THE SLOPE INEQUALITY FOR LEFSCHETZ FIBRATIONS

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ABSTRACT

THE SLOPE INEQUALITY FOR LEFSCHETZ FIBRATIONS

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In this thesis, we produce Lefschetz fibrations over the two-sphere which have smaller slope compared to known examples. The study is motivated by a conjecture of Hain saying that every Lefschetz fibration over the two-sphere with slope λ_f satisfies the slope inequality $4 - 4/g \leq \lambda_f$. Monden recently constructed Lefschetz fibrations with slope which violate this lower bound. In the thesis, we establish new examples having slope less than these. The total spaces of our Lefschetz fibrations are simply-connected. Finally, we try to obtain Lefschetz fibrations with even smaller slope.

Keywords: Lefschetz Fibrations, Slope Inequality, Mapping Class Groups

ÖZ

LEFSCHETZ LİF DEMETLERİ İÇİN EĞİM EŞİTSİZLİĞİ

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Bu tezde, bilinen örneklere nazaran eğimi daha küçük olan iki boyutlu küre üzerinde Lefschetz liflemeleri ürettik. Bu çalışma Hain'in bir sanısından motive olmuştur, öyle ki iki boyutlu küre üzerinde eğimi λ_f olan her Lefschetz liflemesi eğim eşitsizliğini $4 - 4/g \leq \lambda_f$ sağlar. Monden yakınlarda eğimi bu alt sınırı sağlamayan Lefschetz liflemeleri inşa etmiştir. Bu tezde, eğimi bunlardan daha küçük olan yeni örnekler kurduk. Lefschetz liflemelerimizin uzayı basit bağlantılıdır. Son olarak, daha küçük eğimi olan Lefschetz liflemeleri elde etmeye çalıştık.

Anahtar Kelimeler: Lefschetz Liflemeleri, Eğim Eşitsizliği, Gönderim Sınıfları Grupları

To my family

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TABLE OF CONTENTS

LIST OF FIGURES

FIGURES

CHAPTER 1

INTRODUCTION

The topological perspective of Lefschetz fibrations on 4-manifolds was enlightened by Matsumoto in [36]. Although Lefschetz's original work on pencils in [31] was mainly topological, they are mostly studied in the realm of algebraic geometry such that pencil of curves on algebraic surfaces can be blown up at their base points to obtain Lefschetz fibrations. In low dimensional topology, Lefschetz fibrations are very important as they have relations with several distinct subbranches. Therefore, this enables us to approach a problem from distinct aspects at the same time.

In this respect, the most useful relation is between Lefschetz fibrations and symplectic 4-manifolds which is shown by the works of Donaldson and Gompf. In the late 1990s, Donaldson [10, 11] showed that for every closed symplectic 4-manifold X, there exists a Lefschetz pencil structure on it whose closed fibers are symplectic submanifolds so that by blowing it up at its base points, we get a Lefschetz fibration over the two-sphere. Thereby, Lefschetz fibrations can be used as a tool to study topology of symplectic 4-manifolds. On the other hand, Gompf [21, 49] showed that if a closed oriented 4-manifold X admits a Lefschetz pencil or a Lefschetz fibration of genus g (over a Riemann surface) provided that the homology class of the generic fiber is nontrivial, it has a symplectic structure on it so that the fibers are symplectic submanifolds. In fact, if the fiber genus $g > 1$, the homology class of a generic fiber of the Lefschetz fibration is not torsion in $H_2(X;\mathbb{Z})$, so such an X admits a symplectic structure with symplectic fibers.

The other relation of Lefschetz fibrations with mapping class groups is through a combinatorial aspect. Indeed, this is a way to describe symplectic 4-manifolds in terms of mapping class groups. That is, for a given Lefschetz fibration over the two-sphere, there exists an associated product consisting of positive Dehn twists which is equal to identity of the mapping class group of generic fiber. Conversely, a product of positive Dehn twists gives a Lefschetz fibration over two-disk. If this product is equal to the identity, then it gives a Lefschetz fibration over $\mathbb{S}^2$. We can explicitly construct a Lefschetz fibration over the two-sphere with assigned vanishing cycles for such a given positive factorization of the identity. Moreover, we can change this factorization by using relations in mapping class groups so that the associated 4-manifold is changed topologically. A famous one for this is lantern substitution which topologically corresponds to a rational blow-down [2, 14].

Besides these, there are results on the commutator lengths of some elements in the mapping class group of oriented genus-$g(\geq 1)$ surfaces by using properties of symplectic 4-manifolds admitting Lefschetz fibrations and algebraic structure of mapping class groups [15, 28]. Furthermore, by using a relation in mapping class groups, it is deduced that the number of (irreducible) singular fibers in a genus one Lefschetz fibration must be divisible by 12. It will be interesting to find a global invariant for Lefschetz fibrations by means of mapping class group factorizations. More questions related to this issue can be found in [5]. A last result to mention here is as follows: For every finitely presented group G, there exists a Lefschetz fibration of genus g admitting G as the fundamental group of its total space [10, 20]. Korkmaz [29] defined genus of G to be the minimal genus of the Lefschetz fibration with fundamental group G.

Basically, a Lefschetz fibration is a smooth map from a 4-manifold onto a Riemann surface with finitely many critical points. In this thesis, we only deal with Lefschetz fibrations whose base space is the two-sphere. There are several ways to examine Lefschetz fibrations. A classification of Lefschetz fibrations over the two-sphere was made by Harer [22] in which he used Hatcher-Thurston's presentation of the mapping class group. In the thesis, we focus on geography problem of Lefschetz fibrations over $\mathbb{S}^2$ which has been inspired by that for complex surfaces fibered over curves from which the slope inequality is derived.

The very beginning of the study for the slope inequality goes back to the Severi inequality. Pardini stated the Severi inequality in [46] as follows: if S is a smooth minimal complex projective surface of maximal Albanese dimension, then $K_S^2 \geq 4\chi(\mathcal{O}_S)$ holds. S is a surface having maximal Albanese dimension means that the image of its Albanese map is a surface. Likewise, if $K_S^2 < 4\chi(\mathcal{O}_S)$ holds, then S is fibered over a curve of genus $b_1(S)/2$ [62, 41].

In 1932, Severi [53] stated the second case as a theorem but his proof was not correct [8]. After the statement was raised as a conjecture by Reid [50] and Catanese [8] independently; Reid [50], Horikawa [23] and Xiao [63] proved the inequality with some own restrictions and assuming that the surfaces are double covers of ruled surfaces. In 1987, Xiao solved the conjecture for more general surfaces, i.e., not admitting such double covers but having relatively minimal genus-$g(\geq 2)$ fibrations $f : S \to B$ onto a smooth curve with positive genus b and connected fibres. That is, if $f : S \to B$ is a holomorphic fibration of genus g over a curve B of positive genus b satisfying $K_S^2 < 4\chi(\mathcal{O}_S)) + 4(b-1)(g-1)$ (i.e.,$\lambda(f) < 4$) then $b = b_1(S)/2$. In 1990, the conjecture was proved by Manetti [34] assuming that the surface has ample canonical bundle. Finally, Pardini [46] proved the conjecture by using slope inequality for holomorphic fibrations over $\mathbb{CP}^1$.

In complex geometry, the slope of a fibration is defined as follows [62] :

For a relatively minimal fibration $f : S \to B$ of genus g with $g \geq 2$, let $f_*\omega_{S|B}$ be a locally free sheaf of rank g and of degree $\chi(\mathcal{O}_S) - (b-1)(g-1)$. Every locally free quotient of $f_*\omega_{S/B}$ has a non-negative degree and if f is not locally trivial then $\chi_f := deg f_*\omega_{S/B} > 0$. Let $K_{S/B} \equiv K_S - f^*K_B$ be a relative canonical divisor on S then the *slope* of f is defined by

$$\lambda(f) = \frac{K_{S/B}^2}{\chi_f}$$

where

$$K_{S/B}^2 = K_S^2 - 8(b-1)(g-1),$$

$$\chi_f = \chi(\mathcal{O}_S) - (b-1)(g-1).$$

It was first found by Horikawa [23] and Persson [48] for hyperelliptic pencils and proved by Xiao [62] for general case. Xiao concluded that $4(g-1)/g \leq \lambda(f) \leq 12$

and $4 - 4/g \leq \lambda(f)$ is called the *slope inequality*. He also said $\lambda(f)$ is the unique number which satisfies

$$K_S^2 = \lambda(f)\chi(\mathcal{O}_S) + (8 - \lambda(f))(b-1)(g-1).$$

A common procedure to obtain sharper bounds for the slope inequality to put restriction on the general fiber [16]. For example, if $g = 3$ and f is a non-hyperelliptic fibration then $K_{S/B}^2 \geq 3\chi_f$ holds while the slope inequality says for the lower bound is $8/3$ [4]. Also, it was claimed by Xiao that fibrations on the lower bound are hyperelliptic ones especially with only nonseperating vanishing cycles whereas on the upper bound are non-hyperelliptic. Hence the lower bound for the slope of non-hyperelliptic fibrations is bigger [4]. Upper bound can be shown by using Noether condition.

Conjecture 1.0.1 ([62]). *If $\lambda(f) = 4 - 4/g$, then f is hyperelliptic.*

Xiao also mentioned in [62] that he could not obtain a non-hyperelliptic fibration with $\lambda(f) < 4$ and $g \geq 6$.

In Chapter (4), we construct non-hyperelliptic Lefschetz fibrations over the two-sphere with slope less than $4 - 4/g$. Monden has examples in [41] which are also non-hyperelliptic.

The slope inequality for relatively minimal Lefschetz fibrations of genus g over the two-sphere is defined in a similar manner to the complex case. By using the numerical invariants χ_f and K_f^2 which are basically depend on the Euler characteristic e and the signature σ of the corresponding Lefschetz fibration, the slope λ_f can be calculated.

A conjecture on the slope inequality for Lefschetz fibrations over the two-sphere was raised by Hain and in [3], it is stated that it is a symplectic version of Moriwaki inequality which is called a sharper slope inequality [42]. Then, Endo and Nagami restated the conjecture as follows:

Conjecture 1.0.2 ([16, Conjecture 4.12]). *Every smooth Lefschetz fibration $f : X \to \mathbb{S}^2$ of genus $g \geq 2$ over the two-sphere satisfies the slope inequality $\lambda_f \geq 4 - 4/g$.*

If the answer is affirmative, there will be restriction on the factorization of identity which defines a Lefschetz fibration. For our constructed Lefschetz fibrations not satisfying the slope inequality, the number of singular fibers so that the length of the factorization is increased.

Research Results

In [41], Monden gave examples of Lefschetz fibrations over the two-sphere violating the slope inequality. We improve his result by exhibiting different examples of Lefschetz fibrations with smaller slope.

Theorem 4.2.1 For each $g \geq 2$, there exists a Lefschetz fibration of genus g over $\mathbb{S}^2$ with slope

$$\lambda_f = 4 - \frac{4}{g} - \frac{4(g-2)}{g(g+2)}.$$

The associated 4-manifold is simply-connected.

If we consider odd and even cases separately, we can get Lefschetz fibrations with smaller slopes.

Theorem 4.2.4 For $g \geq 4$ and even, there exists a Lefschetz fibration of genus g over $\mathbb{S}^2$ with slope

$$\lambda_f = 4 - \frac{4}{g} - \frac{8(g-2)(g-1)}{g(g^2+8)}.$$

Theorem 4.2.6 For $g \geq 5$ and odd, there exists a Lefschetz fibration of genus g over $\mathbb{S}^2$ with slope

$$\lambda_f = 4 - \frac{4}{g} - \frac{8(g-2)(g-1)}{g(g^2+11)}.$$

If we compare the Lefschetz fibrations in terms of their slopes, we obtain the following observation :

Conclusion. For even $g \geq 4$;

$$\lambda_f = 4 - \frac{4}{g} - \frac{8(g-2)(g-1)}{g(g^2+8)}$$

and for odd $g \geq 5$;

$$\lambda_f = 4 - \frac{4}{g} - \frac{8(g-2)(g-1)}{g(g^2+11)}$$

are the smallest known slopes for Lefschetz fibrations of genus g over $\mathbb{S}^2$.

This thesis is organized as follows:

In Chapter 2, we give the basic definitions and necessary facts about mapping class groups. This will be followed by the relations among mapping classes.

In Chapter 3, in addition to basics of Lefschetz fibrations, we introduce the signature cocyle which is defined with second cohomology class of mapping class group of the fiber and it is used to compute the signature of the constructed Lefschetz fibrations.

In Chapter 4, we first state the results of Monden. Then we present our theorems and expain how we construct our Lefschetz fibrations to prove them.

In Chapter 5, we review geography problem of symplectic 4-manifolds. Since the total spaces of Lefschetz fibrations provide examples of symplectic 4-manifolds, we combine our results to the geography of related symplectic 4-manifolds.

CHAPTER 2

MAPPING CLASS GROUPS

In this chapter, we review the definition and basic facts of mapping class groups. We exhibit some relations among Dehn twists which are building blocks for our results. For more details of the content, one can consult [17, 27].

2.1 Basic Definitions and Facts

Let Σ_g be a closed oriented connected smooth surface of genus $g \geq 2$ and $c_0, c_1, \dots,$ c_n be a sequence of simple closed curves on this surface as in Figure (2.1).

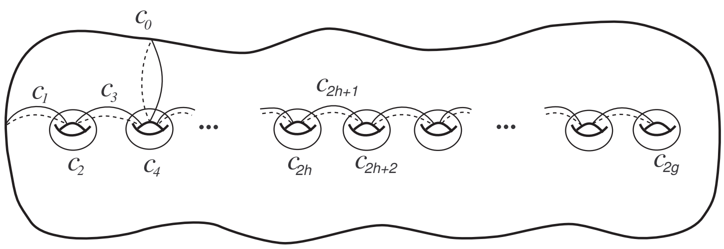

Figure 2.1: A surface of genus g.

Definition 2.1.1. *Let f and h be two self-diffeomorphisms of Σ_g. We say that f and h are homotopic if there is a continuous function $F : [0,1] \times \Sigma_g \to \Sigma_g$ satisfying*

$$F(0,x) = f(x) \quad \textit{and} \quad F(1,x) = h(x) \quad \textit{for all} \quad x \in \Sigma_g.$$

An *isotopy* is a homotopy F such that the map $f_t : \Sigma_g \to \Sigma_g$ defined by $f_t(x) = F(t,x)$ is a diffeomorphism for each $t \in [0,1]$.

Definition 2.1.2. *Let $Diff^+(\Sigma_g)$ denote the group of all orientation-preserving self-diffeomorphisms of Σ_g and $Diff_0^+(\Sigma_g)$ be the normal subgroup of $Diff^+(\Sigma_g)$ con-*

sisting of elements which are isotopic to the identity. The mapping class group of Σ_g *is*

$$\mathrm{Mod}(\Sigma_g) = Diff^+(\Sigma_g)/Diff_0^+(\Sigma_g).$$

In other words, $\mathrm{Mod}(\Sigma_g)$ is the group of all isotopy classes of orientation-preserving self-diffeomorphisms of Σ_g. Indeed, $\mathrm{Mod}(\Sigma_g)$ can be defined in different ways depending on purpose. For an example, one can consider homeomorphisms instead of diffeomorphisms or homotopy classes instead of isotopy classes with some additional restrictions.

The surface in Figure (2.1) may have boundary components or marked points. Let $\Sigma_{g,r}^n$ denote the surface with r boundary components and n marked points. Then the corresponding mapping class group $\mathrm{Mod}(\Sigma_{g,r}^n)$ is the group consisting of the isotopies fixing each marked point and the points on the boundary.

In the following, we introduce a diffeomorphism which is building block in the theory of mapping class groups.

Definition 2.1.3. *Let* a *be a simple closed unoriented curve on* Σ_g*. We first cut the surface* Σ_g *along* a *and twist one of the boundary components by* 360 *degrees to the right. After gluing it back, we get a self-diffeomorphism of the surface. The isotopy class of such a diffeomorphism is called a positive (or right-handed) Dehn twist. On the other hand, if we use rotation about left side, the diffeomorphism will be called as left-handed Dehn twist.*

A positive Dehn twist about a is denoted by t_a and determined by the isotopy class of a. In general, we denote the diffeomorphism and its isotopy class by the same notation. So t_a is also an element of $\mathrm{Mod}(\Sigma_g)$. Similarly, a simple closed curve and its isotopy class are denoted by the same letter. Sometimes a Dehn twist along a curve a can be represented by A instead of t_a. We use this notation in Chapter (4).

Indeed, the group of self-diffeomorphisms of a compact connected surface is a topological group with compact open topology and it is an infinite dimensional group. If we consider them up to isotopy, we obtain the mapping class group of the surface which is a finitely generated (discrete) group. After works of Dehn and Lickorish, Humphries [24] showed that $\mathrm{Mod}(\Sigma_g)$ is generated by $2g + 1$ Dehn twists all of which are about nonseparating curves in Figure (2.1). Also, he showed that $\mathrm{Mod}(\Sigma_g)$ can not be generated by $2g$ or less Dehn twists. More details are given by Korkmaz in [30].

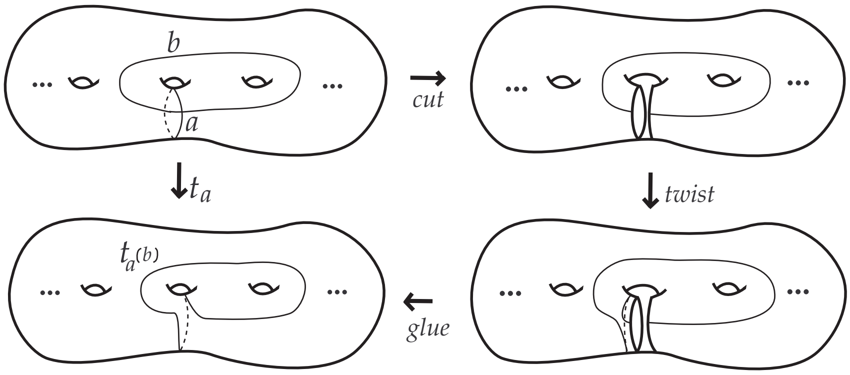

Figure 2.2: Effect of a positive Dehn twist about a nonseparating curve a.

Most of the properties of Dehn twists are studied by their action on simple closed curves. In this sense, we give the following example.

Example 2.1.4. *Let a and b are two nonseparating curves on Σ_g. The image of b under t_a is showed in Figure (2.2). First, we choose a regular neighborhood of a, then apply Dehn twist on this neighborhood and extend t_a to the remaining of the surface as identity. Here, we perform Dehn twist once but it can be applied several times since Dehn twists have infinite order in* $\mathrm{Mod}(\Sigma_g)$.

Note that in the above example, a and b intersect each other only once. For instance, if the curve a is separating that we apply Dehn twist about, the picture of a Dehn twist may look like as in Figure (2.3). Recall that a curve on a surface is called *nonseparating* if we cut the surface along that curve, it is still connected. Otherwise, the curve is *separating*.

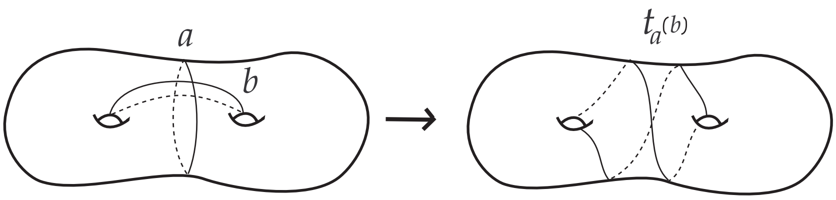

Figure 2.3: Effect of a positive Dehn twist about a separating curve a.

Now, we can mention about topological type of simple closed curves on Σ_g. It follows from classification of surfaces that there exists an orientation-preserving diffeomorphism taking one simple closed curve to another if and only if the corresponding cut surfaces are diffeomorphic. A cut surface is a surface obtained from Σ_g cutting along a curve a so that it has two boundary components corresponds to a. The diffeomorphism between two different cut surfaces defines an equivalence relation on simple closed curves called *topological type*. If both a and b are nonseparating curves, the two cut surfaces have the same Euler characteristic and number of boundary components where each have two boundary components a and b, respectively. Thus, there is only one nonseparating simple closed curve on a surface up to diffeomorphism. So that the topological type for nonseparating curves is unique whereas for separating curves there are $[\frac{g}{2}]$ different types. It is determined by how the curve separates the surface.

Fact 2.1.5. *Let $f : \Sigma_g \to \Sigma_g$ be a diffeomorphism and a be a simple closed curve on Σ_g. Then*

$$f t_a f^{-1} = t_{f(a)}. \qquad (*)$$

If a and b are nonseparating simple closed curves on Σ_g, then by (*), it can be deduced that t_a and t_b are conjugate in $\mathrm{Mod}(\Sigma_g)$.

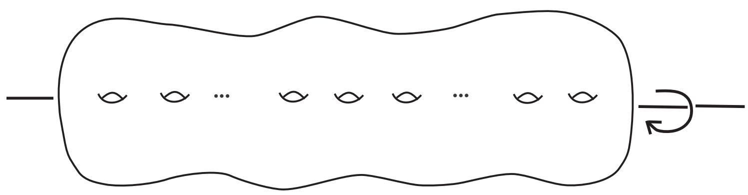

Figure 2.4: Hyperelliptic involution is a rotation by π about the indicated axis.

Definition 2.1.6. *The hyperelliptic mapping class group $\mathcal{H}_g$ of Σ_g is the subgroup of $\mathrm{Mod}(\Sigma_g)$ consisting of all mapping classes commuting with the isotopy class of a hyperelliptic involution $\imath : \Sigma_g \to \Sigma_g$.*

For $g = 1, 2$; $\mathrm{Mod}(\Sigma_g) = \mathcal{H}_g$.

Convention. The composition $t_c t_d$ for two diffeomorphisms t_c and t_d means that first we apply t_d and then t_c.

2.2 Relations in the Mapping Class Group

In the following, we review relations which are important for the presentation of the mapping class groups. For our purpose, it is an interesting issue to see the topological meaning of them in the low dimensional topology. We will use some of these relations to obtain new examples of Lefschetz fibrations. More details can be found in [16, 17, 30].

- **Commutativity relation:** Let a and b be two disjoint simple closed curves on Σ_g. Then,

$$t_a t_b = t_b t_a.$$

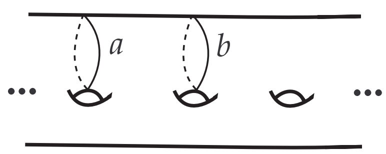

Figure 2.5: Dehn twists about disjoint curves commute.

To prove the above equality, consider a curve b on Σ_g disjoint from a. So, we can determine the support of the Dehn twist t_a not containing b. Then we have $t_a(b) = b$. By the equality (*), we have

$$t_a t_b = t_a t_b t_a^{-1} t_a = t_{t_a(b)} t_a = t_b t_a.$$

Indeed, the converse is also true. Then one can deduce from this, if f leaves the curve c invariant, then

$$t_c f = f t_c.$$

- **Braid relation:** If a and b are two simple closed curves intersecting transversely at one point, then

$$t_a t_b t_a = t_b t_a t_b.$$

By using the proof in Figure (2.6) which is a part of Σ_g, the equality

$$t_a t_b t_a = t_a t_b t_a t_b^{-1} t_a^{-1} t_a t_b = t_{t_a t_b(a)} t_a t_b = t_b t_a t_b$$

is satisfied.

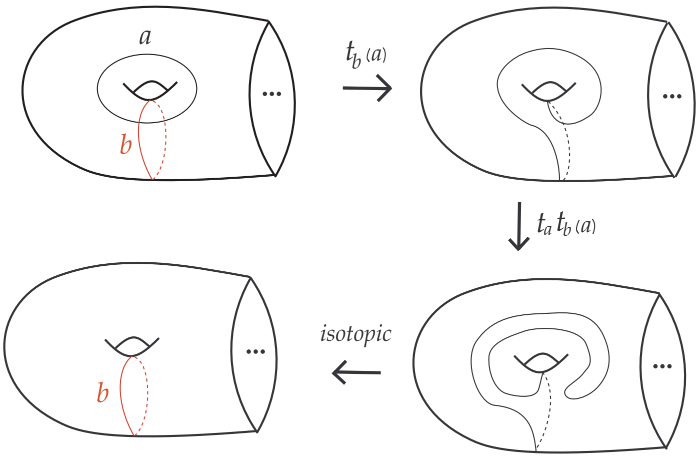

Figure 2.6: The proof of $t_a t_b(a) = b$.

- **Chain relation:**

  A *chain* of length n is an ordered n-tuple $(c_1, \ldots, c_n)$ of simple closed curves on Σ_g satisfying the followings :

  (i) c_i and c_{i+1} intersect transversely at one point for $1 \leq i \leq n-1$;

  (ii) c_i and c_j are disjoint if $|i-j| > 1$.

  If the length of the chain is even, say $n = 2h$, a tubular neighborhood of $c_1 \cup c_2 \cup \cdots \cup c_{2h}$ is a genus-h subsurface Σ of Σ_g and has one boundary component. Let d be a simple closed curve parallel to the boundary component of Σ. The relation

$$(t_{c_1} \cdots t_{c_{2h}})^{4h+2} = t_d$$

  is called *even chain relation* [61].

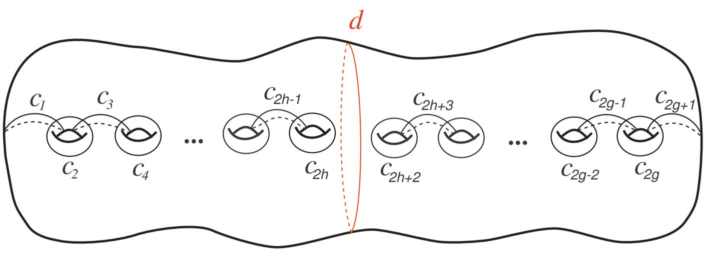

Figure 2.7: Curves in the even chain relation.

  If the length of the chain is odd, say $n = 2h+1$, a tubular neighborhood of $c_1 \cup c_2 \cup \cdots \cup c_{2h+1}$ is a genus-h subsurface Σ of Σ_g and has two boundary components. We denote simple closed curves parallel to two boundary components by d_1 and d_2. The relation

$$(t_{c_1} \cdots t_{c_{2h+1}})^{2h+2} = t_{d_1} t_{d_2}$$

is called *odd chain relation* [61].

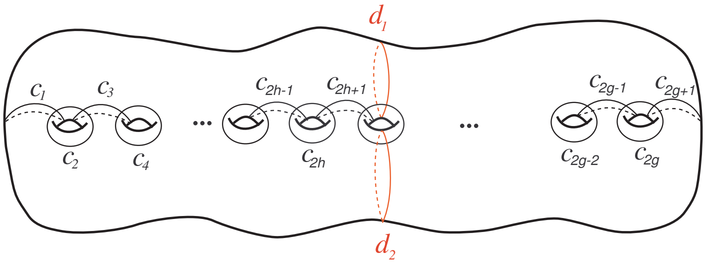

Figure 2.8: Curves in the odd chain relation.

If $h = 1$, the chain relations have special names: one-holed torus relation and two-holed torus relation, respectively.

- **Hyperelliptic Relation:** Consider all curves in the chain $(c_1, \ldots, c_{2g+1})$ on Σ_g in Figure (2.8). The relation

$$(t_{c_1} \cdots t_{c_{2g}} t_{c_{2g+1}}^2 t_{c_{2g}} \cdots t_{c_1})^2 = 1$$

or

$$(t_{c_{2g+1}} \cdots t_{c_2} t_{c_1}^2 t_{c_2} \cdots t_{c_{2g+1}})^2 = 1$$

is called *hyperelliptic relation* [6].

For the proofs of the last three relation, we may use Alexander Lemma. That is, we cut the surface along a set consisting of arcs and simple closed curves whose complement is a union of two-disks. If the action of the two sides of the relation are isotopic on this set (preserving orientation of curves and arcs), then they are equal. Recall that Alexander Lemma says if a diffeomorphism $\mathbb{D}^2 \to \mathbb{D}^2$ is identity on $\partial\mathbb{D}^2$, then it is isotopic to the identity.

- **Lantern Relation:**

Let Σ_0^4 denote a sphere with 4 boundary components. Let d_1, d_2, d_3, d_4 be the curves parallel to these boundaries and x_1, x_2, x_3 interior curves as in Figure (2.9). Then in $\mathrm{Mod}(\Sigma_0^4)$, the Dehn twists satisfy the lantern relation

$$t_{d_1} t_{d_2} t_{d_3} t_{d_4} = t_{x_1} t_{x_2} t_{x_3}.$$

Since each d_i is disjoint from each other, corresponding Dehn twists can be written in any order. Note that

$$t_{x_1}t_{x_2}t_{x_3} = t_{x_2}t_{x_3}t_{x_1} = t_{x_3}t_{x_1}t_{x_2}.$$

If Σ_0^4 is embedded in Σ_g, we can see these curves in Σ_g. This relation first discovered by Dehn [9] and rediscovered by Johnson [25].

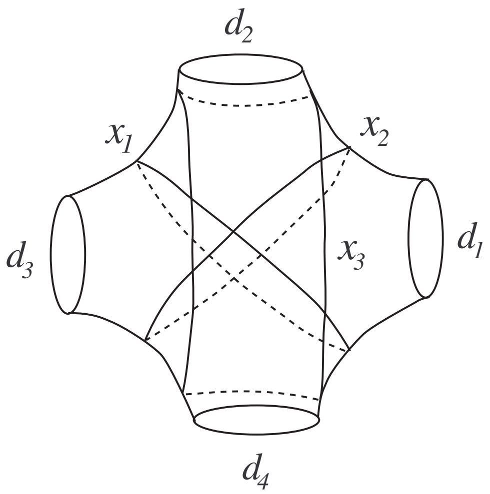

Figure 2.9: Curves for lantern relation.

The proof again follows from Alexander Lemma [30, 17].

CHAPTER 3

LEFSCHETZ FIBRATIONS

In this chapter, we give the definition and the most basic properties of Lefschetz fibrations. We then show that how Lefschetz fibrations can be descibed topologically. This will be followed by the explanation of the relationship between Lefschetz fibrations and mapping class groups. For more details of the content one can see [39, 49] among many others.

3.1 Basics for Lefschetz Fibrations

First, we introduce the definition of a Lefschetz fibration on 4-manifolds.

Definition 3.1.1. *Let X be a closed connected oriented smooth 4-manifold. A Lefschetz fibration on the 4-manifold X is a smooth surjective map $f : X \to \mathbb{S}^2$ with finite set of critical values $P = \{p_1, \dots, p_n\}$ such that around each critical point $c_i \in f^{-1}(p_i)$ and critical value p_i, there are local orientation-preserving complex coordinate charts on which f is of the form $f(z_1, z_2) = z_1^2 + z_2^2$.*

Indeed in the Definition (3.1.1), X may have boundary and $\mathbb{S}^2$ can be replaced by any compact oriented genus-g surface. But some extra conditions must be added to the definition. On the other hand, assumption for orientation-preserving charts above is crucial to consider symplectic structure on the 4-manifold on which Lefschetz fibration is defined.

A *regular or generic fiber* of a Lefschetz fibration is a smooth closed connected oriented genus-g surface Σ_g whose genus is called the *genus of the Lefschetz fibration*

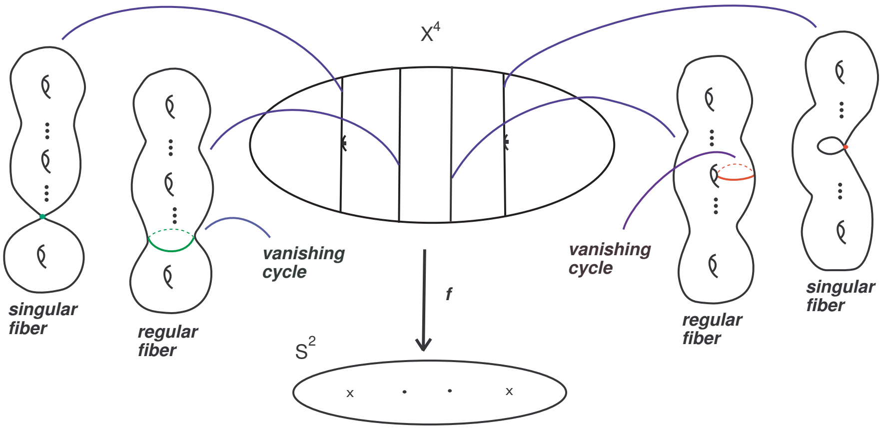

Figure 3.1: An overview for construction of Lefschetz fibrations.

and we call f a *Lefschetz fibration of genus g* or a *genus-g Lefschetz fibration*.

Each critical point c_i is called *vanishing cycle* which corresponds to an embedded circle in a nearby regular fiber. The *singular fiber* is obtained by collapsing c_i to a point. On the local model $(z_1, z_2) \mapsto z_1^2 + z_2^2$, a singular fiber $\Sigma_0 = \{z_1^2 + z_2^2 = 0\}$ is obtained from the regular fiber $\Sigma_\epsilon = \{z_1^2 + z_2^2 = \epsilon\}(\epsilon > 0)$ by collapsing the vanishing cycle $\{(x_1, x_2) \in \mathbb{R}^2, x_1^2 + x_2^2 = \epsilon\} = \Sigma_\epsilon \cap \mathbb{R}^2$ [5]. Up to diffeomorphism, the topology of a neighborhood of a singular fiber is determined by that vanishing cycle .

We will assume that f is injective on critical points, that is, each singular fiber contains only one critical point. We can always have this property by a slight perturbation of f.

It follows that removing critical values and corresponding singular fibers, Lefschetz fibration turns into a smooth fiber bundle over a connected base space $\mathbb{S}^2 - P$ with fibers diffeomorphic to Σ_g. Thus a Lefschetz fibration has all but finitely many fibers with the same diffeomorphism type.

The singular fiber $f^{-1}(p_i)$ is called *reducible* or *irreducible* depending on the corresponding vanishing cycle separates (homologically trivial) or does not separate (homologically nontrivial) the generic fiber. The number of irreducible singular fibers and reducible singular fibers with different topological types gives the combinatorial data of the concerned Lefschetz fibration.

In this study, we only consider *relatively minimal* Lefschetz fibrations which means that no fiber contains (-1)-sphere (a 2-sphere with self intersection number -1). In other words, we ignore fibrations with homotopically trivial vanishing cycles. Otherwise, we can blow down the (-1)-sphere while the rest of the fibration structure is preserved.

As a topological operation we mean by blowing up of X at a point $x \in X$ taking out a 4-ball around x and instead we glue a regular neighborhood of the exceptional sphere (i.e. smoothly embedded sphere with self-intersection -1) in $\overline{\mathbb{CP}^2}$, so we obtain $X\sharp\overline{\mathbb{CP}^2}$. Similarly, if X contains an exceptional sphere M, we may replace a tubular neighborhood of M by a 4-ball, so that $X = Y\sharp\overline{\mathbb{CP}^2}$.

3.2 Monodromy of a Lefschetz Fibration over $\mathbb{S}^2$

We can describe Lefschetz fibrations by using their monodromy representations. The monodromy representation determines a Lefschetz fibration f up to isomorphism for $g \geq 2$ [5, 36, 43].

Let $P = \{p_1, \ldots, p_n\}$ denotes the set of critical values as before. For a fixed base point $p \in \mathbb{S}^2 - P$, we fix an identification for the regular fiber $f^{-1}(p)$ with Σ_g. Let $\gamma : [0,1] \to \mathbb{S}^2$ be an arc with $\gamma([0,1]) = \delta_i$ where δ_i is a loop based at the base point p on $\mathbb{S}^2$ and surrounds only p_i. The restriction of f to the preimage of this loop is a Σ_g-bundle over S^1 with monodromy a positive Dehn twist t_{d_i} along the corresponding vanishing cycle d_i. If we perform this step for all critical values, we get the monodromy representation of a Lefschetz fibration $f : X \to \mathbb{S}^2$

$$\varphi : \pi_1(\mathbb{S}^2 - P) \to \mathrm{Mod}(\Sigma_g).$$

More precisely, we choose a set of sufficiently small disjoint loops $\delta_1, \ldots, \delta_n$ and each encircling only one p_i positively. The loops are ordered cyclically by travelling counterclockwise around the basepoint p and any two of them intersect only at p. The set $\delta_1, \ldots, \delta_n$ generates the free group $\pi_1(\mathbb{S}^2 - P)$ and we have the presentation

$$\langle \delta_1, \ldots, \delta_n | \delta_1\delta_2 \cdots \delta_n = 1 \rangle$$

of $\pi_1(\mathbb{S}^2 - P)$.

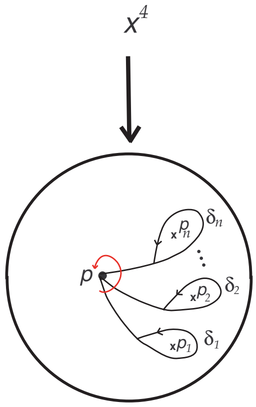

Figure 3.2: A choice of basis for $\pi_1(\mathbb{S}^2 - P)$.

This can be seen from the handlebody decompositon as follows [19]: Consider $\mathbb{S}^2$ as a union of two two-disks $\mathbb{D}^2$ and put all the critical values in one of these hemispheres. We start with $\Sigma_g \times \mathbb{D}^2$ and attach 2-handles with framing -1 to the neighborhood of the generic fiber along the vanishing cycles in distict Σ_g fibers relative to the product framing in $\Sigma_g \times \partial\mathbb{D}^2$ [26]. If the corresponding monodromy on the boundary of $\mathbb{D}^2$ is isotopic to identity, we glue $\Sigma_g \times \mathbb{D}^2$ so that we have a Lefschetz fibration over $\mathbb{S}^2$. The extension is unique if $g \geq 2$ and as a result we get a closed manifold. The monodromy of the Lefschetz fibration is $t_{d_n} \cdots t_{d_2} t_{d_1} = 1$ where $\varphi(\delta_i) = t_{d_i}$. Note that the map φ given by $\varphi(\delta_i) = t_{d_i}$ is an antihomomorphism since in $\pi_1(\mathbb{S}^2 - P)$ elements are read from left to right but in $\mathrm{Mod}_g(\Sigma_g)$ vice versa.

The choice of the loops δ_i in $\pi_1(\mathbb{S}^2 - P)$ is not unique, so any two choices differ by a sequence of *Hurwitz moves* which means to change consecutive factors:

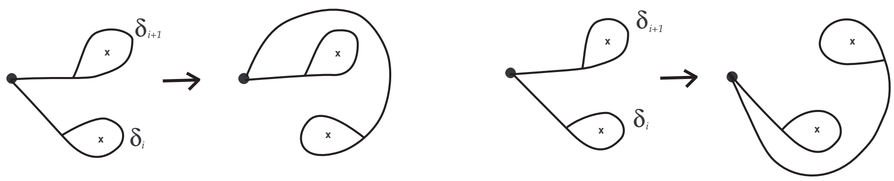

Figure 3.3: Hurwitz moves.

$$\delta_1, \ldots, \delta_i, \delta_{i+1}, \ldots, \delta_n \sim \delta_1, \ldots, \delta_{i+1}, \delta_{i+1}^{-1}\delta_i\delta_{i+1}, \ldots, \delta_n.$$

$$\delta_1, \ldots, \delta_i, \delta_{i+1}, \ldots, \delta_n \sim \delta_1, \ldots, \delta_i\delta_{i+1}\delta_i^{-1}, \delta_i, \ldots, \delta_n.$$

The picture in $\mathrm{Mod}_g(\Sigma_g)$ is

$$t_{d_n} \cdots t_{d_{i+1}} t_{d_i} \cdots t_{d_1} \sim t_{d_n} \cdots t_{t_{d_{i+1}}(d_i)} t_{d_{i+1}} \cdots t_{d_1}.$$
$$t_{d_n} \cdots t_{d_{i+1}} t_{d_i} \cdots t_{d_1} \sim t_{d_n} \cdots t_{d_i} t_{t_{d_i}^{-1}(d_{i+1})} \cdots t_{d_1}.$$

Although the vanishing cycles depend on the choice of paths δ_i, their topological type does not depend on this choice. We say that two factorizations are *Hurwitz equivalent* if one is obtained from the other by applying Hurwitz moves. Since at the beginning, we identify the regular fiber with any Σ_g, to eliminate this dependence, we see the Dehn twists in $\mathrm{Mod}(\Sigma_g)$ of any abstract genus-g surface. This gives us another equivalence relation on the set of factorizations, called *global conjugation*:

$$t_{d_n} \cdots t_{d_2} t_{d_1} \sim t_{\phi(d_n)} \cdots t_{\phi(d_2)} t_{\phi(d_1)}$$

for all $\phi \in \mathrm{Mod}(\Sigma_g)$.

We may also cyclically permute the δ_i's and have the same fibration.

A factorization $\varrho = t_{d_n} \cdots t_{d_2} t_{d_1} = 1$ is called *positive relator* and determines the topology of the associated Lefschetz fibration $f : X \to \mathbb{S}^2$. Conversely, such a positive relator determines a Lefschetz fibration over $\mathbb{S}^2$ with the vanishing cycles $d_1, d_2, \ldots, d_n$.

Definition 3.2.1. *Let $f_i : X_i \to \mathbb{S}^2$ $(i = 1, 2)$ be two genus-g Lefschetz fibrations. We say that f_1 and f_2 are isomorphic if there are orientation-preserving diffeomorphisms $g : X_1 \to X_2$ and $g' : \mathbb{S}^2 \to \mathbb{S}^2$ such that the diagram*

$$\begin{array}{ccc} X_1 & \xrightarrow{g} & X_2 \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_2} \\ \mathbb{S}^2 & \xrightarrow[g']{} & \mathbb{S}^2 \end{array}$$

commutes, i.e. $f_2 g = g' f_1$.

For $g \geq 2$, there is a one-to-one corresponce between the isomorphism classes of Lefschetz fibrations over $\mathbb{S}^2$ and the positive relators up to Hurwitz equivalences and global conjugations [26, 36]. Namely, we have:

Theorem 3.2.2 ([36, Theorem 2.4])**.** *Let $f : X \to B$ and $f' : X' \to B'$ be two Lefschetz fibrations of genus g (≥ 2) over connected bases. Then they are isomorphic if* *and only if one is obtained from the other by a finite sequence of Hurwitz equivalences and global conjugations.*

The correspondence holds for $g \geq 2$ because Teichmuller theory implies that the identity component of $Diff^+(\Sigma_g)$ is contractible and the extension of fibration from base $\mathbb{D}^2$ to $\mathbb{S}^2$ is unique.

Theorem 3.2.3 ([12])**.** *For $g \geq 2$, the components of $Diff^+(\Sigma_g)$ are contractible. If $g = 1$, the identity component of $Diff^+(T^2)$ contains T^2 as a deformation retract.*

3.3 Some Useful Definitions and Properties

Definition 3.3.1. *A section of a Lefschetz fibration $f : X \to \mathbb{S}^2$ is a map $s : \mathbb{S}^2 \to X$ satisfying $fs =$id$_{\mathbb{S}^2}$.*

We will use the same notation for the map section s and its image $s(\mathbb{S}^2)$.

From the definition of a section, we see that if a Lefschetz fibration admits a section, the point of the fiber which has intersection with the section must be fixed. Thus, we have a lift $\pi_1(\mathbb{S}^2 - P) \to \mathrm{Mod}(\Sigma_g^1)$, where $\mathrm{Mod}(\Sigma_g^1)$ is the mapping class group of a surface with one marked point. Moreover, there exists a canonical homomorphism from $\mathrm{Mod}(\Sigma_{g,1})$ to $\mathrm{Mod}(\Sigma_g^1)$ where $\mathrm{Mod}(\Sigma_{g,1})$ is the mapping class group of a surface with one boundary component. The kernel of this homomorphism is isomorphic to $\mathbb{Z}$ generated by t_δ where δ is a simple closed curve parallel to the boundary. Conversely, if we have such a lift, the Lefschetz fibration has a section. If the product of positive Dehn twists is equal to t_δ^n in $\mathrm{Mod}(\Sigma_{g,1})$, this section has self intersection $-n$ [39]. This is negative because of a theorem which says that for a genus-g (> 0) Lefschetz fibration over $\mathbb{S}^2$ with section s, the homological self-intersection of s satisfies $[s(\mathbb{S}^2)]^2 < 0$ [57, 58]. Moreover by using sections, it is deduced that a product of positive Dehn twists in $\mathrm{Mod}(\Sigma_{g,1})$ can not be equal to the identity [39].

It is an open question whether every Lefschetz fibration of genus g over $\mathbb{S}^2$ has a section or not. If $g = 1$, Lefschetz fibrations have sections and there are some partial results for $g = 2$ [59]. On the other hand, if we obtain a Lefschetz fibration from

Lefschetz pencil by blowing up, it has at least one section depending on the number of blow ups.

Sections are important for Lefschetz fibrations because they are useful for the calculation of their fundamental groups.

Lemma 3.3.2 ([49, page 398]). *Let $f : X \to \mathbb{S}^2$ be a Lefschetz fibration of genus g with global monodomy $t_{d_1} t_{d_2} \cdots t_{d_n} = 1$. Assume that f has a section. Then the fundamental group of X is isomorphic to the fundamental group of Σ_g divided out by the normal closure of the simple closed curves $d_1, d_2, \ldots, d_n$ considered as elements in $\pi_1(\Sigma_g)$.*

On the other hand if it is not clear from the monodromy of a Lefschetz fibration whether it has a section or not, $\pi_1(X)$ has the presentation as in Lemma (3.3.2) with possible one more nontrivial relation [40].

As we mentioned in the introduction, there are several ways to obtain Lefschetz fibrations such as blowing up Lefschetz pencils or construction from a given positive factorization of the identity in $\mathrm{Mod}(\Sigma_g)$. The following operation is another way to obtain new Lefschetz fibration from the given ones.

Definition 3.3.3. *Let $f_i : X_i \to \mathbb{S}^2$ $(i = 1, 2)$ be two Lefschetz fibrations of genus g with regular fiber Σ_g. Choose a regular point $b_i \in \mathbb{S}^2$ $(i = 1, 2)$ from each base space and identify tubular neighborhoods $\nu\Sigma_g$ of each $f_i^{-1}(b_i) \cong \Sigma_g$ with $D^2 \times \Sigma_g$ in each X_i. Let $\psi : (X_1 - \nu\Sigma_g) \to (X_2 - \nu\Sigma_g)$ be a fiber-preserving, orientation-reversing diffeomorphism of the boundaries. Then the fiber sum of X_1 and X_2 is defined as follows:*

$$f_1 \sharp_\psi f_2 : X_1 \sharp_\psi X_2 \to \mathbb{S}^2,$$

where $X_1 \sharp_\psi X_2 = (X_1 - \nu\Sigma_g) \bigcup_\psi (X_2 - \nu\Sigma_g)$.

In the Definition (3.3.3), if ψ is the identity map, the fiber sum is called trivial fiber sum and we dropped it from the notation. For different diffeomorphisms ψ, we get different Lefschetz fibrations. Indeed there are examples in [45] for $X_1 \sharp_\psi X_2$ having different first homologies.

Definition 3.3.4 ([44]). *Let $f : X \to \mathbb{S}^2$ be a Lefschetz fibration of genus g with* *global monodromy* $t_{c_1}t_{c_2}\cdots t_{c_n} = 1$*. Then* f *is called a hyperelliptic Lefschetz fibration of genus* g *iff there exists* $\phi \in \mathrm{Mod}(\Sigma_g)$ *such that* $\phi t_{c_i}\phi^{-1} \in \mathcal{H}_g$ *for all* i*,* $1 \leq i \leq n$*.*

So for $g = 1$ and 2, all Lefschetz fibrations are hyperelliptic.

Definition 3.3.5 ([39])**.** *A Lefschetz fibration* $f : X \to \mathbb{S}^2$ *is called holomorphic if* X *is a complex surface and for a suitable complex structure on* $\mathbb{S}^2$*,* f *is a holomorphic map.*

Fiber sums of holomorphic Lefschetz fibrations need not to be holomorphic.

Now, we will give two invariants which are crucial to identify Lefschetz fibrations: the Euler characteristic and the signature. It is a fact that for fiber bundles Euler characteristic is multiplicative.

Theorem 3.3.6. *Let* $f : X \to \mathbb{S}^2$ *be a Lefschetz fibration of genus* g *and* n *be the number of its singular fibers. If* Σ_g *is the regular fiber of* f *, then the Euler characteristic of* X *is*

$$e(X) = 4 - 4g + n.$$

Proof. If we remove n critical values from $\mathbb{S}^2$ and corresponding n singular fibers from X, we have the following surface bundle:

$$X - \bigcup_{i=1}^{n} f^{-1}(p_i) \to \mathbb{S}^2 - \bigcup_{i=1}^{n} p_i.$$

Then since the Euler characteristic is multiplicative on surface bundles, we have

$$e(X - \bigcup_{i=1}^{n} f^{-1}(p_i)) = e(\Sigma_g) \cdot e(\mathbb{S}^2 - \bigcup_{i=1}^{n} p_i) = e(\Sigma_g) \cdot (e(\mathbb{S}^2) - n).$$

Thus,

$$e(X) - \sum_{i=1}^{n} e(f^{-1}(p_i)) = e(\Sigma_g) \cdot e(\mathbb{S}^2) - \sum_{i=1}^{n} e(\Sigma_g).$$

As we stated that the handlebody construction of a Lefschetz fibration is obtained by attaching 2-handles $h_i \approx \mathbb{D}^2 \times \mathbb{D}^2$ to the total space of the trivial fibration $\Sigma_g \times \mathbb{D}^2$ along the vanishing cycles d_i. In other words, to obtain singular fibers we attach the boundary of $\mathbb{D}^2$ to Σ_g along d_i. Thus, we have $e(f^{-1}(p_i)) = e(\Sigma_g \cup \mathbb{D}^2) = e(\Sigma_g) + 1$. Then

$$e(X) - \sum_{i=1}^{n}(e(\Sigma_g) + 1) = e(\Sigma_g) \cdot e(\mathbb{S}^2) - \sum_{i=1}^{n} e(\Sigma_g)$$

implies that $e(X) = e(\Sigma_g) \cdot e(\mathbb{S}^2) + n$. Substituting $e(\Sigma_g) = 2 - 2g$ and $e(\mathbb{S}^2) = 2$, we get $e(X) = 4 - 4g + n$. □

Unlike the Euler characteristic, the signature depends on fibration itself and its calculation is not so easy.

3.4 Signature of Lefschetz Fibrations over $\mathbb{S}^2$

If a Lefschetz fibration is over the two-sphere, Özbağcı [44] and Smith [54] gave formulas for the signature of the Lefschetz fibration. Özbağcı uses direct computations whereas Smith uses properties of Lefschetz fibrations. For hyperelliptic Lefschetz fibrations Endo [13] gave a local signature formula which is a generalization of the σ-number [35] and the fractional signature [36]. Recently, Endo and Nagami [16] showed that the signature of a Lefschetz fibration over the two-sphere can be calculated by using the signatures of relations contained in its monodromy. We use this method to calculate the signatures of our Lefschetz fibrations.

3.4.1 Signature Cocycle

In the following, we give some theorems and definitions relavent to the signature calculation. More details can be found in [16].

Let $\mathcal{S}$ be the set consisting of all isotopy classes of simple closed curves on Σ_g and $\mathcal{F}$ be the free group generated by the set $\mathcal{S}$.

Definition 3.4.1 ([16]). *There exists a homomorphism*

$$\varpi : \mathcal{F} \to \mathrm{Mod}(\Sigma_g)$$

which sends the isotopy class of a simple closed curve a to the positive Dehn twist t_a. The homomorphism ϖ is onto. Each element of the kernel of ϖ is denoted by $Ker\varpi = \mathcal{R}$ and called as a relator in the generators $\mathcal{S}$ of $\mathrm{Mod}(\Sigma_g)$.

Luo [33] showed that $\mathcal{R}$ is normally generated by all comutativity, all braid, all 2-chain and all lantern relations [16].

By using the evaluation map $H_2(\mathrm{Mod}(\Sigma_g)) \to \mathbb{Z}$ for the cohomology class of a 2-cocycle of $\mathrm{Mod}(\Sigma_g)$, a signature of a relator can be defined as follows.

Let $\tau_g : \mathrm{Mod}(\Sigma_g) \times \mathrm{Mod}(\Sigma_g) \to \mathbb{Z}$ be the signature cocycle of $\mathrm{Mod}(\Sigma_g)$ satisfying

$$\tau_g(c, 1) = \tau_g(1, c) = \tau_g(c, c^{-1}) = 0$$

for all $c \in \mathrm{Mod}(\Sigma_g)$. Then there is a homomorphism $c_g : \mathcal{R} \to \mathbb{Z}$ which induces the homomorphism $H_2(\mathrm{Mod}(\Sigma_g)) \to \mathbb{Z}$.

Definition 3.4.2 ([16])**.** *Let $\varrho \in \mathcal{R}$ be a relator. The signature of ϱ is defined by the function*

$$I_g(\varrho) := -c_g(\varrho) - s(\varrho)$$

where $s(\varrho)$ is the sum of the total exponent of separating simple closed curves in the relator ϱ. Note that, I_g is a function from the set of relators to $\mathbb{Z}$.

From now on for convenience, we use the definition of a relator as follows.

Definition 3.4.3. *A product of Dehn twists $\varrho = t_{d_1}^{\epsilon_1} \cdots t_{d_n}^{\epsilon_n}$ in $\mathrm{Mod}(\Sigma_g)$ is a relator if $\varrho = 1$, where $e_i = \pm 1$. Note that, if $e_1 = \cdots = e_n = +1$, then ϱ is called the positive relator as we defined before.*

Definition 3.4.4 ([1])**.** *Let $d_1, d_2, \ldots, d_n$ and $e_1, e_2, \ldots, e_m$ be simple closed curves on Σ_g and the Dehn twists about them in $\mathrm{Mod}(\Sigma_g)$ satisfy the following relation:*

$$t_{e_1} t_{e_2} \cdots t_{e_m} = t_{d_1} t_{d_2} \cdots t_{d_n}.$$

Let $\varrho = U \cdot t_{d_1} t_{d_2} \cdots t_{d_n} \cdot V$ be a positive relator where U and V are product of (positive) Dehn twists. Then, we obtain a new positive relator as follows:

$$\varrho' = U \cdot t_{e_1} t_{e_2} \cdots t_{e_m} \cdot V.$$

If $R = t_{e_1} t_{e_2} \cdots t_{e_m} t_{d_n}^{-1} \cdots t_{d_2}^{-1} t_{d_1}^{-1} (= 1)$, we say that ϱ' is obtained by applying a *R-substitution* to ϱ. Moreover, this definition can be defined on surfaces with boundary components.

In [14], it is stated that substitution for commutativity and braid relation do not change the associated manifold while 2-chain and lantern relation do change the associated manifold.

Definition 3.4.5. *Let $\varrho = t_{d_1}^{\epsilon_1} \cdots t_{d_n}^{\epsilon_n}$ be a positive relator. If there exists an orientation-preserving homeomorphism $f : \Sigma_g \to \Sigma_g$ such that $\varrho^{'} = t_{f(d_1)}^{\epsilon_1} \cdots t_{f(d_n)}^{\epsilon_n}$, we say ϱ and $\varrho^{'}$ are topologically equivalent or have the same topological type.*

Lemma 3.4.6 ([16, Lemma 3.5])**.** *In $\mathrm{Mod}(\Sigma_g)$, the signature I_g of relators has the following properties:*

1. $I_g(\varrho^{-1}) = -I_g(\varrho)$,
2. $I_g(W\varrho W^{-1}) = I_g(\varrho)$,
3. $I_g(\varrho_1\varrho_2) = I_g(\varrho_1) + I_g(\varrho_2)$,
4. $I_g(\varsigma^{'}) = I_g(\varsigma) + I_g(\varrho)$ *if $\varsigma^{'}$ is obtained by applying a ϱ-substitution to ς,*
5. $I_g(\varrho^{'}) = I_g(\varrho)$ *if ϱ and $\varrho^{'}$ are topologically equivalent,*

where $\varrho, \varrho_1, \varrho_2, \varsigma, \varsigma'$ are relators and W is a product of Dehn twists.

Theorem 3.4.7 ([16, Theorem 4.2])**.** *Let $f : X \to \mathbb{S}^2$ be a Lefschetz fibration of genus g with monodromy $t_{d_1} \cdots t_{d_n} = 1$. Then the signature of X is*

$$\sigma(X) = I_g(t_{d_1} \cdots t_{d_n}).$$

Theorem 3.4.8 ([16, Theorem 4.3])**.** *Let $f : X \to \mathbb{S}^2$ and $f' : X' \to \mathbb{S}^2$ be Lefschetz fibrations of genus g with positive relators ς, ς', respectively. Assume that a ϱ-substitution to ς gives ς' for some relator ϱ. Then we have*

$$\sigma(X') = \sigma(X) + I_g(\varrho).$$

3.4.2 Signature of Some Relators in Mapping Class Group

We do not give the details of the calculations. More about these are in the paper of Endo and Nagami in [16, Section 3].

1. **Hyperelliptic Relator**: Let $(c_1, \dots, c_{2g+1})$ be a chain of curves on Σ_g as in Figure (3.4). Then

$$h_g := (t_{c_1} t_{c_2} \cdots t_{c_{2g}} t^2_{c_{2g+1}} t_{c_{2g}} \cdots t_{c_2} t_{c_1})^2 = 1$$

is called the hyperelliptic relator. The signature of h_g is $I_g(h_g) = -4(g+1)$.

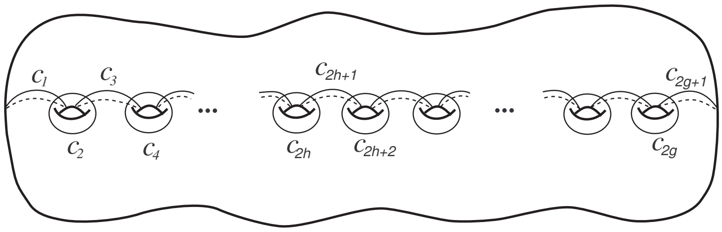

Figure 3.4: The curves $c_1, ..., c_{2g+1}$.

2. **Odd Chain Relator**: Let $(c_1, \dots, c_{2h+1})$ be a chain of curves on Σ_g as in Figure (3.5). Then

$$\mathscr{C}_{2h+1} := (t_{c_1} \cdots t_{c_{2h+1}})^{2h+2} t_{d_1}^{-1} t_{d_2}^{-1} = 1$$

is called odd chain relator. The signature of $\mathscr{C}_{2h+1}$ is $I_g(\mathscr{C}_{2h+1}) = -2h(h+2)$. In a monodromy, we may replace $t_{d_1} t_{d_2}$ by $(t_{c_1} \cdots t_{c_{2h+1}})^{2h+2}$ and call this operation a $\mathscr{C}_{2h+1}$-substitution.

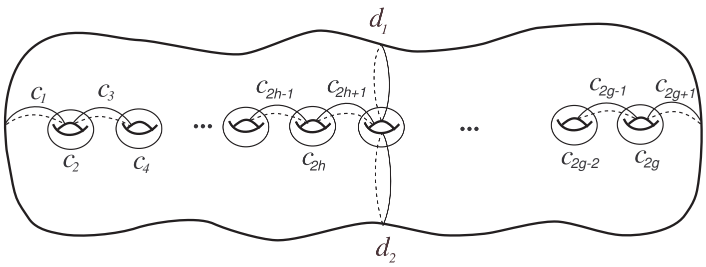

Figure 3.5: The curves d_1 and d_2.

In particular, $I_g(\mathscr{C}_3) = -6$.

3. **Even Chain Relator**: Let $(c_1, \dots, c_{2h})$ be a chain of curves on Σ_g as in Figure (3.6). Then

$$\mathscr{C}_{2h} := (t_{c_1} \cdots t_{c_{2h}})^{4h+2} t_d^{-1} = 1$$

is called even chain relator. The signature of $\mathscr{C}_{2h}$ is $I_g(\mathscr{C}_{2h}) = -4h(h+1)+1$. Interchanging t_d by $(t_{c_1} \cdots t_{c_{2h}})^{4h+2}$ in a relator is called a $\mathscr{C}_{2h}$-substitution.

In Section 2.2 we gave the braid and lantern relations. Let T and L denote the relators for braid and lantern respectively. Then $I_g(T) = 0$ and $I_g(L) = +1$.

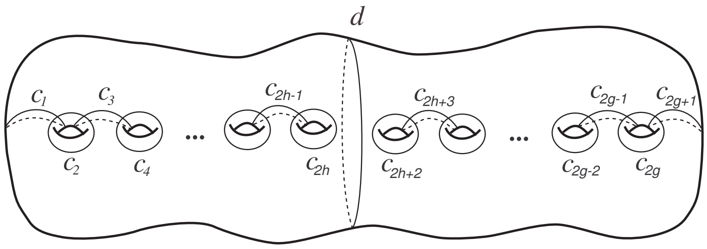

Figure 3.6: The curves $c_1, ..., c_{2g+1}$ and d.

3.5 The Slope Inequality for Lefschetz Fibrations over $\mathbb{S}^2$

Let X be a closed oriented smooth 4-manifold and $f : X \to \mathbb{S}^2$ $(g \geq 2)$ be a (nontrivial) relatively minimal genus-g Lefschetz fibration. We denote the Euler characteristic of X by $e(X)$ and the signature of X by $\sigma(X)$. If X is a symplectic manifold, it is endowed with an almost complex structure which gives to its tangent bundle a complex vector bundle structure. Thus, Chern classes $c_j(X) \in H^{2j}(X;\mathbb{Z}) \cong \mathbb{Z}$ for $j = 1, 2$ can be defined [7]. Then these Chern classes give us two Chern numbers

$$c_1^2(X) := \langle c_1(X) \cup c_1(X), [X] \rangle \quad \text{and} \quad c_2(X) := \langle c_1(X), [X] \rangle .$$

It is well known that $c_2(X)$ is $e(X)$ since top Chern class is always equal to Euler characteristic. By using Hirzebruch signature theorem, the equality $c_1^2(X) = 2e(X) + 3\sigma(X)$ can be shown. According to calculations in [7]; in dimension 4, Hirzebruch signature theorem says that

$$3\sigma(X) = \langle p_1(TX), [X] \rangle$$

where $p_1(TX)$ is the first Pontrjagin class of X. Since Pontrjagin classes of a complex vector bundle S are determined by its Chern classes:

$$p_1(S) = c_1^2(S) - 2c_2(S).$$

So,

$$\begin{aligned} 3\sigma(X) &= \langle p_1(TX), [X] \rangle \\ &= \left\langle c_1^2(X), [X] \right\rangle - 2 \left\langle c_2(X), [X] \right\rangle \\ &= \left\langle c_1^2(X), [X] \right\rangle - 2e(X) \end{aligned}$$

which gives

$$c_1^2(X) = 2e(X) + 3\sigma(X).$$

Note that both Chern numbers are topological invariants. Moreover by using $\sigma(X)$ and $e(X)$, the ratio $(\sigma(X)+e(X))/4$ is defined and called holomorphic Euler characteristic $\chi_h(X)$ and since X is a symplectic manifold, $\chi_h(X) \in \mathbb{Z}$ (It is a well defined integer even for an almost complex manifold- proved by Noether formula). Then the *slope* λ_f is defined by the quotient

$$\lambda_f = \frac{K_f^2}{\chi_f},$$

where $K_f^2 := c_1^2(X)+8(g-1)$ and $\chi_f := \chi_h(X)+(g-1)$. The inequality $4-4/g \leq \lambda_f$ is called the *slope inequality*.

Conjecture 3.5.1 (Hain [16, Conjecture 4.12]). *The slope inequality $\lambda_f \geq 4 - 4/g$ is satisfied by every smooth Lefschetz fibration $f : X \to \mathbb{S}^2$ of genus $g \geq 2$ over the two-sphere.*

Indeed in [3], this conjecture is stated as a question (Question5.10) which is a symplectic version of the Moriwaki inequality. According to calculations in [54],[56] and [43],

$$\chi_f > 0, \quad K_f^2 \geq 4g-4 \quad \text{and} \quad \lambda_f \leq 10.$$

It is shown that if f is a holomorphic Lefschetz fibration then $\lambda_f \geq 4 - 4/g$ holds [62]. Also, hyperelliptic Lefschetz fibrations satisfy the slope inequality. This can be proved by using signature formula for genus-g Lefschetz fibrations which is shown by Matsumoto [36, 35] for $g = 1, 2$ and by Endo [13] for $g \geq 3$. So, any genus-g Lefschetz fibration for $g = 1, 2$ satisfies the slope inequality. Indeed, the slope of any hyperelliptic Lefschetz fibration with only nonseparating vanishing cycles is equal to $4 - 4/g$ [41]. Moreover, Monden [41] obtained lower bounds for the slope of Lefschetz fibrations with $b_2^+ = 1$.

Actually, it is interesting to find non-holomorphic Lefschetz fibrations which imply the difference between the geography of complex surfaces fibered over curves and that of Lefschetz fibrations. Özbağcı and Stipsicz [45] obtained examples of non-holomorphic genus-2 Lefschetz fibrations by fiber summing the holomorphic genus-2 Lefschetz fibration which is constructed by Matsumoto [36]. Moreover, Fintushel and Stern [18] constructed non-holomorphic symplectic Lefschetz fibrations which also does not satisfy the Noether inequality. More examples for non-holomorphicity are in [16, 41].

Remark 3.5.2. *In [38] Miyachi and Shiga constructed Lefschetz fibrations over* Σ_g *having slope a negative rational number.*

CHAPTER 4

CONSTRUCTION OF LEFSCHETZ FIBRATIONS

In this section, we establish and prove our main theorems. We construct Lefschetz fibrations $f : X \to \mathbb{S}^2$ of genus g that do not satisfy the slope inequality. Then by using this structure, we give examples of minimal symplectic 4-manifolds admitting Lefschetz fibrations with small slope. Finally, we present some results of Lefschetz fibrations with even smaller slopes.

4.1 Background

A common way to find a lower bound for the slope is to put restriction on the genus of regular fiber. However, Monden releases this dependence.

Theorem 4.1.1 ([41, Theorem 3.1])**.** *For each $g \geq 3$, there exists a genus-g Lefschetz fibration f over $\mathbb{S}^2$ with slope $\lambda_f = 4 - 4/g - 1/(3g)$ such that the total space is simply connected.*

For the proof, first he takes fiber sum of some chosen Lefschetz fibrations. Then he follows substitution technique in the associated monodromy representation. Applying this procedure several times, he obtaines the following results:

Corollary 4.1.2 ([41, Corollary 3.6])**.** *For each $g \geq 3$ and nonnegative integers m and l, there exists a genus-g Lefschetz fibration $f_{m,l} : X_{m,l} \to \mathbb{S}^2$ with slope*

$$\lambda_{f_{m,l}} = 4 - 4/g - 1/((m+3)g)$$

such that the symplectic 4-manifold $X_{m,l}$ is simply connected. Moreover, if $(m, l) \neq (0, 0)$, then $X_{m,l}$ is minimal.

Corollary 4.1.3 ([41, Corollary 3.7]). *For each $g \geq 3$, $m \geq 1$ and $l \geq 0$, there exists a genus-g Lefschetz fibration $f'_{m,l} : Y_{m,l} \to \mathbb{S}^2$ with slope*

$$\lambda_{f'_{m,l}} = 4 - 4/g - 1/(2g) + 1/(2 \cdot 3^{m-1} g)$$

such that the symplectic 4-manifold $Y_{m,l}$ is simply connected. Moreover, if $l \geq 1$, then $Y_{m,l}$ is minimal.

The Lefschetz fibrations $f_{m,l}$ $(m \geq 0)$ and $f'_{m,l}$ $(m \geq 2)$ constructed to prove above corollaries are non-holomorphic. This is because of the following proposition due to Xiao [62]:

Proposition 4.1.4 ([41, Proposition 4.1]). *If a Lefschetz fibration $f : X \to \mathbb{S}^2$ is holomorphic, then it satisfies the slope inequality $\lambda_f \geq 4 - 4/g$.*

4.2 Main Results and Their Proofs

We go one step further than Monden; that is, we cover the case of genus $g = 2$. However, since all Lefschetz fibrations for $g = 1, 2$ are hyperelliptic, they already satisfy the slope inequality.

Theorem 4.2.1. *For each $g \geq 2$, there exists a Lefschetz fibration $f : X \to \mathbb{S}^2$ of genus g with slope*

$$\lambda_f = 4 - \frac{4}{g} - \frac{4(g-2)}{g(g+2)}$$

such that the 4-manifold X is simply connected.

Proof. Consider the curves shown on the genus-g surface $\Sigma_{g,1}$ illustrated in Figure (4.1). For $1 \leq i \leq 2g+1$, let C_i (resp. C'_{2g+1}) denote the Dehn twist about the curve c_i (resp. c'_{2g+1}). Then the Dehn twist δ about the boundary parallel curve can be written as

$$\delta = (C_1 \Psi C_1)^2 \tag{4.1}$$

where $\Psi = C_2 C_3 \cdots C_{2g} C_{2g+1} C'_{2g+1} C_{2g} \cdots C_3 C_2$. By conjugating both sides with C_1 and C_1^{-1}, we rewrite the equality in (4.1) as

$$\delta = (C_1 \Psi C_1)^2 = (\Psi C_1^2)^2 = (C_1^2 \Psi)^2.$$

Note that, the product $C_1\Psi C_1$ gives a hyperelliptic involution $\imath$ when the boundary of $\Sigma_{g,1}$ is capped off. If we cap off $\partial\Sigma_{g,1}$, we get a homomorphism $\mathrm{Mod}(\Sigma_{g,1}) \to \mathrm{Mod}(\Sigma_g)$ from which we obtain a positive factorization of the identity in $\mathrm{Mod}(\Sigma_g)$. Thus, the relation in (4.1) is a lift of hyperelliptic relation $(C_1\Psi C_1)^2 = 1$ from $\mathrm{Mod}(\Sigma_g)$ to $\mathrm{Mod}(\Sigma_{g,1})$. Here, a curve on $\Sigma_{g,1}$ and its image on Σ_g after capping off are denoted by the same letter.

Let e_1 and e_2 be the boundary components of a regular neighborhood of $c_1 \cup c_2 \cup c_3$. Then consider the diffeomorphism in $\mathrm{Mod}(\Sigma_{g,1})$

$$\phi_i = U E_i C_1 U,$$

sending c_1 to e_i for $i = 1, 2$, where U is the Dehn twist about the simple closed curve u intersecting each one of c_1, e_1 and e_2 transversely only once. We assume each ϕ_i fixes the boundary component pointwise. Note that, we have $C_1^{\phi_i} = E_i$.

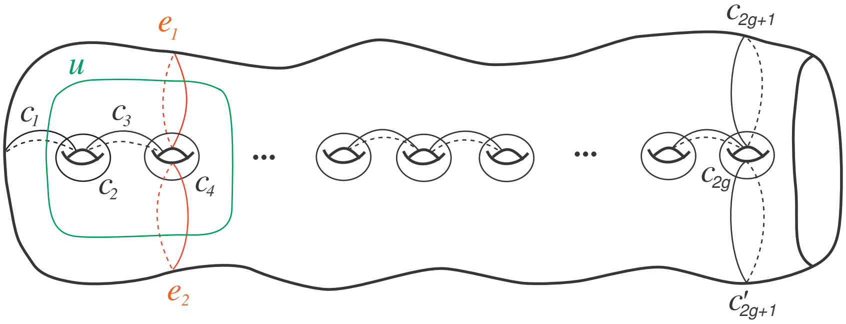

Figure 4.1: The curves $c_1, \ldots, c_{2g+1}, c'_{2g+1}, e_1$ and e_2.

Let $H = (C_1\Psi C_1)^2$ and $H^{\phi_i} = \phi_i H \phi_i^{-1}$ be the conjugation of H with ϕ_i, so that $\delta = H^{\phi_i}$. Then a factorization δ^2 in $\mathrm{Mod}(\Sigma_{g,1})$ is given as follows:

$$\begin{aligned}
\delta^2 &= H^{\phi_1} H^{\phi_2} \\
&= (\Psi C_1^2 \Psi C_1^2)^{\phi_1} (C_1^2 \Psi C_1^2 \Psi)^{\phi_2} \\
&= \Psi^{\phi_1} E_1^2 \Psi^{\phi_1} E_1^2 E_2^2 \Psi^{\phi_2} E_2^2 \Psi^{\phi_2} \\
&= \Psi^{\phi_1} \Psi^{(E_1^2 \phi_1)} E_1^4 E_2^4 \Psi^{(E_2^{-2}\phi_2)} \Psi^{\phi_2},
\end{aligned}$$

which may be written as

$$\delta^2 = E_1^4 E_2^4 \Psi^{(E_2^{-2}\phi_2)} \Psi^{\phi_2} \Psi^{\phi_1} \Psi^{(E_1^2\phi_1)}. \tag{4.2}$$

On the other hand in $\mathrm{Mod}(\Sigma_{g,1})$, the relation $(C_1C_2C_3)^4 = E_1E_2$ holds. That is, $\mathscr{C}_3 := (C_1C_2C_3)^4E_1^{-1}E_2^{-1}$. We apply this $\mathscr{C}_3$-substitution four times in (4.2) and get

$$\delta^2 = (C_1C_2C_3)^{16}\Psi^{(E_2^{-2}\phi_2)}\Psi^{\phi_2}\Psi^{\phi_1}\Psi^{(E_1^2\phi_1)}.$$

Capping off the boundary component gives the following factorization of the identity in $\mathrm{Mod}(\Sigma_g)$.

$$1 = (C_1C_2C_3)^{16}\Psi^{(E_2^{-2}\phi_2)}\Psi^{\phi_2}\Psi^{\phi_1}\Psi^{(E_1^2\phi_1)}. \tag{4.3}$$

The corresponding Lefschetz fibration $f : X \to \mathbb{S}^2$ of genus g admits a (-2)-section. Note that the number of vanishing cycles of f is $16g + 48$. The Lefschetz fibration $f : X \to \mathbb{S}^2$ has the following topological invariants:

By using Theorem (3.3.6), the Euler characteristic of X is

$$e(X) = 2(2 - 2g) + 16g + 48 = 12g + 52.$$

By using Theorem (3.4.7) and Theorem (3.4.8), the signature of X is computed as

$$\begin{aligned}\sigma(X) &= 2I_g(H^{\phi_i}) + 4I_g(\mathscr{C}_3)\\ &= 2(-4(g+1)) + 4(-6)\\ &= -8g - 32.\end{aligned}$$

It follows that

$$K_f^2 = c_1^2(X) + 8(g-1) = 3\sigma(X) + 2e(X) + 8(g-1) = 8g,$$
$$\chi_f = \chi_h(X) + (g-1) = \frac{\sigma(X) + e(X)}{4} + (g-1) = 2g + 4.$$

Thus one computes the slope as

$$\lambda_f = 4 - \frac{4}{g} - \frac{4(g-2)}{g(g+2)}.$$

We now show that X is simply connected.

In (4.3), the product Ψ^{ϕ_i} contains the Dehn twists about the following set of vanishing cycles;

$$\mathcal{V}_1 = \{\phi_i(c_2), c_3, \phi_i(c_4), \phi_i(c_5), c_6, c_7, \ldots, c_{2g+1}, c'_{2g+1}\}.$$

Similarly, the products $\Psi^{(E_1{}^2\phi_1)}$ and $\Psi^{(E_2^{-2}\phi_2)}$ have the following sets of vanishing cycles;

$$\mathcal{V}_2 = \{E_1^2(\phi_1(c_2)), c_3, E_1^2(\phi_1(c_4)), E_1^2(\phi_1(c_5)), c_6, c_7, \ldots, c_{2g+1}, c'_{2g+1}\}$$

and

$$\mathcal{V}_3 = \{E_2^{-2}(\phi_2(c_2)), c_3, E_2^{-2}(\phi_2(c_4)), E_2^{-2}(\phi_2(c_5)), c_6, c_7, \ldots, c_{2g+1}, c'_{2g+1}\}$$

respectively.

Since the Lefschetz fibration f with fiber Σ_g has a section, we may use Lemma (3.3.2) to compute the fundamental group of the total space X; that is, $\pi_1(X)$ is isomorphic to the group $\pi_1(\Sigma_g)$ divided out by the normal closure of the vanishing cycles

$$\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2 \cup \mathcal{V}_3 \cup \{c_1, c_2, c_3\}.$$

Thus the group $\pi_1(X)$ has a presentation with generators $a_1, b_1, a_2, b_2, \ldots, a_g, b_g$ and relations

- $\Pi_{i=1}^g [a_i, b_i] = 1$;
- $v = 1$ for each vanishing cycle $v \in \mathcal{V}$.

We show that all generators are trivial in $\pi_1(X)$.

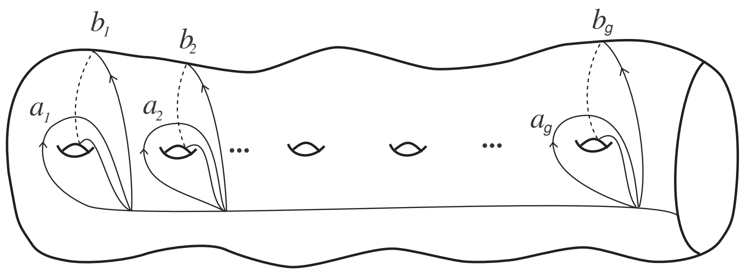

Figure 4.2: The choice of basis for the fundamental group of X.

Since the curves $c_6, c_8, c_{10}, \ldots, c_{2g}$ in $\mathcal{V}_1$ are in the free homotopy class of $a_3, a_4, a_5, \ldots, a_g$, we have

$$a_3 = a_4 = a_5 = \cdots = a_g = 1$$

in $\pi_1(X)$.

For $7 \leq i = 2k+1 \leq (2g-1)$, the vanishing cycle $c_i \in \mathcal{V}_1$ is freely homotopic to $b_k^{-1} a_{k+1} b_{k+1} a_{k+1}^{-1}$. It follows that $b_k = b_{k+1}$, and hence

$$b_3 = b_4 = b_5 = \cdots = b_g.$$

On the other hand, since $c_{2g+1} \in \mathcal{V}_1$ is freely homotopic to b_g, we have $b_g = 1$. Thus,

$$b_3 = b_4 = b_5 = \cdots = b_g = 1.$$

The vanishing cycles c_1 and c_2 are freely homotopic to b_1 and a_1 respectively. Thus,

$$b_1 = a_1 = 1.$$

Also since c_3 is freely homotopic to $b_1^{-1} a_2 b_2 a_2^{-1}$, we get

$$b_2 = 1.$$

Finally, $\phi_1(c_2) \in \mathcal{V}_1$ is freely homotopic to $a_1 b_1 a_1^{-1} a_2$, which implies $a_2 = 1$, concluding that X is simply connected.

□

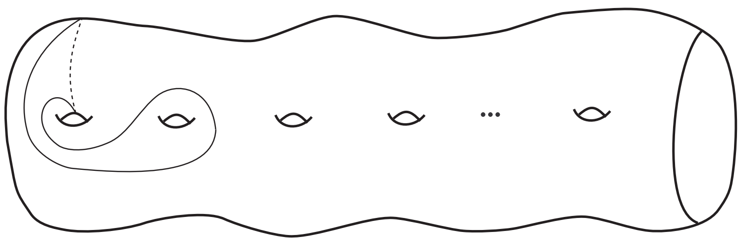

Figure 4.3: The vanishing cycle $\phi_1(c_2)$.

Remark 4.2.2. *The Lefschetz fibration f has slope $\lambda_f = 4 - 4/g$ if $g = 2$ and violates the slope inequality, that is, $\lambda_f < 4 - 4/g$ if $g \geq 3$. If we compare the slope λ_f we found above with the slopes of the Lefschetz fibrations constructed by Monden in [41], λ_f is smaller than all: i.e., $\lambda_f < 4 - 4g - 1/(3g)$ and $\lambda_f < 4 - 4/g - 1/(2g) + 1/(2 \cdot 3^{m-1} g)$ for $g \geq 3$ and for all m.*

Remark 4.2.3. *If we apply $\mathscr{C}_3$-substitution three times to the factorization in (4.2), the slope of the corresponding Lefschetz fibration becomes $\lambda = 4 - 14/(2g+3)$. If the $\mathscr{C}_3$-substitution is applied twice, we get the slope $\lambda = 4 - 6/(g+1)$ and for once $\mathscr{C}_3$-substitution the slope is $\lambda = 4 - 10/(2g+1)$. The Lefschetz fibrations with these slopes do not satisfy the slope inequality. Also their slopes are less than the ones constructed by Monden in [41].*

For $g \geq 2$, if we apply Hurwitz moves and global conjugations to the monodromy of a genus-g Lefschetz fibration, its isomorphism class does not change by results of Kas and Matsumoto [26, 36]. Thus, the Euler characteristic and the signature of the associated 4-manifold are unaffected under these modifications. On the other hand, if we take fiber sum of two Lefschetz fibrations, these invariants are changed so that the slope can be calculated as follows [41].

Lemma 4.2.4. *Let $g \geq 2$ and let $f_1 : X_1 \to \mathbb{S}^2$ and $f_2 : X_2 \to \mathbb{S}^2$ be two nontrivial relatively minimal genus-g Lefschetz fibrations with regular fiber Σ_g. Then any fiber sum $f : X_1 \sharp X_2 \to \mathbb{S}^2$ of f_1 and f_2 has slope $\lambda_f = (K^2_{f_1} + K^2_{f_2})/(\chi_{f_1} + \chi_{f_2})$.*

Proof. The Euler characteristic and the signature of the total space of fiber sum are

$$\begin{aligned} e(X_1 \sharp X_2) &= e(X_1) + e(X_2) + 4(g-1) \text{ and} \\ \sigma(X_1 \sharp X_2) &= \sigma(X_1) + \sigma(X_2) \end{aligned}$$

respectively. It follows that the slope λ_f is K^2_f/χ_f where

$$\begin{aligned} K^2_f &= c_1^2(X_1 \sharp X_2) + 8(g-1) \\ &= 3\sigma(X_1 \sharp X_2) + 2e(X_1 \sharp X_2) + 8(g-1) \\ &= 3[\sigma(X_1) + \sigma(X_2)] + 2[e(X_1) + e(X_2) + 4(g-1)] + 8(g-1) \\ &= 3\sigma(X_1) + 2e(X_1) + 8(g-1) + 3\sigma(X_2) + 2e(X_2) + 8(g-1) \\ &= K^2_{f_1} + K^2_{f_2} \\ \chi_f &= \chi_h(X_1 \sharp X_2) + (g-1) \\ &= \frac{\sigma(X_1 \sharp X_2) + e(X_1 \sharp X_2)}{4} + (g-1) \\ &= \frac{\sigma(X_1) + \sigma(X_2) + e(X_1) + e(X_2) + 4(g-1)}{4} + (g-1) \\ &= \frac{\sigma(X_1) + e(X_1)}{4} + (g-1) + \frac{\sigma(X_2) + e(X_2)}{4} + (g-1) \\ &= \chi_{f_1} + \chi_{f_2} \end{aligned}$$

The lemma follows. $\square$

Corollary 4.2.5. *Let $g \geq 2$ and let $f_1 : X_1 \to \mathbb{S}^2$ and $f_2 : X_2 \to \mathbb{S}^2$ be two nontrivial relatively minimal Lefschetz fibrations of genus g with same slope $\lambda = \lambda_{f_1} = \lambda_{f_2}$. Then any fiber sum $f : X_1 \sharp X_2 \to \mathbb{S}^2$ of f_1 and f_2 has slope λ.*

Next theorem is shown by Stipsicz in [56].

Theorem 4.2.6. *Let $f : X \to \mathbb{S}^2$ be a relatively minimal nontrivial Lefschetz fibration of genus g. Then the total space $X \sharp X$ of the fiber sum $f \sharp f : X \sharp X \to \mathbb{S}^2$ of f with itself is a minimal symplectic 4-manifold.*

Now, by using these results we give an example of minimal symplectic 4-manifold on which we build genus-g Lefschetz fibration with small slope.

Corollary 4.2.7. *For every $g \geq 2$, there exists a minimal symplectic 4-manifold admitting a genus-g Lefschetz fibration with slope*

$$\lambda = 4 - \frac{4}{g} - \frac{4(g-2)}{g(g+2)}.$$

Proof. Let f be the Lefschetz fibration with the slope

$$\lambda_f = 4 - \frac{4}{g} - \frac{4(g-2)}{g(g+2)}$$

given in Theorem (4.2.1). Then by Theorem (4.2.6), the total space of a fiber sum of f with itself is a minimal symplectic 4-manifold with slope λ. $\square$

Remark 4.2.8. *One can obtain infinitely many Lefschetz fibrations of genus g by taking fiber sum of those in Corollary (4.2.7) so that all have the same slope as above.*

Now, our aim is to obtain Lefschetz fibrations even with smaller slopes than those given in Theorem (4.2.1).

4.2.1 The case of even genus g

Theorem 4.2.9. *For $g \geq 4$ and even, there exists a Lefschetz fibration $f : M \to \mathbb{S}^2$ of genus g with slope*

$$\lambda_f = 4 - \frac{4}{g} - \frac{8(g-2)(g-1)}{g(g^2+8)}.$$

Proof. Let $g = 2r$. Consider the factorization

$$H = (C_1 C_2 C_3 \cdots C_{2g} C_{2g+1}^2 C_{2g} \cdots C_3 C_2 C_1)^2, \tag{4.4}$$

where the Dehn twists are about the curves on Σ_g illustrated in Figure (4.4). Note that H is a factorization of the identity. Let ϕ_3 be a diffeomorphism on the surface such that

$$\phi_3 = \begin{cases} c_1 \mapsto d_1, \\ c_3 \mapsto d_2, \\ \quad \vdots \\ c_{2i-1} \mapsto d_i, \\ \quad \vdots \\ c_{2g-3} \mapsto d_{g-1}, \\ c_{2g-1} \mapsto d_g. \end{cases}$$

For instance, one can take

$$\phi_3 = C_2 D_1 C_1 C_2 \cdot C_4 D_2 C_3 C_4 \cdots C_{2g-2} D_{g-1} C_{2g-3} C_{2g-2} \cdot C_{2g} D_g C_{2g-1} C_{2g}.$$

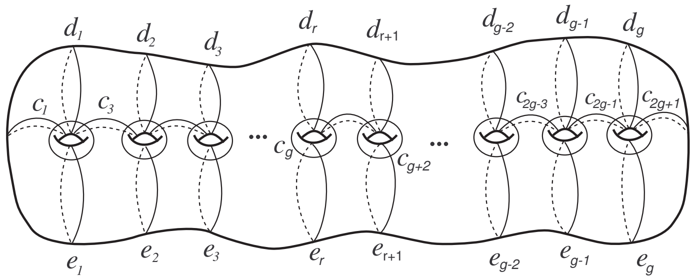

Figure 4.4: The curves $c_1, c_3, ..., c_{2g-1}$, $d_1, ..., d_g$ and $e_1, ..., e_g$.

By conjugating H by ϕ_3 and applying Hurwitz moves, we may write

$$H^{\phi_3} = (D_1 D_2 \cdots D_r D_{r+1} \cdots D_g)^4 L_1.$$

Moreover, by conjugating H^{ϕ_3} by the hyperelliptic involution $\imath$, mapping d_i to e_i, we get

$$H^{\imath\phi_3} = (E_1 E_2 \cdots E_r E_{r+1} \cdots E_g)^4 L_2.$$

Note that each L_i ($i = 1, 2$) is a product of $(4g+4)$ positive Dehn twists. Then by appyling more Hurwitz moves, we have a relator as follows:

$$H^{\phi_3} H^{\imath\phi_3} = (D_1E_1)^4(D_2E_2)^4 \cdots (D_rE_r)^4(D_{r+1}E_{r+1})^4 \cdots (D_{g-1}E_{g-1})^4(D_gE_g)^4 L_1^{'} L_2.$$

For $1 \leq i \leq r-1$, we replace $D_{i+1}E_{i+1}$ by $(C_1C_2 \cdots C_{2i+1})^{2i+2}$ four times. Note that this is a $\mathscr{C}_{2i+1}$-substitution. Similarly, for $1 \leq j \leq r-1$ we replace $D_{g-j}E_{g-j}$ by $(C_{2g+1}C_{2g} \cdots C_{2(g-j)+1})^{2j+2}$ four times. This is a $\mathscr{C}_{2j+1}$-substitution.

The number of (nonseparating) positive Dehn twists in the new relator is

$$2(4g+4) + 16 + \sum_{i=1}^{r-1} \big(4(2i+1)(2i+2)\big) + \sum_{j=1}^{r-1} \big(4(2j+1)(2j+2)\big).$$

Let $f : M \to \mathbb{S}^2$ be the Lefschetz fibration corresponding to this new relator. Then we compute the following topological invariants:

The Euler characteristic of M is

$$\begin{aligned} e(M) &= -4(g-1) + \Big[2(4g+4) + 16 + \sum_{i=1}^{r-1} \big(4(2i+1)(2i+2)\big) \\ &\qquad + \sum_{j=1}^{r-1} \big(4(2j+1)(2j+2)\big)\Big] \\ &= (4g^3 + 6g^2 + 8g + 36)/3. \end{aligned}$$

The signature of M is

$$\begin{aligned} \sigma(M) &= I_g(H^{\phi_3}) + I_g(H^{\imath\phi_3}) + \sum_{i=1}^{r-1} 4I_g(\mathscr{C}_{2i+1}) + \sum_{j=1}^{r-1} 4I_g(\mathscr{C}_{2j+1}) \\ &= -8(g+1) + \sum_{i=1}^{r-1} 4(-2i(i+2)) + \sum_{j=1}^{r-1} 4(-2j(j+2)) \\ &= (-2g^3 - 6g^2 - 4g - 24)/3. \end{aligned}$$

The slope is then computed as

$$\lambda_f = 4 - \frac{4}{g} - \frac{8(g-2)(g-1)}{g(g^2+8)}.$$

□

4.2.2 The case of odd genus g

Theorem 4.2.10. *For $g \geq 5$ and odd, there exists a Lefschetz fibration $f : N \to \mathbb{S}^2$ of genus g with slope*

$$\lambda_f = 4 - \frac{4}{g} - \frac{8(g-2)(g-1)}{g(g^2+11)}.$$

Proof. Let $g = 2g+1$. Similar to the even case, we start with the factorization

$$H = (C_1C_2C_3 \cdots C_{2g}C_{2g+1}^2C_{2g} \cdots C_3C_2C_1)^2, \tag{4.5}$$

where the Dehn twists are about the curves on Σ_g illustrated in Figure (4.5). Let ϕ_3 be a diffeomorphism satisfying $\phi_3(c_{2i-1}) = d_i$ for $1 \leq i \leq g$, as in the proof of Theorem (4.2.9).

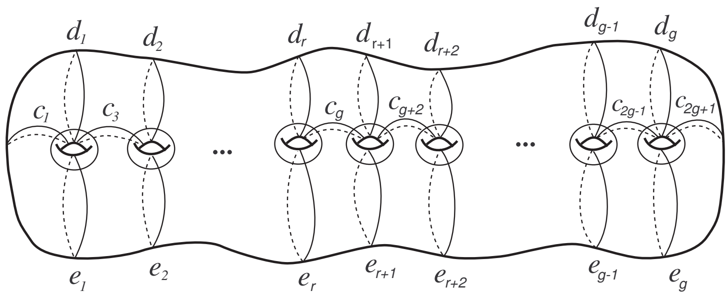

Figure 4.5: The curves $c_1, c_3, ..., c_{2g-1}$, $d_1, ..., d_g$ and $e_1, ..., e_g$.

By conjugating H by ϕ_3 and applying Hurwitz moves, we obtain

$$H^{\phi_3} = (D_1D_2 \cdots D_rD_{r+1}D_{r+2} \cdots D_{g-1}D_g)^4S_1.$$

Furthermore, by conjugating H^{ϕ_3} by the hyperelliptic involution $\imath$, we get

$$H^{\imath\phi_3} = (E_1E_2 \cdots E_rE_{r+1}E_{r+2} \cdots E_{g-1}E_g)^4S_2.$$

Each S_i $(i = 1, 2)$ is a product of $(4g+4)$ positive Dehn twists. Then by applying more Hurwitz moves, we have the following relator:

$$H^{\phi_3}H^{\imath\phi_3} = (D_1E_1)^4(D_2E_2)^4\cdots(D_rE_r)^4(D_{r+1}E_{r+1})^4(D_{r+2}E_{r+2})^4\cdots(D_{g-1}E_{g-1})^4 (D_gE_g)^4S_1'S_2.$$

For $1 \leq i \leq r$, we replace $D_{i+1}E_{i+1}$ by $(C_1C_2\cdots C_{2i+1})^{2i+2}$ four times. Note that this is a $\mathscr{C}_{2i+1}$-substitution. Similarly, for $1 \leq j \leq r-1$ we replace $D_{g-j}E_{g-j}$ by $(C_{2g+1}C_{2g}\cdots C_{2(g-j)+1})^{2j+2}$ four times. This is a $\mathscr{C}_{2j+1}$-substitution.

The number of (nonseparating) positive Dehn twists in the new relator is

$$2(4g+4)+16+\sum_{i=1}^{r}\big(4(2i+1)(2i+2)\big)+\sum_{j=1}^{r-1}\big(4(2j+1)(2j+2)\big).$$

Then the last Lefschetz fibration $f : N \to \mathbb{S}^2$ which is obtained after substitutions has the following topological invariants:

The Euler characteristic of N

$$\begin{aligned} e(N) &= -4(g-1)+\Big[2(4g+4)+16+\sum_{i=1}^{r}\big(4(2i+1)(2i+2)\big) \\ &\qquad +\sum_{j=1}^{r-1}\big(4(2j+1)(2j+2)\big)\Big] \\ &= (4g^3+6g^2+20g+42)/3. \end{aligned}$$

The signature of N

$$\begin{aligned} \sigma(N) &= I_g(H^{\phi_3})+I_g(H^{\imath\phi_3})+\sum_{i=1}^{r}4I_g(\mathscr{C}_{2i+1})+\sum_{j=1}^{r-1}4I_g(\mathscr{C}_{2j+1}) \\ &= -8(g+1)+\sum_{i=1}^{r}4(-2i(i+2))+\sum_{j=1}^{r-1}4(-2j(j+2)) \\ &= (-2g^3-6g^2-10g-30)/3. \end{aligned}$$

Then the slope is

$$\lambda_f = 4-\frac{4}{g}-\frac{8(g-2)(g-1)}{g(g^2+11)}.$$

□

Corollary 4.2.11. *For $g \geq 4$, there exists a minimal simply connected symplectic 4-manifold X admitting a Lefschetz fibration of genus g with slope*

$$\lambda_1 = 4 - \frac{4}{g} - \frac{8(g-2)(g-1)}{g(g^2+8)}$$

if g is even, and

$$\lambda_2 = 4 - \frac{4}{g} - \frac{8(g-2)(g-1)}{g(g^2+11)}.$$

if g is odd.

Proof. Suppose first that g is even. Consider the Lefschetz fibration $f : M \to \mathbb{S}^2$ with slope

$$\lambda_1 = 4 - \frac{4}{g} - \frac{8(g-2)(g-1)}{g(g^2+8)}$$

given in Theorem (4.2.9). If we denote by V the corresponding monodromy factorization, all c_i, except $c_{2r}, c_{2r+1}, c_{2r+2}$, are vanishing cycles. It follows that all b_i and a_i, except a_r, a_{r+1}, are trivial in $\pi_1(M)$.

Let ϕ be a diffeomorphism of Σ_g with $\phi(c_1) = c_{2r}$ and $\phi(c_{2g+1}) = c_{2r+2}$, and let X be the Lefschetz fibration with monodromy factorization VV^{ϕ}.

Now, it follows that the 4-manifold X is simply connected, and minimal by Theorem (4.2.6). Note also that its slope is equal to λ_1 by Corollary (4.2.5).

The case where g is odd is proved similarly. □

CHAPTER 5

GEOGRAPHY PROBLEM

Lefschetz fibrations provide many examples in the class of symplectic 4-manifolds. Since these manifolds have fibering structure on them, we can get results on their diffeomorphism types.

An even dimensional smooth manifold X is called a *symplectic manifold* together with a closed nondegenerate 2-form ω on it. A symplectic 4-manifold X is called *minimal* if it does not contain any (-1)-sphere.

Given for any topological coordinates (c_1^2, χ_h), the existence of a simply-connected closed oriented smooth 4-manifold with additional structure, for an example symplectic structure, is called geography problem.

The geography problem for symplectic 4-manifolds introduced by McCarthy and Wolfson [37]. Gompf improved this work in [21]. Indeed, it is the symplectic version of the geography problem for complex surfaces of general type which was raised by Persson [48]. Since a simply connected complex surface is Kähler and so symplectic, the region occupied by complex surfaces are already covered by some symplectic 4-manifolds. However, there exist some symplectic 4-manifolds outside the prescribed region for complex surfaces. Indeed, there are examples not satisfying the Noether inequality ($2\chi_h - 6 \leq c_1^2$) [55, 21, 47]. Since $12|c_1^2 + c_2$ holds for an almost complex manifold, it also holds for symplectic manifolds. From this, b_2^+ is odd for simply connected symplectic manifolds (A closed simply connected smooth 4-manifold admits an almost complex structure iff b_2^+ is odd). Also the existence of a symplectic structure implies that $b_2^+ > 0$ without assuming simply connectedness. For simply connected case $\chi_h \geq 1$ and a remarkable result of Taubes shows that minimal simply connected symplectic 4-manifolds satisfy $c_1^2 \geq 0$ [60, 32].

In the geography problem for symplectic 4-manifolds, they are considered being minimal. Although many of the Lefschetz fibrations are not minimal, Fintushel and Stern's example is so. In [51], Sato showed that the geography of non-minimal relatively minimal genus-2 Lefschetz fibrations is finite by using the theory of pseudo-holomorphic curves and the Taube's structure theorem on the Gromov invariants. He gave the possible number of (ir)reducible singular fibers [52]. Korkmaz and Baykur constructed genus-2 Lefschetz fibrations having this number of singular fibers [40].

Fintushel and Stern construct Lefschetz fibrations in [18] that do not satisfy the Noether inequality. On the other hand, no example of a Lefschetz fibration violating the Bogomolov-Miyaoka-Yau (BMY) inequality ($c_1^2 \leq 9\chi_h$) has been found.

Our constructed Lefschetz fibrations and the associated symplectic manifolds in Corollary (4.2.7) are minimal and simply-connected. They have the following invariants:

$$\begin{aligned} e &= 28g + 100, \\ \sigma &= -16g - 64, \\ c_1^2 &= 8g + 8, \\ \chi_h &= 3g + 9, \\ b_2^+ &= 6g + 17. \end{aligned}$$

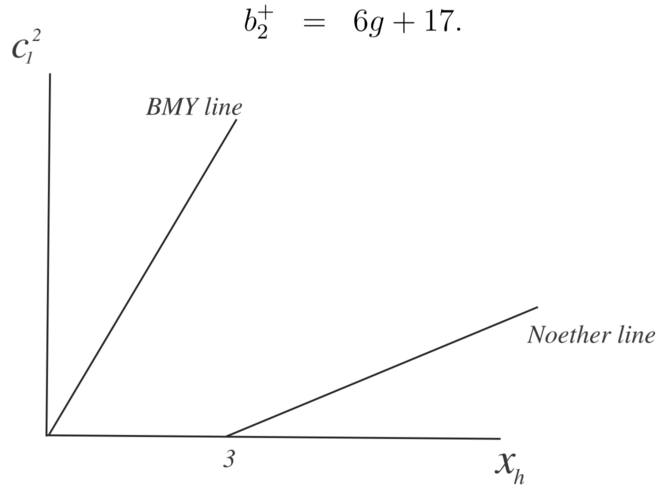

Figure 5.1: The BMY line is $c_1^2 = 9\chi_h$ and the Noether line is $c_1^2 = 2\chi_h - 6$.

Thus, they are in the region which is bounded by the BMY line and the Noether line.

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