UNBOUNDED p-CONVERGENCE IN LATTICE-NORMED VECTOR LATTICES

A THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES
OF
MIDDLE EAST TECHNICAL UNIVERSITY

BY

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IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY
IN
MATHEMATICS

MAY 2017

Approval of the thesis:

UNBOUNDED p-CONVERGENCE IN LATTICE-NORMED VECTOR LATTICES

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ABSTRACT

UNBOUNDED p-CONVERGENCE IN LATTICE-NORMED VECTOR LATTICES

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May 2017, 69 pages

The main aim of this thesis is to generalize unbounded order convergence, unbounded norm convergence and unbounded absolute weak convergence to lattice-normed vector lattices (LNVLs). Therefore, we introduce the follwing notion: a net (x_α) in an LNVL (X, p, E) is said to be *unbounded p-convergent* to $x \in X$ (shortly, x_α *up-converges* to x) if $p(|x_\alpha - x| \wedge u) \xrightarrow{o} 0$ in E for all $u \in X_+$. Throughout this thesis, we study general properties of up-convergence. Besides, we introduce several notions in lattice-normed vector lattices which correspond to notions from vector and Banach lattice theory. Finally, we study briefly the mixed-normed spaces and use them for an investigation of up-null nets and up-null sequences in lattice-normed vector lattices.

Keywords: Vector Lattice, Lattice-Normed Vector Lattice, up-Convergence, uo-Convergence, un-Convergence, uaw-Convergence, Mixed-Normed Space

ÖZ

KAFES(LATTİCE)-NORMLU VEKTÖR KAFESLERDE SINIRSIZ p-YAKINSAMA

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Doktora, Matematik Bölümü
Tez Yöneticisi : Prof. Dr. Eduard Emel'yanov

Mayıs 2017 , 69 sayfa

Bu tezin asıl amacı sınırsız sıralı yankınsama, sınırsız norm yakınsama ve sınırsız mutlak yakınsamaların kafes-normlu vektör kafeslere genellemektir. Böylece, aşağıdaki tanımları yaptık: kafes-normlu vektör kafes (X, p, E) deki bir x_α neti, E de her $u \in X_+$ için $p(|x_\alpha - x| \wedge u) \xrightarrow{o} 0$ şartını sağlarsa x_α neti x elemanına *sınırsız p-yakınsaktır* denir. Bu tezde, up-yakınsamanın genel özelliklerini çalıştık. Buna ilaveten, kafes-normlu vektör kafeslerde vektör ve Banach kafes teorisiyle ilgili çeşitli kavramlar tanımladık. Son olarak, karışık normlu alanları kısaca inceledik ve bunları kafes normlu vektör kafeslerinde up-null ağlar ve up-null dizilerin incelenmesi için kullandık.

Anahtar Kelimeler: Vektör Kafes, Kafes-Normlu Vektör Kafes, up-Yakınsama, uo-Yakınsama, un-Yakınsama, uaw-Yakınsama, Karışık-Normlu Uzay

To my first teacher: my mother Ayda
To my first mathematics teacher: my father Ali
To my brothers, relatives and friends

ACKNOWLEDGMENTS

In the first place, I would like to express my sincere gratitude to my supervisor Prof. Dr. Eduard Yu. Emelyanov for the continuous support of my Ph.D. study and research, for his patience, motivation, enthusiasm, and immense knowledge. His guidance helped me in all the time of research and writing of this thesis. I could not have imagined having a better supervisor and mentor for my Ph.D. study.

Besides my supervisor, I would like to thank the rest of my examining committee members. Also, I would like my instructors at Mathematics Department of Middle East Technical University (METU).

My sincere thanks also goes to Prof. Vladimir G. Troitsky for inviting me to the Department of Mathematical and Statistical Sciences at University of Alberta, Edmonton, Canada, to conduct a collaborative research project with him.

During my stay in Turkey, I had many friends from different countries and I would like to thank everyone of them for a good time we spent together and nice memories we had.

Last but not the least, I would like to thank my family: my parents and my brothers for their prayers, wishes and for supporting me throughout my life.

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CHAPTER 1

INTRODUCTION

A net $(x_\alpha)_{\alpha\in A}$ in a vector lattice X is said to be *order convergent* (or *o-convergent*) to a vector $x \in X$ if there is another net $(y_\beta)_{\beta\in B}$ satisfying: (i) $y_\beta \downarrow 0$; and (ii) for each $\beta \in B$ there exists $\alpha_\beta \in A$ such that $|x_\alpha - x| \leq y_\beta$ for each $\alpha \geq \alpha_\beta$. In this case we write, $x_\alpha \xrightarrow{\text{o}} x$. A net (x_α) in a vector lattice X is *unbounded order convergent* to a vector $x \in X$ if $|x_\alpha - x| \wedge u \xrightarrow{\text{o}} 0$ for all $u \in X_+$, in this case we say that the net (x_α) uo-converges to x and we write $x_\alpha \xrightarrow{\text{uo}} x$. The uo-convergence was first defined by H. Nakano (1948) in [34] under the name of "*individual convergence*". Nakano extended the individual ergodic theorem (it is known also as Birkhoff's ergodic theorem) to particular Banach lattices; KB-spaces. The idea of this extension is that the uo-convergence of sequences in $L_1(\mathbb{P})$, where $\mathbb{P}$ is a probability meausre, is equivalent to almost everywhere convergence. Later the name "unbounded order convergence" was proposed by R. DeMarr (1964) in [10]. DeMarr defined the uo-convergence in ordered vector spaces and his main result was that any locally convex space E can be embedded in a particular ordered vector space X so that topological convergence in E is equivalent to uo-convergence in X. The relation between weak and uo-convergences in Banach lattices were investigated by A. Wickstead (1977) in [38]. In [27] S. Kaplan (1997/98) established two characterizations of uo-convergence in order (Dedekind) complete vector lattices having weak units.

Recently, N. Gao and F. Xanthos (2014) studied uo-convergent and uo-Cauchy nets in Banach lattices and used them to characterize Banach lattices with the positive Schur property and KB-spaces. In addition, they applied uo-Cauchy sequences to extend Doob's submartingale convergence theorem to a measure-free setting; see [22]. As a continuation of their work, N. Gao (2014) studied unbounded order convergence in dual spaces of Banach lattices; see [19]. In [21] N. Gao, V. Troitsky, and F. Xanthos (2017) examined more properties of uo-convergence. In particular, they proved the uo-convergence is stable under passing to and from regular sublattices. This fact used to generalize several results in [22, 19]. Also, they used uo-convergence to study the convergence of Cesàro means in Banach lattices. As a result, they obtained an intrinsic version of Komlós' Theorem in Banach lattices and developed a new and unified approach to study Banach-Saks properties and Banach-Saks operators in Banach lattices based on uo-convergence. In [13] E. Emelyanov and M. Marabeh (2016) have used uo-convergence to derive two measure-free versions of Brezis-Lieb lemma in vector lattices. In addition, H. Li and Z. Chen (2017) showed that every norm bounded positive increasing net in an order continuous Banach lattice is uo-Cauchy and that every uo-Cauchy net in an order continuous Banach lattice has a uo-limit in the universal completion; see [30].

Unbounded order convergence is not just limited to mathematics. In fact, unbounded order convergence has been applied in finance. It is known that coherent risk measures play an important role in financial economics and actuarial science. As risk measures have convexity, a lot of efforts have then been dedicated to the general study of representations of proper convex functionals. In [23] N. Gao and F. Xanthos have exploited uo-convergence to derive a w^*-representation theorem of proper convex increasing functionals on particular dual Banach lattices. The work was extended in [20] by establishing representation theorems of convex functionals and risk measures using unbounded order continuous dual of a Banach lattice.

Let X be a normed lattice, then a net (x_α) in X is *unbounded norm convergent* to a vector $x \in X$ (or x_α un-convergent to x) if $|x_\alpha - x| \wedge u \xrightarrow{\|\cdot\|} 0$ for all $u \in X_+$. In this case, we write $x_\alpha \xrightarrow{\mathrm{un}} x$. The unbounded norm convergence was first defined by V. Troitsky (2004) in [36] under the name "*d-convergence*". He studied the relation between the d-convergence and measure of non-compactness. If $X = C_0(\Omega)$ where Ω is a normal topological space, then the un-convergence in X is the same as uniform convergence on compacta; see [36, Example 20]. In addition, if $X = L_p(\mu)$ where $1 \le p < \infty$ and μ is a finite measure, then un-convergence and convergence in measure agree in X; see [36, Example 23].

The name "unbounded norm convergence" was introduced by Y. Deng, M. O'Brien, and V. Troitsky (2016) in [11]. They studied basic properties of un-convergence and investigated its relation with uo- and weak convergences. Finally, they showed that un-convergence is topological.

In [25], M. Kandić, M. Marabeh, and V. Troitsky (2017) have investigated deeply the "*unbounded norm topology*" (or *un-topology*) in Banach lattices. They showed that the un-topology and the norm topology iff the Banach lattice has a strong unit. The un-topology is metrizable iff the Banach lattice has a quasi-interior point. The un-topology in an order continuous Banach lattice is locally convex iff it is atomic. An order continuous Banach lattice X is a KB-space iff its closed unit ball B_X is un-complete. For a Banach lattice X, B_X is un-compact iff X is an atomic KB-space. Also, they studied un-compact operators and the relationship between un-convergence and weak*-convergence.

Quite recently M. Kandić, H. Li, and V. Troitsky (2017) have genralized the concept of unbounded norm convergence as follows: let X be a normed lattice and Y a vector lattice such that X is an order dense ideal in Y, then a net (y_α) *un-converges to $y \in Y$ with respect to X* if $|y_\alpha - y| \wedge x \xrightarrow{\|\cdot\|} 0$ for every $x \in X_+$. They extended several known results about un-convergence and un-topology to this new setting in [24].

One more mode of unbounded convergence has been introduced and studied by O. Zabeti (2016). A net (x_α) in a Banach lattice X is said to be *unbounded absolute weak convergent* (or uaw-convergent) to $x \in X$ if $|x_\alpha - x| \wedge u \xrightarrow{\text{w}} 0$ for all $u \in X_+$; [40]. Zabeti investigated the relations of uaw-convergence with other convergnces. In addition, he obtained a characterization of order continuous and reflexive Banach lattices in terms of uaw-convergence.

Let X be a vector space, E be a vector lattice, and $p : X \to E_+$ be a lattice norm $\big($i.e., $p(x) = 0 \Leftrightarrow x = 0$, $p(\lambda x) = |\lambda| p(x)$ for all $\lambda \in \mathbb{R}$, $x \in X$, and $p(x+y) \leq p(x) + p(y)$ for all $x, y \in X\big)$, then the triple (X, p, E) is called a *lattice-normed space*; abbreviated as *LNS*. The lattice norm p of an LNS (X, p, E) is said to be *decomposable* if, for all $x \in X$ and $e_1, e_2 \in E_+$, from $p(x) = e_1 + e_2$ it follows that there exist $x_1, x_2 \in X$ such that $x = x_1 + x_2$ and $p(x_k) = e_k$ for $k = 1, 2$. In this case, the LNS (X, p, E) is referred as *decomposable LNS*. A net (x_α) in an LNS (X, p, E)

is *p-convergent* to $x \in X$ if $p(x_\alpha - x) \xrightarrow{o} 0$ in E. A net (x_α) in an LNS (X, p, E) is said to be *p-Cauchy* if $p(x_\alpha - x_\beta) \xrightarrow{o} 0$ in E. An LNS (X, p, E) is called *p-complete* if every p-Cauchy net is p-convergent. A decomposable p-complete LNS is called *Banach-Kantorovich space* (or *BKS* for short). Now let (X, p, E) be an LNS such that X is a vector lattice and the lattice norm p is monotone $\big($i.e. $|x| \leq |y| \Rightarrow p(x) \leq p(y)\big)$, then the triple (X, p, E) is called *lattice-normed vector lattice*, abbreviated as *LNVL*. A decomposable p-complete LNVL is called *Banach-Kantorovich lattice* (or *BKL* for short). If X is a vector lattice, then X is the lattice-normed vector lattice $(X, |\cdot|, X)$ where $|x|$ is the absolute value of $x \in X$. Also, any normed lattice $(X, \|\cdot\|)$ is the lattice-normed vector lattice $(X, \|\cdot\|, \mathbb{R})$.

Lattice-normed spaces were first defined by Leonid Kantorovich (1936) in [26]. After that, the theory of lattice-normed spaces was studied and then well-developed by Semën Kutateladze, Anatoly Kusraev and their students in Novosibirsk and Vladikavkaz. Many results from ergodic theory, probability theory etc. have been extended to lattice-normed vector lattices; see, e.g., [9, 14, 15, 16, 17, 18].

It should be noticed that the theory of lattice-normed spaces is well-developed under the condition of decomposability of lattice norm; see, e.g., [8, 12, 28, 29]. In the present thesis, we develop a general approach to lattice-normed vector lattices *without requiring* decomposability of lattice norm. This approach allow us to unify many results in the theory of vector and Banach lattices.

The structure of thesis is as follows. In **Chapter 2**, we provide basic notions and results from vector lattice theory that are needed throughout this study.

Chapter 3 consists of three sections. In the first section, we review the definition of unbounded order convergence (uo-convergence) and some of its properties. Furthermore, a characterization of uo-convergence in atomic vector lattices is given. The second section represents an application of uo-convergence by deriving two variants of the Brezis-Lieb lemma in vector lattices, the results of this section are published in [13]. In the last section, we recall definitions of unbounded norm convergence (un-convergence) and unbounded absolute weak convergence (uaw-convergence) and some basic results.

In **Chapter 4** we study many notions related to LNVLs in parallel to the theory of Banach lattices. For instance, an LNVL (X, p, E) is said called *op-continuous* if $X \ni x_\alpha \xrightarrow{o} 0$ implies $x_\alpha \xrightarrow{p} 0$; a *$p$-KB-space* if, for any $0 \leq x_\alpha \uparrow$ with $p(x_\alpha) \leq e \in E$, there exists $x \in X$ satisfying $x_\alpha \xrightarrow{p} x$. We give a characterization of op-continuity in Theorem 6, and study several properties of p-KB-spaces, e.g. in Proposition 15 and in Proposition 16. A vector $e \in X$ is said to be a *p-unit* if, for any $x \in X_+$, $p(x - ne \wedge x) \xrightarrow{o} 0$. Any p-unit is a weak unit, whereas strong units are p-units. For a normed lattice $(X, \|\cdot\|)$, a vector in X is a p-unit in $(X, \|\cdot\|, \mathbb{R})$ iff it is a quasi-interior point of the normed lattice $(X, \|\cdot\|)$.

The main part of this thesis is **Chapter 5** in which we define the unbounded p-convergence (up-convergence) in lattice-normed vector lattices (LNVLs). The up-convergence generalizes uo-convergence, un-convergence and unbounded absolute weak convergence. Within Chapter 5 some basic theory of unbounded p-convergence in LNVLs is developed in parallel to uo- and un-convergences. For example, it is enough to check the uo-convergence at a weak unit, while the un-convergence needs to be checked only at a quasi-interior point. Similarly, in LNVLs, up-convergence needs to be examined at a p-unit by Theorem 9. Moreover we introduce and study *up-regular* sublattices. Majorizing sublattices and projection bands are examples of up-regular sublattices by Theorem 10. Also some further investigation of up-regular sublattices is carried out in certain LNVLs in subsection 5.2.2.

Finally, in **Chapter 6** we study properties of mixed-normed LNVLs in Proposition 28, in Theorem 13, and in Theorem 14. We also prove that in a certain LNVL, the up-null nets are "p-almost disjoint" (see Theorem 15). These results generalize correspondent results from [21, 11].

The results of Chapters 4, 5 and 6 appear in the preprint [4].

CHAPTER 2

PRELIMINARIES

For the convenience of the reader, we present in this chapter the general background needed in the thesis.

Let "$\leq$" be an order relation on a real vector space X. Then X is called an *ordered vector space*, if it satisfies the following conditions: (i) $x \leq y$ implies $x + z \leq y + z$ for all $z \in X$; and (ii) $x \leq y$ implies $\lambda x \leq \lambda y$ for all $\lambda \in \mathbb{R}_+$.

For an ordered vector space X we let $X_+ := \{x \in X : x \geq 0\}$. The subset X_+ is called the *positive cone* of X. For each x and y in an ordered vector space X we let $x \vee y := \sup\{x, y\}$ and $x \wedge y := \inf\{x, y\}$. If $x \in X_+$ and $x \neq 0$, then we write $x > 0$.

An ordered vector space X is said to be a *vector lattice* (or a *Riesz space*) if for each pair of vectors $x, y \in X$ the $x \vee y$ and $x \wedge y$ both exist in X. Let X be a vector lattice and $x \in X$ then $x^+ := x \vee 0$, $x^- := (-x) \vee 0$ and $|x| := (-x) \vee x$ are the *positive part*, *negative part* and *absolute value* of x, respectively. Two elements x and y of a vector lattice X are *disjoint* written as $x \perp y$ if $|x| \wedge |y| = 0$. For a nonempty set A of X then its *disjoint complement* A^d is defined by $A^d := \{x \in X : x \perp a \text{ for all } a \in A\}$ and we write A^{dd} for $(A^d)^d$. A sequence (x_n) in a vector lattice is called *disjoint* if $x_n \perp x_m$ for all $n \neq m$. A subset S of a vector lattice X is *bounded from above* (respectively, *bounded from below*) if there is $x \in X$ with $s \leq x$ (respectively, $x \leq s$) for all $s \in S$. If $a, b \in X$, then the subset $[a, b] := \{x \in X : a \leq x \leq b\}$ is called *order interval* in X. A subset S of X is said to be *order bounded* if it is bounded from above and below or equivalently there is $u \in X_+$ so that $S \subseteq [-u, u]$. If a net (x_α) in

X is increasing and $x = \sup_\alpha x_\alpha$, then we write $x_\alpha \uparrow x$. The notation $x_\alpha \downarrow x$ means the net net (x_α) in X is decreasing and $x = \inf_\alpha x_\alpha$. A vector lattice X is said to be *Archimedean* if $\frac{1}{n}x \downarrow 0$ holds for each $x \in X_+$. Throughout this thesis, all vector lattices are assumed to be Archimedean.

A vector lattice X is called *order complete* or *Dedekind complete* if every order bounded above subset has a supremum, equivalently if $0 \leq x_\alpha \uparrow \leq u$ then there is $x \in X$ such that $x_\alpha \uparrow x$.

A vector subspace Y of a vector lattice X is said to be a $sublattice$ of X if for each y_1 and y_2 in Y we have $y_1 \vee y_2 \in Y$. A sublattice Y of X is *order dense* in X if for each $x > 0$ there is $0 < y \in Y$ with $0 < y \leq x$ and Y is said to be *majorizing* in X if for each $x \in X_+$ there exists $y \in Y$ such that $x \leq y$.

A linear operator $T : X \to Y$ between vector lattices is called *lattice homomorphism* if $|Tx| = T|x|$ for all $x \in X$. A one-to-one lattice homomorphism is referred as a *lattice isomorphism*. Two vector lattices X and Y are said to be *lattice isomorphic* when there is a lattice isomorphism from X onto Y.

If X is a vector lattice, then there is a (unique up to lattice isomorphism) order complete vector lattice X^δ that contains X as a majorizing order dense sublattice. We refer to X^δ as the *order completion* of X.

A subset Y of X is said to be *solid* if for $x \in X$ and $y \in Y$ such that $|x| \leq |y|$ it follows that $x \in Y$. A solid vector subspace of a vector lattice is referred as *ideal*. Let A be a nonempty subset of X then I_A the *ideal generated by* A is the smallest ideal in X that contains A. This ideal is given by

$$I_A := \{x \in X : \exists a_1, \dots, a_n \in A \text{ and } \lambda \in \mathbb{R}_+ \text{with } |x| \leq \lambda \sum_{j=1}^{n} |a_j|\}.$$

For $x_0 \in X$ then I_{x_0} the ideal generated by x_0 is referred as a *principal ideal*. This ideal has the form $I_{x_0} := \{x \in X : \exists \lambda \in \mathbb{R}_+ \text{with } |x| \leq \lambda |x_0|\}$.

A net $(x_\alpha)_{\alpha \in A}$ in a vector lattice X is said to be *order convergent* (or *o-convergent*) to a vector $x \in X$ if there is another net $(y_\beta)_{\beta \in B}$ satisfying: (i) $y_\beta \downarrow 0$; and (ii) for each $\beta \in B$ there exists $\alpha_\beta \in A$ such that $|x_\alpha - x| \leq y_\beta$ for each $\alpha \geq \alpha_\beta$. In this case we write $x_\alpha \xrightarrow{o} x$.

It follows from condition (ii) that an order convergent net has an order bounded tail, whereas an order convergent sequence is order bounded. For a net (x_α) in a vector lattice X and $x \in X$ we have $|x_\alpha - x| \xrightarrow{o} 0$ iff $x_\alpha \xrightarrow{o} x$ iff $|x_\alpha| \xrightarrow{o} |x|$. Thus without loss of generality we can only deal with order null nets in X_+. For an order bounded net (x_α) in an order complete vector lattice we have, $x_\alpha \xrightarrow{o} x$ iff $\inf_\alpha \sup_{\beta \geq \alpha} |x_\beta - x| = 0$. A net $(x_\alpha)_{\alpha \in A}$ in X is said to be *order Cauchy* (or *o-Cauchy*) if the double net $(x_\alpha - x_{\alpha'})_{(\alpha,\alpha') \in A \times A}$ is order convergent to 0. A linear operator $T : X \to Y$ between vector lattices is said to be *order continuous* if $x_\alpha \xrightarrow{o} 0$ in X implies $Tx_\alpha \xrightarrow{o} 0$ in Y. Order convergence is the same in a vector lattice and in its order completion.

Lemma 1. *[21, Corollary 2.9] For any net (x_α) in a vector lattice X, $x_\alpha \xrightarrow{o} 0$ in X iff $x_\alpha \xrightarrow{o} 0$ in X^δ.*

A subset A of X is called *order closed* (*o-closed*) if for any net (a_α) in A such that $a_\alpha \xrightarrow{o} x$ it follows that $x \in A$. An order closed ideal is a *band*. For $x_0 \in X$ the *principal band* generated by x_0 is the smallest band that includes x_0. We denote this band by B_{x_0} and it is described as $B_{x_0} := \{x \in X : |x| \wedge n|x_0| \uparrow |x|\}$. A band B in a vector lattice X is said to be a *projection band* if $X = B \oplus B^d$. If B is a projection band, then each $x \in X$ can be written uniquely as $x = x_1 + x_2$ where $x_1 \in B$ and $x_2 \in B^d$. The projection $P_B : X \to X$ defined by $P_B(x) := x_1$ is called the *band projection* corresponding to the band projection B. If P is a band projection then it is a lattice homomorphism and $0 \leq P \leq I$; i.e., $0 \leq Px \leq x$ for all $x \in X_+$. So band projections are order continuous.

A vector lattice X equipped with a norm $\|\cdot\|$ is said to be a *normed lattice* if $|x| \leq |y|$ in X implies $\|x\| \leq \|y\|$. If a normed lattice is norm complete, then it is called a *Banach lattice*. A normed lattice $(X, \|\cdot\|)$ is called *order continuous* if a net $x_\alpha \downarrow 0$ in X implies $\|x_\alpha\| \downarrow 0$ or equivalently $x_\alpha \xrightarrow{o} 0$ in X implies $\|x_\alpha\| \to 0$. A normed lattice $(X, \|\cdot\|)$ is called *σ-order continuous* if a sequence $x_n \downarrow 0$ in X implies $\|x_n\| \downarrow 0$ or equivalently $x_n \xrightarrow{o} 0$ in X implies $\|x_n\| \to 0$. Every order continuous normed lattice is σ-order continuous. A normed lattice $(X, \|\cdot\|)$ is called a *KB-space* if for $0 \leq x_\alpha \uparrow$ and $\sup_\alpha \|x_\alpha\| < \infty$ we get that the net (x_α) is norm convergent.

Let X be a vector lattice. A vector $0 < e \in X$ is called a *weak unit* if $B_e = X$, where B_e denotes the band generated by e.

Equivalently, e is a weak unit iff from $|x| \wedge e = 0$ it follows that $x = 0$. A vector $0 < e \in X$ is said to be a *strong unit* if $I_e = X$, where I_e denotes the ideal generated by e. That is, e is a strong unit iff for each $x \in X$ there is $\lambda > 0$ such that $|x| \leq \lambda e$.

Now assume that X is a normed lattice. Then a vector $0 < e \in X$ is called a *quasi-interior point* if $\overline{I_e} = X$, where I_e denotes the ideal generated by e. It can be shown that e is a quasi-interior point iff for every $x \in X_+$ we have $\|x - x \wedge ne\|$ as $n \to \infty$. Clearly, strong unit $\Rightarrow$ quasi-interior point $\Rightarrow$ weak unit. If a normed lattice is σ-order continuous then each weak unit is a quasi-interior point.

An element $a > 0$ in a vector lattice X is called an *atom* whenever for every $x \in [0, a]$ there is some real $\lambda \geq 0$ such that $x = \lambda a$. It is known that B_a the band generated by a is a projection band and $B_a = I_a = span\{a\}$, where I_a is the ideal generated by a. A vector lattice X is called *atomic* if the band generated by its atoms is X. For any $x > 0$ there is an atom a such that $a \leq x$. For any atom a, let P_a be the band projection corresponding to B_a. Then $P_a(x) = f_a(x)a$ where f_a is the biorthogonal functional corresponding to a. Since band projections are lattice homomorphisms and are order continuous, then so f_a for any atom a.

Example 1.

- *Let* $c_0 := \{x = (x_n)_{n\in\mathbb{N}} : \lim\limits_{n\to\infty} x_n = 0\}$, $\|x\|_\infty := \sup\limits_{n\in\mathbb{N}} |x_n|$.
- *Let* c *denote the space of all convergent sequences with the* ∞*-norm.*
- *For each* $1 \leq p < \infty$*, let* $\ell_p := \{x = (x_n)_{n\in\mathbb{N}} : \sum\limits_{n=1}^{\infty} |x_n|^p < \infty\}$*;* $\|x\|_p := (\sum\limits_{n=1}^{\infty} |x_n|^p)^{1/p}$.
- *Let* $\ell_\infty := \{x = (x_n)_{n\in\mathbb{N}} : \sup\limits_{n\in\mathbb{N}} |x_n| < \infty\}$ *; i.e., the space of all bounded sequences,* $\|x\|_\infty := \sup\limits_{n\in\mathbb{N}} |x_n|$.

All the above spaces are Banach lattices under the coordinatewise ordering. That is, given two sequences $x = (x_n)$ *and* $y = (y_n)$*. Then* $x \leq y$ *iff* $x_n \leq y_n$ *for all* $n \in \mathbb{N}$.

CHAPTER 3

UNBOUNDED CONVERGENCES

In this chapter, we recall definitions and some properties of unbounded order convergence, unbounded norm convergence, and unbounded absolute weak convergence. Also, we employ unbounded order convergence to obtain two versions of Brezis-Lieb lemma in vector lattices.

3.1 Unbounded order convergence in vector lattices

Unbounded order convergence was originally defined by H. Nakano in [34] under the name of "individual convergence". Independently R. DeMarr introduced the notion "unbounded order convergence" in [10]. Later the relation between weak and unbounded order convergence was examined by A. W. Wickstead [38]. After that two *nice* characterizations of uo-convergence in order complete vector lattices were obtained by S. Kaplan [27]. Recently many researchers investigated uo-convergence and its applications; [22, 19, 21, 23, 13, 30, 20].

Definition 1. *Let X be a vector lattice. A net (x_α) in X is said to be* ***unbounded order convergent*** *to a vector x if for any $u \in X_+$, $|x_\alpha - x| \wedge u \xrightarrow{\mathrm{o}} 0$. In this case, we say the net (x_α) uo-converges to x, and write $x_\alpha \xrightarrow{\mathrm{uo}} x$.*

Clearly, order convergence implies uo-convergence. The converse need not be true.

Example 2. *Consider the sequence (e_n) of standard unit vectors in c_0. Let $u = (u_1, u_2, \dots)$ in $(c_0)_+$. Then there is $n_0 \in \mathbb{N}$ such that $u_n < 1$ for any $n \geq n_0$. Let $y_n := \sum\limits_{k=n}^{\infty} u_k$ then $y_n \downarrow 0$ in c_0.*

For each $n \geq n_0$, we have $e_n \wedge u \leq y_n$. Hence, $e_n \xrightarrow{\text{uo}} 0$. The sequence (e_n) is not order bounded in c_0, and so $e_n \not\xrightarrow{o} 0$ in c_0.

For order bounded nets uo-convergence and o-convergence coincide. In [27] S. Kaplan showed that for order complete vector lattices uo-convergence can be checked at a weak unit. Later this result was generalized by N. Gao et al. in [21].

Proposition 1. *[21, Corollary 3.5] Let X be a vector lattice with a weak unit e. Then for any net (x_α) in X, $x_\alpha \xrightarrow{\text{uo}} 0$ iff $|x_\alpha| \wedge e \xrightarrow{\text{o}} 0$.*

Clearly not every vector lattice has a weak unit, but instead each vector lattice has a *complete disjoint system* $\{e_\gamma\}_{\gamma\in\Gamma} \subseteq X_+$, that is (i) $e_\gamma \wedge e_{\gamma'} = 0$ for $\gamma \neq \gamma'$; and (ii) if $x \wedge e_\gamma = 0$ for all $\gamma \in \Gamma$, then $x = 0$. As a generalization of Proposition 1 above, we will show that uo-convergence can be evaluated at elements of a complete disjoint system.

Proposition 2. *Let $\{e_\gamma\}_{\gamma\in\Gamma}$ be a complete disjoint system in a vector lattice X. Then, $x_\alpha \xrightarrow{uo} 0$ in X iff $|x_\alpha| \wedge e_\gamma \xrightarrow{o} 0$ in X for any $\gamma \in \Gamma$.*

Proof. The forward implication is trivial. For the converse implication let X^δ be the order completion of X, and assume that $|x_\alpha| \wedge e_\gamma \xrightarrow{o} 0$ in X for any $\gamma \in \Gamma$. Then by Lemma 1 $|x_\alpha| \wedge e_\gamma \xrightarrow{o} 0$ in X^δ for any $\gamma \in \Gamma$. Let $u \in X_+$, and $\gamma \in \Gamma$ then

$$\big(\inf_\alpha \sup_{\beta\geq\alpha}(|x_\beta| \wedge u)\big) \wedge e_\gamma = \big(\inf_\alpha \sup_{\beta\geq\alpha}(|x_\beta| \wedge e_\gamma)\big) \wedge u = 0 \wedge u = 0.$$

in X^δ. So, $\Big(\inf_\alpha \sup_{\beta\geq\alpha}(|x_\beta| \wedge u)\Big) \wedge e_\gamma = 0$ for each $\gamma \in \Gamma$. Since $\{e_\gamma\}_{\gamma\in\Gamma}$ is a complete disjoint system in X then it can be easily seen that it forms a complete disjoint system in X^δ as well. So $\inf_\alpha \sup_{\beta\geq\alpha}(|x_\beta| \wedge u) = 0$ in X^δ. Hence, $|x_\alpha| \wedge u \xrightarrow{o} 0$ in X^δ. By applying Lemma 1 again we get $|x_\alpha| \wedge u \xrightarrow{o} 0$ in X. □

A sublattice Y of a vector lattice X is called *regular* if for any subset A of $.Y$ if $\inf A$ exists in Y then $\inf A$ exists in X and the two infimums are equal. We recall a characterization of regular sublattices via order convergence.

Lemma 2. *[2, Theorem 1.20] For a sublattice Y of a vector lattice X the following statements are equivalent.*

1. Y *is regular.*

2. *If* (y_α) *in* Y *and* $y_\alpha \downarrow 0$ *in* Y*, then* $y_\alpha \downarrow 0$ *in* X.

3. *If* (y_α) *in* Y *and* $y_\alpha \xrightarrow{\text{o}} y$ *in* Y*, then* $y_\alpha \xrightarrow{\text{o}} y$ *in* X.

Remark 1.

1. *Every ideal is regular. Indeed, let* I *be an ideal in a vector latttice* X*, and* (y_α) *be a net in* I *such that* $y_\alpha \downarrow 0$ *in* I. *We show* $y_\alpha \downarrow 0$ *in* X. *Note first that* $y_\alpha \downarrow$ *in* X. *If* $x \in X$ *such that* $0 \leq x \leq y_\alpha$ *for any* α*, then* $x \in I$ *because* I *is ideal, and so* $x = 0$. *Thus* $y_\alpha \downarrow 0$ *in* X.

2. *Theorem 1.23 in [2] assures that every order dense sublattice is regular.*

3. *Any vector lattice* X *is regular in its order completion* X^δ.

4. *If* Y *is a regular sublattice of a vector lattice* X*, then order convergence in* X *need not imply order convergence in* Y. *For example,* c_0 *is an ideal of* ℓ_∞*, and so* c_0 *is regular, yet* $e_n \xrightarrow{\text{o}} 0$ *in* ℓ_∞ *but not in* c_0.

Recently, an interesting characterization of regular sublattices in terms of uo-convergence was established in [21].

Theorem 1. *[21, Theorem 3.2] Let* Y *be a sublattice of a vector lattice* X. *The following statements are equivalent.*

1. Y *is regular.*

2. *For any net* (y_α) *in* Y, $y_\alpha \xrightarrow{\text{uo}} 0$ *in* Y *iff* $y_\alpha \xrightarrow{\text{uo}} 0$ *in* X.

Given a measure space (Ω, Σ, μ), we write $L_0(\mu)$ for the vector lattice of real-valued measurable functions on Ω modulo almost everywhere ($a.e.$). For $f, g \in L_0(\mu)$, $f \leq g$ means $f(t) \leq g(t)$ for $a.e.$ $t \in \Omega$.

Proposition 3. *[21, Proposition 3.1] For a sequence* (f_n) *in* $L_0(\mu)$*, the following are equivalent:*

1. (f_n) *is* uo*-convergent;*

2. (f_n) *is* uo*-Cauchy;*

3. (f_n) *converges* $a.e.$*;*

4. (f_n) *is order convergent;*

5. (f_n) *is order Cauchy.*

In this case, (f_n) *is order bounded and the limits in 1, 3 and 4 are the same.*

Let (Ω, Σ, μ) be a measure space. Then for each $0 < p \leq \infty$, $L_p(\mu)$ is an ideal in $L_0(\mu)$, and so it is regular in $L_0(\mu)$. Hence combining Theorem 1 and Proposition 3 implies that $f_n \xrightarrow{\text{uo}} 0$ in $L_p(\mu)$ iff $f_n \to 0$ $a.e.$, for any sequence (f_n) in $L_p(\mu)$. Therefore uo-convergence is a generalization of $a.e.$-convergence. It is known that $a.e.$-convergence is not topological in general, i.e., there may not be a topology such that convergence with respect to this topology is the same as $a.e.$-convergence; see for example [35].

In what follows we show uo-convergence in atomic vector lattices is "*coordinatewise*" but first we characterize order convergence in atomic order complete vector lattices. We begin with the following technical lemma.

Lemma 3. *Let* X *and* Y *be vector lattices. If* $T : X \to Y$ *is an order continuous lattice homomorphism and* A *a subset of* X *such that* $\sup A$ *exists in* X*, then* $T(\sup A) = \sup T(A)$.

Proof. Note that $\{a_1 \vee \cdots \vee a_n : n \in \mathbb{N}, a_1, \ldots, a_n \in A\} \uparrow \sup A$. So $T\big(\{a_1 \vee \cdots \vee a_n : n \in \mathbb{N}, a_1, \ldots, a_n \in A\}\big) \uparrow T(\sup A)$. Furthermore, $T\big(\{a_1 \vee \cdots \vee a_n : n \in \mathbb{N}, a_1, \ldots, a_n \in A\}\big) = \{T(a_1 \vee \cdots \vee a_n) : n \in \mathbb{N}, a_1, \ldots, a_n \in A\} = \{Ta_1 \vee \cdots \vee Ta_n : n \in \mathbb{N}, a_1, \ldots, a_n \in A\} \uparrow \sup T(A)$. Hence $T(\sup A) = \sup T(A)$. $\square$

Lemma 4. *If* X *is an atomic order complete vector lattice and* (x_α) *is an order bounded net such that* $f_a(x_\alpha) \to 0$ *for any atom* a*, then* $x_\alpha \xrightarrow{\text{o}} 0$.

Proof. Suppose the contrary, then $\inf_\alpha \sup_{\beta \geq \alpha} |x_\beta| > 0$, so there is an atom a such that $a \leq \inf_\alpha \sup_{\beta \geq \alpha} |x_\beta|$. Hence $a \leq \sup_{\beta \geq \alpha} |x_\beta|$ for any α.

Let f_a be the biorthogonal functional corresponding to a, then it follows from Lemma 3 that $1 = f_a(a) \leq f_a\big(\sup_{\beta \geq \alpha} |x_\beta|\big) = \sup_{\beta \geq \alpha} |f_a(x_\beta)|$ for each α. Thus

$\limsup_\alpha |f_a(x_\alpha)| \geq 1$ which is a contradiction. □

Corollary 1. *If X is an atomic vector lattice and (x_α) is an order bounded net such that $f_a(x_\alpha) \to 0$ for any atom a, then $x_\alpha \xrightarrow{o} 0$.*

Proof. If $a \in X$ is an atom then a is an atom in X^δ and since X is atomic, the order completion X^δ is again atomic; see, e.g., [32, Exercise 37.23]. Thus (x_α) is an order bounded net such that $f_a(x_\alpha) \to 0$ for any atom a, then it follows from Lemma 4 that $x_\alpha \xrightarrow{o} 0$ in X^δ. Now Lemma 1 implies that $x_\alpha \xrightarrow{o} 0$ in X.

□

Next we characterize uo-convergence in atomic vector lattices.

Proposition 4. *Let X be an atomic vector lattice. For any atom a, let f_a be the biorthogonal functional of a. Then, $x_\alpha \xrightarrow{uo} 0$ in X iff $f_a(x_\alpha) \to 0$ for any atom $a \in X$.*

Proof. The "only if" part. Let (x_α) be a net in X such that $x_\alpha \xrightarrow{uo} 0$. Let $a \in X$ be an atom. Then $|x_\alpha| \wedge a \xrightarrow{o} 0$ in X. Since f_a is order continuous lattice homomorphism, then $|f_a(x_\alpha)| \wedge 1 \to 0$ in $\mathbb{R}$. Hence, $f_a(x_\alpha) \to 0$.

The "if" part. Assume there is a net (x_α) in X such that $f_a(x_\alpha) \to 0$ for any atom $a \in X$. Given $u \in X_+$ then we have $f_a(|x_\alpha| \wedge u) \to 0$ for each atom $a \in X$. Since the net $(|x_\alpha| \wedge u)$ is order bounded, then it follows from Corollary 1 that $x_\alpha \wedge u \xrightarrow{o} 0$. Hence, $x_\alpha \xrightarrow{uo} 0$.

□

We end up this section by a list of results that will be generalized in Chapters 5 and 6.

Remark 2.

1. *For a sequence (x_n) in a vector lattice X, if $x_n \xrightarrow{uo} 0$, then $\inf_k |x_{n_k}| = 0$ for any increasing sequence (n_k) of natural numbers.*

2. *Let X be an order continuous Banach lattice. Assume $x_\alpha \xrightarrow{\text{uo}} x$. Then $\|x\| \leq \liminf_\alpha \|x_\alpha\|$; see [22, Lemma 3.6].*

3. *Let X be a normed lattice. Suppose (x_α) is a uo-Cauchy net and $x_\alpha \xrightarrow{\|\cdot\|} x$. Then $x_\alpha \xrightarrow{\text{uo}} x$; see [22, Remark 4.1(2)].*

4. *Let X be an order continuous Banach lattice. Assume (x_α) is almost order bounded and $x_\alpha \xrightarrow{\text{uo}} x$. Then $x_\alpha \xrightarrow{\|\cdot\|} x$; see [22, Proposition 3.7].*

5. *In an order continuous Banach lattice, each almost order bounded uo-Cauchy net converges uo- and in norm to the same limit; see [22, Proposition 4.2].*

6. *Let X be an order continous Banach lattice and (x_n) a norm bounded sequence. If $x_n \xrightarrow{\text{uo}} 0$, then there is a subsequence (x_{n_k}) and a disjoint sequence (d_k) in X such that $x_{n_k} - d_k \xrightarrow{\|\cdot\|} 0$; see [21, Lemma 6.7].*

7. *Let B be a projection band and P the corresponding band projection. If $x_\alpha \xrightarrow{\text{uo}} x$ in X then $Px_\alpha \xrightarrow{\text{uo}} Px$ in both X and B; see [22, Lemma 3.3].*

3.2 An application of uo-convergence

In this section we first recall the Brezis-Lieb lemma and since almost everywhere convergence of sequences in L_p spaces is equivalent to uo-convergence, then we provide two variants of the Brezis-Lieb lemma in vector lattices.

3.2.1 The Brezis-Lieb lemma

The Brezis-Lieb lemma [6, Theorem 2] has numerous applications mainly in calculus of variations (see for example [7, 31]). We begin with its statement. Let $j : \mathbb{C} \to \mathbb{C}$ be a continuous function with $j(0) = 0$. In addition, let j satisfy the following hypothesis: for every sufficiently small $\varepsilon > 0$, there exist two continuous, nonnegative functions φ_ε and ψ_ε such that

$$|j(a + b) - j(a)| \leq \varepsilon\varphi_\varepsilon(a) + \psi_\varepsilon(b) \tag{3.1}$$

for all $a, b \in \mathbb{C}$. The following result has been stated and proved by H. Brezis and E. Lieb in [6].

Theorem 2. (*Brezis-Lieb lemma, [6, Theorem 2]*). *Let* (Ω, Σ, μ) *be a measure space. Let the mapping* j *satisfy the above hypothesis, and let* $f_n = f + g_n$ *be a sequence of measurable functions from* Ω *to* $\mathbb{C}$ *such that*:

1. $g_n \xrightarrow{\text{a.e.}} 0\,;$
2. $j \circ f \in L_1(\mu)$;
3. $\int \varphi_\varepsilon \circ g_n d\mu \leq C < \infty$ *for some* C *independent of* ε *and* n;
4. $\int \psi_\varepsilon \circ f d\mu < \infty$ *for all* $\varepsilon > 0$.

Then, as $n \to \infty$,

$$\int |j(f+g_n) - j(g_n) - j(f)| d\mu \to 0\,. \tag{3.2}$$

Here we reproduce its proof from [6, Theorem 2] with some additional remarks.

Proof. Fix $\varepsilon > 0$ and let

$$W_{\varepsilon,n} = \big[|j \circ f_n - j \circ g_n - j \circ f| - \varepsilon \varphi_\varepsilon \circ g_n\big]^+.$$

As $n \to \infty$, $W_{\varepsilon,n} \xrightarrow{\text{a.e.}} 0$. On the other hand,

$$|j \circ f_n - j \circ g_n - j \circ f| \leq |j \circ f_n - j \circ g_n| + |j \circ f| \leq \varepsilon \varphi_\varepsilon \circ g_n + \psi_\varepsilon \circ f + |j \circ f|.$$

Therefore $0 \leq W_{\varepsilon,n} \leq \psi_\varepsilon \circ f + |j \circ f| \in L_1(\mu)$. By dominated convergence theorem,

$$\lim_{n\to\infty} \int W_{\varepsilon,n} d\mu = 0. \tag{3.3}$$

However,

$$|j \circ f_n - j \circ g_n - j \circ f| \leq W_{\varepsilon,n} + \varepsilon \varphi_\varepsilon \circ g_n \tag{3.4}$$

and thus

$$I_n := \int |j \circ f_n - j \circ g_n - j \circ f| d\mu \leq \int \big(W_{\varepsilon,n} + \varepsilon \varphi_\varepsilon \circ g_n\big) d\mu.$$

Consequently, $\limsup I_n \leq \varepsilon C$. Now let $\varepsilon \to 0$. □

Remark 3. *(i) The conditions (3.3) and (3.4) mean that the sequence* $|j \circ f_n - j \circ g_n|$ *eventually lies in the set* $[-|j \circ f|, |j \circ f|] + \frac{3\varepsilon C}{2} B_{L_1}$*, where* B_{L_1} *is the closed unit ball of* $L_1(\mu)$*. In other words, the sequence* $|j \circ f_n - j \circ g_n|$ *is almost order bounded. Recall that a subset* A *in a normed lattice* $(X, \|\cdot\|)$ *is said to be almost order bounded if for any* $\varepsilon > 0$*, there is* $u_\varepsilon \in X_+$ *such that* $A \subseteq [-u_\varepsilon, u_\varepsilon] + \varepsilon B_X$*, where* B_X *is the closed unit ball of* X*.*

(ii) The superposition operator $J_j : L_0(\mu) \to L_0(\mu)$*,* $J_j(f) := j \circ f$ *induced by the mapping* j *in the proof above can be replaced by a mapping* $J : L_0(\mu) \to L_0(\mu)$ *satisfying some reasonably mild conditions for keeping the statement of the Brezis-Lieb lemma.*

(iii) Theorem 2 is equivalent to its partial case when the $\mathbb{C}$*-valued functions are replaced by* $\mathbb{R}$*-valued ones.*

The next lemma will be used to prove the coming theorem and a version of Brezis-Lieb lemma for arbitrary strictly positive linear functionals.

Lemma 5. (*[22, Proposition 3.7]*). *Let* X *be an order continuous Banach lattice. Assume* (x_α) *is almost order bounded and* $x_\alpha \xrightarrow{\text{uo}} x$*. Then* $x_\alpha \xrightarrow{\|\cdot\|} x$*.*

The following result is motivated by the proof of [6, Theorem 2].

Theorem 3. (*Brezis-Lieb lemma for mappings on* L_0). *Let* (Ω, Σ, μ) *be a measure space,* $f_n = f + g_n$ *be a sequence in* $L_0(\mu)$ *such that* $g_n \xrightarrow{\text{a.e.}} 0$*, and* $J : L_0(\mu) \to L_0(\mu)$ *be a mapping satisfying* $J(0) = 0$*, that preserves almost everywhere convergence and such that the sequence* $J(f_n) - J(g_n)$ *is almost order bounded. Then, as* $n \to \infty$*,*

$$\int |J(f + g_n) - \big(J(g_n) + J(f)\big)| d\mu \to 0\,. \tag{3.5}$$

Proof. Again, as in the proof of the Brezis-Lieb lemma above, denote $I_n := \int |J(f + g_n) - \big(J(f) + J(g_n)\big)|$. By the conditions, the sequence

$$J(f + g_n) - \big(J(f) + J(g_n)\big) = \big(J(f_n) - J(g_n)\big) - J(f)$$

a.e.-converges to 0 and is almost order bounded. Therefore by Lemma 5, $\lim\limits_{n\to\infty} I_n = 0$. $\square$

Recall that a collection $(f_\alpha) \subseteq L_1(\mu)$ is said to be *uniformly integrable* or *equi-integrable* if for each $\varepsilon > 0$ there is $\delta > 0$ such that $\int_E |f_\alpha| d\mu < \varepsilon$ for all α whenever $\mu(E) < \delta$. Since almost order boundedness and uniform integrability are equivalent in finite measure spaces, the following corollary is immediate.

Corollary 2. $\big($*Brezis-Lieb lemma for uniform integrable sequence* $J(f_n) - J(g_n)\big)$. *Let* (Ω, Σ, μ) *be a finite measure space,* $f_n = f + g_n$ *be a sequence in* $L_0(\mu)$ *such that* $g_n \xrightarrow{\text{a.e.}} 0$, *and* $J : L_0(\mu) \to L_0(\mu)$ *be a mapping satisfying* $J(0) = 0$, *that preserves almost everywhere convergence and such that the sequence* $J(f_n) - J(g_n)$ *is uniformly integrable. Then*

$$\lim_{n\to\infty} \int |J(f + g_n) - \big(J(g_n) + J(f)\big)| d\mu = 0 \,.$$

3.2.2 Two variants of the Brezis-Lieb lemma in vector lattices

In this subsection we give two variants of the Brezis-Lieb lemma in the vector lattice setting by replacing $a.e.$-convergence by uo-convergence, integral functionals by strictly positive functionals and the continuity of the scalar function j (in Theorem 2) by the so called *σ-unbounded order continuity* of the mapping $J : X \to Y$ between vector lattices X and Y.

Recall that in L_p spaces $(1 \leq p \leq \infty)$, uo-convergence of sequences is the same as the almost everywhere convergence (see; e.g., [21, Remark 3.4]). Therefore, in order to obtain versions of Brezis-Lieb lemma in vector lattices, we replace almost everywhere convergence by uo-convergence.

Definition 2. *A mapping* $f : X \to Y$ *between vector lattices is said to be* ***σ-unbounded order continuous*** *(in short,* ***σuo-continuous****) if* $x_n \xrightarrow{\text{uo}} x$ *in* X *implies* $f(x_n) \xrightarrow{\text{uo}} f(x)$ *in* Y.

Clearly the above definition is parallel to the well-known notion of σ-order continuous mappings between vector lattices.

Let Y be a vector lattice and l be a strictly positive linear functional on Y. Define the following norm on Y:

$$\|y\|_l := l(|y|). \tag{3.6}$$

Then the $\|\cdot\|_l$-completion $(Y_l, \|\cdot\|_l)$ of $(Y, \|\cdot\|_l)$ is an AL-space, and so it is order continuous Banach lattice. The following result is a measure-free version of Theorem 3.

Proposition 5. (*A Brezis-Lieb lemma for strictly positive linear functionals*). *Let X be a vector lattice and Y_l be the AL-space constructed above. Let $J : X \to Y_l$ be σuo-continuous with $J(0) = 0$, and (x_n) be a sequence in X such that:*

1. $x_n \xrightarrow{\text{uo}} x$ *in* X;
2. *the sequence* $(J(x_n) - J(x_n - x))_{n\in\mathbb{N}}$ *is almost order bounded in* Y_l

Then

$$\lim_{n\to\infty} \|J(x_n) - J(x_n - x) - J(x)\|_l = 0. \tag{3.7}$$

Proof. Since $x_n \xrightarrow{\text{uo}} x$ and J is σuo-continuous, then $J(x_n) \xrightarrow{\text{uo}} J(x)$ and $J(x_n - x) \xrightarrow{\text{uo}} J(0) = 0$. Thus $J(x_n) - J(x_n - x) \xrightarrow{\text{uo}} J(x)$. It follows from Lemma 5 that $\lim_{n\to\infty} \|J(x_n) - J(x_n - x) - J(x)\|_l = 0$. □

The next result is another measure-free version of Theorem 3.

Proposition 6. (*A Brezis-Lieb lemma for σuo-continuous linear functionals*). *Let X, Y be vector lattices and l be a σuo-continuous functional on Y. Assume further $J : X \to Y$ is a σuo-continuous mapping with $J(0) = 0$ and (x_n) is a sequence in X such that $x_n \xrightarrow{\text{uo}} x$. Then*

$$\lim_{n\to\infty} l(J(x_n) - J(x_n - x) - J(x)) = 0\,. \tag{3.8}$$

Proof. Since $x_n \xrightarrow{\text{uo}} x$ and J is σuo-continuous, then $J(x_n) \xrightarrow{\text{uo}} J(x)$ and $J(x_n - x) \xrightarrow{\text{uo}} J(0) = 0$. Thus $\big(J(x_n) - J(x_n - x) - J(x)\big) \xrightarrow{\text{uo}} 0$. But l is σuo-continuous, so $l\big(J(x_n) - J(x_n - x) - J(x)\big) \xrightarrow{\text{uo}} 0$. Since in $\mathbb{R}$ the uo-convergence, the o-convergence, and the standard convergence are all equivalent, then $\lim_{n\to\infty} l\big(J(x_n) - J(x_n - x) - J(x)\big) = 0$. □

We emphasize that in opposite to Proposition 6, in Proposition 5 we do not suppose the functional l to be σuo-continuous.

3.3 Unbounded norm and unbounded absolute weak convergences

Unbounded norm convergence was first known as *d-convergence* and it was defined in [36]. The relation between unbounded norm convergence with other types of convergences was studied in [11].

Definition 3. *[11] Let X be a normed lattice. Then a net (x_α) in X is said to be* ***unbounded norm convergent*** *to a vector x if $\big\||x_\alpha - x| \wedge u\big\| \to 0$ for every $u \in X_+$. In this case, we say (x_α) un-converges to x and write $x_\alpha \xrightarrow{\text{un}} x$.*

Clearly, norm convergence implies un-convergence. The converse need not be true.

Example 3. *Consider the sequence (e_n) of standard unit vectors in c_0. Let $u = (u_1, u_2, \ldots)$ be an element in $(c_0)_+$. Let $0 < \varepsilon < 1$ then there is $n_\varepsilon \in \mathbb{N}$ such that $u_n < \varepsilon$ for all $n \geq n_\varepsilon$. Thus for $n \geq n_\varepsilon$, $\|ne_n \wedge u\|_\infty = u_n < \varepsilon$. Hence $ne_n \xrightarrow{\text{un}} 0$. The sequence (ne_n) is not norm bounded, and so it can not be norm convergent.*

For order bounded nets, un-convergence and norm convergence coincide. If the norm of a Banach lattice is order continuous then uo-convergence implies un-convergence. We have seen in Proposition 1 Section 3.1, that it is enough to evaluate uo-convergence at a weak unit. Similarly, it suffices to evaluate un-convergence at a quasi-interior point.

Proposition 7. *[11, Lemma 2.11] Let X be a normed lattice with a quasi-interior point e. Then for any net (x_α) in X, $x_\alpha \xrightarrow{\text{un}} 0$ iff $\big\||x_\alpha| \wedge e\big\| \to 0$.*

The following result shows that un-convergence is an abstraction of convergence in measure.

Proposition 8. *[11, Corollary 4.2] Let (f_n) be a sequence in $L_p(\mu)$ where $1 \leq p < \infty$ and μ is a finite measure. Then $f_n \xrightarrow{\text{un}} 0$ iff $f_n \xrightarrow{\mu} 0$.*

Let Y be a sublattice of a Banach lattice X. Clearly, if (y_α) is a net in Y and $y_\alpha \xrightarrow{\text{un}} 0$ in X, then $y_\alpha \xrightarrow{\text{un}} 0$ in Y. The converse need not be true.

Example 4. *Let (e_n) be the sequence of standard unit vectors in c_0. Then $e_n \xrightarrow{\text{un}} 0$ in c_0, but this does not hold in ℓ_∞. Indeed, let $u = (1, 1, 1, \ldots)$ then $e_n \wedge u = e_n$ and $\|e_n\|_\infty = 1 \not\to 0$.*

We see from Example 4 that generally un-convergence does not pass from a regular sublattice to the whole space unlike the uo-convergence; see Theorem 1. Nevertheless, un-convergence can be lifted from particular sublattices to the whole space.

Theorem 4. *[25, Theorem 4.3] Let Y be a sublattice of a normed lattice X and (y_α) a net in Y such that $y_\alpha \xrightarrow{\text{un}} 0$ in Y. The following statements hold.*

1. *If Y is majorizing in X, then $y_\alpha \xrightarrow{\text{un}} 0$ in X.*
2. *If Y is norm dense in X, then $y_\alpha \xrightarrow{\text{un}} 0$ in X.*
3. *Y is a projection band in X, then $y_\alpha \xrightarrow{\text{un}} 0$ in X.*

Since every Archimedean vector lattice X is majorizing in its order completion X^δ, we have the following result.

Corollary 3. *[25, Corollary 4.4] If X is a normed lattice and $x_\alpha \xrightarrow{\text{un}} x$ in X, then $x_\alpha \xrightarrow{\text{un}} x$ in the order completion X^δ of X.*

Corollary 4. *[25, Corollary 4.5] If X is a KB-space and $x_\alpha \xrightarrow{\text{un}} 0$ in X, then $x_\alpha \xrightarrow{\text{un}} 0$ in X^{**}.*

Example 4 shows that the assumption that X is a KB-space cannot be removed.

Corollary 5. *[25, Corollary 4.6] Let Y be a sublattice of an order continuous Banach lattice X. If $y_\alpha \xrightarrow{\text{un}} 0$ in Y then $y_\alpha \xrightarrow{\text{un}} 0$ in X.*

Next we consider unbounded absolute weak convergence which was defined and studied in [40].

Definition 4. *Let X be a Banach lattice. A net (x_α) in X is said to be **unbounded absolute weakly convergent** to a vector x if for any $u \in X_+$, $|x_\alpha - x| \wedge u \xrightarrow{\text{w}} 0$. In this case we say the net (x_α) uaw-converges to x, and write $x_\alpha \xrightarrow{\text{uaw}} x$.*

Let X be a Banach lattice. If $x_\alpha \xrightarrow{|\sigma|(X,X^*)} 0$, then $x_\alpha \xrightarrow{\text{uaw}} 0$, where $|\sigma|(X, X^*)$ denotes the absolute weak topology on X. It was pointed out in [40, Example 3] that the converse need not be true. For order bounded nets uaw-convergence and absolute weak convergence are equivalent.

As in the case of un-convergence the following result illustrates that uaw-convergence can only be evaluated at a quasi-interior point.

Proposition 9. *[40, Lemma 6] Let X be a Banach lattice with a quasi-interior point e. Then for any net (x_α) in X, $x_\alpha \xrightarrow{\text{uaw}} 0$ iff $|x_\alpha| \wedge e \xrightarrow{\text{w}} 0$.*

The following result of O. Zabeti will be extended in Chapter 5.

Proposition 10. *[40, Proposition 15] Suppose X is an order continuous Banach lattice and I is an ideal of X. For a net $(x_\alpha) \subseteq I$, if $x_\alpha \xrightarrow{\text{uaw}} 0$ in I then $x_\alpha \xrightarrow{\text{uaw}} 0$ in X.*

Similar to the situation in Corollary 5 uaw-convergence on atomic order continuous Banach lattices can transfer from a sublattice to the whole space.

Proposition 11. *[40, Proposition 16] Suppose X is an order continuous Banach lattice and Y is a sublattice of X. If $y_\alpha \xrightarrow{\text{uaw}} 0$ in Y then $y_\alpha \xrightarrow{\text{uaw}} 0$ in X.*

Next result shows that uo-, un- and uaw-convergences all agree on order continuous Banach lattices.

Proposition 12. *[40, Corollary 14] Suppose X is an order continuous Banach lattice. Then uo-convergence un-convergence and uaw-convergence are agree iff X is atomic.*
Thus if X is an atomic order continuous Banach lattice, (x_α) is a net in X and f_a is the biorthogonal functional corresponding to an atom $a \in X$. Then $x_\alpha \xrightarrow{\text{uo}} 0$ iff $x_\alpha \xrightarrow{\text{un}} 0$ iff $x_\alpha \xrightarrow{\text{uaw}} 0$ iff $f_a(x_\alpha) \to 0$ for any atom $a \in X$.

It should be noticed that both un-convergence and uaw-convergence are induced by toplolgies known as *un-topology* and *uaw-topology* respectively. So unlike uo-convergence the un-convergence and the uaw-convergence are both topological. The un-topology and uaw-topology were ivestigated in [24, 25, 40].

We end up this section by a list of results that will be generalized in Chapters 5 and 6.

Remark 4.

1. *If (x_α) is an increasing net in a normed lattice X and $x_\alpha \xrightarrow{\mathrm{un}} x$ then $x_\alpha \uparrow x$ and $x_\alpha \xrightarrow{\|\cdot\|} x$; see [25, Lemma 1.2 (ii)].*

2. *Let X be a normed lattice. If $x_\alpha \xrightarrow{\mathrm{un}} x$, then $\|x\| \leq \liminf_\alpha \|x_\alpha\|$; see [11, Lemma 2.8].*

3. *Let X be a normed lattice. If $x_\alpha \xrightarrow{\mathrm{un}} x$ and (x_α) almost order bounded, then $x_\alpha \xrightarrow{\|\cdot\|} x$; see [11, Lemma 2.9].*

4. *Let X be a Banach lattice. Assume $x_\alpha \xrightarrow{\mathrm{un}} 0$. Then there is an increasing sequence (α_k) of indices and a disjoint sequence (d_k) satisfying $x_{\alpha_k} - d_k \xrightarrow{\|\cdot\|} 0$; see [11, Theorem 3.2].*

5. *Let (x_α) be a net in an order continuous Banach lattice X such that $x_\alpha \xrightarrow{\mathrm{un}} 0$. Then there exists an increasing sequence of indices (α_k) such that $x_{\alpha_k} \xrightarrow{\mathrm{un}} 0$; see [11, Corollary 3.5].*

6. *Let (x_n) be a sequence in a Banach lattice X. If $x_n \xrightarrow{\mathrm{un}} 0$ then there is a subsequence (x_{n_k}) such that $x_{n_k} \xrightarrow{\mathrm{uo}} 0$ as $k \to \infty$; see [11, Proposition 4.1].*

7. *A sequence in an order continuous Banach lattice X is un-null iff every subsequence has a further subsequence which is uo-null; see [11, Theorem 4.4].*

8. *Suppose that X is atomic and order continuous, and (x_n) is an order bounded sequence in X. If $x_n \xrightarrow{\|\cdot\|} 0$ then $x_n \xrightarrow{\mathrm{o}} 0$; see [11, Lemma 5.1].*

CHAPTER 4

P-NOTIONS IN LATTICE-NORMED VECTOR LATTICES

Within this chapter, we review lattice-normed spaces and some related properties. Also, we provide few more notions and study their general properties. Many of these notions will be used in the subsequent chapters.

4.1 Lattice-normed vector lattices

Right through this section we recall primary concepts related to lattice-normed vector lattice.

Definition 5. *[28, 2.1.1, p. 45] Let X be a vector space and E a vector lattice. A mapping $p : X \to E_+$ is called* ***lattice norm*** *if it satisfies the following conditions:*

1. $p(x) = 0 \Leftrightarrow x = 0$;
2. $p(\lambda x) = |\lambda| p(x)$ *for all* $\lambda \in \mathbb{R}$ *and* $x \in X$;
3. $p(x + y) \leq p(x) + p(y)$ *for all* $x, y \in X$.

The triple (X, p, E) is called ***lattice-normed space*** *and it is abbreviated as* ***LNS****.*

Let (X, p, E) be an LNS and Y be a vector subspace of X. Then Y is understood to be the LNS (Y, p, E).

Given an LNS (X, p, E). The lattice norm p is called ***decomposable*** if, for all $x \in X$ and $e_1, e_2 \in E_+$, from $p(x) = e_1 + e_2$ it follows that there exist $x_1, x_2 \in X$ such that $x = x_1 + x_2$ and $p(x_k) = e_k$ for $k = 1, 2$. A lattice-normed space with a decomposable lattice norm is referred as ***decomposable lattice-normed space***.

It should be noticed that theory of LNSs is well-developed under the decomposability condition; see, e.g., [28, 29, 8, 12]. Throughout this thesis we do not assume LNSs to be decomposable.

Definition 6. *Let (X, p, E) be an LNS. If X is a vector lattice and the lattice norm p is monotone (i.e., $|x| \leq |y| \Rightarrow p(x) \leq p(y)$), then the triple (X, p, E) is called **lattice-normed vector lattice**, abbreviated as **LNVL**.*

Let (X, p, E) be an LNVL and Y be a sublattice of X. Then Y is understood to be the LNVL (Y, p, E).

While dealing with LNVLs we keep in mind all the time the following two examples.

Example 5. *Any vector lattice X is the lattice-normed vector lattice $(X, |\cdot|, X)$, where $|x|$ denotes the absolute value of x.*

Example 6. *Any normed lattice $(X, \|\cdot\|)$ is the lattice-normed vector lattice $(X, \|\cdot\|, \mathbb{R})$.*

Definition 7. *A net (x_α) in an LNS (X, p, E) is **p-convergent to** $x \in X$ if $p(x_\alpha - x) \xrightarrow{o} 0$ in E. In this case, we write $x_\alpha \xrightarrow{p} x$.*

The p-convergence is also known as *bo*-convergence; see, e.g., [28, 2.1.5, p. 48]. The lattice operations in an LNVL X are p-continuous in the following sense.

Lemma 6. *Let $(x_\alpha)_{\alpha \in A}$ and $(y_\beta)_{\beta \in B}$ be two nets in an LNVL (X, p, E). If $x_\alpha \xrightarrow{p} x$ and $y_\beta \xrightarrow{p} y$, then $(x_\alpha \vee y_\beta)_{(\alpha,\beta) \in A \times B} \xrightarrow{p} x \vee y$. In particular, $x_\alpha \xrightarrow{p} x$ implies that $x_\alpha^- \xrightarrow{p} x^-$.*

Proof. There exist two nets $(z_\gamma)_{\gamma \in \Gamma}$ and $(w_\lambda)_{\lambda \in \Lambda}$ in E satisfying $z_\gamma \downarrow 0$ and $w_\lambda \downarrow 0$, and for all $(\gamma, \lambda) \in \Gamma \times \Lambda$ there are $\alpha_\gamma \in A$ and $\beta_\lambda \in B$ such that $p(x_\alpha - x) \leq z_\gamma$ and $p(y_\beta - y) \leq w_\lambda$ for all $(\alpha, \beta) \geq (\alpha_\gamma, \beta_\lambda)$. It follows from the inequality $|a \vee b - a \vee c| \leq |b - c|$ that

$$\begin{aligned} p(x_\alpha \vee y_\beta - x \vee y) &= p(|x_\alpha \vee y_\beta - x_\alpha \vee y + x_\alpha \vee y - x \vee y|) \\ &\leq p(|x_\alpha \vee y_\beta - x_\alpha \vee y|) + p(|x_\alpha \vee y - x \vee y|) \\ &\leq p(|y_\beta - y|) + p(|x_\alpha - x|) \leq w_\lambda + z_\gamma \end{aligned}$$

for all $\alpha \geq \alpha_\gamma$ and $\beta \geq \beta_\lambda$. Since $(w_\lambda + z_\gamma) \downarrow 0$, then $p(x_\alpha \vee y_\beta - x \vee y) \xrightarrow{o} 0$. $\square$

Let (X, p, E) be an LNS and A be a subset of X. Then A is called ***p-closed*** in X if, for any net (x_α) in A that is p-convergent to $x \in X$, we have $x \in A$. The following well-known property is a direct consequence of Lemma 6.

Assertion 1. *The positive cone X_+ in any LNVL X is p-closed.*

Assertion 1 implies the following well-known fact.

Proposition 13. *Any monotone p-convergent net in an LNVL o-converges to its p-limit.*

Proof. It is enough to show that if $(X, p, E) \ni x_\alpha \uparrow$ and $x_\alpha \xrightarrow{p} x$, then $x_\alpha \uparrow x$. Fix arbitrary α. Then $x_\beta - x_\alpha \in X_+$ for $\beta \geq \alpha$. By Assertion 1, $x_\beta - x_\alpha \xrightarrow{p} x - x_\alpha \in X_+$. Therefore $x \geq x_\alpha$. Since α is arbitrary, then x is an upper bound of x_α. If $y \geq x_\alpha$ for all α, then, again by Assertion 1, $y - x_\alpha \xrightarrow{p} y - x \in X_+$, or $y \geq x$. Thus $x_\alpha \uparrow x$. $\square$

Definition 8. *[28, 2.1.2, p. 46] Given an LNS (X, p, E).*

1. *Two vectors x and y in X are called **p-disjoint**, abbreviated as $x \perp_{\mathbf{p}} y$, if $p(x) \perp p(y)$.*
2. *A subset B of X is called **p-band** if*

$$B = M^{\perp_{\mathbf{p}}} = \{x \in X : (\forall m \in M)\, x \perp_{\mathbf{p}} m\}$$

   for some nonempty $M \subseteq X$.

Lemma 7. *Let (X, p, E) be an LNVL.*

1. *If $x, y \in X$ and $x \perp_{\mathbf{p}} y$, then $x \perp y$; i.e., p-disjointness implies disjointness.*
2. *If $B \subseteq X$ is p-band, then B is an ideal of X.*

Proof.

1. Assume $x \perp_{\mathbf{p}} y$ and $0 \leq z \leq |x| \wedge |y|$. Then $p(z) \leq p(|x| \wedge |y|) \leq p(x) \wedge p(y) = 0$ and hence $z = 0$. Thus $x \perp y$.

2. Let $y \in B$ and $x \in X$ such that $|x| \leq |y|$. Since B is p-band then there is a subset M of X such that for any $b \in B$, we have $b \perp_{\mathbf{p}} m$ for all $m \in M$. So, $p(x) \wedge p(m) \leq p(y) \wedge p(m) = 0$ for every $m \in M$. Thus, $x \in M^{\perp_{\mathbf{p}}} = B$, and so B is ideal.

□

The following example shows that there may be many bands which are not p-bands.

Example 7. *Consider the LNVL $(\mathbb{R}^2, \|\cdot\|, \mathbb{R})$. Then $\{0\}$, x-axis, y-axis and $\mathbb{R}^2$ are all bands in $\mathbb{R}^2$, while only $\{0\}$ and $\mathbb{R}^2$ are p-bands.*

Next we provide an example of a p-band that is not a band.

Example 8. *Consider the LNVL (c, p, c) with*

$$p(x) := |x| + (\lim_{n\to\infty} |x_n|) \cdot \mathbb{1} \quad (x = (x_n) \in c),$$

where $\mathbb{1}$ denotes the sequence identically equals to 1. Take $M = \{e_1\}$. We claim the p-band $M^{\perp_{\mathbf{p}}} = \{x \in c_0 : x_1 = 0\}$ is not a band. Indeed, the sequence (y_n) given by $y_n = \sum\limits_{k=2}^{n+1} e_k$ is in $M^{\perp_{\mathbf{p}}}$ and it is order convergent to $(0, 1, 1, 1, \dots) \notin M^{\perp_{\mathbf{p}}}$.

Remark 5.

1. *Every band is p-closed. Indeed, given a band B in an LNVL (X, p, E). If $B \ni x_\alpha \xrightarrow{p} x$, then, by Lemma 6, $|x_\alpha| \wedge |y| \xrightarrow{p} |x| \wedge |y|$ for any $y \in B^d$. Since $|x_\alpha| \wedge |y| = 0$ for all α, then $|x| \wedge |y| = 0$, and so $x \in B^{dd} = B$.*

2. *Every p-band is p-closed. Indeed, let $B = M^{\perp_{\mathbf{p}}}$ for some nonempty $M \subseteq X$, and $B \ni x_\alpha \xrightarrow{p} x_0 \in X$. Take any $m \in M$. It follows from*

$$p(x_0) \wedge p(m) \leq (p(x_0 - x_\alpha) + p(x_\alpha)) \wedge p(m) \leq$$

$$p(x_0 - x_\alpha) \wedge p(m) + p(x_\alpha) \wedge p(m) = p(x_0 - x_\alpha) \wedge p(m) \xrightarrow{o} 0,$$

that $p(x_0) \wedge p(m) = 0$. Since $m \in M$ is arbitrary, then $x_0 \in B$.

4.2 Several basic p-notions in LNVLs

Notions and results of this section are direct analogies of well-known facts of the theory of normed lattices.

Definition 9. *Let $X = (X, p, E)$ be an LNS.*

1. *A net $(x_\alpha)_{\alpha \in A}$ in X is said to be* ***p-Cauchy*** *if the net $(x_\alpha - x_{\alpha'})_{(\alpha,\alpha') \in A \times A}$ p-converges to 0.*
2. *X is called* ***p-complete*** *if every p-Cauchy net in X is p-convergent.*
3. *X is called* ***sequentially p-complete*** *if every p-Cauchy sequence in X is p-convergent.*
4. *A subset Y of X is said to be* ***p-bounded*** *if there exists $e \in E$ such that $p(y) \leq e$ for all $y \in Y$.*

A p-Cauchy net, p-completeness, and p-boundedness in LNSs are also known as *a bo-fundamental net*, *bo-completeness*, and *norm-boundedness* respectively (see, e.g., [28, 2.1.5, p.48]). We continue with more notions.

Definition 10. *Let $X = (X, p, E)$ be an LNVL.*

1. *X is called* ***op-continuous*** *if $x_\alpha \xrightarrow{o} 0$ implies that $p(x_\alpha) \xrightarrow{o} 0$.*
2. *X is called a* ***p-KB-space*** *if every p-bounded increasing net in X_+ is p-convergent.*
3. *The lattice norm p is said to be* ***additive on $\mathbf{X}_+$*** *if $p(x+y) = p(x) + p(y)$ for all $x, y \in X_+$.*

Remark 6.

1. *Clearly, any LNVL $(X, |\cdot|, X)$ is op-continuous.*
2. *In Definition 10.2 we do not require p-completeness of X.*
3. *It is easy to see that a p-KB-space $(X, \|\cdot\|, \mathbb{R})$ is always p-complete (see, e.g. [39, Exercise 95.4]). Therefore the notion of p-KB-space coincides with the notion of KB-space.*

4. *Clearly, an LNVL $X = (X, |\cdot|, X)$ is a p-KB-space iff X is order complete.*

5. *Notice that, for a p-KB-space $X = (X, p, E)$ the vector lattice $p(X)^{dd}$ need not to be order complete. To see this, take a KB-space $(X, \|\cdot\|)$ and $E = C[0,1]$. Then the LNVL (X, p, E) with $p(x) := \|x\| \cdot \mathbb{1}_{[0,1]}$ is clearly a p-KB-space, yet $p(X)^{dd} = E$ is not order complete.*

6. *Recently in [5, Definition 2] the authors introduced notions in lattice-normed ordered vector spaces (LNOVSs) which are similar to the notions given in Definitions 9 and 10.*

We call a vector lattice ***countably atomic*** if it is atomic and it has a countable complete disjoint system of atoms. When the norming lattice is countably atomic then sequentially p-complete LNVLs can be characterized as follows.

Theorem 5. *Let (X, p, E) be an LNVL such that E is countably atomic. Fix a maximal disjoint system of atoms $A = \{a_1, a_2, a_3, \ldots\} \subseteq E$. Then X is sequentially p-complete iff for every sequence (x_n) in X such that the sequence $r_n = \sum\limits_{k=1}^{n} p(x_k)$ is order bounded and for any $a_i \in A$ if the sequence $s_n := \sum\limits_{k=1}^{n} f_{a_i}(p(x_k))$ is convergent then the sequence $t_n = \sum\limits_{k=1}^{n} x_k$ is p-convergent; here f_{a_i} denotes the biorthogonal functional of the atom a_i.*

Proof. $(\Longrightarrow)$ Assume (x_n) is a sequence in X such that the sequence $r_n = \sum_{k=1}^{n} p(x_k)$ is order bounded. Let $a_i \in A$ and suppose the sequence $s_n := \sum_{k=1}^{n} f_{a_i}(p(x_k))$ is convergent. We claim that the sequence $t_n = \sum_{k=1}^{n} x_k$ is p-Cauchy. Indeed, for each $n \geq m$,

$$\begin{aligned}
f_{a_i}(p(t_n - t_m)) &= f_{a_i}(p(\sum_{k=1}^{n} x_k - \sum_{k=1}^{m} x_k)) \\
&= f_{a_i}(p(\sum_{k=m+1}^{n} x_k)) \\
&\leq \sum_{k=m+1}^{n} f_{a_i}(p(x_k)) \\
&= \sum_{k=1}^{n} f_{a_i}(p(x_k)) - \sum_{k=1}^{m} f_{a_i}(p(x_k))) \\
&= s_n - s_m
\end{aligned}$$

Since (s_n) is convergent then it is Cauchy. Then it follows from Corollary 1 in Section 3.1, that the sequence $\big(p(t_n - t_m)\big)_{(n,m)\in\mathbb{N}\times\mathbb{N}}$ is order null in E, and so (t_n) is p-Cauchy. Since X is sequentially p-complete, then we get (t_n) is p-convergent.

($\Longleftarrow$) Suppose (x_n) is a p-Cauchy sequence in X. Then for any $k \in \mathbb{N}$ there is $n_k \in \mathbb{N}$ such that $f_{a_1}(p(x_n - x_m)) < 2^{-k}$ for all $n, m \geq n_k$. We can select n_k's so that $n_1 < n_2 < n_3 < \dots$. Thus (x_{n_k}) is a subsequence of x_n. Let $y_1 := x_{n_1}$ and for $k \geq 2$ let $y_k := x_{n_k} - x_{n_{k-1}}$. Clearly, $\sum\limits_{l=1}^{k} y_l = x_{n_k}$ and $f_{a_1}(p(y_k)) = f_{a_1}(p(x_{n_k} - x_{n_{k-1}})) < 2^{-k+1}$. Hence $\sum\limits_{k=1}^{\infty} f_{a_1}(p(y_k)) \leq f_{a_1}(p(y_1)) + \sum\limits_{k=1}^{\infty} 2^{-k+1} = f_{a_1}(p(y_1)) + 1 < \infty$. Similarly, we can find a further subsequence $(x_{n_{k_j}})$ of (x_{n_k}) such that $\sum\limits_{j=1}^{\infty} f_{a_2}(p(y_{k_j})) < \infty$ where (y_{k_j}) is the subsequence of (y_k) corresponding to $(x_{n_{k_j}})$.

Therefore by a standard diagonal argument we can find a subsequence (x_{n_k}) of (x_n) such that for any $i \in \mathbb{N}$, $\sum\limits_{k=1}^{\infty} f_{a_i}(p(y_k)) < \infty$ where $y_1 := x_{n_1}$ and for $k \geq 2$ let $y_k := x_{n_k} - x_{n_{k-1}}$. By the hypothesis it follows that $x_{n_k} = \sum\limits_{i=1}^{k} y_i$ is p-convergent. Since x_n is p-Cauchy then it can be readily shown that (x_n) p-converges to the same p-limit of (x_{n_k}). $\square$

Lemma 8. *For an LNVL (X, p, E), the following statements are equivalent.*

1. *X is op-continuous;*
2. *$x_\alpha \downarrow 0$ in X implies $p(x_\alpha) \downarrow 0$.*

Proof. The implication 1$\Longrightarrow$2 is trivial.

2$\Longrightarrow$1. Let $x_\alpha \xrightarrow{o} 0$, then there exists a net $z_\beta \downarrow 0$ in X such that, for any β there exists α_β so that $|x_\alpha| \leq z_\beta$ for all $\alpha \geq \alpha_\beta$. Hence $p(x_\alpha) \leq p(z_\beta)$ for all $\alpha \geq \alpha_\beta$. By condition 2. we have $p(z_\beta) \downarrow 0$. Therefore $p(x_\alpha) \xrightarrow{o} 0$ or $x_\alpha \xrightarrow{p} 0$. $\square$

It follows from Lemma 8 that the *op*-continuity in LNVLs is equivalent to the order continuity of the lattice norm in the sense of [28, 2.1.4, p.48]. In the case of a p-complete LNVL, we have a further equivalent condition for *op*-continuity.

Theorem 6. *For a p-complete LNVL (X, p, E), the following statements are equivalent.*

1. *X is op-continuous;*

2. *if $0 \leq x_\alpha \uparrow\leq x$ holds in X, then (x_α) is a p-Cauchy net;*

3. *$x_\alpha \downarrow 0$ in X implies $p(x_\alpha) \downarrow 0$.*

Proof. 1$\Longrightarrow$2. Let $0 \leq x_\alpha \uparrow\leq x$ in X. By [3, Lemma 12.8], there exists a net (y_β) in X such that $(y_\beta - x_\alpha)_{\alpha,\beta} \downarrow 0$. So $p(y_\beta - x_\alpha) \xrightarrow{o} 0$, and hence the net (x_α) is p-Cauchy.

2$\Longrightarrow$3. Assume that $x_\alpha \downarrow 0$ in X. Fix arbitrary α_0, then, for $\alpha \geq \alpha_0$, $x_\alpha \leq x_{\alpha_0}$, and so $0 \leq (x_{\alpha_0} - x_\alpha)_{\alpha\geq\alpha_0} \uparrow\leq x_{\alpha_0}$. By condition 2 the net $(x_{\alpha_0} - x_\alpha)_{\alpha\geq\alpha_0}$ is p-Cauchy, i.e. $p(x_{\alpha'} - x_\alpha) \xrightarrow{o} 0$ as $\alpha_0 \leq \alpha, \alpha^{'} \to \infty$. Since X is p-complete, then there is $x \in X$ satisfying $p(x_\alpha - x) \xrightarrow{o} 0$ as $\alpha_0 \leq \alpha \to \infty$. By Proposition 13, $x_\alpha \downarrow x$ and hence $x = 0$. As a result, $x_\alpha \xrightarrow{p} 0$ and the monotonicity of p implies $p(x_\alpha) \downarrow 0$.

3$\Longrightarrow$1. It is just the implication 2$\Longrightarrow$1 of Lemma 8. $\square$

Corollary 6. *Let (X, p, E) be an op-continuous and p-complete LNVL, then X is order complete.*

Proof. Assume $0 \leq x_\alpha \uparrow\leq u$, then by Theorem 6.2, (x_α) is a p-Cauchy net and since X is p-complete, then there is x such that $x_\alpha \xrightarrow{p} x$. It follows from Proposition 13 that $x_\alpha \uparrow x$, and so X is order complete. $\square$

Corollary 7. *Any p-KB-space is op-continuous.*

Proof. Let $x_\alpha \downarrow 0$. Take any α_0 and let $y_\alpha := x_{\alpha_0} - x_\alpha$ for $\alpha \geq \alpha_0$. Clearly, $0 \leq y_\alpha \uparrow\leq x_{\alpha_0}$. Hence $p(y_\alpha) \uparrow\leq p(x_{\alpha_0})$ for $\alpha \geq \alpha_0$. Since X is a p-KB-space, there exists $y \in X$ such that $p(y_\alpha - y) \xrightarrow{o} 0$. Since $y_\alpha \uparrow$ and $y_\alpha \xrightarrow{p} y$, Proposition 13 ensures that

$$y = \sup_{\alpha\geq\alpha_0} y_\alpha = \sup_{\alpha\geq\alpha_0} (x_{\alpha_0} - x_\alpha) = x_{\alpha_0},$$

and hence $y_\alpha = x_{\alpha_0} - x_\alpha \xrightarrow{p} x_{\alpha_0}$ or $x_\alpha \xrightarrow{p} 0$. Again by Proposition 13 we get $p(x_\alpha) \downarrow 0$. So by Lemma 8, X is op-continuous. $\square$

Proposition 14. *Any p-KB-space is order complete.*

Proof. Let X be a p-KB-space and $0 \leq x_\alpha \uparrow \leq z \in X$. Then $p(x_\alpha) \leq p(z)$. Hence the net (x_α) is p-bounded and therefore $x_\alpha \xrightarrow{p} x$ for some $x \in X$. By Proposition 13, $x_\alpha \uparrow x$. $\square$

Next we get the sequential p-completeness of a p-KB-space under the assumption that the norming lattice is countably atomic.

Theorem 7. *Let (X, p, E) be a p-KB-space. If E is countably atomic, then X is sequentially p-complete.*

Proof. Let (x_n) be a sequence in X such that the sequence $r_n = \sum_{k=1}^{n} p(x_k)$ is order bounded. Fix a countable maximal disjoint system of atoms $A = \{a_1, a_2, a_3, ...\} \subseteq E$. Assume for any $i \in \mathbb{N}$ the sequence $s_n := \sum_{k=1}^{n} f_{a_i}(p(x_k))$ is convergent. Put $y_n := x_1^+ + ... + x_n^+$, then $0 \leq y_n \uparrow$ and by the assumption it is p-bounded. Since X is p-KB-space, then there is $y \in X$ such that $y_n \xrightarrow{p} y$. Similarly, if $z_n := x_1^- + ... + x_n^-$, then there is $z \in X$ such that $z_n \xrightarrow{p} z$. Therefore the sequence $t_n = \sum_{k=1}^{n} x_k$ is p-convergent to $y - z$. Hence X is sequentially p-complete by Theorem 5. $\square$

Proposition 15. *Let (X, p, E) be a p-KB-space, and $Y \subseteq X$ be an order closed sublattice. Then (Y, p, E) is also a p-KB-space.*

Proof. Let $Y_+ \ni y_\alpha \uparrow$ and $p(y_\alpha) \leq e \in E_+$ for all α. Since X is a p-KB-space, there exists $x \in X_+$ such that $y_\alpha \xrightarrow{p} x$. By Proposition 13, we have $y_\alpha \uparrow x$, and so $x \in Y$, because Y is order closed. Thus (Y, p, E) is a p-KB-space. $\square$

It is clear from the proof of Proposition 15, that every p-closed sublattice Y of a p-KB-space X is also a p-KB-space.

Proposition 16. *Let $X = (X, p, E)$ be a p-complete LNVL, E be atomic, and p be additive on X_+. Then X is a p-KB-space.*

Proof. Let a net (x_α) in X_+ be increasing and p-bounded by $e \in E_+$. If the net (x_α) is not p-Cauchy, then by Corollary 1 there is an atom $a \in E$ such that $f_a(p(x_\alpha - x_{\alpha'})) \not\to$

0, where f_a is the biorthogonal functional of a. Then there exists $\varepsilon > 0$ and a strictly increasing sequence (α_n) of indices such that

$$f_a(p(x_{\alpha_n} - x_{\alpha_{n-1}})) \geq \varepsilon > 0 \quad (\forall n \in \mathbb{N}).$$

Thus

$$\begin{aligned} n\epsilon &\leq \sum_{k=2}^{n+1} f_a(p(x_{\alpha_k} - x_{\alpha_{k-1}})) \\ &= f_a\Big(\sum_{k=2}^{n+1} p(x_{\alpha_k} - x_{\alpha_{k-1}}\Big) = f_a\Big(p\Big(\sum_{k=2}^{n+1} x_{\alpha_k} - x_{\alpha_{k-1}}\Big)\Big) \\ &= f_a(p(x_{\alpha_{n+1}} - x_{\alpha_1})) \leq 2f_a(e). \end{aligned}$$

Thus $n\epsilon \leq 2f_a(e)$ for all $n \in \mathbb{N}$, and hence $\epsilon \leq 0$; a contradiction. So, the net (x_α) is p-Cauchy and since X is p-complete, then it is p-convergent. Therefore X is p-KB-space. □

The next example shows that p-completeness in Proposition 16 can not be removed.

Example 9. *For the LNVL $(c_0, |\cdot|, \ell_\infty)$ the norming lattice ℓ_∞ is atomic and its lattice norm is additive on $(c_0)_+$. We claim that $(c_0, |\cdot|, \ell_\infty)$ is not p-complete. Indeed, consider the sequence $x_n = \sum_{i=1}^n e_i$, where e_n's are the standard unit vectors of c_0. For each $n \in \mathbb{N}$ put $y_n = \sum_{i=n}^\infty e_i$, then $y_n \downarrow 0$ in ℓ_∞. If $n_0 \in \mathbb{N}$, then for all $n > m \geq n_0$, $|x_n - x_m| = \sum_{i=m+1}^n e_i \leq y_{n_0}$. Thus (x_n) is p-Cauchy. Clearly, (x_n) is not p-convergent. Moreover, note that $0 \leq x_n \uparrow$ and (x_n) is p-bounded by $\mathbb{1} = (1, 1, \ldots) \in \ell_\infty$. Since (x_n) is not p-convergent then $(c_0, |\cdot|, \ell_\infty)$ is not a p-KB-space.*

Example 10. *Let $X = (X, \|\cdot\|)$ be a normed lattice. Consider the closed unit ball B_{X^*} of the dual Banach lattice X^*. Let E be the vector lattice of all bounded real-valued functions on B_{X^*}. Define an E-valued norm p on X by*

$$p(x)[f] := |f|(|x|) \quad (f \in B_{X^*})$$

for any $x \in X$.

(i) p is a lattice norm.

Proof. Assume $x = 0$. Then $p(0)[f] = 0$ for any $f \in B_{X^*}$. So, $p(0) = 0$. If $p(x) = 0$, then $p(x)[f] = |f|(|x|) = 0$ for all $f \in B_{X^*}$. Hence Hahn-Banach theorem implies that $x = 0$.

Let $\lambda \in \mathbb{R}$ and $x \in X$. Then for $f \in B_{X^*}$, $p(\lambda x)[f] = |f|(|\lambda x|) = |\lambda||f|(|x|) = |\lambda|p(x)[f]$. Thus $p(\lambda x) = |\lambda|p(x)$.

If $x, y \in X$, then

$p(x + y)[f] = |f|(|x + y|) \leq |f|(|x|) + |f|(|y|) = p(x)[f] + p(y)[f] = \big(p(x)+p(y)\big)[f]$ for each $f \in B_{X^*}$. It follows that $p(x+y) \leq p(x)+p(y)$. $\square$

(ii) *The triple (X, p, E) is an LNVL.*

Proof. From (i) it remains to show that the lattice norm p is monotone. Suppose $|x| \leq |y|$. Let $f \in B_{X^*}$, then $p(x)[f] = |f|(|x|) \leq |f|(|y|) = p(y)[f]$. Thus $p(x) \leq p(y)$. $\square$

(iii) *If X is an order continuous Banach lattice, then (X, p, E) is op-continuous.*

Proof. Assume $x_\alpha \downarrow 0$, we show $p(x_\alpha) \downarrow 0$. We claim that $p(x_\alpha) \downarrow 0$ iff $p(x_\alpha)[f] \downarrow 0$ for all $f \in B_{X^*}$.

For the necessity, let $p(x_\alpha) \downarrow 0$ and $f \in B_{X^*}$. Trivially, $|f|(x_\alpha)$ is decreasing. If there exists $z_f \in \mathbb{R}$ such that $0 \leq z_f \leq |f|(x_\alpha)$ for all α, then

$$0 \leq z_f \leq |f|(x_\alpha) \leq \|f\|\|x_\alpha\| \downarrow 0.$$

Hence $z_f = 0$ and $p(x_\alpha)[f] = |f|(x_\alpha) \downarrow 0$.

For the sufficiency, let $p(x_\alpha)[f] \downarrow 0$ for every $f \in B_{X^*}$. Since p is monotone and $x_\alpha \downarrow$, then $p(x_\alpha) \downarrow$. If $0 \leq \varphi \leq p(x_\alpha)$ for all α, then

$$0 \leq \varphi(f) \leq p(x_\alpha)[f] = |f|(x_\alpha) \quad (\forall f \in B_{X^*}).$$

So by the assumption, we get $\varphi(f) = 0$ for all $f \in B_{X^*}$, and hence $\varphi = 0$. Therefore $p(x_\alpha) \downarrow 0$. $\square$

(iv) *If X is a KB-space, then (X, p, E) is a p-KB-space.*

Proof. Suppose that $0 \leq x_\alpha \uparrow$ and $p(x_\alpha) \leq \varphi \in E$. As

$$\begin{aligned} \|x_\alpha\| &= \sup_{f \in B_{X^*}} |f(x_\alpha)| \\ &\leq \sup_{f \in B_{X^*}} |f|(x_\alpha) \\ &= \sup_{f \in B_{X^*}} p(x_\alpha)[f] \\ &\leq \varphi[f] \\ &\leq \|\varphi\|_\infty < \infty \quad (\forall \alpha), \end{aligned}$$

and since X is a KB-space, we get $\|x_\alpha - x\| \to 0$ for some $x \in X_+$. So, for any $f \in B_{X^*}$, we have $|f|(|x_\alpha - x|) \to 0$ or $p(x_\alpha - x)[f] \to 0$. Thus $p(x_\alpha - x) \xrightarrow{o} 0$ in E and hence $x_\alpha \xrightarrow{p} x$. $\square$

Let Ω be a non-empty subset, then $\mathbb{R}^\Omega$ denotes the vector space of all real-valued functions on Ω which is also a vector lattice under the pointwise ordering: $f \leq g$ in $\mathbb{R}^\Omega$ iff $f(t) \leq g(t)$ for all $t \in \Omega$.

Example 11. *Let X be a vector lattice, $X^\#$ be the algebraic dual of X, and Y be a sublattice of $X^\#$ such that $\langle X, Y \rangle$ is a dual system. Define $p : X \to \mathbb{R}^Y$ by $p(x)[y] := |y|(|x|)$. Then $(X, p, \mathbb{R}^Y)$ is an LNVL.*

Recall that a vector lattice X is called *perfect* if the natural embedding $x \to \hat{x}$ given by

$$\hat{x}(f) := f(x), \ \ f \in X_n^\sim$$

from X into $(X_n^\sim)_n^\sim$ is one-to-one and onto, where $X_n^\sim$ denotes the order continuous dual of X [3, p. 63]. If X is a perfect vector lattice, then $X_n^\sim$ separates the points of X [3, Theorem 1.71(1)].

Proposition 17. *Let X be a perfect vector lattice, $Y = X_n^\sim$ and $p : X \to \mathbb{R}^Y$ be defined as $p(x)[f] := |f|(|x|)$, where $f \in Y$. Then the LNVL $(X, p, \mathbb{R}^Y)$ is a p-KB-space.*

Proof. Assume $0 \leq x_\alpha \uparrow$ in X and $p(x_\alpha) \leq \varphi \in \mathbb{R}^Y$. Then, for all $f \in Y$, we have $p(x_\alpha)[f] \leq \varphi(f)$ or $|f|(x_\alpha) \leq \varphi(f)$. So, for all $f \in Y$, $\sup_\alpha |f|(x_\alpha) < \infty$, and hence, by [3, Theorem 1.71(2)], there is $x \in X$ with $x_\alpha \uparrow x$. An argument similar to Example 10 (iii) above shows that X is *op*-continuous. Therefore $x_\alpha \xrightarrow{p} x$. $\square$

4.3 p-Fatou space, p-density and p-units

In the present section, we continue introducing basic notions in LNVLs.

Definition 11. *An LNVL (X, p, E) is called* ***p-Fatou space*** *if $0 \leq x_\alpha \uparrow x$ in X implies $p(x_\alpha) \uparrow p(x)$.*

Note that an LNVL (X, p, E) is a p-Fatou space iff p is order semicontinuous [28, 2.1.4, p.48]. Clearly any op-continuous LNVL (X, p, E) is a p-Fatou space. The LNVL (c, p, c) in Example 8 is not a p-Fatou space. Indeed, let $y_n := \sum_{k=1}^{n} e_k$. Then $0 \leq y_n \uparrow \mathbb{1}$ but $p(y_n) = y_n \uparrow \mathbb{1} \neq p(\mathbb{1}) = (2, 2, 2, \ldots)$.

Next we will show the p-Fatou property ensures that each p-band is a band.

Proposition 18. *Let B be a p-band in a p-Fatou space (X, p, E). Then B is a band in X.*

Proof. Let $B = M^{\perp_{\mathrm{p}}} = \{x \in X : (\forall m \in M)\ p(x) \perp p(m)\}$ for some nonempty $M \subseteq X$. Since by Lemma 7.2 B is an ideal in X then to show that B is a band it is enough to prove that if $B_+ \ni b_\alpha \uparrow x \in X$, then $x \in B$. As X is a p-Fatou space, then $p(b_\alpha) \uparrow p(x)$. By order continuity of lattice operations in E, we obtain that

$$0 = p(b_\alpha) \wedge p(m) \xrightarrow{o} p(x) \wedge p(m) \quad (\forall m \in M).$$

Therefore $p(x) \wedge p(m) = 0$ for all $m \in M$, and hence $x \in B$. $\square$

Now we give the following definition of a p-dense subset in an LNS, which is motivated by the notion of a dense subset of a normed space.

Definition 12. *Given an LNS (X, p, E) and $A \subseteq X$. A subset B of A is said to be* ***p-dense*** *in A if for any $a \in A$ and for any $0 \neq u \in p(X)$ there is $b \in B$ such that $p(a - b) \leq u$.*

Remark 7.

1. *If $(X, \|\cdot\|)$ is a normed lattice, $p = \|\cdot\|$ and $E = \mathbb{R}$, then clearly a subset Y of X is p-dense iff Y is norm dense.*

2. *Consider the LNVL (X,p,E) with $p=|\cdot|$, $E=X$, and let Y be a sublattice X. If Y is p-dense in X, then Y is order dense. Indeed, let $0\neq x\in X_+$, then there is $y\in Y$ such that $|y-\frac{1}{2}x|\leq\frac{1}{3}x$ which implies $0<\frac{1}{6}x\leq y\leq\frac{5}{6}x$, and so $0<y\leq x$.*

3. *c is order dense in ℓ_∞. Note that c is a norm closed subspace of ℓ_∞, and so it is not p-dense in the LNVL $(\ell_\infty,\|\cdot\|_\infty,\mathbb{R})$. Also, it is not p-dense in $(\ell_\infty,|\cdot|,\ell_\infty)$. Indeed, let $u=(1,0,1,0,\ldots)$ and $x=(1,-1,1,-1,\ldots)$. Then there is no y in c such that $|y-x|\leq u$.*

The following notion is motivated by the notion of a weak order unit in a vector lattice $X=(X,|\cdot|,X)$ and by the notion of a quasi-interior point in a normed lattice $X=(X,\|\cdot\|,\mathbb{R})$.

Definition 13. *Let (X,p,E) be an LNVL. A vector $e\in X$ is called a **p-unit** if for any $x\in X_+$ we have $p(x-x\wedge ne)\xrightarrow{o}0$.*

Remark 8. *Let $X=(X,p,E)$ be an LNVL.*

1. *If $X\neq\{0\}$, then for any p-unit e in X it holds that $e>0$. Indeed, let e be a p-unit in $X\neq\{0\}$. Trivially $e\neq 0$. Suppose $e^->0$. Then, for $x:=e^-$, we obtain that*

$$p(x-x\wedge ne)=p(e^--(e^-\wedge n(e^+-e^-)))=$$

$$p(e^--(e^-\wedge n(-e^-)))=p(e^--(-ne^-))=p((n+1)e^-)=$$

$$(n+1)p(e^-)\not\xrightarrow{o}0$$

as $n\to\infty$. This is impossible because e is a p-unit. Therefore, $e^-=0$ and $e>0$.

2. *Let $e\in X$ be a p-unit. Given $0<\lambda\in\mathbb{R}_+$ and $z\in X_+$. Observe that, for $x\in X_+$, $p(x-n\lambda e\wedge x)=\lambda p(\frac{x}{\lambda}-ne\wedge\frac{x}{\lambda})$ and $p(x-n(e+z)\wedge x)\leq p(x-x\wedge ne)$, from which it follows easily that λe and $e+z$ are p-units.*

3. *If $e\in X$ is a strong unit, then e is a p-unit. Indeed, let $x\in X_+$, then there is $k\in\mathbb{N}$ such that $x\leq ke$, so $x-x\wedge ne=0$ for any $n\geq k$.*

4. *If $e \in X$ is a p-unit, then e is a weak unit. Assume $x \wedge e = 0$, then $x \wedge ne = 0$ for any $n \in \mathbb{N}$. Since e is a p-unit, then $p(x) = 0$ and hence $x = 0$.*

5. *If X is op-continuous, then clearly every weak unit of X is a p-unit.*

6. *In $X = (X, |\cdot|, X)$, the lattice norm $p(x) = |x|$ is always order continuous. Therefore the notions of p-unit and of weak unit coincide in X.*

7. *If $X = (X, \|\cdot\|)$ is a normed lattice, $p = \|\cdot\|$, $E = \mathbb{R}$, and $e \in X$, then e is a p-unit iff e is a quasi-interior point of X.*

In the proof of the following proposition, we use the same technique as in the proof of [1, Lemma 4.15].

Proposition 19. *Let (X, p, E) be an LNVL, $e \in X_+$, and I_e be the ideal generated by e in X. If I_e is p-dense in X, then e is a p-unit.*

Proof. Let $0 \neq u \in p(X)$. Let $x \in X_+$, then there exists $y \in I_e$ such that $p(x - y) \leq u$. Since $|y^+ \wedge x - x| \leq |y^+ - x| = |y^+ - x^+| \leq |y - x|$, then by replacing y by $y^+ \wedge x$, we may assume without loss of generality that there is $y \in I_e$ such that $0 \leq y \leq x$ and $p(x - y) \leq u$. Thus, for any $m \in \mathbb{N}$, there is $y_m \in I_e$ such that $0 \leq y_m \leq x$ and

$$p(x - y_m) \leq \frac{1}{m} u.$$

Since $y_m \in I_e$, then there exists $k = k(m) \in \mathbb{N}$ such that $0 \leq y_m \leq ke$, and so $0 \leq y_m \leq ke \wedge x$.

For $n \geq k$, $x - x \wedge ne \leq x - x \wedge ke \leq x - y_m$, and so $p(x - x \wedge ne) \leq p(x - y_m) \leq \frac{1}{m} u$. Hence, $p(x - x \wedge ne) \xrightarrow{o} 0$. Thus, e is a p-unit. $\square$

CHAPTER 5

UNBOUNDED P-CONVERGENCE

Throughout this chapter, we introduce the unbounded p-convergence (up-convergence) in lattice normed vector lattices (LNVLs) and investigate several properties of it. A variant of unbounded p-convergence will be introduced as well. Finally, we give the notion of up-regular sublattices and we relate this notion with a vector lattice and its order completion.

5.1 Unbounded p-convergence

The unbounded p-convergence (up-convergence) in LNVLs generalizes the uo-convergence in vector lattices (see Definition 1), the un-convergence (see Definition 3) and the uaw-convergence (see Definition 4) in Banach lattices.

5.1.1 Main definition and its motivation

Let (X, p, E) be an LNVL. The following definition is motivated by its special case when it is reduced to the un-convergence for a normed lattice $(X, p, E) = (X, \|\cdot\|, \mathbb{R})$ $= (X, \|\cdot\|)$.

Definition 14. *A net $(x_\alpha) \subseteq X$ is said to be **unbounded p-convergent** to $x \in X$ (shortly, x_α **up-converges** to x or $x_\alpha \xrightarrow{up} x$), if*

$$p(|x_\alpha - x| \wedge u) \xrightarrow{o} 0 \quad (\forall u \in X_+).$$

It is immediate to see that up-convergence coincides with un-convergence in the case when p is the norm in a normed lattice, and with uo-convergence in the case when $X = E$ and $p(x) = |x|$. It is clear that $x_\alpha \xrightarrow{p} x$ implies $x_\alpha \xrightarrow{up} x$, and for order bounded nets up-convergence and p-convergence agree. It should be also clear that if an LNVL X is op-continuous, then uo-convergence in X implies up-convergence. The uaw-convergence is also a particular case of up-convergence as it follows from the next example.

Example 12. *As in Example 11 of Section 4.2, let X be a vector lattice, $X^\#$ be the algebraic dual of X, and Y be a sublattice of $X^\#$ such that $\langle X, Y\rangle$ is a dual system. Define $p : X \to \mathbb{R}^Y$ by $p(x)[y] := |y|(|x|)$. Then $x_\alpha \xrightarrow{up} 0$ in X iff for every $u \in X_+$, $|x_\alpha| \wedge u \xrightarrow{|\sigma|(X,Y)} 0$.*

Proof. $x_\alpha \xrightarrow{up} 0$ in X iff for all $u \in X_+$, $p(|x_\alpha| \wedge u) \xrightarrow{o} 0$ in $\mathbb{R}^Y$ iff for every $u \in X_+$, $p(|x_\alpha| \wedge u)[y] \to 0$ for all $y \in Y$ iff for every $u \in X_+$, $|y|(|x_\alpha| \wedge u) \to 0$ for all $y \in Y$ iff for every $u \in X_+$, $|x_\alpha| \wedge u \xrightarrow{|\sigma|(X,Y)} 0$. □

In particular, if X is a Banach lattice, $Y = X^*$, the topological dual of X, $E = \mathbb{R}^Y$ and $p : X \to E$ as defined above then $x_\alpha \xrightarrow{up} 0$ in X iff $x_\alpha \xrightarrow{uaw} 0$.

5.1.2 Basic results on up-convergence

We begin with the next list of properties of up-convergence which follows directly from Lemma 6 in Section 4.1.

Lemma 9. *Let $x_\alpha \xrightarrow{up} x$ and $y_\alpha \xrightarrow{up} y$ in an LNVL (X, p, E). Then*

1. *$ax_\alpha + by_\alpha \xrightarrow{up} ax + by$ for any $a, b \in \mathbb{R}$, in particular, if $x_\alpha = y_\alpha$, then $x = y$;*
2. *$x_{\alpha_\beta} \xrightarrow{up} x$ for any subnet (x_{α_β}) of (x_α);*
3. *$|x_\alpha| \xrightarrow{up} |x|$;*
4. *if $x_\alpha \geq y_\alpha$ for all α, then $x \geq y$.*

Lemma 10. *Let (x_α) be a monotone net in an LNVL (X, p, E) such that $x_\alpha \xrightarrow{up} x$, then $x_\alpha \xrightarrow{o} x$.*

Proof. The proof of Proposition 13 applies here as well. $\square$

The following result is a p-version of Remark 4.1 in Section 3.3.

Theorem 8. *Let (x_α) be a monotone net in an LNVL (X, p, E) which up-converges to x. Then $x_\alpha \xrightarrow{p} x$.*

Proof. Without loss of generality we may assume that $0 \le x_\alpha \uparrow$. From Lemma 10 it follows that $0 \le x_\alpha \uparrow x$ for some $x \in X_+$. So $0 \le x - x_\alpha \le x$ for all α. Since, for each $u \in X_+$, we know that

$$p((x - x_\alpha) \wedge u) \xrightarrow{o} 0.$$

In particular, for $u = x$, we obtain that

$$p(x - x_\alpha) = p((x_\alpha - x) \wedge x) \xrightarrow{o} 0.$$

$\square$

The following result is a generalization of Remark 2.1 in Section 3.1.

Lemma 11. *Assume $x_\alpha \xrightarrow{up} 0$ in an LNVL (X, p, E). Then $\inf_\beta |y_\beta| = 0$ for any subnet (y_β) of the net (x_α).*

Proof. Let (y_β) be a subnet of (x_α). Clearly, $y_\beta \xrightarrow{up} 0$. If $0 \le z \le |y_\beta|$ for all β, then $p(z) = p(z \wedge |\, y_\beta|) \xrightarrow{o} 0$, and so $z = 0$. Hence $\inf_\beta |y_\beta| = 0$. $\square$

The following two results are analogies of Remark 4.2 (Section 3.3) and of Remark 2.2 (Section 3.1) respectively, but first we recall a technical inequality that will be used in the next lemma and in Proposition 21. Let X be a vector lattice and $a, b, c \in X_+$, then $|a \wedge c - b \wedge c| \le |a - b| \wedge c$. Indeed, we know from Birkhoff's inequality that $|a \wedge c - b \wedge c| \le |a - b|$ (see, e.g., [3, Theorem 1.9.(2)]). Also, we have $a \wedge c \le c \Rightarrow a \wedge c - b \wedge c \le c - b \wedge c \le c$. So $a \wedge c - b \wedge c \le c$ and similarly we have $b \wedge c - a \wedge c \le c$. Hence we get $|a \wedge c - b \wedge c| \le c$. Combining the former inequality with Birkhoff's inequality we obtain the required inequality.

Lemma 12. *Let (X, p, E) be an LNVL. Assume that E is order complete and $x_\alpha \xrightarrow{up} x$, then $p(|x| - |x| \wedge |x_\alpha|) \xrightarrow{o} 0$ and $p(x) = \liminf_\alpha p(|x| \wedge |x_\alpha|)$. Moreover, if (x_α) is p-bounded, then $p(x) \leq \liminf_\alpha p(x_\alpha)$.*

Proof. Note that

$$|x| - |x| \wedge |x_\alpha| = \big||x_\alpha| \wedge |x| - |x| \wedge |x|\big| \leq \big||x_\alpha| - |x|\big| \wedge |x| \leq |x_\alpha - x| \wedge |x|.$$

Since $x_\alpha \xrightarrow{up} x$, we get $p(|x| - |x| \wedge |x_\alpha|) \xrightarrow{o} 0$. Thus

$$p(x) = p(|x|) \leq p(|x| - |x| \wedge |x_\alpha|) + p(|x| \wedge |x_\alpha|).$$

So $p(x) \leq \liminf_\alpha p(|x| \wedge |x_\alpha|)$. Hence $p(x) = \liminf_\alpha p(|x| \wedge |x_\alpha|)$. □

Lemma 13. *Let (X, p, E) be an op-continuous LNVL. Assume that E is order complete and $x_\alpha \xrightarrow{uo} x$, then $p(|x| - |x| \wedge |x_\alpha|) \xrightarrow{o} 0$ and $p(x) = \liminf_\alpha p(|x| \wedge |x_\alpha|)$. Moreover, if (x_α) is p-bounded, then $p(x) \leq \liminf_\alpha p(x_\alpha)$.*

We finish this subsection with the following technical lemma which generalizes Remark 2.3 in Section 3.1.

Lemma 14. *Given an LNVL (X, p, E). If $x_\alpha \xrightarrow{p} x$ and (x_α) is an o-Cauchy net, then $x_\alpha \xrightarrow{o} x$. Moreover, if $x_\alpha \xrightarrow{p} x$ and (x_α) is uo-Cauchy, then $x_\alpha \xrightarrow{uo} x$.*

Proof. Since (x_α) is order Cauchy, then $x_\alpha - x_\beta \xrightarrow{o} 0$, where the order limit is taken over α and β. So there exists $z_\gamma \downarrow 0$ such that, for every γ, there is α_γ satisfying

$$|x_\alpha - x_\beta| \leq z_\gamma, \quad \forall \alpha, \beta \geq \alpha_\gamma. \tag{5.1}$$

By taking p-limit over β in (5.1) and applying Lemma 6 in Section 4.1, we get $|x_\alpha - x| \leq z_\gamma$ for all $\alpha \geq \alpha_\gamma$. Thus $x_\alpha \xrightarrow{o} x$.

For the uo-convergence, the similar argument is used, so the proof is omitted. □

5.1.3 up-Convergence and p-units

We have seen in Chapter 3 that uo-convergence can be only evaluated at a weak unit (see Proposition 1, Section 3.1) whereas both un-convergence and uaw-convergence can only be checked at a quasi-interior point (see respectively, Proposition 7 and Proposition 9, Section 3.3). In what follows we show that up-convergence can be evaluated at a p-unit, but first we recall useful characterizations of order convergence.

For any order bounded net (x_α) in an order complete vector lattice X, $x_\alpha \xrightarrow{o} x$ iff $\limsup\limits_{\alpha}|x_\alpha - x| = \inf_\alpha \sup_{\beta\geq\alpha}|x_\beta - x| = 0$; see for example [21, Remark 2.2]. Moreover, for each net (x_α) in a vector lattice X, $x_\alpha \xrightarrow{o} 0$ in X iff $x_\alpha \xrightarrow{o} 0$ in X^δ (the order completion of X); see, Lemma 1 in Chapter 2.

Theorem 9. *Let (X, p, E) be an LNVL and $e \in X_+$ be a p-unit. Then $x_\alpha \xrightarrow{up} 0$ iff $p(|x_\alpha| \wedge e) \xrightarrow{o} 0$ in E.*

Proof. The "only if" part is trivial. For the "if" part, let $u \in X_+$, then

$$|x_\alpha| \wedge u \leq |x_\alpha| \wedge (u - u \wedge ne) + |x_\alpha| \wedge (u \wedge ne) \leq (u - u \wedge ne) + n(|x_\alpha| \wedge e),$$

and so

$$p(|x_\alpha| \wedge u) \leq p(u - u \wedge ne) + np(|x_\alpha| \wedge e)$$

holds in E^δ for any α and any $n \in \mathbb{N}$. Hence

$$\limsup_{\alpha} p(|x_\alpha| \wedge u) \leq p(u - u \wedge ne) + n \limsup_{\alpha} p(|x_\alpha| \wedge e)$$

holds in E^δ for all $n \in \mathbb{N}$. Since $p(|x_\alpha| \wedge e) \xrightarrow{o} 0$ in E, then $p(|x_\alpha| \wedge e) \xrightarrow{o} 0$ in E^δ, and so $\limsup\limits_{\alpha} p(|x_\alpha| \wedge e) = 0$ in E^δ. Thus

$$\limsup_{\alpha} p(|x_\alpha| \wedge u) \leq p(u - u \wedge ne)$$

holds in E^δ for all $n \in \mathbb{N}$. Since e is a p-unit, we have that $\limsup\limits_{\alpha} p(|x_\alpha| \wedge u) = 0$ in E^δ or $p(|x_\alpha| \wedge u) \xrightarrow{o} 0$ in E^δ. It follows from Lemma 1 that $p(|x_\alpha| \wedge u) \xrightarrow{o} 0$ in E and hence $x_\alpha \xrightarrow{up} 0$. $\square$

5.1.4 up-Convergence and sublattices

Given an LNVL (X, p, E), a sublattice Y of X, and a net $(y_\alpha) \subseteq Y$. Then $y_\alpha \xrightarrow{up} 0$ in Y has the meaning that

$$p(|y_\alpha| \wedge u) \xrightarrow{o} 0 \quad (\forall u \in Y_+).$$

The following lemma is a p-version of Remark 2.7, Section 3.1.

Lemma 15. *Let (X, p, E) be an LNVL, B be a projection band of X, and P_B be the corresponding band projection. If $x_\alpha \xrightarrow{up} x$ in X, then $P_B(x_\alpha) \xrightarrow{up} P_B(x)$ in both X and B.*

Proof. It is known that P_B is a lattice homomorphism and $0 \leq P_B \leq I$. Since $|P_B(x_\alpha) - P_B(x)| = P_B|x_\alpha - x| \leq |x_\alpha - x|$, then it follows easily that $P_B(x_\alpha) \xrightarrow{up} P_B(x)$ in both X and B. $\square$

A subset A of a vector lattice X is said to be *uo-closed* in X, if for any net $(x_\alpha) \subseteq A$ and $x \in X$ with $x_\alpha \xrightarrow{\mathrm{uo}} x$ in X, one has $x \in A$.

Proposition 20. *[21, Proposition 3.15] Let X be a vector lattice and Y a sublattice of X. Then Y is uo-closed in X if and only if it is o-closed in X.*

Let (X, p, E) be an LNVL and A be a subset of X. Then A is called ***up-closed*** in X if, for any net (x_α) in A that is up-convergent to $x \in X$, we have $x \in A$. Clearly, every band is up-closed. We present a p-version of Proposition 20 with a similar proof.

Proposition 21. *Let $X = (X, p, E)$ be an LNVL and Y be a sublattice of X. Suppose that either X is op-continuous or Y is a p-KB-space in its own right. Then Y is up-closed in X iff it is p-closed in X.*

Proof. Only the sufficiency requires a proof. Let Y be p-closed in X and (y_α) be a net in Y with $y_\alpha \xrightarrow{up} x \in X$. Without loss of generality, we may assume $(y_\alpha) \subseteq Y_+$ because the lattice operations in X are p-continuous. Note that, for every $z \in X_+$, $|y_\alpha \wedge z - x \wedge z| \leq |y_\alpha - x| \wedge z$. So $p(y_\alpha \wedge z - x \wedge z) \leq p(|y_\alpha - x| \wedge z) \xrightarrow{o} 0$. In particular, $Y \ni y_\alpha \wedge y \xrightarrow{p} x \wedge y$ in X for any $y \in Y_+$. Since Y is p-closed, $x \wedge y \in Y$ for any $y \in Y_+$. Since for any $0 \leq z \in Y^d$ and for any α, we have $y_\alpha \wedge z = 0$, then

$$|x \wedge z| = |y_\alpha \wedge z - x \wedge z| \leq |y_\alpha - x| \wedge z \xrightarrow{p} 0.$$

Therefore $x \wedge z = 0$, and hence $x \in Y^{dd}$. Since Y^{dd} is the band generated by Y in X, there is a net $(z_\beta)_{\beta \in B}$ in the ideal I_Y generated by Y such that $0 \leq z_\beta \uparrow x$ in X. Take for every β an element $w_\beta \in Y$ with $z_\beta \leq w_\beta$. Then $x \geq w_\beta \wedge x \geq z_\beta \wedge x = z_\beta \uparrow x$ in X, and so $w_\beta \wedge x \xrightarrow{o} x$ in X.

Case 1: If X is op-continuous, then $w_\beta \wedge x \xrightarrow{p} x$. Since $w_\beta \wedge x \in Y$ and Y is p-closed, we get $x \in Y$.

Case 2: Suppose Y is a p-KB-space in its own right. Let Δ be the collection of all finite subsets of the index set B. For each $\delta = \{\beta_1, \dots, \beta_n\} \in \Delta$ let $y_\delta := (w_{\beta_1} \vee \dots \vee w_{\beta_n}) \wedge x$. Clearly, $y_\delta \in Y$, $0 \leq y_\delta \uparrow$, and the net (y_δ) is p-bounded in Y. Since Y is a p-KB-space, then there is $y_0 \in Y$ such that $y_\delta \xrightarrow{\mathrm{p}} y_0$ in Y and trivially in X. Since (y_δ) is monotone then it follows from Proposition 13 of Section 4.1 that $y_\delta \uparrow y_0$ in X. Also, we have $y_\delta \xrightarrow{\mathrm{o}} x$ in X. Thus, $x = y_0 \in Y$. $\square$

5.1.5 p-Almost order bounded sets

Recall again that a subset A in a normed lattice $(X, \|\cdot\|)$ is said to be *almost order bounded* if for any $\varepsilon > 0$, there is $u_\varepsilon \in X_+$ such that $\big\|(|a|-u_\varepsilon)^+\big\| = \big\||a|-u_\varepsilon \wedge |a|\big\| \leq \varepsilon$ for any $a \in A$. Similarly we have:

Definition 15. *Given an LNVL (X, p, E). A subset A of X is called a **p-almost order bounded** if, for any $w \in E_+$, there is $x_w \in X_+$ such that $p\big((|a| - x_w)^+\big) = p\big(|a| - x_w \wedge |a|\big) \leq w$ for any $a \in A$.*

It is clear that p-almost order boundedness notion in LNVLs is a generalization of almost order boundedness in normed lattices. In the LNVL $(X, |\cdot|, X)$, a subset of X is p-almost order bounded, iff it is order bounded in X.

The following result is a p-version of Remark 4.3, Section 3.3 and it is also similar to Remark 2.4, Section 3.1.

Proposition 22. *If (X, p, E) is an LNVL, (x_α) is p-almost order bounded, and $x_\alpha \xrightarrow{up} x$, then $x_\alpha \xrightarrow{p} x$.*

Proof. Assume (x_α) is p-almost order bounded. Let $w \in E_+$. Then there exists

$x_w \in X_+$ satisfying

$$p(|x_\alpha| - |x_\alpha| \wedge x_w) = p((|x_\alpha| - x_w)^+) \leq w$$

for all α. Let $z_w := x_w + |x|$. Then

$$p(|x_\alpha - x| - |x_\alpha - x| \wedge z_w) = p((|x_\alpha - x| - z_w)^+) \leq p((|x_\alpha| - x_w)^+) \leq w.$$

Hence, the net $(|x_\alpha - x|)$ is also p-almost order bounded. But $x_\alpha \xrightarrow{up} x$, so $\limsup\limits_\alpha p(|x_\alpha - x| \wedge z_w) = 0$ in E^δ. Thus, for each α,

$$p(x_\alpha - x) = p(|x_\alpha - x|) \leq p(|x_\alpha - x| - |x_\alpha - x| \wedge z_w) + p(|x_\alpha - x| \wedge z_w) \leq w + p(|x_\alpha - x| \wedge z_w)$$

Hence

$$\limsup_\alpha p(x_\alpha - x) \leq w + \limsup_\alpha p(|x_\alpha - x| \wedge z_w) \leq w$$

holds in E^δ. But $w \in E_+$ is arbitrary, so $\limsup\limits_\alpha p(x_\alpha - x) = 0$ in E^δ. Therefore $p(x_\alpha - x) \xrightarrow{o} 0$ in E^δ, and so in E (see Lemma 1). $\square$

The following proposition is a p-version of Remark 2.5, Section 3.1.

Proposition 23. *Given an op-continuous and p-complete LNVL $X = (X, p, E)$. Then every p-almost order bounded uo-Cauchy net is uo- and p-convergent to the same limit.*

Proof. Let (x_α) be a p-almost order bounded uo-Cauchy net. Then the net $(x_\alpha - x_{\alpha'})$ is p-almost order bounded and is uo-converges to 0. Since X is op-continuous, then $x_\alpha - x_{\alpha'} \xrightarrow{up} 0$ and, by Proposition 22, we get $x_\alpha - x_{\alpha'} \xrightarrow{p} 0$. Thus (x_α) is p-Cauchy, and so is p-convergent. By Lemma 14 in Subsection 5.1.2 we get that (x_α) is also uo-convergent to its p-limit. $\square$

5.1.6 rup-Convergence

In this subsection, we introduce the notions of rup-convergence and of an rp-unit. Recall that a net $(x_\alpha)_{\alpha \in A}$ in a vector lattice X is *relatively uniform convergent* (or *ru-convergent*, for short) to $x \in X$ if there is $y \in X_+$, such that, for any $\varepsilon > 0$, there exists $\alpha_0 \in A$ satisfying $|x_\alpha - x| \leq \varepsilon y$ for any $\alpha \geq \alpha_0$, [32, Theorem 16.2]. In this case we write $x_\alpha \xrightarrow{ru} x$.

Definition 16. *Let (X, p, E) be an LNVL. A net $(x_\alpha) \subseteq X$ is said to be* ***relatively unbounded p-convergent*** *(**rup-convergent**) to $x \in X$ if*

$$p(|x_\alpha - x| \wedge u) \xrightarrow{ru} 0 \quad (\forall u \in X_+).$$

In this case we write $x_\alpha \xrightarrow{rup} x$.

Definition 17. *Given an LNVL (X, p, E). A vector $e \in X$ is called an* ***rp-unit*** *if, for any $x \in X_+$, we have $p(x - x \wedge ne) \xrightarrow{ru} 0$.*

Obviously, every rp-unit is a p-unit. So, by Remark 8.1, Section 4.3, if $e \in X \neq \{0\}$ is an rp-unit then $e > 0$. But not every p-unit is an rp-unit.

Example 13. *Let's take $X = (C_b(\mathbb{R}), |\cdot|, C_b(\mathbb{R}))$. Consider $e = e(t) = e^{-|t|}$. If $f \in C_b(\mathbb{R})$ such that $f \wedge e = 0$ in $C_b(\mathbb{R})$, then $f = 0$. Hence e is a weak unit for $C_b(\mathbb{R})$ or a p-unit for $X = (C_b(\mathbb{R}), |\cdot|, C_b(\mathbb{R}))$. However, e is not an rp-unit. Indeed, if $\mathbb{1}$ is the constant function on $\mathbb{R}$, then $\mathbb{1} - \mathbb{1} \wedge ne \xrightarrow{ru} 0$ in $C_b(\mathbb{R})$ iff $\mathbb{1} - \mathbb{1} \wedge ne \to 0$ uniformly in $C_b(\mathbb{R})$. But $\|\mathbb{1} - \mathbb{1} \wedge ne\|_\infty = 1$ for all $n \in \mathbb{N}$. So $\|\mathbb{1} - \mathbb{1} \wedge ne\|_\infty \not\to 0$ as $n \to \infty$. Thus the sequence $(\mathbb{1} \wedge ne)_{n \in \mathbb{N}}$ does not converge uniformly to $\mathbb{1}$ on $C_b(\mathbb{R})$.*

We have seen in Theorem 9, Subsection 5.1.3 that it is enough to check up-convergence at a p-unit. A similar relation is included in the following result in the case of rup-convergence and rp-unit.

Proposition 24. *Let (X, p, E) be an LNVL with an rp-unit e. Then $x_\alpha \xrightarrow{rup} 0$ iff $p(|x_\alpha| \wedge e) \xrightarrow{ru} 0$.*

Proof. The "only if" part is trivial. For the "if" part let $u \in X_+$, then

$$|x_\alpha| \wedge u \leq |x_\alpha| \wedge (u - u \wedge ne) + |x_\alpha| \wedge (u \wedge ne) \leq (u - u \wedge ne) + n(|x_\alpha| \wedge e),$$

and so

$$p(|x_\alpha| \wedge u) \leq p(u - u \wedge ne) + np(|x_\alpha| \wedge e) \tag{5.2}$$

holds for any α and any $n \in \mathbb{N}$.

Given $\varepsilon > 0$. Since e is an rp-unit, then there is $y \in E_+$ and $n_0 \in \mathbb{N}$ such that

$$p(u - u \wedge n_0 e) \leq \frac{\varepsilon}{2} y.$$

It follows from $p(|x_\alpha| \wedge e) \xrightarrow{ru} 0$ that there exists $z \in E_+$ and α_0 such that

$$p(|x_\alpha| \wedge e) \leq \frac{\varepsilon}{2n_0} z$$

for any $\alpha \geq \alpha_0$. Take $w := y + z$, then from (5.2) we get

$$p(|x_\alpha| \wedge u) \leq \varepsilon w$$

for any $\alpha \geq \alpha_0$. Therefore $p(|x_\alpha| \wedge u) \xrightarrow{ru} 0$. $\square$

Clearly, rup-convergence implies up-convergence, but the converse need not be true.

Example 14. *Consider the LNVL $(\ell_\infty, |\cdot|, \ell_\infty)$ and the sequence (x_n) in ℓ_∞, given by $x_n = \sum\limits_{i=n}^{\infty} e_i$, where e_i's are the standard unit vectors in ℓ_∞.*

- *Clearly, $x_n \xrightarrow{\text{uo}} 0$ in ℓ_∞, and so $x_n \xrightarrow{\text{up}} 0$ in $(\ell_\infty, |\cdot|, \ell_\infty)$.*
- *We show (x_n) is not rup-Cauchy sequence in $(\ell_\infty, |\cdot|, \ell_\infty)$, and so it is not rup-convergent.*
- *Since $\mathbb{1}$ is a strong unit in ℓ_∞, then it is enough to show that $|x_n - x_m| = |x_n - x_m| \wedge \mathbb{1} \overset{\text{ru}}{\not\to} 0$ as $n, m \to \infty$.*
- *Obviously that for all $n \neq m$, $|x_n - x_m| \not\leq \frac{1}{2}\mathbb{1}$.*
- *If $|x_n - x_m| \xrightarrow{\text{ru}} 0$, then there exists $0 \leq u \in \ell_\infty$ such that for all $\varepsilon > 0$ there is $n_\varepsilon \in \mathbb{N}$ satisfying $|x_n - x_m| \leq \varepsilon u$ for each $m > n \geq n_\varepsilon$.*
- *Since $\mathbb{1}$ is a strong unit in ℓ_∞, then there is $\lambda > 0$ such that $u \leq \lambda\mathbb{1}$.*
- *Take $\varepsilon_0 = \frac{1}{2\lambda} > 0$, then there is $n_0 \in \mathbb{N}$ such that for every $m > n \geq n_0$, $|x_n - x_m| \leq \varepsilon_0 u \leq \varepsilon_0 \lambda \mathbb{1} = \frac{1}{2}\mathbb{1}$; a contradiction.*
- *Therefore the sequence (x_n) can not be rup-convergent.*

5.2 up-Regular sublattices

The up-convergence passes obviously to any sublattice of X. As it was remarked in [11, p.3], in opposite to uo-convergence (see Theorem 1, Section 3.1), the un-convergence does not pass even from regular sublattices. These two facts motivate the following notion in LNVLs.

Definition 18. *Let (X, p, E) be an LNVL and Y be a sublattice of X. Then Y is called* ***up-regular*** *if, for any net (y_α) in Y, $y_\alpha \xrightarrow{up} 0$ in Y implies $y_\alpha \xrightarrow{up} 0$ in X. Equivalently, Y is up-regular in X when $y_\alpha \xrightarrow{up} 0$ in Y iff $y_\alpha \xrightarrow{up} 0$ in X for any net (y_α) in Y.*

It is clear that if Y is a regular sublattice of a vector lattice X, then Y is up-regular in the LNVL $(X, |\cdot|, X)$; see Theorem 1, Section 3.1. The converse does not hold in general.

Example 15. *Let $X = B([0,1])$ be the space of all real-valued bounded functions on $[0,1]$ and $Y = C[0,1]$. First of all X under the pointwise ordering (i.e., $f \leq g$ in X iff $f(t) \leq g(t)$ for all $t \in [0,1]$) is a vector lattice and if we equip X with the ∞-norm, then it becomes a Banach lattice.*
We claim that the sublattice $Y = (Y, |\cdot|, Y)$ is a up-regular sublattice of $X = (X, |\cdot|, X)$. Let (f_α) be a net in Y such that $f_\alpha \xrightarrow{\mathrm{up}} 0$ in Y. That is $|f_\alpha| \wedge g \xrightarrow{\mathrm{o}} 0$ in X for any $g \in Y_+$. In particular, we have $|f_\alpha| \wedge \mathbb{1} \xrightarrow{\mathrm{o}} 0$ in X, where $\mathbb{1}$ denotes the constant function one. Since $\mathbb{1}$ is a strong unit in X, then it is a p-unit for the LNVL $(X, |\cdot|, X)$. It follows from Theorem 9 in Subsection 5.1.3 that $f_\alpha \xrightarrow{\mathrm{up}} 0$ in X. However, the sublattice Y is not regular in X. Indeed, for each $n \in \mathbb{N}$ let f_n be a continous function on $[0,1]$ defined as:

- *f_n is zero on the intervals $[0, \frac{1}{2} - \frac{1}{n+2}]$ and $[\frac{1}{2} + \frac{1}{n+2}, 1]$,*
- *$f_n(\frac{1}{2}) = 1$,*
- *f_n is linear on the intervals $[\frac{1}{2} - \frac{1}{n+2}, \frac{1}{2}]$ and $[\frac{1}{2}, \frac{1}{2} + \frac{1}{n+2}]$.*

Then $f_n \downarrow 0$ in $C[0,1]$ but $f_n \downarrow \chi_{\frac{1}{2}}$ in $B([0,1])$. So by Lemma 2, Section 3.1 we have that Y is not regular in X.

Consider the sublattice c_0 of ℓ_∞. Then $(c_0, \|\cdot\|_\infty, \mathbb{R})$ is not up-regular in the LNVL $(\ell_\infty, \|\cdot\|_\infty, \mathbb{R})$. Indeed, (e_n) is un-convergent in c_0 but not in ℓ_∞. However, $(c_0, |\cdot|, \ell_\infty)$ is up-regular in the LNVL $(\ell_\infty, |\cdot|, \ell_\infty)$.

5.2.1 Several basic results

We begin with the following result which is a p-version of Theorem 4, Section 3.3.

Theorem 10. *Let Y be a sublattice of an LNVL $X = (X, p, E)$.*

1. *If Y is majorizing in X, then Y is up-regular.*
2. *If Y is p-dense in X, then Y is up-regular.*
3. *If Y is a projection band in X, then Y is up-regular.*

Proof. Let $(y_\alpha) \subseteq Y$ and assume that $y_\alpha \xrightarrow{up} 0$ in Y. Let $0 \neq x \in X_+$.

1. If Y is majorizing in X, then there is $y \in Y$ such that $x \leq y$. It follows from
$$0 \leq |y_\alpha| \wedge x \leq |y_\alpha| \wedge y \xrightarrow{p} 0,$$
that $y_\alpha \xrightarrow{up} 0$ in X.

2. If Y is p-dense in X, then for $0 \neq u \in p(X)$ there is $y \in Y$ such that $p(x-y) \leq u$. Since
$$|y_\alpha| \wedge x \leq |y_\alpha| \wedge |x-y| + |y_\alpha| \wedge |y|,$$
then
$$p(|y_\alpha| \wedge x) \leq p(|y_\alpha| \wedge |x-y|) + p(|y_\alpha| \wedge |y|) \leq u + p(|y_\alpha| \wedge |y|).$$
But $0 \neq u \in p(X)$ is arbitrary and $|y_\alpha| \wedge |y| \xrightarrow{p} 0$, then $|y_\alpha| \wedge x \xrightarrow{p} 0$. Hence $y_\alpha \xrightarrow{up} 0$ in X.

3. Suppose that Y is a projection band in X, then $X = Y \oplus Y^d$. Thus $x = x_1 + x_2$ with $x_1 \in Y_+$ and $x_1 \in Y_+^d$. Since $|y_\alpha| \wedge x_2 = 0$, we have
$$p(|y_\alpha| \wedge x) = p(|y_\alpha| \wedge (x_1 + x_2)) = p(|y_\alpha| \wedge x_1) \xrightarrow{o} 0.$$
Hence $y_\alpha \xrightarrow{up} 0$ in X.

□

The following result deals with a particular case of Example 11, Section 4.2 and is motivated by Proposition 10, Section 3.3.

Theorem 11. *Let X be a vector lattice and $Y = X_n^\sim$ be the order continuous dual. Assume $X_n^\sim$ separates the points of X. Define $p : X \to \mathbb{R}^Y$ by $p(x)[y] = |y|(|x|)$. Then any ideal of X is up-regular in $(X, p, \mathbb{R}^Y)$.*

Proof. Let I be an ideal of X and (x_α) be a net in I such that $x_\alpha \xrightarrow{up} 0$ in I. We show $x_\alpha \xrightarrow{up} 0$ in X. By Example 12 in Subsection 5.1.1, this is equivalent to show $|x_\alpha| \wedge u \xrightarrow{|\sigma|(X,Y)} 0$ for any $u \in X_+$. First note that if $v \in I^d$, then $|x_\alpha| \wedge |v| = 0$, and so, for any $w \in (I \oplus I^d)_+$, we have $|x_\alpha| \wedge w \xrightarrow{|\sigma|(X,Y)} 0$. Note also that $I \oplus I^d$ is order dense (see, e.g., [3, Theorem 1.36.(2)]). Let $u \in X_+$ and $y \in Y$, then there is a net (w_β) in $(I \oplus I^d)_+$ such that $w_\beta \uparrow u$, and so $|y|(w_\beta \wedge u) \uparrow |y|(u)$. Given $\varepsilon > 0$. There is β_0 such that

$$|y|(u) - |y|(w_{\beta_0} \wedge u) < \frac{\varepsilon}{2}.$$

Also, there is α_0 such that

$$|y|(|x_\alpha| \wedge w_{\beta_0}) < \frac{\varepsilon}{2}$$

for all $\alpha \geq \alpha_0$. Taking into account the Birkhoff's inequality $|a \wedge c - b \wedge c| \leq |a - b|$ (see, e.g., [3, Theorem 1.9.(2)]), we have for any $\alpha \geq \alpha_0$,

$$|y|(|x_\alpha| \wedge u) = |y|(|x_\alpha| \wedge u) - |y|(|x_\alpha| \wedge u \wedge w_{\beta_0}) + |y|(|x_\alpha| \wedge u \wedge w_{\beta_0})$$

$$\leq |y|(u) - |y|(w_{\beta_0} \wedge u) + |y|(|x_\alpha| \wedge w_{\beta_0}) < \varepsilon.$$

Since $u \in X_+$ and $y \in Y$ are arbitrary, we get $|x_\alpha| \wedge u \xrightarrow{|\sigma|(X,Y)} 0$ for any $u \in X_+$, and this completes the proof. $\square$

Next we give an extension of Proposition 11, Section 3.3.

Corollary 8. *Let X be a vector lattice and $Y = X_n^\sim$. Assume that $X_n^\sim$ separates the points of X. Define $p : X \to \mathbb{R}^Y$ by $p(x)[y] = |y|(|x|)$. Then any sublattice of X is up-regular in the LNVL $(X, p, \mathbb{R}^Y)$.*

Proof. Let X_0 be a sublattice of X and (x_α) be a net in X_0 such that $x_\alpha \xrightarrow{up} 0$ in X_0. Let I_{X_0} be the ideal generated by X_0 in X. Then X_0 is majorizing in I_{X_0} and, by Theorem 10.1, we get $x_\alpha \xrightarrow{up} 0$ in I_{X_0}. Theorem 11 implies that $x_\alpha \xrightarrow{up} 0$ in X. $\square$

5.2.2 Order completion

In this subsection we discuss the interactions ofup-regularity between the vector lattice and its order completion.

Proposition 25. *Let (X^δ, p, E) be an LNVL, where X^δ is the order completion of X. For any sublattice Y of X, if Y^δ is up-regular in X^δ, then Y is up-regular in $X = (X, p|_X, E)$.*

Proof. Take a net $(y_\alpha) \subseteq Y$ such that $y_\alpha \xrightarrow{up} 0$ in Y. Then $p(|y_\alpha| \wedge u) \xrightarrow{o} 0$ for all $u \in Y_+$. Let $0 \leq w \in Y^\delta$ and, since Y is majorizing in Y^δ, there exists $y \in Y$ such that $0 \leq w \leq y$. Therefore we obtain $y_\alpha \xrightarrow{up} 0$ in Y^δ. Since Y^δ is up-regular in X^δ, the net (y_α) is up-convergent to 0 in X^δ, and so in X. □

Proposition 26. *Let (X^δ, p, E) be an LNVL. For any sublattice $Y \subseteq X$, if Y is up-regular in X, then Y is up-regular in X^δ.*

Proof. Suppose Y is up-regular in X. Since X majorizes X^δ then it follows from Theorem 10.1 in Subsection 5.2.1 that X is up-regular in X^δ. Hence, Y is up-regular in X^δ. □

Theorem 12. *Let (X, p, E) be an LNVL. Define $p_L^\delta : X^\delta \to E^\delta$ and $p_U^\delta : X^\delta \to E^\delta$ as follows: $p_L^\delta(z) = \sup\limits_{0 \leq x \leq |z|} p(x)$ and $p_U^\delta(z) = \inf\limits_{|z| \leq x} p(x)$ for all $z \in X^\delta$ (clearly, both p_U^δ and p_L^δ are extensions of p). Then:*

1. *p_L^δ is a monotone E^δ-valued norm;*

2. *p_U^δ is a monotone E^δ-valued seminorm;*

3. *if X is op-continuous, then p_U^δ is p-continuous (i.e., $z_\gamma \downarrow 0$ in X^δ implies $p_U^\delta(z_\gamma) \downarrow 0$ in E^δ);*

4. *if X is op-continuous, then $p_U^\delta = p_L^\delta$.*

Proof.

1. Let $0 \neq z \in X^\delta$. Since X is order dense in X^δ, there is $x \in X$ such that $0 < x \leq |z|$, and so $p_L^\delta(z) \geq p(x) > 0$.

   Let $0 \neq \lambda \in \mathbb{R}$ and $z \in X^\delta$, then

   $$p_L^\delta(\lambda z) = \sup_{0 \leq x \leq |\lambda z|} p(x) = \sup_{0 \leq \frac{1}{|\lambda|} x \leq |z|} p(x) = |\lambda| \sup_{0 \leq \frac{1}{|\lambda|} x \leq |z|} p(|\lambda|^{-1} x) = |\lambda| p_L^\delta(z).$$

   Let $z, w \in X^\delta$, we show $p_L^\delta(z+w) \leq p_L^\delta(z) + p_L^\delta(w)$. Suppose $0 \leq x \leq |z+w|$, then $0 \leq x \leq |z| + |w|$. By the Riesz decomposition property, there exist $x_1, x_2 \in X$ such that $0 \leq x_1 \leq |z|$, $0 \leq x_2 \leq |w|$, and $x = x_1 + x_2$. So

   $$p(x) = p(x_1 + x_2) \leq p(x_1) + p(x_2) \leq p_L^\delta(z) + p_L^\delta(w).$$

   Thus $p_L^\delta(z+w) = \sup\limits_{0 \leq x \leq |z+w|} p(x) \leq p_L^\delta(z) + p_L^\delta(w)$.

   Now, we prove the monotonicity of the lattice norm p_L^δ. If $|z| \leq |w|$ then for any $x \in X$ with $0 \leq x \leq |z|$, we get $0 \leq x \leq |w|$. So $\sup\limits_{0 \leq x \leq |z|} p(x) \leq \sup\limits_{0 \leq x \leq |w|} p(x)$ or $p_L^\delta(z) \leq p_L^\delta(w)$.

2. Let $0 \neq \lambda \in \mathbb{R}$ and $z \in X^\delta$, then

   $$p_U^\delta(\lambda z) = \inf_{|\lambda z| \leq x} p(x) = \inf_{|z| \leq \frac{1}{|\lambda|} x} p(x) = |\lambda| \inf_{|z| \leq \frac{1}{|\lambda|} x} p(|\lambda|^{-1} x) = |\lambda| p_U^\delta(z).$$

   Next we show that p_U^δ satisfies the triangle inequality. Let $z, w \in X^\delta$ and $x_1, x_2 \in X$ be such that $|z| \leq x_1$ and $|w| \leq x_2$, then $|z+w| \leq |z| + |w| \leq x_1 + x_2$. So

   $$p_U^\delta(z+w) = \inf_{|z+w| \leq x} p(x) \leq p(x_1 + x_2) \leq p(x_1) + p(x_2).$$

   Thus $p_U^\delta(z+w) - p(x_1) \leq p(x_2)$ for any $x_2 \in X$ with $|w| \leq x_2$. Hence $p_U^\delta(z+w) - p(x_1) \leq p_U^\delta(w)$ or $p_U^\delta(z+w) - p_U^\delta(w) \leq p(x_1)$, which holds for all $x_1 \in X$ with $|z| \leq x_1$. Therefore $p_U^\delta(z+w) - p_U^\delta(w) \leq p_U^\delta(z)$ or $p_U^\delta(z+w) \leq p_U^\delta(w) + p_U^\delta(z)$.

   Finally we check that p_U^δ is monotone. If $|z| \leq |w|$, then for any $x \in X$ with $0 < |w| \leq x$, we have $|z| \leq x$. So $\inf\limits_{|w| \leq x} p(x) \geq \inf\limits_{|z| \leq x} p(x)$ or $p_U^\delta(z) \leq p_U^\delta(w)$.

3. Assume $z_\gamma \downarrow 0$ in X^δ. Let $A = \{a \in X : z_\gamma \leq a \text{ for some } \gamma\}$. Then $\inf A = 0$. Indeed, if $0 \leq x \leq a$ for all $a \in A$, then $0 \leq x \leq A_\gamma$ for all γ, where

$A_\gamma = \{a \in X : z_\gamma \le a\}$. So, by [21, Lemma 2.7], we have $x \le z_\gamma$. Thus $x = 0$.

Clearly, A dominates the net (z_γ). We claim that A is directed downward. Let $a_1, a_2 \in A$ then there are γ_1 and γ_2 such that $z_{\gamma_1} \le a_1$ and $z_{\gamma_2} \le a_2$. So there is γ_3 with $\gamma_3 \ge \gamma_1$ and $\gamma_3 \ge \gamma_2$. Hence $z_{\gamma_3} \le z_{\gamma_1}$ and $z_{\gamma_3} \le z_{\gamma_2}$. From which it follows that $z_{\gamma_3} \le z_{\gamma_1} \wedge z_{\gamma_2} \le a_1 \wedge a_2$, and so $a_1 \wedge a_2 \in A$ with $a_1 \wedge a_2 \le a_1$ and $a_1 \wedge a_2 \le a_2$. Therefore $A \downarrow 0$. Since X is *op*-continuous, then $p(A) \downarrow 0$ and, by the definition of p_U^δ, we get that $p(A)$ dominates the net $(p_U^\delta z_\gamma)$. Therefore $p_U^\delta z_\gamma \downarrow 0$.

4. Let $z \in X^\delta$, then $|z| = \sup\{x \in X : 0 \le x \le |z|\}$. By item 3 above, we have p_U^δ is p-continuous. Thus,

$$p_U^\delta(z) = p_U^\delta(|z|) = \sup\{p_U^\delta(x) : x \in X, 0 \le x \le |z|\}$$
$$= \sup\{p(x) : x \in X, 0 \le x \le |z|\} = p_L^\delta(z).$$

□

Proposition 27. *Let (X, p, E) be an LNVL. Then, for every net (x_α) in X,*

$$x_\alpha \xrightarrow{up} 0 \ \text{ in } (X, p, E) \ \text{ iff } \ x_\alpha \xrightarrow{up} 0 \ \text{ in } (X^\delta, p^\delta, E^\delta),$$

where $p^\delta = p_L^\delta$.

Proof. $(\Rightarrow)$. Assume $x_\alpha \xrightarrow{up} 0$ in (X, p, E). Then $p(|x_\alpha| \wedge x) \xrightarrow{o} 0$ in E for all $x \in X_+$, and so $p(|x_\alpha| \wedge x) \xrightarrow{o} 0$ in E^δ for all $x \in X_+$, by Lemma 1, Chapter 2. Hence

$$p^\delta(|x_\alpha| \wedge x) \xrightarrow{o} 0 \tag{5.3}$$

in E^δ for all $x \in X_+$. Let $u \in X_+^\delta$, then there exists $x_u \in X_+$ such that $u \le x_u$, as X majorizes X^δ. From (5.3) it follows that $p^\delta(|x_\alpha| \wedge u) \xrightarrow{o} 0$ in E^δ. Since $u \in X_+^\delta$ is arbitrary, then $x_\alpha \xrightarrow{up} 0$ in $(X^\delta, p^\delta, E^\delta)$.

$(\Leftarrow)$. Suppose (x_α) in X such that $x_\alpha \xrightarrow{up} 0$ in $(X^\delta, p^\delta, E^\delta)$. Then for all $u \in X_+^\delta$, $p^\delta(|x_\alpha| \wedge u) \xrightarrow{o} 0$ in E^δ. In particular, for all $x \in X_+$, $p(|x_\alpha| \wedge x) = p^\delta(|x_\alpha| \wedge x) \xrightarrow{o} 0$ in E^δ. Again, by lemma 1 in Chapter 2, we get $p(|x_\alpha| \wedge x) \xrightarrow{o} 0$ in E for all $x \in X_+$. Hence $x_\alpha \xrightarrow{up} 0$ in (X, p, E). □

CHAPTER 6

MIXED NORMED SPACES

Mixed norms provide an efficient tool to study several classes of operators; see for example [28, Chapter 7]. In this short chapter, we exploit mixed-normed spaces to generalize many results mentioned in Remark 2, Section 3.1 and Remark 4, Section 3.3.

6.1 Mixed norms

In this section, we study LNVLs with mixed lattice norms. Let (X, p, E) be an LNS and $(E, \|\cdot\|_E)$ be a normed lattice. The ***mixed norm*** on X is defined by

$$p\text{-}\|x\|_E = \|p(x)\|_E \quad (\forall x \in X).$$

In this case, the normed space $(X, p\text{-}\|\cdot\|_E)$ is called a ***mixed-normed space*** (see for example [28, 7.1.1, p.292])

The next proposition shows relations of LNVLs with their corresponding mixed-normed spaces.

Proposition 28. *Let (X, p, E) be an LNVL, $(E, \|\cdot\|_E)$ be a Banach lattice, and $(X, p\text{-}\|\cdot\|_E)$ be the mixed-normed space. The following statements hold:*

1. *if (X, p, E) is op-continuous and E is order continuous, then $(X, p\text{-}\|\cdot\|_E)$ is order continuous normed lattice;*
2. *if a subset Y of X is p-bounded (respectively, p-dense) in (X, p, E), then Y is norm bounded (respectively, norm dense) in $(X, p\text{-}\|\cdot\|_E)$;*

3. *if $e \in X$ is a p-unit and E is σ-order continuous, then e is a quasi-interior point of $(X, p\text{-}\|\cdot\|_E)$;*

4. *if (X, p, E) is a p-Fatou space and E is order continuous, then $p\text{-}\|\cdot\|_E$ is a Fatou norm, [39, p.387];*

5. *if Y is a p-almost order bounded subset of X, then Y is almost order bounded in $(X, p\text{-}\|\cdot\|_E)$.*

Proof.

1. Assume $x_\alpha \downarrow 0$ in X. Then $p(x_\alpha) \downarrow 0$ in E. Since $(E, \|\cdot\|_E)$ is order continuous, we get $\|p(x_\alpha)\|_E \downarrow 0$ or $p\text{-}\|x_\alpha\|_E \downarrow 0$.

2. Let Y be a p-bounded subset of X. There is $e \in E_+$ so that $p(y) \leq e$ for all $y \in Y$. So, $\|p(y)\|_E \leq \|e\|_E < \infty$ for all $y \in Y$ or $\sup\limits_{y \in Y} p\text{-}\|y\|_E < \infty$. Hence Y is norm bounded in $(X, p\text{-}\|\cdot\|_E)$.

   Now suppose Y is p-dense. Given $\varepsilon > 0$. Let $x \in X$. There is $0 \neq u \in p(X)$ such that $\|u\|_E = \varepsilon$. Thus, there exists a vector $y \in Y$ such that $p(x - y) \leq u$. So $\|p(x - y)\|_E \leq \|u\|_E$ or $p\text{-}\|x - y\|_E \leq \varepsilon$.

3. Let $x \in X_+$. Since e is a p-unit, then it follows that $p(x - x \wedge ne) \xrightarrow{o} 0$ in E as $n \to \infty$. The σ-order continuity of E implies that $\|p(x - x \wedge ne)\|_E \to 0$, and so $p\text{-}\|x - x \wedge ne\|_E \to 0$. Thus e is a quasi-interior point of $(X, p\text{-}\|\cdot\|_E)$.

4. Suppose $0 \leq x_\alpha \uparrow x$ in X. Since (X, p, E) is a p-Fatou space then $p(x_\alpha) \uparrow p(x)$. Now the order continuity of E assures that $\|p(x_\alpha)\|_E \uparrow \|p(x)\|_E$, and so $p\text{-}\|x_\alpha\|_E \uparrow p\text{-}\|x\|_E$.

5. Given $\varepsilon > 0$. There is $w \in E_+$ with $\|w\|_E = \varepsilon$. Since Y is a p-almost order bounded subset of X then there exists $x_0 \in X_+$ such that $p(|y| - |y| \wedge x_0) \leq w$ for all $y \in Y$. From which it follows that $p\text{-}\big\||y| - |y| \wedge x_0\big\|_E \leq \|w\|_E = \varepsilon$. Hence Y is almost order bounded in $(X, p\text{-}\|\cdot\|_E)$.

$\square$

Theorem 13. *Let (X, p, E) and (E, m, F) be two p-KB-spaces. Then the LNVL $(X, m \circ p, F)$ is also a p-KB-space.*

Proof. Let $0 \leq x_\alpha \uparrow$ and $m\big(p(x_\alpha)\big) \leq g \in F$. Since $0 \leq p(x_\alpha) \uparrow$ and p-bounded in (E, m, F) which is a p-KB-space, then there exists $y \in E$ such that $m\big(p(x_\alpha) - y)\big) \xrightarrow{o} 0$ in F. Hence $p(x_\alpha) \uparrow y$. Thus the net (x_α) is increasing and p-bounded. Since (X, p, E) is a p-KB-space, then there exists $x \in X$ such that $p(x_\alpha - x) \xrightarrow{o} 0$ in E. From Corollary 7, Section 4.2 we know that any p-KB-space is op-continuous. In particular, (E, m, F) is op-continuous, and so $m\big(p(x_\alpha - x)\big) \xrightarrow{o} 0$; i.e., $m \circ p(x_\alpha - x) \xrightarrow{o} 0$. Thus, $(X, m \circ p, F)$ is a p-KB-space. $\square$

Corollary 9. *Let (X, p, E) be a p-KB-space and $(E, \|\cdot\|_E)$ be a KB-space. Then $(X, p\text{-}\|\cdot\|_E)$ is a KB-space.*

The following well-known technical lemma is a particular case of Lemma 14, Subsection 5.1.2.

Lemma 16. *Given a Banach lattice $(X, \|\cdot\|)$. If $x_\alpha \xrightarrow{\|\cdot\|} x$ and (x_α) is order Cauchy, then $x_\alpha \xrightarrow{o} x$.*

Recall that a Banach lattice is called ***un-complete*** if every un-Cauchy net is un-convergent, [25, p. 270].

Theorem 14. *Let (X, p, E) be an LNVL and $(E, \|\cdot\|_E)$ be an order continuous Banach lattice. If $(X, p\text{-}\|\cdot\|_E)$ is a un-complete Banach lattice, then X is up-complete.*

Proof. Let (x_α) be a up-Cauchy net in X. So, for every $u \in X_+$, $p(|x_\alpha - x_\beta| \wedge u) \xrightarrow{o} 0$. Since E is order continuous, then for every $u \in X_+$, $\|p(|x_\alpha - x_\beta| \wedge u)\|_E \to 0$ or $p\text{-}\big\||x_\alpha - x_\beta| \wedge u\big\|_E \to 0$ for each $u \in X_+$; i.e., (x_α) is un-Cauchy in $(X, p\text{-}\|\cdot\|_E)$. Since $(X, p\text{-}\|\cdot\|_E)$ is un-complete, then there exists $x \in X$ such that $x_\alpha \xrightarrow{un} x$ in $(X, p\text{-}\|\cdot\|_E)$. That is, for any $u \in X_+$, $\|p(|x_\alpha - x| \wedge u)\|_E \to 0$. Next we show the net $(p(|x_\alpha - x| \wedge u))$ is order Cauchy in E. Indeed,

$$\big|p(|x_\alpha - x| \wedge u) - p(|x_\beta - x| \wedge u)\big| \leq p\big(\big||x_\alpha - x| \wedge u - |x_\beta - x| \wedge u\big|\big) \leq p(|x_\alpha - x_\beta| \wedge u) \xrightarrow{o} 0.$$

Now, Lemma 16 above, implies that $p(|x_\alpha - x| \wedge u) \xrightarrow{o} 0$. $\square$

6.2 up-Null nets and up-null sequences in mixed-normed spaces

The following theorem is a p-version of Remark 4.4, Section 3.3 and a generalization of Remark 2.6 in Section 3.1 as we take $(X, p, E) = (X, \|\cdot\|, \mathbb{R})$.

Theorem 15. *Let $X = (X, p, E)$ be an op-continuous and p-complete LNVL, $(E, \|\cdot\|_E)$ an order continuous Banach lattice, and $X \ni x_\alpha \xrightarrow{up} 0$. Then there exists an increasing sequence (α_k) of indices and a disjoint sequence (d_k) in X such that $(x_{\alpha_k} - d_k) \xrightarrow{p} 0$ as $k \to \infty$.*

Proof. Consider the mixed norm $p\text{-}\|x\|_E = \|p(x)\|_E$. Since $p(|x_\alpha| \wedge u) \xrightarrow{o} 0$ for all $u \in X_+$ and $(E, \|\cdot\|_E)$ is order continuous, then $p\text{-}\||x_\alpha| \wedge u\|_E = \|p(|x_\alpha| \wedge u)\|_E \to 0$ that means $x_\alpha \xrightarrow{un} 0$ in $(X, p\text{-}\|\cdot\|_E)$. By Remark 4.4, Section 3.3, there exists an increasing sequence (α_n) of indices and a disjoint sequence (d_n) in X such that $p\text{-}\|x_{\alpha_n} - d_n\|_E \to 0$. It follows from [28, 7.1.3 (1), p. 294] that $(X, p\text{-}\|\cdot\|_E)$ is a Banach lattice. So, by [37, Theorem VII.2.1] there is a further subsequence (α_{n_k}) such that $|x_{\alpha_{n_k}} - d_{n_k}| \xrightarrow{o} 0$ in X. By *op*-continuity of X, we get $p(x_{\alpha_{n_k}} - d_{n_k}) \xrightarrow{o} 0$. □

The next corollary is a p-version of Remark 4.5, Section 3.3.

Corollary 10. *Let (X, p, E) be an op-continuous and p-complete LNVL, E be an order continuous Banach lattice, and $X \ni x_\alpha \xrightarrow{up} 0$. Then there is an increasing sequence (α_k) of indices such that $x_{\alpha_k} \xrightarrow{up} 0$.*

Proof. Let (α_k) and (d_k) be as in Theorem 15 above. Since the sequence (d_k) is disjoint, then $d_k \xrightarrow{uo} 0$ by [21, Corollary 3.6]. As X is *op*-continuous, then $d_k \xrightarrow{up} 0$. Since

$$p(|x_{\alpha_k} - d_k| \wedge u) \leq p(x_{\alpha_k} - d_k) \xrightarrow{o} 0 \quad (\forall u \in X_+),$$

then $x_{\alpha_k} - d_k \xrightarrow{up} 0$. Since $d_k \xrightarrow{up} 0$, then $x_{\alpha_k} \xrightarrow{up} 0$. □

Next proposition extends Remark 4.6 in Section 3.3 to LNVLs.

Proposition 29. *Let (X, p, E) be a p-complete LNVL, $(E, \|\cdot\|_E)$ be an order continuous Banach lattice, and $X \ni x_n \xrightarrow{up} 0$. Then there exist a subsequence (x_{n_k}) of (x_n) such that $x_{n_k} \xrightarrow{uo} 0$ as $k \to \infty$.*

Proof. Suppose $x_n \xrightarrow{up} 0$, then for all $u \in X_+$ $p(|x_n| \wedge u) \xrightarrow{o} 0$, and so $\|p(|x_n| \wedge u)\|_E \to 0$ as E is order continuous. Thus $|x_n| \wedge u \xrightarrow{p\text{-}\|\cdot\|_E} 0$; i.e., $x_n \xrightarrow{un} 0$ in $(X, p\text{-}\|\cdot\|_E)$. It follows from [28, 7.1.2, p. 293] that the mixed-normed space $(X, p\text{-}\|\cdot\|_E)$ is a Banach lattice, and so by Remark 4.6 in Section 3.3 there is a subsequence (x_{n_k}) of (x_n) such that $x_{n_k} \xrightarrow{uo} 0$ as $k \to \infty$. $\square$

Next result is a p-version of Remark 4.7, Section 3.3.

Proposition 30. *Let $X = (X, p, E)$ be an op-continuous and p-complete LNVL such that $(E, \|\cdot\|_E)$ is an order continuous atomic Banach lattice. Then a sequence in X is up-null iff every subsequence has a further subsequence which uo-converges to zero.*

Proof. The forward implication follows from Proposition 29. Conversely, let (x_n) be a sequence in X, and assume that $x_n \not\xrightarrow{up} 0$. Then, by Corollary 1 in Section 3.1, there is an atom $a \in E_+$, $u \in X_+$, $\varepsilon_0 > 0$ and a subsequence (x_{n_k}) of (x_n) satisfying $f_a\big(p(|x_{n_k}| \wedge u)\big) \geq \varepsilon_0$ for all k, where f_a denotes the biorthogonal functional corresponding to a. By the hypothesis, there exists a further subsequence $(x_{n_{k_j}})$ of (x_{n_k}) which uo-converges to zero. By the op-continuity of X, we get $p(|x_{n_{k_j}}| \wedge u) \xrightarrow{o} 0$, and so $f_a\big(p(|x_{n_{k_j}}| \wedge u)\big) \to 0$, which is a contradiction. $\square$

Our last result is a p-version of Remark 4.8, Section 3.3.

Proposition 31. *Let (X, p, E) be an op-continuous p-complete LNVL and $(E, \|\cdot\|_E)$ be an order continuous Banach lattice. If X is atomic and (x_n) is an order bounded sequence such that $x_n \xrightarrow{p} 0$ in X, then $x_n \xrightarrow{o} 0$.*

Proof. The mixed-normed space $(X, p\text{-}\|\cdot\|_E)$ is an atomic order continuous Banach lattice such that $x_n \xrightarrow{p\text{-}\|\cdot\|_E} 0$, and so $x_n \xrightarrow{o} 0$ by Remark 4.8, Section 3.3. $\square$

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EDUCATION

Degree	Institution	Year of Graduation
M.S.	Arab American University of Jenin, Applied Mathematics	2011
B.S.	An-Najah National University, Mathematics	2009
High School	Kufr-Ra'I Secondary Boys School	2005

PROFESSIONAL EXPERIENCE

Year	Place	Enrollment
11/03/2016-28/05/2016	University of Alberta	Researcher
22/08/2011–31/05/2012	Bethlehem University	Instructor of Mathematics

PUBLICATIONS

1. A. Aydın, E. Yu. Emelyanov, N. Erkurşun-Özcan and **M. A. A. Marabeh**. Compact-Like Operators in Lattice-Normed Spaces, preprint, arXiv: 1701.03073v2 [math.FA].

2. A. Aydın, E. Yu. Emelyanov, N. Erkurşun-Özcan and **M. A. A. Marabeh**. Unbounded p-Convergence in Lattice-Normed Vector Lattices, preprint, arXiv: 1609.05301v2 [math.FA].

3. M. Kandic, **M.A.A. Marabeh** and V.G. Troitsky. Unbounded Norm Topology in Banach Lattices, Journal of Mathematical Analysis and Applications, Volume 451, Issue 1(2017), pp. 259-279.

4. E. Yu. Emelyanov and **M. A. A. Marabeh**. Two Measure-Free Versions of the Brezis-Lieb Lemma, Vladikavkaz Mathematical Journal (VMJ), Vol. 18, Issue 1(2016), pp. 21-25.

5. Saed F. Mallak, **Mohammad Mara'Beh** and Abdelhalim Zaiqan. Further Particular Classes of Ergodic Finite Fuzzy Markov Chains. Advances in Fuzzy Mathematics (AFM), Volume 6, Number 2(2011), pp. 269-281.

6. Saed F. Mallak, **Mohammad Mara'Beh** and Abdelhalim Zaiqan. A Particular Class of Ergodic Finite Fuzzy Markov Chains. Advances in Fuzzy Mathematics (AFM), Volume 6, Number 2(2011), pp. 253-268.

Honors and Awards

1. METU Scientific Research Project (BAP), 2016.

2. METU Graduate Courses Performance Award for being the most successful student in the PhD program of the Department of Mathematics in the 2013-2014 Academic Year.

3. Full PhD Scholarship from "Türkiye Bursları" to study mathematics at Middle East Technical University (METU), October 2012 – Present.

4. Scholarship from An-Najah National University covers 50% of tuition fees during my undergraduate superiority, 2005 – 2009.

Talks Given

1. Unbounded Norm Topology in Banach Lattices, Graduate Seminar, March 07, 2017, Department of Mathematics, Middle East Technical University (METU),

Ankara, Turkey.

2. Un-compact operators, POSITIVITY Conference, December 3-4, 2016, Department of Mathematics, Abant İzzet Baysal University, Bolu, Turkey.

3. Unbounded Norm Topology in Banach Lattices, Seminar, November 17, 2016, Department of Mathematics, Bilkent University, Ankara, Turkey.

4. Unbounded Order Continuous Operators, Alberta Mathematics Dialouge, April 28-29, 2016, Department of Mathematics and Computing, Mount Royal University, Calgaray, Canada.

5. Unbounded Order Continuous Operators, General Seminar, 06/01/2016, Department of Mathematics, Hacettepe University, Ankara, Turkey.