SPONTANEOUS SYMMETRY BREAKING AND HIGGS MECHANISM

A THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES
OF
MIDDLE EAST TECHNICAL UNIVERSITY

IŞINSU KAHRAMAN

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR
THE DEGREE OF MASTER OF SCIENCE
IN
PHYSICS

JANUARY 2017

Approval of the thesis:

SPONTANEOUS SYMMETRY BREAKING AND HIGGS MECHANISM

submitted by **IŞINSU KAHRAMAN** in partial fulfillment of the requirements for the degree of **Master of Science in Physics Department, Middle East Technical University** by,

Prof. Dr. Gülbin Dural Ünver
Dean, Graduate School of **Natural and Applied Sciences**

Prof. Dr. Sadi Turgut
Head of Department, **Physics**

Assoc. Prof. Dr. Ismail Turan
Supervisor, **Physics Department, METU**

Examining Committee Members:

Prof. Dr. Ali Ulvi Ylmazer
Physics Engineering Department, Ankara University

Assoc. Prof. Dr. Ismail Turan
Physics Department, METU

Prof. Dr. Tahmasib Aliyev
Physics Department, METU

Prof. Dr. Altu Özpineci
Physics Department, METU

Assoc. Prof. Dr. Levent Selbuz
Physics Engineering Department, Ankara University

Date:

I hereby declare that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.

Name, Last Name: IŞINSU KAHRAMAN

Signature :

ABSTRACT

SPONTANEOUS SYMMETRY BREAKING AND HIGGS MECHANISM

Kahraman, Işınsu
M.S., Department of Physics
Supervisor : Assoc. Prof. Dr. Ismail Turan

January 2017, 62 pages

The relevance of Higgs mechanism to nature has been verified recently by two experiments, CMS and ATLAS at Large Hadron Collider. Therefore, a detailed understanding of the mechanism is important more than ever.The details of Higgs potential, its stability and mass generation mechanism are going to be explored and the Higgs particle which is the quantum fluctuation of the field will be discussed within the sponteneous symmetry breaking notion. The one-loop corrections to the effective potential for various toy models as well as the Standard model are discussed within two different methods.

Keywords: Spontaneous Symmetry Breaking, Higgs Mechanism, Effective Potential

ÖZ

KENDİLİĞİNDEN SİMETRİ KIRILMASI VE HIGGS MEKANİZMASI

Kahraman, Işınsu
Yüksek Lisans, Fizik Bölümü
Tez Yöneticisi : Doç. Dr. Ismail Turan

Ocak 2017 , 62 sayfa

Higgs mekanizmasının doğayla olan ilgisi günümüzde büyük hadron çarpıştırıcısında yapılan CMS ve ATLAS adlı iki deney tarafından doğrulanmıştır. Bu yüzden, bu mekanizmayı anlamak herzamankinden daha önemli bir hale gelmiştir. Higgs potansiyelinin ayrıntıları, stabilizasyonu ve kütle kazanımı mekanizması kendiliğinden kırılma mekanizması çerçevesinde incelendi. Etkin potansiyele bir ilmek mertebesinde katkılar bazı basit modeller ve Standart model için iki ayri metod ile tartışldı.

Anahtar Kelimeler: Kendiliğinden Simetri Kırılması, Higgs Mekanizması, Etkin Potansiyel

To my lovely family...

ACKNOWLEDGMENTS

I would first like to express my gratitude to my supervisor Assoc.Prof. Ismail Turan for the useful comments, remarks and engagement through the learning process of my master thesis. His door was always open whenever I ran into any trouble or had a question about my research or anything related. I would also like to thank Research Assistant Dr. Selçuk Bilmiş for his endless help at solving all the problems about my computer issues.

Finally, I must express my very profound gratitude to my family and to my friends for providing me with unfailing support and continuous encouragement throughout my years of study and through the process of researching and writing this thesis. This accomplishment would not have been possible without them.

TABLE OF CONTENTS

LIST OF FIGURES

FIGURES

LIST OF ABBREVIATIONS

SM	Standard Model
GWS	Glashow-Weinberg-Salam
VEV	Vacuum Expectation Value
SSB	Spontaneous Symmetry Breaking
QED	Quantum Electrodynamics
QCD	Quantum Chromodynamics
EW	Electroweak
RGE	Renormalization Group Equations
MS	Minimal Subtraction
1PI	One Particle Irreducible

CHAPTER 1

INTRODUCTION

The symmetry principle in physics plays very elegant and profound role to unleash the secrets of nature at the subatomic scale. Nature seems to obey certain symmetry principles and in particle physics they turn out to be one of the most powerful guiding criteria in especially model building business. One of such examples is the gauge symmetry principle whose presence is right at the core of the so-called Standard Model (SM) [1–3].

The SM is the best model we have in order to describe the interactions among the fundamental particles in the subatomic world. Despite its remarkable achievements to predict vast number of experimental findings, for obvious reasons, the SM has been considered to be incomplete and must be a low energy manifestation of a fundamental theory. Therefore, the efforts for unveiling the mystery of the so-called *new physics* have been the prime subject of both experimental and theoretical programs.

Large Hadron Collider (LHC) at CERN in Geneva is the leading on-going collaboration to test the SM and to search for new physics signals. The machine has been operational for about seven years and the discovery of the first spinless fundamental scalar, the Higgs boson, had been discovered in 2012 [4], which is not considered part of *new physics* but rather completes the SM. After that, nothing has come out as far as *new physics* is concerned. While, as is, it is a troublesome for popular new physics scenarios on the table, the current situation may be considered that the best we have is still the SM. Thus, better understanding of the SM is really vital.

The mathematical construct of the SM relies on its gauge symmetry formulation. It is described by the gauge group $SU(3)_C \times SU(2)_L \times U(1)_Y$. So, any term in the Lagrangian density must be invariant under the gauge transformations of these groups. The pitfall is adding the mass terms for fermions and bosons, which break the gauge symmetry explicitly [5, 6]. The need for the so-called Higgs mechanism [7, 8] comes as a remedy to overcome this problem.

The mechanism envisages Higgs doublet interacting with the SM particles in a gauge invariant manner but once the doublet gets a non-zero vacuum expectation value (VEV), the fermions and vector bosons acquire mass as the symmetry gets broken spontaneously. That is, $SU(2)_L \times U(1)_Y \rightarrow U(1)_{\mathrm{em}}$. For the mechanism to work certain conditions need to be satisfied. The scalar potential, self interactions of the Higgs doublet, should be of the form allowing spontaneous symmetry breaking in true minimum. Various corrections to the scalar potential like one-loop contributions, improvements from Renormalization Group Equations (RGE), thermal corrections etc. would be essential to discuss [9, 10].

On the other hand, there are still loopholes in the mechanism. the Higgs particle must have a mass whose origin remains not addressed. Why its mass should be at the electroweak scale is another issue. For the fermion masses, what determines the hierarchy among the Yukawa couplings is still definitely a valid question even after 2012. Anyhow, the Higgs particle has been found with a mass $m_h \simeq 125$ GeV and the mechanism becomes part of reality.[1] Therefore, in-dept exploration of the mechanism is another mission that seems timely to pursue.

In this study, we first discuss the idea of symmetry in general. They are going to be classified as global and local symmetries, together with their comparison [11]. Their link with the gauge symmetry is mentioned. Then, within an abelian gauge theory, the idea behind breaking the global and local symmetries will be presented separately with a clear instructive toy model, namely the so-called φ^4 theory. A brief discussion of the Goldstone theorem is also included. The Higgs mechanism as a spontaneously broken local symmetry is just motivated and the above ideas are extended to a non-abelian framework.

In the next part, as a way to reach the Higgs sector of the Electroweak theory (also known as the Glashow-Weinberg-Salam (GWS) Model [1–3],We), the realization of the mechanism has been summarized in abelian and non-abelian cases one at a time and as an illustration the so-called Georgi-Glashow model is put at the spot, taking us a step closer to the GWS framework. At the end, the relevant part of GWS model, the Higgs sector, is expanded to show the mass generation as well as the scalar potential etc.

In the last part of the study, we concentrate on the scalar potential, $V(\Phi)$, of the GWS model and follow the path to calculate one-loop corrections. There are two main approach to carry out the computation; one is the method by Coleman and Weinberg (let us call it Coleman-Weinberg method [12]), and the other one by Lee and Sciaccaluga (call it Lee-Sciaccaluga method). The former involves the computation Feynman diagrams involving all number of external legs at one loop, which is rather

[1] The Higgs boson is being the second heaviest particle after the top quark in the SM.

cumbersome. On the other hand, The latter with a mathematical trick allows to get the result by only computing one-loop tadpole diagrams, which is a lot faster and simpler. The only price to pay is the find out the modified masses, vertices ans propagators under the scalar field shift and at the end an additional integration is required, both of which are, however, straightforward procedures. Therefore, we prefer to concentrate on the Lee-Sciaccaluga approach.

Again, in order to gain some experience first, some toy models are used. It is well known that loop corrections in quantum field theory have ill-defined behavior, resulting in infinities from the loop momenta integrals. The workaround is the renormalization procedure together with a method of regularization. We have discussed two independent regularization methods, the cut-off regularization and the dimensional regularization (see for example [13]). Again the final calculation is done in the dimensional regularization since it is more attractive. For the renormalization scheme, the so-called modified Minimal Subtraction ($\overline{MS}$) scheme [14] is used. At the end a conclusion is included.

CHAPTER 2

SYMMETRIES

The discussion of symmetries in particle physics content is a good starting point on the way to construct the Higgs sector of the Glashow-Weinberg-Salam model. They can be classified as global and local symmetries.

2.1 Global and Local Symmetries

A global symmetry is defined by a constant parameter throughout space-time. At every space-time point there is same amount of transformation. The rotational symmetry of a stick could be considered an example of global symmetry: the stick is symmetric under a rotation being the same at every point and time. The fields under this rotation relates are physically distinct; the orientation of a field becomes a measurable quantity.

On the other hand, the local symmetry is the one described a parameter which is both space and time dependent. That is at every single point and time we do a different transformation. Another difference between the global and local symmetries is that the global one is deterministic while the local one is indeterministic. This is to do with the time dependence of the transformation parameter. Local symmetry indeed relates the physically equivalent states which enforce the local symmetry as a gauge symmetry. To get unique results from a theory one has to settle one of these physically equivalent situations, known as the gauge-fixing procedure. Spontaneous symmetry breaking does indeed break the gauge invariance.

2.2 Spontaneously Broken Symmetries

The problem of breaking the gauge symmetry explicitly by including mass terms for the SM fermions and vector bosons motivates the idea of spontaneous symmetry breaking as a favorable mechanism. That is, introducing gauge-invariant interactions among a new scalar and the SM particles and then through a nonzero VEV of the scalar field one can induce mass terms. Let us now explore the idea through simple examples.

2.2.1 Spontaneously Broken Global $U(1)$ Symmetry:

For the discussion of global symmetry breaking, a simple massless ϕ^4 theory with the $U(1)$ symmetry can be used. For the scalar field ϕ, one can write

$$\begin{aligned} \phi &= \phi_1 + i\phi_2 \\ \phi &\to \phi' = e^{i\alpha}\phi \end{aligned} \tag{2.1}$$

where α is the constant of the global symmetry. Then the Lagrangian is

$$\begin{aligned} \mathcal{L} &= \partial_\mu\phi(\partial^\mu\phi)^* - V(\phi) \\ V(\phi) &= (\phi\phi^*)^2 = |\phi|^4 \end{aligned} \tag{2.2}$$

Then under the global symmetry Eqn. (2.1), the Lagrangian transforms as

$$\begin{aligned}\mathcal{L}(\phi,\partial_\mu\phi)\rightarrow\mathcal{L}(e^{i\alpha}\phi,\partial_\mu e^{i\alpha}\phi) &= \partial_\mu(e^{i\alpha}\phi)\underbrace{(\partial^\mu e^{i\alpha}\phi)^*}_{e^{-i\alpha}(\partial^\mu\phi)^*}-[e^{i\alpha}\phi(e^{i\alpha}\phi)^*]^2\\ &= e^{i\alpha}e^{-i\alpha}(\partial_\mu\phi)(\partial^\mu\phi)^*-[e^{i\alpha}e^{-i\alpha}\phi\phi^*]^2\\ &= \mathcal{L}(\phi,\partial_\mu\phi)\end{aligned}$$

This shows that the massless ϕ^4 theory has a global $U(1)$ symmetry.

A similar discussion can be carried out for the massive ϕ^4 theory with the potential $V(\phi)=-\frac{1}{2}\mu^2|\phi|^2+\frac{1}{4}\lambda|\phi|^4$. I set the derivative of the potential to zero. The extrema of the potential can be analyzed based on μ^2 parameter:

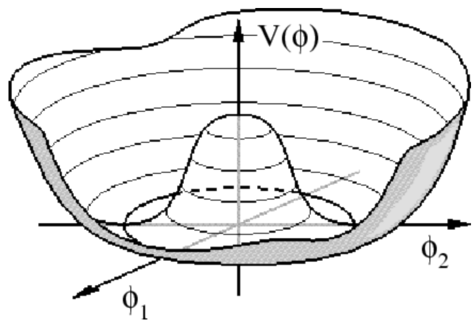

Figure 2.1: The Mexican hat-shaped potential in the (ϕ_1,ϕ_2) plane.

- For the case with $\mu^2>0$, the minimum is found to be at $|\phi|=0$, and we have a paraboloid-shaped potential in the (ϕ_1,ϕ_2) plane. Since in the unbroken case the ground state is thus at $|\phi|=0$, perturbations around this ground state are expressed in small values of the field ϕ. The situation is however different if the symmetry is broken in the ground state. There is a unique ground state at $|\phi|=0$, sharing the $U(1)$ symmetry of the Lagrangian.

- For the case $\mu^2<0$, the configuration $|\phi|=0$ describes now a local maximum, rather than a minimum. The ground state (minimum) of this system is degenerate which means there are multiple states with the same vacuum energy. The different orientations in the complex plane define different states and each ground state is asymmetric under the $U(1)$ symmetry of the Lagrangian. That is, applying the $U(1)$ transformation to any of the vacuum states will rotate it to a different orientation that describes a different physical state. For this case we have a so-called Mexican hat-shaped potential, like in Fig. 2.1. The details will be discussed later.

There is an important point to remind here. Once a global symmetry is broken spontaneously, there are massless bosons, known as Goldstone bosons, that appear in the theory, the situation summarized within the so-called Goldstone theorem.

2.2.1.1 Goldstone Theorem

The Goldstone theorem claims the existence of massless bosons in a theory with a spontaneously broken global symmetry. In other words, it states that for every broken continuous global (not gauge) symmetry there must be a massless particle. The most general proof of the Goldstone theorem is formulated in the domain of quantum mechanics.

To see why this always happens, let us take a theory with a number of fields φ^i with the Lagrangian which involves some terms with derivatives of φ^i and a potential $V(\varphi^i)$. The derivative terms would be zero if the fields are taken to be constant, that is, the kinetic part becomes irrelevant and the Lagrangian will only contain the potential $V(\varphi^i)$. If the minimization condition at $\varphi^i = \varphi^i_0$ is used,

$$\left[\frac{\partial V(\varphi^i)}{\partial \varphi^i}\right]_{\varphi^i=\varphi^i_0} = 0.$$

Then, by expanding the potential $V(\varphi^i)$ about its minimum in Taylor series, one gets

$$V(\varphi) = V(\varphi_0) + \left[\frac{\partial V(\varphi^i)}{\partial \varphi^i}\right]_{\varphi_0} (\varphi^i - \varphi_0^i) + \frac{1}{2}(\varphi - \varphi_0)^i(\varphi - \varphi_0)^j \left[\frac{\partial^2 V}{\partial \varphi^i \partial \varphi^j}\right]_{\varphi_0} + ...$$

Here since the first derivative of the potential vanishes, the leading term involves

$$m_{ij}^2 = \left[\frac{\partial^2 V}{\partial \varphi^i \partial \varphi^j}\right]_{\varphi_0}$$

which defines a symmetric matrix, being the elements of the mass square matrix of the fields. The masses of the fields are given by the eigenvalues of this symmetric matrix. Now it is possible to show that every continous symmetry of $V(\varphi^i)$ which is not a symmetry of the ground state φ_0, will make one of eigenvalues of the mass term of the fields go to zero.

If $V(\varphi^i)$ has a continuous symmetry, it means that it will be invariant under the transformation;

$$\varphi^i \longrightarrow \varphi^i + \beta F^i(\varphi)$$

where β is an infinitesimal parameter and F^i is a function of all the fields. This means that $V(\varphi^i)$ satisfies;

$$\begin{aligned} V(\varphi^i) &= V(\varphi^i + \beta F^i) , \\ \beta F^i \frac{V(\varphi^i + \beta F^i) - V(\varphi^i)}{\beta F^i} &= \beta F^i \frac{\partial V}{\partial \varphi^i} = 0 . \end{aligned}$$

I will take the derivative of this with respect to φ^j ;

$$\begin{aligned} \frac{\partial}{\partial \varphi^j}\left[F^i \frac{\partial V}{\partial \varphi^i}\right] &= 0 , \\ \frac{F^i}{\varphi^j}\frac{\partial V}{\partial \varphi^i} + F^i \frac{\partial^2 V}{\partial \varphi^j \partial \varphi^i} &= 0 , \end{aligned}$$

and by setting $\varphi^i = \varphi_0^i$, we get

$$\left[\frac{\partial F^i}{\partial \varphi^j}\frac{\partial V}{\partial \varphi^i}\right]_{\varphi=\varphi_0} + F^i(\varphi_0)\left[\frac{\partial^2 V}{\partial \varphi^i \partial \varphi^j}\right]_{\varphi=\varphi_0} = 0$$

where the first term is zero since at $\varphi^i = \varphi_0^i$ there is a minimum of V. That is;

$$\left[\frac{\partial V}{\partial \varphi^i}\right]_{\varphi^i=\varphi_0^i} = 0 \; .$$

Then we get;

$$F^i(\varphi_0)\left[\frac{\partial^2 V}{\partial \varphi^i \partial \varphi^j}\right]_{\varphi_0} = F^i(\varphi_0) m_{ij}^2(\varphi_0) = 0 \; .$$

At this point, there are two cases to discuss;

(1) If φ_0^i obey the continuous symmetry of $V(\varphi)$ for all i. That is,

$$\varphi^i \to \varphi^i + \beta F^i(\varphi) = \varphi'^i.$$

Then, it means $\varphi'^i = \varphi_0^i$. From here since $\beta \neq 0$, $F^i(\varphi_0) = 0$. Therefore, under this condition it is trivially satisfied.

(2) If otherwise is true, that is, φ_0^i breaks the symmetry of $V(\varphi)$ for some i, then $F^i(\varphi_0) \neq 0$ to guarantee the spontaneous breaking of the symmetry. Then there exists at least one term with a non-zero $F^i(\varphi_0)$. Hence its coefficient $m_{ij}^2(\varphi_0)$ should vanish. In the mass-square matrix, the corresponding row is totally null, giving a vanishing eigenvalue, which is one of the masses. This is indeed the promise of the Goldstone theorem in simple terms.

Higgs mechanism will now be presented as the spontaneous breaking of a gauge symmetry.[1]

[1] The Goldstone bosons that appeared upon the breaking of a global symmetry in the last section will not be found for spontaneously broken gauge symmetry: the Goldstone theorem breaks down.

2.3 Spontaneously Broken Local $U(1)$ Symmetry

Again let us take the massless complex ϕ^4 theory (with $\phi = \phi_1 + i\phi_2$). Under the local $U(1)$ transformation

$$\phi \to \phi' = e^{i\alpha}\phi$$

where α is now a spacetime varying function, that is, $\alpha = \alpha(x)$. Then the potential transforms as

$$V(\phi) = (\phi\phi^*)^2 \ \to V(\phi') = V(e^{i\alpha(x)}\phi)$$

$$\begin{aligned}
V(e^{i\alpha(x)}\phi) &= [e^{i\alpha(x)}\phi\,\underbrace{(e^{i\alpha(x)}\phi)^*}_{e^{-i\alpha}\phi^*}]^2 \\
&= [e^{i\alpha(x)}e^{-i\alpha(x)}\phi\phi^*]^2 \\
&= (\phi\phi^*)^2 \\
&= V(\phi)
\end{aligned} \tag{2.3}$$

The Kinetic term transforms in the following manner

$$\text{Kinetic Term}\,,\ KT(\phi):\quad (\partial_\mu\phi)\,(\partial^\mu\phi)^* \ \xrightarrow{U(1)}\ \underbrace{(\partial_\mu\phi')}_{\partial_\mu e^{i\alpha(x)}\phi}\ \ \underbrace{(\partial^\mu\phi')^*}_{\partial^\mu e^{-i\alpha(x)}\phi^*}$$

$$\begin{aligned}
(\partial_\mu\phi')(\partial^\mu\phi')* &= \underbrace{(\partial_\mu e^{i\alpha(x)}\phi)}_{i(\partial_\mu\alpha)e^{i\alpha(x)}\phi+e^{i\alpha(x)}(\partial_\mu\phi)}\quad (\partial^\mu e^{i\alpha(x)}\phi)^* \\
&= e^{i\alpha(x)}[i(\partial_\mu\alpha(x))\phi + \partial_\mu\phi]\,\underbrace{(e^{i\alpha(x)}[i(\partial^\mu\alpha(x))\phi + \partial^\mu\phi])^*}_{e^{-i\alpha}[-i(\partial^\mu\alpha)\phi^*+\partial^\mu\phi^*]} \\
&= (\partial_\mu\phi + i\phi\partial_\mu\alpha)(\partial^\mu\phi^* - i\phi^*\partial^\mu\alpha)
\end{aligned}$$

$$\begin{aligned}
KT(\phi') &= KT(\phi) + i\phi(\partial_\mu\alpha)(\partial^\mu\phi^*) - i\phi^*(\partial_\mu\alpha)\partial^\mu\phi + \phi\phi^*\partial_\mu\alpha\partial^\mu\alpha \\
&\neq KT(\phi)
\end{aligned} \tag{2.4}$$

Even though the potential $V(\phi)$ is invariant under local $U(1)$ transformation, the same does not hold for the kinetic energy part so that we have overall the Lagrangian is not invariant, $\mathcal{L}(\phi, \partial^\mu \phi) \neq \mathcal{L}(\phi', \partial^\mu \phi')$.

The common practice to restore the local symmetry is to introduce a new vector field, known as the gauge field. Let us call the gauge field $A_\mu(x)$ such that under the above transformation is extended as;

$$\begin{aligned} A_\mu \to A'_\mu &= A_\mu + \frac{1}{q}\, \partial_\mu \alpha(x) \,, \\ \phi \to \phi' &= e^{i\alpha(x)} \phi \,. \end{aligned} \tag{2.5}$$

Here the parameter q is known as the coupling constant. For later convenience, let us define the so-called covariant derivative D_μ;

$$D_\mu = \partial_\mu - iqA_\mu \,.$$

Under the $U(1)$ local symmetry given in (2.5), the derivative of the scalar field transforms

$$\begin{aligned} \partial_\mu \phi' &= \partial_\mu (e^{i\alpha(x)} \phi) \\ &= e^{i\alpha(x)} (\partial_\mu \phi) + i e^{i\alpha(x)} \phi (\partial_\mu \alpha(x)) \\ &\neq e^{i\alpha(x)} (\partial_\mu \phi) \end{aligned}$$

However, once the usual derivative is replaced with the covariant derivative we get

$$\begin{aligned} (D_\mu \phi)' &= \partial_\mu \phi' - iqA'_\mu \phi' \\ &= \partial_\mu e^{i\alpha(x)} \phi - iq(A_\mu + \frac{1}{q}(\partial_\mu \alpha(x))) e^{i\alpha(x)} \phi \\ &= e^{i\alpha(x)} \partial_\mu \phi + i e^{i\alpha(x)} \phi \partial_\mu \alpha(x) - iq e^{i\alpha(x)} A_\mu \phi - i e^{i\alpha(x)} \phi (\partial_\mu \alpha(x)) \\ &= e^{i\alpha(x)} \underbrace{(\partial_\mu \phi - iqA_\mu \phi)}_{D_\mu \phi} \\ &= e^{i\alpha(x)} (D_\mu \phi) \,. \end{aligned}$$

The above calculation shows that, unlike the ∂_μ derivative of ϕ, the covariant derivative D_μ of the scalar field ϕ transforms same as the scalar field itself under the gauge transformation in (2.5). Thus the kinetic term gives

$$
\begin{aligned}
KT(\phi, A_\mu) = (D_\mu\phi)(D^\mu\phi)^* \to KT(\phi', A'_\mu) &= (D_\mu\phi)'(D^\mu\phi)'^* \\
&= e^{i\alpha(x)}(D_\mu\phi)[e^{i\alpha(x)}(D_\mu\phi)]^* \\
&= e^{i\alpha(x)}e^{-i\alpha(x)}(D_\mu\phi)(D^\mu\phi)^* \\
&= (D_\mu\phi)(D^\mu\phi)^* \\
&= KT(\phi, A_\mu)\ .
\end{aligned}
$$

After having discussed the individual parts of the Lagrangian for the scalar field ϕ, there is also the kinetic energy of the gauge field A_μ, given as $KT(A_\mu) \equiv -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$. Here $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is called the field strength tensor. The Lagrangian can be expressed as $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + D_\mu\phi(D^\mu\phi)^* - V(\phi)$. The only part that has not been tested under the gauge transformation is the kinetic term of A_μ. Let us start with the transformation of the field strength tensor $F_{\mu\nu}$:

$$
\begin{aligned}
F_{\mu\nu} \to F'_{\mu\nu} &= \partial_\mu A'_\nu - \partial_\nu A'_\mu \\
&= \partial_\mu A_\nu + \frac{1}{q}\partial_\nu\alpha(x) - \partial_\nu A_\mu + \frac{1}{q}\partial_\mu\alpha(x) \\
&= \partial_\mu A_\nu - \partial_\nu A_\mu + \frac{1}{q}\partial_\mu\partial_\nu\alpha(x) - \frac{1}{q}\partial_\nu\partial_\mu\alpha(x) \\
&= F_{\mu\nu}\ .
\end{aligned}
$$

Together with this at hand, we now safely say that the Lagrangian of the $U(1)$ gauged massless scalar theory is gauge invariant. There is also a very important massage to take at this point. It is simply the fact that *none of the free theories is gauge invariant*.

What are these interactions? Let us check the kinetic term of the scalar field

$$
\begin{aligned}
(D_\mu\phi)(D^\mu\phi)^* &= (\partial_\mu - iqA_\mu)\phi[(\partial^\mu - iqA^\mu)\phi]^* \\
&= (\partial_\mu\phi - iqA_\mu\phi)(\partial^\mu\phi^* + iqA^\mu\phi^*) \\
&= (\partial_\mu\phi)(\partial^\mu\phi)^* + iq\underbrace{[(\partial_\mu\phi)\phi^*A^\mu - (\partial_\mu\phi^*)\phi A^\mu]}_{\text{3-point interaction}} + \underbrace{q^2A_\mu A^\mu\phi\phi^*}_{\text{4-point interaction}}
\end{aligned}
$$

Thus, gauge invariance dictates not only interaction but also the form of interactions among the fields. For the example at hand, we get $\phi{-}\phi{-}A_\mu$ as well as $\phi{-}\phi{-}A_\mu{-}A_\nu$ types of 3-point and 4-points interactions with a unique vertex factors, respectively.

Note also that we have assumed the gauge field A_μ massless since the mass term, $\frac{1}{2}m_A^2A_\mu A^\mu$, indeed violates the gauge invariance explicitly. That is;

$$
\begin{aligned}
\frac{1}{2}m_A^2A'_\mu A^{\mu\prime} &= \frac{1}{2}m_A^2\left[A_\mu + \frac{1}{q}\partial_\mu\alpha(x)\right]\left[A^\mu + \frac{1}{q}\partial^\mu\alpha(x)\right] \\
&= \frac{1}{2}m_A^2\left[A_\mu A^\mu + \frac{1}{q}A_\mu\partial^\mu\alpha(x) + \frac{1}{q}A^\mu\partial_\mu\alpha(x)\right. \\
&\quad \left. + \frac{1}{q^2}\partial_\mu\alpha(x)\partial^\mu\alpha(x)\right] \\
&= \frac{1}{2}m_A^2(A_\mu A^\mu + \frac{2}{q}A_\mu\partial^\mu\alpha(x) + \frac{1}{q^2}\partial_\mu\alpha(x)\partial^\mu\alpha(x)) \\
&\neq \frac{1}{2}m_A^2A_\mu A^\mu\ .
\end{aligned}
$$

We understand that if one is willing to formulate a theory with massive gauge bosons, a mechanism is needed not to break the gauge invariance explicitly. It is rather needed to do this spontaneously. Hence the idea of spontaneous symmetry breaking and the Higgs mechanism come to the rescue here.

To grasp the idea behind the spontaneous symmetry breaking, let us now take a massive scalar field ϕ being symmetric under a discrete symmetry (Z_2) $\phi \to -\phi$, rather than a continuous symmetry. The scalar potential becomes $V(\phi) = -\frac{1}{2}\mu^2\phi^2 + \frac{\beta}{4}\phi^4$ with the mass of ϕ, $m^2 = -\mu^2$. $V(\phi)$ is clearly invariant under $\phi \to -\phi$.

Now the minima of $V(\phi)$ can be found

$$\begin{aligned}
\left[\frac{\partial V(\phi)}{\partial \phi}\right]_{\phi=\phi_0} &= -\frac{1}{2}\,2\phi_0\mu^2 + 4\,\frac{\beta}{4}\phi_0^3 = -\phi_0\mu^2 + \beta\phi_0^3 = \phi_0\,\underbrace{(-\mu^2 + \beta\phi_0^2)}_{=0} = 0 \\
-\mu^2 + \beta\phi_0^2 &= 0 \\
\phi_0^2 &= \frac{\mu^2}{\beta} \\
|\phi_0| &= \pm\frac{\mu}{\sqrt{\beta}} = \pm v \qquad (2.6)
\end{aligned}$$

where v is the vacuum expectation value (VEV) of the field ϕ. If one expands $\phi(x)$ about its minimum;

$$\phi(x) = v + \sigma(x),$$

The Lagrangian in terms of the so-called fluctuation field $\sigma(x)$ becomes;

$$\begin{aligned}
L &= \frac{1}{2}(\partial_\mu\phi)^2 + \frac{1}{2}\mu^2\phi^2 - \frac{\beta}{4}\phi^4 \\
&= \frac{1}{2}(\partial_\mu(v+\sigma))^2 + \frac{1}{2}\mu^2(v+\sigma)^2 - \frac{\beta}{4}(v+\sigma)^4 \\
&= \frac{1}{2}((\partial_\mu v) + (\partial_\mu\sigma))^2 + \frac{1}{2}\mu^2(v^2 + 2v\sigma + \sigma^2) \\
&\quad -\frac{\beta}{4}(v^4 + 4v^3\sigma + 6v^2\sigma^2 + 4v\sigma^3 + \sigma^4) \\
&= \frac{1}{2}((\partial_\mu v)^2 + (\partial_\mu\sigma)^2 + 2(\partial_\mu v)(\partial_\mu\sigma)) + \frac{1}{2}\mu^2v^2 + \mu^2v\sigma + \frac{1}{2}\mu^2\sigma^2 \\
&\quad -\frac{\beta}{4}(v^4 + 4v^3\sigma + 6v^2\sigma^2 + 4v\sigma^3 + \sigma^4) \\
&= \frac{1}{2}(\partial_\mu\sigma)^2 + \frac{1}{2}\mu^2v^2 + \mu^2v\sigma + \frac{1}{2}\mu^2\sigma^2 - \frac{\beta}{4}(v^4 + 4v^3\sigma + 6v^2\sigma^2 + 4v\sigma^3 + \sigma^4) \\
&= \frac{1}{2}(\partial_\mu\sigma)^2 + \frac{1}{2}\mu^2(v^2\sigma^2) + \mu^2v\sigma - \frac{\beta v^4}{4} - \underbrace{\frac{4\beta v^3\sigma}{4}}_{=\beta v^3\sigma} - \underbrace{\frac{6\beta v^2\sigma^2}{4}}_{=\frac{3\beta v^2\sigma^2}{2}} - \underbrace{\frac{4\beta v\sigma^3}{4}}_{=\beta v\sigma^3} - \frac{\beta\sigma^4}{4} \\
&= \frac{1}{2}(\partial_\mu\sigma)^2 - \frac{\beta\sigma^4}{4} - \mu^2\sigma^2 - \sqrt{\beta}\mu\sigma^3 + \frac{\mu^4}{4\beta} \qquad (2.7)
\end{aligned}$$

where in the last line the relation from (2.6) is used to eliminate the VEV of ϕ field. The last term can be dropped since it is not dynamical.

As seen from (2.7) that the initial symmetry $\phi \to \phi$ has been observed by the fluctuation field (odd power of σ has been induced). The new Lagrangian is the Lagrangian of a simple scalar field σ with mass $\sqrt{2}\mu$ together with additional self interactions like σ^3 and σ^4 types. By expanding the Lagrangian around the minimum, the new field σ does not have Z_2 symmetry and indeed there is no external effect to break, thus the breaking becomes spontaneous. The symmetry of the system as whole remained but it was "hidden" by the ground state. The current example has two possible vacua, $|\phi_0| = \pm v$ but in the general case there are infinitely many possibilities.

Let us finalize this chapter by generalizing our earlier discussion about the gauge symmetry in abelian case to an nonabelian one.[2]

To deal with non-abelian transformations in the gauge symmetry invariance is rather cumbersome. To simplify the discussion an explicit group like $SU(2)$ will be used. Let us tackle the problem as follows. If T_j is the generator of group $SU(2)$ and obeys the commutation relation $[T_j, T_k] = i\epsilon_{jkl}T_l$. If L_j are the $n \times n$ matrices representing $SU(2)$ generators and $\alpha_j(x)(j = 1, 2...N)$ are arbitrary functions of spacetime, then

$$\begin{aligned} \varphi(x) \longrightarrow \varphi'(x) &= e^{-iL\cdot\alpha}\varphi(x) \\ &= U(\alpha)\varphi(x) \end{aligned}$$

where $U(\alpha) = e^{iL\cdot\alpha}$

$$\begin{aligned} \partial_\mu\varphi(x) &\longrightarrow \partial_\mu\varphi'(x) = U(\alpha)\partial_\mu\varphi(x) + [\partial_\mu U(\alpha)]\varphi(x) \\ D_\mu\varphi(x) &\longrightarrow D'_\mu\varphi'(x) = U(\alpha)D_\mu\varphi(x) \end{aligned}$$

where D_μ is the covariant derivative expressed as

$$D_\mu\varphi(x) = [\partial_\mu + igL \cdot W_\mu(x)]\varphi(x)$$

[2] The discussion of the global symmetry case in the nonabelain group is rather straightforward and we skip it.

Here $W^i_\mu(x)\,, i=1,2,3$ are the nonabelian vector gauge fields.

$$\begin{aligned}
D'_\mu\varphi'(x) &= [\partial_\mu + igL\cdot W^{'}_\mu(x)]\varphi'(x)\\
&= \partial_\mu\varphi^{'}(x) + igL\cdot W^{'}_\mu(x)\varphi^{'}(x)\\
&= U(\alpha)\partial_\mu\varphi(x) + [\partial_\mu U(\alpha)]\varphi(x) + igL\cdot W^{'}_\mu(x)\underbrace{\varphi^{'}(x)}_{U(\alpha)\varphi(x)}\\
&= U(\alpha)\partial_\mu\varphi(x) + [\partial_\mu U(\alpha)]\varphi(x) + igL\cdot W^{'}_\mu(x)U(\alpha)\varphi(x)\\
D'_\mu\varphi'(x) &= U(\alpha)D_\mu\varphi(x) = U(\alpha)[\partial_\mu + igL\cdot W_\mu(x)]\varphi(x)\,.
\end{aligned}$$

Using the followings

$$U(\alpha)[\partial_\mu + igLW_\mu(x)]\varphi(x)\partial_\mu - igW^a_\mu T^a U(\alpha)\partial_\mu\varphi(x)$$

$$= U(\alpha)\partial_\mu\varphi(x) + U(\alpha)igL\cdot W_\mu\varphi(x) - [\partial_\mu U(\alpha)]\varphi(x) + igL\cdot W^{'}_\mu(x)U(\alpha)\varphi(x)$$

$$[\partial_\mu U(\alpha)]\varphi(x) + ig\cdot LW^{'}_\mu(x)U(\alpha)\varphi(x) = U(\alpha)igL\cdot W_\mu(x)\varphi(x)$$

and by multiplying each term in this equation with $\frac{i}{g}$ we do the following steps

$$\begin{aligned}
&\frac{i}{g}[\partial_\mu U(\alpha)]\varphi(x) + \frac{i}{g}igL\cdot W^{'}_\mu(x)U(\alpha)\varphi(x) = \frac{i}{g}U(\alpha)igL\cdot W_\mu(x)\varphi(x)\\
&\frac{i}{g}[\partial_\mu U(\alpha)]\varphi(x) - L\cdot W^{'}_\mu(x)U(\alpha)\varphi(x) = -U(\alpha)L\cdot W_\mu(x)\varphi(x)\\
&\frac{i}{g}[\partial_\mu U(\alpha)]\varphi(x) + U(\alpha)L\cdot W_\mu(x)\varphi(x) = L\cdot W^{'}_\mu(x)U(\alpha)\varphi(x)\\
&LW^{'}_\mu(x)U(\alpha)\varphi(x) = \frac{i}{g}[\partial_\mu U(\alpha)]\varphi(x) + U(\alpha)L\cdot W_\mu(x)\varphi(x)\\
&L\cdot W^{'}_\mu(x)U(\alpha) = \frac{i}{g}[\partial_\mu U(\alpha)] + U(\alpha)L\cdot W_\mu(x)
\end{aligned}$$

By multiplying each term in this equation with the inverse matrix $U^{-1}(\alpha)$ one gets

$$L\cdot W^{'}_\mu(x) = U(\alpha)[\frac{i}{g}U^{-1}(\alpha)(\partial_\mu U(\alpha)) + L\cdot W_\mu(x)]U^{-1}(\alpha)$$

To see the effect of transformation on the W^j_μ more directly, let us take an infinitesimal transformation $U(\alpha) = 1 - iL\cdot\alpha + O(\alpha^2)$;

$$\begin{aligned}
L\cdot W_\mu'(x) &= \underbrace{(1 - iL\cdot\alpha\Big[(L\cdot W_\mu(x))(1+iL\cdot\alpha)}_{=L\cdot W_\mu + i(L\cdot W_\mu)L\cdot\alpha - i(L\cdot\alpha)(L\cdot W_\mu)} \\
&\quad + \frac{i}{g}\underbrace{(\partial_\mu U(\alpha))}_{=\partial_\mu(1-iL\cdot\alpha) = -iL\cdot\partial_\mu\alpha}\Big](1+iL\cdot\alpha) \\
L\cdot(W_\mu' - W_\mu) &= i[L\cdot W_\mu, L\cdot\alpha] + \frac{1}{g}L.\partial_\mu\alpha \\
L\cdot\delta W_\mu &= i\alpha_j W_\mu^k[L_k, L_j] + \frac{1}{g}L_j\partial_\mu\alpha_j \\
L_k\delta W_\mu^k &= c_{klm}\alpha_l W_\mu^m L_k + \frac{1}{g}L_k\partial_\mu\alpha_k \\
\delta W_\mu^j(x) &= c_{jkl}\alpha_k(x)W_\mu^l(x) + \frac{1}{g}\partial_\mu\alpha_j(x)
\end{aligned}$$

The kinetic energy of the nonabelian gauge fields is $-\frac{1}{4}G^j_{\mu\nu}G^{j,\mu\nu}$ where $G^j_{\mu\nu} = \partial_\mu W^j_\nu - \partial_\nu W^j_\mu + gc_{jkl}W^k_\mu W^l_\nu$ is the generalized field strength tensor. $c_{jkl} = (c_j)_{kl}$ is a tensor with $[c_j, c_k] = c_{jkl}c_l$ obeying the algebra $[c_j, c_k] = c_{jkl}c_l$. The change in the field strength tensor is $\delta G^j_{\mu\nu} = c_{jkl}\alpha_k G^l_{\mu\nu}$.

The non-abelian gauge fields W_μ are self-coupled through the term $G^j_{\mu\nu}G^{j,\mu\nu}$ which appears in the Lagrangian. As in the abelian case (the photon case), however, mass terms for the W^j_μ can not be tolerated since $W_\mu\cdot W^\mu$ is not gauge invariant. The Lagrangian can be summarized as $L \to L' = L + L_{int}\big[\varphi^j, (\partial_\mu + igL\cdot W_\mu)\varphi^j\big]$.

CHAPTER 3

THE HIGGS MECHANISM

The spontaneous symmetry breaking has been discussed in the previous chapter for various cases. In this chapter we concentrate on the mass generation through the Higgs mechanism. We split the discussion into the abelian and non-abelian cases.

3.1 The Mechanism in Abelian Gauge Theories

To construct the gauge invariant Lagrangian first replace the ordinary four derivative with the covariant derivative in the kinetic terms and introduce a massless gauge field A^μ as we did it in the previous chapter.

$$\mathcal{L} = \frac{1}{2}|(\partial_\mu + iqA_\mu)\varphi|^2 + \frac{1}{2}\mu^2|\varphi|^2 - \frac{\beta}{4}|\varphi|^4 - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}$$

where $\partial_\mu + iqA_\mu = D_\mu$ is the usual covariant derivative. Then expanding the complex field φ around its local minimum, $v = \frac{\mu}{\sqrt{\beta}}$, we get

$$\varphi = \frac{\mu}{\sqrt{\beta}} + \sigma + i\pi \ .$$

Here σ is the Charge Conjugation and Partiy (CP) even scalar boson while π is the CP-odd one which appears to be a Goldstone boson. Plugging this into the Lagrangian, gives

$$\mathcal{L} = \frac{1}{2}\left|(\partial_\mu + iqA_\mu)(\frac{\mu}{\sqrt{\beta}} + \sigma + i\pi)\right|^2 + \frac{1}{2}\mu^2\left|\frac{\mu}{\sqrt{\beta}} + \sigma + i\pi\right|^2 - \frac{\beta}{4}\left|\frac{\mu}{\sqrt{\beta}} + \sigma + i\pi\right|^4$$
$$+\text{Kinetic Terms} \,.$$

There is a problem with this Lagrangian. The Goldstone boson π which emerges as a consequence of Goldstone theorem. In this case, it represents a unphysical particle, so it should not appear in the final result. It means that it should disappear in a suitable gauge. Let us examine it here. If we impose the Lagrangian to be invariant under the gauge transformation $\varphi \to e^{i\theta(x)}\varphi$ together with the following choice for θ

$$\theta = -tan^{-1}(\frac{\varphi_2}{\varphi_1})$$

Then, the complex field φ becomes real. That is, $\varphi \to (\cos\theta + i\sin\theta)\varphi$ and further it is rewritten

$$\begin{aligned}(\cos\theta + i\sin\theta)\varphi &= \frac{\varphi_1 - i\varphi_2}{\sqrt{\varphi_1^2+\varphi_2^2}}(\varphi_1 + i\varphi_2) \\ &= \sqrt{\varphi_1^2+\varphi_2^2}\,.\end{aligned}$$

When the complex field φ becomes a real field, the imaginary part of φ, which is precisely the Goldstone boson π, vanishes in this particular gauge (known as the unitary gauge).

Let us go back to the Lagrangian and expand all the terms including the kinetic parts;

$$
\begin{aligned}
\mathcal{L} &= \frac{1}{2}\big[(\partial_\mu\sigma)^2 + (\partial_\mu\pi)^2\big] - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2}(2\mu^2)\sigma^2 + \frac{1}{2}\Big[\frac{(q\mu)^2}{\beta}\Big](A_\mu)^2 \\
&\quad + q\left[(\partial_\mu\pi)\sigma - (\partial_\mu\sigma)\pi + \frac{\mu}{\sqrt{\beta}}(\partial_\mu\pi)\right]A_\mu + \frac{q^2}{2}(\sigma^2 + \pi^2 + \frac{2\mu}{\sqrt{\beta}}\sigma)(A_\mu)^2 \\
&\quad - \frac{\beta}{4}(\sigma^4 + \pi^4) - \frac{\beta}{2}(\sigma\pi)^2 - \mu\sqrt{\beta}(\sigma^3 + \sigma\pi^2) \\
&= \frac{1}{2}(\partial_\mu\sigma)^2 - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2}(2\mu^2)\sigma^2 + \frac{1}{2}\left[\frac{(q\mu)^2}{\beta}\right](A_\mu)^2 + \frac{q^2}{2}\sigma^2(A_\mu)^2 \\
&\quad + \frac{q^2}{2}\frac{2\mu}{\sqrt{\beta}}\sigma(A_\mu)^2 - \frac{\beta}{4}\sigma^4 - \mu\sqrt{\beta}\sigma^3 \\
&= \frac{1}{2}(\partial_\mu\sigma)^2 - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \mu^2\sigma^2 + \frac{1}{2}\left[\frac{(q\mu)^2}{\beta}\right](A_\mu)^2 + \frac{q^2}{2}\sigma^2(A_\mu)^2 \\
&\quad + \frac{q^2}{2}\frac{2\mu}{\sqrt{\beta}}\sigma(A_\mu)^2 - \frac{\beta}{4}\sigma^4 - \mu\sqrt{\beta}\sigma^3 \\
&= \frac{1}{2}(\partial_\mu\sigma)^2 - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2}(2\mu^2)\sigma^2 + \frac{1}{2}\left[\frac{(q\mu)^2}{\beta}\right](A_\mu)^2 + \frac{q^2}{2}\sigma^2(A_\mu)^2 \\
&\quad + \frac{q^2}{2}(\sigma^2 + \frac{2\mu}{\sqrt{\beta}}\sigma)(A_\mu)^2 - \frac{\beta}{4}\sigma^4 - \mu\sqrt{\beta}\sigma^3 \,.
\end{aligned}
$$

The massive scalar σ is actually the Higgs particle and A^μ is the massive gauge field with 3 degrees of freedom. Addition to the 2 transverse degrees of freedom, the longitudinal polarization comes out as the third degree of freedom since the gauge field A^μ absorbs the Goldstone boson, thus acquiring a mass giving a longitudinal mode as the third degree of freedom. This is the Abelian example of the Higgs mechanism. However, what is now commonly referred to as the Higgs mechanism is its generalization to non-abelian gauge case and indeed extending it to the gauge group of the Standard Model would be possible.

3.2 The Mechanism in Non-Abelian Gauge Theories

One now can apply the Higgs mechanism to a non-abelian gauge symmetry [12]. Given a system of scalar field $.\varphi$, which are invariant under the transformation; $\varphi_i \longrightarrow$

$(1 + i\alpha^a t^a)_{ij}\varphi_i$ It is shown that by imposing local gauge symmetry and by expanding the field φ_i about the vacuum expectation values, the gauge bosons will gain mass. If the fields φ_i are taken to be real, then the matrices t^a will be pure imaginary and written as $t^a_{ij} = iT^a_{ij}$ where T^a are real and antisymmetric. The covariant derivative in the real φ case becomes

$$D_\mu\varphi = (\partial_\mu - igA^a_\mu t^a)\varphi = (\partial_\mu + gA^a_\mu T^a)\varphi$$

The kinetic energy term of the fields;

$$\frac{1}{2}(D_\mu\varphi_i)^2 = \frac{1}{2}(\partial_\mu\varphi_i)^2 + gA^a_\mu(\partial_\mu\varphi_i T^a_{ij}\varphi_j) + \frac{1}{2}g^2 A^a_\mu A^{b\mu}(T^a\varphi)_i(T^b\varphi)_i$$

where the last term has the structure of a gauge boson mass;

$$\frac{1}{2}m^2_{ab}A^a_\mu A^{b\mu}$$

If the fields φ_i are expanded about their vacuum expectation values $\langle\varphi^i\rangle = \langle\varphi^i_0\rangle$, the mass matrix will be $m^2_{ab} = g^2(T^a\varphi_0)_i(T^b\varphi_0)_i$. All diagonal elements of this mass matrix will have the form;

$$m^2_{aa} = g^2(T^a\varphi_0)^2$$

This means that, all gauge bosons will acquire a positive mass. T^a, the generator of the symmetry group will not influence the vacuum expectation value, thereby leaving the initial symmetry unbroken. In this case, the generator T^a will not contribute to the mass term and the corresponding gauge boson will remain massless.

3.2.0.1 An Example: The Georgi-Glashow Model

Before discussing the Higgs mechanism in reality, as a illustrative simpler example, let us discuss the so-called Georgi-Glashow model, proposing a framework which explore the Higgs mechanism to create both massive and massless bosons, for the weak and electromagnetic interactions. In the theory, an $SU(2)$ gauge field coupled to a scalar field φ which transforms as a vector of $SU(2)$. First, we will consider the case that all three gauge bosons end up being massive and the gauge field transforms as a spinor. The covariant derivative is;

$$D_\mu \varphi = (\partial_\mu - igA^a_\mu T^a)\varphi$$

where $T^a = \frac{\sigma^a}{2}$. In order to find the mass term, we will first compute the kinetic energy;

$$\begin{aligned}
\frac{1}{2}(D_\mu \varphi_i)^2 &= \frac{1}{2}(|\partial_\mu \varphi - igA^a_\mu T^a \varphi||\partial_\mu \varphi - igA^{b\mu} T^b \varphi|) \\
&= \frac{1}{2}(|\partial^2_\mu \varphi^2 - \partial_\mu \varphi igA^{b\mu} T^b \varphi - igA^a_\mu T^a \varphi \partial_\mu \varphi + i^2 g^2 A^a_\mu A^{b\mu} T^a T^b \varphi^2|) \\
&= \frac{1}{2}(\partial_\mu \varphi)^2 + \frac{1}{2} ig \partial_\mu \varphi (A^{b\mu} T^b + A^a_\mu T^a)\varphi + \frac{1}{2} g^2 A^a_\mu A^{b\mu} T^a T^b \varphi^2 \\
&= \frac{1}{2}(\partial_\mu \varphi)^2 + \frac{1}{2} ig (\partial_\mu \varphi) \underbrace{(A^{b\mu} T^b + A^a_\mu T^a)}_{=2A^a_\mu T^a} \varphi + \underbrace{\frac{1}{2} g^2 A^a_\mu A^{b\mu} T^a T^b \varphi^2}_{=\text{the mass term}}
\end{aligned}$$

Vacuum expectation value of φ is

$$\langle \varphi \rangle_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix} .$$

$$\begin{aligned}
\text{The mass term} &= \frac{1}{2} g^2 A^a_\mu A^{b\mu} T^a T^b \left(\frac{1}{\sqrt{2}} v\right)^2 \\
&= \frac{1}{4} g^2 A^a_\mu A^{b\mu} T^a T^b v^2
\end{aligned}$$

Using $[T^a, T^b] = \frac{1}{2}\delta^{ab}$ and for $a = b$ $[T^a, T^b] = \frac{1}{2}$,

$$
\begin{aligned}
\text{The mass term} &= \frac{1}{4}g^2 A^a_\mu A^{a\mu}\frac{1}{2}v^2 \\
&= \frac{1}{8}g^2 v^2 A^a_\mu A^{a\mu} \\
&= \frac{1}{8}(gv)^2 A^a_\mu A^{a\mu} \\
&= \frac{1}{2}(\frac{1}{2}gv)^2 A^a_\mu A^{a\mu} \\
&= \frac{1}{2}m^2_{aa} A^a_\mu A^{a\mu}
\end{aligned}
$$

Here the mass term, m_{aa} becomes $m_{aa} = m_a = \frac{1}{2}gv$.

The three fields all acquire the same mass. This means that the vacuum expectation value breaks the symmetry of all generators T^a. Now, I will consider the case that a real-valued field φ transforms as a vector of $U(2)$ and only 2 gauge bosons end up being massive and the third one remains massless. The covariant derivative is;

$$
(D\varphi)_a = \partial_\mu \varphi_a + g\epsilon_{abc} A^b_\mu \varphi_c
$$

The kinetic energy term is;

$$
\begin{aligned}
(D\varphi)^2_a &= \frac{1}{2}((\partial_\mu\varphi_a + g\epsilon_{abc}A^b_\mu\varphi_c)(\partial_\mu\varphi_a + g\epsilon_{abc}A^b_\mu\varphi_c)) \\
&= \frac{1}{2}((\partial_\mu\varphi_a)^2 + (\partial_\mu\varphi_a g\epsilon_{abc}A^b_\mu\varphi_c) + (g\epsilon_{abc}A^b_\mu\varphi_c\partial_\mu\varphi_a) + (g\epsilon_{abc}A^b_\mu\varphi_c)^2) \\
&= \frac{1}{2}(\partial_\mu\varphi_a)^2 + \frac{1}{2}2g\epsilon_{abc}A^b_\mu\varphi_c(\partial_\mu\varphi_a) + \frac{1}{2}(g\epsilon_{abc}A^b_\mu\varphi_c)^2 \\
&= \frac{1}{2}(\partial_\mu\varphi_a)^2 + g\epsilon_{abc}(\partial_\mu\varphi_a)A^b_\mu\varphi_c + \underbrace{\frac{1}{2}g^2(e_{abc}A^b_\mu\varphi_c)^2}_{\text{the mass term}}
\end{aligned}
$$

The vacuum expectation value of the field φ_a is;

$$
\langle\varphi_a\rangle = \langle\varphi_0\rangle_a = v\delta_{a3}
$$

$$
\begin{aligned}
\text{The mass term} &= \frac{1}{2}g^2(e_{abc}A^b_\mu V\delta_{c3})^2 \\
&= \frac{1}{2}g^2V^2\underbrace{(e_{abc}A^b_\mu\delta_{c3})^2}_{=(e_{ab3}A^b_\mu)^2} \\
&= \frac{1}{2}(gV)^2(e_{ab3}A^b_\mu)^2 \\
&= \frac{1}{2}(gV)^2[(A^1_\mu)^2+(A^2_\mu)^2]\,.
\end{aligned}
$$

This mass term shows that two gauge bosons will acquire the mass;

$$m_1 = m_2 = gV$$

and the other boson remains massless. The allowed vacuum states of the Georgi-Glashow model lie on the sphere. If the vacuum state φ_0 points in the z-direction, it will transform under rotations in x and y direction but it will remain invariant under rotations about the z-axis. Therefore, φ_0 will remain invariant under the generator T^3 (unbroken) while breaking the symmetry of T^1 and T^2, meaning that the corresponding gauge fields will remain massless. At the beginning, this model was considered as a serious candidate for electroweak interactions since it involves two massive bosons which are W-bosons and a massless one which is a photon. However, now it is known that the theory, which gives the most experimentally correct description of the weak interactions, is the Glashow-Weinberg-Salam Model.

3.3 The Scalar Sector of the GSW Model

Glashow, Weinberg and Salam introduced a model describing electroweak interactions, which experiments later proved to be correct up to very high precision. Addition to the same kind of $SU(2)$ gauge symmetry which have introduced in the Georgi-Glashow model, $U(1)$ gauge symmetry is also introduced in this model. If the scalar field has charge $+\frac{1}{2}$ under this $U(1)$ symmetry, its complete gauge transformation will be given by;

$$\varphi \rightarrow e^{i\alpha^a T^a} e^{\frac{i\beta}{2}} \varphi$$

where $T^a = \frac{\alpha^a}{2}$. Due to gauge invariance requirements the Higgs field needs to be a doublet under the SU(2) and the vacuum expectation value of the field is;

$$\langle \varphi \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix} .$$

There will be only one linear combination of generators that does not break the symmetry, leading to a massless gauge boson, that is;

$$\alpha^1 = \alpha^2 = 0 \; , \; \alpha^3 = \beta$$

The vacuum expectation value remains invariant under this transformation;

$$\begin{aligned} \langle \varphi \rangle \longrightarrow \frac{1}{\sqrt{2}} e^{i\beta T^z} e^{\frac{i\beta}{2}} \langle \varphi \rangle &= \frac{1}{\sqrt{2}} e^{\frac{i\beta}{2}} \begin{pmatrix} e^{\frac{i\beta}{2}} & 0 \\ 0 & e^{\frac{-i\beta}{2}} \end{pmatrix} \begin{pmatrix} 0 \\ v \end{pmatrix} \\ &= \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\beta} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ v \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} 0 \\ v \end{pmatrix} . \end{aligned}$$

The covariant derivative is;

$$D_\mu \varphi = (\partial_\mu - igA^a_\mu T^a - \frac{1}{2} ig' B_\mu) \varphi$$

The kinetic energy term is;

$$
\begin{aligned}
\frac{1}{2}\left|D_\mu\varphi\right|^2 &= \frac{1}{2}(D_\mu\varphi)^\dagger(D^\mu\varphi) \\
&= \frac{1}{2}\begin{pmatrix}0 & v\end{pmatrix}(\partial_\mu + igA^a_\mu T^a + \frac{1}{2}ig'B_\mu)(\partial^\mu + igA^{b\mu}T^b + \frac{1}{2}ig'B^\mu)\begin{pmatrix}0 \\ v\end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix}0 & v\end{pmatrix}\Big[g^2A^a_\mu A^{b\mu}T^aT^b + \frac{1}{2}gg'A^a_\mu T^a B^\mu \\
&\quad + \frac{1}{2}gg'A^{b\mu}T^bB_\mu + \frac{1}{4}g'^2B^2_\mu + ...\Big]\begin{pmatrix}0 \\ v\end{pmatrix} \\
&= \frac{v^2}{8}\Big[g^2(A^1_\mu)^2 + g^2(A^2_\mu)^2 + g^2(A^3_\mu)^2 - 2gg'A^3_\mu B^\mu + g'^2B^2_\mu + ...\Big] \\
&= \frac{v^2}{8}\Big[g^2(A^1_\mu)^2 + g^2(A^2_\mu)^2 + (-gA^3_\mu + g'B_\mu)^2 + ...\Big] \\
&= \left[\frac{gv}{2}\right]^2(W^\pm_\mu)^2 + \frac{1}{2}\left[\frac{v\sqrt{g^2+g'^2}}{2}\right]^2(Z^0_\mu)^2 + ...
\end{aligned}
$$

where $W^\pm_\mu = \frac{1}{\sqrt{2}}(A^1_\mu \pm iA^2_\mu)$ and $Z^0_\mu = \frac{gA^3_\mu - g'B_\mu}{\sqrt{g^2+g'^2}}$. These are the three vector bosons which acquire the masses $m_W = \frac{gv}{2}$ and $m_Z = \frac{v}{2}\sqrt{g^2+g'^2}$. The fourth vector field which remains massless is

$$A_\mu = \frac{g'A^3_\mu + gB_\mu}{\sqrt{g^2+g'^2}}.$$

We will identify the three massive gauge bosons as the weak force carriers $W^\pm$ and Z^0, and the massless field A_μ as the photon. Now, we will relate the masses m_W and m_Z to the electron charge e and the weak mixing angle θ_w. To do this, we will consider the coupling of the vector fields to fermions. Let us start with the covariant derivative of a fermion belonging to an $SU(2)$ representation with $U(1)$ charge Y

$$D_\mu = \partial_\mu - igA^a_\mu T^a - ig'\frac{Y}{2}B_\mu$$

We will express this in terms of $W_\mu^\pm$, Z_μ and A_μ

$$\begin{aligned} D_\mu &= \partial_\mu - \frac{ig}{\sqrt{2}}(W_\mu^+ T^+ + W_\mu^- T^-) - \frac{i}{\sqrt{g^2+g'^2}} Z_\mu (g^2 T^3 - g'^2 \frac{Y}{2}) \\ &\quad - \frac{igg'}{\sqrt{g^2+g'^2}} A_\mu (T^3 + \frac{Y}{2}) \, . \end{aligned}$$

Last term shows clearly that A_μ^j couples to the gauge generator $(T^3 + \frac{Y}{2})$, which is the one that remains unbroken, leaving the photon massless. If I identify A_μ as the electromagnetic field then the coefficient of the last term should be the electron charge e;

$$e = \frac{gg^{'}}{\sqrt{g^2+g'^2}} \, .$$

Now, we will define the weak mixing angle θ_w;

$$\begin{aligned} \cos\theta_w &= \frac{g}{\sqrt{g^2+g'^2}} \, , \\ \sin\theta_w &= \frac{g^{'}}{\sqrt{g^2+g'^2}} \end{aligned}$$

The gauge boson masses become;

$$\begin{aligned} m_W &= m_Z \cos\theta_w \, , \\ m_A &= 0 \, . \end{aligned}$$

The electric charge is $e = g\sin\theta_W$. $W^\pm$ and Z bosons were first observed in CERN in 1983, confirming the GWS model. The GWS model also known as the unified electroweak model, combined with quantum chromodynamics constitutes the Standard Model of particle physics.

In conclusion, the Higgs mechanism allows the massless gauge field to become massive. The GWS model, has an application of the Higgs mechanism to an $SU(2)XU(1)$ gauge theory which allows for the weak force carriers $W^\pm$ and Z to acquire masses, while leaving the photon massless. The Higgs particle associated with the Higgs field was observed that two CERN experiments which independently arrived at the same discovery of a new particle, Higgs boson, in June 2012.

CHAPTER 4

THE EFFECTIVE POTENTIAL

The aim of this chapter is to compute the scalar potential of various theories up to one-loop order, including massless and massive ϕ^4 theory as well as the GWS model [1–3]. The minimum of the potential is expected to be stable in order to be a useful theory to consider. For that matter, the loop corrections turn out to be significant.

It is also known that loop corrections give divergent integrals, resulting infinities. However, it has been shown that the theories we consider are renormalizable. That is, one can observed these infinities into the paramaters of the model, which are known to be finite experimentally. To make the infinities obvious, different regularization methods are used. In this chapter we follow two different methods; the cut-off regularization method and the dimensional regularization method [13].

There are also two main approaches to compute one-loop corrections. The one by Coleman and Weinberg (call it the Coleman-Weinberg method) and one by Lee and Sciaccaluga (call it the Lee-Sciaccaluga method). After a brief mention of the Coleman-Weinberg method, we will concentrate on the Lee-Sciaccaluga method since it is presumably a lot simpler and faster to get the result.

4.1 The Coleman-Weinberg Method

The details are given in the original paper [12] and in [9] as well. In short, suppose a single real scalar field ϕ, the generating functional of one particle irreducible (1PI) diagrams is written such that the so-called effective action for the classical field ϕ_c can be written as

$$\Gamma(\phi_c) = \sum_{m=1}^{\infty} \frac{1}{m!} \int \Gamma^{(m)}(x_1, ...x_m) d^4x_1 ... d^4x_m \; \phi_c(x_1)...\phi_c(x_m) \tag{4.1}$$

where $\Gamma^{(m)}(x_1, ...x_m)$ are the sum of all 1PI Feynman diagrams with m number of external legs. Clearly, since m runs from 1 to ∞, the method has the burden of calculating diagrams with all possible number of external legs, turning into a sum of infinite number of terms.

It is further possible to show that

$$\Gamma(\phi_c) = -\int \left[V(\phi_c) - \frac{1}{2} Z(\partial_\mu \phi)^2 \right] d^4x \; . \tag{4.2}$$

Here $V(\phi_c)$ is defined to be the effective scalar potential and Z is the normalization constant. If we express the effective potential in terms of the effective action we get

$$V(\phi_c) = -\sum_{m=1}^{\infty} \frac{1}{m!} \, \phi_c^m \, \Gamma^m(p_i = 0) \tag{4.3}$$

where ϕ_i is the momenta of the external legs and for the calculation of the effective potential what we need is to calculate the 1PI Feynman diagrams with $p_i \neq 0$ and then set them to be zero at the end. The disadvantage of the method is that if one needs especially loop corrections higher than one-loop, it gets so cumbersome since it involves all possible diagrams with arbitrary number of external legs.

4.2 The Lee-Sciaccaluga Method

The idea is to do the following trick [9,10,15]. Instead of expanding the action around $\phi_c = 0$, think of doing it around a non-zero arbitrary point shifted $\phi_c = \omega$. Then, if the shifted potential is called $V'(\phi_c)$, we get

$$\begin{aligned} V'(\phi_c) &= -\sum_{m=1}^{\infty} \frac{1}{m!}(\phi_c + \omega)^m \Gamma^m(p_i = 0) \\ &= -\sum_{m=1}^{\infty} \frac{1}{m!} \phi_c^m \Gamma'^m(\omega) \end{aligned}$$

where $\Gamma'^m(\omega)$ is nothing but the 1PI of the Feynman diagrams for the shifted theory with zero external momenta. Like in the previous method, the effective potential $V'(\phi_c)$ can further be expressed as

$$V'(\phi_c) = -\sum_{k=0}^{\infty} \frac{1}{k!} \underbrace{\left(\sum_{m=0}^{\infty} \frac{1}{m!} \Gamma'_m(0) \frac{m!}{(m-k)!} w^{m-k} \right)}_{=\Gamma'_k(\omega)} \phi_c^k$$

where $\Gamma'_1(w)$ is 1PI tadpole diagrams of the shifted theory. Then finally one writes

$$\begin{aligned} \int_0^{\phi_c} d\omega \Gamma'_1(w) &= \sum_{m=0}^{\infty} \frac{1}{m!} \Gamma'_m(0) m \underbrace{\int_0^{\phi_c} d\omega \omega^{m-1}}_{=\frac{\omega^m}{m}} \\ &= \underbrace{\sum_{m=0}^{\infty} \frac{1}{m!} \Gamma'_m(0) \phi_c^m}_{=-V(\phi_c)} \\ V(\phi_c) &= -\int_0^{\phi_c} d\omega \Gamma'_1(\omega) \end{aligned}$$

As compared to the previous method one needs an additional integral to get V but one only need 1PI tadpole diagrams, nothing else. In that calculation, the shifted vertex factors, masses etc should be used.

To summarize the procedure;

(1) Determine the masses, propagators and the vertex factors of the shifted theory.

(2) Evaluate the tadpole diagrams to find $\Gamma_1'(\omega)$ of the shifted theory.

(3) Integrate $\Gamma_1'(\omega)$ with respect to ω and then set $\omega = \phi_c$ to get the minus of effective potential at one-loop.

4.3 One-loop Effective Potential with Cut-off Regularization

Let us consider the Lagrangian for a single, massless real scalar field [9]

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \underbrace{\frac{g\phi^n}{n!}}_{\text{zero-loop potential}} . \tag{4.4}$$

The effective potential through one loop is;

$$V = \frac{1}{n!} g\phi_c^n + i \int \frac{d^4k}{(2\pi)^4} \sum_{r=1}^{\infty} \frac{1}{2r} \left(\frac{g\phi_c^{n-2}/(n-2)!}{k^2 + i\epsilon} \right)^r . \tag{4.5}$$

By rotating this to the Euclidean space;

$$V = \frac{1}{n!} g\phi_c^n + \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} ln \left(1 + \frac{g\phi_c^{n-2}}{(n-2)!k^2} \right) . \tag{4.6}$$

For the general polynomial potential in the Lagrangian, $U(\phi)$, the one loop potential is;

$$V(\phi_c) = U(\phi_c) + \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \ln \left(1 + \frac{U''(\phi_c)}{k^2} \right)$$

Since the integral is divergent at large k values, impose cut-off at $k^2 = \Lambda^2$;

$$V_{\text{loop}} = \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \ln \left(1 + \frac{U''(\phi_c)}{k^2} \right)$$

where $d^4k = k^3dkd\omega = 2\pi^2k^3dk$ and $k^3dk = k^2kdk = \frac{1}{2}k^2dk^2$ so $d^4k = 2\pi^2\frac{1}{2}k^2dk^2 = \pi^2k^2dk^2$

$$
\begin{aligned}
V_{\text{loop}} &= \frac{\pi^2}{(2\pi)^4}\int_0^\Lambda \underbrace{k^2dk^2}_{=d(\frac{k^4}{2})}\ln\left(1+\frac{U''(\phi_c)}{k^2}\right) \\
&= \frac{1}{32\pi^2}\int \underbrace{d(k^4)\ln\left(1+\frac{U''(\phi_c)}{k^2}\right)}_{=dk^4\ln\left(1+\frac{U''(\phi_c)}{k^2}\right)} - k^4\underbrace{d\left[\ln\left(1+\frac{U''(\phi_c)}{k^2}\right)\right]}_{\frac{-U''/k^4}{1+U''/k^2}dk^2} \\
&= \frac{1}{32\pi^2}\left[k^4\ln\left(1+\frac{U''(\phi_c)}{k^2}\right)\right] + \underbrace{\int dk^2\frac{U''}{1+U''/k^2}}_{=II}
\end{aligned}
$$

$\left[k^4\ln\left(1+\frac{U''(\phi_c)}{k^2}\right)\right]_0^{\Lambda^2} = \Lambda^4\ln\left(1+\frac{U''(\phi_c)}{k^2}\right) - 0$ since k^4 goes to zero faster than $\ln\left(1+\frac{U''(\phi_c)}{k^2}\right)$.

$$
V_{\text{loop}} = \frac{1}{32\pi^2}\left[\Lambda^4\ln\left(1+\frac{U''(\phi_c)}{k^2}\right) + II\right]
$$

$$
\begin{aligned}
II &= \int dk^2\frac{U''}{1+U''/k^2} \\
&= U''\int dk^2\frac{k^2}{k^2+U''} \\
&= U''\int dy\frac{y-U''}{y} \\
&= \left[U''y - (U'')^2\underbrace{\int dy\frac{1}{y}}_{=\ln|y|}\right]
\end{aligned}
$$

where $k^2 + U'' = y$ and $dk^2 = dy$ Then;

$$
\begin{aligned}
II &= [U''y - (U'')^2\ln y] \\
&= [U''(k^2+U'') - U''^2\ln(|k^2+U''|)] \\
&= \Lambda^2U'' + (U'')^2 - (U'')^2\underbrace{(\ln(\Lambda^2+U'') - \ln U'')}_{=\ln\left(\frac{\Lambda^2}{U''}+1\right)}
\end{aligned}
$$

So;

$$\begin{aligned}
V_{\text{loop}} &= \frac{1}{32\pi^2}\Lambda^4 \ln\left(1+\frac{U''(\phi_c)}{k^2}\right) + \Lambda^2 U'' + (U'')^2 - (U'')^2 \qquad \underbrace{\underbrace{\ln\left(\frac{\Lambda^2}{U''}+1\right)}_{= \ln\left(\frac{\Lambda^2}{U''}\left(1+\frac{U''}{\Lambda^2}\right)\right)}}_{=\ln \frac{\Lambda^2}{U''}+\ln\left(1+\frac{U''}{\Lambda^2}\right)} \\
&= \frac{1}{32\pi^2}[\Lambda^4 - (U'')^2]\ln\left(1+\frac{U''}{\Lambda^2}\right) - (U'')^2 \underbrace{\ln\left(\frac{\Lambda^2}{U''}\right)}_{=-\ln\left(\frac{U''}{\Lambda^2}\right)} + \Lambda^2 U'' + (U'')^2
\end{aligned}$$

Expanding $\ln(1+x)$ around $x=0$;

$$\ln(1+x) = +x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}... = +x - \frac{x^2}{2} + h.o.t$$

Then;

$$\begin{aligned}
V_{\text{loop}} &= \frac{1}{64\pi^2}\left[\underbrace{[\Lambda^4 - (U'')^2]}_{=\Lambda^4\left(1-\frac{(U'')^2}{\Lambda^4}\right)} \underbrace{\left(\frac{U''}{\Lambda^2} - \frac{1}{2}\left(\frac{U''}{\Lambda^2}\right)^2\right)}_{=\frac{2\Lambda^2 U'' - (U'')^2}{2\Lambda^4}} + \Lambda^2 U'' + U''^2 + (U'')^2 \ln\left(\frac{U''}{\Lambda^2}\right)\right] \\
&= \frac{1}{64\pi^2}\left[\underbrace{\frac{2\Lambda^2 U'' - (U'')^2}{2}}_{=\Lambda^2 U'' - \frac{1}{2}(U'')^2} - \frac{1}{2}\left(\underbrace{\frac{2\Lambda^2 (U'')^3}{\Lambda^4}}_{\text{h.o.t}} - \underbrace{\frac{(U'')^4}{\Lambda^4}}_{\text{h.o.t}}\right)\right. \\
&\quad \left. + \Lambda^2 U'' + (U'')^2 + (U'')^2 \ln\left(\frac{U''}{\Lambda^2}\right)\right] \\
&= \frac{1}{64\pi^2}\left[2\Lambda^2 U'' + \frac{1}{2}(U'')^2 + (U'')^2 \ln\left(\frac{U''}{\Lambda^2}\right)\right] \\
&= \frac{1}{32\pi^2}\Lambda^2 U'' + \frac{1}{64\pi^2}\left[(U'')^2 \ln\left(\frac{U''}{\Lambda^2}\right) + \frac{1}{2}(U'')^2\right]
\end{aligned}$$

Finally, we find;

$$V_{\text{loop}}(\phi_c) = U(\phi_c) + \frac{\Lambda^2}{32\pi^2}U'' + \frac{(U'')^2}{64\pi^2}\left[\ln(U''/\Lambda^2) - \frac{1}{2}\right] . \qquad (4.7)$$

For example the massive case with ϕ^4 theory $(n=4)$ from the above result can be deduced as

$$\begin{aligned}
V(\phi_c) &= U(\phi_c) + \underbrace{\frac{\Lambda^2}{32\pi^2}U'' + \frac{(U'')^2}{64\pi^2}\left[\ln\left(\frac{U''}{\Lambda^2}\right) - \frac{1}{2}\right]}_{=V_{\text{loop}}} \\
V_{\text{loop}} &= \frac{\Lambda^2}{32\pi^2}U'' + \frac{(U'')^2}{64\pi^2}\left[\ln\left(\frac{U''}{\Lambda^2}\right) - \frac{1}{2}\right] \\
&= \frac{\Lambda^2}{32\pi^2}(\mu^2 + 3\lambda\phi_c^2) + \frac{(\mu^2 + 3\lambda\phi_c^2)^2}{64\pi^2}\left[\ln\left(\frac{\mu^2 + 3\lambda\phi_c^2}{\Lambda^2}\right) - \frac{1}{2}\right]
\end{aligned}$$

where the potential explicitly are

$$\begin{aligned}
U(\phi_c) &= \frac{1}{2}\mu^2\phi_c^2 + \frac{\lambda}{4}\phi_c^4 \\
U(\phi_c)' &= \mu^2\phi_c + \lambda\phi_c^3 \\
U(\phi_c)'' &= \mu^2 + 3\lambda\phi_c^2
\end{aligned}$$

For the massive ϕ^4 theory, the Lagrangian of a self-interacting scalar theory is

$$L = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}\mu^2\phi^2 - \frac{\lambda\phi^4}{4}$$

$$V(\phi) = \frac{1}{2}\mu^2\phi^2 + \frac{\lambda\phi^4}{4}$$

The renormalization procedure is that one has to absorb the divergent part into the parameters of the model, μ^2, λ and ϕ itself. The renormalization conditions are;

$$\begin{aligned}
\mu_R^2 = -\Gamma^2(p_i = 0) &= \left[\frac{d^2V}{d\phi_c^2}\right]_{\phi_c=0} \\
\lambda_R &= \left[\frac{d^4V}{d\phi_c^4}\right]_{\phi_c=0}, \\
\left[\frac{\partial\Gamma^{(2)}}{\partial p^2}\right]_{p^2=m^2} &= 1
\end{aligned}$$

where μ_R^2 is renormalized mass-squared and λ_R is the renormalization coupling.

4.4 The ϕ^4 Theory in the Lee-Sciaccaluga Method with Cut-Off

To find the effective potential, firstly we need to evaluate the tadpole diagram. The potential is;

$$V_0(\phi_c) = \frac{1}{2}\mu^2\phi_c^2 + \frac{1}{4}\lambda\phi_c^4$$

We will take its derivative with respect to ϕ_c

$$\begin{aligned}\frac{\partial V_0}{\phi_c} &= \mu^2\phi_c + \lambda\phi_c^3 \\ &= (\mu^2 + \lambda\phi_c^2)\phi_c \\ &= 0\end{aligned}$$

To satisfy this; $\mu^2 + \lambda\phi_c^2 = 0$ From this;

$$\phi_c = 0$$

or

$$\phi_c^2 = -\frac{\mu^2}{\lambda}$$

The shifted potential is;

$$\begin{aligned}V_0(\phi_c - w) &= \frac{1}{2}\mu^2(\phi_c - w)^2 + \frac{1}{4}\lambda(\phi_c - w)^4 \\ &= \frac{1}{2}\mu^2(\phi_c^2 + w^2 - 2w\phi_c) \\ &\quad + \frac{1}{4}\lambda(\phi_c^4 + w^4 + 2w^2\phi_c^2 + 4w^2\phi_c^2 - 4w\phi_c^3 - 4w^3\phi_c) \\ &= \underbrace{\frac{1}{2}\mu^2\phi_c^2 + \frac{1}{4}\lambda\phi_c^4}_{=V_0(\phi_c)} - \mu^2 w\phi_c + \underbrace{\frac{1}{2}\mu^2 w^2 + \frac{1}{4}\lambda w^4}_{=constant} \\ &\quad + \frac{1}{4}\lambda 6w^2\phi_c^2 - \lambda w\phi_c^3 - \lambda w^3\phi_c \\ &= \frac{1}{2}(\mu^2 + 3\lambda w^2)\phi_c^2 - \underbrace{(\mu^2 + \lambda w^2)}_{=0} w\phi_c - \lambda w\phi_c^3 + \frac{1}{4}\lambda\phi_c^4 \\ &= \frac{1}{2}(\mu^2 + 3\lambda w^2)\phi_c^2 - \lambda w\phi_c^3 + \frac{1}{4}\lambda\phi_c^4\end{aligned}$$

The original mass squared term μ^2 will be replaced by $(\mu^2 + 3\lambda w^2)$. Here we have 3-point term $(-\lambda w \phi_c^3)$ and the vertex is $(-3! i\lambda w)$. Firstly, we will write the one-loop tadpole diagram;

$$\begin{aligned}
\frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2}(-3!i\lambda w) &= \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2-\mu^2-3\lambda w^2}(-3!i\lambda w) \\
&= \frac{1}{2}\int \frac{id^4k_E}{(2\pi)^4}\frac{i}{-(k_E^2+\mu^2+3\lambda w^2)}(-3!i\lambda w) \\
&= -\frac{i}{2}\int \frac{d^4k}{(2\pi)^4}\frac{3!\lambda w}{(k^2+\mu^2+3\lambda w^2)}
\end{aligned}$$

Secondly, we will multiply the tadpole diagram with i;

$$i\left(-\frac{i}{2}\int \frac{d^4k}{(2\pi)^4}\frac{3!\lambda w}{k^2+\mu^2+3\lambda w^2}\right) = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\frac{3!\lambda w}{k^2+\mu^2+3\lambda w^2}$$

Next, we will integrate this with respect to w;

$$\int\left(\frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\frac{3!\lambda w}{k^2+\mu^2+3\lambda w^2}\right)dw = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\underbrace{\left(\int \frac{6\lambda w dw}{k^2+\mu^2+3\lambda w^2}\right)}_{=\ln(k^2+\mu^2+3\lambda w^2)}$$

Finally, we will set this to $w = \phi_c$;

$$\frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\ln(k^2+\mu^2+3\lambda w^2) = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\ln(k^2+\mu^2+3\lambda \phi_c^2)$$

If one allows massless vector bosons introduced to the theory: we will first find the kinetic term. From the kinetic term, we will find the mass-squared and the vertex point;

$$L = \frac{1}{2}(D_\mu\phi_c)(D^\mu\phi_c) - V(\phi)$$

$$D_\mu = \partial_\mu + ieA_\mu$$

$$\begin{aligned}
K.E &= \frac{1}{2}(D_\mu\phi_c)(D^\mu\phi_c) \\
&= \frac{1}{2}(\partial_\mu\phi_c + ieA_\mu\phi_c)(\partial^\mu\phi_c + ieA^\mu\phi_c) \\
&= \frac{1}{2}\partial_\mu\partial^\mu\phi_c\phi_c + \frac{1}{2}\partial_\mu\phi_c ieA^\mu\phi_c + \frac{1}{2}ieA_\mu\phi_c\partial^\mu\phi_c + \frac{1}{2}ieA_\mu\phi_c ieA^\mu\phi_c \\
&= \frac{1}{2}\partial_\mu\partial^\mu\phi_c\phi_c + \frac{1}{2}\partial_\mu ieA^\mu\phi_c\phi_c + \frac{1}{2}ieA_\mu\partial^\mu\phi_c\phi_c - \frac{1}{2}e^2A_\mu A^\mu\phi_c\phi_c
\end{aligned}$$

$$\phi_c' = \phi_c - w$$

$$\begin{aligned}
K.E &= \frac{1}{2}\partial_\mu\partial^\mu\phi_c\phi_c + \frac{1}{2}\partial_\mu ieA^\mu\phi_c\phi_c + \frac{1}{2}ieA_\mu\partial^\mu\phi_c\phi_c - \frac{1}{2}e^2A_\mu A^\mu \underbrace{(\phi_c'+w)(\phi_c'+w)}_{(\phi_c'^2+w^2+2w\phi_c')} \\
&= \frac{1}{2}\partial_\mu\partial^\mu\phi_c\phi_c + \frac{1}{2}\partial_\mu ieA^\mu\phi_c\phi_c \\
&\quad + \frac{1}{2}ieA_\mu\partial^\mu\phi_c\phi_c - \frac{1}{2}e^2A_\mu A^\mu\phi_c'^2 - \frac{1}{2}e^2A_\mu A^\mu w^2 - \frac{1}{2}e^2A_\mu A^\mu 2w\phi_c' \\
&= \frac{1}{2}\partial_\mu\partial^\mu\phi_c\phi_c + \frac{1}{2}\partial_\mu ieA^\mu\phi_c\phi_c + \frac{1}{2}ieA_\mu\partial^\mu\phi_c\phi_c \\
&\quad - \underbrace{\frac{1}{2}e^2A_\mu A^\mu\phi_c'^2}_{\text{scalar-vector interaction term}} - \underbrace{\frac{1}{2}e^2w^2A_\mu A^\mu}_{m_A^2} - \underbrace{e^2wA_\mu A^\mu\phi_c'}_{\text{scalar-vector-vector vertex}}
\end{aligned}$$

After the shift $\phi_c' = \phi_c - w$, the scalar-vector interaction term $\frac{1}{2}e^2\phi_c'^2A^\mu A_\mu$ acquire a scalar-vector-vector vertex $-ie^2wg_{\mu\nu}$ and the photon acquires a mass-squared given by e^2w^2. Now, we will write the tadpole diagram by using this mass-squared and this vertex point which I found above;

$$\begin{aligned}
\frac{dV_{\text{vect}}}{dw} &= i\int\frac{d^4k}{(2\pi)^4}(-ie^2wg_{\mu\nu})\frac{-i}{k^2-m^2}\left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right) \\
&= \underbrace{i^4}_{=1}(-1)\int\frac{d^4k_E}{(2\pi)^4}\frac{e^2w}{k_E^2+e^2w^2}\underbrace{\left(g_{\mu\nu}g^{\mu\nu} - \frac{g_{\mu\nu}k^\mu k^\nu}{-k_E^2}\right)}_{-2-1=-3} \\
&= \int\frac{d^4k}{(2\pi)^4}\frac{3e^2w}{(k^2+e^2w^2)}
\end{aligned}$$

where $k^2 = k_0^2 - k'^2 = -(k_E)^2$ $d^4k = id^4k_E$ and $k^\mu k^\nu = -k_E^2$ Next, we will integrate it with respect to w;

$$\begin{aligned}
\int\left(\int\frac{d^4k}{(2\pi)^4}\frac{3e^2w}{(k^2+e^2w^2)}\right)dw &= \int\frac{d^4k}{(2\pi)^4}3\underbrace{\left(\int\frac{e^2wdw}{(k^2+e^2w^2)}\right)}_{=\frac{1}{2}\ln(k^2+e^2w^2)} \\
&= \frac{3}{2}\int\frac{d^4k}{(2\pi)^4}\ln(k^2+e^2w^2)
\end{aligned}$$

Finally, we will set it to $w = \phi_c$;

$$V_{\text{vect}} = \frac{3}{2}\int\frac{d^4k}{(2\pi)^4}\ln(k^2+e^2\phi_c^2)$$

The equation;

$$V = \frac{1}{2}\mu^2\phi_c^2 + \frac{1}{4}\lambda\phi_c^4 + \frac{(\mu^2+3\lambda\phi_c^2)^2}{64\pi^2}\ln(\mu^2+3\lambda\phi_c^2) + a\phi_c^2 + b\phi_c^4$$

The solution of this equation;

$$V = \frac{1}{4}\lambda_R\phi_c^4 + \frac{9\lambda_R^2}{64\pi^2}\phi_c^4\left[\ln\left(\frac{\phi_c^2}{M^2}\right) - \frac{25}{6}\right]$$

By using this solution, we will find the effective potential of the vector boson.

$$3\lambda_R = e^2$$

$$\lambda_R = \frac{e^2}{3}$$

The equation becomes;

$$\begin{aligned} 0 + 3\frac{9}{64\pi^2}\left(\frac{e^2}{3}\right)^2\phi_c^4\left[\ln\left(\frac{\phi_c^2}{M^2}\right) - \frac{25}{6}\right] &= 3\frac{9}{64\pi^2}\frac{e^4}{9}\phi_c^4\left[\ln\left(\frac{\phi_c^2}{M^2}\right) - \frac{25}{6}\right] \\ &= \frac{3e^4}{64\pi^2}\phi_c^4\left[\ln\left(\frac{\phi_c^2}{M^2}\right) - \frac{25}{6}\right] \\ &= \frac{3e^4}{64\pi^2}\phi_c^4\ln\left(\frac{\phi_c^2}{M^2}\right) \end{aligned}$$

This is the equation of effective potential of the vector boson.

Additionally, if we allow fermions to couple: From the Yukawa Lagrangian, we will find the tadpole diagram contribution. We will find the tadpole diagram. The Yukawa Lagrangian is;

$$L_{\text{yuk}} = -g_{\text{yuk}}\phi'_c\varphi'\varphi + g_{\text{yuk}}w\varphi'\varphi$$

Then, the tadpole diagram is;

$$\begin{aligned} i\int\frac{d^4k}{(2\pi)^4}(-ig_{\text{yuk}})\frac{iTr[(k+m_f)]}{k^2-(g_{\text{yuk}}w)^2}(-1) &= i\int\frac{(id^4k_E)}{(2\pi)^4}(-ig_{\text{yuk}})\frac{i4g_{\text{yuk}}w}{-(k_E^2+g_{\text{yuk}}^2w^2)}(-1) \\ &= -i\int\frac{id^4k}{(2\pi)^4}(-ig_{\text{yuk}})\frac{4ig_{\text{yuk}}w}{-(k^2+g_{\text{yuk}}^2w^2)} \\ &= -\int\frac{d^4k}{(2\pi)^4}\frac{4g_{\text{yuk}}^2w}{(k^2+g_{\text{yuk}}^2w^2)} \end{aligned}$$

$$\frac{dV_{fermion}}{dw} = -\int \frac{d^4k}{(2\pi)^4} \frac{4g_{\text{yuk}}^2 w}{(k^2 + g_{\text{yuk}}^2 w^2)}$$

We will take the integral with respect to w to find the effective potential of the fermion loop contribution;

$$\begin{aligned} V_{fermion} &= -\int \frac{d^4k}{(2\pi)^4} 2 \underbrace{\int \frac{2g_{\text{yuk}}^2 w dw}{(k^2 + g_{\text{yuk}}^2 w^2)}}_{=\ln(k^2+g_{\text{yuk}}^2 w^2)} \\ &= -2\int \frac{d^4k}{(2\pi)^4} \ln(k^2 + g_{\text{yuk}}^2 w^2) \end{aligned}$$

Now we will use the following equation to find the solution of this equation;

$$V = \frac{(U'')^2}{64\pi^2} \ln\left(\frac{\phi_c^2}{M^2}\right)$$

$$U'' = g_{\text{yuk}}^2 w^2 = g_{\text{yuk}}^2 \phi_c^2$$

Then the effective potential of the fermion loop is;

$$V_{fermion} = (-4)\left(\frac{(g_{\text{yuk}}^2 \phi_c^2)^2}{64\pi^2}\right) \ln\left(\frac{\phi_c^2}{M^2}\right) = -\frac{1}{16} g_{\text{yuk}}^4 \phi_c^4 \ln\left(\frac{\phi_c^2}{M^2}\right)$$

Additionally depending on the gauge used, there will be Goldstone contributions.

$$\begin{aligned} V_0 &= \frac{1}{2}\mu^2\phi_c^2 + \frac{1}{4}\lambda\phi_c^4 \\ &= \frac{1}{2}\mu^2\phi_c\phi_c^* + \frac{1}{4}\lambda(\phi_c\phi_c^*)^2 \end{aligned}$$

$$\begin{aligned} \phi_c\phi_c^* &= (\phi_c - w + iG)(\phi_c - w - iG) \\ &= (\phi_c - w)^2 + G^2 - i(\phi_c - w)G + i(\phi_c - w)G \end{aligned}$$

$$\begin{aligned} (\phi_c\phi_c^*)^2 &= [(\phi_c - w)^2 + G^2]^2 \\ &= (\phi_c - w)^4 + G^4 + 2G^2(\phi_c - w)^2 \end{aligned}$$

$$
\begin{aligned}
V_0(\phi_c \to \phi_c - w) &= \frac{1}{2}\mu^2[(\phi_c - w)^2 + G^2]^2 + \frac{1}{4}\lambda(\phi_c - w)^4 + \frac{1}{4}\lambda G^4 + \frac{1}{2}\lambda G^2(\phi_c - w)^2 \\
&= \frac{1}{2}\mu^2(\phi_c - w)^2 + \frac{1}{4}\lambda(\phi_c - w)^4 \\
&\quad + \underbrace{\frac{1}{2}\mu^2 G^2 + \frac{1}{4}\lambda G^4 + \frac{1}{2}\lambda G^2(\phi_c^2 - 2w\phi_c + w^2)}_{V_{Goldstone}}
\end{aligned}
$$

$$
\begin{aligned}
V_{Goldstone} &= \frac{1}{2}\mu^2 G^2 + \frac{1}{4}\lambda G^4 + \frac{1}{2}\lambda G^2(\phi_c^2 - 2w\phi_c + w^2) \\
&= \frac{1}{2}(\mu^2 + \lambda w^2)G^2 + \frac{1}{4}\lambda G^4 + \frac{1}{2}\lambda G^2\phi_c^2 - \lambda w G^2\phi_c
\end{aligned}
$$

$$
m_G^2 = \mu^2 + \lambda w^2
$$

$$
\text{Vertex Factor} = -3i(-\lambda w) = 3i\lambda w
$$

"3" factor comes for counting permutations.

$$
\frac{dV_{Goldstone}}{dw} = i\int \frac{id^4k_E}{(2\pi)^4}(3i\lambda w)\frac{-i}{-k_E^2 - m_G^2}
$$

$$
k_E = k
$$

$$
\begin{aligned}
V_{Goldstone} &= 3\int \frac{d^4k}{(2\pi)^4}\int dw\left(\frac{\lambda w}{k^2 + \mu^2 + \lambda w^2}\right) \\
&= \frac{3}{2}\int \frac{d^4k}{(2\pi)^4}\ln(k^2 + \mu^2 + \lambda w^2)
\end{aligned}
$$

We know that;

$$
V(\phi_c) = U(\phi_c) + \frac{A^2}{32\pi^2}U'' + \frac{(U'')^2}{64\pi^2}\left[\ln\left(\frac{U''}{A^2}\right) - \frac{1}{2}\right]
$$

The result is;

$$
V = \frac{1}{4}\lambda_R\phi_c^4 + \frac{9\lambda_R^2}{64\pi^2}\phi_c^4\left[\ln\left(\frac{\phi_c^2}{M^2}\right) - \frac{25}{6}\right]
$$

We can use this to find the solution of equation.

$$
U'' = 3\lambda w^2 = 3\lambda\phi_c^2
$$

Finally;

$$V_{Goldstone} = 3\frac{(\lambda\phi_c^2)^2}{64\pi^2}\ln\left(\frac{\phi_c^2}{M^2}\right)$$

To sum up;

$$V = V_0 + V_{Scalar} + V_{Boson} + V_{Fermion} + V_{Goldstone}$$

$$V_0 = \frac{1}{2}\mu^2\phi_c^2 + \frac{1}{4}\lambda\phi_c^4$$

$$V_{Scalar} =$$

$$V_{Boson} = \frac{3e^4}{64\pi^2}\phi_c^4\ln\left(\frac{\phi_c^2}{M^2}\right)$$

$$V_{Fermion} = -\frac{1}{16}g_{\text{yuk}}^4\phi_c^4\ln\left(\frac{\phi_c^2}{M^2}\right)$$

$$V_{Goldstone} = 3\frac{(\lambda\phi_c^2)^2}{64\pi^2}\ln\left(\frac{\phi_c^2}{M^2}\right)$$

So adding them all up, the final result is;

$$V = \frac{1}{2}\mu^2\phi_c^2 + \frac{1}{4}\lambda\phi_c^4 + \frac{3e^4}{64\pi^2}\phi_c^4\ln\left(\frac{\phi_c^2}{M^2}\right) - \frac{1}{16}g_{\text{yuk}}^4\phi_c^4\ln\left(\frac{\phi_c^2}{M^2}\right) + 3\frac{(\lambda\phi_c^2)^2}{64\pi^2}\ln\left(\frac{\phi_c^2}{M^2}\right) \ .$$

4.5 The ϕ^4 Theory in the Lee-Sciaccaluga Method with Dimensional Regularization

Again doing similar steps first:

$$V(\phi) = \frac{1}{2}m^2\phi^2 + \frac{1}{4}\lambda\phi^4$$

$$
\begin{aligned}
V'(\phi) &= V(\phi+w) \\
&= \frac{1}{2}m^2(\phi+w)^2 + \frac{1}{4}\lambda(\phi+w)^4 \\
&= \underbrace{(m^2w+\lambda w^3)\phi}_{\text{tree level tadpole interaction}} + \underbrace{\frac{1}{2}(m^2+3\lambda w^2)\phi^2}_{\text{mass term}} + \underbrace{\lambda w\phi^3}_{\text{cubic term}} + \underbrace{\frac{1}{4}\lambda\phi^4}_{\text{quartic term}}
\end{aligned}
$$

where $\Gamma'_{1,\text{tree}}(w,0) = -(m^2w+\lambda w^3)$

$$
\begin{aligned}
V_{\text{tree}}(\phi_c) &= -\int_0^{\phi_c} dw\Gamma'_{1,\text{tree}}(w,0) \\
&= m^2\frac{w^2}{2} + \lambda\frac{w^4}{4}
\end{aligned}
$$

At $\phi_c = w$ this equation becomes;

$$
\text{Vertex Factor} = -3\lambda w
$$

$$
V_{\text{tree}}(\phi_c) = \frac{1}{2}m^2\phi^2 + \frac{1}{4}\lambda\phi^4
$$

$$
\begin{aligned}
\text{Propagator} &= \frac{i}{k^2-m^2} \\
&= \frac{i}{k^2-m^2-3\lambda w^2}
\end{aligned}
$$

where $m^2 \to m^2 + 3\lambda w^2$

$$
\Gamma_{1,\text{loop}}(w,0) = -3i\lambda w \int \frac{d^dk}{(2\pi)^d}\frac{1}{k^2-m^2-3\lambda w^2}
$$

$$
-3i\lambda w \int \frac{d^dk}{(2\pi)^d}\frac{1}{k^2-\underbrace{(m^2+3\lambda w^2)}_{=m'^2(w)}} = \int \frac{d^dk}{(2\pi)^d}\frac{1}{k^2-m'(w)^2}
$$

Identity;

$$
\int \frac{d^dk}{(2\pi)^d}\frac{1}{(l^2-\delta)^n} = \frac{(-1)^n i}{(4\pi)^{d/2}}\frac{\Gamma(n-d/2)}{\Gamma(n)}\left(\frac{1}{\delta}\right)^{n-d/2}
$$

In our case $n = 1$ and $\delta = m'^2(w)$ Thus;

$$
\int \frac{d^dk}{(2\pi)^d}\frac{1}{k^2-m'(w)^2} = \frac{-i}{(4\pi)^{d/2}}\frac{\Gamma(1-d/2)}{\Gamma(1)}\left(\frac{1}{m'^2(w)}\right)^{1-d/2}
$$

where $d = 4 - 2\epsilon$ so;

$$\frac{-i}{(4\pi)^{d/2}} \frac{\Gamma(1-d/2)}{\Gamma(1)} \left(\frac{1}{m'^2(w)}\right)^{1-d/2} = \frac{-i}{(4\pi)^{2-/epsilon}} \Gamma(\epsilon - 1) \left(\frac{1}{m'^2(w)}\right)^{\epsilon-1}$$

$$\Gamma(n) = (n-1)\Gamma(n-1)$$

$$\Gamma(\epsilon) = (\epsilon-1)\Gamma(\epsilon-1)$$

$$\Gamma(\epsilon-1) = \frac{1}{\epsilon-1}\Gamma(\epsilon)$$

Expanding $\Gamma(\epsilon)$ around $\epsilon = 0$;

$$\Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma + O(\epsilon)$$

$$\begin{aligned}
\Gamma(\epsilon-1) &= -(1+\epsilon+O(\epsilon^2))\left(\frac{1}{\epsilon} - \gamma + O(\epsilon)\right) \\
&= -\frac{1}{\epsilon} + \gamma - 1 + O(\epsilon) \\
&= -\frac{1}{\epsilon} - 1 + \gamma + O(\epsilon)
\end{aligned}$$

$$\begin{aligned}
a^\epsilon &= e^{\epsilon \ln a} \\
&= 1 + \epsilon \ln a + O(\epsilon^2)
\end{aligned}$$

$$\begin{aligned}
\left(\frac{1}{m'^2}\right)^{\epsilon-1} &= (m'^2)^{1-\epsilon} \\
&= m'^2 (m'^2)^{-\epsilon} \\
&= m'^2 e^{-\epsilon \ln(m'^2)} \\
&= m'^2 (1 - \epsilon \ln(m'^2) + O(\epsilon))
\end{aligned}$$

$$\begin{aligned}
\Gamma(\epsilon-1)\left(\frac{1}{m'^2}\right)^{\epsilon-1} &= \left(-\frac{1}{\epsilon} - 1 + \gamma + O(\epsilon)\right)(m'^2(1 - \epsilon\ln(m'^2)) + ...) \\
&= m'^2\left[-\frac{1}{\epsilon} + \ln(m'^2) + (\gamma - 1) - \epsilon(\gamma-1)\ln(m'^2) + ...\right] \\
&= m'^2\left[-\frac{1}{\epsilon} - 1 + \gamma + \ln(m'^2) + O(\epsilon)\right]
\end{aligned}$$

$$
\begin{aligned}
\int \frac{d^dk}{(2\pi)^d}\frac{1}{k^2-m'(w)^2} &= -\frac{i}{(4\pi)^2}(1+\epsilon\ln(4\pi))(-m'^2)\Big(\frac{1}{\epsilon}+1-\gamma \\
&\quad -\ln(m'^2)+O(\epsilon)\Big) \\
&= \frac{im'^2}{(4\pi)^2}\left[\frac{1}{\epsilon}+1-\gamma-\ln(m'^2)+\ln(4\pi)+O(\epsilon)\right]
\end{aligned}
$$

$$
\begin{aligned}
\Gamma'_{1,\text{loop}} &= \underbrace{-3\lambda w i\frac{im'^2}{(4\pi)^2}}_{\frac{3\lambda w m'^2}{(4\pi)^2}}\left[\frac{1}{\epsilon}+1-\underbrace{\gamma}_{\ln(e^{-\gamma})}-\ln\left(\frac{m'^2}{4\pi}\right)+O(\epsilon)\right] \\
&= \frac{3\lambda w(m^2+3\lambda w^2)}{(4\pi)^2}\left[\frac{1}{\epsilon}-\ln\left(\frac{m'^2}{4\pi e^{-}\gamma}\right)+1+O(\epsilon)\right]
\end{aligned}
$$

As we did before $\lambda\to\lambda\mu^{-2\epsilon}$;

$$
\begin{aligned}
\mu^{-2\epsilon} &= (\mu^2)^{-\epsilon} \\
&= e^{-\epsilon\ln(\mu^2)} \\
&= 1-\epsilon\ln(\mu^2)
\end{aligned}
$$

$$
\begin{aligned}
\Gamma'_{1,\text{loop}} &= \frac{3\lambda w(m^2+3\lambda w^2)}{(4\pi)^2}\underbrace{\mu^{-2\epsilon}}_{1-\epsilon\ln(\mu^2)}\left[\frac{1}{\epsilon}-\ln\left(\frac{m'^2}{4\pi e^{-}\gamma}\right)+1+O(\epsilon)\right] \\
&= \frac{3\lambda w(m^2+3\lambda w^2)}{(4\pi)^2}\frac{1}{\epsilon}-\ln\left(\frac{m'^2}{4\pi e^{\gamma}}\right)+1+O(\epsilon)-\ln(\mu^2)+O(\epsilon)
\end{aligned}
$$

We got;

$$
\Gamma'_{1,\text{loop}} = \frac{3\lambda w(m^2+3\lambda w^2)}{(4\pi)^2}\left[\frac{1}{\epsilon}-\ln\left(\frac{m^2+3\lambda w^2}{4\pi e^{-\gamma}\mu^2}\right)+1\right]
$$

Then one-loop potential is;

$$
\begin{aligned}
V_1(\phi_c) &= -\int_0^{\phi_c} dw\Gamma'_{1,\text{loop}}(w,0) \\
&= -\frac{3\lambda}{(4\pi)^2}\int_0^{\phi_c} dw\left[w(m^2+3\lambda w^2)\left[\frac{1}{\epsilon}-\ln\left(\frac{m^2+3\lambda w^2}{4\pi e^{-\gamma}\mu^2}\right)+1\right]\right]
\end{aligned}
$$

Define $\mu'^2 = 4\pi\epsilon^{-\gamma}\mu^2$;

$$
V_{\text{tot}} = V_{\text{cl}} + V_1 + \delta V^{\text{ms}}
$$

Define $m^2 \to m^2 + \delta m^2$ and $\lambda \to \lambda + \delta\lambda$;

$$V_{\text{cl}} = \frac{1}{2}m^2\phi_c^2 + \frac{1}{4}\lambda\phi_c^4$$

$$V_1 = V_1^{\text{finite}} + V_1^{\text{div}}$$

$$V_{\text{tot}} = V_{\text{cl}} + V_1^{\text{finite}} + V_1^{\text{div}} + \underbrace{\delta V^{\text{ms}}}_{=\frac{1}{2}\delta m^2\phi_c^2+\frac{1}{4}\delta\lambda\phi_c^4}$$

$$V_{\text{tot}} = V_{\text{cl}} + V_1^{\text{ms}}$$

$$V_1^{\text{ms}} = V_1^{\text{finite}} + V_1^{\text{div}} + \frac{1}{2}\delta m^2\phi_c^2 + \frac{1}{4}\delta\lambda\phi_c^4$$

$$\begin{aligned} V_1^{\text{div}} &= -\frac{3\lambda}{(4\pi)^2}\int_0^{\phi_c} dw\frac{w(m^2+3\lambda w^2)}{\epsilon} \\ &= -\frac{3\lambda}{(4\pi)^2}\underbrace{\int_0^{\phi_c} wdw\frac{m^2}{\epsilon}}_{=\frac{\phi_c^2}{2}\frac{m^2}{\epsilon}} - \frac{3\lambda}{(4\pi)^2}3\lambda\underbrace{\int_0^{\phi_c}\frac{w^3dw}{\epsilon}}_{=\frac{1}{4}\phi_c^4\frac{1}{\epsilon}} + \frac{1}{2}\phi_c^2\delta m^2 + \frac{1}{4}\phi_c^4\delta\lambda \\ &= \frac{1}{2}\underbrace{\left[\delta m^2 - \frac{3}{2}\frac{\lambda}{(4\pi)^2}\frac{m^2}{\epsilon}\right]}_{=0} + \frac{1}{4}\phi_c^4\underbrace{\left[\delta\lambda - \frac{9\lambda^2}{(4\pi)^2}\frac{1}{\epsilon}\right]}_{=0} \\ &= 0 \end{aligned}$$

$$\delta m^2 = 3\lambda\frac{m^2}{(4\pi)^2}\frac{1}{\epsilon}$$

$$\delta\lambda = 9\lambda^2\frac{1}{(4\pi)^2}\frac{1}{\epsilon}$$

$$\begin{aligned} V_1^{\text{ms}} &= V_1^{\text{fin}} \\ &= +\frac{3\lambda}{(4\pi)^2}\int_0^{\phi_c} dw\left[w(m^2+3\lambda w^2)\ln\left(\frac{m^2+3\lambda w^2}{\mu'^2}\right) - 1\right] \end{aligned}$$

By taking this integral by Mathematica;

$$V_1^{\text{ms}} = \frac{1}{64\pi^2}(m^2+3\lambda\phi_c^2)^2\left[\ln\left(\frac{m^2+3\lambda\phi_c^2}{\mu^2}\right) - \frac{3}{2}\right]$$

4.6 One-loop Effective Potential of the GWS Model

Higgs Doublet;

$$\begin{aligned}
\varphi &= \begin{pmatrix} G^{\pm} \\ \frac{1}{\sqrt{2}}(H + iG^0) \end{pmatrix} \\
&= \frac{1}{\sqrt{2}} \begin{pmatrix} \varphi_1 + i\varphi_2 \\ \varphi_3 + i\varphi_4 \end{pmatrix}
\end{aligned}$$

Classical Potential;

$$\begin{aligned}
V_0 &= m^2 \varphi^{\dagger}\varphi + \lambda(\varphi^{\dagger}\varphi)^2 \\
&= m^2 \underbrace{\frac{1}{2} + \begin{pmatrix} \phi_1 - i\phi_2 & \phi_3 - i\phi_4 \end{pmatrix} \begin{pmatrix} \varphi_1 + i\varphi_2 \\ \varphi_3 + i\varphi_4 \end{pmatrix}}_{=A} \lambda A^2 \\
&= \frac{1}{2} m^2 \phi_i \phi_i + \frac{1}{4}\lambda(\phi_i\phi_i)^2
\end{aligned}$$

$$\begin{aligned}
V_0' &= V_0(\phi_j \to \phi_j + w_j) \\
&= \frac{1}{2} m^2 \underbrace{(\phi_i + w_i)(\phi_i + w_i)}_{\phi_i^2 + w_i^2 + 2w_i\phi_i} + \frac{1}{4}\lambda \underbrace{(\phi_i + w_i)^2}_{\phi_i^2 + w_i^2 + 2w_i\phi_i} \underbrace{(\phi_j + w_j)^2}_{\phi_j^2 + w_j^2 + 2w_j\phi_j} \\
&= \frac{1}{2} m^2 w_i^2 + m^2 w_i \phi_i + \frac{1}{4}\lambda(w_i^2 2w_j\phi_j + w_j^2 2w_i\phi_i) + \frac{1}{2} m^2 \phi_i^2 \\
&\quad + \frac{1}{4}\lambda(w_j^2\phi_i^2 + w_i^2\phi_j^2 + 4w_iw_j\phi_i\phi_j) + \frac{1}{4}\lambda(2w_j\phi_j\phi_i^2 + 2w_i\phi_i\phi_j^2) \\
&= \underbrace{m^2 w_i\phi_i + \lambda w_i^2 w_j\phi_j}_{=\phi_j w_j(m^2 + \lambda w^2)} + \underbrace{\frac{1}{2}\lambda(w_i^2\phi_j^2 + 2w_iw_j\phi_i\phi_j)}_{=\frac{1}{2}[(m^2+\lambda w^2)\delta_{ij} + 2\lambda w_i w_j]\phi_i\phi_j} + \lambda w_i\phi_i\phi_j^2 + \frac{1}{4}\lambda(\phi_i\phi_i)^2 \\
&= \phi_j w_j(m^2 + \lambda w^2) + \frac{1}{2}[(m^2 + \lambda w^2)\delta_{ij} + 2\lambda w_i w_j]\phi_i\phi_j + \lambda w_i\phi_i\phi_j^2 + \frac{1}{4}\lambda(\phi_i\phi_i)^2
\end{aligned}$$

where $w^2 = w_1^2 + w_2^2 + w_3^2 + w_4^2$

Let us list different Contributions in the GWS model at One-loop as

$$V_1 = V_s + V_v + V_f$$

At the end since $\langle\phi_3\rangle \neq 0$ only, set $w_1 = w_2 = w_4 = 0$ and $w_3 \neq 0$

Masses;

$m(\phi_i) = m^2 + \lambda w_3^2$ for $i = 1, 2, 4$

$m(\phi_i) = m^2 + 3\lambda w_3^2$ for $i = 3$

Vertices;

$$V_s = \underbrace{V_s(\phi = \phi_3)}_{=V_1^{\text{ms}}(\phi=\phi_c)} + 3\,\underbrace{V_s(\phi = \phi_{1,2,4})}_{=V_1^{\text{ms}}(\phi=\phi_c,\lambda\to\frac{\lambda}{3})}$$

$$\begin{aligned} V_s^{\text{ms}} &= \frac{1}{64\pi^2}(m^2+3\lambda\phi_c^2)^2\left[\ln\left(\frac{m^2+3\lambda\phi_c^2}{\mu^2}\right)-\frac{3}{2}\right] \\ &+\frac{3}{64\pi^2}(m^2+\lambda\phi_c^2)^2\left[\ln\left(\frac{m^2+\lambda\phi_c^2}{\mu^2}\right)-\frac{3}{2}\right] \end{aligned}$$

Now the vector boson part;

Vertices come from the kinetic term $(D^\mu\varphi)^\dagger(D_\mu\varphi)$ where $D_\mu\varphi = (\partial_\mu - igW_\mu.\frac{T}{2} - ig'\frac{Y}{2}B_\mu)\varphi$

Since the result is gauge-invariant, let us consider unitary gauge for simplicity;

$$\varphi = \frac{1}{\sqrt{2}}\begin{pmatrix} \varphi_1 + i\varphi_2 \\ \varphi_3 + i\varphi_4 \end{pmatrix} \to \frac{1}{\sqrt{2}}e^{iT.\theta(x)}\begin{pmatrix} 0 \\ \varphi_3 \end{pmatrix}$$

With $\theta(x) = 0$, we have

$$\varphi = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ \varphi_3 \end{pmatrix}$$

in the unitary gauge.

$$\underbrace{\varphi(\phi_i \to \phi_i + w_3)}_{=\varphi'} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ \varphi_3 + w_3 \end{pmatrix}$$

where $Y(\varphi) = 0$

$$\begin{aligned}
D_\mu \varphi' &= \frac{1}{\sqrt{2}} \left[\begin{pmatrix} 0 \\ \partial_\mu \varphi_3 \end{pmatrix} - i\frac{1}{2} \begin{pmatrix} gW_\mu^3 + g'B_\mu & W_\mu^1 - iW_\mu^2 \\ W_\mu^1 + iW_\mu^2 & -gW_\mu^3 + g'B_\mu \end{pmatrix} \begin{pmatrix} 0 \\ \varphi_3 + w_3 \end{pmatrix} \right] \\
&= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ \partial_\mu \varphi_3 \end{pmatrix} - \frac{i}{2\sqrt{2}} \begin{pmatrix} (\phi_3 + w_3)(W_\mu^1 - iW_\mu^2) \\ (\phi_3 + w_3)(-gW_\mu^3 + g'B_\mu) \end{pmatrix}
\end{aligned}$$

$$(D_\mu \varphi')^\dagger = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & \partial_\mu \phi_3 \end{pmatrix} + \frac{i}{2\sqrt{2}} (\phi_3 + w_3) \begin{pmatrix} W_\mu^1 + iW_\mu^2 & -gW_\mu^3 + g'B_\mu \end{pmatrix}$$

Instead of using D_μ in this form, lets express it first in terms of mass eigenstates of the gauge bosons.

$D_\mu = \partial_\mu - igW_\mu.\frac{T}{2} - ig'\frac{Y}{2}B_\mu$

Then, one can write D_μ as;

$$\begin{aligned}
D_\mu &= \partial_\mu - i\frac{g}{\sqrt{2}}(W_\mu^+ T^+ + W_\mu^- T^-) - i\frac{g}{\cos\theta_w} Z_\mu (T^3 - \sin^2\theta_w Q) - ieQA_\mu \\
T^\pm &= \frac{1}{2}(\sigma^1 \pm i\sigma^2) = \sigma^\pm \\
e &= \frac{gg'}{\sqrt{g^2 + g'^2}} \\
\sin\theta_w &= \frac{e}{g} \\
\cos\theta_w &= \frac{e}{g'} \\
\sigma^+ &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\
\sigma^- &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}
\end{aligned}$$

Since $Q=0$ for ϕ_3

$$
\begin{aligned}
D_\mu\varphi' &= \left[\partial_\mu - i\frac{g}{\sqrt{2}}(W_\mu^+T^+ + W_\mu^-T^-) - i\frac{g}{\cos\theta_w}T^3 Z_\mu\right]\varphi' \\
&= \begin{pmatrix} \partial_\mu - \frac{ig}{2\cos\theta_w}Z_\mu & \frac{-ig}{\sqrt{2}}W_\mu^+ \\ \frac{-ig}{\sqrt{2}}W_\mu^- & \partial_\mu + \frac{ig}{2\cos\theta_w}Z_\mu \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ w_3+\phi_3 \end{pmatrix} \\
&= \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{-ig}{\sqrt{2}}(w_3+\phi_3)W_\mu^+ \\ \partial_\mu\phi_3 + \frac{ig}{2\cos\theta_w}(w_3+\phi_3)Z_\mu \end{pmatrix}
\end{aligned}
$$

$$
(D_\mu\varphi')^\dagger = \frac{1}{\sqrt{2}}\begin{pmatrix} \frac{ig}{\sqrt{2}}(w_3+\phi_3)W_\mu^- & \partial_\mu\phi_3 - \frac{ig}{2\cos\theta_w}(w_3+\phi_3)Z_\mu \end{pmatrix}
$$

$$
\begin{aligned}
(D_\mu\varphi')^\dagger(D_\mu\varphi') &= \frac{1}{2}\Big[\frac{g^2}{2}(w_3+\phi_3)^2W_\mu^+W^{-\mu} + (\partial_\mu\phi_3)^2 \\
&\quad -\frac{ig}{2\cos\theta_w}(\partial_\mu\phi_3)(w_3+\phi_3)Z^\mu \\
&\quad +\frac{ig}{2\cos\theta_w}(\partial_\mu\phi_3)(w_3+\phi_3)Z^\mu + \frac{g^2}{4cos\theta_w^2}(w_3+\phi_3)^2Z_\mu Z^\mu\Big] \\
&= \underbrace{\frac{1}{2}(\partial_\mu\phi_3)^2}_{=K.T} + \frac{1}{4}g^2({w_3}^2+{\phi_3}^2+2w_3\phi_3)W_\mu^+W^{-\mu} \\
&\quad +\frac{g^2}{4\cos\theta_w^2}({w_3}^2+{\phi_3}^2+2w_3\phi_3)Z_\mu Z^\mu \\
&= KT + \underbrace{\frac{1}{4}g^2{w_3}^2}_{=m_W^2} W_\mu^+W^{-\mu} + \underbrace{\frac{g^2}{4\cos\theta_w^2}{w_3}^2}_{=\frac{1}{2}m_Z^2} Z_\mu Z^\mu \\
&\quad +\frac{1}{2}g^2w_3\phi_3W_\mu^+W^{-\mu} + \frac{1}{2}\frac{g^2}{4\cos\theta_w^2}w_3\phi_3Z_\mu Z^\mu \\
&\quad +\frac{1}{4}g^2{\phi_3}^2W_\mu^+W^{-\mu} + \frac{1}{4}\frac{g^2}{4\cos\theta_w^2}{\phi_3}^2Z_\mu Z^\mu
\end{aligned}
$$

Before we proceed let us clarify a point about the mass term of $W_\mu^\pm$. For the original $(W_1^\mu, W_2^\mu, W_3^\mu)$ fields,the mass term as usual is of the form (for only W_1^μ and W_2^μ) is $-\frac{1}{2}M_1^2W_{1\mu}W_1^\mu - \frac{1}{2}M_2^2W_{2\mu}W_2^\mu$ one can show that starting from $(D_\mu\varphi)^\dagger D^\mu\varphi$, $M_1^2 = M_2^2 = M^2 = \frac{1}{4}g^2w_3^2$ Now we can check the masses of the physical fields $W_\mu^\pm$. In terms of the original fields Then;

$$\begin{aligned}
\text{MT} &= -\frac{1}{2}M_1^2 W_{1\mu}W_1^\mu - \frac{1}{2}M_2^2 W_{2\mu}W_2^\mu \\
&= -\frac{1}{4}M^2[\underbrace{(W_\mu^+ + W_\mu^-)(W^{+\mu} + W^{-\mu}) - (W_\mu^+ - W_\mu^-)(W^{+\mu} - W^{-\mu})}_{=2W_\mu^+W^{-\mu}+2W_\mu^-W^{+\mu}=4W_\mu^+W^{-\mu}}] \\
&= -\underbrace{M^2}_{=M_W^2} W_\mu^+ W^{-\mu}
\end{aligned}$$

$M_W^2 = M^2 = \frac{1}{4}g^2 w_3^2$

$$\text{Vertex Factor of } W^\pm \text{ for } \phi_3 = i\frac{g^2 w_3}{2}g_{\mu\nu}$$

$$\text{Vertex Factor of } Z \text{ for } \phi_3 = \underbrace{\frac{1}{2}}_{\text{Identical } Z \text{ bosons}} \frac{ig^2 w_3}{2\cos\theta_w}g_{\mu\nu}$$

Propagators of W and Z in Landau gauge;

$$D^{\mu\nu} = \frac{i}{k^2 - m_v^2}\left(-g^{\mu\nu} + \frac{k^\mu k^\nu}{k^2}\right)$$

$$\begin{aligned}
\Gamma_1'^{(w)}(w_3,0) &= i\int \frac{d^dk}{(2\pi)^d}\frac{-ig^2 w_3}{2}g_{\mu\nu}\frac{i}{k^2 - m_w^2(w_3)}\left(-g^{\mu\nu} + \frac{k^\mu k^\nu}{k^2}\right) \\
&= \frac{g^2 w_3}{2}\int \frac{d^dk}{(2\pi)^d}\frac{i}{k^2 - m_w^2(w_3)}\left(-\underbrace{g^{\mu\nu}g_{\mu\nu}}_{=d} + \underbrace{\frac{k^\mu k^\nu g_{\mu\nu}}{k^2}}_{=1}\right) \\
&= i\frac{g^2 w_3}{2}(1-d)\int \frac{d^dk}{(2\pi)^d}\frac{1}{k^2 - m_W^2(w_3)} \\
&= \underbrace{\mu^{2\epsilon}}_{=4-d}\frac{g^2 w_3}{2(4\pi)^{d/2}}\frac{(1-d)\Gamma(1-d/2)}{(m_W^2(w_3))^{\epsilon-1}} \\
&= \frac{1}{2}\underbrace{\frac{g^2 w_3^2}{w_3}}_{=\frac{4m_w^2(w_3)}{w_3}} m_W^2(w_3)\frac{\mu^{2\epsilon}\Gamma(\epsilon-1)(1-d)}{\underbrace{(4\pi)^{d-2}}_{=(4\pi)^2(4\pi)^{-2\epsilon}} m_W^{2\epsilon}(w_3)} \\
&= \frac{2m_w^4(w_3)}{w_3(4\pi)^2}\left[\frac{\mu^2(4\pi)}{m_w^2(w_3)}\right]^\epsilon \Gamma(\epsilon-1)(1-d) \\
&= \frac{2m_w^4(w_3)}{w_3(4\pi)^2}\left(1 + \epsilon\ln\left[\frac{\bar{\mu}^2}{m_w^2(w_3)}\right]\right)\underbrace{(3-2\epsilon)(\frac{1}{\epsilon}+1)}_{=\frac{3}{\epsilon}+1} \\
&= \frac{2m_w^4(w_3)}{w_3(4\pi)^2}\left[\frac{3}{\epsilon} + 1 + 3\ln\left[\frac{\bar{\mu}^2}{m_w^2(w_3)}\right]\right]
\end{aligned}$$

After MS substraction;

$$\Gamma'_{1MS}(w_3) = \frac{2m_W^4(w_3)}{w_3(4\pi)^2}\left(1 - 3\ln\left[\frac{m_W^2(w_3)}{\bar{\mu}^2}\right]\right)$$

Then,

$$\begin{aligned} V_W^{MS}(\phi_c) &= \int_0^{\phi_c} dw_3 \Gamma'_{1MS}(w_3) \\ &= \frac{3}{2}\frac{1}{(4\pi)^2} m_W^4(\phi_c)\left[\ln\left(\frac{m_W^2(\phi_c)}{\bar{\mu}}^2\right) - \frac{5}{6}\right] \end{aligned}$$

In the case of Z-boson in the loop $m_w(\phi_c) \to m_Z(\phi_c)$. Also while $W_\mu - W_\nu - \phi_3$ vertex is $\frac{2m_W^2(w_3)g_{\mu\nu}}{w_3}$, the $Z_\mu - Z_\nu - \phi_3$ vertex is $\frac{m_Z^2(w_3)g_{\mu\nu}}{w_3}$.Thus, in addition to the above substitution, there is an additional factor of $\frac{1}{2}$ in the Z-exchange case.

$$V_Z^{MS}(\phi_c) = \frac{1}{2} V_W^{MS}(m_W(\phi_c) \to m_Z(\phi_c))$$

The last part is due to fermions in the loop. Yukawa Lagrangian in the lepton sector;

$$L_{yuk}^{lep} = -y_l(\bar{L}\varphi R + \bar{R}\varphi^\dagger L)$$

where $R = l_R$ and y_l is the yukawa coupling. One can show that L_{yuk}^{lep} is invariant under $SU(2)_L X U(1)_Y$ transformation. Since

$$\varphi = \begin{pmatrix} 0 \\ \frac{l}{\sqrt{2}}(\phi_3 + w_3) \end{pmatrix},$$

L_{Yuk}^{lep} becomes;

$$\begin{aligned} L_{Yuk}^{lep} &= -\frac{y_l}{\sqrt{2}}(\phi_3 + w_3)(\bar{l_R} l_L + \bar{l_L} l_R) \\ &= -\frac{y_l}{\sqrt{2}}(\phi_3 + w_3)\bar{l}l \\ &= -m_l \bar{l}l - \frac{y_l}{\sqrt{2}}\phi_3 \bar{l}l \end{aligned}$$

where $m_l = \frac{1}{\sqrt{2}} y_l w_3$ and the neutrinos remain massless. In the quark sector, the down-type quarks behave like exactly the charged leptons. Hence,

$$\begin{aligned} L_{Yuk}^{down} &= -\frac{y_d}{\sqrt{2}}(\phi_3 + w_3)(\bar{d_L} d_R + \bar{d_R} d_L) \\ &= m_d \bar{d} d - \frac{y_d}{\sqrt{2}} \phi_3 \bar{d} d \end{aligned}$$

The case with the up-sector is slightly different. If the same Higgs doublet is used, one gets the wrong combination, namely $\bar{u_R} d_L + \bar{d_L} u_R$. In the representation used so far the hypercharge of φ is $Y(\varphi) = 1$. However, one can also use the charge conjugated φ, $\tilde{\varphi}$, as $\tilde{\varphi} = i\tau_2 \varphi* = \begin{pmatrix} \phi_3 - i\phi_4 \\ -(\phi_1 - i\phi_2) \end{pmatrix}$ with $Y(\tilde{\varphi}) = -1$. Here one can show that $\tilde{\varphi}$ can be transformed as φ under $SU(2)_L X U(1)_Y$. Therefore, it should also exists in the Lagrangian and we have;

$$L_{Yuk}^{up} = -y_u \bar{Q}_L \tilde{\varphi} U_R + h.c.$$

where $\tilde{\varphi} = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_3 + w_3 \\ 0 \end{pmatrix}$

Then;

$$\begin{aligned} L_{Yuk}^{up} &= -\frac{y_u}{\sqrt{2}} \\ &= -\frac{y_u(\phi_3 + w_3)}{\sqrt{2}}(\bar{U_L} U_R + h.c.) \\ &= m_u \bar{U} U \end{aligned}$$

where $m_u = \frac{y_u}{\sqrt{2}} w_3$.Thus, we can summarize the mass and the vertex term as;

$$L_{Yuk}^{fer} = -\sum \frac{y_f}{\sqrt{2}}(\phi_3 + w_3)\bar{f} f$$

where $\phi_3 - f - f$ factor is $\frac{-im_f(w_3)}{w_3}$. Fermion propagator is $\frac{-i}{k - m_f(w_3)}$ since $y_{top} = 1$

and the largest, we keep top loop only;

$$\begin{aligned}\Gamma_1'^{top}(w_3) &= i\int \frac{d^dk}{(2\pi)^d}\frac{-im_f(w_3)}{w_3}\frac{-iTr(k+m_f(w_3))}{k^2-m_f^2(w_3)}(-1)\\ &= 3\frac{im_f(w_3)}{w_3}\int\frac{d^dk}{(2\pi)^d}\frac{4m_f(w_3)}{k^2-m_f^2(w_3)}\end{aligned}$$

where 3 comes from the color sum in the top loop.

$$\Gamma_1'^{top}(w_3) = \frac{12im_f^2(w_3)}{w_3}\mu^{4-d}\underbrace{\int\frac{d^dk}{(2\pi)^d}\frac{1}{k^2-m_f^2(w_3)}}_{=I}$$

where $d = 4-2\epsilon$

$$I = \frac{-i}{(4\pi)^2(4\pi)^{-\epsilon}}\frac{\Gamma(\epsilon-1)}{\Gamma(1)}\mu^{2\epsilon}(m_f^2(w_3))^(1-\epsilon)$$

$$\begin{aligned}\Gamma_1'^{top}(w_3) &= \frac{12m_t^4(w_3)}{(4\pi)^2w_3}\mu^{2\epsilon}\Gamma(\epsilon-1)\left(\frac{m_f^2(w_3)}{4\pi}\right)^{-\epsilon}\\ &= -\frac{12m_f^4(w_3)}{(4\pi)^2w_3}\left(\frac{1}{\epsilon}+1\right)\left(\frac{4\pi\mu^2e^{-\gamma_E}}{m_f^2(w_3)}\right)^{\epsilon}\\ &= -\frac{12m_f^4(w_3)}{(4\pi)^2w_3}\left[\frac{1}{\epsilon}+\ln\left[\frac{\bar{\mu}^2}{m_f^2(w_3)}\right]+1+O(\epsilon)\right]\end{aligned}$$

Again;

$$\Gamma_{1MS}' = -\frac{12m_f^4(w_3)}{(4\pi)^2w_3}\left[1-\ln\left(\frac{m_f^2(w_3)}{\bar{\mu}^2}\right)\right]$$

$$V_{top}^{MS}(\phi_c) = \int_0^{\phi_c}\frac{12m_f^4(w_3)}{(4\pi)^2w_3}\left[1-\ln\left(\frac{m_f^2(w_3)}{\bar{\mu}^2}\right)\right]dw_3$$

With the help of Mathematica, we get;

$$V_{top}^{MS}(\phi_c) = -3\frac{m_f^4(\phi_c)}{(4\pi)^2}\left[\ln\left(\frac{m_f^2(\phi_c)}{\bar{\mu}^2}\right)-\frac{3}{2}\right]$$

$$V_{loop}^{MS}(\phi_c) = V_{loop}^{h}(\phi_c)+V_{loop}^{Golds}(\phi_c)+V_{loop}^{W^\pm}(\phi_c)+V_{loop}^{Z}(\phi_c)+V_{loop}^{top}(\phi_c)$$

$$V_{loop}^{h}(\phi_c) = \frac{1}{(8\pi)^2}m_h^4(\phi_c)\left[\ln\left(\frac{m_h^2(\phi_c)}{\bar{\mu}^2}\right)-\frac{3}{2}\right]$$

$$V_{loop}^{Golds}(\phi_c) = \frac{3}{(8\pi)^2} m_G^4(\phi_c) \left[\ln\left(\frac{m_G^2(\phi_c)}{\bar{\mu}^2}\right) - \frac{3}{2}\right]$$

$$V_{loop}^{W^\pm}(\phi_c) = \frac{6}{(8\pi)^2} m_W^4(\phi_c) \left[\ln\left(\frac{m_W^2(\phi_c)}{\bar{\mu}^2}\right) - \frac{5}{6}\right]$$

$$V_{loop}^{Z}(\phi_c) = \frac{1}{2} V_{loop}^{W}(m_W(\phi_c) \to m_Z(\phi_c))$$

$$V_{loop}^{top}(\phi_c) = \frac{-12}{(8\pi)^2} m_t^4(\phi_c) \left[\ln\left(\frac{m_t^2(\phi_c)}{\bar{\mu}^2}\right) - \frac{3}{2}\right]$$

Thus the total final result can be expressed in a compact form as

$$V_{loop}^{MS}(\phi_c) = \frac{N_t}{(8\pi)^2} \sum (-1)^{2s_p}(1+2s_p) m_p^4(\phi_c) \left[\ln\left(\frac{m_p^2(\phi_c)}{\bar{\mu}^2}\right) - a_p\right]$$

Here the parameters are

$$\begin{aligned} s_p &= 0 \quad \text{for } h, G \\ s_p &= \frac{1}{2} \quad \text{for fermions} \\ s_p &= 1 \quad \text{for } W^\pm, Z \text{ bosons .} \end{aligned}$$

For $s_p = 0$ for h, G, $\frac{1}{2}$ for fermions, and 1 for vector bosons. The parameter $a_p = a = \frac{3}{2}$ for scalar and fermions and $a_p = a - \frac{1}{a} = \frac{5}{6}$ for vector bosons. Also $N_t = 12$ for top quark(but 4 for charged leptons) and $N_t = 1$ for all scalar and vector bosons.

CHAPTER 5

CONCLUSION

The role of symmetries in particle physics can not be undeniable for the progress in especially developing/improving theories. Especially after the discovery of the Higgs particle, taken as the evidence of the presence of the Higgs mechanism, breaking the local gauge symmetry invariance spontaneously fits very well with the mechanism. Hence, these recent advances put all of these symmetry-related arguments into more central position.

Motivated from this, we have discussed the global and local symmetries and their spontaneous breaking both in abelian and non-abelian frameworks with various toy models like φ^4 theory, the Georgi-Glashow model etc. Later the Higgs sector in these frameworks have been explored and at the end the Higgs sector of the Glashow-Weinberg-Salam model is presented.

Calculating the one-loop corrections to the scalar potential of the Higgs sector is an important business and it is worth spending time on it. First the discussion has been exercised in various simplified frameworks. Two alternative methods, the Coleman-Weinberg and the Lee-Sciaccaluga methods, are compared. The regularization and renormalization methods have been implemented to get the effective form of one-loop corrected scalar potential.

Concentrating on the scalar potential of the GWS model, the one-loop corrections have been computed within the dimensional regularization method together with the $\overline{MS}$ renormalization scheme. The one-loop tadpole diagrams in the GWS model have the following particles in the loop; scalars (Higgs and Goldstones), vector bosons ($W^{\pm}, Z$) and the fermions (all charged leptons and quarks in principle but the top quark mainly due to its large Yukawa coupling). After getting the analytical form of the potential, a qualitative discussion about its the stability is given. For the discussion make sense, one should also include the RGE improvements, temperature corrections etc, which are all beyond the scope of the present study. However, it can be considered to be the first step along these lines. Additionally, having one doublet at hand, everybody wonders whether there are other scalar doublets and/or singlets in nature. Such extensions will definitely take the current study as a starting point.

REFERENCES

[1] S. L. Glashow. Partial Symmetries of Weak Interactions. *Nucl. Phys.*, 22:579–588, 1961.

[2] Steven Weinberg. A Model of Leptons. *Phys. Rev. Lett.*, 19:1264–1266, 1967.

[3] Abdus Salam. Weak and Electromagnetic Interactions. *Conf. Proc.*, C680519:367–377, 1968.

[4] Serguei Chatrchyan et al. Observation of a new boson at a mass of 125 GeV with the CMS experiment at the LHC. *Phys. Lett.*, B716:30–61, 2012.

[5] Gerard 't Hooft. Renormalization of Massless Yang-Mills Fields. *Nucl. Phys.*, B33:173–199, 1971.

[6] Gerard 't Hooft. Renormalizable Lagrangians for Massive Yang-Mills Fields. *Nucl. Phys.*, B35:167–188, 1971.

[7] Peter W. Higgs. Broken Symmetries and the Masses of Gauge Bosons. *Phys. Rev. Lett.*, 13:508–509, 1964.

[8] F. Englert and R. Brout. Broken Symmetry and the Mass of Gauge Vector Mesons. *Phys. Rev. Lett.*, 13:321–323, 1964.

[9] Marc Sher. Electroweak Higgs Potentials and Vacuum Stability. *Phys. Rept.*, 179:273–418, 1989.

[10] E. A. Reyes Rojas. *The Higgs Boson at LHC and the Vacuum Stability of the Standard Model*. PhD thesis, Colombia, U. Natl., 2015.

[11] Gautam Bhattacharyya. A Pedagogical Review of Electroweak Symmetry Breaking Scenarios. *Rept. Prog. Phys.*, 74:026201, 2011.

[12] Sidney R. Coleman and Erick J. Weinberg. Radiative Corrections as the Origin of Spontaneous Symmetry Breaking. *Phys. Rev.*, D7:1888–1910, 1973.

[13] P. A. Baikov and Vladimir A. Smirnov. Equivalence of recurrence relations for Feynman integrals with the same total number of external and loop momenta. *Phys. Lett.*, B477:367–372, 2000.

[14] C. Ford, I. Jack, and D. R. T. Jones. The Standard model effective potential at two loops. *Nucl. Phys.*, B387:373–390, 1992. [Erratum: Nucl. Phys.B504,551(1997)].

[15] S. Y. Lee and Alain M. Sciaccaluga. Evaluation of Higher Order Effective Potentials with Dimensional Regularization. *Nucl. Phys.*, B96:435–444, 1975.