LANDAU AND DIRAC-LANDAU PROBLEM ON ODD-DIMENSIONAL SPHERES

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ABSTRACT

LANDAU AND DIRAC-LANDAU PROBLEM ON ODD-DIMENSIONAL SPHERES

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In this thesis, solutions of the Landau and Dirac-Landau problems for charged particles on odd-dimensional spheres S^{2k-1} in the background of constant $\mathrm{SO}(2k-1)$ gauge fields are presented. Firstly, reviews of the quantum Hall effect and in particular the Landau problem on the two-dimensional sphere S^2 and and all even-dimensional spheres, S^{2k}, are given. Then, the key ideas in these problems are expanded and adapted to set up the Landau problem on S^{2k-1}. Using group theoretical methods, the energy levels of the appropriate Landau Hamiltonian together with its degeneracies are determined. The corresponding wave functions are given in terms of the Wigner D-functions of the symmetry group $\mathrm{SO}(2k)$ of S^{2k-1}. The explicit local forms of the lowest Landau level wave functions are constructed for a particular set of $\mathrm{SO}(2k-1)$ gauge field background charges. We access the constant $\mathrm{SO}(2k-2)$ gauge field backgrounds on the equatorial S^{2k-2} and obtain the differential geometric structures on the latter by forming the relevant projective modules. Finally, we examine the Dirac-Landau problem on S^{2k-1} and obtain the energy spectrum, degeneracies and number of zero

modes of the gauged Dirac operator on S^{2k-1}.

Keywords: Quantum Hall Effect, Odd-Dimensional Spheres, Landau Problem, Dirac-Landau Problem

ÖZ

TEK BOYUTLU KÜRELER ÜZERİNDE LANDAU VE DIRAC-LANDAU PROBLEMİ

COŞKUN, ÜMİT HASAN
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Bu tezde S^{2k-1} tek boyutlu küreleri üzerinde ve sabit $\mathrm{SO}(2k-1)$ arkaplan ayar alanları etkisindeki yüklü parçacıklar için Landau ve Dirac-Landau problemlerinin çözümleri sunuldu. Başlangıç olarak, kuantum Hall etkisi bilhassa Landau probleminin iki küre, S^2, ve tüm çift boyutlu küreler, S^{2k}, üzerindeki incelemeleri verildi. Akabinde, bu problemlerdeki kilit fikirler genişletilip uyarlanılarak S^{2k-1} üzerinde Landau problem kuruldu. Grup teori teknikleri kullanılarak uygun Landau Hamiltonianlarının enerji seviyeleri ile birlikte eşenerjili durumları da belirlendi. Bu enerji seviyelerine tekabül eden dalga fonksiyonları S^{2k-1}'in simetri grubu olan $\mathrm{SO}(2k)$ grubunun Wigner D-fonksiyonları türünden verildi. $\mathrm{SO}(2k-1)$ arkaplan ayar yüklerinin belirli durumları için en düşük Landau seviyelerindeki dalga fonksiyonlarının açık yerel biçimleri kuruldu. Ekvatoral S^{2k-2}'ler üzerindeki sabit $\mathrm{SO}(2k-2)$ arkaplan ayar alanlarına ulaştık ve ilgili projektif modülleri kurarak bu arkaplan ayar alanlarının diferansiyel geometrik yapılarını elde ettik. Son olarak, S^{2k-1} üzerinde arkaplan ayar alanını içeren Dirac-Landau problemini inceledik ve S^{2k-1} üzerindeki Dirac operatörünün enerji spektrumunu, eşenerjili durumlarının ve sıfır modlarının sayısınını elde ettik.

Anahtar Kelimeler: Kuantum Hall Etkisi, Tek-Boyutlu Küreler, Landau Problemi, Dirac-Landau Problemi

To my family

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CHAPTER 1

INTRODUCTION

Quantum Hall effect (QHE) is one of the most striking phenomena in condensed matter physics in its breath and depth and has been an important research subject ever since its experimental discovery [1]. QHE can be summarized as the quantization of the resistivity ρ or the conductivity σ in planar many-body electron systems in very low temperatures and very high external magnetic fields. The quantization of the Hall resistance was first observed in 1980 by von Klitzing, Dorda and Pepper [1]. von Klitzing was awarded the 1985 Nobel prize for this discovery.

The theoretical framework to explain QHE on two dimensional systems was first given by Laughlin [2]. In his work, Laughlin developed a many-body wave function, which is called the Laughlin-wave function, for two dimensional electron systems under the influence of an external perpendicular magnetic field. His explanation was a phenomenological model for both the integer filling factors $\nu \in \mathbf{Z}_+$ and also for the fractional filling factors $\nu = \frac{1}{m}$, where m is an odd integer. Robert Laughlin was awarded the 1998 Nobel Prize in Physics for this theoretical discovery. In QHE, although the system is invariant under the translations the Hamiltonian is not invariant under such a transformation. Hamiltonian changes under a translation up to a gauge transformation, in other words it remains invariant under a combination of a translation accompanied by a gauge transformation. Such a combination can be employed to define the so called the magnetic translations.

The formalism for Landau quantization on two dimensional sphere S^2 is given by Haldane in 1983 [3]. Haldane set up and solved the problem of an electron confined to move on a 2-sphere, S^2 and subject to a fixed magnetic field (in magnitude) background which is provided by a Dirac monopole placed at the center of the sphere. In this model, the degeneracies are finite at each Landau level because of the compact geometry and the magnetic translations are easily found to be left translations commuting with the Hamiltonian and covariant derivatives. Haldane was able to construct many particle wave functions both at integer and at $\nu = \frac{1}{m}$ (m is odd integer) fractional filling factors, and the latter was turned out to be very useful on understanding some properties of FQHE [4].

In the past decade or so, formulation of QHE on higher-dimensional manifolds and investigations on its several aspects have been a continually appearing theme in contemporary theoretical physics. After the pioneering work of Zhang & Hu [5] in formulating the QHE problem on S^4, a multitude of articles have explored the formulation of QHE on various higher-dimensional manifolds, such as $\mathbb{C}P^N$, the even-dimensional spheres S^{2k}, complex Grassmann manifolds $\boldsymbol{Gr}_2(\mathbb{C}^N)$ as well as on a particular flag manifold [6, 7, 8, 9, 10, 11]. One motivation for their study is to understand the generalization of the massless excitations, (chiral bosons) which are known to be present at the edge of the two-dimensional quantum Hall samples (see, for instance, [12]). However, it turns out that, not only photons and gravitons, but somewhat undesirably even higher massless spin states occur at the the edges, which are effectively described by chiral and gauged Wess-Zumino-Witten (WZW) theories and therefore has interesting physical content in their own right [13, 14]. There are also strong motivations emerging from physics of D-branes and strings, as certain configurations with open strings ending on D-branes have low energy limits, which are effectively described by the QHE on spheres [15, 16]. Relation between the matrix algebras describing fuzzy spaces, such as the fuzzy sphere S_F^2, higher-dimensional fuzzy spheres S_F^{2k}, fuzzy complex projective spaces $\mathbb{C}P_F^N$ and the Hilbert spaces of the lowest Landau level (LLL) of QHE on aforementioned manifolds have also been explored in the literature to shed further light into the geometrical structure of the LLL [17], while in the present work we will have the opportunity to present yet another facet of this relationship in a particular example. Thus, expanding upon these concrete developments and along with the novel motivations emerging from the physics of TIs, in this thesis, our ultimate goal is to investigate the formulation of QHE on all odd-dimensional spheres S^{2k-1}.

As we have already noted, QHE problem on S^3 is solved by Nair & Daemi [18] and a complementary treatment is recently given in Hasebe's work [19][1]. The clear path for the construction of QHE over compact higher dimensional manifolds appear to be closely linked to the coset space realization of such spaces. Indeed odd spheres can also be realized as coset manifolds as $S^{2k-1} \equiv \frac{\mathrm{SO}(2k)}{\mathrm{SO}(2k-1)} \equiv \frac{\mathrm{Spin}(2k)}{\mathrm{Spin}(2k-1)}$. In their approach Nair & Daemi took advantage of the fact that S^3 can also be realized as $S^3 \equiv \frac{\mathrm{SU}(2)\times\mathrm{SU}(2)}{\mathrm{SU}(2)_D}$ owing to the isomorphisms $\frac{\mathrm{SU}(2)\times\mathrm{SU}(2)}{\mathbf{Z}_2} = \mathrm{SO}(4)$ and $\frac{\mathrm{SU}(2)}{\mathbf{Z}_2} = \mathrm{SO}(3)$, and they subsequently constructed the Landau problem for a charged particle on S^3 under the influence of a constant $\mathrm{SU}(2)_D$ gauge field background carrying an irreducible representation (IRR) of the latter. This quick approach is not immediately applicable to higher dimensional odd spheres. Nevertheless, coset space realization of S^{2k-1} in terms of the $\mathrm{SO}(2k)/\mathrm{SO}(2k-1)$ can be used to handle this problem.

A brief summary of our results and their organization in this thesis is in order. In chapter 2, we give a review of the classical and quantum Hall effects based on the lecture notes [23, 4, 12]. In chapter 3, we give a short review of the Haldane's Landau quantization on S^2 [3] and the Landau problem on even-dimensional spheres, S^{2k}, constructed by Y. Kimura and K. Hasebe [6]. Building upon the ideas of [6], we set up and solve the Landau problem for charged particles on odd-dimensional spheres S^{2k-1} in the background of constant $\mathrm{SO}(2k-1)$ gauge fields carrying the irreducible representation $\left(\frac{I}{2}, \frac{I}{2}, \cdots, \frac{I}{2}\right)$. In particular, we determine the spectrum of the Hamiltonian, the degeneracy of the Landau levels and give the eigenstates in terms of the Wigner $\mathcal{D}$-functions, and for odd values of I the explicit local form of the wave functions in the lowest Landau level. In this section, we also demonstrate in detail, how the essential differential geometric structure of the Landau problem on the equatorial S^{2k-2} is captured

[1] Other recent developments in solving Landau problem and Dirac-Landau problem in flat higher dimensional spaces are reported in [20, 21, 22].

by constructing the relevant projective modules and the related $\mathrm{SO}(2k-2)$ valued curvature two-forms. We illustrate our general results on the examples of S^3 and S^5 for concreteness and in the latter case identify an exact correspondence between the union of Hilbert spaces of LLL's with I ranging from 0 to $I_{max} = 2K$ or $I_{max} = 2K+1$ to the Hilbert spaces of the fuzzy $\mathbb{C}P^3$ at level K or that of winding number ± 1 line bundles over $\mathbb{C}P^3$ at level K, respectively. In section 3 we determine the spectrum of the Dirac operator on S^{2k-1} in the same gauge field background together with its degeneracies and also compute the number of its zero modes. Some relevant formulas from the representation theory of groups is given in a short appendix for completeness.

CHAPTER 2

BASIC ASPECTS OF PLANAR QUANTUM HALL EFFECT

In this chapter, we will discuss and develop the essential aspects of classical and (integer) quantum Hall effects (IQHE). Firstly, we start from an elementary level and review the classical dynamics of an electron subject external electromagnetic field. We use these developments to give a brief description of the classical Hall effect and explain the physical meaning of the conductivity and resistivity tensors in classical Hall effect. Subsequently, we study the quantization of the classical dynamics of an electron confined to a plane and subject to an external magnetic field. This is known as the Landau problem in the literature. This is followed by a qualitative discussion of IQHE. Our discussion in this chapter is not exhaustive and only oriented to provide the reader with sufficient details to view the discussions in the ensuing chapters in a more concrete and broader perspective. We closely follow the lecture notes of D. Tong on the subject, which are available from the web [24] and also the books of Ezawa [12], Jain [4] and the review of Girvin [23] in this chapter.

2.1 Classical Hall Effect

2.1.1 Cyclotron Motion

In this section, we briefly recall the Lagrangian and the Hamiltonian formulations describing the dynamics of a charged particle in external electromagnetic fields. Lagrangian formulation gives us a clear path to obtain the equations of motions for a charged particle in electromagnetic field. We will see that electrons perform cyclotron motion when they are subject to a pure perpendicular magnetic field. Hamiltonian formulation is used to pass to the quantum mechanical description of this system in a rather straightforward manner as we will see in section 2.2.1.

The Lagrangian of a point particle with mass m and charge q moving with the velocity $\dot{\vec{x}}$ in an electromagnetic field with the electric and magnetic fields $\vec{E} = -\vec{\nabla}\phi$ and $\vec{B} = \vec{\nabla} \times \vec{A}$ respectively, is given by

$$L = \frac{1}{2}m\dot{\vec{x}}^2 + q\dot{\vec{x}} \cdot \vec{A} - q\phi, \tag{2.1}$$

where $\vec{A}$ is the vector potential and ϕ is the scalar potential. The j-th component of the conjugate momentum obtained from (2.1) is,

$$p_j = \frac{\partial L}{\partial \dot{x}_j} = m\dot{x}_j + qA_j. \tag{2.2}$$

The equations of motion can be obtained by the Euler-Lagrange equations. First, we need the partial derivatives of the Lagrangian (2.1) with respect to x_j,

$$\frac{\partial L}{\partial x_j} = q\dot{x}_i\frac{\partial A_i}{\partial x_j} - q\frac{\partial \phi}{\partial x_j}, \tag{2.3}$$

and the equation of motion is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_j}\right) - \frac{\partial L}{\partial x_j} = q\dot{x}_i\frac{\partial A_i}{\partial x_j} - q\frac{\partial \phi}{\partial x_j} - m\ddot{x}_j - q\dot{x}_i\frac{\partial A_j}{\partial x_i} = 0, \tag{2.4}$$

which implies

$$\begin{aligned} m\ddot{x}_j &= q\dot{x}_i\left(\frac{\partial A_i}{\partial x_j} - \frac{\partial A_j}{\partial x_i}\right) - q\frac{\partial \phi}{\partial x_j}, \\ &= q\dot{x}_i\epsilon_{ijk}B_k + qE_j, \\ &= q(\dot{\vec{x}} \times \vec{B} + \vec{E})_j, \end{aligned} \tag{2.5}$$

or,

$$m\ddot{\vec{x}} = q(\dot{\vec{x}} \times \vec{B} + \vec{E}). \tag{2.6}$$

This is just the Lorentz force equation as we may have already anticipated.

The corresponding Hamiltonian to (2.1) is calculated as

$$\begin{aligned} H &= p_i \dot{x}_i - L \\ &= \frac{1}{2m}(\vec{p} - q\vec{A})^2 + q\phi \\ &= \frac{1}{2m}\vec{\pi}^2 + q\phi, \end{aligned} \tag{2.7}$$

where we have introduced the notation $\vec{\pi} := m\dot{\vec{x}} = \vec{p} - q\vec{A}$ for the mechanical momentum. This is the momentum that can be experimentally measured in the system as opposed to the conjugate momentum $\vec{p}$. We will elaborate on the physical meaning of $\vec{\pi}$'s later in section 2.2.1 when we discuss the Landau problem.

Now, let us consider a simple configuration in which the charged particle is an electron of charge $-e$ ($e > 0$) with mass m. We consider that there is an external magnetic field in the z direction, $\vec{B} = B\hat{k}$, and no electric field. The Lorentz force equation (2.6) now becomes,

$$m\ddot{\vec{x}} = -q\dot{\vec{x}} \times \vec{B} = -eB\dot{\vec{x}} \times \hat{k}. \tag{2.8}$$

If the electron is confined to move on the xy-plane the equation (2.8) can be written as two coupled second order ordinary differential equations,

$$m\ddot{x} = -eB\dot{y} \quad \text{and} \quad m\ddot{y} = eB\dot{x}. \tag{2.9}$$

We may write,

$$\begin{aligned} \ddot{x} + i\ddot{y} &= \frac{eB}{m}(-\dot{y} + i\dot{x}) \\ &= i\frac{eB}{m}(\dot{x} + i\dot{y}). \end{aligned} \tag{2.10}$$

Introducing a complex variable $z = x + iy$ and $\omega_c := \frac{eB}{m}$, we may express (2.10) as

$$\ddot{z} = i\omega_c \dot{z}. \tag{2.11}$$

Integrating with respect to time yields

$$\dot{z} = \dot{z}_0 e^{i(\omega_c t + \alpha)}, \tag{2.12}$$

where $\dot{z}_0$ and α are determined by initial conditions. We may express this equation (2.12) as

$$\dot{x}(t) = \dot{x}(0)\cos(\omega_c t + \alpha) \quad \text{and} \quad \dot{y}(t) = \dot{y}(0)\sin(\omega_c t + \alpha), \tag{2.13}$$

where α is therefore the initial phase and with the notation $\vec{v} = (\dot{x}(t), \dot{y}(t))$, we see that

$$\begin{aligned} v(t) &= \sqrt{\dot{x}^2(t) + \dot{y}^2(t)} \\ &= \sqrt{\dot{x}^2(0) + \dot{y}^2(0)} \\ &=: v(0) = v_0, \end{aligned} \tag{2.14}$$

is the magnitude of the velocity of the particle on the xy-plane which stays constant throughout the motion. Integrating once more gives

$$x(t) \; = x_0 + R\sin(\omega_c + \alpha) \quad \text{and} \quad y(t) \; = y_0 + R\cos(\omega_c + \alpha). \tag{2.15}$$

Thus, the particle is performing a circular motion with frequency ω_c about circle of radius R centered at (x_0, y_0). The radius of the cyclotron orbit is related to initial velocity of the particle. We immediately see that it is given by

$$R = \frac{v_0}{\omega_c} = \frac{v_0 m}{eB}. \tag{2.16}$$

2.1.2 Classical Hall Effect

The classical Hall effect, discovered in 1879 by E. H. Hall [25], is, at the microscopic level, also related to the cyclotron motion of the charged particles in magnetic fields. In the classical Hall effect, a basic experimental setup includes a strip of a metal or a semiconductor and a pure perpendicular magnetic field to the strip. If we make a current I_x to flow along x-direction, charge carriers in the sample begin to divert to one side and accumulate on that side. Specifically, an electron moving in the $-x$ direction will be deflected to the $-y$ direction by the magnetic force. As a result, one edge of the strip becomes positively charged while the electrons accumulate on the opposite edge and an electric field forms due to this polarization. In a short amount of time, electric and magnetic forces will counterbalance and the system will arrive its equilibrium state which means

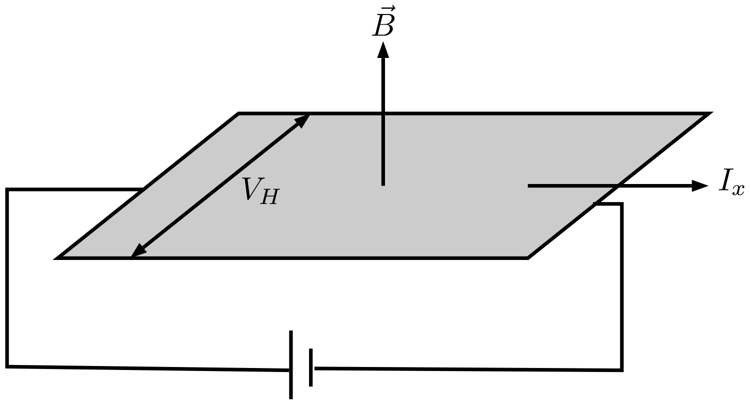

Figure 2.1: Basic setup of the Hall effect

the net force vanishes. In this equilibrium state, we have

$$\vec{F}_N = 0 = m\ddot{\vec{x}} = -e(\vec{E} + \dot{\vec{x}} \times \vec{B}), \tag{2.17}$$

which gives

$$\dot{x} = -\frac{E_y}{B}. \tag{2.18}$$

The relation between the velocity and the current is given by

$$J_i = -en\dot{x}_i, \tag{2.19}$$

where n is the charge density of the sample. From (2.18) and (2.19) we obtain

$$\frac{E_y}{J_x} = \frac{B}{en}. \tag{2.20}$$

(2.20) is a particular form of the Ohm's Law relating electric field to the current via the resistivity tensor ρ, $E_i = \rho_{ij} J_j$ as we will discuss in more detail in the next subsection. Let us however note that for a sample of side length L, we may write

$$R_{xy} = \frac{V_y}{I_x} = \frac{E_y L}{J_x L} = \frac{B}{en}, \tag{2.21}$$

where R_{xy} is called the Hall resistance. Normally resistivity and resistance differ by a factor related to geometry. We see that the Hall resistance R_{xy} and Hall resistivity are equal.

Let us also write that the potential V_y measured across the Hall sample is known as the Hall voltage, and we may write $V_H = V_y$ for notational clarity.

2.1.3 The Drude Model

Previously, we have not incorporated the effect of an electric field in the dynamics of charges in a constant perpendicular magnetic field. Such an electric field accelerates the charged particles and creates a current in the direction of the electric field. In this section, we briefly present the Drude model to explain the conductivity of a metal subject to electromagnetic fields. In this model, we have a linear friction term in the form

$$\vec{F}_D = -\frac{m}{\tau}\dot{\vec{x}}, \tag{2.22}$$

where τ is called scattering time, is also introduced. τ has the meaning of average time between scattering events. We follow [24] here to construct the Drude model. We add the electric field term $-e\vec{E}$ and the linear friction term in (2.22) in equation (2.6), which yields

$$m\ddot{\vec{x}} = -e\dot{\vec{x}} \times \vec{B} - e\vec{E} - \frac{m}{\tau}\dot{\vec{x}}. \tag{2.23}$$

This is called the Drude equation [26].

For steady state solution, i.e. non-accelerating electron solutions ($\ddot{\vec{x}} = 0$), equation (2.23) reduces to

$$\begin{aligned} 0 &= -eB\varepsilon_{ij}\dot{x}_j - eE_i - \frac{m}{\tau}\dot{x}_i \\ &= \frac{B}{n}\varepsilon_{ij}J_j - eE_i + \frac{m}{ne\tau}J_i \\ \implies & \left(\varepsilon_{ij}\frac{eB\tau}{m} + \delta_{ij}\right)J_j = \frac{e^2 n\tau}{m}E_i, \end{aligned} \tag{2.24}$$

where n is the density of electrons and we have used $J_i = -ne\dot{x}_i$ in arriving at equation (2.24). This equation can be interpreted as the Ohm's law. This latter can be written in the form

$$\vec{J} = \sigma\vec{E} \quad \text{or} \quad J_i = \sigma_{ij}E_j, \tag{2.25}$$

where σ_{ij} is called the conductivity tensor. Inverting equation (2.24) we get

$$\begin{aligned} \sigma_{ij} &= \frac{e^2 n\tau}{m}\frac{1}{m^2 + eB\tau^2}\left(m^2\delta_{ij} - eB\tau m\varepsilon_{ij}\right), \\ &= \frac{nme^2\tau}{m^2 + (eB\tau)^2}\left(-\varepsilon_{ij}\frac{eB\tau}{m} + \delta_{ij}\right). \end{aligned} \tag{2.26}$$

Physically, the conductivity tensor should be a rotationally invariant object, as this macroscopic quantity could not depend on how we choose our coordinate frame in the laboratory. In two dimensions, this fact simply implies that σ_{ij} can only be composed of the rotational invariant tensors δ_{ij} and ε_{ij} in two dimensions, thus it must have the form

$$\begin{aligned} \sigma_{ij} &= \sigma_D \delta_{ij} + \sigma_{OD} \varepsilon_{ij} \\ &= \begin{pmatrix} \sigma_D & \sigma_{OD} \\ -\sigma_{OD} & \sigma_D \end{pmatrix}, \end{aligned} \tag{2.27}$$

where σ_D stands for the diagonal conductivity $\sigma_D = \sigma_{xx} = \sigma_{yy}$ and σ_{OD} stands for the off-diagonal conductivity $\sigma_{OD} = \sigma_{xy} = \sigma_{yx}$ with $\sigma_{OD} = \frac{nme^2\tau}{m^2+(eB\tau)^2} \frac{eB\tau}{m}$ and $\sigma_D = \frac{nme^2\tau}{m^2+(eB\tau)^2}$.

By definition, the resistivity is just the inverse of the conductivity, $\rho_{ij} = (\sigma^{-1})_{ij}$. Thus, the resistivity tensor is

$$\rho_{ij} = (\sigma^{-1})_{ij} = \frac{m}{ne^2\tau} \left(\delta_{ij} + \frac{eB\tau}{m} \varepsilon_{ij} \right). \tag{2.28}$$

While the Hall resistivity ρ_{xy} takes the form

$$\begin{aligned} R_{xy} = \rho_{xy} &= \frac{m}{ne^2\tau} \left(\delta_{xy} + \frac{eB\tau}{m} \varepsilon_{xy} \right) \\ &= \frac{m}{ne^2\tau} \left(\frac{eB\tau}{m} \right) = \frac{B}{en}. \end{aligned} \tag{2.29}$$

Comparing with (2.29), we see that for classical Hall system $R_{xy} = \rho_{xy}$ and conclude that the Hall resistivity obtained from the more elementary model we have given in section 2.1.2 and from the Drude model are exactly the same. This result indicates the fact that the Hall resistivity only depends on the magnetic field and the charge density.

2.2 Integer Quantum Hall Effect

Our discussion in this section is oriented to sketch some of the essential features of IQHE. We will not attempt to give a full construction of IQHE in this section. Many excellent references can be consulted to obtain a broader and a complete understanding of IQHE [24, 4, 23].

2.2.1 Landau Problem

In this section, we review Landau quantization which is the quantization of the cyclotron orbits of charged particles in magnetic fields. This quantization process discretizes the energy levels of the particles and corresponding energy levels are called Landau levels. Understanding Landau quantization is central for obtaining a complete picture of QHE.

The Poisson brackets of position and conjugate momentum variables are readily given by

$$\{x_i, p_j\} = \delta_{ij} \quad \text{and} \quad \{x_i, x_j\} = \{p_i, p_j\} = 0, \tag{2.30}$$

while a straightforward calculation gives

$$\begin{aligned} \{\pi_i, \pi_j\} = \{m\dot{x}_i, m\dot{x}_j\} = \{p_i + eA_i, p_j + eA_j\} &= -e\left(\frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j}\right) \\ &= -e\varepsilon_{ijk}B_k \end{aligned} \tag{2.31}$$

In order to proceed with the quantization of the classical system described in section 2.1, we can start by replacing the classical quantities by their corresponding quantum operators. Thus, the Poisson brackets can now be replaced by the canonical commutation relations

$$[x_i, p_j] = i\hbar\delta_{ij} \quad \text{and} \quad [x_i, x_j] = [p_i, p_j] = 0. \tag{2.32}$$

Using the above commutation relations, the commutation relation of the kinematical momentum components π_x and π_y now reads

$$[\pi_x, \pi_y] = -ie\hbar B. \tag{2.33}$$

We can define new operators out of the π_x and π_y operators, so called the annihilation and creation operators, as

$$a = \frac{1}{\sqrt{2e\hbar B}}(\pi_x - i\pi_y) \quad \text{and} \quad a^\dagger = \frac{1}{\sqrt{2e\hbar B}}(\pi_x + i\pi_y), \tag{2.34}$$

with

$$[a, a^\dagger] = 1. \tag{2.35}$$

The quantum mechanical Hamiltonian

$$H = \frac{\vec{\pi}^2}{2m} = \frac{(\vec{p} + eA)^2}{2m}, \tag{2.36}$$

can be cast in the form

$$\begin{aligned} H = \frac{\vec{\pi}^2}{2m} &= \frac{\pi_x^2 + \pi_y^2}{2m} - \frac{\hbar\omega_c}{2} + \frac{\hbar\omega_c}{2} \\ &= \frac{1}{2m}(\pi_x^2 + \pi_y^2 - e\hbar B) + \frac{\hbar e B}{2m} \\ &= \frac{\hbar e B}{m}\left[\frac{1}{2e\hbar B}(\pi_x^2 + \pi_y^2 - i[\pi_x, \pi_y]) + \frac{1}{2}\right] \\ &= \hbar\omega_c\left(a^\dagger a + \frac{1}{2}\right), \end{aligned} \tag{2.37}$$

which is the same as that of the harmonic oscillator Hamiltonian with frequency ω_c. We can choose to use the Fock space or the occupation number basis, i.e. the Hilbert space is

$$\mathscr{H} = \text{span}\{|0\rangle, |1\rangle, |2\rangle, \ldots, |n\rangle, \ldots\}. \tag{2.38}$$

States $|n\rangle$ are in one to one correspondence with the eigenvalues, n, of the number operator $N = a^\dagger a$. Using (2.35) and the fact that the vacuum state $|0\rangle$ is annihilated by a, i.e. $a|0\rangle = 0$, we can show that

$$a^\dagger |n\rangle = \sqrt{n+1}\,|n+1\rangle \quad \text{and} \quad a|n\rangle = \sqrt{n}\,|n-1\rangle. \tag{2.39}$$

Therefore, the state $|n\rangle$ can be built by acting on the ground state n-times with the creation operator:

$$|n\rangle = \frac{1}{\sqrt{n}}(a^\dagger)^n |0\rangle. \tag{2.40}$$

Spectrum of the Hamiltonian is simply

$$E_n = \hbar\omega_c\left(n + \frac{1}{2}\right). \tag{2.41}$$

These energy levels are called the Landau levels.

Let us also define the guiding-center coordinates

$$X = x + \frac{1}{eB}\pi_y \quad \text{and} \quad Y = y - \frac{1}{eB}\pi_x, \tag{2.42}$$

which satisfy the commutation relations

$$[X, Y] = il_B^2, \tag{2.43}$$

and where $l_B = \sqrt{\frac{\hbar}{eB}}$ is called the magnetic length which is the natural length scale in QHE and we will comment on its physical meaning further in the later sections. We may also define a relative position vector with respect to guiding-center coordinates (X, Y) as

$$\vec{R} = \left(-\frac{1}{eB}\pi_y, \frac{1}{eB}\pi_x\right). \tag{2.44}$$

Heisenberg equations of motions tell us that

$$\frac{dX}{dt} = \frac{1}{i\hbar}[X, H] = 0, \qquad \frac{dY}{dt} = \frac{1}{i\hbar}[Y, H] = 0, \tag{2.45}$$

with the obvious solutions that X and Y are constant of motion, while

$$\frac{d\pi_x}{dt} = \frac{1}{i\hbar}[\pi_x, H] = \omega_c\pi_y, \qquad \frac{d\pi_y}{dt} = \frac{1}{i\hbar}[\pi_y, H] = -\omega_c\pi_x. \tag{2.46}$$

We may write

$$\begin{aligned} \frac{d\pi_x}{dt} + i\frac{d\pi_y}{dt} &= \omega_c\pi_y - i\omega_c\pi_x \\ &= -i\omega_c(\pi_x + i\pi_y). \end{aligned} \tag{2.47}$$

Introducing a complex variable $u = \pi_x + i\pi_y$, we may express (2.47) as

$$\frac{du}{dt} = -i\omega_c u. \tag{2.48}$$

Integrating with respect to time yields

$$u = e^{-i\omega_c t}u_0 \tag{2.49}$$

where u_0 is determined by initial conditions. Thus we have

$$\pi_x = \cos(\omega_c t)u_0 \quad \text{and} \quad \pi_y = \sin(\omega_c t)u_0. \tag{2.50}$$

This means that there is cyclotron motion with respect frequency $\omega_c = \frac{eB}{m} = \frac{\hbar}{ml_B^2}$ about the guiding-center, and the guiding center coordinates remain at rest. This picture also justifies the name "guiding center" for (X, Y).

The corresponding wave functions to the energy levels (2.41) can be found by specifying a gauge choice for the external perpendicular magnetic field. We will first choose the Landau gauge given by

$$\vec{A} = xB\hat{j}. \tag{2.51}$$

Clearly, this choice of gauge gives the perpendicular constant magnetic field as

$$\vec{\nabla} \times \vec{A} = -\frac{\partial A_y}{\partial z}\hat{i} + \frac{\partial A_y}{\partial x}\hat{k} = B\hat{k}. \tag{2.52}$$

This choice of gauge breaks the transitional symmetry in the x direction as well as the rotational symmetry of the Hamiltonian and it is therefore appropriate for rectangularly shaped Hall samples.

In this gauge, Hamiltonian (2.7) becomes

$$H = \frac{1}{2m}[p_x^2 + (p_y^2 + exB)^2]. \tag{2.53}$$

We immediately gather that $[H, p_y] = 0$, as there is no dependence on the y-coordinate in the Hamiltonian. This implies that H and p_y can be diagonalized simultaneously. As the eigenstates for p_y are of the form e^{iky} with the eigenvalue $k = p_y/\hbar$, we may write down the energy eigenstates as

$$\psi(x, y) = e^{iky}\varphi(x). \tag{2.54}$$

Substituting (2.54) in (2.53) we get a one-dimensional Schrödinger equation by

$$\begin{aligned} H\psi(x, y) &= H(e^{iky}\varphi(x)) \\ &= e^{iky}\left[p_x^2\varphi(x) + e^2B^2(x + kl_B^2)^2\varphi(x)\right] = Ee^{iky}\varphi(x) \\ &\implies H_x\varphi(x) = E\varphi(x) \end{aligned} \tag{2.55}$$

with the effective Hamiltonian

$$H_x = \frac{1}{2m}\left[p_x^2 + e^2B^2(x + kl_B^2)^2\right]. \tag{2.56}$$

The effective Hamiltonian (2.56) is just in the form of the Hamiltonian for a one-dimensional harmonic oscillator, whose central position is shifted to $x = -kl_B^2$. The energy levels are then given by (2.41) as expected. The ground state wave function, or the lowest Landau level (LLL) wave function, whose energy can be obtained by setting $n = 0$ in (2.41) as $E_{LLL} = \hbar\omega_c/2$ is

$$\psi_{0,k} \sim e^{iky}e^{-\frac{(x+kl_B^2)^2}{2l_B^2}}, \tag{2.57}$$

up to an overall normalization. n-th Landau level states up to a normalization factor can be obtained by multiplying (2.57) with the Hermite polynomials

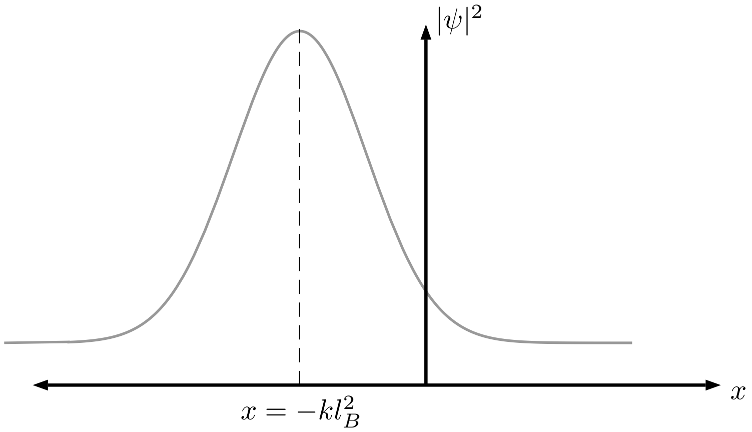

Figure 2.2: The probability density $|\psi_{LLL}|^2$ peaks around $x = -kl_B^2$

$H_n(x + kl_b^2)$, we have

$$\psi_{0,k} \sim e^{iky} e^{-\frac{(x+kl_B^2)^2}{2l_B^2}} H_n(x + kl_B^2). \tag{2.58}$$

Landau gauge provides a useful way to estimate the degeneracy at any Landau level. We observe that the energy levels depend only on the quantum number $n \in \mathbf{Z}_+$ while the wave functions are depending on both n and the wave number k in the y-direction. In order to determine the degeneracy at any Landau level, we may pick a rectangular region on the (x, y) plane. Let us consider that this rectangular region has sides of length L_x and L_y respectively. We intend to find the number of quantum states that are bounded in the region $\mathcal{R}$, $0 \leq x \leq L_x$, $0 \leq y \leq L_y$. Assuming periodic boundary conditions in the y-direction, i.e. imposing

$$\psi(x, y + L_y) = \psi(x, y), \tag{2.59}$$

we have

$$e^{ik(y+L_y)} = e^{iky}, \tag{2.60}$$

which implies that $e^{ikL_y} = 1$ and therefore k is quantized in units of $\frac{2\pi}{L_y}$. We may write

$$k = \frac{2\pi}{L_y} m, \quad m \in \mathbf{Z}, \tag{2.61}$$

We further see from (2.57) and (2.58) that the Landau levels wave functions are localized around $x = -kl_B^2$. We may write $dx = -l_B^2 dk$ and therefore the interval $0 \leq x \leq L_x$ corresponds to having quantized values of k in the range $-\frac{L_x}{l_B^2} \leq k \leq 0$. Consequently, total number of states in the region $\mathcal{R}$ can be estimated as

$$N := \frac{\int_{-L_x/l_B}^{0} dk}{\frac{2\pi}{L_y}} = \frac{L_y}{2\pi}\int_{-L_x/l_B^2}^{0} dk = \frac{L_x L_y}{2\pi l_B^2} = \frac{eBA}{2\pi\hbar}, \tag{2.62}$$

where $A = L_x L_y$ is the area of the region $\mathcal{R}$. We also infer from this result that the density of states, i.e. the number of states per unit area, is the same at all Landau levels and it is given by

$$\rho = \frac{1}{2\pi l_B^2}. \tag{2.63}$$

We will see that ρ plays an important role in obtaining an intuitive understanding of both the integer and fractional quantum Hall effects.

Up to now, we have derived the energy eigenvalues and the corresponding wave functions of the Landau problem by using the Landau gauge. It is also instructive to consider the problem in the so called the symmetric gauge.

For this purpose, the vector potential can be chosen as

$$\vec{A} = \frac{1}{2}B(-y\hat{i} + x\hat{j}) \tag{2.64}$$

which is called the symmetric gauge. Clearly, it yields $\vec{\nabla} \times \vec{A} = B\hat{k}$. In fact, we can see that $\vec{A}_{sym}$ and $\vec{A}_{Landau}$ are indeed related by a gauge transformation given as

$$\vec{A}_{sym} = \vec{A}_{Landau} + \vec{\nabla}\Lambda \tag{2.65}$$

with $\Lambda = -\frac{1}{2}Bxy$.

We can also introduce a second pair of annihilation and creation operators as

$$b = \sqrt{\frac{eB}{2\hbar}}(X + iY) \quad \text{and} \quad b^\dagger = \sqrt{\frac{eB}{2\hbar}}(X - iY), \tag{2.66}$$

where

$$[a, a^\dagger] = [b, b^\dagger] = 1 \quad \text{and} \quad [a, b] = 0. \tag{2.67}$$

We have another number operator for the operator b defined by

$$M = b^\dagger b. \tag{2.68}$$

Since $[a, b] = 0$ they are diagonalizable simultaneously and any general Fock state $|n, m\rangle$ satisfies

$$N |n, m\rangle = n |n, m\rangle \quad \text{and} \quad M |n, m\rangle = m |n, m\rangle . \tag{2.69}$$

$a, a^\dagger$ and $b, b^\dagger$ operator pairs act on the state $|n, m\rangle$ and give

$$\begin{aligned} a |n, m\rangle &= \sqrt{n} |n-1, m\rangle, \quad a^\dagger |n, m\rangle = \sqrt{n+1} |n+1, m\rangle , \\ b |n, m\rangle &= \sqrt{m} |n, m-1\rangle, \quad b^\dagger |n, m\rangle = \sqrt{m+1} |n, m+1\rangle . \end{aligned} \tag{2.70}$$

If we define the ground state as $|0, 0\rangle$, which means we have $a |0, 0\rangle = b |0, 0\rangle = 0$, then the state $|n, m\rangle$ can be constructed as

$$|n, m\rangle = \frac{(a^\dagger)^n (b^\dagger)^m}{\sqrt{n!m!}} |0, 0\rangle . \tag{2.71}$$

In contrast to the Landau gauge the symmetric gauge violates the transitional symmetry in the (x, y) plane. Nevertheless, it leaves the rotation symmetry of the Hamiltonian invariant. That is, we have that $[H, L] = 0$, where L is the angular momentum operator given by

$$\begin{aligned} L &= x p_y - y p_x \\ &= b^\dagger b - a^\dagger a. \end{aligned} \tag{2.72}$$

Then we may simultaneously diagonalize H and L, and we distinguish the degenerate energy eigenstates at a given Landau level using the angular momentum quantum number.

The construction of the wave functions in the symmetric gauge at the lowest Landau level, $(n = 0)$, deserves special emphasis. The effect of the annihilation operator on the lowest Landau level states is $a |0, m\rangle = 0$. This relation can also be given as a first order differential equation that is required to be satisfied by the ground state wave functions. First observe that we may write,

$$\begin{aligned} a &= \frac{1}{\sqrt{2e\hbar B}} (\pi_x - i\pi_y) \\ &= \frac{1}{\sqrt{2e\hbar B}} \left[-i\hbar \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) + \frac{eB}{2} (-y - ix) \right] \end{aligned} \tag{2.73}$$

Let us introduce the complex variables z and $\bar{z}$[1]

$$z = x - iy, \quad \bar{z} = x + iy. \tag{2.74}$$

[1] If we choose to work with $z = x + iy$ and $\bar{z} = x - iy$, we should set $B < 0$ to obtain holomorphic functions instead of anti-holomorphic ones.

The corresponding holomorphic and anti-holomorphic derivatives are

$$\partial := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) \quad \text{and} \quad \bar{\partial} := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right). \tag{2.75}$$

These satisfy $\partial z = \bar{\partial}\bar{z} = 1$ and $\partial\bar{z} = \bar{\partial}z = 0$ as can easily be verified. Annihilation and creation operators can be reexpressed as

$$a = -i\sqrt{2}\left(l_B\bar{\partial} + \frac{z}{4l_B}\right) \quad \text{and} \quad a^\dagger = -i\sqrt{2}\left(l_B\partial - \frac{\bar{z}}{4l_B}\right). \tag{2.76}$$

Thus we have the first order differential equation

$$-i\sqrt{2}\left(l_B\bar{\partial} + \frac{z}{4l_B}\right)\psi_{LLL} = 0, \tag{2.77}$$

whose solution can be written as

$$\psi_{LLL}(z,\bar{z}) = f(z)e^{-|z|^2/4l_B^2}, \tag{2.78}$$

where $f(z)$ is an holomorphic function. The degenerate states at LLL can be obtained by employing the other set of annihilation-creation operators, b and $b^\dagger$. These operators have the differential forms

$$b = -i\sqrt{2}\left(l_B\partial + \frac{\bar{z}}{4l_B}\right) \quad \text{and} \quad b^\dagger = -i\sqrt{2}\left(l_B\bar{\partial} - \frac{z}{4l_B}\right). \tag{2.79}$$

Using the fact that

$$b\psi_{LLL;m=0} = 0, \tag{2.80}$$

we find that

$$\psi_{LLL;m=0} \sim e^{-|z|^2/4l_B^2}, \tag{2.81}$$

up to a normalization factor. With a repeated application of $b^\dagger$ on (2.81) degenerate states at the LLL can be obtained as

$$\psi_{LLL;m} \sim \left(\frac{z}{l_B}\right)^m e^{-|z|^2/4l_B^2}, \tag{2.82}$$

where the factor of $\left(\frac{z}{l_B}\right)$ results from the $m-$times repeated application of $b^\dagger$ on $\psi_{LLL;m=0}$. These eigenstates also provide a basis for the angular momentum operator,

$$L\psi_{LLL;m} = m\hbar\psi_{LLL;m}. \tag{2.83}$$

Another crucial point we need to mention is that by a repeated application of creation operator $a^\dagger = -i\sqrt{2}\left(l_B\partial - \frac{\bar{z}}{4l_B}\right)$ on the LLL wave function (2.82), we can determine the higher Landau level wave functions.

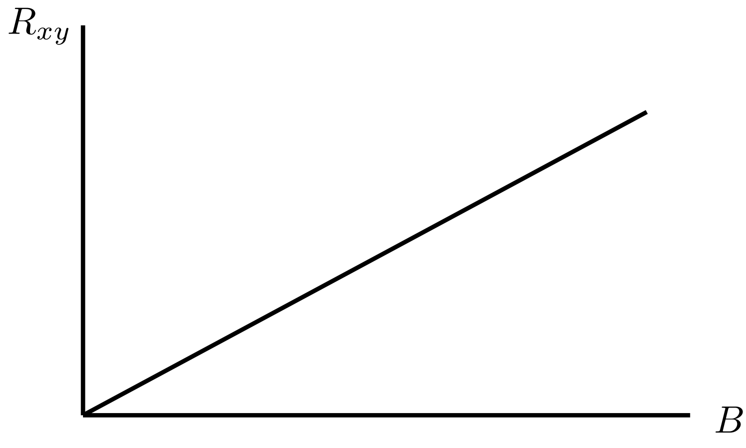

Figure 2.3: Dependence of R_{xy} on B field

2.2.2 Brief Explanation of Integer Quantum Hall Effect

Quantum Hall effect is a very curious phenomenon which is observed in two-dimensional electronic systems in the presence of external strong perpendicular magnetic fields ($\sim$ 10 T) and at low temperatures ($\sim$ 4 K). The reason behind such a configuration is the fact that magnetoresistance of the material becomes negligible at low temperatures and strong magnetic fields [27]. It was discovered that independent of all the microscopic details, the size, the shape of the sample and exactly what the sample was made of, when a certain current through the sample is put, a certain voltage is measured which is called the Hall voltage. The ratio of the Hall voltage to the current as units of Ohms was found to be universal by K. von Klitzing, G. Dorda and M. Pepper in 1980 paper which was the experimental discovery of the integer quantum Hall effect (IQHE) [1]. They performed an experiment on a metal-oxide-semiconductor (MOSFET) by trapping the electrons between silicon and silicon oxide. They showed that the Hall resistivity takes discrete values given by

$$R_H = \left(\frac{\hbar}{e^2}\right)\frac{1}{\nu}, \tag{2.84}$$

where ν is an integer and the term $\hbar/e^2$ is called the von Klitzing constant which is approximately 25812.807 Ohms. ν is called the filling factor whose physical meaning is the relation between the number of electrons in the system and number of the available quantum states in a Landau level. For instance, $\nu = 1$ corresponds to a configuration in which each degenerate state in the LLL is occupied by only one electron.

Previously, we have mentioned that the mechanism behind the IQHE can be explained by the Landau quantization. Depending on the magnetic field B, each Landau level can accommodate only limited number of electrons. When the total number of the electrons in the sample and the capacity of the Landau levels balance, which means that some Landau levels are completely field, the sample exhibits the IQHE phenomena.

Another astonishing observation is that unlike the classical Hall resistivity behavior, the quantized Hall resistivity is found to have plateaus at the each discrete values of the Hall resistivity given in (2.84). These plateaus can be

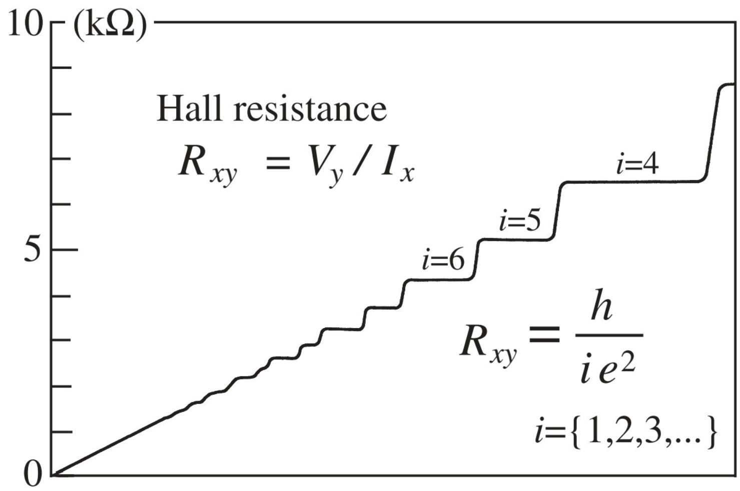

Figure 2.4: Plateau behaviour of the Hall resistivity.[2]

explained by the impurities in the sample.

A short time after the discovery of the IQHE, At lower temperatures (≤ 2 K), Horst L. Störmer, Daniel C. Tsui, and Arthur C. Gossard showed that the filling factor ν can also take fractional values and this was the discovery of the FQHE [29]. For fractional filling factor such as, $\nu = 1/m$, m is odd integer, it means that an electron occupies more than one Landau site. This is the fractional QHE whose detailed discussion is beyond the scope of this thesis. The

[2] *Note.* From "*Metrology and microscopic picture of the integer quantum hall effect*" by J. Weis and K. von Klitzing, 2011, *Philosophical Transactions of the Royal Society of London A: Mathematical, Physical and Engineering Sciences* 369 (2011) 3954 [28].

Landau quantization alone is not sufficient to explain FQHE. Together with the Landau quantization we must consider the correlations between the electrons. The explanation of FQHE is more complex because each fractional number case needs a different mechanism to explain. The interested reader can consult the references [12], [4] and [23].

CHAPTER 3

LANDAU PROBLEM ON S^2 AND EVEN SPHERES

This chapter is a review of the Landau problem on S^2 and on even-dimensional spheres, S^{2k}, and the formulations needed in the construction of following chapters. The discussion in this chapter is based on Haldane's original paper [3], the article of Greiter [30] and Master thesis of F. Ballı [31]. Formulations of QHE on S^{2k} follow the original paper of Hasebe and Kimura [6].

3.1 Landau Problem on S^2

The dynamics of a charged particle on the surface of a sphere S^2 with radius R, in the presence of a magnetic monopole background, was first solved by Haldane in 1983. He proposed a translationally invariant model of fractional quantum Hall effect developed by Laughlin. In the system proposed by Haldane, electrons are confined to move on the two-sphere in the presence of a perpendicular magnetic field which can be created by placing a magnetic monopole in the center of the two-sphere. In natural units ($c = \hbar = 1$), the magnetic field created by this monopole magnetic charge is given by

$$\vec{B} = \frac{g}{R^2}\hat{r} = \frac{n}{2eR^2}\hat{r}, \tag{3.1}$$

where e and g are electron and magnetic monopole charges respectively, and also we have implemented the Dirac quantization rule given by

$$eg = \hbar\frac{n}{2}, \quad n \in \mathbf{Z}. \tag{3.2}$$

The Hamiltonian which describes the dynamics of an electron with mass m confined on S^2 with radius R in terms of the kinematical angular momentum operator is given by

$$H = \frac{\vec{\Lambda}^2}{2mR^2}, \tag{3.3}$$

where $\vec{\Lambda}$ is the kinematical angular momentum of the charged particles given by

$$\vec{\Lambda} = \vec{r} \times \vec{\pi} = \vec{r} \times (-i\hbar\vec{\nabla} + e\vec{A}). \tag{3.4}$$

We can compare H in (3.3) with that of particle on S^2, when there is no external magneti field. In this case the Hamiltonian is

$$H_0 = \frac{L^2}{2mR^2} = \frac{1}{2mR^2}(\vec{r} \times -i\hbar\vec{\nabla})^2, \tag{3.5}$$

where L_i are the orbital angular momentum operators satisfying the commutation relations

$$[L_i, L_j] = i\varepsilon_{ijk}L_k. \tag{3.6}$$

The spectrum and corresponding wave function of this Hamiltonian are easily obtained, since we already know that $L^2Y_{lm} = l(l+1)\hbar^2Y_{lm}$, where $Y_{lm}(\theta, \phi)$ are the spherical harmonics. Thus, we have that the spectrum of H_0 is given by $\frac{l(l+1)\hbar^2}{2mR^2}$ and corresponding wave functions are just the spherical harmonics.

The commutator of the Cartesian components of $\vec{\Lambda}$ can be calculated as

$$[\Lambda_i, \Lambda_j] = i\varepsilon_{ijk}\hbar(\Lambda_k - \hbar\frac{n}{2}\hat{r}_k). \tag{3.7}$$

The commutator of Λ with $\hat{r}$ can easily be calculated as

$$[\Lambda_i, \hat{r}_j] = i\hbar\varepsilon_{ijk}\hat{r}_k. \tag{3.8}$$

Observe that the dynamical angular momentum is parallel to the surface of the sphere

$$\hat{r} \cdot \vec{\Lambda} = \vec{\Lambda} \cdot \hat{r} = 0. \tag{3.9}$$

We see that $\vec{\Lambda}$ itself does not satisfy angular momentum algebra. We introduce a total angular momentum operator $\vec{J}$ by

$$\vec{J} = \vec{\Lambda} + \hbar\frac{n}{2}\hat{r}, \tag{3.10}$$

and $\vec{J}$ satisfies angular momentum algebra

$$[J_i, J_j] = i\hbar\varepsilon_{ijk}J_k. \tag{3.11}$$

It is possible to calculate the total angular momentum of the electric charge magnetic monopole system. Using (3.1) and $\vec{E} = \frac{e}{4\pi r^2}\hat{r}$ and the fact that

$$\vec{J}_{em} = \int \vec{r} \times (\vec{E} \times \vec{B}) d^3x. \tag{3.12}$$

We can show that

$$\vec{J}_{em} = eg\hat{r} = \frac{n\hbar}{2}\hat{r}, \tag{3.13}$$

where the last equality is due to the Dirac quantization condition given in (3.2). Thus the radial part of the total angular momentum in (3.10) is that of the electromagnetic field. Using (3.9) and commutation relations of $\vec{\Lambda}$ we can further show that

$$\begin{aligned} [J_i, \Lambda_j] &= i\hbar\varepsilon_{ijk}\Lambda_k, \\ [J_i, \hat{r}_j] &= i\hbar\varepsilon_{ijk}\hat{r}_k, \end{aligned} \tag{3.14}$$

Also note that the radial component of $\vec{J}$ is

$$\vec{J} \cdot \hat{r} = \hat{r} \cdot \vec{J} = \hbar\frac{n}{2}. \tag{3.15}$$

With the help of (3.10) and (3.15) we can write $\vec{\Lambda}^2$ in the following form

$$\vec{\Lambda}^2 = \vec{J}^2 - \left(\hbar\frac{n}{2}\right)^2. \tag{3.16}$$

Let us understand the physical content of this result. From (3.15) we infer that $\vec{J}$ should be able to take the value $n/2$. Thus, for this reason, we may always express

$$j = \frac{n}{2} + k, \tag{3.17}$$

where k is an arbitrary positive integer. Note that, since $\vec{J}$ fulfill the SU(2) algebra we can write

$$\begin{aligned} \vec{J}^2 &= \hbar^2 j(j+1) \\ &= \hbar^2\left(k + \frac{n}{2}\right)\left(k + 1 + \frac{n}{2}\right). \end{aligned} \tag{3.18}$$

Consequently, the energy eigenvalues can be expressed as

$$\begin{aligned} E_k &= \frac{\hbar}{2mR^2}\left(|J|^2 - \left(\frac{n}{2}\right)^2\right) \\ &= \frac{\hbar}{2mR^2}\left(k + \frac{n}{2}\right)\left(k + 1 + \frac{n}{2}\right) \\ &= \frac{\hbar eB}{2m}(2k+1) + \frac{\hbar}{2mR^2}k(k+1). \end{aligned} \tag{3.19}$$

where we have used (3.1) to write the last line. E_k is the energy of $k^{\rm th}$- Landau level. In other words k labels the energy levels of the charged particle in this problem. The LLL is obtained by setting $k = 0$ and we have

$$E_{LLL} = \frac{eB}{2m}, \tag{3.20}$$

which is the same as the LLL energy in the planar case as given in (2.41) in the previous chapter.

We may also consider taking the planar limit by taking $R \to \infty$ and $n \to \infty$ while keeping the ration $\frac{n}{R^2} = 2eB$ fixed. From (3.19) we see that this yields energy spectrum $E = \frac{\hbar eB}{m}(k + 1/2) = \hbar\omega_c(k + 1/2)$ which is the same as that of the planar problem obtained in (2.41).

Let us also note that for a given j value the z-component J_z of $\vec{J}$, can take $2j + 1$ different values, $-j, -j + 1, \ldots, j - 1, j$. Thus for $j = \frac{n}{2} + k$ there are $2j + 1 = n + 2k + 1$ different values of J_z. This gives the number of degenerate eigenstates of the Hamiltonian at a given Landau level k.

Our next task is to obtain the energy eigenstates of Landau levels. Up to this point, we have not chosen a gauge yet. We may follow Haldane's original approach here and choose the latitudinal gauge

$$\vec{A} = -\frac{\hbar n}{2eR}\cot\theta\hat{\varphi}. \tag{3.21}$$

By establishing an analogy between spin and the radial component of $\vec{J}$, We introduce spinor coordinates $\xi = \begin{pmatrix} u \\ v \end{pmatrix}$ for the particle,

$$u = \cos\left(\frac{1}{2}\theta\right)e^{\frac{1}{2}i\varphi}, \qquad v = \sin\left(\frac{1}{2}\theta\right)e^{-\frac{1}{2}i\varphi}, \tag{3.22}$$

On the other hand, in these coordinates angular momentum operator J_i can be expressed as

$$J_i = \frac{1}{2}(\sigma_i)_{\gamma\rho}\xi_\gamma\partial_\rho, \qquad \gamma, \rho : 1, 2. \tag{3.23}$$

where σ_i are the Pauli sigma matrices. In explicit form, we have

$$J_x = \frac{1}{2}(u\partial_v + v\partial_u), \;\; J_y = -i\frac{1}{2}(u\partial_v - v\partial_u), \;\; J_z = \frac{1}{2}(u\partial_u - v\partial_v). \tag{3.24}$$

Equation (3.15) suggests us to solve the following eigenvalue equation satisfied by the LLL wave functions $\Psi(u,v)^{(n)}$,

$$(\hat{r}\cdot\vec{J})\Psi(u,v)^{(n)} = \hbar\frac{n}{2}\Psi(u,v)^{(n)}. \tag{3.25}$$

We know that u and v transform under the action of spin 1/2 IRR of SU(2) group. A general element g of SU(2) in spin 1/2 IRR can be parametrized as

$$g = \begin{pmatrix} \bar{\alpha} & \bar{\beta} \\ -\beta & \alpha \end{pmatrix}, \quad |\alpha|^2 + |\beta|^2 = 1. \tag{3.26}$$

Left action of g on ξ gives

$$\begin{pmatrix} \bar{\alpha} & \bar{\beta} \\ -\beta & \alpha \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}. \tag{3.27}$$

Partial derivatives with respect to u and v can be obtained via Jacobian transformation of η_1 and η_2

$$\begin{aligned}\begin{pmatrix} \partial_u \\ \partial_v \end{pmatrix} &= J(\eta_1,\eta_2)\begin{pmatrix} \partial_{\eta_1} \\ \partial_{\eta_2} \end{pmatrix} \\ &= \begin{pmatrix} \frac{\partial\eta_1}{\partial u} & \frac{\partial\eta_2}{\partial v} \\ \frac{\partial\eta_1}{\partial u} & \frac{\partial\eta_2}{\partial v} \end{pmatrix}\begin{pmatrix} \partial_{\eta_1} \\ \partial_{\eta_2} \end{pmatrix} \\ &= \begin{pmatrix} \bar{\alpha} & -\beta \\ -\bar{\beta} & \alpha \end{pmatrix}\begin{pmatrix} \partial_{\eta_1} \\ \partial_{\eta_2} \end{pmatrix}.\end{aligned} \tag{3.28}$$

This allows us to rewrite (3.25) as

$$\frac{1}{2}(\eta_1\partial_{\eta_1} - \eta_2\partial_{\eta_2}) = \hbar\frac{n}{2}\Psi(u,v)^{(n)}. \tag{3.29}$$

There are two solutions of (3.29) with eigenvalues $n/2$ and $-n/2$. Choosing $n/2$ gives the solution

$$\Psi(u,v)^{(n)} = \eta_1^n = (\bar{\alpha}u + \bar{\beta}v)^n \tag{3.30}$$

3.2 Landau Problem on Even Spheres S^{2k}

A k-sphere is a generalization of the circle. The k-sphere is a k-dimensional manifold and it can be embedded in the $(k+1)$-dimensional Euclidean space,

$\mathbb{R}^{k+1}$. For $(X_1, X_2, ..., X_k, X_{k+1}) \in \mathbb{R}^{k+1}$, S^k is the set of points satisfying the constraint

$$\sum_{r=1}^{k+1} X_\alpha^2 = R^2, \quad r : 1, \dots, k+1, \tag{3.31}$$

where R is the "radius" of the $k-$sphere. From the above definition, an even dimensional sphere, S^{2k}, can be defined by embedding it in $\mathbb{R}^{2k+1}$ and the coordinates X_i satisfy

$$\sum_{i=1}^{2k+1} X_i^2 = R^2, \qquad i : 1, 2, ..., 2k+1. \tag{3.32}$$

Just as SO(3) is the group of symmetry, i.e. isometry group of S^2, SO$(2k+1)$ is the isometry group, i.e. group under whose action (3.32) remains invariant. For our purposes, another useful definition of even-dimensional spheres is given as the coset spaces

$$S^{2k} = \frac{\mathrm{SO}(2k+1)}{\mathrm{SO}(2k)}. \tag{3.33}$$

The holonomy group of the even sphere manifolds S^{2k} is SO$(2k)$. This is the subgroup of the north pole of S^{2k} remains unrotated. For instance, the holonomy group of S^2 is simply SO(2) =U(1) which is a one parameter subgroup of SO(3) corresponding to a rotation about a fixed axis. For S^4, we have SO(4) as the isometry group which is a six parameter group, under whose action a given point on S^4 will remain unrotated.

We follow the work of Hasebe and Kimura [6] to briefly discuss the Landau problem on S^{2k}. For this purpose, we introdue the Hopf spinor $\Psi = \begin{pmatrix} \Psi_+ \\ \Psi_- \end{pmatrix}$, which can be built as,

$$\begin{aligned} \Psi_+ &= \sqrt{\frac{R + X_{2k+1}}{2R}}\Psi^{(2k-2)}, \\ \Psi_- &= \frac{1}{\sqrt{2R(R+X_{2k+1})}}(X_{2k} - iX_i\gamma^i)\Psi^{(2k-2)}, \end{aligned} \tag{3.34}$$

where γ^i's are generalized Clifford algebra gamma matrices in $(2k-1)$-dimensional space[1].

[1] Gamma matrices in different dimensions are discussed in more detail in the beginning of the next chapter

In (3.34), $\psi^{(2k-2)}$ is a spinor involving coordinates of S^{2k-2}. This can be built iteratively starting with the Hopf spinor on S^2. Hence, there are such of spinors Ψ on S^{2k} down to $\Psi^{(2)}$ on S^2 which can be shown as,

$$\Psi \to \Psi^{(2k-2)} \to \Psi^{(2k-4)} \to \cdots \to \Psi^{(2)}. \tag{3.35}$$

The spinor lives on S^{2k} is characterized as a section of the Spin$(2k)$ bundle on S^{2k} and the spin connection one form may be written as

$$A := \Psi^\dagger d\Psi = \Psi^{(2k-2)\dagger} \cdot iA_i dX_i \cdot \Psi^{(2k-2)}. \tag{3.36}$$

Using (3.34 A may be evaluated to be

$$A_a = -\frac{1}{R(R+X_{2k+1})}\Xi^+_{ab}X_b, \qquad A_{2k+1} = 0, \tag{3.37}$$

where Ξ_{ab}'s are $Spin(2k)$ generators,

$$\Xi_{ab} = -i\frac{1}{4}[\Gamma_a, \Gamma_b], \qquad a, b : 1, .., 2k. \tag{3.38}$$

Ξ_{ab} is a reducible representation of Spin$(2k)$ and it can be decomposed into two irreducible representations of Spin$(2k)$

$$\Xi_{ab} = \begin{pmatrix} \Xi^+_{ab} & 0 \\ 0 & \Xi^-_{ab} \end{pmatrix}, \tag{3.39}$$

This configuration corresponds to a non-abelian magnetic monopole placed at the center of S^{2k} and the Spin$(2k)$ gauge connection on S^{2k} is generated by this monopole. The corresponding curvature tensor is given by

$$R^2 F_{ab} = -(X_a A_b - X_b A_a - \Xi^+_{ab}), \;\; R^2 F_{a2k+1} = (R + X_{2k+1})A_a, \tag{3.40}$$

and one can show that $R^4 F^2_{ab} = \Xi^{+2}_{ab}$. As a result, there is a connection between the SO$(2k+1)$ spinor IRR index I and the monopole charge g. Let us make this observation more concrete.

The magnitude of the angular momentum of the monopole is actually the eigenvalue of the Spin$(2k)$ quadratic Casimir operator, $R^4 F^2_{ab} = \Xi^{+2}_{ab}$.

For a more general background gauge field we may consider the I-fold symmetric tensor product of $(1/2, \ldots, 1/2)$ which is simply the $(I/2, \ldots, I/2)$ IRR of

SO($2k$). Due to (3.40), magnitude of this background field is fixed a related to the Casimir of the SO($2k$) in this IRR. The monopole charge g and the index I are related by $g = I/2$.

The Hamiltonian for the charged particles on S^{2k} can be written as

$$H = \frac{\hbar^2}{2MR^2} \sum_{i<j} \Lambda_{ij}^2, \tag{3.41}$$

where Λ_{ij} is called covariant angular momentum,

$$\Lambda_{ij} = -i(X_i D_j - X_j D_i), \tag{3.42}$$

and $D_i = \partial_i + iA_j$ is the covariant derivative as usual. The commutators of the covariant angular momentum is Λ_{ij} is

$$\begin{aligned} [\Lambda_{ij}, \Lambda_{kl}] &= i[\delta_{ik}\Lambda_{jl} + \delta_{jl}\Lambda_{ik} - \delta_{jk}\Lambda_{il} - \delta_{il}\Lambda_{jk}] \\ &-i[X_i X_k F_{jl} + X_j X_l F_{ik} - X_j X_k F_{il} - X_i X_l F_{jk}]. \end{aligned} \tag{3.43}$$

This shows that Λ_{ij}'s do not satisfy the SO($2k+1$) algebra because of the existence of the gauge field F_{ij} which contributes to the total angular momentum of the configuration.

The covariant angular momentum Λ_{ij} is parallel to the tangent space of the sphere S^{2k} but the monopole angular momentum $R^2 F_{ij}$ is orthogonal to the S^{2k}. This can be verified by calculating, $\Lambda_{ij} F_{ij} = F_{ij} \Lambda_{ij} = 0$.

The total angular momentum is

$$L_{ij} = \Lambda_{ij} + R^2 F_{ij}, \tag{3.44}$$

specifically,

$$L_{ab} = L_{ab}^{(0)} + \Xi_{ab}^{+}, \;\; L_{a2k+1} = L_{a2k+1}^{(0)} + RA_a + X_a A_{2k+1}.$$

We can show that this new operator satisfies the SO($2k+1$) algebra; i.e.

$$[L_{ij}, L_{kl}] = i(\delta_{ik} L_{jl} + \delta_{jl} L_{ik} - \delta_{jk} L_{il} - \delta_{il} L_{jk}), \tag{3.45}$$

In addition, the Hamiltonian (3.41) commutes with the L_{ij} due to the $SO(2k+1)$ symmetry in the system. Now, we rewrite the Hamiltonian in terms of the total angular momentum and the monopole angular momentum as

$$H = \frac{1}{2MR^2}\sum_{i<j}(L_{ij}^2 - R^4 F_{ij}^2) \tag{3.46}$$

We see from the Hamiltonian above, the energy spectrum is simply the difference between quadratic Casimirs of Spin$(2k+1)$ IRR of $(n+I/2, I/2, \ldots, I/2)$ and SO$(2k)$ IRR of $(I/2, \ldots, I/2)$. Then, the energy spectrum reads,

$$E(n,I) = \frac{\hbar^2}{2MR^2}[n^2 + n(I+2k-1) + \frac{1}{2}Ik], \tag{3.47}$$

where $n, I = 0, 1, 2, \cdots$. n labels the Landau level.

The lowest Landau level (LLL) is obtained by setting $n = 0$. The energy and the degeneracy of the LLL are given as

$$E_{LLL} = \frac{\hbar^2}{2M}\frac{I}{2R^2}k, \tag{3.48}$$

$$d(I) = \frac{(I+2k-1)!!}{(2k-1)!!(I-1)!!}\prod_{l=1}^{k-1}\frac{(I+2l)!l!}{(I+l)!(2l)!} \approx I^{\frac{1}{2}k(k+1)}. \tag{3.49}$$

The wave functions of the LLL are given as the SO$(2k+1)$ spinors carrying the $(I/2, \ldots, I/2)$ IRR.

CHAPTER 4

LANDAU PROBLEM ON ODD SPHERES

In this chapter, we present our original results on the formalism of Landau problem over odd spheres. Developments in this chapter are based on the article [32], published in Phys Rev D, coauthored with S. Kürkçüoğlu and G.C. Toğa.

4.1 Landau Problem on Odd Spheres S^{2k-1}

Our main problem is to solve non-relativistic dynamics of charged particles on odd spheres, S^{2k-1}, in the presence of a constant background gauge field. Via group theoretical methods we will determine energy spectrum and corresponding wave functions. First, we need to introduce some necessary definitions.

In previous chapter we have given the general definition of higher dimensional spheres. Using the coset definition of spheres we can define S^{2k-1} as

$$S^{2k-1} = \frac{\mathrm{SO}(2k)}{\mathrm{SO}(2k-1)}. \tag{4.1}$$

Representation theory of SO$(2k)$ can be constructed by exploiting the relationship between the spin group Spin$(2k)$ and SO$(2k)$. Since Spin$(2k)$ is the universal covering space of SO$(2k)$ the representation theory of Spin$(2k)$ will be sufficient for our purposes. The generators of Spin$(2k)$ can be constructed by $2^k \times 2^k$ dimensional Γ matrices. The Clifford algebra is

$$\{\Gamma_a, \Gamma_b\} = 2\delta_{ab}, \quad a, b : 1, 2, \ldots, 2k, \tag{4.2}$$

where Γ_a are the generators of the Clifford algebra in $2k$ dimensions. The explicit forms of these matrices are

$$
\begin{aligned}
\Gamma_\mu &= \begin{pmatrix} 0 & -i\gamma_\mu \\ i\gamma_\mu & 0 \end{pmatrix}, \\
\Gamma_{2k} &= \begin{pmatrix} 0 & 1_{2^{k-1}\times 2^{k-1}} \\ 1_{2^{k-1}\times 2^{k-1}} & 0 \end{pmatrix}, \qquad \mu : 1, 2, \ldots, 2k-1, \\
\Gamma_{2k+1} &= \begin{pmatrix} -1_{2^{k-1}\times 2^{k-1}} & 0 \\ 0 & 1_{2^{k-1}\times 2^{k-1}} \end{pmatrix},
\end{aligned}
\tag{4.3}
$$

where γ_μ's are the generators of Clifford algebra in $(2k-1)$ dimensions and μ runs from 1 to $2k-1$. We are now able to construct the generators of SO$(2k)$ and SO$(2k-1)$ in terms of these gamma matrices defined above. The generator of SO$(2k)$ may be written by

$$
\Xi_{ab} = -\frac{i}{4}[\Gamma_a, \Gamma_b], \qquad a, b : 1, 2, ..., 2k, \tag{4.4}
$$

and the generator for SO$(2k-1)$ is given by

$$
\Sigma_{\mu\nu} = -\frac{i}{4}[\gamma_\mu, \gamma_\nu], \qquad \mu, \nu : 1, 2, \ldots, 2k-1. \tag{4.5}
$$

While $\Sigma_{\mu\nu}$ forms an irreducible representation (IRR) of SO$(2k-1)$, Ξ_{ab} form reducible representation and it can be decomposed into two fundamental IRR's

$$
\begin{aligned}
\Xi_{ab} &= \Xi^+_{ab} \oplus \Xi^-_{ab} \\
&= \begin{pmatrix} \Xi^+_{ab} & 0 \\ 0 & \Xi^-_{ab} \end{pmatrix}
\end{aligned}
\tag{4.6}
$$

and

$$
\Xi^\pm_{\mu\nu} = \Sigma_{\mu\nu}, \qquad \mp\Xi^\pm_{2k\mu} = \frac{1}{2}\gamma_\mu \tag{4.7}
$$

We introduce eigenspinors of our Hamiltonian

$$
\Psi = \frac{1}{\sqrt{2R(R + X_{2k})}} \left[(R + X_{2k})1_{2^k} + X_\mu \Gamma^\mu\right] \phi \tag{4.8}
$$

where $\phi = \frac{1}{\sqrt{2}} \begin{pmatrix} \tilde{\phi} \\ \tilde{\phi} \end{pmatrix}$ with $\tilde{\phi}$ being a normalized 2^{k-1} component spinor. With these definition and conventions Ψ is normalized, i.e. $\Psi^\dagger\Psi = 1$. Then we introduce a Hopf-like projection by

$$
\frac{X_a}{R} = \Psi^\dagger \Gamma_a \Psi \tag{4.9}
$$

this map together with Ψ provides us fractionalization of X_a. The spinor Ψ can be used to build the spin connection over S^{2k-1} as

$$A = \tilde{\phi}(-iA_a dx^a)\tilde{\phi}, \tag{4.10}$$

this A is also called SO$(2k-1)$ gauge field. The components of this gauge field given in the appendix B.1 are

$$A_\mu = -\frac{1}{R(R+X_{2k})}\Sigma_{\mu\nu}X_\nu, \qquad A_{2k} = 0. \tag{4.11}$$

Components of field strength can be calculated using commutator of covariant derivatives, which are defined as $D_a = \partial_a + iA_a$,

$$F_{ab} = -i[D_a, D_b] = \partial_a A_b - \partial_b A_a + i[A_a, A_b] \tag{4.12}$$

Calculations presented in appendix B.2 yields,

$$F_{\mu\nu} = \frac{1}{R^2}(X_\nu A_\mu - X_\mu A_\nu + \Sigma_{\mu\nu}), \qquad F_{2k\mu} = -\frac{R+X_{2k}}{R^2}A_\mu \tag{4.13}$$

Taking square of (4.13) gives

$$R^4\sum_{a<b}F_{ab}^2 - \sum_{\mu<\nu}\Sigma_{\mu\nu}^2 = 0 \tag{4.14}$$

The second term in (4.14) is proportional to identity due to Schur's lemma. In fact, it is the eigenvalue of the Casimir operator of SO$(2k-1)$ Lie algebra. As a result, we can choose the constant gauge field as the $I-$fold symmetric tensor product of the fundamental spinor representation of SO$(2k-1)$,

$$\begin{aligned}\left(\frac{I}{2}\right) &\equiv \left(\frac{I}{2}, \ldots, \frac{I}{2}\right) \\ &= \left(\frac{1}{2}, \ldots, \frac{1}{2}\right) \otimes_s \cdots \otimes_s \left(\frac{1}{2}, \ldots, \frac{1}{2}\right),\end{aligned} \tag{4.15}$$

where $\otimes_s$ denotes symmetric tensor product.

The Hamiltonian which describes the dynamics of a charge particle on S^{2k-1} in the presence of a constant SO$(2k-1)$ gauge field can be written down as

$$H = \frac{\hbar}{2MR^2}\sum_{a<b}\Lambda_{ab}^2, \tag{4.16}$$

where Λ_{ab} are defined as

$$\Lambda_{ab} = -i(X_a D_b - X_b D_a). \tag{4.17}$$

It is important to note that commutators of Λ_{ab} which are calculated as

$$\begin{aligned}[\Lambda_{ab}, \Lambda_{cd}] =& i(\delta_{ac}\Lambda_{bd} + \delta_{bd}\Lambda_{ac} - \delta_{bc}\Lambda_{ad} - \delta_{ad}\Lambda_{bc}) \\ & - i(X_a X_c F_{bd} + X_b X_d F_{ac} - X_b X_c F_{ad} - X_a X_d F_{bc}),\end{aligned} \tag{4.18}$$

does not satisfy SO$(2k)$ commutation relations, due to the existence of the background gauge field. This issue can be fixed by defining a new operator L_{ab} which combines Λ_{ab} and the spin angular momentum of the background gauge field,

$$L_{ab} = \Lambda_{ab} + R^2 F_{ab}. \tag{4.19}$$

Specifically we can write

$$L_{\mu\nu} = L^{(0)}_{\mu\nu} + \Sigma_{\mu\nu}, \qquad L_{2k\mu} = L^{(0)}_{2k\mu} - RA_\mu, \tag{4.20}$$

where $L^{(0)}_{ab} = -i(X_a\partial_b - X_b\partial_a)$ generates SO$(2k)$ over S^{2k-1}. Now, the commutators of L_{ab}

$$[L_{ab}, L_{cd}] = i(\delta_{ac}L_{bd} + \delta_{bd}L_{ac} - \delta_{bc}L_{ad} - \delta_{ad}L_{bc}), \tag{4.21}$$

satisfies the SO$(2k)$ commutation relations as expected.

Now, the Hamiltonian (4.16) takes the following form,

$$H = \frac{\hbar}{2MR^2}\left(\sum_{a<b} L^2_{ab} - \sum_{\mu<\nu} \Sigma^2_{\mu\nu}\right), \tag{4.22}$$

where we have used the fact that

$$\Lambda_{ab}F_{ab} = F_{ab}\Lambda_{ab} = 0. \tag{4.23}$$

Our next task is to obtain the energy spectrum of this Hamiltonian. We already know that $\Sigma_{\mu\nu}$ is in the $\left(\frac{I}{2}\right)$ IRR of SO$(2k-1)$. Hence, we can determine the IRR of SO$(2k)$ by looking at its restriction into IRR of the subgroup SO$(2k-1)$. These restriction rules are called branching rules. An SO$(2k)$ IRR can be labeled by integers or half odd integers $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ with $\lambda_1 \geq \lambda_2 \geq \ldots |\lambda_k|$. $\lambda_i \geq 0$ for $i = 1, 2, \ldots, k-1$ and additionally λ_k can have negative values. Two IRRs of SO$(2k)$ labeled by $(\lambda_1, \lambda_2 \ldots, \lambda_k)$ and $(\lambda_1, \lambda_2 \ldots, -\lambda_k)$ are conjugate to each other. On the other hand, IRRs of SO$(2k-1)$ are labeled by $k-1$ integers $(\mu_1, \mu_2, \ldots, \mu_{k-1})$ satisfying $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{k-1}$. Like previous case, μ_i are integers or half odd integers. However, the last index μ_{k-1} must be a positive number. In these specific IRRs, the branching rule from SO$(2k)$ to SO$(2k-1)$ state that

$$\lambda_1 \geq \mu_1 \lambda_2 \geq \cdots \geq \mu_{k-1} \geq |\lambda_k|. \tag{4.24}$$

Subsequently, $\left(\frac{I}{2}\right)$ IRR of SO$(2k-1)$ appears in this branching if the following conditions satisfy,

$$\lambda_1 \geq \frac{I}{2}, \quad \lambda_2 = \lambda_3 = \cdots = \lambda_{k-1} = \frac{I}{2}, \quad |\lambda_k| \leq \frac{I}{2}. \tag{4.25}$$

Hence, it is convenient to write $\lambda_1 = n + \frac{I}{2}$ ($n \in \mathbb{Z}_{\geq 0}$), and we can set $\lambda_k = s$ for some $|s| \leq \frac{I}{2}$. As a result the most general IRR of SO$(2k)$ is $\left(n + \frac{I}{2}, \frac{I}{2}, \ldots, \frac{I}{2}, s\right)$. This general IRR can be decomposed into SO$(2k-1)$ IRRs as

$$\left(n + \frac{I}{2}, \frac{I}{2}, \cdots, \frac{I}{2}, s\right) = \bigoplus_{\mu_1 = \frac{I}{2}}^{n+\frac{I}{2}} \bigoplus_{\mu_2 = s}^{\frac{I}{2}} \left(\mu_1, \frac{I}{2}, \cdots, \frac{I}{2}, \mu_2\right). \tag{4.26}$$

Squared terms in (4.22) are quadratic Casimirs of SO$(2k)$ and SO$(2k-1)$ in the IRRs $\left(n + \frac{I}{2}, \frac{I}{2}, \cdots, \frac{I}{2}, s\right)$ and $\left(\frac{I}{2}, \frac{I}{2}, \cdots, \frac{I}{2}\right)$, respectively. Hence the spectrum of the Hamiltonian can be calculated as

$$\begin{aligned} E =& \frac{\hbar}{2MR^2} \left(C^2_{SO(2k)} \left(n + \frac{I}{2}, \frac{I}{2}, \cdots, \frac{I}{2}, s\right) - C^2_{SO(2k-1)} \left(\frac{I}{2}, \frac{I}{2}, \cdots, \frac{I}{2}\right)\right) \\ =& \frac{\hbar}{2MR^2} \left(n^2 + s^2 + n(I + 2k - 2) + \frac{I}{2}(k-1)\right). \end{aligned} \tag{4.27}$$

The lowest Landau level (LLL) is not unique, it depends on the background charge I. If I is even the LLL can be obtained by setting $n = 0$ and $s = 0$, if I is odd we must set $n = 0$ and $s = \pm\frac{1}{2}$, thus we have

$$E_{LLL} = \begin{cases} \frac{\hbar}{2MR^2} \frac{I}{2}(k-1) & \text{for even } I\,, \\ \frac{\hbar}{2MR^2} \left(\frac{I}{2}(k-1) + \frac{1}{4}\right) & \text{for odd } I \end{cases} \tag{4.28}$$

Here n and s are the quantum numbers which are labeling the Landau levels.

We know that each Landau level is highly degenerate. The number of the degeneracy can be found by calculating the dimension of the IRR $\left(n + \frac{I}{2}, \frac{I}{2}, \cdots, \frac{I}{2}, s\right)$ of SO$(2k)$. The degeneracy of the Landau level labeled by (n, s) is then

$$d(n, s) = \prod_{i<j}^{k} \left(\frac{m_i - m_j}{g_i - g_j}\right) \prod_{i<j}^{k} \left(\frac{m_i + m_j}{g_i + g_j}\right), \tag{4.29}$$

where $g_i = k - i$ and $m_1 = n + \frac{I}{2} + g_1$, $m_i = \frac{I}{2} + g_i$ $(i = 2, \dots, k-1)$, and $m_k = s + g_k$. For the LLL, in a large I limit degeneracy goes like

$$\begin{aligned} d(0,0) &\to I^{\frac{(k-1)(k+2)}{2}} \\ d(0, \pm\frac{1}{2}) &\to I^{\frac{(k-1)(k+2)}{2}} . \end{aligned} \tag{4.30}$$

The thermodynamic limit which can be taken by the following. As I,$R \to \infty$ while the "magnetic length" scale keeping fixed $l_M = \frac{R}{\sqrt{I}}$, we have

$$E(n,s) \longrightarrow \frac{\hbar}{2M\ell_M^2}\left(n + \frac{1}{2}(k-1)\right) , \quad E_{LLL} = \frac{\hbar}{2M\ell_M^2}\frac{k-1}{2}, \tag{4.31}$$

and we see that the LLL energy has the same form as in the standart integer quantum hall effect in 2D up to an overall constant.

We have found the energy levels. Now our next task is to find corresponding wave functions. In fact, the wave functions can be written in terms of Wigner $\mathcal{D}$-functions $\mathcal{D}^{\left(n+\frac{I}{2},\frac{I}{2},\dots,\frac{I}{2},s\right)}(g)_{[L][R]}$ of SO$(2k)$ with $\left(n + \frac{I}{2}, \frac{I}{2}, \dots, \frac{I}{2}, s\right)$ IRR. The $[L]$ and $[R]$ indices represent the states in the current IRR of SO$(2k)$ with respect to the IRRs of SO$(2k-1)$ found in the branching. However, instead of working with Wigner $\mathcal{D}$-functions we will use 2^{k-1}-component spinors from (4.8)

$$\Psi^\pm = \frac{1}{2}\frac{1}{\sqrt{R(R+X_{2k})}}((R+X_{2k})\mathbb{I}_{2^{k-1}} \mp iX_\mu\gamma^\mu)\tilde{\phi} , \qquad \Psi = \begin{pmatrix} \Psi^+ \\ \Psi^- \end{pmatrix} . \tag{4.32}$$

In fact, they are the LLL wave functions for $I = 1$. Ψ^+ and Ψ^- correspond to $s = \frac{1}{2}$ and $s = -\frac{1}{2}$ respectively. A more compact notation for $\Psi^\pm$ is

$$\Psi_a^\pm := K_{\alpha\beta}^\pm \tilde{\phi}_\beta, \tag{4.33}$$

and they satisfy that

$$L_{\mu\nu}\Psi_\alpha^\pm = K_{\alpha\beta}^\pm (\Sigma_{\mu\nu})_{\beta\gamma}\tilde{\phi}_\gamma , \quad L_{2k\mu}\Psi_\alpha^\pm = K_{\alpha\beta}^\pm (\mp\frac{1}{2}\gamma_\mu)_{\beta\gamma}\tilde{\phi}_\gamma . \tag{4.34}$$

Using these identities we calculate that

$$\begin{aligned} \sum_{a<b} L_{ab}^2 \Psi^\pm &= \sum_{\mu<\nu}\left(\Sigma_{\mu\nu}^2 + \frac{1}{4}\gamma_\mu^2\right)\Psi^\pm , \\ &= \sum_{\mu<\nu}\left(\Sigma_{\mu\nu}^2 + \frac{1}{2}\left(k - \frac{1}{2}\right)\right)\Psi^\pm , \end{aligned} \tag{4.35}$$

What are the LLL wave functions for a general odd I ? It can be constructed as the I-fold symmetric product of $\Psi_a^{\pm}$

$$\Psi^I = \sum_{\alpha_1, \cdots \alpha_I} f_{\alpha_1 \cdots \alpha_I} \Psi_{\alpha_1} \cdots \Psi_{\alpha_I}, \quad (\alpha = 1, 2, \ldots, 2^{k-1}) \tag{4.36}$$

where $f_{\alpha_1 \ldots \alpha_I}$ are total symmetric coefficients in its indices. They also obey $\Gamma^{\alpha}_{\alpha_1 \alpha_2} f_{\alpha_1 \alpha_2 \ldots \alpha_I} = 0$, $f_{\alpha\alpha\alpha_3 \ldots \alpha_I} = 0$ so they eliminate the representations different than symmetric that appear in the $I-$fold tensor product of SO$(2k-1)$.

For many particle systems the LLL wave function is simply Slater determinant of Ψ_I. For a system with N particles, the LLL wave function is

$$\Psi_N^I = \sum_{\alpha_1, \cdots \alpha_I} \epsilon_{\alpha_1 \cdots \alpha_I} \Psi^I_{\alpha_1}(x_1) \cdots \Psi^I_{\alpha_I}(x_N), \tag{4.37}$$

where $\epsilon_{\alpha_1 \cdots \alpha_I}$ is the generalized Levi-Civita symbol.

4.2 The Equatorial S^{2k-2}

The results we have found for the odd spheres can lead us to explore the known results of the Landau problems on even spheres, S^{2k-2}. First we calculate that

$$(K^{\pm})^2 = \frac{1}{R}(X_{2k} \mathbb{I}_{2^{k-1}} \mp i X_\mu \gamma^\mu). \tag{4.38}$$

The equatorial S^{2k-2} can be obtained by identifying one of the coordinates of S^{2k-1},customarily the last coordinate x_{2k}, to zero. Thus, on the equatorial S^{2k-2} $K^{\pm}$ takes the form

$$(K_0^{\pm})^2 := (K^{\pm})^2 \Big|_{x_{2k}=0} = \mp i \frac{1}{R} X_\mu \gamma^\mu, \tag{4.39}$$

where R is the radius of S^{2k-2}. Now, we are able to introduce an idempotent on S^{2k-2} as

$$Q = i(K_0^{\pm})^2, \quad Q^\dagger = Q, \quad Q^2 = \mathbb{I}_{2^{k-1}}. \tag{4.40}$$

Using this idempotent we can construct the rank-1 projection operators as

$$\mathcal{P}_{\pm} = \frac{\mathbb{I}_{2^{k-1}} \pm Q}{2} \tag{4.41}$$

and observe that $\mathcal{P}_{\pm}^2 = \mathcal{P}_{\pm}$.
Let us denote the algebra of functions on S^{2k-2} as $\mathcal{A}$. The free $\mathcal{A}$-module can be

written as $\mathcal{A}^{2^{k-1}} = \mathcal{A} \otimes \mathbb{C}^{2^{k-1}}$ and the projective modules $\mathcal{P}_{\pm}\mathcal{A}^{2^{k-1}}$. This means we may write

$$\mathcal{A}^{2^{k-1}} = \mathcal{P}_{+}\mathcal{A}^{2^{k-1}} \oplus \mathcal{P}_{-}\mathcal{A}^{2^{k-1}} \,, \tag{4.42}$$

where $\mathcal{P}_{+}\mathcal{A}^{2^{k-1}}$ and $\mathcal{P}_{-}\mathcal{A}^{2^{k-1}}$ are of dimension 2^{k-2}.

1-rank projectors can be used to construct higher rank projectors. Specifically, rank I projectors can be constructed by

$$\mathcal{P}_{\pm}^{I} = \prod_{i=1}^{I} \frac{\mathbb{I} \pm \mathcal{Q}_i}{2} \,, \quad \mathcal{Q}_i = \mathbb{I}_{2^{k-1}} \otimes \mathbb{I}_{2^{k-1}} \otimes \cdots \otimes Q \otimes \cdots \otimes \mathbb{I}_{2^{k-1}} \,, \tag{4.43}$$

where $\mathcal{Q}_i$ is the I-fold tensor product of identity matrices except that its ith entry is $\mathcal{Q}$. $\mathcal{P}_{\pm}^{I}$ and $\mathcal{Q}_i$ act on the free module $\mathcal{A}_I^{2^{k-1}} = \mathcal{A} \otimes \mathbb{C}_I^{2^{k-1}}$, where $\mathbb{C}_I^{2^{k-1}}$ is the I-fold tensor product of $\mathbb{C}^{2^{k-1}}$, $\mathbb{C}_I^{2^{k-1}} = \mathbb{C}^{2^{k-1}} \otimes \cdots \otimes \mathbb{C}^{2^{k-1}}$. Without going into technical details, we may state that $\mathcal{P}_{\pm}$ are unitarily equivalent to projections $\mathcal{P}'^{I}_{\pm}$ which project to the $\left(\frac{I}{2}, \frac{I}{2}, \cdots, \frac{I}{2}\right)$ IRR of SO$(2k-2)$. The unitary equivalence of $\mathcal{P}_{\pm}$ and $\mathcal{P}'^{I}_{\pm}$ can be demonstrated building upon the ideas presented in [33]. However, we will not consider this here.

The branching of the $\left(\frac{I}{2}, \frac{I}{2}, \cdots, \frac{I}{2}\right)$ IRR of SO$(2k-2)$ under the IRR of the SO$(2k-2)$ reads

$$\left(\frac{I}{2}, \frac{I}{2}, \cdots, \frac{I}{2}\right) = \bigoplus_{\mu=-\frac{I}{2}}^{\frac{I}{2}} \left(\frac{I}{2}, \frac{I}{2}, \cdots, \frac{I}{2}, \mu\right) . \tag{4.44}$$

We see that in the right hand side of (4.44) the $\left(\frac{I}{2}, \frac{I}{2}, \ldots, \pm\left|\frac{I}{2}\right|\right)$ IRRs of SO$(2k-2)$ appears so $\mathcal{P}_{\pm}^{I}$ are the projections to this IRR.

The connection two forms constructed from $\mathcal{P}_{\pm}^{I}$ are

$$\mathcal{F}_{\pm} = \mathcal{P}_{\pm}^{I}\, d\left(\mathcal{P}_{\pm}^{I}\right) d\left(\mathcal{P}_{\pm}^{I}\right) . \tag{4.45}$$

Hence, together with (4.44), we can say that $\mathcal{F}_{\pm}$ are just the SO$(2k-2)$ background gauge fields on even spheres S^{2k-2}. Lastly, an important topological number, which is Chern number, is

$$c_{k-1}^{\pm} = \frac{1}{k!(2\pi)^k} \int_{S^{2k-2}} \mathcal{P}_{\pm}^{I} \left(d\left(\mathcal{P}_{\pm}^{I}\right)\right)^{2k-2} . \tag{4.46}$$

where $c_{k-1}^{\pm}$ is the $(k-1)^{\text{th}}$ Chern number with $c_{k-1} \equiv c_{k-1}^{+} > 0$ and $c_{k-1}^{-} = -c_{k-1}$. In fact, the degeneracy of the LLL on S^{2k-2} is related with this Chern number. The correspondence between these two numbers is $c_{k-1}(I) = d_{LLL}^{S^{2k-2}}(k-1, I-1)$.

4.3 Landau Problem on S^3

The energy levels of the Landau problem on three sphere can be obtained by setting $k = 2$ in (4.27),

$$E_{n,s} = \frac{\hbar}{2MR^2}\left(n^2 + 2n + In + \frac{I}{2} + s^2\right), \tag{4.47}$$

and the degeneracy of the $(n, s)^{\text{th}}$ Landau level can be easily calculated by the dimension of the $\left(n + \frac{I}{s}, s\right)$ IRR of SO(4)

$$d(n, s) = (n + \frac{I}{2} + s + 1)(n + \frac{I}{2} - s + 1) = (n + \frac{I}{2} + 1)^2 - s^2\,. \tag{4.48}$$

The LLL is then

$$E_{LLL} = \frac{\hbar}{2MR^2}\frac{I}{2}, \quad \text{I even}, \qquad E_{LLL} = \frac{\hbar}{2MR^2}\left(\frac{I}{2} + \frac{1}{4}\right), \quad \text{I odd}, \tag{4.49}$$

and their degeneracies are

$$\begin{aligned} d(n = 0, s = 0) &= (\frac{I}{2} + 1)^2 \\ d(n = 0, s = \pm\frac{1}{2}) &= d(0, +1/2) + d(0, -1/2) \\ &= \frac{1}{2}(I + 1)(I + 3). \end{aligned} \tag{4.50}$$

These results are first considered by Nair and Daemi and are all in agreement with the result of their work.

We can also obtain equatorial sphere S^2 by setting $x_4 = 0$. The idempotent now takes the form $Q = \vec{\sigma} \cdot \hat{X}$ and corresponding projectors are $\mathcal{P}_\pm = \frac{\mathbb{I}_2 \pm \sigma \cdot \hat{X}}{2}$, with $\hat{X} = \frac{X}{R}$. This yields the Abelian Dirac monopole connection $B_\mu = \frac{1}{2}\varepsilon_{\mu\nu\rho}F_{\nu\rho} = \frac{I}{2}\frac{X_i}{R^3}$. Finally, the first Chern number $c_1(I) = I$ corresponds the zero modes of the Dirac operator on two sphere in them monopole background.

4.4 Landau Problem on S^5

Another illustration is 5-sphere defined as a coset space $S^5 \equiv \text{SO}(6)/\text{SO}(5)$. Setting $k = 3$ gives the energy levels as

$$E = \frac{\hbar}{2MR^2}(n^2 + 4n + In + I + s^2)\,, \tag{4.51}$$

and its degeneracy is just the dimensiın of the $\left(n+\frac{I}{2}, \frac{I}{2}, s\right)$ IRR of SO(6) as

$$d(n,s) = \frac{1}{12}(n+1)^2(n+I+3)\left((n+\frac{I}{2}+2)^2 - s^2\right)\left((\frac{I}{2}+1)^2 - s^2\right) . \quad (4.52)$$

For the LLL,

$$E_{LLL} = \frac{\hbar}{2MR^2}I\,, \quad \text{I even}\,, \qquad E_{LLL} = \frac{\hbar}{2MR^2}\left(I+\frac{1}{4}\right)\,, \quad \text{I odd} \qquad (4.53)$$

$$d(n=0,s=0) = \frac{1}{3\cdot 2^6}(I+2)^2(I+3)(I+4)^2\,, \quad \text{I even}\,, \qquad (4.54)$$

and

$$d(n=0,s=\pm\frac{1}{2}) = d(0,+1/2)+d(0,-1/2) = \frac{1}{3\cdot 2^5}(I+1)(I+3)^3(I+5)\,, \quad \text{I odd}\,. \qquad (4.55)$$

The equatorial four sphere S^4 can be obtained by setting $X_6 = 0$ of S^5. The idempotent and the corresponding projectors now take form $Q = \frac{\gamma_\mu X_\mu}{R}$ and $\mathcal{P}_\pm = \frac{\mathbb{I}_4 \pm Q}{2}$ on the equatorial S^4. The field strength is $F_{ij} = \frac{1}{R^2}(X_j A_i - X_i A_j + \Sigma^+_{ij})$, $F_{5i} = -\frac{R+X_5}{R^2}A_i$, $i = (1\,, \cdots, 4)$, where $A_i = -\frac{1}{R(R+X_5)}\Sigma^+_{ij}X_j$, $A_5 = 0$, and $\Sigma^+_{ij} = -i\frac{1}{4}[\sigma_i\,, \sigma_j]$.

4.5 Dirac-Landau Problem on S^{2k-1}

In this section, our aim is to determine the spectrum of the Dirac operator for charged particles on S^{2k-1} under the influence of a constant $SO(2k-1)$ gauge field background.

Let us briefly recall the situation in the absence of a background gauge field. In this case, Dirac operator for odd-dimensional spheres S^{2k-1} is well-known. It can be expressed in the form [34]

$$\mathcal{D}^\pm = \frac{1}{2}(1 \mp \Gamma_{2k+1})\sum_{a<b}(-\Xi_{ab}L^{(0)}_{ab} + k - \frac{1}{2})\,, \qquad (4.56)$$

where $L^{(0)}_{ab}$ is given after (4.19) and carriers the $(n, 0, \cdots, 0)$ IRR of $SO(2k)$ and Ξ_{ab} given in (4.4) carries the reducible representation $\left(\frac{1}{2}, \frac{1}{2}, \cdots, \frac{1}{2}\right) \oplus \left(\frac{1}{2}, \frac{1}{2}, \cdots, -\frac{1}{2}\right)$ of $SO(2k)$. The projectors $\mathscr{P}^\mp = \frac{1}{2}(1 \mp \Gamma_{2k+1})$ allows us to pick either of the two inequivalent representations. To obtain the spectrum of $\mathcal{D}^{\pm}$, we simply need to observe that

$$(n,0,\cdots,0)\otimes\left(\frac{1}{2},\frac{1}{2},\cdots,\pm\frac{1}{2}\right)=\left(n+\frac{1}{2},\frac{1}{2},\cdots,\pm\frac{1}{2}\right)\oplus\left(n-\frac{1}{2},\frac{1}{2},\cdots,\mp\frac{1}{2}\right), \tag{4.57}$$

Since the $\left(\frac{1}{2},\frac{1}{2},\cdots,\pm\frac{1}{2}\right)$ IRRs of $SO(2k)$ are conjugates, both representations yield the same spectrum for the Dirac operator $\mathcal{D}^{\pm}$ as expected, which is found to be [34]

$$E_{\uparrow}=n+k-\frac{1}{2}\,,\quad E_{\downarrow}=-(n+k-\frac{3}{2})\,, \tag{4.58}$$

for the spin up and spin down states, respectively. Using the notation $j_{\uparrow\downarrow}=n\pm\frac{1}{2}$, we can express the spectrum of $\mathcal{D}^{\pm}$ more compactly as $E_{\uparrow\downarrow}=\pm(j_{\uparrow\downarrow}+k-1)$.

Let us now consider the gauged Dirac operator, which can be written by replacing $L_{ab}^{(0)}$ with $\Lambda_{ab}=L_{ab}-R^2F_{ab}$ as

$$\mathcal{D}_G^{\pm}=\frac{1}{2}(1\mp\Gamma_{2k+1})\sum_{a<b}\left(-\Xi_{ab}(L_{ab}-R^2F_{ab})+k-\frac{1}{2}\right)\,. \tag{4.59}$$

It is not possible to obtain the spectrum $\mathcal{D}_G$ in the same manner as that of the zero gauge field background case. There is, however, a well-known formula on symmetric spaces that relates the square of the gauged Dirac operator to the gauged Laplacian, the Ricci scalar of the manifold under consideration and a Zeeman energy term related to the curvature of the background gauge field [35]. Furthermore, on a symmetric coset space, say $K\equiv G/H$, a particular gauge field background which is compatible with the isometries of K generated by G (in the sense that the Lie derivative of the gauge field strength along a Killing vector of K is a gauge transformation of the field strength) is given by taking the gauge group as the holonomy group H and identifying the gauge connection with the spin connection. Then the square of the Dirac operator can be expressed as [35]

$$(i\mathcal{D}_G^{\pm})^2=C^2(G)-C^2(H)+\frac{\mathcal{R}}{8}\,, \tag{4.60}$$

where $\mathcal{R}$ is the Ricci scalar of the manifold K and $C^2(G)$ and $C^2(H)$ are quadratic Casimirs of G, H, respectively, and $C^2(H)$ is evaluated in the IRR of H characterizing the background gauge field, while $C^2(G)$ is evaluated in certain IRRs of G containing the fixed combinations of the background isospin of the gauge field and the intrinsic spin of the fermion. These considerations fit perfectly with our problem for odd spheres S^{2k-1} under fixed SO$(2k-1)$ gauge field backgrounds, since in the present problem we have taken the gauge group as the holonomy group SO$(2k-1)$ of the odd-spheres and the gauge connection has already been identified with the spin connection and taken explicitly in the IRR of SO$(2k-1)$, which is the I-fold symmetric tensor product of the fundamental spinor representation $\left(\frac{1}{2}, \cdots \frac{1}{2}\right)$. Therefore, we can write

$$(i\mathcal{D}_G^{\pm})^2 = C^2_{SO(2k)}\,(n+J, J, \cdots, J, \pm\tilde{s}) - C^2_{SO(2k-1)}\left(\frac{I}{2}, \frac{I}{2}, \cdots, \frac{I}{2}\right) + \frac{1}{4}(2k^2-3k+1) \tag{4.61}$$

where $2(2k^2-3k+1)$ is nothing but the Ricci scalar of the sphere S^{2k-1} and J takes on the values $J = \frac{I}{2} + \frac{1}{2}$ $(I \geq 0)$ and $J = \frac{I}{2} - \frac{1}{2}$ $(I \geq 1)$ corresponding to the spin up and spin down states, respectively and $|\tilde{s}| \leq J$. We find

$$\mathcal{E}_{\uparrow} = n(n+2k-1) + I(n+k-1) + k(k-1) + \tilde{s}^2\,, \quad I \geq 0\,, \tag{4.62}$$
$$\mathcal{E}_{\downarrow} = n(n+I+2k-3) + \tilde{s}^2\,, \quad I \geq 1 \tag{4.63}$$

It is readily seen that the spectrum for conjugate SO$(2k)$ IRRs coincide with $\tilde{s} \to -\tilde{s}$.

Degeneracy of $\mathcal{E}_{\uparrow}$ and $\mathcal{E}_{\downarrow}$ are given by the dimensions of the IRRs $(n+J, J, \cdots, J, s \pm \frac{1}{2})$ with $J = \frac{I}{2} + \frac{1}{2}$ and $J = \frac{I}{2} - \frac{1}{2}$, respectively. They can be computed from (4.29) with $g_i = k - i$ and $m_1 = n + J + g_1$, $m_i = J + g_i$ $(i = 2, \cdots, k-1)$ and $m_k = \tilde{s} + g_k$.

The Hamiltonian for the Dirac-Landau problem may be taken as $H = \frac{1}{2MR^2}(i\mathcal{D}_G^{\pm})^2$. For even I, we see that the LLL is given by taking $n = 0$ and $\tilde{s} = \pm\frac{1}{2}$ in (4.63) yielding $\mathcal{E}_{\downarrow}^{LLL} = \frac{1}{4}$ with the same degeneracy for both of the operators $(i\mathcal{D}_G^{\pm})^2$ and given as $d(n=0, \tilde{s}=\frac{1}{2}) = d(n=0\,, \tilde{s}=-\frac{1}{2})$, which can be computed from (4.29) using the facts given in the previous paragraph. For odd I, we see that LLL is given by taking $n = 0$ and $\tilde{s} = 0$ in (4.63) yielding $\mathcal{E}_{\downarrow}^{LLL} = 0$. These are the zero modes of the Dirac operators $\mathcal{D}_G^{\pm}$ with the degeneracy $d(n=0, \tilde{s}=0)$.

For S^3, we find that the LLL degeneracy for even I is given as

$$d(n=0,I) = \begin{array}{l} \dfrac{I(I+2)}{4} \text{ for even } I \\ \dfrac{(I+1)^2}{4} \text{ for odd } I, \end{array} \tag{4.64}$$

which is the number of zero modes of Dirac operators $\mathcal{D}_G^{\pm}$. These match with results of [18]. Another example is S^5, with the LLL degeneracy for even I given as $\frac{1}{3\cdot 2^6} I(I+2)^3(I+4)$, and for odd I it is $\frac{1}{3\cdot 2^6}(I+1)^2(I+2)(I+3)^2$.

We may recall that on even dimensional manifolds, Atiyah-Singer index theorem relates the number of zero modes, i.e. index of the Dirac operator to Chern classes, which are integers of topological significance [36]. On odd dimensional manifolds, however, there is known such general index theorem. One possible candidate for a topological number on these manifolds could be conceived as the Chern-Simons forms. Nevertheless, for odd spheres it is not too hard to see that these vanish identically when evaluated for the $\mathrm{SO}(2k-1)$ connection. Thus, it remains an open question to find out if and how the zero modes of $\mathcal{D}_G^{\pm}$ are related to a number of topological origin.

Finally, let us also note that setting $I=0$ in (4.62), we have $\tilde{s} = \pm\frac{1}{2}$ and we find $\mathcal{E}_{\uparrow} = (n+k-\frac{1}{2})^2$, which matches with the known result for $\mathcal{D}^{\pm}$ given in (4.58). Explicitly, we have $E_{\uparrow} = \sqrt{\mathcal{E}_{\uparrow}}$, while $E_{\downarrow} = -\sqrt{\mathcal{E}_{\uparrow}}$ with $n \to n-1$ and $\tilde{s} \to -\tilde{s}$. The latter are necessary to match the IRR $\left(n+\frac{1}{2}, \frac{1}{2}, \cdots, \frac{1}{2}, \pm\frac{1}{2}\right)$ with the second summand in (4.57).

CHAPTER 5

CONCLUSION

In this thesis, we have given the reviews of QHE on simple planar systems, two sphere S^2 and even-dimensional spheres S^{2k}. Then, by adapting the methods used by Hasebe and Kimura, we have solved the Landau problem and the Dirac-Landau problem for charged particles on odd-dimensional spheres S^{2k-1} in the background of constant $\mathrm{SO}(2k-1)$ gauge fields. First, using group theoretical arguments, we have determined the spectrum of the Schrödinger Hamiltonian together with its degeneracies at each Landau level. We gave the corresponding eigenstates in terms of the Wigner $\mathcal{D}$-functions in general, while for odd values of I an explicit local form of the LLL eigenstates is also obtained. We have noticed a peculiar relation between the Landau problem on S^{2k-1} and that on the equatorial S^{2k-2}, which allowed us to give the background $\mathrm{SO}(2k-2)$ gauge fields over S^{2k-2} by constructing the relevant projective modules. Additionally, for the Landau problem on S^5, we were able to demonstrate an exact correspondence between the union of Hilbert spaces of LLL's with I ranging from 0 to $I_{max} = 2K$ or $I_{max} = 2K+1$ to the Hilbert spaces of the fuzzy $\mathbb{C}P^3$ or that of winding number ± 1 line bundles over $\mathbb{C}P^3$ at level K, respectively. This correspondence also means that the quantum number $s = \pm\frac{1}{2}$ for the LLL over S^5 is actually related to the winding number $\kappa = \pm 1$ of the monopole bundles over $\mathbb{C}P^3_F$ via $s = \frac{\kappa}{2}$, which permits us to give, in a sense, a topological meaning to the ± 1 values of $2s$. In the last section, we have determined the spectrum of the Dirac operators on S^{2k-1} in the same gauge field background together with their degeneracies and found the number of their zero modes as well. Our results are in agreement with the spectra of the ungauged Dirac operators on S^{2k-1} for

$I = 0$ and generalizes it to all constant spin connection $\mathrm{SO}(2k-1)$ backgrounds.

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APPENDIX A

SOME REPRESENTATION THEORY

A.1 Branching Rules

Irreducible representations of SO($\mathcal{N}$) and SO($\mathcal{N}-1$)can be given in terms of the highest weight labels $[\lambda] \equiv (\lambda_1\,, \lambda_2\,, \cdots\,, \lambda_{\mathcal{N}-1}\,, \lambda_{\mathcal{N}})$ and $[\mu] \equiv (\mu_1\,, \mu_2\,, \cdots\,,\ \mu_{\mathcal{N}-1})$ respectively. Branching of the IRR $[\lambda]$ of SO($\mathcal{N}$) under SO($\mathcal{N}-1$) IRRs follows from the rule [37]

$$[\lambda] = \bigoplus_{\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \lambda_{k-1} \geq \mu_{k-1} \geq |\lambda_k|} [\mu], \qquad \text{for} \qquad \mathcal{N} = 2k \tag{A.1}$$

$$[\lambda] = \bigoplus_{\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \lambda_{k-1} \geq \mu_{k-1} \geq \lambda_k \geq |\mu_k|} [\mu], \qquad \text{for} \qquad \mathcal{N} = 2k+1 \tag{A.2}$$

A.2 Quadratic Casimir operators of SO$(2k)$ and SO$(2k-1)$ Lie algebras

Eigenvalues for the quadratic Casimir operators of SO($2k$) and SO($2k-1$) in the IRRs $[\lambda] \equiv (\lambda_1, \lambda_2 \cdots \lambda_k)$, $[\mu] \equiv (\mu_1, \mu_2 \cdots \mu_{k-1})$, respectively are given as [37]:

$$\begin{aligned} C_2^{SO(2k)}[\lambda] &= \sum_{i=1}^{k} \lambda_i(\lambda_i + 2k - 2i) \\ C_2^{SO(2k-1)}[\mu] &= \sum_{i=1}^{k-1} \mu_i(\mu_i + 2k - 1 - 2i)\,. \end{aligned} \tag{A.3}$$

Eigenvalues of quadratic Casimir operators of some specific IRRs are given as

$$\begin{aligned} C_2^{SO(4)}\left(n+\frac{I}{2},s\right) &= \frac{I^2}{4}+In+I+n^2+2n+s^2 \\ C_2^{SO(3)}\left(\frac{I}{2}\right) &= \frac{I^2}{4}+\frac{I}{2} \\ C_2^{SO(6)}\left(n+\frac{I}{2},\frac{I}{2},s\right) &= \frac{I^2}{2}+In+3I+n^2+4n+s^2 \\ C_2^{SO(5)}\left(\frac{I}{2},\frac{I}{2}\right) &= \frac{I^2}{2}+2I \end{aligned} \tag{A.4}$$

A.3 Relationship between Dynkin and Highest weight labels

Throughout this thesis highest weight labels (HW) has been used to label the irreducible representations of Lie algebras. Another more common way to label the IRRs is Dykin labeling (index). The relationship between Dykin labels and highest weight labels are as follows. For a SO(5) IRR , the labels are given as

$$(p,q)_{Dynkin} \equiv (\lambda_1,\lambda_2)_{HW}$$

and the relation between these labels are given by

$$\lambda_1 = p+\frac{q}{2} \qquad \lambda_2 = \frac{q}{2}$$

Then for instance, we have $(I/2, I/2)_{HW}$ corresponds to $(0, I)_{Dynkin}$.

For a SO(6) IRR , the labels are related by

$$(p,q,r)_{Dynkin} \equiv (\lambda_1,\lambda_2,\lambda_3)_{HW}$$

and the relation between these labels are given by

$$\lambda_1 = q+\frac{p+r}{2} \qquad \lambda_2 = \frac{p+r}{2} \qquad \lambda_3 = \frac{p-r}{2}$$

For SO(4) IRRs the labels are related by

$$(p,q)_{Dynkin} \equiv (\lambda_1,\lambda_2)_{HW}$$

and the relation between these labels are given by

$$\lambda_1 = \frac{p+q}{2} \qquad \lambda_2 = \frac{p-q}{2}$$

For instance, $(n+I/2, s)_{HW}$ corresponds to $(\frac{1}{2}(n+I/2+s), \frac{1}{2}(n+I/2-s))_{Dynkin}$.

APPENDIX B

DETAILS OF SOME COMPUTATIONS IN CHAPTER 4

B.1 Calculation of Gauge Field A_μ

We may express (4.8) as

$$\Psi = \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2R(R + X_{2k})}} \left[(R + X_{2k})1_{2^k} + \Gamma_\mu X_\mu\right] \begin{pmatrix} \tilde{\phi} \\ \tilde{\phi} \end{pmatrix}. \tag{B.1}$$

Writing out the spinor indices explicitly we have

$$\Psi_\alpha = \frac{1}{2} \frac{1}{\sqrt{R(R + X_{2k})}} \left[(R + X_{2k})\tilde{\phi}_\alpha + X_\mu (\Gamma_\mu)_{\alpha\alpha'} \tilde{\phi}_{\alpha'}\right]. \tag{B.2}$$

$$\Psi^\dagger_\alpha = \frac{1}{2} \frac{1}{\sqrt{R(R + X_{2k})}} \left[(R + X_{2k})\tilde{\phi}^\dagger_\alpha + X_\mu (\Gamma_\mu)_{\alpha''\alpha} \tilde{\phi}^\dagger_{\alpha''}\right]. \tag{B.3}$$

The SO($2k - 1$) gauge field is given by (4.10). Showing the spinor indices explicitly we may write

$$-iA_a = \Psi^\dagger_\alpha (\partial_a \Psi)_\alpha, \tag{B.4}$$

or

$$-iA_\mu = \Psi^\dagger_\alpha (\partial_\mu \Psi)_\alpha, \quad -iA_{2k} = \Psi^\dagger_{2k} (\partial_{2k} \Psi)_\alpha. \tag{B.5}$$

Let us first note that we have

$$\partial_a R = \frac{X_a}{R}, \quad \partial_a X_{2k} = \delta_{a2k}, \quad \partial_a X_\mu = \delta_{a\mu}. \tag{B.6}$$

With these we have

$$\partial_a \frac{1}{\sqrt{2R(R + X_{2k})}} = -\frac{1}{(2R(R + X_{2k}))^{3/2}} (2X_a + \frac{1}{R} X_a X_{2k} + R\delta_{a2k}), \tag{B.7}$$

and

$$\begin{aligned}\partial_a \Psi_\alpha =& -\frac{1}{\sqrt{2}}\frac{1}{(2R(R+X_{2k}))^{3/2}}(2X_a+\frac{1}{R}X_aX_{2k}+R\delta_{a2k})(R+X_{2k})\tilde{\phi}_\alpha \\ &-\frac{1}{\sqrt{2}}\frac{1}{(2R(R+X_{2k}))^{3/2}}(2X_a+\frac{1}{R}X_aX_{2k}+R\delta_{a2k})X_\mu(\Gamma_\mu)_{\alpha\alpha'}\tilde{\phi}_{\alpha'} \\ &+\frac{1}{\sqrt{2}}\frac{1}{(2R(R+X_{2k}))^{3/2}}(\frac{X_a}{R}+\delta_{a2k})\tilde{\phi}_\alpha \\ &+\frac{1}{\sqrt{2}}\frac{1}{(2R(R+X_{2k}))^{3/2}}\delta_{a\mu}(\Gamma_\mu)_{\alpha\alpha'}\tilde{\phi}_{\alpha'}.\end{aligned} \tag{B.8}$$

With these we can now compute A_a, we have

$$\begin{aligned}\Psi^\dagger\partial_a\Psi =& \frac{1}{2}\Bigg[-\frac{1}{(2R)^2}(2X_a+\frac{1}{R}X_aX_{2k}+R\delta_{a2k})+\frac{1}{2R}\left(\frac{X_a}{R}+\delta_{a2k}\right) \\ &-\frac{1}{(2R)^2(R+X_{2k})^2}(2X_a+\frac{1}{R}X_aX_{2k}+R\delta_{a2k})X_\mu X_\mu+\frac{1}{(2R)(R+X_{2k})}X_\mu\Bigg]\tilde{\phi}^\dagger_{2k}\alpha\tilde{\phi}_\alpha \\ &-\frac{1}{2}\Bigg[\frac{2}{(2R)^2(R+X_{2k})}(2X_a+\frac{1}{R}X_aX_{2k}+R\delta_{a2k})X_\mu-\frac{1}{2R}\delta_{a\mu} \\ &-\frac{1}{2R(R+X_{2k})}(\frac{X_a}{R}+\delta_{a2k})X_\mu\Bigg](\Gamma_\mu)_{\alpha\alpha'}\tilde{\phi}^\dagger_\alpha\tilde{\phi}_{\alpha'} \\ &+\frac{1}{2}\Bigg[-\frac{1}{(2R)^2(R+X_{2k})^2}(2X_a+\frac{1}{R}X_aX_{2k}+R\delta_{a2k})X_\mu X_\nu\frac{1}{2}[\Gamma_\mu,\Gamma_\nu]_{\alpha\alpha'}\tilde{\phi}^\dagger_\alpha\tilde{\phi}_{\alpha'} \\ &+\frac{1}{2R(R+X_{2k})}X_\nu\delta_{a\mu}\frac{1}{2}[\Gamma_\mu,\Gamma_\nu]_{\alpha\alpha'}\tilde{\phi}^\dagger_\alpha\tilde{\phi}_{\alpha'}\Bigg].\end{aligned}$$

After straightforward manipulation the terms in the first square bracket in the above expression an be shown to vanish for both $a=\mu$ and $a=2k$ components. We also have that

$$\tilde{\phi}^\dagger_\alpha(\Gamma_\mu)_{\alpha\alpha'}\tilde{\phi}_{\alpha'} = -i\tilde{\phi}^\dagger\gamma_\mu\tilde{\phi}+i\tilde{\phi}^\dagger\gamma_\mu\tilde{\phi}=0 \tag{B.9}$$

upon using Γ_μ given in (4.3) and that

$$X_\mu X_\nu[\Gamma_\mu,\Gamma_\nu]_{\alpha\alpha'}=0 \tag{B.10}$$

since $X_\mu X_\nu$ is symmetric under the exchange of μ,ν while $[\Gamma_\mu,\Gamma_\nu]$ is antisymmetric.

Thus we have

$$\Psi^\dagger\partial_a\Psi = \frac{1}{4R(R+X_{2k})}X_\nu\delta_{a\mu}\frac{1}{2}[\Gamma_\mu,\Gamma_\nu]_{\alpha\alpha'}\tilde{\phi}^\dagger_\alpha\tilde{\phi}_{\alpha'} \tag{B.11}$$

from which we immediately see that $A_{2k} = 0$ since $\mu \neq 2k$. Finally for $a = \mu$ we find

$$\begin{aligned}
\Psi^\dagger \partial_a \Psi &= \frac{1}{4R(R+X_{2k})} X_\nu (\tilde{\phi}^\dagger, \tilde{\phi}) \begin{pmatrix} \frac{1}{2}[\gamma_\mu, \gamma_\nu] & 0 \\ 0 & \frac{1}{2}[\gamma_\mu, \gamma_\nu] \end{pmatrix} \begin{pmatrix} \tilde{\phi} \\ \tilde{\phi} \end{pmatrix} \\
&= \tilde{\phi}^\dagger \left(\frac{i}{R(R + X_{2k} \Sigma_{\mu\nu} X_\nu)} \right) \tilde{\phi} \\
&= \tilde{\phi}^\dagger (-iA_\mu) \tilde{\phi}
\end{aligned} \tag{B.12}$$

and therefore $A_\mu = -\frac{1}{R(R+X_{2k})} \Sigma_{\mu\nu} X_\nu$ as we wanted to show.

B.2 Calculation of Field Strength $F_{\mu\nu}$ and $F_{2k\mu}$

Field strength is given by

$$F_{ab} = \partial_a A_b - \partial_b A_a + i[A_a, A_b], \quad a, b : 1, 2, \ldots, 2k. \tag{B.13}$$

First, we give calculation of $F_{2k\mu}$,

$$F_{2k\mu} = \partial_{2k} A_\mu - \partial_\mu A_{2k} + i[A_{2k}, A_\mu] = \partial_{2k} A_\mu, \tag{B.14}$$

where μ runs from 1 to $2k-1$. The second and the third term on the right hand side of (B.14) vanishes because $A_{2k} = 0$. Thus, we have

$$\begin{aligned}
F_{2k\mu} = \partial_{2k} A_\mu &= -\Sigma_{\mu\nu} \left[\partial_{2k} \left(\frac{1}{R(R+X_{2k})} \right) X_\nu + \frac{1}{R(R+X_{2k})} \partial_{2k} X_\nu \right] \\
&= \Sigma_{\mu\nu} \left[\frac{\frac{X_{2k}}{R}(R+X_{2k}) + R + X_{2k}}{R^2 (R+X_{2k})^2} \right] X_\nu \\
&= -\frac{R+X_{2k}}{R^2} A_\mu.
\end{aligned} \tag{B.15}$$

Second, we calculate $F_{\mu\nu}$,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu], \quad \mu, \nu : 1, 2, \ldots, 2k-1. \tag{B.16}$$

We need to calculate

$$\begin{aligned}
i[A_\mu, A_\nu] &= i \frac{1}{R^2 (R+X_{2k})^2} [\Sigma_{\mu\sigma} X_\sigma, \Sigma_{\nu\rho} X_\rho] \\
&= i \frac{X_\sigma X_\rho}{R^2 (R+X_{2k})^2} [\Sigma_{\mu\sigma}, \Sigma_{\nu\rho}] \\
&= \frac{X_\sigma X_\rho}{8R^2 (R+X_{2k})^2} (\delta_{\mu\nu} \Sigma_{\sigma\rho} + \delta_{\sigma\nu} \Sigma_{\rho\mu} + \delta_{\mu\rho} \Sigma_{\nu\sigma} + \delta_{\sigma\rho} \Sigma_{\mu\nu}) \\
&= \frac{1}{8R^2 (R+X_{2k})^2} (X_\nu X_\rho \Sigma_{\rho\mu} + X_\sigma X_\mu \Sigma_{\nu\sigma} + (R^2 - X_{2k}^2) \Sigma_{\mu\nu}),
\end{aligned} \tag{B.17}$$

and we have

$$\begin{aligned}
\partial_\mu A_\nu &= \partial_\mu \left(-\frac{1}{R(R+X_{2k})} \Sigma_{\nu\rho} X_\rho \right) \\
&= \Sigma_{\nu\rho} \left(\frac{X_\rho}{R^3(R+X_{2k})^2} \frac{X_\mu}{R} (2R + X_{2k}) - \frac{1}{R(R+X_{2k})} \delta_{\mu\rho} \right) \\
&= \frac{2R+X_{2k}}{R^3(R+X_{2k})^2} X_\rho X_\mu \Sigma_{\nu\rho} - \frac{1}{R(R+X_{2k})} \Sigma_{\nu\mu}.
\end{aligned} \tag{B.18}$$

Similarly, we have

$$\begin{aligned}
\partial_\nu A_\mu &= \partial_\nu \left(-\frac{1}{R(R+X_{2k})} \Sigma_{\mu\gamma} X_\gamma \right) \\
&= \Sigma_{\mu\gamma} \left(\frac{X_\gamma}{R^3(R+X_{2k})^2} \frac{X_\nu}{R} (2R + X_{2k}) - \frac{1}{R(R+X_{2k})} \delta_{\nu\gamma} \right) \\
&= \frac{2R+X_{2k}}{R^3(R+X_{2k})^2} X_\gamma X_\nu \Sigma_{\mu\gamma} - \frac{1}{R(R+X_{2k})} \Sigma_{\mu\nu}.
\end{aligned} \tag{B.19}$$

Finally, we obtain

$$\begin{aligned}
F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu] \\
&= \frac{1}{R^2} (X_\nu A_\mu - X_\mu A_\nu + \Sigma_{\mu\nu}).
\end{aligned} \tag{B.20}$$