FREE VIBRATION ANALYSIS OF FUNCTIONALLY GRADED RECTANGULAR NANO-PLATES CONSIDERING SPATIAL VARIATION OF THE NONLOCAL PARAMETER

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ABSTRACT

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This study presents a new nonlocal elasticity based analysis method for free vibrations of functionally graded rectangular nano-plates. The method allows taking into account spatial variation of the nonlocal parameter. Governing partial differential equations and associated boundary conditions are derived by employing the variational approach and applying Hamilton's principle. All required material properties are assumed to be functions of thickness coordinate in the derivations. Displacement field is expressed in a unified way to be able to produce numerical results pertaining to three different plate theories, namely Kirchhoff, Mindlin, and third-order shear deformation theories. The equations are solved numerically by means of the generalized differential quadrature method. Proposed procedures are verified through comparisons made to the results available in the literature. Further numerical results are generated by considering functionally graded simply-supported and cantilever nano-plates undergoing free vibrations. These findings demonstrate influences of factors such as dimensionless plate length, plate theory, nonlocal parameter ratio, and power-law index upon natural vibration frequencies.

Keywords: Nonlocal elasticity, Free vibration, Higher order shear deformation plate theory, Functionally graded material, Generalized differential quadrature method

ÖZ

LOKAL OLMAYAN PARAMETRENIN UZAYSAL DEĞİŞİMİNİ GÖZ ÖNÖNDE BULUNDURARAK FONKSIYONEL DERECELENDİRİLMİŞ DIKDÖRTGEN NANO-PLAKLARIN SERBEST TİTREŞİM ANALİZLERİ

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Bu çalışma fonksiyonel derecelendirilmiş dikdörtgen nano-plakların serbest titreşimleri analizi için yeni lokal olmayan elastisite teorisi bazında bir analiz yöntemini sunmaktadır. Bu yöntem lokal olmayan parametrenin uzaysal değişiminin göz önünde bulundurulmasına imkan tanımaktadır. Yönetici kısmı diferansiyel denklemler ve ilgili sınır koşulları varyasyonel yaklaşım kullanılarak ve Hamilton prensibi uygulanarak türetilmiştir. Derivasyonda gereken tüm malzeme özelliklerinin kalınlık koordinatı fonksiyonları olmaları varsayılmaktadır. Kirchhoff, Mindlin, ve üçüncü dereceden kesme deformasyon teorisi adlarında üç farklı plak teorileri ile ilgili sayısal sonuçları üretebilmek için yerdeğiştirme alanı birleştirilmiş bir yöntem ile ifade edilmiştir. Denklemler genelleştirilmiş diferansiyel kare yapma metodu ile sayısal olarak çözülmüştür. Önerilen yöntemler literatürde mevcut sonuçlar ile yapılan karşılaştırmalar vasıtasıyla doğrulanmıştır. Daha fazla sayısal sonuçlar serbest titreşim altında basit mesnetli ve ankastre fonksiyonel derecelendirilmiş nano-plaklar göz önünde bulundurularak üretilmiştir. Bu sonuçlar boyutsuz plak uzunluğu, plak teorisi, lokal olmayan parametre oranı, ve üstel indeks gibi faktörlerin doğal titreşim frekanslar üzerinde etkilerini göstermektedir.

Anahtar Kelimeler: Lokal olmayan elastisite teorisi, Serbest titreşim, Yüksek mertebeden kesme deformasyon plak teorisi, Fonksiyonel derecelendirilmiş malzeme, Genelleştirilmiş diferansiyel kare yapma metodu

To My Parents

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LIST OF SYMBOLS

a	Length of nano-plate
b	Width of nano-plate
c	Index of ceramic phase
E	Modulus of elasticity
f	Shape function for plate theories
h	Thickness of nano-plate
K	Kinetic energy
m	Index of metallic phase
n	Power-law index
n_x, n_y	Components of unit outward normal vector
N_x	Number of grid points in x direction
N_y	Number of grid points in y direction
$N_{\alpha\beta}, M_{\alpha\beta}, P_{\alpha\beta}, R_{\alpha\beta}$	Local stress resultants
U	Strain energy
u	Displacement component in x direction
u_0	Displacement component of mid-plane in x direction
V	Volume fraction
v	Displacement component in y direction
v_0	Displacement component of mid-plane in y direction
w	Displacement component in z direction
w_0	Displacement component of mid-plane in z direction
ϕ_x	Rotation of transverse normal about y direction
ϕ_y	Rotation of transverse normal about x direction
ε_{ij}	Strain tensor
σ_{ij}	Nonlocal stress tensor
t_{ij}	Local stress tensor
μ	Nonlocal parameter
ν	Poisson's ratio
ρ	Mass density

CHAPTER 1

INTRODUCTION

1.1 Introduction

Design of small scale structures as a key step in development of micro electromechanical systems (MEMS) and nano electromechanical systems (NEMS) has recently attracted the attention of scientific community. Nano-scale particles, beams, plates and tubes are examples of such structures. Nano-surgery, drug delivery, design of non-rejectable artificial organs and diagnosis are some of the applications for micro- and nanoscale structures and devices in medicine. These small scale devices are also used in nano-computers and biotechnology applications such as genome synthesis [1]. Micro- and nano-scale structures, devices, and systems have other specific applications in high frequency resonators [2], actuators [3] and sensors [4]. The aircraft and space vehicles can be controlled by displacing the control surfaces as well as by changing the geometry of wing and control surfaces. For example, ailerons, elevators, canards, fins, flaps, rudders, stabilizers and tips of advanced aircraft can be controlled by micro-scale actuators. Furthermore, micro- and nano-scale sensors can be used to measure the aerodynamic loads, vibrations, temperature, pressure, velocity, acceleration and noise. For comprehensive and accurate design of small-scale devices, structural characteristics of micro and nano-scale structures should be examined in detail. Free vibration analysis and computation of natural frequencies play an important role in design and optimization of micro and nano-scale devices such as sensors, resonators and oscillators. One of many different factors that can lead to bias in micro-mechanical gyroscopes [5, 6] is sensitive element and its dynamic behavior. Vibrations that occur at the excitation frequency is an example of such interferences.

The understanding of the mechanical response of small scale plates as a branch of small scale structures with two-dimensional shape is an important subject from design perspective of small scale devices. Design of a micro-valve for micro-fluidic applications [7], nano-plate resonator [8] and an oscillator based on a Graphene-Aluminum Nitride nano plate resonator [9] are examples of technological applications for MEMS and NEMS devices made of micro and nano-scale plates. Classical continuum theories fail to predict the size effect observable at small-scales and molecular dynamics based simulations need to confront an immense computational effort. Furthermore, experimental study of nano-scale structures is quite difficult due to problems rising in small scales for controlling each and every parameter. Consequently, higher order continuum theories have been commonly utilized to examine behavior of small-scale elements. Among such theories, nonlocal elasticity [10, 11], strain gradient [12, 13, 14], and couple stress [15, 16] theories are prevalent. In structural study of nano-scale plates, nonlocal elasticity theory is combined with plate deformation theories to obtain the governing equations and associated boundary conditions.

The main objective of this study is to present a nonlocal elasticity based method for free vibration of functionally graded rectangular nano-plates. Eringen's differential form of the nonlocal constitutive equation is used to consider the size effect in formulation. The method allows taking into account the spatial variation of nonlocal parameter.

1.2 Previous Works

Nonlocal elasticity theory along with different beam theories, has been widely used to investigate the small-scale effects on free vibration, bending and buckling of homogeneous nano-beams. Using the differential form of the nonlocal constitutive equation, Reddy [17] reformulated the Euler–Bernoulli, Timoshenko and Reddy beam theories. Bending solutions are proposed by Wang et al. [18] based on Eringen's nonlocal elasticity theory and Timoshenko beam theory. Aydogdu [19] proposed a generalized nonlocal beam theory to study bending, buckling and free vibration of nanobeams. A new higher order nonlocal shear deformation beam theory is presented by Thai [20] to study static and free vibration problems of nanobeams. In another work, Thai et al. [21] put forward a sinusoidal shear deformation beam theory for nonlocal free vibration and static analysis of nano-beams. Natural vibration frequencies of a rotating nonuniform cantilever nanobeam is investigated by Ruiz et al. [22]. Ritz method is used by Ghannadpour et al. [23] to solve the nonlocal static and free vibration problems of Euler–Bernoulli nano-beams. Studies on nonlinear models for bending [24] and buckling [25] of classical and shear deformable nano-beams involve the consideration of von Karman nonlinear strains in the formulation. Moreover, A nonlocal finite element model for Euler–Bernoulli beams is developed by Sciarra [26].

Investigation of small scale effects on static and free vibration analysis of homogeneous nano-plates is carried out by many research groups combining the nonlocal elasticity theory with different shear deformation plate theories. Using the differential form of the nonlocal constitutive equation, Lu et al. [27] reformulated the Kirchhoff and the Mindlin plate theories. In a similar manner, Aghababaei et al. [28] used third-order shear deformation plate theory along with nonlocal theory to solve static and free vibration problems of a rectangular nano-plate. The works by Wang et L. [29], Alzahrani et al. [30] and Alibeigloo et al. [31] are examples of studies on static bending of homogeneous rectangular nano-plates utilizing nonlocal elasticity theory to capture the size effects. In various studies on the nonlocal natural frequencies of homogeneous nano-plates, first order [32] and two-variable refined plate theories [33] are also used to incorporate the effect of shear deformation in the formulation. In another study on small scale plates, nonlocal vibration under uniaxial pre-stressed conditions is investigated by Murmu et al. [34]. In a similar manner, nonlocal plate theories are utilized in different studies on buckling analysis of nanoplates possessing constant [35, 36, 37, 38] and variable thicknesses [39], thermal effects [40, 41] and linearly varying in-plane load [42].

Functionally graded materials (FGMs) are a special class of composites, which possess smooth spatial variations in the volume fractions of the constituent phases. They find applications in a wide variety of technological fields including thermal barrier coatings, solid oxide fuel cells, high performance cutting tools, and biomedical materials. Deployment of functionally graded components in small-scale systems has recently become feasible with advances in fabrication technologies such as magnetron sputtering [43], chemical vapor deposition and plasma enhanced chemical vapor deposition [44], and modified soft lithography [45]. These developments are accompanied by theoretical and computational studies directed towards understanding mechanical behavior of small-scale FGM composite structures.

Nonlocal elasticity theory has been widely used to investigate the small-scale effects on free vibration, bending and buckling of small-sized FGM beams and plates. Finite element approaches, based on the Euler–Bernoulli beam theory are presented by Eltaher et al. [46, 47] to solve for free vibration, bending and buckling of nonlocal FGM nano-beams. Nonlocal free vibration [48], bending [49] and buckling [50] of FGM nano-scale beams are also investigated using Timoshenko beam theory. Finite element analysis for bending and buckling of small-scale FGM beams based on the nonlocal Timoshenko beam theory is also conducted by Eltaher et al. [51]. Rahmani et al. [52] studied buckling of FGM nano-beams using nonlocal Reddy beam theory. Reddy et al. [53] put forward a nonlinear finite element approach based on the Euler–Bernoulli and Timoshenko beam theories for structural analysis of nano-scale FGM beams. Nonlocal nonlinear free vibration of FGM Euler–Bernoulli nano-beams is examined by Nazemnezhad et al. [54] and Hosseini-Hashemi et al. [55] introduced the surface effects into the same problem. Forced vibration [56] and thermo-mechanical vibration [57] analyses are also considered using nonlocal elasticity theory.

In nonlocal free vibration analysis of small-scale FGM rectangular plates, Kirchhoff, Mindlin, second order and third-order shear deformation theories are respectively used by Zare et al. [58], Natarajan et al. [59], Nami et al. [60] and Daneshmehr et al. [61]. An analytical approach is proposed by Salehipour et al. [62] to solve three dimensional nonlocal elasticity problem of small-scale plates in free vibration. Nonlocal third order plate theory is utilized in buckling analysis of nano-scale FGM plates under mechanical [63] and thermal loads [64]. Moreover, nonlocal bending problem is considered for plates embedded under distributed nano-particles [65] and rested on a Winkler-Pasternak elastic foundation [66]. A computational approach based on isogeometric analysis [67] and a differential quadrature-based method [68] are also introduced to solve for bending and free vibration of these structures.

1.3 Motivation and Scope of Study

In all work mentioned in the previous section, the nonlocal parameter of the nonlocal elasticity theory is assumed to be constant. However, the nonlocal parameter is essentially a material property and thus varies as a function of spatial coordinates in a functionally graded composite structure. The primary objective in this study, is to reveal the influence of the spatial variation of the nonlocal parameter upon free vibration behavior of small-scale rectangular functionally graded plates.

In Chapter 2, a set of governing partial differential equations and boundary conditions are derived by employing the nonlocal elasticity theory and variational principles. All material properties, including the nonlocal parameter, are assumed to be functions of the thickness coordinate in the derivations. Displacement field is expressed in a unified way to be able to produce numerical results for Kirchhoff, Mindlin, and third-order plate theories.

In Chapter 3, generalized differential quadrature method (GDQM) is introduced. The governing equations and related boundary equations are solved numerically by means of the generalized differential quadrature method. MATLAB software is used to develop necessary computer programs to implement the numerical solutions.

In Chapter 4, Developed procedures are verified through comparisons made to the findings available in the literature. Simply-supported and cantilever nano-plates are considered in parametric analyses. Results presented for these configurations illustrate influences of material and geometric parameters upon natural vibration frequencies.

CHAPTER 2

FORMULATION

2.1 Problem Definition

In this study, a functionally graded rectangular nano-plate is considered to examine the small-scale and shear deformation effects on natural frequencies of the system. The geometry of the rectangular nano-plate is depicted in Figure 2.1. The plate is of thickness h and assumed to possess property variations in thickness direction.

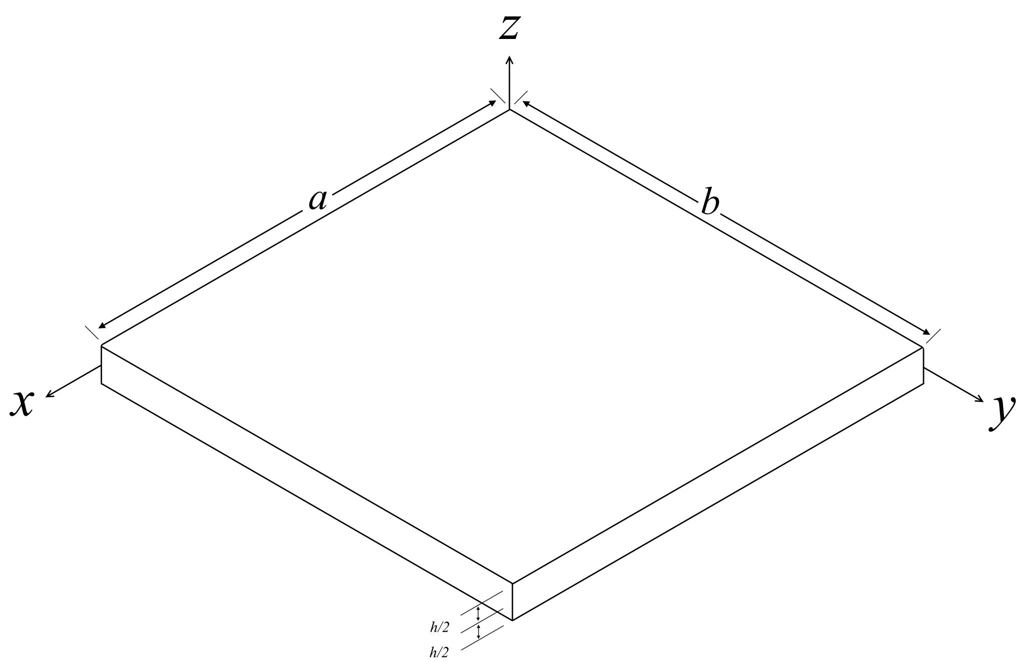

Figure 2.1: The geometry of the functionally graded rectangular nanoplate

2.2 Shear Deformation Plate Theories

The classical plate theory is based on the Kirchhoff hypothesis. This theory neglects the effect of transverse shear deformation. On the other hand, higher order shear deformation plate theories take into consideration the effect of transverse shear deformation by relaxing some of the restrictions imposed by Kirchhoff hypothesis on classical plate theory. In this study, displacement field of the nano-plate is expressed in a unified manner to be able to generate results according to different plate theories. The generalized form of the displacement field is given as:

$$u(x,y,z,t) = u_0(x,y,t) - zw_{,x} + f(z)(\phi_x + w_{,x}), \tag{2.1}$$

$$v(x,y,z,t) = v_0(x,y,t) - zw_{,y} + f(z)(\phi_y + w_{,y}), \tag{2.2}$$

$$w(x,y,z,t) = w_0(x,y,t), \tag{2.3}$$

In this representation u, v and w are displacement components in x, y and z directions, respectively; u_0, v_0 and w_0 are displacements of a point on the mid-plane; ϕ_x and ϕ_y are the rotations of a transverse normal about y and x axes, respectively; and a comma stands for differentiation. The function $f(z)$ represents the shape function determining the distribution of the transverse shear stress and strain through the thickness of the plate. The shape functions f indicating Kirchhoff, Mindlin and third order plate theories are defined as:

$$\text{Kirchhoff plate theory: } f(z) = 0 \tag{2.4}$$

$$\text{Mindlin plate theory: } f(z) = z \tag{2.5}$$

$$\text{Third order plate theory: } f(z) = z\left(1 - \frac{4z^2}{3h^2}\right) \tag{2.6}$$

Suitable shape functions f should approximately satisfy parabolic shear deformation distribution. Additionally, the boundary conditions should be satisfied on the bottom and top surfaces of the plate. According to theory of elasticity, strain field corresponding to these displacements is then found in the form:

$$\varepsilon_{xx} = \frac{1}{2}(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial x}) = u_{0,x} - zw_{,xx} + f(\phi_{x,x} + w_{,xx}), \tag{2.7}$$

$$\varepsilon_{yy} = \frac{1}{2}(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial y}) = v_{0,y} - zw_{,yy} + f(\phi_{y,y} + w_{,yy}), \tag{2.8}$$

$$\varepsilon_{zz} = \frac{1}{2}(\frac{\partial w}{\partial z} + \frac{\partial w}{\partial z}) = 0, \tag{2.9}$$

$$\varepsilon_{xy} = \frac{1}{2}(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}) = \frac{1}{2}\left\{(u_{0,y} + v_{0,x}) - 2zw_{,xy} + f(\phi_{x,y} + 2w_{,xy} + \phi_{y,x})\right\}, \tag{2.10}$$

$$\varepsilon_{yz} = \frac{1}{2}(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}) = \frac{1}{2}f'(\phi_y + w_{,y}), \tag{2.11}$$

$$\varepsilon_{xz} = \frac{1}{2}(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}) = \frac{1}{2}f'(\phi_x + w_{,x}). \tag{2.12}$$

According to this generalized form of the strain field, zero and constant shear strains are obtained for Kirchhoff and Mindlin plate theories, respectively. The reason for using cubic terms in displacement field of the third order plate theory is to have quadratic variation of the transverse shear strains through the thickness of the plate.

2.3 Nonlocal Elasticity Theory

In nonlocal elasticity theory, stress at a point is expressed as a function of the strain field in the material domain as follows:

$$\sigma_{ij} = \iiint_V \alpha(|x' - x|, \tau) t_{ij}(x') \, dV(x') \tag{2.13}$$

where α is the nonlocal modulus or kernel function, $|x' - x|$ represents the distance, τ is a material property that depends on internal and external characteristic lengths (such as the lattice spacing and wavelength); σ_{ij} and t_{ij} stand for nonlocal and local stress tensors, respectively. Eringen [69] proposed an equivalent differential form of the nonlocal constitutive equation in the form:

$$(1 - \mu\nabla^2)\sigma_{ij} = t_{ij} \tag{2.14}$$

In this equation, μ is the nonlocal parameter defined by

$$\mu = (e_0 l)^2, \tag{2.15}$$

where l is internal characteristic length and e_0 is a material property found through experimental characterization. Because of its dependence on l and e_0, μ is a material property [70] and should be expressed as a function of the z-coordinate as well. In this study, all material properties including the nonlocal parameter are functions of the thickness coordinate and their spatial variations along the thickness direction are described by

$$E(z) = E_c V_c(z) + E_m V_m(z), \tag{2.16}$$

$$\nu(z) = \nu_c V_c(z) + \nu_m V_m(z), \tag{2.17}$$

$$\rho(z) = \rho_c V_c(z) + \rho_m V_m(z), \tag{2.18}$$

$$\mu(z) = \mu_c V_c(z) + \mu_m V_m(z). \tag{2.19}$$

The subscripts c and m stand for ceramic and metallic phases; V_c and V_m are volume fractions. E, ν and ρ are respectively modulus of elasticity, Poisson's ratio and mass density. Spatial variations of the volume fractions are represented according to the power law as follows:

$$V_c(z) = (\frac{1}{2} + \frac{z}{h})^n, \tag{2.20}$$

$$V_m(z) = 1 - V_c(z). \tag{2.21}$$

The power-law index n is a non-negative variable parameter which defines property distribution profiles. When n is less than 1 the nano-plate is ceramic-rich, whereas if n is greater than unity plate has a metal-rich profile.

The relation between local stress tensor t_{ij} and strain tensor ε_{ij} is expressed by

$$\begin{Bmatrix} t_{xx} \\ t_{yy} \\ t_{xy} \\ t_{xz} \\ t_{yz} \end{Bmatrix} = \begin{pmatrix} Q_{11} & Q_{12} & 0 & 0 & 0 \\ Q_{21} & Q_{22} & 0 & 0 & 0 \\ 0 & 0 & Q_{66} & 0 & 0 \\ 0 & 0 & 0 & Q_{66} & 0 \\ 0 & 0 & 0 & 0 & Q_{66} \end{pmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ 2\varepsilon_{xy} \\ 2\varepsilon_{xz} \\ 2\varepsilon_{yz} \end{Bmatrix}, \tag{2.22}$$

where,

$$Q_{11} = Q_{22} = \frac{E(z)}{1 - \nu(z)^2} \tag{2.23}$$

$$Q_{12} = Q_{21} = \frac{E(z)\upsilon(z)}{1 - \nu(z)^2} \tag{2.24}$$

$$Q_{66} = \frac{E(z)}{2(1 + \nu(z))} \tag{2.25}$$

2.4 Derivation of the Governing Equations of Nano-plate Using Nonlocal Balance Law

According to nonlocal balance law:

$$\sigma_{ij,j} + f_i = \rho \ddot{u}_i \quad i, j = x, y, z \tag{2.26}$$

f_i and u_i are respectively body force and displacement in i direction. In order to obtain equations of motion in the absence of body forces, f_i is set to be zero. Multiplying both sides of the Equation 2.26 by nonlocal operator, we have:

$$t_{ij,j} = (1 - \mu\nabla^2)\rho \ddot{u}_i \tag{2.27}$$

For $i = x$, using Equations 2.1-2.3 and integrating the balance Equation 2.27 through the thickness of the plate and simplifying, first governing equation of the nano-plate is derived as:

$$\begin{aligned}\frac{\partial N_{xx}}{\partial x} + \frac{\partial N_{xy}}{\partial y} = &\left\{ I_0 \frac{\partial^2 u_0}{\partial t^2} - I_1 \frac{\partial^3 w}{\partial t^2 \partial x} + I_3 (\frac{\partial^2 \phi_x}{\partial t^2} + \frac{\partial^3 w}{\partial t^2 \partial x}) \right\} \\ &-\nabla^2 \left\{ L_0 \frac{\partial^2 u_0}{\partial t^2} - L_1 \frac{\partial^3 w}{\partial t^2 \partial x} + L_3 (\frac{\partial^2 \phi_x}{\partial t^2} + \frac{\partial^3 w}{\partial t^2 \partial x}) \right\}\end{aligned} \tag{2.28}$$

For $i = y$, using Equations 2.1-2.3 and integrating the balance Equation 2.27 through the thickness of the plate and simplifying, second governing equation of the nano-plate is derived as:

$$\begin{aligned}\frac{\partial N_{yy}}{\partial y} + \frac{\partial N_{xy}}{\partial x} = &\left\{ I_0 \frac{\partial^2 v_0}{\partial t^2} - I_1 \frac{\partial^3 w}{\partial t^2 \partial y} + I_3 (\frac{\partial^2 \phi_y}{\partial t^2} + \frac{\partial^3 w}{\partial t^2 \partial y}) \right\} \\ &-\nabla^2 \left\{ L_0 \frac{\partial^2 v_0}{\partial t^2} - L_1 \frac{\partial^3 w}{\partial t^2 \partial y} + L_3 (\frac{\partial^2 \phi_y}{\partial t^2} + \frac{\partial^3 w}{\partial t^2 \partial y}) \right\}\end{aligned} \tag{2.29}$$

For $i = x$, using Equations 2.1-2.3 and multiplying both sides of the Equation 2.27 by $f(z)$ and integrating through the thickness of the plate and using integration by parts, third governing equation of the nano-plate is derived as:

$$\begin{aligned}\frac{\partial P_{xx}}{\partial x} + \frac{\partial P_{xy}}{\partial y} - R_{xz} = &\left\{ I_3 \frac{\partial^2 u_0}{\partial t^2} - I_4 \frac{\partial^3 w}{\partial t^2 \partial x} + I_5 (\frac{\partial^2 \phi_x}{\partial t^2} + \frac{\partial^3 w}{\partial t^2 \partial x}) \right\} \\ &-\nabla^2 \left\{ L_3 \frac{\partial^2 u_0}{\partial t^2} - L_4 \frac{\partial^3 w}{\partial t^2 \partial x} + L_5 (\frac{\partial^2 \phi_x}{\partial t^2} + \frac{\partial^3 w}{\partial t^2 \partial x}) \right\}\end{aligned} \tag{2.30}$$

For $i = y$, using Equations 2.1-2.3 and multiplying both sides of the Equation 2.27 by $f(z)$ and integrating through the thickness of the plate and using integration by parts, forth governing equation of the nano-plate is derived as:

$$\begin{aligned}\frac{\partial P_{yy}}{\partial y}+\frac{\partial P_{xy}}{\partial x}-R_{yz}=&\left\{I_3\frac{\partial^2 v_0}{\partial t^2}-I_4\frac{\partial^3 w}{\partial t^2\partial y}+I_5(\frac{\partial^2\phi_y}{\partial t^2}+\frac{\partial^3 w}{\partial t^2\partial y})\right\}\\&-\nabla^2\left\{L_3\frac{\partial^2 v_0}{\partial t^2}-L_4\frac{\partial^3 w}{\partial t^2\partial y}+L_5(\frac{\partial^2\phi_y}{\partial t^2}+\frac{\partial^3 w}{\partial t^2\partial y})\right\}\end{aligned}\tag{2.31}$$

For $i=z$, using Equations 2.1-2.3 and integrating the balance Equation 2.27 through the thickness of the plate and simplifying:

$$\frac{\partial N_{xz}}{\partial x}+\frac{\partial N_{yz}}{\partial y}=I_0\frac{\partial^2 w}{\partial t^2}-\nabla^2 L_0\frac{\partial^2 w}{\partial t^2}\tag{2.32}$$

For $i=x$, using Equations 2.1-2.3 and multiplying both sides of the Equation 2.27 by z and integrating through the thickness of the plate and using integration by parts:

$$\begin{aligned}\frac{\partial M_{xx}}{\partial x}+\frac{\partial M_{xy}}{\partial y}-N_{xz}=&\left\{I_1\frac{\partial^2 u_0}{\partial t^2}-I_2\frac{\partial^3 w}{\partial t^2\partial x}+I_4(\frac{\partial^2\phi_x}{\partial t^2}+\frac{\partial^3 w}{\partial t^2\partial x})\right\}\\&-\nabla^2\left\{L_1\frac{\partial^2 u_0}{\partial t^2}-L_2\frac{\partial^3 w}{\partial t^2\partial x}+L_4(\frac{\partial^2\phi_x}{\partial t^2}+\frac{\partial^3 w}{\partial t^2\partial x})\right\}\end{aligned}\tag{2.33}$$

For $i=y$, using Equations 2.1-2.3 and multiplying both sides of the Equation 2.27 by z and integrating through the thickness of the plate and using integration by parts:

$$\begin{aligned}\frac{\partial M_{yy}}{\partial y}+\frac{\partial M_{xy}}{\partial x}-N_{yz}=&\left\{I_1\frac{\partial^2 v_0}{\partial t^2}-I_2\frac{\partial^3 w}{\partial t^2\partial y}+I_4(\frac{\partial^2\phi_y}{\partial t^2}+\frac{\partial^3 w}{\partial t^2\partial y})\right\}\\&-\nabla^2\left\{L_1\frac{\partial^2 v_0}{\partial t^2}-L_2\frac{\partial^3 w}{\partial t^2\partial y}+L_4(\frac{\partial^2\phi_y}{\partial t^2}+\frac{\partial^3 w}{\partial t^2\partial y})\right\}\end{aligned}\tag{2.34}$$

Combining Equations 2.32-2.34 and using Equations 2.30-2.31, fifth governing equation of the nano-plate is derived as:

$$\begin{aligned}&\frac{\partial^2 M_{xx}}{\partial x^2}+\frac{\partial^2 M_{yy}}{\partial y^2}+2\frac{\partial^2 M_{xy}}{\partial x\partial y}-\frac{\partial^2 P_{xx}}{\partial x^2}-\frac{\partial^2 P_{yy}}{\partial y^2}-2\frac{\partial^2 P_{xy}}{\partial x\partial y}+\frac{\partial R_{yz}}{\partial y}+\frac{\partial R_{xz}}{\partial x}=\\&I_0\frac{\partial^2 w}{\partial t^2}+(I_1-I_3)(\frac{\partial^3 u_0}{\partial t^2\partial x}+\frac{\partial^3 v_0}{\partial t^2\partial y})+(-I_2+2I_4-I_5)(\frac{\partial^4 w}{\partial t^2\partial x^2}+\frac{\partial^4 w}{\partial t^2\partial y^2})\\&+(I_4-I_5)(\frac{\partial^3\phi_x}{\partial t^2\partial x}+\frac{\partial^3\phi_y}{\partial t^2\partial y})-\nabla^2\left\{L_0\frac{\partial^2 w}{\partial t^2}+(L_1-L_3)(\frac{\partial^3 u_0}{\partial t^2\partial x}+\frac{\partial^3 v_0}{\partial t^2\partial y})\right\}\\&-\nabla^2\left\{+(-L_2+2L_4-L_5)(\frac{\partial^4 w}{\partial t^2\partial x^2}+\frac{\partial^4 w}{\partial t^2\partial y^2})+(L_4-L_5)(\frac{\partial^3\phi_x}{\partial t^2\partial x}+\frac{\partial^3\phi_y}{\partial t^2\partial y})\right\}\end{aligned}\tag{2.35}$$

In the governing equations, $N_{\alpha\beta}$, $M_{\alpha\beta}$, $P_{\alpha\beta}$, $N_{\alpha z}$ and $R_{\alpha z}$ are stress resultants defined by the following equations

$$\left\{\begin{array}{c}N_{\alpha\beta}\\M_{\alpha\beta}\\P_{\alpha\beta}\end{array}\right\}=\int_{-h/2}^{h/2}t_{\alpha\beta}\left\{\begin{array}{c}1\\z\\f\end{array}\right\}dz,\quad\alpha=x,y,\quad\beta=x,y,\tag{2.36}$$

$$\left\{ \begin{array}{c} N_{\alpha z} \\ R_{\alpha z} \end{array} \right\} = \int_{-h/2}^{h/2} t_{\alpha z} \left\{ \begin{array}{c} 1 \\ f' \end{array} \right\} dz, \quad \alpha = x, y, \tag{2.37}$$

Coefficient terms in the governing equations of the nano-plate are defined as

$$\left\{ \begin{array}{c} I_0 \\ I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \end{array} \right\} = \int_{-h/2}^{h/2} \rho(z) \left\{ \begin{array}{c} 1 \\ z \\ z^2 \\ f \\ zf \\ f^2 \end{array} \right\} dz, \tag{2.38}$$

$$\left\{ \begin{array}{c} L_0 \\ L_1 \\ L_2 \\ L_3 \\ L_4 \\ L_5 \end{array} \right\} = \int_{-h/2}^{h/2} \mu(z)\rho(z) \left\{ \begin{array}{c} 1 \\ z \\ z^2 \\ f \\ zf \\ f^2 \end{array} \right\} dz. \tag{2.39}$$

2.5 Derivation of Governing Equations and Boundary Conditions for Free Vibration of Nano-plate Using Hamilton's Principle

For an FGM composite nano-plate undergoing free vibrations, Hamilton's principle requires that

$$\delta \int_{t_1}^{t_2} (K - U) dt = 0, \tag{2.40}$$

where U is strain energy and K is kinetic energy. Variations of the energy terms are written as:

$$\delta U = \iiint_V \left(\sigma_{xx}\delta\varepsilon_{xx} + \sigma_{yy}\delta\varepsilon_{yy} + 2\sigma_{xy}\delta\varepsilon_{xy} + 2\sigma_{xz}\delta\varepsilon_{xz} + +2\sigma_{yz}\delta\varepsilon_{yz} \right) dV \tag{2.41}$$

$$\delta K = \iiint_V \rho(z)(\dot{u}\delta\dot{u} + \dot{v}\delta\dot{v} + \dot{w}\delta\dot{w})\, dV \tag{2.42}$$

Using Equations 2.1-2.3, 2.7-2.12, 2.14, 2.40, 2.41-2.42 and variational principles, governing partial differential equations are derived as follows:

$$\begin{aligned}\frac{\partial N_{xx}}{\partial x}+\frac{\partial N_{xy}}{\partial y}&=\left\{I_0\frac{\partial^2 u_0}{\partial t^2}-I_1\frac{\partial^3 w}{\partial t^2\partial x}+I_3(\frac{\partial^2\phi_x}{\partial t^2}+\frac{\partial^3 w}{\partial t^2\partial x})\right\}\\ &-\nabla^2\left\{L_0\frac{\partial^2 u_0}{\partial t^2}-L_1\frac{\partial^3 w}{\partial t^2\partial x}+L_3(\frac{\partial^2\phi_x}{\partial t^2}+\frac{\partial^3 w}{\partial t^2\partial x})\right\}\end{aligned}\tag{2.43}$$

$$\begin{aligned}\frac{\partial N_{yy}}{\partial y}+\frac{\partial N_{xy}}{\partial x}&=\left\{I_0\frac{\partial^2 v_0}{\partial t^2}-I_1\frac{\partial^3 w}{\partial t^2\partial y}+I_3(\frac{\partial^2\phi_y}{\partial t^2}+\frac{\partial^3 w}{\partial t^2\partial y})\right\}\\ &-\nabla^2\left\{L_0\frac{\partial^2 v_0}{\partial t^2}-L_1\frac{\partial^3 w}{\partial t^2\partial y}+L_3(\frac{\partial^2\phi_y}{\partial t^2}+\frac{\partial^3 w}{\partial t^2\partial y})\right\}\end{aligned}\tag{2.44}$$

$$\begin{aligned}&\frac{\partial^2 M_{xx}}{\partial x^2}+\frac{\partial^2 M_{yy}}{\partial y^2}+2\frac{\partial^2 M_{xy}}{\partial x\partial y}-\frac{\partial^2 P_{xx}}{\partial x^2}-\frac{\partial^2 P_{yy}}{\partial y^2}-2\frac{\partial^2 P_{xy}}{\partial x\partial y}+\frac{\partial R_{yz}}{\partial y}+\frac{\partial R_{xz}}{\partial x}=\\ &I_0\frac{\partial^2 w}{\partial t^2}+(I_1-I_3)(\frac{\partial^3 u_0}{\partial t^2\partial x}+\frac{\partial^3 v_0}{\partial t^2\partial y})+(-I_2+2I_4-I_5)(\frac{\partial^4 w}{\partial t^2\partial x^2}+\frac{\partial^4 w}{\partial t^2\partial y^2})\\ &+(I_4-I_5)(\frac{\partial^3\phi_x}{\partial t^2\partial x}+\frac{\partial^3\phi_y}{\partial t^2\partial y})-\nabla^2\left\{L_0\frac{\partial^2 w}{\partial t^2}+(L_1-L_3)(\frac{\partial^3 u_0}{\partial t^2\partial x}+\frac{\partial^3 v_0}{\partial t^2\partial y})\right\}\\ &-\nabla^2\left\{+(-L_2+2L_4-L_5)(\frac{\partial^4 w}{\partial t^2\partial x^2}+\frac{\partial^4 w}{\partial t^2\partial y^2})+(L_4-L_5)(\frac{\partial^3\phi_x}{\partial t^2\partial x}+\frac{\partial^3\phi_y}{\partial t^2\partial y})\right\}\end{aligned}\tag{2.45}$$

$$\begin{aligned}\frac{\partial P_{xx}}{\partial x}+\frac{\partial P_{xy}}{\partial y}-R_{xz}&=\left\{I_3\frac{\partial^2 u_0}{\partial t^2}-I_4\frac{\partial^3 w}{\partial t^2\partial x}+I_5(\frac{\partial^2\phi_x}{\partial t^2}+\frac{\partial^3 w}{\partial t^2\partial x})\right\}\\ &-\nabla^2\left\{L_3\frac{\partial^2 u_0}{\partial t^2}-L_4\frac{\partial^3 w}{\partial t^2\partial x}+L_5(\frac{\partial^2\phi_x}{\partial t^2}+\frac{\partial^3 w}{\partial t^2\partial x})\right\}\end{aligned}\tag{2.46}$$

$$\begin{aligned}\frac{\partial P_{yy}}{\partial y}+\frac{\partial P_{xy}}{\partial x}-R_{yz}&=\left\{I_3\frac{\partial^2 v_0}{\partial t^2}-I_4\frac{\partial^3 w}{\partial t^2\partial y}+I_5(\frac{\partial^2\phi_y}{\partial t^2}+\frac{\partial^3 w}{\partial t^2\partial y})\right\}\\ &-\nabla^2\left\{L_3\frac{\partial^2 v_0}{\partial t^2}-L_4\frac{\partial^3 w}{\partial t^2\partial y}+L_5(\frac{\partial^2\phi_y}{\partial t^2}+\frac{\partial^3 w}{\partial t^2\partial y})\right\}\end{aligned}\tag{2.47}$$

These equations are the same as those obtained in Section 2.4 using nonlocal balance law. In order to solve the equations numerically, they should be written in terms of displacements as follows:

$$\begin{aligned}&A_0\frac{\partial^2 u_0}{\partial x^2}+C_0\frac{\partial^2 u_0}{\partial y^2}+(B_0+C_0)\frac{\partial^2 v_0}{\partial x\partial y}+(A_3-A_1)\frac{\partial^3 w}{\partial x^3}+(B_3-B_1-2C_1+2C_3)\frac{\partial^3 w}{\partial x\partial y^2}\\ &+A_3\frac{\partial^2\phi_x}{\partial x^2}+C_3\frac{\partial^2\phi_x}{\partial y^2}+(B_3+C_3)\frac{\partial^2\phi_y}{\partial x\partial y}=\left\{I_0\frac{\partial^2 u_0}{\partial t^2}-I_1\frac{\partial^3 w}{\partial t^2\partial x}+I_3(\frac{\partial^2\phi_x}{\partial t^2}+\frac{\partial^3 w}{\partial t^2\partial x})\right\}\\ &-\nabla^2\left\{L_0\frac{\partial^2 u_0}{\partial t^2}-L_1\frac{\partial^3 w}{\partial t^2\partial x}+L_3(\frac{\partial^2\phi_x}{\partial t^2}+\frac{\partial^3 w}{\partial t^2\partial x})\right\},\end{aligned}\tag{2.48}$$

$$
\begin{aligned}
C_0\frac{\partial^2 v_0}{\partial x^2}+A_0\frac{\partial^2 v_0}{\partial y^2}+(B_0+C_0)\frac{\partial^2 u_0}{\partial x\partial y}+(A_3-A_1)\frac{\partial^3 w}{\partial y^3}+(B_3-B_1-2C_1+2C_3)\frac{\partial^3 w}{\partial y\partial x^2}\\
+C_3\frac{\partial^2 \phi_y}{\partial x^2}+A_3\frac{\partial^2 \phi_y}{\partial y^2}+(B_3+C_3)\frac{\partial^2 \phi_x}{\partial x\partial y}=\left\{I_0\frac{\partial^2 v_0}{\partial t^2}-I_1\frac{\partial^3 w}{\partial t^2\partial y}+I_3(\frac{\partial^2 \phi_y}{\partial t^2}+\frac{\partial^3 w}{\partial t^2\partial y})\right\}\\
-\nabla^2\left\{L_0\frac{\partial^2 v_0}{\partial t^2}-L_1\frac{\partial^3 w}{\partial t^2\partial y}+L_3(\frac{\partial^2 \phi_y}{\partial t^2}+\frac{\partial^3 w}{\partial t^2\partial y})\right\},
\end{aligned}
\tag{2.49}
$$

$$
\begin{aligned}
(A_1-A_3)(\frac{\partial^3 v_0}{\partial y^3}+\frac{\partial^3 u_0}{\partial x^3})+(2A_4-A_2-A_5)(\frac{\partial^4 w}{\partial x^4}+\frac{\partial^4 w}{\partial y^4})+(A_4-A_5)(\frac{\partial^3 \phi_x}{\partial x^3}+\frac{\partial^3 \phi_y}{\partial y^3})\\
+(B_1-B_3+2C_1-2C_3)(\frac{\partial^3 v_0}{\partial x^2\partial y}+\frac{\partial^3 u_0}{\partial x\partial y^2})+2(2B_4-B_2-B_5+4C_4-2C_2-2C_5)\frac{\partial^4 w}{\partial x^2\partial y^2}\\
+(B_4-B_5+2C_4-2C_5)(\frac{\partial^3 \phi_y}{\partial x^2\partial y}+\frac{\partial^3 \phi_x}{\partial x\partial y^2})+C_6(\frac{\partial \phi_x}{\partial x}+\frac{\partial \phi_y}{\partial y})+C_6(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2})=\\
I_0\frac{\partial^2 w}{\partial t^2}+(I_1-I_3)(\frac{\partial^3 u_0}{\partial t^2\partial x}+\frac{\partial^3 v_0}{\partial t^2\partial y})+(-I_2+2I_4-I_5)(\frac{\partial^4 w}{\partial t^2\partial x^2}+\frac{\partial^4 w}{\partial t^2\partial y^2})\\
+(I_4-I_5)(\frac{\partial^3 \phi_x}{\partial t^2\partial x}+\frac{\partial^3 \phi_y}{\partial t^2\partial y})-\nabla^2\left\{L_0\frac{\partial^2 w}{\partial t^2}+(L_1-L_3)(\frac{\partial^3 u_0}{\partial t^2\partial x}+\frac{\partial^3 v_0}{\partial t^2\partial y})\right\}\\
-\nabla^2\left\{+(-L_2+2L_4-L_5)(\frac{\partial^4 w}{\partial t^2\partial x^2}+\frac{\partial^4 w}{\partial t^2\partial y^2})+(L_4-L_5)(\frac{\partial^3 \phi_x}{\partial t^2\partial x}+\frac{\partial^3 \phi_y}{\partial t^2\partial y})\right\},
\end{aligned}
\tag{2.50}
$$

$$
\begin{aligned}
A_3\frac{\partial^2 u_0}{\partial x^2}+C_3\frac{\partial^2 u_0}{\partial y^2}+(B_3+C_3)\frac{\partial^2 v_0}{\partial x\partial y}+(A_5-A_4)\frac{\partial^3 w}{\partial x^3}+(B_5-B_4-2C_4+2C_5)\frac{\partial^3 w}{\partial x\partial y^2}\\
+A_5\frac{\partial^2 \phi_x}{\partial x^2}+C_5\frac{\partial^2 \phi_x}{\partial y^2}+(B_5+C_5)\frac{\partial^2 \phi_y}{\partial x\partial y}-C_6\phi_x-C_6\frac{\partial w}{\partial x}=\\
\left\{I_3\frac{\partial^2 u_0}{\partial t^2}-I_4\frac{\partial^3 w}{\partial t^2\partial x}+I_5(\frac{\partial^2 \phi_x}{\partial t^2}+\frac{\partial^3 w}{\partial t^2\partial x})\right\}\\
-\nabla^2\left\{L_3\frac{\partial^2 u_0}{\partial t^2}-L_4\frac{\partial^3 w}{\partial t^2\partial x}+L_5(\frac{\partial^2 \phi_x}{\partial t^2}+\frac{\partial^3 w}{\partial t^2\partial x})\right\},
\end{aligned}
\tag{2.51}
$$

$$
\begin{aligned}
C_3\frac{\partial^2 v_0}{\partial x^2}+A_3\frac{\partial^2 v_0}{\partial y^2}+(B_3+C_3)\frac{\partial^2 u_0}{\partial x\partial y}+(A_5-A_4)\frac{\partial^3 w}{\partial y^3}+(B_5-B_4-2C_4+2C_5)\frac{\partial^3 w}{\partial y\partial x^2}\\
+C_5\frac{\partial^2 \phi_y}{\partial x^2}+A_5\frac{\partial^2 \phi_y}{\partial y^2}+(B_5+C_5)\frac{\partial^2 \phi_x}{\partial x\partial y}-C_6\phi_y-C_6\frac{\partial w}{\partial y}=\\
\left\{I_3\frac{\partial^2 v_0}{\partial t^2}-I_4\frac{\partial^3 w}{\partial t^2\partial y}+I_5(\frac{\partial^2 \phi_y}{\partial t^2}+\frac{\partial^3 w}{\partial t^2\partial y})\right\}\\
-\nabla^2\left\{L_3\frac{\partial^2 v_0}{\partial t^2}-L_4\frac{\partial^3 w}{\partial t^2\partial y}+L_5(\frac{\partial^2 \phi_y}{\partial t^2}+\frac{\partial^3 w}{\partial t^2\partial y})\right\}.
\end{aligned}
\tag{2.52}
$$

in which, the coefficient terms are given as

$$\begin{Bmatrix} A_0 \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \end{Bmatrix} = \int_{-h/2}^{h/2} Q_{11} \begin{Bmatrix} 1 \\ z \\ z^2 \\ f(z) \\ zf(z) \\ (f(z))^2 \end{Bmatrix} dz \tag{2.53}$$

$$\begin{Bmatrix} B_0 \\ B_1 \\ B_2 \\ B_3 \\ B_4 \\ B_5 \end{Bmatrix} = \int_{-h/2}^{h/2} Q_{12} \begin{Bmatrix} 1 \\ z \\ z^2 \\ f(z) \\ zf(z) \\ (f(z))^2 \end{Bmatrix} dz \tag{2.54}$$

$$\begin{Bmatrix} C_0 \\ C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \end{Bmatrix} = \int_{-h/2}^{h/2} Q_{66} \begin{Bmatrix} 1 \\ z \\ z^2 \\ f(z) \\ zf(z) \\ (f(z))^2 \\ (f'(z))^2 \end{Bmatrix} dz \tag{2.55}$$

The main advantage of Hamilton's principle is that the governing equations and the boundary conditions can be found simultaneously. The boundary conditions are obtained as:

$$u_0 = 0, \quad \text{or} \quad N_{xx}n_x + N_{xy}n_y = 0, \tag{2.56}$$

$$v_0 = 0, \quad \text{or} \quad N_{xy}n_x + N_{yy}n_y = 0, \tag{2.57}$$

$$\phi_x = 0, \quad \text{or} \quad P_{xx}n_x + P_{xy}n_y = 0, \tag{2.58}$$

$$\phi_y = 0, \quad \text{or} \quad P_{xy}n_x + P_{yy}n_y = 0, \tag{2.59}$$

$$\tfrac{\partial w}{\partial x} = 0, \quad \text{or} \quad (M_{xx} - P_{xx})n_x + (M_{xy} - P_{xy})n_y = 0, \tag{2.60}$$

$$\tfrac{\partial w}{\partial y} = 0, \quad \text{or} \quad (M_{xy} - P_{xy})n_x + (M_{yy} - P_{yy})n_y = 0, \tag{2.61}$$

$w = 0$, or

$$(\frac{\partial M_{xx}}{\partial x}+\frac{\partial M_{xy}}{\partial y}-\frac{\partial P_{xx}}{\partial x}-\frac{\partial P_{xy}}{\partial y}+R_{xz})n_x+(\frac{\partial M_{yy}}{\partial y}+\frac{\partial M_{xy}}{\partial x}-\frac{\partial P_{yy}}{\partial y}-\frac{\partial P_{xy}}{\partial x}+R_{yz})n_y =$$
$$\left\{(I_1 - I_3)\frac{\partial^2 u_0}{\partial t^2} + (-I_2 + 2I_4 - I_5)\frac{\partial^3 w}{\partial x \partial t^2} + (I_4 - I_5)\frac{\partial^2 \phi_x}{\partial t^2}\right\} n_x$$
$$+ \left\{(I_1 - I_3)\frac{\partial^2 v_0}{\partial t^2} + (-I_2 + 2I_4 - I_5)\frac{\partial^3 w}{\partial y \partial t^2} + (I_4 - I_5)\frac{\partial^2 \phi_y}{\partial t^2}\right\} n_y$$
$$- \nabla^2 \left\{(L_1 - L_3)\frac{\partial^2 u_0}{\partial t^2} + (-L_2 + 2L_4 - L_5)\frac{\partial^3 w}{\partial x \partial t^2} + (L_4 - L_5)\frac{\partial^2 \phi_x}{\partial t^2}\right\} n_x$$
$$- \nabla^2 \left\{(L_1 - L_3)\frac{\partial^2 v_0}{\partial t^2} + (-L_2 + 2L_4 - L_5)\frac{\partial^3 w}{\partial y \partial t^2} + (L_4 - L_5)\frac{\partial^2 \phi_y}{\partial t^2}\right\} n_y, \quad (2.62)$$

where n_x and n_y are the components of the unit outward normal vector.

2.6 Derivation of Governing Equations for Bending and Buckling

Considering the exact elasticity solution for static problems of plates, time-dependent terms are set to be zero in the governing equations found in Section 2.4. Additionally, boundary conditions associated with the bending and buckling loads should be treated as traction type of boundary conditions in the solution. In order to overcome this complexity of a full three-dimensional elasticity solution, two-dimensional equilibrium equations are used in the literature to solve the static problems of plates. In order to transform the initial three-dimensional elastic problem in a two-dimensional one, the distribution of the nonlocal stresses along the thickness of the plate has been replaced with the resulting nonlocal internal actions defined on the reference surface. Using this method, the equations of equilibrium for bending and buckling of nanoplate are obtained as:

$$\frac{\partial {N^{nl}}_{xx}}{\partial x} + \frac{\partial {N^{nl}}_{xy}}{\partial y} = 0 \quad (2.63)$$

$$\frac{\partial {N^{nl}}_{yy}}{\partial y} + \frac{\partial {N^{nl}}_{xy}}{\partial x} = 0 \quad (2.64)$$

$$\frac{\partial^2 {M^{nl}}_{xx}}{\partial x^2} + \frac{\partial^2 {M^{nl}}_{yy}}{\partial y^2} + 2\frac{\partial^2 {M^{nl}}_{xy}}{\partial x \partial y} - \frac{\partial^2 {P^{nl}}_{xx}}{\partial x^2} - \frac{\partial^2 {P^{nl}}_{yy}}{\partial y^2} - 2\frac{\partial^2 {P^{nl}}_{xy}}{\partial x \partial y}$$
$$+ \frac{\partial {R^{nl}}_{yz}}{\partial y} + \frac{\partial {R^{nl}}_{xz}}{\partial x} + q + T_{xx}\frac{\partial^2 w}{\partial x^2} + T_{yy}\frac{\partial^2 w}{\partial y^2} = 0 \quad (2.65)$$

$$\frac{\partial {P^{nl}}_{xx}}{\partial x} + \frac{\partial {P^{nl}}_{xy}}{\partial y} - {R^{nl}}_{xz} = 0 \tag{2.66}$$

$$\frac{\partial {P^{nl}}_{yy}}{\partial y} + \frac{\partial {P^{nl}}_{xy}}{\partial x} - {R^{nl}}_{yz} = 0 \tag{2.67}$$

In which, q is distributed bending force, T_{xx} and T_{yy} are buckling loads. Nonlocal stress resultants are defined by:

$$\left\{ \begin{array}{c} {N^{nl}}_{\alpha\beta} \\ {M^{nl}}_{\alpha\beta} \\ {P^{nl}}_{\alpha\beta} \end{array} \right\} = \int_{-h/2}^{h/2} \sigma_{\alpha\beta} \left\{ \begin{array}{c} 1 \\ z \\ f \end{array} \right\} dz, \quad \alpha = x, y, \quad \beta = x, y, \tag{2.68}$$

$$\left\{ \begin{array}{c} {N^{nl}}_{\alpha z} \\ {R^{nl}}_{\alpha z} \end{array} \right\} = \int_{-h/2}^{h/2} \sigma_{\alpha z} \left\{ \begin{array}{c} 1 \\ f' \end{array} \right\} dz, \quad \alpha = x, y, \tag{2.69}$$

Only the local stress resultants can be written in terms of displacements. Therefore, in order to be able to solve the governing equations for bending and buckling, we need to express the equations in terms of local stress resultants. For this purpose, assuming a constant nonlocal parameter through the thickness of the plate, the following equations can be obtained using Equations 2.14,2.36,2.37,2.68,2.69:

$$\left\{ \begin{array}{c} N_{\alpha\beta} \\ M_{\alpha\beta} \\ P_{\alpha\beta} \end{array} \right\} = (1 - \mu\nabla^2) \left\{ \begin{array}{c} {N^{nl}}_{\alpha\beta} \\ {M^{nl}}_{\alpha\beta} \\ {P^{nl}}_{\alpha\beta} \end{array} \right\}, \quad \alpha = x, y, \quad \beta = x, y, \tag{2.70}$$

$$\left\{ \begin{array}{c} N_{\alpha z} \\ R_{\alpha z} \end{array} \right\} = (1 - \mu\nabla^2) \left\{ \begin{array}{c} {N^{nl}}_{\alpha z} \\ {R^{nl}}_{\alpha z} \end{array} \right\}, \quad \alpha = x, y, \tag{2.71}$$

Multiplying both sides of the equilibrium Equations 2.63-2.67 by nonlocal operator and using Equations 2.70-2.71, nonlocal governing equations for bending and buckling of nanoplate are described in terms of local stress resultants as follows:

$$\frac{\partial N_{xx}}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0 \tag{2.72}$$

$$\frac{\partial N_{yy}}{\partial y} + \frac{\partial N_{xy}}{\partial x} = 0 \tag{2.73}$$

$$\begin{aligned} &\frac{\partial^2 M_{xx}}{\partial x^2} + \frac{\partial^2 M_{yy}}{\partial y^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} - \frac{\partial^2 P_{xx}}{\partial x^2} - \frac{\partial^2 P_{yy}}{\partial y^2} - 2\frac{\partial^2 P_{xy}}{\partial x \partial y} \\ &\qquad + \frac{\partial R_{yz}}{\partial y} + \frac{\partial R_{xz}}{\partial x} + (1 - \mu\nabla^2)(q + T_{xx}\frac{\partial^2 w}{\partial x^2} + T_{yy}\frac{\partial^2 w}{\partial y^2}) = 0 \end{aligned} \tag{2.74}$$

$$\frac{\partial P_{xx}}{\partial x} + \frac{\partial P_{xy}}{\partial y} - R_{xz} = 0 \tag{2.75}$$

$$\frac{\partial P_{yy}}{\partial y} + \frac{\partial P_{xy}}{\partial x} - R_{yz} = 0 \tag{2.76}$$

Equations 2.72-2.76 are only valid under the assumption of constant nonlocal parameter. If one wants to solve bending and buckling problems assuming a variable nonlocal parameter, equilibrium equations given by Equations 2.63-2.67 need to be considered. But these equations contain nonlocal stress resultants and they could not be written in terms of displacements. In order to solve these equations, another method needs to be developed which is out of the scope of this study.

CHAPTER 3

NUMERICAL SOLUTION

In this chapter, numerical solution technique is described in details. Generalized differential quadrature method is used to solve the governing partial differential equations and associated boundary conditions. MATLAB software is used to generate necessary computer programs to implement the numerical solutions.

3.1 Generalized Differential Quadrature Method (GDQM)

In some complex problems, analytical solutions of governing equations are unattainable. In such cases, a numerical method is used to solve the governing system of equations. As a result, adopting a suitable numerical technique plays a crucial role in approaching different engineering problems. GDQM is a powerful numerical technique for solving the governing partial differential equations of the systems with single and regular shape domains, like rectangular plates. This method is applicable to wide variety of problems with different boundary conditions using a small number of grid points which makes GDQM a powerful approach with respect to time efficiency and versatility.

Bellman and Casti [71] proposed DQM for evaluating the global and partial derivatives of any smooth function. According to this method, derivatives of a function g are approximated in a neighborhood of a point of its domain using a weighted linear sum of the values assumed by the same function in all the points of the domain along the direction of derivation:

$$\frac{\partial^n g(x,t)}{\partial x^n}|_{x_i} = \sum_{j=1}^{N} c_{ij}^{(n)} g\left(x_j, t\right), \quad i = 1, 2, ..., N \tag{3.1}$$

where $c_{ij}^{(n)}$ are weighting coefficients for the nth-order derivative and N is the number of nodes. Shu [72] proposed a generalized approach of calculating weighting coefficients using Lagrange interpolating polynomials. This method of weighting coefficients calculation is based on a recursive formula which eliminates the need for solving any algebraic system. The following equations can be used to calculate the coefficients for the derivatives of first and higher orders:

$$c_{ij}^{(1)} = \frac{M^{(1)}(x_i)}{(x_i - x_j) M^{(1)}(x_j)}, \quad i \neq j \tag{3.2}$$

$$c_{ii}^{(1)} = \frac{M^{(2)}(x_i)}{2M^{(1)}(x_i)}, \quad i = j \tag{3.3}$$

$$c_{ij}^{(m)} = m\left(c_{ii}^{(m-1)} c_{ij} - \frac{c_{ij}^{(m-1)}}{x_i - x_j}\right), \quad i \neq j, i, j = 1, 2, ..., N, m = 2, 3, ..., N-1 \tag{3.4}$$

$$c_{ii}^{(m)} = -\sum_{j=1, j \neq i}^{N} c_{ij}^{(m)}, \quad i = 1, 2, ..., N \tag{3.5}$$

Where M is defined by

$$M(x) = \prod_{j=1}^{N} (x - x_j) = (x - x_1)(x - x_2)...(x - x_N) \tag{3.6}$$

3.2 Numerical Solution for Simply-supported and Cantilever Nano-plates

In parametric analyses, we consider two different types of nano-plate configurations: A nano-plate simply-supported over all edges; and a cantilever nano-plate fixed at $x = 0$. When the interpolants are algebraic polynomials [73, 74], as $N \to \infty$, effective nodal point distributions should satisfy the density condition given by

$$density \sim \frac{N}{\pi\sqrt{1 - x^2}} \tag{3.7}$$

This means that the points are clustered near $x = \pm 1$. In this study, for both simply-supported and cantilever nano-plates, nodal points are identified as Chebyshev-Gauss-Lobatto points, which are given by

$$x_i = \frac{1}{2}\left\{1 - \cos\left(\frac{\pi(i-1)}{N-1}\right)\right\}, \quad i = 1, 2, ..., N. \tag{3.8}$$

This set of nodal grid points satisfies the condition given by Equation 3.7. More discussion on the accuracy of polynomial interpolation and choice of grid points is given in References [73, 74].

Applying the representation in Equation 3.1 to the differential operators, series forms of the governing equations are derived as:

$$
\begin{aligned}
&A_0 \sum_{k=1}^{N_x} c_{ik}{}^{(2)} u_{0_{k,j}} + C_0 \sum_{k=1}^{N_y} c_{jk}{}^{(2)} u_{0_{i,k}} + (B_0 + C_0) \sum_{m=1}^{N_y} c_{jm}{}^{(1)} \sum_{k=1}^{N_x} c_{ik}{}^{(1)} v_{0_{k,m}} \\
&+(A_3 - A_1) \sum_{k=1}^{N_x} c_{ik}{}^{(3)} w_{k,j} \\
&+(B_3 - B_1 - 2C_1 + 2C_3) \sum_{m=1}^{N_y} c_{jm}{}^{(2)} \sum_{k=1}^{N_x} c_{ik}{}^{(1)} w_{k,m} + A_3 \sum_{k=1}^{N_x} c_{ik}{}^{(2)} \phi_{x_{k,j}} \\
&+C_3 \sum_{k=1}^{N_y} c_{jk}{}^{(2)} \phi_{x_{i,k}} + (B_3 + C_3) \sum_{m=1}^{N_y} c_{jm}{}^{(1)} \sum_{k=1}^{N_x} c_{ik}{}^{(1)} \phi_{y_{k,m}} = I_0 \ddot{u}_0 \\
&+(I_3 - I_1) \sum_{k=1}^{N_x} c_{ik}{}^{(1)} \ddot{w}_{k,j} + I_3 \ddot{\phi}_x - L_0 \sum_{k=1}^{N_x} c_{ik}{}^{(2)} \ddot{u}_{0_{k,j}} \\
&-(L_3 - L_1) \sum_{k=1}^{N_x} c_{ik}{}^{(3)} \ddot{w}_{k,j} - L_3 \sum_{k=1}^{N_x} c_{ik}{}^{(2)} \ddot{\phi}_{x_{k,j}} - L_0 \sum_{k=1}^{N_y} c_{jk}{}^{(2)} \ddot{u}_{0_{i,k}} \\
&-(L_3 - L_1) \sum_{m=1}^{N_y} c_{jm}{}^{(2)} \sum_{k=1}^{N_x} c_{ik}{}^{(1)} \ddot{w}_{k,m} - L_3 \sum_{k=1}^{N_y} c_{jk}{}^{(2)} \ddot{\phi}_{x_{i,k}},
\end{aligned}
\tag{3.9}
$$

$$
\begin{aligned}
&C_0 \sum_{k=1}^{N_x} c_{ik}{}^{(2)} v_{0_{k,j}} + A_0 \sum_{k=1}^{N_y} c_{jk}{}^{(2)} v_{0_{i,k}} + (B_0 + C_0) \sum_{m=1}^{N_y} c_{jm}{}^{(1)} \sum_{k=1}^{N_x} c_{ik}{}^{(1)} u_{0_{k,m}} \\
&+(A_3 - A_1) \sum_{k=1}^{N_y} c_{jk}{}^{(3)} w_{i,k} \\
&+(B_3 - B_1 - 2C_1 + 2C_3) \sum_{m=1}^{N_y} c_{jm}{}^{(1)} \sum_{k=1}^{N_x} c_{ik}{}^{(2)} w_{k,m} + C_3 \sum_{k=1}^{N_x} c_{ik}{}^{(2)} \phi_{y_{k,j}} \\
&+A_3 \sum_{k=1}^{N_y} c_{jk}{}^{(2)} \phi_{y_{i,k}} + (B_3 + C_3) \sum_{m=1}^{N_y} c_{jm}{}^{(1)} \sum_{k=1}^{N_x} c_{ik}{}^{(1)} \phi_{x_{k,m}} = I_0 \ddot{v}_0 \\
&+(I_3 - I_1) \sum_{k=1}^{N_y} c_{jk}{}^{(1)} \ddot{w}_{i,k} + I_3 \ddot{\phi}_y - L_0 \sum_{k=1}^{N_x} c_{ik}{}^{(2)} \ddot{v}_{0_{k,j}} \\
&-(L_3 - L_1) \sum_{k=1}^{N_y} c_{jk}{}^{(3)} \ddot{w}_{i,k} - L_3 \sum_{k=1}^{N_x} c_{ik}{}^{(2)} \ddot{\phi}_{y_{k,j}} - L_0 \sum_{k=1}^{N_y} c_{jk}{}^{(2)} \ddot{v}_{0_{i,k}} \\
&-(L_3 - L_1) \sum_{m=1}^{N_y} c_{jm}{}^{(1)} \sum_{k=1}^{N_x} c_{ik}{}^{(2)} \ddot{w}_{k,m} - L_3 \sum_{k=1}^{N_y} c_{jk}{}^{(2)} \ddot{\phi}_{y_{i,k}},
\end{aligned}
\tag{3.10}
$$

$$
\begin{aligned}
&(A_1-A_3)(\sum_{k=1}^{N_y} c_{jk}{}^{(3)} v_{0_{i,k}} + \sum_{k=1}^{N_x} c_{ik}{}^{(3)} u_{0_{k,j}}) + (2A_4 - A_2 - A_5)(\sum_{k=1}^{N_x} c_{ik}{}^{(4)} w_{k,j} \\
&+ \sum_{k=1}^{N_y} c_{jk}{}^{(4)} w_{i,k}) + (A_4 - A_5)(\sum_{k=1}^{N_x} c_{ik}{}^{(3)} \phi_{x_{k,j}} + \sum_{k=1}^{N_y} c_{jk}{}^{(3)} \phi_{y_{i,k}}) \\
&+(B_1 - B_3 + 2C_1 - 2C_3)(\sum_{m=1}^{N_y} c_{jm}{}^{(1)} \sum_{k=1}^{N_x} c_{ik}{}^{(2)} v_{0_{k,m}} + \sum_{m=1}^{N_y} c_{jm}{}^{(2)} \sum_{k=1}^{N_x} c_{ik}{}^{(1)} u_{0_{k,m}}) \\
&+2(2B_4 - B_2 - B_5 + 4C_4 - 2C_2 - 2C_5) \sum_{m=1}^{N_y} c_{jm}{}^{(2)} \sum_{k=1}^{N_x} c_{ik}{}^{(2)} w_{k,m} \\
&+(B_4 - B_5 + 2C_4 - 2C_5)(\sum_{m=1}^{N_y} c_{jm}{}^{(1)} \sum_{k=1}^{N_x} c_{ik}{}^{(2)} \phi_{y_{k,m}} + \sum_{m=1}^{N_y} c_{jm}{}^{(2)} \sum_{k=1}^{N_x} c_{ik}{}^{(1)} \phi_{x_{k,m}}) \\
&+C_6(\sum_{k=1}^{N_x} c_{ik}{}^{(1)} \phi_{x_{k,j}} + \sum_{k=1}^{N_y} c_{jk}{}^{(1)} \phi_{y_{i,k}}) + C_6(\sum_{k=1}^{N_x} c_{ik}{}^{(2)} w_{k,j} + \sum_{k=1}^{N_y} c_{jk}{}^{(2)} w_{i,k}) = \\
&I_0 \ddot{w} + (I_1 - I_3)(\sum_{k=1}^{N_x} c_{ik}{}^{(1)} \ddot{u}_{0_{k,j}} + \sum_{k=1}^{N_y} c_{jk}{}^{(1)} \ddot{v}_{0_{i,k}}) \\
&+(-I_2 + 2I_4 - I_5)(\sum_{k=1}^{N_x} c_{ik}{}^{(2)} \ddot{w}_{k,j} + \sum_{k=1}^{N_y} c_{jk}{}^{(2)} \ddot{w}_{i,k}) \\
&+(I_4 - I_5)(\sum_{k=1}^{N_x} c_{ik}{}^{(1)} \ddot{\phi}_{x_{k,j}} + \sum_{k=1}^{N_y} c_{jk}{}^{(1)} \ddot{\phi}_{y_{i,k}}) - L_0 \sum_{k=1}^{N_x} c_{ik}{}^{(2)} \ddot{w}_{k,j} \\
&-(L_1 - L_3)(\sum_{k=1}^{N_x} c_{ik}{}^{(3)} \ddot{u}_{0_{k,j}} + \sum_{m=1}^{N_y} c_{jm}{}^{(1)} \sum_{k=1}^{N_x} c_{ik}{}^{(2)} \ddot{v}_{0_{k,m}}) \\
&-(-L_2 + 2L_4 - L_5)(\sum_{k=1}^{N_x} c_{ik}{}^{(4)} \ddot{w}_{k,j} + \sum_{m=1}^{N_y} c_{jm}{}^{(2)} \sum_{k=1}^{N_x} c_{ik}{}^{(2)} \ddot{w}_{k,m}) \\
&-(L_4 - L_5)(\sum_{k=1}^{N_x} c_{ik}{}^{(3)} \ddot{\phi}_{x_{k,j}} + \sum_{m=1}^{N_y} c_{jm}{}^{(1)} \sum_{k=1}^{N_x} c_{ik}{}^{(2)} \ddot{\phi}_{y_{k,m}}) \\
&-L_0 \sum_{k=1}^{N_y} c_{jk}{}^{(2)} \ddot{w}_{i,k} - (L_1 - L_3)(\sum_{m=1}^{N_y} c_{jm}{}^{(2)} \sum_{k=1}^{N_x} c_{ik}{}^{(1)} \ddot{u}_{0_{k,m}} + \sum_{k=1}^{N_y} c_{jk}{}^{(3)} \ddot{v}_{0_{i,k}}) \\
&-(-L_2 + 2L_4 - L_5)(\sum_{m=1}^{N_y} c_{jm}{}^{(2)} \sum_{k=1}^{N_x} c_{ik}{}^{(2)} \ddot{w}_{k,m} + \sum_{k=1}^{N_y} c_{jk}{}^{(4)} \ddot{w}_{i,k}) \\
&-(L_4 - L_5)(\sum_{m=1}^{N_y} c_{jm}{}^{(2)} \sum_{k=1}^{N_x} c_{ik}{}^{(1)} \ddot{\phi}_{x_{k,m}} + \sum_{k=1}^{N_y} c_{jk}{}^{(3)} \ddot{\phi}_{y_{i,k}}),
\end{aligned}
\tag{3.11}
$$

$$
\begin{aligned}
&A_3 \sum_{k=1}^{N_x} c_{ik}{}^{(2)} u_{0_{k,j}} + C_3 \sum_{k=1}^{N_y} c_{jk}{}^{(2)} u_{0_{i,k}} + (B_3 + C_3) \sum_{m=1}^{N_y} c_{jm}{}^{(1)} \sum_{k=1}^{N_x} c_{ik}{}^{(1)} v_{0_{k,m}} \\
&+(A_5 - A_4) \sum_{k=1}^{N_x} c_{ik}{}^{(3)} w_{k,j} + (B_5 - B_4 - 2C_4 + 2C_5) \sum_{m=1}^{N_y} c_{jm}{}^{(2)} \sum_{k=1}^{N_x} c_{ik}{}^{(1)} w_{k,m} \\
&+A_5 \sum_{k=1}^{N_x} c_{ik}{}^{(2)} \phi_{x_{k,j}} + C_5 \sum_{k=1}^{N_y} c_{jk}{}^{(2)} \phi_{x_{i,k}} + (B_5 + C_5) \sum_{m=1}^{N_y} c_{jm}{}^{(1)} \sum_{k=1}^{N_x} c_{ik}{}^{(1)} \phi_{y_{k,m}} \\
&-C_6 \phi_x - C_6 \sum_{k=1}^{N_x} c_{ik}{}^{(1)} w_{k,j} = I_3 \ddot{u}_0 + (I_5 - I_4) \sum_{k=1}^{N_x} c_{ik}{}^{(1)} \ddot{w}_{k,j} + I_5 \ddot{\phi}_x \\
&-L_3 \sum_{k=1}^{N_x} c_{ik}{}^{(2)} \ddot{u}_{0_{k,j}} - (L_5 - L_4) \sum_{k=1}^{N_x} c_{ik}{}^{(3)} \ddot{w}_{k,j} - L_5 \sum_{k=1}^{N_x} c_{ik}{}^{(2)} \ddot{\phi}_{x_{k,j}} \\
&-L_3 \sum_{k=1}^{N_y} c_{jk}{}^{(2)} \ddot{u}_{0_{i,k}} - (L_5 - L_4) \sum_{m=1}^{N_y} c_{jm}{}^{(2)} \sum_{k=1}^{N_x} c_{ik}{}^{(1)} \ddot{w}_{k,m} - L_5 \sum_{k=1}^{N_y} c_{jk}{}^{(2)} \ddot{\phi}_{x_{i,k}},
\end{aligned}
\tag{3.12}
$$

$$
\begin{aligned}
&C_3 \sum_{k=1}^{N_x} c_{ik}{}^{(2)} v_{0_{k,j}} + A_3 \sum_{k=1}^{N_y} c_{jk}{}^{(2)} v_{0_{i,k}} + (B_3 + C_3) \sum_{m=1}^{N_y} c_{jm}{}^{(1)} \sum_{k=1}^{N_x} c_{ik}{}^{(1)} u_{0_{k,m}} \\
&+(A_5 - A_4) \sum_{k=1}^{N_y} c_{jk}{}^{(3)} w_{i,k} + (B_5 - B_4 - 2C_4 + 2C_5) \sum_{m=1}^{N_y} c_{jm}{}^{(1)} \sum_{k=1}^{N_x} c_{ik}{}^{(2)} w_{k,m} \\
&+C_5 \sum_{k=1}^{N_x} c_{ik}{}^{(2)} \phi_{y_{k,j}} + A_5 \sum_{k=1}^{N_y} c_{jk}{}^{(2)} \phi_{y_{i,k}} + (B_5 + C_5) \sum_{m=1}^{N_y} c_{jm}{}^{(1)} \sum_{k=1}^{N_x} c_{ik}{}^{(1)} \phi_{x_{k,m}} \\
&-C_6 \phi_y - C_6 \sum_{k=1}^{N_y} c_{jk}{}^{(1)} w_{i,k} = I_3 \ddot{v}_0 + (I_5 - I_4) \sum_{k=1}^{N_y} c_{jk}{}^{(1)} \ddot{w}_{i,k} + I_5 \ddot{\phi}_y \\
&-L_3 \sum_{k=1}^{N_x} c_{ik}{}^{(2)} \ddot{v}_{0_{k,j}} - (L_5 - L_4) \sum_{k=1}^{N_y} c_{jk}{}^{(3)} \ddot{w}_{i,k} - L_5 \sum_{k=1}^{N_x} c_{ik}{}^{(2)} \ddot{\phi}_{y_{k,j}} \\
&-L_3 \sum_{k=1}^{N_y} c_{jk}{}^{(2)} \ddot{v}_{0_{i,k}} - (L_5 - L_4) \sum_{m=1}^{N_y} c_{jm}{}^{(1)} \sum_{k=1}^{N_x} c_{ik}{}^{(2)} \ddot{w}_{k,m} - L_5 \sum_{k=1}^{N_y} c_{jk}{}^{(2)} \ddot{\phi}_{y_{i,k}}.
\end{aligned} \tag{3.13}
$$

N_x and N_y above are number of nodal points in x- and y-directions, respectively.

For a simply-supported nano-plate, boundary conditions at $y = 0$ and $y = b$ read:

$$u_0 = v_0 = w = \phi_x = 0, \tag{3.14}$$

$$A_3 \sum_{k=1}^{N_y} c_{jk}{}^{(1)} v_{0_{i,k}} + (A_5 - A_4) \sum_{k=1}^{N_y} c_{jk}{}^{(2)} w_{i,k} + A_5 \sum_{k=1}^{N_y} c_{jk}{}^{(1)} \phi_{y_{i,k}} = 0, \tag{3.15}$$

$$A_1 \sum_{k=1}^{N_y} c_{jk}{}^{(1)} v_{0_{i,k}} + (A_4 - A_2) \sum_{k=1}^{N_y} c_{jk}{}^{(2)} w_{i,k} + A_4 \sum_{k=1}^{N_y} c_{jk}{}^{(1)} \phi_{y_{i,k}} = 0, \tag{3.16}$$

and at $x = 0$ and $x = a$, we have

$$u_0 = v_0 = w = \phi_y = 0, \tag{3.17}$$

$$A_3 \sum_{k=1}^{N_x} c_{ik}{}^{(1)} u_{0_{k,j}} + (A_5 - A_4) \sum_{k=1}^{N_x} c_{ik}{}^{(2)} w_{k,j} + A_5 \sum_{k=1}^{N_x} c_{ik}{}^{(1)} \phi_{x_{k,j}} = 0, \tag{3.18}$$

$$A_1 \sum_{k=1}^{N_x} c_{ik}{}^{(1)} u_{0_{k,j}} + (A_4 - A_2) \sum_{k=1}^{N_x} c_{ik}{}^{(2)} w_{k,j} + A_4 \sum_{k=1}^{N_x} c_{ik}{}^{(1)} \phi_{x_{k,j}} = 0. \tag{3.19}$$

For the cantilever nano-plate fixed at $x = 0$, boundary conditions at the cantilever edge are:

$$u_0 = v_0 = w = \phi_x = \phi_y = \frac{\partial w}{\partial x} = 0. \tag{3.20}$$

The conditions at $x = a$ are derived as:

$$
\begin{aligned}
&A_0 \sum_{k=1}^{N_x} c_{ik}{}^{(1)} u_{0_{k,j}} + (A_3 - A_1) \sum_{k=1}^{N_x} c_{ik}{}^{(2)} w_{k,j} + A_3 \sum_{k=1}^{N_x} c_{ik}{}^{(1)} \phi_{x_{k,j}} \\
&+B_0 \sum_{k=1}^{N_y} c_{jk}{}^{(1)} v_{0_{i,k}} + (B_3 - B_1) \sum_{k=1}^{N_y} c_{jk}{}^{(2)} w_{i,k} + B_3 \sum_{k=1}^{N_y} c_{jk}{}^{(1)} \phi_{y_{i,k}} = 0,
\end{aligned} \tag{3.21}
$$

$$
\begin{aligned}
&C_0\Big(\sum_{k=1}^{N_y} c_{jk}{}^{(1)} u_{0_{i,k}} + \sum_{k=1}^{N_x} c_{ik}{}^{(1)} v_{0_{k,j}}\Big) + 2(C_3 - C_1) \sum_{m=1}^{N_y} c_{jm}{}^{(1)} \sum_{k=1}^{N_x} c_{ik}{}^{(1)} w_{k,m} \\
&+C_3\Big(\sum_{k=1}^{N_y} c_{jk}{}^{(1)} \phi_{x_{i,k}} + \sum_{k=1}^{N_x} c_{ik}{}^{(1)} \phi_{y_{k,j}}\Big) = 0,
\end{aligned}
\tag{3.22}
$$

$$
\begin{aligned}
&A_3 \sum_{k=1}^{N_x} c_{ik}{}^{(1)} u_{0_{k,j}} + (A_5 - A_4) \sum_{k=1}^{N_x} c_{ik}{}^{(2)} w_{k,j} + A_5 \sum_{k=1}^{N_x} c_{ik}{}^{(1)} \phi_{x_{k,j}} \\
&+B_3 \sum_{k=1}^{N_y} c_{jk}{}^{(1)} v_{0_{i,k}} + (B_5 - B_4) \sum_{k=1}^{N_y} c_{jk}{}^{(2)} w_{i,k} + B_5 \sum_{k=1}^{N_y} c_{jk}{}^{(1)} \phi_{y_{i,k}} = 0,
\end{aligned}
\tag{3.23}
$$

$$
\begin{aligned}
&C_3\Big(\sum_{k=1}^{N_y} c_{jk}{}^{(1)} u_{0_{i,k}} + \sum_{k=1}^{N_x} c_{ik}{}^{(1)} v_{0_{k,j}}\Big) + 2(C_5 - C_4) \sum_{m=1}^{N_y} c_{jm}{}^{(1)} \sum_{k=1}^{N_x} c_{ik}{}^{(1)} w_{k,m} \\
&+C_5\Big(\sum_{k=1}^{N_y} c_{jk}{}^{(1)} \phi_{x_{i,k}} + \sum_{k=1}^{N_x} c_{ik}{}^{(1)} \phi_{y_{k,j}}\Big) = 0,
\end{aligned}
\tag{3.24}
$$

$$
\begin{aligned}
&(A_1 - A_3) \sum_{k=1}^{N_x} c_{ik}{}^{(2)} u_{0_{k,j}} + (-A_2 + 2A_4 - A_5) \sum_{k=1}^{N_x} c_{ik}{}^{(3)} w_{k,j} \\
&+(A_4 - A_5) \sum_{k=1}^{N_x} c_{ik}{}^{(2)} \phi_{x_{k,j}} + (B_1 - B_3) \sum_{m=1}^{N_y} c_{jm}{}^{(1)} \sum_{k=1}^{N_x} c_{ik}{}^{(1)} v_{0_{k,m}} \\
&+(-B_2 + 2B_4 - B_5) \sum_{m=1}^{N_y} c_{jm}{}^{(2)} \sum_{k=1}^{N_x} c_{ik}{}^{(1)} w_{k,m} \\
&+(B_4 - B_5) \sum_{m=1}^{N_y} c_{jm}{}^{(1)} \sum_{k=1}^{N_x} c_{ik}{}^{(1)} \phi_{y_{k,m}} \\
&+(C_1 - C_3)\Big(\sum_{k=1}^{N_y} c_{jk}{}^{(2)} u_{0_{i,k}} + \sum_{m=1}^{N_y} c_{jm}{}^{(1)} \sum_{k=1}^{N_x} c_{ik}{}^{(1)} v_{0_{k,m}}\Big) \\
&+2(-C_2 + 2C_4 - C_5) \sum_{m=1}^{N_y} c_{jm}{}^{(2)} \sum_{k=1}^{N_x} c_{ik}{}^{(1)} w_{k,m} \\
&+(C_4 - C_5)\Big(\sum_{k=1}^{N_y} c_{jk}{}^{(2)} \phi_{x_{i,k}} + \sum_{m=1}^{N_y} c_{jm}{}^{(1)} \sum_{k=1}^{N_x} c_{ik}{}^{(1)} \phi_{y_{k,m}}\Big) \\
&+C_6\Big(\phi_x + \sum_{k=1}^{N_x} c_{ik}{}^{(1)} w_{k,j}\Big) = (I_1 - I_3)\ddot{u}_0 \\
&+(-I_2 + 2I_4 - I_5) \sum_{k=1}^{N_x} c_{ik}{}^{(1)} \ddot{w}_{k,j} + (I_4 - I_5)\ddot{\phi}_x \\
&-(L_1 - L_3) \sum_{k=1}^{N_x} c_{ik}{}^{(2)} \ddot{u}_{0_{k,j}} - (-L_2 + 2L_4 - L_5) \sum_{k=1}^{N_x} c_{ik}{}^{(3)} \ddot{w}_{k,j} \\
&-(L_4 - L_5) \sum_{k=1}^{N_x} c_{ik}{}^{(2)} \ddot{\phi}_{x_{k,j}} - (L_1 - L_3) \sum_{k=1}^{N_y} c_{jk}{}^{(2)} \ddot{u}_{0_{i,k}} \\
&-(-L_2 + 2L_4 - L_5) \sum_{m=1}^{N_y} c_{jm}{}^{(2)} \sum_{k=1}^{N_x} c_{ik}{}^{(1)} \ddot{w}_{k,m} \\
&-(L_4 - L_5) \sum_{k=1}^{N_y} c_{jk}{}^{(2)} \ddot{\phi}_{x_{i,k}},
\end{aligned}
\tag{3.25}
$$

$$
\begin{aligned}
&(A_1-A_3)\sum_{k=1}^{N_x}c_{ik}{}^{(1)}u_{0_{k,j}}+(-A_2+2A_4-A_5)\sum_{k=1}^{N_x}c_{ik}{}^{(2)}w_{k,j}\\
&+(A_4-A_5)\sum_{k=1}^{N_x}c_{ik}{}^{(1)}\phi_{x_{k,j}}+(B_1-B_3)\sum_{k=1}^{N_y}c_{jk}{}^{(1)}v_{0_{i,k}}\\
&+(-B_2+2B_4-B_5)\sum_{k=1}^{N_y}c_{jk}{}^{(2)}w_{i,k}+(B_4-B_5)\sum_{k=1}^{N_y}c_{jk}{}^{(1)}\phi_{y_{i,k}}=0.
\end{aligned}
\tag{3.26}
$$

And, the conditions at $y = 0$ and $y = b$ are of the forms:

$$
\begin{aligned}
&C_0(\sum_{k=1}^{N_y}c_{jk}{}^{(1)}u_{0_{i,k}}+\sum_{k=1}^{N_x}c_{ik}{}^{(1)}v_{0_{k,j}})+2(C_3-C_1)\sum_{m=1}^{N_y}c_{jm}{}^{(1)}\sum_{k=1}^{N_x}c_{ik}{}^{(1)}w_{k,m}\\
&+C_3(\sum_{k=1}^{N_y}c_{jk}{}^{(1)}\phi_{x_{i,k}}+\sum_{k=1}^{N_x}c_{ik}{}^{(1)}\phi_{y_{k,j}})=0,
\end{aligned}
\tag{3.27}
$$

$$
\begin{aligned}
&B_0\sum_{k=1}^{N_x}c_{ik}{}^{(1)}u_{0_{k,j}}+(B_3-B_1)\sum_{k=1}^{N_x}c_{ik}{}^{(2)}w_{k,j}+B_3\sum_{k=1}^{N_x}c_{ik}{}^{(1)}\phi_{x_{k,j}}\\
&+A_0\sum_{k=1}^{N_y}c_{jk}{}^{(1)}v_{0_{i,k}}+(A_3-A_1)\sum_{k=1}^{N_y}c_{jk}{}^{(2)}w_{i,k}+A_3\sum_{k=1}^{N_y}c_{jk}{}^{(1)}\phi_{y_{i,k}}=0,
\end{aligned}
\tag{3.28}
$$

$$
\begin{aligned}
&C_3(\sum_{k=1}^{N_y}c_{jk}{}^{(1)}u_{0_{i,k}}+\sum_{k=1}^{N_x}c_{ik}{}^{(1)}v_{0_{k,j}})+2(C_5-C_4)\sum_{m=1}^{N_y}c_{jm}{}^{(1)}\sum_{k=1}^{N_x}c_{ik}{}^{(1)}w_{k,m}\\
&+C_5(\sum_{k=1}^{N_y}c_{jk}{}^{(1)}\phi_{x_{i,k}}+\sum_{k=1}^{N_x}c_{ik}{}^{(1)}\phi_{y_{k,j}})=0,
\end{aligned}
\tag{3.29}
$$

$$
\begin{aligned}
&B_3\sum_{k=1}^{N_x}c_{ik}{}^{(1)}u_{0_{k,j}}+(B_5-B_4)\sum_{k=1}^{N_x}c_{ik}{}^{(2)}w_{k,j}+B_5\sum_{k=1}^{N_x}c_{ik}{}^{(1)}\phi_{x_{k,j}}\\
&+A_3\sum_{k=1}^{N_y}c_{jk}{}^{(1)}v_{0_{i,k}}+(A_5-A_4)\sum_{k=1}^{N_y}c_{jk}{}^{(2)}w_{i,k}+A_5\sum_{k=1}^{N_y}c_{jk}{}^{(1)}\phi_{y_{i,k}}=0,
\end{aligned}
\tag{3.30}
$$

$$
\begin{aligned}
&(B_1 - B_3)\sum_{m=1}^{N_y} c_{jm}{}^{(1)}\sum_{k=1}^{N_x} c_{ik}{}^{(1)} u_{0_{k,m}} \\
&+(-B_2 + 2B_4 - B_5)\sum_{m=1}^{N_y} c_{jm}{}^{(1)}\sum_{k=1}^{N_x} c_{ik}{}^{(2)} w_{_{k,m}} \\
&+(B_4 - B_5)\sum_{m=1}^{N_y} c_{jm}{}^{(1)}\sum_{k=1}^{N_x} c_{ik}{}^{(1)} \phi_{x_{k,m}} + (A_1 - A_3)\sum_{k=1}^{N_y} c_{jk}{}^{(2)} v_{0_{i,k}} \\
&+(-A_2 + 2A_4 - A_5)\sum_{k=1}^{N_y} c_{jk}{}^{(3)} w_{i,k} + (A_4 - A_5)\sum_{k=1}^{N_y} c_{jk}{}^{(2)} \phi_{y_{i,k}} \\
&+(C_1 - C_3)(\sum_{m=1}^{N_y} c_{jm}{}^{(1)}\sum_{k=1}^{N_x} c_{ik}{}^{(1)} u_{0_{k,m}} + \sum_{k=1}^{N_x} c_{ik}{}^{(2)} v_{0_{k,j}}) \\
&+2(-C_2 + 2C_4 - C_5)\sum_{m=1}^{N_y} c_{jm}{}^{(1)}\sum_{k=1}^{N_x} c_{ik}{}^{(2)} w_{k,m} \\
&+(C_4 - C_5)(\sum_{m=1}^{N_y} c_{jm}{}^{(1)}\sum_{k=1}^{N_x} c_{ik}{}^{(1)} \phi_{x_{k,m}} + \sum_{k=1}^{N_x} c_{ik}{}^{(2)} \phi_{y_{k,j}}) \\
&+C_6(\phi_y + \sum_{k=1}^{N_y} c_{jk}{}^{(1)} w_{_{i,k}}) = (I_1 - I_3)\ddot{v}_0 \\
&+(-I_2 + 2I_4 - I_5)\sum_{k=1}^{N_y} c_{jk}{}^{(1)} \ddot{w}_{_{i,k}} + (I_4 - I_5)\ddot{\phi}_y \\
&-(L_1 - L_3)\sum_{k=1}^{N_x} c_{ik}{}^{(2)} \ddot{v}_{0_{k,j}} \\
&-(-L_2 + 2L_4 - L_5)\sum_{m=1}^{N_y} c_{jm}{}^{(1)}\sum_{k=1}^{N_x} c_{ik}{}^{(2)} \ddot{w}_{k,m} \\
&-(L_4 - L_5)\sum_{k=1}^{N_x} c_{ik}{}^{(2)} \ddot{\phi}_{y_{k,j}} \\
&-(L_1 - L_3)\sum_{k=1}^{N_y} c_{jk}{}^{(2)} \ddot{v}_{0_{i,k}} - (-L_2 + 2L_4 - L_5)\sum_{k=1}^{N_y} c_{jk}{}^{(3)} \ddot{w}_{_{i,k}} \\
&-(L_4 - L_5)\sum_{k=1}^{N_y} c_{jk}{}^{(2)} \ddot{\phi}_{y_{i,k}},
\end{aligned} \tag{3.31}
$$

$$
\begin{aligned}
&(B_1 - B_3)\sum_{k=1}^{N_x} c_{ik}{}^{(1)} u_{0_{k,j}} + (-B_2 + 2B_4 - B_5)\sum_{k=1}^{N_x} c_{ik}{}^{(2)} w_{k,j} \\
&+(B_4 - B_5)\sum_{k=1}^{N_x} c_{ik}{}^{(1)} \phi_{x_{k,j}} + (A_1 - A_3)\sum_{k=1}^{N_y} c_{jk}{}^{(1)} v_{0_{i,k}} \\
&+(-A_2 + 2A_4 - A_5)\sum_{k=1}^{N_y} c_{jk}{}^{(2)} w_{i,k} + (A_4 - A_5)\sum_{k=1}^{N_y} c_{jk}{}^{(1)} \phi_{y_{i,k}} = 0.
\end{aligned} \tag{3.32}
$$

The series form of the governing equations and boundary conditions given above can be used to numerically solve the free vibration problem of rectangular nano-plate. In what follows below, we provide the matrix form of the governing equations and boundary conditions for plates undergoing free vibrations. In this case, displacement vector d is defined as

$$
d = d^* e^{i\Omega t} \tag{3.33}
$$

where Ω is natural frequency and d^* is mode shape vector expressed as

$$d^* = \left\{ \{u_i{}^*\}^T, \{v_i{}^*\}^T, \{w_i{}^*\}^T, \{\phi_{x_i}{}^*\}^T, \{\phi_{y_i}{}^*\}^T \right\}^T, i = 1, 2, ..., N_x \times N_y. \quad (3.34)$$

By substituting Equation 3.33 into governing equations and boundary conditions, one can obtain:

$$D_b d^{*e} + D_d d^{*i} - \Omega^2 M d^{*i} = 0 \quad (3.35)$$

$$B_b d^{*e} + B_d d^{*i} = 0 \quad (3.36)$$

In equations above, D_b and B_b are coefficient matrices associated with exterior points of the domain, whereas D_d and B_d are related to interior nodes. d^{*i} is the mode shape vector for interior points where the governing equations are applied and d^{*e} is the mode shape vector for exterior nodes where the boundary equations are applied. For both simply-supported and cantilever nano-plates, governing equations and boundary conditions are consolidated into the following matrix form:

$$(K - \Omega^2 M) d^{*i} = 0, \quad (3.37)$$

where M is the mass matrix and K is the stiffness matrix given by

$$K = -D_b B_b{}^{-1} B_d + D_d \quad (3.38)$$

Equation 3.37 is solved to determine natural frequencies and the corresponding mode shape vectors. In order to implement the numerical method described in this chapter, a computer program is developed using MATLAB software. In the following chapter, we give detailed results for free vibration of simply supported and cantilever nano-plates.

CHAPTER 4

RESULTS

In parametric analyses, we examine free vibrations of ceramic-metal functionally graded composite nano-plates, whose constituents are silicon nitride (Si3N4) and stainless steel. Properties for this material pair are given by

$$E_c = 348.43\text{GPa}, \nu_c = 0.3, \rho_c = 2370\text{kg/m}^3, \tag{4.1}$$

$$E_m = 201.04\text{GPa}, \nu_m = 0.3, \rho_m = 8166\text{kg/m}^3. \tag{4.2}$$

Nonlocal parameter of the metallic phase is taken as $\mu_m = 2nm^2$ which is a reference value adopted in various studies in the literature [59, 61]. The degree of variation in the nonlocal parameter is quantified by the ratio μ_c/μ_m. When the nonlocal parameter is assumed to be constant, μ_c/μ_m is equal to unity, whereas when μ varies across the thickness $\mu_c/\mu_m \neq 1$. We set μ_c/μ_m as 2 in a number of parametric analyses. In remaining cases, it is varied to be able to assess the influence of the nonlocal parameter variation.

4.1 Comparison Results

To be able to verify theoretical and computational developments provided in previous chapters, comparison results are provided in Tables 4.1 and 4.2. In Table 4.1, comparison results are given for dimensionless natural frequencies of a homogeneous plate according to different plate theories. It can be seen that the results of this study are in good agreement with those provided by Aghababaei et al. [28]. In Table 4.2 we provide comparisons to the results given in the article by Zare et al. [58]. This article presents an analytical method for obtaining results regarding the free vibrations of functionally graded rectangular nano-plates under the assumption of constant nonlocal parameter. Material properties used in [58] are the same as those given by Equations 4.1 and 4.2. Comparisons of first three dimensionless natural frequencies of a simply-supported functionally graded nano-plate are given in Table 4.2. Dimensionless natural frequency is defined as

$$\omega = \Omega h \sqrt{\frac{2\left(1+\nu_c\right)\rho_c}{E_c}}. \tag{4.3}$$

Table 4.1: Comparisons of dimensionless natural frequencies calculated for a simply-supported homogeneous nano-plate. $a = 10\text{nm}$, $a/h = 10$, $a/b = 1$, $\nu = 0.3$, $E = 30 \times 10^6$, $N_x = N_y = 11$.

$\mu_c = \mu_m = \mu$	Plate Theory		ω_{11}	ω_{22}	ω_{33}
0	TSDT	Present study	0.0934	0.3448	0.6947
		Aghababaei et al. [28]	0.0935	0.3458	0.7020
	MPT	Present study	0.0921	0.3395	0.6811
		Aghababaei et al. [28]	0.0930	0.3414	0.6889
	KPT	Present study	0.0955	0.3831	0.8665
		Aghababaei et al. [28]	0.0963	0.3853	0.8669

Table 4.2: Comparisons of dimensionless first three natural frequencies calculated for a simply-supported functionally graded nano-plate possesing a constant nonlocal parameter μ. $n = 5$, $a = 10\text{nm}$, $a/h = 20$, $N_x = N_y = 11$.

a/b	$\mu_c = \mu_m = \mu$ (nm^2)		ω_1	ω_2	ω_3
1	0	Present study	0.0114	0.0285	0.0285
		Zare et al. [58]	0.0114	0.0281	0.0281
	1	Present study	0.0104	0.0233	0.0233
		Zare et al. [58]	0.0104	0.0230	0.0230
	4	Present study	0.0085	0.0165	0.0165
		Zare et al. [58]	0.0085	0.0165	0.0165
2	0	Present study	0.0285	0.0454	0.0732
		Zare et al. [58]	0.0281	0.0443	0.0704
	1	Present study	0.0233	0.0340	0.0484
		Zare et al. [58]	0.0230	0.0330	0.0466
	4	Present study	0.0165	0.0223	0.0296
		Zare et al. [58]	0.0165	0.0218	0.0286

Both our results and those provided in [58] are generated by using Kirchhoff plate theory. Natural frequencies computed in the present study are in very good agreement with those given by Zare et al. [58]. Different solution methods that are employed to obtain frequency results is the source of small differences found between the results of present study and those given by Zare et al. [58]. It can be seen that by increasing the nonlocal parameter, natural frequencies decrease. This is due to the softening effect of the nonlocal elasticity theory on nano-scale structures. The symmetry of the problem in case of $a/b = 1$ results in the same second and third natural frequencies for a given configuration of the simply-supported nano-plate.

4.2 Free Vibration Results

In Figures 4.1-4.14 and Tables 4.4-4.7, results of our parametric analyses for functionally graded rectangular nano-plates are provided taking into consideration the spatial variation of nonlocal parameter through the thickness of plate. In Table 4.3, the results of convergence study on dimensionless first natural frequencies of simply-supported and cantilever nano-plates are provided. It can be seen that the first dimensionless natural frequency converges at $N_x = N_y = 11$ for simply-supported nano-plate. However, for cantilever nano-plate, more number of grid points are needed for better accuracy and the frequencies converge at $N_x = N_y = 13$. Based on this convergence study, for generating all of the following results in this section with the exception of figures for the mode shapes of nano-plates, the number of grid points for simply-supported and cantilever nano-plates are chosen as $N_x = N_y = 11$ and $N_x = N_y = 13$, respectively. To be able to generate smooth mode shapes of nano-plates, $N_x = N_y = 33$ is adopted as the number of grid points for generating Figures 4.3 and 4.4.

Table 4.3: Convergence study on dimensionless first natural frequencies of simply-supported and cantilever nano-plates. $a/\sqrt{\mu_m} = 10$, $a/b = 1$, $a/h = 20$, $n = 2$, $\mu_m = 2\mathrm{nm}^2$, $\mu_c/\mu_m = 2$.

$N = N_x = N_y$	9	11	13	15
Simply-supported	0.0112	0.0113	0.0113	0.0113
Cantilever	0.00220	0.00223	0.00225	0.00225

Figure 4.1 depicts dimensionless first natural frequencies of simply-supported and cantilever nano-plates as a function of the dimensionless plate length $a/\sqrt{\mu_m}$. Dimensionless natural frequency is defined by Equation 4.3. The frequencies are calculated for three different plate theories, namely, Kirchhoff plate theory (KPT), Mindlin plate theory (MPT), and third-order shear deformation theory (TSDT). For both simply-supported and cantilever nano-plates, ω_1 increases with a corresponding increase in $a/\sqrt{\mu_m}$ and approaches a constant at larger values of dimensionless plate length. The constants are equal to the dimensionless natural frequencies of the plate computed in accordance with the classical continuum theory by setting $\mu_c = \mu_m = 0$. As expected, size effect is more dominant for smaller values of the ratio $a/\sqrt{\mu_m}$. Differences of the results predicted by different plate theories seem to be larger for cantilever nano-plate compared to those found for simply-supported one. This is due to different scales that need to be used in preparation of these figures. When the percentage differences are calculated, it is revealed that the differences for simply-supported nano-plate are greater. The higher degree of constraint in boundaries results in the larger differences found for the simply-supported plate. Cantilever plate contains one clamped and three free edges, but the simply-supported plate is constrained from all sides which leads to larger shear effect. This is the fundamental reason behind the differences found between the predictions of different plate theories.

Similar trends can be observed regarding the curves obtained for second natural frequencies of the simply-supported and cantilever plates as given in Figure 4.2. Comprehensive examination of Figures 4.1 and 4.2 reveals the fact that differences between the predictions of plate theories get larger as mode number increases. This is the result of different assumptions on through-the-thickness distribution of transverse shear deformation for different plate theories. The effects of transverse shear deformation and rotary inertia are more influential in higher modes which causes larger differences between the results obtained for different plate theories. It is mathematically proved by Wang et al. [75] that the differences in plate theories increase for the higher modes of vibration. Third-order shear deformation theory produces more accurate results due to its quadratic transverse shear strain distribution.

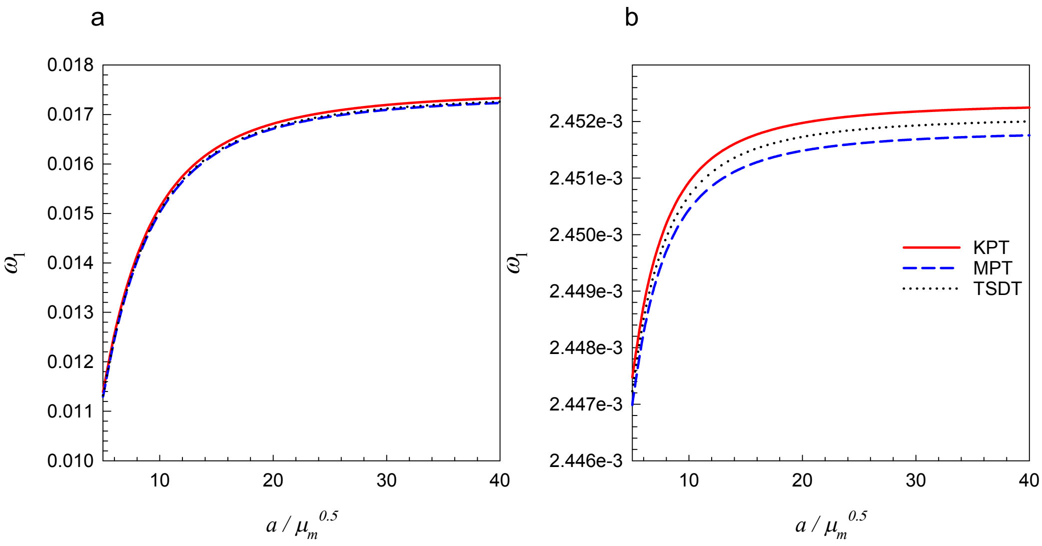

Figure 4.1: Dimensionless first natural frequency ω_1 as a function of $a/\sqrt{\mu_m}$ for three different plate theories: (a) Simply-supported nano-plate; (b) cantilever nano-plate. $a/b = 4/3$, $a/h = 20$, $\mu_m = 2\text{nm}^2$, $\mu_c/\mu_m = 2$, $n = 2$.

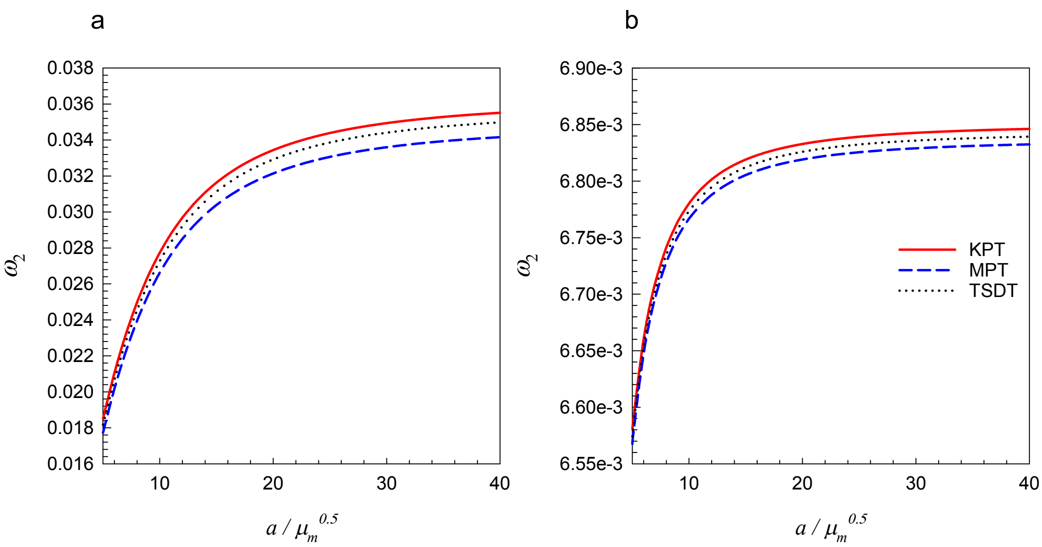

Figure 4.2: Dimensionless second natural frequency ω_2 as a function of $a/\sqrt{\mu_m}$ for three different plate theories: (a) Simply-supported nano-plate; (b) cantilever nano-plate. $a/b = 4/3$, $a/h = 20$, $\mu_m = 2\text{nm}^2$, $\mu_c/\mu_m = 2$, $n = 2$.

The first and second mode shapes of simply-supported and cantilever nano-plates are provided in Figure 4.3. Moreover, third and forth mode shapes for the same configuration of the plates are given in Figure 4.4. In the generation of the mode shapes and the remaining sets of results presented in this section, third-order shear deformation plate theory is used for higher degree of accuracy.

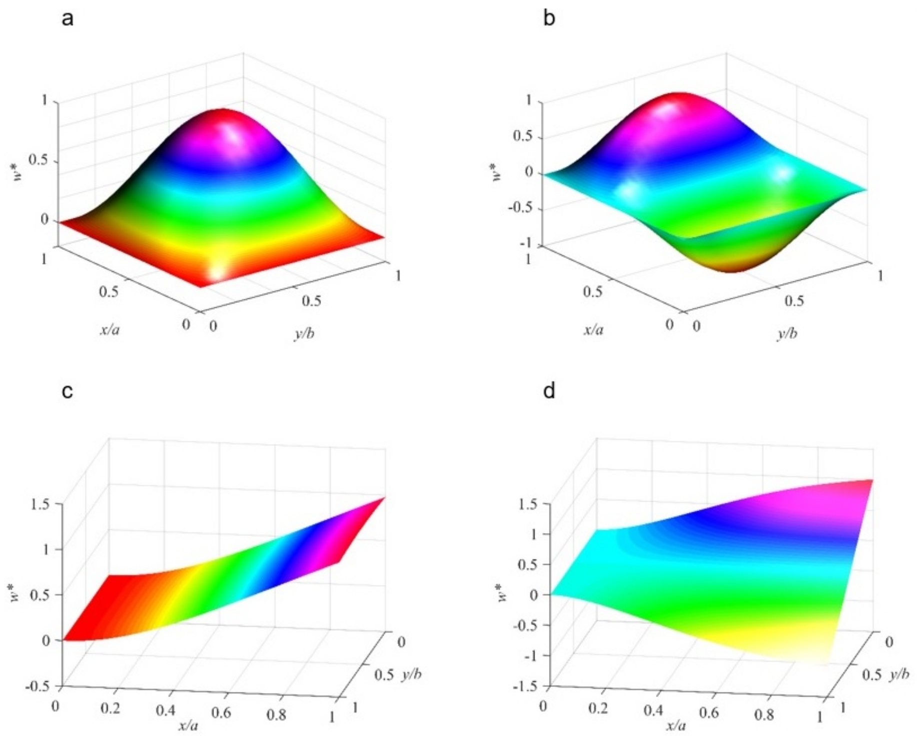

Figure 4.3: First two mode shapes of simply-supported and cantilever nano-plates: (a) First mode shape of simply supported nano-plate, $\omega_1 = 0.015$, (b) second mode-shape of simply-supported nano-plate, $\omega_2 = 0.027$, (c) first mode shape of cantilever nano-plate, $\omega_1 = 2.451 \times 10^{-3}$, (d) second mode shape of cantilever nano-plate, $\omega_2 = 6.774 \times 10^{-3}$. $a/\sqrt{\mu_m} = 10$, $a/b = 4/3$, $a/h = 20$, $\mu_m = 2\text{nm}^2$, $\mu_c/\mu_m = 2$, $n = 2$.

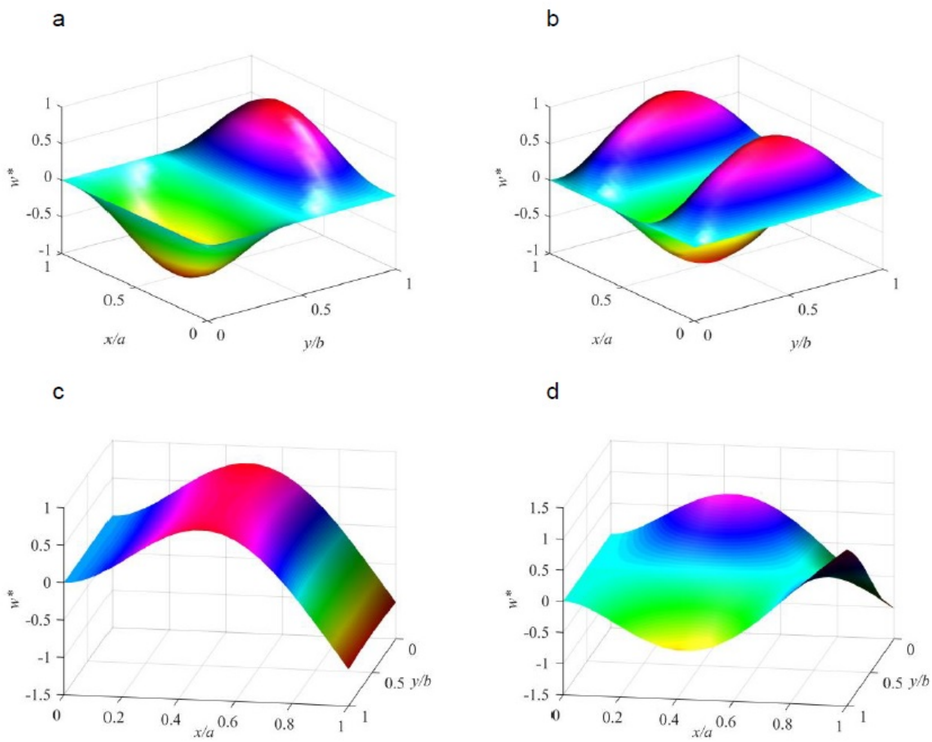

Figure 4.4: Third and fourth mode shapes of simply-supported and cantilever nano-plates: (a) third mode shape of simply supported nano-plate, $\omega_3 = 0.035$, (b) fourth mode-shape of simply-supported nano-plate, $\omega_4 = 0.043$, (c) third mode shape of cantilever nano-plate, $\omega_3 = 1.439 \times 10^{-2}$, (d) fourth mode shape of cantilever nano-plate, $\omega_4 = 2.166 \times 10^{-2}$. $a/\sqrt{\mu_m} = 10$, $a/b = 4/3$, $a/h = 20$, $\mu_m = 2\mathrm{nm}^2$, $\mu_c/\mu_m = 2$, $n = 2$.

Figures 4.5-4.8 depict the influence of the nonlocal parameter ratio μ_c/μ_m on natural frequencies ω_1, ω_2, ω_3 and ω_4, respectively. It can be seen that for both simply-supported and cantilever nano-plates, nonlocal parameter ratio has a significant impact on the dimensionless natural frequency. These results prove the fact that variation of the nonlocal parameter needs to be taken into account to be able to produce more accurate numerical results regarding the structural behavior of FGM nano-plates. All of the dimensionless natural frequencies get smaller as the ratio μ_c/μ_m is increased from 0.5 to 4.0. Natural frequencies merge at a single value as the dimensionless plate length $a/\sqrt{\mu_m}$ increases. This constant value is the frequency obtained by applying the classical continuum theory and setting $\mu_c = \mu_m = 0$.

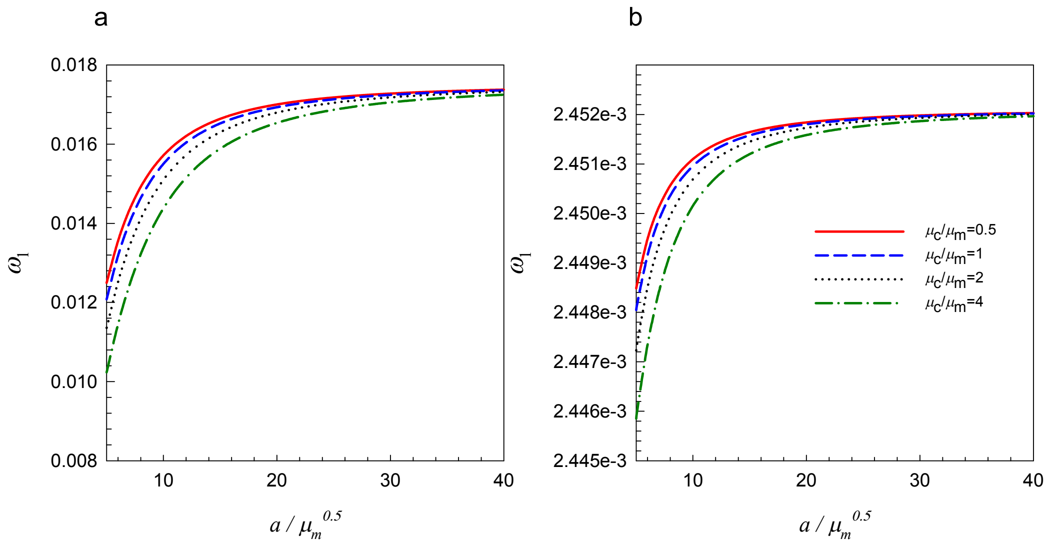

Figure 4.5: Dimensionless first natural frequency ω_1 versus $a/\sqrt{\mu_m}$ according to different nonlocal conditions: (a) Simply-supported nano-plate; (b) cantilever nano-plate. $a/b = 4/3$, $a/h = 20$, $\mu_m = 2\text{nm}^2$, $n = 2$.

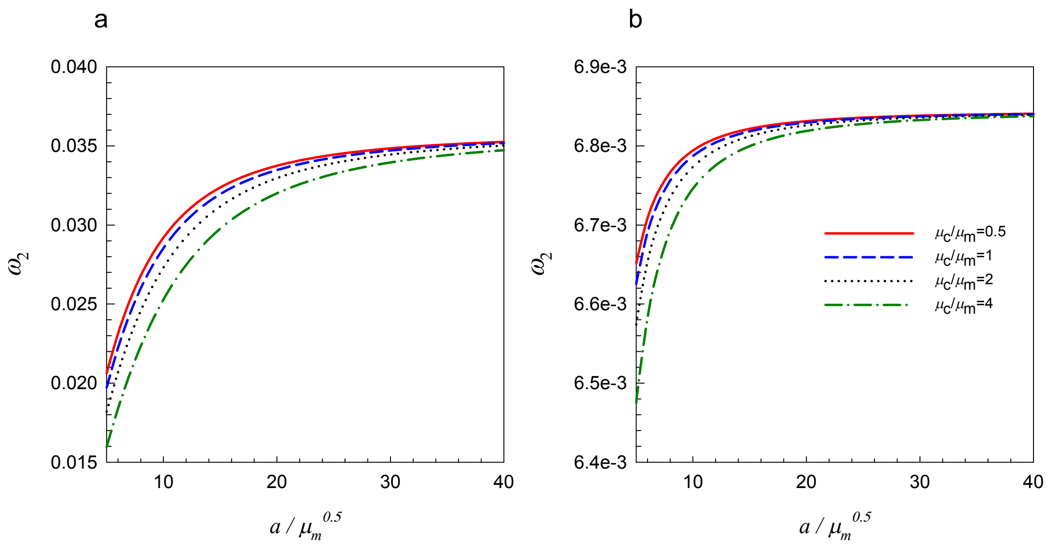

Figure 4.6: Dimensionless second natural frequency ω_2 versus $a/\sqrt{\mu_m}$ according to different nonlocal conditions: (a) Simply-supported nano-plate; (b) cantilever nano-plate. $a/b = 4/3$, $a/h = 20$, $\mu_m = 2\text{nm}^2$, $n = 2$.

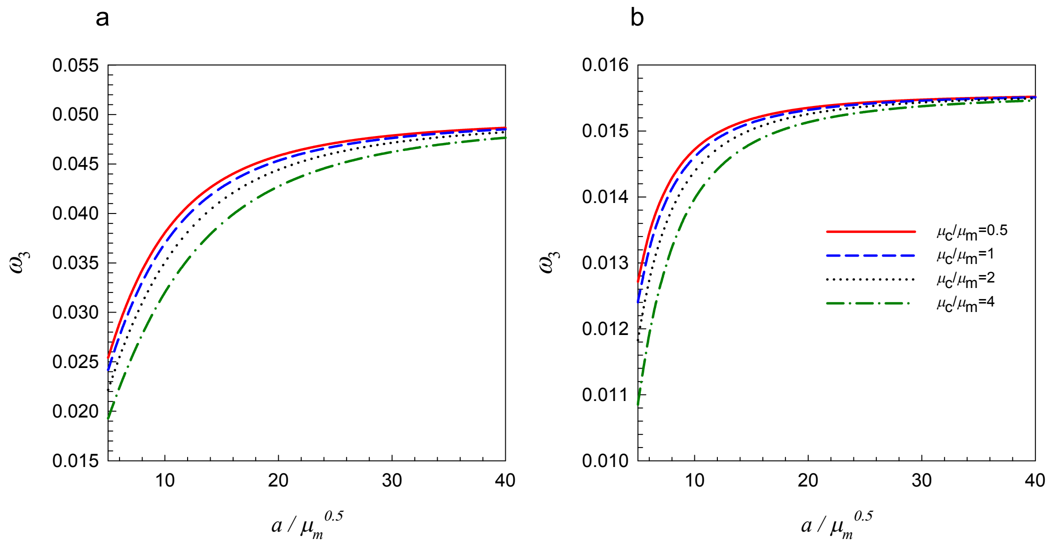

Figure 4.7: Dimensionless third natural frequency ω_3 versus $a/\sqrt{\mu_m}$ according to different nonlocal conditions: (a) Simply-supported nano-plate; (b) cantilever nano-plate. $a/b = 4/3$, $a/h = 20$, $\mu_m = 2\text{nm}^2$, $n = 2$.

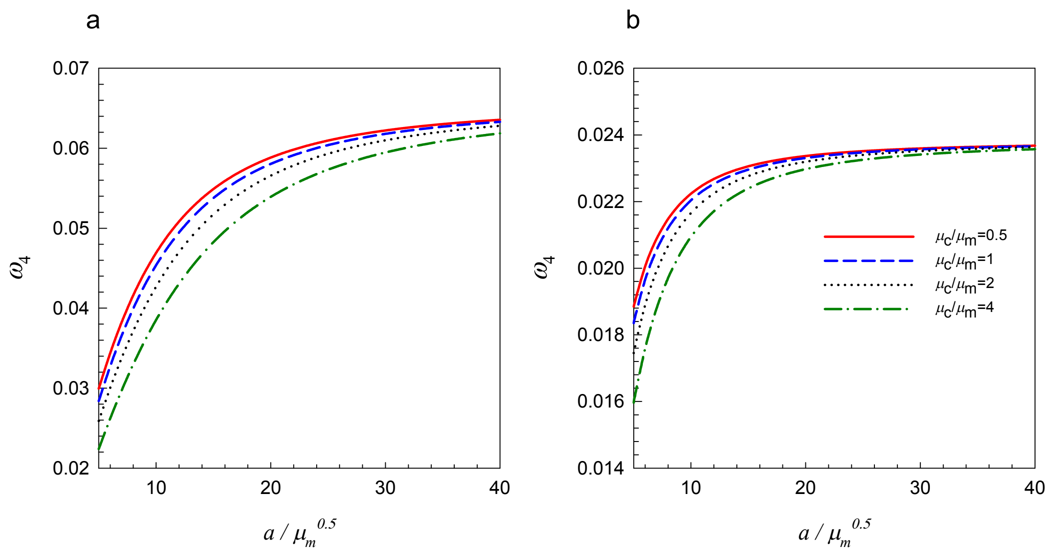

Figure 4.8: Dimensionless fourth natural frequency ω_4 versus $a/\sqrt{\mu_m}$ according to different nonlocal conditions: (a) Simply-supported nano-plate; (b) cantilever nano-plate. $a/b = 4/3$, $a/h = 20$, $\mu_m = 2\text{nm}^2$, $n = 2$.

Further results regarding the influence of μ_c/μ_m on dimensionless natural frequencies ω_1 and ω_2 are provided in Tables 4.4-4.7. Dependence on μ_c/μ_m is examined by considering different values of dimensionless plate length $a/\sqrt{\mu_m}$ and aspect ratio a/b. In all cases, dimensionless frequencies are decreasing functions of nonlocal parameter ratio μ_c/μ_m. Note that in Tables 4.4 and 4.5 power law index is set to be $n = 2$, but in Tables 4.6 and 4.7, it takes the value $n = 5$.

Table 4.4: Dimensionless first natural frequencies of simply-supported and cantilever nano-plates. $a/h = 20$, $n = 2$, $\mu_m = 2\text{nm}^2$.

a/b	μ_c/μ_m	Simply-supported nano-plate		Cantilever nano-plate	
		$a/\sqrt{\mu_m} = 5$	$a/\sqrt{\mu_m} = 10$	$a/\sqrt{\mu_m} = 5$	$a/\sqrt{\mu_m} = 10$
1/2	0.5	6.6127e-3	7.5184e-3	1.9457e-3	1.9575e-3
	1	6.4742e-3	7.4665e-3	1.9454e-3	1.9574e-3
	2	6.2215e-3	7.3657e-3	1.9450e-3	1.9573e-3
	4	5.7938e-3	7.1757e-3	1.9444e-3	1.9570e-3
1	0.5	9.7128e-3	1.1667e-2	2.2529e-3	2.2547e-3
	1	9.4402e-3	1.1545e-2	2.2526e-3	2.2546e-3
	2	8.9575e-3	1.1314e-2	2.2522e-3	2.2545e-3
	4	8.1795e-3	1.0889e-2	2.2514e-3	2.2541e-3
2	0.5	1.8897e-2	2.6108e-2	2.4577e-3	2.4597e-3
	1	1.8094e-2	2.5561e-2	2.4574e-3	2.4596e-3
	2	1.6754e-2	2.4562e-2	2.4569e-3	2.4594e-3
	4	1.4778e-2	2.2871e-2	2.4561e-3	2.4590e-3

Table 4.5: Dimensionless second natural frequencies of simply-supported and cantilever nano-plates. $a/h = 20$, $n = 2$, $\mu_m = 2\text{nm}^2$.

a/b	μ_c/μ_m	Simply-supported nano-plate		Cantilever nano-plate	
		$a/\sqrt{\mu_m} = 5$	$a/\sqrt{\mu_m} = 10$	$a/\sqrt{\mu_m} = 5$	$a/\sqrt{\mu_m} = 10$
1/2	0.5	9.6723e-3	1.1618e-2	3.2856e-3	3.3026e-3
	1	9.4009e-3	1.1497e-2	3.2825e-3	3.3018e-3
	2	8.9204e-3	1.1267e-2	3.2764e-3	3.3001e-3
	4	8.1457e-3	1.0844e-2	3.2649e-3	3.2968e-3
1	0.5	1.8760e-2	2.5917e-2	5.3673e-3	5.4606e-3
	1	1.7963e-2	2.5374e-2	5.3502e-3	5.4561e-3
	2	1.6634e-2	2.4383e-2	5.3165e-3	5.4471e-3
	4	1.4673e-2	2.2706e-2	5.2510e-3	5.4291e-3
2	0.5	2.5295e-2	3.7799e-2	9.1438e-3	9.3824e-3
	1	2.4063e-2	3.6737e-2	9.1006e-3	9.3708e-3
	2	2.2058e-2	3.4858e-2	9.0159e-3	9.3476e-3
	4	1.9203e-2	3.1828e-2	8.8526e-3	9.3018e-3

Table 4.6: Dimensionless first natural frequencies of simply-supported and cantilever nano-plates. $a/h = 20$, $n = 5$, $\mu_m = 2\text{nm}^2$.

a/b	μ_c/μ_m	Simply-supported nano-plate		Cantilever nano-plate	
		$a/\sqrt{\mu_m} = 5$	$a/\sqrt{\mu_m} = 10$	$a/\sqrt{\mu_m} = 5$	$a/\sqrt{\mu_m} = 10$
1/2	0.5	5.9650e-3	6.8347e-3	1.8325e-3	1.8535e-3
	1	5.9076e-3	6.8129e-3	1.8313e-3	1.8485e-3
	2	5.7974e-3	6.7698e-3	1.8308e-3	1.8460e-3
	4	5.5943e-3	6.6861e-3	1.8305e-3	1.8448e-3
1	0.5	8.6895e-3	1.0541e-2	1.9299e-3	1.9368e-3
	1	8.5773e-3	1.0490e-2	1.9295e-3	1.9352e-3
	2	8.3655e-3	1.0391e-2	1.9293e-3	1.9344e-3
	4	7.9851e-3	1.0201e-2	1.9292e-3	1.9340e-3
2	0.5	1.6674e-2	2.3319e-2	1.9612e-3	1.9736e-3
	1	1.6348e-2	2.3093e-2	1.9605e-3	1.9706e-3
	2	1.5751e-2	2.2660e-2	1.9601e-3	1.9692e-3
	4	1.4729e-2	2.1863e-2	1.9600e-3	1.9685e-3

Table 4.7: Dimensionless second natural frequencies of simply-supported and cantilever nano-plates. $a/h = 20$, $n = 5$, $\mu_m = 2\text{nm}^2$.

a/b	μ_c/μ_m	Simply-supported nano-plate		Cantilever nano-plate	
		$a/\sqrt{\mu_m} = 5$	$a/\sqrt{\mu_m} = 10$	$a/\sqrt{\mu_m} = 5$	$a/\sqrt{\mu_m} = 10$
1/2	0.5	8.6631e-3	1.0508e-2	2.9891e-3	3.0061e-3
	1	8.5514e-3	1.0458e-2	2.9878e-3	3.0057e-3
	2	8.3403e-3	1.0359e-2	2.9852e-3	3.0050e-3
	4	7.9612e-3	1.0170e-2	2.9800e-3	3.0036e-3
1	0.5	1.6580e-2	2.3186e-2	4.7770e-3	4.8635e-3
	1	1.6257e-2	2.2961e-2	4.7700e-3	4.8616e-3
	2	1.5663e-2	2.2532e-2	4.7564e-3	4.8580e-3
	4	1.4649e-2	2.1740e-2	4.7293e-3	4.8507e-3
2	0.5	2.2246e-2	3.3636e-2	8.1831e-3	8.4087e-3
	1	2.1748e-2	3.3200e-2	8.1652e-3	8.4038e-3
	2	2.0845e-2	3.2375e-2	8.1297e-3	8.3942e-3
	4	1.9131e-2	3.0895e-2	8.0599e-3	8.3750e-3

In Figures 4.9-4.12, ω_1, ω_2, ω_3 and ω_4 are presented as functions of the power-law index n and dimensionless plate length $a/\sqrt{\mu_m}$. The index n controls the variation of the ceramic volume fraction as indicated by Equation 2.20. The nano-plate is ceramic-rich if $n < 1$, and metal-rich if $n > 1$. All of the dimensionless natural frequencies decrease as the exponent n is increased from 0.5 to 8. Thus, ceramic-rich functionally graded nano-plates possess larger natural frequencies. This is the expected result since the ceramic phase of the FGM composite nano-plate has larger modulus of elasticity and lower density compared to the metallic phase. In all cases, the constant values obtained for larger values of dimensionless plate length are equal to the frequencies predicted by the classical continuum theory.

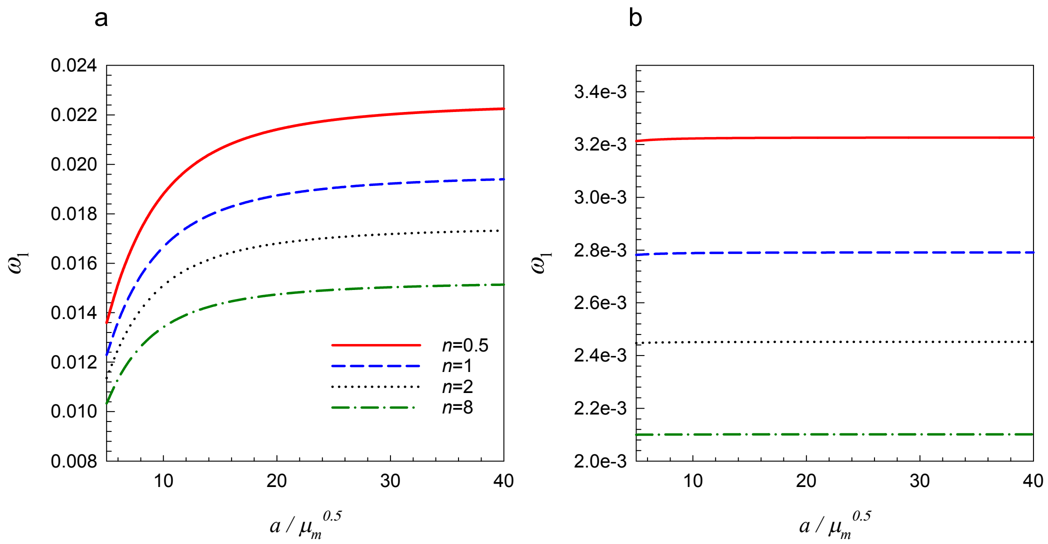

Figure 4.9: Dimensionless first natural frequency ω_1 versus $a/\sqrt{\mu_m}$ for various values of the power-law index n: (a) Simply-supported nano-plate; (b) cantilever nano-plate. $a/b = 4/3$, $a/h = 20$, $\mu_m = 2\text{nm}^2$, $\mu_c/\mu_m = 2$.

Further numerical results regarding the effects of the power-law index n and the nonlocal parameter ratio μ_c/μ_m on the first four dimensionless natural frequencies of a simply-supported FGM nano-plate are presented in Figures 4.13 and 4.14. Curves obtained for different values of n show the fact that as the power-law index n gets smaller, the natural frequencies increase. Due to differences in the properties of the metallic and ceramic phases, ceramic-rich functionally graded nano-plates possess larger natural frequencies than the metal-rich nano-plates. Nonlocal parameter ratio μ_c/μ_m is also shown to have important effect on natural frequencies. Sensitivity of the frequencies to the change in the nonlocal parameter ratio points to the fact that nonlocal parameter variation should be taken into account in dynamic analysis of FGM nano-scale plates.

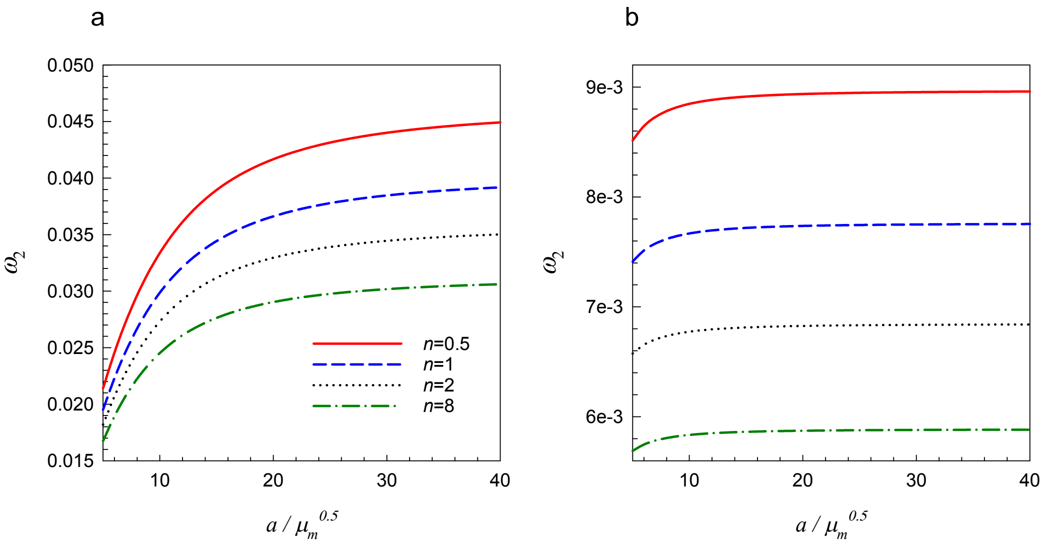

Figure 4.10: Dimensionless second natural frequency ω_2 versus $a/\sqrt{\mu_m}$ for various values of the power-law index n: (a) Simply-supported nano-plate; (b) cantilever nano-plate. $a/b = 4/3$, $a/h = 20$, $\mu_m = 2\mathrm{nm}^2$, $\mu_c/\mu_m = 2$.

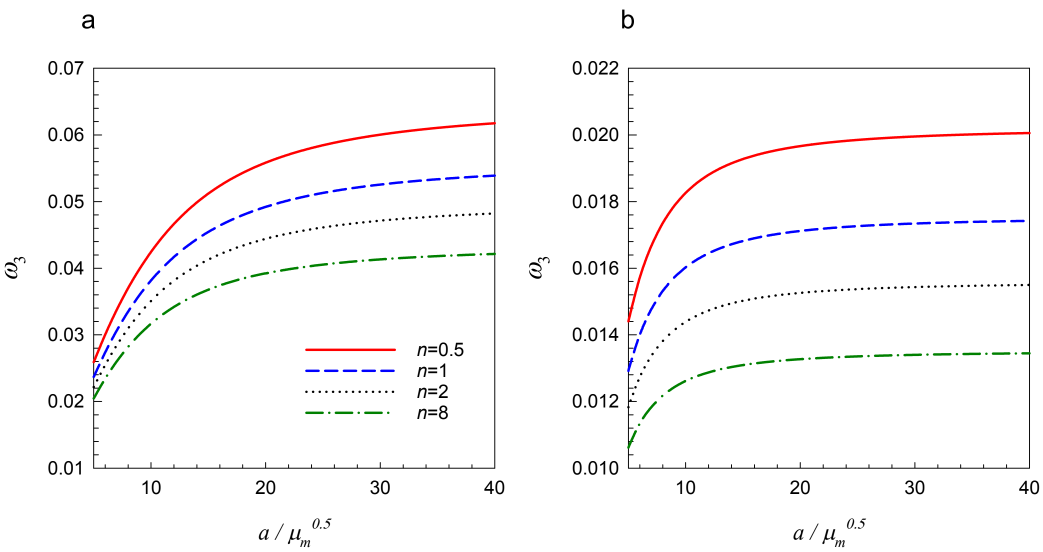

Figure 4.11: Dimensionless third natural frequency ω_3 versus $a/\sqrt{\mu_m}$ for various values of the power-law index n: (a) Simply-supported nano-plate; (b) cantilever nano-plate. $a/b = 4/3$, $a/h = 20$, $\mu_m = 2\mathrm{nm}^2$, $\mu_c/\mu_m = 2$.

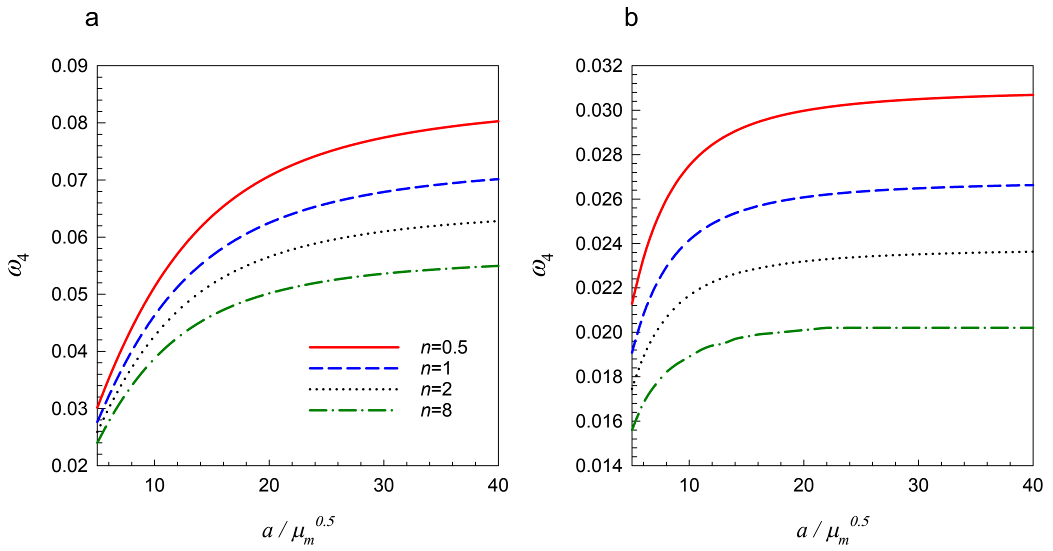

Figure 4.12: Dimensionless fourth natural frequency ω_4 versus $a/\sqrt{\mu_m}$ for various values of the power-law index n: (a) Simply-supported nano-plate; (b) cantilever nano-plate. $a/b = 4/3$, $a/h = 20$, $\mu_m = 2\text{nm}^2$, $\mu_c/\mu_m = 2$.

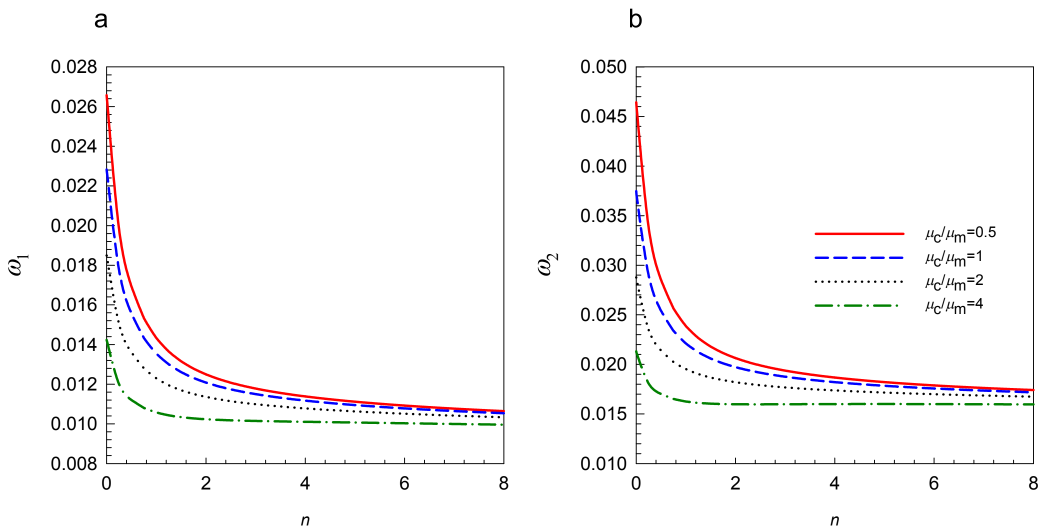

Figure 4.13: Dimensionless natural frequencies of a simply-supported nano-plate as functions of n and μ_c/μ_m: (a) First natural frequency ω_1; (b) Second natural frequency ω_2. $a/\sqrt{\mu_m} = 5$, $a/b = 4/3$, $a/h = 20$, $\mu_m = 2\text{nm}^2$.

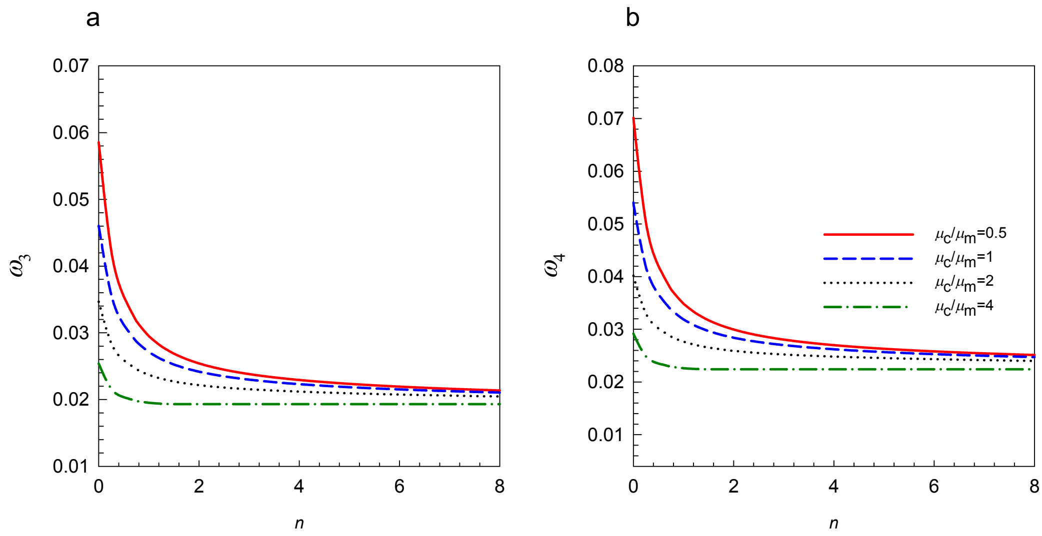

Figure 4.14: Dimensionless natural frequencies of a simply-supported nano-plate as functions of n and μ_c/μ_m: (a) Third natural frequency ω_3; (b) Fourth natural frequency ω_4. $a/\sqrt{\mu_m} = 5$, $a/b = 4/3$, $a/h = 20$, $\mu_m = 2\text{nm}^2$.

CHAPTER 5

CONCLUSION AND FUTURE WORK

In this work, a nonlocal elasticity based method for free vibration analysis of functionally graded rectangular nano-plates is presented. Eringen's differential form of the nonlocal constitutive equation is employed in order to account for the effects observable in small scales. The novelty of this study is the consideration of spatial variation of the nonlocal parameter in free vibration formulation of functionally graded rectangular nano-plates. Hamilton's principle is applied to derive governing equations and associated boundary conditions. In this derivation, all material properties, including the nonlocal parameter, are assumed to be functions of the thickness coordinate. The variation of these properties is computed according to the power law. In order to generate results for three different plate theories, generalized displacement field is used in the formulation. This method is capable of producing results for both classical and shear deformation plate theories, namely, Kirchhoff, Mindlin, and third order shear deformation theories. Generalized differential quadrature method is employed to solve governing partial differential equations. For this purpose, series forms of the governing equations and boundary conditions are obtained by means of differential quadrature method. For discretizing the rectangular domain of the problem, Chebyshev-Gauss-Lobatto points are used as nodal points along the length and width of the plate. In order to implement this numerical method, computer programs are developed in MATLAB software. Proposed procedure is verified through comparisons made to the results available in the literature for free vibration analysis of a functionally graded rectangular nano-plate possessing a constant nonlocal parameter along the thickness direction. Numerical results for simply-supported and cantilever nano-plates are provided to be able to investigate the effects of shear deformation, dimensionless plate length, nonlocal parameter ratio, and power-law index upon the natural frequencies.

Analysis of natural frequencies calculated according to three different plate theories show the effect of transverse shear deformation on the stiffness of the plate. It can be seen that Kirchhoff plate theory overpredicts natural frequencies. This shows the fact that in the analysis of relatively thick plates, it is crucial to take into account the effect of shear strains by employing shear deformation plate theories.

For all cases, it can be seen that the softening effect of nonlocal theory is dominant for smaller plate length. As dimensionless plate length is increased, vibration frequencies increase and then approach constant values. The constant value obtained at large values of $a/\sqrt{\mu_m}$ is equal to the natural frequency values computed for a plate by means of the classical continuum theory. Therefore, in free vibration analysis of nano-plates, it is necessary to employ higher order continuum theories to capture the size effect observable in small scales.

The spatial variation of nonlocal parameter is quantified by the nonlocal parameter ratio μ_c/μ_m. The influence of the nonlocal parameter ratio on the natural frequencies of nano-plate is revealed to be significant. An increase in nonlocal parameter ratio produces softening effect and the dimensionless natural frequencies decrease. This shows the fact that assuming a constant nonlocal parameter along the thickness of the plate, can reduce the accuracy of results for natural frequencies of the nano-plate. Thus, reliable results regarding the free vibration of functionally graded rectangular nano-plates can be produced by taking into account the spatial variation of nonlocal parameter. The method presented in this work could be useful in design and optimization of functionally graded nano-scale plates.

In the literature on nano-structures, nonlocal theory has been successfully applied to different problems of nano-beams and plates. However, this theory is highly dependent on the optimized value of small-scale parameter. Except for a few studies on evaluation of nonlocal parameter by matching the results obtained from molecular dynamic simulations, the exact value of nonlocal parameter for many problems and materials is unknown. To be able to generate more accurate results for design and development of functionally graded nano-structures, it is necessary to conduct more research on the determination of the exact value of nonlocal parameter for different materials.

In some applications, vibrational energy is dissipated by damping effect of fluid or air phases. The dissipation of energy by damping elements changes the response of the system under vibration. For accurate estimation of the response of a system, damping effect should be considered in mathematical modeling and analysis.

Free vibration study of nano-structures plays an important role in design and development of essential components for MEMS and NEMS, nano-machines, nano-sensors, resonators and oscillators. The method developed in this study can be used to produce more accurate results regarding the free vibration of functionally graded rectangular small-scale plates.

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