CHATTERING AND SINGULAR PERTURBATION IN DISCONTINUOUS DYNAMICS

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ABSTRACT

CHATTERING AND SINGULAR PERTURBATION IN DISCONTINUOUS DYNAMICS

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The main purpose of this dissertation is to address the chattering and singularity phenomena in discontinuous dynamical systems. The study describes models of singular impulsive differential equations such that in the system, not only the differential equation is singularly perturbed, but also the impulsive function is singular. Tikhonov Theorem is extended for the impulsive differential equations. Interestingly, in some models described here, a solution of the problem approaches more than one root of the differential equation as the parameter decreases to zero.

Wilson-Cowan neuron model is studied with impulse function in which the membrane time constant is considered as both the singularity and bifurcation parameter. A new technique of analysis of the phenomenon is suggested. This allows to consider the existence of solutions of the model and bifurcation in ultimate neural behavior is observed through numerical simulations. The bifurcations are reasoned by impulses and singularity in the model and they concern the structure of attractors, which consist of newly introduced sets in the phase space such that medusas and rings.

Moreover, the singular impact moments are introduced and they are utilized for the problems with chattering solutions. The singular impulse moments gives the advantages that the chattering arising in models, e.g., a bouncing ball, an inverted pendulum and a hydraulic relief valve, can be analyzed through the singularity point of view. The presence of chattering is shown exclusively by examination of the right hand side of impact models. Criteria for the sets of initial data which always lead to chattering are established.

Keywords: Singular Perturbation, Chattering, Impulsive Singularity, Tikhonov Theorem, Wilson-Cowan Model, Medusa Bifurcation

ÖZ

SÜREKSİZ DİNAMİKLERDE TINLAMA VE TEKİL PERTÜRBASYON

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Bu tezin amacı, süreksiz dinamik sistemlerde tınlama ve tekillik olgusunu ele almaktır. Çalışma tekil impalsif diferansiyel denklemlerin yeni modellerini açıklamıştır, şöyle ki; sistemde diferansiyel denklem pertürbe edilirken, aynı zamanda impalsif fonksiyonda tekildir. Tikhonov Teoremi, impalsif diferansiyel denklemler için genişletilmiştir. İlginçtir ki, burada açıklanan bazı modellerde, problemin çözümü, parametre sıfıra giderken diferansiyel denklemin birden fazla sabit noktasına yaklaşmaktadır.

Wilson-Cowan nöron modeli, membran zaman sabitinin hem singülarite hem de çatallanma parametresi olarak kabul edildiği impuls fonksiyonu ile incelenmiştir. Olgunun yeni bir analiz tekniği önerilmektedir. Bu, modelin çözümlerinin varlığını göz önüne almaya ve son sinirsel davranışta çatallanmanın sayısal simülasyonlar yoluyla gözlemlenmesine izin verir. Çatallanmalar, modeldeki dürtüler ve tekilliklerden kaynaklanır ve fazör alanında medusalar ve halkalar gibi yeni tanıtılan kümelerden oluşan çekicilerin yapısını ilgilendirir.

Ayrıca, tekil etki zamanları tanıtılmış ve bunlar tınlama çözümleri ile ilgili problemler için kullanılmıştır. Tekil dürtü zamanları, modellerde ortaya çıkan tınlama, zıplayan bir top, ters çevrilmiş bir sarkaç ve hidrolik bir tahliye vanası gibi örneklerde tekillik bakış açısı ile analiz edilebilmesinin avantajını verir. Tınlamanın varlığı, yalnızca darbe modellerinin sağ tarafının incelenmesi ile gösterilmiştir. Her zaman tınlamaya

yol açan başlangıç veri kümeleri için kriterler oluşturulmuştur.

Anahtar Kelimeler: Tekil Pertürbasyon, Tınlama, İmpalsif Tekillik, Tikhonov Teoremi, Wilson-Cowan Modeli, Medusa Çatallanması

To my family

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CHAPTER 1

INTRODUCTION AND PRELIMINARIES

Perturbation theory is a key theme in mathematics and its applications to the natural and engineering sciences [13, 31, 66, 75, 86, 115]. It has enormous numbers of mathematical methods used to obtain approximate solution to problems that have no analytical solution. The techniques work by reducing a difficult problem to a simple problem that has an analytical solution. The simplified problem is then "perturbed" to make the conditions that the perturbed solution really fulfills closer to the real problem. These problems have a small positive parameter. This parameter influences the problem in a manner that the solution changes quickly in some region of the problem domain and gradually in other parts.

Now, one can ask: What is a singular perturbation? Singular perturbation theory deals with the investigation of issues including a parameter for which the solutions of the issue at a limiting value of the parameter are distinctive in character from the limit of the solutions to the general issue; to be specific, the limit is singular. In contrast to singular perturbation, for regular perturbation problems, the solutions of the general problem approach to the solutions of the limit problem as the parameter tends to the limit value. The traditional view of the singular problem is that a differential equation (plus other conditions) having a small parameter that is multiplying the highest derivatives. The contemporary view is that it is called a singular problem if, in an appropriate norm, the difference of the perturbed problem and degenerated one does not tend to zero when the small parameter goes to zero.

The history of singular perturbation dates back to 1904. Ludwig Prandtl, a professor of mechanics at the Technical University of Hanover, made a presentation to the Third

International Congress of Mathematicians in Heidelberg, Germany, entitled “Über Flüssigkeitsbewegung bei sehr kleiner Reibung” (On Fluid Motion with Small Friction). His seven page report was published in the proceedings one year later. In the 1930s, a few early papers concerning singularly perturbed boundary value problems mathematically showed up. However, they did not have a long-term effect. For example, Y-W. Chen (1935) was a graduate student in mathematics at Göttingen, who had a thesis topic on an ordinary differential equation model with boundary layer behavior. After he completed his thesis, Chen, however, never did further study on this subject. In the late 1930’s the Japanese mathematician M. Nagumo studied a singularly perturbed initial value problem $\varepsilon \ddot{x} + f(t, x, \dot{x}, \varepsilon) = 0$. A particular problem of this frame was proposed to Nagumo by a chemist.

Friedrichs and Wasow (1946) were the first persons who use the term "singular perturbations" in a paper. Wasow continued to study on a variety of asymptotic problems, while Friedrichs was most influential in advancing singular perturbations through his lectures.

The singular perturbations were studied around the world. The efficient investigation of singular perturbation problems started in the Soviet Union in the late of 1940s. The basic studies were done by A. N. Tikhonov in 1948 and 1952 [116, 117]. Tikhonov and his students at Moscow State University, particularly Adelaida Vasil’eva, developed comprehensive expansion strategies for broad types of differential equations. In her PhD thesis, Vasil’eva investigated the derivatives of the solution of problem with respect to small parameter which ultimately led to the construction of an asymptotic expansion of the solution. The singular perturbation approach of Tikhonov and Vasil’eva was first applied to optimal control and regulator design by Kokotovic and Sannuti [98, 99] and, more specifically, to flight-path optimization by Kelley and Edelbaum [46].

The basic Tikhonov results were independently obtained later by Norman Levinson, from MIT [73]. Levinson’s approach was more geometric, aimed at describing relaxation oscillations, as occur for the van der Pol equation.

After the 1960s, singular perturbation theory has expanded. The topic is now a part of studies in applied mathematics and in many fields of engineering. The detailed history of singular perturbation can be found in [89, 90].

This perturbation method has vast applications in many fields such as: chemical kinetics and combustion, mathematical biology, fluid dynamics, condensed matter physics and control theory [16, 17, 18, 21, 24, 25, 26, 27, 28, 29, 37, 38, 40, 41, 42, 47, 50, 53, 54, 55, 57, 61, 67, 70, 71, 81, 82, 83, 84, 85, 91, 92, 93, 95, 96, 100, 101, 102, 103, 104, 107, 109, 111, 113, 114, 123]. For a general theory, we refer the reader to [63, 78, 90, 119, 121] and the references therein for nice examples and applications.

In 1990s, the regular perturbation theory for the differential equations with impulses were investigated by Akhmet. Results were published in three papers [9, 10, 11] and they are widely described in the book [2]. In this thesis, we will study the singular perturbation theory for the impulsive differential equations.

1.1 A Short Description of Singular Perturbation

In this thesis, real numbers, natural numbers, integers and Euclidean norm are denoted by $\mathbb{R}$, $\mathbb{N}$, $\mathbb{Z}$ and $\|.\|$, respectively.

1.1.1 Motivation

We start with a basic example in order to understand the singularly perturbed problem. Consider the initial value problem

$$\mu\dot{x} + 2x = 2, \quad \text{with} \quad x(0,\mu) = x_0, \tag{1.1}$$

where μ is a small real number. The solution of this problem is

$$x(t,\mu) = 1 + (x_0 - 1)e^{-\frac{2t}{\mu}}.$$

For $\mu < 0$ and $x_0 \neq 1$, the solution will become unbounded as $\mu \to 0$ for any $t > 0$. However, for $\mu > 0$, the solution tends to 1 for any $t > 0$ as $\mu \to 0$. In Figure 1.1, the solution $x(t,\mu)$ is obtained for initial condition $x_0 = 0$ and for some values of parameter μ.

As the parameter μ tends to zero from the right, we have the limit

$$x(t,\mu) \to \begin{cases} x_0 & \text{for } t = 0, \\ 1 & \text{for } t > 0. \end{cases}$$

If $x_0 \neq 1$, then the limit is discontinuous at $t = 0$. So, convergence is not uniform at $t = 0$.

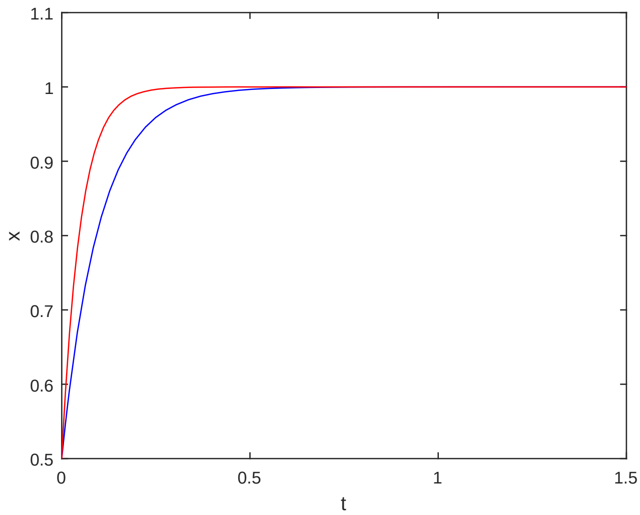

Figure 1.1: The solution of system (1.1) with initial value $x_0 = 0.5$, for blue=0.2, red=0.1. It is obviously seen that as the parameter μ decreases, the solution $x(t,\mu)$ is getting closer to 1. There is a layer at the neighborhood of $t = 0$ on which the convergence fails. However, for fix $t > 0$, the convergence is uniform.

1.1.2 Definition of a Singular Problem

Let us describe generally the definition of singularity. Consider

- Problem $P(\mu)$: the problem with small parameter μ,
- Problem $P(0)$: the reduced (degenerated) problem.

The problem $P(0)$ is a simplified model of $P(\mu)$ taking $\mu = 0$. Denote the solution of $P(0)$ by $z(t,0)$ and the solution of $P(\mu)$ by $z(t,\mu)$.

Definition 1.1.1 *[119] $P(\mu)$ is called regularly perturbed problem in a domain D if*

$$\sup_D \|z(t,\mu) - z(t,0)\| \to 0 \text{ when } \mu \to 0.$$

Otherwise, it is called singularly perturbed problem.

It follows from the definition that for a regularly perturbed problem the solution $z(t,0)$ of $P(0)$ will be close to the solution $z(t,\mu)$ of $P(\mu)$ in the entire domain D for all sufficiently small μ. However, if the problem $P(\mu)$ is singularly perturbed, then $z(t,\mu)$ will not be close to $z(t,0)$ for all small μ at least in some part of domain D.

1.2 A Brief Description of Differential Equations with Impulses

Nature offers numerous cases of frameworks where states of the systems can be changed suddenly. At this point, mathematicians propose discontinuous differential equations to describe the real life problems adequately. There are different types of them. In this thesis, we use the differential equations with impulse effects or with the other name: Impulsive differential equations. Consider, for example, biological structures involving thresholds such as drug resistance models in medicine. These models exhibit impulse effects depending on the dosage of the drug [108]. Moreover, examples of such systems arise in mechanics, e.g., the behavior of a bouncing bead, the behavior of clock mechanisms [4, 22, 23, 62, 88]. Consider a free falling bead in the uniform gravity force field with a fixed horizontal base. After colliding with the base the bead bounces back with the velocity whose norm is equal to the norm of the pre-impact velocity multiplied by r, where r is the restitution coefficient, $0 < r < 1$. Then, after some time interval the bead will fall on the base again and the norm of its velocity will be equal to the norm of bouncing velocity in the previous collision multiplied by r. The process continues with these collisions. It can be seen that after the each collision the velocity of the bead changes abruptly. This example shows how the bouncing bead can be examined in this theory. The theory of impulsive differential equations are well developed and also it has many applications in: neuroscience, physics, mechanics,etc. The detailed theory and applications can be found in [2, 3, 6, 7, 97].

Now, let us give a general description of the impulsive differential equations. There are two types of abrupt changes in the state of a mathematical system, namely, the impulse effects at prescribed moments and the impulse effects at non-prescribed moments. The following model represents the first one.

$$\begin{aligned} \dot{x} &= f(t,x), \\ \Delta x|_{t=\theta_i} &= I_i(x), \end{aligned} \tag{1.2}$$

where $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, $\{\theta_i\}$ is a sequence of real numbers with the set of indexes $\mathbb{A}$ which is either finite or infinite, $I : \mathbb{A} \times \mathbb{R}^n \to \mathbb{R}^n$, $\Delta x|_{t=\theta_i} := x(\theta_i+) - x(\theta_i)$, and $x(\theta_i+) = \lim_{t \to \theta_i^+} x(t)$. Let us give details of the system (1.2). When $t \neq \theta_i,$ a phase portrait of system (1.2) is characterized by differential equation counterpart of (1.2); it has jump at $t = \theta_i$ and satisfies difference equation $x(\theta_i+) - x(\theta_i) = I_i(x(\theta_i)).$

The discontinuity moments occur at non-prescribed time in the second one. This makes the theoretical analysis difficult. Fortunately, Akhmet [2] offers a useful tool- $B-$equivalence method- which gives the advantages to transform the models with non-prescribed moments to the ones with prescribed moments. Let us consider systems of the form

$$\begin{aligned} \dot{x} &= f(t,x), \\ \Delta x|_{t=\eta_i(x)} &= I_i(x), \end{aligned} \tag{1.3}$$

where $\eta_i(x)$ are surfaces of discontinuity. One can easily see that impulse time in (1.3) by its own nature depend on solutions. Consequently, jump moments can be very different.

In this thesis, we will address systems with both impulse action at prescribed and non-prescribed moments.

1.3 The Organization of the Thesis

The thesis is organized as follows:

Chapter 1. Introduction and preliminaries: We give some background of the general theory of singular perturbation and impulsive differential equations.

Chapter 2. Singularly perturbed differential equations with singular impulse functions

- **Section 1. Tikhonov theorem for differential equations with singular impulses:** The most general form of Tikhonov theorem for the impulsive systems is obtained. The singularity in this chapter arises from both differential equations and the impulsive functions.

- **Section 2. A differential equation with singular impulses and multi-stable roots:** Singularly perturbed differential equations with both small parameter in front of the derivative and impulse function as in previous section is discussed with a new approach. This approach is as follows: the solution approaches more than one root of the differential equation as the parameter decreases to zero.

Chapter 3. Bifurcation analysis of Wilson-Cowan model with singular impulses: The theory developed in Chapter 2 is complemented with numerical simulations in Wilson-Cowan model. In the coupled Wilson-Cowan models, a new attractor composed of a new concept, namely medusa, is observed.

Chapter 4. Analysis of impact chattering: The concept of impact chattering is introduced. Definition of chattering is given. Moreover, sufficient conditions are provided for the existence of the impact chattering.

Chapter 5. Chattering as a singular problem: The singular impulse moments are defined. Based on this definition, new singularly perturbed models are proposed. Also, the systems with chattering solutions are discussed in the view of singular problem.

Conclusion and future works: Main contributions of the research and future works are listed.

CHAPTER 2

SINGULARLY PERTURBED DIFFERENTIAL EQUATIONS WITH SINGULAR IMPULSE FUNCTIONS

In this chapter, we state and prove the impulsive analogous of Tikhonov theorem with singular impulsive function. Moreover, in the second section, the solution of the new constructed model will approach more than one stable equilibrium when the parameter decreases.

2.1 Tikhonov Theorem for Differential Equations with Singular Impulses

2.1.1 Introduction

The singularly perturbed differential equations arise in various fields of chemical kinetics [102], mathematical biology [57, 91], fluid dynamics [37] and in a variety models for control theory [52, 67]. These problems depend on a small positive parameter such that the solution varies rapidly in some regions and varies slowly in other regions.

The contribution of our work relates to a new Tikhonov theorem for singularly perturbed impulsive systems. This theorem expresses the limiting behavior of solutions of the singularly perturbed system. It is a powerful instrument for analysis of singular perturbation problems. It has been studied for many types of differential equations; partial differential equations [64], singularly perturbed differential inclusions [120], functional-differential inclusions [45], discontinuous differential equations [33, 34, 35, 105, 106].

Impulse effects exist in a wide diversity of evolutionary processes that exhibit abrupt changes in their states [2, 3, 6]. In many systems, in addition to singular perturbation, there are also impulse effects [33, 34, 35, 105, 106]. Chen et al. [35] derived a sufficient condition that guarantees robust exponential stability for sufficiently small singular perturbation parameter by applying the Lyapunov function method and using a two-time scale comparison principle. In [105, 106], authors proposed Lyapunov function method to set up the exponential stability criteria for singularly perturbed impulsive systems. This method can be efficiently used to overcome the impulsive perturbation such that the stability of the original system can be ensured. In [33], Lyapunov function method was further extended to study the exponential stability of singularly perturbed stochastic time-delay systems with impulse effect. The results in [33, 105, 106] only guarantee the systems under consideration to be exponentially stable for sufficiently small positive parameter.

Various types of singular perturbation problems are discussed in many books [19, 90, 118, 119, 121]. Consider the following model of singularly perturbed differential equation

$$\begin{aligned} \mu \dot{z} &= f(z, y, t), \\ \dot{y} &= g(z, y, t), \end{aligned} \tag{2.1}$$

where μ is a small positive real number. In the literature, the results based on this system are known as Tikhonov Theorem [90, 117, 121]. Bainov and Covachev [19] first extended the impulsive analogous of Tikhonov Theorem concerning system (2.1) in the form of

$$\begin{aligned} \mu \dot{z} &= f(z, y, t), \quad \Delta z|_{t=t_i} = I_i(z(t_i)), \\ \dot{y} &= g(z, y, t), \quad \Delta y|_{t=t_i} = J_i(y(t_i)), \end{aligned} \tag{2.2}$$

where $i = 1, 2, ..., p$ and $0 < t_1 < t_2 < ... < t_p < T$. However, only the differential equation in their problem is singularly perturbed.

In this study, we consider differential equations where impulses are also singularly perturbed which are different from [19]. The following system is our focus of discussion

$$\begin{aligned} \mu \frac{dz}{dt} &= F(z, y, t), \quad \mu \Delta z|_{t=\theta_i} = I_i(z, y, \mu) \\ \frac{dy}{dt} &= f(z, y, t), \quad \Delta y|_{t=\eta_j} = J_j(z, y), \end{aligned} \tag{2.3}$$

where z, F and I_i are m-dimensional vector valued functions, y, f and J_j are n-dimensional vector valued functions, $0 < \theta_1 < \theta_2 < \cdots < \theta_p < T, \theta_i, i = 1, 2, \ldots, p,$ and $\eta_j, j = 1, 2, \ldots, k,$ are distinct discontinuity moments in $(0, T)$.

The main novelty of this section is the extension of Tikhonov Theorem such that system (2.3) has the small parameter in impulse function, the discontinuity moments are different for each dependent variables. The singularity in the impulsive part of the system can be treated through perturbation theory methods.

2.1.2 A Particular Case of the Main Theorem

Before carrying out our main investigation, let us consider a particular case of the main theorem. This case is useful by its geometric clarity. We introduce the following problem

$$\begin{aligned} \mu \frac{dz}{dt} &= F(z), \\ \mu \Delta z|_{t=\theta_i} &= I_i(z, \mu), \end{aligned} \tag{2.4}$$

with $z(0, \mu) = z_0$, where $z \in \mathbb{R}^m$, $t \in [0, T]$, $F(z)$ is a continuously differentiable function on D and $I_i(z, \mu), i = 1, 2, \ldots, p,$ is a continuous function for $(z, \mu) \in D \times [0, 1]$, D is the domain $D = \{0 \leq t \leq T, \|z\| < d\}$, the impulse moments $\theta_i, i = 1, 2, \ldots, p,$ are defined above.

The parameter in the impulsive equation makes it possible that $\frac{I_i(z,\mu)}{\mu}, i = 1, 2, \ldots, p,$ blow up at impulse moments as $\mu \to 0$. This is why, a deep analysis and convenient conditions for the limiting processes with $\mu \to 0$ have to be provided.

2.1.2.1 Singularity with a Single Layer

Let us take $\mu = 0$ in (2.4). Then, one has $0 = F(z) = I_i(z, 0), i = 1, 2, \ldots, p.$ It is the degenerate system since its order is less than the order of (2.4). Consider an isolated real root $z = \varphi$ of $F(z) = 0$ and $I_i(z, 0) = 0$.

Now, introduce a new variable $\tau = \frac{t}{\mu}$ and $x = z - \varphi$ for the first equation in (2.4) to obtain

$$\frac{dx}{d\tau} = F(x + \varphi). \tag{2.5}$$

The following condition is required.

(C1) Suppose that there is a positive definite function $V(x)$ such that $V(0) = 0$ and whose derivative with respect to τ along system (2.5) is negative definite.

This condition implies that the zero solution of (2.5) is uniformly asymptotically stable. Moreover, for the impulsive function we need the following condition.

(C2)

$$\lim_{(z,\mu)\to(\varphi,0)} \frac{I_i(z,\mu)}{\mu} = 0, i = 1, 2, \ldots, p,$$

which prevents impulsive function to blow up as the parameter μ decays to zero. This condition is the counterpart of (C1) considering impulsive function. Condition (C2) plays a similar role to the condition (C1) in the proof of the next theorem.

Theorem 2.1.1 *Suppose that conditions (C1) and (C2) are true. If the initial value z_0 is located in the domain of attraction of the root φ, then solution $z(t, \mu)$ of* (2.4) *with $z(0, \mu) = z_0$ exists on $[0, T]$ and it satisfies the limit*

$$\lim_{\mu\to 0} z(t, \mu) = \varphi \quad \textit{for} \quad 0 < t \leq T. \tag{2.6}$$

Proof. In this proof, we will follow the idea of the proof of [118, Theorem 7.3]. Consider the first interval $[0, \theta_1]$. Let $z_0 \in D$ such that it is in the domain of attraction of φ. Then, for fix $\mu > 0$, the differential equation

$$\frac{dz}{dt} = \frac{F(z)}{\mu} \tag{2.7}$$

with initial value $z(0, \mu) = z_0$ has a unique solution $z(t, \mu)$ since $F(z) \in C^1(D)$. Then rescale the time as $t = \tau\mu$ and substitute $x = z - \varphi$ in (2.7) to get

$$\frac{dx}{d\tau} = F(x + \varphi). \tag{2.8}$$

$x = 0$ is an equilibrium of this differential equation. By condition (C1), equation (2.8) has a positive definite function $V(x)$ whose derivative with respect to τ is negative definite and $V(0) = 0$. Hence, by the Lyapunov's second method, one can say that the zero solution of (2.8) is uniformly asymptotically stable as $\tau \to \infty$. Therefore, $\forall\, \varepsilon > 0$ and for sufficiently small μ on $0 < t \le \theta_1$ one has $\|z(t,\mu) - \varphi\| < \varepsilon$, i.e,

$$\lim_{\mu \to 0} z(t,\mu) = \varphi \quad \text{for} \quad 0 < t \le \theta_1.$$

Now, consider the next interval $(\theta_1, \theta_2]$. From condition (C2), we have

$$\lim_{\mu \to 0} z(\theta_1+,\mu) = \lim_{\mu \to 0} \left\{ z(\theta_1,\mu) + \frac{I(z(\theta_1,\mu),\mu)}{\mu} \right\} = \varphi.$$

It means that $z(\theta_1+,\mu)$ is in the domain of attraction of the root φ. Repeating the same processes as for the previous interval, one obtains

$$\lim_{\mu \to 0} z(t,\mu) = \varphi \quad \text{for} \quad \theta_1 < t \le \theta_2.$$

Similarly, one can show that $z(t,\mu) \to \varphi$ as $\mu \to 0$ for $t \in (\theta_i, \theta_{i+1}], i = 2, \ldots, p-1$ and $t \in (\theta_p, T]$. As a result, limit (2.6) is true and the theorem is proved.

The convergence is not uniform at $t = 0$ since $z(0,\mu) = z_0$ and $z_0 \ne \varphi$ for all $\mu > 0$. We can say that the region of nonuniform convergence is $O(\mu)$ thick, since for $t > 0$, $\|z(t,\mu) - \varphi\|$ can be made arbitrarily close to zero by choosing μ small enough. The interval of nonuniform convergence is called an initial layer. This theorem implies that there is a single initial layer.

Example. Consider the system

$$\begin{aligned} \mu\dot{x}_1 &= -x_1 + x_2, \quad \mu\Delta x_1|_{t=\theta_i} = -2\mu x_1, \\ \mu\dot{x}_2 &= -x_1 - x_2, \quad \mu\Delta x_2|_{t=\theta_i} = \mu \sin(x_2^{1/3} + \mu), \end{aligned} \tag{2.9}$$

with initial value $(x_1(0,\mu), x_2(0,\mu))$, where $\theta_i = i/3, i = 1, 2, \ldots, 10$. Let us take $\mu = 0$ in this system. Then

$$\begin{aligned} 0 &= -x_1 + x_2, \quad 0 = 0, \\ 0 &= -x_1 - x_2, \quad 0 = 0. \end{aligned}$$

and so $(x_1, x_2) = (0, 0)$ is the root. Substitute $\tau = \frac{t}{\mu}$ into the differential equations part of (2.9) to obtain

$$\begin{aligned} \frac{dx_1}{d\tau} &= -x_1 + x_2, \\ \frac{dx_2}{d\tau} &= -x_1 - x_2, . \end{aligned} \tag{2.10}$$

We take the positive definite function $V(x_1, x_2) = x_1^2 + x_2^2$. Then

$$\frac{dV}{d\tau} = 2x_1(-x_1 + x_2) + 2x_2(-x_1 - x_2) = -2(x_1^2 + x_2^2) = -2V.$$

Hence, $V(x_1, x_2)$ has a negative definite derivative with respect to τ along (2.10). Now, let us check the condition (C2). Denote $x = (x_1, x_2)$. Then

$$\lim_{(x,\mu)\to(0,0)} \frac{I(x,\mu)}{\mu} = 0$$

since $\lim_{(x_1,\mu)\to(0,0)} -2x_1 = 0$ and $\lim_{(x_2,\mu)\to(0,0)} \sin(x_2^{1/3} + \mu) = 0$. Therefore, by Theorem 2.1.1, if the initial value $(x_1(0,\mu), x_2(0,\mu))$ of (2.9) is in the domain of attraction of the root $(0,0)$, then solution $(x_1(t,\mu), x_2(t,\mu))$ of (2.9) tends to $(0,0)$ as $\mu \to 0$ for $0 < t \leq T$. It is clearly seen in Figure 2.1 that the solution of system (2.9) with initial $(1.5, -1.5)$ tends to $(0,0)$ as $\mu \to 0$.

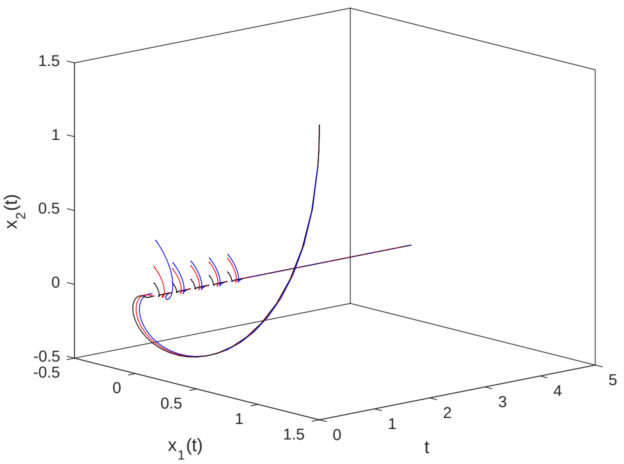

Figure 2.1: Blue, red and black lines represents the solution of system (2.9) with initial value $(1.5, -1.5)$ for $\mu = 0.07$, $\mu = 0.05$ and $\mu = 0.03$, respectively.

2.1.2.2 Singularity with Multi-Layers

In the previous subsection, it is shown that there is a single initial layer. Using an impulse function, the convergence can be nonuniform near several points, that is to say that multi-layers emerge. These layers will occur on the neighborhoods of $t = 0$ and $t = \theta_i, i = 1, 2, \ldots, p$.

Again, we consider system (2.4) with the same properties. In addition, we need the following condition

(C3)

$$\lim_{(z,\mu)\to(\varphi,0)} \frac{I_i(z,\mu)}{\mu} = I_i^0 \neq 0, i = 1,2,\ldots,p,$$

and assume that $\varphi + I_i^0$ is in the domain of attraction of the root φ.

By the virtue of this condition, after the each impulse moment, the difference $\|z(\theta_i+,\mu)-\varphi\|$ does not go to zero as $\mu \to 0$. Hence, convergence is not uniform.

Theorem 2.1.2 *Suppose that conditions (C1) and (C3) hold. If the initial value z_0 is located in the domain of attraction of the root φ, then the solution $z(t,\mu)$ of* (2.4) *with $z(0,\mu) = z_0$ exists on $[0,T]$ and the limit*

$$\lim_{\mu\to 0} z(t,\mu) = \varphi \tag{2.11}$$

is true for $t \in \bigcup_{i=0}^{p-1} (\theta_i, \theta_{i+1}] \cup (\theta_p, T]$, where $\theta_0 = 0$.

Proof. Proof is similar to the proof of Theorem 2.1.1 with the exception that singularity with multi-layers appears near $t = 0$ and $t = \theta_i, i = 1,2,\ldots,p$.

By condition (C3), after the each discontinuity moment θ_i, the solution $z(t,\mu)$ is not close to the root φ. In other words, the difference $\|z(\theta_i+,\mu)-\varphi\|$ cannot be arbitrarily small as $\mu \to 0$. Hence, one can obtain multi-layers up to the number $p+1$.

Let us illustrate the theorem with the following example.

Example. Consider the following impulsive differential equation with small parameter:

$$\begin{aligned} \mu\dot{z} &= -z - z^3, \\ \mu\Delta z|_{t=\theta_i} &= \mu z^{1/3} + \sin\mu + 0.1\mu, \end{aligned} \tag{2.12}$$

where $\theta_i = i/3, i = 1,2,...,10$. Let us take $\mu = 0$ in this system. Then we have the algebraic equation $0 = -z - z^3$. It has solution $z = 0$. Now, introduce $t = \tau\mu$ in the first equation of (2.12) to obtain

$$\frac{dz}{d\tau} = -z - z^3 \tag{2.13}$$

Using the Lyapunov function $V(z) = z^2$, it can be shown that $z = 0$ is a uniformly asymptotically stable solution of (2.13). Moreover, condition (C3) is satisfied since

$$\lim_{(z,\mu)\to(0,0)} \frac{\mu z^{1/3} + \sin\mu + \mu 0.1}{\mu} = 1.1.$$

Choose the initial value $z(0,\mu) = 0.6$. Then the solution $z(t,\mu)$ of system (2.12) with this initial value has multi-layers at $t = 0$ and $t = \theta_i+, i = 1, 2, \ldots, 10$. Clearly, in Figure 2.2, it can be seen that multi-layers occur.

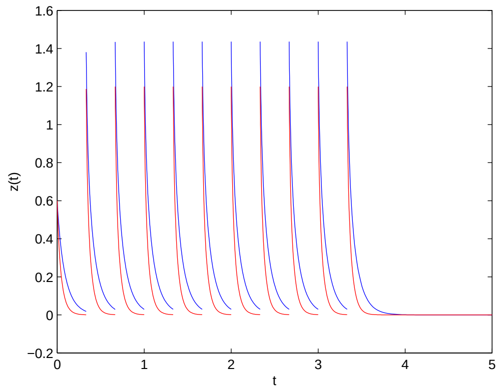

Figure 2.2: Solution $z(t,\mu)$ of system (2.12) with initial value $z(0,\mu) = 0.6$ for different values of parameter μ. Blue and red line represent for $\mu = 0.1, \mu = 0.05$, respectively. It is seen that at $t = 0$ and at each $\theta_i, i = 1, 2, ..., 10$, the convergence is nonuniform, i.e., multi-layers exist.

Let us generalize the Theorem 2.1.2. Consider the following impulsive system

$$\begin{aligned} \mu\frac{dz}{dt} &= F(z), \\ \mu\Delta z|_{t=\theta_i} &= I_i(z,\mu), \\ \mu\Delta z|_{t=\tau_j^i} &= J_j(z,\mu), \end{aligned} \tag{2.14}$$

where the impulse moments $\tau_j^i, j = 1, 2, \ldots, p_j$ are such that $\theta_i < \tau_1^i < \tau_2^i < \cdots < \tau_{p_j}^i < \theta_{i+1}$, $i = 1, 2, \ldots, p-1$ and $\theta_p < \tau_1^p < \tau_2^p < \cdots < \tau_{p_j}^p < T$. Assume $J_j(\varphi, 0) = 0, j = 1, 2, \ldots, p_j$, and the following condition holds for (2.14)

(C4)

$$\lim_{(z,\mu)\to(\varphi,0)} \frac{J_j(z,\mu)}{\mu} = 0, j = 1, 2, \ldots, p_j.$$

Now, we can assert the following theorem.

Theorem 2.1.3 *Suppose that conditions (C1), (C3) and (C4) hold. If the initial value* z_0 *is located in the domain of attraction of the root* φ*, then the solution* $z(t,\mu)$ *of* (2.14) *with* $z(0,\mu) = z_0$ *exists on* $[0,T]$ *and the limit*

$$\lim_{\mu \to 0} z(t,\mu) = \varphi \tag{2.15}$$

is true for $t \in \bigcup_{i=0}^{p-1} (\theta_i, \theta_{i+1}] \cup (\theta_p, T]$*, where* $\theta_0 = 0$.

2.1.3 Main Result

Now, we turn to main problem (2.3).

2.1.3.1 Singularity with a Single Layer

Define the initial conditions (for simplicity, we set $t_0 = 0$.)

$$z(0,\mu) = z^0, y(0,\mu) = y^0, \tag{2.16}$$

where z^0 and y^0 will be assumed to be independent of μ, and let us investigate the solution $z(t,\mu)$, $y(t,\mu)$ of (2.3) and (2.16) on segment $0 \le t \le T$.

In system (2.3), take $\mu = 0$, then we obtain

$$\begin{aligned} &0 = F(\bar{z},\bar{y},t), \quad 0 = I_i(\bar{z},\bar{y},0), i = 1,2,\ldots,p, \\ &\frac{d\bar{y}}{dt} = f(\bar{z},\bar{y},t), \quad \Delta\bar{y}|_{t=\eta_j} = J_j(\bar{z},\bar{y}), j = 1,2,\ldots,k, \end{aligned} \tag{2.17}$$

which we call as degenerate system due to the fact that its order is less than the order of (2.3). Therefore, for the system (2.17) the number of initial conditions must be set less than the number of initial conditions for (2.3). We naturally insert the initial condition for y, i.e., put

$$\bar{y}(0) = y^0, \tag{2.18}$$

and drop the initial condition for z. Now, the question is that whether there will be a solution $z(t,\mu)$ and $y(t,\mu)$ of problem (2.3), (2.16) for small μ which is close to the solution $\bar{z}(t), \bar{y}(t)$ of the degenerate problem (2.17), (2.18).

To solve system (2.17), it is necessary to find $\bar{z}$ from $0 = F(\bar{z}, \bar{y}, t)$ and $0 = I_i(\bar{z}, \bar{y}, 0), i = 1, 2, \ldots, p$. Then choose one of the root $\bar{z} = \varphi(\bar{y}, t)$ such that $0 = F(\varphi(\bar{y}, t), \bar{y}, t)$ and $0 = I_i(\varphi(\bar{y}, \theta_i), \bar{y}, 0)$, and substitute into (2.17) with initial value (2.18) to obtain

$$\begin{aligned}
&\frac{d\bar{y}}{dt} = f(\varphi(\bar{y}, t), \bar{y}, t), \quad \Delta\bar{y}|_{t=\eta_j} = J_j(\varphi(\bar{y}, t), \bar{y}), \\
&\bar{y}(0) = y^0.
\end{aligned} \tag{2.19}$$

We need the following conditions in this section:

A1. The functions $F(z, y, t)$, $f(y, z, t)$, and $J_j(z, y)$, $j = 1, 2, \ldots, k$, are continuous in some domain $H = \{(y, t) \in \bar{N} = \{0 \leq t \leq T, \|y\| \leq c\}, \|z\| < d\}$, $I_i(z, y, \mu), i = 1, 2, \ldots, p$, is continuous in $H \times [0, 1]$ and they are Lipschitz continuous with respect to z and y.

A2. Algebraic equations $0 = F(z, y, t)$ and $0 = I_i(z, y, 0)$ have a root $z = \varphi(y, t)$ such that $F(\varphi(y, t), y, t) = 0$ and $I_i(\varphi(y(\theta_i), \theta_i), y(\theta_i), 0) = 0, i = 1, 2, \ldots, p$, in domain $\bar{N}$ such that:

1. $\varphi(y, t)$ is a piecewise continuous function in $\bar{N}$,
2. $(\varphi(y, t), y, t) \in$ H,
3. The root $\varphi(y, t)$ is isolated in $\bar{N}$, i.e., $\exists\, \epsilon > 0$: $F(z, y, t) \neq 0$ and/or $I_i(z, y, \mu) \neq 0, i = 1, 2, \ldots, p$, for $0 < \|z - \varphi(y, t)\| < \epsilon$, $(y, t) \in \bar{N}$.

A3. 1. System (2.19) has a unique solution $\bar{y}(t)$ on $0 \leq t \leq T$, and $(\bar{y}(t), t) \in \bar{N}$ for $0 \leq t \leq T$. Moreover, $f(\varphi(y, t), y, t)$ and $J_j(\varphi(y, t), y)$ are Lipschitz with respect to $y \in \bar{N}$.

2. $\varphi(\bar{y}(\eta_j+), \eta_j+) = \varphi(\bar{y}(\eta_j), \eta_j), j = 1, 2, \ldots, k$.

Now, setting $x = z - \varphi$ and $t = \tau\mu$, we introduce the system

$$\frac{dx}{d\tau} = F(x + \varphi(y, t), y, t), \quad \tau \geq 0, \tag{2.20}$$

where y and t are considered as parameters, $x = 0$ is an isolated stationary point of (2.20) for $(y, t) \in \bar{D}$.

A4. Suppose that there is a positive definite function $V(x, y, \tau)$ whose derivative with respect to τ along the system (2.20) is negative definite in the region H.

Consider the adjoint system

$$\frac{d\tilde{z}}{d\tau} = F(\tilde{z}, y^0, 0), \quad \tau \geq 0, \tag{2.21}$$

with initial condition

$$\tilde{z}(0) = z^0. \tag{2.22}$$

Since z^0 maybe, in general, far from stationary point $\varphi(y^0, 0)$, then the solution $\tilde{z}(\tau)$ of equations (2.21) and (2.22) need not tend to $\varphi(y^0, 0)$ as $\tau \to \infty$. Assume that

A5. the solution $\tilde{z}(\tau)$ of equations (2.21) and (2.22) satisfies the conditions

1. $\tilde{z}(\tau) \to \varphi(y^0, 0)$ as $\tau \to \infty$,
2. $(\tilde{z}(\tau), y^0, 0) \in H$ for $\tau \geq 0$.

In this case, z^0 is said to belong to the basin of attraction of the stationary point $\tilde{z} = \varphi(y^0, 0)$. By virtue of the asymptotic stability of this point all points near it will belong to its basin of attraction.

A6. Assume also

$$\lim_{(z,y,\mu)\to(\varphi(\bar{y}(\theta_i),\theta_i),\bar{y}(\theta_i),0)} \frac{I_i(z, y, \mu)}{\mu} = 0, i = 1, 2, \ldots, p.$$

Now, we state and prove the modified Tikhonov Theorem.

Theorem 2.1.4 *Suppose that conditions $A1 - A6$ hold. Then, for sufficiently small μ, solutions $z(t, \mu)$ and $y(t, \mu)$ of problem* (2.3) *with initial conditions* (2.16) *exist on $0 \leq t \leq T$, are unique, and satisfy*

$$\lim_{\mu \to 0} y(t, \mu) = \bar{y}(t) \quad \text{for} \quad 0 \leq t \leq T \tag{2.23}$$

and

$$\lim_{\mu \to 0} z(t, \mu) = \bar{z}(t) = \varphi(\bar{y}(t), t) \quad \text{for} \quad 0 < t \leq T. \tag{2.24}$$

Before proving this theorem, we will consider the following auxiliary system:

$$\begin{aligned} \mu\frac{dz}{dt} &= F(z, y, t), \\ \frac{dy}{dt} &= f(z, y, t), \quad \Delta y|_{t=\eta_j} = J_j(z, y), \end{aligned} \tag{2.25}$$

where this system has the same properties as (2.3).

In system (2.25), take $\mu = 0$, then we obtain

$$\begin{aligned} &0 = F(\bar{z}, \bar{y}, t), \\ &\frac{d\bar{y}}{dt} = f(\bar{z}, \bar{y}, t), \quad \Delta\bar{y}|_{t=\eta_j} = J_j(\bar{z}, \bar{y}), \end{aligned} \tag{2.26}$$

which is the degenerate system of (2.25).

To solve system (2.26), it is necessary to find $\bar{z}$ from $0 = F(\bar{z}, \bar{y}, t)$. Then choose one of the root $\bar{z} = \varphi(\bar{y}, t)$ and substitute into (2.26) with initial value (2.18) to obtain

$$\begin{aligned} &\frac{d\bar{y}}{dt} = f(\varphi(\bar{y}, t), \bar{y}, t), \quad \Delta\bar{y}|_{t=\eta_j} = J_j(\varphi(\bar{y}, t), \bar{y}), \\ &\bar{y}(0) = y^0. \end{aligned} \tag{2.27}$$

Now, introduce the adjoint system

$$\frac{d\tilde{z}}{d\tau} = F(\tilde{z}, y, t) \quad \tau \geq 0, \tag{2.28}$$

where y and t are considered as parameters, $\tilde{z} = \varphi(y, t)$ is an isolated stationary point of (2.28) for $(y, t) \in \bar{N}$.

Suppose that

B. the stationary point $\tilde{z} = \varphi(y, t)$ of (2.20) is uniformly asymptotically stable with respect to $(y, t) \in \bar{N}$, i.e. $\forall \varepsilon > 0 \, \exists \, \delta(\varepsilon) > 0$ such that if $\|\tilde{z}(0) - \varphi(y, t)\| < \delta(\varepsilon)$ then $\|\tilde{z}(\tau) - \varphi(y, t)\| < \varepsilon$ and $\tilde{z}(\tau) \to \varphi(y, t)$ as $\tau \to \infty$.

If this condition is true, then the root $\tilde{z} = \varphi(y, t)$ is said to be stable in $\bar{N}$.

Lemma 2.1.1 *Suppose that for system* (2.25) *conditions A1-A3, A5 and B are true, then, for sufficiently small* μ, *solutions* $z(t, \mu)$ *and* $y(t, \mu)$ *of problem* (2.25) *with initial conditions* (2.16) *exist on* $0 \leq t \leq T$, *are unique, and satisfy*

$$\lim_{\mu \to 0} y(t, \mu) = \bar{y}(t) \quad \textit{for} \quad 0 \leq t \leq T \tag{2.29}$$

and

$$\lim_{\mu \to 0} z(t, \mu) = \bar{z}(t) = \varphi(\bar{y}(t), t) \quad \textit{for} \quad 0 < t \leq T. \tag{2.30}$$

Proof. First, consider the interval $[0, \eta_1]$. On this interval, Lemma 2.1.1 is a type of Tikhonov Theorem [119, Theorem 2.1] and all conditions are satisfied. Therefore, by [119, Theorem 2.1], for sufficiently small μ, solutions $z(t,\mu), y(t,\mu)$ of (2.3) and (2.16) exist on $[0, \eta_1]$ and satisfies

$$\begin{aligned} &\lim_{\mu\to 0} y(t,\mu) = \bar{y}(t) \quad \textit{for} \quad 0 \le t \le \eta_1, \\ &\lim_{\mu\to 0} z(t,\mu) = \bar{z}(t) = \varphi(\bar{y}(t), t) \quad \textit{for} \quad 0 < t \le \eta_1. \end{aligned} \tag{2.31}$$

Now, consider the second interval $(\eta_1, \eta_2]$. For this interval the initial values are $z_1 = z(\eta_1+, \mu), y_1 = y(\eta_1+, \mu)$. Since $\lim_{\mu\to 0} y(\eta_1, \mu) = \bar{y}(\eta_1)$ and $\lim_{\mu\to 0} z(\eta_1, \mu) = \varphi(\bar{y}(\eta_1), \eta_1)$, z_1 is in the basin of attraction of $\varphi(\bar{y}(t), t)$ and $y_1 \in N$. Again, all conditions of Tikhonov Theorem are satisfied and by [119, Theorem 2.1]

$$\begin{aligned} &\lim_{\mu\to 0} y(t,\mu) = \bar{y}(t) \quad \textit{for} \quad \eta_1 < t \le \eta_2, \\ &\lim_{\mu\to 0} z(t,\mu) = \bar{z}(t) = \varphi(\bar{y}(t), t) \quad \textit{for} \quad \eta_1 < t \le \eta_2. \end{aligned}$$

Similarly, for the next intervals $(\eta_i, \eta_{i+1}], i = 2, 3, \ldots, k-1$, and $(\eta_k, T]$ one can show that as $\mu \to 0$, $\lim_{\mu\to 0} y(t,\mu) = \bar{y}(t)$ and $\lim_{\mu\to 0} z(t,\mu) = \bar{z}(t) = \varphi(\bar{y}(t), t)$. Lemma is proved.

Remark. At discontinuity moments $\eta_j, j = 1, 2, \ldots, k$, layers do not emerge. This is because,

$$\lim_{\mu\to 0} z(\eta_j+, \mu) = \varphi(\bar{y}(\eta_j), \eta_j) = \bar{z}(\eta_j), j = 1, 2, \ldots, k.$$

Proof of Theorem 2.1.4. Consider the interval $[0, \theta_1]$. Hence, on this interval, Theorem 2.1.4 is the form of Lemma 2.1.1. Condition A4 is corresponding to the assumption that uniformly asymptomatically stability of the root $\varphi(y, \tau\mu)$ as $\tau \to \infty$, i.e. condition B is satisfied. Obviously, all conditions of Lemma 2.1.1 are true. Consequently, for sufficiently small μ, solutions $z(t,\mu), y(t,\mu)$ of (2.25) and (2.16) exist and satisfy

$$\begin{aligned} &\lim_{\mu\to 0} y(t,\mu) = \bar{y}(t) \quad \textit{for} \quad 0 \le t \le \theta_1, \\ &\lim_{\mu\to 0} z(t,\mu) = \bar{z}(t) = \varphi(\bar{y}(t), t) \quad \textit{for} \quad 0 < t \le \theta_1. \end{aligned} \tag{2.32}$$

Now, consider the next interval $(\theta_1, \theta_2]$. Condition A6 implies that

$$\lim_{\mu\to 0} z(\theta_1+, \mu) = \lim_{\mu\to 0} \left\{ z(\theta_1, \mu) + \frac{I_1(z(\theta_1, \mu), y(\theta_1, \mu), \mu)}{\mu} \right\} = \varphi(\bar{y}(\theta_1), \theta_1).$$

Hence, condition A5 is true. Repeating the same processes as for the previous interval, one can demonstrates that $z(t,\mu) \to \varphi(\bar{y}(t),t)$ and $y(t,\mu) \to \bar{y}(t)$ as $\mu \to 0$ for $(\theta_1, \theta_2]$. Thus, recurrently it can be proven that for $t \in (\theta_i, \theta_{i+1}], i = 1, 2, \ldots, p-1$ and $t \in (\theta_p, T]$ it is true that $z(t,\mu) \to \varphi(\bar{y}(t),t)$ and $y(t,\mu) \to \bar{y}(t)$ as $\mu \to 0$. Therefore limits (2.23) and (2.24) are true. Theorem is proved.

Example for Lemma 2.1.1. Consider the system

$$
\begin{aligned}
&\mu \frac{dz}{dt} = z(1-z-2y), \\
&\frac{dy}{dt} = y(1-2z-y), \quad \Delta y|_{t=\eta_j} = y^2 - y + z,
\end{aligned}
\tag{2.33}
$$

with initial conditions $z(0,\mu) = 1$ and $y(0,\mu) = 2$, where $\eta_j = j/3, j = 1, 2, \ldots, 5$. Let us take $\mu = 0$ in this problem. Then, the first equation becomes $0 = z(1-z-2y)$. It has the solutions $z = 0$ and $z = 1-2y$. Consider the zero solution $z = 0$. Now, we check the conditions of Lemma 2.1.1.

$$
\frac{\partial}{\partial z} z(1-z-2y)|_{z=0} = 1 - 2y < 0
$$

if $y > 1/2$. Therefore, if $y > 1/2$, $z = 0$ is uniformly asymptotically stable. Substitute $z = 0$ into the second line of (2.33) to obtain

$$
\frac{d\bar{y}}{dt} = \bar{y}(1-\bar{y}), \quad \Delta \bar{y}|_{t=\eta_j} = \bar{y}^2 - \bar{y}, \tag{2.34}
$$

with initial value $\bar{y}(0) = 2$. This system has a unique solution $\bar{y}(t)$. Thus, by Lemma 2.1.1, solutions $z(t,\mu), y(t,\mu)$ of (2.33) with $z(0,\mu) = 1$ and $y(0,\mu) = 2$ tends to $0, \bar{y}(t)$, respectively, as $\mu \to 0$ for $0 < t \leq T$. Obviously, in Figure 2.3, it can be seen that when μ decreases to zero, solutions $z(t,\mu), y(t,\mu)$ approaches to $0, \bar{y}(t)$, respectively.

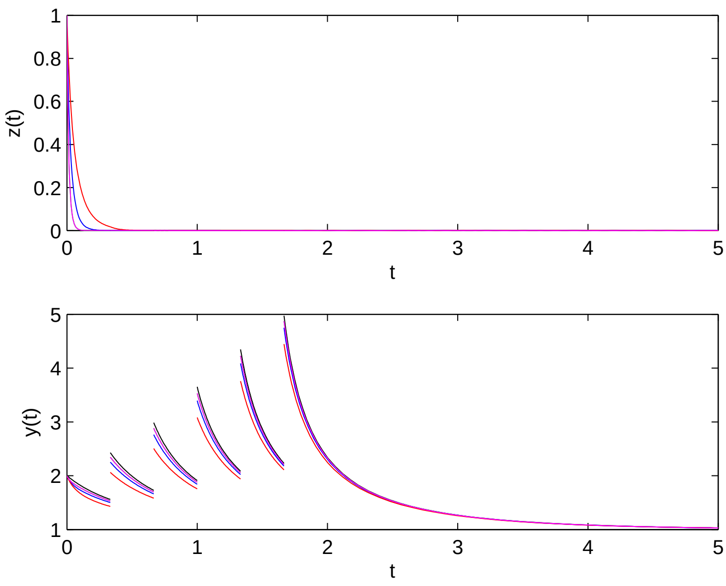

Figure 2.3: Black, magenta, blue and red lines are the coordinates of system (2.33) with initial values $z(0,\mu) = 1$ and $y(0,\mu) = 2$ for different values of $\mu : 0, 0.05, 0.1, 0.2$, respectively.

2.1.3.2 Singularity with Multi-Layers

In the previous subsection, it is shown that the convergence is not uniform at $t = 0$. That is, an initial layer is obtained by Tikhonov Theorem. To get multi-layers by Tikhonov Theorem we need another condition for the impulse function. These layers will occur on the neighborhoods of $t = 0$ and $t = \theta_i, i = 1, 2, \ldots, p$.

Again, we consider system (2.3) with the same properties. In addition, we need the following condition

A7.
$$\lim_{(z,y,\mu)\to(\varphi(\bar{y}(\theta_i),\theta_i),\bar{y}(\theta_i),0)} \frac{I_i(z,y,\mu)}{\mu} = I_i^0 \neq 0$$
and assume that $\varphi(\bar{y}(\theta_i),\theta_i) + I_i^0, i = 1, 2, \ldots, p$, is in the basin of attraction of $\varphi(\bar{y}(t), t)$.

This condition implies that after each impulse moment, the difference $\|z(\theta_i+,\mu) - \varphi\|$ does not go to zero as $\mu \to 0$. Hence, convergence is not uniform.

Theorem 2.1.5 *Suppose that conditions A1-A5 and A7 hold. Then, for sufficiently small* μ*, solutions* $z(t,\mu)$ *and* $y(t,\mu)$ *of problem* (2.3) *with initial conditions* (2.16) *exist on* $0 \le t \le T$*, are unique, and satisfy*

$$\lim_{\mu\to 0} y(t,\mu) = \bar{y}(t) \quad \textit{for} \quad 0 \le t \le T$$

and

$$\lim_{\mu\to 0} z(t,\mu) = \bar{z}(t) = \varphi(\bar{y}(t),t)$$

is true for $t \in \bigcup_{i=0}^{p-1}(\theta_i,\theta_{i+1}] \cup (\theta_p,T]$*, where* $\theta_0 = 0$.

Proof. Proof is similar to the proof of Theorem 2.1.4 with the exception that singularity with multi-layers appears near $t = 0$ and $t = \theta_i, i = 1,2,\ldots,p$.

Now, let us generalize this theorem. Consider the following impulsive system

$$\begin{aligned} \mu\frac{dz}{dt} &= F(z,y,t), \quad \mu\Delta z|_{t=\theta_i} = I_i(z,y,\mu) \quad \mu\Delta z|_{t=\tau_j^i} = \tilde{J}_i(z,y,\mu) \\ \frac{dy}{dt} &= f(z,y,t), \quad \Delta y|_{t=\eta_j} = J_j(z,y), \end{aligned} \tag{2.35}$$

where τ_j^i is defined in Section 2.1.2.2. Additionally, we need $\tilde{J}_i(\varphi(\bar{y}(\tau_j^i),\tau_j^i,),\bar{y}(\tau_j^i),0) = 0, i = 1,2,\ldots,p, j = 1,2,\ldots,p_j$ and the following condition

A8.

$$\lim_{(z,y,\mu)\to(\varphi(\bar{y}(\theta_i),\theta_i),\bar{y}(\theta_i),0)} \frac{\tilde{J}_i(z,y,\mu)}{\mu} = 0, i = 1,2,\ldots,p.$$

Now, we can assert our theorem.

Theorem 2.1.6 *Suppose that conditions A1-A5 and A7-A8 hold. Then, for sufficiently small* μ*, solutions* $z(t,\mu)$ *and* $y(t,\mu)$ *of problem* (2.35) *with initial conditions* (2.16) *exist on* $0 \le t \le T$*, are unique, and satisfy*

$$\lim_{\mu\to 0} y(t,\mu) = \bar{y}(t) \quad \textit{for} \quad 0 \le t \le T$$

and

$$\lim_{\mu\to 0} z(t,\mu) = \bar{z}(t) = \varphi(\bar{y}(t),t)$$

is true for $t \in \bigcup_{i=0}^{p-1}(\theta_i,\theta_{i+1}] \cup (\theta_p,T]$*, where* $\theta_0 = 0$.

2.1.4 Conclusion

In this section, we have introduced a new type of singular impulsive differential equations. The main novelty of this part is that singularity in the impulsive part of the systems can be treated through perturbation methods.

In the book of Bainov and Covachev [19], and several papers cited in the book, they considered singular impulsive systems with small parameter involved only in the differential equations of the systems, but not in the impulsive equations of them while we insert a small parameter into the impulse equation such that the singularity concept has been significantly extended for discontinuous dynamics. Thus, the most general Tikhonov theorem for the impulsive case has been obtained.

2.2 A Differential Equation with Singular Impulses and Multi-stable Roots

2.2.1 Introduction

The singularly perturbed differential equations have been investigated from the beginning of 20^{th} century due to their extensive applications in various fields of chemical kinetics [102], mathematical biology [57, 91], fluid dynamics [37] and in a variety models for control theory [52, 67]. This problem contains a small parameter such that the problem cannot be approximated by setting the parameter value to zero. In other words, the solution of this problem varies rapidly in some regions and varies slowly in other regions.

It is well known that one of the basic instruments in differential equations is impulse effect so that the role of discontinuity is understood better for the real world problems. It exists in a wide diversity of evolutionary processes that exhibit abrupt changes in their states [2, 3, 6, 7]. Hence, it has received considerable attention from many researchers. In many systems, in addition to singular perturbation, there are also impulse effects [33, 34, 35, 105, 106]. Chen et al. [35] derived a sufficient condition that guarantees robust exponential stability for sufficiently small singular perturbation parameter by applying the Lyapunov function method and using a two-time scale comparison principle. In [105, 106], authors proposed Lyapunov function method to set up the exponential stability criteria for singularly perturbed impulsive systems. This method can be efficiently used to overcome the impulsive perturbation such that the stability of the original system can be ensured. In [33], Lyapunov function method was further extended to study the exponential stability of singularly perturbed stochastic time-delay systems with impulse effect. The results in [33, 105, 106] only guarantee the systems under consideration to be exponentially stable for sufficiently small positive parameter.

In this section, we develop the singularly perturbed problem to singularly perturbed differential equations with both small parameter in front of the derivative and impulse function. The intrinsic idea of the section is that in our model solution approaches more than one root of the differential equation as the parameter decreases to zero. Another novelty here is that the system has two impulse functions one of which is singular. This provides new theoretical opportunities.

In this section, we propose the following singular impulsive differential equations with a positive small real parameter μ:

$$\begin{aligned} &\mu\frac{dz}{dt} = F(z), \\ &\mu\Delta z|_{t=\eta_i} = I(z,\mu), \quad \Delta z|_{t=\theta_i} = J(z) \end{aligned} \tag{2.36}$$

with $z(0,\mu) = z^0$, where $z \in \mathbb{R}^n$, $t \in [0,T]$, $F(z)$ is continuously differentiable on D, $I(z,\mu)$ is continuous on $D \times [0,1]$ and $J(z)$ is continuous on D , D is the domain $D = \{0 \le t \le T, \|z\| < d\}$, $\theta_i, i = 1,2,\ldots,p$, and $\eta_j, j = 1,2,\ldots,\bar{p}$, are distinct discontinuity moments in $(0,T)$.

The parameter in the impulsive equation makes it possible that $\frac{I(z,\mu)}{\mu}$ blow up at impulse moments as $\mu \to 0$. This is why, a deep analysis and convenient conditions for the limiting processes with $\mu \to 0$ have to be researched.

2.2.2 Main Result

Let us take $\mu = 0$ in (2.36). Then, one has

$$\begin{aligned} &0 = F(z), \\ &0 = I(z,0), \quad \Delta z|_{t=\theta_i} = J(z) \end{aligned}$$

It is the degenerate system since its order is less than the order of (2.36). Assume that $F(z) = 0$ has the roots $\varphi_1, \varphi_2, ...\varphi_k, \varphi_{k+1}, \ldots, \varphi_l$ such that $I(\varphi_j, 0) = 0, j = 1, 2, ..., l$ and all of them are real and isolated in $\bar{D}$.

The following conditions are required for system (2.36).

(C1) Jacobian matrix $\frac{\partial F}{\partial z}\Big|_{z=\varphi_j}, j = 1, 2, \ldots, k$, is Hurwitzian.

This condition implies that the roots $\varphi_j, j = 1, 2, \ldots, k$ are stable solutions of the differential equation in (2.36). Furthermore, we need the following conditions for the impulsive functions.

(C2) For each $j \in \{1, 2, \ldots, k\}$ there exists $i \in \{1, 2, \ldots, k\}$ such that

$$\varphi_j + J(\varphi_j) = \varphi_i.$$

That is, after the each impulse moment θ_i, the solution $z(t, \mu)$ will be close to another stable equilibrium.

(C3)

$$\lim_{\substack{z \to \varphi_j \\ \mu \to 0}} \frac{I(z, \mu)}{\mu} = 0, \quad j = 1, 2, \ldots, k.$$

In the denominator of the limit we have a small parameter μ which goes to zero. In order to avoid a blow up we need the last condition. Also, the zero value of the limit gives us the advantage that the solution stays in the domain of attractions of the stable roots.

Denote D_j as the domain of attraction of root $\varphi_j, j = 1, 2, \ldots, k$, such that $D_i \cap D_j = \emptyset$ if $i \neq j$ and $D_j \subset D, j = 1, 2, \ldots, k$. Moreover, $z_j(t)$ will be used for denoting the solution of

$$0 = F(z), 0 = I(z, 0)$$

such that if the initial value $z^0 \in D_j$, then $z_j(t) = \varphi_j$ for $t \in (0, \theta_1]$ and it alternates to the other stable roots by condition (C2) for the next intervals $(\theta_i, \theta_{i+1}], i = 1, 2, \ldots, p-1$.

Theorem 2.2.1 *Suppose that conditions (C1)-(C3) are true. If the initial value z^0 is located in the domain of attraction D_j of the root $\varphi_j, j = 1, 2, \ldots, k$, then the solution $z(t, \mu)$ of* (2.36) *with $z(0, \mu) = z^0$ exists on $[0, T]$ and it satisfies the limit*

$$\lim_{\mu \to 0} z(t, \mu) = z_j(t) \quad \textit{for} \quad 0 < t \leq T, \tag{2.37}$$

where $j = 1, 2, \ldots, k(k-1)^p$.

Proof 2.2.1 *In this proof, we will show that the theorem is true for two stable roots, namely φ_1, φ_2. Consider the interval $[0, \theta_1]$. Without loss of generality, assume that $\eta_i \in (0, \theta_1), i = 1, 2, \ldots, n, n < \bar{p}$. Let $z^0 \in D_1$ such that it is in the domain of attraction of φ_1. Then, for fixed $\mu > 0$, the differential equation on the interval $[0, \eta_1]$*

$$\frac{dz}{dt} = \frac{F(z)}{\mu} \tag{2.38}$$

with initial value $z(0, \mu) = z^0$ has a unique solution $z(t, \mu)$, since $F(z) \in C^1(D)$. Introduce a new variable $\tau = \frac{t}{\mu}$ in (2.38) *to obtain*

$$\frac{dz}{d\tau} = F(z), \quad \tau > 0. \tag{2.39}$$

Then, by condition (C1), one gets the following limit

$$\lim_{\tau \to \infty} z(\tau, \mu) = \varphi_1,$$

that is,

$$\lim_{\mu \to 0} z(t, \mu) = \varphi_1 \quad \textit{for} \quad 0 < t \leq \eta_1.$$

Next, let us consider the interval $(\eta_1, \eta_2]$. From condition (C3), we have

$$\lim_{\mu \to 0} z(\eta_1+, \mu) = \lim_{\mu \to 0} \left\{ z(\eta_1, \mu) + \frac{I(z(\eta_1, \mu), \mu)}{\mu} \right\} = \varphi_1.$$

Hence, $z(\eta_1+, \mu)$ is in the domain of attraction of φ_1, $z(\eta_1+, \mu) \in D_1$. So, from condition (C1),

$$\lim_{\mu \to 0} z(t, \mu) = \varphi_1 \quad \textit{for} \quad \eta_1 < t \leq \eta_2.$$

Repeating the same technique as for the previous intervals, one has the limit

$$\lim_{\mu \to 0} z(t, \mu) = \varphi_1 \quad \textit{for} \quad 0 < t \leq \theta_1.$$

For two stable roots condition (C2) is of the form

$$\varphi_1 + J(\varphi_1) = \varphi_2,$$

or

$$\varphi_2 + J(\varphi_2) = \varphi_1.$$

It follows that $z(\theta_1+,\mu) = z(\theta_1,\mu) + J(z(\theta_1,\mu))$ *is in the neighborhood of* φ_2*, i.e., it is in* D_2*, the domain of attraction of* φ_2*. Now, consider next the interval* $(\theta_1,\theta_2]$*. As in the previous interval, we have*

$$\lim_{\mu\to 0} z(t,\mu) = \varphi_2 \quad \textit{for} \quad \theta_1 < t \le \theta_2$$

and $z(\theta_2+,\mu) = z(\theta_2,\mu) + J(z(\theta_2,\mu))$ *is in the neighborhood of* φ_1*. Recursively, one can prove that*

$$\lim_{\mu\to 0} z(t,\mu) = z_1(t),$$

where

$$z_1(t) = \begin{cases} \varphi_1 \textit{ if } t \in (0,\theta_1] \cup (\theta_2,\theta_3] \cup \dots \\ \varphi_2 \textit{ if } t \in (\theta_1,\theta_2] \cup (\theta_3,\theta_4] \cup \dots \end{cases}.$$

Note that if $z^0 \in D_2$*, then we will have*

$$\lim_{\mu\to 0} z(t,\mu) = z_2(t),$$

where

$$z_2(t) = \begin{cases} \varphi_2 \textit{ if } t \in (0,\theta_1] \cup (\theta_2,\theta_3] \cup \dots \\ \varphi_1 \textit{ if } t \in (\theta_1,\theta_2] \cup (\theta_3,\theta_4] \cup \dots \end{cases}.$$

Therefore, theorem is proved.

The convergence is not uniform at $t = 0$ since $z(0,\mu) = z^0$ and $z^0 \neq \varphi_1$ for all $\mu > 0$. We can say that the region of nonuniform convergence is $O(\mu)$ thick, since for $t > 0$, $\|z(t,\mu) - \varphi_1\|$ can be made arbitrarily close to zero by choosing μ small enough. The interval of nonuniform convergence is called an initial layer. This theorem implies that there is a single initial layer.

Example 2.2.1 *Let us consider the following one dimensional nonlinear system*

$$\begin{aligned}&\mu \dot{z} = -z(1-z)(2-z),\\ &\mu \Delta z|_{t=\eta_i} = -\mu z^{1/2}(1-z^2)(2-z) - \mu^2, \quad \Delta z|_{t=\theta_i} = 2-2z,\end{aligned} \tag{2.40}$$

with $z(0,\mu) = z^0$ *where* $\theta_i = \frac{2i}{3}, \eta_i = \frac{2i-1}{3}, i = 1,2,3,4,5,$ *and* $F(z) = -z(1-z)(2-z), I(z,\mu) = -\mu z^{1/2}(1-z^2)(2-z) - \mu^2, J(z) = 2-2z$. *Now, we check the conditions of Theorem 2.2.1. Take* $\mu = 0$ *to obtain* $0 = F(z)$. *It has the roots* $\varphi_1 = 0, \varphi_2 = 1, \varphi_3 = 2,$ *such that* $I(0,0) = I(1,0) = I(2,0) = 0.$

$$\left.\frac{\partial F}{\partial z}\right|_{z=\varphi_j} = (3z^2+6z-2)|_{z=\varphi_j} < 0, j = 1,3.$$

Moreover,

$$0 + J(0) = 2, 2 + J(2) = 0,$$

and

$$\lim_{\substack{z\to\varphi_j\\ \mu\to 0}} \frac{I(z,\mu)}{\mu} = \lim_{\substack{z\to\varphi_j\\ \mu\to 0}} (-z^{1/2}(1-z^2)(2-z) - \mu) = 0, \quad j = 1,3.$$

All conditions are satisfied. Thus, by Theorem 2.2.1, if the initial value z^0 *is located in the domain of attraction* D_j, *of the root* $\varphi_j, j = 1,3,$ *then solution* $z(t,\mu)$ *of* (2.40) *with* $z(0,\mu) = z^0$ *exists on* $[0,T]$ *and it satisfies the limit*

$$\lim_{\mu\to 0} z(t,\mu) = z_j(t) \quad \textit{for} \quad 0 < t \le T, \tag{2.41}$$

where $j = 1,2$. *If* $z^0 \in D_1$, *then*

$$z_1(t) = \begin{cases} 0 \textit{ if } t \in (0,\theta_1] \cup (\theta_2,\theta_3] \cup \dots \\ 2 \textit{ if } t \in (\theta_1,\theta_2] \cup (\theta_3,\theta_4] \cup \dots \end{cases}$$

and if $z^0 \in D_2$, *then*

$$z_2(t) = \begin{cases} 2 \textit{ if } t \in (0,\theta_1] \cup (\theta_2,\theta_3] \cup \dots \\ 0 \textit{ if } t \in (\theta_1,\theta_2] \cup (\theta_3,\theta_4] \cup \dots \end{cases}.$$

To demonstrate the results via simulations, choose $z(0,\mu) = 2.2$ *which is in the domain of the attraction of root* $z = 2$. *In Figure 2.4, the result of Theorem 2.2.1 is obviously seen.*

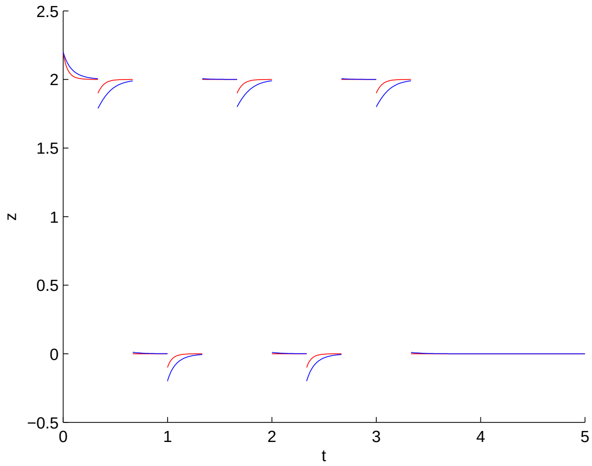

Figure 2.4: Blue and red lines represents the coordinates of solution of system (2.40) with initial $z(0, \mu) = 2.2$ for $\mu = 0.2$ and $\mu = 0.1$, respectively.

CHAPTER 3

BIFURCATION ANALYSIS OF WILSON-COWAN MODEL WITH SINGULAR IMPULSES

3.1 Introduction

Wilson and Cowan [124] proposed a model for describing the dynamics of localized populations of excitatory and inhibitory neurons. This model is a coarse-grained description of the overall activity of a large-scale neural network, employing just two differential equations [65]. It is used in the developing of multi-scale mathematical model of cortical electric activity with realistic mesoscopic connectivity [122]. On the other hand, sudden changes and the instantaneous perturbations in a neural network at a certain time, which are identified by external elements, are examples of impulsive phenomena which may influence the evolutionary process of the neural network [1]. In fact, the existence of impulse is often a source of richness for a model. That is to say, the impulsive neural networks will be an appropriate description of symptoms of sudden dynamic changes. Therefore, the models considered in this chapter have impulsive moments.

The singularly perturbed problems depend on a small positive parameter, which is in front of the derivative, such that the solution varies rapidly in some regions and varies slowly in other regions. They arise in the various processes and phenomena such as chemical kinetics, mathematical biology, neural networks, fluid dynamics and in a variety models for control theory [102, 57, 91, 96, 37, 67, 52]. In this chapter, we will investigate the Wilson-Cowan model with singular impulsive function in which singular perturbation method has been used to analyze the dynamics of neuron models.

Local bifurcations are ubiquitous in mathematical biology [69] and mathematical neuroscience [59, 49, 44], because they provide a framework for understanding behavior of the biological networks modeled as dynamical systems. Moreover, a local bifurcation can affect the global dynamic behavior of a neuron [59]. There are many neuronal models to consider the bifurcation analysis, for instance, the bifurcation for Wilson-Cowan model is discussed in the book of Hoppensteadt and Izhikevich [59] in which they consider the model of the following type

$$\dot{x} = -x + S(\rho + cx),$$

where $x \in \mathbb{R}$ is the activity of the neuron, $\rho \in \mathbb{R}$ is the external input to the neuron, the feedback parameter $c \in \mathbb{R}$ characterizes the non-linearity of the system, and S is a sigma shaped function. This system consists only one neuron or one population of neurons. When the bifurcation parameter ρ changes, the saddle-node bifurcation occurs. In our research, we will discuss two and four of population of neurons. These systems have impulses at prescribed moments of time. We will observe the local bifurcation in these models.

The attractors observed in our simulations do not resemble any attractors which have already been observed in the literature. This is why, we need to introduce a new terminology to describe an ultimate behavior of motion in the model. We call the recently introduced components of constructed attractors as medusas and rings. This “zoological" approach to dynamics is not unique in differential equations. For example, canards are cycles of singularly perturbed differential equations [68, 112, 39]. They were discovered in the van der Pol oscillator by Benoit et al [20]. This phenomenon explains the very fast transition upon variation of a parameter from a small amplitude limit cycle to a relaxation oscillation [68]. The fast transition is called canard explosion and happens within an exponentially small range of the control parameter. Because this phenomenon is hard to detect it was nicknamed a canard, after the French newspaper slang word for hoax. Furthermore, the shape of these periodic orbits in phase space resemble a duck; hence the name “canard," the French word for duck. So the notion of a canard cycle was born and the chase after these creatures began [32]. It is important to note that both canards and medusas appear in the singularly perturbed systems.

Bifurcation occurred in this chapter cannot be reduced to the existing local bifurcations in the literature, namely, saddle-node, pitchfork, Hopf bifurcations, etc. First of all, we are talking about the change of an attractor set in the four subpopulations of neurons of Wilson-Cowan model with impulses depending on the change of the small parameter. This time the bifurcation parameter is also the parameter of the singularity. Moreover, it is a parameter of the singularity not only in the differential equations of the model, but also in the impulsive part of it. Thus, the cause of bifurcation is not the change of eigenvalues, but it relates to the singular compartment and the impulsive dynamics of the model. This is why, theoretical approvement of the observed bifurcations has not been done here. However, we see that the abrupt changes in the phase portrait through simulations. Additionally, we notice that in the numerical study attractors of the model can be described through the new picture's elements which we call as medusa, medusa without ring and rings, which, in general, may not be considered invariant for solutions of the model despite that the elements are introduced for the first time. We are confident that they are very generic for differential equations with impulses and they will give a big benefit in the next investigations of discontinuous neural networks.

We will start by defining the membrane time constant since it will be used as the parameter of singularity and bifurcation.

3.2 Membrane Time Constant

The role of the membrane time constant is important in Wilson-Cowan models. In these models the frequency of the oscillation is determined primarily by the membrane time constants [110]. Let us define the membrane time constant μ for a simple circuit. Suppose that the membrane is characterized by a single membrane capacitance C in series with a single voltage-independent membrane resistance R, see Figure 3.1. Then, by Ohm's law the dynamics of the potential V across this circuit in response to a current injection I changes as

$$RC\frac{dV}{dt} = -V + IR,$$

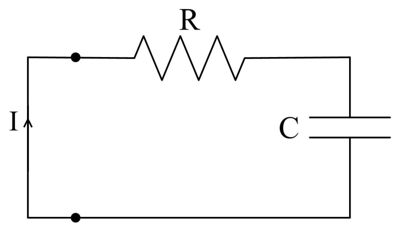

Figure 3.1: A simple RC circuit.

which has the solution

$$V(t) = IR(1 - e^{-\frac{t}{RC}})$$

The membrane time constant, here, is defined by the product of the membrane resistance and membrane capacitance $\mu = RC$. The potential $V(t)$ is governed by exponential decay toward the steady-state $V = IR$ as $\mu \to 0$. The membrane time constant is used to understand how quickly a neuron's voltage level changes after it receives an input signal.

3.3 Singular Model with Singular Impulsive Function

The dynamics of excitatory and inhibitory neurons are described as follows [124]

$$\begin{aligned} \mu_e \frac{dE}{dt} &= -E + (k_e - r_e E) S_e(c_1 E - c_2 I + P), \\ \mu_i \frac{dI}{dt} &= -I + (k_i - r_i I) S_i(c_3 E - c_4 I + Q), \end{aligned} \tag{3.1}$$

where $E(t)$ and $I(t)$ are the proportion of excitatory and inhibitory cells firing per unit time at time t, respectively, c_1 and c_2 are the connectivity coefficients, which are both positive, represent the average number of excitatory and inhibitory synaptic inputs per cell, $P(t)$ represents the external input to the excitatory subpopulation, the quantities c_3, c_4 and $Q(t)$ are defined similarly for the inhibitory subpopulation. The nonzero quantities μ_e and μ_i represent the membrane time constants while k_e, k_i, r_e and r_i are associated with the refractory terms. Moreover, $S_e(x)$ is the sigmoid function of the following form

$$S_e(x) = \frac{1}{1 + \exp[-a_e(x - \theta_e)]} - \frac{1}{1 + \exp(a_e \theta_e)}, \tag{3.2}$$

where θ_e is the position of the maximum slope of $S_e(x)$ and $\max[\dot{S}_e(x)] = a_e/4$, and S_i is defined similarly.

Since the external inputs influence the neurons' activities, $E(t)$ and $I(t)$ can change abruptly. It is natural to consider the previous continuous dynamics in the way that the membrane time constants proceed to be involved in the electrical processes and the impulsive equations have the form

$$\begin{aligned} \Delta E|_{t=\theta_i} &= \bar{K}(E, I), \\ \Delta I|_{t=\theta_i} &= \bar{J}(E, I), \end{aligned} \tag{3.3}$$

where the impulse moments θ_i are distinct, $\theta_i \in (0, T)$ and the equality $\Delta E|_{t=\theta_i} = E(\theta+) - E(\theta-)$ denotes the jump operator in which $t = \theta$ is the time when the external input influence $E(t)$, $E(\theta-)$ is the pre-impulse value and $E(\theta+)$ is the post-impulse value. Moreover, if one considers the impulsive equations as the limit cases of the differential equations, then at some moments impulsive changes of the activities can depend on the membrane time constants, similar to the ones for the system (3.1). More precisely, we will also study the equations of the form

$$\begin{aligned} \mu_e \Delta E|_{t=\eta_j} &= K(E, I, \mu_e), \\ \mu_i \Delta I|_{t=\eta_j} &= J(E, I, \mu_i), \end{aligned} \tag{3.4}$$

where the moments η_j and θ_i are, in general, different. Finally, gathering all the dynamics details formulated above, our single Wilson-Cowan model with impulses has the following form

$$\begin{aligned} \mu_e \frac{dE}{dt} &= -E + (k_e - r_e E) S_e(c_1 E - c_2 I + P), \\ \mu_i \frac{dI}{dt} &= -I + (k_i - r_i I) S_i(c_3 E - c_4 I + Q), \\ \Delta E|_{t=\theta_i} &= \bar{K}(E, I), \\ \Delta I|_{t=\theta_i} &= \bar{J}(E, I), \\ \mu_e \Delta E|_{t=\eta_j} &= K(E, I, \mu_e), \\ \mu_i \Delta I|_{t=\eta_j} &= J(E, I, \mu_i), \end{aligned} \tag{3.5}$$

with the initial activity $(E(0), I(0)) = (E_0, I_0)$.

Define the function $F(E, I) = \begin{pmatrix} -E + (k_e - r_e E) S_e(c_1 E - c_2 I + P) \\ -I + (k_i - r_i I) S_i(c_3 E - c_4 I + Q) \end{pmatrix}$.

Suppose that $E, I \in \mathbb{R}$, $t \in [0,T]$, $F(E,I)$ is continuously differentiable on D, $K(E,I,\mu_e), J(E,I,\mu_i)$ are continuous on $D\times[0,1]$ and $\bar{K}(E,I), \bar{J}(E,I)$ are continuous on D, D is the domain $D = \{0 \le t \le T, |E| < d, |I| < d\}$, $\theta_i, i = 1,2,\dots,p$, and $\eta_j, j = 1,2,\dots,\bar{p}$, are distinct discontinuity moments in $(0,T)$.

Substituting $\mu_e = \mu_i = 0$ in (3.1) and (3.4), we obtain $F(E,I) = 0$ and

$$\begin{aligned} 0 &= K(E,I,0), \\ 0 &= J(E,I,0). \end{aligned} \tag{3.6}$$

Assume that equations $F(E,I) = 0$ and (3.6) have the steady states

$$(E_1,I_1),(E_2,I_2),...(E_k,I_k),(E_{k+1},I_{k+1}),\dots,(E_l,I_l)$$

such that all of them are real and isolated in $\bar{D}$. They are considered to be states of low level background activities since such activities seem ubiquitous in neural tissue. $E(t)$ and $I(t)$ will be used to refer the activities in the respective subpopulations.

The following conditions are required for system (3.1).

(C1) Jacobian matrices of $F(E,I)$ at the points $(E_1,I_1),(E_2,I_2),...,(E_k,I_k)$ are Hurwitz matrices (they have eigenvalues whose real parts are negative).

This condition implies that the states $(E_1,I_1),(E_2,I_2),...,(E_k,I_k)$ are stable steady states of the differential equation (3.1). Moreover, for the impulsive functions we need the following conditions.

(C2) For each $j \in \{1,2,\dots,k\}$ there exists $i \in \{1,2,\dots,k\}$ such that

$$\begin{pmatrix} E_j \\ I_j \end{pmatrix} + \begin{pmatrix} \bar{K}(E_j,I_j) \\ \bar{J}(E_j,I_j) \end{pmatrix} = \begin{pmatrix} E_i \\ I_i \end{pmatrix}.$$

That is, after the each impulse moment θ_j the activity $(E(t),I(t))$ will be close to another stable steady state $\begin{pmatrix} E_i \\ I_i \end{pmatrix}$.

(C3)

$$\lim_{\substack{(E,I)\to(E_j,I_j) \\ \mu_{e,i}\to 0}} \begin{pmatrix} \frac{K(E,I,\mu_e)}{\mu_e} \\ \frac{J(E,I,\mu_i)}{\mu_i} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad j = 1,2,\dots,k.$$

In the denominator of the limit we have small parameters μ_e and μ_i which decay to zero. In order to avoid a blow up we need the last condition. In addition, the zero value of the limit gives us the privilege that the activities stay in the domain of attractions of the stable steady states.

Denote D_j as the domain of attraction of stable steady state $(E_j, I_j), j = 1, 2, \ldots, k,$ such that $D_i \cap D_j = \emptyset$ if $i \neq j$ and $D_j \subset D, j = 1, 2, \ldots, k$. Also, $z_j(t)$ will be used for denoting the solution of $F(E, I) =$ and (3.6) such that if the initial value $(E_0, I_0) \in D_j$, then $z_j(t) = (E_j, I_j)$ for $t \in (0, \theta_1]$ and it alternates to the other stable steady states by condition (C2) for the next intervals $(\theta_i, \theta_{i+1}], i = 1, 2, \ldots, p-1$.

Theorem 3.3.1 *Suppose that conditions (C1)-(C3) are true. If the initial value (E_0, I_0) is located in the domain of attraction D_j of the steady state $(E_j, I_j), j = 1, 2, \ldots, k,$ then the solution $(E(t), I(t))$ of* (3.5) *with (E_0, I_0) exists on $[0, T]$ and it satisfies the limit*

$$\lim_{\mu_{e,i} \to 0} (E(t), I(t)) = z_j(t) \quad \textit{for} \quad 0 < t \leq T, \tag{3.7}$$

where $j = 1, 2, \ldots, k(k-1)^p$.

The proof follows from the proof in [5].

Example. Now, let us take the external forces $P(t) = Q(t) = 0$, $\mu_e = \mu_i = \mu$, and other coefficients in (3.1) as follows: $c_1 = 12, c_2 = 4, c_3 = 13, c_4 = 11, a_e = 1.2, a_i = 1, \theta_e = 2.8, \theta_i = 4, r_e = 1, r_i = 1, k_e = 0.97, k_i = 0.98$. Then, one obtains

$$\begin{aligned} \mu \frac{dE}{dt} &= -E + (0.97 - E) S_e(12E - 4I), \\ \mu \frac{dI}{dt} &= -I + (0.98 - I) S_i(13E - 11I). \end{aligned} \tag{3.8}$$

Taking $\mu = 0$, one has the three equilibria (see Figure 3.2), namely

$$\begin{pmatrix} E \\ I \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0.44234 \\ 0.22751 \end{pmatrix} \text{ and } \begin{pmatrix} 0.18816 \\ 0.067243 \end{pmatrix}.$$

We have $F(E, I) = \begin{pmatrix} -E + (0.97 - E) S_e(12E - 4I) \\ -I + (0.98 - I) S_i(13E - 11I) \end{pmatrix}$. Then, the Jacobian matri-

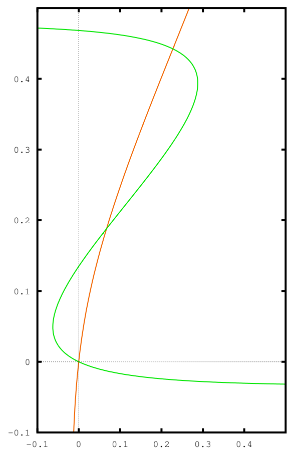

Figure 3.2: E-I phase plane of (3.8). The green represents $-E+(0.97-E)S_e(12E-4I)=0$ and the red represents $-I+(0.98-I)S_i(13E-11I)=0$.

ces of $F(E,I)$ on the steady states are

$$\begin{pmatrix}-0.5468 & -0.1511\\ 0.2250 & -1.1904\end{pmatrix},\ \begin{pmatrix}-0.9895 & -0.2829\\ 2.1299 & -31045\end{pmatrix}$$

and

$$\begin{pmatrix}1.0001 & -0.7469\\ 0.9879 & -1.9096\end{pmatrix},$$

respectively. All eigenvalues of the first two matrices are negative, but last one has a positive eigenvalue. Therefore, the first two steady states are stable.

We extend model (3.8) with the following impulse functions

$$\begin{aligned}\Delta E|_{t=\theta_i} &= -2E+0.44234,\\ \Delta I|_{t=\theta_i} &= -2I+0.22751.\end{aligned}\tag{3.9}$$

$$\begin{aligned}\mu\Delta E|_{t=\eta_i} &= -\mu E^{1/2}(E-0.44234)^2-\sin(\mu^2)I,\\ \mu\Delta I|_{t=\eta_i} &= -\mu I^{1/3}(I-0.22751)^3-\sin(\mu^2)E,\end{aligned}\tag{3.10}$$

where $\theta_i=\frac{2i}{3},\eta_i=\frac{2i-1}{3},i=1,2,\ldots,20$. Let us check the conditions of Theorem 3.3.1. We have shown that the states $\begin{pmatrix}0\\0\end{pmatrix},\begin{pmatrix}0.44234\\0.22751\end{pmatrix}$ are stable. Moreover, they satisfy the equations (3.10) if $\mu=0$. Condition (C2) holds since

$$\begin{pmatrix}0\\0\end{pmatrix}+\begin{pmatrix}0.44234\\0.22751\end{pmatrix}=\begin{pmatrix}0.44234\\0.22751\end{pmatrix}$$

and

$$\begin{pmatrix} 0.44234 \\ 0.22751 \end{pmatrix} + \begin{pmatrix} -0.88468 + 0.44234 \\ -0.45502 + 0.22751 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Lastly, let us check the condition (C3):

$$\lim_{\substack{(E,I)\to(E_j,I_j) \\ \mu\to 0}} \begin{pmatrix} -E^{1/2}(E-0.44234)^2 - \frac{1}{\mu}\sin(\mu^2)I \\ -I^{1/3}(I-0.22751)^3 - \frac{1}{\mu}\sin(\mu^2)E \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad j = 1, 2.$$

Clearly, all conditions are satisfied. Therefore, if the initial value (E_0, I_0) is in the domain of attraction of the steady state $(0, 0)$ then the activities $(E(t), I(t))$ approaches to the steady states as $\mu \to 0$, that is to say,

$$\lim_{\mu\to 0}(E(t,\mu), I(t,\mu)) = \begin{cases} (0,0) \text{ if } t \in (0,\theta_1] \cup (\theta_2,\theta_3] \cup \ldots \\ (0.44234, 0.22751) \text{ if } t \in (\theta_1,\theta_2] \cup (\theta_3,\theta_4] \cup \ldots \end{cases},$$

and if it is in the domain of attraction of the steady state $(0.44234, 0.22751)$, then

$$\lim_{\mu\to 0}(E(t,\mu), I(t,\mu)) = \begin{cases} (0.44234, 0.22751) \text{ if } t \in (0,\theta_1] \cup (\theta_2,\theta_3] \cup \ldots \\ (0,0) \text{ if } t \in (\theta_1,\theta_2] \cup (\theta_3,\theta_4] \cup \ldots \end{cases}.$$

To demonstrate the results via simulation, we take $(E_0, I_0) = (0.25, 0)$ which is in the domain of attraction of $(0.44234, 0.22751)$. Obviously, the results of the theorem can be seen in Figure 3.3.

3.4 Bifurcation of New Attractor Composed of Medusa

In discontinuous dynamics, we will show that a new type of attractor consisting medusa, medusa without ring, and rings exist. The technique used to obtain the new attractor is as follows. We need a pair of coupled Wilson-Cowan models in which each system has an excitatory and an inhibitory subpopulation. The first system admits stable steady states and it has singular impulses. The second one has a limit cycle. Also, in the latter system, the membrane time constants are equals to 1 whereas in the former it is the singularity parameter.

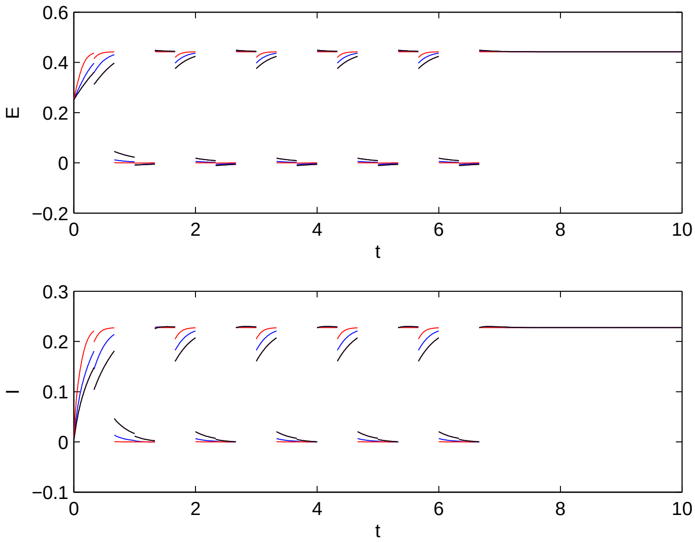

Figure 3.3: Coordinates of (3.8), (3.9) and (3.10) with the initial value $(0.25, 0)$, where red, blue and black lines corresponds to value of $\mu = 0.1, 0.2, 0.3$, respectively.

The first Wilson-Cowan model with impulsive singularity is of the following form:

$$\begin{aligned}
\mu_e \frac{dE}{dt} &= -E + (0.97 - E)S_e(13E - 4I), \\
\mu_i \frac{dI}{dt} &= -I + (0.98 - I)S_i(22E - 2I), \\
\Delta E|_{t=\theta_i} &= 6.741E^2 - 3.58612E + 0.45064, \\
\Delta I|_{t=\theta_i} &= 6.6087I^2 - 3.85682I + 0.49, \\
\mu_e \Delta E|_{t=\eta_i} &= \mu_e E^{1/2}(E - 0.20353)(E - 0.45604)^2 - \sin(\mu_e^2), \\
\mu_i \Delta I|_{t=\eta_i} &= \mu_i I^{1/3}(I - 0.18691)(I - 0.49)^2 - \sin(\mu_i^2),
\end{aligned} \tag{3.11}$$

where the sigmoid functions are

$$\begin{aligned}
S_e(x) &= \frac{1}{1 + \exp[-1.5(x - 2.5)]} - \frac{1}{1 + \exp(3.75)}, \\
S_i(x) &= \frac{1}{1 + \exp[-6(x - 4.3)]} - \frac{1}{1 + \exp(25.8)}.
\end{aligned}$$

and impulse moments are $\theta_i = 2i + 4.95, \eta_i = 2i - 1 + 4.95, i = 1, 2, \ldots, 50$. The differential equations in (3.11) have three stable states

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0.20353 \\ 0.18691 \end{pmatrix}, \begin{pmatrix} 0.45064 \\ 0.49 \end{pmatrix}$$

and two unstable steady states

$$\begin{pmatrix} 0.096205 \\ 0 \end{pmatrix}, \begin{pmatrix} 0.37647 \\ 0.49 \end{pmatrix}.$$

The second model, which has a limit cycle, is of the form

$$\begin{aligned} \frac{de}{dt} &= -e + (0.97 - e)\tilde{S}_e(16e - 12i + 1.25), \\ \frac{di}{dt} &= -i + (0.98 - i)\tilde{S}_i(15e - 3i), \end{aligned} \tag{3.12}$$

where

$$\begin{aligned} \tilde{S}_e(x) &= \frac{1}{1+\exp[-1.3(x-4)]} - \frac{1}{1+\exp(5.2)}, \\ \tilde{S}_i(x) &= \frac{1}{1+\exp[-2(x-3.7)]} - \frac{1}{1+\exp(7.4)}. \end{aligned}$$

We couple system (3.11) and (3.12) as follows

$$\begin{aligned} \mu_e\frac{dE}{dt} &= -E + (0.97 - E)S_e(13E - 4I), \\ \mu_i\frac{dI}{dt} &= -I + (0.98 - I)S_i(22E - 2I), \\ \frac{de}{dt} &= -e + (0.97 - e)\tilde{S}_e(16e - 12i + 1.25), \\ \frac{di}{dt} &= -i + (0.98 - i)\tilde{S}_i(15e - 3i), \\ \mu_e\Delta E|_{t=\eta_i} &= \mu_e E^{1/2}(E - 0.20353)(E - 0.45604)^2 e - \sin(\mu_e^2), \\ \mu_i\Delta I|_{t=\eta_i} &= \mu_i I^{1/3}(I - 0.18691)(I - 0.49)^2 i - \sin(\mu_i^2), \\ \Delta E|_{t=\theta_i} &= 6.741E^2 - 3.58612E + 0.45064, \\ \Delta I|_{t=\theta_i} &= 6.6087I^2 - 3.85682I + 0.49, \\ \Delta e|_{t=\eta_i} &= E(E - 0.20353)^2(E - 0.45604), \\ \Delta i|_{t=\eta_i} &= I(I - 0.18691)(I - 0.49). \end{aligned} \tag{3.13}$$

It is already known that differential equations in (3.11) have three stable steady states. Suppose that the membrane time constants in (3.13) are equal such that $\mu_e = \mu_i = \mu$ and the initial condition is $(0.4656, 0.1101, 0.1101, 0.04766)$. Clearly, in Figure 3.4, one can observe that a medusa exist for the value of parameter $\mu = 0.05$. Note that this is a single trajectory and its form looks like a medusa.

Figure 3.4 is formed as follows. The (E,e,i)-coordinates which start at the given initial value approaches to the cycle. It moves around the cycle until the impulse moment η_1.

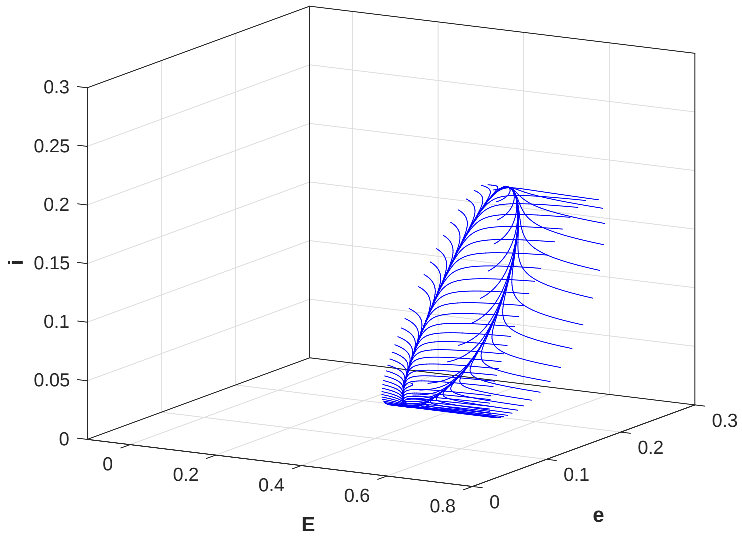

Figure 3.4: (E,e,i)-coordinates of system (3.13) for the initial value $(0.4656, 0.1101, 0.1101, 0.04766)$ and the parameter $\mu = 0.05$.

When the time reaches $t = \eta_1$, because of the impulse function the coordinate jump to $(E(\eta_1+, \mu), e(\eta_1+, \mu), i(\eta_1+, \mu))$. Again it will approach the cycle and move until the impulse moment $t = \theta_1$. Then the coordinate jumps to $(E(\theta_1+, \mu), e(\theta_1+, \mu), i(\theta_1+, \mu))$ and it will approach to the cycle. The (E,e,i)-coordinate moves in this pattern and finally the medusa in Figure 3.4 is observed. The pattern is visualized in Figure 3.5.

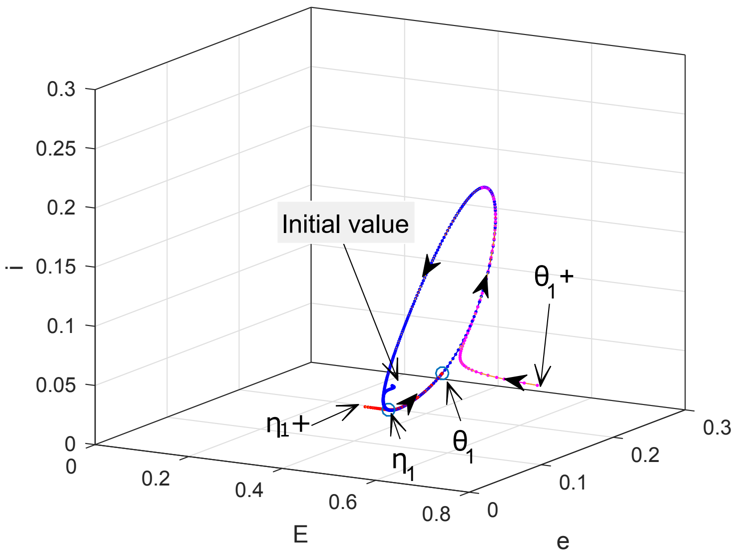

Figure 3.5: Formation of Figure 3.4.

In the following figures, we will see that for different values of the parameter μ and for the different values of the initial conditions in various domain of attractions, we will obtain different medusas and rings. First of all, consider the system (3.13) with the initial values $(-0.01, 0, 0.17, 0.25)$, $(0.21, 0.20, 0.20, 0.15)$, $(0.5, 0.5, 0.3, 0.3)$, and with the parameter $\mu = 0.9$ to get Figure 3.6. In this figure, there is a medusa without ring, a medusa and a cycle. Indeed, they are a single trajectory, which is disconnected in the geometrical sense, but it is connected in the dynamics sense.

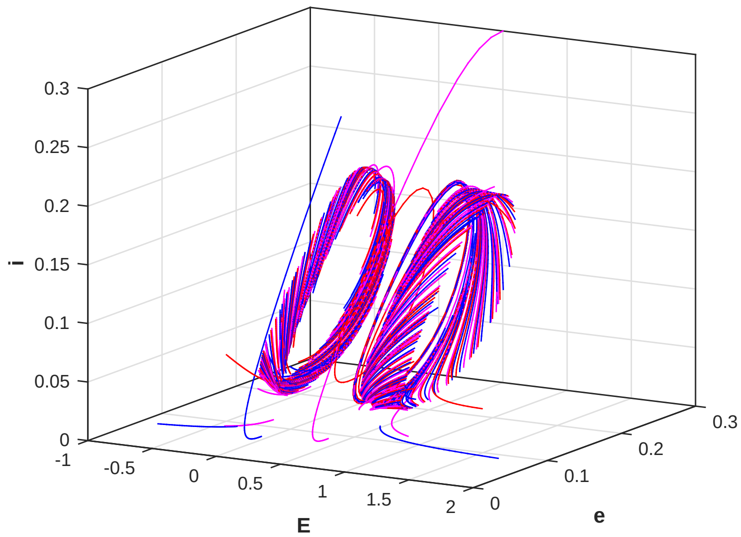

Figure 3.6: (E,e,i)-coordinates of system (3.13) for the initial values $(-0.01, 0, 0.17, 0.25)$, $(0.21, 0.20, 0.20, 0.15)$, $(0.5, 0.5, 0.3, 0.3)$, and for the parameter $\mu = 0.9$. Blue, red and magenta trajectories correspond to each initial value, respectively. It is seen that two medusas one of which is without ring and one cycle are formed. The cycle is between two medusas.

Next, we change the parameter to $\mu = 0.2$ and use the initial activations.

In Figure 3.7, one medusa and two different rings are emerged. Geometrically, the attractor is disconnected. However, it is connected in the dynamics sense since it is a single attractor with three parts. There does not exist any limit cycle. The cycles which look like limits cycles are just parts of the whole trajectory.

Let us consider Figure 3.8. In this figure, the initial activations are the same as in Figure 3.7. The parameter is fixed and $\mu = 0.1$. Although the initial values are different, the trajectories eventually obtain the shape of the same medusa. There is an alone red trajectory. It is a part of the whole red trajectory. Therefore, it is neither a

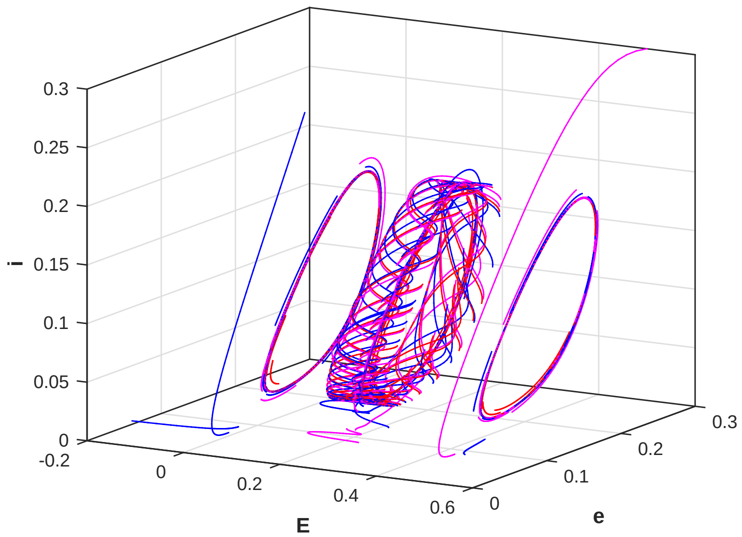

Figure 3.7: Attractor consists of one medusa and two different rings. Blue, red and magenta trajectories represent solutions in the coordinates (E,e,i) for the given initial values $(-0.01, 0, 0.17, 0.25)$, $(0.21, 0.20, 0.20, 0.15)$, $(0.5, 0.5, 0.3, 0.3)$, respectively, and $\mu = 0.2$.

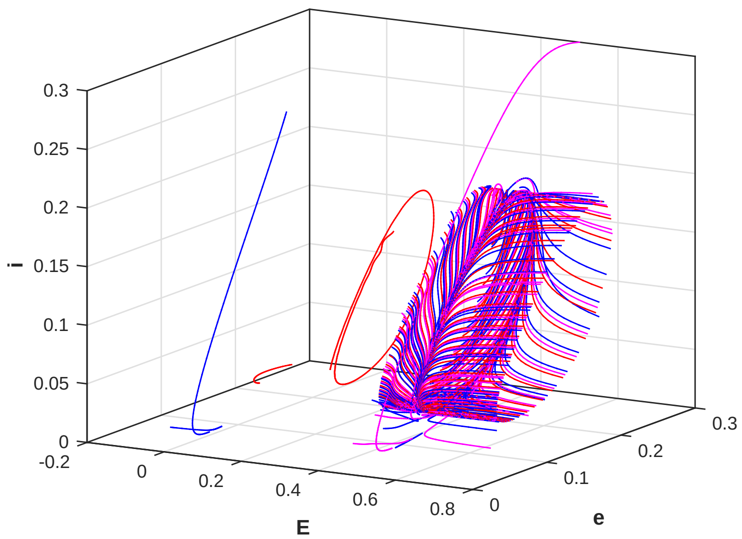

Figure 3.8: Attractor consist only one medusa. Blue, red and magenta trajectories represent solutions in the coordinates (E,e,i) for the given initial values $(-0.01, 0, 0.17, 0.25)$, $(0.21, 0.20, 0.20, 0.15)$, $(0.5, 0.5, 0.3, 0.3)$, respectively, and $\mu = 0.1$. The alone red cycle is not an attractor. It is just a part of the trajectory.

limit cycle nor a ring since the trajectory never comes to the neighborhood of it.

Finally, fix the parameter $\mu = 0.05$. In Figure 3.9, any trajectory from the different initial values blue $(-0.01, 0, 0.17, 0.25)$, red $(0.21, 0.20, 0.20, 0.15)$ and magenta $(0.5, 0.5, 0.3, 0.3)$ ultimately gets the form of red or blue medusa. The blue and the magenta trajectories converges to the same medusa. This is why, we will say that the attractor consists of two disjoint medusas. They are disjoint since there is not a single trajectory which makes two medusas.

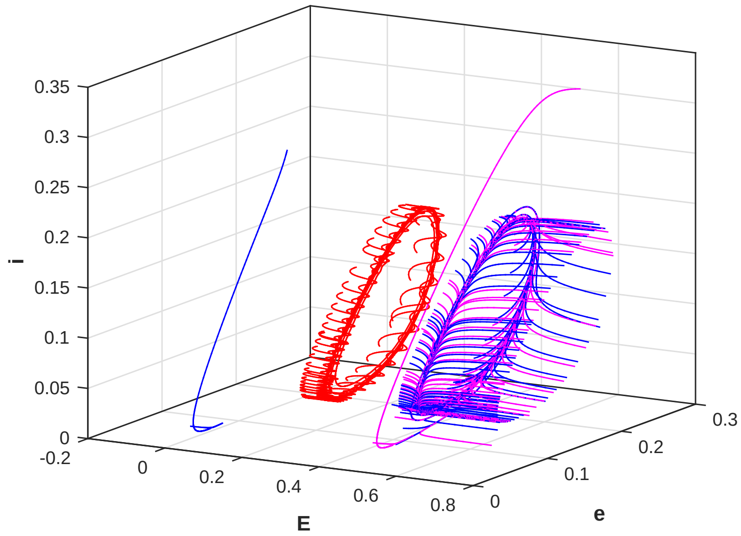

Figure 3.9: Trajectories of system (3.13) in coordinates (E,e,i) for different initial values $(-0.01, 0, 0.17, 0.25)$, $(0.21, 0.20, 0.20, 0.15)$, $(0.5, 0.5, 0.3, 0.3)$ and for the fixed parameter $\mu = 0.05$. Blue, red and magenta trajectories represent solutions for the given initial values, respectively.

Note that as the parameter decreases the form of the trajectory becomes horizontal through the E-coordinate.

In conclusion, we see that the neuron populations' dynamics have the following properties: in the case $\mu = 0.9$, two medusas one of which is without ring and a cycle are obtained. When $\mu = 0.2$ one medusa and two rings emerge and when $\mu = 0.1$ one medusa emerges. Finally, if $\mu = 0.05$ two medusas emerge. These results demonstrate that for different values of small parameter μ the qualitative changes in the behavior of trajectories of (3.13) occur ultimately. Therefore, we have a bifurcation. It is important to note that this bifurcation occurs because of the singularity and impulses. This is why, one cannot explain the bifurcations in this chapter through the traditional types of bifurcations, saddle-node, pitchfork, Hopf bifurcation, etc. For example, the change of the numbers of medusas and rings in the local phase portrait depend on the impulsive jumps' sizes. Bifurcation, here, also depends on the positions of cycles for the unperturbed system.

3.5 Conclusion

It is the first time that only a single small parameter μ causes not only to the singularity, but also to the bifurcation. The singularity in this chapter is a new kind such that it emerges both from the differential equation part and in the impulsive function. It is also important that the small parameter μ is a natural parameter which comes from the membrane time constant in Wilson-Cowan neuron model. We have shown the existence of bifurcation through the simulations. Theoretical proofs are not given since it is difficult to analyze the discontinuous dynamics of the model in which a single parameter causes both singularity and bifurcation. Therefore, bifurcation is not occurred by the change of eigenvalues, but it relates to the singular compartment and the impulsive dynamics of the model.

New type of attractor, which consists medusa, medusa without ring and rings, is defined. The name comes from the similarity of the form of trajectory and medusa.

CHAPTER 4

ANALYSIS OF IMPACT CHATTERING

4.1 Introduction

The implementation of sliding mode control is often irritated by high frequency oscillations known as “chattering” in system outputs issued by dynamics from actuators and sensors ignored in system modeling [72]. In study [62], chattering is considered as a special type of oscillation characterized by very small amplitudes that are decreasing with time. In impacting systems, it is understood as an infinite number of discontinuity moments occurring in a finite time period, for instance, a ball bouncing to rest on a horizontal surface [51]. It is asserted in [51] that chattering resembles with the inelastic collapse. The balls dissipate their energy through an infinite number of collisions in a finite time interval. Budd and Dux [30] showed that chattering can occur for a periodically forced, single degree of freedom impact oscillator with a restitution law. They demonstrated that chattering can form part of a periodic motion, and this relates to certain types of chaotic behavior. However, they studied through an example. Using the solution, they proved the existence of chattering for a linear system.

Nordmark and Piiroinen [88] considered simulation problems for chattering as well as analysis of stability of the limit cycle, which is chattering by solving the first variational equations. Moreover, they used the mappings, which are constructed with the help of a solution, in simulation schemes. Similar to the one in paper [30], it was shown that the existence of chattering for a linear system. Nonetheless, in both papers [30, 88], they do not consider the conditions which guarantee the appearance of chattering. In this study, we consider the chattering as a motion with infinite number of discontinuities in a finite time. This is the first time that sufficient conditions are provided for the chattering based on properties not on maps derived with the help of solutions, but, on conditions for the right-hand side of impulsive systems. Our models essentially are nonlinear (see, for example, Example 4.2.1). Since this is the first result in this direction, the models under consideration are respectively simple. Nevertheless, this is a class of mechanical models which can be significantly enlarged in the future investigations by consideration of large ensembles of impact oscillators and weakening conditions of the present chapter. We consider models with vibrating surface of impacts as well as analyzed problems of Pyragas controllability and existence of continuous chattering for a model connected unilaterally to a system with an impact chattering. An interesting problem of the regular perturbation of a system with chattering is discussed.

A particular feature of system with impacts is the existence of the chattering. We have two different types of it, namely complete and incomplete chattering [30, 88]. Complete chattering is the phenomenon wherein a system an infinite number of discontinuities in a finite time occurs, where the velocity tends to zero uniformly. Incomplete chattering bears on a sequence of the impacts that initially has the same behavior as complete chattering, but it ends after a large but finite number of impacts [88]. In section 3, we will discuss the transient chattering for systems with small parameter considering the transformation of the incomplete chattering to the complete one when the parameter diminishes to zero.

It was first found by Arnold [15] that the significant characteristic property of chatter vibration is that it is not generated by external periodic forces, but rather it is generated in the dynamic process itself. Therefore, it is important to emphasize that the systems under investigation in this chapter are autonomous.

Consider the problem of impact interaction of a body falling in the uniform gravity force field with a fixed horizontal base. After colliding with the base the body bounces back with the velocity whose norm is equal to the norm of the pre-impact velocity multiplied by r, where r is the restitution coefficient, $0 < r < 1$. Then, after some time interval the body will fall on the base again and the norm of its velocity will be equal to the norm of bouncing velocity in the previous collision multiplied by r. The process cannot end in a finite number of collisions. Thus, the considered phenomenon consists in following: after the initial collision a series of repeated collisions of attenuated to zero, which ends in a finite time with establishing a long contact between interacted bodies. Arising this contact results in decreasing number of degrees of freedom of the system by a unit or more. So, it is reasonable to call this phenomenon the impact chattering.

It is shown by investigations and observations that the impact chattering meets in operating almost every mechanism and machine of impact-oscillating type [80]. Various problems of impact chattering are far from trivial, and their solutions cannot be obtained in closed form for rather general case. As for the use of approximate analytical and numerous methods, it is simplified essentially if one proceeds from the conception about infinite number of impacts inside a finite time range. For example, the existence of impact chattering was investigated in [80]. They simply consider the free falling of a bead on an immobile base and on a vibrating table with constant velocity. In this chapter, we consider a more general system and prove the existence of impact chattering.

The chattering phenomena are unwanted in engineering since it is an appearance of infinite discontinuities in a short period of time and this makes theoretical analysis of mechanical models difficult. We have a research plan to consider theoretical and mathematical complexities connected to chattering, and we approach the problem from one of the two possible points of view. The first one is when mechanical models change such that the theoretical chattering disappears [8]. The other point of view, which is considered in this chapter, is that we approximate a model with infinite moments of discontinuities with those having a finite number of impacts.

This chapter is organized as follows. First of all, the impact model is stated. In this model each collision is assumed instantaneous, and it comes to rest after an infinite number of impulse moments in a finite time. The existence of chattering is proved. Asymptotic approximation of solutions with chattering are discussed in Section 4.3. Then, we show that the chattering occurs for a bead bouncing on a sinusoidally vibrating table in Section 4.4. The modified Moon-Holmes model with a small perturbation is discussed in Section 4.5. Using the continuous dependence on parameters and initial value for the impulsive differential equations with non-fixed moments, it is shown that the solution of the modified Moon-Holmes model is chattering. Following that, the appearance of continuous chattering by perturbation method is demonstrated in Section 4.6. Finally, by Pyragas control method the chattering solution is controlled to be periodic.

4.2 Existence of Chattering

An impacting system admits a chattering if there is a solution with infinite impulse moments in a finite time. Moreover, we will say that a perturbed system admits a transient chattering, if a number of impacts increases to infinity on a fixed interval as the small parameter tends to zero.

A mechanism with a rigid flat surface of impacts and the constant coefficient of restitution r, $0 < r < 1$, can be modeled by the following impulsive system

$$\begin{aligned} \ddot{x} &= f(x, \dot{x}), \\ \Delta \dot{x}|_{x=\varphi} &= -(1+r)\dot{x}, \end{aligned} \tag{4.1}$$

where $x(t)$ is the coordinate of the bead which is over the impact surface $x = \varphi$, $\dot{x}(t)$ is its velocity, $f(u, v)$ is a continuous function on the domain $H = \{0 < \varphi \leq u \leq h, |v| \leq \bar{h}\}$ for fixed positive numbers $h, \bar{h}$, and it satisfies the local Lipschitz condition in its variables on H. The equality $\Delta \dot{x}(\theta) = \dot{x}(\theta+) - \dot{x}(\theta-)$ denotes the jump operator in which $t = \theta$ is the time when the bead reaches the rigid obstacle, $\dot{x}(\theta-)$ is the pre-impact velocity and $\dot{x}(\theta+)$ is the post-impact velocity.

In system (4.1), we need the following conditions.

(C1) There is a positive number m such that $f(u, v) < -m$ for all $(u, v) \in H$,

(C2) $f(u, v) = f(u, -v)$ for all $(u, v) \in H$.

Conditions on function $f(u, v)$ and compactness of domain H imply that there exists a positive number M such that $f(u, v) \geq -M$ for all $(u, v) \in H$.

Theorem 4.2.1 *If conditions (C1), (C2) are satisfied and the following inequality*

$$M\sqrt{\frac{2(h-\varphi)}{m}} < \bar{h} \tag{4.2}$$

is valid, then all solutions with initial value $(x(0), \dot{x}(0)) = (x_0, 0)$, $\varphi < x_0 < h$, *of system* (4.1) *are chattering.*

Proof 4.2.1 *Consider an initial value* $(x_0, 0) \in H$, $\varphi < x_0 < h$. *Denoting* $x_1 = x, x_2 = \dot{x}$ *present the system* (4.1) *as*

$$\begin{aligned} \dot{x_1} &= x_2, \\ \dot{x_2} &= f(x_1, x_2), \\ \Delta x_2|_{x_1=\varphi} &= -(1+r)x_2. \end{aligned} \tag{4.3}$$

The solution of system (4.3) *starting at* $(x_0, 0)$ *is*

$$x_1(t) = x_0 + \int_0^t (t-s) f(x_1(s), x_2(s)) ds, \tag{4.4a}$$

$$x_2(t) = \int_0^t f(x_1(s), x_2(s)) ds, \tag{4.4b}$$

while it is continuous. By equation (4.4a) *and condition* $(C1)$, *the coordinate* $x_1(t)$ *decreases to* φ *such that there exists a moment* θ_1 *where* $x_1(\theta_1) = \varphi$ *and* $x_2(\theta_1) < 0$. *Moreover,* $x_1(\theta_1+) = \varphi$ *and* $x_2(\theta_1+) = -r x_2(\theta_1) > 0$.

Let us show that the solution is continuable to $+\infty$ *and it remains in the domain* H. *First of all, consider the interval* $[0, \theta_1]$. *From conditions (C1) and (C2), it implies that* $x_1(t) \le x_0 < h, t \in [0, \theta_1]$. *Using* (4.4a) *and inequality* $\varphi < x_0 < h$ *we get*

$$|h-\varphi| > |\varphi - x_0| = \left| \int_0^{\theta_1} (\theta_1 - s) f(x_1(s), x_2(s)) ds \right| \ge \int_0^{\theta_1} (\theta_1 - s) m ds = m\frac{\theta_1^2}{2},$$

which implies that $\theta_1 < \sqrt{\frac{2(h-\varphi)}{m}}$.

Consequently, from (4.4b) *and condition* (4.2)

$$|x_2(\theta_1)| = \left| \int_0^{\theta_1} f(x_1(s), x_2(s)) ds \right| \le M\theta_1 < \bar{h}.$$

Thus, we obtain that $\varphi \le x_1(t) < h$ *and* $|x_2(t)| < \bar{h}$ *for* $t \in [0, \theta_1]$.

Applying the same arguments as for θ_1 one can show that there is an intersection moment θ_2 such that $x_1(\theta_2) = \varphi$ and $x_1(t) > \varphi, t \in (\theta_1, \theta_2)$. In this interval, we have

$$x_1(t) = \varphi + x_2(\theta_1+)(t-\theta_1) + \int_{\theta_1}^{t} (t-s) f(x_1(s), x_2(s)) ds, \tag{4.5a}$$

$$x_2(t) = x_2(\theta_1+) + \int_{\theta_1}^{t} f(x_1(s), x_2(s)) ds, \tag{4.5b}$$

By condition $(C2)$, $x_2(\theta_1+)$ is the maximum value of $|x_2(t)|$ for $t \in (\theta_1, \theta_2]$. Thus, $|x_2(t)| \leq r|x_2(\theta_1)| < r\bar{h} < \bar{h}$. Moreover, from conditions $(C1)$ and $(C2)$, there exists a moment ξ_1, $\theta_1 < \xi_1 < \theta_2$, such that $x_2(\xi_1) = 0$ and $x_1(\xi_1)$ is the maximum value of $x_1(t)$ on $(\theta_1, \theta_2]$. Thus, $x_1(t) \leq x_1(\xi_1) < x_0 < h$, and the trajectory of $x(t)$ is in H for $t \in [\theta_1, \theta_2]$. Next, recursively, it can be shown that there exists an increasing sequence $\theta_i, i = 1, 2, \ldots,$ such that $x_1(\theta_i) = \varphi, i = 1, 2, \ldots,$ and the orbit of $x(t)$ is in H for all $t \geq 0$.

Now, we will show that the sequence θ_i converges. The solution of system (4.3) *is defined by*

$$x_1(t) = \varphi + x_2(\theta_i+)(t-\theta_i) + \int_{\theta_i}^{t} (t-s) f(x_1(s), x_2(s)) ds, \tag{4.6a}$$

$$x_2(t) = x_2(\theta_i+) + \int_{\theta_i}^{t} f(x_1(s), x_2(s)) ds. \tag{4.6b}$$

on the interval $(\theta_i, \theta_{i+1}], i = 1, 2, \ldots.$

Using condition (C1), it can be shown that there exists a moment ξ_i, $\theta_i < \xi_i < \theta_{i+1}$, such that $x_2(\xi_i) = 0$. Also, utilizing condition (C2), we obtain $\xi_i = \frac{\theta_i + \theta_{i+1}}{2}$. The solution on the interval $(\theta_{i+1}, \theta_{i+2}]$ is

$$\begin{aligned} x_1(t) &= \varphi + r x_2(\theta_i+)(t-\theta_{i+1}) + \int_{\theta_{i+1}}^{t} (t-s) f(x_1(s), x_2(s)) ds, \\ x_2(t) &= r x_2(\theta_i+) + \int_{\theta_{i+1}}^{t} f(x_1(s), x_2(s)) ds. \end{aligned} \tag{4.7}$$

From $x_2(\xi_i) = 0$ and $x_2(\xi_{i+1}) = 0$, we get

$$x_2(\theta_i+) = -\int_{\theta_i}^{\xi_i} f(x_1(s), x_2(s)) ds, \tag{4.8}$$

$$r x_2(\theta_i+) = -\int_{\theta_{i+1}}^{\xi_{i+1}} f(x_1(s), x_2(s)) ds. \tag{4.9}$$

Let us divide (4.9) *by* (4.8) *in order to get*

$$r = \frac{\int_{\theta_{i+1}}^{\xi_{i+1}} f(x_1(s), x_2(s))ds}{\int_{\theta_i}^{\xi_i} f(x_1(s), x_2(s))ds}.$$

Using mean value theorem, we have

$$r = \frac{(\xi_{i+1} - \theta_{i+1})f(x_1(s^*), x_2(s^*))}{(\xi_i - \theta_i)f(x_1(s^{**}), x_2(s^{**}))}, \tag{4.10}$$

for some s^{**} *and* s^* *in* (θ_i, θ_{i+1}) *and* $(\theta_{i+1}, \theta_{i+2})$ *respectively.*

Then,

$$\frac{\theta_{i+2} - \theta_{i+1}}{\theta_{i+1} - \theta_i} = \frac{\xi_{i+1} - \theta_{i+1}}{\xi_i - \theta_i} < \frac{M_i}{m_i} r, i = 1, 2, 3...., \tag{4.11}$$

where $M_i = \max\limits_{[\theta_i, \theta_{i+1}]} |f(x_1(t), x_2(t))|$ *and* $m_i = \min\limits_{[\theta_i, \theta_{i+1}]} |f(x_1(t), x_2(t))|$. *Since* $r < 1$, $\max\limits_{[\theta_i, \theta_{i+1}]} |x_1(t)| \to \varphi$ *and* $\max\limits_{[\theta_i, \theta_{i+1}]} |x_2(t)| \to 0$ *as* $i \to \infty$. *Moreover, continuity of* $f(u, v)$ *implies that* $\frac{M_i}{m_i} \to 1$ *as* $i \to \infty$. *This and* (4.11) *prove the convergence. The theorem is proved.*

Example 4.2.1 *Consider the following non-linear system*

$$\begin{aligned} &\ddot{x} + \cos(\dot{x}) + x^3 = 0, \\ &\Delta\dot{x}|_{x=2} = -(1 + r)\dot{x}, \end{aligned} \tag{4.12}$$

in the domain $2 \leq x \leq 2.5, |\dot{x}| < 7$. *We have* $f(x, \dot{x}) = -\cos(\dot{x}) - x^3 \leq -7$, $f(x, \dot{x}) = f(x, -\dot{x})$, $f(x, \dot{x}) \geq -16.625$ *in the domain. Condition* (4.2) *is true since* $16.625\sqrt{1/7} \approx 6.28 < 7$. *That is, we are in circumstances of Theorem 4.2.1 and if we choose* $r = 0.8$, $x(0) = 2.1, \dot{x}(0) = 0$, *the solution of system* (4.12) *is chattering. The simulation of this solution can be seen in Figure 4.1.*

4.3 Asymptotics

Solutions of the system (4.1) admit infinitely many jumps, and this makes, in general, impossible to find an exact solution or adequately to simulate it. So, in this section we suggest considering degenerate equation to find the perturbed system approximately. In order to increase the precision of approximation we follow the idea of asymptotic

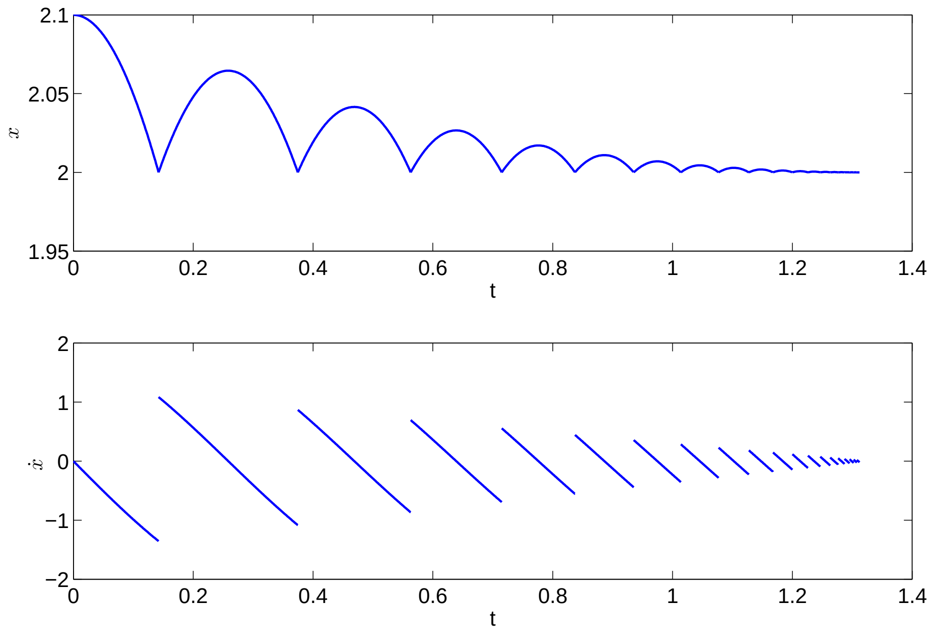

Figure 4.1: The graphs of coordinates $x(t)$ and $\dot{x}(t)$ with initials $x(0) = 2.1$ and $\dot{x}(0) = 0$ of system (4.12) with $r = 0.8$.

approximations. Consider the system

$$\begin{aligned} \ddot{x} &= f(x, \dot{x}), \\ \Delta \dot{x}|_{\substack{x=\varphi \\ i<[\frac{1}{r}]}} &= -(1+r)\dot{x}, \end{aligned} \tag{4.13}$$

where i is the index of impacts θ_i, $[.]$ denotes the greatest integer function, with additional condition that the number of impulsive moments has to be not more than $\left[\frac{1}{r}\right]$, i.e., $\theta_i, i = 1, 2, ..., \left[\frac{1}{r}\right]$. One can guarantee for the fixed value of the parameter r, the incomplete chattering occurs only. The number of impacts increases unboundedly as the parameter tends to zero. For this reason, we say that system (4.13) admits the transient chattering. Assume that this system satisfy all conditions of Theorem 4.2.1. For time $t > \theta_{[\frac{1}{r}]}$, the system is only governed by $\ddot{x} = f(x, \dot{x})$. Condition $(C1)$ implies that on the interval $[\theta_{[\frac{1}{r}]}, \theta_\infty]$, the bead stays on the position $x = \varphi$.

For each its solution, system (4.13) has finite number of discontinuity moments. That is why, one can find an exact solution of the problem or at least it is possible to make proper simulations. One can easily see that solutions of the last system and system (4.1) with identical initial data coincide on the interval $[0, \theta_{[\frac{1}{r}]})$. They are different only in the interval $[\theta_{[\frac{1}{r}]}, \theta_\infty]$. The length of the last interval diminishes to 0 as $r \to 0$. Consequently, the solutions of system (4.13) are asymptotic approximations for the solutions of system (4.1).

4.4 The Dynamics of Repeated Impacts Against a Sinusoidally Vibrating Table

In this section, we consider a mechanical model consisting of a bead bouncing on a vibrating table, which is investigated in the papers of Holmes and Guckenheimer [58, 56]. It is demonstrated that the model can generate chaos [58]. In this chapter, we show that in the mechanism one can observe another type of complex dynamics, namely chattering.

Consider a bouncing bead colliding with a sinusoidally vibrating table. Assume that the table is so massive that it does not react to collisions with the bouncing bead and it moves according to law $X(t) = X_0 \sin \omega t$. The change of the velocity of the bouncing bead at the impact moment is given by the relation $r = \frac{\dot{X}_+ - \dot{x}_+}{\dot{x}_- - \dot{X}_-}$, where r is the restitution coefficient, $0 < r < 1$, $\dot{X}_-, \dot{X}_+, \dot{x}_-, \dot{x}_+$ are the velocities of the table and the bouncing bead before and after impact, respectively. Since the collision does not affect the velocity of the table, we can write $\dot{X}_- = \dot{X}_+$. Then the model will be as follows

$$\begin{aligned} \ddot{x} &= -g, \\ \Delta \dot{x}|_{x=X} &= -(1+r)(\dot{x} - \dot{X}), \\ X(t) &= X_0 \sin(\omega t), \end{aligned} \tag{4.14}$$

where g is the gravitational acceleration ($g \approx 9.8 m/s^2$).

Now, let us consider a general form. Instead of gravitational constant g, take a function $f(u, v)$. Then, the model will be of the form

$$\begin{aligned} \ddot{x} &= f(x, \dot{x}), \\ \Delta \dot{x}|_{x=X} &= -(1+r)(\dot{x} - \dot{X}), \\ X(t) &= X_0 \sin \omega t, \end{aligned} \tag{4.15}$$

where function $f(u, v)$ is a continuous function on the domain $G = \{X_0/10 \le u \le h, |v| \le \bar{h}\}$, for fixed positive numbers $h, \bar{h}$, and it satisfies the local Lipschitz condition in its variables on G. Also, this system satisfies the conditions (C1), (C2) defined in the first section for all $(u, v) \in G$. By conditions on function $f(u, v)$ and compactness of the domain G, we have a positive number M such that $f(u, v) \geq -M$ for all $(u, v) \in G$.

Next, consider the graph of the function $X(t) = X_0 \sin \omega t$. The slope of the graph is $\dot{X}(t) = X_0 \omega \cos \omega t$. It is easily seen that if ω is small and t is near $\pi/2\omega$, the graph is close to a horizontal line. Consequently, for sufficiently small ω and for time t near $\pi/2\omega$ if the following inequality

$$M\sqrt{\frac{2(h - X_0/10)}{m}} < \bar{h} \tag{4.16}$$

is true and conditions (C1), (C2) are satisfied, according to Theorem 4.2.1 there is chattering for solutions whose integral curves are near to the point $P(\pi/2\omega, X_0)$. Finally, to demonstrate the result through simulation, we continue with the bouncing bead on the sinusoidally vibrating table.

Example 4.4.1 *Let us return to the bouncing bead on the sinusoidally vibrating table with the same properties of system* (4.15)*. Then the model will be as follows*

$$\begin{aligned} \ddot{x} &= -g, \\ \Delta \dot{x}|_{x=X} &= -(1 + r)(\dot{x} - \dot{X}), \\ X(t) &= X_0 \sin(\omega t), \end{aligned} \tag{4.17}$$

where $t \geq 0$. *Let us take* $\varphi = X_0 = 1$ *and consider the domain* $0.1 \leq x \leq 2, |\dot{x}| < 7$. *Then, we have* $f(x, \dot{x}) = -g < 0$, $|f(x, \dot{x})| = |-g| = g = M = m$ *and* $M\sqrt{\frac{2(h-X_0/10)}{m}} = \sqrt{37.24} < 7$. *If we choose the initial conditions* $x(2\pi/\omega) = 1.9, \dot{x}(2\pi/\omega) = 0$, *where* $r = 0.9$, $\omega = 0.29$, *it can be seen that the conditions of Theorem 4.2.1 are satisfied and consequently, this solution is chattering. In Figure 4.2, one can observe the coordinates of system* (4.17) *which supports our theoretical result.*

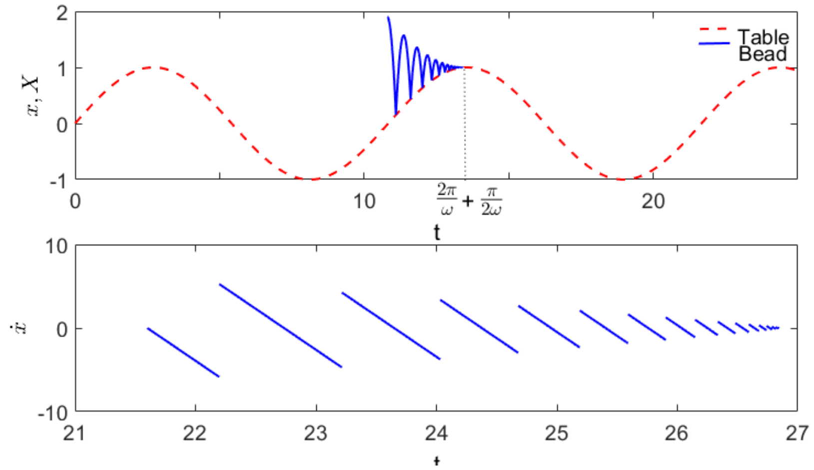

Figure 4.2: The graph of the coordinates of system (4.17).

4.5 The Modified Moon-Holmes Model

The main task of this section is to consider the modified Moon-Holmes Model. Moon and Holmes [79] showed that the Duffing equation in the form

$$\ddot{x} + \delta\dot{x} - x + x^3 = \gamma \cos wt$$

provides the simplest possible model for the forced vibrations of a cantilever beam in the nonuniform field of two permanent magnets. Such an equation describes the dynamics of a buckled beam or plate when only one mode of vibration is considered. We modify the model as adding a rigid obstacle over the magnet and in front of the beam such that the beam collides the obstacle and from Newton Law of impacts it bounces back. (The system is sketched in Figure 4.3.) The suggested model has the form of the following impulsive system

$$\begin{aligned} \ddot{x} &= -\delta\dot{x} + x - x^3 + \gamma \cos wt, \\ \Delta\dot{x}|_{x=\varphi} &= -(1+r)\dot{x}, \end{aligned} \tag{4.18}$$

where x is the distance from the wall to the end of the beam, φ is the position of the obstacle, r is the restitution coefficient. Now, if the coefficients γ and δ are equal to zero, one obtains

$$\begin{aligned} \ddot{x} &= x - x^3, \\ \Delta\dot{x}|_{x=\varphi} &= -(1+r)\dot{x}. \end{aligned} \tag{4.19}$$

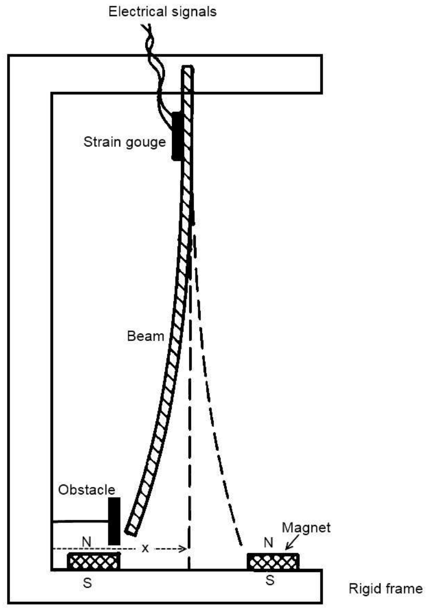

Figure 4.3: The magneto-elastic beam with the obstacle.

For this system, choose $\varphi = 1.1$ and for the domain H let $h = 1.5, \bar{h} = 3$. One can see that function $f(x, \dot{x}) = x - x^3$ satisfies conditions (C1) and (C2), i.e. $-1.875 \leq f(u, v) \leq -0.331$ for all $(u, v) \in H$ and $f(u, v)$ is an even function in v. Moreover, condition (4.2) is valid since $1.875\sqrt{\frac{2(1.5-1.1)}{0.331}} \approx 2,91 < 3$. Therefore, by Theorem 4.2.1 all solutions of system (4.19) with initial values $(x(0), \dot{x}(0)) = (x_0, 0)$, $\varphi < x_0 < h$, are chattering. Obviously, system (4.18) does not satisfy condition (C2). But, one can easily notice that for sufficiently small δ and γ, by the continuous dependence on parameters and initial value for the impulsive differential equations with non-fixed moments [2], the solutions of (4.18) with the same initial conditions of (4.19) are chattering as well. For the numerical simulation, let $(x(0), \dot{x}(0)) = (1.3, 0)$ and $r = 0.9$. Then, one can see that Figure 4.4 supports our theoretical discussion.

4.6 Continuous Chattering

In this section, we demonstrate the continuous chattering which is understood as infinitely many oscillations in finite time. Let us observe how continuous chattering appears if a mechanical model is perturbed with a discontinuous one. For this reason,

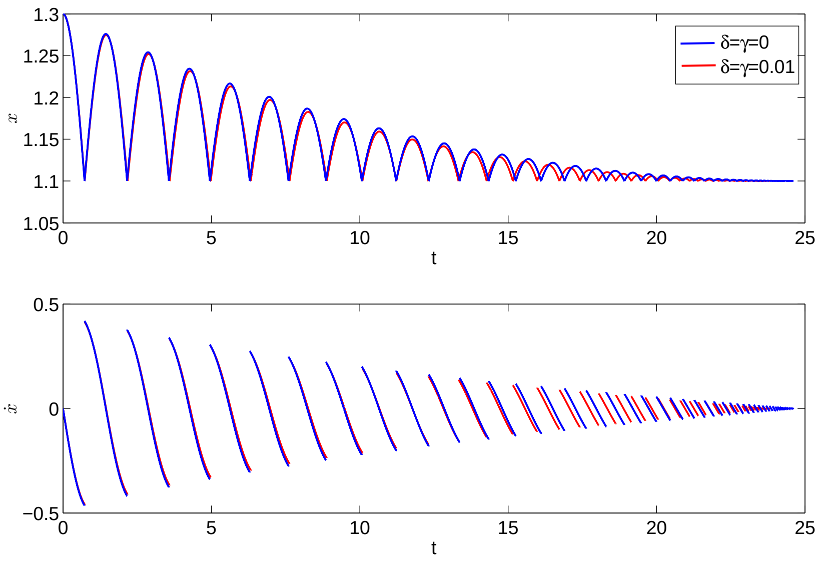

Figure 4.4: The coordinates of systems (4.18) and (4.19) with $w = 0.1$. It can be seen that the solution of perturbed system (4.18) is also chattering.

we couple system (4.1) with the following equation of a mass-spring-damper equation

$$m\ddot{y} + c\dot{y} + ky = 0, \tag{4.20}$$

with mass m, spring constant k, and viscous damper of damping coefficient c. If the characteristic equation of system (4.20) has roots with negative real parts, then it admits asymptotically stable equilibrium. By the argument of periodicity theorem for system with stable equilibrium, one can expect that in system (4.20), continuous chattering appears if it is perturbed by a chattering solution of (4.1). Thus, let us write the coupled system taking $f(x, \dot{x}) = -g$ in (4.1), $m = 1, c = 3, k = 2$ in (4.20) and $x = x_1, \dot{x} = x_2, y = x_3, \dot{y} = x_4$, in the form

$$\begin{aligned}
&\dot{x}_1 = x_2, \\
&\dot{x}_2 = -g, \\
&\Delta|_{x_1=1} = -(1+r)x_2 \\
&\dot{x}_3 = x_4, \\
&\dot{x}_4 = -2x_3 - 3x_4 + 20x_2^2.
\end{aligned} \tag{4.21}$$

with initial conditions $x_1(0) = 6, x_2(0) = 0, x_3(0) = 10, x_4(0) = -1000$ and $r = 0.9$. Since the first coupling is unilateral, the second equation does not influence

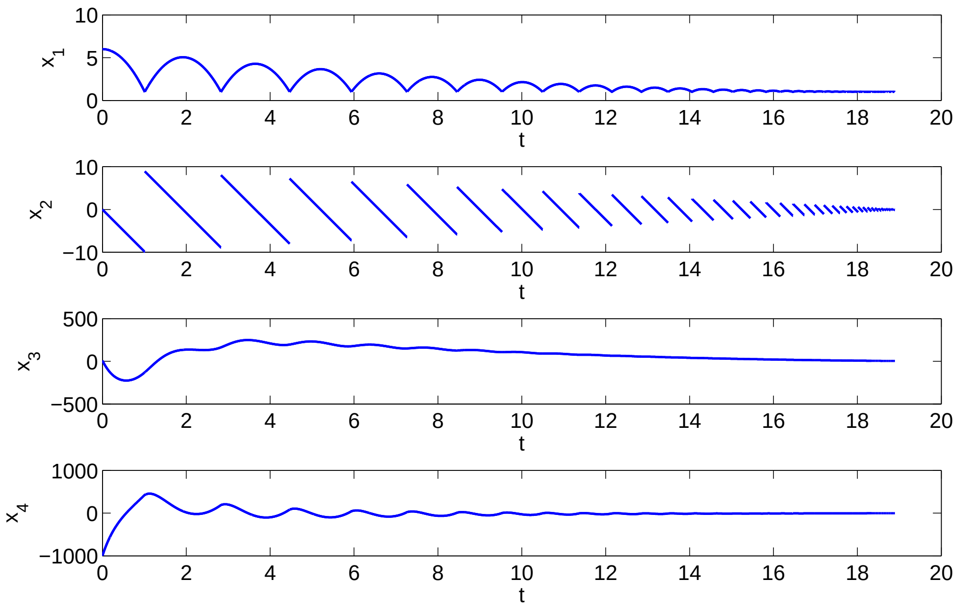

Figure 4.5: The graphs of the coordinates of system (4.21).

the first one. That is why, its dynamics are the same as in Figure 4.5. But, for the second coupling in Figure 4.5, we can see the effect of perturbation which we call as continuous chattering.

4.7 Pyragas Control

There are many papers which are searching methods to minimize and control different types of chattering [33, 12]. Also, the problem definitely has to be analyzed for the impact chattering. One can accept that the control of impact chattering is a concrete perturbation, which brings the system under control to a regular motion. That is, equilibria or periodic motions. In the circumstances of the present research, it is desired that a family of chattering solutions has to be regularized, if all of them are not possible. We will discuss, in this part of the chapter, system (4.12) of Example 4.2.1. It was shown that any solution of this system, which starts in a domain, is chattering. Let us apply the control of the form $C[x_1(t-\tau) - x_1(t)]$ to the system. It is applied, for instance, to stabilize periodic motions of chaotic dynamics, and it is called Pyragas control [94]. Now, we will apply the control to depress the chattering in the system. Let us construct the following system denoting $x_1 = x$ and $x_2 = \dot{x}$

$$\begin{aligned} \dot{x_1} &= x_2, \\ \dot{x_2} &= -x_1^3 - \cos x_2 + C[x_1(t-\tau) - x_1(t)] \\ \Delta x_2|_{x_1=2} &= -(1+r)x_2. \end{aligned} \tag{4.22}$$

We performed a series of simulations of the system with fixed $C = -30$, $\tau = 1$ and $r = 0.6$. Consider $x_2(0)$ as it was requested to prove family of chattering solutions in Theorem 4.2.1. For the initial first coordinate $x_1(0)$ we tried values starting from 2.5 to 200.

$x_1(0)$	2.5	3	5	10	100	200
Period T	1.22	1.22	1.22	1.22	1.22	1.22
Amplitude	2.933	2.933	2.933	2.933	2.933	2.933

Table 4.1: Periods and amplitudes of the first coordinate $x_1(t)$ of system (4.22) for different values of $x_1(0)$.

For all these solutions the ultimate periodicity has been approved with period $T = 1.22$. Observe that the period is different from the delay term $\tau = 1$. One can see from the table that the amplitudes are equal to 2.933 for all values of $x_1(0)$ as well. At the same time, the chattering has not been decaying for the solution with $x_1(0) = 2.1$. These all demonstrate that the control problem can be solved for the chattering, but certain conditions have to be determined to specify the controllable domains and conditions for the stability of the arranged periodic motions. We suppose that these problems will be researched in next studies.

For $x_1(0) = 3$, the periodic orbit $x_1(t)$ can be seen in Figure 4.6, which shows the effectiveness of the control.

4.8 Conclusion

In this chapter, we have considered the mechanical models with impacts. For these models, the chattering phenomenon, which is defined as a motion with infinitely many discontinuities in a finite time, is studied. The sufficient conditions are determined for

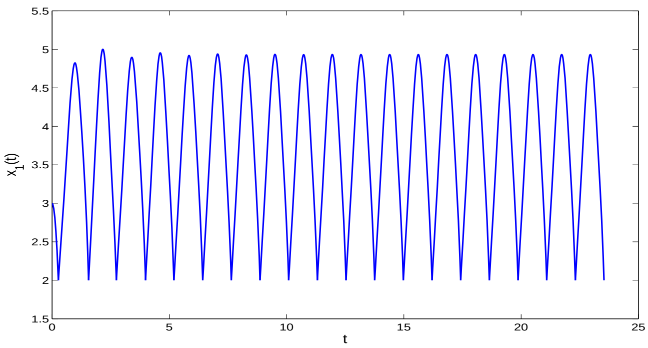

Figure 4.6: Simulation of the first coordinate $x_1(t)$ of controlled system (4.22) with initial conditions $x_1(0) = 3, x_2(0) = 0$.

the existence of the chattering. Asymptotics are discussed to find an approximation solution and to simulate the chattering solution. We study the famous example: the bouncing bead on a sinusoidally vibrating table which generates chaos [58]. It is shown that this mechanism has chattering solutions. Furthermore, we modify the Moon-Holmes model [79], which yields chaos also, with an obstacle to obtain an impacting model. We demonstrate that this model provides chattering. Perturbing a continuous mechanical process by a discontinuous one having chattering solutions, continuous chattering, which is defined as the appearance of infinitely many oscillations in a finite time, is constructed.

The application of results of paper [88] is to prove the existence of a unique chattering solution of the bouncing ball, see Section 3.5. At the same time, by simulation it is proven that a double pendulum admits chattering. Our method, in some sense, is wider than the one in paper [88]. For example, by Theorem 4.2.1 in this chapter, we have verified that there are infinitely many chattering motions with initial values in an interval. Thus, the present result is complement to that one accomplished in [88]. However, our approach does not work for the double pendulum, since condition $(C2)$ is not valid for the model. Nevertheless, in our next investigation, we plan to extend the method without condition $(C2)$.

CHAPTER 5

CHATTERING AS A SINGULAR PROBLEM

5.1 Introduction

Investigations and observations show that the impact chattering meets in operating almost every mechanism and machine of impact-oscillating type [80, 87, 125, 126]. Chattering is an important feature of impact systems [30, 36]. It is known as an infinite number of discontinuity moments occurring in a finite time period. It is asserted in [51] that chattering resembles with the inelastic collapse. The balls dissipate their energy through an infinite number of collisions in a finite time interval. Budd and Dux [30] showed that chattering can occur for a periodically forced, single degree of freedom impact oscillator with a restitution law. They demonstrated that chattering can form part of a periodic motion, and this relates to certain types of chaotic behavior. Nordmark and Piiroinen [88] considered simulation problems for chattering as well as analysis of stability of the limit cycle, which is chattering by solving the first variational equations. Moreover, they used the mappings, which are constructed with the help of a solution, in simulation schemes. Similar to the one in paper [30], it was shown that the existence of chattering for a linear system.

In paper [4], authors consider the mechanical models with Newton's Law of impacts. They provided sufficient conditions for the presence of chattering by examination of the right hand side of the impact models. The criteria for the sets of initial data which always lead to chattering were established. Moreover, they subject the Moon-Holmes model to regular impact perturbations for the chattering generation. Using the chattering solutions, they generated the continuous chattering and they applied

Pyragas control to the system in order to depress the chattering.

Two different types of chattering, namely complete and incomplete chattering [30, 88, 126] exist in impact systems. Complete chattering is the phenomenon wherein a system an infinite number of discontinuities in a finite time occurs, where the velocity tends to zero uniformly. Incomplete chattering bears on a sequence of the impacts that initially has the same behavior as complete chattering, but it ends after a large but finite number of impacts [88]. It is important to note that, in paper [126], authors showed that in an electrically driven impact microactuator, as the excitation voltage is increased the complete chattering is observed.

In this chapter, we will study three important models in mechanics; an inverted pendulum [14, 43], a bouncing ball [48, 56, 76] and a hydraulic relief valve model [60], all of which has chattering solutions. Moreover, we will study a spring-mass system for the small mass. This model will have a chattering solution with the singular properties as well.

In other respects, singular perturbation problems are common in many areas of the science since they give a high level overview of certain problems that appear in the modeling of real-world problems by differential equations [37, 52, 57, 67, 74, 77, 86, 91, 102]. These problems depend on a small positive parameter such that the solution varies rapidly in some regions and varies slowly in other regions. In the book [86], author, in order to show the difficulty that arises when a small parameter multiplies the highest derivative, studies a second order differential equation, and one can understand the nature of the singular perturbation through that discussion. On the other hand, impulse effects exist in a wide diversity of evolutionary processes that exhibit abrupt changes in their states [2, 3, 6]. In many systems, in addition to singular perturbation, there also have impulse effects [33, 34, 35, 105, 106]. Chen et al. [35] derived a sufficient condition that guarantees robust exponential stability for sufficiently small singular perturbation parameter by applying the Lyapunov function method and using a two-time scale comparison principle. In [105, 106], authors proposed Lyapunov function method to set up the exponential stability criteria for singularly perturbed impulsive systems. This method can be efficiently used to overcome the impulsive perturbation such that the stability of the original system can be ensured. In [33], Lyapunov function method was further extended to study the exponential stability of singularly perturbed stochastic time-delay systems with impulse effect. However, the stability criteria in [33, 105, 106] are all based on Lyapunov functions. There is no systematic procedure supplied therein for constructing the appropriate Lyapunov functions. The results in [33, 105, 106] only guarantee the systems under consideration to be exponentially stable for a sufficiently small positive parameter.

In this chapter, we introduce a new type of singularity. The systems under consideration have singularity which appears through moments of impacts. More precisely, we say that the impact moments are singular if they are infinite and there exist accumulation points for the moments. Since there exist an infinite number of discontinuity moments in a finite time, the possibility of the blow up of solutions occurs here. This is why, the phenomenon has to be accepted as a singular one. We will consider the system where the singularity presents only in the discontinuity moments as well as the case when it occurs not only in the moments but also in the differential equation.

The main goal of a singular perturbation problem's investigation is the analysis of possibility to approximate a solution $z(t,\mu)$ of a problem $P(\mu)$ having small parameter with a solution $\bar{z}(t)$ of the degenerate problem $P(0)$ wherein the parameter is zero. In what follows, we generally use the notations $z(t,\mu)$ and $\bar{z}(t)$ for the solutions of perturbed $P(\mu)$ and degenerate $P(0)$ problems, respectively. It is important to say that in the present research, we are fully consistent with the paradigm of the singular perturbation considering solutions of chattering problem with solutions of models without chattering.

In continuous dynamics, a parameter dependent problem $P(\mu)$ is *singular* if the convergence of a solution $z(t,\mu)$ to a solution $\bar{z}(t)$ of degenerate equation $P(0)$ is not uniform [90, 119]. In the present chapter, we provide a rigorous argument that the problem under investigation is singular from this fundamental point of view. However, during the analysis, we have found that there are additional arguments to be a singular problem which are usually not mentioned in the literature. They are:

1. The solutions $z(t,\mu)$ and $\bar{z}(t)$ are from different functional spaces. In our case, they are functions with infinitely many discontinuity moments and those with a finite number of discontinuities.

2. The set of discontinuity moments is in an interval which shrinks to a point.

Possibly, in the future, the first and the second features can be considered as sufficient conditions for a problem with discontinuities to be a singular one.

5.2 Preliminaries: Chattering in Mechanical Models

Consider the problem of impact interaction of a body falling in the uniform gravity force field with a fixed horizontal base. After colliding with the base the body bounces back with the velocity whose norm is equal to the norm of the pre-impact velocity multiplied by μ, where μ is the restitution coefficient, $0 < \mu < 1$. Then, after some time interval the body will fall on the base again and the norm of its velocity will be equal to the norm of bouncing velocity in the previous collision multiplied by μ. The process cannot end in a finite number of collisions. Thus, the considered phenomenon consists in following: after the initial collision a series of repeated collisions of attenuated to zero, which ends in a finite time with establishing a long contact between interacted bodies. Arising this contact results in decreasing number of degrees of freedom of the system by a unit or more.

In this section, we will demonstrate that some mechanical models with chattering solutions.

5.2.1 A Bouncing Ball

The most famous model in mechanics is a bouncing ball model [48, 56, 76]. Therefore, first of all, we start with a bouncing ball model. A ball is dropped from a height h_0 without initial velocity. The ball falls vertically onto a smooth horizontal surface. During the free fall, we assume that the ball is subjected only to gravity. Besides, during each bounce the collision is assumed instantaneous, i.e., the duration of contact is zero, and inelastic, i.e., a part of the kinetic energy of the ball dissipated. Therefore, the ball's velocity after a collision is smaller than before the collision, and consequently the height of bounces decreases with time. Let μ be the ratio between the ball velocity after and before the impact. This ratio is between the ball and the surface, which is assumed constant for all impacts. Thus, if v_n is the ball velocity before the nth bounce, we have the following expression

$$v_{n+1} = \mu v_n, \quad n = 1, 2, 3, ...$$

Consider Fig. 5.1, it is easy to show that the ball first strikes the surface after a time

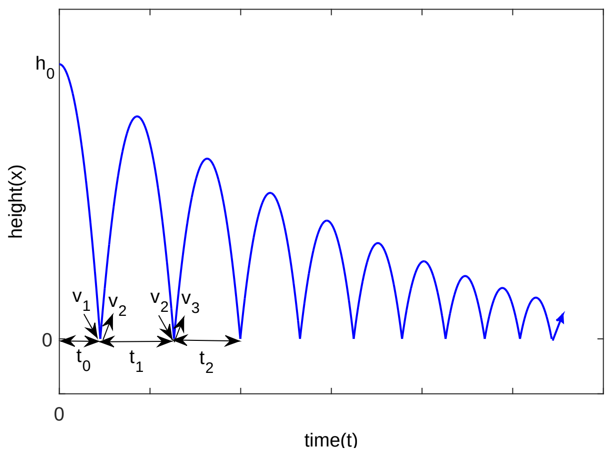

Figure 5.1: Representation of a bouncing ball.

$t_0 = \sqrt{2h_0/g}$, with a velocity $v_1 = \sqrt{2h_0 g}$ where g is the acceleration of gravity. Let t_n be the time of flight of the ball between the nth and $n+1$th bounces. Let us compute the time $t_n, n = 1, 2, 3, ...$:

$$t_n = \frac{2v_n}{g} = \mu^n \sqrt{\frac{8h_0}{g}}, \quad n = 1, 2, 3, ...$$

Now, we describe the system as follows:

$$\begin{aligned} &\ddot{x} = -g, \quad \Delta \dot{x} \mid_{x=0} = -(1+\mu)\dot{x}, \\ &x(0) = h_0, \quad \dot{x}(0) = 0, \end{aligned} \tag{5.1}$$

where $x \geq 0$. In Fig. 5.1 and from the above assumption, we can calculate the impact moments as: $\theta_0 = \sqrt{2h_0/g}$, $\theta_{i+1} = \theta_i + \mu^{i+1}\sqrt{\frac{8h_0}{g}}$, $i = 0, 1, 2, \ldots$. Hence,

$$\theta_\infty = \theta_0 + \sum_{i=1}^{\infty} t_n = \theta_0 + t_1 \sum_{i=1}^{\infty} \mu^{n-1} = \frac{1+\mu}{1-\mu}\sqrt{\frac{2h_0}{g}},$$

since $\mu < 1$. As a result, the ball admits infinitely many impacts and stays on the surface without bouncing for $t > \theta_\infty$.

It is easily seen that if one fix the moments θ_i and take $x = y, \dot{x} = z$ in (5.1), then the motion of the bouncing ball satisfies the following equations.

$$
\begin{aligned}
&\dot{z} = -g, \quad \Delta z|_{t=\theta_i} = -(1+\mu)z, \\
&\dot{y} = z, \\
&z(0,\mu) = 0, \quad y(0,\mu) = h_0,
\end{aligned}
\tag{5.2}
$$

where $y \geq 0$. It can be seen in Fig. 5.2 that solutions of system (5.2) with initial

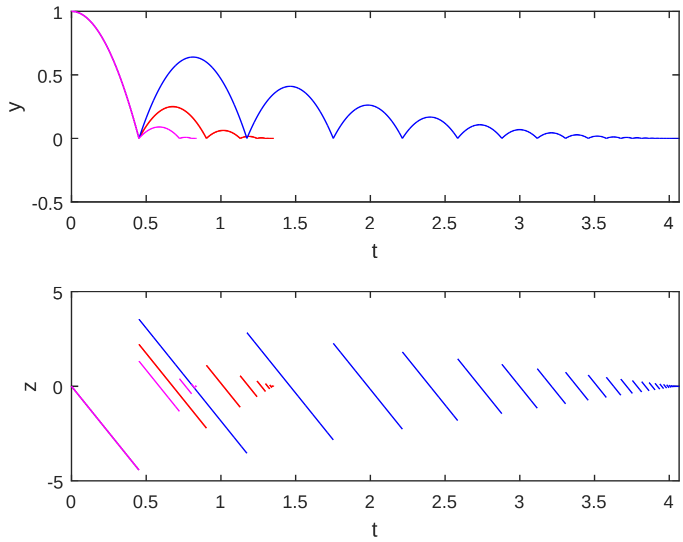

Figure 5.2: Solution of system (5.2) with initial values $z(0,\mu) = 1, y(0,\mu) = 0$ for different values of μ (blue, red and magenta represent the coordinates of (5.2) for $\mu = 0.8$, $\mu = 0.5$ and $\mu = 0.3$, respectively). It is obviously seen that as the parameter μ decreases to zero, the time of the ball to rest also decreases.

values $z(0,\mu) = 1, y(0,\mu) = 0$ for different values of μ have many impact moments. It is obvious that as the parameter μ decreases to zero, the time of the ball to rest decreases and the impact moments tend to the first impact moment. That is, as $\mu \to 0$, the solution $(z(t,\mu), y(t,\mu))$ of (5.2) ultimately looks like in Fig. 5.3. There arise questions: *1) are the functions in Fig. 5.3 are the limits of the solutions; 2) what is the type of the convergence; 3) is there a model such that the functions are their solutions?* In what follows, we will answer the questions, and moreover, we specify relations between the original model and the model with zero value of the parameter as *a singular perturbation*, such that one can approximate the solution of (5.2) by solutions of a degenerate equation. Moreover, we shall show that the interval $(\theta_0, \theta_\infty)$

is a boundary layer of the problem. Similar discussion can be made for the next two mechanical models.

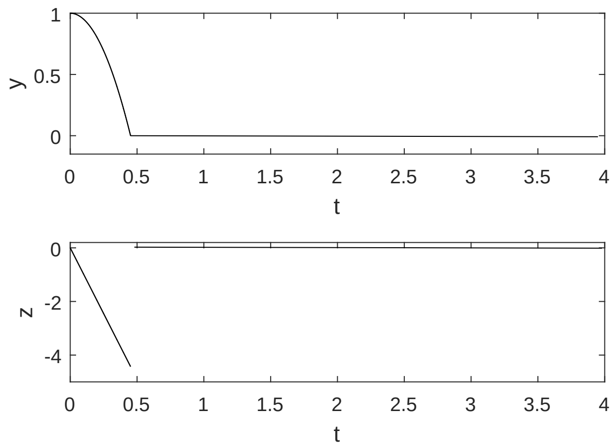

Figure 5.3: Ultimate form of the solution of system (5.2) as $\mu \to 0$.

5.2.2 An Inverted Pendulum

Next, we will consider the inverted pendulum. It is used in the modeling of various engineering applications, such as rings, printers, machine tools, dynamics of rigid standing structures and rolling railway wheel set [14, 43]. The model in [43] will be discussed which has a lateral obstacle for the chattering. The inverted pendulum has impact against the rigid flat wall with a constant restitution coefficient μ. The mechanical model can be observed in Fig. 5.4. The dynamics of the inverted

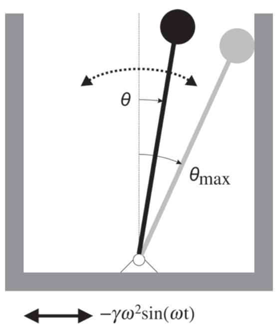

Figure 5.4: The impacting inverted pendulum [43].

pendulum between the lateral walls is described by the equations

$$\begin{aligned} &\ddot{x} + 2\delta\dot{x} - x = \gamma \sin(\omega t), |x| < 1, \\ &\Delta\dot{x}|_{|x|=1} = -(1+\mu)\dot{x}, \end{aligned} \tag{5.3}$$

where $x = \theta/\theta_{max}$ is the normalized angle (Fig. 5.4), δ is the viscous damping $(0 < \delta < 1)$, $f(t) = \gamma \sin(\omega t)$ is the harmonic excitation representing the horizontal acceleration of the base. During the motion of the impacting pendulum, we will take the wall at the position $x = 1$ as an impacting surface, $0 \leq x \leq 1$, $\gamma = 0.001, \omega = 5, \delta = -0.005$ and $\mu = 0.9$. Denote $x = y, \dot{x} = z$. Then, system (5.3) will be

$$\begin{aligned} &\dot{z} = 0.01z + y + 0.001\sin(5t), \quad \Delta z|_{y=1} = -(1+\mu)z, \\ &\dot{y} = z, \end{aligned} \tag{5.4}$$

where $0 \leq y \leq 1$. In Fig. 5.5, one can observe that the pendulum performs many strikes in finite time if the initial values are $z(0,\mu) = 0, y(0,\mu) = 0$. (The detailed mathematical investigations are presented in paper [43]. In particular, it was shown that there are infinitely many strikes.)

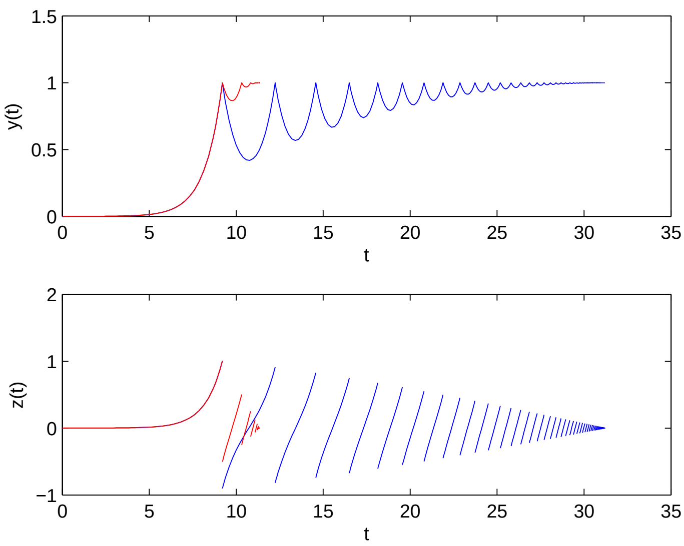

Figure 5.5: Solutions of (5.4) with initial values $z(0,\mu) = 0, y(0,\mu) = 0$. Here, red and blue lines represent the solutions for $\mu = 0.5$ and $\mu = 0.8$, respectively.

Moreover, Fig. 5.5 tells us that when μ decreases solutions of (5.4) get closer to functions demonstrated in Fig. 5.6.

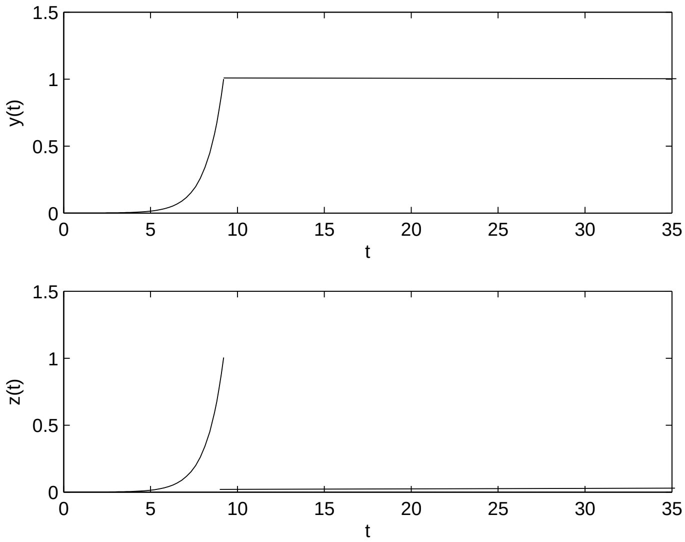

Figure 5.6: Demonstration of the solution of (5.4) with initial values $z(0,\mu) = 0, y(0,\mu) = 0$ as $\mu \to 0$ in ultimate situation.

5.2.3 A Hydraulic Pressure Relief Valve

In this subsection, a mathematical model describing the dynamics of a single stage relief valve embedded within a simple hydraulic circuit, which is derived in [60], will be discussed. The equation of motion for the valve poppet system, which is

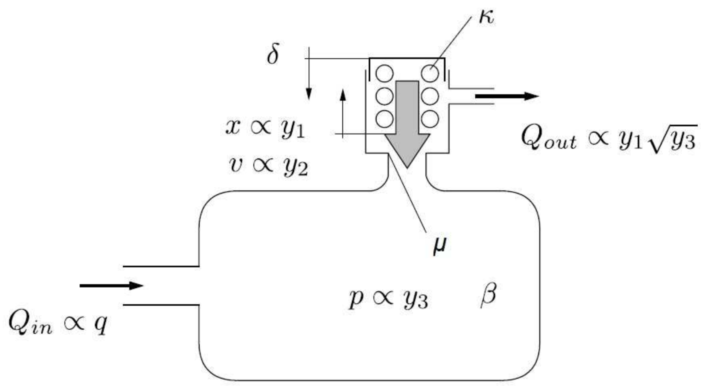

Figure 5.7: Sketch of the physical system. Here $y_{1,2,3}$ stand for the dimensionless displacement, velocity and pressure, q is the dimensionless flow rate entering the system, δ and κ are the (dimensionless) spring precompression and damping coefficients, μ is the restitution coefficient between the seat and the valve body, β is a measure of the compressibility parameter of the fluid and the elastic hoses [60].

described in Fig. 5.7, is of the form:

$$
\begin{aligned}
\dot{y}_1 &= y_2, \\
\dot{y}_2 &= -\kappa y_2 - (y_1 + \delta) + y_3, \\
\dot{y}_3 &= \beta(q - y_1\sqrt{y_3}), \\
\Delta y_2|_{y_1=0} &= -(1+\mu)y_2,
\end{aligned}
\tag{5.5}
$$

with the initial values $y_1(0,\mu) = 10$, $y_2(0,\mu) = 0$, $y_3(0,\mu) = 10$, where y_1 is the position and y_2 is the velocity of the poppet, y_3 is the pressure in the chamber. $y_1 > 0$ if the valve is open and $y_1 = 0$ if it is closed. It is assumed that $y_3 > 0$, that is, the reservoir pressure is above the ambient pressure, hence the flow direction is always outwards from the reservoir. δ and κ are the spring precompression and damping coefficients, μ is the restitution coefficient between the seat and the valve body, β is a measure of the compressibility parameter of the fluid and the elastic hoses.

In [60], it is shown that in the case of low flow rates, i.e., for small values of q, and $y_3 < \delta$ chattering motion exists. Therefore, it is reasonable to discuss this model as an example for the chattering. Fig. 5.8 represents the solutions $y_1(t,\mu), y_2(t,\mu), y_3(t,\mu)$

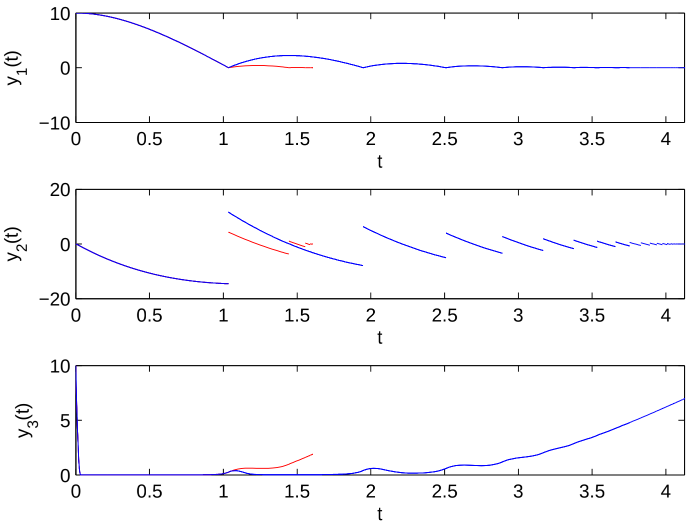

Figure 5.8: Solutions of system (5.5) for $y_1(0,\mu) = 10$, $y_2(0,\mu) = 0$, $y_3(0,\mu) = 10$, where $\beta = 20, q = 0.3, \kappa = 1.25, \delta = 20$. Here, blue and red represent the solutions for $\mu = 0.8$ and $\mu = 0.3$, respectively.

for different values of μ. Clearly, we can see that as the restitution coefficient μ decays to zero, they approach to functions $y_1(t), y_2(t), y_3(t),$ which are represented in Fig. 5.9.

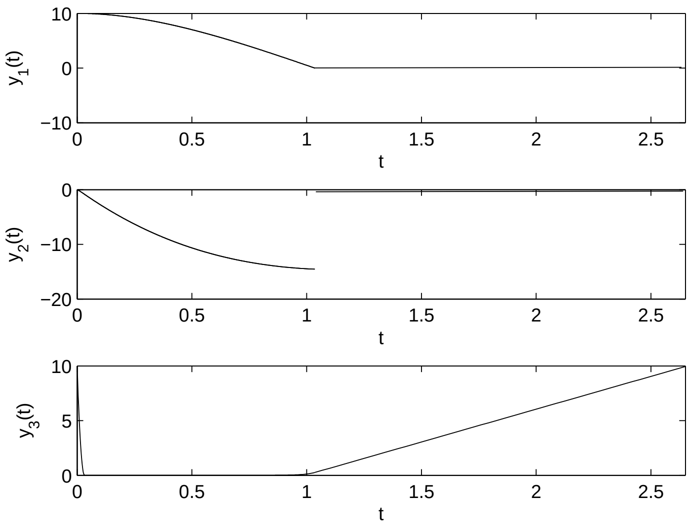

Figure 5.9: Representation of solution of system (5.5) for $y_1(0,\mu) = 10, y_2(0,\mu) = 0,$ $y_3(0,\mu) = 10,$ where $\beta = 20, q = 0.3, \kappa = 1.25, \delta = 20$ as $\mu \to 0$ in ultimate situation.

5.3 Main Results

5.3.1 Singularity in Impact Moments

We start to consider the following impulsive system

$$\frac{dz}{dt} = F(z,y,t), \quad \Delta z|_{t=\theta_i(\mu)} = I(z,y) \tag{5.6a}$$

$$\frac{dy}{dt} = f(z,y,t), \quad \Delta y|_{t=\xi_j} = J(z,y) \tag{5.6b}$$

where functions F, f, I and J are m-dimensional vector valued functions, $z, y \in \mathbb{R}^m$, $t \in [0,T]$, $\theta_1(\mu) = d_1(\mu)$, $\theta_{i+1}(\mu) = \theta_i(\mu) + d_{i+1}(\mu), i \geq 1$, $\sum_{i=1}^{\infty} d_i(\mu)$ is uniformly convergent and $d_i(0) = 0$, $0 < \xi_1 < \xi_2 < \cdots < \xi_k < T$, and $\xi_i, 1 \leq j \leq k$, is fixed. Consequently, limit $\lim_{i\to\infty} \theta_i(\mu) = \theta_\infty(\mu)$ exists and model (5.6) has infinitely many discontinuity moments in finite time.

Let us take $\mu = 0$ in (5.6). Then we obtain

$$\frac{d\bar{z}}{dt} = F(\bar{z}, \bar{y}, t), \tag{5.7a}$$

$$\frac{d\bar{y}}{dt} = f(\bar{z}, \bar{y}, t), \quad \Delta\bar{y}|_{t=\xi_j} = J(\bar{z}, \bar{y}). \tag{5.7b}$$

We will call system (5.7) as a degenerate equation for system (5.6).

Define the initial conditions (for simplicity, we set $t_0 = 0$ and it is not a jump moment.)

$$z(0, \mu) = z^0, y(0, \mu) = y^0, \tag{5.8}$$

where z^0 and y^0 are assumed to be independent of μ. Let us investigate the solution $z(t, \mu), y(t, \mu)$ of (5.6) and (5.8) on segment $0 \leq t \leq T$.

Define the domain $H = \{0 \leq t \leq T, |y| < a, |z| < b\}$. Let $\tilde{J}(z, y) = z + I(z, y), z, y \in H$. Assume that $\tilde{J}(z, y)$ satisfies the Lipschitz condition:

(D1) $\|\tilde{J}(z_1, y) - \tilde{J}(z_2, y)\| < L\|z_1 - z_2\|, 0 < L < 1, z_1, z_2, y \in H.$

We write $x = \tilde{J}_\infty(z, y), z, y \in H$, if the limit

$$\lim_{n\to\infty} \underbrace{\tilde{J}(\tilde{J}(\ldots\tilde{J}(z, y)\ldots, y), y)}_{n-times} = x$$

exists.

Fix $z^0, y^0 \in H$ such that

(D2) $\tilde{J}_\infty(z^0, y^0) = \varphi.$

Consider the following initial conditions for (5.7)

$$\bar{z}(0) = \varphi, \bar{y}(0) = y^0. \tag{5.9}$$

We need the following conditions:

(D3) Functions $F(z, y, t), f(z, y, t), I(z, y)$ and $J(z, y)$ are continuous in each argument, and $F(z, y, t), f(y, z, t)$ are continuously differentiable with respect to z and y in the domain H.

(D4) Functions $F(z,y,t), f(z,y,t)$ are bounded on H i.e., $\|F(z,y,t)\| \leq M < \infty$ and $\|f(z,y,t)\| \leq m < \infty$ for $(z,y,t) \in H$.

Theorem 5.3.1 *If conditions (D1)-(D4) are true, then solutions $z(t,\mu)$ and $y(t,\mu)$ of problem* (5.6) *with initial conditions* (5.8) *exist on $0 \leq t \leq T$, are unique, and satisfy*

$$\lim_{\mu \to 0} y(t,\mu) = \bar{y}(t) \textit{ for } 0 \leq t \leq T \tag{5.10}$$

and

$$\lim_{\mu \to 0} z(t,\mu) = \bar{z}(t) \textit{ for } 0 < t \leq T, \tag{5.11}$$

where $\bar{z}(t), \bar{y}(t)$ are solutions of (5.7) *and* (5.9).

In general, the initial condition z^0 is not equal to φ. This is why, the solution of (5.6) does not converge to the solution of (5.7) uniformly and the problem is singularly perturbed.

Proof. Let $z^0, y^0 \in H$. Then the existence and uniqueness of solutions $z(t,\mu)$ and $y(t,\mu)$ of (5.6) with (5.8) follow from [2, Theorem 2.3.2 and Theorem 2.3.4] since condition (D3) holds.

Now, consider the following system

$$\frac{d\tilde{z}}{dt} = 0, \quad \Delta\tilde{z}|_{t=\theta_i(\mu)} = I(\tilde{z}, y^0) \tag{5.12a}$$

$$\frac{d\bar{y}}{dt} = f(\bar{z}, \bar{y}, t), \quad \Delta\bar{y}|_{t=\xi_j} = J(\bar{z}, \bar{y}). \tag{5.12b}$$

Solution $\tilde{z}(t)$ of (5.12a) with initial value $\tilde{z}(0) = z^0$, for each $\mu > 0$ is equal to φ if $t \geq \theta_\infty(\mu)$.

$\theta_\infty(\mu) \to 0$ as $\mu \to 0$. Therefore, there exists $\mu_0 > 0$ such that $\theta_\infty(\mu) < \xi_1$ for $0 \leq \mu < \mu_0$. Define the recursive formula

$$T_n(\mu) = LT_{n-1}(\mu) + d_n(\mu), n \geq 2,$$

where $T_1(\mu) = Md_1(\mu)$.

Then, for $0 \leq t \leq \theta_\infty(\mu)$ compare (5.6a) and (5.12a). If $t \in [0, \theta_1(\mu)]$, we obtain

$$\|z(t,\mu) - \tilde{z}(t)\| = \left\| z^0 + \int_0^t F(z,y,s)ds - z^0 \right\|$$

$$\leq \int_0^t M ds \leq Mt \leq Md_1(\mu),$$

and for $t \in (\theta_1(\mu), \theta_2(\mu)]$:

$$\begin{aligned}
\|z(t,\mu) - \tilde{z}(t)\| = \bigg\| & z^0 + \int_0^{\theta_1} F(z,y,s)ds + I(z^0 \\
& + \int_0^{\theta_1} F(z,y,s)ds, y^0) + \int_{\theta_1}^t F(z,y,s)ds \\
& - (z^0 + I(z^0,y^0))\bigg\| \leq LMd_1(\mu) + Md_2(\mu).
\end{aligned}$$

For $t \in (\theta_2(\mu), \theta_3(\mu)]$:

$$\begin{aligned}
\|z(t,\mu) - \tilde{z}(t)\| = \bigg\| & z^0 + \int_0^{\theta_1} F(z,y,s)ds + I(z^0 \\
& + \int_0^{\theta_1} F(z,y,s)ds, y^0) + \int_{\theta_1}^{\theta_2} F(z,y,s)ds \\
& + I\bigg(z^0 + \int_0^{\theta_1} F(z,y,s)ds + I(z^0 + \\
& + \int_0^{\theta_1} F(z,y,s)ds, y^0) \\
& + \int_{\theta_1}^{\theta_2} F(z,y,s)ds, y^0 \bigg) + \int_{\theta_2}^t F(z,y,s)ds \\
& - (z^0 + I(z^0,y^0) + I(z^0 + I(z^0,y^0),y^0))\bigg\| \\
& \leq L^2Md_1(\mu) + LMd_2(\mu) + Md_3(\mu).
\end{aligned}$$

By induction, one can show that for $t \in (\theta_{n-1}(\mu), \theta_n(\mu)]$, $\|z(t,\mu) - \tilde{z}(t)\| \leq T_n(\mu)$,

$$T_n(\mu) = \sum_{i=1}^n L^{n-i} M d_i(\mu) < M \sum_{i=1}^n d_i(\mu) = M\theta_n(\mu).$$

$T_n(\mu) \to 0$ as $\mu \to 0$. Therefore, for $t \in (\theta_{n-1}(\mu), \theta_n(\mu)]$, $z(t,\mu)$ is in the neighborhood of $\tilde{z}(t)$. Moreover, as $n \to \infty$, we have $z(\theta_\infty(\mu),\mu) \to \tilde{z}(\theta_\infty(\mu)) = \varphi$. At time $t = \theta_\infty(\mu)$,

$$\bar{z}(\theta_\infty(\mu)) = \varphi + \int_0^{\theta_\infty(\mu)} F(\bar{z},\bar{y},s)ds \text{ and } \tilde{z}(\theta_\infty(\mu)) = \varphi,$$

where $\bar{z}(t)$ is the solution of (5.7) and (5.9). Hence,

$$\|\bar{z}(\theta_\infty(\mu)) - \tilde{z}(\theta_\infty(\mu))\| \leq M\theta_\infty(\mu).$$

Since $\theta_\infty(\mu) \to 0$ as $\mu \to 0$, $\tilde{z}(\theta_\infty(\mu)) \to \bar{z}(\theta_\infty(\mu))$, and so $z(\mu, \theta_\infty(\mu)) \to \bar{z}(\theta_\infty(\mu))$.

Now, if $\theta_\infty(\mu) < t \leq T$, consider the systems (5.6a) and (5.7a). By continuous dependence $z(t,\mu)$ is the neighborhood of $\bar{z}(t)$.

Similarly, if $t \in [0, \theta_1(\mu)]$, we get

$$\begin{aligned}\|y(t,\mu) - \bar{y}(t)\| = \|y^0 + \int_0^t f(z,y,s)ds - y^0 \\ - \int_0^t f(\bar{z},\bar{y},s)ds\| \leq 2mt \leq 2md_1(\mu).\end{aligned}$$

For $t \in (\theta_1(\mu), \theta_2(\mu)]$:

$$\begin{aligned}\|y(t,\mu) - \bar{y}(t)\| &= \|y(\theta_1(\mu),\mu) + \int_{\theta_1(\mu)}^t f(z,y,s)ds - \bar{y}(\theta_1(\mu)) \\ &\quad - \int_{\theta_1(\mu)}^t f(\bar{z},\bar{y},s)ds\| \\ &\leq 2md_1(\mu) + 2m(t - \theta_1(\mu)) \\ &\leq 2md_1(\mu) + 2md_2(\mu).\end{aligned}$$

By induction, for $t \in (\theta_{n-1}(\mu), \theta_n(\mu)]$, we have $\|y(t,\mu) - \bar{y}(t)\| \leq \sum_{i=1}^{n} 2md_i(\mu) = 2m\theta_n(\mu)$. Since $\theta_n(\mu) \to 0$ as $\mu \to 0$, $y(t,\mu) \to \bar{y}(t)$. Thus, $y(\theta_\infty(\mu),\mu) \to \bar{y}(\theta_\infty(\mu))$ as $\mu \to 0, n \to \infty$. Now, we examine the solution for $t > \theta_\infty(\mu)$. It is readily seen that by continuous dependence $y(t,\mu)$ is in the neighborhood of $\bar{y}(t)$ on $(\theta_\infty(\mu), T]$. Theorem is proved.

This is the time to explain, on the basis of the above theorem and proof, why the problem investigated in this chapter is a singularly perturbed problem. Indeed, a perturbation is singular if the convergence is not uniform [90, 119]. In our case, solution $(z(t,\mu), y(t,\mu))$ converges to $(\bar{z}(t), \bar{y}(t))$ uniformly on each interval $[\varepsilon, T], \varepsilon > 0$, but there is no convergence at the point $t = 0$ since $z^0 \neq \varphi$. This is why, the convergence is not uniform and it is the sufficient argument to say that in Theorem 5.3.1 a singular problem is considered. Another remarkable fact in our research is that the region in which the impact moments are placed shrinks to a single point when the parameter diminishes, i.e, $[0, \theta_\infty(\mu)] \to 0$ as $\mu \to 0$. From the above research, one can make a conclusion that $[0, \theta_\infty(\mu)]$ is a boundary layer. Finally, it should be emphasized that the singularity in this chapter is not a consequence of the small parameter multiplied by the derivative, but it is caused by singularity in impact moments.

In the next section, we will combine the singular perturbation through the small parameter multiplying the highest derivative and singularity in discontinuity moments.

5.3.2 Singularity in Impact Moments and Small Parameter Multiplying the Derivative

In the previous subsection, we show the singularity emerging from impact moments. Now, in addition to this, we will demonstrate the singularity in both from impact moments and from the small parameter in front of the derivative which can be described as follows

$$\mu\frac{dz}{dt} = F(z,y,t), \quad \Delta z|_{t=\theta_i(\mu)} = I(z,y), \tag{5.13a}$$

$$\frac{dy}{dt} = f(z,y,t), \quad \Delta y|_{t=\xi_j} = J(z,y), \tag{5.13b}$$

where all functions, discontinuity moments, domain, initial conditions are defined in subsection 5.3.1. This system is different from system (5.6) as follows: we have a small parameter multiplying the derivative and singularity in impact moments. Hence, additional condition is needed.

(D5) Suppose that $\lim\limits_{\mu\to 0^+}\dfrac{\theta_i(\mu)}{\mu} = 0, i = 1, 2, \ldots.$

Let us investigate the solution $z(t,\mu)$, $y(t,\mu)$ of (5.13) and (5.8) on segment $0 \le t \le T$.

Take $\mu = 0$ in (5.13). Then we obtain

$$\begin{aligned} &0 = F(\bar{z},\bar{y},t),\\ &\frac{d\bar{y}}{dt} = f(\bar{z},\bar{y},t) \quad \Delta\bar{y}|_{t=\xi_j} = J(\bar{z},\bar{y}). \end{aligned} \tag{5.14}$$

Assume that $0 = F(\bar{z},\bar{y},t)$ has a root $\bar{z} = \varphi$ such that condition (D2) is true. Hence, we can write

$$\begin{aligned} &\frac{d\bar{y}}{dt} = f(\varphi,\bar{y},t), \quad \Delta\bar{y}|_{t=\xi_j} = J(\varphi,\bar{y}),\\ &\bar{y}(0) = y^0. \end{aligned} \tag{5.15}$$

Introduce the adjoint system

$$\frac{d\tilde{z}}{d\tau} = F(\tilde{z}, y, t), \tag{5.16}$$

where y and t are considered as parameters, $\tilde{z} = \varphi$ is an isolated stationary point of (5.16) for $y, t \in H$. We need the following condition, also,

(D6) the stationary point $\tilde{z} = \varphi$ of (5.16) is uniformly asymptotically stable.

Theorem 5.3.2 *If conditions (D1)-(D6) are true, then solutions $z(t, \mu)$ and $y(t, \mu)$ of problem* (5.13) *with initial conditions* (5.8) *exist on $0 \le t \le T$, are unique, and satisfy*

$$\lim_{\mu \to 0} y(t, \mu) = \bar{y}(t) \textit{ for } 0 \le t \le T \tag{5.17}$$

and

$$\lim_{\mu \to 0} z(t, \mu) = \varphi \textit{ for } 0 < t \le T, \tag{5.18}$$

where $\bar{y}(t)$ is the solution of (5.15).

Proof. Let $z^0, y^0 \in H$. Similarly, the existence and uniqueness of solutions $z(t, \mu)$ and $y(t, \mu)$ of (5.13) with (5.8) follow from [2, Theorem 2.3.2 and Theorem 2.3.4] since condition (D3) holds.

Now, consider the following system

$$\frac{d\hat{z}}{dt} = 0, \quad \Delta\hat{z}|_{t=\theta_i} = I(\hat{z}, y^0) \tag{5.19a}$$

$$\frac{d\bar{y}}{dt} = f(\varphi, \bar{y}, t), \Delta\bar{y}|_{t=\xi_j} = J(\varphi, \bar{y}) \tag{5.19b}$$

which has the same discontinuity moments, impulse function and initial condition as equation (5.13). Solution $\hat{z}(t)$ of (5.19) with initial value $\hat{z}(0) = z^0$, for each $\mu > 0$ is equal to φ if $t \ge \theta_\infty$. Consequently, one can say

$$\lim_{\mu \to 0} \hat{z}(t) = \varphi, \quad 0 < t \le T.$$

$\theta_\infty(\mu) \to 0$ as $\mu \to 0$. Therefore, there exists $\mu_0 > 0$ such that $\theta_\infty(\mu) < \xi_1$ for $0 \le \mu < \mu_0$. Define the recursive formula

$$H_n(\mu) = LH_{n-1}(\mu) + \frac{d_n(\mu)}{\mu}, n \ge 2,$$

where $H_1(\mu) = M\frac{d_1(\mu)}{\mu}$.

Then, for $0 \leq t \leq \theta_\infty$ compare (5.13a) and (5.19a). If $t \in [0, \theta_1]$, we obtain

$$\begin{aligned}\|z(t,\mu) - \hat{z}(t)\| &= \left\| z^0 + \int_0^t \frac{F(z,y,s)}{\mu} ds - z^0 \right\| \\ &\leq \int_0^t \frac{M}{\mu} ds \leq \frac{M}{\mu} t \leq M\frac{d_1(\mu)}{\mu},\end{aligned}$$

and for $t \in (\theta_1, \theta_2]$:

$$\begin{aligned}\|z(t,\mu) - \hat{z}(t)\| = \Bigg\| z^0 &+ \int_0^{\theta_1} \frac{F(z,y,s)}{\mu} ds + I(z^0 \\ &+ \int_0^{\theta_1} \frac{F(z,y,s)}{\mu} ds, y^0) + \\ &+ \int_{\theta_1}^t \frac{F(z,y,s)}{\mu} ds - (z^0 + I(z^0, y^0)) \Bigg\| \\ &\leq LM\frac{d_1(\mu)}{\mu} + M\frac{d_2(\mu)}{\mu}.\end{aligned}$$

For $t \in (\theta_2, \theta_3]$:

$$\begin{aligned}\|z(t,\mu) - \hat{z}(t)\| = \Bigg\| z^0 &+ \int_0^{\theta_1} \frac{F(z,y,s)}{\mu} ds + I(z^0 \\ &+ \int_0^{\theta_1} \frac{F(z,y,s)}{\mu} ds, y^0) + \int_{\theta_1}^{\theta_2} \frac{F(z,y,s)}{\mu} ds \\ &+ I\Bigg(z^0 + \int_0^{\theta_1} \frac{F(z,y,s)}{\mu} ds + I(z^0 + \\ &+ \int_0^{\theta_1} \frac{F(z,y,s)}{\mu} ds, y^0) \\ &+ \int_{\theta_1}^{\theta_2} \frac{F(z,y,s)}{\mu} ds, y^0 \Bigg) + \int_{\theta_2}^t \frac{F(z,y,s)}{\mu} ds \\ &- (z^0 + I(z^0, y^0) + I(z^0 + I(z^0, y^0), y^0)) \Bigg\| \\ &\leq L^2 M\frac{d_1(\mu)}{\mu} + LM\frac{d_2(\mu)}{\mu} + M\frac{d_3(\mu)}{\mu}.\end{aligned}$$

By induction, for $t \in (\theta_{n-1}, \theta_n]$, $\|z(t,\mu) - \hat{z}(t)\| \leq H_n(\mu)$, and

$$H_n(\mu) = \sum_{i=1}^n L^{n-i} M\frac{d_i(\mu)}{\mu} < M\sum_{i=1}^n \frac{d_i(\mu)}{\mu} = M\frac{\theta_n(\mu)}{\mu}.$$

It follows from condition (D5) that $H_n(\mu) \to 0$ as $\mu \to 0$. Therefore, for $t \in (\theta_{n-1}, \theta_n]$, $z(t,\mu)$ is in the neighborhood of $\hat{z}(t)$. Moreover, as $n \to \infty$, we have $z(\theta_\infty, \mu) \to \hat{z}(\theta_\infty) = \varphi$.

Now, if $\theta_\infty < t \le T$, consider the systems (5.13a) and (5.19a). By condition (D6), $z(t,\mu)$ is the neighborhood of $\tilde{z}(t)$.

Similarly, if $t \in [0,\theta_1]$, we get

$$\begin{aligned}\|y(t,\mu)-\bar{y}(t)\| = \|y^0 + \int_0^t f(z,y,s)ds - y^0 \\ - \int_0^t f(\varphi,\bar{y},s)ds\| \le 2mt \le 2md_1(\mu).\end{aligned}$$

For $t \in (\theta_1,\theta_2]$:

$$\begin{aligned}\|y(t,\mu)-\bar{y}(t)\| &= \|y(\theta_1,\mu) + \int_{\theta_1}^t f(z,y,s)ds - \bar{y}(\theta_1) \\ &- \int_{\theta_1}^t f(\varphi,\bar{y},s)ds\| \\ &\le 2md_1(\mu) + 2m(t-\theta_1) \le 2md_1(\mu) \\ &+ 2md_2(\mu).\end{aligned}$$

It follows from the induction that for $t \in (\theta_{n-1},\theta_n]$ we have $\|y(t,\mu)-\bar{y}(t)\| \le \sum_{i=1}^{n} 2md_i(\mu) = 2m\theta_n(\mu)$. Since $\theta_n(\mu) \to 0$ as $\mu \to 0$, $y(t,\mu) \to \bar{y}(t)$. Thus, $y(\theta_\infty,\mu) \to \bar{y}(\theta_\infty)$ as $\mu \to 0, n \to \infty$. Now, we examine the solution for $t > \theta_\infty$. It is readily seen that by continuous dependence $y(t,\mu)$ is in the neighborhood of $\bar{y}(t)$ on $(\theta_\infty,T]$. Theorem is proved.

5.4 Examples

The first example is a scalar one.

Example 1. Consider the initial value problem

$$\begin{aligned}&\frac{dz}{dt} = z^2, \\ &\Delta z|_{t=\theta_i(\mu)} = -0.3z, \quad z(0) = z^0,\end{aligned} \tag{5.20}$$

where $\theta_1(\mu) = h\mu, \theta_{i+1}(\mu) = \theta_i(\mu) + h\mu^{i+1}i \ge 1$, and $z \in \mathbb{R}$. Here, $J(x) = x + I(x) = 0.7x$. Let us check the conditions of Theorem 5.3.1.

$$\lim_{n\to\infty} \underbrace{J(J(\ldots J(z_0)))}_{n-times} = \lim_{n\to\infty} (0.7)^n z_0 = 0 = \varphi.$$

Also, $|J(x) - J(y)| \le 0.7|x - y|$. The system corresponding (5.7) and (5.9) is

$$\frac{d\bar{z}}{dt} = \bar{z}^2, \quad \bar{z}(0) = 0. \tag{5.21}$$

Therefore, for any $z^0 \in \mathbb{R}$, solution $z(t,\mu)$ of (5.20) approaches to solution $\bar{z} = \varphi = 0$ of (5.21) as $\mu \to 0$. It can be seen in Fig. 5.10 that solution $z(t,\mu)$ of (5.20) approaches $\bar{z} = 0$ as $\mu \to 0$ for the initial value $z^0 = 1$.

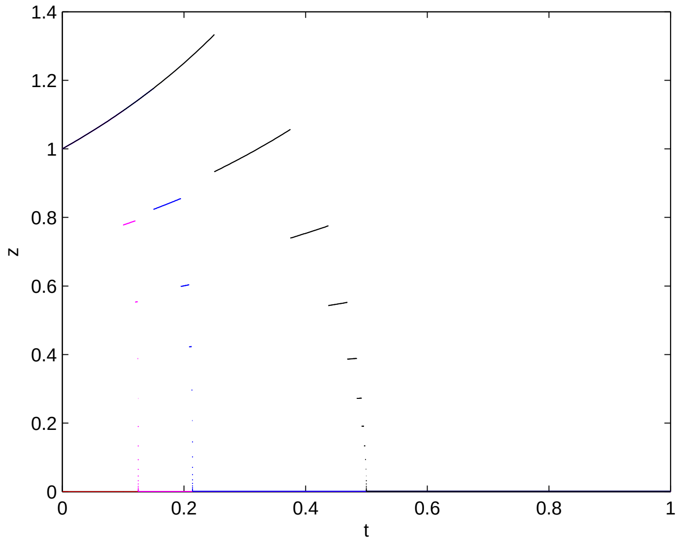

Figure 5.10: Black, blue, magenta and red represent the solution $z(t,\mu)$ of (5.20) for values of $\mu = 0.5, 0.3, 0.2, 0$, respectively, with initial value $z^0 = 1$.

Example 2. Let us consider the second example as follows.

$$\begin{aligned} \frac{dz}{dt} &= z^2 - 5z + y, \quad \Delta z|_{t=\theta_i(\mu)} = -0.8z + y^2, \\ \frac{dy}{dt} &= yz, \quad \Delta y|_{t=\xi_j} = z, \end{aligned} \tag{5.22}$$

where $0 < \mu < 1$, $\theta_1(\mu) = \sin \mu^2, \theta_{i+1}(\mu) = \theta_i(\mu) + (\sin \mu)^{i+2}, i > 1, \xi_j = 0.3 + j/24, j = 12, 13, 14, 15, 16$, with initial conditions $z(0,\mu) = 1, y(0,\mu) = 1$, on the domain $H = \{0 \le t \le 1.5, \|z\| < 2, \|y\| < 10\}$. It is easily seen that the conditions of Theorem 5.3.1 are satisfied. That is, $J(z, y) = z + I(z, y) = 0.2z + y^2$ satisfies

$$\|J(z_1, y) - J(z_2, y)\| \le 0.2\|z_1 - z_2\|,$$

$$J_\infty(1,1) = \lim_{n\to\infty}\left((0.2)^n + \sum_{i=1}^{n-1}(0.2)^i\right) = 5/4.$$

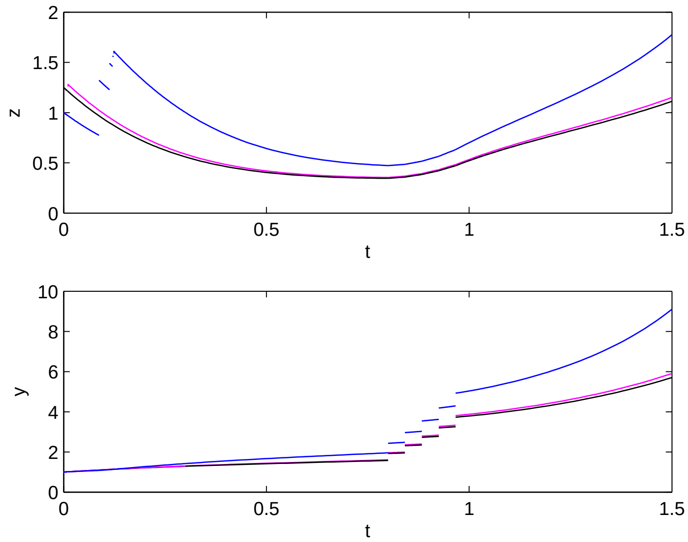

Figure 5.11: Magenta and blue represent the coordinates $z(t,\mu), y(t,\mu)$ of (5.22) with initial conditions $z(0,\mu) = 1, y(0,\mu) = 1$, for $\mu = 0.1$, $\mu = 0.3$, respectively, and black represents the coordinates $\bar{z}(t), \bar{y}(t)$ of (5.23) with $\bar{z}(0) = 5/4, \bar{y}(0) = 1$.

Now, consider the following system which is obtained by taking $\mu = 0$ in (5.22) with new initial conditions:

$$\begin{aligned} \frac{d\bar{z}}{dt} &= \bar{z}^2 - 5\bar{z} + \bar{y}, \\ \frac{d\bar{y}}{dt} &= \bar{y}\bar{z}, \quad \Delta\bar{y}|_{t=\xi_j} = \bar{z}, \end{aligned} \tag{5.23}$$

with $\bar{z}(0) = 5/4, \bar{y}(0) = 1$. By Theorem 5.3.1, solutions $z(t,\mu), y(t,\mu)$ of (5.22) with $z(0,\mu) = 1, y(0,\mu) = 1$, tend to solutions $\bar{z}(t), \bar{y}(t)$ of (5.23) as $\mu \to 0$, respectively, which are illustrated in Fig. 5.11.

5.5 Asymptotic Approximations

Solutions of the systems defined in this chapter admit infinitely many jumps, and this makes, in general, impossible to find an exact solution or adequately to simulate it. So, in this section we suggest a model with finitely many impacts to find the solution of the perturbed system approximately. In order to increase the precision of approximation we follow the idea of asymptotic approximations. In systems, we will take finitely many discontinuity moments to find an asymptotic approximation. That is, the discussed systems are equipped with the same properties except infinite impact moments.

Let us discuss the asymptotic approximation for each proposed system. Through this section, [.] denotes the greatest integer function.

The solution of system (5.6) with (5.8) has the following asymptotic representation

$$z(t,\mu)=\begin{cases} z_m(t) & \text{if } 0\le t\le \theta_{m+1}(\mu),\\ z_m(t)+\bar{z}(t) & \\ -\tilde{J}_m(z^0,y^0)+\varepsilon_1(t,\mu) & \text{if } \theta_{m+1}(\mu)<t\le T,\end{cases}$$

$$y(t,\mu)=\begin{cases} y_m(t) & \text{if } 0\le t\le \theta_{m+1}(\mu),\\ \bar{y}(t)+\tilde{\varepsilon}_1(t,\mu) & \text{if } \theta_{m+1}(\mu)<t\le T,\end{cases}$$

where $\tilde{J}_m(z,y)$ is defined in Section 5.3.1, $\varepsilon_1(t,\mu)\to 0$ and $\tilde{\varepsilon}_1(t,\mu)\to 0$ as $\mu\to 0$, $\bar{z}(t),\bar{y}(t)$ are the solutions of (5.7) and (5.9), $z_m(t),y_m(t)$ are the solutions of

$$\frac{dz_m}{dt}=F(z_m,y_m,t),\quad \Delta z_m|_{t=\theta_i(\mu)}=I(z_m,y_m), \tag{5.24a}$$

$$\frac{dy_m}{dt}=f(z_m,y_m,t),\quad \Delta y_m|_{t=\xi_j}=J(z_m,y_m) \tag{5.24b}$$

with initial conditions $z_m(0)=z^0, y_m(0)=y^0$, $1\le i\le m$. Here, assume that (5.24) has the same properties as (5.6) except infinite impact moments $\theta_i(\mu)$. Therefore, solutions $z(t,\mu),y(t,\mu)$ of (5.6) and $z_m(t),y_m(t)$ of (5.24) with the same initial values are equal on the interval $[0,\theta_{m+1}(\mu)]$. If $t\in(\theta_{m+1}(\mu),T]$,

$$\begin{aligned}\|\varepsilon_1(t,\mu)\| &= \|z(t,\mu)-z_m(t)-\bar{z}(t)+\tilde{J}_m(z^0,y^0)\|\\ &\le \|z(t,\mu)-\bar{z}(t)\|+\|z_m(t)-\tilde{J}_m(z^0,y^0)\|\end{aligned}$$

Hence, by Theorem 5.3.1, $\|z(t,\mu)-\bar{z}(t)\|<\frac{\varepsilon}{2}$ as $\mu\to 0$. Moreover, $\|z_m(t)-\tilde{J}_m(z^0,y^0)\|<\frac{\varepsilon}{2}$ as $\mu\to 0$ since $\theta_m(\mu)\to 0$ as $\mu\to 0$. Therefore, we have $\|\varepsilon_1(t,\mu)\|<\varepsilon$.

Consider, again, $t\in(\theta_{m+1}(\mu),T]$. In (5.6b), let us substitute the asymptotic approximation of $z(t,\mu)$. Then,

$$\begin{aligned}&\frac{dy}{dt}=f(z_m+\bar{z}-\tilde{J}_m(z^0,y^0)+\varepsilon_1(t,\mu),y,t),\\ &\Delta y|_{t=\xi_j}=J(z_m+\bar{z}-\tilde{J}_m(z^0,y^0)+\varepsilon_1(t,\mu),y),\end{aligned} \tag{5.25}$$

where $y(0,\mu) = y^0$. System (5.7b) with $\bar{y}(0) = y^0$ is the degenerated problem of (5.25). Thus, by regular perturbation theory for impulsive systems [2], $\|y(t,\mu) - \bar{y}(t)\| = \|\tilde{\varepsilon}_1(t,\mu)\| < \varepsilon$ for sufficiently small μ if $t \in (\theta_{m+1}(\mu), T]$. Consequently, we have proven the asymptotic representation for the solution of problem (5.6) with (5.8). Note that if we increase the impact moments we obtain a better approximation.

The following is the final asymptotic approximation which is for solution of (5.13) and (5.8).

$$z(t,\mu) = \begin{cases} z_m(t) & \text{if } 0 \le t \le \theta_{m+1}(\mu), \\ z_m(t) + \tilde{J}_{[\frac{1}{\mu}]}(z^0, y^0) & \\ -\tilde{J}_m(z^0, y^0) + \varepsilon_2(t,\mu) & \text{if } \theta_{m+1}(\mu) < t \le T, \end{cases}$$

$$y(t,\mu) = \begin{cases} y_m(t) & \text{if } 0 \le t \le \theta_{m+1}(\mu), \\ \bar{y}(t) + \tilde{\varepsilon}_2(t,\mu) & \text{if } \theta_{m+1}(\mu) < t \le T, \end{cases}$$

where $\varepsilon_2(t,\mu) \to 0$ and $\tilde{\varepsilon}_2(t,\mu) \to 0$ as $\mu \to 0$, $\bar{y}(t)$ is the solution of (5.15), $z_m(t), y_m(t)$ are the solutions of

$$\mu \frac{dz_m}{dt} = F(z_m, y_m, t), \quad \Delta z_m|_{t=\theta_i(\mu)} = I(z_m, y_m), \tag{5.26a}$$

$$\frac{dy_m}{dt} = f(z_m, y_m, t), \quad \Delta y_m|_{t=\xi_j} = J(z_m, y_m) \tag{5.26b}$$

with initial conditions $z_m(0) = z^0, y_m(0) = y^0$, $1 \le i \le m$. Suppose that (5.26) has the same properties as (5.13) except infinite impact moments $\theta_i(\mu)$s. Therefore, solutions $z(t,\mu), y(t,\mu)$ of (5.13) and $z_m(t), y_m(t)$ of (5.26) with the same initial conditions are equal on the interval $[0, \theta_{m+1}]$. Now, we need to show for $\theta_{m+1}(\mu) < t \le T$ that the asymptotic representation is true. If $t \in (\theta_{m+1}(\mu), T]$,

$$\begin{aligned} \|\varepsilon_2(t,\mu)\| &= \|z(t,\mu) - z_m(t) - \tilde{J}_{[\frac{1}{\mu}]}(z^0, y^0) + \tilde{J}_m(z^0, y^0)\| \\ &\le \|z(t,\mu) - \tilde{J}_{[\frac{1}{\mu}]}(z^0, y^0)\| + \|z_m(t) - \tilde{J}_m(z^0, y^0)\| \end{aligned}$$

Here, since $\tilde{J}_{[\frac{1}{\mu}]}(z^0, y^0) \to \varphi$ as $\mu \to 0$, by Theorem 5.3.2, $\|z(t,\mu) - \tilde{J}_{[\frac{1}{\mu}]}(z^0, y^0)\| < \frac{\varepsilon}{2}$ as $\mu \to 0$. Also, $\|z_m(t) - \tilde{J}_m(z^0, y^0)\| < \frac{\varepsilon}{2}$ as $\mu \to 0$ since $\theta_m(\mu) \to 0$ as $\mu \to 0$. Thus, we have $\|\varepsilon_2(t,\mu)\| < \varepsilon$.

Consider $\theta_{m+1}(\mu) < t \leq T$ and substitute the asymptotic value of $z(t,\mu)$ into (5.13b) to obtain

$$\begin{aligned} \frac{dy}{dt} &= f(z_m + \tilde{J}_{[\frac{1}{\mu}]}(z^0, y^0) - \tilde{J}_m(z^0, y^0) + \varepsilon_2(t,\mu), y, t), \\ \Delta y|_{t=\xi_j} &= J(z_m + \tilde{J}_{[\frac{1}{\mu}]}(z^0, y^0) - \tilde{J}_m(z^0, y^0) + \varepsilon_2(t,\mu), y), \end{aligned} \tag{5.27}$$

where $y(0,\mu) = y^0$. System (5.27) with $y(0,\mu) = y^0$ is the regularly perturbed problem of (5.15). As a result, by regular perturbation theory for impulsive systems [2], $\|y(t,\mu) - \bar{y}(t)\| = \|\tilde{\varepsilon}_2(t,\mu)\| < \varepsilon$ for sufficiently small μ if $t \in (\theta_{m+1}(\mu), T]$. Finally, the asymptotic representation of the solution of problem (5.13) with (5.8) has been proven.

Remark 5.5.1 *The precise asymptotic properties of $\varepsilon_1, \tilde{\varepsilon}_1, \varepsilon_2$ and $\tilde{\varepsilon}_2$ cannot be described in the section, since one needs concrete asymptotics for the sequence $d_i(\mu)$. Nevertheless, in the bouncing ball model in the next section the asymptotics for $d_i(\mu)$, ε_1 and for $\tilde{\varepsilon}_1$ will be found.*

5.6 Chattering in the View of Singularity

In this section, we will discuss the models in Section 5.2 as singular models defined in Section 5.3.

5.6.1 A Bouncing Ball

Consider again bouncing ball system (5.2) on $[\theta_0, T]$ since the parameter μ does not affect the solution on $[0, \theta_0]$. That is, we intend to apply Theorem 5.3.1 for a singular perturbation problem on the interval $[\theta_0, T]$, considering the moment $t = \theta_0$ instead of $t = 0$, discussed in the theorem. One can find that in the notations of the theorem $\tilde{J}(z,y) = z - (1+\mu)z = -\mu z$, $F(z,y,t) = -g$, $f(z,y,t) = z$, $t_0 = \theta_0 = \sqrt{2h_0/g}$, and impact moments $\theta_{i+1} = \theta_i + \mu^{i+1}\sqrt{\frac{8h_0}{g}}$, $i = 0, 1, 2, \ldots$, $\theta_\infty = \frac{1+\mu}{1-\mu}\sqrt{\frac{2h_0}{g}}$. Clearly, $\tilde{J}(z,y)$ has Lipschitz constant $\mu < 1$ and $\tilde{J}_\infty(z,y) = 0$ for any $z, y \in \mathbb{R}$. That is, $\varphi = 0$. Hence, conditions (D1) and (D2) are satisfied. Also, the functions $F(z,y,t) = -g$, $f(z,y,t) = z$, and $I(z,y) = -(1+\mu)z$ are continuously differentiable for any $z \in \mathbb{R}$,

$F(z, y, t)$ and $f(z, y, t)$ are bounded in a finite domain H. It implies that conditions (D3) and (D4) hold as well. Take $\mu = 0$ in (5.2) and $\bar{y}(\theta_0) = 0$, Then

$$\begin{aligned} \dot{\bar{z}} &= -g, \\ \dot{\bar{y}} &= \bar{z}, \\ \bar{y}(\theta_0) &= 0, \quad \bar{z}(\theta_0) = 0. \end{aligned} \tag{5.28}$$

which is the degenerate system of (5.2). One should emphasize that the last system is considered as a model of the ball over the table, which is placed on the level $y = 0$. That is, the motion with zero initial values is an equilibrium, since the table is an obstacle for the ball to fall. As a result, the system has the solution $\bar{z}(t) = 0, \bar{y}(t) = 0$. Conditions of Theorem 5.3.1 are satisfied. Therefore, solutions of (5.2) with initial value $(z(\theta_0, \mu), y(\theta_0, \mu))$ tend to solutions of (5.28) as $\mu \to 0$ on the interval $[\theta_0, T]$. Moreover, the sequence $d_i(\mu) = \mu^i \sqrt{\frac{8h_0}{g}}$ is described and the interval $[\theta_0, \theta_\infty]$ shrinks to the single point θ_0 as $\mu \to 0$. This interval is the boundary layer.

Solution of (5.2) on the interval $[0.4515, 4]$ has the initial value $(z(0.4515, \mu), y(0.4515, \mu)) = (-4.429, 0)$. It is difficult to simulate the solution for small value of μ. Hence, we demonstrate the singularity by handmade picture and the coordinate $z(t, \mu)$ on the interval $[0.4515, 2]$ look likes in Fig. 5.12. and the limit position of the coordinate

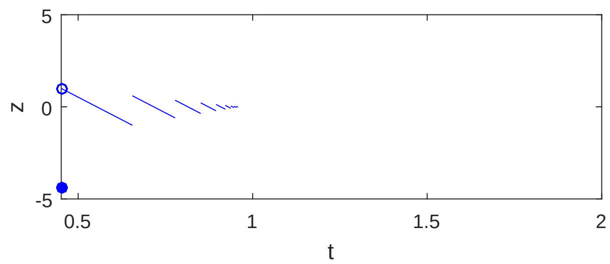

Figure 5.12: A sketch of the coordinate $z(t, \mu)$ of the solution of (5.2) on $[\theta_0, 2]$, where $z(\theta_0, \mu) = -4.429$ and $\theta_0 = 0.4515$ for a small value of μ.

as $\mu \to 0$ is pictured in Fig. 5.13.

It is useful to consider degenerate model for small value of μ according to Theorem 1. One can see from Fig. 5.12 and Fig. 5.13 that the $z(t, \mu)$ coordinate of solution of (5.2) converge to the function in Fig. 5.13. However, the $\bar{z}(t)$ coordinate of the solution of (5.28) graphically is represented just by a line. As a result, the convergence

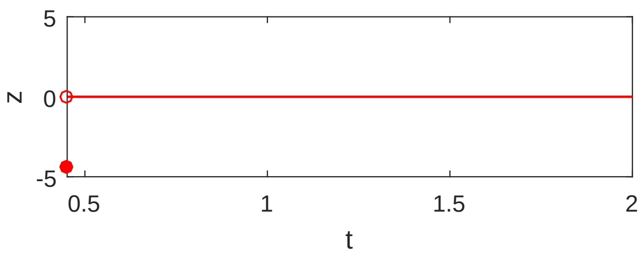

Figure 5.13: The limit of the coordinate $z(t,\mu)$ of the solution of (5.2) on $[\theta_0, 4]$, where $z(\theta_0,\mu) = -4.429$ and $\theta_0 = 0.4515$. This figure shows that the uniform convergence of the coordinate z fails at θ_0.

is not uniform and it is a singular problem. However, approximation opportunity is approved by Theorem 5.3.1.

The chattering problem is difficult to solve explicitly. Theorem 5.3.1 says that if the parameter μ is sufficiently small, then we can accept the solution of non-perturbed system (5.28) as an approximate solution of system (5.2).

Now, we find an asymptotic approximation for the bouncing ball. In the notation of Section 5.5, the solution of system (5.2) on $[0.4515, 4]$ with $(z(0.4515,\mu), y(0.4515,\mu)) = (-4.429, 0)$ has the following asymptotic representation

$$z(t,\mu) = \begin{cases} z_m(t) & \text{if } 0 \le t \le \theta_{m+1}(\mu), \\ -\tilde{J}_m(-4.429,0)+ & \\ +\varepsilon_1(t,\mu) & \text{if } \theta_{m+1}(\mu) < t \le 4, 2 \end{cases}$$

$$y(t,\mu) = \begin{cases} y_m(t) & \text{if } 0 \le t \le \theta_{m+1}(\mu), \\ \tilde{\varepsilon}_1(t,\mu) & \text{if } \theta_{m+1}(\mu) < t \le 4, \end{cases}$$

where $\varepsilon_1(t,\mu) \to 0$ and $\tilde{\varepsilon}_1(t,\mu) \to 0$ as $\mu \to 0$, since $(\bar{z}(t), \bar{y}(t)) = (0,0)$ is the solution of degenerate equation (5.28), $(z_m(t), y_m(t))$ is the solution of

$$\begin{aligned} \frac{dz_m}{dt} &= -g, \quad \Delta z_m|_{t=\theta_i} = -(1+\mu)z_m, \\ \frac{dy_m}{dt} &= z_m, \end{aligned} \tag{5.29}$$

with initial condition $(z_m(0), y_m(0)) = (-4.429, 0)$, $1 \le i \le m$. We know from Section 5.5 that on the interval $[0, \theta_{m+1}]$ the solution and the asymptotic approximation coincide. Moreover, $(z_m(t), y_m(t)) = (0,0)$ for $t \in (\theta_{m+1}, 4]$ since there is no impact moments in this interval, i.e., the ball stays on the surface. Therefore, the errors on the interval $(\theta_{m+1}, 4]$ are

$$\begin{aligned}\|\varepsilon_1(t,\mu)\| &= \|z(t,\mu) + \tilde{J}_m(-4.429, 0)\| \\ &= \|z(t,\mu) + 4.429(-1)^{m+1}\mu^m\| \\ &\leq \|z(t,\mu)\| + 4.429\mu^m \\ &\leq 8.878\mu^m\end{aligned}$$

and

$$\|\tilde{\varepsilon}_1(t,\mu)\| \leq \mu^{2m}\frac{4.429^2}{2g}.$$

5.6.2 An Inverted Pendulum

Now, let us discuss the inverted pendulum model. Consider again system (5.4) on $[9.205, 15]$ since the fist impact moment is 9.205 and the solution is not affected by the parameter μ on $[0, 9.205]$. Let $\tilde{J}(z,y) = z - (1+\mu)z = -\mu z$. Obviously, $\tilde{J}(z,y)$ has Lipschitz constant $\mu < 1$ and $\tilde{J}_\infty(z,y) = 0$ for any $z, y \in \mathbb{R}$, i.e., $\varphi = 0$. Let us consider the restitution coefficient as $\mu = 0$ in (5.4) with the initial condition $(\bar{z}(9.205), \bar{y}(9.205)) = (0, 1)$.

$$\begin{aligned}&\dot{\bar{z}} = 0.01\bar{z} + \bar{y} + 0.001\sin(5t), \\ &\dot{\bar{y}} = \bar{z}, \\ &\bar{z}(9.205) = 0, \bar{y}(9.205) = 1.\end{aligned} \tag{5.30}$$

This system is the degenerate equations of (5.4). It means that at the position $y = 1$ the pendulum has zero velocity. Therefore, it admits the equilibrium solution $(\bar{z}(t), \bar{y}(t)) = (0, 1)$. Obviously, conditions (D1)-(D4) of Theorem 5.3.1 are satisfied. Therefore, solutions $z(t,\mu), y(t,\mu)$ of (5.4) with initial $(z(9.205,\mu), y(9.205,\mu)) = (1, 1)$ tend to solutions $\bar{z}(t), \bar{y}(t)$ of (5.30) with initial $(\bar{z}(9.205), \bar{y}(9.205)) = (0, 1)$ as $\mu \to 0$. Note that the convergence of $z(t,\mu) \to \bar{z}(t)$ is not uniform on $[9.205, 15]$.

The solution of (5.4) on the interval $[9.205, 15]$ has the initial value $(z(9.205,\mu), y(9.205,\mu)) = (1, 1)$. We represent the $z(t,\mu)$ coordinate of the solution for small value of μ on the interval $[9.205, 15]$ in Fig. 5.14. Moreover, the limit position of the coordinate $z(t,\mu)$ as $\mu \to 0$ is demonstrated in Fig. 5.15. It can be seen that the convergence is not uniform on the interval $[9.205, 15]$. So, it is a singularly perturbed problem.

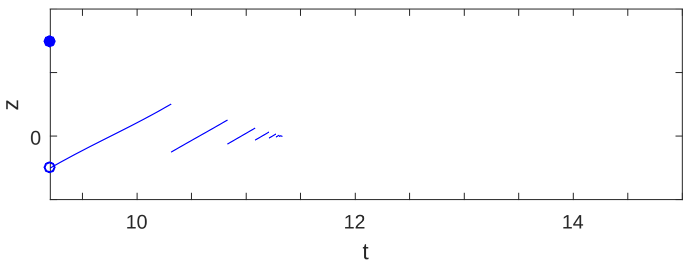

Figure 5.14: A sketch of the coordinate $z(t,\mu)$ of the solution of (5.4) on $[9.205, 15]$ for a small value of μ, where $z(9.205,\mu) = 1$ and 9.205 is the first impact moment of system (5.4)

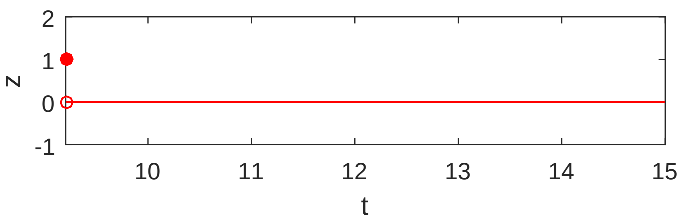

Figure 5.15: The limit of the coordinate $z(t,\mu)$ of the solution of (5.4) with initial value $(z(9.205,\mu), y(9.205,\mu)) = (1,1)$ on $[9.205, 15]$ as $\mu \to 0$.

5.6.3 A Hydraulic Relief Valve

Our third model is the hydraulic relief valve. Consider the system (5.5) on the interval $[1.038, 4]$ on which the parameter μ has an effect on the solutions. Take $\mu = 0$ in (5.5). Then, we obtain

$$\begin{aligned}
\dot{\bar{y}}_1 &= \bar{y}_2, \\
\dot{\bar{y}}_2 &= -\kappa\bar{y}_2 - (\bar{y}_1 + \delta) + \bar{y}_3, \qquad (5.31)\\
\dot{\bar{y}}_3 &= \beta(q - \bar{y}_1\sqrt{\bar{y}_3}).
\end{aligned}$$

Define $y = (y_1, y_3)$ and $\tilde{J}(y_2, y) = y_2 + I(y_2, y) = -\mu y_2$, where $I(y_2, y) = -(1+\mu)y_2$. Clearly, $\tilde{J}_\infty(y_2, y) = 0$, for any $y_1, y_2, y_3 \in \mathbb{R}$, since $0 < \mu < 1$. Also, $\tilde{J}(y_2, y)$ has a Lipschitz constant $\mu < 1$, i.e., $\varphi = 0$. For this model take the initial values $\bar{y}_1(1.038) = 0, \bar{y}_2(1.038) = 0, \bar{y}_3(1.038) = 0$. Hence, system (5.31) with these initials is the degenerate equations of (5.5) with initials $y_1(1.038,\mu) = 0$, $y_2(1.038,\mu) = -14.54$, $y_3(1.038,\mu) = 0$. Conditions of Theorem 5.3.1 are satisfied. Therefore, in (5.5), if we choose $\beta = 20, q = 0.3, \kappa = 1.25, \delta = 20$, then solutions $y_1(t,\mu), y_2(t,\mu), y_3(t,\mu)$ of (5.5) with the given initial conditions tend to solutions

$\bar{y}_1(t), \bar{y}_2(t), \bar{y}_3(t)$ of (5.31) as $\mu \to 0$ on the interval $[1.038, 4]$.

Similar to the previous mechanical models, we represent the $z(t, \mu)$ coordinate of the solution for a small value of μ on the interval $[1.038, 4]$ in Fig. 5.16. In addition,

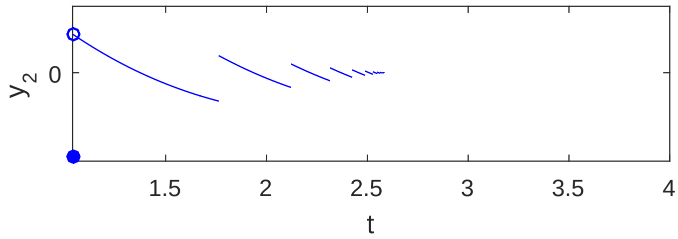

Figure 5.16: A sketch of the coordinate $y_2(t, \mu)$ of the solution of (5.5) on $[1.038, 4]$ for a small value of μ, where $z(1.038, \mu) = -14.54$, and 1.038 is the first impact moment of system (5.5).

the limit position of the coordinate y_2 as $\mu \to 0$ is shown in Fig. 5.17. One can

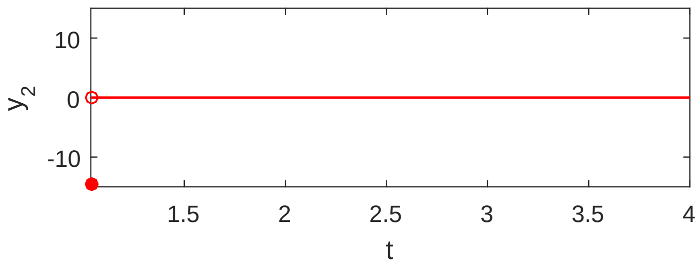

Figure 5.17: The limit of the coordinate $y_2(t, \mu)$ of the solution of (5.5) with initial values $(y_1(1.038, \mu), y_2(1.038, \mu), y_3(1.038, \mu))$=$(0, -14.54, 0)$ on the interval $[1.038, 4]$ as $\mu \to 0$.

figure out that the convergence of the coordinate $y_2(t, \mu)$ to $\bar{y}_2(t)$ is not uniform on the interval $[1.038, 4]$, which implies that this is a singularly perturbed problem as well.

5.6.4 A Spring-Mass System with a Small Mass

Now, we study a model for Theorem 5.3.2. Consider a small mass connected to a spring with a coefficient k. Assume that the surface on which mass is placed has no friction. It is released from a position x_0 without initial velocity and it moves onto a smooth vertical surface. During the process, we assume that the mass is subjected

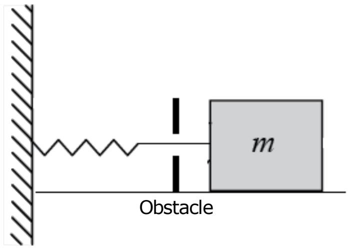

Figure 5.18: A Spring-mass system with an obstacle.

only to the spring's coefficient. The mathematical model of this problem is as follows.

$$\mu\ddot{x} + kx = 0, \tag{5.32}$$

$$\Delta\dot{x}|_{x=0} = -(1+\mu)\dot{x}, \tag{5.33}$$

where k is the spring coefficient, μ is the mass as well as the restitution coefficient. From the equation (5.33), one obtains

$$\dot{x}^{+} = -\mu\dot{x}^{-} \text{ if } x = 0. \tag{5.34}$$

Fix a small $\mu > 0$. One can find from the simple calculation that any solution $x(t,\mu)$ of (5.32) and (5.34) with initial value $x(0,\mu) = x_0, \dot{x}(0,\mu) = 0$ is chattering and the moments of impacts depend on μ. A simulation of this solution with $x(0,\mu) = 0.5, \dot{x}(0,\mu) = 0$ is presented in Fig. 5.19. Consequently, one can obtain that the solution satisfies the system

$$\begin{aligned} &\mu\dot{z} = -ky, \quad \Delta z|_{t=\theta_i(\mu)} = -(1+\mu)z, \\ &\dot{y} = z, \end{aligned} \tag{5.35}$$

where $y = x, z = \dot{x}$. It is easy to verify that system (5.35) satisfies the conditions of Theorem 5.3.2, and the degenerate system (5.14) admits the form

$$\begin{aligned} 0 &= -k\bar{y}, \\ \dot{\bar{y}} &= \bar{z}, \end{aligned} \tag{5.36}$$

such that $(0,0)$ is the solution of this problem. It is asymptotically stable as the mass is pressed to the wall motionless. Thus, one can observe the consequences of the singular perturbation through the simulations. They are presented in Fig. 5.20. It is important to say that, in Fig. 5.20 (a), the limit process is very similar to the that one can see in simulations of continuous dynamics with singular perturbation. This confirms one more time that the singular problem is under discussion.

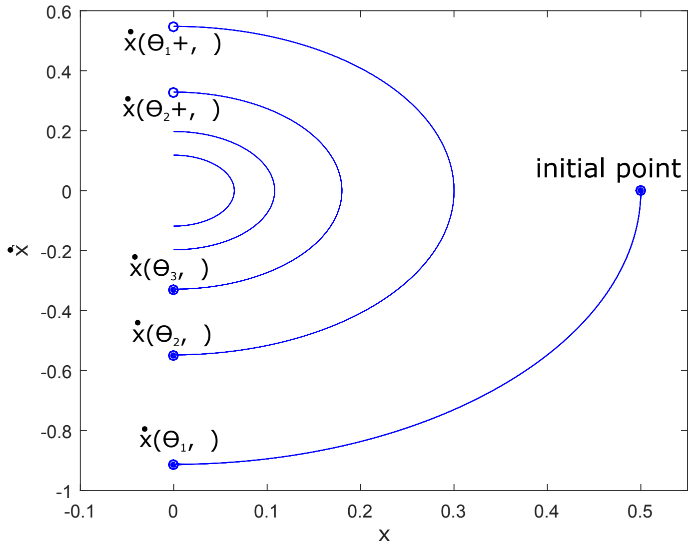

Figure 5.19: The solution of (5.35) with 5 impacts and with the initial $(0.5, 0)$ where $k = 2$, $\mu = 0.6$.

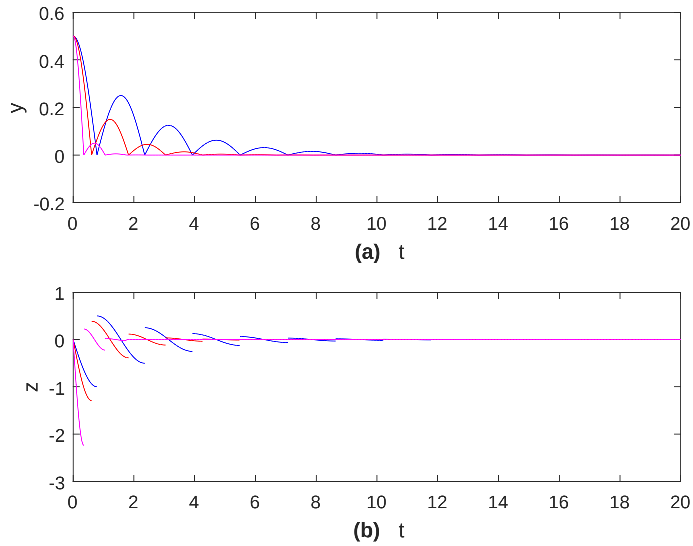

Figure 5.20: The coordinates of (5.32) and (5.34) with the initial value $(0.5, 0)$ where $k = 2$. Blue, red and magenta lines represent the coordinates for $\mu = 0.5$, $\mu = 0.3$ and $\mu = 0.1$, respectively.

5.7 Conclusion

In this chapter, the chattering through the singularity point of view has been discussed. The chattering property is known as the appearance of an infinite number of impact moments occurring in a finite time. The singularity of impact moments has been introduced. The chattering problem has been discussed as a singularly perturbed problem. Three important mechanical models; a bouncing ball, an inverted pendulum and a hydraulic relief valve models have been studied for the chattering problem as a singular one. Additionally, the spring-mass system with the small mass and with the chattering solution has been discussed as a singularly perturbed problem. Models with chattering are sophisticated for analysis because of infinitely many impact moments accumulated in finite time. The result of this chapter which formulates the chattering as a singular problem will help researchers to consider degenerate systems without chattering to approximate initial models and reducing the complexity. We consider the system with singularity in impact moments as well as the system with both singularity in impact moments and small parameter multiplying the derivative.

CHAPTER 6

CONCLUSION AND FUTURE WORKS

In this dissertation, we studied singular perturbation problem and chattering phenomenon in discontinuous dynamical systems. We introduced the infinite number of impulse moments depending on a small parameter and it was called as the singular impulse moments. This definition gives a useful and convenient way to handle the chattering problems in mechanics. It is the first time that the existence of chattering is proved by considering the right hand side of the system. Moreover, the singular impulse functions were used in the models, which can cause solutions to blow up at jump moments as the parameter goes to zero. A new type of bifurcation, namely medusa bifurcation, was obtained by the small parameter. This intrinsic idea is that the parameter causes both the bifurcation and the singularity and it is the membrane time constant.

After introducing some background of the general theory of singular perturbation and impulsive differential equations in Chapter 1, in the first section of Chapter 2 we studied a new Tikhonov theorem for singularly perturbed impulsive systems. It is the first time that the singularity arise both from differential equations and the impulsive functions. It was shown that singularity in the impulsive part of the systems can be treated through perturbation methods. Lyapunov's second method was used to show the stability in the rescaled time. Then, the theoretical results were obtained in this chapter by means of simulation results. The second section of Chapter 2 concerned with developing the singularly perturbed problem to singularly perturbed differential equations with both small parameter in front of the derivative and impulse function. The intrinsic idea of this chapter is that in the models, the solution approaches more than one root of the differential equation as the parameter decreases to zero. Moreover, the system has two impulse functions one of which is singular. This provides new theoretical opportunities. Appropriate examples with numerical simulations were given to illustrate the theoretical results.

Chapter 3 is based on Wilson-Cowan model. Using the theoretical novelty of Chapter 2, it was shown that only a single small parameter causes not only to the singularity, but also to the bifurcation. This parameter is a natural one such that it comes from the membrane time constant in Wilson-Cowan neuron model. As in the previous chapter, the singularity in this chapter emerges both from the differential equation part and in the impulsive function. We have shown that the existence of bifurcation through the simulations since it is difficult to analyze the discontinuous dynamics of the model in which a single parameter causes both singularity and bifurcation. New type of attractor, which consists of medusa, medusa without ring and rings, was defined. The name comes from the similarity of the form of trajectory and medusa.

In Chapter 4, we analyzed the chattering in impact models and the existence of impact chattering was proved for the autonomous systems.

Chapter 5 dealt with chattering in the view of singular problem. We implemented the definition of singular impulse moments. In the beginning of the chapter, we gave three important mechanical models, namely a bouncing ball, an inverted pendulum and a hydraulic relief valve, in which chattering solutions arise. Then, we generalized the results obtained in the first part through singular perturbation theory. Besides, these three mechanical models were discussed in the singularity point of view. Additionally, the spring-mass system with the small mass and with the chattering solution has been discussed as a singularly perturbed problem. Illustration of theoretical results were obtained through numerical simulations.

This study suggests many aspects of singular perturbation theory in discontinuous dynamical systems such that it could be further developed. We describe some ideas for possible future work.

- The models in Chapter 4 are respectively simple. Nevertheless, this is a class of mechanical models which can be significantly enlarged in the future investigations by consideration of large ensembles of impact oscillators and weakening conditions.

- Issuing from the numerical simulations one can see that results of Chapter 3 can be further developed. In particular, the existence of medusa bifurcation can be proved analytically.
- Chapter 3 interestingly presents a very powerful small parameter. We have shown that it causes both singular perturbation and bifurcation. The relation between the bifurcation and singular perturbation should be demonstrated through this parameter.
- Application of theories established in this thesis to mechanical models such as bouncing ball, inverted pendulum, hydraulic relief valve, neural networks models are subjects to be addressed. Moreover, other real world problems need to be mentioned.
- The concept of singular perturbation and chaos are related themes. It remains to be investigated that the relations between singular perturbation theory and chaos.

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PUBLICATIONS

1. M. Akhmet, S. Çağ, Analysis of Impact Chattering, Miskolc Mathematical Notes, Vol. 17 (2016), No. 2, 707–721.

2. M. Akhmet, S. Çağ, Tikhonov Theorem for Differential Equations with Singular Impulses, Turkish Journal of Mathematics (accepted).

3. M. Akhmet, S. Çağ, Chattering as a Singular Problem (submitted).

4. M. Akhmet, S. Çağ, A Differential Equation with Singular Impulses and Multistable Roots (submitted).

5. M. Akhmet, S. Çağ, Bifurcation Analysis of Wilson-Cowan Model with Singular Impulses (submitted).

CONFERENCES AND WORKSHOPS

- 1st National Workshop on Complex Dynamical Systems and Their Applications, October 2012, Ankara, Turkey. **Member of organizing committee.**

- 2nd International Workshop on Complex Dynamical Systems and Their Applications, October 2013, İstanbul, Turkey. **Attendee.**

- International Conference on Nonlinear Differential and Difference Equations: Recent Developments and Applications, May 2014, Antalya, Turkey. **Attendee.**

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- International Workshop on Dynamics of Multi-Level Systems (DYMULT) 2015, Max Planck Institute for the Physics of Complex Systems, Dresden, Germany. **Attendee.**

SEMINARS

- On Singular Perturbation, 2012, Department of Mathematics, METU.

PROJECTS

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