ASYMPTOTIC INTEGRATION OF IMPULSIVE DIFFERENTIAL EQUATIONS

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ABSTRACT

ASYMPTOTIC INTEGRATION OF IMPULSIVE DIFFERENTIAL EQUATIONS

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The main objective of this thesis is to investigate asymptotic properties of the solutions of differential equations under impulse effect, and in this way to fulfill the gap in the literature about asymptotic integration of impulsive differential equations. In this process our main instruments are fixed point theorems; lemmas on compactness; principal and nonprincipal solutions of impulsive differential equations and Cauchy function for impulsive differential equations.

The thesis consists of five chapters. In Chapter 1, the statement of the problem and a review of related literature are given. Chapter 2 contains preliminary concepts about impulsive differential equations and necessary theorems from functional analysis. Moreover, it provides characterizations of principal and nonprincipal solutions of impulsive differential equations with continuous solutions, and new results about existence of principal and nonprincipal solutions for impulsive differential equations with discontinuous solutions. In Chapter 3, new results, stating asymptotic properties of the solutions, are expressed via principal and nonprincipal solutions. In the first section, impulsive differential equations with discontinuous solutions are considered. By dividing these equations into two groups according to the type of impulse effects, various asymptotic representations for the solutions of each group are given. Moreover, sufficient conditions for existence of positive and monotone solutions are obtained. A new lemma consisting of compactness criteria for sets of piecewise continuous functions is also presented. The second section is devoted to impulsive differential equations with continuous solutions, and for these equations, both analogous results to previous theorems and a general asymptotic formula depending on a parameter is obtained. Several subsidiary examples are placed at the end of the chapter. In Chapter 4, asymptotic representation of solutions for impulsive differential equations with discontinuous solutions is produced with the help of Cauchy functions, and an example is presented. The last chapter contains a summary of the thesis and suggests some open problems for further studies.

Keywords: Asymptotic Integration, Impulsive Differential Equations, Principal and Nonprincipal Solutions, Asymptotic Representation of Solutions, Fixed Point Theory, Cauchy Function.

ÖZ

İMPALSİF DİFERENSİYEL DENKLEMLERİN ASİMPTOTİK İNTEGRASYONU

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Bu tezin esas amacı, impals etkisi altındaki diferensiyel denklemlerin asimptotik özelliklerini incelemek ve bu yolla impalsif diferensiyel denklemlerin asimptotik integrasyonu konusunda literatürdeki açığı kapatmaktır. Bu süreçte temel araçlarımız sabit nokta teoremleri; kompaktlık üzerine lemmalar; impalsif diferensiyel denklemlerin küçük ve büyük çözümleri ve impalsif diferensiyel denklemler için Cauchy fonksiyonudur.

Tez beş bölümden oluşmaktadır. 1. bölümde problemin ifadesi ve literatürdeki ilgili çalışmalar verilmiştir. 2. bölüm impalsif diferensiyel denklemlerle ilgili temel kavramlar ve fonksiyonel analizden gerekli teoremleri içermektedir. Ayrıca, sürekli çözüme sahip impalsif diferensiyel denklemler için küçük ve büyük çözümlerin karakterizasyonlarını ve süreksiz çözüme sahip impalsif diferensiyel denklemler için de küçük ve büyük çözümlerin varlığı ile ilgili yeni sonuçları kapsamaktadır. 3. bölümde küçük ve büyük çözümler aracılığıyla çözümlerin asimptotik özelliklerini ifade eden yeni sonuçlar sunulmuştur. Birinci kısımda süreksiz çözümlü impalsif diferensiyel denklemler ele alınmıştır. Bu denklemler impals etkilerinin tipine göre iki gruba ayrılıp her iki grup için çözümlerin çeşitli asimptotik gösterimleri verilmiştir. Ayrıca, pozitif ve monoton çözümlerin varlığı için yeter koşullar elde edilmiştir. Parçalı sürekli fonksiyonlar kümeleri için de kompaktlık kriterlerini veren bir lemmaya da yer verilmiştir. İkinci kısım sürekli çözümlü impalsif diferensiyel denklemlere ayrılmış ve bu denklemler için hem önceki sonuçların benzerleri hem de bir parametreye bağlı genel bir asimptotik gösterim elde edilmiştir. Bölümün sonunda destekleyici örneklere yer verilmiştir. 4. bölümde, Cauchy fonksiyonları yardımıyla, süreksiz çözümlü impalsif diferensiyel denklemler için çözümlerin asimptotik gösterimi ortaya konmuş ve bir örnek verilmiştir. Son bölüm tezin özetini içermekte ve sonraki çalışmalar için bazı açık problemler önermektedir.

Anahtar Kelimeler: Asimptotik İntegrasyon, İmpalsif Diferensiyel Denklemler, Küçük ve Büyük Çözümler, Çözümlerin Asimptotik Temsilleri, Sabit Nokta Teorisi, Cauchy Fonksiyonu.

To my beloved ones,
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CHAPTER 1

INTRODUCTION AND RELATED LITERATURE

Mathematical characterization of evolution of a real process usually depends on the sudden changes or perturbations in its state. The period of time that passes while these perturbations materialize is usually imperceptible in comparison with the time passing during the process. In other words, it is assumed that the perturbations are momentary and the state of the process changes through jump. For example, a bouncing bead on a rigorous surface experiences abrupt changes in its velocity after each strike, and ordinary differential equations are inadequate for mathematical modeling of this phenomenon. For such real world problems having instantaneous changes in their state impulsive differential equations are used [5, 6, 9, 33, 56, 46]. Since the abrupt changes in a state results in discontinuities in the solution trajectory, it is seen that the solutions of impulsive differential equations have some jump discontinuities, and this leads to a more fruitful and richer theory as compared to the theory of ordinary differential equations. On the other hand, existence of jump points causes complex and cumbersome calculations, so it is more difficult and challenging to deal with impulsive differential equations. Nevertheless, differential equations under impulse effect attract the attention of scientists due to the expanding application area in various disciplines and necessity in modeling real processes, and this leads to a requirement of developing the theory. As a response, many mathematical studies have been conducted and the number of mathematicians dealing with the theory of impulsive differential equations has increased in the last decades.

1.1 Problem Statement

We present an example in order to illustrate our main problem. Consider the second order, nonhomogeneous ordinary differential equation

$$x'' = \frac{\ln t}{t^3} \tag{1.1}$$

which has the general solution

$$x(t) = \frac{2\ln t + 3}{4t} + c_1 t + c_2, \quad c_1, c_2 \in \mathbb{R}.$$

It is easy to see that

$$x(t) = c_1 t + c_2 + o(1), \quad t \to \infty.$$

There are numerous papers which have been written about asymptotic representation of solutions of differential equations, however, in a vast majority of them the differential equations of the form $x'' = f(t,x)$ or $x'' = f(t,x,x')$ are studied [2, 26, 27, 30, 32, 35, 38, 39, 40, 55]. In 2012, Ertem and Zafer, by noticing that $v(t) = t$ and $u(t) = 1$ are linearly independent solutions of $x'' = 0$, applied this idea to the ordinary differential equations of the form

$$(p(t)x')' + q(t)x = f(t,x). \tag{1.2}$$

By a successful generalization of the previous results they proved that there exist solutions $x(t)$ of (1.2) satisfying the asymptotic formulas:

$$x(t) = av(t) + bu(t) + o(u(t)), \quad t \to \infty$$

or

$$x(t) = av(t) + bu(t) + o(v(t)), \quad t \to \infty,$$

where $u(t)$ and $v(t)$ are linearly independent solution of the corresponding homogeneous equation $(p(t)x')' + q(t)x = 0$.

In this thesis, we present asymptotic formulas of solutions of second order impulsive differential equations, and so we improve some of the existing results to differential equations under impulse effect. For example, consider the nonhomogeneous impulsive differential equation

$$\begin{cases} x'' = \frac{3}{2}t^2, & t \neq i, \\ \Delta x' - \frac{1}{2}x = -\left(\frac{i}{2}\right)^4, & t = i, \quad i = 1, 2, \ldots \end{cases} \tag{1.3}$$

whose general solution is

$$x(t) = \frac{t^4}{8} + c_1 v(t) + c_2 u(t), \quad i-1 < t \le i$$

where

$$u(t) = 2^{1-i}(i-t+1), \quad i-1 < t \le i, \quad i = 1, 2, \ldots$$

and

$$v(t) = 2^{i-2}(t-i+2), \quad i-1 < t \le i, \quad i = 1, 2, \ldots$$

are linearly independent solutions of the homogeneous equation

$$\begin{cases} x'' = 0, & t \ne i, \\ \Delta x' - \frac{1}{2}x = 0, & t = i, \quad i = 1, 2, \ldots \end{cases}$$

that corresponds to (1.3). Since $v(t) \le 2^t$, it follows that

$$x(t) = c_1 v(t) + c_2 u(t) + o(v(t)), \quad t \to \infty.$$

Although many studies exist in the literature dealing with asymptotic representation of the solutions of ordinary differential equations, for differential equations under impulse effects analogous results are almost never found. Based on this deficiency in the literature, we consider the following second order, nonlinear impulsive differential equation:

$$\begin{cases} (p(t)x')' + q(t)x = f(t,x), & t \ne \theta_i, \\ \Delta x + p_i x + \tilde{p}_i x' = f_i(x), & t = \theta_i, \\ \Delta p(t)x' + q_i x + \tilde{q}_i x' = \tilde{f}_i(x), & t = \theta_i, \end{cases}$$

where $\{p_i\}, \{\tilde{p}_i\}, \{q_i\}, \{\tilde{q}_i\}$ are sequences of real numbers, $f \in \mathrm{C}([t_0,\infty) \times \mathbb{R}, \mathbb{R})$, $f_i, \tilde{f}_i \in \mathrm{C}([t_0,\infty), \mathbb{R})$. Our aim is to give asymptotic integration results for this second order impulsive differential equations of both continuous solutions and discontinuous solutions.

1.2 Thesis Structure

The present chapter is introductory. Chapter 2 contains preliminary concepts about impulsive differential equations as well as basic theorems and lemmas from functional analysis. Also, characterization of principal and nonprincipal solutions for impulsive differential equations with continuous solutions is presented. Then, new theorems stating existence of principal and nonprincipal solutions for impulsive differential equations with discontinuous solutions are expressed and proved. Chapter 3 consists of many results on asymptotic integration of impulsive differential equations via principal and nonprincipal solutions. In the first section, impulsive differential equations with discontinuous solutions are considered and various asymptotic representations of the solutions are expressed. These equations are divided into two groups according to types of impulse conditions, namely the impulsive differential equations with separated impulse conditions and impulsive differential equations with mixed impulse conditions. Positive solutions for both type of equations are also investigated. The second section is devoted to asymptotic integration of a general form of impulsive differential equations with continuous solutions. Besides, existence of monotone positive solutions is proved. Finally, some examples which illustrate our results are placed. In chapter 4 we present asymptotic representation of impulsive differential equations of discontinuous solutions by using Cauchy functions, and give an example to support our findings. Chapter 5, the last chapter of the thesis includes a brief summary of the thesis. We also propose some open problems and explain the challenge of dealing with impulsive differential equations having discontinuous solutions.

1.3 Literature Review

Although there is hardly any work about asymptotic representation of solutions for differential equations under impulse effect, there is a great deal of literature concerning asymptotic representation of solutions for differential equations without any impulse effects. Here we will mention only some of these results which are somehow related to our study.

1.3.1 Ordinary Differential Equations

In 2002, Mustafa and Rogovchenko [38] considered the second order nonlinear ordinary differential equation

$$x'' + f(t, x, x') = 0, \tag{1.4}$$

where $t \geq t_0 \geq 1$ and proved that there is a solution $x(t)$ of equation (1.4) such that the following asymptotic representation holds:

$$x(t) = at + o(t), \quad t \to \infty.$$

Then, in 2003 Yin [55] studied the same differential equation and proved that there exists a solution $x(t)$ of (1.4) which is monotone, positive and satisfies the following asymptotic property:

$$\lim_{t \to \infty} \frac{x(t)}{t} = c.$$

Lipovan [35] considered the ordinary differential equation

$$x'' = f(t, x), \quad t \geq 1 \tag{1.5}$$

and proved that for arbitrary real constants a and b, a solution of (1.5) satisfy the following asymptotic property:

$$x(t) = at + b + o(1), \quad t \to \infty.$$

In 2006, Mustafa and Rogovchenko [39] have generalized Lipovan's result by using a parameter $c \in [0, 1]$. They proved that under sufficient conditions on the function $f(t, x)$ a solution $x(t)$ satisfies one of the following asymptotic formulas:

$$\begin{aligned} x(t) &= at + o(t), \quad t \to \infty, \\ x(t) &= at + b + o(1), \quad t \to \infty, \\ x(t) &= at + o(t^c), \quad t \to \infty, \quad c \in (0, 1). \end{aligned} \tag{1.6}$$

In most of the papers which are related to asymptotic integration of ordinary differential equations the authors have focused on the equations of the form $x'' = f(t, x, x')$ or $x'' = f(t, x)$. For a general approach to this subject we refer to the papers [17, 18, 19] by Ertem and Zafer. They considered the second order, nonlinear differential equation of the form

$$(p(t)x')' + q(t)x = f(t,x) \tag{1.7}$$

and improved some of the studies mentioned above. For example, in the paper [18] they proved the existence of solutions $x(t)$ satisfying either the asymptotic formula

$$x(t) = av(t) + bu(t) + o(u(t)), \quad t \to \infty$$

or

$$x(t) = av(t) + bu(t) + o(v(t)), \quad t \to \infty,$$

where u and v are special solutions that are called principal and nonprincipal solutions, respectively, of the homogeneous equation $(p(t)x')' + q(t)x = 0$. Also, they considered the following equation [19]

$$(p(t)x')' + q(t)x = f(t,x,x')$$

and proved that it has a solution $x(t)$ such that for some $c > 0$,

$$\lim_{t\to\infty} \frac{x(t)}{v(t)} = c.$$

For further monographs about asymptotic integration of ordinary differential equations we may refer to the papers [12, 27, 30, 32, 58] and the books [4, 26]. An extensive literature may be found in the survey paper by Agarwal et. al. [2] and the Ph.D. thesis by Ertem [16].

1.3.2 Impulsive Differential Equations

Analogues results for second order differential equations under impulse effect are hardly found, nevertheless we may mention some studies in which authors have derived asymptotic formulas or asymptotic constancy of the solutions. For instance, the second order impulsive differential equation

$$\begin{cases} x'' = f(t,x,x'), & t \neq t_i, t \geq a > 0, \\ \Delta x = g_1(t,x,x'), & t = t_i, \\ \Delta x' = g_2(t,x,x'), & t = t_i, \ i = 1,2,\ldots \end{cases} \tag{1.8}$$

was considered by Pinto [42] and it was proven that there exist $\delta_1, \delta_2 \in \mathbb{R}$ such that (1.8) have a solution $x(t)$ satisfying

$$x(t) = (\delta_1 + \delta_2)t + o(t), \quad t \to \infty$$

and

$$x'(t) = \delta_2 + o(1/t), \quad t \to \infty.$$

For the equation

$$\begin{cases} (p(t)x')' = f(t, x, x'), & t \neq t_i, t \geq 0, \\ \Delta x = I_i(x), & t = t_i, \\ \Delta x' = J_i(x'), & t = t_i,\ i = 1, 2, \ldots, \end{cases} \tag{1.9}$$

Cheng and Yan [11] proved that for any $b \in \mathbb{R}$ (1.9) has a solution $x(t)$ satisfying

$$p(t)x'(t) = b + o(1), \quad t \to \infty.$$

Gonzalez and Pinto [43] considered the system of impulsive differential equations

$$\begin{cases} x' = f(t, x), & t \neq t_i, \\ \Delta x = \phi_i x, & t = t_i,\ i = 1, 2, \ldots \end{cases}$$

and showed that every solution $x(t)$ with initial condition $x(a) = x_0$, $a \geq t_0$ is defined on $[a, \infty)$ and satisfies

$$\lim_{t \to \infty} x(t) = \xi \tag{1.10}$$

for some $\xi \in \mathbb{R}^n$. Note that, if

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \tag{1.11}$$

where $x_1(t) = x(t)$ and $x_2(t) = x'(t)$, then (1.10) is equivalent to

$$x(t) = \xi_1 + o(1), \quad x'(t) = \xi_2 + o(1), \quad t \to \infty$$

where $\xi_1, \xi_2 \in \mathbb{R}$. There are various analogous results given by Bainov and Simeonov [8] for first order linear and nonlinear impulsive differential systems. For example, for the linear impulsive differential system

$$\begin{cases} x' = A(t)x + f(t), & t \neq t_i, \\ \Delta x = A_i x + f_i, & t = t_i,\ i = 1, 2, \ldots \end{cases} \tag{1.12}$$

they proved that a solution that has the asymptotic representation

$$x(t) = \xi + o(1), \quad t \to \infty$$

exists, where ξ is an arbitrary real constant vector. By considering the perturbed impulsive differential system

$$\begin{cases} y' = [A(t) + B(t)]y + f(t) + g(t), & t \neq t_i, \\ \Delta y = [A_i + B_i]y + f_i + g_i, & t = t_i, \ i = 1, 2, \dots \end{cases}$$

they showed that there exists a solution $y(t)$ with the asymptotic formula

$$y(t) = x(t) + o(1), \quad t \to \infty,$$

where $x(t)$ is a solution of (1.12). Further results can be found in [11, 28, 34, 48, 54] and the references therein.

CHAPTER 2

PRELIMINARIES

In this section we give basic definitions and concepts of impulsive differential equations and recall some theorems from functional analysis.

2.1 Impulsive Differential Equations

Impulsive differential equations generally describe phenomena subject to instantaneous changes in a state. For example releasing pesticides in a farm at specified moments causes a sudden decrease in the number of the pests [53]. The sudden changes that take place in a state are called as pulse phenomena, and the moments are the impulse moments. There are many counterparts of evolution processes which have pulse phenomena, e.g., oscillation of a pendulum under external percussive effects [31]; controlling a satellite orbit by the radial acceleration [56]; a disorder in the system of cellular neural networks [22]; optimization problems of the predator-prey relationships which are under external impulsive effects [36]; changes in prices driven by demand shocks or another interference [50], the drug concentration in the blood after taking medicine [14], etc. These are all modeled by using impulsive differential equations due to the inadequacy of ordinary differential equations. It is seen that the theory of impulsive differential equations is favorable due to fruitfulness and accuracy in applications. Many contributions have been made to the theory especially after late 1980s. The qualitative behavior of impulsive differential equations is widely investigated in books of Samoilenko, Perestyuk [47]; Laksmikantham, Bainov, Simeonov [33] and Bainov, Simeonov [6, 7]. Referring to these books, we give some fundamental concepts.

Impulsive differential equations can be divided into two classes according to the type of impulse moments, namely the equations with fixed moments of impulses and the equations with nonfixed (variable) moments of impulses. In this thesis we will deal with the equations with fixed moments. Let $S \subseteq \mathbb{R}^n$, $x(t) \in S$ be the point representing the state of a process at time t, $\mathbb{I} \subseteq \mathbb{Z}$ denote the set of indices, and $\{\theta_i\}_{i\in\mathbb{I}}$ denote the sequence of moments of impulses such that it is strictly increasing and has no accumulation points, i.e., $\theta_i < \theta_{i+1}$ for all $i \in \mathbb{I}$ and $\lim_{i\to\infty} \theta_i = \infty$. The jump operator Δx at an impulse point θ_i is defined as $\Delta x|_{t=\theta_i} := x(\theta_i+) - x(\theta_i-)$, where $x(\theta_i\pm) = \lim_{h\to 0^+} x(\theta_i \pm h)$. Define $\mathrm{PLC}(I)$ as the space of piecewise left continuous functions on the interval I, where I is an arbitrary subinterval of $\mathbb{R}$ and the discontinuities are of the first kind. $\mathrm{PLC}^1(I)$ is defined similarly. Then the mathematical model of a real process having impulses at fixed moments is given by the following impulsive differential equation:

$$\begin{cases} x' = f(t,x), \ t \neq \theta_i, \\ \Delta x = f_i(x), \ t = \theta_i. \end{cases} \tag{2.1}$$

Here, without loss of generality it is assumed that $x(\theta_i-) = x(\theta_i)$, that is the solutions are left continuous. A solution $x(t)$ of (2.1) is a function that belongs to the space $\mathrm{PLC}^1(I)$ and satisfies

1) $x'(t) = f(t, x(t))$ where $t \neq \theta_i$,

2) for $t = \theta_i$ the jump condition $\Delta x|_{t=\theta_i} = f_i(x)$, i.e., $x(\theta_i+) = x(\theta_i) + f_i(x(\theta_i))$ is fulfilled.

If the mathematical model has variable moments of impulses, i.e., the points of jump depend on the solution then we have the following impulsive differential equation:

$$\begin{cases} x' = f(t,x), \ t \neq \theta_i(x), \\ \Delta x = f_i(x), \ t = \theta_i(x). \end{cases} \tag{2.2}$$

In equation (2.2) the discontinuities depend on the solution, hence each solution has different points of jumps and each element of the sequence $\{\theta_i(x)\}$, $i =$

$1, 2, \ldots$ is called the state dependent moment of discontinuity. Nevertheless the solutions are still piecewise left continuous functions.

Let

$$x(t_0) = x_0, \quad t_0 \neq \theta_i \tag{2.3}$$

be given for some $t_0 \in I$.

Theorem 2.1.1 (Local existence theorem [5]) *Define* $\Pi_i x := f_i(x) + x$ *and suppose* $f(t, x)$ *is continuous on* $I \times S$ *and* $\Pi_i S \subseteq S$*. Then, for any point* $(t_0, x_0) \in I \times G$ *there is* $\alpha > 0$ *such that a solution* $x(t, t_0, x_0)$ *of* (2.1)-(2.3) *exists on the interval* $(t_0 - \alpha, t_0 + \alpha)$*.*

Theorem 2.1.2 (Uniqueness theorem [5]) *Assume that* $f(t, x)$ *satisfies a local Lipschitz condition, and every equation*

$$x = \nu + f_i(x), \quad i \in \mathbb{I},\, x \in S$$

has at most one solution with respect to ν*. Then, any solution* $x(t, t_0, x_0)$*,* $(t_0, x_0) \in I \times S$ *of* (2.1)-(2.3) *is unique. That is, if* $y(t, t_0, x_0)$ *is another solution of* (2.1)-(2.3)*, and the two solutions are defined at some* $t \in I$*; then* $x(t, t_0, x_0) = y(t, t_0, x_0)$*.*

2.2 Fixed Point Theorems

Theorem 2.2.1 (Schauder fixed point theorem [1]) *Let* S *be a bounded; closed and convex subset of a normed linear space* Y*. Then every compact, continuous map* $T : S \to S$ *has at least one fixed point.*

Theorem 2.2.2 (Banach fixed point theorem [1]) *Let* (X, d) *be a complete metric space and* $T : X \to X$ *satisfy*

$$d(T(x), T(y)) \leq L d(x, y) \text{ for all } x, y \in X,$$

where $L \in [0, 1)$*. Then* T *has a unique fixed point* $u \in X$*.*

Lemma 2.2.1 (Kolmogorov-M. Riesz-Frechét [10, 20, 23, 45]) *Let $\Omega \subset \mathbb{R}^n$ and $S \subset L_p(\mathbb{R}^n)$ with $1 \leq p < \infty$. Then S is compact in $L_p(\Omega)$ if, and only if the following properties hold:*

(i) there exists a constant $M > 0$ such that $\int_S |f|^p \mathrm{d}x < M^p$;

(ii) $\lim_{|h| \to 0} \|\tau_h f - f\|_p = 0$ uniformly on S, i.e., $\forall \epsilon > 0 \; \exists \delta > 0$ such that

$$\int_S |f(x+h) - f(x)|^p \mathrm{d}x < \epsilon \; \forall f \in S \text{ with } |h| < \delta,$$

where $(\tau_h f)(x) = f(x+h)$, $x \in \mathbb{R}^n$.

Lemma 2.2.2 (Frechét [21, 23]) *A subset S of l^p, where $1 \leq p < \infty$, is totally bounded if, and only if,*

(i) it is pointwise bounded;

(ii) for every $\epsilon > 0$ there is some $n \in \mathbb{N}$ so that, for every $x \in S$, $\sum\limits_{k>n} |x_k|^p < \epsilon^p$.

Lemma 2.2.3 ([19]) *Suppose that $v(t) > 0$ and $v'(t) \geq 0$ for $t > 0$. Let*

$$S \in \Omega := \{y \in C^1[0,\infty) | \lim_{t\to\infty} y(t) \& \lim_{t\to\infty} z(t) \text{ exist}\}$$

where $z(t) := y(t) + \frac{v(t)}{v'(t)} y'(t)$. Then S is compact in Ω if the following conditions hold:

i) S is uniformly bounded in Ω, i.e., there exists $L > 0$ such that $\|y\|_\Omega \leq L$ and $\|z\|_\Omega \leq L$ for all $y \in S$;

ii) S is equicontinuous, i.e., $\forall \epsilon > 0 \; \exists \delta_\epsilon > 0$ such that

$$|t_1 - t_2| < \delta_\epsilon \Rightarrow |y(t_1) - y(t_2)| < \epsilon \text{ and } |z(t_1) - z(t_2)| < \epsilon;$$

iii) S is equiconvergent, i.e., given $\epsilon > 0$, there is $\tau(\epsilon) > 0$ such that

$$|y(t) - l_y| < \epsilon \text{ and } |z(t) - l_z| < \epsilon$$

for any $t \geq \tau(\epsilon)$ and $y \in S$, where $l_y = \lim\limits_{t\to\infty} y(t)$ and $l_z = \lim\limits_{t\to\infty} z(t)$.

Lemma 2.2.4 ([13, 54]) *Let*

$$S \in V := \{f \in PLC[t_0, \infty) | \lim_{t\to\infty} f(t) \ \textit{exists}\} \tag{2.4}$$

Then S is compact in V if the following properties hold:

(i) S is bounded in V;

(ii) the functions belonging to S are piecewise equicontinuous on any interval of $[t_0, \infty)$;

(iii) the functions belonging to S are equiconvergent, that is, given $\epsilon > 0$, there is $\tau(\epsilon) > 0$ such that $|f(t) - \ell_f| < \epsilon$ for any $t \geq \tau(\epsilon)$ and $f \in S$, where $l_f = \lim_{t\to\infty} f(t)$.

2.3 Principal and Nonprincipal Solutions

The concept of principal and nonprincipal solutions was first introduced in 1936 by Leighton and Morse [37] to study singular quadratic functionals. It has been applied by many authors to several topics in ordinary differential equations, such as Sturmian theory; oscillation problems; factorizations; etc., see [3, 15, 24, 29, 44]. For various characterizations see the monographs [25, 29, 57].

2.3.1 Impulsive Differential Equations with Continuous Solutions

Applications of principal and nonprincipal solutions to impulsive differential equations had not appeared until the year 2010 due to the lack of theorems about existence of such solutions for differential equations under impulse effect. In 2010, Özbekler and Zafer [41] considered the second order impulsive differential equations of the form

$$\begin{cases} (p(t)x')' + q(t)x = 0, & t \neq \theta_i, \\ \Delta p(t)x' + q_i x = 0, & t = \theta_i, \end{cases} \tag{2.5}$$

with the domain

$$\Omega = \{x : [t_0, \infty) \to \mathbb{R} | \ x \in \mathrm{C}[t_0, \infty), x', (p(t)x')' \in \mathrm{PLC}[t_0, \infty)\},$$

and proved that if (2.5) has a positive solution on $[t_0, \infty)$, then principal and nonprincipal solutions exist.

Theorem 2.3.1 ([41]) *Let $p(t) > 0, q(t) \in PLC[t_0, \infty)$. If (2.5) has a positive solution or is non-oscillatory at infinity, then there exist linearly independent solutions u and v such that*

$$\lim_{t\to\infty} \frac{u(t)}{v(t)} = 0; \tag{2.6}$$

$$\int_{t_0}^{\infty} \frac{1}{p(t)u^2(t)} \mathrm{d}t = \infty, \quad \int_{t_0}^{\infty} \frac{1}{p(t)v^2(t)} \mathrm{d}t < \infty; \tag{2.7}$$

$$\frac{v'(t)}{v(t)} > \frac{u'(t)}{u(t)}, \quad t \geq t_0. \tag{2.8}$$

The solution u is said to be principal, and v is a nonprincipal solution.

Remark 2.3.1 *If v is given to be a nonprincipal solution of (2.5) at infinity then*

$$u(t) = v(t) \int_{t}^{\infty} \frac{1}{p(s)v^2(s)} \mathrm{d}s \tag{2.9}$$

is a principal solution at infinity, and if u is given to be a principal solution at infinity then

$$v(t) = u(t) \int_{t_0}^{t} \frac{1}{p(s)u^2(s)} \mathrm{d}s \tag{2.10}$$

is a nonprincipal solution at infinity.

2.3.2 Impulsive Differential Equations with Discontinuous Solutions

For our purpose, we need new results about principal and nonprincipal solutions of differential equations having impulse effect not only on the derivative of the solution but also on the solution in itself. Consider the second order, homogeneous impulsive differential equation

$$\begin{cases} (p(t)x')' + q(t)x = 0, & t \neq \theta_i, \\ \Delta x + p_i x + \tilde{p}_i x' = 0, & t = \theta_i, \\ \Delta p(t)x' + q_i x + \tilde{q}_i x' = 0, & t = \theta_i, \end{cases} \tag{2.11}$$

where $p(t) > 0, q(t) \in \mathrm{PLC}[t_0, \infty)$. We shall divide the impulse effects into two cases, namely, if the homogeneous impulse effects fully depend on the solution or fully depend on the derivative of the solution we shall call them separated impulse conditions, otherwise mixed impulse conditions. Also, for the sake of clarity we define and use the following notations throughout the thesis:

$\underline{i}(t) := \inf\{i|\ \theta_i \geq t\}$, $\bar{i}(t) := \sup\{i|\ \theta_i \leq t\}$, where $\lim_{t\to\infty} \underline{i}(t) = \lim_{t\to\infty} \bar{i}(t) = \infty$.

CASE I.

If $\tilde{p}_i = q_i = 0$, then (2.11) turns into the impulsive differential equation with separated impulse conditions

$$\begin{cases} (p(t)x')' + q(t)x = 0, & t \neq \theta_i, \\ \Delta x + p_i x = 0, & t = \theta_i, \\ \Delta p(t)x' + \tilde{q}_i x' = 0, & t = \theta_i, \end{cases} \tag{2.12}$$

where $x, x', (p(t)x')' \in \mathrm{PLC}[t_0, \infty)$.

Theorem 2.3.2 *Let $\{p_i\}, \{\tilde{q}_i\}$ be real sequences satisfying*

$$\frac{1}{p_i} + \frac{p(\theta_i)}{\tilde{q}_i} = 1. \tag{2.13}$$

If equation (2.12) *has a positive solution then there exist two linearly independent solutions u and v which satisfy the properties* (2.6)-(2.8)*, i.e., u is principal and v is a nonprincipal solution of* (2.12).

Proof. Let $u(t)$ be a solution of (2.12) that is positive on $[t_0, \infty)$. Define $v(t) = u(t) \int_{t_0}^{t} \frac{1}{p(s)u^2(s)} \mathrm{d}s$. Then $v(t)$ solves (2.12) if (2.13) is satisfied. Suppose

$$\int_{t_0}^{\infty} \frac{1}{p(t)u^2(t)} \mathrm{d}t = \infty, \tag{2.14}$$

then

$$\lim_{t\to\infty} \frac{u(t)}{v(t)} = \lim_{t\to\infty} \frac{1}{\displaystyle\int_{t_0}^{t} \frac{1}{p(t)u^2(t)}} \mathrm{d}t = 0$$

and $\left(\dfrac{u}{v}\right)'(t) = \dfrac{W(v,u)}{pv^2}(t) = \dfrac{-1}{p(t)v^2(t)}$. From

$$\Delta\left(\frac{u}{v}\right)(\theta_i) = \frac{(1-p_i)u(\theta_i)}{(1-p_i)v(\theta_i)} - \frac{u(\theta_i)}{v(\theta_i)} = 0$$

we obtain

$$-\int_{t_0}^{\infty}\frac{1}{p(t)v^2(t)}\mathrm{d}t=\int_{t_0}^{\infty}\left(\frac{u}{v}\right)'(t)$$
$$=\lim_{t\to\infty}\frac{u(t)}{v(t)}-\frac{u(t_0)}{v(t_0)}-\sum_{i=\underline{i}(t_0)}^{\infty}\Delta\left(\frac{u}{v}\right)(\theta_i)$$
$$=-\frac{u(t_0)}{v(t_0)}, \tag{2.15}$$

and so $\int_{t_0}^{\infty}\frac{1}{p(t)v^2(t)}\mathrm{d}t<\infty$. Moreover,

$$\frac{v'(t)}{v(t)}=\frac{u'(t)\int_{t_0}^{t}\frac{1}{p(s)u^2(s)}\mathrm{d}s+\frac{1}{p(t)u(t)}}{v(t)}=\frac{\frac{u'(t)}{u(t)}v(t)+\frac{1}{p(t)u(t)}}{v(t)}=\frac{u'(t)}{u(t)}+\frac{1}{p(t)u(t)v(t)}.$$

Since $p(t)u(t)v(t)>0$, we obtain

$$\frac{v'(t)}{v(t)}>\frac{u'(t)}{u(t)}.$$

Suppose $\int_{t_0}^{\infty}\frac{1}{p(t)u^2(t)}\mathrm{d}t<\infty$. In this case, we may define

$$\tilde{u}(t)=u(t)\int_{t}^{\infty}\frac{1}{p(s)u^2(s)}\mathrm{d}s.$$

So

$$\int_{t_0}^{\infty}\frac{1}{p(t)\tilde{u}^2(t)}\mathrm{d}t=\int_{t_0}^{\infty}\frac{1}{p(t)u^2(t)}\left(\int_{t}^{\infty}\frac{1}{p(s)u^2(s)}\mathrm{d}s\right)^{-2}\mathrm{d}t$$
$$=\int_{t_0}^{\infty}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{t}^{\infty}\frac{1}{p(s)u^2(s)}\mathrm{d}s\right)^{-1}\mathrm{d}t$$
$$=\lim_{r\to\infty}\left(\int_{r}^{\infty}\frac{1}{p(s)u^2(s)}\mathrm{d}s\right)^{-1}-\left(\int_{t_0}^{\infty}\frac{1}{p(s)u^2(s)}\mathrm{d}s\right)^{-1}$$
$$-\sum_{i=\underline{i}(t_0)}^{\infty}\Delta\left(\int_{\theta_i}^{\infty}\frac{1}{p(s)u^2(s)}\mathrm{d}s\right)^{-1}$$
$$=\infty.$$

Now we put $\tilde{v}(t)=\tilde{u}(t)\int\limits_{t_0}^{t}\frac{1}{p(s)\tilde{u}^2(s)}\mathrm{d}s$. The remaining part of the proof can be done by setting above $\tilde{u}$ instead of u, and $\tilde{v}$ instead of v. □

Remark 2.3.2 *Assume that $u(t)$ is a solution of* (2.12) *that is positive on* $[t_0, \infty$. *Then, by using the variation of parameters method one obtains that*

$$v(t) = u(t) \int_{t_0}^{t} \frac{\prod_{i=\underline{i}(t_0)}^{\bar{i}(s)} (1-p_i)^2}{p(s)u^2(s)} \mathrm{d}s$$

is also a solution of (2.12) *if and only if* $p_i = \tilde{q}_i$.

CASE II.

This time take $\tilde{p}_i = \tilde{q}_i = 0$ in (2.11) to have the impulsive differential equation with mixed impulse conditions

$$\begin{cases} (p(t)x')' + q(t)x = 0, & t \neq \theta_i, \\ \Delta x + p_i x = 0, & t = \theta_i, \\ \Delta p(t)x' + q_i x = 0, & t = \theta_i, \end{cases} \tag{2.16}$$

where $x, x', (p(t)x')' \in \mathrm{PLC}[t_0, \infty)$.

Theorem 2.3.3 *If equation* (2.16) *has a positive solution then there exist two linearly independent solutions u and v which satisfy the properties* (2.6)-(2.8), *i.e., u is principal and v is a nonprincipal solution of* (2.16).

Proof. Let $u(t)$ be a solution of (2.16) that is positive on $[t_0, \infty)$. Then, we first prove that a second solution is of the form

$$v(t) = u(t) \int_{t_0}^{t} \frac{\prod_{i=\underline{i}(t_0)}^{\bar{i}(s)} (1-p_i)}{p(s)u^2(s)} \mathrm{d}s.$$

For $t \neq \theta_l$,

$$v'(t) = u'(t) \int_{t_0}^{t} \frac{\prod_{i=\underline{i}(t_0)}^{\bar{i}(s)} (1-p_i)}{p(s)u^2(s)} \mathrm{d}s + \frac{\prod_{i=\underline{i}(t_0)}^{\bar{i}(t)} (1-p_i)}{p(t)u(t)}$$

and

$$
\begin{aligned}
(p(t)v'(t))' + q(t)v(t) =& [(p(t)u'(t))' + q(t)u(t)] \int\limits_{t_0}^{t} \frac{\prod\limits_{i=\underline{i}(t_0)}^{\bar{i}(s)} (1-p_i)}{p(s)u^2(s)} \mathrm{d}s \\
&+ u'(t) \frac{\prod\limits_{i=\underline{i}(t_0)}^{\bar{i}(t)} (1-p_i)}{u^2(t)} - u'(t) \frac{\prod\limits_{i=\underline{i}(t_0)}^{\bar{i}(t)} (1-p_i)}{u^2(t)} \\
=& 0. \qquad (2.17)
\end{aligned}
$$

For $t = \theta_l$,

$$
\Delta v + p_l v = (\Delta u + p_l u) \int\limits_{t_0}^{\theta_l} \frac{\prod\limits_{i=\underline{i}(t_0)}^{\bar{i}(s)} (1-p_i)}{p(s)u^2(s)} \mathrm{d}s = 0 \qquad (2.18)
$$

and

$$
\begin{aligned}
\Delta p(t)v' + q_l v =& (\Delta p(t)u' + q_l u) \int\limits_{t_0}^{\theta_l} \frac{\prod\limits_{i=\underline{i}(t_0)}^{\bar{i}(s)} (1-p_l)}{p(s)u^2(s)} \mathrm{d}s + (1-p_l) \frac{\prod\limits_{i=\underline{i}(t_0)}^{l-1} (1-p_i)}{u(\theta_l+)} \\
&- \frac{\prod\limits_{i=\underline{i}(t_0)}^{l-1} (1-p_i)}{u(\theta_l)} \\
=& 0. \qquad (2.19)
\end{aligned}
$$

From (2.17), (2.18) and (2.19) we conclude that $v(t)$ is a solution of (2.16).

Suppose that

$$
\int\limits_{t_0}^{\infty} \frac{1}{p(s)u^2(s)} \mathrm{d}s = \infty. \qquad (2.20)
$$

Then,

$$
\int\limits_{t_0}^{\infty} \frac{\prod\limits_{i=\underline{i}(t_0)}^{\bar{i}(s)} (1-p_i)}{p(s)u^2(s)} \mathrm{d}s = \infty.
$$

Clearly

$$
\lim_{t \to \infty} \frac{u(t)}{v(t)} = 0. \qquad (2.21)
$$

In addition,

$$\frac{v'(t)}{v(t)} = \frac{u'(t)}{u(t)} + \frac{\prod\limits_{i=\underline{i}(t_0)}^{\bar{i}(t)} (1-p_i)}{p(t)u^2(t) \int\limits_{t_0}^{t} \frac{\prod_{i=\underline{i}(t_0)}^{\bar{i}(s)} (1-p_i)}{p(s)u^2(s)} \mathrm{d}s}.$$

Since the latter term is positive it follows that

$$\frac{v'(t)}{v(t)} \geq \frac{u'(t)}{u(t)}. \tag{2.22}$$

On the other hand,

$$\left(\frac{u}{v}\right)'(t) = \frac{-\prod\limits_{i=\underline{i}(t_0)}^{\bar{i}(t)} (1-p_i)}{p(t)v^2(t)}.$$

Proceeding as in (2.15) one has

$$-\int\limits_{t_0}^{\infty} \frac{\prod\limits_{i=\underline{i}(t_0)}^{\bar{i}(t)} (1-p_i)}{p(t)v^2(t)} \mathrm{d}t = -\frac{u(t_0)}{v(t_0)},$$

which implies that

$$\int\limits_{t_0}^{\infty} \frac{1}{p(t)v^2(t)} \mathrm{d}t < \infty.$$

Hence, all the properties (2.6)-(2.8) are satisfied. Finally, if (2.20) does not hold we define

$$\tilde{u}(t) := u(t) \int\limits_{t}^{\infty} \frac{\prod\limits_{i=\underline{i}(s)}^{\infty} (1-p_i)^{-1}}{p(s)u^2(s)} \mathrm{d}s.$$

Then

$$\begin{aligned}
\int\limits_{t_0}^{\infty} \frac{\prod\limits_{i=\underline{i}(t)}^{\infty} (1-p_i)^{-1}}{p(t)\tilde{u}^2(t)} \mathrm{d}t &= \int\limits_{t_0}^{\infty} \frac{\prod\limits_{i=\underline{i}(t)}^{\infty} (1-p_i)^{-1}}{p(t)u^2(t)} \left(\int\limits_{t}^{\infty} \frac{\prod\limits_{i=\underline{i}(s)}^{\infty} (1-p_i)^{-1}}{p(s)u^2(s)} \mathrm{d}s \right)^{-2} \mathrm{d}s \\
&= \int\limits_{t_0}^{\infty} \frac{\mathrm{d}}{\mathrm{d}t} \left(\int\limits_{t}^{\infty} \frac{\prod\limits_{i=\underline{i}(s)}^{\infty} (1-p_i)^{-1}}{p(s)u^2(s)} \mathrm{d}s \right)^{-1} \mathrm{d}t \\
&= \infty.
\end{aligned}$$

Thus, by defining

$$\tilde{v}(t) := \tilde{u}(t) \int\limits_{t_0}^{t} \frac{\prod\limits_{i=\underline{i}(t_0)}^{\bar{i}(s)} (1 - p_i)}{p(s)\tilde{u}^2(s)} \mathrm{d}s,$$

and following the steps done for u and v, we complete the proof. □

Corollary 2.3.1 *If we take* $p_i = 0$ *in* (2.16) *it turns into* (2.5). *Hence the last theorem generalizes Theorem 2.3.1.*

CHAPTER 3

ASYMPTOTIC INTEGRATION VIA PRINCIPAL AND NONPRINCIPAL SOLUTIONS

Long time behavior of the solutions of differential equations has an important place in applications to real life problems. In many physical or chemical problems, engineering applications and biological phenomenon the properties of the solutions as $t \to \infty$ are investigated [51]. The present chapter is devoted to asymptotic representation of solutions of impulsive differential equations as $t \to \infty$.

Consider the second order impulsive differential equation

$$\begin{cases} (p(t)x')' + q(t)x = f(t,x), & t \neq \theta_i, \\ \Delta x + p_i x + \tilde{p}_i x' = f_i(x), & t = \theta_i, \\ \Delta p(t)x' + q_i x + \tilde{q}_i x' = \tilde{f}_i(x), & t = \theta_i, \end{cases} \tag{3.1}$$

which has impulse effects both on the solution and on its derivative. In the next sections we investigate sufficient conditions for existence of solutions of impulsive differential equations, which are special cases of equation (3.1), to satisfy prescribed asymptotic behavior.

3.1 Impulsive Differential Equations With Discontinuous Solutions

In this section we give various asymptotic formulas for the solutions of impulsive differential equations with separated and mixed type impulse conditions by means of principal and nonprincipal solutions. We are first interested in differential equations having impulse effect not only on the derivative of the solution but also on the solution itself.

3.1.1 Asymptotic Representation of Solutions

CASE I.

By taking $p(t) = 1$ and $\tilde{p}_i = q_i = 0$ in equation (3.1) we obtain the impulsive differential equation with separated impulse conditions

$$\begin{cases} x'' + q(t)x = f(t,x), & t \neq \theta_i, \\ \Delta x + p_i x = f_i(x), & t = \theta_i, \\ \Delta x' + \tilde{q}_i x' = \tilde{f}_i(x), & t = \theta_i. \end{cases} \tag{3.2}$$

We shall give asymptotic representations depending on the principal and non-principal solutions of the corresponding homogeneous equation

$$\begin{cases} x'' + q(t)x = 0, & t \neq \theta_i, \\ \Delta x + p_i x = 0, & t = \theta_i, \\ \Delta x' + \tilde{q}_i x' = 0, & t = \theta_i. \end{cases} \tag{3.3}$$

Suppose that (2.13) hold, i.e., $1/p_i + 1/\tilde{q}_i = 1$.Let u and v be principal and nonprincipal solution of (3.3), and suppose without any loss of generality that they are positive on $[T, \infty)$.

Theorem 3.1.1 *Assume that* $T \geq 1$, $f \in C([1,\infty)\times\mathbb{R},\mathbb{R})$, $f_i, \tilde{f}_i \in C([1,\infty),\mathbb{R})$, *and there exist functions* $g_j \in C(\mathbb{R}_+,\mathbb{R}_+)$, $j = 1,2,3$, $h_k \in C([1,\infty),\mathbb{R}_+)$, $k = 1,2$ *and real sequences* $\{\bar{h}_i\}, \{\tilde{h}_i\}$ *such that*

$$|f(t,x)| \leq h_1(t) g_1\left(\frac{|x|}{v(t)}\right) + h_2(t), \quad t \geq T, \tag{3.4}$$

$$|f_i(x)| \leq \bar{h}_i g_2\left(\frac{|x|}{v(\theta_i)}\right), \quad \theta_i \geq T, \tag{3.5}$$

$$|\tilde{f}_i(x)| \leq \tilde{h}_i g_3\left(\frac{|x|}{v(\theta_i)}\right), \quad \theta_i \geq T, \tag{3.6}$$

where

$$\int_T^\infty v(s)(h_1(s) + h_2(s))\mathrm{d}s + \sum_{i=\underline{i}(T)}^{\infty} |v'(\theta_i+)|\bar{h}_i + v(\theta_i+)\tilde{h}_i < \infty. \tag{3.7}$$

Then, for any given $a, b \in \mathbb{R}$, equation (3.2) *has a solution $x(t)$ such that*

$$x(t) = av(t) + bu(t) + o(u(t)), \quad t \to \infty. \tag{3.8}$$

Proof. Define the operator

$$\begin{aligned}(Fy)(t) =& bu(t) + u(t)\Bigg\{ \int\limits_t^\infty f(s, y(s) + av(s))u(s) \int\limits_t^s \frac{1}{u^2(r)}\mathrm{d}r\mathrm{d}s \\ &+ \sum_{i=\underline{i}(t)}^\infty \Bigg((1-p_i)\tilde{f}_i(y(\theta_i) + av(\theta_i))u(\theta_i) \int\limits_t^{\theta_i} \frac{1}{u^2(s)}\mathrm{d}s \\ &- (1-\tilde{q}_i)f_i(y(\theta_i) + av(\theta_i))\Bigg(u'(\theta_i) \int\limits_t^{\theta_i} \frac{1}{u^2(s)}\mathrm{d}s + \frac{1}{u(\theta_i)} \Bigg)\Bigg)\Bigg\}\end{aligned} \tag{3.9}$$

on the Banach space [49]

$$Y = \left\{ y \in \mathrm{PLC}([T, \infty), \mathbb{R}) | \, \frac{|y(t)|}{v(t)} \le M_y \right\}. \tag{3.10}$$

Let

$$S = \{ y \in Y | \, \|y(t) - bu(t)\| \le 1 \}.$$

It is easy to show $S \subset Y$, and it is a closed, bounded and convex set. Our aim is to prove that the operator F has a fixed point. We will do the proof step by step by using the Schauder fixed point theorem.

Step 1.

Each fixed point y of the operator F is a solution of

$$\begin{cases} y'' + q(t)y = f(t, y(t) + av(t)), & t \ne \theta_i, \\ \Delta y + p_i y = f_i(y + av(\theta_i)), & t = \theta_i, \\ \Delta y' + \tilde{q}_i y' = \tilde{f}_i(y + av(\theta_i)), & t = \theta_i, \end{cases} \tag{3.11}$$

for $t \ge T$.

Proof. Denote

$$\Psi(t) := \int_t^\infty f(s, y(s)+av(s))u(s)\int_t^s \frac{1}{u^2(r)}\mathrm{d}r\mathrm{d}s$$
$$+ \sum_{i=\underline{i}(t)}^{\infty}\Bigg((1-p_i)\tilde{f}_i(y(\theta_i)+av(\theta_i))u(\theta_i)\int_t^{\theta_i}\frac{1}{u^2(s)}\mathrm{d}s$$
$$-(1-\tilde{q}_i)f_i(y(\theta_i)+av(\theta_i))\left(u'(\theta_i)\int_t^{\theta_i}\frac{1}{u^2(s)}\mathrm{d}s+\frac{1}{u(\theta_i)}\right)\Bigg).$$

Then

$$\Psi'(t) = -f(t,y(t)+av(t))u(t)\int_t^t\frac{1}{u^2(r)}\mathrm{d}r$$
$$-\Bigg\{\int_t^\infty f(s,y(s)+av(s))u(s)\left(\frac{1}{u^2(t)}\right)\mathrm{d}s$$
$$+\sum_{i=\underline{i}(t)}^{\infty}\Bigg((1-p_i)\tilde{f}_i(y(\theta_i)+av(\theta_i))u(\theta_i)\frac{1}{u^2(t)}$$
$$-(1-\tilde{q}_i)f_i(y(\theta_i)+av(\theta_i))u'(\theta_i)\frac{1}{u^2(t)}\Bigg)\Bigg\}$$
$$= -\frac{1}{u^2(t)}\Bigg\{\int_t^\infty f(s,y(s)+av(s))u(s)\mathrm{d}s$$
$$+\sum_{i=\underline{i}(t)}^{\infty}(1-p_i)\tilde{f}_i(y(\theta_i)+av(\theta_i))u(\theta_i)-(1-\tilde{q}_i)f_i(y(\theta_i)+av(\theta_i))u'(\theta_i)\Bigg\}.$$

So, we may write that

$$y(t) := u(t)(b+\Psi(t)). \tag{3.12}$$

Let $t \neq \theta_l$. Then, it implies that

$$y'(t) = u'(t)(b+\Psi(t)) + u(t)\Psi'(t)$$
$$= u'(t)(b+\Psi(t)) - \frac{1}{u(t)}\Bigg\{\int_t^\infty f(s,y(s)+av(s))u(s)\mathrm{d}s$$
$$+\sum_{i=\underline{i}(t)}^{\infty}(1-p_i)\tilde{f}_i(y(\theta_i)+av(\theta_i))u(\theta_i)-(1-\tilde{q}_i)f_i(y(\theta_i)+av(\theta_i))u'(\theta_i)\Bigg\},$$

and so

$$\begin{aligned} y''(t)+q(t)y(t) =&[u''(t)+q(t)u(t)](b+\Psi(t))+f(t,y(t)+av(t))\\ =&f(t,y(t)+av(t)). \end{aligned}$$

Let us check the jump points $t=\theta_l$. First of all,

$$\begin{aligned} \Psi(\theta_l+) :=& \int\limits_{\theta_l}^{\infty} f(s,y(s)+av(s))u(s)\int\limits_{\theta_l}^{s}\frac{1}{u^2(r)}\mathrm{d}r\mathrm{d}s\\ &+\sum_{i=l+1}^{\infty}\Bigg((1-p_i)\tilde{f}_i(y(\theta_i)+av(\theta_i))u(\theta_i)\int\limits_{\theta_l}^{\theta_i}\frac{1}{u^2(s)}\mathrm{d}s\\ &-(1-\tilde{q}_i)f_i(y(\theta_i)+av(\theta_i))\Bigg(u'(\theta_i)\int\limits_{\theta_l}^{\theta_i}\frac{1}{u^2(s)}\mathrm{d}s+\frac{1}{u(\theta_i)}\Bigg)\Bigg)\\ =&\Psi(\theta_l)+(1-\tilde{q}_l)f_l(y(\theta_l)+av(\theta_l))\frac{1}{u(\theta_l)}. \end{aligned}$$

From (2.13) it follows that

$$\begin{aligned} \Delta y|_{t=\theta_l} =&u(\theta_l+)(b+\Psi(\theta_l+))-u(\theta_l)(b+\Psi(\theta_l))\\ =&(1-p_l)u(\theta_l)\left(b+\Psi(\theta_l)+(1-\tilde{q}_l)f_l(y(\theta_l)+av(\theta_l))\frac{1}{u(\theta_l)}\right)\\ &-u(\theta_l)(b+\Psi(\theta_l))\\ =&-p_lu(\theta_l)(b+\Psi(\theta_l))+(1-p_l)(1-\tilde{q}_l)f_l(y(\theta_l)+av(\theta_l)). \end{aligned}$$

Hence

$$\Delta y|_{t=\theta_l}+p_ly((\theta_l))=f_l(y(\theta_l)+av(\theta_l)).$$

Finally,

$$\begin{aligned} \Delta y'|_{t=\theta_l} =& u'(\theta_l+)(b+\Psi(\theta_l+))-u'(\theta_l)(b+\Psi(\theta_l))\\ &-\left(\frac{1}{u(\theta_l+)}-\frac{1}{u(\theta_l)}\right)\Bigg\{\int\limits_{\theta_l}^{\infty} f(s,y(s)+av(s))u(s)\mathrm{d}s\\ &+\sum_{i=l}^{\infty}\Bigg((1-p_i)\tilde{f}_i(y(\theta_i)+av(\theta_i))u(\theta_i)-(1-\tilde{q}_i)f_i(y(\theta_i)+av(\theta_i))u'(\theta_i)\Bigg)\Bigg\}\\ &+\frac{1}{u(\theta_l+)}\Bigg((1-p_l)\tilde{f}_l(y(\theta_l)+av(\theta_l))u(\theta_l)\\ &-(1-\tilde{q}_l)f_l(y(\theta_l)+av(\theta_l))u'(\theta_l)\Bigg). \end{aligned}$$

Then, by using (2.13) we obtain

$$\begin{aligned}\Delta y'|_{t=\theta_l} =&(1-\tilde{q}_l)u'(\theta_l)\Bigg(b+\Psi(\theta_l)+(1-\tilde{q}_l)f_l(y(\theta_l)+av(\theta_l))\frac{1}{u(\theta_l)}\Bigg)\\ &-\tilde{q}_l\left(\frac{1}{u(\theta_l)}\right)\Bigg\{\int_{\theta_l}^{\infty}f(s,y(s)+av(s))u(s)\mathrm{d}s\\ &+\sum_{i=l}^{\infty}\Bigg((1-p_i)\tilde{f}_i(y(\theta_i)+av(\theta_i))u(\theta_i)\\ &-(1-\tilde{q}_i)f_i(y(\theta_i)+av(\theta_i))u'(\theta_i)\Bigg\}\\ &+\tilde{f}_l(y(\theta_l)+av(\theta_l))-\frac{1}{u(\theta_l)}(1-\tilde{q}_l)^2f_l(y(\theta_l)+av(\theta_l))u'(\theta_l)\Bigg),\end{aligned}$$

which leads to

$$\Delta y'|_{t=\theta_l}+\tilde{q}_l y'((\theta_l))=\tilde{f}_l(y(\theta_l)+av(\theta_l)).$$

Step 2. $F:S\to S$ is completely continuous.

Proof. **(i)** $F(S)\subset S$:

Let $y\in S$, then

$$\left|\frac{y(t)+av(t)}{v(t)}\right|\le|a|+|b|+1.$$

Since $g_j\in\mathrm{C}(\mathbb{R}_+,\mathbb{R}_+)$ there exist constants $K_j>0$, $j=1,2,3$ such that $\max_{0\le\tau\le M_0}g_j(\tau)=K_j$, where $M_0:=|a|+|b|+1$. This implies that

$$|f(s,y(s)+av(s))|\le K_1h_1(s)+h_2(s), \tag{3.13}$$

$$|f_i(y(\theta_i)+av(\theta_i))|\le K_2\bar{h}_i \tag{3.14}$$

$$|\tilde{f}_i(y(\theta_i)+av(\theta_i))|\le K_3\tilde{h}_i. \tag{3.15}$$

Choose T sufficiently large that

$$\int_T^{\infty}v(s)h_1(s)\mathrm{d}s+\sum_{i=\underline{i}(T)}^{\infty}(|v'(\theta_i+)|\bar{h}_i+v(\theta_i+)\tilde{h}_i)\le\frac{1}{2K}, \tag{3.16}$$

$$\int_T^{\infty}v(s)h_2(s)\mathrm{d}s\le\frac{1}{2}, \tag{3.17}$$

where $K:=\max\{K_1,K_2,K_3\}$.

For the sake of brevity, let us define $g(s) := f(s, y(s)+av(s))$, $\bar{g}(\theta_i) := f_i(y(\theta_i)+av(\theta_i))$ and $\tilde{g}(\theta_i) := \tilde{f}_i(y(\theta_i)+av(\theta_i))$. Then, (3.16) and (3.17) imply that

$$\begin{aligned}
\frac{|(Fy)(t)-bu(t)|}{v(t)} &\leq \frac{|(Fy)(t)-bu(t)|}{u(t)} \\
&\leq \int_t^{\infty} |g(s)|v(s)\mathrm{d}s + \sum_{i=\underline{i}(t)}^{\infty} (|\bar{g}(\theta_i)|v'(\theta_i+)||\tilde{g}(\theta_i)|v(\theta_i+)) \\
&\leq \int_T^{\infty} (K_1h_1(s)+h_2(s))v(s)\mathrm{d}s + \sum_{i=\underline{i}(T)}^{\infty} (K_2\bar{h}_i|v'(\theta_i+)| + K_3\tilde{h}_i v(\theta_i+)) \\
&\leq K\left\{\int_T^{\infty} h_1(s)v(s)\mathrm{d}s + \sum_{i=\underline{i}(T)}^{\infty} (\bar{h}_i|v'(\theta_i+)| + \tilde{h}_i v(\theta_i+))\right\} + \int_T^{\infty} h_2(s)v(s)\mathrm{d}s \\
&\leq 1.
\end{aligned}$$

By taking supremum over $[T,\infty)$ we obtain $\|Fy - bu\| \leq 1$.

Fix $t_1 \in [1,\infty)$ with $t < t_1$, then

$$\begin{aligned}
(Fy)(t)-(Fy)(t_1) = [u(t)-u(t_1)]\Bigg\{ & b + \int_{t_1}^{\infty} g(s)u(s)\int_{t_1}^{s}\frac{1}{u^2(r)}\mathrm{d}r\mathrm{d}s \\
& + \sum_{i=\underline{i}(t_1)}^{\infty}\Bigg((1-p_i)\tilde{g}(\theta_i)u(\theta_i)\int_{t_1}^{\theta_i}\frac{1}{u^2(s)}\mathrm{d}s \\
& \qquad - (1-\tilde{q}_i)\bar{g}(\theta_i)\Bigg(u'(\theta_i)\int_{t_1}^{\theta_i}\frac{1}{u^2(s)}\mathrm{d}s + \frac{1}{u(\theta_i)}\Bigg)\Bigg)\Bigg\} \\
& + u(t)\Bigg\{\int_t^{t_1} g(s)u(s)\int_t^{s}\frac{1}{u^2(r)}\mathrm{d}r\mathrm{d}s \\
& + \sum_{i=\underline{i}(t)}^{\bar{i}(t_1)}\Bigg((1-p_i)\tilde{g}(\theta_i)u(\theta_i)\int_t^{\theta_i}\frac{1}{u^2(s)}\mathrm{d}s \\
& \qquad - (1-\tilde{q}_i)\Bigg(\bar{g}(\theta_i)u'(\theta_i)\int_t^{\theta_i}\frac{1}{u^2(s)}\mathrm{d}s + \frac{1}{u(\theta_i)}\Bigg)\Bigg) \\
& \quad + \int_t^{t_1}\frac{1}{u^2(s)}\mathrm{d}s\Bigg(\int_{t_1}^{\infty} g(s)u(s)\mathrm{d}s \\
& \qquad + \sum_{i=\underline{i}(t_1)}^{\infty}\Bigg((1-p_i)\tilde{g}(\theta_i)u(\theta_i) - (1-\tilde{q}_i)\bar{g}(\theta_i)u'(\theta_i)\Bigg)\Bigg)\Bigg\}.
\end{aligned}$$

From (3.13)-(3.15) it follows that

$$
\begin{aligned}
&|(Fy)(t)-(Fy)(t_1)| \le |u(t)-u(t_1)| \\
&\quad\times\left(|b| + \int_{t_1}^{\infty}(K_1h_1(s)+h_2(s))v(s)\mathrm{d}s + \sum_{i=\underline{i}(t_1)}^{\infty}(K_2\bar{h}_i|v'(\theta_i+)| + K_3\tilde{h}_i v(\theta_i+))\right) \\
&\quad + u(t)\left\{\int_{t}^{t_1}(K_1h_1(s)+h_2(s))v(s)\mathrm{d}s + \sum_{i=\underline{i}(t)}^{\bar{i}(t_1)}(K_2\bar{h}_i|v'(\theta_i+)| + K_3\tilde{h}_i v(\theta_i))\right. \\
&\quad + \int_{t}^{t_1}\frac{1}{u^2(s)}\mathrm{d}s\left(\int_{t_1}^{\infty}(K_1h_1(s)+h_2(s))v(s)\mathrm{d}s\right. \\
&\quad \left.\left. + \sum_{i=\underline{i}(t_1)}^{\infty}(K_2\bar{h}_i|v'(\theta_i+)| + K_3\tilde{h}_i v(\theta_i))\right)\right\}.
\end{aligned}
$$

So, by the condition (3.7) it is seen that Lebesgue dominated convergence theorem and Weierstrass-M test are applicable, hence we obtain

$$\lim_{t\to t_1-}(Fy)(t) = (Fy)(t_1).$$

Similarly one can show that $\lim_{t\to t_1+}(Fy)(t) = (Fy)(t_1)$ for $t_1 \neq \theta_k$ and $\lim_{t\to\theta_k+}(Fy)(t)$ exist for all $k = 1, 2, \ldots$ Therefore $F(S) \subset S$.

(ii) *F is continuous:*

Pick $\{y_n\} \in S$ such that $\lim_{n\to\infty} y_n = y \in S$. We need to show that

$$\|Fy_n - Fy\| \to 0, \quad n \to \infty. \tag{3.18}$$

For simplicity denote $g_n(s) = f(s, y_n(s) + av(s))$, $\bar{g}_n(\theta_i) = f_i(y_n(\theta_i) + av(\theta_i))$ and $\tilde{g}_n(\theta_i) = \tilde{f}_i(y_n(\theta_i) + av(\theta_i))$. From (3.13)-(3.15) we have

$$
\begin{aligned}
\frac{|(Fy_n)(t)-(Fy)(t)|}{v(t)} \leq & \frac{|(Fy_n)(t)-(Fy)(t)|}{u(t)} \\
\leq & \int_t^\infty |g_n(s)-g(s)|u(s)\int_t^s \frac{1}{u^2(r)}\mathrm{d}r\mathrm{d}s \\
& + \sum_{i=\underline{i}(t)}^\infty \Bigg(|(1-p_i)(\tilde{g}_n(\theta_i)-\tilde{g}(\theta_i))|u(\theta_i)\int_t^{\theta_i}\frac{1}{u^2(s)}\mathrm{d}s \\
& + |(1-\tilde{q}_i)(\bar{g}_n(\theta_i)-\bar{g}(\theta_i))| \left| u'(\theta_i)\int_{t_1}^{\theta_i}\frac{1}{u^2(s)}\mathrm{d}s+\frac{1}{u(\theta_i)}\right| \Bigg) \\
\leq & \int_t^\infty |g_n(s)-g(s)|v(s)\mathrm{d}s \\
& + \sum_{i=\underline{i}(t)}^\infty (|\tilde{g}_n(\theta_i)-\tilde{g}(\theta_i)|v(\theta_i+)+|\bar{g}_n(\theta_i)-\bar{g}(\theta_i)||v'(\theta_i+)|) \\
\leq & \int_T^\infty 2(K_1h_1(s)+h_2(s))v(s)\mathrm{d}s \\
& + \sum_{i=\underline{i}(T)}^\infty 2(K_2\bar{h}_i|v'(\theta_i+)|+K_3\tilde{h}_i v(\theta_i+)),
\end{aligned}
$$

which, in view of (3.7), is finite. By means of the Lebesgue dominated convergence theorem and Weierstrass-M test, (3.18) holds, so F is a continuous operator.

(iii) *F is relatively compact:*

Take an arbitrary sequence $\{y_n\} \in S$. We want to prove that there exists a subsequence $\{y_{n_k}\} \in S$ so that Fy_{n_k} is convergent. Let $F = F_1 + F_2$, where

$$
(F_1y_n)(t) = bu(t) + u(t)\int_t^\infty g_n(s)u(s)\int_t^s \frac{1}{u^2(r)}\mathrm{d}r\mathrm{d}s, \quad t \geq T
$$

and

$$
\begin{aligned}
(F_2y_n)(t) = u(t)\sum_{i=\underline{i}(t)}^\infty \Bigg(& (1-p_i)\tilde{g}_n(\theta_i)u(\theta_i)\int_t^{\theta_i}\frac{1}{u^2(s)}\mathrm{d}s \\
& - (1-\tilde{q}_i)\bar{g}_n(\theta_i)\left(u'(\theta_i)\int_t^{\theta_i}\frac{1}{u^2(s)}\mathrm{d}s+\frac{1}{u(\theta_i)}\right)\Bigg), \quad t \geq T.
\end{aligned}
$$

Define

$$f_n(s) := g_n(s)u(s)\int\limits_t^s \frac{1}{u^2(r)}\mathrm{d}r$$

$$\tilde{f}_n(\theta_i) := (1-p_i)\tilde{g}_n(\theta_i)u(\theta_i)\int\limits_t^{\theta_i} \frac{1}{u^2(s)}\mathrm{d}s$$

$$\bar{f}_n(\theta_i) := (1-\tilde{q}_i)\bar{g}_n(\theta_i)\left(u'(\theta_i)\int\limits_t^{\theta_i} \frac{1}{u^2(s)}\mathrm{d}s + \frac{1}{u(\theta_i)}\right).$$

From (3.13) and (3.17) there exists a constant $c_1 > 0$ such that

$$\|f_n\|_{L^1([T,\infty))} \le c_1, \quad n \ge 1.$$

Thus, the first hypothesis of Lemma 2.2.1 holds. Define

$$(\tau_h f)(s) = f(s+h).$$

We next show that

$$\|\tau_h f_n - f_n\|_{L^1([T,\infty))} \to 0, \quad h \to 0. \tag{3.19}$$

From (3.13) we have

$$\begin{aligned}\int\limits_T^\infty |(\tau_h f_n)(s) - f_n(s)|\mathrm{d}s &\le \int\limits_T^\infty |f_n(s+h)|\mathrm{d}s + \int\limits_T^\infty |f_n(s)|\mathrm{d}s \\ &= \int\limits_{T+h}^\infty |f_n(s)|\mathrm{d}s + \int\limits_T^\infty |f_n(s)|\mathrm{d}s \\ &\le \int\limits_T^\infty 2|f_n(s)|\mathrm{d}s \\ &\le \int\limits_T^\infty 2(K_1h_1(s) + h_2(s))v(s)\mathrm{d}s.\end{aligned}$$

By virtue of Lebesgue dominated convergence theorem, we deduce that (3.19) holds. So, by Lemma 2.2.1 there exists a subsequence $\{f_{n_k}\}$ which is convergent in $L^1([T,\infty))$. However, from continuity, its limit is

$$g(s)u(s)\int\limits_t^s \frac{1}{u^2(r)}\mathrm{d}r =: z(s),$$

i.e.,

$$\int_T^\infty |z(s)|\mathrm{d}s = \lim_{n\to\infty} \int_T^\infty |f_{n_k}(s)|\mathrm{d}s.$$

Define

$$w(t) := bu(t) + u(t) \int_t^\infty z(s)\mathrm{d}s,$$

then we see that

$$\frac{|(F_1 y_{n_k})(t) - w(t)|}{v(t)} \le \int_T^\infty |f_{n_k}(s) - z(s)|\mathrm{d}s.$$

The condition (3.7) implies that Lebesgue dominated convergence theorem is applicable, hence

$$\lim_{k\to\infty} \|F_1 y_{n_k} - F_1 y\| = 0.$$

We will apply Lemma 2.2.2 to show that F_2 is a compact operator. From (3.14) and (3.15) it can be seen that

$$|\bar{f}_n(\theta_i)| \le |\bar{g}_n||v'(\theta_i+)| \le K_2 \bar{h}_i$$

and

$$|\tilde{f}_n(\theta_i)| \le |\tilde{g}_n| v(\theta_i+) \le K_3 \tilde{h}_i.$$

In view of (3.16), each element of the sets $\{\bar{f}_n\}$, $\{\tilde{f}_n\}$ is pointwise bounded. Let $\epsilon > 0$ be given and choose $j \in \mathbb{N}$ sufficiently large so that

$$\sum_{i=j}^\infty \bar{h}_i |v'(\theta_i+)| < \frac{\epsilon}{K_2}$$

and

$$\sum_{i=j}^\infty \tilde{h}_i v(\theta_i+) < \frac{\epsilon}{K_3}.$$

Then we have

$$\sum_{i=j}^\infty |\tilde{f}_n(\theta_i)| < \epsilon, \qquad \sum_{i=j}^\infty |\bar{f}_n(\theta_i)| < \epsilon,$$

and so all hypotheses of Lemma 2.2.2 hold. Thus the sets $\{\bar{f}_n\}$, $\{\tilde{f}_n\}$ are compact in $\ell^1([T,\infty))$, i.e., there exists subsequences $\{\bar{f}_{n_k}\}$, $\{\tilde{f}_{n_k}\}$ which are convergent in $\ell^1([T,\infty))$, say

$$\lim_{k\to\infty} \sum_{i=\underline{i}(T)}^\infty |\bar{f}_{n_k} - \bar{z}_i| = 0, \qquad \lim_{k\to\infty} \sum_{i=\underline{i}(T)}^\infty |\tilde{f}_{n_k} - \tilde{z}_i| = 0.$$

Finally, let

$$\tilde{w}(t) := u(t) \sum_{i=\underline{i}(t)}^{\infty} (\bar{z}_i + \tilde{z}_i),$$

then by using (3.15) and the Minkowski's inequality we obtain

$$\frac{|(F_2 y_{n_k})(t) - \tilde{w}(t)|}{v(t)} \leq \sum_{i=\underline{i}(T)}^{\infty} (|\tilde{f}_{n_k}(\theta_i) - \tilde{z}_i| + |\bar{f}_{n_k} - \bar{z}_i|).$$

We take supremum over $[T, \infty)$, then with the help of Weierstrass-M test we conclude that $(F_2 y_{n_k}) \to \tilde{w}$, i.e., F_2 has a convergent subsequence in S.

Since linear combination of two compact operators is compact, all hypotheses of Schauder fixed point theorem are satisfied. This proves that the operator (3.9) has a fixed point $y \in X$, which is a solution to equation (3.11). By means of the conditions (3.4)-(3.7), we conclude that

$$\lim_{t\to\infty} \frac{x(t) - av(t) - bu(t)}{u(t)} = 0,$$

and so the asymptotic representation (3.8) is valid. □

Theorem 3.1.2 *Assume that* (3.4), (3.5) *and* (3.6) *hold. If*

$$\int_T^{\infty} u(s)(h_1(s) + h_2(s))\mathrm{d}s + \sum_{i=\underline{i}(T)}^{\infty} |u'(\theta_i+)|\bar{h}_i + u(\theta_i+)\tilde{h}_i < \infty \tag{3.20}$$

and

$$\limsup_{t\to\infty} \frac{u(t)}{v(t)} \left\{ \int_T^t v(s)(h_1(s) + h_2(s))\mathrm{d}s + \sum_{i=\underline{i}(T)}^{\bar{i}(t)} |v'(\theta_i+)|\bar{h}_i + v(\theta_i+)\tilde{h}_i \right\} = 0, \tag{3.21}$$

then, for any given $a, b \in \mathbb{R}$, *equation* (3.2) *has a solution* $x(t)$ *such that*

$$x(t) = av(t) + bu(t) + o(v(t)), \quad t \to \infty. \tag{3.22}$$

Proof. Let

$$S = \{y \in Y| \, \|y(t) - av(t)\| \leq 1\},$$

where Y is given by (3.10). Consider the operator $F : S \to Y$ defined by

$$(Fy)(t) = av(t) - u(t)\Bigg\{\int_T^t f(s, y(s) + bu(s))v(s)\mathrm{d}s$$

$$+ \sum_{i=\underline{i}(T)}^{\bar{i}(t)} \Bigg((1-p_i)\tilde{f}_i(y(\theta_i) + bu(\theta_i))v(\theta_i) - (1-\tilde{q}_i)f_i(y(\theta_i) + bu(\theta_i))v'(\theta_i) \Bigg)\Bigg\}$$

$$- v(t)\Bigg\{\int_t^\infty f(s, y(s) + bu(s))u(s)\mathrm{d}s$$

$$+ \sum_{i=\underline{i}(t)}^{\infty} \Bigg((1-p_i)\tilde{f}_i(y(\theta_i) + bu(\theta_i))u(\theta_i) - (1-\tilde{q}_i)f_i(y(\theta_i) + bu(\theta_i))u'(\theta_i) \Bigg)\Bigg\}.$$

Let $y(t)$ be a fixed point of F. We need to show that $y(t)$ is a solution of

$$\begin{cases} y'' + q(t)y = f(t, y(t) + bu(t)), & t \neq \theta_i, \\ \Delta y + p_i y = f_i(y + bu(\theta_i)), & t = \theta_i, \\ \Delta y' + \tilde{q}_i y' = \tilde{f}_i(y + bu(\theta_i)), & t = \theta_i, \end{cases} \tag{3.23}$$

for $t \geq T$. Denote

$$\varphi(t) := \int_T^t f(s, y(s) + bu(s))v(s)\mathrm{d}s$$

$$+ \sum_{i=\underline{i}(T)}^{\bar{i}(t)} \Bigg((1-p_i)\tilde{f}_i(y(\theta_i) + bu(\theta_i))v(\theta_i) - (1-\tilde{q}_i)f_i(y(\theta_i) + bu(\theta_i))v'(\theta_i) \Bigg)$$

and

$$\psi(t) := \int_t^\infty f(s, y(s) + bu(s))u(s)\mathrm{d}s$$

$$+ \sum_{i=\underline{i}(t)}^{\infty} \Bigg((1-p_i)\tilde{f}_i(y(\theta_i) + bu(\theta_i))u(\theta_i) - (1-\tilde{q}_i)f_i(y(\theta_i) + bu(\theta_i))u'(\theta_i) \Bigg).$$

Thus

$$y(t) = av(t) - u(t)\varphi(t) - v(t)\psi(t).$$

For $t \neq \theta_l$,

$$\begin{aligned} y'(t) =& av'(t) - u'(t)\varphi(t) - u(t)f(t, y(t) + bu(t))v(t) \\ & - v'(t)\psi(t) + v(t)f(t, y(t) + bu(t))u(t) \\ =& av'(t) - u'(t)\varphi(t) - v'(t)\psi(t). \end{aligned}$$

Since $W(u,v) = uv' - vu' = 1$, it follows that

$$
\begin{aligned}
y''(t) + q(t)y(t) =& a(v''(t) + q(t)v(t)) - (u''(t) + q(t)u(t))\varphi(t) \\
& -(v''(t) + q(t)v(t))\psi(t) + [v'(t)u(t) - u'(t)v(t)]f(t, y(t) + bu(t)) \\
=& f(t, y(t) + bu(t)).
\end{aligned}
$$

For $t = \theta_l$,

$$
\begin{aligned}
\varphi(\theta_l+) =& \int_T^{\theta_l} f(s, y(s) + bu(s))v(s)\mathrm{d}s \\
& + \sum_{i=\underline{i}(T)}^{l} \Big((1-p_i)\tilde{f}_i(y(\theta_i) + bu(\theta_i))v(\theta_i) - (1-\tilde{q}_i)f_i(y(\theta_i) + bu(\theta_i))v'(\theta_i) \Big) \\
=& \varphi(\theta_l) + (1-p_l)\tilde{f}_l(y(\theta_l) + bu(\theta_l))v(\theta_l) - (1-\tilde{q}_l)f_l(y(\theta_l) + bu(\theta_l))v'(\theta_l)
\end{aligned}
$$

and

$$
\begin{aligned}
\psi(\theta_l+) =& \int_{\theta_l}^{\infty} f(s, y(s) + bu(s))u(s)\mathrm{d}s \\
& + \sum_{i=l+1}^{\infty} \Big((1-p_i)\tilde{f}_i(y(\theta_i) + bu(\theta_i))u(\theta_i) - (1-\tilde{q}_i)f_i(y(\theta_i) + bu(\theta_i))u'(\theta_i) \Big) \\
=& \psi(\theta_l) - (1-p_l)\tilde{f}_l(y(\theta_l) + bu(\theta_l))u(\theta_l) + (1-\tilde{q}_l)f_l(y(\theta_l) + bu(\theta_l))u'(\theta_l).
\end{aligned}
$$

Then, with the help of (2.13) we have

$$
\begin{aligned}
\Delta y|_{t=\theta_l} =& a[v(\theta_l+) - v(\theta_l)] \\
& - u(\theta_l+)\varphi(\theta_l+) + u(\theta_l)\varphi(\theta_l) - v(\theta_l+)\psi(\theta_l+) + v(\theta_l)\psi(\theta_l) \\
=& -ap_lv(\theta_l) \\
& - (1-p_l)u(\theta_l)[\varphi(\theta_l) + (1-p_l)\tilde{f}_l(y(\theta_l) + bu(\theta_l))v(\theta_l) \\
& - (1-\tilde{q}_l)f_l(y(\theta_l) + bu(\theta_l))v'(\theta_l)] + u(\theta_l)\varphi(\theta_l) \\
& - (1-p_l)v(\theta_l)[\psi(\theta_l) - (1-p_l)\tilde{f}_l(y(\theta_l) + bu(\theta_l))u(\theta_l) \\
& + (1-\tilde{q}_l)f_l(y(\theta_l) + bu(\theta_l))u'(\theta_l)] + v(\theta_l)\psi(\theta_l) \\
=& -ap_lv(\theta_l) + p_l[u(\theta_l)\varphi(\theta_l) + v(\theta_l)\psi(\theta_l)] \\
& + f_l(y(\theta_l) + bu(\theta_l))[u(\theta_l)v'(\theta_l) - v(\theta_l)u'(\theta_l)].
\end{aligned}
$$

Since $W(u,v) = uv' - vu' = 1$, we obtain

$$
\Delta y|_{t=\theta_l} + p_l y((\theta_l)) = f_l(y(\theta_l) + bu(\theta_l)).
$$

Finally,

$$\begin{aligned}\Delta y'|_{t=\theta_l} =&a[v'(\theta_l+) - v'(\theta_l)]\\
&- u'(\theta_l+)\varphi(\theta_l+) + u'(\theta_l)\varphi(\theta_l) - v'(\theta_l+)\psi(\theta_l+) + v'(\theta_l)\psi(\theta_l)\\
=&- a\tilde{q}_l v'(\theta_l)\\
&- (1-\tilde{q}_l)u'(\theta_l)[\varphi(\theta_l) + (1-p_l)\tilde{f}_l(y(\theta_l) + bu(\theta_l))v(\theta_l)\\
&- (1-\tilde{q}_l)f_l(y(\theta_l) + bu(\theta_l))v'(\theta_l)] + u'(\theta_l)\varphi(\theta_l)\\
&- (1-\tilde{q}_l)v'(\theta_l)[\psi(\theta_l) - (1-p_l)\tilde{f}_l(y(\theta_l) + bu(\theta_l))u(\theta_l)\\
&+ (1-\tilde{q}_l)f_l(y(\theta_l) + bu(\theta_l))u'(\theta_l)] + v'(\theta_l)\psi(\theta_l)\\
=&- a\tilde{q}_l v'(\theta_l) + \tilde{q}_l[u'(\theta_l)\varphi(\theta_l) + v'(\theta_l)\psi(\theta_l)]\\
&+ \tilde{f}_l(y(\theta_l) + bu(\theta_l))[u(\theta_l)v'(\theta_l) - v(\theta_l)u'(\theta_l)],\end{aligned}$$

and so

$$\Delta y'|_{t=\theta_l} + \tilde{q}_l y((\theta_l)) = \tilde{f}_l(y(\theta_l) + bu(\theta_l)).$$

Therefore $y(t)$ is a solution of (3.23). The next step is to prove that F is completely continuous. By using (3.4)-(3.6) for a sufficiently large T we can write

$$\begin{aligned}&\frac{|(Fy)(t) - av(t)|}{v(t)} \le \frac{u(t)}{v(t)}\Bigg\{\int_T^t |f(s, y(s) + bu(s))|v(s)\mathrm{d}s\\
&+ \sum_{i=\underline{i}(T)}^{\bar{i}(t)}\Bigg(|(1-p_i)\tilde{f}_i(y(\theta_i) + bu(\theta_i))v(\theta_i)| + |(1-\tilde{q}_i)f_i(y(\theta_i) + bu(\theta_i))v'(\theta_i)|\Bigg)\Bigg\}\\
&+ \Bigg\{\int_t^\infty |f(s, y(s) + bu(s))|u(s)\mathrm{d}s\\
&+ \sum_{i=\underline{i}(t)}^{\infty}\Bigg(|(1-p_i)\tilde{f}_i(y(\theta_i) + bu(\theta_i))|u(\theta_i) + |(1-\tilde{q}_i)f_i(y(\theta_i) + bu(\theta_i))u'(\theta_i)|\Bigg)\Bigg\}\\
&\le \frac{u(t)}{v(t)}\int_T^t (K_1h_1(s) + h_2(s))v(s)\mathrm{d}s + \sum_{i=\underline{i}(T)}^{\bar{i}(t)}(K_2\bar{h}_i|v'(\theta_i+)| + K_3\tilde{h}_i v(\theta_i+))\\
&+ \int_T^\infty (K_1h_1(s) + h_2(s))u(s)\mathrm{d}s + \sum_{i=\underline{i}(T)}^{\infty}(K_2\bar{h}_i|u'(\theta_i+)| + K_3\tilde{h}_i u(\theta_i+)).\end{aligned}$$

The conditions (3.20) and (3.21) allow us to choose T sufficiently large so that

$$\frac{|(Fy)(t) - av(t)|}{v(t)} \le 1, \quad t \ge T.$$

Then, by taking supremum over $[T,\infty)$ we obtain $\|Fy - av\| \le 1$.

Since the remaining steps of the proof are similar to that of Theorem 3.1.1 we omit the details. $\square$

Theorem 3.1.3 *Assume that* (3.4), (3.5) *and* (3.6) *hold, and*

$$\begin{cases} |f(t,x_1) - f(t,x_2)| \le \dfrac{k(t)}{v(t)}|x_1 - x_2|, \\ |f_i(x_1) - f_i(x_2)| \le \dfrac{k_i}{v(\theta_i)}|x_1 - x_2|, \\ \left|\tilde{f}_i(x_1) - \tilde{f}_i(x_2)\right| \le \dfrac{\tilde{k}_i}{v(\theta_i)}|x_1 - x_2|, \end{cases} \tag{3.24}$$

where $k \in C([t_0,\infty),[0,\infty))$, $\{k_i\},\{\tilde{k}_i\}$ *satisfy*

$$\int_T^\infty u(s)k(s)\mathrm{d}s + \sum_{i=\underline{i}(T)}^{\infty} |u'(\theta_i+)|k_i + u(\theta_i+)\tilde{k}_i + \frac{k_i}{u(\theta_i+)} < \infty. \tag{3.25}$$

Suppose also that there exists a function $\beta \in C([t_0,\infty),[0,\infty))$ *such that*

$$\frac{1}{u^2(t)}\left\{ \int_t^\infty u(s)(h_1(s) + h_2(s))\mathrm{d}s + \sum_{i=\underline{i}(t)}^{\infty} |u'(\theta_i+)|\bar{h}_i + u(\theta_i+)\tilde{h}_i \right\} \le \beta(t), \quad t \ge T \tag{3.26}$$

and

$$\int_T^t \beta(s)\mathrm{d}s + \sum_{i=\underline{i}(T)}^{\bar{i}(t)} \frac{\bar{h}_i}{u(\theta_i+)} = o((v(t))^\mu), \quad t \to \infty, \quad \mu \in (0,1). \tag{3.27}$$

Then, for any given $a, b \in \mathbb{R}$, *equation* (3.2) *has a unique solution* $x(t)$ *such that*

$$x(t) = av(t) + bu(t) + o(u(t)(v(t))^\mu), \quad t \to \infty. \tag{3.28}$$

Proof. Define the space

$$X = \left\{ x \in \mathrm{PLC}([T,\infty),\mathbb{R}) | \frac{|x(t) - av(t) - bu(t)|}{v(t)} \le 1 \right\}$$

and the operator

$$(Fx)(t) = av(t) + bu(t) - u(t)\Bigg\{\int_T^t \frac{1}{u^2(s)}\Bigg(\int_s^\infty u(r)f(r,x(r))\mathrm{d}r - \sum_{i=\underline{i}(s)}^{\infty}(1-p_i)u(\theta_i)\tilde{f}_i(x(\theta_i)) - (1-\tilde{q}_i)u'(\theta_i)f_i(x(\theta_i))\Bigg)\mathrm{d}s - \sum_{i=\underline{i}(T)}^{\bar{i}(t)} \frac{1}{(1-p_i)u(\theta_i)}f_i(x(\theta_i))\Bigg\}, \quad x \in X.$$

It can be shown that X is a complete metric space with the metric

$$d(x_1,x_2) = \sup_{t\in[T,\infty)} \frac{1}{v(t)}|x_1(t)-x_2(t)|, \quad x_1,x_2 \in X.$$

We want to show that F has a unique fixed point with the help of the Banach fixed point theorem. For the sake of clarity, we will do the proof step by step.

Step 1.

Each fixed point of F is a solution for (3.2).

Proof. Denote

$$\alpha(t) := \int_T^t \frac{1}{u^2(s)}\Bigg(\int_s^\infty u(r)f(r,x(r))\mathrm{d}r + \sum_{i=\underline{i}(s)}^{\infty}(1-p_i)u(\theta_i)\tilde{f}_i(x(\theta_i)) - (1-\tilde{q}_i)u'(\theta_i)f_i(x(\theta_i))\Bigg)\mathrm{d}s.$$

Then,

$$\alpha'(t) = \frac{1}{u^2(t)}\Bigg(\int_t^\infty u(r)f(r,x(r))\mathrm{d}r - \sum_{i=\underline{i}(t)}^{\infty}(1-p_i)u(\theta_i)\tilde{f}_i(x(\theta_i)) - (1-\tilde{q}_i)u'(\theta_i)f_i(x(\theta_i))\Bigg).$$

Suppose x is a fixed point of the operator F. Then

$$x(t) = av(t) + bu(t) - u(t)\Bigg\{\alpha(t) - \sum_{i=\underline{i}(T)}^{\bar{i}(t)} \frac{1}{(1-p_i)u(\theta_i)}f_i(x(\theta_i))\Bigg\}.$$

For $t \neq \theta_l$,

$$\begin{aligned}
x'(t) =& av'(t) + bu'(t) - u'(t)\left\{\alpha(t) - \sum_{i=\underline{i}(T)}^{\bar{i}(t)} \frac{1}{(1-p_i)u(\theta_i)} f_i(x(\theta_i))\right\} - u(t)\alpha'(t) \\
=& av'(t) + bu'(t) - u'(t)\left\{\alpha(t) - \sum_{i=\underline{i}(T)}^{\bar{i}(t)} \frac{1}{(1-p_i)u(\theta_i)} f_i(x(\theta_i))\right\} \\
& - \frac{1}{u(t)}\left\{\int_t^{\infty} u(r)f(r,x(r))\mathrm{d}r \right. \\
& \left. - \sum_{i=\underline{i}(t)}^{\infty} (1-p_i)u(\theta_i)\tilde{f}_i(x(\theta_i)) - (1-\tilde{q}_i)u'(\theta_i)f_i(x(\theta_i))\right\}.
\end{aligned}$$

Thus,

$$\begin{aligned}
x''(t) + q(t)x(t) =& a(v''(t) + q(t)v(t)) + b(u''(t) + q(t)u(t)) + f(t,x(t)) \\
& - (u''(t) + q(t)u(t))\left\{\alpha(t) - \sum_{i=\underline{i}(T)}^{\bar{i}(t)} \frac{1}{(1-p_i)u(\theta_i)} f_i(x(\theta_i))\right\} \\
=& f(t,x(t)).
\end{aligned}$$

Since $\alpha(t)$ is a continuous function we have $\alpha(\theta_l+) = \alpha(\theta_l)$, so

$$\begin{aligned}
\Delta x|_{t=\theta_l} =& a\Delta v + b\Delta u - u(\theta_l+)\left\{\alpha(\theta_l) - \sum_{i=\underline{i}(T)}^{l-1} \frac{1}{(1-p_i)u(\theta_i)} f_i(x(\theta_i)) \right. \\
& \left. - \frac{1}{(1-p_l)u(\theta_l)} f_l(x(\theta_l))\right\} \\
& + u(\theta_l)\left\{\alpha(\theta_l) - \sum_{i=\underline{i}(T)}^{l-1} \frac{1}{(1-p_i)u(\theta_i)} f_i(x(\theta_i))\right\} \\
=& a\Delta v + b\Delta u - \Delta u\left\{\alpha(\theta_l) - \sum_{i=\underline{i}(T)}^{l-1} \frac{1}{(1-p_i)u(\theta_i)} f_i(x(\theta_i))\right\} \\
& + \frac{u(\theta_l+)}{(1-p_l)u(\theta_l)} f_l(x(\theta_l)),
\end{aligned}$$

and this implies that

$$\Delta x|_{t=\theta_l} + p_l x(\theta_l) = f_l(x(\theta_l)).$$

Finally,

$$
\begin{aligned}
u(\theta_l+)\alpha'(\theta_l+) =& \frac{1}{u(\theta_l+)}\Bigg(\int\limits_{\theta_l}^{\infty} u(r)f(r,x(r))\mathrm{d}r \\
& - \sum_{i=l}^{\infty}(1-p_i)u(\theta_i)\tilde{f}_i(x(\theta_i)) - (1-\tilde{q}_i)u'(\theta_i)f_i(x(\theta_i)) \\
& + (1-p_l)u(\theta_l)\tilde{f}_l(x(\theta_l)) - (1-\tilde{q}_l)u'(\theta_l)f_l(x(\theta_l))\Bigg) \\
=& \frac{1}{(1-p_l)u(\theta_l)}\Bigg(\int\limits_{\theta_l}^{\infty} u(r)f(r,x(r))\mathrm{d}r \\
& - \sum_{i=l}^{\infty}(1-p_i)u(\theta_i)\tilde{f}_i(x(\theta_i)) - (1-\tilde{q}_i)u'(\theta_i)f_i(x(\theta_i))\Bigg) \\
& + \tilde{f}_l(x(\theta_l)) - \frac{(1-\tilde{q}_l)u'(\theta_l)}{(1-p_l)u(\theta_l)}f_l(x(\theta_l)) \\
=& \frac{1}{1-p_l}u(\theta_l)\alpha'(\theta_l) + \tilde{f}_l(x(\theta_l)) - \frac{(1-\tilde{q}_l)u'(\theta_l)}{(1-p_l)u(\theta_l)}f_l(x(\theta_l)),
\end{aligned}
$$

and from (2.13)

$$
\begin{aligned}
\Delta x'|_{t=\theta_l} =& a\Delta v' + b\Delta u' - \Delta u'\left\{\alpha(\theta_l) - \sum_{i=\underline{i}(T)}^{l-1}\frac{1}{(1-p_i)u(\theta_i)}f_i(x(\theta_i))\right\} \\
& - \frac{u'(\theta_l+)}{(1-p_l)u(\theta_l)}f_l(x(\theta_l)) \\
& - \frac{1}{1-p_l}u(\theta_l)\alpha'(\theta_l) + \tilde{f}_l(x(\theta_l)) + \frac{(1-\tilde{q}_l)u'(\theta_l)}{(1-p_l)u(\theta_l)}f_l(x(\theta_l)) + u(\theta_l)\alpha'(\theta_l) \\
=& -\tilde{q}_l[av(\theta_l) + bu(\theta_l) - u'(\theta_l)\alpha(\theta_l) - u(\theta_l)\alpha'(\theta_l)] + \tilde{f}_l(x(\theta_l)).
\end{aligned}
$$

Thus,

$$\Delta x'|_{t=\theta_l} + \tilde{q}_l x'(\theta_l) = \tilde{f}_l(x(\theta_l)).$$

So, we conclude that $x(t)$ is a solution of the equation (3.2).

Step 2.

$F : X \to X$ is a contraction mapping.

Proof. **(i)** $F : X \to X$:

For $x \in X$, we have

$$\frac{|x(t)|}{v(t)} \leq |a| + |b| + 1 := M_0.$$

Since $g_j \in \mathrm{C}(\mathbb{R}_+, \mathbb{R}_+)$ there exist constants $K_j > 0$ such that $\max\limits_{0\le\tau\le M_0} g_j(\tau) = K_j$, $j = 1, 2, 3$. From (3.26) we have

$$\int_T^\infty u(s)(h_1(s) + h_2(s))\mathrm{d}s + \sum_{i=\underline{i}(T)}^{\infty} |u'(\theta_i+)|\bar{h}_i + u(\theta_i+)\tilde{h}_i \le u^2(T)\beta(T).$$

So, for T is sufficiently large, we may write that

$$\int_T^\infty u(s)h_1(s)\mathrm{d}s + \sum_{i=\underline{i}(T)}^{\infty} |u'(\theta_i+)|\bar{h}_i + u(\theta_i+)\tilde{h}_i \le \frac{1}{4K}, \tag{3.29}$$

$$\int_T^\infty u(s)h_2(s)\mathrm{d}s \le \frac{1}{4}, \tag{3.30}$$

where $K := \max\{K_1, K_2, K_3\}$. Also, from (3.27)

$$\sum_{i=\underline{i}(T)}^{\bar{i}(t)} \frac{\bar{h}_i}{u(\theta_i+)} \le \frac{(v(t))^\mu}{2K}. \tag{3.31}$$

So, by using (3.4)-(3.6) one can write

$$\begin{aligned}
|(Fx)(t) - av(t) - bu(t)| \le{}& u(t)\Bigg\{\int_T^t \frac{1}{u^2(s)}\Bigg(\int_s^\infty u(r)|f(r,x(r))|\mathrm{d}r \\
&+ \sum_{i=\underline{i}(s)}^{\infty} |u'(\theta_i+)f_i(x(\theta_i))| + u(\theta_i+)|\tilde{f}_i(x(\theta_i))|\Bigg)\mathrm{d}s + \sum_{i=\underline{i}(T)}^{\bar{i}(t)} \frac{1}{u(\theta_i+)}|f_i(x(\theta_i))|\Bigg\} \\
\le{}& u(t)\Bigg\{\int_T^t \frac{1}{u^2(s)}\Bigg(\int_T^\infty u(r)|f(r,x(r))|\mathrm{d}r \\
&+ \sum_{i=\underline{i}(T)}^{\infty} |u'(\theta_i+)f_i(x(\theta_i))| + u(\theta_i+)|\tilde{f}_i(x(\theta_i))|\Bigg)\mathrm{d}s + \sum_{i=\underline{i}(T)}^{\bar{i}(t)} \frac{1}{u(\theta_i+)}|f_i(x(\theta_i))|\Bigg\} \\
\le{}& u(t)\Bigg\{\int_T^t \frac{1}{u^2(s)}\Bigg(\int_T^\infty u(r)\left(h_1(r)g_1\left(\frac{|x(r)|}{v(r)}\right) + h_2(r)\right)\mathrm{d}r \\
&+ \sum_{i=\underline{i}(T)}^{\infty} |u'(\theta_i+)|\bar{h}_i g_2\left(\frac{|x(\theta_i)|}{v(\theta_i)}\right) + u(\theta_i+)\tilde{h}_i g_3\left(\frac{|x(\theta_i)|}{v(\theta_i)}\right)\Bigg)\mathrm{d}s \\
&+ \sum_{i=\underline{i}(T)}^{\bar{i}(t)} \frac{\bar{h}_i}{u(\theta_i+)} g_2\left(\frac{|x(\theta_i)|}{v(\theta_i)}\right)\Bigg\}.
\end{aligned}$$

With the help of (3.29) and (3.30) it follows that

$$
\begin{aligned}
|(Fx)(t)-av(t)-bu(t)| \leq u(t) \Bigg\{ \int_T^t \frac{1}{u^2(s)} \Bigg(\int_T^\infty u(r)\left(K_1 h_1(r)+h_2(r)\right) \mathrm{d}r \\
+ \sum_{i=\underline{i}(T)}^{\infty} K_2 |u'(\theta_i+)| \bar{h}_i + K_3 u(\theta_i+) \tilde{h}_i \Bigg) \mathrm{d}s + \sum_{i=\underline{i}(T)}^{\bar{i}(t)} \frac{K_2}{u(\theta_i+)} \bar{h}_i \Bigg\} \\
\leq \frac{1}{2} u(t) \Bigg\{ \int_T^t \frac{1}{u^2(s)} \mathrm{d}s + (v(t))^{\mu} \Bigg\}.
\end{aligned}
$$

Then, from (2.9) it follows that

$$
|(Fx)(t)-av(t)-bu(t)| \leq v(t),
$$

and so

$$
\frac{|(Fx)(t)-av(t)-bu(t)|}{v(t)} \leq 1.
$$

Next we show that $Fx \in \mathrm{PLC}[T,\infty)$. Let $t_1 \in [T,\infty)$ with $t < t_1$. Then we have

$$
\begin{aligned}
&|(Fx)(t)-(Fx)(t_1)| \\
&\leq |a||v(t)-v(t_1)| + |u(t)-u(t_1)| \Bigg\{ |b| + \int_T^t \frac{1}{u^2(s)} \Bigg(\int_s^\infty u(r)|f(r,x(r))| \mathrm{d}r \\
&+ \sum_{i=\underline{i}(s)}^{\infty} |u'(\theta_i+) f_i(x(\theta_i))| + u(\theta_i+)|\tilde{f}_i(x(\theta_i))| \Bigg) \mathrm{d}s + \sum_{i=\underline{i}(T)}^{\bar{i}(t)} \frac{1}{u(\theta_i+)} |f_i(x(\theta_i))| \Bigg\} \\
&+ u(t_1) \Bigg\{ \int_t^{t_1} \frac{1}{u^2(s)} \Bigg(\int_s^\infty u(r)|f(r,x(r))| \mathrm{d}r \\
&+ \sum_{i=\underline{i}(s)}^{\infty} |u'(\theta_i+) f_i(x(\theta_i))| + u(\theta_i+)|\tilde{f}_i(x(\theta_i))| \Bigg) + \sum_{i=\underline{i}(t)}^{\bar{i}(t_1)} \frac{1}{u(\theta_i+)} |f_i(x(\theta_i))| \Bigg\}
\end{aligned}
$$

From (3.4)-(3.6) one can write

$$
\begin{aligned}
&|(Fx)(t)-(Fx)(t_1)| \\
&\le |a||v(t)-v(t_1)| + |u(t)-u(t_1)| \Bigg\{ |b| + \int_T^t \Bigg(\frac{1}{u^2(s)} \int_T^\infty u(r)\left(h_1(r)K_1 + h_2(r)\right) \mathrm{d}r \\
&\quad + \sum_{i=\underline{i}(T)}^{\infty} K_2|u'(\theta_i+)|\bar{h}_i + K_3 u(\theta_i+)\tilde{h}_i \Bigg) \mathrm{d}s + \sum_{i=\underline{i}(T)}^{\bar{i}(t)} \frac{1}{u(\theta_i+)} |f_i(x(\theta_i))| \Bigg\} \\
&\quad + u(t_1) \Bigg\{ \int_t^{t_1} \frac{1}{u^2(s)} \Bigg(\int_T^\infty u(r)\left(K_1 h_1(r) + h_2(r)\right) \mathrm{d}r \\
&\quad + \sum_{i=\underline{i}(T)}^{\infty} K_2|u'(\theta_i+)|\bar{h}_i + K_3 u(\theta_i+)\tilde{h}_i \Bigg) \mathrm{d}s + \sum_{i=\underline{i}(t)}^{\bar{i}(t_1)} K_2 \frac{\bar{h}_i}{u(\theta_i+)} \Bigg\},
\end{aligned}
$$

which, as a consequence of (3.26) and (3.27), is finite. So, taking limit as $t \to t_1-$ we obtain

$$(Fx)(t) \to (Fx)(t_1).$$

In a similar way, one can show that

$$\lim_{t \to t_1+} (Fx)(t) = (Fx)(t_1) \text{ for } t_1 \neq \theta_i$$

and

$$\lim_{t \to \theta_i+} (Fx)(t) \text{ exist for all } i = 1, 2, \ldots$$

Hence we conclude that $F(X) \subset X$.

(ii) *F is a contraction mapping:*

For a sufficiently large T, in view of (3.25), we can write that

$$\int_T^\infty u(s)k(s)\mathrm{d}s + \sum_{i=\underline{i}(T)}^{\infty} |u'(\theta_i+)|k_i + u(\theta_i+)\tilde{k}_i + \frac{k_i}{u(\theta_i+)} < \mu. \tag{3.32}$$

Pick $x_1, x_2 \in X$. With the help of (3.4) and (3.25) we have

$$
\begin{aligned}
&|(Fx_1)(t) - (Fx_2)(t)| \\
\leq & u(t)\Bigg\{\int_T^t \frac{1}{u^2(s)}\Bigg(\int_s^\infty u(r)|f(r,x_1(r)) - f(r,x_2(r))|\mathrm{d}r \\
&+ \sum_{i=\underline{i}(s)}^{\infty} |u'(\theta_i+)||f_i(x_1(\theta_i)) - f_i(x_2(\theta_i))| + u(\theta_i+)|\tilde{f}_i(x_1(\theta_i)) - \tilde{f}_i(x_2(\theta_i))|\Bigg)\mathrm{d}s \\
&+ \sum_{i=\underline{i}(T)}^{\bar{i}(t)} \frac{1}{u(\theta_i+)}|f_i(x_1(\theta_i)) - f_i(x_2(\theta_i))|\Bigg\} \\
\leq & u(t)\Bigg\{\int_T^t \frac{1}{u^2(s)}\Bigg(\int_s^\infty u(r)\frac{k(r)}{v(r)}|x_1(r) - x_2(r)|\mathrm{d}r \\
&+ \sum_{i=\underline{i}(s)}^{\infty} |u'(\theta_i+)|\frac{k_i}{v(\theta_i)}|x_1(\theta_i) - x_2(\theta_i)| + u(\theta_i+)\frac{\tilde{k}_i}{v(\theta_i)}|x_1(\theta_i) - x_2(\theta_i)|\Bigg)\mathrm{d}s \\
&+ \sum_{i=\underline{i}(T)}^{\bar{i}(t)} \frac{1}{u(\theta_i+)}\frac{k_i}{v(\theta_i)}|x_1(\theta_i) - x_2(\theta_i)|\Bigg\} \\
\leq & d(x_1,x_2)u(t)\Bigg\{\int_T^t \frac{1}{u^2(s)}\Bigg(\int_s^\infty u(r)k(r)\mathrm{d}r \\
&+ \sum_{i=\underline{i}(s)}^{\infty} |u'(\theta_i+)|k_i + u(\theta_i+)\tilde{k}_i\Bigg)\mathrm{d}s + \sum_{i=\underline{i}(T)}^{\bar{i}(t)} \frac{k_i}{u(\theta_i+)}\Bigg\} \\
\leq & \mu d(x_1,x_2)v(t),
\end{aligned}
$$

which implies that

$$d(Fx_1, Fx_2) \leq \mu d(x_1, x_2).$$

Therefore, $F : X \to X$ is a contraction mapping. Hence we conclude that the operator F has a unique fixed point.

Step 3.

Any solution $x(t)$ satisfies the asymptotic formula (3.28).

Proof. By means of the conditions of the theorem we observe that

$$|x(t) - av(t) - bu(t)| \leq u(t)\left\{\int_T^t \frac{1}{u^2(s)}\left(\int_T^\infty u(r)\left(K_1 h_1(r) + h_2(r)\right) \mathrm{d}r\right.\right.$$

$$\left.\left.+ \sum_{i=\underline{i}(T)}^{\infty} K_2|u'(\theta_i+)|\bar{h}_i + K_3 u(\theta_i+)\tilde{h}_i\right)\mathrm{d}s + \sum_{i=\underline{i}(T)}^{\bar{i}(t)} \frac{K_2}{u(\theta_i+)}\bar{h}_i\right\}$$

$$\leq Ku(t)\left\{\int_T^t \frac{1}{u^2(s)}\mathrm{d}s + \sum_{i=\underline{i}(T)}^{\bar{i}(t)} \frac{\bar{h}_i}{u(\theta_i+)}\right\}$$

$$\leq Ku(t)\left\{\int_T^t \beta(s)\mathrm{d}s + \sum_{i=\underline{i}(T)}^{\bar{i}(t)} \frac{\bar{h}_i}{u(\theta_i+)}\right\}.$$

From (3.27) and the above inequality, it follows that

$$\lim_{t\to\infty} \frac{|x(t) - av(t) - bu(t)|}{u(t)(v(t))^{\mu}} \to 0, \quad t \to \infty.$$

□

Theorem 3.1.4 *Suppose that* (3.4)-(3.6) *and* (3.24) *hold. If*

$$\int_T^\infty v(s)k(s)\mathrm{d}s + \sum_{i=\underline{i}(T)}^{\infty} |v'(\theta_i+)|k_i + v(\theta_i+)\tilde{k}_i + \sum_{i=\underline{i}(T)}^{\infty} \frac{k_i}{v(\theta_i+)} < \infty, \qquad (3.33)$$

and there exists a function $\beta \in C([t_0, \infty), [0, \infty))$ *such that*

$$\frac{1}{v^2(t)}\left\{\int_t^\infty v(s)(h_1(s) + h_2(s))\mathrm{d}s\right.$$

$$\left.+ \sum_{i=\underline{i}(t)}^{\infty} |v'(\theta_i+)|\bar{h}_i + v(\theta_i+)\tilde{h}_i\right\} \leq \beta(t), \quad t \geq T, \qquad (3.34)$$

where

$$\int_T^t \beta(s)\mathrm{d}s + \sum_{i=\underline{i}(T)}^{\bar{i}(t)} \frac{\bar{h}_i}{v(\theta_i+)} = o((u(t))^{\mu}), \quad t \to \infty, \quad \mu \in (0, 1), \qquad (3.35)$$

then, for any given $a, b \in \mathbb{R}$, *equation* (3.2) *has a unique solution* $x(t)$ *such that*

$$x(t) = av(t) + bu(t) + o(v(t)(u(t))^{\mu}), \quad t \to \infty. \qquad (3.36)$$

Proof. Consider the operator F defined on X by

$$
\begin{aligned}
(Fx)(t) =& av(t) + bu(t) \\
&+ v(t)\Bigg\{\int_t^\infty \frac{1}{v^2(s)}\Bigg(\int_s^\infty v(r)f(r,x(r))\mathrm{d}r \\
&+ \sum_{i=\underline{i}(s)}^{\infty}(1-p_i)v(\theta_i)\tilde{f}_i(x(\theta_i)) - (1-\tilde{q}_i)v'(\theta_i)f_i(x(\theta_i))\mathrm{d}s\Bigg) \\
&+ \sum_{i=\underline{i}(t)}^{\infty}\frac{1}{(1-p_i)v(\theta_i)}f_i(x(\theta_i))\Bigg\}.
\end{aligned}
$$

The proof follows the steps similar to that of Theorem 3.1.3 with similar computations, hence we omit it. □

CASE II.

In this part we give asymptotic formulas for the solutions of the following impulsive differential equation with mixed impulse conditions

$$
\begin{cases}
x'' + q(t)x = f(t,x), & t \neq \theta_i, \\
\Delta x + p_i x = f_i(x), & t = \theta_i, \\
\Delta x' + q_i x = \tilde{f}_i(x), & t = \theta_i.
\end{cases}
\tag{3.37}
$$

We will utilize the principal and nonprincipal solutions of the related homogeneous equation

$$
\begin{cases}
x'' + q(t)x = 0, & t \neq \theta_i, \\
\Delta x + p_i x = 0, & t = \theta_i, \\
\Delta x' + q_i x = 0, & t = \theta_i.
\end{cases}
\tag{3.38}
$$

Let u and v be principal and nonprincipal solution of (3.38), and suppose without any loss of generality that they are positive on $[T,\infty)$.

Theorem 3.1.5 *Assume that* $T \geq 1$, $f \in C([1,\infty)\times\mathbb{R},\mathbb{R})$, $f_i, \tilde{f}_i \in C([1,\infty),\mathbb{R})$, *and there exist functions* $g_j \in C(\mathbb{R}_+,\mathbb{R}_+)$, $j = 1,2,3,$ $h_k \in C([1,\infty),\mathbb{R}_+)$,

$k = 1, 2$ *such that* (3.4)-(3.6) *hold, where*

$$\int_T^{\infty} \prod_{k=\underline{i}(T)}^{\bar{i}(s)} (1-p_k)^{-1} v(s)(h_1(s)+h_2(s))\mathrm{d}s$$
$$+ \sum_{i=\underline{i}(T)}^{\infty} \prod_{k=\underline{i}(T)}^{i} (1-p_k)^{-1} \left(|v'(\theta_i+)|\bar{h}_i + v(\theta_i+)\tilde{h}_i\right) < \infty. \tag{3.39}$$

Then, for any given $a, b \in \mathbb{R}$, *equation* (3.37) *has a solution* $x(t)$ *such that*

$$x(t) = av(t) + bu(t) + o(u(t)), \quad t \to \infty. \tag{3.40}$$

Proof. Define the operator

$$\begin{aligned}(Fy)(t) =& bu(t) + u(t)\Bigg\{ \int_t^{\infty} f(s, y(s)+av(s))u(s) \int_t^{s} \frac{\prod_{k=\underline{i}(r)}^{\bar{i}(s)} (1-p_k)^{-1}}{u^2(r)} \mathrm{d}r\mathrm{d}s \\
&+ \sum_{i=\underline{i}(t)}^{\infty} \Bigg((1-p_i)u(\theta_i) \int_t^{\theta_i} \frac{\prod_{k=\underline{i}(s)}^{i} (1-p_k)^{-1}}{u^2(s)} \mathrm{d}s \tilde{f}_i(y(\theta_i)+av(\theta_i)) \\
&- \left[\frac{1}{(1-p_i)u(\theta_i)} + (u'(\theta_i) - q_i u(\theta_i)) \int_t^{\theta_i} \frac{\prod_{k=\underline{i}(s)}^{i} (1-p_k)^{-1}}{u^2(s)} \mathrm{d}s \right] \\
&\times f_i(y(\theta_i)+av(\theta_i)) \Bigg) \Bigg\}\end{aligned}$$

on the Banach space

$$Y = \left\{ y \in \mathrm{PLC}([T,\infty), \mathbb{R}) | \, \frac{|y(t)|}{v(t)} \le M_y \right\}.$$

Set

$$S = \{ y \in Y | \, \|y(t) - bu(t)\| \le 1 \}.$$

Let y be a fixed point of the operator F. We claim that $y(t)$ is a solution of

$$\begin{cases} y'' + q(t)y = f(t, y(t)+av(t)), & t \ne \theta_i; \\ \Delta y + p_i y = f_i(y + av(\theta_i)), & t = \theta_i; \\ \Delta y' + q_i y = \tilde{f}_i(y + av(\theta_i)), & t = \theta_i, \end{cases} \tag{3.41}$$

for $t \geq T$. To see this, denote

$$\Phi(t) := \int_t^\infty f(s, y(s) + av(s))u(s) \int_t^s \frac{\prod_{k=\underline{i}(r)}^{\bar{i}(s)} (1-p_k)^{-1}}{u^2(r)} \mathrm{d}r\mathrm{d}s$$

$$+ \sum_{i=\underline{i}(t)}^\infty \Bigg((1-p_i)u(\theta_i) \int_t^{\theta_i} \frac{\prod_{k=\underline{i}(s)}^{i} (1-p_k)^{-1}}{u^2(s)} \mathrm{d}s \tilde{f}_i(y(\theta_i) + av(\theta_i))$$

$$- \left[\frac{1}{(1-p_i)u(\theta_i)} + (u'(\theta_i) - q_i u(\theta_i)) \int_t^{\theta_i} \frac{\prod_{k=\underline{i}(s)}^{i} (1-p_k)^{-1}}{u^2(s)} \mathrm{d}s \right]$$

$$\times f_i(y(\theta_i) + av(\theta_i)) \Bigg).$$

Now, according to the convention $\prod_{i=\underline{i}(t)}^{\bar{i}(t)} \alpha_i = 1$ we have

$$\Phi'(t) = -\Bigg\{ f(t, y(t) + av(t))u(t) \int_t^t \frac{\prod_{k=\underline{i}(r)}^{\bar{i}(t)} (1-p_k)^{-1}}{u^2(r)} \mathrm{d}r$$

$$+ \int_t^\infty f(s, y(s) + av(s))u(s) \frac{\prod_{k=\underline{i}(t)}^{\bar{i}(t)} (1-p_k)^{-1}}{u^2(t)} \mathrm{d}s$$

$$+ \sum_{i=\underline{i}(t)}^\infty \Bigg((1-p_i)\tilde{f}_i(y(\theta_i) + av(\theta_i))u(\theta_i) \frac{\prod_{k=\underline{i}(t)}^{i} (1-p_k)^{-1}}{u^2(t)}$$

$$- (u'(\theta_i) - q_i u(\theta_i)) f_i(y(\theta_i) + av(\theta_i)) \frac{\prod_{k=\underline{i}(t)}^{i} (1-p_k)^{-1}}{u^2(t)} \Bigg) \Bigg\}$$

$$= -\frac{1}{u^2(t)} \Bigg\{ \int_t^\infty f(s, y(s) + av(s))u(s)\mathrm{d}s$$

$$+ \sum_{i=\underline{i}(t)}^\infty \prod_{k=\underline{i}(t)}^{i} (1-p_k)^{-1} \Bigg((1-p_i)\tilde{f}_i(y(\theta_i) + av(\theta_i))u(\theta_i)$$

$$- (u'(\theta_i) - q_i u(\theta_i)) f_i(y(\theta_i) + av(\theta_i)) \Bigg) \Bigg\}.$$

Thus, we obtain

$$y(t) := u(t)(b + \Phi(t)). \tag{3.42}$$

Let $t \neq \theta_l$. Then, we have

$$y'(t) = u'(t)(b + \Phi(t)) + u(t)\Phi'(t)$$

and

$$\begin{aligned} y''(t) + q(t)y(t) =& [u''(t) + q(t)u(t)](b + \Phi(t)) + f(t, y(t) + av(t)) \\ =& f(t, y(t) + av(t)). \end{aligned}$$

For $t = \theta_l$,

$$\begin{aligned} \Phi(\theta_l+) :=& \int_{\theta_l}^{\infty} f(s, y(s) + av(s))u(s) \int_{\theta_l}^{s} \frac{\prod_{k=\underline{i}(r)}^{\bar{i}(s)} (1-p_k)^{-1}}{u^2(r)} \mathrm{d}r\mathrm{d}s \\ & + \sum_{i=l+1}^{\infty} \Bigg((1-p_i)\tilde{f}_i(y(\theta_i) + av(\theta_i))u(\theta_i) \int_{\theta_l}^{\theta_i} \frac{\prod_{k=\underline{i}(s)}^{i} (1-p_k)^{-1}}{u^2(s)} \mathrm{d}s \\ & - \left[\frac{1}{(1-p_i)u(\theta_i)} + (u'(\theta_i) - q_i u(\theta_i)) \int_{\theta_l}^{\theta_i} \frac{\prod_{k=\underline{i}(s)}^{i} (1-p_k)^{-1}}{u^2(s)} \mathrm{d}s \right] \\ & \times f_i(y(\theta_i) + av(\theta_i)) \Bigg) \\ =& \Phi(\theta_l) + \frac{1}{(1-p_l)u(\theta_l)} f_l(y(\theta_l) + av(\theta_l)). \end{aligned} \tag{3.43}$$

Hence

$$\begin{aligned} \Delta y|_{t=\theta_l} =& u(\theta_l+)(b + \Phi(\theta_l+)) - u(\theta_l)(b + \Phi(\theta_l)) \\ =& (1-p_l)u(\theta_l) \left(b + \Phi(\theta_l) + \frac{1}{(1-p_l)u(\theta_l)} f_l(y(\theta_l) + av(\theta_l)) \right) \\ & - u(\theta_l)(b + \Phi(\theta_l)) \\ =& -p_l u(\theta_l)(b + \Phi(\theta_l)) + f_l(y(\theta_l) + av(\theta_l)), \end{aligned}$$

and so

$$\Delta y|_{t=\theta_l} + p_l y(\theta_l) = f_l(y(\theta_l) + av(\theta_l)).$$

Clearly $\prod\limits_{k=l+1}^{i}(1-p_k)^{-1} = (1-p_{l+1})\prod\limits_{k=l}^{i}(1-p_k)^{-1}$, so

$$\begin{aligned}
u(\theta_l+)\Phi'(\theta_l+) = &-\frac{1}{u(\theta_l+)}\Bigg\{\int\limits_{\theta_l}^{\infty} f(s,y(s)+av(s))u(s)\mathrm{d}s \\
&+\sum_{i=l}^{\infty}(1-p_l)\prod_{k=l}^{i}(1-p_k)^{-1}\Bigg((1-p_i)\tilde{f}_i(y(\theta_i)+av(\theta_i))u(\theta_i) \\
&\qquad -(u'(\theta_i)-q_iu(\theta_i))f_i(y(\theta_i)+av(\theta_i))\Bigg)\Bigg\} \\
&+\frac{1}{u(\theta_l+)}\prod_{k=l+1}^{l}(1-p_k)^{-1}\Bigg((1-p_l)\tilde{f}_l(y(\theta_l)+av(\theta_l))u(\theta_l) \\
&\qquad -(u'(\theta_l)-q_lu(\theta_l))f_l(y(\theta_l)+av(\theta_l))\Bigg) \\
=&u(\theta_l)\Phi'(\theta_l)+\tilde{f}_l(y(\theta_l)+av(\theta_l))-\frac{(u'(\theta_l)-q_lu(\theta_l))}{(1-p_l)u(\theta_l)}f_l(y(\theta_l)+av(\theta_l)).
\end{aligned}$$

From (3.43) it follows that

$$\begin{aligned}
\Delta y'|_{t=\theta_l} =&u'(\theta_l+)(b+\Phi(\theta_l+))-u'(\theta_l)(b+\Phi(\theta_l))+u(\theta_l+)\Phi'(\theta_l+)-u(\theta_l)\Phi'(\theta_l) \\
=&-q_lu(\theta_l)(b+\Phi(\theta_l))\tilde{f}_l(y(\theta_l)+av(\theta_l))
\end{aligned}$$

and hence

$$\Delta y'|_{t=\theta_l}+q_ly((\theta_l))=\tilde{f}_l(y(\theta_l)+av(\theta_l)).$$

Thus, a fixed point y is a solution.

The next step is to show that F has a fixed point. For this, we use the Schauder fixed point theorem. First, it should be proven that $F(S)\subset S$. Let $y(t)\in S$. Then,

$$\begin{aligned}
\frac{|(Fy)(t)-bu(t)|}{u(t)} \le &\Bigg\{\int\limits_{t}^{\infty}\prod_{k=\underline{i}(T)}^{\bar{i}(s)}(1-p_k)^{-1}|f(s,y(s)+av(s))|v(s)\mathrm{d}s \\
&+\sum_{i=\underline{i}(t)}^{\infty}\prod_{k=\underline{i}(T)}^{i}(1-p_k)^{-1}\Bigg(|v'(\theta_i+)f_i(y(\theta_i)+av(\theta_i))| \\
&\qquad +v(\theta_i+)|\tilde{f}_i(y(\theta_i)+av(\theta_i))|\Bigg)\Bigg\}.
\end{aligned}$$

We skip the remaining part of the proof because of the similarities with that of Theorem 3.1.1. □

Theorem 3.1.6 *Assume that* (3.4)-(3.6) *hold. If*

$$\int_T^\infty \prod_{k=\underline{i}(T)}^{\bar{i}(s)} (1-p_k)^{-1} u(s)(h_1(s)+h_2(s))\mathrm{d}s$$
$$+\sum_{i=\underline{i}(T)}^{\infty} \prod_{k=\underline{i}(T)}^{i} (1-p_k)^{-1} \left(|u'(\theta_i+)|\bar{h}_i + u(\theta_i+)\tilde{h}_i\right) < \infty \tag{3.44}$$

and

$$\limsup_{t\to\infty} \frac{u(t)}{v(t)} \Bigg\{ \prod_{k=\underline{i}(T)}^{\bar{i}(s)} (1-p_k)^{-1} v(s)(h_1(s)+h_2(s))\mathrm{d}s$$
$$+\sum_{i=\underline{i}(T)}^{\bar{i}(t)} \prod_{k=\underline{i}(T)}^{i} (1-p_k)^{-1} \left(|v'(\theta_i+)|\bar{h}_i + v(\theta_i+)\tilde{h}_i\right) \Bigg\} = 0, \tag{3.45}$$

then, for any given $a, b \in \mathbb{R}$, *equation* (3.37) *has a solution* $x(t)$ *such that*

$$x(t) = av(t) + bu(t) + o(v(t)), \quad t \to \infty. \tag{3.46}$$

Proof. Consider the operator defined on Y by

$$\begin{aligned}(Fy)(t) =& av(t) - u(t)\Bigg\{ \int_T^t \prod_{k=\underline{i}(T)}^{\bar{i}(s)} (1-p_k)^{-1} f(s, y(s)+bu(s))v(s) \\ &+ \sum_{i=\underline{i}(T)}^{\bar{i}(t)} \Bigg(\prod_{k=\underline{i}(T)}^{i} (1-p_k)^{-1} [(1-p_i)v(\theta_i)\tilde{f}_i(y(\theta_i)+bu(\theta_i)) \\ &- (v'(\theta_i) - q_i v(\theta_i)) f_i(y(\theta_i)+bu(\theta_i))] \Bigg) \Bigg\} \\ &- v(t)\Bigg\{ \int_t^\infty \prod_{k=\underline{i}(T)}^{\bar{i}(s)} (1-p_k)^{-1} f(s, y(s)+bu(s))u(s) \\ &+ \sum_{i=\underline{i}(t)}^{\infty} \Bigg(\prod_{k=\underline{i}(T)}^{i} (1-p_k)^{-1} [(1-p_i)u(\theta_i)\tilde{f}_i(y(\theta_i)+bu(\theta_i)) \\ &- (u'(\theta_i) - q_i u(\theta_i)) f_i(y(\theta_i)+bu(\theta_i))] \Bigg) \Bigg\},\end{aligned}$$

and let

$$S = \{y \in Y \,|\, \|y(t) - av(t)\| \le 1\}.$$

By following the same processes as in the proofs of Theorems 3.1.2 and 3.1.5 it can be shown that any fixed point of F is a solution of

$$\begin{cases} y'' + q(t)y = f(t, y(t) + bu(t)), & t \neq \theta_i, \\ \Delta y + p_i y = f_i(y + bu(\theta_i)), & t = \theta_i, \\ \Delta y' + q_i y = \tilde{f}_i(y + bu(\theta_i)), & t = \theta_i, \end{cases} \tag{3.47}$$

and that F has a fixed point which satisfy the asymptotic formula (3.46). □

Theorem 3.1.7 *Assume that* (3.4)- (3.6) *and* (3.24) *hold, where*

$$\int_T^\infty \prod_{k=\underline{i}(T)}^{\bar{i}(s)} (1-p_k)^{-1} u(s)k(s)\mathrm{d}s + \sum_{i=\underline{i}(T)}^{\infty} \prod_{k=\underline{i}(T)}^{i} (1-p_k)^{-1} \left(|u'(\theta_i+)|k_i + u(\theta_i+)\tilde{k}_i\right) + \frac{k_i}{u(\theta_i+)} < \infty. \tag{3.48}$$

Suppose also that there exists functions $\beta \in C([t_0,\infty),[0,\infty))$ *such that*

$$\frac{1}{u^2(t)}\left\{\int_t^\infty \prod_{k=\underline{i}(T)}^{\bar{i}(s)} (1-p_k)^{-1} u(s)(h_1(s)+h_2(s))\mathrm{d}s + \sum_{i=\underline{i}(t)}^{\infty} \prod_{k=\underline{i}(T)}^{i} (1-p_k)^{-1} \left(|u'(\theta_i+)|\bar{h}_i + u(\theta_i+)\tilde{h}_i\right)\right\} \leq \beta(t), \quad t \geq T, \tag{3.49}$$

where

$$\int_T^t \beta(s)\mathrm{d}s + \sum_{i=\underline{i}(T)}^{\bar{i}(t)} \frac{\bar{h}_i}{u(\theta_i+)} = o((v(t))^\mu), \quad t\to\infty, \quad \mu \in (0,1). \tag{3.50}$$

Then, for any given $a, b \in \mathbb{R}$, *equation* (3.37) *has a unique solution* $x(t)$ *such that*

$$x(t) = av(t) + bu(t) + o(u(t)(v(t))^\mu), \quad t\to\infty. \tag{3.51}$$

Proof. For

$$x \in X = \left\{x \in \mathrm{PLC}([T,\infty),\mathbb{R})| \ \frac{|x(t)-av(t)-bu(t)|}{v(t)} \leq 1\right\}, \tag{3.52}$$

we define the operator

$$
\begin{aligned}
(Fx)(t) =& av(t) + bu(t) \\
& - u(t)\Bigg(\int\limits_t^\infty \frac{1}{u^2(s)} \Bigg\{ \int\limits_s^\infty \prod_{k=\underline{i}(s)}^{\bar{i}(r)} (1-p_k)^{-1} u(r) f(r, x(r)) \mathrm{d}r \\
& + \sum_{i=\underline{i}(s)}^{\infty} \prod_{k=\underline{i}(s)}^{i} (1-p_k)^{-1} \Big((u'(\theta_i) - q_i u(\theta_i)) f_i(x(\theta_i)) \\
& - (1-p_i) u(\theta_i) \tilde{f}_i(x(\theta_i)) \Big) \Bigg\} + \sum_{i=\underline{i}(t)}^{\infty} \frac{1}{(1-p_i)u(\theta_i)} f_i(x(\theta_i)) \Bigg).
\end{aligned}
$$

By using the hypotheses of the theorem it can be shown that each fixed point of the operator F is a solution of (3.37) and by means of the Banach fixed point theorem F has a fixed point. The procedure is quite similar to that of Theorem 3.1.3, and hence we skip the details. □

Theorem 3.1.8 *Suppose that* (3.4)-(3.6) *and* (3.24) *hold, where*

$$
\begin{aligned}
& \int_T^\infty \prod_{k=\underline{i}(T)}^{\bar{i}(s)} (1-p_k)^{-1} v(s) k(s) \mathrm{d}s \\
& + \sum_{i=\underline{i}(T)}^{\infty} \prod_{k=\underline{i}(T)}^{i} (1-p_k)^{-1} \left(|v'(\theta_i+)| k_i + v(\theta_i+) \tilde{k}_i \right) + \frac{k_i}{v(\theta_i+)} < \infty. \qquad (3.53)
\end{aligned}
$$

Suppose also that there exist $\beta \in C([t_0, \infty), [0, \infty))$ *such that*

$$
\begin{aligned}
& \frac{1}{v^2(t)} \Bigg\{ \int_t^\infty \prod_{k=\underline{i}(T)}^{\bar{i}(s)} (1-p_k)^{-1} v(s) (h_1(s) + h_2(s)) \mathrm{d}s \\
& + \quad \sum_{i=\underline{i}(t)}^{\infty} \prod_{k=\underline{i}(T)}^{i} (1-p_k)^{-1} \left(|v'(\theta_i+)| \bar{h}_i + v(\theta_i+) \tilde{h}_i \right) \Bigg\} \le \beta(t), \quad t \ge T, \qquad (3.54)
\end{aligned}
$$

where

$$
\int_T^t \beta(s) \mathrm{d}s + \sum_{i=\underline{i}(T)}^{\bar{i}(t)} \frac{\bar{h}_i}{v(\theta_i+)} = o((u(t))^\mu), \quad t \to \infty, \quad \mu \in (0,1). \qquad (3.55)
$$

Then, for any given $a, b \in \mathbb{R}$*, equation* (3.37) *has a unique solution* $x(t)$ *such that*

$$
x(t) = av(t) + bu(t) + o(v(t)(u(t))^\mu), \quad t \to \infty. \qquad (3.56)
$$

Proof. For $x \in X$, where X is defined by (3.52), we define the operator

$$
\begin{aligned}
(Fx)(t) &= av(t) + bu(t) \\
&- v(t)\Bigg(\int_t^\infty \frac{1}{v^2(s)}\Bigg\{\int_s^\infty \prod_{k=\underline{i}(s)}^{\bar{i}(r)} (1-p_k)^{-1} v(r) f(r, x(r)) \mathrm{d}r \\
&+ \sum_{i=\underline{i}(s)}^{\infty} \prod_{k=\underline{i}(s)}^{i} (1-p_k)^{-1}\Big((v'(\theta_i) - q_i v(\theta_i)) f_i(x(\theta_i)) - (1-p_i) v(\theta_i) \tilde{f}_i(x(\theta_i))\Big)\Bigg\} \\
&+ \sum_{i=\underline{i}(t)}^{\infty} \frac{1}{(1-p_i)v(\theta_i)} f_i(x(\theta_i))\Bigg).
\end{aligned}
$$

Applying the steps as in the proof of Theorem 3.1.3 one can complete the proof.
□

3.1.2 Existence of Positive Solutions

This section is devoted to existence of positive solutions with prescribed asymptotic behavior. First, we give a compactness criterion for the space $\mathrm{PLC}^1[0,\infty)$, which is an improvement of Lemmas 2.2.3 and 2.2.4.

Lemma 3.1.1 *Suppose that*

$$v(t) > 0,\ v'(t) > 0,\ \text{and } u'(t) \geq 0 \text{ for all } t \geq t_0 \text{ where } t_0 \geq 0. \tag{3.57}$$

Define

$$\Omega := \{w \in PLC^1[0,\infty) |\ \lim_{t\to\infty} w(t) \text{ and } \lim_{t\to\infty} \rho(t) \text{ exist}\},$$

where $\rho(t) = w(t) + w'(t)v(t)/v'(t)$. *Then* $S \subset \Omega$ *is compact if the following conditions hold:*

i) S *is bounded in* Ω, *i.e., there exists* $L > 0$ *such that* $\|w\| \leq L$ *and* $\|\rho\| \leq L$ *for all* $w \in S$;

ii) the functions belonging to S *are piecewise equicontinuous on any subinterval of* $[0,\infty)$, *i.e.,* $\forall \epsilon > 0\ \exists \delta_\epsilon > 0$ *such that*

$$|w(t_1) - w(t_2)| < \epsilon \text{ and } |\rho(t_1) - \rho(t_2)| < \epsilon \text{ whenever } |t_1 - t_2| < \delta_\epsilon$$

where $t_1, t_2 \in (\theta_k, \theta_{k+1}] \cap [0,\infty)$;

iii) the functions belonging to S are equiconvergent, i.e., given $\epsilon > 0$, there is $\tau(\epsilon) > 0$ such that

$$|w(t) - l_w| < \epsilon \text{ and } |\rho(t) - l_\rho| < \epsilon$$

for any $t \geq \tau(\epsilon)$ and $w \in S$, where $l_w = \lim_{t\to\infty} w(t)$ and $l_\rho = \lim_{t\to\infty} \rho(t)$.

Proof. Consider the space $V := \{f \in \mathrm{PLC}[t_0, \infty)| \lim_{t\to\infty} f(t) \text{ exists}\}$ and take a sequence $\{w_n\} \in S$. Clearly, $\{w_n\} \in V$. Then, by Lemma 2.2.2 there exists a subsequence $\{w_{n_k}\}$ in V such that $\lim_{k\to\infty} w_{n_k} = w \in V$. Let

$$\rho_n(t) := w_n(t) + \frac{v(t)}{v'(t)} w_n'(t) \in \Omega.$$

Clearly $\{\rho_n\} \in V$ so, again by Lemma 2.2.2, there exists a subsequence $\{\rho_{n_l}\} \in V$ such that $\lim_{l\to\infty} \rho_{n_l} = \rho \in V$. The next step is to show that $w \in \Omega$. For this purpose we define $\overline{w^i_{n_k}}(t)$ as

$$\overline{w^i_{n_k}}(t) := \begin{cases} w^i_{n_k}(t), & t \in (\theta_i, \theta_{i+1}], \\ w^i_{n_k}(\theta_i+), & t = \theta_i, \end{cases} \tag{3.58}$$

where $w^i_{n_k}(t)$ is continuous on $(\theta_i, \theta_{i+1}]$ and $w^i_{n_k}(\theta_i+)$ exists for all $i = 1, 2, \ldots$ Hence $\overline{w^i_{n_k}}(t) \in \mathrm{C}[\theta_i, \theta_{i+1}]$. Since $\mathrm{C}[\theta_i, \theta_{i+1}]$ is a compact space, there exists a function $\overline{w^i}(t) \in C[\theta_i, \theta_{i+1}]$ such that $\overline{w^i_{n_k}} \to \overline{w^i}$ in $\mathrm{C}[\theta_{i-1}, \theta_i]$ as $k \to \infty$. So $\overline{w^i}(t)$ is continuous for $t \in (\theta_i, \theta_{i+1}]$ and $\lim_{t\to\theta_i+} \overline{w^i}(t)$ exists. Thus, we may write

$$\overline{w^i}(t) = \begin{cases} w^i(t), & t \in (\theta_i, \theta_{i+1}], \\ w^i(\theta_i+), & t = \theta_i, \end{cases}$$

for all $i \geq 1$. Hence $w_{n_k} \to w$ as $k \to \infty$, and $w \in \mathrm{PLC}[t_0, \infty)$. On the other hand, if we define $\tilde{\rho}_{n_k} := v'\rho_{n_k}$ we obtain

$$\begin{aligned} \int_{t_0}^t \tilde{\rho}_{n_k}(s)\mathrm{d}s &= \int_{t_0}^t v'(s)\rho_{n_k}(s)\mathrm{d}s \\ &= \int_{t_0}^t (v(s)w_{n_k}(s))'\mathrm{d}s \\ &= \int_{t_0}^{\theta_1} (v(s)\overline{w^0_{n_k}}(s))'\mathrm{d}s + \int_{\theta_1}^{\theta_2} (v(s)\overline{w^1_{n_k}}(s))'\mathrm{d}s + \ldots \\ &\quad + \int_{\theta_k}^t (v(s)\overline{w^k_{n_k}}(s))'\mathrm{d}s \\ &= v(t)w_{n_k}(t) - v(t_0)w_{n_k}(t_0), \end{aligned}$$

where $t \in [t_0, t_*]$, for some $t_* \geq t_0$. Then,

$$\int_{t_0}^{t} \tilde{\rho}(s)\mathrm{d}s = \lim_{n\to\infty} \int_{t_0}^{t} \tilde{\rho}_{n_k}(s)\mathrm{d}s = v(t)w(t) - v(t_0)w(t_0), \quad t \in [t_0, t^*],$$

where $\tilde{\rho} = v'\rho$. This shows that $w \in \mathrm{PLC}^1([t_0,\infty),\mathbb{R})$ and so, by the fundamental theorem of calculus we have $\tilde{\rho}(t) = (v(t)w(t))'$ which implies that

$$\rho(t) = w(t) + \frac{v(t)}{v'(t)}w'(t).$$

Therefore, $w \in \Omega$, and hence S is relatively compact in Ω. $\square$

In the next theorem we state sufficient conditions for existence of positive solutions and asymptotic representation of solutions of the impulsive differential equation with separated impulse conditions

$$\begin{cases} x'' + q(t)x = f(t,x,x'), & t \neq \theta_i, \\ \Delta x + p_i x = f_i(x,x'), & t = \theta_i, \\ \Delta x' + \tilde{q}_i x' = \tilde{f}_i(x,x'), & t = \theta_i, \end{cases} \tag{3.59}$$

where $f \in \mathrm{C}([t_0,\infty)\times\mathbb{R}\times\mathbb{R},\mathbb{R})$, $f_i, \tilde{f}_i \in \mathrm{C}(\mathbb{R}\times\mathbb{R},\mathbb{R})$, $t_0 > 0$.

Theorem 3.1.9 *Assume that* (3.57) *holds and*

$$\begin{cases} |f(t,x,x')| \leq F(t,|x|,|x'|), & t \geq t_0 \\ |f_i(x,x')| \leq F_i(|x|,|x'|), & \\ |\tilde{f}_i(x,x')| \leq \tilde{F}_i(|x|,|x'|), & \end{cases} \tag{3.60}$$

where $F \in C([t_0,\infty)\times[0,\infty)\times[0,\infty),[0,\infty))$ is nondecreasing in its second and third variables for each fixed t, and $F_i, \tilde{F}_i \in C([0,\infty)\times[0,\infty),[0,\infty))$ are nondecreasing in both variables. Let $u(t)$ and $v(t)$ be principal and nonprincipal solutions of equation (3.3)*, respectively. If, for $c > 0$*

$$\int_{t_0}^{\infty} u(t)F(t,2cv(t),2cv'(t))\mathrm{d}t + \sum_{i=\underline{i}(t_0)}^{\infty} \big(u(\theta_i+)\tilde{F}_i(2cv(\theta_i),2cv'(\theta_i)) + u'(\theta_i+)F_i(2cv(\theta_i),2cv'(\theta_i))\big) < c \tag{3.61}$$

and

$$\lim_{t\to\infty}\frac{u'(t)}{v'(t)}\left\{\int_{t_0}^{t} v(s)F(s,2cv(s),2cv'(s))\mathrm{d}s\right.$$
$$\left.+\sum_{i=\underline{i}(t_0)}^{\bar{i}(t)}\left\{v(\theta_i+)\tilde{F}_i(2cv(\theta_i),2cv'(\theta_i))+v'(\theta_i+)F_i(2cv(\theta_i),2cv'(\theta_i))\right\}\right\}=0, \tag{3.62}$$

then, equation (3.59) *has a solution* $x(t)$ *such that*

$$\begin{cases} x(0)=0, \\ x(t)>0 \text{ for } t>0, \\ x'(t)>0 \text{ for } t\neq\theta_i, \quad i=1,2,\ldots \\ \lim\limits_{t\to\infty}\dfrac{x(t)}{v(t)}=c. \end{cases} \tag{3.63}$$

Proof. By the help of the conditions (3.60) and (3.61) there exists $\delta\in(0,c)$, $c>0$ such that

$$\int_{t_0}^{\infty} u(t)F(t,(2c-\delta)v(t),(2c-\delta)v'(t))\mathrm{d}t$$
$$+\sum_{i=\underline{i}(t_0)}^{\infty}\left\{u(\theta_i+)\tilde{F}_i(2cv(\theta_i),2cv'(\theta_i))+u'(\theta_i+)F_i(2cv(\theta_i),2cv'(\theta_i))\right\}\le c-\delta. \tag{3.64}$$

Consider the vector space

$$\Omega=\{y\in \mathrm{PLC}^1[t_0,\infty)|\ \lim_{t\to\infty}y(t) \text{ and } \lim_{t\to\infty}z(t) \text{ exist}\},$$

where $z(t)=y(t)+\frac{v(t)}{v'(t)}y'(t)$. Then, Ω is a Banach space with the norm

$$\|y\|=\max\left\{\sup_{t\ge t_0}|y(t)|,\sup_{t\ge t_0}|z(t)|\right\}.$$

The set $S=\{y\in\Omega|\ \delta\le y(t)\le 2c-\delta,\ \delta\le z(t)\le 2c-\delta\}$ is closed, bounded and convex. For simplicity, let us denote $g(s):=f(s,y(s)v(s),(y(s)v(s))')$, $\bar{g}(\theta_i):=f_i(y(\theta_i)v(\theta_i),(y(\theta_i)v(\theta_i))')$ and $\tilde{g}(\theta_i):=\tilde{f}_i(y(\theta_i)v(\theta_i),(y(\theta_i)v(\theta_i))')$, and define the operator T on the set S by

$$(Ty)(t) = c - \frac{u(t)}{v(t)} \left\{ \int_{t_0}^{t} v(s)g(s)\mathrm{d}s + \sum_{i=\underline{i}(t_0)}^{\bar{i}(t)} \left\{ v(\theta_i+)\tilde{g}(\theta_i) + v'(\theta_i+)\bar{g}(\theta_i) \right\} \right\}$$
$$- \left\{ \int_{t}^{\infty} u(s)g(s)\mathrm{d}s + \sum_{i=\underline{i}(t)}^{\infty} \left\{ u(\theta_i+)\tilde{g}(\theta_i) + u'(\theta_i+)\bar{g}(\theta_i) \right\} \right\}. \quad (3.65)$$

Let y be a fixed point of the operator (3.65), then in a similar way to the proof of Theorem 3.1.2 it can be shown that $y(t)/v(t)$ is a solution of equation (3.59). We will use Schauder fixed point theorem in order to show that T has a fixed point in S. For this purpose we first show that T maps S onto S, then we prove that T is completely continuous.

(i) *$T(S) \subset S$:*

Pick any $t_1 \in [t_0, \infty)$, and suppose that $t \neq \theta_l$, $l = 1, 2, \ldots$ with $t < t_1$. Then

$$|(Ty)(t) - (Ty)(t_1)| \leq \left| \frac{u(t)}{v(t)} - \frac{u(t_1)}{v(t_1)} \right|$$
$$\times \left\{ \int_{t_0}^{t} v(s)|g(s)|\mathrm{d}s + \sum_{i=\underline{i}(t_0)}^{\bar{i}(t)} \left\{ v(\theta_i+)|\tilde{g}(\theta_i)| + v'(\theta_i+)|\bar{g}(\theta_i)| \right\} \right\}$$
$$+ \frac{u(t_1)}{v(t_1)} \left\{ \int_{t}^{t_1} v(s)|g(s)|\mathrm{d}s + \sum_{i=\underline{i}(t)}^{\bar{i}(t_1)} \left\{ v(\theta_i+)|\tilde{g}(\theta_i)| + v'(\theta_i+)|\bar{g}(\theta_i)| \right\} \right\}$$
$$+ \int_{t}^{t_1} u(s)|g(s)|\mathrm{d}s + \sum_{i=\underline{i}(t)}^{\bar{i}(t_1)} \left\{ u(\theta_i+)|\tilde{g}(\theta_i)| + u'(\theta_i+)|\bar{g}(\theta_i)| \right\}.$$

Taking limit as $t \to t_1^-$ we obtain $(Ty)(t) \to (Ty)(t_1-)$ for all $t, t_1 \in [t_0, \infty)$. In a similar manner it can be obtained that $\lim_{t \to t_1+}(Ty)(t) = (Ty)(t_1+)$ for all $t, t_1 \in [t_0, \infty)$ for which $t \neq \theta_l$, and $\lim_{t \to \theta_l+}(Ty)(t)$ exist. On the other hand, for $t \neq \theta_l$, $l = 1, 2, \ldots$

$$(Ty)'(t) = -\left(\frac{u(t)}{v(t)} \right)' \left\{ \int_{t_0}^{t} v(s)g(s)\mathrm{d}s + \sum_{i=\underline{i}(t_0)}^{\bar{i}(t)} \left\{ v(\theta_i+)\tilde{g}(\theta_i) + v'(\theta_i+)\bar{g}(\theta_i) \right\} \right\}.$$

For $t < t_1$ one can write

$$
\begin{aligned}
|(Ty)'(t)-(Ty)'(t_1)| \le & \left|\frac{W(v(t),u(t))}{v^2(t)}-\frac{W(v(t_1),u(t_1))}{v^2(t_1)}\right| \\
& \times \left\{\int_{t_0}^{t} v(s)g(s)\mathrm{d}s + \sum_{i=\underline{i}(t_0)}^{\bar{i}(t)} \left\{v(\theta_i+)|\tilde{g}(\theta_i)| + v'(\theta_i+)|\bar{g}(\theta_i)|\right\}\right\} \\
& + \frac{W(v(t_1),u(t_1))}{v^2(t_1)}\left\{\int_{t}^{t_1} v(s)g(s)\mathrm{d}s + \sum_{i=\underline{i}(t)}^{\bar{i}(t_1)} \left\{v(\theta_i+)|\tilde{g}(\theta_i)| + v'(\theta_i+)|\bar{g}(\theta_i)|\right\}\right\} \\
= & \frac{1}{v^2(t_1)}\left\{\int_{t}^{t_1} v(s)g(s)\mathrm{d}s + \sum_{i=\underline{i}(t)}^{\bar{i}(t_1)} \left\{v(\theta_i+)|\tilde{g}(\theta_i)| + v'(\theta_i+)|\bar{g}(\theta_i)|\right\}\right\},
\end{aligned}
$$

where $W(v,u)$ is the Wronskian determinant. This follows that

$$\lim_{t\to t_1-}(Ty)'(t) = (Ty)'(t_1).$$

In a similar way, it can be shown that

$$\lim_{t\to t_1+}(Ty)'(t) = (Ty)'(t_1) \text{ for } t \neq \theta_l$$

and

$$\lim_{t\to \theta_l+}(Ty)'(t) \text{ exist for all } l = 1, 2, \ldots$$

Hence $(Ty)(t) \in \mathrm{PLC}^1[t_0,\infty)$.

By using (2.9) and (3.65) we have

$$
\begin{aligned}
& |(Ty)(t)-c| \\
& \quad \le \frac{u(t)}{v(t)}\left\{\int_{t_0}^{t} v(s)|g(s)|\mathrm{d}s + \sum_{i=\underline{i}(t_0)}^{\bar{i}(t)} \left\{v(\theta_i+)|\tilde{g}(\theta_i)| + v'(\theta_i+)|\bar{g}(\theta_i)|\right\}\right\} \\
& \quad + \int_{t}^{\infty} u(s)|g(s)|\mathrm{d}s + \sum_{i=\underline{i}(t)}^{\infty} \left\{u(\theta_i+)|\tilde{g}(\theta_i)| + u'(\theta_i+)|\bar{g}(\theta_i)|\right\} \\
& \quad = \int_{t}^{\infty} \frac{1}{v^2(r)}\mathrm{d}r \left\{\int_{t_0}^{t} v(s)|g(s)|\mathrm{d}s + \sum_{i=\underline{i}(t_0)}^{\bar{i}(t)} \left\{v(\theta_i+)|\tilde{g}(\theta_i)| + v'(\theta_i+)|\bar{g}(\theta_i)|\right\}\right\} \\
& \quad + \int_{t}^{\infty} u(s)|g(s)|\mathrm{d}s + \sum_{i=\underline{i}(t)}^{\infty} \left\{u(\theta_i+)|\tilde{g}(\theta_i)| + u'(\theta_i+)|\bar{g}(\theta_i)|\right\}.
\end{aligned}
$$

Then, from (3.60) and (3.64),

$$
\begin{aligned}
|(Ty)(t)-c| \leq & \int_{t_0}^{t} u(s)|g(s)|\mathrm{d}s + \sum_{i=\underline{i}(t_0)}^{\bar{i}(t)} \left\{u(\theta_i+)|\tilde{g}(\theta_i)| + u'(\theta_i+)|\bar{g}(\theta_i)|\right\} \\
& + \int_{t}^{\infty} u(s)|g(s)|\mathrm{d}s + \sum_{i=\underline{i}(t)}^{\infty} \left\{u(\theta_i+)|\tilde{g}(\theta_i)| + u'(\theta_i+)|\bar{g}(\theta_i)|\right\} \\
= & \int_{t_0}^{\infty} u(s)|g(s)|\mathrm{d}s + \sum_{i=\underline{i}(t_0)}^{\infty} \left\{u(\theta_i+)|\tilde{g}(\theta_i)| + u'(\theta_i+)|\bar{g}(\theta_i)|\right\} \\
\leq & \int_{t_0}^{\infty} G(s)u(s)\mathrm{d}s + \sum_{i=\underline{i}(t_0)}^{\infty} \left\{u(\theta_i+)\tilde{G}_i + u'(\theta_i+)\bar{G}_i\right\} \\
\leq & c-\delta,
\end{aligned}
$$

where $G(s) := F(s,(2c-\delta)v(s),(2c-\delta)v'(s))$, $\tilde{G}_i := \tilde{F}_i((2c-\delta)v(\theta_i),(2c-\delta)v'(\theta_i))$ and $\bar{G}_i := F_i((2c-\delta)v(\theta_i),(2c-\delta)v'(\theta_i))$. Introduce the operator

$$
(Fy)(t) = (Ty)(t) + \frac{v(t)}{v'(t)}(Ty)'(t).
$$

With the help of property (2.8) and hypothesis (3.57) one can see that

$$
\begin{aligned}
|(Fy)(t)-c| \leq & \frac{u'(t)}{v'(t)} \left\{ \int_{t_0}^{t} v(s)|g(s)|\mathrm{d}s + \sum_{i=\underline{i}(t_0)}^{\bar{i}(t)} \left\{v(\theta_i+)|\tilde{g}(\theta_i)| + v'(\theta_i+)|\bar{g}(\theta_i)|\right\} \right\} \\
& + \int_{t}^{\infty} u(s)|g(s)|\mathrm{d}s + \sum_{i=\underline{i}(t)}^{\infty} \left\{u(\theta_i+)|\tilde{g}(\theta_i)| + u'(\theta_i+)|\bar{g}(\theta_i)|\right\} \\
\leq & \frac{u(t)}{v(t)} \left\{ \int_{t_0}^{t} v(s)|g(s)|\mathrm{d}s + \sum_{i=\underline{i}(t_0)}^{\bar{i}(t)} \left\{v(\theta_i+)|\tilde{g}(\theta_i)| + v'(\theta_i+)|\bar{g}(\theta_i)|\right\} \right\} \\
& + \int_{t}^{\infty} u(s)|g(s)|\mathrm{d}s + \sum_{i=\underline{i}(t)}^{\infty} \left\{u(\theta_i+)|\tilde{g}(\theta_i)| + u'(\theta_i+)|\bar{g}(\theta_i)|\right\} \\
\leq & c-\delta.
\end{aligned}
$$

(ii) *T is continuous:*

Take $y_n \in S$ such that $y_n \to y$ as $n \to \infty$. Then, for the sake of brevity, denote $g_n(s) = f(s, y_n(s)v(s), (y_n(s)v(s))')$, $\bar{g}_n(\theta_i) = \bar{f}_i(y_n(\theta_i)v(\theta_i), (y_n(\theta_i)v(\theta_i))')$, $\tilde{g}_n(\theta_i) = \tilde{f}_i(y_n(\theta_i)v(\theta_i), (y_n(\theta_i)v(\theta_i))')$. By means of (3.60), (3.61) and (3.64) we obtain

$$
\begin{aligned}
&|(Ty_n)(t)-(Ty)(t)| \\
&\quad\le \frac{u(t)}{v(t)}\Bigg\{\int_{t_0}^{t} v(s)|g_n(s)-g(s)|\mathrm{d}s \\
&\qquad + \sum_{i=\underline{i}(t_0)}^{\bar{i}(t)} \big\{v(\theta_i+)|\tilde{g}_n(\theta_i)-\tilde{g}(\theta_i)| + v'(\theta_i+)|\bar{g}_n(\theta_i)-\bar{g}(\theta_i)|\big\}\Bigg\} \\
&\qquad + \int_{t}^{\infty} u(s)|g_n(s)-g(s)|\mathrm{d}s \\
&\qquad + \sum_{i=\underline{i}(t)}^{\infty} \big\{v(\theta_i+)|\tilde{g}_n(\theta_i)-\tilde{g}(\theta_i)| + v'(\theta_i+)|\bar{g}_n(\theta_i)-\bar{g}(\theta_i)|\big\} \\
&\quad\le 2\frac{u(t)}{v(t)}\Bigg\{\int_{t_0}^{t} v(s)G(s)\mathrm{d}s + \sum_{i=\underline{i}(t_0)}^{\bar{i}(t)} \big\{v(\theta_i+)\tilde{G}_i + v'(\theta_i+)\bar{G}_i\big\}\Bigg\} \\
&\qquad + 2\Bigg\{\int_{t}^{\infty} u(s)G(s)\mathrm{d}s + \sum_{i=\underline{i}(t)}^{\infty} \big\{u(\theta_i+)\tilde{G}_i + u'(\theta_i+)\bar{G}_i\big\}\Bigg\} \\
&\quad\le 2(c-\delta).
\end{aligned}
$$

Therefore, applying Lebesgue dominated convergence theorem and Weierstrass-M test we get

$$
\lim_{n\to\infty} |(Ty_n)(t)-(Ty)(t)| = 0. \tag{3.66}
$$

On the other hand,

$$
\begin{aligned}
&|(Fy_n)(t)-(Fy)(t)| \\
&\quad\le \frac{u'(t)}{v'(t)}\Bigg\{\int_{t_0}^{t} v(s)|g_n(s)-g(s)|\mathrm{d}s \\
&\qquad + \sum_{i=\underline{i}(t_0)}^{\bar{i}(t)} \big\{v(\theta_i+)|\tilde{g}_n(\theta_i)-\tilde{g}(\theta_i)| + v'(\theta_i+)|\bar{g}_n(\theta_i)-\bar{g}(\theta_i)|\big\}\Bigg\} \\
&\qquad + \int_{t}^{\infty} u(s)|g_n(s)-g(s)|\mathrm{d}s \\
&\qquad + \sum_{i=\underline{i}(t)}^{\infty} \big\{u(\theta_i+)|\tilde{g}_n(\theta_i)-\tilde{g}(\theta_i)| + u'(\theta_i+)|\bar{g}_n(\theta_i)-\bar{g}(\theta_i)|\big\}.
\end{aligned}
$$

Then, with the help of the property (2.8) and the condition (3.60) we have

$$\begin{aligned}
&|(Fy_n)(t)-(Fy)(t)| \\
&\leq 2\frac{u(t)}{v(t)}\left\{\int_{t_0}^{t} v(s)G(s)\mathrm{d}s + \sum_{i=\underline{i}(t_0)}^{\bar{i}(t)} \{v(\theta_i+)\tilde{G}_i + v'(\theta_i+)\bar{G}_i\}\right\} \\
&\quad + 2\left\{\int_{t}^{\infty} u(s)G(s)\mathrm{d}s + \sum_{i=\underline{i}(t)}^{\infty} \{u(\theta_i+)\tilde{G}_i + u'(\theta_i+)\bar{G}_i\}\right\} \\
&\leq 2(c-\delta),
\end{aligned}$$

which implies that

$$\lim_{n\to\infty} |(Fy_n)(t)-(Fy)(t)| = 0. \tag{3.67}$$

From (3.66) and (3.67) we conclude that

$$\lim_{n\to\infty} \|(Ty_n)-(Ty)\| = 0.$$

(iii) *T is relatively compact:*

We will make use of Lemma 3.1.1.

(1) $T(S)$ is bounded in Ω:

We previously proved that $|(Ty)(t)| \leq 2c-\delta$ and $|(Fy)(t)| \leq 2c-\delta$, for all $t \geq t_0$ and all $y \in S$. Therefore $T(S)$ is uniformly bounded.

(2) $T(S)$ is piecewise equicontinuous:

Fix $\epsilon > 0$. Then, with the help of the properties (2.6)-(2.8) and the condition (3.61), there exist t_ϵ, t_ϵ^1, t_ϵ^2 with $t_\epsilon^2 \geq t_\epsilon^1 \geq t_\epsilon \geq t_0$ such that

$$\int_{t_\epsilon}^{\infty} u(s)G(s)\mathrm{d}s + \sum_{i=\underline{i}(t_\epsilon)}^{\infty} \{u(\theta_i+)\tilde{G}_i + u'(\theta_i+)\bar{G}_i\} < \frac{\epsilon}{6}; \tag{3.68}$$

$$\begin{aligned}
&\int_{t_0}^{t_\epsilon} v(s)G(s)\int_{t_\epsilon^1}^{\infty} \frac{1}{v^2(r)}\mathrm{d}r\mathrm{d}s \\
&\quad + \sum_{i=\underline{i}(t_0)}^{\bar{i}(t_\epsilon)} \{v(\theta_i+)\tilde{G}_i + v'(\theta_i+)\bar{G}_i\}\int_{t_\epsilon^1}^{\infty} \frac{1}{v^2(s)}\mathrm{d}s < \frac{\epsilon}{2}
\end{aligned} \tag{3.69}$$

and

$$\frac{u'(t)}{v'(t)}\left\{\int_{t_0}^{t_\epsilon} v(s)G(s)\mathrm{d}s+\sum_{i=\underline{i}(t_0)}^{\bar{i}(t_\epsilon)}\left\{v(\theta_i+)\tilde{G}_i+v'(\theta_i+)\bar{G}_i\right\}\right\}<\frac{\epsilon}{6},\ t\geq t_\epsilon^2. \tag{3.70}$$

From (2.9) one has

$$u(s)-\frac{u(t)}{v(t)}v(s)=v(s)\int_s^t\frac{1}{v^2(r)}\mathrm{d}r,$$

so the operator (3.65) can be rewritten as

$$\begin{aligned}(Ty)(t)=c-&\left\{\int_{t_0}^{\infty}u(s)g(s)\mathrm{d}s+\sum_{i=\underline{i}(t_0)}^{\infty}\left\{u(\theta_i+)\tilde{g}(\theta_i)+u'(\theta_i+)\bar{g}(\theta_i)\right\}\right\}\\&+\int_{t_0}^{t}v(s)g(s)\int_s^t\frac{1}{v^2(r)}\mathrm{d}r\mathrm{d}s\\&+\sum_{i=\underline{i}(t_0)}^{\bar{i}(t)}\left\{v(\theta_i+)\tilde{g}(\theta_i)+v'(\theta_i+)\bar{g}(\theta_i)\right\}\int_{\theta_i}^{t}\frac{1}{v^2(s)}\mathrm{d}s-\frac{\bar{g}(\theta_i)}{v(\theta_i+)}.\end{aligned}$$

Let $t_1,t_2\in(\theta_k,\theta_{k+1}]\subset[t_0,\nu]$, where $\nu\in(t_0,\infty)$ is arbitrary. Without any loss of generality, set $t_1<t_2$. Then,

$$\begin{aligned}|(Ty)(t_1)-(Ty)(t_2)|=&\Bigg|-\int_{t_0}^{t_1}v(s)g(s)\int_{t_1}^{t_2}\frac{1}{v^2(r)}\mathrm{d}r\mathrm{d}s\\&-\sum_{i=\underline{i}(t_0)}^{\bar{i}(t_1)}\left\{v(\theta_i+)\tilde{g}(\theta_i)+v'(\theta_i+)\bar{g}(\theta_i)\right\}\int_{t_1}^{t_2}\frac{1}{v^2(s)}\mathrm{d}s\\&-\int_{t_1}^{t_2}v(s)g(s)\int_s^{t_2}\frac{1}{v^2(r)}\mathrm{d}r\mathrm{d}s\Bigg|\\\leq&\int_{t_0}^{t_1}v(s)G(s)\int_{t_1}^{t_2}\frac{1}{v^2(r)}\mathrm{d}r\mathrm{d}s\\&+\sum_{i=\underline{i}(t_0)}^{\bar{i}(t_1)}\left\{v(\theta_i+)\tilde{G}_i+v'(\theta_i+)\bar{G}_i\right\}\int_{t_1}^{t_2}\frac{1}{v^2(s)}\mathrm{d}s\\&+\int_{t_1}^{t_2}v(s)G(s)\int_s^{t_2}\frac{1}{v^2(r)}\mathrm{d}r\mathrm{d}s.\end{aligned}$$

Define

$$M_1 := \max_{t_0 \le s \le t_\epsilon} \frac{1}{v^2(s)}, \quad M_2 := \max_{t_\epsilon \le s \le t_\epsilon^1} \frac{1}{v^2(s)}, \quad M_3 := \max_{t_0 \le s \le t_\epsilon} G(s)u(s).$$

If $t_0 \le t_1 \le t_2 \le t_\epsilon$, then

$$\begin{aligned}\int_{t_0}^{t_1} v(s)G(s) \int_{t_1}^{t_2} \frac{1}{v^2(r)} \mathrm{d}r\mathrm{d}s &\le \int_{t_0}^{t_\epsilon} v(s)G(s) \int_{t_1}^{t_2} \frac{1}{v^2(r)} \mathrm{d}r\mathrm{d}s \\ &\le M_1 |t_1 - t_2| \int_{t_0}^{t_\epsilon} v(s)G(s)\mathrm{d}s. \end{aligned} \tag{3.71}$$

Similarly,

$$\begin{aligned}&\sum_{i=\underline{i}(t_0)}^{\bar{i}(t_1)} \left\{ v(\theta_i+)\tilde{G}_i + v'(\theta_i+)\bar{G}_i \right\} \int_{t_1}^{t_2} \frac{1}{v^2(s)} \mathrm{d}s \\ &\qquad \le M_1 |t_1 - t_2| \sum_{i=\underline{i}(t_0)}^{\bar{i}(t_\epsilon)} \left\{ v(\theta_i+)\tilde{G}_i + v'(\theta_i+)\bar{G}_i \right\} \end{aligned} \tag{3.72}$$

and

$$\begin{aligned}\int_{t_1}^{t_2} v(s)G(s) \int_{s}^{t_2} \frac{1}{v^2(r)} \mathrm{d}r\mathrm{d}s &\le \int_{t_1}^{t_2} v(s)G(s) \int_{s}^{\infty} \frac{1}{v^2(r)} \mathrm{d}r\mathrm{d}s \\ &= \int_{t_1}^{t_2} u(s)G(s)\mathrm{d}s \le M_3 |t_1 - t_2|. \end{aligned} \tag{3.73}$$

If $t_\epsilon \le t_1 \le t_2 \le t_\epsilon^1$, then

$$\begin{aligned}&\int_{t_0}^{t_1} v(s)G(s) \int_{t_1}^{t_2} \frac{1}{v^2(r)} \mathrm{d}r\mathrm{d}s \\ &\qquad \le \int_{t_0}^{t_\epsilon} v(s)G(s) \int_{t_1}^{t_2} \frac{1}{v^2(r)} \mathrm{d}r\mathrm{d}s + \int_{t_\epsilon}^{t_1} v(s)G(s) \int_{t_1}^{t_2} \frac{1}{v^2(r)} \mathrm{d}r\mathrm{d}s \\ &\qquad \le M_2 |t_1 - t_2| \int_{t_0}^{t_\epsilon} v(s)G(s)\mathrm{d}s + \int_{t_\epsilon}^{\infty} u(s)G(s)\mathrm{d}s; \end{aligned} \tag{3.74}$$

$$
\begin{aligned}
\sum_{i=\underline{i}(t_0)}^{\bar{i}(t_1)} & \big\{v(\theta_i+)\tilde{G}_i + v'(\theta_i+)\bar{G}_i\big\} \int_{t_1}^{t_2} \frac{1}{v^2(s)} \mathrm{d}s \\
& \le M_2|t_1 - t_2| \sum_{i=\underline{i}(t_0)}^{\bar{i}(t_\epsilon)} \big\{v(\theta_i+)\tilde{G}_i + v'(\theta_i+)\bar{G}_i\big\} \\
& \quad + \sum_{i=\underline{i}(t_\epsilon)}^{\infty} \big\{u(\theta_i+)\tilde{G}_i + u'(\theta_i+)\bar{G}_i\big\}(\theta_i); \qquad (3.75)
\end{aligned}
$$

and

$$
\begin{aligned}
\int_{t_1}^{t_2} & v(s)G(s) \int_{s}^{t_2} \frac{1}{v^2(r)} \mathrm{d}r\mathrm{d}s \\
& \le \int_{t_1}^{t_2} v(s)G(s) \int_{s}^{\infty} \frac{1}{v^2(r)} \mathrm{d}r\mathrm{d}s \le \int_{t_\epsilon}^{\infty} u(s)G(s)\mathrm{d}s. \qquad (3.76)
\end{aligned}
$$

If $t_\epsilon^1 \le t_1 \le t_2$, then

$$
\begin{aligned}
\int_{t_0}^{t_1} & v(s)G(s) \int_{t_1}^{t_2} \frac{1}{v^2(r)} \mathrm{d}r\mathrm{d}s \\
& \le \int_{t_0}^{t_\epsilon} v(s)G(s) \int_{t_1}^{t_2} \frac{1}{v^2(r)} \mathrm{d}r\mathrm{d}s + \int_{t_\epsilon}^{t_1} v(s)G(s) \int_{t_1}^{t_2} \frac{1}{v^2(r)} \mathrm{d}r\mathrm{d}s \\
& \le \int_{t_0}^{t_\epsilon} v(s)G(s) \int_{t_\epsilon^1}^{\infty} \frac{1}{v^2(r)} \mathrm{d}r\mathrm{d}s + \int_{t_\epsilon}^{\infty} u(s)G(s)\mathrm{d}s. \qquad (3.77)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\sum_{i=\underline{i}(t_0)}^{\bar{i}(t_1)} & \big\{v(\theta_i+)\tilde{G}_i + v'(\theta_i+)\bar{G}_i\big\} \int_{t_1}^{t_2} \frac{1}{v^2(s)} \mathrm{d}s \\
& \le \sum_{i=\underline{i}(t_0)}^{\bar{i}(t_\epsilon)} \big\{v(\theta_i+)\tilde{G}_i + v'(\theta_i+)\bar{G}_i\big\} \int_{t_\epsilon^1}^{\infty} \frac{1}{v^2(s)} \mathrm{d}s \\
& \quad + \sum_{i=\underline{i}(t_\epsilon)}^{\infty} \big\{u(\theta_i+)\tilde{G}_i + u'(\theta_i+)\bar{G}_i\big\} \qquad (3.78)
\end{aligned}
$$

and

$$\int_{t_1}^{t_2} v(s)G(s)\int_{s}^{t_2}\frac{1}{v^2(r)}\mathrm{d}r\mathrm{d}s \le \int_{t_1}^{t_2} v(s)G(s)\int_{s}^{\infty}\frac{1}{v^2(r)}\mathrm{d}r\mathrm{d}s$$
$$\le \int_{t_\epsilon}^{\infty} u(s)G(s)\mathrm{d}s. \tag{3.79}$$

Now, by using the property (2.8), and hypotheses (3.57) and (3.60) we obtain

$$\begin{aligned}
&|(Fy)(t_1)-(Fy)(t_2)|\\
&=\Bigg|-\frac{u'(t_1)}{v'(t_1)}\Bigg\{\int_{t_0}^{t_1} v(s)g(s)\mathrm{d}s+\sum_{i=\underline{i}(t_0)}^{\bar{i}(t_1)}\big\{v(\theta_i+)\tilde{g}(\theta_i)+v'(\theta_i+)\bar{g}(\theta_i)\big\}\Bigg\}\\
&\quad+\frac{u'(t_2)}{v'(t_2)}\Bigg\{\int_{t_0}^{t_2} v(s)g(s)\mathrm{d}s+\sum_{i=\underline{i}(t_0)}^{\bar{i}(t_2)}\big\{v(\theta_i+)\tilde{g}(\theta_i)+v'(\theta_i+)\bar{g}(\theta_i)\big\}\Bigg\}\\
&\quad-\int_{t_1}^{t_2} u(s)g(s)\mathrm{d}s\Bigg|\\
&\le\left|\frac{u'(t_1)}{v'(t_1)}-\frac{u'(t_2)}{v'(t_2)}\right|\Bigg\{\int_{t_0}^{t_1} v(s)|g(s)|\mathrm{d}s\\
&\quad+\sum_{i=\underline{i}(t_0)}^{\bar{i}(t_1)}\big\{v(\theta_i+)|\tilde{g}(\theta_i)|+v'(\theta_i+)|\bar{g}(\theta_i)|\big\}\Bigg\}\\
&\quad+\frac{u'(t_2)}{v'(t_2)}\int_{t_1}^{t_2} v(s)|g(s)|\mathrm{d}s+\int_{t_1}^{t_2} u(s)|g(s)|\mathrm{d}s\\
&\le\left|\frac{u'(t_1)}{v'(t_1)}-\frac{u'(t_2)}{v'(t_2)}\right|\Bigg\{\int_{t_0}^{t_1} v(s)g(s)\mathrm{d}s\\
&\quad+\sum_{i=\underline{i}(t_0)}^{\bar{i}(t_1)}\big\{v(\theta_i+)\tilde{G}_i+v'(\theta_i+)\bar{G}_i\big\}\Bigg\}+2\int_{t_1}^{t_2} u(s)G(s)\mathrm{d}s.
\end{aligned}$$

If $t_0 \le t_1 \le t_2 \le t_\epsilon$, then

$$\int_{t_0}^{t_1} v(s)G(s)\mathrm{d}s+\sum_{i=\underline{i}(t_0)}^{\bar{i}(t_1)}\big\{v(\theta_i+)\tilde{G}_i+v'(\theta_i+)\bar{G}_i\big\}$$
$$\le\int_{t_0}^{t_\epsilon} v(s)G(s)\mathrm{d}s+\sum_{i=\underline{i}(t_0)}^{\bar{i}(t_\epsilon)}\big\{v(\theta_i+)\tilde{G}_i+v'(\theta_i+)\bar{G}_i\big\} \tag{3.80}$$

and

$$\int_{t_1}^{t_2} u(s)G(s)\mathrm{d}s \leq M_3|t_1 - t_2|. \tag{3.81}$$

Moreover u'/v' is continuous, so there exists $\delta_1 > 0$ such that $|t_1 - t_2| < \delta_1$ implies that

$$\left|\frac{u'(t_1)}{v'(t_1)} - \frac{u'(t_2)}{v'(t_2)}\right| \left\{ \int_{t_0}^{t_\epsilon} v(s)G(s)\mathrm{d}s + \sum_{i=\underline{i}(t_0)}^{\bar{i}(t_\epsilon)} v(\theta_i)\tilde{G}_i \right\} < \frac{\epsilon}{3}. \tag{3.82}$$

If $t_\epsilon \leq t_1 \leq t_2 \leq t_\epsilon^2$, then

$$\begin{aligned}
&\left|\frac{u'(t_1)}{v'(t_1)} - \frac{u'(t_2)}{v'(t_2)}\right| \left\{ \int_{t_0}^{t_1} v(s)G(s)\mathrm{d}s + \sum_{i=\underline{i}(t_0)}^{\bar{i}(t_1)} \left\{ v(\theta_i+)\tilde{G}_i + v'(\theta_i+)\bar{G}_i \right\} \right\} \\
&\quad \leq \int_{t_0}^{t_\epsilon} v(s)G(s)\mathrm{d}s + \sum_{i=\underline{i}(t_0)}^{\bar{i}(t_\epsilon)} \left\{ v(\theta_i+)\tilde{G}_i + v'(\theta_i+)\bar{G}_i \right\} \\
&\qquad + 2\max\left\{ \frac{u'(t_1)}{v'(t_1)}, \frac{u'(t_2)}{v'(t_2)} \right\} \left\{ \int_{t_\epsilon}^{t_1} v(s)G(s)\mathrm{d}s \right. \\
&\qquad \left. + \sum_{i=\underline{i}(t_\epsilon)}^{\bar{i}(t_1)} \left\{ v(\theta_i+)\tilde{G}_i + v'(\theta_i+)\bar{G}_i \right\} \right\} \\
&\quad \leq \left|\frac{u'(t_1)}{v'(t_1)} - \frac{u'(t_2)}{v'(t_2)}\right| \int_{t_0}^{t_\epsilon} v(s)G(s)\mathrm{d}s \\
&\qquad + 2\left\{ \int_{t_\epsilon}^{\infty} u(s)G(s)\mathrm{d}s + \sum_{i=\underline{i}(t_\epsilon)}^{\infty} \left\{ u(\theta_i+)\tilde{G}_i + u'(\theta_i+)\bar{G}_i \right\} \right\}
\end{aligned} \tag{3.83}$$

and

$$\int_{t_1}^{t_2} u(s)G(s)\mathrm{d}s \leq \int_{t_\epsilon}^{\infty} u(s)G(s)\mathrm{d}s. \tag{3.84}$$

Again from continuity of u'/v', there exists $\delta_2 > 0$ such that $|t_1 - t_2| < \delta_2$ implies that

$$\begin{aligned}
&\left|\frac{u'(t_1)}{v'(t_1)} - \frac{u'(t_2)}{v'(t_2)}\right| \left\{ \int_{t_0}^{t_\epsilon} v(s)G(s)\mathrm{d}s + \sum_{i=\underline{i}(t_0)}^{\bar{i}(t_\epsilon)} \left\{ v(\theta_i+)\tilde{G}_i + v'(\theta_i+)\bar{G}_i \right\} \right\} \\
&\qquad < \frac{\epsilon}{3}.
\end{aligned} \tag{3.85}$$

If $t_\epsilon^2 \le t_1 \le t_2$, then

$$\begin{aligned}
&\left|\frac{u'(t_1)}{v'(t_1)} - \frac{u'(t_2)}{v'(t_2)}\right|\left\{\int_{t_0}^{t_1} v(s)G(s)\mathrm{d}s + \sum_{i=\underline{i}(t_0)}^{\bar{i}(t_1)} \big\{v(\theta_i+)\tilde{G}_i + v'(\theta_i+)\bar{G}_i\big\}\right\} \\
\le & 2\max\left\{\frac{u'(t_1)}{v'(t_1)}, \frac{u'(t_2)}{v'(t_2)}\right\} \\
& \times \left\{\int_{t_0}^{t_\epsilon} v(s)G(s)\mathrm{d}s + \sum_{i=\underline{i}(t_0)}^{\bar{i}(t_\epsilon)} \big\{v(\theta_i+)\tilde{G}_i + v'(\theta_i+)\bar{G}_i\big\}\right\} \\
& + 2\max\left\{\frac{u'(t_1)}{v'(t_1)}, \frac{u'(t_2)}{v'(t_2)}\right\} \\
& \times \left\{\int_{t_\epsilon}^{t_1} v(s)G(s)\mathrm{d}s + \sum_{i=\underline{i}(t_\epsilon)}^{\bar{i}(t_1)} \big\{v(\theta_i+)\tilde{G}_i + v'(\theta_i+)\bar{G}_i\big\}\right\} \\
\le & 2\max\left\{\frac{u'(t_1)}{v'(t_1)}, \frac{u'(t_2)}{v'(t_2)}\right\} \\
& \times \left\{\int_{t_0}^{t_\epsilon} v(s)G(s)\mathrm{d}s + \sum_{i=\underline{i}(t_0)}^{\bar{i}(t_\epsilon)} \big\{v(\theta_i+)\tilde{G}_i + v'(\theta_i+)\bar{G}_i\big\}\right\} \\
& + 2\int_{t_\epsilon}^{\infty} u(s)G(s)\mathrm{d}s \qquad (3.86)
\end{aligned}$$

and

$$\int_{t_1}^{t_2} u(s)G(s)\mathrm{d}s \le \int_{t_\epsilon}^{\infty} u(s)G(s)\mathrm{d}s. \qquad (3.87)$$

Denote

$$M_1^\epsilon := \int_{t_0}^{t_\epsilon} v(s)G(s)\mathrm{d}s + \sum_{i=\underline{i}(t_0)}^{\bar{i}(t_\epsilon)} \big\{v(\theta_i+)\tilde{G}_i + v'(\theta_i+)\bar{G}_i\big\},$$

$$M_2^\epsilon := \int_{t_0}^{t_\epsilon} u(s)G(s)\mathrm{d}s + \sum_{i=\underline{i}(t_0)}^{\bar{i}(t_\epsilon)} \big\{u(\theta_i+)\tilde{G}_i + u'(\theta_i+)\bar{G}_i\big\},$$

and let

$$\delta_\epsilon = \min\left\{\frac{\epsilon}{2M_1M_1^\epsilon}, \frac{\epsilon}{2M_1M_1^\epsilon}, \frac{\epsilon}{3M_3}, \delta_1, \delta_2\right\}.$$

By using (3.68)-(3.87), we conclude that $|(Ty)(t_1)-(Ty)(t_2)| < \epsilon$ and $|(Fy)(t_1)-(Fy)(t_2)| < \epsilon$ whenever $|t_1 - t_2| < \delta_\epsilon$ for all $t \ge t_0$ and all $y \in \Omega$. Therefore, $T(S)$ is piecewise equicontinuous.

(3) $T(S)$ is equiconvergent:

Let $\epsilon > 0$ be given and $t_1, t_2 \geq \max\{t_\epsilon^1,\, t_\epsilon^2\}$, where $t_1 \leq t_2$. In view of (3.68) and (3.69) one has

$$
\begin{aligned}
|(Ty)(t_1) - (Ty)(t_2)| \leq & \int_{t_0}^{t_1} v(s)G(s) \int_{t_1}^{t_2} \frac{1}{v^2(r)} \mathrm{d}r\mathrm{d}s \\
& + \sum_{i=\underline{i}(t_0)}^{\bar{i}(t_1)} \left\{ v(\theta_i+)\tilde{G}_i + v'(\theta_i+)\bar{G}_i \right\} \int_{t_1}^{t_2} \frac{1}{v^2(s)} \mathrm{d}s + \int_{t_1}^{t_2} v(s)G(s) \int_{s}^{t_2} \frac{1}{v^2(r)} \mathrm{d}r\mathrm{d}s \\
& + \sum_{i=\underline{i}(t_1)}^{\bar{i}(t_2)} \left\{ v(\theta_i+)\tilde{G}_i + v'(\theta_i+)\bar{G}_i \right\} \int_{\theta_i}^{t_2} \frac{1}{v^2(s)} \mathrm{d}s \\
\leq & \int_{t_0}^{t_\epsilon} v(s)G(s) \int_{t_\epsilon^1}^{\infty} \frac{1}{v^2(r)} \mathrm{d}r\mathrm{d}s + \sum_{i=\underline{i}(t_0)}^{\bar{i}(t_\epsilon)} \left\{ v(\theta_i+)\tilde{G}_i + v'(\theta_i+)\bar{G}_i \right\} \int_{t_\epsilon^1}^{\infty} \frac{1}{v^2(s)} \mathrm{d}s \\
& + 2\left\{ \int_{t_\epsilon}^{\infty} u(s)G(s)\mathrm{d}s + \sum_{i=\underline{i}(t_\epsilon)}^{\infty} \left\{ u(\theta_i+)\tilde{G}_i + u'(\theta_i+)\bar{G}_i \right\} \right\} \\
< & \epsilon
\end{aligned}
$$

and

$$
\begin{aligned}
|(Fy)(t_1) - (Fy)(t_2)| \leq & \left| \frac{u'(t_1)}{v'(t_1)} - \frac{u'(t_2)}{v'(t_2)} \right| \\
& \times \left\{ \int_{t_0}^{t_1} v(s)G(s)\mathrm{d}s + \sum_{i=\underline{i}(t_0)}^{\bar{i}(t_1)} \left\{ v(\theta_i+)\tilde{G}_i + v'(\theta_i+)\bar{G}_i \right\} \right\} \\
& + 2\left\{ \int_{t_1}^{t_2} u(s)G(s)\mathrm{d}s + \sum_{i=\underline{i}(t_1)}^{\bar{i}(t_2)} \left\{ u(\theta_i+)\tilde{G}_i + u'(\theta_i+)\bar{G}_i \right\} \right\} \\
\leq & 2\max\left\{ \frac{u'(t_1)}{v'(t_1)}, \frac{u'(t_2)}{v'(t_2)} \right\} \\
& \times \left\{ \int_{t_0}^{t_\epsilon} v(s)G(s)\mathrm{d}s + \sum_{i=\underline{i}(t_0)}^{\bar{i}(t_\epsilon)} \left\{ v(\theta_i+)\tilde{G}_i + v'(\theta_i+)\bar{G}_i \right\} \right\} \\
& + 4\left\{ \int_{t_\epsilon}^{\infty} u(s)G(s)\mathrm{d}s + \sum_{i=\underline{i}(t_\epsilon)}^{\infty} \left\{ u(\theta_i+)\tilde{G}_i + u'(\theta_i+)\bar{G}_i \right\} \right\} \\
< & \epsilon
\end{aligned}
$$

for all $y \in S$. By means of Lemma 3.1.1 we conclude that T is a compact operator, and from Schauder fixed point theorem T has a fixed point $y \in S$.

Put

$$x(t) = v(t)y(t), \quad t \geq t_0,$$

then

$$\begin{aligned} x(t) = cv(t) - u(t)\Bigg\{ &\int_{t_0}^{t} v(s)f(s, x(s), x'(s))\mathrm{d}s \\ &+ \sum_{i=\underline{i}(t_0)}^{\bar{i}(t)} \left\{ v(\theta_i+)\tilde{f}_i(x(\theta_i), x'(\theta_i)) + v'(\theta_i+)f_i(x(\theta_i), x'(\theta_i)) \right\} \Bigg\} \\ - v(t)\Bigg\{ &\int_{t}^{\infty} u(s)f(s, x(s), x'(s))\mathrm{d}s \\ &+ \sum_{i=\underline{i}(t)}^{\infty} \left\{ u(\theta_i+)\tilde{f}_i(x(\theta_i), x'(\theta_i)) + u'(\theta_i+)f_i(x(\theta_i), x'(\theta_i)) \right\} \Bigg\} \end{aligned}$$

is a solution of (3.59). By the definition of the set S, we have $x(t) \geq \delta > 0$ and, for $t \neq \theta_i$,

$$z(t) = y(t) + \frac{v(t)}{v'(t)}y'(t) = \frac{x'(t)}{v'(t)} \geq \delta > 0.$$

It follows that $x'(t) > 0$ for $t \neq \theta_i$. From (3.61)-(3.62) we further have

$$\lim_{t \to \infty} \frac{x(t)}{v(t)} = c.$$

□

Corollary 3.1.1 *If, in addition to hypotheses of Theorem 3.1.9*

$$\Delta x = -p_i x + f_i(x) > 0,$$

then there exists an increasing solution $x(t)$ of (3.59) *which satisfies* (3.63).

Corollary 3.1.2 *If $p_i = 0$ and $f_i(x, x') = 0$ in* (3.59)*, then under the hypotheses of Theorem 3.1.9, there exists an increasing solution $x(t)$ of* (3.59) *which satisfies* (3.63).

The next theorem is analogous to Theorem 3.1.9 in which both positivity and asymptotic representation of solutions are given for

$$\begin{cases} x'' + q(t)x = f(t, x, x'), & t \neq \theta_i, \\ \Delta x + p_i x = f_i(x, x'), & t = \theta_i, \\ \Delta x' + q_i x = \tilde{f}_i(x, x'), & t = \theta_i, \end{cases} \tag{3.88}$$

where $f \in \mathrm{C}([t_0, \infty) \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $f_i, \tilde{f}_i \in \mathrm{C}(\mathbb{R} \times \mathbb{R}, \mathbb{R})$, $t_0 > 0$.

Theorem 3.1.10 *Assume that* (3.57) *and* (3.60) *hold, where* $F \in C([t_0, \infty) \times [0, \infty) \times [0, \infty), [0, \infty))$ *is nondecreasing in its second and third variables for each fixed* t, *and* $F_i, \tilde{F}_i \in C([0, \infty), [0, \infty))$ *are nondecreasing in their both variables. Let* $u(t)$ *and* $v(t)$ *be principal and nonprincipal solutions of* (3.38), *respectively. If, for* $c > 0$,

$$\int_{t_0}^{\infty} \prod_{k=\underline{i}(t_0)}^{\bar{i}(t)} (1 - p_k)^{-1} u(t) F(t, 2cv(t), 2cv'(t)) \mathrm{d}t + \sum_{i=\underline{i}(t_0)}^{\infty} \prod_{k=\underline{i}(t_0)}^{i} (1 - p_k)^{-1} \times \big(u(\theta_i+) \tilde{F}_i(2cv(\theta_i), 2cv'(\theta_i)) + u'(\theta_i+) F_i(2cv(\theta_i), 2cv'(\theta_i)) \big) < c \tag{3.89}$$

and

$$\lim_{t\to\infty} \frac{u'(t)}{v'(t)} \Bigg\{ \int_{t_0}^{t} \prod_{k=\underline{i}(t_0)}^{\bar{i}(s)} (1 - p_k)^{-1} v(s) F(s, 2cv(s), 2cv'(s)) \mathrm{d}s + \sum_{i=\underline{i}(t_0)}^{\bar{i}(t)} \prod_{k=\underline{i}(t_0)}^{i} (1 - p_k)^{-1} \times \big(v(\theta_i+) \tilde{F}_i(2cv(\theta_i), 2cv'(\theta_i)) + v'(\theta_i+) F_i(2cv(\theta_i), 2cv'(\theta_i)) \big) \Bigg\} = 0, \tag{3.90}$$

then equation (3.88) *has a solution* $x(t)$ *such that* (3.63) *holds.*

Proof. By the help of (3.60) and (3.89), there exists $\delta \in (0, c)$ such that

$$\int_{t_0}^{\infty} \prod_{k=\underline{i}(t_0)}^{\bar{i}(t)} (1-p_k)^{-1} u(t) F(t, (2c-\delta)v(t), (2c-\delta)v'(t)) \mathrm{d}t$$
$$+ \sum_{i=\underline{i}(t_0)}^{\infty} \prod_{k=\underline{i}(t_0)}^{i} (1-p_k)^{-1} \big\{ u(\theta_i+) \tilde{F}_i(2cv(\theta_i), 2cv'(\theta_i))$$
$$+ u'(\theta_i+) F_i(2cv(\theta_i), 2cv'(\theta_i)) \big\} \leq c - \delta, \quad c > 0.$$

Denote $g(s) := f(s, y(s)v(s), (y(s)v(s))')$, $\bar{g}(\theta_i) := f_i(y(\theta_i)v(\theta_i), (y(\theta_i)v(\theta_i))')$ and $\tilde{g}(\theta_i) := \tilde{f}_i(y(\theta_i)v(\theta_i), (y(\theta_i)v(\theta_i))')$. Define the operator T on the set S by

$$Ty(t) = c - \frac{u(t)}{v(t)} \left\{ \int_{t_0}^{t} \prod_{k=\underline{i}(t_0)}^{\bar{i}(s)} (1-p_k)^{-1} v(s) g(s) \mathrm{d}s \right.$$
$$\left. + \sum_{i=\underline{i}(t_0)}^{\bar{i}(t)} \prod_{k=\underline{i}(t_0)}^{i} (1-p_k)^{-1} \big\{ v(\theta_i+) \tilde{g}(\theta_i) + v'(\theta_i+) \bar{g}(\theta_i) \big\} \right\}$$
$$- \left\{ \int_{t}^{\infty} \prod_{k=\underline{i}(t_0)}^{\bar{i}(s)} (1-p_k)^{-1} u(s) g(s) \mathrm{d}s \right.$$
$$\left. + \sum_{i=\underline{i}(t)}^{\infty} \prod_{k=\underline{i}(t_0)}^{i} (1-p_k)^{-1} \big\{ u(\theta_i+) \tilde{g}(\theta_i) + u'(\theta_i+) \bar{g}(\theta_i) \big\} \right\}.$$

By following the steps in the proof of Theorem 3.1.9 we can easily complete the proof. □

Corollary 3.1.3 *If, in addition to hypotheses of Theorem 3.1.10,*

$$\Delta x = -p_i x + f_i(x) > 0,$$

then there exists an increasing solution $x(t)$ of (3.88) *which satisfies* (3.63).

Corollary 3.1.4 *If $p_i = 0$ and $f_i(x, x') = 0$ in* (3.88)*, then under the hypotheses of Theorem 3.1.10, there exists an increasing solution $x(t)$ of* (3.88) *which satisfies* (3.63).

3.2 Impulsive Differential Equations With Continuous Solutions

In this section we provide analogues of the results obtained in the previous section for impulsive differential equations whose solutions are continuous. If we take $p_i = \tilde{p}_i = 0$ and $f_i(x) = 0$ in equation (3.1), i.e., if there is no impulse effect on the solution $x(t)$, then we obtain the nonlinear impulsive differential equation

$$\begin{cases} (p(t)x')' + q(t)x = f(t,x), & t \neq \theta_i, \\ \Delta p(t)x' + q_i x = \tilde{f}_i(x), & t = \theta_i, \end{cases} \tag{3.91}$$

where $f \in \mathrm{C}([t_0,\infty) \times \mathbb{R}, \mathbb{R})$, $\tilde{f}_i \in \mathrm{C}(\mathbb{R}, \mathbb{R})$, $t_0 > 0$.

We shall give theorems on asymptotic integration of (3.91) with the help of principal and nonprincipal solutions of the corresponding homogeneous equation

$$\begin{cases} (p(t)x')' + q(t)x = 0, & t \neq \theta_i, \\ \Delta p(t)x' + q_i x = 0, & t = \theta_i. \end{cases} \tag{3.92}$$

Let u and v be principal and nonprincipal solution of (3.92), and suppose without any loss of generality that they are positive on $[T,\infty)$.

3.2.1 Asymptotic Representation of Solutions

This part consists of four theorems. Since the methods used and the calculations are similar to the proofs of the theorems given above we only give sketch of the proof of each theorem and skip the details.

Theorem 3.2.1 *Assume that* $T \geq 1$, $f \in C([1,\infty) \times \mathbb{R}, \mathbb{R})$, $\tilde{f}_i \in C([1,\infty), \mathbb{R})$, *and there exist functions* $g_j \in C(\mathbb{R}_+, \mathbb{R}_+)$, $j = 1,2$, $h_k \in C([1,\infty), \mathbb{R}_+)$, $k = 1,2$ *and a real sequence* $\tilde{h}_i$ *such that*

$$|f(t,x)| \leq h_1(t) g_1\left(\frac{|x|}{v(t)}\right) + h_2(t), \quad t \geq T \tag{3.93}$$

and

$$\left|\tilde{f}_i(x)\right| \leq \tilde{h}_i g_2\left(\frac{|x|}{v(\theta_i)}\right), \quad \theta_i \geq T, \tag{3.94}$$

where

$$\int_T^\infty v(s) h_k(s) \mathrm{d}s + \sum_{i=\underline{i}(T)}^{\infty} v(\theta_i) \tilde{h}_i < \infty, \; k = 1,2. \tag{3.95}$$

Then, for any given $a, b \in \mathbb{R}$, *equation* (3.91) *has a solution* $x(t)$ *such that*

$$x(t) = av(t) + bu(t) + o(u(t)), \quad t \to \infty. \tag{3.96}$$

Proof. Consider the Banach space

$$Y = \left\{ y \in \mathrm{PLC}([T,\infty),\mathbb{R}) \middle| \frac{|y(t)|}{v(t)} \leq M_y \right\} \tag{3.97}$$

with the norm

$$\|y\| = \sup_{t\in[T,\infty)} \frac{|y(t)|}{v(t)},$$

and define the set

$$S = \{y \in Y | \, \|y(t) - bu(t)\| \leq 1\}.$$

It is easy to show that S is a closed, bounded and convex set. Let $F : S \to Y$ be defined by

$$\begin{aligned}(Fy)(t) = bu(t) + u(t)\Bigg\{ &\int_t^\infty f(s, y(s) + av(s))u(s) \int_t^s \frac{1}{p(r)u^2(r)} \mathrm{d}r\mathrm{d}s \\ &+ \sum_{i=\underline{i}(t)}^{\infty} \tilde{f}_i(y(\theta_i) + av(\theta_i))u(\theta_i) \int_t^{\theta_i} \frac{1}{p(s)u^2(s)} \mathrm{d}s \Bigg\}.\end{aligned} \tag{3.98}$$

Let y be a fixed point of the operator (3.98), then $y(t)$ is a solution of

$$\begin{cases} (p(t)y')' + q(t)y = f(t, y(t) + av(t)), & t \neq \theta_i, \\ \Delta p(t)y' + q_i y = \tilde{f}_i(y + av(\theta_i)), & t = \theta_i. \end{cases} \tag{3.99}$$

By using Schauder fixed point theorem and the lemmas 2.2.1, 2.2.2 it can be shown that F has a fixed point. □

Theorem 3.2.2 *Assume that* (3.93) *and* (3.94) *hold. If*

$$\int_T^\infty u(s)h_k(s)\mathrm{d}s + \sum_{i=\underline{i}(T)}^{\infty} u(\theta_i)\tilde{h}_i < \infty, \; k = 1, 2 \tag{3.100}$$

and

$$\limsup_{t\to\infty} \frac{u(t)}{v(t)} \left\{ \int_T^t v(s)h_k(s)\mathrm{d}s + \sum_{i=\underline{i}(T)}^{\bar{i}(t)} v(\theta_i)\tilde{h}_i \right\} < \infty, \; k = 1, 2, \tag{3.101}$$

then, for any given $a, b \in \mathbb{R}$, *equation* (3.91) *has a solution* $x(t)$ *such that*

$$x(t) = av(t) + bu(t) + o(v(t)), \quad t \to \infty. \tag{3.102}$$

Proof. Consider the Banach space (3.97) and define the closed, bounded and convex set

$$S = \{y \in Y | \, \|y(t) - av(t)\| \le 1\}.$$

Then $F : S \to Y$ be defined by

$$\begin{aligned}
&(Fy)(t) = av(t) \\
&- u(t) \left\{ \int\limits_T^t f(s, y(s) + bu(s))v(s) + \sum_{i=\underline{i}(T)}^{\bar{i}(t)} \tilde{f}_i(y(\theta_i) + bu(\theta_i))v(\theta_i) \right\} \\
&- v(t) \left\{ \int\limits_t^\infty f(s, y(s) + bu(s))u(s) + \sum_{i=\underline{i}(t)}^{\infty} \tilde{f}_i(y(\theta_i) + bu(\theta_i))u(\theta_i) \right\}. \qquad (3.103)
\end{aligned}$$

Each fixed point of F is a solution of the equation

$$\begin{cases} (p(t)y')' + q(t)y = f(t, y + bu(t)), & t \neq \theta_i \\ \Delta p(t)y' + q_i y = \tilde{f}_i(y + bu(\theta_i)), & t = \theta_i. \end{cases} \qquad (3.104)$$

Schauder fixed point theorem can be applied to show that F has a fixed point. □

Theorem 3.2.3 *Assume that* (3.93) *and* (3.94) *hold, and that*

$$\begin{cases} |f(t, x_1) - f(t, x_2)| \le \frac{k(t)}{v(t)} |x_1 - x_2|, \\ \left|\tilde{f}_i(x_1) - \tilde{f}_i(x_2)\right| \le \frac{\tilde{k}_i}{v(\theta_i)} |x_1 - x_2|, \end{cases} \qquad (3.105)$$

where $k \in C([t_0, \infty), [0, \infty))$ *and* $\tilde{k}_i$ *satisfy*

$$\int_T^\infty u(s)k(s)\mathrm{d}s + \sum_{i=\underline{i}(T)}^{\infty} u(\theta_i)\tilde{k}_i < \infty. \qquad (3.106)$$

Suppose also that there exists a function $\beta \in C([t_0, \infty), [0, \infty))$ *such that*

$$\frac{1}{p(t)u^2(t)} \left\{ \int_t^\infty u(s)h_j(s)\mathrm{d}s + \sum_{i=\underline{i}(t)}^{\infty} u(\theta_i)\tilde{h}_i \right\} \le \beta(t), \quad t \ge T, \ j = 1, 2 \qquad (3.107)$$

and

$$\int_T^t \beta(s)\mathrm{d}s = o((v(t))^\mu), \quad t \to \infty, \quad \mu \in (0, 1). \qquad (3.108)$$

Then, for any given $a, b \in \mathbb{R}$, *equation* (3.91) *has a unique solution* $x(t)$ *such that*

$$x(t) = av(t) + bu(t) + o(u(t)(v(t))^\mu), \quad t \to \infty. \qquad (3.109)$$

Proof. Consider the complete metric space

$$Z = \left\{ x \in \mathrm{PLC}([T,\infty),\mathbb{R}) \mid \frac{|x(t) - av(t) - bu(t)|}{v(t)} \leq 1 \right\}$$

with the metric

$$d(x_1, x_2) = \sup_{t\in[T,\infty)} \frac{1}{v(t)} |x_1(t) - x_2(t)|, \quad x_1, x_2 \in Z.$$

Define the operator F on Z by

$$(Fx)(t) = av(t) + bu(t) - u(t) \int_T^t \frac{1}{p(s)u^2(s)} \left\{ \int_s^\infty u(r) f(r, x(r)) \mathrm{d}r \right.$$
$$\left. + \sum_{i=\underline{i}(s)}^{\infty} u(\theta_i) \tilde{f}_i(x(\theta_i)) \right\} \mathrm{d}s. \quad (3.110)$$

By following the same procedure as in proof of Theorem 3.1.3 one can show that each fixed point of F solves (3.91), and complete the proof with the help of the Banach fixed point theorem.

□

Theorem 3.2.4 *Suppose that* (3.93), (3.94) *and* (3.105) *hold, where*

$$\int_T^\infty v(s)k(s)\mathrm{d}s + \sum_{i=\underline{i}(T)}^{\infty} v(\theta_i)\tilde{k}_i < \infty. \quad (3.111)$$

Suppose also that there exists a function $\beta \in C([t_0,\infty),[0,\infty))$ *such that*

$$\frac{1}{p(t)v^2(t)} \left\{ \int_t^\infty v(s)h_j(s)\mathrm{d}s + \sum_{i=\underline{i}(t)}^{\infty} v(\theta_i)\tilde{h}_i \right\} \leq \beta(t), \quad t \geq T,\ j = 1,2 \quad (3.112)$$

and

$$\int_T^t \beta(s)\mathrm{d}s = o((u(t))^\mu), \quad t \to \infty, \quad \mu \in (0,1). \quad (3.113)$$

Then, for any given $a, b \in \mathbb{R}$, *equation* (3.91) *has a unique solution* $x(t)$ *such that*

$$x(t) = av(t) + bu(t) + o(v(t)(u(t))^\mu), \quad t \to \infty. \quad (3.114)$$

Proof. We define the operator F on the metric space Z by

$$(Fx)(t) = av(t) + bu(t) + v(t) \int_t^\infty \frac{1}{p(s)v^2(s)} \left\{ \int_s^\infty v(r) f(r, x(r)) \mathrm{d}r + \sum_{i=\underline{i}(s)}^{\infty} v(\theta_i) \tilde{f}_i(x(\theta_i)) \right\} \mathrm{d}s. \quad (3.115)$$

Let x be a fixed point of F, then $x(t)$ solves (3.91). By means of the Banach fixed point theorem it can be shown that F has a unique fixed point. □

3.2.2 Generalization of Asymptotic Representations via a Parameter

Theorem 3.2.5 *Assume that* (3.93) *and* (3.94) *hold, where*

$$\int_T^\infty (u(s))^{1-c}(v(s))^c h_k(s)\mathrm{d}s + \sum_{i=\underline{i}(T)}^{\infty} (u(\theta_i))^{1-c}(v(\theta_i))^c \tilde{h}_i < \infty, \ k = 1, 2 \quad (3.116)$$

for some $c \in [0, 1]$. *Then, for any given* $a, b \in \mathbb{R}$, *equation* (3.91) *has a solution* $x(t)$ *such that*

$$\begin{cases} x(t) = av(t) + o(v(t)), \quad t \to \infty, & \text{if } c = 0, \\ x(t) = av(t) + bu(t) + o(u(t)), \quad t \to \infty, & \text{if } c = 1, \\ x(t) = av(t) + o((u(t))^c (v(t))^{1-c}), \quad t \to \infty, & \text{if } c \in (0, 1). \end{cases} \quad (3.117)$$

Proof. Consider the Banach space

$$X = \left\{ x \in \mathrm{PLC}([T, \infty), \mathbb{R}) | \ \frac{|x(t)|}{v(t)} \text{ is bounded} \right\}$$

with the norm $\|x\| = \sup_{t \in [T,\infty)} \dfrac{|x(t)|}{v(t)}$. Let

$$S = \left\{ x \in X | \ \frac{|x(t)|}{v(t)} \le |a| + 2|b| + 1, \quad t \ge T \right\}.$$

It is easy to show that S is a closed, bounded and convex set. Define the operator $F : S \to X$ by

$$\begin{aligned}(Fx)(t) = &[a - b(\mathrm{sgn}c - 1)]v(t) - cv(t)\int_t^\infty \frac{(u(s))^{c-1}}{p(s)(v(s))^{c+1}}\\ &\times \left\{ b - \int_s^\infty (u(\tau))^{1-c}(v(\tau))^c f(\tau, x(\tau))\mathrm{d}\tau - \sum_{i=\underline{i}(s)}^\infty (u(\theta_i))^{1-c}(v(\theta_i))^c \tilde{f}_i(x) \right\}\mathrm{d}s\\ &+ (1-c)u(t)\int_T^t \frac{(u(s))^{c-2}}{p(s)(v(s))^c}\\ &\times \left\{ b - \int_s^\infty (u(\tau))^{1-c}(v(\tau))^c f(\tau, x(\tau))\mathrm{d}\tau - \sum_{i=\underline{i}(s)}^\infty (u(\theta_i))^{1-c}(v(\theta_i))^c \tilde{f}_i(x) \right\}\mathrm{d}s.\end{aligned} \tag{3.118}$$

Step 1.

Each fixed point x of the operator F is a solution of

$$\begin{cases} (p(t)x')' + q(t)x = f(t, x(t)), & t \neq \theta_i, \\ \Delta p(t)x' + q_i x = \tilde{f}_i(x), & t = \theta_i \end{cases} \tag{3.119}$$

for $t \geq T$.

To see this we denote

$$\begin{aligned}H(t) :=& b - \int_t^\infty (u(\tau))^{1-c}(v(\tau))^c f(\tau, x(\tau))\mathrm{d}\tau - \sum_{i=\underline{i}(t)}^\infty (u(\theta_i))^{1-c}(v(\theta_i))^c \tilde{f}_i(x)\\ \Phi(t) :=& \int_t^\infty \frac{(u(s))^{c-1}}{p(s)(v(s))^{c+1}} H(s)\mathrm{d}s\\ \Psi(t) :=& \int_T^t \frac{(u(s))^{c-2}}{p(s)(v(s))^c} H(s)\mathrm{d}s.\end{aligned}$$

Then, we may write that

$$x(t) := [a - b(\mathrm{sgn}c - 1)]v(t) - cv(t)\Phi(t) + (1-c)u(t)\Psi(t). \tag{3.120}$$

Let $t \neq \theta_l$. Then

$$\begin{aligned}x'(t) =& [a - b(\mathrm{sgn}c - 1)]v'(t) - cv'(t)\Phi(t) - cv(t)\Phi'(t)\\ &+ (1-c)u'(t)\Psi(t) + (1-c)u(t)\Psi'(t),\end{aligned}$$

where

$$\Phi'(t) := -\frac{(u(t))^{c-1}}{p(t)(v(t))^{c+1}}H(t)$$
$$\Psi'(t) := \frac{(u(t))^{c-2}}{p(t)(v(t))^{c}}H(t).$$

From

$$H'(t) = (u(t))^{1-c}(v(t))^{c}f(t, x(t))$$

one has

$$\begin{aligned}(p(t)\Phi'(t))' =& \big\{-(c-1)u'(t)(u(t))^{c-2}(v(t))^{-c-1} \\ &+ (c+1)v'(t)(u(t))^{c-1}(v(t))^{-c-2}\big\}H(t) \\ &- (v(t))^{-1}f(t, x(t))\end{aligned}$$

and

$$\begin{aligned}(p(t)\Psi'(t))' =& \big\{(c-2)u'(t)(u(t))^{c-3}(v(t))^{-c} - cv'(t)(u(t))^{c-2}(v(t))^{-c-1}\big\}H(t) \\ &+ (u(t))^{-1}f(t, x(t)).\end{aligned}$$

Then

$$\begin{aligned}(p(t)x'(t))' + q(t)x(t) =& [a - b(\mathrm{sgn}c - 1)](p(t)v'(t))' - c(p(t)v'(t))'\Phi(t) \\ &- 2cp(t)v'(t)\Phi'(t) - cv(t)(p(t)\Phi'(t))' \\ &+ (1-c)(p(t)u'(t))'\Psi(t) + 2(1-c)p(t)u'(t)\Psi'(t) \\ &+ (1-c)u(t)(p(t)\Psi'(t))' + [a - b(\mathrm{sgn}c - 1)]q(t)v(t) \\ &- cq(t)v(t)\Phi(t) + (1-c)q(t)u(t)\Psi(t).\end{aligned}$$

Being solutions of (3.92), u and v satisfy $(p(t)u')' + q(t)u = (p(t)v')' + q(t)v = 0$, so

$$
\begin{aligned}
(p(t)x'(t))' + q(t)x(t) = & -2cp(t)v'(t)\Phi'(t) - cv(t)(p(t)\Phi'(t))' \\
& + 2(1-c)p(t)u'(t)\Psi'(t) + (1-c)u(t)(p(t)\Psi'(t))' \\
= & \big[2cv'(t)(u(t))^{c-1}(v(t))^{-c-1} + c(c-1)u'(t)(u(t))^{c-2}(v(t))^{-c} \\
& - c(c+1)v'(t)(u(t))^{c-1}(v(t))^{-c-1}\big] H(t) + cf(t, x(t) \\
& + \big[2(1-c)u'(t)(u(t))^{c-2}(v(t))^{-c} \\
& + (1-c)(c-2)u'(t)(u(t))^{c-2}(v(t))^{-c} \\
& - c(1-c)v'(t)(u(t))^{c-1}(v(t))^{-c-1}\big] H(t) + (1-c)f(t, x(t)) \\
= & f(t, x(t)).
\end{aligned}
$$

At the jump points $t = \theta_l$ we have

$$
\begin{aligned}
\Phi(\theta_l+) &= \Phi(\theta_l), \\
\Psi(\theta_l+) &= \Psi(\theta_l), \\
H(\theta_l+) &= H(\theta_l) + (u(\theta_l))^{1-c}(v(\theta_l))^{c}\big)\tilde{f}_l(x(\theta_l)), \\
H(\theta_l+) &= H(\theta_l) + (u(\theta_l))^{1-c}(v(\theta_l))^{c}, \\
p(\theta_l+)\Phi'(\theta_l+) &= p(\theta_l)\Phi'(\theta_l) - (v(\theta_l))^{-1}\big)\tilde{f}_l(x(\theta_l)), \\
p(\theta_l+)\Psi'(\theta_l+) &= p(\theta_l)\Psi'(\theta_l) + (u(\theta_l))^{-1}\big)\tilde{f}_l(x(\theta_l)).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\Delta p(t)x'|_{t=\theta_l} + q_l x(\theta_l) = & [a - b(\mathrm{sgn}c - 1) - c\Phi(\theta_l)][\Delta p(t)v'(t) + q_l v(\theta_l)] \\
& + (1-c)\Psi(\theta_l)[\Delta p(t)u'(t) + q_l u(\theta_l)] - cv(\theta_l)\Delta p(t)\Phi'(t) \\
& + (1-c)u(\theta_l)\Delta p(t)\Psi'(t) \\
= & -cv(\theta_l)(-(v(\theta_l))^{-1}\tilde{f}_l(x(\theta_l))) + (1-c)u(\theta_l)(u(\theta_l))^{-1}\tilde{f}_l(x(\theta_l)) \\
= & \tilde{f}_l(x(\theta_l)).
\end{aligned}
$$

Step 2. $F : S \to S$ is completely continuous. We proceed as follows.

(i) $F(S) \subset S$: Take $x \in S$, then

$$\frac{|(Fx)(t)|}{v(t)} \leq |a - b(\text{sgn}c - 1)| + c \int_t^\infty \frac{(u(s))^{c-1}}{p(s)(v(s))^{c+1}}$$
$$\times \left| b - \int_s^\infty (u(\tau))^{1-c}(v(\tau))^c f(\tau, x(\tau))\mathrm{d}\tau - \sum_{i=\underline{i}(s)}^\infty (u(\theta_i))^{1-c}(v(\theta_i))^c \tilde{f}_i(x) \right| \mathrm{d}s$$
$$+ (1-c)\frac{u(t)}{v(t)} \int_T^t \frac{(u(s))^{c-2}}{p(s)(v(s))^c}$$
$$\times \left| b - \int_s^\infty (u(\tau))^{1-c}(v(\tau))^c f(\tau, x(\tau))\mathrm{d}\tau - \sum_{i=\underline{i}(s)}^\infty (u(\theta_i))^{1-c}(v(\theta_i))^c \tilde{f}_i(x) \right| \mathrm{d}s.$$

From (3.93) and (3.94),

$$\frac{|(Fx)(t)|}{v(t)} \leq |a - b(\text{sgn}c - 1)| + c \int_t^\infty \frac{(u(s))^{c-1}}{p(s)(v(s))^{c+1}}$$
$$\times \left\{ |b| + \int_T^\infty (u(\tau))^{1-c}(v(\tau))^c \left(g\left(\frac{|x|}{v(\tau)}\right) h_1(\tau) + h_2(\tau) \right) \mathrm{d}\tau \right.$$
$$\left. + \sum_{i=\underline{i}(T)}^\infty (u(\theta_i))^{1-c}(v(\theta_i))^c \tilde{g}\left(\frac{|x|}{v(\theta_i)}\right) \tilde{h}_i \right\} \mathrm{d}s + (1-c)\frac{u(t)}{v(t)} \int_T^t \frac{(u(s))^{c-2}}{p(s)(v(s))^c}$$
$$\times \left\{ |b| + \int_T^\infty (u(\tau))^{1-c}(v(\tau))^c \left(g\left(\frac{|x|}{v(\tau)}\right) h_1(\tau) + h_2(\tau) \right) \mathrm{d}\tau \right.$$
$$\left. + \sum_{i=\underline{i}(T)}^\infty (u(\theta_i))^{1-c}(v(\theta_i))^c \tilde{g}\left(\frac{|x|}{v(\theta_i)}\right) \tilde{h}_i \right\} \mathrm{d}s.$$

Define

$$K_1 := \max_{0 \leq w \leq M_0} g(w), \quad \text{and} \quad K_2 := \max_{0 \leq w \leq M_0} \tilde{g}(w),$$

where $M_0 = |a| + 2|b| + 1$. Then

$$
\begin{aligned}
\frac{|(Fx)(t)|}{v(t)} \leq & |a-b(\operatorname{sgn} c-1)| + c\int_t^\infty \frac{(u(s))^{c-1}}{p(s)(v(s))^{c+1}} \\
& \times \Bigg\{ |b| + \int_T^\infty (u(\tau))^{1-c}(v(\tau))^c (K_1 h_1(\tau)+h_2(\tau))\mathrm{d}\tau \\
& \qquad + \sum_{i=\underline{i}(T)}^{\infty} (u(\theta_i))^{1-c}(v(\theta_i))^c K_2 \tilde{h}_i \Bigg\} \mathrm{d}s \\
& + (1-c)\frac{u(t)}{v(t)} \int_T^t \frac{(u(s))^{c-2}}{p(s)(v(s))^c} \\
& \times \Bigg\{ |b| + \int_T^\infty (u(\tau))^{1-c}(v(\tau))^c \left(K_1 h_1(\tau)+h_2(\tau)\right)\mathrm{d}\tau \\
& \qquad + \sum_{i=\underline{i}(T)}^{\infty} (u(\theta_i))^{1-c}(v(\theta_i))^c K_2 \tilde{h}_i \Bigg\} \mathrm{d}s.
\end{aligned}
$$

By using (3.116), for T is sufficiently large we may write that

$$
\int_T^\infty (u(s))^{1-c}(v(s))^c h_k(s)\mathrm{d}s + \sum_{i=\underline{i}(T)}^{\infty} (u(\theta_i))^{1-c}(v(\theta_i))^c \tilde{h}_i \leq 1, \;\; k=1,2.
$$

So, with the help of (2.9) we have

$$
\begin{aligned}
\frac{|(Fx)(t)|}{v(t)} \leq & |a-b(\operatorname{sgn} c-1)| + (|b|+1) \\
& \times \left(c\int_t^\infty \frac{(u(s))^{c-1}}{p(s)(v(s))^{c+1}}\mathrm{d}s + (1-c)\frac{u(t)}{v(t)} \int_T^t \frac{(u(s))^{c-2}}{p(s)(v(s))^c}\mathrm{d}s \right) \\
= & |a-b(\operatorname{sgn} c-1)| + (|b|+1) \left(c\int_t^\infty \frac{(v(s))^{c-1} \left(\int_s^\infty \frac{1}{p(\tau)v^2(\tau)}\mathrm{d}\tau \right)^{c-1}}{p(s)(v(s))^{c+1}} \mathrm{d}s \right. \\
& \left. + (1-c)\frac{u(t)}{v(t)} \int_T^t \left(\frac{u(s)}{v(s)} \right)^c \frac{1}{p(s)u^2(s)}\mathrm{d}s \right).
\end{aligned}
$$

The Wronskian $W(v(t), u(t)) = -1/p(t)$ implies that

$$\frac{|(Fx)(t)|}{v(t)} = |a - b(\mathrm{sgn} c - 1)| + (|b| + 1)$$
$$\times \left(-\int_t^\infty \frac{d}{\mathrm{d}s} \left(\int_s^\infty \frac{1}{p(\tau)v^2(\tau)} \mathrm{d}\tau \right)^c \mathrm{d}s + \frac{u(t)}{v(t)} \int_T^t \frac{d}{\mathrm{d}s} \left(\frac{u(s)}{v(s)} \right)^{c-1} \mathrm{d}s \right)$$
$$\leq |a - b(\mathrm{sgn} c - 1)| + (|b| + 1) \left(\left(\int_t^\infty \frac{1}{p(\tau)v^2(\tau)} \mathrm{d}\tau \right)^c + \left(\frac{u(t)}{v(t)} \right)^c \right).$$

Then, the property (2.6) allows us to write $(u(t)/v(t))^c \leq 1/2$, $t \geq T$. Hence

$$\frac{|(Fx)(t)|}{v(t)} \leq |a - b(\mathrm{sgn} c - 1)| + (|b| + 1) \leq |a| + 2|b| + 1.$$

To prove that $Fx \in \mathrm{PLC}[t_0, \infty)$, let $t_1 \in [1, \infty)$ with $t < t_1$. Then

$$|(Fx)(t) - (Fx)(t_1)| \leq |v(t) - v(t_1)| \Bigg\{ |a - b(\mathrm{sgn} c - 1)| + c \int_t^\infty \frac{(u(s))^{c-1}}{p(s)(v(s))^{c+1}}$$
$$\times \left| b - \int_s^\infty (u(\tau))^{1-c}(v(\tau))^c f(\tau, x(\tau)) \mathrm{d}\tau - \sum_{i=\underline{i}(s)}^\infty (u(\theta_i))^{1-c}(v(\theta_i))^c \tilde{f}_i(x) \right| \mathrm{d}s \Bigg\}$$
$$+ c v(t_1) \int_t^{t_1} \frac{(u(s))^{c-1}}{p(s)(v(s))^{c+1}}$$
$$\times \left| b - \int_s^\infty (u(\tau))^{1-c}(v(\tau))^c f(\tau, x(\tau)) \mathrm{d}\tau - \sum_{i=\underline{i}(s)}^\infty (u(\theta_i))^{1-c}(v(\theta_i))^c \tilde{f}_i(x) \right| \mathrm{d}s$$
$$+ (1 - c)|u(t) - u(t_1)| \int_T^t \frac{(u(s))^{c-2}}{p(s)(v(s))^c}$$
$$\times \left| b - \int_s^\infty (u(\tau))^{1-c}(v(\tau))^c f(\tau, x(\tau)) \mathrm{d}\tau - \sum_{i=\underline{i}(s)}^\infty (u(\theta_i))^{1-c}(v(\theta_i))^c \tilde{f}_i(x) \right| \mathrm{d}s$$
$$+ (1 - c)u(t_1) \int_t^{t_1} \frac{(u(s))^{c-2}}{p(s)(v(s))^c}$$
$$\times \left| b - \int_s^\infty (u(\tau))^{1-c}(v(\tau))^c f(\tau, x(\tau)) \mathrm{d}\tau - \sum_{i=\underline{i}(s)}^\infty (u(\theta_i))^{1-c}(v(\theta_i))^c \tilde{f}_i(x) \right| \mathrm{d}s.$$

Hence by using (3.93), (3.94) and (3.116) and taking limit as $t \to t_1-$ we obtain $(Fx)(t) \to (Fx)(t_1)$ for any $t_1 \in [1, \infty)$. Secondly we assume $t > t_1$ and by a similar proof, we can show that $\lim_{t\to t_1+}(Fx)(t) = (Fx)(t_1)$ for $t_1 \neq \theta_i$ and $\lim_{t\to\theta_i+}(Fx)(t)$ exist for all $i = 1, 2, \ldots$ Hence $F(S) \subset S$.

(ii) *F is continuous:*

Consider a sequence $\{x_n\} \in S$ such that $x_n \to x \in S$ as $n \to \infty$. We want to show that

$$\|Fx - Fx_n\| \to 0, \quad n \to \infty. \tag{3.121}$$

From (3.93) and (3.94) one has

$$|f(s, x_n(s))| \leq K_1 h_1(s) + h_2(s) \tag{3.122}$$

and

$$|\tilde{f}_i(x_n(\theta_i))| \leq K_2 \tilde{h}_i. \tag{3.123}$$

Thus,

$$\begin{aligned}
&\frac{|(Fx_n)(t) - (Fx)(t)|}{v(t)} \\
\leq& c\int_t^\infty \frac{(u(s))^{c-1}}{p(s)(v(s))^{c+1}}\left\{\int_s^\infty (u(\tau))^{1-c}(v(\tau))^c|f(\tau, x(\tau)) - f(\tau, x_n(\tau))|\mathrm{d}\tau\right. \\
&\qquad\left.+ \sum_{i=\underline{i}(s)}^\infty (u(\theta_i))^{1-c}(v(\theta_i))^c|\tilde{f}_i(x) - \tilde{f}_i(x_n)|\right\}\mathrm{d}s \\
&+ (1-c)\frac{u(t)}{v(t)}\int_T^t \frac{(u(s))^{c-2}}{p(s)(v(s))^c}\left\{\int_s^\infty (u(\tau))^{1-c}(v(\tau))^c|f(\tau, x(\tau)) - f(\tau, x_n(\tau))|\mathrm{d}\tau\right. \\
&\qquad\left.+ \sum_{i=\underline{i}(s)}^\infty (u(\theta_i))^{1-c}(v(\theta_i))^c|\tilde{f}_i(x) - \tilde{f}_i(x)|\right\}\mathrm{d}s \\
\leq& c\int_t^\infty \frac{(u(s))^{c-1}}{p(s)(v(s))^{c+1}}\left\{\int_s^\infty (u(\tau))^{1-c}(v(\tau))^c 2(K_1 h_1(s) + h_2(s))\mathrm{d}\tau\right. \\
&\qquad\left.+ \sum_{i=\underline{i}(s)}^\infty (u(\theta_i))^{1-c}(v(\theta_i))^c 2K_2\tilde{h}_i\right\}\mathrm{d}s \\
&+ (1-c)\frac{u(t)}{v(t)}\int_T^t \frac{(u(s))^{c-2}}{p(s)(v(s))^c}\left\{\int_s^\infty (u(\tau))^{1-c}(v(\tau))^c 2(K_1 h_1(s) + h_2(s))\mathrm{d}\tau\right. \\
&\qquad\left.+ \sum_{i=\underline{i}(s)}^\infty (u(\theta_i))^{1-c}(v(\theta_i))^c 2K_2\tilde{h}_i\right\}\mathrm{d}s,
\end{aligned}$$

which, in view of (3.116), is finite. Thus, by means of the Lebesgue dominated convergence theorem, 3.121 holds, and so F is continuous.

(iii) *F is relatively compact:*

Take an arbitrary sequence $\{x_n\} \in S$. We want to prove that there exists a subsequence $\{x_{n_k}\} \in S$ so that Fx_{n_k} is convergent. If we define
$f_n(\tau) := (u(\tau))^{1-c}(v(\tau))^c f(\tau, x_n(\tau)),\ \tilde{f}_n(\theta_i) := (u(\theta_i))^{1-c}(v(\theta_i))^c \tilde{f}_i(x_n)$
we get

$$\begin{aligned}(Fx_n)(t) &= [a - b(\mathrm{sgnc} - 1)]v(t) \\ &\quad - cv(t)\int_t^\infty \frac{(u(s))^{c-1}}{p(s)(v(s))^{c+1}}\left\{b - \int_s^\infty f_n(\tau)\mathrm{d}\tau - \sum_{i=\underline{i}(s)}^\infty \tilde{f}_n(\theta_i)\right\}\mathrm{d}s \\ &\quad + (1-c)u(t)\int_T^t \frac{(u(s))^{c-2}}{p(s)(v(s))^c}\left\{b - \int_s^\infty f_n(\tau)\mathrm{d}\tau - \sum_{i=\underline{i}(s)}^\infty \tilde{f}_n(\theta_i)\right\}\mathrm{d}s.\end{aligned}$$

From (3.93) and (3.116) one can find a constant $c_1 > 0$ such that

$$\|f_n\|_{L^1([T,\infty))} \le c_1,\ n \ge 1.$$

Therefore the first hypothesis of Lemma 2.2.1 holds. Define $(g_h f)(s) := f(s+h)$. From (3.93) one has

$$\begin{aligned}\|g_h f_n - f_n\|_{L^1([T,\infty))} &\le \int_T^\infty |f_n(\tau + h)|\mathrm{d}\tau + \int_T^\infty |f_n(\tau)|\mathrm{d}\tau \\ &\le \int_T^\infty 2|f_n(\tau)|\mathrm{d}\tau \\ &\le \int_T^\infty 2(u(\tau))^{1-c}(v(\tau))^c(K_1h_1(\tau) + h_2(\tau))\mathrm{d}\tau.\end{aligned}$$

By virtue of Lebesgue dominated convergence theorem we deduce that

$$\|(f_n)(\tau + h) - f_n(\tau)\| \to 0,\ h \to 0.$$

So, by Lemma 2.2.1 there exists a subsequence $\{f_{n_k}\}$ which is convergent in $L^1([T,\infty))$, say

$$\int_T^\infty |z(\tau)|\mathrm{d}\tau := \lim_{k\to\infty}\int_T^\infty |f_{n_k}(\tau)|\mathrm{d}\tau.$$

On the other hand, from (3.94) it can be seen that

$$|\tilde{f}_n(\theta_i)| \le K_2 \tilde{h}_i.$$

In view of (3.116), $\{\tilde{f}_n\}$ is pointwise bounded. Let $\epsilon > 0$ be given and choose $j \in \mathbb{N}$ sufficiently large so that

$$\sum_{i=j}^{\infty} (u(\theta_i))^{1-c} (v(\theta_i))^c \tilde{h}_i < \frac{\epsilon}{K_2}.$$

Then

$$\sum_{i=j}^{\infty} |\tilde{f}_n(\theta_i)| < \epsilon,$$

and so all hypotheses of Lemma 2.2.2 hold. Thus the set $\{\tilde{f}_n\}$ is compact in $\ell^1([T,\infty))$ i.e., there exists a subsequence $\{\tilde{f}_{n_k}\}$ which converges to an element of $\ell^1([T,\infty))$, say $\tilde{z}_i$. So

$$\begin{aligned}
&|(Fx_{n_k})(t) - (Fx)(t)| \\
\le& cv(t) \int_t^{\infty} \frac{(u(s))^{c-1}}{p(s)(v(s))^{c+1}} \left\{ \int_s^{\infty} |f_{n_k}(\tau) - z(\tau)| \mathrm{d}\tau + \sum_{i=\underline{i}(s)}^{\infty} |\tilde{f}_{n_k}(\theta_i) - \tilde{z}_i| \right\} \mathrm{d}s \\
&+ (1-c)u(t) \int_T^{t} \frac{(u(s))^{c-2}}{p(s)(v(s))^{c}} \left\{ \int_s^{\infty} |f_{n_k}(\tau) - z(\tau)| \mathrm{d}\tau + \sum_{i=\underline{i}(s)}^{\infty} |\tilde{f}_{n_k}(\theta_i) - \tilde{z}_i| \right\} \mathrm{d}s.
\end{aligned}$$

By using (3.93), (3.94), (3.116), the Lebesgue dominated convergence theorem and Weierstrass-M test, we conclude that

$$\|Fx_{n_k} - Fx\| \to 0, \quad k \to \infty.$$

Thus all the hypotheses of Schauder fixed point theorem hold. This proves that the operator (3.118) has a fixed point $x \in X$ which is a solution of equation (3.91).

Step 3. The asymptotic formulae (3.117) hold.
Proof.

If $c = 0$, then we obtain the integral equation

$$\begin{aligned} x(t) =&(a+b)v(t) \\ &+ u(t)\int_T^t \frac{1}{p(s)u^2(s)}\left(b - \int_s^\infty u(\tau)f(\tau,x(\tau))\mathrm{d}\tau - \sum_{i=\underline{i}(s)}^\infty u(\theta_i)\tilde{f}_i(x)\right)\mathrm{d}s \\ =&av(t) - u(t)\int_T^t \frac{1}{p(s)u^2(s)}\left(\int_s^\infty u(\tau)f(\tau,x(\tau))\mathrm{d}\tau - \sum_{i=\underline{i}(s)}^\infty u(\theta_i)\tilde{f}_i(x)\right)\mathrm{d}s. \end{aligned}$$

From (3.93), (3.94) and (3.116) we can write

$$\lim_{t\to\infty}\int_t^\infty u(\tau)f(\tau,x(\tau))\mathrm{d}\tau - \sum_{i=\underline{i}(t)}^\infty u(\theta_i)\tilde{f}_i(x) = 0,$$

so, an application of l'Hôpital's rule gives that

$$\frac{x(t)-av(t)}{v(t)} \to 0, \quad t\to\infty.$$

If $c = 1$ then

$$x(t) = av(t) - v(t)\int_t^\infty \frac{1}{p(s)v^2(s)}\left(b - \int_s^\infty v(\tau)f(\tau,x(\tau))\mathrm{d}\tau - \sum_{i=\underline{i}(s)}^\infty v(\theta_i)\tilde{f}_i(x)\right)\mathrm{d}s.$$

With the help of (3.93), (3.94), (3.116) and l'Hôpital's rule we obtain

$$\begin{aligned} \lim_{t\to\infty}\frac{x(t)-av(t)}{u(t)} =& -\lim_{t\to\infty}\frac{v(t)}{u(t)}\int_t^\infty \frac{1}{p(s)v^2(s)} \\ &\times\left(b - \int_s^\infty v(\tau)f(\tau,x(\tau))\mathrm{d}\tau - \sum_{i=\underline{i}(s)}^\infty v(\theta_i)\tilde{f}_i(x)\right)\mathrm{d}s \\ =&b \end{aligned}$$

which allows us to write

$$\lim_{t\to\infty}\frac{x(t)-av(t)-bu(t)}{u(t)} = 0.$$

Finally, if $c \in (0,1)$ we have from l'Hôpital's rule, (3.93), (3.94), (3.116) and the properties (2.6) and (2.7)

$$\begin{aligned}
&\lim_{t\to\infty} \frac{1}{(u(t))^c(v(t))^{1-c}} cv(t) \int_t^\infty \frac{(u(s))^{c-1}}{p(s)(v(s))^{c+1}} \Bigg\{ b - \int_s^\infty (u(\tau))^{1-c}(v(\tau))^c f(\tau, x(\tau))\mathrm{d}\tau \\
&\qquad\qquad - \sum_{i=\underline{i}(s)}^\infty (u(\theta_i))^{1-c}(v(\theta_i))^c \tilde{f}_i(x) \Bigg\} \mathrm{d}s \\
&= c \lim_{t\to\infty} \left(\frac{v(t)}{u(t)}\right)^c \int_t^\infty \frac{(u(s))^{c-1}}{p(s)(v(s))^{c+1}} \Bigg\{ b - \int_s^\infty (u(\tau))^{1-c}(v(\tau))^c f(\tau, x(\tau))\mathrm{d}\tau \\
&\qquad\qquad - \sum_{i=\underline{i}(s)}^\infty (u(\theta_i))^{1-c}(v(\theta_i))^c \tilde{f}_i(x) \Bigg\} \mathrm{d}s \\
&= \lim_{t\to\infty} \left(b - \int_t^\infty (u(\tau))^{1-c}(v(\tau))^c f(\tau, x(\tau))\mathrm{d}\tau - \sum_{i=\underline{i}(t)}^\infty (u(\theta_i))^{1-c}(v(\theta_i))^c \tilde{f}_i(x) \right) \\
&= b. \qquad (3.124)
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\lim_{t\to\infty} \frac{1}{(u(t))^c(v(t))^{1-c}} (1-c)u(t) \int_T^t \frac{(u(s))^{c-2}}{p(s)(v(s))^c} \\
&\quad \times \Bigg\{ b - \int_s^\infty (u(\tau))^{1-c}(v(\tau))^c f(\tau, x(\tau))\mathrm{d}\tau - \sum_{i=\underline{i}(s)}^\infty (u(\theta_i))^{1-c}(v(\theta_i))^c \tilde{f}_i(x) \Bigg\} \mathrm{d}s \\
&= \lim_{t\to\infty} (1-c) \left(\frac{u(t)}{v(t)}\right)^{1-c} \int_T^t \frac{(u(s))^{c-2}}{p(s)(v(s))^c} \\
&\quad \times \Bigg\{ b - \int_s^\infty (u(\tau))^{1-c}(v(\tau))^c f(\tau, x(\tau))\mathrm{d}\tau - \sum_{i=\underline{i}(s)}^\infty (u(\theta_i))^{1-c}(v(\theta_i))^c \tilde{f}_i(x) \Bigg\} \mathrm{d}s \\
&= \lim_{t\to\infty} \left(b - \int_t^\infty (u(\tau))^{1-c}(v(\tau))^c f(\tau, x(\tau))\mathrm{d}\tau - \sum_{i=\underline{i}(t)}^\infty (u(\theta_i))^{1-c}(v(\theta_i))^c \tilde{f}_i(x) \right) \\
&= b. \qquad (3.125)
\end{aligned}$$

Therefore, combining (3.124) and (3.125) we obtain

$$\lim_{t\to\infty} \frac{x(t) - (a - b(\mathrm{sgn} c - 1))v(t)}{(u(t))^c(v(t))^{1-c}} \to 0, \quad t \to \infty.$$

□

3.2.3 Existence of Monotone Positive Solutions

Consider

$$\begin{cases} (p(t)x')' + q(t)x = f(t, x, x'), & t \neq \theta_i, \\ \Delta p(t)x' + q_i x = \tilde{f}_i(x, x'), & t = \theta_i, \end{cases} \tag{3.126}$$

where $f \in \mathrm{C}([t_0, \infty) \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $\tilde{f}_i \in \mathrm{C}(\mathbb{R} \times \mathbb{R}, \mathbb{R})$, $t_0 > 0$.

The homogeneous equation associated to (3.126) is

$$\begin{cases} (p(t)x')' + q(t)x = 0, & t \neq \theta_i, \\ \Delta p(t)x' + q_i x = 0, & t = \theta_i. \end{cases} \tag{3.127}$$

We give the following result for equation (3.126) which states not only asymptotic representation but also positivity and monotonicity of the solution.

Theorem 3.2.6 *Assume that* (3.57) *holds and*

$$\begin{cases} |f(t, x, x')| \leq F(t, |x|, |x'|), & t \geq t_0 \\ \left|\tilde{f}_i(x, x')\right| \leq \tilde{F}_i(|x|, |x'|), \end{cases} \tag{3.128}$$

where $F \in C([t_0, \infty) \times [0, \infty) \times [0, \infty), [0, \infty))$ is nondecreasing with respect to its second and third variable for each fixed t, and $\tilde{F}_i \in C([0, \infty) \times [0, \infty), [0, \infty))$ is nondecreasing with respect to its both variables. If, for $c > 0$,

$$\int_{t_0}^{\infty} u(t)F(t, 2cv(t), 2cv'(t))\mathrm{d}t + \sum_{i=\underline{i}(t_0)}^{\infty} u(\theta_i)\tilde{F}_i(2cv(\theta_i), 2cv'(\theta_i)) < c, \tag{3.129}$$

and as $t \to \infty$,

$$\frac{u'(t)}{v'(t)} \left\{ \int_{t_0}^{t} v(s)F(s, 2cv(s), 2cv'(s))\mathrm{d}s + \sum_{i=\underline{i}(t_0)}^{\bar{i}(t)} v(\theta_i)\tilde{F}_i(2cv(\theta_i), 2cv'(\theta_i)) \right\} \to 0 \tag{3.130}$$

hold, then, equation (3.126) *has an is increasing solution such that* (3.63) *holds.*

Proof. From (3.128) and (3.129) there exists a $\delta \in (0, c)$ such that

$$\int_{t_0}^{\infty} u(t)F(t, (2c-\delta)v(t), (2c-\delta)v'(t))\mathrm{d}t$$

$$+ \sum_{i=\underline{i}(t_0)}^{\infty} u(\theta_i)\tilde{F}_i((2c-\delta)v(\theta_i), (2c-\delta)v'(\theta_i)) \leq c - \delta, \quad c > 0. \tag{3.131}$$

Consider the Banach space

$$\Omega = \{y \in \mathrm{PLC}^1[t_0, \infty) |\ \lim_{t\to\infty} y(t) \text{ and } \lim_{t\to\infty} z(t) \text{ exist}\}$$

with the norm

$$\|y\| = \max\left\{\sup_{t\geq t_0} |y(t)|, \sup_{t\geq t_0} |z(t)|\right\},$$

where

$$z(t) = y(t) + \frac{v(t)}{v'(t)} y'(t).$$

We introduce the set $S = \{y \in \Omega |\ \delta \leq y(t) \leq 2c - \delta,\ \delta \leq z(t) \leq 2c - \delta\}$. It can be easily shown that S is a closed, bounded and convex set. Denote

$$g(s) := f(s, y(s)v(s), (y(s)v(s))') \text{ and } \tilde{g}(\theta_i) := \tilde{f}_i(y(\theta_i)v(\theta_i), (y(\theta_i)v(\theta_i))'),$$

and define the operator T on the set S by

$$\begin{aligned} Ty(t) = c - \frac{u(t)}{v(t)} &\left\{ \int_{t_0}^{t} v(s)g(s)\mathrm{d}s + \sum_{i=\underline{i}(t_0)}^{\bar{i}(t)} v(\theta_i)\tilde{g}(\theta_i) \right\} \\ &- \left\{ \int_{t}^{\infty} u(s)g(s)\mathrm{d}s + \sum_{i=\underline{i}(t)}^{\infty} u(\theta_i)\tilde{g}(\theta_i) \right\}. \end{aligned} \tag{3.132}$$

By following the steps in Theorem 3.1.9 it can be shown that the operator T has a fixed point. Then, from (3.128), (3.129) and (3.130) it is seen that $\lim_{t\to\infty} y(t) = c$.

Define

$$x(t) := y(t)v(t),$$

then clearly $x(t)$ is a solution of (3.126), and

$$\lim_{t\to\infty} \frac{x(t)}{v(t)} = c.$$

From the definition of the set S, we have further $x(t) > 0$ and $x'(t) > 0$. Since $x(t)$ is continuous, i.e., there is no impulse effect on $x(t)$, we conclude that it is strictly increasing.

□

3.3 Examples

In this section we give some examples to illustrate our results.

Example 3.3.1 Consider the equation

$$\begin{cases} x'' = \frac{x \ln t}{t^6}, & t \neq \frac{i}{2}, \\ \Delta x' - \frac{8(-1)^i}{i+1-(-1)^i} x = \frac{2x^2 \sin x}{i^6}, & t = \frac{i}{2}, \quad i = 1, 2, \ldots \end{cases} \tag{3.133}$$

where $t \geq 1$. Clearly,

$$\begin{cases} x'' = 0, & t \neq \frac{i}{2}, \\ \Delta x' - \frac{8(-1)^i}{i+1-(-1)^i} x = 0, & t = \frac{i}{2}, \quad i = 1, 2, \ldots \end{cases} \tag{3.134}$$

is the corresponding homogeneous equation. A solution of (3.134) can be calculated as

$$v(t) = (2(-1)^{i-1} + 1)t + 2\lfloor \frac{i}{2} \rfloor (-1)^i, \quad \frac{i-1}{2} < t \leq \frac{i}{2}. \tag{3.135}$$

Let $T = 1$. Then

$$\begin{aligned} \int_1^\infty \frac{1}{v^2(s)} \mathrm{d}s &= \sum_{i=1}^\infty \int_{i-1}^{i} \frac{1}{((2(-1)^{i-1} + 1)s + 2\lfloor \frac{i}{2} \rfloor (-1)^i)^2} \mathrm{d}s \\ &= \sum_{k=1}^\infty \int_k^{\frac{2k+1}{2}} \frac{1}{(3s - 2k)^2} \mathrm{d}s + \sum_{k=2}^\infty \int_{\frac{2k-1}{2}}^{k} \frac{1}{(-s + 2k)^2} \mathrm{d}s \\ &= \frac{1}{2} \sum_{k=1}^\infty \frac{1}{k^2 + \frac{3k}{2}} + \frac{1}{2} \sum_{k=2}^\infty \frac{1}{k^2 + \frac{k}{2}} \end{aligned}$$

is finite. So, $v(t)$ is a nonprincipal solution. Clearly, $|f(t,x)| \leq |x|/t^5$ and since $v(t) < 2t$ we may choose $h_1(t) = 2t^{-4}$, $g_1(x) = x$ and $h_2(t) = 0$. Moreover, since $|\tilde{f}_i(x)| \leq 2x^2/i^6$ we may take $\tilde{h}_i = 1/4i^4$ and $g_2(x) = 8x^2$. It can be easily seen that (3.93) and (3.94) are satisfied. Moreover we have

$$\int_1^\infty v(s)h_1(s)\mathrm{d}s + \sum_{\frac{i}{2}=1}^\infty v(\frac{i}{2})\tilde{h}_i$$

$$= \sum_{\frac{i}{2}=1}^\infty \int_{\frac{i-1}{2}}^{\frac{i}{2}} \{((2(-1)^{i-1}+1)s + 2\lfloor\frac{i}{2}\rfloor(-1)^i)\frac{2}{s^4}\}\mathrm{d}s$$

$$+ \sum_{\frac{i}{2}=1}^\infty ((2(-1)^{i-1}+1)\frac{i}{2} + 2\lfloor\frac{i}{2}\rfloor(-1)^i)\frac{1}{2^6 i^4}$$

$$= 2\sum_{i=1}^\infty \left(\int_i^{i+\frac{1}{2}} (3s-2i)s^{-4}\mathrm{d}s + \int_{i+\frac{1}{2}}^{i+1} (-s+2(i+1))s^{-4}\mathrm{d}s \right)$$

$$+ \frac{1}{2^6}\sum_{k=1}^\infty \left(\frac{k}{(2k)^4} + \frac{k+3/2}{(2k+1)^4} \right)$$

$$= 2\sum_{i=1}^\infty \left(\frac{3i^2+6i-5}{12(i+1)^4} - \frac{8i^2-8i-20}{3(2i+1)^4} + \frac{5}{6i^2} \right)$$

$$+ \frac{1}{2^6}\sum_{k=1}^\infty \left(\frac{1}{164k^3} + \frac{1}{2(2k+1)^3} + \frac{1}{(2k+1)^4} \right),$$

which is convergent. Therefore, all hypotheses of Theorem 3.2.1 are satisfied. So, for any given real numbers a and b, there exists a solution $x(t)$ of the equation (3.133) such that

$$x(t) = av(t) + bu(t) + o(u(t)), \quad t \to \infty,$$

where $v(t)$ is given by (3.135) and $u(t)$ is to be computed by using the relation (2.9).

Example 3.3.2 Consider the equation

$$\begin{cases} x'' = \frac{x + \ln t \sin x}{t2^t}, & t \neq i, \\ \Delta x' - \frac{1}{2}x = \frac{x^2}{4^i}, & t = i, \quad i = 1, 2, \ldots \end{cases} \tag{3.136}$$

where $t \geq 1$. The corresponding homogeneous equation is

$$\begin{cases} x'' = 0, & t \neq i, \\ \Delta x' - \frac{1}{2}x = 0, & t = i, \quad i = 1, 2, \ldots \end{cases} \tag{3.137}$$

and

$$u(t) = 2^{1-i}(i - t + 1), \quad v(t) = 2^{i-2}(t - i + 2), \quad i - 1 < t \leq i \tag{3.138}$$

are two linearly independent solutions of (3.137) Let $T = 1$. Observe that

$$\int_1^\infty \frac{1}{u^2(s)} \mathrm{d}s = \sum_{i=2}^\infty \int_{i-1}^i \frac{1}{2^{2(1-i)}(i - s + 1)^2} \mathrm{d}s = \sum_{i=2}^\infty 2^{2i-3} = \infty,$$

and that

$$\int_1^\infty \frac{1}{v^2(s)} \mathrm{d}s = \sum_{i=2}^\infty \int_{i-1}^i \frac{1}{2^{2(i-2)}(s - i + 2)^2} \mathrm{d}s = \sum_{i=2}^\infty \frac{1}{2^{2i-3}} = \frac{2}{3}.$$

Thus, $u(t)$ is the principal and $v(t)$ is a nonprincipal solution of (3.136). Now, $|f(t,x)| \leq \dfrac{|x|}{t2^t} + \dfrac{1}{2^t}$, and $|\tilde{f}_i(x)| \leq \dfrac{x^2}{4^i}$.

Since $v(t) \leq 2^t$ we can write $h_1(t) = 1/t$, $g_1(x) = 2x$, $h_2(t) = 1/2^t$, $\tilde{h}_i = 1/4$, and $g_2(x) = x^2$. Then

$$\int_1^\infty u(s)(h_1(s) + h_2(s))\mathrm{d}s + \sum_{i=1}^\infty u(\theta_i)\tilde{h}_i$$
$$= \sum_{i=2}^\infty \int_{i-1}^i 2^{1-i}(i - s + 1)\left(\frac{1}{s} + \frac{1}{2^s}\right)\mathrm{d}s + \sum_{i=1}^\infty 2^{-1-i}$$

is convergent. Also,

$$\limsup_{t\to\infty} \frac{u(t)}{v(t)} \left\{ \int_T^t v(s)(h_1(s) + h_2(s))\mathrm{d}s + \sum_{i=\underline{i}(T)}^{\bar{i}(t)} v(\theta_i)\tilde{h}_i \right\}$$
$$= \limsup_{t\to\infty} \frac{i - t + 1}{2^{2i-3}(t - i + 2)} \left\{ \sum_{i=1}^{\bar{i}(t)} \int_{i-1}^i 2^{i-2}(s - i + 2)\left(\frac{1}{s} + \frac{1}{2^s}\right)\mathrm{d}s + \sum_{i=1}^{\bar{i}(t)} 2^{i-3} \right\}$$
$$\leq \limsup_{t\to\infty} \frac{i - t + 1}{2^{2i-3}(t - i + 2)} 2^{i+1}$$
$$< \infty.$$

Therefore (3.93), (3.94), (3.100) and (3.101) are satisfied. From Theorem 3.2.2 we conclude that there exist a solution of equation (3.136) satisfying the asymptotic representation (3.102), where a and b are arbitrary constants.

Note that (3.95) does not hold. So, Theorem 3.2.1 can not be used for equation (3.133).

Example 3.3.3 Consider the equation

$$\begin{cases} x'' = \frac{x^3}{(t^2+x^2)t^4 8^t}, & t \neq i, \\ \Delta x' - \frac{1}{2}x = \frac{x}{8^i}, & t = i, \quad i = 1, 2, \ldots \end{cases} \tag{3.139}$$

where $t \geq 1$. Take $T = 1$. Observe that the corresponding homogeneous equation is (3.137), and its principal and nonprincipal solutions are given by (3.138). Clearly,

$$|f(t,x)| \leq \frac{|x|}{t^4 8^t}, \quad |\tilde{f}_i(x)| \leq \frac{|x|}{8^i}.$$

Since $v(t) \leq 2^t$ we can write $h_1(t) = t^{-4}2^{-2t}$, $g_1(x) = x$, $h_2(t) = 0$, $\tilde{h}_i = 2^{-2i}$, and $g_2(x) = x/2$. Let $k(t) = h_1(t)$, $\tilde{k}_i = \tilde{h}_i/2$ and

$$\beta(t) = \frac{1}{(i-t+1)^2 2^i}, \; i-1 \leq t < i, \; i = 1, 2, \ldots$$

We may write that

$$|f(t,x_1) - f(t,x_2)| \leq \frac{1}{2^{3t}t^4}|x_1 - x_2| \leq \frac{k(t)}{v(t)}|x_1 - x_2|$$

and

$$|\tilde{f}(x_1) - \tilde{f}(x_2)| \leq \frac{1}{2^{3i}}|x_1 - x_2| = \frac{\tilde{k}_i}{v(\theta_i)}|x_1 - x_2|.$$

So

$$\begin{aligned} &\int_1^\infty u(s)k(s)\mathrm{d}s + \sum_{i=1}^\infty u(\theta_i)\tilde{k}_i \\ &\qquad = \sum_{i=2}^\infty \int_{i-1}^i \left(2^{1-i}(i-s+1)\frac{1}{2^{2s}s^4} \right) \mathrm{d}s + \sum_{k=1}^\infty \frac{1}{2^{3i}} \end{aligned}$$

is convergent. Moreover,

$$\begin{aligned} &\frac{1}{u^2(t)}\left\{ \int_t^\infty u(s)h_j(s)\mathrm{d}s + \sum_{i=\underline{i}(t)}^\infty u(\theta_i)\tilde{h}_i \right\} \\ &\quad = \frac{2^{2i}}{4(i-t+1)^2}\left\{ \sum_{i=\underline{i}(t)}^\infty \int_{i-1}^i \frac{2^{1-i}(i-s+1)}{2^{2s}s^4}\mathrm{d}s + \sum_{i=\underline{i}(t)}^\infty \frac{2}{2^{3i}} \right\} \\ &\quad \leq \frac{1}{(i-t+1)^2}\frac{1}{2^i} \\ &\quad = \beta(t), \end{aligned}$$

where

$$\int_1^t \beta(s)\mathrm{d}s = \sum_{2\le i<t} \int_{i-1}^{i} \frac{1}{(i-s+1)^2 2^i}\mathrm{d}s \le \frac{i}{2^i}.$$

Notice that

$$\frac{\int_1^t \beta(s)\mathrm{d}s}{(v(t))^\mu} \le \frac{i}{2^i(2^{i-2}(t-i+2))^\mu} \to 0, \quad t\to\infty.$$

By Theorem 3.2.3, for arbitrary $a, b \in \mathbb{R}$ there exist a solution $x(t)$ of equation (3.139) such that

$$x(t) = av(t) + bu(t) + o(u(t)(v(t))^\mu) \quad t\to\infty.$$

Example 3.3.4 Consider the equation

$$\begin{cases} x'' = \frac{x}{t^3 4^t}, & t \ne i, \\ \Delta x' - \frac{1}{2}x = \frac{x^2 \sin x}{i^2(1+x^2)4^i}, & t = i, \quad i = 1, 2, \dots \end{cases} \tag{3.140}$$

where $t \ge 1$. The principal and nonprincipal solutions of the associated homogeneous equation are given by (3.138). Then,

$$|f(t,x)| \le \frac{|x|}{t^3 4^t}, \quad |\tilde{f}_i(x)| \le \frac{|x|}{i^2 4^i}.$$

Choose $T = 1$, $h_1(t) = t^{-3}2^{-t}$, $g_1(x) = x$, $h_2(t) = 0$, $\tilde{h}_i = i^{-2}2^{-i}$, and $g_2(x) = x/2$. Let $k(t) = h_1(t)$, $\tilde{k}_i = \tilde{h}_i/2$ and

$$\beta(t) = \frac{1}{4(i-t+1)^2}, \; i-1 \le t < i, \quad i = 1, 2, \dots$$

Then,

$$|f(t,x_1) - f(t,x_2)| \le \frac{1}{2^{2t}t^3}|x_1 - x_2| \le \frac{k(t)}{v(t)}|x_1 - x_2|$$

and

$$|\tilde{f}(x_1) - \tilde{f}(x_2)| \le \frac{1}{i^2 2^{2i}}|x_1 - x_2| = \frac{\tilde{k}_i}{v(\theta_i)}|x_1 - x_2|.$$

We see that

$$\int_1^\infty v(s)k(s)\mathrm{d}s + \sum_{i=1}^\infty v(\theta_i)\tilde{k}_i = \sum_{i=2}^\infty \int_{i-1}^{i} \left(2^{i-2}(s-i+2)\frac{1}{s^3 2^s}\right)\mathrm{d}s + \sum_{k=1}^\infty \frac{1}{4i^2}$$

is convergent. Moreover,

$$\frac{1}{v^2(t)}\left\{\int_t^\infty v(s)h_j(s)\mathrm{d}s+\sum_{i=\underline{i}(t)}^\infty v(\theta_i)\tilde{h}_i\right\}$$

$$=\frac{2^4}{4^i(t-i+2)^2}\left\{\sum_{i=\underline{i}(t)}^\infty\int_{i-1}^i\frac{2^{i-2}(s-i+2)}{2^s s^3}\mathrm{d}s+\sum_{i=\underline{i}(t)}^\infty\frac{1}{2i^2}\right\}$$

$$\leq\frac{1}{4^i(t-i+1)^2}$$

$$=\beta(t).$$

Clearly,

$$\int_1^t\beta(s)\mathrm{d}s=o((u(t))^\mu)\to 0,\quad t\to\infty.$$

Hence, for arbitrary real numbers a and b, there exists a solution $x(t)$ of equation such that

$$x(t)=av(t)+bu(t)+o(v(t)(u(t))^\mu),\quad t\to\infty.$$

Note that, if we apply (3.107) we obtain

$$\frac{1}{u^2(t)}\left\{\int_t^\infty u(s)h_j(s)\mathrm{d}s+\sum_{i=\underline{i}(t)}^\infty u(\theta_i)\tilde{h}_i\right\}$$

$$=\frac{2^{2i}}{4(i-t+1)^2}\left\{\sum_{i=\underline{i}(t)}^\infty\int_{i-1}^i\frac{2^{1-i}(i-s+1)}{2^s s^3}\mathrm{d}s+\sum_{i=\underline{i}(t)}^\infty\frac{2^{1-i}}{2^i i^2}\right\}.$$

Since the series in the brackets are convergent we may find a constant $k>0$ such that

$$k:=\sum_{i=\underline{i}(t)}^\infty\int_{i-1}^i\frac{2^{1-i}(i-s+1)}{2^s s^3}\mathrm{d}s+\sum_{i=\underline{i}(t)}^\infty\frac{2^{1-i}}{2^i i^2}.$$

However,

$$\int_1^t\beta(s)\mathrm{d}s=\sum_{2\leq i<t}\int_{i-1}^i\frac{2^{2i}}{4(i-s+1)^2}k\mathrm{d}s=\sum_{2\leq i<t}\frac{k}{4}2^{2i-1}=\frac{k}{6}(2^{2\lfloor t\rfloor}-1),$$

and so

$$\lim_{t\to\infty}\frac{\frac{k}{6}(2^{2\lfloor t\rfloor}-1)}{(v(t))^\mu}\neq 0,\ \mu\in(0,1).$$

Hence Theorem 3.2.3 cannot be applied here.

Example 3.3.5 Consider the equation

$$\begin{cases} x'' = \frac{x^2+\ln t \sin x}{t^2 e^{2t}}, & t \neq i, \\ \Delta x' - \frac{1}{2}x = \frac{\ln x}{i^3 2^i}, & t = i, \quad i = 1, 2, \ldots \end{cases} \tag{3.141}$$

where $t \geq 1$. Then,

$$|f(t,x)| \leq \left(\frac{x}{t}\right)^2 e^{-2t} + \frac{1}{te^{2t}}, \quad |\tilde{f}_i(x)| \leq \frac{|x|}{i^3 2^i}.$$

Let $h_1(t) = e^{-2t}$, $g_1(x) = x^2$, $h_2(t) = e^{-2t}/t$, $\tilde{h}_i = i^{-2}2^{-i}$, $g_2(x) = x$ and take $T = 1$. Then, for $c \in [0,1]$, we observe that

$$\begin{aligned}
&\int_1^\infty u(s)^{1-c} v(s)^c (h_1(s) + h_2(s)) \mathrm{d}s + \sum_{i=1}^\infty u(\theta_i)^{1-c} v(\theta_i)^c \tilde{h}_i \\
&= \sum_{i=2}^\infty \int_{i-1}^i (2^{1-i}(i-s+1))^{1-c} (2^{i-2}(s-i+2))^c \left(e^{-2s} + \frac{1}{se^{2s}}\right) \mathrm{d}s \\
&\quad + \sum_{k=1}^\infty (2^{1-i})^{1-c} (2^{i-1})^c \frac{1}{i^2 2^i} \\
&\leq \sum_{i=2}^\infty \int_{i-1}^i 2^{i+2} \left(e^{-2s} + \frac{1}{se^{2s}}\right) \mathrm{d}s + \sum_{k=1}^\infty 2^{i+1} \frac{1}{i^2 2^i} \\
&\leq \sum_{i=2}^\infty 24 \left(\frac{2}{e^2}\right)^i \mathrm{d}s + \sum_{k=1}^\infty \frac{2}{i^2}
\end{aligned}$$

is convergent. Hence, for arbitrary $a, b \in \mathbb{R}$, there exist a solution $x(t)$ of equation (3.141) such that

$$x(t) = av(t) + o((u(t))^c (v(t))^{1-c}), \quad t \to \infty,$$

where u and v are the principal and nonprincipal solutions given by (3.138).

Example 3.3.6 Consider the equation

$$\begin{cases} x'' = \frac{x'+t^2 \cos x}{8(1+t^2)e^t}, & t \neq i, \\ \Delta x' - \frac{1}{2}x = \frac{x \ln i + \sin x}{8i^4}, & t = i, \quad i = 1, 2, \ldots \end{cases} \tag{3.142}$$

where $t \geq 1$. Clearly,

$$|f(t,x,y)| \leq \frac{|x| + |y|}{8e^t} =: F(t, |x|, |y|)$$

and

$$|\tilde{f}_i(x,y)| \le \frac{|x|}{8i^3} + \frac{|y|}{8i^4} =: \tilde{F}_i(|x|,|y|).$$

Observe that the homogeneous equation associated to (3.142) is again (3.137), and the functions given by (3.138) are its principal and nonprincipal solutions. Let $t_0 = 1$. Then

$$\begin{aligned}
&\int_1^\infty u(t)F(t,2cv(t),2cv'(t))\mathrm{d}t + \sum_{i=1}^\infty u(\theta_i)\tilde{F}_i(2cv(\theta_i),2cv'(\theta_i)) \\
&\qquad = \frac{c}{8}\sum_{i=2}^\infty \int_{i-1}^{i} \frac{(i-t+1)(t-i+3)}{8e^t}\mathrm{d}t + \frac{c}{8}\sum_{i=1}^\infty \left(\frac{2}{i^3}+\frac{1}{i^4}\right) \\
&\qquad < c
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{t\to\infty}\frac{u'(t)}{v'(t)}\left\{\int_1^t v(s)F(s,2cv(s),2cv'(s))\mathrm{d}s + \sum_{i=1}^{\bar{i}(t)} v(\theta_i)\tilde{F}_i(2cv(\theta_i),2cv'(\theta_i))\right\} \\
&= \lim_{t\to\infty}\frac{8c}{2^{2i}}\left\{\sum_{i=2}^{\lfloor t\rfloor} 2^{2i-4}\int_{i-1}^{i}\frac{(s-i+2)(s-i+3)}{8e^s}\mathrm{d}s + \sum_{i=1}^{\lfloor t\rfloor} 2^{2i-4}\left(\frac{1}{i^3}+\frac{1}{2i^4}\right)\right\} \\
&= 0.
\end{aligned}$$

So, by Theorem 3.2.6, equation (3.142) has a solution $x(t)$ which is increasing and satisfies

$$x(0)=0,\ x(t)>0 \text{ for } t>0,\ x'(t)>0 \text{ for } t\ge 0 \text{ and } \lim_{t\to\infty}\frac{x(t)}{v(t)} = c.$$

CHAPTER 4

ASYMPTOTIC INTEGRATION VIA CAUCHY FUNCTIONS

In this chapter we shall give asymptotic representation of solutions for

$$\begin{cases} x'' = f(t,x), & t \neq \theta_i, \\ \Delta x - p_i x = f_i(x), & t = \theta_i, \\ \Delta x' - \tilde{q}_i x' = \tilde{f}_i(x), & t = \theta_i, \end{cases} \tag{4.1}$$

by making use of the solutions of associated homogeneous equation

$$\begin{cases} x'' = 0, & t \neq \theta_i, \\ \Delta x - p_i x = 0, & t = \theta_i, \\ \Delta x' - \tilde{q}_i x' = 0, & t = \theta_i. \end{cases} \tag{4.2}$$

4.1 Introduction

In this section we state and prove two preparatory lemmas.

Lemma 4.1.1 *Let*

$$x_1(t) = t\prod_{i=1}^{\bar{i}(t)}(1+\tilde{q}_i) + \sum_{i=1}^{\bar{i}(t)}\left(\theta_i(p_i - \tilde{q}_i)\prod_{j=1}^{i-1}(1+\tilde{q}_j)\prod_{j=i+1}^{\bar{i}(t)}(1+p_j)\right), \tag{4.3}$$

$$x_2(t) = \prod_{i=1}^{\bar{i}(t)}(1+p_i). \tag{4.4}$$

The set $\{x_1(t), x_2(t)\}$ *is a fundamental set of solutions of* (4.2).

Proof. The functions are linearly independent, so all we need to show is that they are solutions. Obviously,

$$x_1'(t) = \prod_{i=1}^{\bar{i}(t)} (1+\tilde{q}_i), \quad x_1''(t) = 0, \quad x_2''(t) = 0, \quad t \neq \theta_k.$$

Next, we look at the impulse condition at $t = \theta_k$. By a direct calculation, we have

$$\begin{aligned}
\Delta x_1|_{t=\theta_k} =& x_1(\theta_k+) - x_1(\theta_k-) \\
=& \theta_k \prod_{i=1}^{k}(1+\tilde{q}_i) + \sum_{i=1}^{k}\left(\theta_i(p_i-\tilde{q}_i)\prod_{j=1}^{i-1}(1+\tilde{q}_j)\prod_{j=i+1}^{k}(1+p_j)\right) \\
&- \theta_k \prod_{i=1}^{k-1}(1+\tilde{q}_i) - \sum_{i=1}^{k-1}\left(\theta_i(p_i-\tilde{q}_i)\prod_{j=1}^{i-1}(1+\tilde{q}_j)\prod_{j=i+1}^{k-1}(1+p_j)\right) \\
=& (1+\tilde{q}_k)\theta_k \prod_{i=1}^{k-1}(1+\tilde{q}_i) + \theta_k(p_k-\tilde{q}_k)\prod_{j=1}^{i-1}(1+\tilde{q}_j)(1+p_k)\prod_{j=i+1}^{k-1}(1+p_j) \\
&+ \sum_{i=1}^{k-1}\left(\theta_i(p_i-\tilde{q}_i)\prod_{j=1}^{i-1}(1+\tilde{q}_j)(1+p_k)\prod_{j=i+1}^{k-1}(1+p_j)\right) \\
&- \theta_k \prod_{i=1}^{k-1}(1+\tilde{q}_i) - \sum_{i=1}^{k-1}\left(\theta_i(p_i-\tilde{q}_i)\prod_{j=1}^{i-1}(1+\tilde{q}_j)\prod_{j=i+1}^{k-1}(1+p_j)\right) \\
=& \tilde{q}_k\theta_k \prod_{i=1}^{k-1}(1+\tilde{q}_i) + \theta_k p_k \prod_{j=1}^{k-1}(1+\tilde{q}_j) - \theta_k\tilde{q}_k \prod_{j=1}^{k-1}(1+\tilde{q}_j) \\
&+ p_k \sum_{i=1}^{k-1}\left(\theta_i(p_i-\tilde{q}_i)\prod_{j=1}^{i-1}(1+\tilde{q}_j)\prod_{j=i+1}^{k-1}(1+p_j)\right) \\
=& p_k\theta_k \prod_{i=1}^{k-1}(1+\tilde{q}_i) + p_k \sum_{i=1}^{k-1}\left(\theta_i(p_i-\tilde{q}_i)\prod_{j=1}^{i-1}(1+\tilde{q}_j)\prod_{j=i+1}^{k-1}(1+p_j)\right) \\
=& p_k x_1(\theta_k),
\end{aligned}$$

and

$$\begin{aligned}
\Delta x_1'|_{t=\theta_k} &= x_1'(\theta_k+) - x_1'(\theta_k-) \\
&= \prod_{i=1}^{k}(1+\tilde{q}_i) - \prod_{i=1}^{k-1}(1+\tilde{q}_i) \\
&= (1+\tilde{q}_k)\prod_{i=1}^{k-1}(1+\tilde{q}_i) - \prod_{i=1}^{k-1}(1+\tilde{q}_i) \\
&= \tilde{q}_k x_1'(\theta_k).
\end{aligned}$$

In the same manner,

$$\begin{aligned}\Delta x_2|_{t=\theta_k} &= \prod_{i=1}^{k}(1+p_i) - \prod_{i=1}^{k-1}(1+p_i)\\ &= (1+p_k)\prod_{i=1}^{k-1}(1+p_i) - \prod_{i=1}^{k-1}(1+p_i)\\ &= p_k x_2(\theta_k),\end{aligned}$$

and $\Delta x_2'|_{t=\theta_k} = 0 = \tilde{q}_k x_2'(\theta_k)$.

Consequently, $x_1(t)$ and $x_2(t)$ are two linearly independent solutions of (4.2). □

Definition 4.1.1 *We say that a function $K(t,s)$ is a Cauchy function for (4.2) if $K(t,s)$ is a solution of (4.2) for each fixed s, $K(s,s) = 0$, and $K_t(s,s) = 1$.*

Lemma 4.1.2 *Let $x_1(t)$ and $x_2(t)$ be solutions of (4.2) defined by (4.3) and (4.4). Then the function $K(t,s)$ defined by*

$$K(t,s) = c_1(s)x_1(t) + c_2(s)x_2(t), \tag{4.5}$$

where

$$\begin{aligned}c_1(s) &= \prod_{i=1}^{\bar{i}(s)}(1+\tilde{q}_i)^{-1},\\ c_2(s) &= -\sum_{i=1}^{\bar{i}(s)}\left(\theta_i(p_i - \tilde{q}_i)\prod_{j=1}^{i}(1+p_j)^{-1}\prod_{j=i}^{\bar{i}(s)}(1+\tilde{q}_j)^{-1}\right)\end{aligned}$$

is a Cauchy function for (4.2).

Proof. Clearly, $K(t,s)$ is a solution of (4.2) for each fixed s, and $K(s,s) = 0$ in view of the convention that $\sum_{i=\underline{i}(s)}^{\bar{i}(s)} x_i = 0$ and $\prod_{i=\underline{i}(s)}^{\bar{i}(s)} x_i = 1$. Finally,

$$K_t(s,s) = \prod_{i=\underline{i}(s)}^{\bar{i}(s)}(1+\tilde{q}_i)^{-1} = 1.$$

□

4.2 Main Result

Theorem 4.2.1 *Let* $\{x_1(t), x_2(t)\}$ *be a fundamental set of solutions of* (4.2)*. Suppose that there exist functions* $g_j \in C(\mathbb{R}_+, \mathbb{R}_+)$, $j = 1, 2, 3$, $h_k \in C([1, \infty), \mathbb{R}_+)$, $k = 1, 2$ *satisfying*

$$|f(t,x)| \leq h_1(t)\, g_1\left(\frac{|x|}{x_2(t)}\right) + h_2(t); \quad t \geq T, \tag{4.6}$$

$$|f_i(x)| \leq \bar{h}_i\, g_2\left(\frac{|x|}{x_2(\theta_i)}\right); \qquad \theta_i \geq T, \tag{4.7}$$

$$\left|\tilde{f}_i(x)\right| \leq \tilde{h}_i\, g_3\left(\frac{|x|}{x_2(\theta_i)}\right); \qquad \theta_i \geq T, \tag{4.8}$$

where

$$\begin{aligned}&\int_T^\infty |K(t,s)|[h_1(s) + h_2(s)]\mathrm{d}s \\ &+ \sum_{i=\underline{i}(T)}^{\infty} \prod_{k=\underline{i}(t)}^{i} (1+p_k)^{-1}\bar{h}_i + |K(t,\theta_i+)|\tilde{h}_i < \infty\end{aligned} \tag{4.9}$$

with p_i, $\tilde{q}_i > 0$*. Then, for any given* $a, b \in \mathbb{R}$*, equation* (4.1) *has a solution* $x(t)$ *such that*

$$x(t) = ax_1(t) + bx_2(t) + o(x_2(t)), \quad t \to \infty. \tag{4.10}$$

Proof. We will use the Schauder fixed point theorem. The process is a little long, we first define the space

$$X = \left\{ y \in \mathrm{PLC}([T,\infty),\mathbb{R}) \middle| \frac{|y(t)|}{x_2(t)} \text{ is bounded} \right\}. \tag{4.11}$$

It is not difficult to see that X is a Banach space with the norm

$$\|y\| = \sup_{t\in[T,\infty)} \frac{|y(t)|}{x_2(t)},$$

see [49].

Let $S = \{y \in X | \, \|y(t) - bx_2(t)\| \leq 1\}$. Clearly, S is closed, bounded and convex subset of X. Define the operator $F : S \to X$ by

$$(Fy)(t) := bx_2(t) - \int_t^\infty K(t,s) f(s, y(s) + ax_1(s))\mathrm{d}s$$
$$+ \sum_{i=\underline{i}(t)}^{\infty} \left\{ \prod_{k=\underline{i}(t)}^{i} (1+p_k)^{-1} f_i(y(\theta_i) + ax_1(\theta_i)) - K(t, \theta_i +) \tilde{f}_i(y(\theta_i) + ax_1(\theta_i)) \right\}, \tag{4.12}$$

where $t \geq T$ and $K(t,s)$ is the Cauchy function defined by (4.5).

Step 1.

Each fixed point y of the operator F is a solution of

$$\begin{cases} y'' = f(t, y(t) + ax_1(t)), & t \neq \theta_i, \\ \Delta y - p_i y = f_i(y + ax_1(\theta_i)), & t = \theta_i, \\ \Delta y' - \tilde{q}_i y' = \tilde{f}_i(y + ax_1(\theta_i)), & t = \theta_i \end{cases} \tag{4.13}$$

for $t \geq T$.

Proof. Let y be a fixed point. This means that

$$y(t) := bx_2(t) - \int_t^\infty K(t,s) f(s, y(s) + ax_1(s))\mathrm{d}s$$
$$+ \sum_{i=\underline{i}(t)}^{\infty} \left\{ \prod_{k=\underline{i}(t)}^{i} (1+p_k)^{-1} f_i(y(\theta_i) + ax_1(\theta_i)) - K(t, \theta_i +) \tilde{f}_i(y(\theta_i) + ax_1(\theta_i)) \right\}$$

Let $t \neq \theta_l$. Then

$$y'(t) = bx_2'(t) - K(t,t) f(t, y(t) + ax_1(t)) - \int_t^\infty \prod_{i=\underline{i}(t)}^{\bar{i}(s)} (1+\tilde{q}_i)^{-1} f(s, y(s) + ax_1(s))\mathrm{d}s$$
$$- \sum_{i=\underline{i}(t)}^{\infty} \prod_{k=\underline{i}(t)}^{i} (1+\tilde{q}_k)^{-1} \tilde{f}_i(y(\theta_i) + ax_1(\theta_i)).$$

In view of $K(t,t) = 0$ it follows that

$$y''(t) = bx_2''(t) + \prod_{i=\underline{i}(t)}^{\bar{i}(t)} (1+\tilde{q}_i)^{-1} f(t, y(t) + ax_1(t)) = f(t, y(t) + ax_1(t)).$$

For $t=\theta_l$, by using Lemma 4.1.2 we have

$$\begin{aligned}
\Delta y|_{t=\theta_l} =& b[x_2(\theta_l+)-x_2(\theta_l)] - \int\limits_{\theta_l}^{\infty}[K(\theta_l+,s)-K(\theta_l,s)]f(s,y(s)+ax_1(s))\mathrm{d}s \\
&+\sum_{i=l}^{\infty}\left\{\left[\prod_{k=l+1}^{i}(1+p_k)^{-1}-\prod_{k=l}^{i}(1+p_k)^{-1}\right]f_i(y(\theta_i)+ax_1(\theta_i))\right. \\
&\left.-[K(\theta_l+,\theta_i+)-K(\theta_l,\theta_i+)]\tilde{f}_i(y(\theta_i)+ax_1(\theta_i))\right\} \\
&+\prod_{k=l+1}^{l}(1+p_k)^{-1}f_l(y(\theta_l)+ax_1(\theta_l))-K(\theta_l+,\theta_l+)\tilde{f}_l(y(\theta_l)+ax_1(\theta_l)) \\
=& bp_lx_2(\theta_l)-p_l\int\limits_{\theta_l}^{\infty}K(\theta_l,s)f(s,y(s)+ax_1(s))\mathrm{d}s \\
&+p_l\sum_{i=l}^{\infty}\left\{\prod_{k=l}^{i}(1+p_k)^{-1}f_i(y(\theta_i)+ax_1(\theta_i))\right. \\
&\left.-K(\theta_l,\theta_i+)\tilde{f}_i(y(\theta_i)+ax_1(\theta_i))\right\}+f_l(y(\theta_l)+ax_1(\theta_l)) \\
=& p_ly(\theta_l)+f_l(y(\theta_l)+ax_1(\theta_l))
\end{aligned}$$

and

$$\begin{aligned}
\Delta y'|_{t=\theta_l} =& b[x_2'(\theta_l+)-x_2'(\theta_l)] \\
&-\int\limits_{\theta_l}^{\infty}\left[\prod_{i=l+1}^{\bar{i}(s)}(1+\tilde{q}_i)^{-1}-\prod_{i=l}^{\bar{i}(s)}(1+\tilde{q}_i)^{-1}\right]f(s,y(s)+ax_1(s))\mathrm{d}s \\
&-\sum_{i=l}^{\infty}\left[\prod_{k=l+1}^{i}(1+\tilde{q}_k)^{-1}-\prod_{k=l}^{i}(1+\tilde{q}_k)^{-1}\right]\tilde{f}_i(y(\theta_i)+ax_1(\theta_i)) \\
&+\prod_{k=l}^{l}(1+\tilde{q}_k)^{-1}\tilde{f}_l(y(\theta_l)+ax_1(\theta_l)).
\end{aligned}$$

From Lemma 4.1.1 it follows that

$$\begin{aligned}
\Delta y'|_{t=\theta_l} =& b\tilde{q}_lx_2'(\theta_l)-\tilde{q}_l\int\limits_{\theta_l}^{\infty}\prod_{i=l}^{\bar{i}(s)}(1+\tilde{q}_i)^{-1}f(s,y(s)+ax_1(s))\mathrm{d}s \\
&-\tilde{q}_l\sum_{i=l}^{\infty}\prod_{k=l}^{i}(1+\tilde{q}_k)^{-1}\tilde{f}_i(y(\theta_i)+ax_1(\theta_i))+\tilde{f}_l(y(\theta_l)+ax_1(\theta_l)) \\
=& \tilde{q}_ly'(\theta_l)+\tilde{f}_l(y(\theta_l)+ax_1(\theta_l)).
\end{aligned}$$

Step 2.

$F: S \to S$ is completely continuous.

(i) *$F(S) \subset S$:*

We need to show that $(Fy)(t)$ is continuous on $(\theta_i, \theta_{i+1}]$ and $(Fy)(\theta_i+)$ exists for each i for which $\theta_i \geq 0$, and that $\|Fy - bx_2\| \leq 1$.

Since $p_i > 0$ clearly $x_2(t) \geq 1$, and so

$$\frac{|y(t) + ax_1(t)|}{x_2(t)} \leq 1 + |a| + |b| := M_0.$$

Since $g_j \in \mathrm{C}(\mathbb{R}_+, \mathbb{R}_+)$ there exist constants $K_j > 0$ such that $\max\limits_{0 \leq \tau \leq M_0} g_j(\tau) = K_j$, $j = 1, 2, 3$, which implies that

$$|f(s, y(s) + ax_1(s))| \leq K_1 h_1(s) + h_2(s), \tag{4.14}$$

$$|f_i(y(\theta_i) + ax_1(\theta_i))| \leq K_2 \bar{h}_i, \tag{4.15}$$

and

$$|\tilde{f}_i(y(\theta_i) + ax_1(\theta_i))| \leq K_3 \tilde{h}_i. \tag{4.16}$$

Fix t_1, and let $t < t_1$, then

$$\begin{aligned}|(Fy)(t) - (Fy)(t_1)| = \Bigg| & b[x_2(t) - x_2(t_1)] \\ & - \int_t^{t_1} K(t,s) f(s, y(s) + ax_1(s)) \mathrm{d}s \\ & + \int_{t_1}^{\infty} [K(t_1, s) - K(t,s)] f(s, y(s) + ax_1(s)) \mathrm{d}s \\ & + \sum_{i=\underline{i}(t)}^{\bar{i}(t_1)} \left\{ \prod_{k=\underline{i}(t)}^{i} (1+p_k)^{-1} f_i(y(\theta_i) + ax_1(\theta_i)) - K(t, \theta_i+) \tilde{f}_i(y(\theta_i) + ax_1(\theta_i)) \right\} \\ & + \sum_{i=\underline{i}(t_1)}^{\infty} \left\{ \left[\prod_{k=\underline{i}(t)}^{i} (1+p_k)^{-1} - \prod_{k=\underline{i}(t_1)}^{i} (1+p_k)^{-1} \right] f_i(y(\theta_i) + ax_1(\theta_i)) \right. \\ & \left. - [K(t, \theta_i+) - K(t_1, \theta_i+)] \tilde{f}_i(y(\theta_i) + ax_1(\theta_i)) \right\} \Bigg|.\end{aligned}$$

Apply the conditions (4.14)-(4.15) to have

$$
\begin{aligned}
|(Fy)(t)-(Fy)(t_1)| \le & |b|[x_2(t)-x_2(t_1)] + \int_t^{t_1} |K(t,s)|[K_1h_1(s)+h_2(s)]\mathrm{d}s \\
& + \int_{t_1}^{\infty} [|K(t_1,s)|+|K(t,s)|][K_1h_1(s)+h_2(s)]\mathrm{d}s \\
& + \sum_{i=\underline{i}(t)}^{\bar{i}(t_1)} \left\{ \prod_{k=\underline{i}(t)}^{i} (1+p_k)^{-1}K_2\bar{h}_i + |K(t,\theta_i+)|K_3\tilde{h}_i \right\} \\
& + \sum_{i=\underline{i}(t_1)}^{\infty} \left\{ \left[\prod_{k=\underline{i}(t)}^{i} (1+p_k)^{-1} + \prod_{k=\underline{i}(t_1)}^{i} (1+p_k)^{-1} \right] M_2K_2\bar{h}_i \right. \\
& \left. + [|K(t,\theta_i+)| + K(t_1,\theta_i+)]|K_3\tilde{h}_i \right\}.
\end{aligned}
$$

The condition (4.9) allows us to use the Lebesgue dominated convergence theorem and Weierstrass-M test. Thus, we obtain

$$\lim_{t\to t_1-}(Fy)(t) = (Fy)(t_1).$$

In a similar way we can show that $\lim_{t\to t_1+}(Fy)(t) = (Fy)(t_1)$ for $t_1 \neq \theta_k$, and $\lim_{t\to\theta_k+}(Fy)(t)$ exist for each $k = 1, 2, \dots$. If we choose T sufficiently large so that

$$\int_T^{\infty} |K(t,s)|h_1(s)\mathrm{d}s \le \frac{1}{4K_1}, \qquad \int_T^{\infty} |K(t,s)|h_2(s)\mathrm{d}s \le \frac{1}{4}, \tag{4.17}$$

$$\sum_{i=\underline{i}(T)}^{\infty} \prod_{k=\underline{i}(t)}^{i} \bar{h}_i \le \frac{1}{4K_2}, \qquad \sum_{i=\underline{i}(T)}^{\infty} |K(t,\theta_i+)|\tilde{h}_i \le \frac{1}{4K_3} \tag{4.18}$$

then, from $x_1(t) \ge 1$ and (4.17)-(4.18) we can write

$$
\begin{aligned}
\frac{|(Fy)(t)-bx_2(t)|}{x_2(t)} & \le \int_t^{\infty} |K(t,s)|\big\{K_1h_1(s)+h_2(s)\big\}\mathrm{d}s \\
& + \sum_{i=\underline{i}(t)}^{\infty} \{ \prod_{k=\underline{i}(t)}^{i} (1+p_k)^{-1}K_2\bar{h}_i + |K(t,\theta_i+)|K_3\tilde{h}_i \} \\
& \le 1.
\end{aligned}
$$

Taking supremum over $[T,\infty)$ it follows that $\|Fy - bx_2\| \le 1$, i.e., $F(S) \subset S$.

(ii) *F is continuous:*

Let $\{y_n\} \in S$ such that $y_n \to y \in S$ as $n \to \infty$. Thus, with the help of the inequalities (4.14)-(4.15) one has

$$|(Fy_n)(t) - (Fy)(t)| \le \int_t^\infty 2|K(t,s)|\{K_1 h_1(s) + h_2(s)\}\mathrm{d}s$$
$$+ 2\sum_{i=\underline{i}(t)}^{\infty}\{\prod_{k=\underline{i}(t)}^{i}(1+p_k)^{-1}K_2\bar{h}_i + |K(t,\theta_i+)|K_3\tilde{h}_i\} \tag{4.19}$$

which, in view of the condition (4.9), is finite. Thus, by means of the Lebesgue dominated convergence theorem and Weierstrass-M test, we have

$$\lim_{n\to\infty}(Fy_n)(t) = (Fy)(t), \quad n \to \infty,$$

which shows that $F : S \to X$ is a continuous operator.

(iii) *F is relatively compact:*

Let $\{y_n\} \in S$. We want to prove that there exists a subsequence $\{y_{n_k}\} \in S$ so that Fy_{n_k} is convergent. We decompose F as $F = F_1 + F_2$, where

$$(F_1 y_n)(t) = bx_2(t) - \int_t^\infty K(t,s)f(s, y_n(s) + ax_1(s))\mathrm{d}s,$$

$$(F_2 y_n)(t) = \sum_{i=\underline{i}(t)}^{\infty}\left\{\prod_{k=\underline{i}(t)}^{i}(1+p_k)^{-1}f_i(y_n(\theta_i) + ax_1(\theta_i))\right.$$
$$\left. - K(t,\theta_i+)\tilde{f}_i(y_n(\theta_i) + ax_1(\theta_i))\right\}.$$

We will show that both F_1 and F_2 are compact. Denote

$$f_n(s) := f(s, y_n(s) + ax_1(s)).$$

From (4.6) and (4.9) there exists a constant $c_1 > 0$ such that $\|f_n\|_{L^1([T,\infty))} \le c_1$, so the first hypothesis of Lemma 2.2.1 holds. Now we define $(\tau_h f)(s) := f(s+h)$, and see from (4.6) that

$$\int_T^\infty |(\tau_h f_n)(s) - f_n(s)|\mathrm{d}s \;\le\; \int_T^\infty |f_n(s+h)|\mathrm{d}s + \int_T^\infty |f_n(s)|\mathrm{d}s$$
$$= \int_{T+h}^\infty |f_n(s)|\mathrm{d}s + \int_T^\infty |f_n(s)|\mathrm{d}s$$
$$\le \int_T^\infty 2|f_n(s)|\mathrm{d}s \le \int_T^\infty 2[K_1 h_1(s) + h_2(s)]\mathrm{d}s.$$

By virtue of Lebesgue dominated convergence theorem we deduce that

$$\|\tau_h f_n - f_n\|_{L^1([T,\infty))} \to 0, \quad h \to 0. \tag{4.20}$$

So, by Lemma 2.2.1 there exists a subsequence $\{f_{n_k}\}$ which is convergent in $L^1([T,\infty))$. From the continuity of f_{n_k}, its limit is $f(t, y(t) + ax_1(t))$, i.e.,

$$\|f_{n_k} - f\|_{L^1([T,\infty))} \to 0, \quad k \to \infty.$$

Let

$$z(t) := K(t,s)f(s, y(s) + ax_1(s)), \quad t \ge T.$$

Then

$$\|z_{n_k} - z\|_{L^1[T,\infty)} = \int_T^\infty |K(t,s)(f_{n_k}(s) - f(s))|\mathrm{d}s.$$

Due to (4.6) it follows that

$$\|z_{n_k} - z\|_{L^1[T,\infty)} \le \int_T^\infty 2|K(t,s)|[(K_1 h_1(s) + h_2(s)]\mathrm{d}s.$$

Using (4.9) and Lebesgue dominated convergence theorem, we obtain

$$\lim_{k\to\infty} z_{n_k} = z \in L^1[T,\infty).$$

But

$$(F_1 y_{n_k})(t) = bx_2(t) - \int_t^\infty z_{n_k}(s)\mathrm{d}s.$$

Hence we conclude that $(F_1 y_{n_k})(t) \to (F_1 y)(t)$ as $k \to \infty$.

To show that F_2 is also compact, we will employ Lemma 2.2.2. Put $\bar{f}_n(\theta_i) = f_i(y_n(\theta_i) + ax_1(\theta_i))$ and $\tilde{f}_n(\theta_i) = \tilde{f}_i(y_n(\theta_i) + ax_1(\theta_i))$. From (4.7) and (4.8) it can be seen that each element of the sets $\{\bar{f}_n\}$ and $\{\tilde{f}_n\}$ are pointwise bounded. Let $\epsilon > 0$ be given and choose $j \in \mathbb{N}$ sufficiently large so that

$$\sum_{i=j}^{\infty} \prod_{k=\underline{i}(t)}^{i} (1+p_k)^{-1} \bar{h}_i < \frac{\epsilon}{K_2}, \qquad \sum_{i=j}^{\infty} |K(t, \theta_i +)| \tilde{h}_i < \frac{\epsilon}{K_3}.$$

Then we see that

$$\sum_{i=j}^{\infty} |\bar{f}_n(\theta_i)| < \epsilon, \qquad \sum_{i=j}^{\infty} |\tilde{f}_n(\theta_i)| < \epsilon,$$

and so all hypotheses of Lemma 2.2.2 hold. Thus $\{\bar{f}_n\}$ and $\{\tilde{f}_n\}$ are compact in $\ell^1[T, \infty)$. Therefore, there exist subsequences $\{\bar{f}_{n_l}\}$ and $\{\tilde{f}_{n_l}\}$ which are convergent in $\ell^1[T, \infty)$, and from the continuity they converge to $f_i(y(\theta_i) + ax_1(\theta_i))$ and $\tilde{f}_i(y(\theta_i) + ax_1(\theta_i))$, respectively. Let

$$w(t) = \prod_{k=\underline{i}(t)}^{i} (1+p_k)^{-1} f_i(y(\theta_i) + ax_1(\theta_i)) - K(t, \theta_i +) \tilde{f}_i(y(\theta_i) + ax_1(\theta_i)).$$

By means of Minkowski's inequality and the conditions (4.7), (4.8) we obtain

$$\begin{aligned}
\|(F_2 y_{n_l}) - w\|_{\ell^1([T,\infty))} &\leq \sum_{i=\underline{i}(T)}^{\infty} \left| \prod_{k=\underline{i}(t)}^{i} (1+p_k)^{-1} \bar{f}_{n_l} + K(t, \theta_i +) \tilde{f}_{n_l} - w(\theta_i) \right| \\
&\leq \sum_{i=\underline{i}(T)}^{\infty} \left(\prod_{k=\underline{i}(t)}^{i} (1+p_k)^{-1} |\bar{f}_{n_l} - f_i| + |K(t, \theta_i +)| |\tilde{f}_{n_l} - \tilde{f}_i| \right) \\
&\leq \sum_{i=\underline{i}(T)}^{\infty} \left(2 \prod_{k=\underline{i}(t)}^{i} (1+p_k)^{-1} K_2 \bar{h}_i + 2 |K(t, \theta_i +)| K_3 \tilde{h}_i \right).
\end{aligned}$$

The condition (4.9) allows us to use the Weierstrass-M test, so we conclude that $\{F_2 y_{n_l}\}$ is a convergent subsequence of $\{F_2 y_n\}$. Therefore, F_2 is compact in S. Hence $F = F_1 + F_2$ is completely continuous.

By the Schauder fixed point theorem, we may conclude that the operator F defined by (4.12) has a fixed point $y \in X$. In view of (4.6)-(4.9), we easily see that $\|y - bx_2\| \to 0$ as $t \to \infty$, or

$$\lim_{t \to \infty} \frac{x(t) - ax_1(t) - bx_2(t)}{x_2(t)} = 0.$$

$\square$

Corollary 4.2.1 Suppose that (4.6)-(4.8) are satisfied, and (4.9) is replaced by

$$\int_T^\infty s[h_1(s) + h_2(s)]\mathrm{d}s + \sum_{i=\underline{i}(T)}^{\infty} (\bar{h}_i + \theta_i \tilde{h}_i) < \infty, \tag{4.21}$$

and $p_i, \tilde{q}_i > 0$ satisfy

$$\sum_{i=\underline{i}(T)}^{\infty} (p_i + \tilde{q}_i) < \infty. \tag{4.22}$$

Then (4.10) can be replaced by

$$x(t) = ax_1(t) + bx_2(t) + o(x_1(t)), \quad t \to \infty. \tag{4.23}$$

Proof. Define

$$\tilde{X} = \left\{ y \in \mathrm{PLC}([T,\infty),\mathbb{R}) | \frac{|y(t)|}{x_1(t)} \text{ is bounded} \right\}. \tag{4.24}$$

It is a Banach space with the norm

$$\|y\| = \sup_{t\in[T,\infty)} \frac{|y(t)|}{x_1(t)}.$$

Let $\tilde{S} = \{y \in \tilde{X} | \, \|y(t) - ax_1(t)\| \le 1\}$. Define the operator $\tilde{F} : \tilde{S} \to \tilde{X}$ by

$$(\tilde{F}y)(t) := ax_1(t) - \int_t^\infty K(t,s) f(s, y(s) + bx_2(s))\mathrm{d}s$$
$$+ \sum_{i=\underline{i}(t)}^{\infty} \left\{ \prod_{k=\underline{i}(t)}^{i} (1 + p_k)^{-1} f_i(y(\theta_i) + bx_2(\theta_i)) - K(t, \theta_i+) \tilde{f}_i(y(\theta_i) + bx_2(\theta_i)) \right\}.$$

where $t \ge T$ and $K(t,s)$ is the Cauchy function defined by (4.5). The condition (4.22) implies that

$$\prod_{i=\underline{i}(T)}^{\infty} (1 + p_i) < \infty, \quad \prod_{i=\underline{i}(T)}^{\infty} (1 + \tilde{q}_i) < \infty, \text{ and } \prod_{i=\underline{i}(T)}^{\infty} (1 + p_i)^{-1} \le 1.$$

So there exists a constant M such that $K(t,s) \le Ms$. This shows that (4.9) can be replaced by (4.21).

The remaining part of the proof follows the same steps to that of Theorem 4.2.1, so we omit the details. □

Remark 4.2.1 Suppose (4.22) holds. Then $x_1(t)$ and $x_2(t)$ given by (4.3) and (4.4), respectively, turn out to be nonprincipal and principal solutions of the homogeneous equation (4.2). Namely, they satisfy the properties (2.6)-(2.8).

Indeed,

$$\begin{aligned}\frac{x_1(t)}{x_2(t)} =& t\prod_{i=1}^{\bar{i}(t)}(1+\tilde{q}_i)(1+p_i)^{-1} \\ &+\sum_{i=1}^{\bar{i}(t)}\left(\theta_i(p_i-\tilde{q}_i)\prod_{0\le\theta_j<\theta_i}(1+\tilde{q}_j)\prod_{j=1}^{i-1}(1+p_j)^{-1}\right).\end{aligned}$$

Since $p_i, \tilde{q}_i > 0$, from (4.22) we conclude that

$$\lim_{t\to\infty}\frac{x_1(t)}{x_2(t)} = \infty.$$

Moreover,

$$\int_T^\infty \frac{\mathrm{dt}}{x_2^2(t)} = \sum_{i=\underline{i}(T)}^{\infty}\int_{\theta_{i-1}}^{\theta_i}\frac{\mathrm{dt}}{\prod\limits_{k=0}^{i-1}(1+p_k)^2} = \sum_{i=\underline{i}(T)}^{\infty}\frac{\theta_i-\theta_{i-1}}{\prod\limits_{k=0}^{i-1}(1+p_k)^2}.$$

From (4.22) we have

$$\lim_{i\to\infty}\frac{1}{\prod\limits_{k=0}^{i-1}(1+p_k)^2} \neq 0,$$

which implies that

$$\int_T^\infty \frac{\mathrm{dt}}{x_2^2(t)} = \infty,$$

and

$$\begin{aligned}&\int_T^\infty \frac{\mathrm{dt}}{x_1^2(t)} \\ &= \sum_{i=\underline{i}(T)}^{\infty}\int_{\theta_{i-1}}^{\theta_i}\frac{\mathrm{dt}}{\left(t\prod\limits_{i=1}^{\bar{i}(t)}(1+\tilde{q}_i)+\sum\limits_{i=1}^{\bar{i}(t)}\left(\theta_i(p_i-\tilde{q}_i)\prod\limits_{j=1}^{i-1}(1+\tilde{q}_j)\prod\limits_{j=i+1}^{\bar{i}(t)}(1+p_j)\right)\right)^2}.\end{aligned}$$

Let $f(t) = \dfrac{1}{t^2}$ and $g(t) = \dfrac{1}{x_1^2(t)}$. From (4.22),

$$\lim_{t\to\infty}\frac{f(t)}{g(t)} < \infty.$$

So, by means of limit comparison test we conclude that

$$\int_T^\infty \frac{\mathrm{dt}}{x_1^2(t)} < \infty.$$

Finally,

$$x_1'(t) = \prod_{0\le\theta_i<t} (1+\tilde{q}_i) \quad \text{and} \quad x_2'(t) = 0.$$

So, $\tilde{q}_i > 0$ implies that

$$\frac{x_1'(t)}{x_1(t)} > \frac{x_2'(t)}{x_2(t)}.$$

Hence the hypotheses (2.6)-(2.8) are satisfied and we conclude that $x_2(t)$ is the principal and $x_1(t)$ is a nonprincipal solution of equation (4.2).

Remark 4.2.2 If (4.22) holds, then, being principal and nonprincipal solutions of (4.2), u and v satisfy $\lim_{t\to\infty} \frac{x_2(t)}{x_1(t)} = 0$. So, it can be easily seen that, if

$$\lim_{t\to\infty} \frac{x(t) - ax_1(t) - bx_2(t)}{x_2(t)} = 0,$$

then

$$\lim_{t\to\infty} \frac{x(t) - ax_1(t) - bx_2(t)}{x_1(t)} = 0.$$

Hence, the conclusion of Theorem 4.2.1

$$x(t) = ax_1(t) + bx_2(t) + o(x_2(t)), \quad t \to \infty$$

implies the conclusion of Corollary 4.2.1

$$x(t) = ax_1(t) + bx_2(t) + o(x_1(t)), \quad t \to \infty.$$

Remark 4.2.3 When the impulse effects in equation (4.1) are dropped, i.e., $p_i = \tilde{q}_i = 0$ and $f_i(x) = \tilde{f}_i(x) = 0$, we obtain the classical asymptotic representation

$$x(t) = at + b + o(1), \quad t \to \infty.$$

4.3 An Example

Example 4.3.1 Consider the equation

$$\begin{cases} x'' = \frac{x^2\ln(1/t^2)+t^3e^{-t}}{t^4+1}, & t \neq i, \\ \Delta x - \frac{1}{4i^2-1}x = \ln\left(\frac{1}{i^4}\right)\sin x, & t = i, \\ \Delta x' - \frac{1}{4i^2-1}x' = \frac{1}{(i^2+1)^2e^{x^2}}, & t = i, \quad i = 1, 2, \ldots \end{cases} \tag{4.25}$$

where $t \geq 1$. The corresponding homogeneous equation is

$$\begin{cases} x'' = 0, & t \neq i, \\ \Delta x - \frac{1}{4i^2-1}x = 0, & t = i, \\ \Delta x' - \frac{1}{4i^2-1}x' = 0, & t = i, \quad i = 1, 2, \ldots . \end{cases} \tag{4.26}$$

The functions

$$x_1(t) = t\prod_{j=1}^{i-1}\left(\frac{4j^2}{4j^2-1}\right), \quad t \in (i-1, i], \quad i = 1, 2, \ldots \tag{4.27}$$

and

$$x_2(t) = \prod_{j=1}^{i-1}\left(\frac{4j^2}{4j^2-1}\right), \quad t \in (i-1, i], \quad i = 1, 2, \ldots \tag{4.28}$$

are linearly independent solutions of (4.26).

Let $T = 1$. First we observe that $p_i = \tilde{q}_i = 1/(4i^2-1)$ and, from the Wallis' product [52]

$$\prod_{i=1}^{\infty}\left(1 + \frac{1}{4i^2-1}\right) = \frac{\pi}{2}.$$

On the other hand,

$$|f(t,x)| \leq \frac{x^2}{t^2(t^4+1)} + \frac{e^{-t}}{t}, \quad |f_i(x)| \leq \frac{|x|}{i^4}, \quad |\tilde{f}_i(x)| \leq \frac{1}{(i^2+1)^2e^{x^2}}.$$

So, we may choose $h_1(t) = 1/(t^2(t^4+1))$, $h_2(t) = e^{-t}/t$, $g_1(x) = (2x/\pi)^2$ and $\bar{h}_i = i^{-4}$, $\tilde{h}_i = 1/(i^2+1)^2$, $g_2(x) = 2x/\pi$, $g_3(x) = e^{-(2x/\pi)^2}$. Then, it can be easily seen that (4.6)-(4.8) are satisfied. Since

$$c_1(s) = \prod_{j=1}^{\lfloor s\rfloor}\left(\frac{4j^2-1}{4j^2}\right), \quad c_2(s) = 0$$

we have further that

$$|K(t,s)| = |c_1(s)x_1(t) + c_2(s)x_2(t)| < \frac{2t}{\pi} < s.$$

This implies that

$$\int_1^\infty s(h_1(s)+h_2(s))\mathrm{d}s + \sum_{i=1}^\infty \theta_i(\bar{h}_i + \tilde{h}_i)$$
$$= \sum_{i=2}^\infty \int_{i-1}^{i} \left(\frac{1}{s(s^4+1)} + e^{-s}\right)\mathrm{d}s + \sum_{i=2}^\infty i\left(\frac{1}{i^4} + \frac{1}{(i^2+1)^2}\right)$$

is convergent. Since all hypotheses of Theorem 4.2.1 are satisfied, we conclude that there exist a solution $x(t)$ of (4.25) such that for arbitrary real numbers a and b

$$x(t) = ax_1(t) + bx_2(t) + o(x_2(t)), \quad t \to \infty.$$

CHAPTER 5

CONCLUSION

In this thesis we mainly dealt with asymptotic integration of impulsive differential equations with continuous or discontinuous solutions. Our main instruments while doing this study are principal and nonprincipal solutions and Cauchy functions. By means of these intruments, we have displayed various asymptotic integration results for differential equations under impulse effect, so the gap in the literature of this matter is partially fulfilled.

In the first chapter the main problem has been introduced and some related results in the literature have been compiled. However, it can be seen that there is no counterparts of our study although the existence of principal and nonprincipal solutions is known for impulsive differential equations with continuous solutions.

In the second chapter we have mentioned preliminary concepts about impulsive differential equations and recalled some theorems on the fixed point theory. Also, we have proved the existence of principal and nonprincipal solutions for impulsive differential equations with discontinuous solutions by dividing them into two types according to impulse effects, namely separated impulse conditions and mixed impulse conditions. However, in the separated case we encountered some constraint on the impulse conditions. Therefore, our results are open to be improved if the existence of principal and nonprincipal solutions could be proved by removing the restriction which appear on the impulse effect. Furthermore, for the general case, i.e., in problem (2.11) neither of the sequences p_i, q_i, $\tilde{p}_i$, $\tilde{q}_i$ is zero, the problem of existence of principal and nonprincipal solutions has not yet been proven.

The third chapter consists of many applications about asymptotic integration of impulsive differential equations. We have presented several results on asymptotic representation of solutions with the help of the principal and nonprincipal solutions. Our results are new even for impulsive differential equations of continuous solutions despite the fact that the existence of principal and nonprincipal solutions for these equations was proven in 2010. When the impulse effects on the solution are dropped, i.e., impulsive differential equations with continuous solutions are obtained, Polya and Trench factorizations are usually adequate tools to convert an impulsive differential equation into an impulsive integral equation. In the literature there is no results about the factorizations of impulsive differential with discontinuous solutions, accordingly, it needs much more effort to deal with the asymptotic integration of this type of impulsive differential equations. For the equations having no impulse effect on the solutions, we have obtained miscellaneous results on asymptotic representations and properties of solutions such as monotonicity of the solutions or representations of solutions depending on a parameter. Finally we have expressed subsidiary examples that support our results.

In the fourth chapter, we first constructed a Cauchy function for discontinuous impulsive differential equations with separated impulse conditions. We should point out that the concept of Cauchy functions for impulsive differential equations is rarely found in the literature even for impulsive differential equations with continuous solutions. With the help of the Cauchy function, asymptotic formulas of the solutions have been obtained, and an example has been presented to support that our results are theoretically applicable.

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