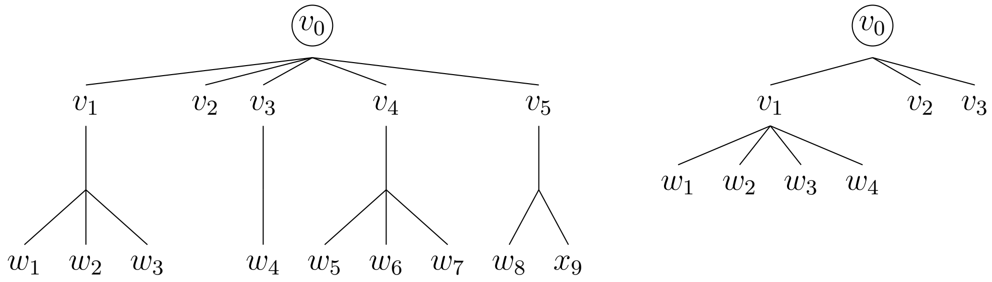

Figure 2.1: Rooted trees where circled v_0 is the root

Notice that in Figure 2.2, the tree in the right hand side is non-rooted because we did not specify any vertex to be the root.

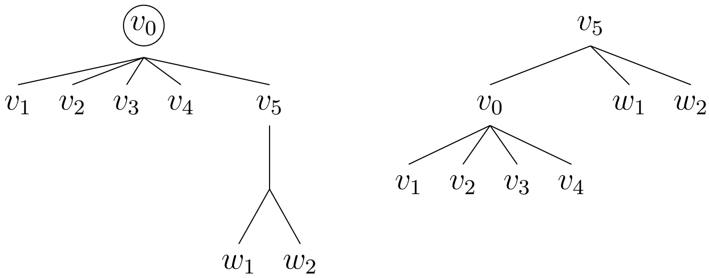

Figure 2.2: Two trees one is rooted, the other is not

- *A rooted tree is **non-degenerate** if the degree of the root is more than one and the degree of every other vertex is more than two. A rooted tree is called **degenerate**, if the degree of the root is* 1 *and starting with the root, we can find a unique path that contains all the vertices of the tree. A degenerate tree looks like a linear line.*

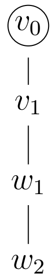

Figure 2.3: A degenerate rooted tree example

Note that, if a tree is non-degenerate, then it should be an infinite tree that is,

- *A tree is called **infinite** if the cardinality of $V(T)$ (or $E(T)$) is infinite.*

- *A **rooted subtree** (T', w) of a rooted tree (T, v_0) is a tree where the vertex and edge sets $V(T')$, $E(T')$ are subsets of $V(T)$, $E(T)$, respectively.*

Throughout the thesis, we will discuss mainly non-degenerate infinite rooted trees. We prefer non-degenerate trees because mostly we will discuss about level preserving homeomorphisms of the boundary of the tree and for a tree being degenerate is meaningless in that concept. A **homeomorphism** between two topological spaces X and Y is a bijective map, f, such that both f and f^{-1} is continuous maps. For an infinite rooted tree (T, v_0), we have the following definitions.

Definition 2.12.

- *An **end** is an infinite path, $(v_0, v_1, v_2, \ldots)$, which is an infinite sequence of distinct vertices starting with the root v_0 such that $\{v_i, v_{i+1}\}$ is in $E(T)$ for all $i \in \mathbb{N} \cup \{0\}$.*

- *We call the set of all ends of the rooted tree (T, v_0) as the **boundary** of the tree and denote it by ∂T.*

- *For every non-negative integer n, we call the set*

$$V_n = \{v \in V(T) \mid d(v, v_0) = n\}$$

*as the $\mathbf{n^{th}}$ **level** (the level number n). The level number 0 consists of the root, v_0 only.*

Let $\{v_0, v_1, \ldots, v\}$ be the path connecting the root with v. If w is a member of the path sequence, then we say that v is **below the vertex** w.

The rooted subtree containing all the vertices below the vertex v on level n, with the vertex v as a root will be denoted by $\mathbf{T_v}$ and said to be a subtree of level n. See the Figure 2.4.

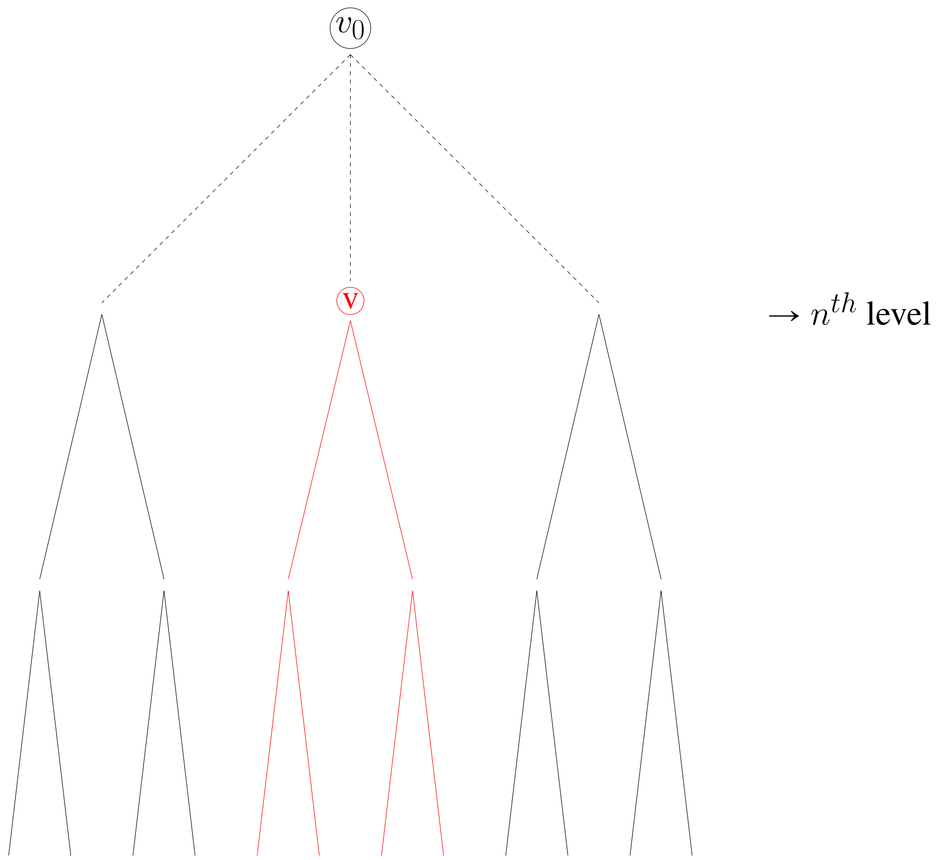

Figure 2.4: The subtree T_v of the tree (T, v_0)

Definition 2.13. *An* ***automorphism*** *of the rooted tree* (T, v_0) *is defined as the bijection of the vertex set* $V(T)$ *that fixes the root and preserves the incidence relation between vertices. The full automorphism group of the tree is denoted by* $Aut(T)$.

Lemma 2.14. *Levels are invariant with respect to automorphisms of the rooted tree.*

Proof. Let $\sigma \in Aut(T)$. As σ fixes the root, $\sigma(v_0) = v_0$. We will use induction on the level numbers. For an element v_1 in V_1, the root v_0 and v_1 is connected by an edge. Now, σ preserves the incidence relation that is $\sigma(v_1)$ and $\sigma(v_0)$ must be connected by an edge. As $\sigma(v_0) = v_0$, $\sigma(v_1)$ is an element of V_1. Hence, σ preserves the first level.

Assume, σ sends the vertices of level less than or equal to $n-1$ to the same levels they belong and let $v_n \in V_n$. As T is a tree, there exist an element $w \in V_{n-1}$ such that w and v_n is connected by an edge. Hence, $\sigma(w)$ and $\sigma(v_n)$ is also connected by an edge. By induction hypothesis, $\sigma(w)$ is a vertex in level $n-1$ that says $\sigma(v_n)$ must be a vertex in level n. □

Let G be a subgroup of $Aut(T)$. If G acts transitively on the levels of the tree, then G is called **spherically transitive**. A rooted tree T is called **spherically homogeneous** (spherically transitive) if the full automorphism group of the tree is spherically transitive.

Lemma 2.15. *A locally finite rooted tree T is spherically transitive if and only if the degree of all vertices of the same level are equal.*

Proof. Let $v_1 \in V_n(T)$ for an arbitrary level n. If the degree of v_1 is k, then we want to show the degree of any vertex in $V_n(T)$ is also k.

Consider a vertex $v \in V_n(T)$ different than v_1. Since T is spherically transitive, $Aut(T)$ is transitive on levels and there exist an automorphism σ sending v_1 to v. Note that, σ preserves the incidence relation between vertices. Hence, the k vertices that are connected to v_1 must be sent to the k vertices so that they are connected to $\sigma(v_1) = v$. Therefore, v has also degree k.

On the other hand, assume the degree of all vertices of the same level are equal. For any two vertieces v and w on the same level n, let $(\varnothing, v_1, v_2, \dots v_{n-1}, v)$ and $(\varnothing, w_1, w_2, \dots w_{n-1}, w)$ be the paths from the root to v and w, respectively.

Consider the subtrees of the first level T_{v_1} and T_{w_1} since the degrees of all vertices are same there exists an isomorphism α between them sending v_1 to w_1. On this isomorphism we can choose the image of $v_2 \in T_{v_1}$ to be $w_2 \in T_{w_1}$ so that the restriction of α to the subtree T_{v_2} is an isomorphism from T_{v_2} to T_{w_2}. Continuing like that we will have isomorphism α sending v to w. Extend α to an automorphism $\bar{\alpha}$ of the tree T as follows;

If z belongs to T_{v_1} or T_{w_1} then $\bar{\alpha}(z) = \alpha(z)$. If z is a vertex not belonging to T_{v_1} and T_{w_1} then $\bar{\alpha}(z) = z$. Hence, we get the result.

□

The above lemma says that in a locally finite spherically transitive tree, the degree of a vertex on level n depends only the level n. Therefore, for a locally finite spherically transitive tree we have the following;

Definition 2.16. *Let* $\Omega = (a_0, a_1, \ldots)$ *be a sequence where* a_0 *is the degree of the root and* $a_n + 1$ *is the degree of any vertex in level* n. *We define the* ***characteristic*** *of the spherically transitive tree* (T, v_0) *as the characteristic of* Ω *which is the Steinitz number* $char(\Omega) = 2^{r_1}3^{r_2}5^{r_3}\ldots$ *where the powers* r_j *are determined by the prime factors of each* a_i*, that is we factorize each* a_i *into prime numbers and* r_j *is the number of the* j^{th} *prime appearing in* a_i*'s. If a prime appears infinitely many, then set the corresponding* $r_j = \infty$. *The sequence* Ω *is called* ***characteristic sequence***.

Example 2.17. *In the Figure 2.5, we have a spherically homogeneous tree with the characteristic sequence $\Omega = (4, 2, 6)$ and the characteristic is $char(\Omega) = 2^4 3$.*

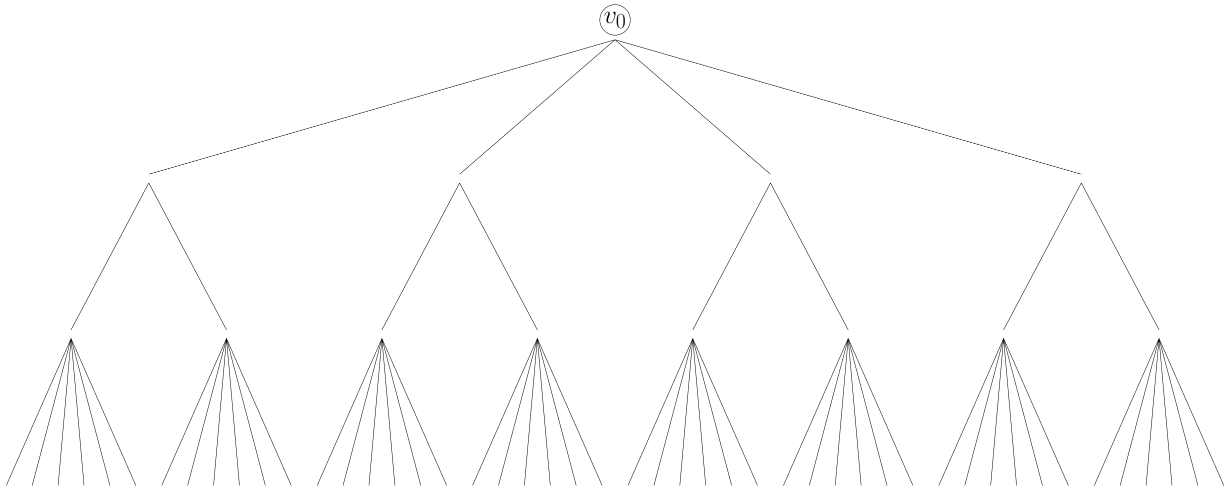

Figure 2.5: A spherically homogeneous rooted tree with characteristic $2^4 3$

If T is a spherically homogeneous rooted tree with characteristic sequence $\Omega = (a_0, a_1, \ldots)$, then we can label the tree T in the following way. The set of vertices consists of all sequences of the form $(\varnothing, i_0, i_1, \ldots, i_n)$ where $i_k \in \{0, 1, \ldots, a_k - 1\}$ $n \geqslant 0$ is an integer. We denote the root with the empty set notation corresponding to the empty sequence. Two vertices are adjacent if and only if they are of the form $(\varnothing, i_0, i_1, \ldots, i_{n-1})$, $(\varnothing, i_0, i_1, \ldots, i_{n-1}, i_n)$. If the characteristic sequence Ω is known, then we denote the tree by T_Ω. See the Figure 2.6. For simplicity unless it is needed, a vertex $(\varnothing, i_0, i_1, \ldots, i_n)$ will be written without $\varnothing$ as $(i_0, i_1, \ldots, i_n)$.

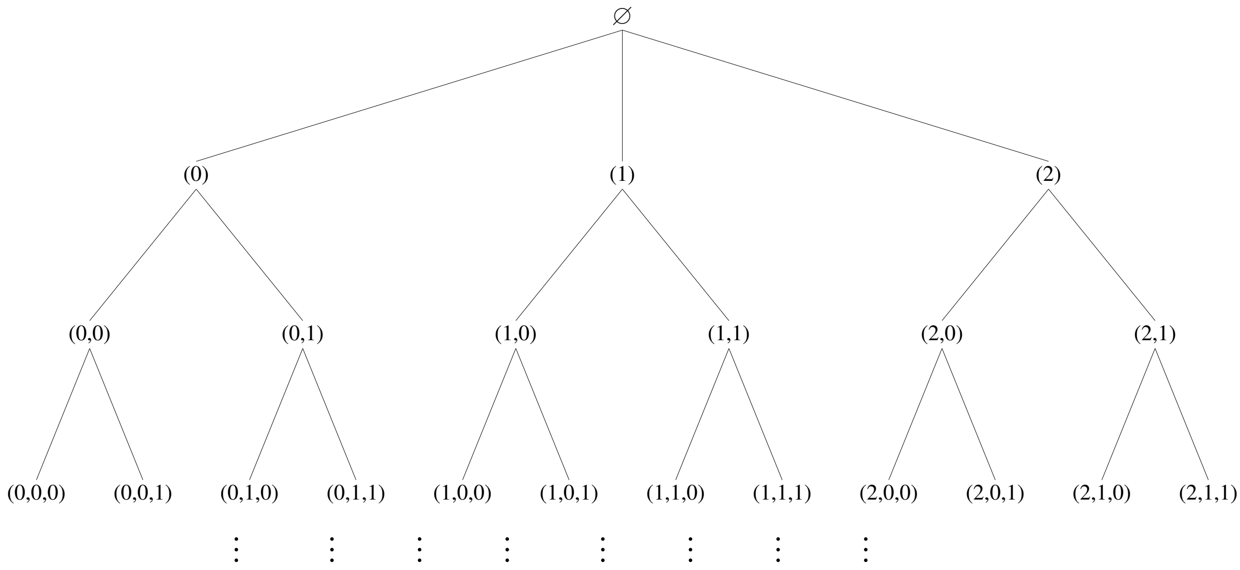

Figure 2.6: The tree T_Ω where $\Omega = (3, 2, 2, \ldots)$

2.2.2 Topology on the boundary of the tree

For readers' convenience we will give some basic definitions on topology and construct the topology on the boundary of the rooted tree.

Definition 2.18. *[14, Page 76] Let X be any set. A collection τ of subsets of X is called **topology** on a set X if it has the following properties:*

1. *$\varnothing$ and X are in τ.*
2. *The union of the elements of any subcollection of τ is in τ.*
3. *The intersection of the elements of any finite subcollection of τ is in τ.*

*A set X for which a topology τ has been specified is called a **topological space**. In this case, we say that a subset U of X is an **open set** of X if U belongs to the collection τ. On the other hand, a subset U of X is called **closed** if the complement $X \backslash U$ is open.*

Definition 2.19. *[14, Page 119] A function*

$$d : X \times X \to \mathbb{R}$$

*is said to be a **metric** on a set X if it has the following properties:*

1. *$d(x,y) \geqslant 0$ for all $x, y \in X$; equality holds if and only if $x = y$.*

2. *$d(x,y) = d(y,x)$ for all $x, y \in X$.*

3. *(Triangle inequality) $d(x,z) \leqslant d(x,y) + d(y,z)$, for all $x, y, z \in X$.*

A **metric space** X is a set together with a metric.

Let T_Ω be a spherically homogeneous tree. On the boundary ∂T_Ω of the tree which is the set of all ends of the tree, we define a metric; for any two ends γ_1, γ_2

$$\rho(\gamma_1, \gamma_2) = \tfrac{1}{n+1}$$

where n is the length of the common parts (edges) of γ_1, γ_2.

Lemma 2.20. *The distance ρ defines a metric on ∂T_Ω.*

Proof. 1) $\rho(\gamma_1, \gamma_2) \geqslant 0$ equality holds if and only if $\gamma_1 = \gamma_2$;

Common parts of any two different ends is bigger than or equal to 0. Hence, the ratio $\frac{1}{n+1}$ is bigger than 0.

If two ends are the same, then as their all edges are common the ratio $\frac{1}{n+1}$ will be equal to 0.

2) $\rho(\gamma_1, \gamma_2) = \rho(\gamma_2, \gamma_1)$; By the definition of ρ it is obvious.

3) $\rho(\gamma_1, \gamma_2) \leqslant \rho(\gamma_1, \gamma_3) + \rho(\gamma_3, \gamma_2)$;

Let the common parts of γ_i, γ_j be n_{ij} for $i, j = 1, 2, 3$ and $i \neq j$.

Now, without loss of generality say $n_{13} \leqslant n_{23}$. Since we start to count the common parts of two ends from the root, any common part of γ_1 with γ_3 will also be the common parts of γ_2 with γ_3. But again as the common parts are counted from the root γ_1 and γ_2 must share the common parts of γ_1 and γ_3. Hence the common parts n_{12} of γ_1 with γ_2 is greater than or equal to the common parts n_{13} of γ_1 with γ_3. We have $n_{13} \leqslant n_{12}$, so $\frac{1}{n_{13}+1} \geqslant \frac{1}{n_{12}+1}$. Thus $\rho(\gamma_1, \gamma_2) \leqslant \rho(\gamma_1, \gamma_3) + \rho(\gamma_3, \gamma_2)$. ☐

Definition 2.21. *A metric d on a set X is called an **ultra-metric** if*

$$d(x,y) \leqslant \textit{max}\,\{d(x,z), d(z,y)\} \quad \textit{for all } x, y, z \in X$$

In fact, the metric ρ is an ultra-metric. By the proof of third property of being metric, we find $\frac{1}{n_{13}+1} \geqslant \frac{1}{n_{12}+1}$ whenever $n_{13} \leqslant n_{23}$, it yields us the more strong condition that

$$\rho(\gamma_1, \gamma_2) \leqslant max\{\rho(\gamma_1, \gamma_3), \rho(\gamma_3, \gamma_2)\}.$$

Definition 2.22. *[14, Page 78] A **basis** for a topology on X is a collection $\mathscr{B}$ of subsets of X (called **basis elements**) such that*

(1) *For each $x \in X$, there is at least one basis element B containing x.*

(2) *If x belongs to the intersection of two basis elements B_1 and B_2, then there is a basis element B_3 containing x such that $B_3 \subset B_1 \cap B_2$.*

Definition 2.23. *[14, Page 119] If X is a metric space with a given metric d, then the collection of the sets*

$$B(x, \epsilon) = \{y \mid d(x, y) < \epsilon\}$$

*for any $\epsilon > 0$ and any $x \in X$ will form a basis for the metric space X. The sets $B(x, \epsilon)$ are called ϵ-ball centered at x, or sometimes simply **balls**.*

The topology induced by the metric ρ has a base of open sets;

$$P_{nv_i} = \{\gamma \in \partial T_\Omega \mid v_i \in V_n(T_\Omega)\} \text{ where } v_i \in \gamma$$

The set P_{nv_i} consists of all ends passing through the vertex v_i on the level n. Observe that P_{nv_i} corresponds to the open ball $B(\gamma, \frac{1}{n+1})$ where γ is any end containing the vertex v_i.

Lemma 2.24. *If $v \in V_n$ is connected by an edge to $w \in V_{n+1}$, then the ball P_{n+1w} will properly be contained in P_{nv}. If v_1 and v_2 are two different vertices in the same level n, then $P_{nv_1} \cap P_{nv_2} = \varnothing$.*

Proof. Since there is only one path connecting root to w by the definition of P_{n+1w}, any end that passes through the vertex w must also pass from the vertex v. Since the tree is non-degenerate, i.e. there exists at least one other vertex in level $n + 1$ that the vertex v connected to, P_{n+1w} is properly contained in P_{nv}.

On the other hand, if the balls in the same level have an intersection, then there must be an end γ passing through both v_1 and v_2 contradicting to the fact that T_Ω is a tree. □

Lemma 2.25. *Every ball of the topology on ∂T_Ω is clopen, that is both closed and open.*

Proof. For an arbitrary $\gamma \in \partial T_\Omega$ we will show $\partial T_\Omega \backslash B(\gamma, \epsilon)$ is an open set in ∂T_Ω.

Let $\gamma_1 \in \partial T_\Omega \backslash B(\gamma, \epsilon)$ that is to say $\rho(\gamma, \gamma_1) \geqslant \epsilon$. Then $B(\gamma, \epsilon) \cap B(\gamma_1, \epsilon) = \varnothing$. (If not, $\sigma \in B(\gamma, \epsilon) \cap B(\gamma_1, \epsilon)$ means $\rho(\gamma, \sigma) < \epsilon$ and $\rho(\sigma, \gamma_1) < \epsilon$, then $\rho(\gamma, \gamma_1) \leqslant max\{\rho(\sigma, \gamma_1), \rho(\sigma, \gamma)\}$, contradiction) so $B(\gamma_1, \epsilon) \subseteq \partial T_\Omega \backslash B(\gamma, \epsilon)$. □

The topology on the boundary of a spherically homogeneous tree coincides with the product topology of discrete spaces. Hence, in the following section we will give the definition and facts about product topology.

2.2.3 The Product Topology [14, Ch. 2 Section 19]

Let $\{X_\alpha \mid \alpha \in I\}$ be a collection of topological spaces. Consider the Cartesian product $\prod_{\alpha \in I} X_\alpha$ and write any element of the Cartesian product as $(x_\alpha) = (x_\alpha)_{\alpha \in I}$.
Define the projection map for all $\beta \in I$ as follows;

$$\Pi_\beta : \prod_{\alpha \in I} X_\alpha \to X_\beta$$

where $\Pi_\beta((x_\alpha)) = x_\beta$, β component of (x_α).

Definition 2.26. *Let X be a set. A collection $\mathscr{S}$ of subsets of X is called a subbasis for a topology on X if the union of members of $\mathscr{S}$ is X.*

Remark 2.27. *In this case, finite intersection of members of $\mathscr{S}$ is a basis for a topology on X, for the proof see [14, Ch. 2 Page 82].*

Hence, the elements $\Pi_\beta^{-1}(U_\beta) = \prod_{\alpha \in I} U_\alpha$ such that $U_\alpha = X_\alpha$ for all $\alpha \in I$ except for β where U_β is open in X_β forms a subbasis. By remark, the finite intersection of the subsets $\Pi_\beta^{-1}(U_\beta)$ forms a basis for a topology on $\prod_{\alpha \in I} X_\alpha$ which is called **product topology**.

Theorem 2.28 (Theorem 19.2, [14]). *Suppose the topology on each space X_α is given by a basis $\mathscr{B}_\alpha$. Then the collection of all sets of the form*

$$\prod_{\alpha\in I} B_\alpha$$

where $B_\alpha \in \mathscr{B}_\alpha$ for finitely many indices α and $B_\alpha = X_\alpha$ for all the remaining indices, will serve as a basis for the product topology.

Definition 2.29. *[14, Page 77] Suppose that τ and τ' are two topologies on a given set X. If $\tau' \supset \tau$ we say that τ' is **finer** than τ.*

Lemma 2.30. *[14, Lemma 13.3] Let $\mathscr{B}, \mathscr{B}'$ are two basis of the topologies τ and τ' on the same set X, respectively. Then the following are equivalent:*

1. *τ' is finer than τ.*

2. *For each $x \in X$ and $B \in \mathscr{B}$, there is a basis element $B' \in \mathscr{B}'$ such that $x \in B' \subset B$.*

Two topologies τ', τ coincides if τ is finer than τ' and τ' is finer than τ.

Lemma 2.31. *The topology defined above by the metric ρ on the boundary, ∂T_Ω of a locally finite spherically homogeneous tree with characteristic sequence $\Omega = (a_0, a_1, \ldots)$ coincides with the product topology of discrete spaces defined on the sets $X_0 = \{\varnothing\}$, $X_j = \{0, 1, 2, \ldots, a_{j-1} - 1\}$.*

Proof. For the spherically homogeneous tree T, we have the characteristic sequence $\Omega = (a_0, a_1, a_2, \ldots)$. Set $X_0 = \{\varnothing\}$ and $X_j = \{0, 1, 2, \ldots a_{j-1} - 1\}$ for all $j \in \mathbb{N}$. Recall that by the explanation after the Figure 2.5, on page 17, we can label the vertices by using the finite sequences of the form $(\varnothing, i_0, i_1 \ldots i_n)$. As the boundary consists of ends we may write any end as an infinite sequence $\gamma = (\varnothing, i_0, i_1, i_2, \ldots)$ where $i_j \in X_{j+1}$ for all $j \in \mathbb{N} \cup \{0\}$. Then the boundary of the spherically homogeneous tree will be the set;

$$\partial T_\Omega = \{\gamma |\ \gamma = (\varnothing, i_0, i_1, i_2, \ldots),\ i_j \in X_{j+1},\ \forall\, j \in \mathbb{N} \cup \{0\}\ \}$$

The basis elements for ∂T_Ω are ;

$$P_{nv} = \{\gamma = (\varnothing, i_0, i_1, \ldots, i_{n-1}, j_n, \ldots) \mid j_t \in X_{t+1} \text{ for all } t \geqslant n\}$$

where v is the vertex of level n defined by the labeling $(\varnothing, i_0, i_1, \ldots i_{n-1})$.

Consider the sets $X_0 = \{\varnothing\}$ and $X_i = \{0, 1, \ldots, a_{i-1} - 1\}$ for all $i \in \mathbb{N}$ and the discrete topology on X_i with the metric

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

and the singletons are the basis elements for X_i.

$$B(x, \epsilon) = \begin{cases} x & \text{if } \epsilon < 1 \\ X_i & \text{if } \epsilon \geqslant 1 \end{cases}$$

If we define the product topology $\prod_{i=0}^{\infty} X_i$, then the elements of product topology are infinite sequences $(\varnothing, i_0, i_1, \ldots)$ where $i_j \in X_{j+1}$.

Moreover, the basis elements are $X_0 \times \prod_{k=0}^{\infty} B(i_k, \epsilon_k)$ where $\epsilon_k < 1$ for only finitely many k. Observe that both ∂T_Ω and the product topology $\prod_{i=0}^{\infty} X_i$ consists of the same type of elements, namely infinite sequences.

To conclude the proof we need to prove that for any basis element from the first topology there exists a basis element from the other one that contains it, and vice versa. However, observe that any basis element P_{nv} consist of elements of the form $\gamma = (\varnothing, i_0, i_1, \ldots, i_{n-1}, j_n, \ldots)$ where $(\varnothing, i_0, i_1, \ldots i_{n-1}) = v$ and $j_t \in X_{t+1}$ for all $t \geqslant n$. Note that γ is an element of the product $X_0 \times B(i_0, \epsilon) \times \ldots \times B(i_{n-1}, \epsilon) \times X_{n+1} \times X_{n+2} \times \ldots$ which is a basis element for the product topology where $\epsilon < 1$.

On the other hand, for any basis element $X_0 \times \prod_{k=0}^{\infty} B(i_k, \epsilon_k)$ there exists a number n such that $\epsilon_k > 1$ for all $k > n - 1$ so, the basis element have elements of the form $(\varnothing, i_0, i_1, \ldots i_{n-1}, j_n, \ldots)$ where i_k's are fixed coming from X_{k+1} for all $k \in \{0, 1, \ldots n-2\}$ and $j_t \in X_{t+1}$ for all $t \geqslant n$ which are the elements of the ball P_{nv} on the level n in this basis of the topology ∂T_Ω. Therefore, the topologies coincide. □

Notice that by the above lemma the topology on ∂T_Ω has the same properties with the product topology of the finite discrete sets X_i. It is easy to see that the discrete topology on the finite set is compact. Hence, by the well known Tychonoff's theorem [14, Theorem 37.3] saying that the product of compact spaces is compact, we can conclude that ∂T_Ω is a compact topological space.

However, for readers convenience in the next lemma we will give another proof of compactness. To see compactness we will use a strong theorem from topology:

Theorem 2.32. *[14, Theorem 45.1] A metric space is compact if and only if it is totally bounded and complete.*

Definition 2.33. *[14, Ch. 7, Section 43] Let (X, d) be a metric space. A sequence (x_n) consisting of the elements of X is said to be **Cauchy sequence** if for every $\epsilon > 0$ there is an integer N such that*

$$d(x_n, x_m) < \epsilon \textit{ for all } n, m \geqslant N$$

*The metric space X is **complete** if every Cauchy sequence in the space converges to an element of the metric space itself.*

Definition 2.34. *[14, Ch. 7, Section 45] A metric space is **totally bounded** if for all $\epsilon > 0$, there exists a finite collection of open sets with radius ϵ whose union contains the whole space.*

Definition 2.35. *[14, Ch. 2, page 98] A topological space is **Hausdorff** if for every distinct pair of points x, y in X there exist open balls U and V such that $x \in U$, $y \in V$ and $U \cap V = \varnothing$.*

From now on, unless it is stated otherwise, the spherically homogeneous trees will be considered as locally finite.

Lemma 2.36. *The topological space ∂T_Ω introduced by the metric ρ is compact and Hausdorff.*

Proof. First we will show ∂T_Ω is totally bounded. Let $\epsilon > 0$ be given. Choose the level k on the tree to be $\lceil \frac{1}{\epsilon} - 1 \rceil$ where $\lceil \frac{1}{\epsilon} - 1 \rceil$ denotes the integer part of $\frac{1}{\epsilon} - 1$. (If $\lceil \frac{1}{\epsilon} - 1 \rceil$ is negative or 0 then choose $k = 1$). If there exists m vertices on level k, set the vertices of it as $\{v_1, v_2, \ldots, v_m\}$ and consider the union of the balls P_{kv_i} for all v_i.

Notice that, an arbitrary ball $B(\gamma, \epsilon)$, with radius ϵ centered at γ equals to one of the balls P_{kv_i} of level k where $v_i \in \gamma$.

Now, any end γ in ∂T_Ω must pass through one of the vertices $\{v_1, \dots, v_m\}$, say it passes through v_j then $\gamma \in P_{kv_j}$ so the union of the balls will give the space itself. Hence ∂T_Ω is totally bounded.

To see the compactness, it suffices to show ∂T_Ω is complete.

Let $\{\gamma_i\}_{i \in I}$ be a cauchy sequence in ∂T_Ω, i.e. for all ϵ there exists an integer $N_\epsilon > 0$ such that for all $m, n > N_\epsilon$ the distance $\rho(\gamma_m, \gamma_n) < \epsilon$ so the common parts of the ends are getting bigger and common parts of ends are the finite rooted paths.

For all $\epsilon > 0$ denote the common paths of the sequence depending on N_ϵ by σ_ϵ. When ϵ gets smaller N_ϵ gets bigger so whenever $N_\epsilon < N_{\epsilon'}$, σ_ϵ will be contained in the path $\sigma_{\epsilon'}$ (we will show it by $\sigma_\epsilon < \sigma_{\epsilon'}$). Hence, when N_ϵ tends to infinity we get an infinite sequence of paths

$$\sigma_\epsilon < \sigma_{\epsilon'} < \sigma_{\epsilon''} < \dots$$

which converges to an infinite path σ consisting of all the union of paths σ_ϵ for all $\epsilon > 0$.

If we denote the vertex of any end γ on the level n by $\gamma(n)$ then the limit of the sequence is the end γ where $\gamma(n) = \lim_{i \to \infty} \gamma_i(n)$ which is an element of ∂T_Ω. Hence ∂T_Ω is compact.

It is easy to observe that ∂T_Ω is Hausdorff. For any two ends $\gamma_1, \gamma_2 \in \partial T_\Omega$, $\gamma_1 \neq \gamma_2$ let n be the last level that γ_1, γ_2 has common parts, i.e. after the level n the paths do not have common parts. Then choose $v_1 \in \gamma_1$ and $v_2 \in \gamma_2$ where v_1, v_2 are in the level $n + 1$. Consider the open balls P_{n+1v_1} and P_{n+1v_2} which has an empty intersection and $\gamma_i \in P_{n+1v_i}$, for $i = 1, 2$. Therefore ∂T_Ω is Hausdorff. $\square$

CHAPTER 3

HIERARCHOMORPHISMS

In this chapter, we will introduce the concept of hierarchomorphisms. Hierarchomorphisms are the break point of our tool which will help us seeing the homogeneous symmetric groups, $S(\xi)$, as a subgroup of homeomorphism group of the boundary of a spherically homogeneous tree.

3.1 Large group of hierarchomorphisms

The following definitions and facts, about hierarchomorphisms are due to Neretin's works; [15], [16]. Since for now, we are dealing with locally finite trees, let us assume T is a locally finite tree. A tree T is called **homogeneous** if all vertices have the same degree.

Definition 3.1. *Let T be a locally finite homogeneous rooted tree. Consider a partition of T into finitely many pairwise disjoint subtrees S_i's for $1 \leqslant i \leqslant k$ such that $V(T) = \bigcup_{i=1}^{k} V(S_i)$ Consider collection of elements, g_i, of $Aut(T)$ for all $i = 1, 2, \ldots, k$ satisfying;*

1) $\bigcup_{i=1}^{k} V(g_i(S_i)) = V(T)$.

2) The subtrees $g_i(S_i)$ are pairwise disjoint for all $i = 1, 2, \ldots, k$.

*Thus, the collection of automorphisms defines a bijection $g : V(T) \longrightarrow V(T)$ via $g(a) = g_i(a)$ if $a \in V(S_i)$. Such an element g is called **hierarchomorphism**. Each hierarchomorphism g is defined by $g = \{g_i, S_i\}$, where g_i is a map and S_i's are subtrees of T satisfying the above conditions.*

Lemma 3.2. *[16, Lemma 3.1] All hierarchomorphisms of the tree T tree forms a group.*

Proof. For any hierarchomorphism $g = \{g_i, S_i\}$ the map $g^{-1} = \{g_i^{-1}, g_i(S_i)\}$ is the inverse of g as by definition $g_i(S_i)$'s are mutually disjoint and $\bigcup_{i=1}^{k} V(g_i(S_i)) = V(T)$ and $gg^{-1}(a) = g_i g_i^{-1}(a) = a$ whenever $a \in V(g_i^{-1}(S_i))$.

On the other hand, let $g = \{g_i, S_i\}$ and $h = \{h_j, R_j\}$ be two hierarchomorphism. Then since $g_i(S_i)$ is a subtree, it lies in the finitely many union of R_j's. Considering $\bigcup g_i^{-1}(R_k) \cap S_i$ for all (i, j) we have $hg = \{h_j g_i, g_i^{-1}(R_j) \cap S_i\}$ is a hierarchomorphism. □

The group of all hierarchomorphisms is called **large hierarchomorphism** group of T , see [16, Page 513].

3.2 Spherical hierarchomorphisms

In an infinite rooted tree the concept is much more useful if we define a subgroup of large hierarchomorphism group, namely, spherical hierarchomorphisms.

Definition 3.3. *[12, Page 35] A map u of the vertex set $V(T)$ is a* ***spherical hierarchomorphism****, if u is a bijection permuting the vertices of level n, for some $n \in \mathbb{N}$, and preserves the incidence relation between vertices from the levels of numbers, greater than or equal to n.*

If u is a spherical hierarchomorphism, then for some $n \in \mathbb{N}$, the restriction $u_{|V_n}$ will be a permutation, σ_n, of vertices of level n. Name the vertices as $v_1, v_2, \dots v_{m_n}$. If $u(v_i) = v_j$, then as u preserves the incidence relation between all vertices with level number greater than or equal to n, the subtree T_{v_i} must be sent to the subtree T_{v_j}.

Hence, u can be written as $(\alpha_1, \alpha_2, \dots, \alpha_{m_n})\sigma_n$, where α_i is an automorphism of the subtree T_{v_i} for all $i = 1, 2, \dots m_n$ and σ_n is the permutation of the subtrees $T_{v_1}, T_{v_2}, \dots, T_{v_{m_n}}$.

Note that u is not uniquely written but we can choose a minimal level n making such decomposition possible. A spherical hierarchomorphism is an automorphism of the tree if and only if $n = 0$.

Obviously, the set of all spherical hierarchomorphisms forms a group. For readers convenience in the next lemma we will give the proof.

Lemma 3.4. *All spherical hierarchomorphisms of a spherically homogeneous rooted tree T forms a group and denoted by $LHier_0(T)$.*

Proof. Take any two spherical hierarchomorphisms $u_1, u_2 \in LHier_0(T)$ and let $u_1 = (\alpha_1, \alpha_2, \ldots, \alpha_{m_k})\sigma_k$ and $u_2 = (\beta_1, \beta_2, \ldots, \beta_{m_t})\sigma_t$ where k and t is the minimal level assigned for u_1 and u_2, respectively.

Define $u_1^{-1} = (\alpha_1^{-1}, \alpha_2^{-1}, \ldots, \alpha_{m_k}^{-1})\sigma_k^{-1}$ as α_i, σ_k are bijections, they have inverses and u_1^{-1} is a spherical hierarchomorphism as it permutes the vertices on level k and as all α_i's are automorphisms u_1^{-1} preserves the incidence relation between vertices of level greater than or equal to k.

As for the product of two spherical hierarchomorphisms, define the product u_1u_2 as follows;

Without loss of generality say $t > k$ then u_1u_2 is a bijection that permutes the vertices in the level t and preserves incidence relations between the vertices for all levels greater than or equal to t and $u_1u_2 = (\gamma_1, \gamma_2, \ldots, \gamma_{m_t})\sigma_t'$ where $\gamma_i = \alpha_j|_{T_{v_i}}\beta_i$ and j corresponds to the vertex which lies above the vertex v_i for all $i = 1, 2, \ldots, m_t$ and $\sigma_t' = \sigma_k'\sigma_t$. Notice that, σ_k induces a new permutation σ_k' on level t where the restriction of σ_k' to the level k equals to σ_k. □

Definition 3.5. *[12, Page 35] If for a spherical hierarchomorphism $u = (\alpha_1, \alpha_2, \ldots, \alpha_{m_k})\sigma_k$ all α_i's are the identity automorphisms, then u is called* ***finite****. All finite spherical hierarchomorphisms are denoted by $LHier_{0f}(T)$.*

Obviously, $LHier_{0f}(T)$ is a subgroup of $LHier_0(T)$. If $u_1 = (e, e, \ldots, e)\sigma_k$ and $u_2 = (e, e, \ldots, e)\sigma_t$ where each e refers to the identity automorphism of the corresponding subtree, then $u_1u_2^{-1} = (e, e, \ldots, e)\sigma_k'\sigma_t^{-1}$ is an element of $LHier_{0f}(T)$.

The definition of spherical hierarchomorphisms does not suggest that the structure of the tree must protected. Moreover, an arbitrary spherical hierarchomorphism may not preserve the structure of the boundary, ∂T_Ω, of the tree. However, we are interested in the spherical hierarchomorphisms that has an action on the boundary.

Definition 3.6. *[12, Page 35] 1) The group of transformations of the boundary of T induced by $LHier_0(T)$ is called* ***small spherical hierarchomorphisms*** *and denoted by $LHier(T)$.*

2) The group of transformations of the boundary of T induced by $LHier_{0f}(T)$ is called ***small finite spherical hierarchomorphisms*** *and denoted by $LHier_f(T)$.*

For any small spherical hierarchomorphism, u, we have a level t such that u permutes the vertices of level t. To see the action of u on the boundary, let $\gamma = (\varnothing, a_1, a_2, \ldots)$ be an end, where a_i is a vertex on level i. Then, if a_t is sent to b_t by u, then the path joining the root with a_t is sent to the path joining the root with a_t. Since u acts as an automorphism on each subtree of level t this action will preserve the end structure of γ.

3.3 Local isometries and the group H_Ω

In this section, we will make the connection between the homogeneous symmetric groups, $S(\xi)$ and the group H_Ω where H_Ω is a natural subgroup of the group of local isometries of a spherically homogeneous tree.

Let X be a metric space with the metric ρ.

Definition 3.7. *A bijective map on X is called an* ***isometry*** *if it preserves the distance between elements.*

Definition 3.8. *[12, Section 3.1, Definition 6] Let $\beta : X \longrightarrow X$ be a bijection satisfying the following;*

$\forall x \in X$ there exists a neighborhood V_x of x such that for every $x_1, x_2 \in V_x$ the equality

$$\rho(x_1^\beta, x_2^\beta) = \rho(x_1, x_2)$$

*holds. Then β is called a **local isometry**.*

Definition 3.9. *[12, Section 3.1, Definition 7] For a bijection $\beta : X \longrightarrow X$ if there exists $\delta > 0$ such that*

$$\rho(x_1^\beta, x_2^\beta) = \rho(x_1, x_2)$$

*holds for all $x_1, x_2 \in X$ satisfying $\rho(x_1, x_2) < \delta$, then β is called **uniformly local isometry**.*

Obviously, the two definitions are not the same. However, for a compact metric space it is easy to see that they are equivalent.

Lemma 3.10. *[12, Section 3.1, Lemma 3] If X is compact, then every local isometry is a uniform local isometry.*

For readers convenience, we will give the proof.

Proof. Let α be a local isometry. For all $x_i \in X$, consider open neighborhoods U_{x_i} such that U_{x_i}'s are the neighborhoods of x_i in the definition of local isometry. Now, $X = \bigcup_{i \in I} U_{x_i}$ is an open covering for x. By [14, lemma 27.5], as X is compact there exists $\delta > 0$ such that every subset of X of diameter less than δ is contained in some member of the cover. Hence, for all x_1, x_2 satisfying $\rho(x_1, x_2) < \delta$ we have $x_1, x_2 \in U_{x_j}$ for some $j \in I$. Since α is a local isometry and U_{x_j} is the neighborhood of x_j satisfying $\rho(x_1^\alpha, x_2^\alpha) = \rho(x_1, x_2)$, we have that α is a uniform local isometry. □

Lemma 3.11. *Any local isometry of a metric space X is a homeomorphism.*

Proof. Let g be a local isometry, we need to show that g and g^{-1} is continuous.

For $x \in X$ let $\epsilon > 0$ be given. We must find a $\delta > 0$ such that whenever $\rho(x, y) < \delta$, $\rho(x^g, y^g) < \epsilon$ holds.

Now, as g is local isometry for $x \in X$ there exists a δ_1 such that $\rho(x, y) = \rho(x^g, y^g)$ for all y satisfying $\rho(x, y) < \delta_1$. If $\epsilon < \delta_1$, choose $\delta = \epsilon$ so that whenever $\rho(x, y) < \delta = \epsilon < \delta_1$, we have $\rho(x, y) = \rho(x^g, y^g) < \epsilon$.

On the other hand, if $\epsilon > \delta_1$, choose $\delta = \delta_1$, then whenever $\rho(x,y) < \delta = \delta_1$, we have $\rho(x^g, y^g) = \rho(x,y) < \delta_1 < \epsilon$. Hence, g is continuous at x. As x is arbitrary g is continuous everywhere.

Since g^{-1} is also a local isometry it is also continuous. Therefore, g is a homeomorphism. □

Let us turn our attention to the boundary, ∂T_Ω, of a tree T_Ω. The isometries of ∂T_Ω consists of bijective maps from ∂T_Ω onto itself preserving the distance between elements of ∂T_Ω which we will denote by $Isom(\partial T_\Omega)$. We have the following lemma about $Isom(\partial T_\Omega)$.

Lemma 3.12. *The automorphism group of T_Ω coincides with the isometry group of ∂T_Ω.*

Proof. Let $\sigma \in Isom(\partial T_\Omega)$. Then for any two ends γ_1, γ_2 we know $\rho(\gamma_1, \gamma_2) = \rho(\sigma(\gamma_1), \sigma(\gamma_2))$ where ρ is the metric defined in the Subsection 2.2.2 on page 18. Hence, if γ_1 and γ_2 have n common parts so does $\sigma(\gamma_1)$ and $\sigma(\gamma_2)$.

Claim: σ induce a map from $V(T)$ to itself and it preserves incidence relation between vertices.

As σ sends ends to ends and ends consists of sequences of vertices starting with the root, if $\gamma_1 = (v_0, v_1, \ldots)$, then $\sigma(\gamma_1)$ is an end consisting of the vertices $(\sigma(v_0), \sigma(v_1), \ldots)$. Moreover, as σ preserves the distance, it preserves the incidence relation between vertices. On the other hand clearly, any automorphism of T_Ω induces an isometry of ∂T_Ω. □

The following definition will lead us to the subgroups H_Ω of the homeomorphism group, $Hom(\partial T_\Omega)$ of a spherically homogeneous tree T_Ω which will turn out to be the subgroup of local isometries.

Definition 3.13. *Let T_Ω be a spherically homogeneous rooted tree where $\Omega = (a_0, a_1, \ldots)$. Define $\mathbf{H_n}$ to be the subset of $Hom(\partial T_\Omega)$ which only permutes the balls P_{nv_i} of level n, that is if we think about the vertex labeling given in the Figure 2.6, an element* *in H_n can only change the first n coordinates of vertices $(i_0, i_1, \ldots, i_m)$ and do not change the coordinates i_k for all $k \geqslant n$.*

To understand the action of H_n on the ∂T_Ω we can look at the balls P_{nv_i} as the rooted subtree, T_{v_i}, of T_Ω of level n. Since we have a spherically homogeneous tree, the subtrees T_{v_i}'s of level n are also spherically homogeneous trees with characteristic sequence $\Omega_n = (a_n, a_{n+1}, \ldots)$. Moreover, from the labeling of the trees explained in the Section 2.2.1 with an example 2.6, it is easy to see that two spherically homogeneous trees with the same characteristic sequence is isomorphic with a natural correspondence of the vertices.

Hence, an element of H_n which sends T_{v_i} to T_{v_j} is just a permutation of the balls of level n that cuts the ball P_{nv_i} and glues it on the ball P_{nv_j}.

Lemma 3.14. *H_n is a subgroup of $Hom(\partial T_\Omega)$*

Proof. Let σ be an element of H_n. Then σ can change the coordinates of the ends for only n many coordinates. If $\gamma_1 = (i_0, i_1 \ldots), \gamma_2 = (j_0, j_1, \ldots)$ are two ends such that $\sigma(\gamma_1) = \sigma(\gamma_2)$, then the coordinates $i_k = j_k$ for all $k \geqslant n$ as σ does not change that coordinates. Moreover, σ restricted to level n is a permutation on the vertices of level n hence $i_k = j_k$ for all $k < n$. Clearly, for any $\gamma = (i_0, i_1, \ldots, i_n, \ldots)$ as σ sends the vertex in the level n labeled as $(i_0, \ldots, i_{n-1})$ to the vertex $(j_0, \ldots, j_{n-1})$ there exists $\gamma' = (j_0, \ldots, j_{n-1}, i_n, i_{n+1}, \ldots)$ such that $\sigma(\gamma') = \gamma$. Therefore, σ is a bijective map.

Since any element of H_n sends basis elements, P_{nv_i}, to basis elements and bijective, they are homeomorphisms.

On the other hand, σ^{-1} is also a homeomorphism only permuting the balls of level n and if σ' is another element in H_n, then the product $\sigma'\sigma^{-1}$ will also lie in H_n. □

If we define $f_\Omega(n) = a_0 a_1 \ldots a_{n-1}$, then clearly we have $f_\Omega(n)$ many vertices on the level n and H_n will be isomorphic to the symmetric group, $S_{f_\Omega(n)}$.

Lemma 3.15. *H_n is a subgroup of H_k for all $k \geqslant n$.*

Proof. By definition of H_n, an element in H_n permutes the balls of level n while an element in H_k permutes the balls of level k by just cutting and gluing the balls. Note that an element in H_n sending P_{nv_i} to P_{nv_j} also sends the balls lying below the vertex v_i on level k to the balls lying below the vertex v_j on level k. Hence, H_n lies inside H_k. □

Since the groups, H_n, lies inside each other, define the union of these groups and denote it by $H_\Omega := \bigcup_{n_1}^{\infty} H_n$. Now, in the following lemma we will give the link between H_Ω and the local isometry group $LI(\partial T_\Omega)$ of ∂T_Ω.

Lemma 3.16. *[12, Section 3, Lemma 4] Let g be a local isometry of ∂T_Ω. Then there exist $\alpha \in Aut(T_\Omega)$ and $\beta \in H_\Omega$ such that $g = \alpha\beta$.*

Proof. We know by Lemma 2.36, that ∂T_Ω is compact and by Lemma 3.11, a local isometry in the metric space is a homeomorphism. Moreover, by Lemma 3.10, g is also a uniform local isometry. Hence, choose $\delta > 0$ such that for any γ_1, γ_2 satisfying $\rho(\gamma_1, \gamma_2) < \delta$

$$\rho(\gamma_1^g, \gamma_2^g) = \rho(\gamma_1, \gamma_2)$$

Now, for this δ there exists an integer n such that for arbitrary ends $\gamma_1, \gamma_2 \in P_{nv_i}$ where P_{nv_i} is the ball consisting of ends passing through the vertex v_i on the level n, we have $\rho(\gamma_1, \gamma_2) = \frac{1}{n+1} < \delta$. Hence, g preserves the distance in P_{nv_i} so g sends P_{nv_i} to another ball P_{nv_j} of the level n.

Let $\beta \in H_\Omega$ be a homeomorphism which acts on V_n in the same way as g does. Then $\alpha = g\beta^{-1}$ acts trivially on V_n. So α act as an isometry on each ball of level n. As g and β acts as a permutation on the vertices of level n and they send ends to ends the action of α on the levels less then n is trivial. Hence, α is an isometry on ∂T_Ω and as we know $Aut(T_\Omega) = Isom(\partial T_\Omega)$, the isometry α will be an element of $Aut(T_\Omega)$ as required.

On the other hand, obviously $Aut(T_\Omega) < LI(\partial T_\Omega)$ and $H_\Omega < LI(\partial T_\Omega)$. □

An important property of $LI(\partial T_\Omega)$ is that all finitely generated subgroups of $LI(\partial T_\Omega)$ are residually finite. A group G is called **residually finite** if the intersection of all normal subgroups of finite index is trivial.

Proposition 3.17. *[12, Section 3, Proposition 7] All finitely generated subgroups in the group $LI(\partial T_\Omega)$ are residually finite.*

Proof. Let G be any finitely generated subgroup of $LI(\partial T_\Omega)$, and $\Omega = (a_1, a_2, \ldots)$. Then there exists a number $\delta > 0$ such that for all $g \in G$

$$\rho(x^g, y^g) = \rho(x, y) \text{ for all } x, y \in T_\Omega \text{ satisfying } \rho(x, y) < \delta$$

Choose $n = \lfloor 1/\delta \rfloor$, write $a'_1 = a_1 a_2 \ldots a_n$, $a'_k = a_{n+k-1}$ for all $k \geqslant 2$ and consider the group $Aut(T_{\Omega'})$ where $\Omega' = (a'_1, a'_2, \ldots,)$. After level n, as G preserves the distance it must preserve the incidence relation between the vertices so it acts as an automorphism after the level n. Since the tree $T_{\Omega'}$ has the same subtrees with the subtrees of T_Ω after the level n we can embed G into the group $Aut(T_{\Omega'})$. Hence G is isomorphic to a subgroup of $Aut(T_{\Omega'})$, which is residually finite by [3, Prop 3.5]. □

The connection between H_Ω and small spherical (finite) hierarchomorphisms given in the Definition 3.6 will be given in the following theorem.

Theorem 3.18. *[12, Section 3, Theorem 6] 1. The group $LHier(T_\Omega)$ is isomorphic to the group $LI(\partial T_\Omega)$.*

2. The group $LHier_f(T_\Omega)$ is isomorphic to the group H_Ω.

Proof. We know any element of $LHier(T_\Omega)$ can be written as $u = (\alpha_1, \alpha_2, \ldots, \alpha_{m_k})\sigma_k$ for some level k.

Obviously, $k = 0$ will induce an automorphism of the tree so $Aut(T_\Omega)$ is isomorphic to a subgroup of $LHier(T_\Omega)$ and by definition of H_Ω, $H_\Omega < LHier(T_\Omega)$. Hence, by Lemma 3.16, $LI(\partial T_\Omega)$ is isomorphic to a subgroup of $LHier(T_\Omega)$. On the other hand, any element $u \in LHier(T_\Omega)$ can be written as the product $(\alpha_1, \alpha_2, \ldots, \alpha_{m_k})\sigma_k$ where σ_k induces an element of $H_k \leqslant H_\Omega$ and the sequence of automorphisms α_i induces an automorphism α in an obvious manner; it fixes every vertex in the levels less than or equal to k and acts as α_i's for other vertices. Hence, $LHier(T_\Omega)$ is isomorphic to $LI(\partial T_\Omega)$.

The second statement of the theorem is trivial as any element u of $LHier_f(T_\Omega)$ can be written $u = (e, e, \ldots, e)\sigma_k$ which will induce an element of H_Ω and vice versa. □

CHAPTER 4

AUTOMORPHISM GROUP OF $S(\xi)$

In this chapter, we will give properties of $Aut(S(\xi))$ by giving the connection between the homogeneous symmetric groups, $S(\xi)$ and the group H_Ω where H_Ω is the natural subgroup of the group of local isometries of a spherically homogeneous tree defined in Section 3.3.

4.1 $S(\xi)$ as a subgroup of homeomorphism group of ∂T_Ω

Recall from Definition 3.13 that H_n is the subgroup of $Hom(\partial T_\Omega)$ which is isomorphic to $S_{f_\Omega(n)}$ where $f_\Omega(n) = a_0 a_1 \ldots a_{n-1}$ for all $n \in \mathbb{N}$ where $\Omega = (a_0, a_1, \ldots)$.

Considering the labeling in the spherically homogeneous tree T_Ω, see the Figure 2.6, we can write the vertices of level n for any $n \in \mathbb{N}$ in the following way;

$$V_n = \{(i_0, i_1, \ldots, i_{n-1})\ | i_j \in X_j = \{0, 1, 2, \ldots a_j - 1\} \text{ for all j} \in \{0, 1, \ldots, n-1\}\}$$

By definition, H_n acts as a permutation group on V_n.

Lemma 4.1. *For $n \leqslant k$, (H_n, V_n) is embedded into (H_k, V_k) via strictly diagonal embedding.*

Proof. Note that by Definition 3.13, H_n's are subgroups of the homeomorphism group $Hom(\partial T_\Omega)$ and H_n can change only the first n components of the vertices, that is any element in H_n fixes the components of the vertices after i_{n-1} and H_k fixes the components of the vertices after i_{k-1} so an element that can only change first n components will be an element of H_k for any $k \geqslant n$. Hence, $H_n \leqslant H_k$.

Now, H_n as a subgroup of H_k has an action on V_k. Let $(b_0, b_1, \dots b_{k-1})$ be an element in V_k. The orbit of the action of H_n containing the vertex $(b_0, b_1, \dots b_{k-1})$ is of the form

$$\Delta = \{(i_0, i_1, \dots, i_{n-1}, b_n, b_{n+1}, \dots, b_{k-1}) \mid i_j \in X_j \text{ for all j} \in \{0, 1, \dots n-1\}\}$$

where b_j's are fixed coming from the j^{th} component of $(b_0, b_1, \dots b_{k-1})$.

Define the map $\lambda : V_n \to \Delta$ sending any element $(i_0, i_1, \dots, i_{n-1}) \in V_n$ to $\lambda((i_0, i_1, \dots, i_{n-1})) = (i_0, i_1, \dots, i_{n-1}, b_n, \dots, b_{k-1})$. Obviously, λ is a bijection and λ will induce a permutational isomorphism between the permutation groups (H_n, V_n) and (H_n, Δ) by the following;

Let $\sigma \in H_n$ and $\gamma = (i_0, i_1, \dots, i_{n-1}) \in V_n$. Then since σ acts as a permutation on the set of vertices of level n, $\gamma^\sigma = \delta = (j_0, j_1, \dots, j_{n-1})$ for some vertex $\delta \in V_n$. Now, we have $\lambda(\gamma^\sigma) = \lambda((j_0, j_1, \dots, j_{n-1})) = (j_0, j_1, \dots, j_{n-1}, b_n, b_{n+1}, \dots, b_{k-1})$. On the other hand, we know H_n is a subgroup of H_k so, $\lambda((i_0, i_1, \dots, i_{n-1})^\sigma = (i_0, i_1, \dots, i_{n-1}, b_n, b_{n+1}, \dots b_{k-1})^\sigma = (j_0, j_1, \dots, j_{n-1}, b_n, b_{n+1}, \dots, b_{k-1})$. Hence, the embedding is a strictly diagonal embedding. □

Now, by the results of [9, Theorem 3, (iii)], the group H_Ω which is the union of the subgroups H_n which are isomorphic to $S_{f_\Omega(n)}$, is isomorphic to $S(\Omega)$ where $char(S(\Omega)) = char(\Omega)$.

Similarly, we can define AH_Ω which is the union of subgroups AH_n of H_n which are isomorphic to $Alt_{f_\Omega(n)}$.

4.2 Vertex labeling with respect to the action of $S(\Omega)$

Since H_Ω is isomorphic to $S(\Omega)$ by the above explanation and H_Ω is the union of the symmetric groups acting on the vertices, we can label the vertices so that the corresponding permutations will make sense in the group $S(\Omega)$. We label the vertices of level n in the following way;

Let $\Omega = (a_0, a_1, \dots)$. An arbitrary vertex in level n is of the form $(i_0, i_1, \dots, i_{n-1})$

where $i_k \in X_k$ for all $k \geqslant 0$. Assign a number j to this vertex by setting

$$j = (i_0 + 1) + i_1 f_\Omega(1) + \ldots + i_{n-1} f_\Omega(n-1)$$

On the level n, we have $f_\Omega(n)$ many vertices and with this labeling shown in the following Figure 4.1, we label all the vertices from 1 up to the number $f_\Omega(n)$, respecting the action $S(f_\Omega(n))$.

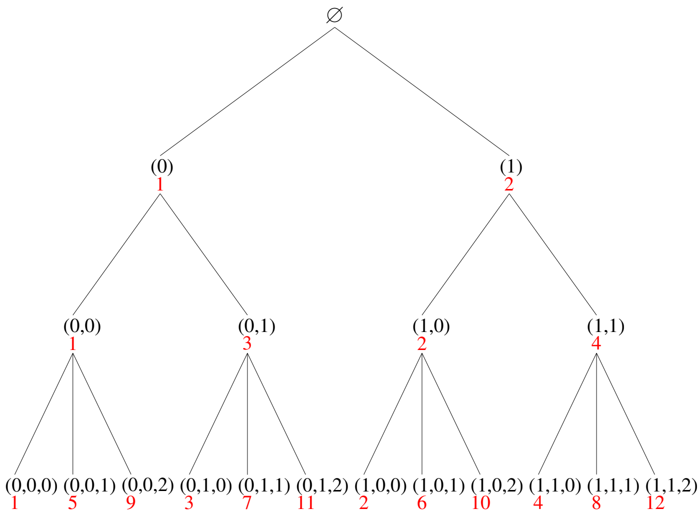

Figure 4.1: Tree with vertices labeled by natural numbers

The following example will give the clue for the readers to understand how $S(\Omega)$ acts on the tree.

Example 4.2. *Given the above tree Figure 4.1, the element of H_Ω in the first level that interchanges the ball $P_{1(0)}$ with $P_{1(1)}$ will act in the second level diagonally and it will be the element sending $P_{2(0,0)}$ to $P_{2(1,0)}$ and $P_{2(0,1)}$ to $P_{2(1,1)}$. In the third level the element will send*

$$P_{3(0,0,0)} \leftrightarrow P_{3(1,0,0)}$$

$$P_{3(0,0,1)} \leftrightarrow P_{3(1,0,1)}$$

$$P_{3(0,0,2)} \leftrightarrow P_{3(1,0,2)}$$

$$P_{3(0,1,0)} \leftrightarrow P_{3(1,1,0)}$$

$$P_{3(0,1,1)} \leftrightarrow P_{3(1,1,1)}$$

$$P_{3(0,1,2)} \leftrightarrow P_{3(1,1,2)}$$

In the other labeling this is to say that in the first level, the element corresponds to the cycle $(1,2)$ *and in the second level, it corresponds to* $(1,2)(3,4)$. *In the third level, the element will be* $(1,2)(3,4)(5,6)(7,8)(9,10)(11,12)$. *Continuing like this we will see the element of* H_Ω *in the form of an element of* $S(\Omega)$.

4.3 Automorphism group of $S(\Omega)$

Let X be a locally compact, Hausdorff space. Denote the group of all homeomorphisms of X by $HomX$. If G is a subgroup of $HomX$, then we will use the following useful theorem from analysis. The theorem which is a work of Rubin [18] will be used often and called as Rubin's theorem.

Theorem 4.3. *[18, Corollary 3.5] Consider a locally compact Hausdorff space,* X. *Let* G_1, G_2 *be subgroups of* $HomX$ *and assume for every open* $\mathcal{O} \subseteq X$, $x \in \mathcal{O}$ *and* $i = 1,2$ *the set* $\{g(x) \mid g \in G_i \text{ and } g|_{X \setminus \mathcal{O}} = identity\}$ *be somewhere dense. Then any isomorphism from* G_1 *to* G_2 *is induced by an element* $h \in HomX$ *via conjugation.*

Lemma 4.4. *[12, Section 2.3, Remark 1] The groups* H_Ω *and* AH_Ω *satisfy the conditions of Rubin's theorem 4.3.*

Proof. The space ∂T_Ω is compact and Hausdorff by Lemma 2.36 so it is locally compact. Let D be any open set in ∂T_Ω and $\gamma \in D$ set

$$A = \{g(\gamma) \mid g \in H_\Omega \text{ and } g|_{\partial T_\Omega \setminus D} = identity\}$$

We will show that A is somewhere dense, that is the interior of the closure of A is nonempty. First of all, g moves γ to the elements of D only as $g|_{\partial T_\Omega \setminus D} = identity$. Therefore, $A \subseteq D$. Claim, $D \subseteq \bar{A}$. Let $D = \bigcup P_{k_i l_i}$. Then choose $\gamma' \in D$ which is not an image of γ under any g. (Obviously any $\gamma' \in A$ lies inside $\bar{A}$). We will show for any neighborhood U of γ', $A \cap U \neq \varnothing$. Any neighborhood of γ' is of the form $P_{k_i l_i}$ for some k_i, l_i and in any case $P_{k_i l_i} \cap A \neq \varnothing$ as $P_{k_i l_i}$ contains an end which has infinitely many common components with γ. Hence, $D \subseteq \bar{A}$ so the interior $int(\bar{A})$ of $\bar{A}$ is nonempty. Similarly, one can show that AH_Ω satisfies the conditions of the theorem. □

Therefore, this theorem says that any automorphism from $AutH_\Omega$ $(AutAH_\Omega)$ is induced by an element of $Hom(\partial T_\Omega)$. Moreover, we have the following result.

Theorem 4.5. *[12, Section 3, Theorem 9] $AutH_\Omega \cong N_{Hom(\partial T_\Omega)}(H_\Omega)$ where $N_{Hom(\partial T_\Omega)}(H_\Omega)$ is the normalizer of H_Ω in $Hom(\partial T_\Omega)$.*

Proof. Let $N_{Hom(\partial T_\Omega)}(H_\Omega) = N$, define a map $\psi : N \to Aut(H_\Omega)$. As any element $h \in N$ satisfies $H_\Omega^h = H_\Omega$, the element h induces an automorphism of H_Ω via conjugation.

ψ is a homomorphism : For any $h_1, h_2 \in N$, $g \in H_\Omega$

$$(g)\psi(h_1h_2) = g^{h_1h_2} = (g^{h_1})^{h_2} = g^{h_1}\psi(h_2) = (g)\psi(h_1)\psi(h_2)$$

ψ is onto : By Rubin's theorem for any automorphism of H_Ω, there exists an element $h \in Hom(\partial T_\Omega)$ inducing the map and so $h \in N$.

ψ is 1-1 :

$$Ker(\psi) = \{h \in N \mid g^h = g \text{ for all } g \in H_\Omega\}$$

If $1 \neq h \in Ker(\psi)$, then there exists an end γ which h does not fix, say $h(\gamma) = \gamma_1 \neq \gamma$. Since H_Ω acts on the boundary, we can find an element g sending γ_1 to $\delta \neq \gamma_1$, and fixing γ. Then since $g^h = g$, we have

$$\gamma = g(\gamma) = h^{-1}gh(\gamma) = h^{-1}g(\gamma_1) = h^{-1}(\delta)$$

But $h^{-1}(\gamma_1) = \gamma$ and h is a bijective map gives a contradiction. □

In the next proposition, by using the result of Rubin's theorem, one can show that the group of automorphisms of H_Ω, hence $Aut(S(\Omega))$ is locally inner. An automorphism α of a group G is **locally inner** if for every finitely generated subgroup H of G, there exist an element $g \in G$ satisfying $h^\alpha = h^g$ for all $h \in H$.

Proposition 4.6. *[12, Section 3, Prop. 10] Let $\alpha \in Aut(H_\Omega)$ $(Aut(AH_\Omega))$. Then if $\alpha(H_n) < H_k$ $(\alpha(AH_n) < AH_k)$ for $1 < n \leqslant k$, then α is induced by an inner automorphism of H_k.*

Proof. Let $\alpha \in Aut(H_\Omega)$ be given such that $\alpha(H_n) \leqslant H_k$. By Rubin's theorem α is induced by a homeomorphism σ of ∂T_Ω.

As σ is a homeomorphism, it sends an open ball to the union of open balls. $\sigma(P_{ni}) = \bigcup_{j=1}^{\infty} P_{k_{i_j} i_j}$. On the other hand, P_{ni} is compact being closed subset of a compact space. Therefore, $\sigma(P_{ni})$ is compact and there exists a finite covering $\bigcup_{j=1}^{r} P_{k_{i_{s_j}} i_{s_j}}$ of $\sigma(P_{ni})$ choose $m_i = max\{k_{i_{s_j}} | 1 \leqslant j \leqslant r\}$. And we can choose m_i for all $1 \leqslant i \leqslant f_\Omega(n)$. Now set $k = max\{m_i \mid 1 \leqslant i \leqslant f_\Omega(n)\}$ so that $\sigma(P_{ni}) = \bigcup_{j=1}^{r(i)} P_{kl_{ij}}$ for all $1 \leqslant i \leqslant f_\Omega(n)$.

<u>Claim:</u> For $i \neq j$, $\sigma(P_{ni})$ and $\sigma(P_{nj})$ do not intersect.

<u>Proof of Claim:</u> If $\sigma(P_{ni}) \cap \sigma(P_{nj}) \neq \varnothing$, then there exist ends $\gamma_i \in P_{ni}$ and $\gamma_j \in P_{nj}$ such that $\sigma(\gamma_i) = \sigma(\gamma_j)$. But σ is an homeomorphism of ∂T_Ω and it is one to one implying that $\gamma_i = \gamma_j$. Since γ_i and γ_j lies in the different balls of the same level and the balls in the same level do not intersect, we get the result.

<u>Claim:</u> $r(i)$ does not depend on i.

<u>Proof of Claim:</u> For an arbitrary $g \in H_\Omega$ consider

$$g^\sigma \sigma(P_{ni}) = \bigcup_{j=1}^{r(i)} g^\sigma(P_{kl_{ij}})$$

Now we will have;

$$g^\sigma \sigma(P_{ni}) = \sigma g \sigma^{-1}(\sigma(P_{n_i})) = \sigma g(P_{ni}) = \bigcup_{j=1}^{r(i)} g^\sigma(P_{kl_{ij}})$$

as H_n is transitive we can choose g so that $g(P_{ni}) = P_{nj}$ where $i \neq j$. Hence,

$$\sigma(P_{nj}) = \sigma g(P_{ni}) = \bigcup_{j=1}^{r(i)} g^\sigma(P_{kl_{ij}})$$

Observe that g^σ is an element of H_k and it sends balls to the balls of the same level. So right hand side is a union of $r(i)$ many balls of level k, and left hand side is $\sigma(P_{nj})$ which is the union of $r(j)$ many balls of level k. Hence $r(i) = r(j) = r$ where $r = \frac{f_\Omega(k)}{f_\Omega(n)}$.

Now, on the level k, σ takes r many balls lying under the vertex i on level n and sends them to r many balls $P_{kl_{ij}}$ for all $1 \leqslant i \leqslant f_\Omega(n)$. Since $\sigma(P_{ni})$'s are all disjoint union of the balls of level k and $H_k \cong S_{f_\Omega(k)}$ is $f_\Omega(k)-$transitive, there exist a $\delta \in H_k$ such that $\sigma\delta^{-1}$ acts trivially on the balls of level k.

Observe that for any $g \in H_n$ $\sigma\delta^{-1}$ centralizes g;

$$g^{\sigma\delta^{-1}}(P_{ks}) = \sigma\delta^{-1}g\delta\sigma^{-1}(P_{ks}) = g(P_{ks})$$

Therefore, $g^\sigma = g^\delta$ for any $g \in H_n$. □

In fact, we can embed any countable residually finite group into the group $Aut(H_\Omega)$. For this purpose, we will construct **level preserving automorphisms** of $Aut(H_\Omega)$. For an increasing sequence $N = \{n_i \geqslant 3 | \, i \in \mathbb{N}\}$, of positive integers define a set $\mathbf{A_N}$ of all automorphisms α of H_Ω that satisfies $\alpha(H_{n_i}) = H_{n_i}$ for all $i \in \mathbb{N}$. We will refer A_N as N-level preserving automorphisms of H_Ω.

Lemma 4.7. *A_N is a subgroup of $Aut(H_\Omega)$.*

Proof. For $\alpha_1, \alpha_2 \in A_N$ and for any $n_i \in N$, since α_1 and α_2 are automorphisms, we obviously have

$$\alpha_1\alpha_2(H_{n_i}) = H_{n_i}$$

$$\alpha_1^{-1}(H_{n_i}) = H_{n_i}$$

□

Proposition 4.8. *[12, Section 3 Prop. 12] For $N = \{n_i \geqslant 3 | \, i \in \mathbb{N}\}$, an increasing sequence of integers A_N is isomorphic to the Cartesian product of the groups $C_{H_{n_k}}(H_{n_{k-1}})$ for all $k \in \mathbb{N}$.*

Proof. First of all if we take any arbitrary element $(c_1, c_2, \ldots, c_i, \ldots)$ inside the Cartesian product of $C_{H_{n_k}}(H_{n_{k-1}})$, then the infinite product $c_1c_2\ldots$ induces an automorphism α defined as follows;

$$g^\alpha = g^{c_1c_2\cdots} \text{ for all } g \in H_\Omega$$

Note that if $g \in H_{n_i}$, then $g^{c_j} = g$ for all $j > i$. So, $g^{c_1c_2\cdots} = g^{c_1c_2\ldots c_i}$ hence, the infinite product $c_1c_2\ldots$ gives a well defined automorphism which belongs to A_N as for any $g \in H_{n_i}$ $g^\alpha = g^{c_1c_2\ldots c_i} \in H_{n_i}$. Indeed, this α is a well defined map as if we take two elements $g = h \in H_\xi$ then $g, h \in H_{n_i}$ for some i hence, $g^\alpha = g^{c_1 c_2 \ldots c_i} = h^{c_1 c_2 \ldots c_i} = h^\alpha$.

Now, take any $\alpha \in A_N$ then as the automorphism is locally inner, $\alpha|_{H_{n_i}}$ is induced by an element of H_{n_i}. If $\alpha|_{H_{n_i}} = g_i$ for some $g_i \in H_{n_i}$ and $g^\alpha = g^{g_i}$. Moreover, the groups satisfy $H_{n_i} \subseteq H_{n_{i+1}}$ so $g^{g_{i+1}} = g^{g_i}$ for all $g \in H_{n_i}$. Hence, $g_{i+1}g_i^{-1} \in C_{H_{n_{i+1}}}(H_{n_i})$. So, $g_{i+1} = c_{i+1}g_i$ for some $c_{i+1} \in C_{H_{n_{i+1}}}(H_{n_i})$. Also observe that $g_1 = c_1 \in H_{n_1} = C_{H_{n_1}}(\{id\})$.

Therefore, we may define the sequences consisting of elements of $C_{H_{n_{i+1}}}(H_{n_i})$ which are of the form $(c_1, c_2, \ldots)$ where $c_{i+1} \in C_{H_{n_{i+1}}}(H_{n_i})$. If we denote $C_{H_{n_i}}(H_{n_{i-1}}) = D_i$, then note that D_i commute pairwise. For any $c_i \in D_i$ and $c_j \in D_j$, without loss of generality if $i > j$ as c_i centralizes $H_{n_{i-1}}$ and $H_{n_j} \subseteq H_{n_{i-1}}$ the equality $c_i c_j = c_j c_i$ holds for all $i, j \in \mathbb{N}$. So all of D_i's are normal subgroups of A_N. Now, if we may prove the intersection $D_i \cap D_1 D_2 \ldots D_{i-1} D_{i+1} \ldots$ is trivial, then we are done. Take an element a_i inside the intersection $D_i \cap D_{i+1} D_{i+2} \ldots$. So we may write a_i as follows;

$$a_i = a_1 a_2 \ldots a_{i-1} a_{i+1} \ldots \quad \text{where } a_k \in D_k \text{ for all } k \in \mathbb{N}$$

as $a_i \neq 1$ there exists j such that $a_j \neq 1$. Choose j to be the smallest such integer then if $i < j$ as $a_i = a_j a_{j+1} \ldots$ and for all $g \in H_{n_{j-1}}$ we have $g a_i = g a_j a_{j+1} \ldots = a_j a_{j+1} \ldots g = a_i g$. Moreover, a_i is also an element of $H_{n_{j-1}}$ but the center of $H_{n_{j-1}}$ is trivial so a_i must be identity.

On the other hand, if $j < i$, then write the equality as $a_j^{-1} = a_{j+1} \ldots a_{i-1} a_i^{-1} a_{i+1} \ldots$ now by the above argument a_j^{-1} commutes with all elements of H_{n_k} for some k bigger than j hence this is a contradiction. □

Observe that the group H_{n_j} is embedded into $H_{n_{j+1}}$ via strictly diagonal embedding and H_{n_j} is isomorphic to the symmetric group $S_{f_\Omega(n_j)}$ where $f_\Omega(n_j) = a_0 a_1 \ldots a_{n_j - 1}$ whenever $\Omega = (a_0, a_1, \ldots)$. In this case the centralizer, $C_{H_{n_j}}(H_{n_{j+1}})$, is isomorphic to the symmetric group, $S_{\frac{f_\Omega(n_j)}{f_\Omega(n_{j-1})}}$, by [2, Page 109, Exercise 4.2.5].

Since n_j's can be chosen so that the ratio $\frac{f_\Omega(n_j)}{f_\Omega(n_{j-1})}$ is arbitrarily large, in the group $Aut(H_\Omega)$ as subgroups we have Cartesian products of finite symmetric groups of any large degree. Hence, we have the following corollary.

Corollary 4.9. *[12, Section 3.2, Cor. 3] Every residually finite group can be embedded into $Aut(H_\Omega)$*

Proof. The equivalent definition of being residually finite is that a group is residually finite if it can be embedded into a direct product of finite groups.

Note that we can embed a direct product of finite groups into the Cartesian product of finite symmetric group by simply embedding each factor of the direct product into a finite symmetric group via right regular representation.

As we can choose the increasing sequence N so that the corresponding group A_N is the Cartesian product of required finite symmetric groups, we can embed a residually finite group into A_N for some increasing sequence N. $\square$

For the groups, $C_{H_{n_k}}(H_{n_{k-1}})$, the action of an element in $C_{H_{n_k}}(H_{n_{k-1}})$ to an end can be seen in the following remark.

Remark 4.10. *Let $c_k \in C_{H_{n_k}}(H_{n_{k-1}})$ and $u = (i_0, i_1, \ldots)$ be an end. First of all since $c_k \in H_{n_k}$, it only changes the first n_k components of u and leaves the rest the same.*

Moreover, we claim that c_k does not change the first n_{k-1} components. Assume not.

Let c_k sends u to the end $c_k(u) = (j_0, j_1, \ldots, j_{n_k-1}, i_{n_k}, \ldots)$ where $j_s \neq i_s$ for some $s \in \{0, 1, \ldots, n_{k-1}-1\}$. Choose $\sigma \in H_{n_{k-1}}$ such that σ fixes the first n_{k-1} components of any end passing through the vertex $(i_0, i_1, \ldots i_{n_{k-1}-1})$ and does not fix the first n_{k-1} components of any end passing through the vertex $(j_0, j_1, \ldots j_{n_{k-1}-1})$. As c_k centralizes $H_{n_{k-1}}$, we must have $c_k^\sigma = c_k$, however;

$$\sigma^{-1}c_k\sigma(u) = \sigma^{-1}c_k\sigma(i_0, i_1, i_2, \ldots) = \sigma^{-1}c_k(i_0, i_1, i_2, \ldots, i_{n_k-1}, i_{n_k} \ldots)$$

$$= \sigma^{-1}(j_0, j_1, \ldots, j_{n_k-1}, i_{n_k}, \ldots) \neq (j_0, j_1, \ldots, j_{n_k-1}, i_{n_k}, \ldots) = c_k(u)$$

Hence, we get a contradiction.

Now, since $C_{H_{n_k}}(H_{n_{k-1}}) \cong S_{\frac{f_\Omega(n_k)}{f_\Omega(n_{k-1})}}$, we can regard the element c_k as being the permutation element of the components of the ends from n_{k-1} to $n_k - 1$ and we can write

$$c_k(u) = (i_0, i_1, \ldots i_{n_{k-1}-1}, c_k(i_{n_{k-1}}, \ldots, i_{n_k-1}), i_{n_k}, \ldots).$$

Lemma 4.11. *For $N = (n_1, n_2, \ldots)$ and $\gamma = (i_0, i_1, \ldots) \in \partial T$*

$$Stab_{A_N}(\gamma) \cong \underset{j=1}{\overset{\infty}{Cr}}\, S_{t_j - 1}$$

where $t_1 = f_\Omega(n_1)$ and $t_j = \frac{f_\Omega(n_j)}{f_\Omega(n_{j-1})}$ for all $j \geqslant 2$

Proof. Since $A_N = H_{n_1} C_{H_{n_2}}(H_{n_1}) \ldots$ any element g is of the form $c_1 c_2 c_3 \ldots$ where $c_j \in C_{H_{n_j}}(H_{n_{j-1}})$. By the Remark 4.10, we know the action on the ends of any element in $C_{H_{n_{j+1}}}(H_{n_j})$, so we can consider

$$g(\gamma) = (c_1(i_0, i_1, \ldots, i_{n_1 - 1}), c_2(i_{n_1}, \ldots, i_{n_2 - 1}), c_3(i_{n_3}, \ldots, i_{n_3 - 1}), \ldots)$$

Hence, an element g lies in $Stab_{A_N}(\gamma)$ if and only if c_j fixes $(i_{n_{j-1}}, \ldots, i_{n_j - 1})$ and considering the fact that $C_{H_{n_{i+1}}}(H_{n_i}) \cong S_{\frac{f_\Omega(n_{i+1})}{f_\Omega(n_i)}}$ we get the result. □

4.4 Orbits and Transitivity

Theorem 4.12. *$N_{Hom(\partial T_\Omega)}(H_\Omega) \cong AutH_\Omega$ is transitive on ∂T_Ω.*

Proof. Let γ and μ be arbitrary ends. Choose a level $k > 3$ so that $\gamma = (i_0, \ldots i_{k-2}, i_{k-1}, \ldots)$ and $\mu = (i_0, \ldots i_{k-2}, j_{k-1}, \ldots)$ are elements of different balls of level k and $i_{k-1} \neq j_{k-1}$. Consider $N = (k, k+1, k+2, \ldots)$ and the group $A_N \cong H_k C_{H_{k+1}}(H_k) C_{H_{k+2}}(H_{k+1}) \ldots$.

Let $\alpha = c_k c_{k+1} \ldots$. Choose $c_k \in H_k$ to be the element sending $(i_0, i_1 \ldots, i_{k-2}, i_{k-1})$ to $(i_0, i_1 \ldots i_{k-2}, j_{k-1})$ and c_{k+s} to be the element in the centralizer $C_{H_{k+s}}(H_{k+s-1})$ that sends the coordinate i_{k+s-1} to j_{k+s-1} for all $s \geqslant 1$. Hence, by the Remark 4.10 we have,

$$\alpha(\gamma) = \alpha(i_0, \ldots i_{k-2}, i_{k-1}, \ldots) = (c_k(i_0, \ldots i_{k-2}, i_{k-1}), c_{k+1}(i_k), c_{k+2}(i_{k+1}), \ldots)$$

$$= (i_0, i_1, \ldots, i_{k-2}, j_{k-1}, j_k, j_{k+1}, \ldots) = \mu.$$

□

Note that, $H_\Omega \trianglelefteq N = N_{Hom(\partial T_\Omega)}(H_\Omega)$ and if N would be highly transitive then since 2-transitive groups are primitive N must be primitive. On the other hand, the action of H_Ω on ∂T_Ω is faithful with uncountably many orbits and it is a well known fact that for a transitive group the orbits of the normal subgroups are blocks of imprimitivity. Hence, N can not be 2-transitive.

Corollary 4.13. *$Aut(H_\Omega)$ is uncountable.*

Proof. As $Aut(H_\Omega)$ is transitive on the boundary and the boundary is uncountable, by the well known orbit-stabilizer theorem, the cardinality of ∂T_Ω equals to the cardinality of the set of left cosets of a stabilizer of an end in $Aut(H_\Omega)$. Hence, $Aut(H_\Omega)$ is uncountable. □

We know that as groups H_Ω and the homogeneous symmetric group, $S(\Omega)$ are isomorphic. The following theorem yields a strong result about these groups.

Theorem 4.14. *[12, 3.3 Theorem 15] The orbits of H_Ω are dense in ∂T_Ω and H_Ω acts faithfully on every of its orbit, $O \subset \partial T_\Omega$. Moreover, (H_Ω, O) and $(S(\Omega), \mathbb{N})$ are permutational isomorphic for every orbit O of H_Ω.*

Proof. Let O be an orbit of the action of H_Ω on ∂T_Ω. We will show that the closure, $\overline{O}$, equals to the whole space ∂T_Ω. Obviously, $\overline{O} \subseteq \partial T_\Omega$. Take $u \in \partial T_\Omega$, to show $u \in \overline{O}$ it suffices to show that for any neighborhood $P_{n(i_0,\ldots,i_{n-1})}$ of u, $P_{n(i_0,\ldots,i_{n-1})} \cap O \neq \emptyset$. Obviously, if $\gamma = (j_0, j_1, \ldots) \in O$ then the elements of O are of the form $h(\gamma) = (i_0, i_1, \ldots i_{k-1}, j_k, j_{k+1}, \ldots)$ where h runs through H_Ω and $0 \leqslant i_s \leqslant a_s - 1$ for $s \leqslant k$ and j_t's are tth components of γ. Of course, as shown in the figure below there exist an element $h \in H_\Omega$ sending γ to an element of $P_{n(i_0,\ldots,i_{n-1})}$. Hence $h(\gamma) \in P_{n(i_0,\ldots,i_{n-1})} \cap O$.

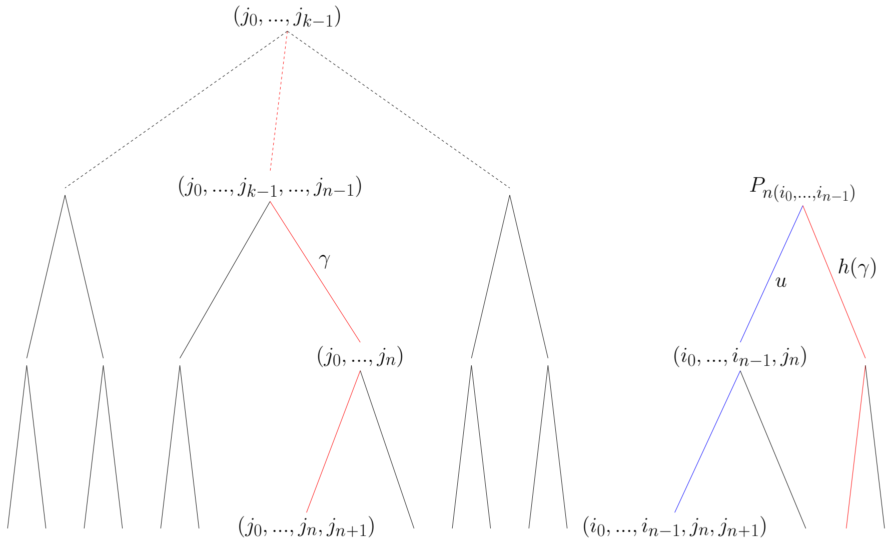

Figure 4.2: An element $h \in H_\Omega$ sending γ to $P_{n(i_0,...,i_{n-1})}$

Notice that depending on the characteristic Ω, H_Ω is either simple or has a unique normal subgroup of index 2. In the first case, as kernel is a normal subgroup and the action is not trivial we have the result. In the second case, if the index of the kernel in H_Ω is 2 then as the kernel of the action is the intersection of all point stabilizers the cardinality of orbits of H_Ω will be less than 2, which is impossible.

Let

$$O_0 = \{(j_0, j_1, \ldots, j_{n-1}, 0, 0, 0, 0 \ldots)|\ 0 \leqslant j_k \leqslant a_k - 1,\ k < n,\ n \in \mathbb{N} \cup \{0\}\}$$

be the orbit containing the end $(0, 0, 0, \ldots,)$. For a vertex $(j_0, j_1, \ldots j_{n-1})$ on a level n enumerate the vertex with i_n as follows;

$$i_n = (j_0 + 1) + j_1 f_\Omega(1) + j_2 f_\Omega(2) + \ldots + j_{n-1} f_\Omega(n-1).$$

Since $0 \leqslant j_k \leqslant a_k - 1$, we number the vertices of level n by numbers 1 to $f_\Omega(n)$. For an end $u = (j_0, j_1, \ldots j_{k-1}, 0, 0, \ldots)$ in O_0 since $j_{k+s} = 0$ for all $s \geqslant 0$, the end u can be identified by i_k. Consider the map $\phi : O_0 \to \mathbb{N}$ sending each end u to its corresponding number i_k.

In fact, the map ϕ is bijective. Let $u_1 = (j_0, j_1, \ldots j_{k-1}, 0, 0, \ldots)$, $u_2 = (t_0, t_1, \ldots t_{n-1}, 0, 0, \ldots)$ be two ends such that $i_k = \phi(u_1) = \phi(u_2) = l_n$. Without loss of generality assume

$k < n$ and assume $u_1 \neq u_2$ so there exist a minimal number $s \leqslant k$ such that $j_s \neq t_s$.

$$i_k = (j_0 + 1) + j_1 f_\Omega(1) + \ldots + j_{s-1} f_\Omega(s-1) + j_s f_\Omega(s) + \ldots + j_{k-1} f_\Omega(k-1)$$

$$l_n = (j_0+1)+j_1 f_\Omega(1)+\ldots+j_{s-1} f_\Omega(s-1)+t_s f_\Omega(s)+\ldots+t_{k-1} f_\Omega(k-1)+\ldots+t_{n-1} f_\Omega(n-1)$$

Look at the equalities in mod $f_\Omega(s+1)$,

$$i_k \equiv (j_0 + 1) + j_1 f_\Omega(1) + \ldots + j_{s-1} f_\Omega(s-1) + j_s f_\Omega(s) \mod f_\Omega(s+1)$$

$$l_n \equiv (j_0 + 1) + j_1 f_\Omega(1) + \ldots + j_{s-1} f_\Omega(s-1) + t_s f_\Omega(s) \mod f_\Omega(s+1)$$

Hence,

$0 = i_k - l_n \equiv j_s f_\Omega(s) - t_s f_\Omega(s) \mod f_\Omega(s+1)$, as $f_\Omega(s+1) = a_s f_\Omega(s)$ and $0 \leqslant j_s, t_s \leqslant a_s - 1$ we get $j_s = t_s$.

The ontoness of map comes as follows, if $n \in \mathbb{N}$, let k be the first level number such that $n \leqslant f_\Omega(k)$. Write $n = j_{k-1} f_\Omega(k-1) + i_{k-1}$ where $j_{k-1} < a_{k-1}$ and $i_{k-1} \leqslant f_\Omega(k-1)$ and then write $i_{k-1} = j_{k-2} f_\Omega(k-2) + i_{k-2}$ where $j_{k-2} < a_{k-2}$ and $i_{k-2} \leqslant f_\Omega(k-2)$, continuing the process we have the numbers $j_s < a_s$ for all $s \geqslant 0$. And under the map ϕ the end $u = (j_0, j_1, \ldots j_{k-1}, 0, 0, \ldots)$ will have n as the image.

Now for a vertex $v = (j_0, \ldots, j_{n-1}) \in V_n$ since there is a corresponding number $1 \leqslant i \leqslant f_\Omega(n)$ denote the ball on the level n passing through the vertex v by P_{ni} instead of P_{nv}. An element $\sigma \in H_n$ acts as a permutation on the balls P_{ni} for $1 \leqslant i \leqslant f_\Omega(n)$ and this action is extended diagonally to the balls of bigger levels. For an end $u \in O_0$ which belongs to P_{ni} for some i, if u is identified by a number j which is bigger than $f_\Omega(n)$ on some level $s \geqslant n$, note that if $P_{sj}^\sigma = P_{sk}$ then $u^\sigma \in P_{sk}$, and u^σ is identified with k.

Consider the isomorphism $\theta_n : H_n \to S(f_\Omega(n))$. As any σ is a permutation of the balls of level n, and the balls are numbered by the numbers 1 to $f_\Omega(n)$, the image $\theta_n(\sigma)$ will be induced by this action naturally.

So, for $u \in P_{nj}$, $\sigma \in H_n$, if $P_{nj}^\sigma = P_{nk}$ and if u is identified with j and $j \leqslant f_\Omega(n)$,

$$k = \phi(u^\sigma) = \phi(u)^{\theta_n(\sigma)} = k.$$

If $u \in P_{nj}$ is identified by $t_j \geqslant f_\Omega(n)$ on some level $s \geqslant n$, then we consider the action of σ on the balls of level s, if $P_{nj}^\sigma = P_{nk}$, then $P_{st_j}^\sigma = P_{st_k}$. Hence,

$$t_k = \phi(u^\sigma) = \phi(u)^{\theta_n(\sigma)} = t_k.$$

Hence, the groups (H_n, O_0) and $(S(f_\Omega(n)), \mathbb{N})$ are permutationally isomorphic. Since $\theta_n|_{H_k} = \theta_k$ for all $k < n$ and we have $(S(\Omega), \mathbb{N}) = \bigcup_{n=1}^{\infty} (S(f_\Omega(n), \mathbb{N})$ and $(H_\Omega, O_0) = \bigcup_{n=1}^{\infty} (H_n, O_0)$ we have the result. □

Remark 4.15. *As $S(\Omega) = \bigcup_{n=1}^{\infty} S(f_\Omega(n))$ where $S(f_\Omega(n))$ is isomorphic to the finite symmetric group on $f_\Omega(n)$ letters and we know that $S(f_\Omega(n))$ is $f_\Omega(n)$ transitive, $S(\Omega)$ acts highly transitively on $\mathbb{N}$. Moreover, as the action (H_Ω, O) is permutationally isomorphic to $(S(\Omega), \mathbb{N})$, H_Ω acts highly transitive on every of its orbits.*

Proposition 4.16. *The group of local isometries of ∂T_Ω is highly transitive on the boundary of the tree.*

Proof. By Lemma 3.16, we know that $LI = LI(\partial T_\Omega)$ is the product of $AutT_\Omega$ and H_Ω. Let $A = \{\gamma_1, \gamma_2, \ldots, \gamma_n\}$ and $B = \{\mu_1, \mu_2, \ldots, \mu_n\}$ be two sets of ends. Choose $m = max\{k, l\}$ where k and l are the minimum levels for A and B, respectively, that all the elements of corresponding sets belongs to different balls.

Now that all γ_i's $i = 1, \ldots, n$ in a different ball we can choose $\alpha \in H_m$ so that $\alpha(\gamma_i)$ and μ_i lies in same ball on level m. Note that on level m all μ_i's $i = 1, \ldots, n$ lie on different balls. Using the fact that, automorphism group of a spherically homogeneous tree is transitive on its boundary [3, Section 6.2] and each ball of level m is canonically isomorphic to a spherically homogeneous tree we can find automorphisms, β_i, of the corresponding balls and extend them trivially to an automorphism of the tree so that we will have $\beta_1\beta_2 \ldots \beta_n\alpha$ is the required map. Note that $\beta_1\beta_2 \ldots \beta_n \in Stab_{Aut(T_\Omega)}(m)$, the level stabilizer of the automorphism group. □

CHAPTER 5

DIAGONAL EMBEDDINGS

In this chapter, we will introduce another technique to construct a different class of subgroups S_χ where $\chi = \langle (1, k_0), (n_1, k_1), \ldots \rangle$, which are locally finite and simple.

Moreover, in the Sections 5.2, 5.3 we will give the structure of the centralizers of elements and finite subgroups.

5.1 The group S_χ

Consider the embedding of finite symmetric groups as follows;

$$d(r, s) : S_n \longrightarrow S_{nr+s}$$

For any $\alpha \in S_n$, $d(r, s)(\alpha) \in S_{nr+s}$ is determined as follows;

$$((k-1)r + i)^{d(r,s)(\alpha)} = (k^\alpha - 1)r + i \text{ where } 1 \leqslant i \leqslant r \;\; 1 \leqslant k \leqslant n$$

Hence, if $\alpha = \left(\begin{smallmatrix} 1 & 2 & \cdots & n \\ i_1 & i_2 & \cdots & i_n \end{smallmatrix}\right)$, then

$$d(r, s)(\alpha) = \left(\begin{smallmatrix} 1 & 2 & \cdots & r & | & \cdots & | & (n-1)r+1 & (n-1)r+2 & \cdots & nr & | & nr+1 & | & \cdots & | & nr+s \\ (i_1-1)r+1 & (i_1-1)r+2 & \cdots & i_1 r & | & \cdots & | & (i_n-1)r+1 & (i_n-1)r+2 & \cdots & i_n r & | & nr+1 & | & \cdots & | & nr+s \end{smallmatrix}\right)$$

Lemma 5.1. *$d(r, s)$ is a diagonal embedding.*

Proof. By the Definition 2.1, to see $d(r, s)$ is a diagonal embedding, first let us determine the forms of arbitrary orbits of $d(r, s)(S_n)$ in the set $\{1, 2, \ldots, nr + s\}$. Since the action is trivial on the points i where $nr + 1 \leqslant i \leqslant nr + s$, the orbit $\Delta_i = \{i^{d(r,s)(\alpha)} \mid \alpha \in S_n\}$ consists of only the point i. The other orbits are of the form

$\Delta_i = \{i, r+i, 2r+i, \cdots, (n-1)r+i\}$ for all $1 \leqslant i \leqslant r$. Notice that the length of the orbits are n.

Define a map, $\lambda : \{1, 2, \ldots, n\} \to \Delta_i$ where $\lambda(j) = (j-1)r + i$.

Now, the action $(S_n, \{1, 2, \ldots, n\})$ is permutationally isomorphic to $(d(r,s)(S_n), \Delta_i)$ as follows;

For any $j \in \{1, 2, \ldots, n\}$ and $\alpha \in S_n$,

$$\lambda(j)^{d(r,s)(\alpha)} = ((j-1)r + i)^{d(r,s)(\alpha)} = (j^\alpha - 1)r + i = \lambda(j^\alpha)$$

Hence, the embedding $d(r,s)$ is a diagonal embedding. □

In the next lemma, we will see that, the composition of two maps of the form $d(r,s)$ is again of the same form.

Lemma 5.2. *[11, Lemma 2.5] Let M_1, M_2, M_3 be arbitrary sets with cardinalities $|M_1| = m_1$, $|M_2| = m_2 = m_1n_1 + r_1$, $|M_3| = m_3 = m_2n_2 + r_2$ if $d(n_1, r_1) : Sym(M_1) \longrightarrow Sym(M_2)$, $d(n_2, r_2) : Sym(M_2) \longrightarrow Sym(M_3)$, then $d(n_1n_2, n_2r_1 + r_2) : Sym(M_1) \longrightarrow Sym(M_3)$ such that*

$$d(n_2, r_2)d(n_1, r_1) = d(n_1n_2, n_2r_1 + r_2)$$

Proof. It is enough to show that for an element $\alpha \in S_{m_1}$ the images $d(n_2, r_2)d(n_1, r_1)(\alpha)$ and $d(n_1n_2, n_2r_1 + r_2)(\alpha)$ are the same permutations. Consider an arbitrary point, say $(k-1)n_2 + i$, on the set $\{1, 2, \ldots, m_2n_2\}$ where $1 \leqslant i \leqslant n_2$ and $1 \leqslant k \leqslant m_2$. Notice that we can write $k = (s-1)n_1 + j$ where $1 \leqslant s \leqslant m_1$ and $1 \leqslant j \leqslant n_1$. Then

$$((k-1)n_2 + i)^{d(n_2,r_2)(d(n_1,r_1)(\alpha))} = (k^{d(n_1,r_1)(\alpha)} - 1)n_2 + i \tag{5.1}$$

Now using $k = (s-1)n_1 + j$ we have;

Equation 5.1 $= ((s-1)n_1 + j)^{(\alpha)d(n_1,r_1)} - 1)n_2 + i = (s^\alpha - 1)n_1n_2 + (j-1)n_2 + i$
On the other hand,

$$((k-1)n_2+i)^{(\alpha)d(n_1n_2,n_2r_1+r_2)} = ((s-1)n_1n_2+(j-1)n_2+i)^{(\alpha)d(n_1n_2,n_2r_1+r_2)} = (s^\alpha-1)n_1n_2+(j-1)n_2+i$$

Moreover, for any point s from the set $\{m_2n_1 + 1, \ldots, m_2n_2 + r_2\}$ we have

$$s^{(\alpha)d(n_1,r_1)d(n_2,r_2)} = s^{(\alpha)d(n_1n_2,n_2r_1+r_2)} = s$$

Hence, we get the result. □

Lemma 5.2 suggests the compatibility of the maps. Therefore, for an infinite sequence of integer tuples $\chi = \langle(1, k_0), (n_1, k_1), \ldots\rangle$, the sequences of diagonal maps,

$$S_{k_0} \xrightarrow{d(n_1,k_1)} S_{n_1k_0+k_1} \xrightarrow{d(n_2,k_2)} S_{(n_1k_0+k_1)n_2+k_2} \xrightarrow{d(n_3,k_3)} \ldots$$

will define a direct limit group S_χ.

For construction and the motivation see [11].

Similar to the case of homogeneous symmetric groups $S(\xi)$, we can regard the group S_χ as a subgroup of homeomorphism group of a rooted tree. For this purpose, we will construct a new rooted tree on which the diagonal direct limit group S_χ acts.

Take empty set as the root. Let $\{1, 2, \ldots k_0\} \cup \{\$\}$ be the vertices of first level. On the second level, let there be n_1 edges coming down from each vertex except \$, and let there be $k_1 + 1$ edges coming down from \$. Now, label the set of vertices of level 2 as follows $V_2 = \{11, 12, \ldots, 1n_1, 21, \ldots, 2n_1, \ldots k_01, \ldots k_0n_1\} \cup \{\$1, \$2, \ldots \$k_1, \$\$\}$. For the third level, from each vertex other than \$\$, let there be n_2 edges coming down and $k_2 + 1$ edges coming down from \$\$. Similarly, write the vertices of level 3 as, $V_3 = \{111, \ldots 11n_2, \ldots 1n_11, \ldots, 1n_1n_2, \ldots k_011, \ldots, k_0n_1n_2, \$11, \ldots, \$1n_2, \ldots \$k_11, \ldots, \$k_1n_2\} \cup \{\$\$1, \$\$2, \ldots, \$\$k_2, \$\$\$\}$.

Continuing like this, for the given infinite sequence of tuples

$$\chi = \langle(1, k_0), (n_1, k_1), (n_2, k_2), \ldots\rangle$$

and the set of vertices labeled as above, we have the corresponding tree T_χ in the following figure;

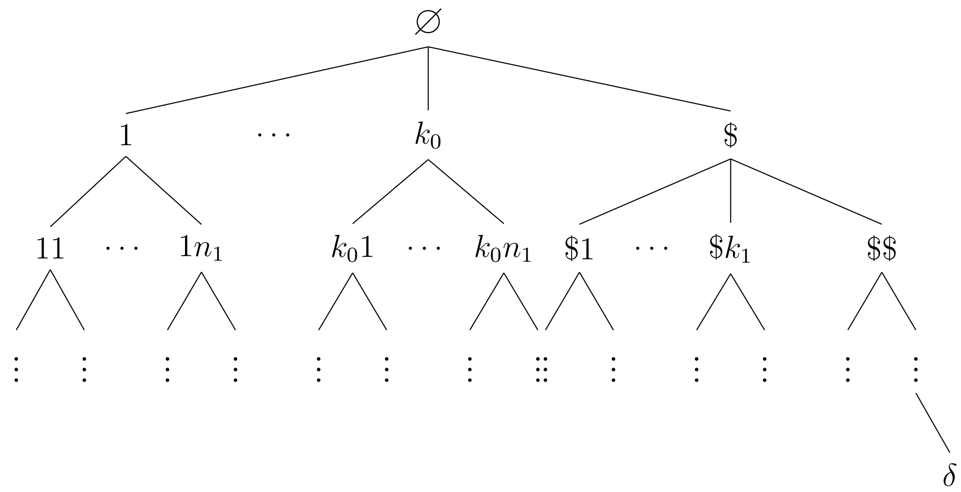

Figure 5.1: Rooted tree T_χ

Denote the number of vertices of level $i+1$ except the vertex $\$\$\ldots\$$ on level $i+1$ by

$$r(\chi, i) := |V_{i+1}(T_\chi)| - 1$$

Note that for any $\chi = \langle(1, k_0), (n_1, k_1), \ldots, \rangle$, we have

$$r(\chi, i) = k_0 n_1 n_2 n_3 \ldots n_i + k_1 n_2 n_3 \ldots n_i + \ldots + k_{i-1} n_i + k_i.$$

Observe that, if in the sequence χ all $k_i = 0$ except k_0, then we have a spherically homogeneous tree with characteristic sequence $(k_0, n_1, n_2, \ldots)$.

Consider the boundary, $\partial T_\chi \backslash \{\delta\}$, of the tree. Let δ be the end where $\delta = (\$, \$, \$, \$, \ldots)$ on T_χ. Similar to the case of spherically homogeneous tree we can define the metric ρ on $\partial T_\chi \backslash \{\delta\}$ such that $\rho(\gamma_1, \gamma_2) = \frac{1}{n+1}$ where n is the common parts of the ends γ_1 and γ_2. The balls of this topology will be also denoted by P_{nv} which are the set of all ends passing through the vertex v on level n. Note that by the construction of the tree, it can be easily seen that all the balls in the same level m except the ball with root $\$\$\ldots\$$ can be identified with a spherically homogeneous tree with characteristic sequence $\Omega_m = (n_m, n_{m+1}, \ldots)$.

In the Section 5.4, the topological properties will be given in further detail.

Now, consider the subgroup $S(\chi, n)$ of $Hom(\partial T_\chi \backslash \{\delta\})$ which only permutes the $r(\chi, n-1)$ balls of level n and acts trivially inside the balls. That is, an element in $S(\chi, n)$ sending the ball P_{nv} to a ball P_{nw} is just takes the ball P_{nv} and glues it on the ball P_{nw}. The construction and the motivation is very similar to those homeomorphisms H_n of the spherically homogeneous rooted tree.

Lemma 5.3. *If $i \leqslant j$, then $S(\chi, i)$ is embedded into $S(\chi, j)$ via diagonal embedding.*

Proof. By definition of the groups, $S(\chi, i)$ and $S(\chi, j)$ are the symmetric groups of the vertices of level i and j, respectively. Since we construct the tree so that S_χ acts on the boundary, it suffices to show $S(\chi, i)$ is embedded into $S(\chi, j)$ via an embedding of the form $d(n, r)$. But by Lemma 5.2, it is enough to show $S(\chi, i)$ is embedded into $S(\chi, i+1)$ via the diagonal embedding of the form $d(n, r)$. Let the embedding map be $f : S(\chi, i) \to S(\chi, i+1)$. Note that $S(\chi, i)$ is isomorphic to $S_{r(\chi, i-1)}$ where $r(\chi, i-1) = k_0 n_1 \dots n_{i-2} + k_1 n_2 \dots n_{i-2} + \dots + k_{i-2}$. When we embed $S(\chi, i)$ into $S(\chi, i+1)$, on level $i+1$ the images of the elements $S(\chi, i)$ acts trivially on the vertices with the label $\underbrace{\$\$\$\dots\$}_{\text{i - many}} t$ where $1 \leqslant t \leqslant k_i$, that is to say we have k_i many fixed points in the embedding.

On the other hand, using the labeling of the vertices of the tree T_χ, see Figure 5.1, we can write a non-trivial orbit of the image $S(\chi, i)$ on the set $V_{i+1} \backslash \{ \underbrace{\$\$\dots\$}_{\text{(i + 1) - many}} \}$ containing the vertex $\underbrace{11\dots1}_{\text{(i + 1) - many}}$ as follows;

$$\begin{aligned}\Delta_{\underbrace{11\dots1}_{\text{(i + 1) - many}}} &= \{ v^{f(\alpha)} \in V_{i+1} \backslash \{ \underbrace{\$\$\dots\$}_{\text{(i + 1) - many}} \} \mid \alpha \in S(\chi, i) \} \\ &= \{ w1 \mid w \in V_i \backslash \{ \underbrace{\$\$\dots\$}_{\text{i - many}} \} \}\end{aligned}$$

Hence with the obvious bijection, λ sending any w to $w1$ between $V_i \backslash \{ \underbrace{\$\$\dots\$}_{\text{i - many}} \}$ and $\Delta_{\underbrace{11\dots1}_{\text{(i + 1) - many}}}$ we can see that, for any $\alpha \in S(\chi, i)$ and $w \in V_i \backslash \{ \underbrace{\$\$\dots\$}_{\text{i - many}} \}$ the image will be $\lambda(w^\alpha) = w^\alpha 1$. On the other hand, by the definition of $S(\chi, i)$, the element $f(\alpha)$ permute only the first i coordinates of vertices of level $i+1$ and on that coordinates it acts as α. Therefore,

$$\lambda(w)^{f(\alpha)} = (w1)^f(\alpha) = w^\alpha 1 = \lambda(w^\alpha)$$

Hence, $(S(\chi, i), V_i \backslash \{ \underbrace{\$\$\dots\$}_{\text{i - many}} \})$ is permutationally isomorphic to $(f(S(\chi, i), \Delta_{\underbrace{11\dots1}_{\text{(i + 1) - many}}}))$.

Note that in the embedding of $S_{r(\chi,i-1)}$ into $S_{r(\chi,i)}$ we have $r(\chi,i) = r(\chi,i-1)n_i + k_i$ with $k_i$many fixed points and n_i many orbits of length $r(\chi,i-1)$. Hence, the embedding f is actually $d(n_i,k_i)$. □

Since we construct the tree by using the infinite sequence of tuples $\chi = \langle(1,k_0),(n_1,k_1),\ldots\rangle$, and the embeddings are diagonal of the form $d(n_i,k_i)$ we can write $S_\chi := \bigcup_{i=1}^{\infty} S(\chi,i)$.

The classification of the groups S_χ is done in [11]. For readers convenience we will give some necessary theorems, lemmas and their proofs to make the group S_χ more understandable.

Lemma 5.4. *[11, Lemma 2.9] Let $d(r,s)$ be the diagonal embedding of S_n into S_{nr+s}, if $f(r,s)$ is an other diagonal embedding of S_n into S_{nr+s}, then $S_n^{f(r,s)}$ and $S_n^{d(r,s)}$ are conjugate by an element of S_{nr+s}.*

Proof. Let Δ_i, $\mathscr{O}_i$ for $1 \leqslant i \leqslant r$ be the orbits of length n and for $r+1 \leqslant i \leqslant r+s$ be the orbits of length 1, of $S_n^{f(r,s)}$, $S_n^{d(r,s)}$ respectively. Since the embeddings are diagonal there exists bijections $\phi_i : \Delta_i \longrightarrow \{1,2,\ldots,n\}$ inducing permutation isomorphism between $(S_n,\{1,2,\ldots,n\})$ and $(f(r,s)(S_n)_{|\Delta_i},\Delta_i)$, and also there exists bijections $\psi_i : \{1,2,\ldots,n\} \longrightarrow \mathscr{O}_i$ inducing permutation isomorphism between $(S_n,\{1,2,\ldots,n\})$ and $(d(r,s)(S_n)_{|\mathscr{O}_i},\mathscr{O}_i)$. By using these bijections construct a new bijection $\pi \in S_{nr+s}$ as follows;

$$For\ j \in \Delta_i,\ \pi(j) = \psi_i\phi_i(j)$$

Now, if we consider the isomorphism $\lambda : S_n^{f(r,s)} \longrightarrow S_n^{d(r,s)}$ sending each $f(r,s)(\alpha)$ to $d(r,s)(\alpha)$ together with π we have,

For any $\alpha \in S_n$ and $j \in \Delta_i$,

$$\pi(j^{f(r,s)(\alpha)}) = \psi_i\phi_i(j^{f(r,s)(\alpha)}) = \psi_i((\phi_i(j))^{\alpha}) = \psi_i\phi_i(j)^{d(r,s)(\alpha)} = \pi(j)^{\lambda(f(r,s)(\alpha))}$$

Since the groups are permutationally isomorphic subgroups of S_{nr+s},, they are conjugate by [2, Exercise 1.6.1]. □

The above lemma will lead to the fact that if there is another direct limit group of symmetric groups embedded via diagonal embedding with respect to χ, then the group must be isomorphic to S_χ where $\chi = \langle(1, k_0), (n_1, k_1), \ldots\rangle$. To see this we need the following lemma.

Lemma 5.5. *[1, Lemma 2.3] Let H and G be a direct limit group of subgroups H_i and G_i via the embeddings θ_i and ψ_i, respectively. Let α_i be an isomorphism between H_i and G_i for all $i \geqslant 1$. If the following diagram*

$$\begin{array}{ccc} H_i & \xrightarrow{\theta_i} & H_{i+1} \\ {\scriptstyle\alpha_i}\downarrow & & \downarrow{\scriptstyle\alpha_{i+1}} \\ G_i & \xrightarrow[\psi_i]{} & G_{i+1} \end{array}$$

is commutative for all $i \geqslant 1$, then the groups H and G are isomorphic.

Proof. Assume the diagram is commutative, that is $\alpha_{i+1}\theta_i = \psi_i\alpha_i$ for all $i \geqslant 1$. Let $\alpha : H \to G$ such that for an element $h_i \in H_i$, the image $\alpha(h_i) = \ldots \psi_{i+1}\psi_i\alpha_i(h_i)$.

Claim: The restriction $\alpha_{i+1|_{H_i}}$ equals to α_i

Notice that H_i is embedded H_{i+1} by the map θ_i, hence the image of h_i in the group H_{i+1} is $\theta_i(h_i)$. Hence $\alpha_{i+1}(h_i) = \alpha_{i+1}\theta_i(h_i) = \psi_i\alpha_i(h_i)$ and this is the image of $\alpha_i(h_i)$ in the group G_{i+1}.

Claim: α is a homomorphism.

It is enough to show that $\alpha(h_ih_{i+1}) = \alpha(h_i)\alpha(h_{i+1})$ for arbitrary elements $h_i \in H_i$, $h_{i+1} \in H_{i+1}$.

$$\begin{aligned} \alpha(h_ih_{i+1}) &= \ldots \psi_{i+1}(\alpha_{i+1}(h_ih_{i+1})) \\ &= \ldots \psi_{i+1}((\alpha_{i+1}(h_i))(\alpha_{i+1}(h_{i+1}))) \\ &= \ldots \psi_{i+1}((\alpha_{i+1}\theta_i(h_i))(\alpha_{i+1}(h_{i+1}))) \\ &= \ldots \psi_{i+1}((\psi_i\alpha_i(h_i))(\alpha_{i+1}(h_{i+1}))) \end{aligned}$$

$$= \dots \psi_{i+1}\psi_i\alpha_i(h_i)(\dots \psi_{i+1}\alpha_{i+1}(h_{i+1}))$$
$$= \alpha(h_i)\alpha(h_{i+1})$$

□

Theorem 5.6. *[11, Proposition 2.10] If H is a diagonal direct limit group of finite symmetric groups, then H is isomorphic to the group S_χ for some sequence χ.*

Proof. Since H is a diagonal direct limit of finite symmetric groups, there exists an infinite sequence $\chi = \langle (1, k_0), (n_1, k_1), \dots \rangle$ corresponding to the embeddings and $H = \bigcup_{i=0}^{\infty} H_i$ where $H_i \cong Sym(r(\chi, i))$ and $Sym(r(\chi, i)) = S_{k_0 n_1 n_2 \dots n_{i-1} + \dots + k_{i-1}}$. Let $f_i = f(n_i, k_i)$ be the diagonal embeddings constructing H, and $d_i = d(n_i, k_i)$ be the embeddings constructing S_χ. In the diagram below

$$\begin{array}{ccccccc}
H_0 & \xrightarrow{f(n_1,k_1)} & H_1 & \xrightarrow{f(n_2,k_2)} & H_2 & \longrightarrow & \cdots \\
\pi_0 \downarrow & & \downarrow \pi_1 & & \downarrow \pi_2 & & \\
S_{k_0} & \xrightarrow[d(n_1,k_1)]{} & S_{n_1k_0+k_1} & \xrightarrow[d(n_2,k_2)]{} & S_{(n_1k_0+k_1)n_2+k_2} & \longrightarrow & \cdots
\end{array}$$

where π_i are the elements conjugating the images as in the case of Lemma 5.4, we have $(f_i(H_{i-1}))^{\pi_i} = d_i(Sym(r(\chi, i-1)))$. Notice that π_i's are not unique and we can choose π_i's so that the diagram is commutative as follows;

Start with an order, $\{a_{0,1}, a_{0,2}, a_{0,3}, \dots, a_{0,k_0}\}$, on k_0 points on which H_0 acts and fix it. Now by considering the image $f(n_1, k_1)((a_{0,1}, a_{0,2}, a_{0,3}, \dots, a_{0,k_0}))$ of the element $(a_{0,1}, a_{0,2}, a_{0,3}, \dots, a_{0,k_0})$ in H_1 fix an order on $n_1k_0 + k_1$ points on which H_1 acts as follows $\{a_{1,1}, a_{1,2}, a_{1,3}, \dots, a_{1,n_1k_0+k_1}\}$ where

$$f(n_1, k_1)((a_{0,1}, a_{0,2}, a_{0,3}, \dots, a_{0,k_0})) = (a_{1,1}, a_{1,2}, a_{1,3}, \dots, a_{1,k_0})(a_{1,k_0+1}, \dots, a_{1,2k_0})$$
$$\dots (a_{1,(n_1-1)k_0+1}, \dots, a_{1,n_1k_0})(a_{1,n_1k_0+1}) \dots (a_{1,n_1k_0+k_1})$$

and continue the enumeration in the same way.

Now fix the numeration in $S_{r(\chi,i)}$ in the same way, that is start with $\{1, 2, \dots, k_0\}$, and enumerate them with $\{c_{0,1}, c_{0,2}, \dots, c_{0,k_0}\}$, respectively then consider the image

$d(n_1, k_1)((1, 2, \ldots, k_0))$ of the element $(1, 2, \ldots, k_0)$ in $S_{r(\chi,1)}$ and chose the enumeration respectively. Now choose for $0 \leqslant i$, π_i so that $\pi_i(a_{i,j}) = c_{i,j}$ where $1 \leqslant j \leqslant r(\chi, i)$.

Claim: $d(n_i, k_i)\pi_{i-1} = \pi_i f(n_i, k_i)$

Proof: Let $\alpha \in H_{i-1}$ so that α is a permutation on the set $\{a_{i-1,j} \mid 1 \leqslant j \leqslant r(\chi, i-1)\}$. Remember that the action of π_i's are conjugation, hence we must show $d(n_i, k_i)(\alpha^{\pi_{i-1}}) = (f(n_i, k_i)\alpha)^{\pi_i}$. If α is as follows,

$$\alpha = \begin{pmatrix} a_{i-1,1} & \cdots & a_{i-1,r(\chi,i-1)} \\ a_{i-1,b_1} & \cdots & a_{i-1,b_{r(\chi,i-1)}} \end{pmatrix} \text{then } \alpha^{\pi_{i-1}} = \begin{pmatrix} c_{i-1,1} & \cdots & c_{i-1,r(\chi,i-1)} \\ c_{i-1,b_1} & \cdots & c_{i-1,b_{r(\chi,i-1)}} \end{pmatrix}$$

and

$$d(n_i, k_i)(\alpha^{\pi}) =$$

$$\begin{pmatrix} c_{i,1} & \cdots & c_{i,r(\chi,i-1)} \\ c_{i,b_1} & \cdots & c_{i,b_{r(\chi,i-1)}} \end{pmatrix} \cdots \begin{pmatrix} c_{i,(n_i-1)r(\chi,i-1)+1} & \cdots & c_{i,(n_i-1)r(\chi,i-1)+r(\chi,i-1)} \\ c_{i,(n_i-1)r(\chi,i-1)+b_1} & \cdots & c_{i,(n_i-1)r(\chi,i-1)+b_{r(\chi,i-1)}} \end{pmatrix}$$

the points which are not seen in the above decomposition are fixed. On the other hand,

$$f(n_i, k_i)(\alpha) =$$

$$\begin{pmatrix} a_{i,1} & \cdots & a_{i,r(\chi,i-1)} \\ a_{i,b_1} & \cdots & a_{i,b_{r(\chi,i-1)}} \end{pmatrix} \cdots \begin{pmatrix} a_{i,(n_i-1)r(\chi,i-1)+1} & \cdots & a_{i,(n_i-1)r(\chi,i-1)+r(\chi,i-1)} \\ a_{i,(n_i-1)r(\chi,i-1)+b_1} & \cdots & a_{i,(n_i-1)r(\chi,i-1)+b_{r(\chi,i-1)}} \end{pmatrix}$$

similarly missing points in the decomposition are fixed ones. Hence, by applying π_i to the above image we can see the desired result as

$d(n_i, k_i)(\alpha^{\pi_{i-1}}) = (f(n_i, k_i)\alpha)^{\pi_i}$. As the diagram is commutative, by Lemma 5.5 the groups are isomorphic. □

In the paper [11], the classification of S_χ is given by using measure theory. We will not get into the details and only give the result about the classification.

For the infinite sequence of tuples $\chi = \langle (1, k_0), (n_1, k_1), \ldots \rangle$ define the characteristic of χ as $char(\chi) = \prod_{i=1}^{\infty} n_i$ and characteristic series of χ as

$$\mu(\chi) = \sum_{i=0}^{\infty} \frac{k_i}{n_1 \ldots n_i} = k_0 + \frac{k_1}{n_1} + \frac{k_2}{n_1 n_2} + \ldots$$

Also, denote by $\mathbf{S}$, the set of all infinite sequences $\chi = \langle (1, k_0), (n_1, k_1), \ldots, \rangle$ where $k_0 > 0$, $k_i \geqslant 0$, $n_i \geqslant 1$ for all $i \geqslant 1$ and by $\mathbf{S_1}$, the subset of the sequences that have $k_0 \geqslant 2$, $n_i \geqslant 2$.

Definition 5.7. *[11, Definition 2] For $\chi_1, \chi_2 \in \mathbf{S}$ we say that χ_1 and χ_2 are $(u, v)-$commensurable for positive integers u, v if*

$1)\mathbf{u}char(\chi_1) = \mathbf{v}char(\chi_2).$

$2)\mu(\chi_1)$ *and* $\mu(\chi_2)$ *are both convergent or divergent together.*

$3)$ *If they are convergent, then* $\mathbf{v}\mu(\chi_2) = \mathbf{u}\mu(\chi_1)$.

$4)$ *The sequences χ_1, χ_2 can have finitely or infinitely many zero members k_i simultaneously.*

Theorem 5.8. *[11, Theorem 3.2] The direct limit groups S_{χ_1} and S_{χ_2} for some $\chi_1, \chi_2 \in \mathbf{S}$ are isomorphic if and only if χ_1 and χ_2 are commensurable.*

The next theorem yields similar results in the case of $S(\xi)$. As done in constructing $S(\xi)$, one can also consider the alternating groups $A(\chi, n)$ which are the subgroups of $S(\chi, n)$ and define the group $A_\chi := \bigcup_{i=1}^{\infty} A(\chi, i)$.

Theorem 5.9. *[11, Theorem 3.1] Let $\chi \in \mathbf{S}$. Then we have the followings;*

- $S_\chi = A_\chi$ *if and only if the characteristic, $char(\chi)$, of χ is divisible by 2^∞.*
- *If $char(\chi)$ is not divisible by 2^∞, then $[S_\chi : A_\chi] = 2$*
- A_χ *is a simple group.*

In the next theorem, we will see the relation between the diagonal direct limit group S_χ and the homogeneous symmetric group $S(\xi)$.

Definition 5.10. *Let G acts on a set X. Then the **stabilizer** of a point x in X is defined to be the set $Stab_G(x) = \{g \in G \mid x^g = x\}$*

Theorem 5.11. *Let H_ξ be the subgroup of $Hom(\partial T_\xi)$ which is isomorphic to the homogeneous symmetric group, $S(\xi)$, for a sequence $\xi = (p_1, p_2, \ldots)$ and let $\gamma \in \partial T_\xi$. Then $Stab_{H_\xi}(\gamma) \cong S_\chi$ where S_χ is the group constructed as the diagonal direct limits of finite symmetric groups with $\chi =< (1, p_1 - 1), (p_2, p_2 - 1), \ldots >$.*

Proof. Let $\gamma \in \partial T_\xi$. Then $Stab_{H_\xi}(\gamma) = \bigcup_{k=1}^{\infty} Stab_{H_k}(\gamma)$. Obviously, $Stab_{H_k}(\gamma) \subseteq Stab_{H_{k+1}}(\gamma)$, and since H_k is isomorphic to finite symmetric group on $f_\xi(k) = p_1 p_2 \ldots p_k$ letters and $Stab_{H_k}(\gamma) \cong Sym(f_\xi(k) - 1)$, by Theorem 5.6, it is enough to show that the embeddings $Stab_{H_k}(\gamma) \longrightarrow Stab_{H_{k+1}}(\gamma)$ are diagonal. $Stab_{H_k}(\gamma)$ acts on the vertices of level k and it fixes the vertex corresponding to γ on this level. Notice that the group, $Stab_{H_k}(\gamma)$ is subgroup of H_ξ and acts on $\partial T_\xi \backslash \partial T_{\gamma(k)}$. The lengths of orbits of $Stab_{H_k}(\gamma)$ in $Stab_{H_{k+1}}(\gamma)$ are either 1 or $p_1 p_2 \ldots p_k$ because $Stab_{H_k}(\gamma)$ acts as full permutation group on $f_\xi(k) - 1$ vertices. Since the embedding of H_k into H_{k+1} is strictly diagonal, the embedding of $Stab_{H_k}(\gamma)$ into $Stab_{H_{k+1}}(\gamma)$ is diagonal. On the other hand $Stab_{H_1}(\gamma) \cong Sym(p_1 - 1)$, $Stab_{H_2}(\gamma) \cong Sym((p_1 - 1)p_2 + p_2 - 1) = Sym(p_1 p_2 - 1)$ and $Stab_{H_k}(\gamma) \cong Sym(p_1 \ldots p_k - 1)$. Hence the stabilizer, $Stab_{H_\xi}(\gamma)$, is isomorphic to S_χ where $\chi = < (1, p_1 - 1), (p_2, p_2 - 1), \ldots >$. □

5.2 Centralizers of Elements in S_χ

In this section, our aim is to obtain the structure of centralizers of arbitrary elements in the locally finite group S_χ. It turns out that the centralizer contains homogeneous monomial groups.

Finite monomial groups are studied by Ore in [17]. In the paper, he investigates some properties of monomial groups and determine all normal subgroups of the class. Starting with the finite monomial groups and using the strictly diagonal embeddings, one can find the homogeneous monomial groups, which is constructed by Kuzucuoğlu, Oliynyk and Suschansky in [10]. In the article [10], they classified all the homogeneous monomial groups by using the lattice of Steinitz numbers and find the structure of centralizer of elements in homogeneous monomial groups.

The monomial group of degree n over a group H is denoted by $\Sigma_n(H)$. By [17], the monomial group is isomorphic to $S_n \ltimes \underbrace{(H \times \ldots \times H)}_{\text{r-many}}$ or in the wreath product notation, $\Sigma_n(H) \cong H \wr S_n$. For any sequence ξ consisting of primes, by taking strictly diagonal embeddings of finite monomial groups $\Sigma_n(H)$ we have the homogeneous monomial groups which is denoted by $\Sigma_\xi(H)$. For the notations and definitions see

[10]. If we take H to be the identity group, then $\Sigma_\xi(1)$ will be the homogeneous symmetric group $S(\xi)$. The centralizers of elements in the homogeneous monomial groups are studied in [10, Theorem 2.6].

Now, we turn our attention to the centralizers of elements in the group S_ξ of diagonal type. Let $\chi = \langle(1, k_0), (n_1, k_1), (n_2, k_2), \ldots\rangle$ and $S_\chi = \bigcup_{i=1}^{\infty} S(\chi, i-1)$. For any element α in S_χ, we have a smallest number so that $\alpha \in S(\chi, n)$. Therefore, we can define the following;

Definition 5.12. *For $\alpha \in S_\chi = \bigcup_{n=1}^{\infty} S(\chi, i)$, let n be the smallest integer such that $\alpha \in S(\chi, n)$. Then the* ***principal beginning*** *α_0 of α is the element in the finite symmetric group $S_{r(\chi,n-1)}$ of which the image in the group S_χ is α.*

Notice that the definition of principal beginning is similar to the case of homogeneous symmetric groups, see Definition 2.3.

Definition 5.13. *The* ***short cycle type*** *of an element $\alpha_0 \in S_n$ is $t(\alpha_0) = (r_1, \ldots, r_t)$ where r_i, $1 \leqslant i \leqslant t$, is the number of $i-$cycles appearing in the cycle decomposition of α_0 and t is taken to be the biggest cycle length that appears in the decomposition.*

Theorem 5.14. *Let $\alpha \in S_\chi$, $\chi = \langle(1, k_0), (n_1, k_1), \ldots\rangle$ and let $\alpha_0 \in S_{r(\chi,l-1)}$ be the principal beginning of α and $t(\alpha_0) = (r_1, r_2, \ldots, r_k)$ be the short cycle type of α_0 where r_1 is the number of fixed vertices other than $\$\$\ldots\$$ in level l. Then the centralizer of α in S_χ;*

$$C_{S_\chi}(\alpha) \cong \operatorname*{Dr}_{i=2}^{k} \Sigma_{\xi_i}(C_i) \times S_{\chi'}$$

where $\xi = (k_0, n_1, n_2, \ldots)$, $char(\xi_i) = \frac{char(\xi)}{k_0 n_1 \ldots n_{l-1}} r_i$ for all $i \geqslant 2$, $\chi' = \langle(1, r_1), (n_l, k_l), \ldots\rangle$ and C_i is the cyclic group of order i.

Proof. Let $\alpha_0 \in S_{r(\chi,l-1)}$ be the principal beginning of α. Now we know the cycle type of α_0 and there are r_i many i cycles and r_1 many fixed points except the vertex $\$\$\$\ldots\$$ in level l.

Note that, since $\alpha_0 = x_{1,0} x_{2,0} \ldots x_{k,0}$ where $x_{i,0}$ is the product of i-cycles in the cycle decomposition of α_0 , and $\alpha = x_1 x_2 \ldots x_k$ where the principal beginning of x_i is $x_{i,0}$ for $1 \leqslant i \leqslant k$, by using the same method as in the paper [5], we have

$$C_{S_\chi}(\alpha) = \operatorname*{Dr}_{i=1}^{k} C_{S_\chi}(x_i)$$

Therefore, it is enough to find the centralizer of an element with a fixed cycle type.

Observe that for an element x with principal beginning $x_0 \in S(\chi, l)$ which is a product of i-cycles $i \geqslant 2$, the embedding of x_0 into $S(\chi, l+1)$ is strictly diagonal. So by [5, Theorem 3] and [10, Corollary 2.7], we have $C_{S_\chi}(x_i) = \Sigma_{\xi_i}(C_i)$ where $char(\xi_i) = \frac{char(\xi)}{k_0 n_1 \ldots n_{l-1}} r_i$ and $\Sigma_{\xi_i}(C_i)$ is the homogeneous monomial group over the cyclic group C_i of order i.

For the centralizer of x_1 which is identity but is formed with the fixed points of α_0 in level l, we have r_1 many fixed points and any element in symmetric group, S_{r_1}, on r_1 vertices will commute with α_0. The embedding of S_{r_1} into $S(\chi, l+1)$ is diagonal and the image is isomorphic to a subgroup of the symmetric group, $S_{r_1 n_l + k_l}$. Continuing like that we will have the diagonal embeddings of finite symmetric groups which is isomorphic to $S_{\chi'}$ where $\chi' =< (1, r_1), (n_l, k_l), \ldots >$. Hence, we get the result.

$$C_{S_\chi}(\alpha) \cong \underset{i=2}{\overset{k}{Dr}} \Sigma_{\xi_i}(C_i) \times S_{\chi'}$$

where $\xi = (k_0, n_1, n_2, \ldots)$, $char(\xi_i) = \frac{char(\xi)}{k_0 n_1 \ldots n_{l-1}} r_i$ for all $i \geqslant 2$, $\chi' = \langle (1, r_1), (n_l, k_l), \ldots \rangle$.

□

Corollary 5.15. *If $k_i = 0$ for all $i > 0$, then as $S_\chi = S(\xi)$ we get*

$$C_{S(\xi)}(\alpha) \cong \underset{i=1}{\overset{k}{Dr}} \Sigma_{\xi_i}(C_i)$$

where $\xi = (k_0, n_1, n_2, \ldots)$, $char(\xi_i) = \frac{char(\xi)}{r(\chi, l-1)} r_i$ for all $i \geqslant 1$.

5.3 Centralizers of Finite Subgroups in S_χ

In this section following the steps that are done in [5], we will determine the structure of centralizer of finite subgroups of S_χ.

Definition 5.16. *For a finite subgroup $F \leqslant S_\chi$, let $F \leqslant S(\chi, k)$ where k is the smallest such level. Then **the type** of F is defined by $t(F) = ((m_1, r_1), (m_2, r_2), \ldots, (m_k, r_k))$ where m_i is the smallest level in which F has an orbit Ω_i on $r(\chi, m_i - 1)$ vertices and it has r_i many equivalent orbits giving equivalent representations of F. Note that m_i's are not necessarily distinct. Without loss of generality, if F has fixed points, then (m_1, r_1) will represent the equivalent orbits of length 1.*

Theorem 5.17. *Let F be a finite group of S_χ with sets of orbits $\Gamma_1, \Gamma_2, \ldots, \Gamma_k$ where Γ_i is the set of all equivalent orbits.*

Let the type of F be $t(F) = ((m_1, r_1), (m_2, r_2), \ldots, (m_k, r_k))$. Then

$$C_{S_\chi}(F) \cong \underset{i=2}{\overset{k}{Dr}} \, \Sigma_\xi(C_{Sym(\Omega_i)}(F_{|\Omega_i})) \times S_{\chi_1}$$

where $char(\xi_i) = r_i \prod_{j=1}^{\infty} n_{m_j}$ and $\chi_1 =< (1, r_1), (n_{m_1}, k_{m_1}), (n_{m_1+1}, k_{m_1+1}), \ldots, >$ and Ω_i is a representative in the equivalence class, Γ_i, of orbits for all $i = 2, 3, \ldots, k$.

Proof. Let Σ be the set of all orbits of F on $\mathbb{N}$. We can define a relation on Σ as follows; $O_i \sim O_j$ if and only if the actions of F on both orbits are equivalent. It is easy to verify that this relation is an equivalence relation. Form the equivalence classes as $\Gamma_1, \ldots, \Gamma_k$.

Observe that $C_{S_\chi}(F)$ leaves each Γ_i invariant and moreover, there exists two orbits Δ_1 and Δ_2 such that $\Delta_1^g = \Delta_2$ for some $g \in C_{S_\chi}(F)$ if and only if Δ_1 and Δ_2 are equivalent orbits. In fact, by [2, Ex. 4.2.4] for two equivalent orbits O_i and O_j in Γ_i the bijection

$$\lambda : O_i \longrightarrow O_j$$

satisfying for any $f \in F$, $x \in O_i$ $\lambda(x^f) = \lambda(x)^f$ can be extended to a bijection c of $\mathbb{N}$ so that $c \in C_{S_\chi}(F)$.

With this observation, we can write $C_{S_\chi}(F)$ as the direct product of the centralizers where for each non-equivalent action of F we have a direct factor. So it is enough to find the structure of the centralizer of F where each orbit in the action of F is equivalent.

Let the type of F be $t(F) = ((m_1, r_1), (m_2, r_2), \ldots, (m_k, r_k))$. Consider the restriction of the action of F on level m_i where $i \geqslant 2$ and consider the equivalence class Γ_i corresponding to (m_i, r_i). Let Ω_i be a representative of this class, we know there are r_i many copies of Ω_i with the same action. By above observation we have $C_{S(\chi, m_i)}(F) \cong C_{Sym(\Omega_i)}(F_{|\Omega_i}) \wr S_{r_i} \cong \Sigma_{r_i}((C_{Sym(\Omega_i)}(F_{|\Omega_i})))$ when we consider the centralizer of F on level m_i+1 it will be isomorphic to $C_{S(\chi, m_i)}(F) \cong C_{Sym(\Omega_i)}(F_{|\Omega_i}) \wr S_{r_i n_{m_i}} \cong \Sigma_{r_i n_{m_i}}((C_{Sym(\Omega_i)}(F_{|\Omega_i})))$ where the embedding is the strictly diagonal embedding. Continuing like this we get the centralizer to be isomorphic to

$$C_{S_\chi}(F_{|\Omega_i}) \cong \Sigma_{\xi_i}(C_{Sym(\Omega_i)}(F_{|\Omega_i}))$$

where $char(\xi_i) = \frac{char(\chi)}{n_1 n_2 \ldots n_{m_{i-1}}} r_i$.

As for the orbits of length 1, on level m_1, we have r_1 many fixed points. Then any element from S_{r_1} will commute with elements of F. After the level m_1, the embedding of S_{r_1} is inherited from the diagonal embedding of the group S_χ. Therefore, in the end we will have a direct factor in the centralizer of F in S_χ which is S_{χ_1} where $\chi_1 =< (1, r_1), (n_{m_1}, k_{m_1}), (n_{m_1+1}, k_{m_1+1}), \ldots >$.

Hence, we get the result. □

5.4 Some topological properties of $\partial T_\chi \backslash \{\delta\}$

Recall that we define the metric ρ on the set $\partial T_\chi \backslash \{\delta\}$ as $\rho(\gamma_1, \gamma_2) = \frac{1}{n+1}$ where n is the common parts of the ends γ_1, γ_2.

The balls which we denote as P_{nv} consist of ends passing through the vertex v on level n.

Lemma 5.18. *[11, Lemma 2.4] Let* $\chi = \langle (1, k_0), (n_1, k_1), \ldots \rangle \in \mathbf{S_1}$*, that is* $k_0 \geqslant 2$ *and* $n_i \geqslant 2$ *for all* i*. Then* $\partial T_\chi \backslash \{\delta\}$ *is locally compact and Hausdorff.*

Proof. Recall the definition of locally compactness. A topological space X is locally compact if every point $x \in X$ has a compact neighborhood.

Let $\gamma \in \partial T_\chi \backslash \{\delta\}$. Then as γ is different from δ, there exists a vertex v on some level m which is different then the vertex $\$\$ \ldots \$$ on the level m. Consider the ball P_{mv}. Obviously, P_{mv} contains the end γ. On the other hand, by construction of the tree T_χ,the subtree P_{mv} can be identified with the spherically homogeneous tree with characteristic sequence $(n_m, n_{m+1}, \ldots)$ which is shown to be compact by Lemma 2.36.

The proof of the property of being Hausdorff, is the same as in the case of ∂T_Ω, see Lemma 2.36. □

Lemma 5.19. *[11, Lemma 2.4] The group S_χ satisfies the Rubin's theorem 4.3.*

Proof. First of all $\partial T_\chi \backslash \{\delta\}$ is locally compact and Hausdorff. Let D be an arbitrary open set and $\gamma \in D$. Then we need to show that the set

$$B = \{g(\gamma) \mid g \in S_\chi \text{ and } g_{|(\partial T_\chi \backslash \{\delta\}) \backslash D} = identity\}$$

is somewhere dense (that is the interior of the closure is nonempty).

Consider an arbitrary ball P_{mv} in D containing γ which is identified with the spherically homogeneous tree T_{Ω_m} where $\Omega_m = (n_m, n_{m+1}, \ldots)$. Then the homogeneous finite symmetric group $S(\Omega_m)$ lies inside the group S_χ.

On the other hand, the set $A = \{g(\gamma) \mid g \in S(\Omega_m) \text{ and } g_{|\partial T_{\Omega_m} \backslash D} = identity\}$ which is somewhere dense by Lemma 4.4, obviously lies in B. Hence, B is also somewhere dense. □

The above lemma tells that any automorphism of the group S_χ is induced by a homeomorphism of the space $\partial T_\chi \backslash \{\delta\}$. Furthermore we have the following result.

Theorem 5.20. *The automorphism group $Aut(S_\chi)$ is isomorphic to the normalizer $N = N_{Hom(\partial T_\chi \backslash \{\delta\})}(S_\chi)$ in $Hom(\partial T_\chi \backslash \{\delta\})$.*

Proof. The proof is the same as in the case of $Aut(H_\Omega)$, see the proof of Theorem 4.5. □

For a metric space X with a metric ρ, recall the definition of uniform local isometry.

Definition 5.21. *A bijection α of the metric space X is called **uniform local isometry** if there exists a positive number δ, satisfying $\rho(x_1^\alpha, x_2^\alpha) = \rho(x_1, x_2)$ for all x_1, x_2 with the property $\rho(x_1, x_2) < \delta$.*

Lemma 5.22. *All finitely generated subgroups of $ULI(\partial T_\chi \backslash \{\delta\})$ are residually finite.*

Proof. Let G be a finitely generated subgroup in $ULI(\partial T_\chi \backslash \{\delta\})$. By definition of uniform local isometry there exist $\delta > 0$ such that for any $g \in G$, if $\rho(x, y) \leqslant \delta$, then $\rho(x^g, y^g) = \rho(x, y)$. Now let $m = \lfloor 1/\delta \rfloor$ and let $\chi' =< (1, r(\chi, m-1)), (n_m, k_m), \ldots >$

. Consider $Aut(T'_\chi)$, since every element of G preserves the distance after the level m we can regard G as a subgroup of isometry group $Aut(T'_\chi)$ and $Aut(T'_\chi)$ is residually finite.(As the intersection of level stabilizers is trivial.) □

Lemma 5.23. *For $g \in ULI(\partial T_\chi \backslash \{\delta\})$, there exist $\alpha \in Aut(T_\chi)$ and $\beta \in S_\chi$ such that $g = \alpha\beta$.*

Proof. The proof is the same as in the case of the spherically homogeneous tree T_Ω, see Lemma 3.16. □

5.5 Level Preserving Automorphism of S_χ

For an increasing sequence $N = \{a_1 \geqslant 3, a_2, \ldots\}$, an automorphism α of S_χ which has the property $\alpha(S(\chi, a_i)) = S(\chi, a_i)$ for all i is called **N-level preserving automorphism** of S_χ. Before the main result we must prove two lemmas.

Lemma 5.24. $C_{S(\chi,j)}(S(\chi,i)) \cap C_{S(\chi,k)}(S(\chi,j)) = 1$ *for* $j \geqslant 3$

Proof. As $S(\chi, j)$ is isomorphic to a symmetric group and the symmetric groups of order bigger than or equal to 6 has identity center, we have the result. □

Lemma 5.25. $C_{S(\chi,j)}(S(\chi,i)) \cong S_{n_i \ldots n_{j-1}} \times S_{\phi(i,j)}$ *where $\phi(i,j)$ is the number of fixed points of the action of $S(\chi,i)$ in the group $S(\chi,j)$*

Proof. Note that when we embed $S(\chi,i)$ to $S(\chi,j)$ by diagonal embedding there are $\phi(i,j)$ many orbits of length 1 and the symmetric group consisting of $\phi(i,j)$ many elements will lie inside the centralizer $C_{S(\chi,j)}(S(\chi,i))$.

Since the embedding is diagonal we have $n_i n_{i+1} \ldots n_{j-1}$ many orbits with $r(\chi, i-1)$ many elements. And by definition of the diagonal embedding the actions on all orbits are equivalent to the action of $S(\chi,i)$ on $r(\chi, i-1)$ elements and the elements permuting the orbits will also centralize the group $S(\chi,i)$. Hence by [2, Chapter 4.2], $C_{S(\chi,j)}(S(\chi,i)) \cong S_{n_i \ldots n_{j-1}} \times S_{\phi(i,j)}$ □

For an increasing sequence, $N = \{a_i \mid a_i > 2\}$, of positive numbers, denote the N-level preserving automorphisms of the group S_χ by B_N. Then we have the following result.

Proposition 5.26. *B_N forms a group and it is isomorphic to the Cartesian product of the centralizers $C_{S(\chi,a_{i+1})}(S(\chi,a_i))$.*

Proof. Let $\alpha \in Aut(S_\chi)$. Then $\alpha_{|S(\chi,a_i)}$ is an automorphism of $S(\chi,a_i)$ since the groups $S(\chi,a_i)$ are isomorphic to a symmetric group of order bigger than 6 (as $a_i > 2$) the automorphism $\alpha_{|S(\chi,a_i)}$ is inner. Hence, there exist $\alpha_i \in S(\chi,a_i)$ such that $\alpha_i^{-1}\alpha_{|S(\chi,a_i)}$ acts as identity on $S(\chi,a_i)$. On the other hand, since the groups satisfy $S(\chi,a_i) \leqslant S(\chi,a_{i+1})$, for any element $g \in S(\chi,a_i)$ we must have $g^{\alpha_i^{-1}\alpha_{i+1}} = g$. Hence, $\alpha_i^{-1}\alpha_{i+1} \in C_{S(\chi,a_{i+1})}(S(\chi,a_i))$.

The rest of the proof is the same as the proof of the Proposition 4.8 for the N-level preserving automorphisms of $S(\xi)$ in Chapter 4. □

CHAPTER 6

HOMOGENEOUS FINITARY SYMMETRIC GROUPS

In this chapter, we will study the tree connection of the groups $FSym(\kappa)(\xi)$ first introduced in [4] and studied in further detail in [5].

6.1 Construction of the group $FSym(\kappa)(\xi)$

For any infinite cardinal κ, we start with the finitary symmetric group, $FSym(\kappa)$ and we regard κ as an ordinal number. Consider the embeddings

$$d^p : FSym(\kappa) \to FSym(\kappa p)$$

where the image of any $\alpha \in FSym(\kappa)$ is given by

$$(\kappa s + i)^{d^p(\alpha)} = \kappa s + i^{\alpha}, \quad i \in \kappa \text{ and } 0 \leqslant s \leqslant p-1.$$

As in the finite case, see Section 2.1, we divide the ordinal κp into p equal parts and in each part the action of $d^p(\alpha)$ is diagonally the same as the action of α.

If $\alpha = \begin{pmatrix} 1 \dots n \\ i_1 \dots i_n \end{pmatrix} \in FSym(\kappa)$, then

$$d^p(\alpha) = \left(\begin{array}{ccc|ccc|c|ccc} 1 & \dots & n & \kappa+1 & \dots & \kappa+n & \dots & \kappa(p-1)+1 & \dots & \kappa(p-1)+n \\ i_1 & \dots & i_n & \kappa+i_1 & \dots & \kappa+i_n & \dots & \kappa(p-1)+i_1 & \dots & \kappa(p-1)+i_n \end{array} \right)$$

with the assumption that the elements in $\kappa(s+1) \backslash (\kappa s + supp(\alpha))$ is fixed for $s = 0, \dots, p-1$.

Let $\xi = (p_1, p_2, \dots)$ be an infinite sequence of not necessarily distinct prime numbers as before and $n_i = p_1 p_2 \dots p_i$. Consider the embedding sequences in the following way;

$$FSym(\kappa) \xrightarrow{d^{p_1}} FSym(\kappa p_1) \xrightarrow{d^{p_2}} FSym(\kappa p_1 p_2) \ldots$$

$$Alt(\kappa) \xrightarrow{d^{p_1}} Alt(\kappa p_1) \xrightarrow{d^{p_2}} Alt(\kappa p_1 p_2) \ldots$$

Then the direct limit group constructed with the above embedding sequences will be $FSym(\kappa)(\xi)$ and $Alt(\kappa)(\xi)$, respectively. If in the direct limit group the image of $FSym(\kappa n_i)$ ($Alt(\kappa n_i)$) is denoted by $FSym(\kappa)(n_i)$ ($Alt(\kappa)(n_i)$), then we can write

$$FSym(\kappa)(\xi) = \bigcup_{i=1}^{\infty} FSym(\kappa)(n_i)$$

$$Alt(\kappa)(\xi) = \bigcup_{i=1}^{\infty} Alt(\kappa)(n_i)$$

The group $FSym(\kappa)(\xi)$ is called **homogeneous finitary symmetric group**. The classification of such groups and the structure of the centralizers of elements as well as the finite groups was done by Kuzucuoğlu, Kegel and myself in [5].

6.2 A non-locally finite spherically homogenous tree

Similar to the finite and diagonal case, in this section we will construct a spherically homogeneous rooted tree (which is not locally finite this time) and the homogeneous finitary symmetric group will act on the boundary of this tree.

Let $\xi = (p_1, p_2, \ldots)$. Using this infinite sequence we will construct the vertex set and the edge set. Denote the root by $\varnothing$ as usual. Since we start the embeddings with $FSym(\kappa)$, let there be κ many vertices in the first level. For the second level, let there be p_1 many vertices coming down from each vertex on the first level. Hence, as an ordinal number we have κp_1 many vertices in the second level. For the third level, do the same thing by using the prime p_2 and so we will have $\kappa p_1 p_2$ many vertices. Continuing like this we will have the tree $T_{\kappa(\xi)}$. See the figure below.

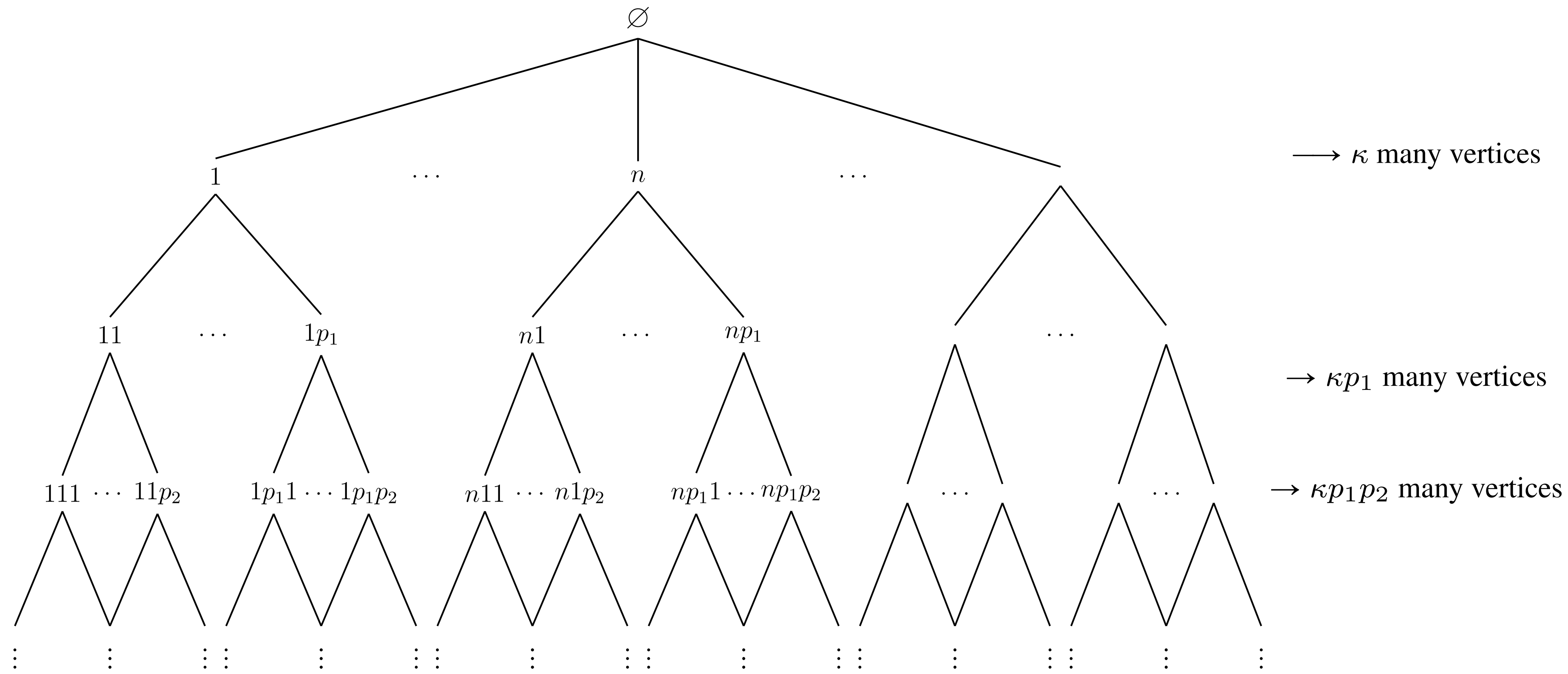

Figure 6.1: Spherically homogeneous rooted tree $T_{\kappa(\xi)}$

Lemma 6.1. *$T_{\kappa(\xi)}$ is a spherically homogeneous tree.*

Proof. Recall that if the full automorphism group of a tree acts transitively on each level of the tree, then the tree is called spherically homogeneous.

Since the tree is rooted, any automorphism will fix the root. Also by definition of the automorphism of a rooted tree, the levels of the vertex must be protected. Let v and w be two vertices of level n. By the construction of the tree, the subtrees $T_{\kappa v}$ and $T_{\kappa w}$ which are the rooted subtrees with root v and w respectively, are isomorphic. Now, we can extend this isomorphism to an automorphism of the tree.

If on level $n-1$ the vertices v and w are adjacent to the same vertex, then the map sending $T_{\kappa v}$ to $T_{\kappa w}$ and fixing all the other vertices will be an automorphism of $T_{\kappa(\xi)}$. If they are not adjacent on level $n-1$, consider the first vertex that they are connected by a path, name the vertex as u. By the construction $T_{\kappa u}$ is a locally finite spherically homogenous tree. Hence, we can find an automorphism of $T_{\kappa u}$ that sends v to w. Extend this automorphism to an automorphism of the tree $T_{\kappa(\xi)}$ by fixing every vertex not belonging to the tree $T_{\kappa u}$. Hence, in any case we find an automorphism of the tree $T_{\kappa(\xi)}$ that sends v to w. □

Consider the boundary $\partial T_{\kappa(\xi)}$ of the tree $T_{\kappa(\xi)}$. Since the tree is spherically homogeneous we can define the metric ρ which is mentioned in the Section 2.2.2. Recall the metric is as follows;

Let γ_1, γ_2 be two ends in $\partial T_{\kappa(\xi)}$. Define $\rho(\gamma_1, \gamma_2) = \frac{1}{n+1}$ where n is the length of common parts of the ends γ_1, γ_2. Recall that the map ρ defines an ultra-metric, for the proof see the proof of Lemma 2.20 and the explanation below it.

With the ultra-metric ρ, $\partial T_{\kappa(\xi)}$ becomes a metric space. Since the trees $T_{\kappa(\xi)}$ and T_Ω are spherically homogeneous, they share some of the topological properties and definitions.

For the space $\partial T_{\kappa(\xi)}$ the basic open sets are;

$$P_{nv_i} = \{\gamma \in \partial T_{\kappa(\xi)} \mid v_i \in V(T_{\kappa(\xi)}),\ v_i \in \gamma\}$$

In other words, basic open set P_{nv_i} includes the ends that passes through the vertex v_i

on the level n.

One can easily see that P_{nv_i} is the boundary of the spherically homogenous rooted tree with root v_i and characteristic sequence $\xi_1 =< p_{n+1}, p_{n+2}, \dots >$.

We will give some properties of this metric space, and these properties will be used in the next section.

Recall that, a space X is locally compact if every element has a compact neighborhood and called Hausdorff if for any two elements $x \neq y$ in the space X there exist open sets U and V satisfying $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

Lemma 6.2. *$\partial T_{\kappa(\xi)}$ is a locally compact and Hausdorff space.*

Proof. Let $\gamma \in \partial T_{\kappa(\xi)}$ where $\xi =< p_1, p_2, \dots >$. Consider the basic open set P_{nv_i} which contains the end γ. By Lemma 2.36, P_{nv_i} is compact. Hence, $\partial T_{\kappa(\xi)}$ is locally compact.

For the Hausdorff property, let γ_1, γ_2 be two ends. Let $\rho(\gamma_1, \gamma_2) = \frac{1}{n+1}$, that is they have n many common parts and after n^{th} level the ends belong to different balls. Choose two balls P_{n+1v}, P_{n+1w} on the $(n+1)^{th}$ level such that $\gamma_1 \in P_{n+1v}$ and $\gamma_2 \in P_{n+1w}$. By the Remark 2.24, two balls on the same level is either the same or disjoint we have $P_{n+1v} \cap P_{n+1w} = \emptyset$. □

6.3 Tree connection of the homogeneous finitary symmetric groups

In this section we will construct a subgroup of the homeomorphism group of $\partial T_{\kappa(\xi)}$ which will be isomorphic to the homogeneous finitary symmetric group.

Definition 6.3. *Define* $\mathbf{FH_n}$ *as the subgroup of the homeomorphism group of $\partial T_{\kappa(\xi)}$ that only permutes the finitely many balls of level n. It is the similar case of H_n of locally finite homogeneous tree T_Ω, see Definition 3.13.*

Since the group FH_n acts transitively on the set of balls on the same level, there exists an element from FH_n that sends the ball P_{nv_i} to P_{nv_j}. It takes the ball P_{nv_i} and glues it on P_{nv_j}. Different from the case of the group H_n, on level n we have κp_{n-1} many vertices, however the group FH_n permutes only finitely many of them. Hence, the group is isomorphic to $FSym(\kappa p_{n-1})$.

Lemma 6.4. *FH_n is a subgroup of $Hom(\partial T_{\kappa(\xi)})$.*

Before the proof, we need to label the vertices and ends. Let $\xi =< p_1, p_2, \ldots >$. For the labeling of the vertices see the Figure 6.1. Enumerate the κ many vertices of first level starting from 1. (With the use of ordinal numbers). For the second level, label the vertices adjacent to the vertex n on the first level by $n1, n2, \ldots np_1$. Continue like this to have labeled vertices. After this labeling, since an end can be expressed by the adjacent vertices, say $\gamma = (\varnothing, v_1, v_2, \ldots)$ where v_i is the vertex on level i, just put the labels of vertices as follows; Since v_i and v_{i+1} are adjacent for all $i \in \mathbb{N}$, if $v_i = a_1a_2\ldots a_i$ where $a_1 \in |\kappa|$, $a_j \in \{1, 2, \ldots p_j\}$ for all $j > 1$, then $v_{i+1} = a_1a_2\ldots a_ia_{i+1}$. Hence, we can write the vertex $\gamma = (\varnothing, a_1, a_1a_2, \ldots, a_1a_2\ldots a_i, \ldots)$ or simply we will write $\gamma = a_1a_2\ldots a_i\ldots$. Now we can prove the above lemma.

Proof. Let α be an element of FH_n. Let $\gamma_1 = a_1a_2\ldots \in P_{nv}$ and $\gamma_2 = b_1b_2\ldots \in P_{nw}$ be two ends in $\partial T_{\kappa(\xi)}$. The element α is one-to-one. Indeed, let $\alpha(\gamma_1) = \alpha(\gamma_2)$. By the labeling of the vertices, we can say $v = a_1a_2\ldots a_n$ and $w = b_1b_2\ldots b_n$. Since α is a permutation on the balls of level n, and it sends γ_1 and γ_2 to the same element we have $v = w$, that is $a_i = b_i$ for all $1 \leqslant i \leqslant n$. Note that by definition α does not change the coordinates of an end after the n^{th} coordinate, hence we have $a_i = b_i$ for all $i > n$. Hence, α is one-to-one.

Let $\gamma = a_1a_2\ldots a_n\ldots$ be an arbitrary element. Since α is a permutation on the balls of level n there exist a ball P_{nw} where $w = b_1b_2\ldots b_n$ and α maps P_{nw} to P_{nv} where $v = a_1a_2\ldots a_n$. Choose the element $\gamma' = b_1b_2\ldots b_na_{n+1}a_{n+2}\ldots$ in the ball P_{nw}, then $\alpha(\gamma') = \gamma$. Hence, α is onto.

Moreover, α sends basis elements (the balls) to basis elements and is bijective, hence α is a homeomorphism.

□

Lemma 6.5. *For any n, FH_n acts on V_n, the vertex set of level n and for any $k > n$ FH_n is embedded into FH_k via strictly diagonal embedding.*

Proof. Recall that V_n is the set of vertices on level n. With the labeling of the vertices, we can write $V_n = \{a_1a_2 \ldots a_n \mid a_1 \in |\kappa| \;\; a_i \in \{1, \ldots p_i\}\}$. Since an element from FH_n permutes the balls of level n and a ball in level n can be emphasized by its root (a vertex in level n) we can regard the action as the action on V_n, that is if α sends P_{nv} to P_{nw}, then α has action on V_n by sending v to w. Since α is a permutation on the balls of level n, it is also a permutation on V_n. Notice that an element α from FH_n may change only the first n terms of an end and leaves the other parts same hence that means for any $k > n$ the element α lies inside the group FH_k.

Let us consider the permutation groups (FH_n, V_n) and (FH_k, V_k). We will show the embedding is strictly diagonal. Since FH_n is a subgroup of FH_k it acts on V_k. An arbitrary orbit containing an element $v = a_1a_2 \ldots a_k$ of V_k is of the form

$$\Delta_v = \{b_1b_2 \ldots b_n a_{n+1} a_{n+2} \ldots a_k \mid b_1 \in |\kappa| \;\; b_i \in \{1, \ldots p_i\}\}$$

where a_j's are fixed coming from the terms of the vertex v. Define a map $\phi : V_n \longrightarrow \Delta_v$ such that for $v = c_1c_2 \ldots c_n$ the image will be $v^\phi = c_1c_2 \ldots c_n a_{n+1} \ldots a_k$. Clearly the map is a bijective map. Let $\alpha \in FH_n$ and $v = c_1 \ldots c_n \in V_n$ and $\alpha(v) = w = b_1 \ldots b_n$. Then we have

$$\alpha(v^\phi) = \alpha(c_1c_2 \ldots c_n a_{n+1} \ldots a_k) = b_1 \ldots b_n a_{n+1} \ldots a_k = (\alpha(v))^\phi$$

Hence, the embedding is a strictly diagonal embedding. □

For the boundary of the tree $\partial T_{\kappa(\xi)}$, the union $\bigcup_{i=1}^{\infty} FH_i$ will determine a direct limit group, which will be denoted by $FH_{\kappa(\xi)}$. In the next theorem, we will give the connection between $FH_{\kappa(\xi)}$ and finitary homogeneous symmetric groups.

Theorem 6.6. *$FH_{\kappa(\xi)}$ is isomorphic to $FSym(\kappa)(\xi)$*

Proof. Since $FH_{\kappa(\xi)}$ is the direct limit of finitary symmetric groups with strictly diagonal type by theorem 4 of [6] it is isomorphic to the homogeneous finitary symmetric group $FSym(\kappa)(\xi)$. □

The construction of the homogeneous finitary symmetric groups was done in [5], and the structure of centralizers of elements and finite groups was given in the same paper

[5]. Moreover, the complete classification of the homogeneous finitary symmetric groups was done in [6]. In the next section by using topological properties we will give some properties about the automorphism groups of these groups.

6.4 Automorphism group of $FH_{\kappa(\xi)}$

Proposition 6.7. *$Aut(FH_{\kappa(\xi)})$ satisfies the Rubin's theorem 4.3.*

Proof. The topological space $\partial T_{\kappa(\xi)}$ is locally compact and Hausdorff by Lemma 6.2. Let D be an open set and $x \in D$. We will show that the set

$$A = \{g(x) \mid g \in FH_{\kappa(\xi)} \text{ and } g_{|\partial T_{\kappa(\xi)} \setminus D} = id\}$$

is somewhere dense, that is $int(\bar{A}) \neq \emptyset$.

Let $x \in D$ choose the smallest n and choose a vertex v such that $x \in P_{nv}$ and $P_{nv} \subset D$. (If D is given as the whole space, choose the ball on the first level that contains x). Observe that P_{nv} can be identified with the boundary of a locally finite spherically homogeneous tree $T_{\xi'}$ where $\xi' =< p_n, p_{n+1}, \ldots >$. On the boundary of $T_{\xi'}$ consider the homogeneous symmetric group $H_{\xi'}$ which is isomorphic to $S(\xi')$. By the Lemma 4.4, $H_{\xi'}$ satisfies the Rubin's theorem. Hence, the set $B = \{g(x) | \ g \in H_{\xi'} \text{ and } g_{|\partial T_{\xi'} \setminus P_{nv}} = id\}$ has its interior of closure nonempty. Notice that the action of $H_{\xi'}$ on the ball P_{nv} is the same as the action of $FH_{\kappa(\xi)}$ on the ball P_{nv}. Hence, $A = B$ and $int(\bar{B}) \neq \emptyset$.

On the other hand, since $P_{nv} \subset D$ we have $B \subset A$ and by topological properties $\emptyset \neq int(\bar{B}) \subset int(\bar{A})$. □

With the help of Rubin's theorem, we conclude that any automorphism of the finitary homogenous symmetric group is induced by an element of the homeomorphism group of $\partial T_{\kappa(\xi)}$.

Theorem 6.8. *The automorphism group of the group $FH_{\kappa(\xi)}$ is isomorphic to the normalizer of itself in the group of homeomorphisms of $\partial T_{\kappa(\xi)}$.*

Proof. We will show $Aut(FH_{\kappa(\xi)}) \cong N_{Hom(\partial T_{\kappa(\xi)})}(FH_{\kappa(\xi)})$. Denote the normalizer group by N. Define a map

$$\psi : N \longrightarrow Aut(FH_{\kappa(\xi)})$$
$$h \longrightarrow \psi_h : FH_{\kappa(\xi)} \longrightarrow FH_{\kappa(\xi)}$$
$$g \longrightarrow h^{-1}gh$$

where $g \in FH_{\kappa(\xi)}$. Since normalizer acts on the group via conjugation, the map ψ is a homomorphism. By Proposition 6.7, any automorphism is induced by a homeomorphism hence the map ψ is onto. To conclude it is enough to show that the centralizer of the group $FH_{\kappa(\xi)}$ in the homeomorphism group is trivial. Assume not. Let $h \neq id$ be a homeomorphism in the centralizer. Choose an end γ so that $h(\gamma) = \gamma_1 \neq \gamma$. Choose two elements g_1, g_2 in $FH_{\kappa(\xi)}$ fixing γ and $\delta_1 = g_1(\gamma_1) \neq g_2(\gamma_1) = \delta_2$. As g_1, g_2 fixes γ, g_1^h, g_2^h must fix it.

$$h^{-1}g_1h(\gamma) = h^{-1}g_1(\gamma_1) = h^{-1}(\delta_1) = \gamma$$

$$h^{-1}g_2h(\gamma) = h^{-1}g_2(\gamma_1) = h^{-1}(\delta_2) = \gamma$$

But h^{-1} is a bijection so $\delta_1 = \delta_2$, contradiction.

□

6.4.1 Level preserving automorphisms

For the group $FSym(\kappa)(\xi)$, where $\xi =< p_1, p_2, \ldots >$ and p_i's are not necessarily distinct primes, we will define subgroups of automorphism group $Aut(FSym(\kappa)(\xi))$ which we call as level preserving automorphisms.

Definition 6.9. *Consider the sequence $N = (n_1, n_2, \ldots)$ associated with ξ where $n_i = p_1 p_2 \ldots p_i$. Let M be a subsequence of N obtained by deleting some of the terms of N.*
*An automorphism α is called M-**level preserving** if $\alpha(FSym(\kappa)(m_i)) = FSym(\kappa)(m_i)$ for all $i \in \mathbb{N}$.*

If α and β are two M-level preserving automorphisms of $FSym(\kappa)(\xi)$, then obviously $\alpha\beta(FSym(\kappa)(m_i)) = \alpha(FSym(\kappa)(m_i)) = FSym(\kappa)(m_i)$. Hence, M-level preserving automorphisms forms a subgroup. We will denote it by $M-Aut(FSym(\kappa)(\xi))$. Before the characterization of M-level preserving automorphisms, we need two lemmas.

Lemma 6.10. *The centralizer $C_{Sym(\kappa)(m_i)}(FSym(\kappa)(m_{i-1}))$ is isomorphic to the finite group $Sym(\frac{m_i}{m_{i-1}})$.*

Proof. Note that $FSym(\kappa)(m_{i-1})$ has trivial center. By the strictly diagonal embedding the group $FSym(\kappa)(m_{i-1})$ acts on $|\kappa m_i|$ with $\frac{m_i}{m_{i-1}}$ equivalent orbits hence by [2, Ch. 4 Section 2], any element permuting the orbits will be in the centralizer. □

Lemma 6.11. *Let $\xi =< p_1, p_2, \ldots >$. Then $Sym(\kappa)(\xi) = \bigcup_{i=1}^{\infty} Sym(\kappa)(n_i)$ is a subgroup of $Aut(FSym(\kappa)(\xi))$ where $n_1 = 1$ and $n_i = p_1 p_2 \ldots p_{i-1}$.*

Proof. Note that we can talk about the strictly diagonal embeddings

$$Sym(\kappa) \xrightarrow{d^{p_1}} Sym(\kappa p_1) \xrightarrow{d^{p_2}} Sym(\kappa p_1 p_2) \ldots$$

and the direct limit group will be $Sym(\kappa)(\xi) = \bigcup_{i=1}^{\infty} Sym(\kappa)(n_i)$ where $n_1 = 1$, $n_i = p_1 p_2 \ldots p_{i-1}$ and $Sym(\kappa)(n_i)$ is the image of $Sym(\kappa n_i)$ in the direct limit group.

For an element $\alpha \in Sym(\kappa)(\xi)$, there exist a smallest n_i such that $\alpha \in Sym(\kappa)(n_i)$.
<u>Claim:</u> α is an automorphism of $FSym(\kappa)(\xi)$.
As $FSym(\kappa)(n_i) \trianglelefteq Sym(\kappa)(n_i)$, we have $FSym(\kappa)(n_i)^{\alpha} = FSym(\kappa)(n_i)$. On the other hand for $j > i$, since $Sym(\kappa)(n_i) \subset Sym(\kappa)(n_j)$, we can regard α as an element of $Sym(\kappa)(n_j)$. Hence, $FSym(\kappa)(n_j)^{\alpha} = FSym(\kappa)(n_j)$. Therefore, $FSym(\kappa)(\xi)^{\alpha} = FSym(\kappa)(\xi)$. In particular, α is an M-level preserving automorphism where $M = (n_i, n_{i+1}, \ldots)$. □

Theorem 6.12. *$M - Aut(FSym(\kappa)(\xi))$ is isomorphic to the Cartesian product*

$$Sym(\kappa) \times \prod_{i=2}^{\infty} C_{Sym(\kappa m_i)}(FSym(\kappa)(m_{i-1}))$$

In particular, $M - Aut(FSym(\kappa)(\xi))$ has cardinality 2^{κ} and every group of cardinality $< \kappa$ can be embedded into $M - Aut(FSym(\kappa)(\xi))$.

Proof. Let α be an M-level preserving automorphism where $M = (m_1, m_2, \ldots)$. As α preserves the levels m_i for all $i \in \mathbb{N}$, $\alpha(FSym(\kappa)(m_1)) = FSym(\kappa)(m_1)$ Therefore, the restriction of α to the group $FSym(\kappa)(m_1)$ is an automorphism of $FSym(\kappa)(m_1)$. By (Baer-Schrier-Ulam) theorem [2, Theorem 8.2.A]

$$Aut(FSym(\kappa)) = Sym(\kappa)$$

Hence, there exists an element $g_1 \in Sym(\kappa)$ such that $g_1^{-1}\alpha$ is an automorphism of $FSym(\kappa)(\xi)$ as by Lemma 6.11, $g_1 \in Aut(FSym(\kappa)(\xi))$ and by the construction it preserves the levels contained in M. Note that $g_1^{-1}\alpha_{|_{FSym(\kappa)(m_1)}}$ is identity on $FSym(\kappa)(m_1)$. Consider the element $g_1^{-1}\alpha$ restricted to $FSym(\kappa)(m_2)$. Since $g_1^{-1}\alpha$ is an automorphism of $FSym(\kappa)(m_2)$, by the same argument above there exists $g_2 \in Sym(\kappa m_2)$ such that $g_2 = g_1^{-1}\alpha_{|_{FSym(\kappa)(m_2)}}$. Note that g_2 centralizes $FSym(\kappa)(m_1)$. Hence, $g_2 \in C_{Sym(\kappa m_2)}(FSym(\kappa)(m_1))$. Now $g_2^{-1}g_1^{-1}\alpha_{|_{FSym(\kappa)(m_2)}}$ centralizes $FSym(\kappa)(m_2)$ and again by construction it is an M-level preserving automorphsim. Hence, there exists $g_3 = g_2^{-1}g_1^{-1}\alpha_{|_{FSym(\kappa)(m_2)}} \in Sym(\kappa m_3)$ which is also an element of the centralizer $C_{Sym(\kappa m_3)}(FSym(\kappa)(m_2))$.

Continuing like this we will have $\alpha = g_1 g_2 \ldots$ satisfying

1. $g_k^{-1}g_{k-1}^{-1}\ldots g_2^{-1}g_1^{-1}\alpha_{|_{FSym(\kappa)(m_k)}} = id_{FSym(\kappa)(m_k)}$
2. $g_k \in C_{Sym(\kappa m_k)}(FSym(\kappa)(m_{k-1}))$
3. $g_k g_n = g_n g_k$ for all $k, n \in \mathbb{N}$.

Note that the orbits of $FSym(\kappa)(m_{k-1})$ on $|\kappa m_k|$ are the same as the orbits of $Sym(\kappa m_{k-1})$ on $|\kappa m_k|$ and $Z(FSym(\kappa)(m_{k-1})) = Z(Sym(\kappa m_{k-1})) = 1$. Hence,

$$C_{Sym(\kappa m_k)}(FSym(\kappa)(m_{k-1})) = C_{Sym(\kappa m_k)}(Sym(\kappa m_{k-1}))$$

Therefore, $g_k g_n = g_n g_k$.

Let us denote $D_k = C_{Sym(\kappa m_k)}(FSym(\kappa)(m_{k-1})) = C_{Sym(\kappa m_k)}(Sym(\kappa m_{k-1}))$. By the third property, 3, above D_i's are normal subgroups of M-level preserving automorphism group. If we can show that $D_k \cap D_1 \ldots D_{k-1}D_{k+1}\ldots = id$ for any $k \in \mathbb{N}$, then we are done. Assume not. Let $a_k = a_1 a_2 \ldots a_{k-1}a_{k+1}\ldots$ be an element in the intersection. Let j be the smallest integer such that $a_j \neq id$.

If $j > k$ and $a_k = a_j a_{j+1} \ldots$. Then for any element $g \in Sym(\kappa m_k)$ we have $g a_k = g a_j a_{j+1} \ldots = a_j a_{j+1} \ldots g = a_k g$. However, $Sym(\kappa m_k)$ has trivial center, it is a contradiction. If $j < k$ and $a_k = a_j \ldots a_{k-1} a_{k+1} \ldots$, then considering the element $a_j^{-1} = a_{j+1} \ldots a_{k-1} a_k^{-1} a_k \ldots$ we turn into the first case and get again a contradiction. $\square$

CHAPTER 7

NORMALIZERS OF FINITE SUBGROUPS

A group acting on a set Ω is called semi-regular if the point stabilizers are identity for all $\alpha \in \Omega$. A transitive semi-regular group is called regular. In this chapter, we will find the structure of a normalizer of a semi-regular finite subgroup of $S(\xi)$, see Chapter 2. For a regular subgroup of $Sym(\Omega)$, the structure of normalizers are well known. However, for readers convenience in the next two lemmas we will give the structure of centralizers and normalizers of regular subgroups.

Lemma 7.1. *The centralizer of the right regular representation of a group G is the left regular representation (and vice versa). Moreover, right regular representation is conjugate to the left regular representation.*

Proof. Let $\rho : G \longrightarrow Sym(G)$ and $\lambda : G \longrightarrow Sym(G)$ be right and left regular representations of G, respectively. We will show that $C_{Sym(G)}(\rho(G)) = \lambda(G)$. Let $\pi \in Sym(G)$ be an element in the centralizer. Then for all $g \in G$ we have $\rho_g \pi = \pi \rho_g$ where ρ_g is the image of g under ρ. In particular, we have $\rho_g \pi(1) = \pi \rho_g(1)$. Hence, $\pi(1)g = \pi(g)$. If $\pi(1) = h^{-1}$ for some $h \in G$, then we have $h^{-1}g = \lambda_h(g) = \pi(g)$ for all $g \in G$ where λ_h is the image of h under the left regular representation. Therefore, $\pi = \lambda_h$. On the other hand, for any element λ_h in the left regular representation, $\rho_g \lambda_h(x) = gxh^{-1} = \lambda_h \rho_g(x)$.
Moreover, if we consider the element t in $Sym(G)$ sending every element to its inverse, then we have $t^{-1}\rho_g t(x) = g^{-1}x = \lambda_g(x)$ for all $x \in G$. Hence, $\rho(G)^t = \lambda(G)$. □

Lemma 7.2. *[2, Corollary 4.2B] Let G be a regular subgroup of $Sym(\Omega)$. Then $N = N_{Sym(\Omega)}(G) \cong G \rtimes Aut(G)$.*

Proof. Consider the map

$$\psi : N \longrightarrow Aut(G)$$
$$n \longmapsto \psi(n) : G \longrightarrow G$$
$$g \mapsto n^{-1}gn$$

It is obvious that ψ is a homomorphism. First, we will show that ψ is onto. Let σ be an automorphism of G. Then since G is regular, for any $\alpha \in \Omega$ the point stabilizer, G_α is identity. Therefore, the two transitive representations of G sending x to x and sending x to x^σ are equivalent by [2, Lemma 1.6]. Hence, the actions are conjugate, that is there exists an element $t \in Sym(\Omega)$ such that $x^t = x^\sigma$. Hence, $\psi(t) = \sigma$.
On the other hand, $Ker\psi = C_{Sym(\Omega)}(G)$ and since G is regular by Lemma 7.1 we have $Ker\psi \cong G$. As ψ is onto we have $N/G \cong Aut(G)$.

Moreover, G is a normal subgroup of N and N acts on Ω. Since G is regular, it is transitive. Then for any $x \in \Omega$ and $n \in N$ there exists $g \in G$ such that $x.g = x.n$. Hence, $ng^{-1} \in Stab_N(x)$. Therefore, $N = GStab_N(x)$ (Frattini Argument) and $Stab_N(x) \cong Aut(F)$. Since $G \cap Aut(G) = \{1\}$, we have $N \cong G \rtimes Aut(G)$. □

In this chapter, we will find the structure of normalizers of finite semi-regular subgroups of $S(\xi)$. Since $S(\xi)$ is the union of finite symmetric groups, first for a finite set Ω, we find the normalizer of a semi-regular subgroup in $Sym(\Omega)$. The structure of normalizers of such groups is mentioned by Kohl in one of his talks, see [8]. However, the proof is not stated so for the readers convenience we will give the proof in the following theorem.

Theorem 7.3. *For a finite semi-regular subgroup F of $Sym(\Omega)$ where $|\Omega| < \infty$, let $\Delta_1, \ldots, \Delta_r$ be the orbits of F on Ω. Then $N_{Sym(\Omega)}(F) \cong F^r \rtimes (Sym(r) \times Aut(F))$*

Proof. By [2, Lemma 1.6B], since on each Δ_i the action of F is transitive and F is semi-regular the actions on the orbits are equivalent. Since the actions are equivalent by [2, Exercises 4.2.5] the centralizer of F is isomorphic to $F^r \rtimes Sym(r)$. We know that the centralizer is a normal subgroup of the normalizer. Hence, $(F^r \rtimes Sym(r)) \lhd N_{Sym(\Omega)}(F)$.

Let $F_{|\Delta_i}$ be the restriction of F on the orbit Δ_i. Then $F_{|\Delta_i}$ is regular hence by Lemma 7.2 $N_{Sym(\Delta_i)}(F_{|\Delta_i}) = F_{|\Delta_i} \rtimes Aut(F_{|\Delta_i})$.

Let $\sigma_i : \Delta_1 \longrightarrow \Delta_i$ be the map inducing equivalent actions for all $1 \leqslant i \leqslant r$. The map σ_1 is the identity map. Hence, for any $\alpha \in \Delta_1$ we have $\alpha^{f\sigma_i} = \alpha^{\sigma_i f}$ by considering σ_i as the transposition in $Sym(\Omega)$ acting as identity outside of $\Delta_1 \cup \Delta_i$. For any $1 \neq f \in F$, since the orbits are equivalent and the group is semi-regular we can write $f = f_{\Delta_1} \ldots f_{\Delta_r}$ where f_{Δ_i} is the restriction of f on the orbit Δ_i. Now for $a \in \Delta_1$, obviously we have $a^{f_{\Delta_1}\sigma_i} = a^{\sigma_i f_{\Delta_i}}$ and since σ_i is a transposition for any $b \in \Delta_i$ we also have $b^{\sigma_i f_{\Delta_1}\sigma_i} = (b^{\sigma_i})^{f_{\Delta_1}\sigma_i} = (b^{\sigma_i})^{\sigma_i f_{\Delta_i}} = b^{f_{\Delta_i}}$. Hence, we have $f_{\Delta_1}^{\sigma_i} = f_{\Delta_i}$.

Therefore, any $f \in F$ can be written as $f = f_{\Delta_1} \ldots f_{\Delta_r}$ where each f_{Δ_i} is determined by $f_{\Delta_1}^{\sigma_i}$. Now, if $\alpha_1 \in Aut(F_{|\Delta_1})$, then we can say that α_1 is a permutation on Δ_1 because of the following;

Let F be a semi-regular subgroup of $Sym(\Omega)$, $|\Omega| < \infty$, with orbits $\Delta_1, \ldots, \Delta_r$. If α is an automorphism of F, for an arbitrary orbit Δ_i, fix an element $a_i \in \Delta_i$. Now, α acts on Δ_i via $\alpha \cdot b = a_i^{\alpha(f_b)}$ where f_b is the unique element such that $a_i^{f_b} = b$, since F is semi-regular, the existence and uniqueness of f_b follows directly. We will show that the action is well-defined and one-to-one. Obviously, if $a = b$, then $f_a = f_b$ by uniqueness of the element sending a_i to a. Hence, $\alpha \cdot a = a_i^{\alpha(f_a)} = a_i^{\alpha(f_b)} = \alpha \cdot b$. So, the action is well-defined. Let $\alpha \cdot a = \alpha \cdot b$. Then there exists f_a, f_b such that $a_i^{f_a} = a$ and $a_i^{f_b} = b$. Hence, $a_i^{\alpha(f_a)} = a_i^{\alpha(f_b)}$ so $a_i^{\alpha(f_a)(\alpha(f_b))^{-1}} = a_i$. But since F is semi-regular the point stabilizers are identity, so we have $\alpha(f_a) = \alpha(f_b)$. Hence, $f_a = f_b$ implying $a = b$, by uniqueness of f_a. Since Δ_i is finite and we can define the action of α within the orbits of F and we see that α is a permutation on each orbit of F. Hence, $\alpha \in Sym(\Omega)$. Also note that, α has an action on each orbit Δ_i. So, we can think $\alpha = \alpha_1\alpha_2 \ldots \alpha_r$, where each α_i is the element of $Sym(\Delta_i)$, inducing an automorphism of $F_{|\Delta_i}$.

Because of the above explanation, if $\alpha_1 \in Aut(F_{|\Delta_1})$, then $\alpha_1^{\sigma_i} \in Aut(F_{|\Delta_i})$ and for any $f = f_{\Delta_1} \ldots f_{\Delta_r} \in F$ we have,

$$f^{\alpha_1\alpha_1^{\sigma_2}\ldots\alpha_1^{\sigma_r}} = f_{\Delta_1}^{\alpha_1} f_{\Delta_2}^{\alpha_1^{\sigma_2}} \ldots f_{\Delta_r}^{\alpha_1^{\sigma_r}} = f_{\Delta_1}^{\alpha_1} f_{\Delta_1}^{\sigma_2\alpha_1^{\sigma_2}} \ldots f_{\Delta_1}^{\sigma_r\alpha_1^{\sigma_r}} = f_{\Delta_1}^{\alpha_1} f_{\Delta_1}^{\alpha_1\sigma_2} \ldots f_{\Delta_1}^{\alpha_1\sigma_r} \in F.$$

Notice that $f_{\Delta_1}^{\alpha_1} \in F_{|\Delta_1}$ so, $\alpha_1\alpha_1^{\sigma_2} \ldots \alpha_1^{\sigma_r}$ induces an automorphism of F which normalizes F and it is uniquely determined by α_1. Indeed, for any automorphism $\beta \in Aut(F_{|\Delta_1})$ there exists an element in the normalizer which is uniquely determined by β.

Conversely, let $n \in N_{Sym(\Omega)}(F)$, since n may permute the orbits of F and we know the centralizer $C_{Sym(\Omega)}(F) \cong F^r \rtimes Sym(r)$ is contained in the normalizer we may multiply n with an element $\pi \in Sym(r)$ in the centralizer so that $n\pi^{-1}$ acts trivial on the set of orbits. Now, $n\pi^{-1} \in N_{Sym(\Omega)}(F)$ can be thought as an element of $Sym(\Delta_1) \times \ldots \times Sym(\Delta_r)$. Let $n\pi^{-1} = m$. Since m acts trivially on the set of orbits we may consider $m = m_{\Delta_1} \ldots m_{\Delta_r}$ where $m_{\Delta_i} = m_{|\Delta_i} \in Sym(\Delta_i)$. Notice that $m_{\Delta_i} \in N_{Sym(\Delta_i)}(F_{|\Delta_i})$ that is, $F_{|\Delta_i}^{m_{\Delta_i}} = F_{|\Delta_i}$ for all $1 \leqslant i \leqslant r$. Now, for each i since $F_{|\Delta_i}$ is regular by Lemma 7.2, we can write $m_{\Delta_i} = \alpha_i n_i$ where $\alpha_i \in Aut(F_{|\Delta_i})$ and $n_i \in F_{|\Delta_i}$. Now, m can be written as $\alpha_1 n_1 \alpha_2 n_2 \ldots \alpha_r n_r$. Moreover, since $\alpha_{i-1} \in Sym(\Delta_{i-1})$ and $n_i \in Sym(\Delta_i)$ we have $\alpha_{i-1} n_i = n_i \alpha_{i-1}$ for all $1 \leqslant i \leqslant r$ hence $m = \alpha_1 \alpha_2 \ldots \alpha_r n_1 n_2 \ldots n_r$. Observe that each $n_i \in F_{|\Delta_i}$ and the element $s := n_1^{-1} n_2^{-1} \ldots n_r^{-1} m = \alpha_1 \alpha_2 \ldots \alpha_r$ is an element of the normalizer.

For any $f \in F$, since $\alpha_1 \alpha_2 \ldots \alpha_r$ normalizes f, we have $f^{\alpha_1 \alpha_2 \ldots \alpha_r} = g$ for some $g \in F$. We know g and f can be written in the form $f_{\Delta_1} f_{\Delta_2} \ldots f_{\Delta_r}$ and $g_{\Delta_1} g_{\Delta_2} \ldots g_{\Delta_r}$. where $f_{\Delta_i}, g_{\Delta_i}$ is the restriction of f and g to the orbit Δ_i, respectively. Hence,

$$f^{\alpha_1 \alpha_2 \ldots \alpha_r} = (f_{\Delta_1} f_{\Delta_2} \ldots f_{\Delta_r})^{\alpha_1 \alpha_2 \ldots \alpha_r} = g_{\Delta_1} g_{\Delta_2} \ldots g_{\Delta_r}$$

Since each α_i is an element of $Sym(\Delta_i)$, we can write $(f_{\Delta_1} f_{\Delta_2} \ldots f_{\Delta_r})^{\alpha_1 \alpha_2 \ldots \alpha_r} = f_{\Delta_1}^{\alpha_1} f_{\Delta_2}^{\alpha_2} \ldots f_{\Delta_r}^{\alpha_r} = g_{\Delta_1} g_{\Delta_2} \ldots g_{\Delta_r}$.

Considering the fact that each f_{Δ_i} and g_{Δ_i} is determined by $f_{\Delta_1}^{\sigma_i}$ and $g_{\Delta_1}^{\sigma_i}$, respectively, we see $f_{\Delta_i}^{\alpha_i} = f_{\Delta_1}^{\sigma_i \alpha_i} = g_{\Delta_1}^{\sigma_i}$. So, $\sigma_i \alpha_i \sigma_i \in N_{Sym(\Delta_1)}(F_{\Delta_1})$.

Observe that $\sigma_i \alpha_i \sigma_i$ and α_1 has the same action on F_{Δ_1}, because for the arbitrary element f, $f_{\Delta_1}^{\alpha_1} = g_{\Delta_1} = f_{\Delta_1}^{\sigma_i \alpha_i \sigma_i}$. Hence, $\alpha_1^{-1} \sigma_i \alpha_i \sigma_i \in C_{Sym(\Delta_1)}(F_{\Delta_1}) \cong F_{\Delta_1}$ and for each $1 \leqslant i \leqslant r$ there exists $t_{1i} \in F_{\Delta_1}$ such that $\alpha_1^{-1} \sigma_i \alpha_i \sigma_i = t_{1i}$. Then multiplying from left by $\sigma_i \alpha_1$ and from right by σ_i we have $\alpha_i = \alpha_1^{\sigma_i} t_{1i}^{\sigma_i}$. Eventually, the element $s = \alpha_1 \alpha_2 \ldots \alpha_r = \alpha_1 \alpha_1^{\sigma_2} t_{12}^{\sigma_2} \ldots \alpha_1^{\sigma_r} t_{1r}^{\sigma_r}$ can be written as $s = \alpha_1 \alpha_1^{\sigma_2} \ldots \alpha_1^{\sigma_r} t_{12}^{\sigma_2} \ldots t_{1r}^{\sigma_r}$, because of the fact that $t_{1i}^{\sigma_i} \in Sym(\Delta_i)$ and $\alpha_1^{\sigma_j} \in Sym(\Delta_j)$ commutes. So, we get $s(t_{1r}^{\sigma_r})^{-1} \ldots (t_{12}^{\sigma_2})^{-1} = \alpha_1 \alpha_1^{\sigma_2} \ldots \alpha_1^{\sigma_r} \in Aut(F)$. Observe that this automorphism is uniquely determined by α_1. Hence, there exist an automorphism $\alpha \in Aut(F)$ such that $s(t_{1r}^{\sigma_r})^{-1} \ldots (t_{12}^{\sigma_2})^{-1}\alpha_1^{-1\sigma_r} \ldots \alpha_1^{-1}\alpha^{-1} = id$. So $n \in F_{|\Delta_1} F_{|\Delta_2} \ldots F_{|\Delta_r} Aut(F) Sym(r)$. Since each $F_{|\Delta_i} \cong F$ and for $i \neq j$ commutes elementwisely, we have $n \in F^r Aut(F) Sym(r)$.

Let $\alpha \in Aut(F)$. Then by Page 83, α can be seen as an element of $Sym(\Omega)$ and can be written as $\alpha_1\alpha_2 \ldots \alpha_r$ where $\alpha_i \in Sym(\Delta_i)$. For any $f \in F$, write f in the form $f = f_{\Delta_1} f_{\Delta_1}^{\sigma_2} \ldots f_{\Delta_1}^{\sigma_r}$. We have $f^\alpha = f_{\Delta_1}^{\alpha_1} f_{\Delta_1}^{\sigma_2\alpha_2} \ldots f_{\Delta_1}^{\sigma_r\alpha_r}$. Now, if $f^\alpha = g_{\Delta_1} g_{\Delta_1}^{\sigma_2} \ldots g_{\Delta_1}^{\sigma_r}$ we have $\alpha_1^{-1}\alpha_i^{\sigma_i}$ is in the centralizer $C_{Sym(\Delta_i)}(F_{|\Delta_i})$, so $\alpha_1^{-1}\alpha_i^{\sigma_i}$ is the identity automorphism of $F_{|\Delta_i}$. Hence, we get that $\alpha_i = \alpha_1^{\sigma_i}$ for all $1 < i \leqslant r$. On the other hand, note that $Sym(r)$ can be generated by σ_i's for all $1 \leqslant i \leqslant r$ where σ_i's are the permutations inducing the equivalency of the actions of F on each orbit. Now, $\alpha^{\sigma_i} = (\alpha_1\alpha_1^{\sigma_2} \ldots \alpha_1^{\sigma_r})^{\sigma_i} = \alpha_1^{\sigma_i}\alpha_1^{\sigma_2\sigma_i} \ldots \alpha_1^{\sigma_r\sigma_i} = \alpha$ for any $1 < i \leqslant r$. Hence, the elements of $Sym(r)$ and $Aut(F)$ commute elementwisely.

Also observe that the intersection $Aut(F) \cap Sym(r)$ is trivial, since by Page 83, $Aut(F)$ acts identity on the set of orbits but the elements in $Sym(r)$ permutes the set of orbits. By the structure of the centralizer, we know $Sym(r)$ acts on F^r and the action of the group $Aut(F)$ on F^r can be seen above. Hence, $N_{Sym(\Omega)}(F) \cong F^r \rtimes (Aut(F) \times Sym(r))$. □

In fact, let F be a finite group acting on Ω where $|\Omega| < \infty$ such that either $F_\alpha = 1$ or $F_\alpha = F$ for any $\alpha \in \Omega$ and let $\Delta_1, \ldots \Delta_r$ be the orbits of length greater than 1. Then it can easily be seen that the restriction of the action of F on the set $\Delta_1 \cup \ldots \cup \Delta_r$ is semi-regular. Moreover, we have the following result.

Corollary 7.4. *For a finite subgroup F of $Sym(\Omega)$ satisfying either $F_\alpha = 1$ or $F_\alpha = F$ we have $N_{Sym(\Omega)} \cong (F^r \rtimes (Sym(r) \times Aut(F))) \times Sym(k)$ where r is the number of orbits of length greater than 1 and k is the number of orbits of length 1.*

Proof. The symmetric group on the set of all points fixed by F centralizes F. Indeed, F can be seen as the subgroup of $Sym(|\Omega| - k)$ which has a semi-regular action then by Theorem 7.3 the result will follow. □

Let F be a semi-regular finite subgroup of homogeneous symmetric group $S(\xi)$, where $\xi =< p_1, p_2, \dots >$. We are interested in the structure of the normalizer of F. Since F is finite we can assume that $F \leqslant S_{n_i}$ for some i where $n_i = p_1 p_2 \dots p_i$.

Theorem 7.5. *Let F be a finite semi-regular subgroup of $S(\xi)$ where $\xi =< p_1, p_2, \dots >$. If F is in S_{n_i} for some $n_i = p_1 p_2 \dots p_i$, then*

$$N_{S(\xi)}(F)/C_{S(\xi)}(F) \cong Aut(F)$$

Proof. Recall that we have strictly diagonal embeddings $d^{p_{i+1}} : S_{n_i} \longrightarrow S_{n_{i+1}}$, and the direct limit group via these embeddings are denoted by $S(\xi) = \bigcup_{i=1}^{\infty} S(n_i)$, where $S(n_i)$ is the image of S_{n_i} in the group $S(\xi)$.

Consider the map $d^{p_{i+1}}$. For simplicity of the notation we will denote it by d_{i+1}. By Theorem 7.3, we know the structure of normalizer of F in S_{n_i} is $F^r \rtimes (Aut(F) \times Sym(r))$. When we embed F into $S_{n_{i+1}}$ via d_{i+1} we also know that the normalizer is $F^{rp_{i+1}} \rtimes (Aut(F) \times Sym(rp_{i+1}))$. We need to show that the embedding of $N_{S_{n_i}}(F)$ to $N_{S_{n_{i+1}}}(F)$ is inherited from the strictly diagonal embedding d_{i+1}. By [5, Theorem 3], we know the embeddings of the centralizers are inherited from d_{i+1} so it is enough to show the quotient groups are embedded via strictly diagonal embeddings.

We know that $N_{S_{n_i}}(F)/C_{S_{n_i}}(F) \cong Aut(F)$ and similarly $N_{S_{n_{i+1}}}(F)/C_{S_{n_{i+1}}}(F) \cong Aut(F)$. Let $\Delta_1, \Delta_2, \dots, \Delta_r$ be the orbits of F in S_{n_i} and for all $2 \leqslant t \leqslant r$ let σ_t be the bijective maps from Δ_1 to Δ_t inducing the equivalency of the actions as before. Since F is embedded via strictly diagonal embedding, for all $1 \leqslant k \leqslant p_{i+1}$ and for all $1 \leqslant t \leqslant r$ we know the structure of the orbits $\Delta_{(k-1)r+t} = \{(k-1)n_i + a | a \in \Delta_t\}$. So for each k, there is a one-to-one correspondence between Δ_t and $\Delta_{(k-1)r+t}$. Now if we define $\Delta_{(k-1)r+1} \cup \Delta_{(k-1)r+2} \cup \dots \Delta_{kr} := \Sigma_k$ where $1 \leqslant k \leqslant p_{i+1}$, and $\Sigma_k = \{(k-1)n_i+1, (k-1)n_i+2, \dots, kn_i\}$, then $\Sigma_1 \cup \Sigma_2 \cup \dots \cup \Sigma_{p_{i+1}}$ will be equal to the all set $\{1, 2, \dots, n_{i+1}\}$. Keeping in mind that, the orbits are constructed via strictly diagonal embedding, for all $1 \leqslant k \leqslant p_{i+1}$, there exist bijections $\beta_k : \Sigma_1 \longrightarrow \Sigma_k$ where Δ_j is mapped to $\Delta_{(k-1)r+j}$ via strictly diagonal embedding. Hence, we have $\sigma_{(k-1)r+t}$ equals to the composition of maps $\sigma_t \beta_k$ for all $1 \leqslant k \leqslant p_{i+1}$ and $1 \leqslant t \leqslant r$, which induces the strictly diagonal embedding.

Recall, in the proof of Theorem 7.3, we see that for $F \subset S_{n_{i+1}}$, $Aut(F) \subset Sym(n_{i+1})$ any automorphism, α, of F can be written in the form $\alpha_1 \alpha_1^{\sigma_2} \dots \alpha_1^{\sigma_{p_{i+1}r}}$ where α_1 is an automorphism of $F_{|\Delta_1}$, the restriction of F on Δ_1. Since each element $\alpha_1^{\sigma_i}$ commutes with each other by rearranging the elements we can write

$$\alpha = \alpha_1 \alpha_1^{\sigma_{r+1}} \dots \alpha_1^{\sigma_{(p_{i+1}-1)r+1}} / \alpha_1^{\sigma_2} \alpha_1^{\sigma_{r+2}} \dots \alpha_1^{\sigma_{(p_{i+1}-1)r+2}} / \dots / \alpha_1^{\sigma_r} \alpha_1^{\sigma_{2r}} \dots \alpha_1^{\sigma_{p_{i+1}r}}$$

Putting $\sigma_t \beta_k$ in the place of $\sigma_{(k-1)r+t}$,

$$\alpha = \alpha_1 \alpha_1^{\beta_2} \dots \alpha_1^{\beta_{p_{i+1}}} / \alpha_1^{\sigma_2} \alpha_1^{\sigma_2 \beta_2} \dots \alpha_1^{\sigma_2 \beta_{p_{i+1}}} / \dots / \alpha_1^{\sigma_r} \alpha_1^{\sigma_r \beta_2} \dots \alpha_1^{\sigma_r \beta_{p_{i+1}}}$$

The slashes are just for the readers to easily follow the writings. In fact, since $\alpha_1^{\sigma_k}$ is an element of $Sym(\Delta_k)$ in S_{n_i}. Consider any part separated with a slash. Each separated part of the element α is of the form $\alpha_1^{\sigma_k} \alpha_1^{\sigma_k \beta_2} \dots \alpha_1^{\sigma_k \beta_{p_{i+1}}}$ for some k. If we denote that element by α_k, then one can see that α_k is an element of $Sym(\Delta_k) \times Sym(\Delta_{r+k}) \times \dots \times Sym(\Delta_{(p_{i+1}-1)r+k})$. Indeed, by the structure of maps β_j, the element $\alpha_1^{\sigma_k \beta_j}$ acts on $\Delta_{(j-1)r+k}$ in the same way as the action of element $\alpha_1^{\sigma_k}$ on Δ_k. Hence, $d_{i+1}(\alpha_1^{\sigma_k}) = \alpha_1^{\sigma_k} \alpha_1^{\sigma_k \beta_2} \dots \alpha_1^{\sigma_k \beta_{p_{i+1}}}$ for all $1 \leqslant k \leqslant r$. Since $d_{(i+1)}$ is a homomorphism we have, $d_{i+1}(\alpha_1 \alpha_1^{\sigma_2} \dots \alpha_1^{\sigma_r}) = \alpha_1 \alpha_1^{\sigma_2} \dots \alpha^{\sigma_{p_{i+1}r}}$.

Therefore, the automorphisms are inherited from the embeddings d_{i+1} but the structure of the quotient group does not change. Therefore, continuing the embeddings we get the automorphism group of F in the group $S(\xi)$ and we get the result. $\square$

7.1 Normalizers of Finite Groups in Finitary Homogeneous Symmetric Groups

For an arbitrary cardinal κ, and a sequence of prime numbers $\xi = < p_1, p_2, \dots >$ recall that in Chapter 6 we constructed $FSym(\kappa)(\xi)$.

Since $FSym(\kappa)(\xi)$ is the union of finitary symmetric group we can not have a finite semi-regular subgroup, however we can extend the idea as follows. Let F be a finite subgroup of $FSym(\kappa)(\xi)$. Since we can write $FSym(\kappa)(\xi)$ as $\bigcup_{i=1}^{\infty} FSym(\kappa n_i)$ and F is finite, we may assume that $F \leqslant FSym(\kappa n_i)$ for some $i \in \mathbb{N}$. Then F acts on the set κn_i with finite support. So we can write $\kappa n_i = \kappa n_i \backslash supp(F) \cup supp(F)$. Assume that the action of F on the set $supp(F)$ is semi-regular.

Then for these type of subgroups of $FSym(\kappa)(\xi)$, we have the following result.

Theorem 7.6. *Let F be a finite subgroup of $FSym(\kappa)(\xi)$ which acts semi-regularly on its support, $supp(F)$. Then if $F \in FSym(\kappa n_i)$ for some $n_i = p_1 p_2 \ldots p_i$ we have*

$$N_{FSym(\kappa)(\xi)}(F)/C_{FSym(\kappa)(\xi)}(F) \cong Aut(F)$$

Proof. The proof follows from the fact that F can be viewed as a finite semi-regular subgroup of the symmetric group on the set of the union of the orbits of length greater than 1. Since the permutations of the orbits of length 1 lies in the centralizer by Theorem 7.3, we have the result. □

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