DISCRETE-TIME SURPLUS PROCESS FOR TAKAFUL INSURANCE WITH MULTIPLE THRESHOLD LEVELS

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ABSTRACT

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Takaful is a type of insurance system whose participants contribute a certain sum of money to a common pool in order to guarantee each other against predefined losses. Over the past few years, the global Takaful industry has maintained solid growth trajectory with significant untapped potential across several markets. For this reason, there have been numerous researches conducted on the efficiency of Takaful system, but only few of them approach its actuarial aspect. This study aims to construct a new risk model representing Takaful framework in order to develop a recursive computational procedure for some ruin-related quantities calculation associated with Takaful insurance. We consider a discrete-time Sparre Andersen model which incorporates multiple threshold levels functioning as trigger points for investment activities, loan undertakings, and dividend payments. We analyze the effectiveness of Takaful insurance by comparing its finite-time ruin probabilities and expected total discounted dividends with its conventional counterpart. Additionally, we investigate several numerical results for different Takaful business models/forms and make observations concerning the impact of some parameters used in our risk model on the ruin-related quantities.

Keywords : Takaful insurance; ruin probability; expected dividend; multiple threshold model; Sparre Andersen model

ÖZ

ÇOKLU SINIR AŞAMALI TAKAFUL SİGORTALARINDA KESİKLİ ZAMAN AŞKIN SÜREÇLERİ

Achlak, Arham
Yüksek Lisans, Aktüerya Bilimleri
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Kasım 2016, 77 sayfa

Takaful, katılımcıların önceden belirlenmiş kayıplarını garanti altına almak için belirli miktarda bir parayı ortak havuzda biriktirme sistemine dayalı bir sigortadır. Son bir kaç yıldır, küresel Takaful piyasası gelişmemiş potensiyel bazı marketlerde sağlam bir büyüme göstermektedir. Bu sebepten dolayı, Takaful sistemi ve etkinliği ile ilgili bir çok araştırma yapılmış ancak bunlardan çok azının aktüeryal bakış açısıyla yapıldığı görülmektedir. Bu çalışmada, Takaful çerçevesinde iflas ile alakalı bazı niceliklerin hesaplanması için özyinemeli hesaplama yönteminin geliştirilip yeni bir risk modeli oluşturulması amaçlanmaktadır. Bu amaçla, yatırım faaliyetlerini, kredi teahhütünü ve temettü ödemelerini tetikleyen, birden fazla eşik seviyesi içeren kesikli zamanlı Sparre Andersen modeli ele alınmıştır. Takaful sigortasının etkinliği kesikli zamanlı iflas olasılıkları ile iskonto edilmiş temmettülerin beklenen değerinin karşılaştırılması yardımıyla yapılmıştır. Ayrıca, farklı Takaful çeşitleri için elde edilen sonuçlar ve gözlemlerin risk modelinde iflas ile alakalı bazı niceliklere etkisi incelenmiştir.

Anahtar Kelimeler: Takaful sigorta; iflas olasılık; beklenen temettü; çok eşikli model; Sparre Andersen model

To Irham and Ikram

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LIST OF ABBREVIATIONS

CE	Common Era
CPI	Consumer Price Index
DEA	Data Envelopment Analysis
GCC	Gulf Cooperation Council
GDP	Gross Domestic Product
i.e.	id est
iid	independent and identically distributed
KD	Kim-Drekic
$\mathbb{N}$	Natural Numbers = $\{0, 1, 2, ...\}$
pmf	probability mass function
$\mathbb{R}$	Real Numbers
$\mathbb{R}^+$	Positive Real Numbers
SA	Sparre Andersen
s.t.	such that
TBR	Treasury Bill Rate
TO	Takaful Operator
UAE	United Arab Emirates
US	United States
$\mathbb{Z}$	Integers
$\mathbb{Z}^+$	Positive Integers

CHAPTER 1

INTRODUCTION

Takaful is a type of insurance system whose participants contribute a certain sum of money into a common pool in order to guarantee each other against predefined losses. Takaful insurance concept is based on Islamic law (Sharia), which explains the responsibility of individuals to cooperate and support one another. Takaful insurance has been introduced as an alternative to conventional insurance, whose practice involves some elements outlawed by Sharia. Accordingly, Takaful is founded on the principle of mutual assistance, solidarity, and ethical operation with the main purpose of helping each other rather than generating profit. Even though Takaful is branded as Islamic insurance, its target market is not necessarily faith-based since it has the attributes to appeal to a larger customer base including non-Muslims. As is the case with conventional insurance, Takaful is separated into two branches: general Takaful (non-life Takaful) and family Takaful (life Takaful). Key features that distinguish Takaful insurance from its conventional counterpart are listed in Table 1.1.

Takaful insurance utilizes risk sharing scheme among its participants through a pooling method as opposed to risk transfer mechanism from the insured to the insurer in conventional insurance. Takaful participants make regular donation/contribution into a common pool referred to as Takaful fund, which is used to reimburse the claims of the participants. The Takaful fund is managed and administered on behalf of the participants by a licensed body referred to as Takaful operator (TO). In other words, the Takaful fund as a pool of donations is not owned by any company but only managed by a separate entity called Takaful operator. Consequently, the Takaful operator charges an agreed fee to cover its management expenses including the costs of sales and marketing, underwriting, and claims management. In practice, Takaful insurance has various business models, which define the relation between the Takaful fund, the Takaful operator, and the participants differently. Some of the most common business models of Takaful insurance are explained in detail in Chapter 2.

Takaful insurance concept has been practiced in various forms since 622 CE [15], but the first Takaful company/operator was established in 1979 in Sudan. Now there are 184 companies/TOs fully engaged in Takaful business [25], 31 companies fully engaged in Retakaful (reinsurance with Takaful concept) [26], and 48 conventional insurers operating Takaful windows [24] including insurance giants, such as Allianz, AXA, Swiss Re, Munich Re, and Hannover Re. Total gross contributions of the global

Table 1.1: Comparison between conventional insurance and Takaful insurance.

	Conventional insurance	**Takaful insurance**
Role of insurance company	insurer	operator/risk manager
Approach to risk	risk transfer	risk sharing
Contractual basis	contract of exchange	contract of donation
Obligation to pay claims	insurance company	Takaful fund
Underwriting surplus	belongs to shareholders	belongs to participants (might be shared with shareholders under some specific arrangements)
Underwriting deficit	borne by shareholders	borne by participants
Investment activities	no restriction	Sharia compliant
Principal source of income for the insurance company	underwriting surplus	fees and/or profit sharing

Takaful industry is estimated to have reached US$23.2 billion in 2015, which represents an increase of 5% since 2014 [3]. In contrast, global gross contributions stood at only US$5 billion in 2006, which indicates that the industry has been growing fast with huge market potential given that the total insurance premium volume in 2015 is equal to US$4.6 trillion [30]. Development and profitability of Takaful operators varies significantly by sectors and markets, depending on the market's economic maturity and regulatory structure [1]. Regardless of volatility in the financial markets, there seems to be growth momentum in three key markets: Saudi Arabia, the United Arab Emirates (UAE), and Malaysia [1]. Moreover, regulatory enhancements in some frontier markets, such as Indonesia and Turkey have created new opportunities for Takaful insurance [1]. Malaysia has become the largest family Takaful market in the world, and with regulatory clarity and a proven model, the country is expected to maintain its leading position [1]. Given its recent growth trend, there is a need for Takaful operators who are able to give leadership to the growing internationalization of the industry [1].

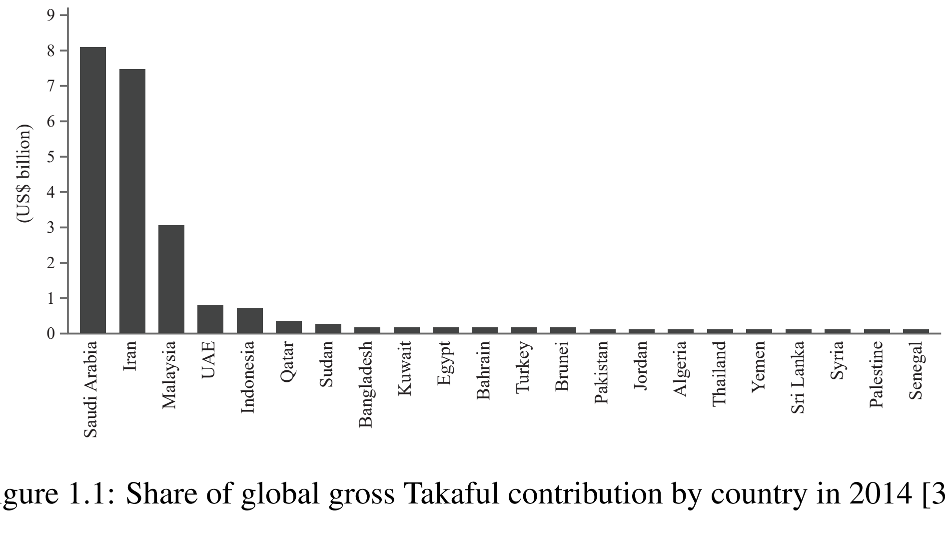


Figure 1.1: Share of global gross Takaful contribution by country in 2014 [3].

Figure 1.1 shows the distribution of Takaful industry's global gross contribution in 2014, where Saudi Arabia (36.6%), Iran (33.6%), and Malaysia (13.6%) are the top three countries accounting for 83.8% of the total global contributions. Even though the industry are centered in a few key markets, there is still significant untapped potential in many countries where Takaful insurance already operates. One of the main reasons is that these countries still have low insurance penetration rates with solid growth prospect due to their young populations. Figure 1.2 gives information on insurance penetration rates in some selected countries in 2015 with an average rate of 1.93% [30] (increased from 1.80% in 2014 [29, 2]). For this reason, there has been great excitement regarding the future prospects of Takaful insurance.

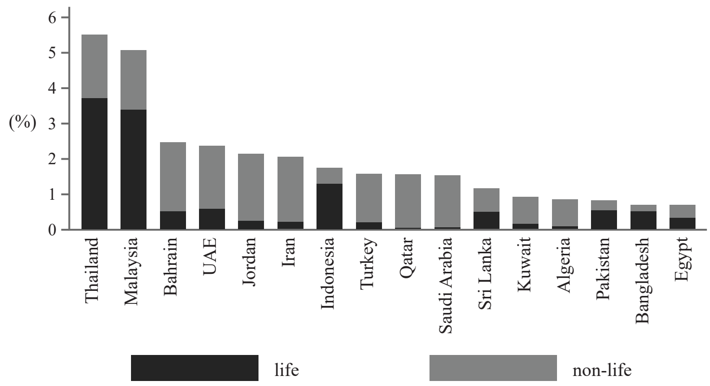


Figure 1.2: Insurance penetration rates (% of GDP) of some countries in 2015 [30].

The largest volumes of Takaful contributions may still be in the Middle East and Southeast Asia, but its dramatic growth has enticed major conventional insurers to buy into the concept. In spite of several setbacks and various obstacles, Takaful insurance also has considerable potential in European and North American markets. Mutual insurance can trace its roots in Europe back almost 1,000 years ago to Icelandic farmers who pooled their resources to provide protection [14]. More recently, the sector grew exponentially during the Industrial Revolution, driven by the migration of people from rural to urban communities [14]. Even though the mutual concept became less popular in the 1990s, the past decade has seen a revival of the model, partly in response to the financial crisis of 2008 [14]. Accordingly, it can help the entrance and growth of Takaful insurance in European market due to its parallels with the mutual sector. Many industry experts argue that the maturity of the insurance industries in Europe could make the expansion of Takaful more dramatic in Europe than in the traditional markets of Middle East and Southeast Asia, if the right products and regulatory balance can be found [17]. In the United States the regulation and limited investment options has been an obstacle for an entrance of a full Takaful operator. Nevertheless, there is one insurance brokerage firm called Zayan Takaful who is offering custom-tailored Takaful products to consumers across the United States. On the other hand, Canada is poised to expand its leadership both at home and abroad in Takaful industry, which reflects both the recent experience of Canadian insurers in Southeast Asia as well as the long history of Canadian mutual insurance [21].

Despite its success, the Takaful industry is facing some challenges, such as lack of consumer awareness in many markets, shortage of human resources with both insurance and Sharia expertise, lack of standardization of both the industry's framework and market regulation, and some problems related to its solvency and capital requirements [18]. In addition, Takaful insurance's recent growth trend has caused many highly inflated projections in the industry with many start-ups developing on overly challenging business plan [18]. Although it is beyond question that Takaful insurance has promising market potential, any future business projections should be reasonable [18]. Moreover, industry results should be analyzed carefully due to lack of credible data, coupled with uncertainty on the future penetration rates of the countries with Takaful insurance potential [18]. Another important problem is that the very presence of many players defeats the main objective of Takaful insurance, which is to create a risk pool that caters to the needs of its participants in order to achieve mutuality [18]. Consequently, this issue may result in some consolidation within the Takaful industry.

1.1 Literature Survey

Since Takaful insurance is getting popular over the past few years, there have been a number of studies analyzing the Takaful concept, especially the features that distinguish Takaful from its conventional counterpart. Pasha and Hussain [23] gave a review of Takaful insurance and comparison between its business models. They concluded that Takaful system is still evolving through debates, discussions, and alternative approaches in order to introduce a uniform consensus based system. Sadeghi [28] examined the emergence and evolution of Takaful from both theoretical and practical prospective. The theoretical studies described the historical evolution of Takaful insurance and the development of Takaful institutions over time. On the other hand, the empirical studies analyzed the extent of public awareness of the Takaful industry, Takaful product offerings, and Takaful product attributes.

Some other studies focus on the development and growth of Takaful insurance in some specific countries where it is well practiced or recently getting attention. For instance, Alhumoudi [7] studied the Takaful concept and its application in Saudi Arabia, where the industry has the most global share. Razak et al. [27] determined the factors that influence people in Malaysia toward the acceptance of Takaful with a conclusion that the most dominant factor was the service quality. In addition, Camp [11] studied the introduction of family Takaful within the Belgian legal and regulatory framework. He deduced that Takaful insurance could enter Belgian market by combining it with existing mutual insurance or following European approach to also cover larger populations in neighboring countries.

Another important aspect of Takaful insurance that attracts researchers is its efficiency in some markets which can be studied by analyzing the performance of Takaful versus its conventional counterpart. Abdou et al. [4] distinguished the performance levels of the Malaysian Takaful and conventional life insurance industries by utilizing financial ratios and macroeconomic variables, such as GDP, CPI and TBR. According to their descriptive studies it can be observed that conventional insurers perform better than Takaful operators in terms of solvency ratios and profitability. However, Takaful operators have more prudent underwriting policies in place, which restrain information asymmetry and minimizes the risk of moral hazard by keeping low claim ratios. Khan and Noreen [19] measured the efficiency of Takaful versus conventional insurance companies in Pakistan by using data envelopment analysis (DEA) approach. Their empirical results showed that Takaful operators are more efficient than conventional insurers due to its high allocative efficiency despite the fact that Takaful operators are relatively new in Pakistan.

In spite of numerous studies on Takaful insurance, there is still few researches conducted from its actuarial aspect. For instance, there has not been much attention paid to its ruin analysis, which is one of the most popular discussions on insurance related topics. To the best of our knowledge, there is not any risk model proposed to comply with the distinct characteristics of Takaful insurance in the literature. The theoretical groundwork of ruin analysis known as the classical risk process or the Cramér-Lundberg model was introduced in 1903 by Lundberg [22] and was republished in 1930s by Cramér [10]. Then, Andersen [8] extended the model in 1957 by allowing claim inter-arrival times to have arbitrary distribution functions [31].

$$U_t^{(SA)} = v + ct - \sum_{i=1}^{N_t} X_i, \qquad t \geq 0. \tag{1.1}$$

The extended model expressed in Equation 1.1 is called Sparre Andersen (SA) model and it gives the financial position or the surplus level of an insurer at time t denoted by $U_t^{(SA)}$, who starts its business at an initial level v. Over time the insurer experiences two opposing cash flows: incoming regular premium from its costumers with constant rate $c > 0$ and outgoing random claim payment with claim number and claim size distribution given as N_t and X_i respectively. The extended model, also known as the renewal risk model defines $\{N_t\}_{t\geq 0}$ to be a renewal process and $\{X_i\}_{i\in\mathbb{Z}^+}$ to be independent and identically distributed (iid) random variables. Furthermore, the model assumes that $X_i > 0$ almost surely and that $\{N_t\}_{t\geq 0}$ and $\{X_i\}_{i\in\mathbb{Z}^+}$ are independent.

The Sparre Andersen model however is not applicable to Takaful insurance system because of several reasons. Firstly, the model does not incorporate investment activities, which is crucial for Takaful insurance since the operational income of the Takaful operator/shareholders comes from the investment return in most business models. Secondly, a Takaful operator is required to distribute dividend payments to its participants when the Takaful fund generates enough surplus, hence this characteristic need to be considered in modeling the Takaful fund. Lastly, the Sparre Andersen model does not have loan undertaking component, whereas the Takaful operator is expected to give interest free loan to the Takaful fund in case of deficit. Consequently, some adjustments in the Sparre Andersen model are required to make it suitable for modeling the surplus process of a Takaful fund.

Recently in the literature, a ruin analysis of a discrete-time dependent Sparre Andersen model with multiple threshold levels has been introduced by Kim and Drekic [20]. These threshold levels function as triggers for external financial activities, such as investment activities, loan undertakings, and dividend payments. Their analysis focused on the development of recursive calculations to quantify finite-time ruin probabilities and expected total discounted dividend payments paid prior to ruin. The model in their analysis is a generalization of a risk model with a single threshold level that was used for another ruin analysis in a discrete-time Sparre Andersen model by Drekic and Mera [13]. The ruin analyses with single threshold based risk model was an extension of the work by Alfa and Drekic [6], in which a discrete-time Sparre Andersen insurance risk model was analyzed as a doubly infinite Markov chain to calculate the time of ruin, the surplus immediately prior to ruin, and the deficit at ruin.

1.2 Aim of the Thesis

The growing trend of Takaful insurance requires proper studies on its financial stability and actuarial modeling in order to understand more of its unique characteristics. As there is limited researches conducted on the aspect of surplus process of Takaful insurance, we aim to construct a new risk model portraying Takaful framework in order to calculate some of its ruin-related quantities. The model should be presented in a form of three interacting components: a Takaful operator, a Takaful fund, and a group of participants whose relationship is defined differently according to the employed business model. The surplus model constructed in this study is obtained by modifying and implementing first time in the literature a surplus model developed by Kim and Drekic [20], where we introduce an additional threshold level and adjust some of the existing threshold levels. By implementing their modified model, we can compare finite-time ruin probabilities and expected total discounted dividends of Takaful insurance from our study with the corresponding quantities of conventional insurance obtained from their numerical results. Moreover, we are also interested in analyzing the behavior of expected total discounted dividend payments of some Takaful business models in order to compare the performance of Takaful operators for each model. To quantify both finite-time ruin probabilities and expected total discounted dividends we focus on the development of some recursive formulas associated with the Takaful risk model.

Together with this introductory chapter, this thesis consists of seven chapters. In Chapter 2 we give more details on Takaful insurance, explain some characteristics that distinguish it from its conventional counterpart, and describe some of the most commonly practiced business models. In Chapter 3 we explain briefly the discrete-time Sparre Andersen model with multiple threshold levels recently introduced by Kim and Drekic [20] and make some modifications in the model to obtain a suitable surplus process for Takaful insurance. Chapter 4 presents our mathematical procedure in constructing a recursive algorithm to calculate finite-time ruin probabilities of Takaful insurance. In Chapter 5 similar methods from the previous chapter are implemented to get other formulas for the expected total discounted dividends calculation for each of the Takaful models we consider in Chapter 2. Particularly, we create three algorithms for three different sources of dividend payments which are investment gain, underwriting surplus, and fees from the participants. Finally, Chapter 6 provides some numerical results via Monte Carlo simulation analyses and Chapter 7 concludes the thesis.

CHAPTER 2

TAKAFUL INSURANCE STRUCTURE

Takaful originates from an Arabic word *kafalah*, which means “mutual guarantee”. Takaful insurance is a cooperative system of reimbursement to its participant in case of loss, organized as Sharia compliant alternative to conventional insurance. The participants agree to guarantee each other’s loss by making regular monetary contributions to a common pool referred to as Takaful fund, which is managed by an entrepreneurial agent referred to as Takaful operator (TO). Therefore, according to Takaful insurance scheme the participants as the insured are essentially the insurers themselves, and hence the TO as the manager of the Takaful fund does not take any profit directly from insurance activities. Instead, the operator is entitled to receive fees from the participants and/or to share in profit generated from investment or underwriting activities of the Takaful fund. Takaful concept has been introduced to overcome some elements/principles in conventional insurance that are outlawed by Sharia because they are considered to be unethical. The prohibited elements are *gharar* (transaction over uncertain things), *maysir* (business activities where gains are obtained from mere chance), and *riba* (the practice of lending money with interest rate). Even though it is branded as Islamic insurance, its target market is not necessarily faith-based since Takaful has the attributes to appeal to a wider customer base including non-Muslims. For instance, its members have opportunities to participate in investment activities associated with the Takaful insurance scheme they are involved in. Another advantage is that the Takaful participants are entitled to the underwriting surplus generated from low number or amount of claim payments, hence Takaful concept can help reduce the level of moral hazard in insurance business.

2.1 Pooling System

Gharar refers to any transaction of probable items whose existence or characterization are not certain, due to insufficient information and knowledge on the outcome of the contract or the nature and quality of the object [32]. Risk trading in conventional insurance scheme is a clear example of *gharar*, since the risks are subject to uncertainty on future claim payments from the insurer. Conventional insurance is designed around the transfer of risk from an insured to an insurer in return for premium payments, where the frequency, timing, and severity of the claims are each subject to varying degree of uncertainty. Sharia forbids the involvement of *gharar* in conventional insurance transaction in order to protects both participants and insurance companies from hazardous or unjust transactions. Consequently, to overcome this issue Takaful insurance prefers the concept of risk sharing rather than risk transfer, where Takaful participants share their risk by making regular monetary contributions into a pooling system. The contributions made by the participants are considered as donations and are utilized to reimburse predefined losses or damages realized by the participant. Therefore the contractual basis in Takaful insurance scheme is not exchange of premium for claim coverage, but donation of money to help the participants who are in need of help. The Takaful operator then manage the risk pool, including its underwriting, administrative, and investment activities in return for some compensation from the participants and/or the Takaful fund. In other words, the Takaful operator does not overtake the participants' risk, but only manage them in return for some compensation, whose amount depends on the business model being practiced. For this reason, there are two funds associated with a Takaful business: the Takaful fund which is the collected premium/contribution, and the fund owned by the Takaful operator consisting the initial capital provided by shareholders and the compensation earned over time from managing the business.

2.2 Profit Sharing

The term *maysir* stands for transaction with an underlying gambling or speculative nature where one party benefits from other parties by mere chance. Conventional insurance scheme involves *maysir* in a sense that the insured are considered to be betting premiums on the condition that the insurer has to pay an amount of money in case of occurrence of specified events. In the context of life insurance, many contracts are designed to ultimately benefit one side of the contract at the expense of the other. For instance, the payment of a small sum of premium by an insured can yield a disproportionately large payout benefiting the insured at the expense of the insurer. Alternatively, the payment stream of premiums for many years can result in no return at all, which benefit the insurer at the cost of the insured. Since Takaful insurance employs a pooling system, the Takaful operator does not own the collected contribution of its participants. Therefore in case of loss, the Takaful operator does not lose anything since the claims are paid from the Takaful fund. On the other hand if there is no loss, the operator does not directly gain anything either since the premium/contribution was paid to the Takaful fund which then can be invested in Sharia compliant instruments. Furthermore, any operating profit generated in each term whether it is investment gain or underwriting surplus is shared among the participants and the operator/shareholders in a predetermined ratio depending on the business model being practiced, hence Takaful insurance can benefit both parties at the same time. In the case of deficit in the Takaful fund, the operator is expected to provide an interest free loan to the fund which can be repaid from future generated profit. Regarding the existence of *maysir* in a conventional insurance contract, some jurists however disagree on the perception and assume the financial motivation is the desire for protection against loss rather then achieving gain from speculation [5].

2.3 Sharia Board

In addition to *gharar* and *maysir*, another element that is commonly found in most capital markets is *riba* or usury, which means charging interest on loans. *Riba* is considered to be unfair and inequitable to the borrowing party and therefore earning interest from lending money is considered unethical. Some insurance companies carry on their business by issuing bonds which is an interest-based financial activity, especially in a long term business, such as life insurance. Contrary to most beliefs, Sharia recognizes the concept of time value of money since the value of almost all commodities increase with time. However, Sharia does not treat money as a commodity but only as a medium of exchange, and therefore charging interest on loan or issuing bond is forbidden. As a result, a Takaful operator employs a Sharia Board, which is an independent body of specialized jurist in Islamic finance that is responsible in monitoring the activities of the operator. The Sharia Board ensures that there is no involvement of *riba* and that all investment to be made in Sharia compliant instruments. In addition to investment activity supervision, the role of the Sharia Board also includes approval of Takaful products, undertaking Sharia audit to ensure products comply with the regulation, and provide guidance to the Takaful operator on its wider social role[16]. For instance, the profit sharing ratio for the Takaful operator and the participants and the amount of fee for the operator must be approved by the Sharia Board. Nevertheless, in practice the duties of Sharia Boards can vary significantly from one country to another.

2.4 Takaful Branches

As is the case with conventional insurance, Takaful is separated into two branches: general Takaful (non-life Takaful) and family Takaful (life Takaful). General Takaful covers losses accruing from property damage or personal accidents. The entire contribution of each participant of general Takaful insurance is treated as donation for protection purposes only and it is paid to the Takaful fund. On the other hand, family Takaful pays benefit to its participant's beneficiary upon the death of the participant. Additionally, family Takaful encourages its participants to save regularly and allows them to have separate account solely for investment activities. In other words, the participants' contributions have savings or investment component in addition to risk protection part. The saving and investment element are not transferred to the risk pool, hence the ownership remains with each participant individually. Since the investment part for generating returns is usually much larger than the protection part, contributions in family Takaful can be looked upon more as an investment than a donation. Consequently, the Takaful operator in family Takaful is especially in charge of the profitable investment of the Takaful fund on behalf of the participants. Besides life insurance plans, family Takaful offers as well medical and health, accident, and education coverage plans. In the following section we describe some of the most common business models of Takaful insurance by considering only the risk protection part. Hence, the following business models can be regarded as the business model utilized in general Takaful insurance.

2.5 Takaful Business Models

By employing a pooling system, the total donations/contributions collected from the participants constitute a separate fund within the fund managed by a Takaful operator. Consequently there are two separate fund inside a Takaful business, the Takaful fund consisting of the contributions and the operator fund owned by the Takaful operator. All operating expenses, including claim payment, underwriting cost, reinsurance treaty, and reserve management are paid from the Takaful fund. Takaful version of reinsurance is called Retakaful, and in general Takaful operators reinsure their business to Retakaful operators. In practice Takaful insurance have several business models, which define the relation between the two funds and the way the Takaful operator get compensated for managing a Takaful business. For our study we focus on five most common models: mudarabah model (mud), modified mudarabah model (mdf), wakalah model (wak), hybrid model (hyb), and waqf model (wqf). To avoid confusion, we need to mention that there is no consistent terminology in the literature with respect to the name of the Takaful business models and it depends on the interpretation of the author.

For each model we provide simple expressions to better understand the shares of investment gain and underwriting surplus that are entitled to the participants and the operator. For simplicity of the expressions we consider a Takaful contract in a single term, which means each participant has made only one contribution. Define S to be the surplus or the level of the Takaful fund at the end of the term before distributing the share of the participants and the operator whose function is given in 2.1, where Q is an initial donation of capital from the shareholders (different from the initial capital in the operator fund), C is the total amount of contributions made by the participants, G is the investment gain generated from investing a fraction of the total contribution, and E is the operating expenses. The initial donation Q is only applicable for waqf model, which means Q is equal to 0 for mudarabah, modified mudarabah, wakalah, and hybrid models. Let also define the compensation of the Takaful operator/shareholders in mudarabah, modified mudarabah, wakalah, hybrid, and waqf models to be O_{mud}, O_{mdf}, O_{wak}, O_{hyb}, and O_{wqf} respectively. The corresponding amount of dividend paid to participants are P_{mud}, P_{mdf}, P_{wak}, P_{hyb}, and P_{wqf} respectively.

$$S = Q + C + G - E, \tag{2.1}$$

Note that the initial capital is not included in Equation 2.1 because it is not part of the Takaful fund. However if the surplus at the end of the term experience a deficit due to negative return from investment activities or insufficient capital to cover claims, then the Takaful operator is expected to give an interest free loan to cover the deficit. In that case the loan can be provided from the initial capital which can be repaid from future generated surplus. Therefore we assume that $G \geq 0$ and $(C - E) \geq 0$. The loan requires careful risk management techniques as there is no exact amount and timing of the loan to be repaid. The interest free loan (referred to as *qard hasan* in Takaful literature) is required by financial regulation in most countries to indicate that the interest of participants are protected in case of deficit in the Takaful fund.

2.5.1 Mudarabah Model

Mudarabah model (pure mudarabah model) is structured on the principle of profit sharing and partnership between a Takaful operator who acts as an entrepreneur and a group of participants as capital providers for a Takaful fund. A fraction of the fund is invested in Sharia compliant instruments whose return is shared between the two parties in a predetermined ratio approved by a Sharia Board. A portion $(x \in (0,1))$ of the investment gain is used to compensate the Takaful operator for managing the business, while the rest $(1-x)$ of the investment gain is shared among the participants. In pure mudarabah model, the whole underwriting surplus, which is the total contribution minus the operating expenses is shared back to the participants.

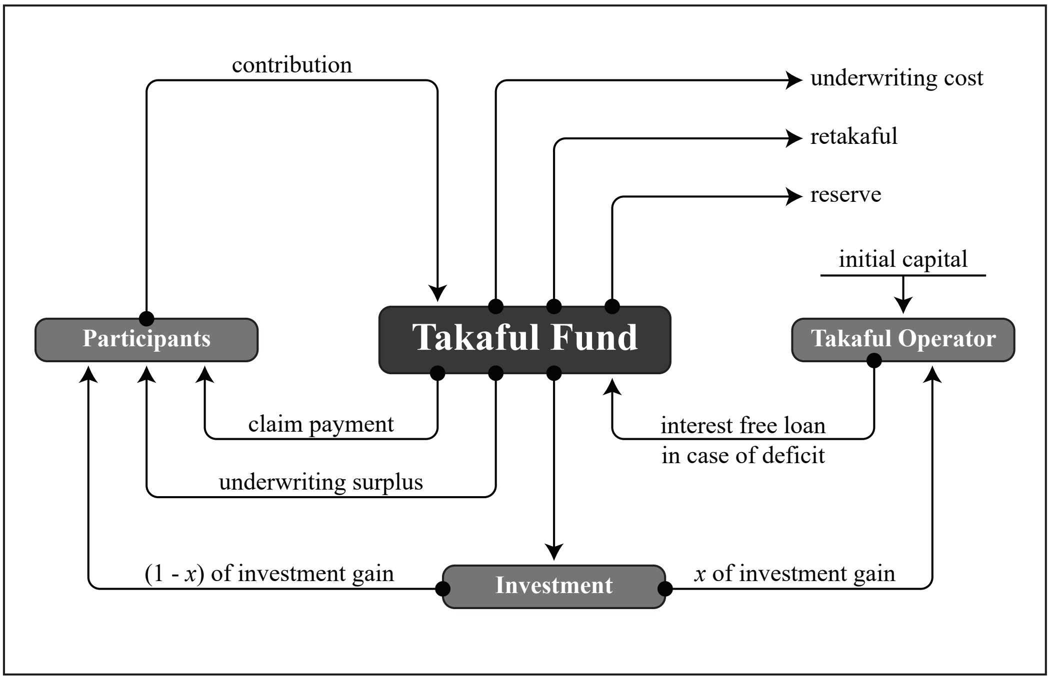


Figure 2.1: The framework of mudarabah model.

Under this Takaful business model, the compensation of the Takaful operator is given as:

$$O_{mud} = xG, \tag{2.2}$$

while the share of the participants is expressed as follows:

$$P_{mud} = (1-x)G + C - E. \tag{2.3}$$

Although positive gain/return is shared, losses from the investment activities are solely borne by the participants as the capital providers. On the other hand, The Takaful operator is expected to provide an interest free loan up to a certain level when the Takaful fund is not enough to meet its obligation to cover claim payments.

2.5.2 Modified Mudarabah Model

Under modified mudarabah model, the relation between a Takaful operator, a Takaful fund, and participants is similar to pure mudarabah model. The only difference is that in addition of some shares $(x \in (0,1))$ of the investment gain, modified mudarabah model allows the Takaful operator to take some shares $(y \in (0,1))$ of the underwriting surplus as an incentive for managing the Takaful business, specifically its underwriting activities. By offering a more decent income for the operator, modified mudarabah model allows the operator to withstand competition with conventional insurers. The modified version also intends to maximize actuarial cost method of the Takaful operator in order to collect optimal amount of contributions from the participants (avoiding overpricing and underpricing). However there are severe objections by some jurists in the acquisition of underwriting surplus by the operator since the concept is not in line with original mudharabah contract.

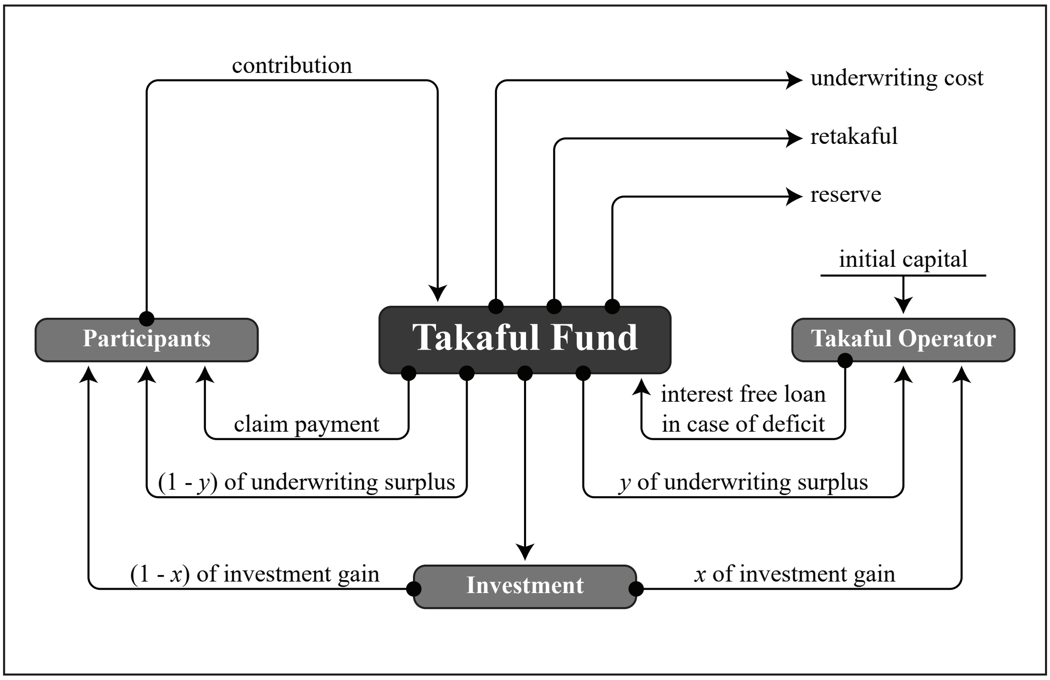


Figure 2.2: The framework of modified mudarabah model.

The compensation of the operator in the modified case is expressed as:

$$O_{mdf} = xG + y(C - E), \tag{2.4}$$

and the corresponding share of the participants is:

$$P_{mdf} = (1 - x)G + (1 - y)(C - E). \tag{2.5}$$

2.5.3 Wakalah Model

Wakalah model is grounded on the principal-agent arrangement, where a Takaful operator acts as an agent of a group of participants, tasked with the administration of a Takaful fund in return for a fixed fee payment. In other words, the operator does not share in the investment gain nor the underwriting surplus but instead receives a fixed up-front fee to cover its management expenses and operational income. Theoretically, under wakalah model the Takaful operator bears no insurance risk, however due to the obligation to provide an interest free loan in case of deficit in the Takaful fund, the operator indeed faces the risk of incapability in recapturing its initial capital if insufficient surplus is generated over time.

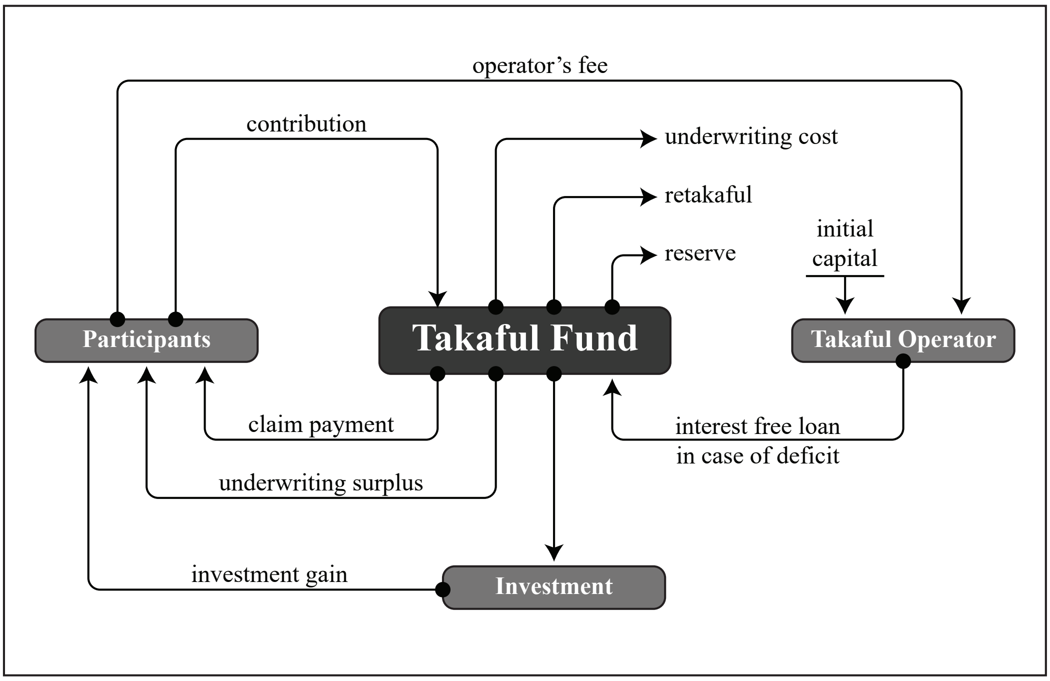


Figure 2.3: The framework of wakalah model.

Generally the fee is a percentage of the total contribution, thus the whole contribution of each participants does not go the Takaful fund. However in order to be in line with Equation 2.1, instead we set that the participants has to provide an additional payment for the Takaful operator's compensation/fee. Therefore the total contributions made by the participants is $C + \acute{C}$, where $\acute{C}$ is the operator's fee and C is the portion transferred to the Takaful fund. In other words, the equations of the operator's compensation and the share owned by the participants are respectively given as follows:

$$O_{wak} = \acute{C}, \tag{2.6}$$

$$P_{wak} = S = C + G - E. \tag{2.7}$$

2.5.4 Hybrid Model

Hybrid model is the combination of mudarabah model and wakalah model. Under this model, a Takaful operator receives a fixed up-front fee $(\acute{C})$ from participants and also gets a share $(x \in (0, 1))$ of the generated investment gain. This business model can be implemented in order to maximize the investment performance of the Takaful operator in wakalah model by granting it some incentives from the investment gain. In terms of underwriting surplus, the operator has no share of it under this model. The hybrid model is getting popular because it avoids the Sharia conflict pertaining to the acquisition of some shares of the underwriting surplus by the operator, but still enables the operator to yield equivalent commercial results to conventional insurers' operation.

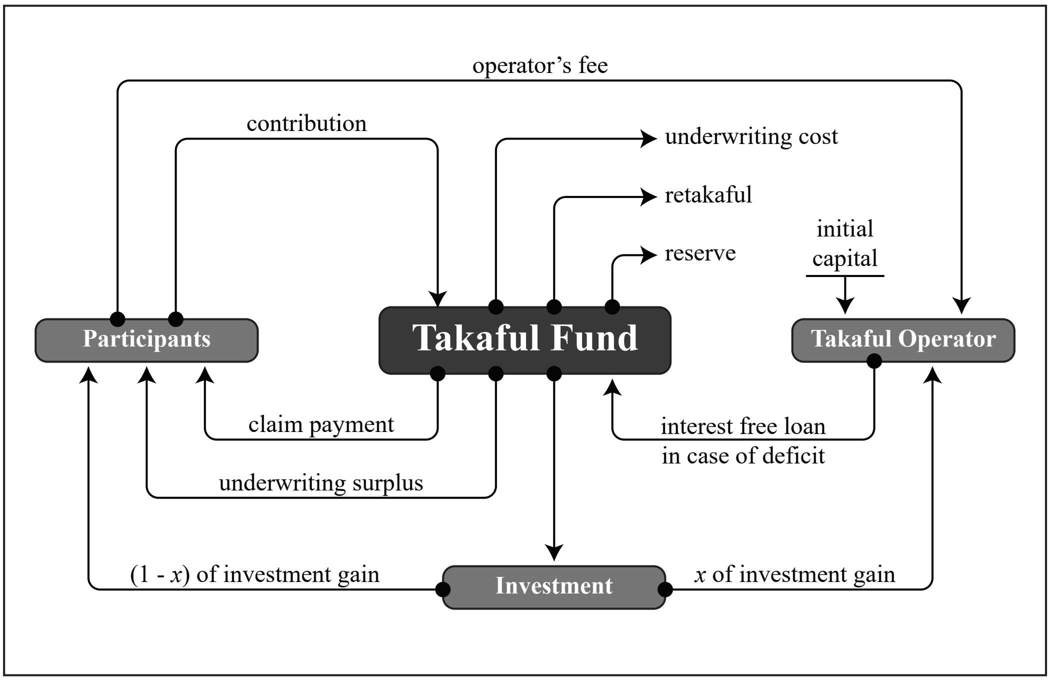


Figure 2.4: The framework of hybrid model.

The compensation of the operator from the hybrid model is given as follows:

$$O_{hyb} = xG + \acute{C}, \tag{2.8}$$

whereas the share of the participant is expressed as:

$$P_{hyb} = (1 - x)G + C - E. \tag{2.9}$$

Similar to the other business models, an interest free loan from the operator is expected when the Takaful fund experiences deficit.

2.5.5 Waqf Model

Under waqf model, a Takaful operator/shareholders establish a separate legal entity referred to as waqf fund and inject an initial donation of capital to the waqf fund in order to extend its financial assistance to a group of participants. A waqf fund is be defined as an inalienable endowment, that can be used for charitable purpose, with no intention of reclaiming the fund. Therefore, the shareholders must relinquish their ownership of the initial waqf donation under this business model. The participants then make their contribution to the waqf fund, which functions as a Takaful fund in a similar way to the previous business models, but as a separate body that belongs to neither the Takaful operator nor the participants. Therefore the return from investment activities is not shared to the participants, but shared between the waqf fund and the Takaful operator as the fund manager. Coupled with the investment gain, the operator also receive some management fees from the participants, while the whole underwriting surplus is entitled to the participants. The initial donation of capital by the shareholders remains at the Takaful fund.

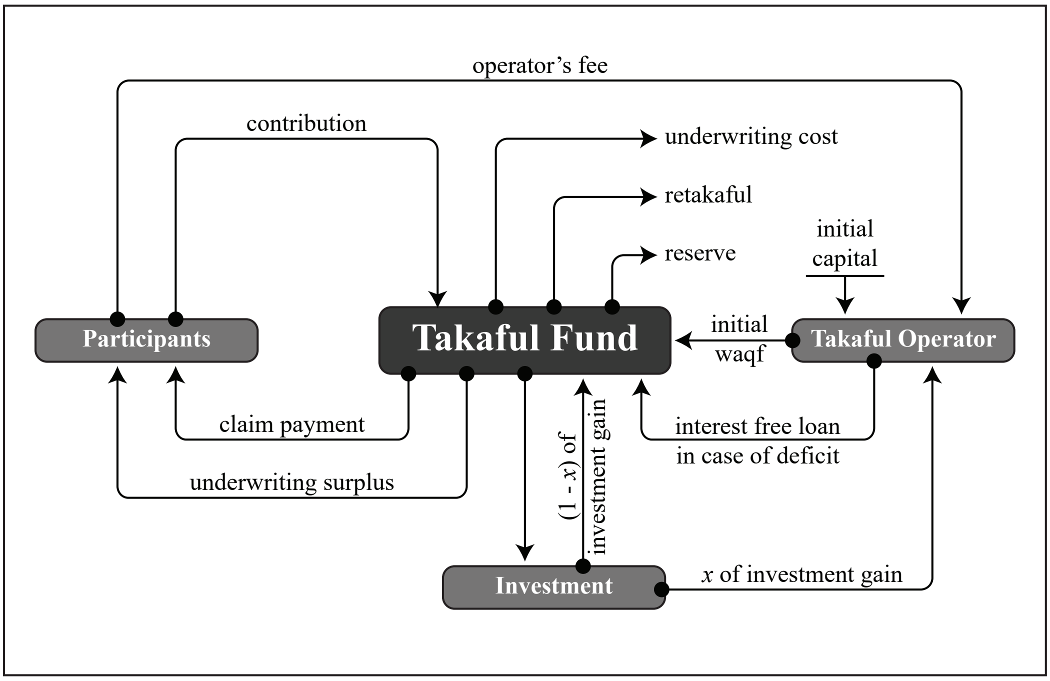


Figure 2.5: The framework of waqf model.

The corresponding compensation for the operators and the share owned by the participants are respectively given as follows:

$$O_{wqf} = xG + \acute{C}, \tag{2.10}$$

$$P_{wqf} = C - E. \tag{2.11}$$

The five Takaful business models described above are practiced for general Takaful contracts. The main feature that distinguish family Takaful contracts from general case is the incorporation of saving and investment fund, where the participants contribute additional fund to be invested by the operator in order to aid future huge benefit payment. For in-depth description on the conceptual basis of the Takaful business models for both general Takaful and family Takaful, we refer to Bisani [9]. Notice that from the Takaful fund's perspective the first four business models (mudarabah, modified mudarabah, wakalah, and hybrid models) are practically the same due to the fact that both investment gain and underwriting surplus are distributed from the fund as a whole. Moreover, we have set the operator's fee not to be extracted from the participants contribution, but instead as an addition payment so that the amount of contribution received by the Takaful fund remain the same for all four models. Therefore, in the following chapters we refer the first four business models as non-waqf model since it does not incorporate initial waqf fund injection unlike waqf model.

The choice of the business models depends on many factors, including target population, regional acceptance, Shariah Board views, regulatory framework, product design, pricing, and marketing [18]. Mudarabah model was first introduced in the Malaysian market in 1984, and is now mainly practiced in the Asia-Pacific region. Nevertheless, the model is less accepted globally because the application of its pure version in short term contracts has low profitability, while its modified version receives strong criticism from many scholars and mostly rejected by the GCC countries. The development of wakalah model started in 1979 in Sudan and is now popular in the Middle East due to its transparency and fixed nature of operator's compensation, hence it provides leverage for the operator to act in the best interests of the participants. However, wakalah model has its limitation since the only source of income for the operator is the wakalah fee, and hence an operator employing this business model may charge high up-front fee, which might look unattractive to the participants. On the other hand, there has been an increasing trend toward the hybrid model which is widely practiced in the Middle East and Malaysia since to some extent it overcome the limitations of the previous business models. Waqf model is commonly practiced by Takaful operators in Pakistan and South Africa. In the literature, there are contradicting opinions concerning this business model. Whilst some researches indicate waqf model as the best business model for Takaful insurance, it is approached with scepticism by some jurists. There is no doubt that the incorporation of a waqf structure in Takaful raises more issues in the governance principles, in which legal or tax aspects can also play an important role toward the initial capital injection of the shareholders.

Although each business model implies distinct right and responsibility between the participants and the Takaful operator, all models pursue the same objective of sharing individual risk collectively, for the purpose of decreasing the level of risk exposure of each individual participants against specific losses. The Takaful industry shows that each business models has its advantages and disadvantages. Consequently, there have been many variations of Takaful business models developed by the practitioners in order to address the limitations. Therefore, it is essential that all Takaful operators have a unified approach toward the development of a uniform model in order to achieve consumer confidence. Uniformity in Takaful insurance business model can help withstand competition with conventional insurance in order to ensure sustainable growth of the global Takaful industry. At national levels, a trend toward standardization can already be noted, whereas the global standard remains a challenge. Actuarial principles and practices in conventional insurance context including enterprise risk management, asset liability management, embedded value calculation, surplus determination, and distribution methodologies have direct application to Takaful insurance [18]. As a result, actuarial expertise and knowledge in the management of mutual insurance is essential to standardize the Takaful practice around the world and in general to enhance Takaful insurance's success.

CHAPTER 3

TAKAFUL RISK MODEL

In this chapter we introduce a new risk model portraying Takaful insurance system, which is obtained by constructing the surplus process of a Takaful fund. To begin with, we recall the Cramér-Lundberg model or commonly known as the classical risk process, which is the theoretical foundation of ruin theory. Its extended version called Sparre Andersen (SA) model is expressed in Equation 3.1, which gives the financial position or the surplus level of an insurer at time t denoted by $U_t^{(SA)}$ who started its business with initial capital v. Over time the insurer experiences two opposing cash flows: incoming regular premium from its costumers with constant rate $c > 0$ and outgoing random claim payment with claim number and claim size distribution given as N_t and X_i respectively. The extended model, also known as the renewal risk model defines $\{N_t\}_{t\geq 0}$ to be a renewal process and $\{X_i\}_{i\in\mathbb{Z}^+}$ to be independent and identically distributed (iid) random variables. Furthermore, the model assumes that $X_i > 0$ almost surely and that $\{N_t\}_{t\geq 0}$ and $\{X_i\}_{i\in\mathbb{Z}^+}$ are independent.

$$U_t^{(SA)} = v + ct - \sum_{i=1}^{N_t} X_i, \qquad t \geq 0. \tag{3.1}$$

The Sparre Andersen model however has limitations in terms of application to the surplus process of a Takaful fund because of several reasons. Firstly, the model does not incorporate investment activities, which is crucial for Takaful insurance since the operational income of the Takaful operator/shareholders comes from the investment return in most business models. In fact, Wakalah model is the only business model described in Chapter 2 in which the operator is not entitled to any share of the investment return. Secondly, a Takaful operator is required to distribute dividend payments to its participants when the Takaful fund generates enough surplus, hence this characteristic need to be considered in modeling the Takaful fund. Lastly, the Sparre Andersen model does not have loan undertaking component, whereas the Takaful operator is expected to give interest free loan to the Takaful fund in case of deficit. Consequently, some adjustments in the Sparre Andersen model are required to make it suitable for modeling the surplus process of a Takaful fund.

3.1 Kim-Drekic (KD) Risk Model

In order to construct a risk model of a Takaful fund, we consider a discrete-time dependent Sparre Andersen model with external financial activities recently introduced in the literature by Kim and Drekic [20]. In addition to $U_t^{(KD)}$, which is the amount of surplus at time t, the authors introduce a separate fund $F_t^{(KD)}$ called external fund, a monetary account an insurer has to better manage its reserve through investment activities and loan undertakings. Both the surplus process $(U_t^{(KD)})$ and the external fund $(F_t^{(KD)})$ are measured in discrete time (i.e., $t \in \mathbb{N}$), and the surplus process is measured in discrete monetary units (i.e., $U_t^{(KD)} \in \mathbb{Z}$). The external financial activities require the risk model to incorporate four threshold levels: a minimum acceptable surplus level (l_1), a trigger point for investment activities (l_2), a trigger point for random dividend payment to the shareholders (l_3), and a borrowing limit of the insurer (β). The threshold levels l_1, l_2, and l_3 belong to the surplus process, while β belongs to the external fund, and they satisfy the condition $\beta \leq 0 \leq l_1 \leq l_2 \leq l_3$.

The relation between the surplus process and the external fund is expressed through deposit and withdrawal activities. Deposit refers to cash outflow from the surplus process to the external fund and is categorized into two types: left limit deposit $(D_{1,t})$ and right limit deposit $(D_{2,t})$. Left limit deposit is made at the beginning of each term when there is enough fund in the surplus process to be invested, whereas right limit deposit is made at the end of each term when the external fund experiences huge deficit due to interest rate accumulation from loan undertakings. On the other hand, withdrawal (W_t) refers to cash inflow from the external fund to the surplus process, which is made at the end of each term when the surplus level goes below a certain level given that the external fund has sufficient level to cover the deficit in the surplus process. In addition, random dividend payment (V_t) to shareholders is made at the beginning of each term when considerable surplus is generated after claims from the previous term are paid. Moreover, it is assumed that for any term, the premiums are received at the beginning of the term and any claims are applied at the end of the term.

$U_t^{(KD)}$ represents the surplus level at the end of the interval $(t-1, t]$, at which point any premium, claim, deposit, withdrawal, and dividend payment corresponding to this time interval has been received/paid out. Similarly, $F_t^{(KD)}$ represents the external fund level at the end of the interval $(t-1, t]$, at which point any deposit, withdrawal, loan undertaking, and investment activity corresponding to the time interval has been made. It is also assumed that the insurer has to pay out all the outstanding debt before resuming its investment activities, and that the insurer first make use of its investment assets to make any adjustment to its surplus level before engaging in loan activities.

To aid in the understanding of KD risk model, let t^- represents the time point at the end of the time interval $(t-1, t]$ immediately after claim is paid but before any withdrawal or right limit deposit is made. In addition, let t^+ represents the time point at the beginning of the time interval $(t, t+1]$ immediately after premium, dividend, and left limit deposit associated with the interval $(t, t+1]$ are paid/received.

Beginning at time 0 with initial surplus level of $v \in \{l_1, l_1+1, l_1+2, ...\}$ and initial external fund amount of $g \in \mathbb{N}$, the insurer's amount of surplus at time t according to KD risk model is expressed in the following equation.

$$U_t^{(KD)} = v + \sum_{i=0}^{t-1} P_i - \sum_{i=0}^{t-1} D_{1,i} + \sum_{i=1}^{t} W_i - \sum_{i=1}^{t} D_{2,i} - \sum_{i=1}^{N_t} X_i, \quad t \in \mathbb{N}. \tag{3.2}$$

P_t is the premium amount received each term at a constant rate of $c \in \mathbb{Z}^+$ minus the random dividend payment denoted by V_t with probability mass function (pmf) $b_i = Pr\{V_t = i\}$, where $i = c_1, c_1+1, ..., c_2$, for some c_1 and c_2 satisfying the condition $d \leq c_1 \leq c_2 \leq c$. Also, $D_{1,t}$ is the left limit deposit, whose amount is either 0 or a constant $d \in \mathbb{Z}^+$ (given that $d \leq c$), depending on the level of $U_t^{(KD)}$. Further, W_t and $D_{2,t}$ respectively represents the withdrawal and the right limit deposit activities whose amounts depend on the level of $U_{t^-}^{(KD)}$. Here N_t is the claim frequency distribution and X_i is the claim severity distribution. It is assumed that N_t is a discrete-time renewal process with independent, positive, integer-valued inter-claim times $\{Y_i, i \in \mathbb{Z}^+\}$, where Y_i is the time between the $(i-1)$-th and i-th claims (with the understanding that the 0-th claim occurs at time 0).

$$P_t = \begin{cases} c & \text{if } U_t^{(KD)} < l_3, \\ c - V_t & \text{if } U_t^{(KD)} \geq l_3. \end{cases} \tag{3.3}$$

$$D_{1,t} = \begin{cases} 0 & \text{if } U_t^{(KD)} < l_2, \\ d & \text{if } U_t^{(KD)} \geq l_2. \end{cases} \tag{3.4}$$

$$W_t = \begin{cases} 0 & \text{if } U_{t^-}^{(KD)} \geq l_1, \\ \min\{l_1 - U_{t^-}^{(KD)}, \max\{0, F_{t^-}^{(KD)} - \beta\}\} & \text{if } U_{t^-}^{(KD)} < l_1. \end{cases} \tag{3.5}$$

$$D_{2,t} = \max\{0, \beta - F_{t^-}^{(KD)}\}. \tag{3.6}$$

The evolution of the external fund can be expressed recursively with respect to an investment return rate which is assumed to be constant $\kappa_1 > 0$ and an interest rate $\kappa_2 > 0$.

$$F_{t^-}^{(KD)} = \begin{cases} F_{(t-1)^+}^{(KD)}(1+\kappa_1) & \text{if } F_{(t-1)^+}^{(KD)} \geq 0, \\ \\ F_{(t-1)^+}^{(KD)}(1+\kappa_2) & \text{if } \beta \leq F_{(t-1)^+}^{(KD)} < 0, \end{cases} \qquad t \in \mathbb{Z}^+. \tag{3.7}$$

3.2 Modified KD Risk Model

The risk model proposed by Kim and Drekic[20] can be used as a reference in modeling a risk model for a Takaful fund because it incorporates investment activities, loan undertakings, and dividend payments which are essential components in Takaful system. Accordingly, in addition to the surplus process U_t, we introduce F_t as a separate monetary account of the Takaful fund which is used by the Takaful operator to better manage the Takaful fund's reserve through investment activities and loan undertakings. Both the surplus process (U_t) and the external fund (F_t) are measured in discrete time (i.e., $t \in \mathbb{N}$), and the surplus process is measured in discrete monetary units (i.e., $U_t \in \mathbb{Z}$). However, some modifications are required to make the model more suitable for an application to the Takaful fund. First of all, in KD risk model the only source of dividends is the surplus process, where the dividends bear a resemblance to underwriting surplus since they represent the excess of fund in the surplus process generated from low total size of claims. In order to implement the definition of underwriting surplus more appropriately, we define the amount of dividend payments from underwriting surplus to be the excess of fund from l_3, instead of using any random variable which is the case with KD risk model. Secondly, in addition to underwriting surplus, under some circumstances investment gain in Takaful insurance is also distributed as dividends to the participants and/or the shareholders. Therefore an additional threshold point (l_5) is introduced in our model, which functions as a trigger point for dividend payment from investment gain. Lastly, the loan undertakings in Takaful insurance bear no interest rate since the loans are from the Takaful operator/shareholders who provide initial capital and receive compensations/dividends over time. Therefore, the utilization of β as the maximum limit of loan undertakings with no interest rate eliminates the use of Equation 3.6 since the level of the external fund would never be below l_4.

U_t represents the surplus level at the end of the interval $(t-1, t]$, at which point any premium/contribution, claim, deposit, withdrawal, or dividend payment from underwriting surplus corresponding to this time interval has been received/paid out. Similarly, F_t represents the external fund level at the end of the interval $(t-1, t]$, at which point any deposit, withdrawal, loan undertaking, investment activity, or dividend payment from investment gain corresponding to the time interval has been made. Here deposit (D_t) refers to cash outflow from the surplus process to the external fund which is made at the beginning of each term when there is enough fund in the surplus process to be invested. Whilst withdrawal (W_t) refers to cash inflow from the external fund to the surplus process which is made at the end of each term when the surplus level goes below a certain level given that the external fund has sufficient level to cover the deficit in the surplus process. Similar assumptions from KD risk model are applied to our proposed risk model for Takaful insurance. That is to say, we assume that for any term, the contributions are received at the beginning of the term and any claims are applied at the end of the term. Moreover, the dividend payment from underwriting surplus $(R_{1,t})$ is made at the beginning of each term when considerable surplus is generated after claims from the previous term are paid. Whereas the dividend payment from investment gain $(R_{2,t})$ is made at the end of each term when the level of the external fund at the beginning of the term is above a certain level. We also assume that the Takaful fund has to pay out all the outstanding debt before resuming its investment activities, and that the Takaful fund first make use of its investment assets to make any adjustment to its surplus level before engaging in loan activities. To aid in the understanding of our risk model, let t^- represents the time point at the end of the time interval $(t-1, t]$ immediately after claim and dividend from investment gain are paid but before any withdrawal is made. In addition, let t^+ represents the time point at the beginning of the time interval $(t, t+1]$ after contribution, dividend from underwriting surplus, and deposit associated with the interval $(t, t+1]$ are paid/received. The time line of cash inflows and outflows experienced by the surplus process and the external fund is given in the following figure, where only activities before time point t are accumulated for U_t and F_t.

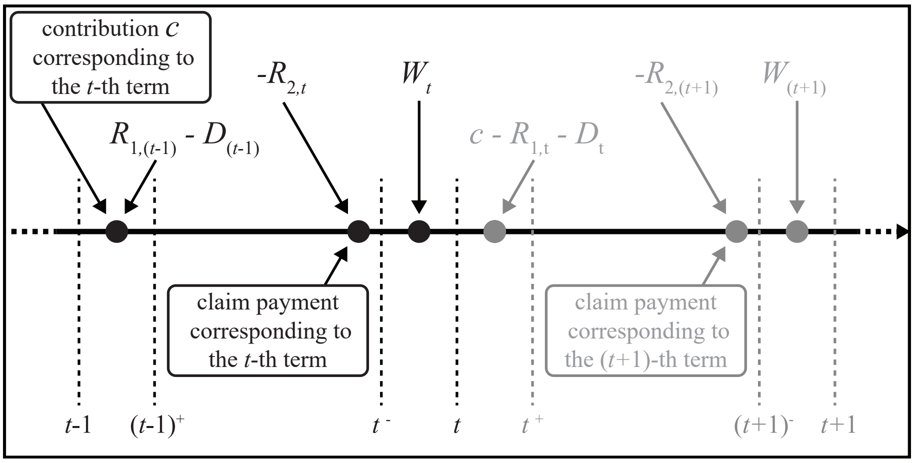


Figure 3.1: The flowchart of all components in U_t and F_t.

The modifications of KD risk model for an application to the surplus process of a Takaful fund results in employment of five threshold levels: l_1, l_2, l_3, l_4, and l_5 which are defined to be integers. The first three threshold levels belong to the surplus process of the Takaful fund and their function are the same as the corresponding threshold levels in KD risk model with an exception of the relation between dividend payments from underwriting surplus and l_3. In other words, l_1 is the minimum acceptable surplus level, l_2 is the trigger point for investment activities, and l_3 is the trigger point for dividend payments from underwriting surplus, which satisfy the condition $0 \leq l_1 \leq l_2 \leq l_3$. On the other hand, l_4 and l_5 belong to the external fund, where l_4 is the maximum limit of loan undertaking (denoted as β in KD risk model), while l_5 is the trigger point for dividend payments from investment gain. Furthermore, they satisfy the condition $l_4 \leq 0 \leq l_5$. Beginning at time 0 with an initial surplus level of $v \in \{l_1, l_1{+}1, l_1{+}2, ...\}$ and initial external fund amount of $g \in \mathbb{N}$, the Takaful fund's amount of surplus at time t is expressible as follow:

$$U_t = v + ct - \sum_{i=0}^{t-1} D_i - \sum_{i=0}^{t-1} R_{1,i} + \sum_{i=1}^{t} W_i - \sum_{i=1}^{N_t} X_i, \quad t \in \mathbb{N}. \tag{3.8}$$

The initial capital provided by the shareholders is not part of the Takaful fund, but it is kept by the operator in case the fund experience deficit and require an interest free loan. We note that both v and g are equal to 0 for non-waqf model, whereas at least one of v and g is higher than 0 for waqf model with $v+g$ being the initial waqf capital provided by the shareholders. The premium/contribution rate is denoted by a constant $c \in \mathbb{Z}^+$ whose amount is the same for all Takaful business models. D_t is the deposit amount whose definition is exactly the same as the left limit deposit $(D_{1,t})$ in KD risk model, whose value is either 0 or a constant $d \in \mathbb{Z}^+$ (where $d \leq c$), depending on the level of U_t. Further, $R_{1,t}$ represents the amount of dividend payments from underwriting surplus whose value is the excess of surplus from l_3.

$$D_t = \begin{cases} 0 & \text{if } U_t < l_2, \\ d & \text{if } U_t \geq l_2. \end{cases} \tag{3.9}$$

$$R_{1,t} = \begin{cases} 0 & \text{if } U_t < l_3, \\ U_t - l_3 & \text{if } U_t \geq l_3. \end{cases} \tag{3.10}$$

W_t stands for the withdrawal amount whose definition is also the same as the one in KD risk model. The surplus process has to be kept at the minimum acceptable surplus level (l_1), hence when its level drops below l_1, some capital is withdrawn from the external fund to make the surplus level stay at l_1 even if the operator has to provide an interest free loan. However, the loan undertakings must not exceed the maximum loan amount threshold level (l_4), hence in some cases the withdrawal amount might not be enough to maintain the surplus level at l_1. Nevertheless, as long as the withdrawal amount is enough to keep the surplus process at some non-negative level, then the process continues. The loan from the operator then can be covered in the following terms where the surplus process reaches l_2 by depositing some capital to the external fund.

$$W_t = \begin{cases} 0 & \text{if } U_{t^-} \geq l_1, \\ \min\{l_1 - U_{t^-}, \max\{0, F_{t^-} - l_4\}\} & \text{if } U_{t^-} < l_1. \end{cases} \tag{3.11}$$

N_t is the primary distribution or the claim frequency distribution which represents the distribution of the number claims occurred by time t. Whereas X_i is the secondary distribution or the claim severity distribution which stands for the distribution of the size of an individual claim, where $\{X_i, i \in \mathbb{Z}^+\}$ are assumed to form an iid sequence of positive, integer-valued random variables. The number of claims process is assumed to be a discrete time renewal process with independent, positive, integer-valued inter-claim times $\{Y_i, i \in \mathbb{Z}^+\}$, where Y_i is the time between the $(i-1)$-th and i-th claims (with the understanding that the 0-th claim occurs at time 0). In particular, $(Y_i, i \in \mathbb{Z}^+)$ forms an iid sequence of positive random variables with common pmf a_k.

$$a_k = Pr\{Y_i = k\}, \quad k = 1, 2, ..., n_a, \tag{3.12}$$

where $n_a \in \mathbb{Z}^+$ represents the upper bound for the inter-claim times, i.e., for $t \in \mathbb{Z}^+$, there is at least one claim occurrence in any time interval $(t, t + n_a]$. Furthermore, the pairs $\{(Y_i, X_i), i \in \mathbb{Z}^+\}$ is assumed to be iid, so that the joint pmf of (Y_i, X_i) can be denoted by $a_k\alpha_j(k)$, where $\alpha_j(k)$ stands for the conditional pmf of X_i given Y_i. Such an organization allows for possible dependence between inter-claim times and claim sizes in Sparre Andersen models [12].

$$Pr\{Y_i = k, X_i = j\} = a_k\alpha_j(k), \tag{3.13}$$

$$\alpha_j(k) = Pr\{X_i = j | Y_i = k\}. \tag{3.14}$$

The evolution of the external fund is given recursively with respect to an investment return rate which is assumed to be at a constant $\kappa > 0$ when its level is greater than or equal to 0. If necessary the external fund can borrow some capital from the operator as much as possible without bringing its level below l_4. When the external fund reaches l_5 some of the investment gain generated in the following term is distributed as dividend whose amount is denoted by $R_{2,t}$. For waqf model we adopt the convention that $l_5 = 0$, because unlike the other business models, the investment gain in waqf model is shared between the shareholders $(x \in (0, 1))$ and the Takaful fund itself $(1 - x)$. Therefore, even if the level of the external fund is above l_5, a portion of the investment gain still remains in the fund.

$$F_{t^-} = \begin{cases} F_{(t-1)^+}(1+\kappa) - R_{2,t} & \text{if } F_{(t-1)^+} \geq 0, \\ F_{(t-1)^+} & \text{if } l_4 \leq F_{(t-1)^+} < 0, \end{cases} \qquad t \in \mathbb{Z}^+, \tag{3.15}$$

$$R_{2,t} = \begin{cases} F_{(t-1)^+}(\pi) & \text{if } F_{(t-1)^+} \geq l_5, \\ 0 & \text{if } F_{(t-1)^+} < l_5, \end{cases} \qquad t \in \mathbb{Z}^+, \tag{3.16}$$

$$\pi = \begin{cases} \kappa & \text{for non-waqf model}, \\ x\kappa & \text{for waqf model}, \end{cases} \tag{3.17}$$

where

$$F_{t^+} = \begin{cases} F_t & \text{if } U_t < l_2, \\ F_t + d & \text{if } U_t \geq l_2, \end{cases} \qquad t \in \mathbb{N}, \tag{3.18}$$

$$F_t = \begin{cases} F_{t^-} - W_t & \text{if } U_{t^-} < l_1, \\ F_{t^-} & \text{if } U_{t^-} \geq l_1, \end{cases} \qquad t \in \mathbb{Z}^+. \tag{3.19}$$

3.3 Illustration of Takaful Risk Model

For illustrative purpose, Figure 3.2 shows a sample evolution of a Takaful fund employing non-waqf model from time $t = 0$ until $t = 8$ using the modified KD risk model. Although the initial surplus process and the minimum acceptable surplus level are equal to 0 in non-waqf model, in the illustration we set that $v = l_1 = 10$ so that we can clearly describe the role of l_1. In order to explain the sample better, we assign numerical values to the parameters including the five threshold levels, the contribution rate, the deposit amount, the investment return rate, and the initial values of both the surplus process and the external fund. There are three claims by the end of time point $t = 8$, where the first claim occurred at the sixth term with the size of 33 unit, the second one occurred at the seventh term with the size of 20 unit, and the third claim occurred at the eighth term with the size of 11 unit.

Table 3.1 gives more details on the evolution of each components of both the surplus process and the external fund in the illustration. Recall that all contributions (c), deposits (D_t), and dividend payments from underwriting surplus $(R_{1,t})$ are paid/received at the beginning of the time interval $(t, t+1]$, while all claims (X_i), withdrawal (W_t), and dividend payments from investment gain $(R_{2,t})$ are paid/received at the end of the time interval $(t-1, t]$. Furthermore, t^- represents the time point at the end of the time interval $(t-1, t]$ immediately after claim payments and investment gain distribution are made but before any withdrawal is made. In addition, t^+ represents the time point at the beginning of the time interval $(t, t+1]$ after contributions, dividend from underwriting surplus, and deposit associated with the interval $(t, t+1]$ are paid/received. Notice that the investment return/gain rate κ is equal to 1 which means any assets invested is doubled in one term. Even though this rate is not practical, we use it for the sake of simplicity and to emphasize the results of the investment activities.

The surplus level at time points $t = 1, 2$ was obtained by adding contribution c to the level at $t-1$ since there was no claim in the corresponding term and the previous surplus level was between l_1 and l_2. i.e.,

$$U_1 = U_0 + c = 10 + 5 = 15,$$

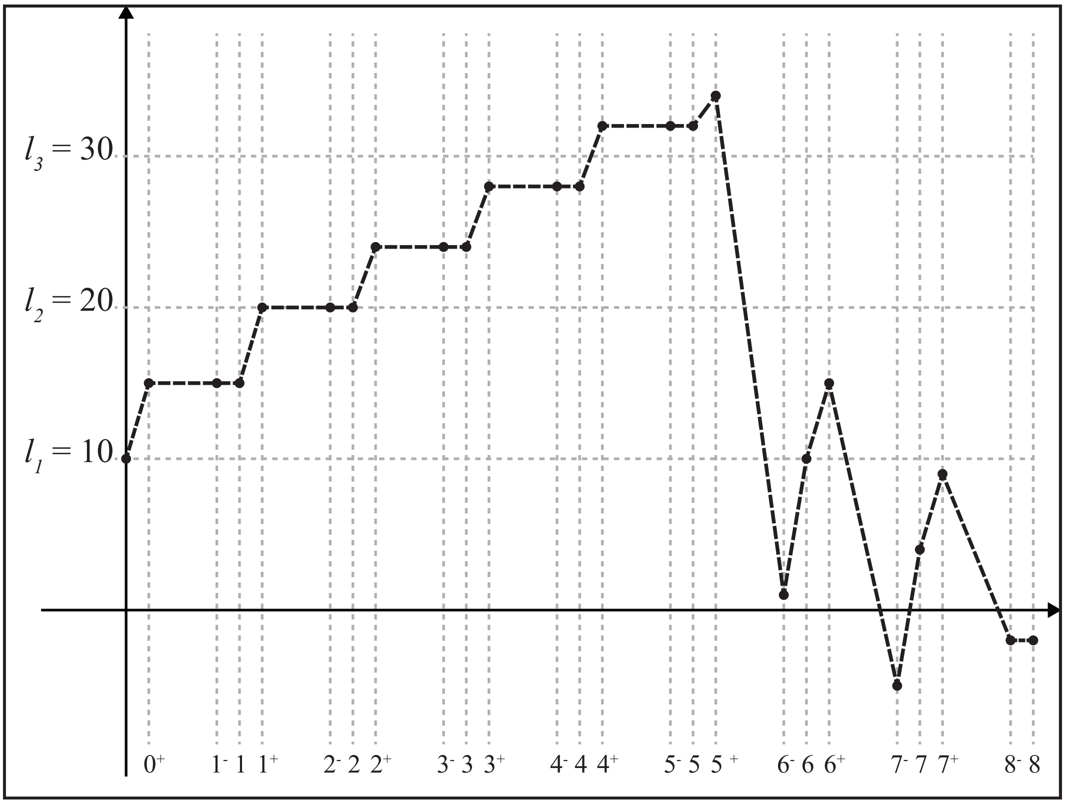
$$U_2 = U_1 + c = 15 + 5 = 20.$$

On the other hand, surplus level at time points $t = 3, 4, 5$ was obtained by adding contribution c and subtracting deposit d from the level at $t-1$ since there was no claim in the corresponding term and the previous surplus level was between l_2 and l_3. i.e.,

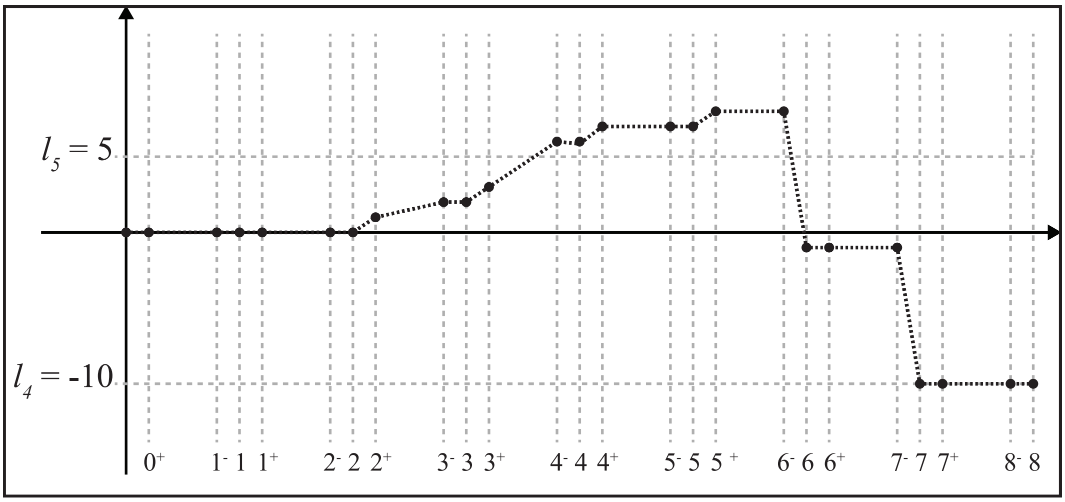
$$U_3 = U_2 - d + c = 20 - 1 + 5 = 24,$$

$$U_4 = U_3 - d + c = 24 - 1 + 5 = 28,$$

$$U_5 = U_4 - d + c = 28 - 1 + 5 = 32,$$



(a) Surplus process with $v = 10$, $c = 5$, and $d = 1$.



(b) External fund with $g = 0$ and $\kappa = 1$.

Figure 3.2: Sample evolution of a Takaful fund using the modified KD risk model for time interval $t \in [0, 8]$.

Table 3.1: Development of the components of the sample surplus process and the sample external fund.

t	D_t	W_t	$R_{1,t}$	$R_{2,t}$	U_t	F_t	(Y_i, X_i)
0	0	0	0	0	10	0	
0^+					15	0	
1^-					15	0	
1	0	0	0	0	15	0	
1^+					20	0	
2^-					20	0	
2	1	0	0	0	20	0	
2^+					24	1	
3^-					24	2	
3	1	0	0	0	24	2	
3^+					28	3	
4^-					28	6	
4	1	0	0	0	28	6	
4^+					32	7	
5^-					32	7	
5	1	0	2	7	32	7	
5^+					34	8	
6^-					1	8	$(Y_1, X_1) = (6, 33)$
6	0	9	0	8	10	-1	
6^+					15	-1	
7^-					-5	-1	$(Y_2, X_2) = (1, 20)$
7	0	9	0	0	4	-10	
7^+					9	-10	
8^-					-2	-10	$(Y_3, X_3) = (1, 11)$
8	0	0	0	0	-2	-10	

In addition to a contribution and a deposit, there was underwriting surplus distribution at the beginning of the sixth term since the surplus level at $t = 5$ was above l_3. However, the first claim of size 33 occurred and brought the surplus level below its minimum acceptable level (l_1), hence a withdrawal amount of 9 was received to keep the level at l_1. i.e.,

$$U_{5^+} = U_5 - (U_5 - 30) - d + c = 30 - 1 + 5 = 34,$$

$$U_{6^-} = U_{5^+} - X_1 = 34 - 33 = 1,$$

$$U_6 = U_{6^-} + W_6 = U_{6^-} + (l_1 - U_{6^-}) = l_1 = 10.$$

Accordingly, at time points $t = 2, 3, 4, 5$, the external fund received deposits from the surplus process. At the end of the corresponding terms the external fund received investment gains with respect to investment return rate κ. Further, some of these investment gains were distributed as dividends at the fifth and the sixth term since the base capitals were above l_5 for these terms. Moreover, the withdrawal received by the surplus process at the sixth term reduced the amount of the external fund to a level below 0, which means the Takaful fund had to get an interest free loan from the Takaful operator to keep its surplus process at the minimum acceptable level. i.e.,

$$\begin{gathered}
F_{2^+} = F_2 + d = 0 + 1 = 1, \\
F_3 = F_{3^-} = F_{2^+}(1+\kappa) = 1 * 2 = 2, \\
F_{3^+} = F_3 + d = 2 + 1 = 3, \\
F_4 = F_{4^-} = F_{3^+}(1+\kappa) = 3 * 2 = 6, \\
F_{4^+} = F_4 + d = 6 + 1 = 7, \\
F_5 = F_{5^-} = F_{4^+}(1+\kappa) - R_{2,5} = F_{4^+}(1+\kappa) - F_{4^+}(\kappa) = 7, \\
F_{5^+} = F_5 + d = 7 + 1 = 8, \\
F_{6^-} = F_{5^+}(1+\kappa) - R_{2,6} = F_{5^+}(1+\kappa) - F_{5^+}(\kappa) = 8, \\
F_6 = F_{6^-} - W_6 = 8 - 9 = -1.
\end{gathered}$$

The second claim of size 20 at the seventh term again brought the surplus level below its minimum acceptable level, and hence another withdrawal was made to cover the deficit. However the withdrawal could not force the external fund level to surpass its maximum borrowing limit (l_4), therefore the Takaful fund continued its process with some positive surplus level even though the minimum acceptable surplus level was not satisfied. i.e.,

$$\begin{gathered}
U_{7^-} = U_6 + c - X_2 = 15 - 20 = -5, \\
U_7 = U_{7^-} + W_7 = U_{7^-} + (F_6 - l_4) = -5 + 9 = 4, \\
F_7 = F_6 - (F_6 - l_4) = l_4 = 10.
\end{gathered}$$

Finally, the occurrence of the third claim of size 11 resulted in negative surplus level at the end of the eighth term, and hence another withdrawal was required to cover the deficit at least to bring the surplus back at 0 level. However, the external fund was already at its minimum level, which means the withdrawal can not be made and the surplus stayed below 0 at time point $t = 8$. i.e.,

$$U_8 = U_7 + c - X_3 = 4 + 5 - 11 = -2.$$

This phenomenon of U_t becomes negative for some $t \in \mathbb{Z}^+$ is defined as ruin occurrence. Its formal definition is given in the next chapter, where we aim to quantify the probability of ruin occurrence with respect to two random variables, namely inter-claim time distribution and claim size distribution. Note that the deposit, withdrawal, and dividend amounts are also stochastic in a sense that they are dependent on the surplus process and the external fund.

CHAPTER 4

FINITE-TIME RUIN PROBABILITY

In this chapter, we develop an algorithm to quantify the finite-time ruin probabilities associated with the Takaful risk model proposed in Chapter 3. First of all, ruin is defined as the event when the level of the surplus process (U_t) falls below 0 for some $t \in \mathbb{Z}^+$. The point in time at which ruin occurs for the first time is denoted by T.

$$T = \begin{cases} \min\{t \in \mathbb{Z}^+ | U_t < 0\} & \text{if } \exists t \in \mathbb{Z}^+ \text{ s.t. } U_t < 0, \\ \infty & \text{if } \forall t \in \mathbb{Z}^+ \ U_t \geq 0. \end{cases} \tag{4.1}$$

Let $\Psi(v, g, \tau)$ denotes the finite-time probability, which is the probability of ruin occurrence at or before time point τ, given that the initial surplus level is equal to v and the initial external fund is equal to g. We note that both v and g are equal to 0 for non-waqf model, whereas at least one of v and g is higher than 0 for waqf model with $v + g$ being the initial waqf capital provided by the shareholders.

$$\begin{gathered} \Psi(v, g, \tau) = Pr\{T \leq \tau | U_0 = v, F_0 = g\}, \\ \tau \in \mathbb{N},\ v \in \{l_1, l_1 + 1, ...\},\ g \in \mathbb{N}. \end{gathered} \tag{4.2}$$

To help us in the computation process of the finite-time ruin probability, we denote $\Theta(u, f, \tau)$ to be the finite-time survival probability, which is the probability that ruin does not occur by time point τ with the initial surplus level and the initial external fund level are equal to u and f respectively. In other words, $\Theta(u, f, \tau)$ is the probability that the surplus process remains at non-negative level at or before time point τ, which implies ruin occurs for the first time after τ.

$$\begin{gathered} \Theta(u, f, \tau) = Pr\{T > \tau | U_0 = u, F_0 = f\}, \\ \tau \in \mathbb{N},\ u \in \mathbb{N},\ f \in \{l_4, l_4 + 1, ...\}. \end{gathered} \tag{4.3}$$

In what follows, we focus on the construction of a recursive computational procedure to formulate the finite-time survival probabilities, which can be utilized to find the finite-time ruin probabilities via the following remark:

$$\Psi(v, g, \tau) = 1 - \Theta(v, g, \tau). \tag{4.4}$$

Furthermore, we also denote $\Theta^*(u, f, \tau, k)$ to be the finite-time conditional survival probability, which is the probability that ruin does not occur until time point τ, given that the first claim X_1 takes place at time $Y_1 = k$, with the initial surplus level and the initial external fund level are equal to u and f respectively. Next, to find $\Theta(u, f, \tau)$ we accumulate the weighted sum of $\Theta^*(u, f, \tau, k)$ for all possible values of k, which is the support set of the inter-claim time distribution.

$$\Theta(u, f, \tau) = \sum_{k=1}^{n_a} a_k \Theta^*(u, f, \tau, k), \tag{4.5}$$

$$\Theta^*(u, f, \tau, k) = Pr\{T > \tau | U_0 = u, F_0 = f, Y_1 = k\}, \quad k = 1, 2, ..., n_a. \tag{4.6}$$

The finite-time conditional survival probabilities depend on the first claim X_1 of size j if it occurs at or before the time point τ. In contrast, X_1 occurrence after τ implies that the process survives until τ, hence the finite-time conditional survival probabilities for $k > \tau$ is equal to 1. Therefore, Equation 4.5 can be rewritten as:

$$\Theta(u, f, \tau) = \sum_{k=1}^{\min\{n_a, \tau\}} a_k \Theta^*(u, f, \tau, k) \ + \sum_{k=\min\{n_a, \tau\}+1}^{n_a} a_k. \tag{4.7}$$

Now to find the quantity of $\{\Theta^*(u, f, \tau, k),\ k = \{1, 2, ..., \min\{n_a, \tau\}\}\}$, we accumulate the weighted sum of $\Theta(u^*, f^*, \tau - k)$, which is the the probability of surviving the time interval $(k, \tau]$ with the level of surplus process and external fund at time k after the first claim payment are equal to u^* and f^* respectively, for all possible values of j that do not cause ruin at time k. Therefore, in this case j is bounded above by the amount of total available fund in both the surplus process and the external fund at time k before the first claim payment plus the loan that the operator can provide. This available fund amount is non-stochastic since we know that the first claim occurs at time k, i.e. it does not depend on the inter-claim time nor the claim size distributions. We denote $\hat{U}_{(t,u)}$ to be the non-stochastic form of the surplus process, which can be defined as the level of U_t under the condition of no claim occurrence. The corresponding non-stochastic form of the external fund is denoted by $\hat{F}_{(t,u,f)}$.

$$\Theta^*(u,f,\tau,k) = \begin{cases} \displaystyle\sum_{j=1}^{\hat{U}_{(k,u)}+\lfloor \hat{F}_{(k,u,f)} \rfloor - l_4} \alpha_j(k)\Theta(u^*,f^*,\tau-k) & \text{if } k \in \{1,2,...,\min\{n_a,\tau\}\}, \\ 1 & \text{if } k \in \{\min\{n_a,\tau\}+1,...,n_a\}, \end{cases} \tag{4.8}$$

where $\lfloor x \rfloor$ referred to as the floor function of x, which is the largest integer less than or equal to x.

$$\lfloor x \rfloor = \max\{z \in \mathbb{Z} | z \leq x\}, \quad x \in \mathbb{R}. \tag{4.9}$$

The floor function in Equation 4.8 is utilized to calculate the integer component of the fund available in the external fund. Such an assumption can be considered as conservative in nature, since any non-integer part of the external fund which can rise due to investment return accumulation is essentially rounded down [20]. Equation 4.10 and Equation 4.11 express the amount of u^* and f^* respectively, which depend on the way the first claim $X_1 = j$ affects $\hat{U}_{(k,u)}$ and $\hat{F}_{(k,u,f)}$. Specifically, their amount depend on whether the claim size is large enough to bring U_{k^-} at some point between 0 and l_1, in which case some amount of fund is withdrawn from F_{k^-}.

$$u^* = \min\{\hat{U}_{(t,u)} - j + \lfloor \hat{F}_{(k,u,f)} \rfloor - l_4, \max\{l_1, \hat{U}_{(k,u)} - j\}\}, \tag{4.10}$$

$$f^* = \max\{l_4, \lfloor \hat{F}_{(k,u,f)} \rfloor + \min\{0, \hat{U}_{(k,u)} - j - l_1\}\}. \tag{4.11}$$

Notice that u^* and f^* split their values into three cases depending on three possible positions of $\hat{U}_{(k,u)} - j$. Firstly, $\hat{U}_{(k,u)} - j \geq l_1$ in which case no withdrawal is required. Secondly, $\hat{U}_{(k,u)} - j < l_1$ and there is enough fund in the external fund to bring the surplus level back at its minimum acceptable level l_1. Lastly, $\hat{U}_{(k,u)} - j < l_1$ while the withdrawal is not enough to cover the deficit. Therefore the pair of u^* and f^* can be rewritten as:

$$(u^*, f^*) = \begin{cases} (\hat{U}_{(k,u)} - j, \lfloor \hat{F}_{(k,u,f)} \rfloor) & \text{if } l_1 - (\hat{U}_{(k,u)} - j) \leq 0, \\ (l_1, \lfloor \hat{F}_{(k,u,f)} \rfloor + \hat{U}_{(k,u)} - j - l_1) & \text{if } 0 < l_1 - (\hat{U}_{(k,u)} - j) \leq \lfloor \hat{F}_{(k,u,f)} \rfloor - l_4, \\ (\hat{U}_{(k,u)} - j + \lfloor \hat{F}_{(k,u,f)} \rfloor - l_4, l_4) & \text{if } l_1 - (\hat{U}_{(k,u)} - j) > \lfloor \hat{F}_{(k,u,f)} \rfloor - l_4. \end{cases} \tag{4.12}$$

$\hat{U}_{(t,u)}$ and $\hat{F}_{(t,u,f)}$ can be expressed in a similar way to the surplus process and the external fund respectively by ignoring claim payments and withdrawal activities. The non-stochastic form of the surplus process is expressible as:

$$\hat{U}_{(t,u)} = u + ct - \sum_{i=0}^{t-1} D_i - \sum_{i=0}^{t-1} R_{1,i}, \quad t \in \mathbb{N}, u \in \mathbb{N}. \tag{4.13}$$

The non-stochastic form of the external fund for non-waqf model is given as:

$$\hat{F}_{(t,u,f)} = \begin{cases} F_{((t-1)^+,u,f)}(1+\kappa) & \text{if } 0 \leq F_{((t-1)^+,u,f)} < l_5, \\ F_{((t-1)^+,u,f)} & \text{if } l_4 \leq F_{((t-1)^+,u,f)} < 0 \text{ or } F_{((t-1)^+,u,f)} \geq l_5, \end{cases} \tag{4.14}$$

and with $x \in (0,1)$ representing the share of investment gain entitled to the operator, the non-stochastic form of the external fund for waqf model is given as:

$$\hat{F}_{(t,u,f)} = \begin{cases} F_{((t-1)^+,u,f)}(1+(1-x)\kappa) & \text{if } F_{((t-1)^+,u,f)} \geq 0, \\ F_{((t-1)^+,u,f)} & \text{if } l_4 \leq F_{((t-1)^+,u,f)} < 0, \end{cases} \tag{4.15}$$

$$\hat{F}_{(t^+,u,f)} = \begin{cases} \hat{F}_{(t,u,f)} & \text{if } \hat{U}_{(t,u)} < l_2, \\ \hat{F}_{(t,u,f)} + d & \text{if } \hat{U}_{(t,u)} \geq l_2, \end{cases} \quad t \in \mathbb{N}, u \in \mathbb{N}, f \in \{l_4, l_4+1, ...\}. \tag{4.16}$$

The expressions of $\hat{U}_{(t,u)}$ and $\hat{F}_{(t,u,f)}$ in the above equations however are given recursively, which means their values for lower t are required to find the values for t, and hence it can be time consuming in the calculation process, given that our finite-time ruin probability formula is also in a recursive form. For this reason, to aid us in the computational process, in the following sections we construct algorithms to compute the non-recursive expressions of $\hat{U}_{(t,u)}$ and $\hat{F}_{(t,u,f)}$ as functions of the five threshold levels, the initial levels v and g, the contribution rate c, the deposit amount d, and the investment gain rate κ.

4.1 Non-Stochastic Form of the Surplus Process ($\hat{U}_{(t,u)}$)

In order to construct a non-recursive function of $\hat{U}_{(t,u)}$, first we need to identify the time point when it reaches some threshold levels associated with the surplus process. Denoted by p_u and q_u are the time points when $\hat{U}_{(t,u)}$ reaches l_2 and l_3 respectively for the first time, given that $U_0 = u$.

$$p_u = \min\{t \in \mathbb{N} | \hat{U}_t \geq l_2\}. \tag{4.17}$$

$$q_u = \min\{t \in \mathbb{N} | \hat{U}_t \geq l_3\}. \tag{4.18}$$

Before the time point $t = p_u$, there is only one regular cash inflow to the non-stochastic surplus process which is the contribution c from the participants received at the beginning of each term.

$$\hat{U}_{(t,u)} = u + ct, \quad t \in \{0, 1, ..., p_u\}. \tag{4.19}$$

Hence, p_u can be expressed as the number of times c is added to u from $t = 0$ until $\hat{U}_{(t,u)}$ reaches l_2 , with 0 amount if u is already greater than or equal to l_2.

$$p_u = \begin{cases} 0 & \text{if } u \geq l_2, \\ \left\lceil \frac{l_2 - u}{c} \right\rceil & \text{if } u < l_2, \end{cases} \tag{4.20}$$

where $\lceil x \rceil$ referred to as the ceiling function of x, which is the smallest integer greater than or equal to x.

$$\lceil x \rceil = \min\{z \in \mathbb{Z} | z \geq x\}, \quad x \in \mathbb{R}. \tag{4.21}$$

From the time point p_u until $\hat{U}_{(t,u)}$ reaches l_3, in addition to the regular cash inflow c, the non-stochastic surplus process also has a regular cash outflow to the non-stochastic external fund which is the deposit of d.

$$\hat{U}_{(t,u)} = u + cp_u + (c - d)(t - p_u), \quad t \in \{p_u + 1, ..., q_u\}. \tag{4.22}$$

Therefore, q_u can be expressed as the number of times $c-d$ is added to $u+cp_u$ from the time point $t=p_u$ until $\hat{U}_{(t,u)}$ reaches l_3, with 0 amount if u is already greater than or equal to l_3.

$$q_u = \begin{cases} 0 & \text{if } u \geq l_3, \\ \left\lceil \frac{\max\{0, l_3-u-cp_u\}}{c-d} \right\rceil + p_u & \text{if } u < l_3. \end{cases} \tag{4.23}$$

After the time point $t=q_u$, due to dividend payments from underwriting surplus, $\hat{U}_{(t,u)}$ reset its level to l_3 before getting contribution c and depositing d. Hence, by combining equations above we get a non-recursive function of the non-stochastic form of the surplus process, which is expressed in the following equation:

$$\hat{U}_{(t,u)} = \begin{cases} u + cp_{(u,t)} + (c-d)(t-p_{(u,t)}) & \text{if } t \leq q_u, \\ l_3 - d + c & \text{if } t > q_u, \end{cases} \qquad t \in \mathbb{N}, \tag{4.24}$$

where

$$p_{(u,t)} = \min\{p_u, t\}. \tag{4.25}$$

The following figure give a sample evolution of $\hat{U}_{(t,u)}$, which represents the amount of fund in the surplus process at time t that can be used to cover the first claim payment.

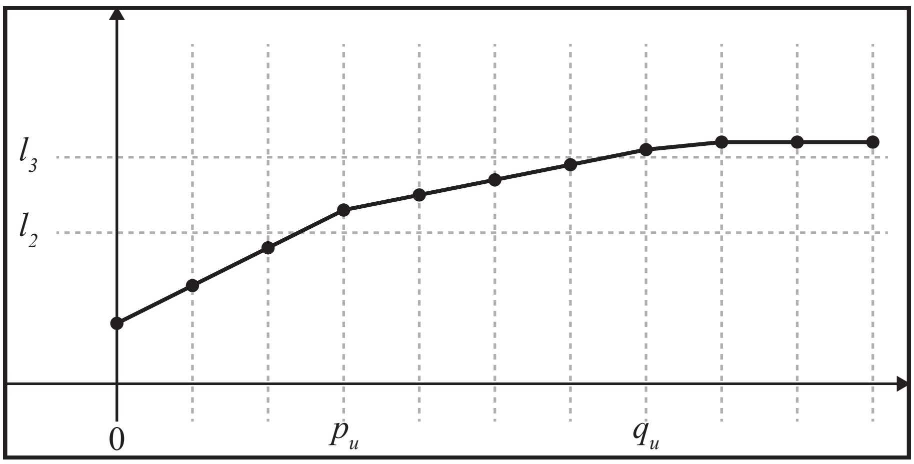


Figure 4.1: Sample evolution of $\hat{U}_{(t,u)}$ with $u=8$, $l_1=0$, $l_2=20$, $l_3=30$, $c=5$, and $d=3$.

4.2 Non-Stochastic Form of the External Fund $(\hat{F}_{(t,u,f)})$

In order to construct a non-recursive function of $\hat{F}_{(t,u,f)}$, first we need to identify the time point when $\hat{F}_{(t^+,u,f)}$ reaches 0 or l_5 for the first time which are denoted by $r_{(u,f)}$ and $s_{(u,f)}$ respectively, given that $U_0 = u$ and $F_0 = f$.

$$r_{(u,f)} = \min\{t \in \mathbb{N} | \hat{F}_{(t^+,u,f)} \geq 0\}. \tag{4.26}$$

$$s_{(u,f)} = \min\{t \in \mathbb{N} | \hat{F}_{(t^+,u,f)} \geq l_5\}. \tag{4.27}$$

When the non-stochastic external fund is at some negative level, there is only one cash inflow from the non-stochastic surplus process which is the deposit d and it is received regularly starting from the time point $t = p_u$.

$$\hat{F}_{(t^+,u,f)} = \begin{cases} f & \text{if } f < 0,\ t \in \{0, 1, ..., p_u - 1\}, \\ f + d(t + 1 - p_u) & \text{if } f < 0,\ t \in \{p_u, p_u + 1, ..., r_{(u,f)}\}. \end{cases} \tag{4.28}$$

Hence, $r_{(u,f)}$ can be expressed as p_u plus the number of times d is added to f until $\hat{F}_{(t^+,u,f)}$ reaches 0 (minus 1 since $\hat{F}_{(t^+,u,f)}$ accumulates d which corresponds to the the $(t+1)$-th term), with 0 amount if f is already greater than or equal to 0.

$$r_{(u,f)} = \begin{cases} 0 & \text{if } f \geq 0, \\ \left\lceil \frac{-f}{d} \right\rceil + p_u - 1 & \text{if } f < 0. \end{cases} \tag{4.29}$$

To help us examine the development of $\hat{F}_{(t^+,u,f)}$ before it reaches $s_{(u,f)}$, we split the support set of f into two disjoint sets, namely $\{l_4, l_4+1, ..., -1\}$ and $\mathbb{N}$. When f is less than 0, the non-stochastic form of the external fund receives deposit d from the time point $t = p_u$ until it reaches 0 level. After that, $\hat{F}_{(t^+,u,f)}$ continues to grow with the deposit stream coupled with the investment gain rate κ until it reaches l_5. For $f < 0$ and $t \in \{0, 1, ..., r_{(u,f)}\}$, the expression of $\hat{F}_{(t^+,u,f)}$ is given in Equation 4.28, and from $t = r_{(u,f)} + 1$ until $s_{(u,f)}$ its level is expressed in Equation 4.30.

$$\hat{F}_{(t^+,u,f)} = \big(f + d(r_{(u,f)} + 1 - p_u) + da_{\overline{t-r_{(u,f)}|}\kappa}\big)(1+\kappa)^{(t-r_{(u,f)})},$$
$$f < 0,\ t \in \{r_{(u,f)} + 1, ..., s_{(u,f)}\}, \tag{4.30}$$

where $a_{\overline{n|}\kappa}$ represents the present value of an annuity formula for n period with constant investment rate κ in our case.

$$a_{\overline{n|}\kappa} = \begin{cases} \frac{1-(1+\kappa)^{-n}}{\kappa} & \text{if } n \in \mathbb{Z}^+, \\ 0 & \text{otherwise.} \end{cases} \tag{4.31}$$

On the other hand, if f is greater than or equal to 0, then $r_{(u,f)}$ is equal to 0 and p_u can be higher than $r_{(u,f)}$. Therefore there are two possible scenarios of the way the non-stochastic form of the external fund reaches l_5. Either it reaches l_5 by only utilizing growth from the investment activities, or it also gets the deposit stream from the non-stochastic surplus process. i.e.,

$$\hat{F}_{(t^+,u,f)} = \begin{cases} f(1+\kappa)^t & \text{if } f(1+\kappa)^{p_u-1} \geq l_5, \\ f(1+\kappa)^t + da_{\overline{t+1-p_u|}\kappa}(1+\kappa)^{(t+1-p_u)} & \text{if } f(1+\kappa)^{p_u-1} < l_5, \end{cases}$$
$$f \geq 0,\ t \in \{0, 1, ..., s_{(u,f)}\}. \tag{4.32}$$

Therefore, by combining Equation 4.30 and Equation 4.32, we can define $s_{(u,f)}$ in the following equation with 0 amount if f is already greater than or equal to l_5.

$$s_{(u,f)} = \begin{cases} 0 & \text{if } f \geq l_5, \\ \min\{i \in \mathbb{Z}^+ | f(1+\kappa)^i \geq l_5\} & \text{if } 0 \leq f < l_5 \text{ and } f(1+\kappa)^{p_u-1} \geq l_5, \\ \min\{i \in \mathbb{Z}^+ | f(1+\kappa)^i + da_{\overline{i+1-p_u|}\kappa}(1+\kappa)^{(i+1-p_u)} \geq l_5\} & \text{if } 0 \leq f < l_5 \text{ and } f(1+\kappa)^{p_u-1} < l_5, \\ \min\big\{i \in \mathbb{Z}^+ \mid \big(f + d(r_{(u,f)} + 1 - p_u) + da_{\overline{i-r_{(u,f)}|}\kappa}\big)(1+\kappa)^{(i-r_{(u,f)})} \geq l_5\big\} & \text{if } f < 0. \end{cases} \tag{4.33}$$

Theorem 4.1. *$s_{(u,f)}$ can be calculated explicitly using the following equation:*

$$s_{(u,f)} = \begin{cases} 0 & \text{if } f \geq l_5, \\ \left\lceil \log_{(1+\kappa)} \left(\frac{l_5}{f}\right) \right\rceil & \text{if } 0 \leq f < l_5 \text{ and } f(1+\kappa)^{p_u - 1} \geq l_5, \\ \left\lceil \log_{(1+\kappa)} \left(\frac{l_5\kappa + d}{f\kappa + d(1+\kappa)^{(1-p_u)}}\right) \right\rceil & \text{if } 0 \leq f < l_5 \text{ and } f(1+\kappa)^{p_u - 1} < l_5, \\ \left\lceil \log_{(1+\kappa)} \left(\frac{(l_5\kappa + d)(1+\kappa)^{r_{(u,f)}}}{f\kappa + d\kappa(r_{(u,f)} + 1 - p_u) + d}\right) \right\rceil & \text{if } f < 0. \end{cases} \tag{4.34}$$

Proof. **Case I:** $f \geq l_5$ (trivial)

Case II: $0 \leq f < l_5$ and $f(1+\kappa)^{p_u-1} \geq l_5$

$$\begin{aligned} & & f(1+\kappa)^i &\geq l_5 \\ \Rightarrow & & (1+\kappa)^i &\geq \frac{l_5}{f} \\ \Rightarrow & & i &\geq \log_{(1+\kappa)}\left(\frac{l_5}{f}\right) \qquad \because \kappa > 0 \end{aligned}$$

$$\begin{aligned} \Rightarrow \quad \min\{i \in \mathbb{Z}^+ | f(1+\kappa)^i \geq l_5\} &= \min\left\{ i \in \mathbb{Z}^+ \middle| i \geq \log_{(1+\kappa)}\left(\frac{l_5}{f}\right)\right\} \\ &= \left\lceil \log_{(1+\kappa)}\left(\frac{l_5}{f}\right)\right\rceil \end{aligned}$$

<u>**Case III:** $0 \leq f < l_5$ and $f(1+\kappa)^{p_u-1} < l_5$</u>

$$f(1+\kappa)^i + da_{\overline{i+1-p_u|}\kappa}(1+\kappa)^{(i+1-p_u)} \geq l_5$$

$$\Rightarrow \quad f(1+\kappa)^i + d\left(\frac{1-(1+\kappa)^{(p_u-i-1)}}{\kappa}\right)(1+\kappa)^{(i+1-p_u)} \geq l_5$$

$$\Rightarrow \quad f(1+\kappa)^i + d\left(\frac{(1+\kappa)^{(i+1-p_u)}-1}{\kappa}\right) \geq l_5$$

$$\Rightarrow \quad f(1+\kappa)^i + \frac{d(1+\kappa)^{(i+1-p_u)}}{\kappa} - \frac{d}{\kappa} \geq l_5$$

$$\Rightarrow \quad f(1+\kappa)^i + \frac{d(1+\kappa)^i}{\kappa(1+\kappa)^{(p_u-1)}} \geq l_5 + \frac{d}{\kappa}$$

$$\Rightarrow \quad (1+\kappa)^i\left(f + \frac{d}{\kappa(1+\kappa)^{(p_u-1)}}\right) \geq l_5 + \frac{d}{\kappa}$$

$$\Rightarrow \quad (1+\kappa)^i \geq \frac{l_5 + \frac{d}{\kappa}}{f + \frac{d}{\kappa(1+\kappa)^{(p_u-1)}}}$$

$$\Rightarrow \quad i \geq \log_{(1+\kappa)}\left(\frac{l_5\kappa + d}{f\kappa + d(1+\kappa)^{(1-p_u)}}\right) \qquad \because \kappa > 0$$

$$\Rightarrow \quad \min\{i \in \mathbb{Z}^+ | f(1+\kappa)^i + da_{\overline{i+1-p_u|}\kappa}(1+\kappa)^{(i+1-p_u)} \geq l_5\}$$

$$= \min\left\{i \in \mathbb{Z}^+ \;\middle|\; i \geq \log_{(1+\kappa)}\left(\frac{l_5\kappa + d}{f\kappa + d(1+\kappa)^{(1-p_u)}}\right)\right\}$$

$$= \left\lceil \log_{(1+\kappa)}\left(\frac{l_5\kappa + d}{f\kappa + d(1+\kappa)^{(1-p_u)}}\right)\right\rceil$$

<u>**Case IV:** $f < 0$</u>

$$\big(f + d(r_{(u,f)} + 1 - p_u) + da_{\overline{i-r_{(u,f)}|}\kappa}\big)(1+\kappa)^{(i-r_{(u,f)})} \geq l_5$$

$$\Rightarrow \quad \frac{\big(f + d(r_{(u,f)} + 1 - p_u)\big)(1+\kappa)^i}{(1+\kappa)^{r_{(u,f)}}} + d\Bigg(\frac{1-(1+\kappa)^{(r_{(u,f)}-i)}}{\kappa}\Bigg)(1+\kappa)^{(i-r_{(u,f)})} \geq l_5$$

$$\Rightarrow \quad \frac{\big(f + d(r_{(u,f)} + 1 - p_u)\big)(1+\kappa)^i}{(1+\kappa)^{r_{(u,f)}}} + d\Bigg(\frac{(1+\kappa)^{(i-r_{(u,f)})}-1}{\kappa}\Bigg) \geq l_5$$

$$\Rightarrow \quad \frac{\big(f + d(r_{(u,f)} + 1 - p_u)\big)(1+\kappa)^i}{(1+\kappa)^{r_{(u,f)}}} + \frac{d(1+\kappa)^{(i-r_{(u,f)})}}{\kappa} - \frac{d}{\kappa} \geq l_5$$

$$\Rightarrow \frac{\big(f + d(r_{(u,f)} + 1 - p_u)\big)(1+\kappa)^i}{(1+\kappa)^{r_{(u,f)}}} + \frac{d(1+\kappa)^i}{\kappa(1+\kappa)^{r_{(u,f)}}} \geq l_5 + \frac{d}{\kappa}$$

$$\Rightarrow (1+\kappa)^i\Bigg(\frac{f + d(r_{(u,f)} + 1 - p_u)}{(1+\kappa)^{r_{(u,f)}}} + \frac{d}{\kappa(1+\kappa)^{r_{(u,f)}}}\Bigg) \geq l_5 + \frac{d}{\kappa}$$

$$\Rightarrow (1+\kappa)^i \geq \frac{(l_5\kappa + d)(1+\kappa)^{r_{(u,f)}}}{\kappa\big(f + d(r_{(u,f)} + 1 - p_u)\big) + d}$$

$$\Rightarrow \qquad i \geq \log_{(1+\kappa)}\left(\frac{(l_5\kappa + d)(1+\kappa)^{r_{(u,f)}}}{\kappa\big(f + d(r_{(u,f)} + 1 - p_u)\big) + d}\right) \qquad \because \kappa > 0$$

$$\Rightarrow \quad \min\big\{i \in \mathbb{Z}^+ \;\big|\; \big(f + d(r_{(u,f)} + 1 - p_u) + da_{\overline{i-r_{(u,f)}|}\kappa}\big)(1+\kappa)^{(i-r_{(u,f)})} \geq l_5\big\}$$

$$= \min\left\{i \in \mathbb{Z}^+ \;\middle|\; i \geq \log_{(1+\kappa)}\left(\frac{(l_5\kappa + d)(1+\kappa)^{r_{(u,f)}}}{f\kappa + d\kappa(r_{(u,f)} + 1 - p_u) + d}\right)\right\}$$

$$= \left\lceil \log_{(1+\kappa)}\left(\frac{(l_5\kappa + d)(1+\kappa)^{r_{(u,f)}}}{f\kappa + d\kappa(r_{(u,f)} + 1 - p_u) + d}\right)\right\rceil$$

□

We define also $r_{(u,f,t)}$ and $s_{(u,f,t)}$ as:

$$r_{(u,f,t)} = \min\{r_{(u,f)}, t\}, \tag{4.35}$$

$$s_{(u,f,t)} = \min\{s_{(u,f)}, t\}. \tag{4.36}$$

4.2.1 $\hat{F}_{(t,u,f)}$ for Non-Waqf Model

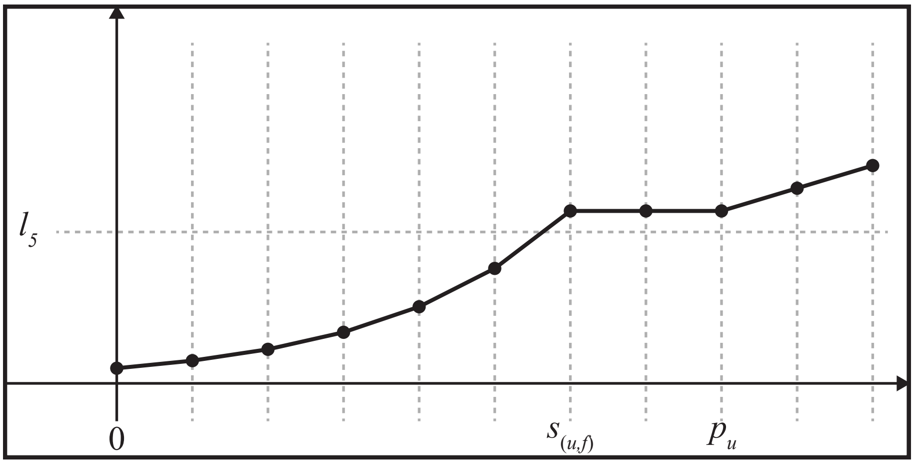
For the case of Takaful fund employing non-waqf model, the investment gain rate κ is not applicable to $\hat{F}_{(t,u,f)}$ after the time point $s_{(u,f)}$ due to investment gain distribution . For this reason, the only cash inflow to $\hat{F}_{(t,u,f)}$ for $t > s_{(u,f)}$ is the deposit stream starting from p_u or $s_{(u,f)}$ whichever occurs later. Therefore, by combining the equations above we get a non-recursive function of the non-stochastic form of the external for non-waqf model, which is expressed in the following equation:

$$\hat{F}_{(t,u,f)} = \begin{cases} f(1+\kappa)^{s_{(u,f,t)}} + d\ddot{a}_{\overline{s_{(u,f,t)}-p_{(u,t)}}|\kappa}(1+\kappa)^{(s_{(u,f,t)}-p_{(u,t)})} & \\ \quad ... + d(t - \max\{p_{(u,t)}, s_{(u,f,t)}\}) & \text{if } f \geq 0, \\ & \\ \big(f + d(r_{(u,f,t)} - p_{(u,t)}) + d\ddot{a}_{\overline{s_{(u,f,t)}-r_{(u,f,t)}}|\kappa}\big)(1+\kappa)^{(s_{(u,f,t)}-r_{(u,f,t)})} & \\ \quad ... + d(t - s_{(u,f,t)}) & \text{if } f < 0, \end{cases} \tag{4.37}$$

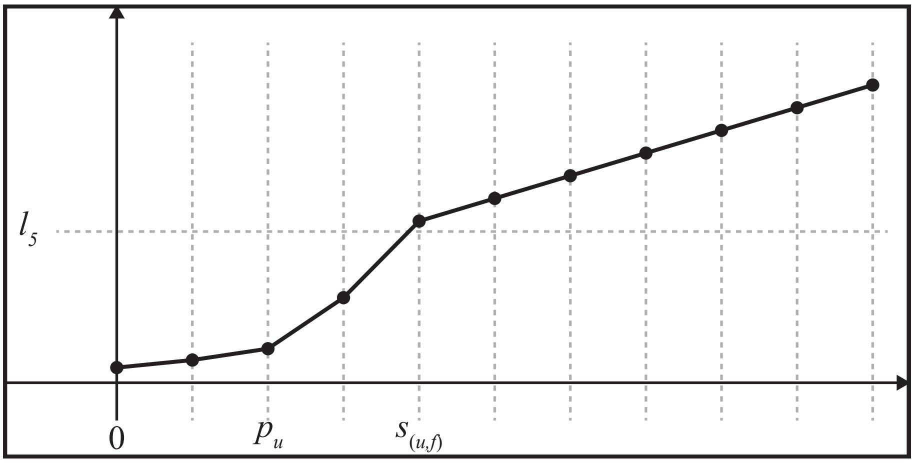
where $\ddot{a}_{\overline{n}|\kappa}$ represents the present value of an annuity due formula for n period with constant investment rate κ in our case.

$$\ddot{a}_{\overline{n}|\kappa} = \begin{cases} \frac{1-(1+\kappa)^{-n}}{\kappa}(1+\kappa) & \text{if } n \in \mathbb{Z}^+, \\ 0 & \text{otherwise.} \end{cases} \tag{4.38}$$

The last terms of both cases in above equation represent the growth of the non-stochastic external fund after $s_{(u,f)}$. We can check that Equation 4.37 satisfies Equations 4.28, 4.30, and 4.32 by making use of the relation between $\hat{F}_{(t,u,f)}$ and $\hat{F}_{(t^+,u,f)}$ as shown in Equation 4.14 and Equation 4.16. The following figures give sample evolution of $\hat{F}_{(t,u,f)}$ representing the amount of capital in the external fund that can be used to cover the first claim payment associated with Takaful business employing non-waqf model.



(a) $s_{(u,f)} \leq p_u$



(b) $s_{(u,f)} > p_u$

Figure 4.2: Sample evolution of $\hat{F}_{(t,u,f)}$ for non-waqf model with $f = 2$, $l_5 = 20$, $\kappa = 0.5$, and $d = 3$.

A Takaful operator employing mudarabah model, modified mudarabah model, or hybrid model might set a low level for l_5 so that the investment gain can be distributed as dividends at early time points. By setting $l_5 = 0$, the non-stochastic external fund can be simplified since $s_{(t,u,f)}$ becomes equal to $r_{(t,u,f)}$. Given also that $s_{(t,u,f)} = 0$ for $f \geq 0$, Equation 4.37 can be reduced to the following equation:

$$\hat{F}_{(t,u,f)} = f + d(t - p_{u,t}), \quad t \in \mathbb{N}. \tag{4.39}$$

Notice that Equation 4.39 is not affected by the rate κ since $l_5 = 0$ implies that all investment gain is distributed as dividends. Moreover, the total fund available to cover the first claim payment for non waqf-model can be simplified to:

$$\begin{aligned}\hat{U}_{(k,u)} + \hat{F}_{(k,u,f)} - l_4 &= \begin{cases} u + cp_{(u,k)} + (c-d)(t - p_{(u,k)}) + f + d(k - p_{(u,k)}) - l_4 & \text{if } k \leq q_u, \\ l_3 - d + c + f + d(k - p_{u,k}) - l_4 & \text{if } k > q_u, \end{cases} \\ &= \begin{cases} u + f + ct - l_4 & \text{if } k \leq q_u, \\ l_3 + c + f + d(k - p_{(u,k)} - 1) - l_4 & \text{if } k > q_u. \end{cases}\end{aligned} \tag{4.40}$$

4.2.2 $\hat{F}_{(t,u,f)}$ for Waqf Model

For waqf model we adopt the convention that $l_5 = 0$, because unlike the other business models, the investment gain in waqf model is shared between the shareholders $(x \in (0,1))$ and the Takaful fund itself $(1-x)$. Therefore, even if the level of the external fund is above l_5, a portion of the investment gain still remains in the fund. To find the non-stochastic form of the external fund for waqf model, we modify Equation 4.37 by using $(1-x)\kappa$ as the growth rate of the external fund instead of κ since the portion $x\kappa$ is distributed as dividends to the shareholders. Additionally, we remove the last terms of both cases in the equation, since the external fund of waqf model always increases with $x\kappa$ as long as its level is not at some negative level. Accordingly, we replace $s_{(u,f,t)}$ from the equation with t, because $s_{(u,f,t)}$ is the time limit for the external fund to be affected by the investment return rate. Hence, for waqf model Equation 4.37 can be reduced to the following equation:

$$\hat{F}_{(t,u,f)} = \begin{cases} f(1+\kappa)^t + d\ddot{a}_{\overline{t-p_{(u,t)}}|\acute{\kappa}}(1+\acute{\kappa})^{(t-p_{(u,t)})} & \text{if } f \geq 0, \\ \\ \big(f + d(r_{(u,f,t)} - p_{(u,t)}) + d\ddot{a}_{\overline{t-r_{(u,f,t)}}|\acute{\kappa}}\big)(1+\acute{\kappa})^{(t-r_{(u,f,t)})} & \text{if } f < 0, \end{cases} \tag{4.41}$$

$$\acute{\kappa} = (1-x)\kappa, \quad x \in (0,1), \tag{4.42}$$

Finally, by combining Equation 4.7 and Equation 4.8, we establish the following recursive formula for the finite-time conditional survival probabilities calculation.

$$\Theta(u,f,\tau) = \sum_{k=1}^{\min\{n_a,\tau\}} a_k \left(\sum_{j=1}^{\hat{U}_{(k,u)}+\lfloor \hat{F}_{(k,u,f)} \rfloor - l_4} \alpha_j(k)\Theta(u^*,f^*,\tau-k) \right) + \sum_{k=\min\{n_a,\tau\}+1}^{n_a} a_k. \tag{4.43}$$

The base case of the recursive formula is $\Theta(u,f,0)$ and its value can be obtained directly by assigning $\tau = 0$, in which the first term equates to 0 since $\min\{n_a,\tau\} = 0$.

$$\Theta(u,f,0) = \sum_{k=1}^{n_a} a_k = 1. \tag{4.44}$$

By using the remark in Equation 4.4 we can find the corresponding finite-time ruin probabilities associated with the risk model for Takaful Insurance.

CHAPTER 5

EXPECTED TOTAL DISCOUNTED DIVIDENDS

In addition to paying the claims of the participants, over time a Takaful fund is expected to generate dividend payments to its Takaful operator/shareholders as the fund manager and to its participants as the capital providers. The dividend payments have three sources and the way they are distributed depends on the business model utilized by the Takaful operator. In particular, the ratios of dividend payments for the shareholders and the participants are different for each source in different models as explained in Chapter 2. The return from investment activities is the main source of dividend payments in Takaful insurance, where in our risk model the amounts of dividends from investment gain depend on the level of the external fund and the threshold level l_5. In order to aid the Takaful fund in covering its claim payments, the external fund is required to grow and reach l_5 first before distributing the investment return as dividends. In mudarabah model, modified mudarabah model, and hybrid model, a portion of the dividends from investment gain is distributed to the shareholders, while the rest is paid to the participants. On the other hand, in wakalah model, the participants are entitled to the whole amount of dividends from investment gain. In waqf model, the investment gain is shared between the shareholders and the Takaful fund itself, hence we adopt the convention that $l_5 = 0$ for this business model. Another source of the dividend payments is the underwriting surplus, which in our risk model is defined as the excess of fund in the surplus process from the threshold level l_3. Most scholars argue that the underwriting surplus can only be distributed to the participants since the fund in the surplus process is essentially the participants' contribution, and hence the shareholders has no right to it. For this reason, the only Takaful business model in which the shareholders receive a portion of the underwriting surplus is modified mudarabah model. Lastly, dividends/compensations for the operator can be in the form of regular fee payment from the participants. In our risk model, the fee payments is presented as an extra regular payment the participants provide at the beginning of each term in addition to the regular contribution to the Takaful fund. The regular fee payment is applied to wakalah model, hybrid model, and waqf model. In this chapter, we focus on the development of algorithms to quantify the expected total discounted dividends from investment gain, expected total discounted dividends from underwriting surplus, and expected total discounted fee payments by using similar approach from Chapter 4. Then these quantities are combined to find the expected total discounted dividends paid to the Takaful operator/shareholders and expected total discounted dividends paid to the participants.

5.1 Expected Total Discounted Dividends from Investment Gain

First, we construct a recursive formula to calculate the expected total discounted dividend payments the external fund generates until time $\tau \in \mathbb{N}$ or until ruin occurs at time $T \in \mathbb{Z}^+$ whichever happens first. Let $R_{2,t}$ represent the amount of dividend from investment gain distributed at time t. Its value is random since it is determined by the level of the external fund, which depends on the inter-claim time and the claim size distributions. Then for some discount rate $\nu \in (0,1)$, we denote $\Gamma(u,f,\tau)$ to be the expected total discounted dividends from investment gain which corresponds to the time interval $(0, \min\{\tau, T\}]$, with the initial surplus level and the initial external fund level are equal to u and f respectively.

$$\Gamma(u,f,\tau) = E\left(\sum_{i=1}^{\min\{\tau,T\}} \nu^i R_{2,i} \middle| U_0 = u, F_0 = f \right), \quad u \in \mathbb{N},\ f \in \{l_4, l_4{+}1, ...\},\ \tau \in \mathbb{N}, \tag{5.1}$$

We also denote $\Gamma^*(u,f,\tau,k)$ to be conditional expected total discounted dividends from investment gain, given that the first claim X_i takes place at time $Y_1 = k$. And to find $\Gamma(u,f,\tau)$ we accumulate the weighted sum of $\Gamma^*(u,f,\tau,k)$ for all possible values of k, which is the support set of the inter-claim time distribution.

$$\Gamma(u,f,\tau) = \sum_{k=1}^{n_a} a_k \Gamma^*(u,f,\tau,k), \tag{5.2}$$

$$\Gamma^*(u,f,\tau,k) = E\left(\sum_{i=1}^{\min\{\tau,T\}} \nu^i R_{2,i} \middle| U_0 = u, F_0 = f, Y_1 = k \right), \quad k = 1,2,...,n_a. \tag{5.3}$$

The value of $\Gamma^*(u,f,\tau,k)$ is independent from the first claim X_1 if it occurs after the time point τ, and we can compute the conditional expectation by accumulating the discounted investment gain (or only a portion x of it for waqf model) starting from $s_{(u,f)}$, which is the earliest time point $\hat{F}_{(t^+,u,f)}$ reaches l_5, until τ. On the other hand, the value of $\Gamma^*(u,f,\tau,k)$ depends on the first claim X_1 of size j if it occurs at or before τ, and we can compute the conditional expectation by accumulating the discounted investment gain (or only a portion x of it for waqf model) starting from $s_{(u,f)}$ until k plus the discounted weighted sum of $\Gamma(u^*, f^*, \tau - k)$, which is the expected total discounted dividends from investment gain with the level of surplus process and external fund at time k after the first claim payment are equal to u^* and f^* respectively, for all possible values of j that do not cause ruin at time k. Therefore, in this case j is bounded above by the amount of total available fund in both the surplus and the external fund at time k before the first claim payment plus the loan that the operator can provide.

$$\Gamma^*(u,f,\tau,k) = \begin{cases} \displaystyle\sum_{i=s_{(u,f)}}^{k-1} \nu^{i+1}\pi\hat{F}_{(i^+,u,f)} + \sum_{j=1}^{\hat{U}_{(k,u)}+\lfloor \hat{F}_{(k,u,f)}\rfloor - l_4} \nu^k \alpha_j(k)\Gamma(u^*,f^*,\tau-k) \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } k \in \{1,2,...,\min\{n_a,\tau\}\}, \\ \displaystyle\sum_{i=s_{(u,f)}}^{\tau-1} \nu^{i+1}\pi\hat{F}_{(i^+,u,f)} \qquad\qquad \text{if } k \in \{\min\{n_a,\tau\}+1,...,n_a\}, \end{cases} \tag{5.4}$$

$$\pi = \begin{cases} \kappa & \text{for non-waqf model,} \\ x\kappa & \text{for waqf model.} \end{cases} \tag{5.5}$$

We recall that for waqf model $s_{(u,f)}$ is equal to $r_{(u,f)}$, which is which is the earliest time point $\hat{F}_{(t^+,u,f)}$ reaches 0, since for this business model we adopt the convention $l_5 = 0$. We recall also that the amount of $\hat{F}_{(t^+,u,f)}$ depends on whether there is a deposit from $\hat{U}_{(t,u)}$ at time t, i.e.,

$$\hat{F}_{(t^+,u,f)} = \begin{cases} \hat{F}_{(t,u,f)} & \text{if } \hat{U}_{(t,u)} < l_2, \\ \hat{F}_{(t,u,f)} + d & \text{if } \hat{U}_{(t,u)} \geq l_2. \end{cases} \tag{5.6}$$

The base condition for the recursive equation is $\Gamma(u,f,0)$ whose value is equal to 0 due to the simple fact that no investment activities has yet been made at this point.

$$\Gamma(u,f,0) = 0. \tag{5.7}$$

5.2 Expected Total Discounted Dividends from Underwriting Surplus

Here we construct a recursive formula to calculate the expected total discounted dividend payments the surplus process generates until time $\tau \in \mathbb{N}$ or until ruin occurs at time $T \in \mathbb{Z}^+$ whichever happens first. Let $R_{1,t}$ represent the amount of dividend from underwriting surplus distributed at time t. Its value is random since it is determined by the level of the surplus process, which depends on the inter-claim time and the claim size distributions. Then for some discount rate $\nu \in (0,1)$, we denote $\Upsilon(u,f,\tau)$ to be the expected total discounted dividends from underwriting surplus which corresponds to the time interval $(0, \min\{\tau, T\}]$, with the initial surplus level and the initial external fund level are equal to u and f respectively.

$$\Upsilon(u,f,\tau) = E\left(\sum_{i=0}^{\min\{\tau,T\}} \nu^i R_{1,i} \middle| U_0 = u, F_0 = f \right), \quad u \in \mathbb{N},\ f \in \{l_4, l_4{+}1, ...\},\ \tau \in \mathbb{N}, \tag{5.8}$$

We also denote $\Upsilon^*(u,f,\tau,k)$ to be conditional expected total discounted dividends from underwriting surplus, given that the first claim X_i takes place at time $Y_1 = k$. And to find $\Upsilon(u,f,\tau)$ we accumulate the weighted sum of $\Upsilon^*(u,f,\tau,k)$ for all possible values of k, which is the support set of the inter-claim time distribution.

$$\Upsilon(u,f,\tau) = \sum_{k=1}^{n_a} a_k \Upsilon^*(u,f,\tau,k), \tag{5.9}$$

$$\Upsilon^*(u,f,\tau,k) = E\left(\sum_{i=0}^{\min\{\tau,T\}} \nu^i R_{1,i} \middle| U_0 = u, F_0 = f, Y_1 = k \right), \quad k = 1,2,...,n_a. \tag{5.10}$$

The value of $\Upsilon^*(u,f,\tau,k)$ is independent from the first claim X_1 if it occurs after the time point τ, and we can compute the conditional expectation by accumulating the discounted underwriting surplus starting from q_u, which is the earliest time point $\hat{U}_{(t,u)}$ reaches l_3, until τ. On the other hand, the value of $\Upsilon^*(u,f,\tau,k)$ depends on the first claim X_1 of size j if it occurs at or before τ, and we can compute the conditional expectation by accumulating the discounted underwriting surplus starting from q_u until k plus the discounted weighted sum of $\Upsilon(u^*, f^*, \tau - k)$, which is the expected total discounted dividends from underwriting surplus with the level of surplus process and external fund at time k after the first claim payment are equal to u^* and f^* respectively, for all possible values of j that do not cause ruin at time k. Therefore, in this case j is bounded above by the amount of total available fund in both the surplus and the external fund at time k before the first claim payment plus the loan that the operator can provide.

$$
\Upsilon^*(u,f,\tau,k) = \begin{cases} \sum_{i=q_u}^{k-1} \nu^i(\hat{U}_{(i,u)} - l_3) + \sum_{j=1}^{\hat{U}_{(k,u)}+\lfloor \hat{F}_{(k,u,f)} \rfloor - l_4} \nu^k \alpha_j(k) \Upsilon(u^*, f^*, \tau - k) \\ \qquad\qquad\qquad\qquad\qquad \text{if } k \in \{1, 2, ..., \min\{n_a, \tau\}\}, \\ \sum_{i=q_u}^{\tau} \nu^i(\hat{U}_{(i,u)} - l_3) \qquad\qquad \text{if } k \in \{\min\{n_a,\tau\}+1, ..., n_a\}. \end{cases} \tag{5.11}
$$

The base condition for the recursive equation is $\Upsilon(u, f, 0)$ whose value is equal to the excess of u from l_3.

$$
\Upsilon(u, f, 0) = (u - l_3)_+. \tag{5.12}
$$

5.3 Expected Total Discounted Fee Payments

Here we construct a recursive formula to calculate the expected total expected total discounted fee payments received by the the operator/shareholders until time $\tau \in \mathbb{N}$ or until ruin occurs at time $T \in \mathbb{Z}^+$ whichever happens first. Let a constant $\acute{c} \in \mathbb{R}^+$ represent the amount one single fee payment the participants pay to the operator at the beginning of each term. Even though $\acute{c}$ is not random, the total discounted fee is stochastic in a sense that ruin may or occur before or after τ. Then for some discount rate $\nu \in (0, 1)$, we denote $\Phi(u, f, \tau)$ to be the expected total discounted fee payments which corresponds to the time interval $(0, \min\{\tau, T\}]$, with the initial surplus level and the initial external fund level are equal to u and f respectively.

$$
\Phi(u,f,\tau) = E\left(\sum_{i=0}^{\min\{\tau,T\}-1} \nu^i \acute{c} \,\middle|\, U_0 = u, F_0 = f \right), \quad u \in \mathbb{N},\ f \in \{l_4, l_4{+}1, ...\},\ \tau \in \mathbb{N}, \tag{5.13}
$$

We also denote $\Phi^*(u,f,\tau,k)$ to be conditional expected total discounted fee payments, given that the first claim X_i takes place at time $Y_1 = k$. And to find $\Phi(u,f,\tau)$ we accumulate the weighted sum of $\Phi^*(u,f,\tau,k)$ for all possible values of k, which is the support set of the inter-claim time distribution.

$$\Phi(u,f,\tau) = \sum_{k=1}^{n_a} a_k \Phi^*(u,f,\tau,k), \tag{5.14}$$

$$\Phi^*(u,f,\tau,k) = E\left(\sum_{i=0}^{\min\{\tau,T\}-1} \nu^i \acute{c} \middle| U_0 = u, F_0 = f, Y_1 = k \right), \quad k = 1,2,...,n_a. \tag{5.15}$$

The value of $\Phi^*(u,f,\tau,k)$ is independent from the first claim X_1 if it occurs after the time point τ, and we can compute the conditional expectation by accumulating the discounted fee payments starting from $t = 0$ until τ. On the other hand, the value of $\Phi^*(u,f,\tau,k)$ depends on the first claim X_1 of size j if it occurs at or before τ, and we can compute the conditional expectation by accumulating discounted fee payments starting from $t = 0$ until k plus the discounted weighted sum of $\Phi(u^*, f^*, \tau - k)$, which is the expected total discounted fee payment with the level of surplus process and external fund at time k after the first claim payment are equal to u^* and f^* respectively, for all possible values of j that do not cause ruin at time k. Therefore, in this case j is bounded above by the amount of total available fund in both the surplus and the external fund at time k before the first claim payment plus the loan that the operator can provide.

$$\Phi^*(u,f,\tau,k) = \begin{cases} \sum_{i=0}^{k-1} \nu^i \acute{c} + \sum_{j=1}^{\hat{U}_{(k,u)} + \lfloor \hat{F}_{(k,u,f)} \rfloor - l_4} \nu^k \alpha_j(k) \Phi(u^*, f^*, \tau - k) & \text{if } k \in \{1,2,...,\min\{n_a,\tau\}\}, \\ \sum_{i=0}^{\tau-1} \nu^i \acute{c} & \text{if } k \in \{\min\{n_a,\tau\}+1,...,n_a\}, \end{cases} \tag{5.16}$$

$$= \begin{cases} \acute{c}\frac{1-\nu^k}{1-\nu} + \sum_{j=1}^{\hat{U}_{(k,u)} + \lfloor \hat{F}_{(k,u,f)} \rfloor - l_4} \nu^k \alpha_j(k) \Phi(u^*, f^*, \tau - k) & \text{if } k \in \{1,2,...,\min\{n_a,\tau\}\}, \\ \acute{c}\frac{1-\nu^\tau}{1-\nu} & \text{if } k \in \{\min\{n_a,\tau\}+1,...,n_a\}. \end{cases}$$

$$\Phi(u,f,0) = 0. \tag{5.17}$$

5.4 Expected Total Discounted Dividend For Different Takaful Business Models

Finally, we develop several algorithms representing the expected total discounted dividends by combining all of the equations above. These formulas are constructed for both shareholders and participants of each Takaful business model introduced in Chapter 2. Let $x \in (0,1)$ be the share of dividends from investment gain that is distributed to shareholders which corresponds to mudarabah model, modified mudarabah model, and hybrid model. Let also $y \in (0,1)$ be the share of dividends from underwriting surplus that is distributed to shareholders which corresponds to modified mudarabah model. Additionally, $\acute{c} \in \mathbb{R}^+$ denotes the operator's fee corresponding to wakalah model, hybrid model, and waqf model. Moreover, under waqf model the operator is entitled to the whole distributed investment gain, however only a portion x is distributed since the rest stays in the Takaful fund.

The following table gives the expected total discounted dividends for different Takaful models utilized by a Takaful operator who started its business with initial level of surplus process at $v \in \{l_1, l_1+1, ...\}$ and initial external fund at $g \in \mathbb{N}$ (with $v = g = 0$ for non-waqf model). Denoted by $O(v, g, \tau)$ is the the expected total discounted dividends from investment gain and underwriting surplus plus the operator's fee distributed to the operator/shareholders at time $t = \tau$. The share of the expected total discounted dividends from investment gain and underwriting surplus minus the operator's fee for the participants is denoted by $P(v, g, \tau)$.

Table 5.1: Expected total discounted dividend payments for different Takaful business models.

Form	$O(v,g,\tau)$	$P(v,g,\tau)$
Mudarabah	$x\Gamma(0,0,\tau)$	$(1-x)\Gamma(0,0,\tau) + \Upsilon(0,0,\tau)$
Mdf. Mud.	$x\Gamma(0,0,\tau) + y\Upsilon(0,0,\tau)$	$(1-x)\Gamma(0,0,\tau) + (1-y)\Upsilon(0,0,\tau)$
Wakalah	$\Phi(0,0,\tau)$	$\Gamma(0,0,\tau) + \Upsilon(0,0,\tau) - \Phi(0,0,\tau)$
Hybrid	$x\Gamma(0,0,\tau) + \Phi(0,0,\tau)$	$(1-x)\Gamma(0,0,\tau) + \Upsilon(0,0,\tau) - \Phi(0,0,\tau)$
Waqf	$\Gamma(v,g,\tau) + \Phi(v,g,\tau)$	$\Upsilon(v,g,\tau) - \Phi(v,g,\tau)$

CHAPTER 6

NUMERICAL RESULTS

In this chapter, the algorithms from Chapter 4 and Chapter 5 are implemented under some specific settings to calculate some ruin-related quantities associated with our proposed risk model. The calculations employ Monte Carlo simulation using a symbolic mathematical computation program named *Wolfram Mathematica*. By examining the numerical results from our simulations, we aim to investigate the behavior of the risk model and make some observations on its application to Takaful insurance. Firstly, we intend to analyze the efficiency of Takaful insurance by comparing its ruin-related quantities obtained from our study with the case of conventional insurance which corresponds to the work by Kim and Drekic [20]. For this reason, all claim-related distributions used in this chapter are taken from the simulation by Kim and Drekic [20], in which the claim size distribution is assumed to be independent from the inter-claim time distribution (i.e., $\alpha_j(k) = \alpha_j$). Secondly, we examine the sensitivity of some parameters in the risk model to analyze their impact on finite-time ruin probabilities and expected total discounted dividend payments. Lastly, we aim to investigate the profitability of Takaful operators employing different business models by comparing the expected total discounted dividends paid to their shareholders. A set of inter-claim time distributions $(a_k = Pr\{Y_i = k\})$ used in our simulations are given as follows:

$$\text{(a)} \quad a_k = \begin{cases} (2/11)(9/11)^{k-1} & \text{if } k = 1, 2, ..., 24, \\ (9/11)^{24} & \text{if } k = 25. \end{cases} \tag{6.1}$$

$$\text{(b)} \quad a_k = 1/10, \quad k = 1, 2, ..., 10. \tag{6.2}$$

$$\text{(c)} \quad a_k = \frac{1}{1-(39/50)^{25}} \binom{25}{k} (11/50)^k (39/50)^{25-k}, \quad k = 1, 2, ..., 25. \tag{6.3}$$

$$\text{(d)} \quad a_k = \begin{cases} (0.645)(1/2)^k + (0.355)(1/12)(11/12)^{k-1} & \text{if } k = 1, 2, ..., 14, \\ (0.645)(1/2)^{14} + (0.355)(1/12)(11/12)^{14} & \text{if } k = 15, \\ (0.355)(1/12)(11/12)^{k-1} & \text{if } k = 16, 17, ..., 49, \\ (0.355)(11/12)^{49} & \text{if } k = 50. \end{cases} \tag{6.4}$$

We note that (a) is the pmf of a truncated geometric distribution with $n_a = 25$; (b) is the pmf of a uniform distribution on $1, 2, ..., 10$; (c) is the pmf of a zero-truncated binomial distribution with $n_a = 25$; and (d) is the pmf of a mixture of two truncated geometric distributions with $n_a = 50$. The means of the four inter-claim time distributions are approximately equal to 5.5, but their variances are different with (c) being the least variable and (d) being the most variable.

As for the the claim size distribution ($\alpha_j = Pr\{X_i = j\}$), we utilize a discretized version of the Pareto distribution with mean 10, whose expression is given as follows:

$$\alpha_j = \left(1 + \frac{j-1}{30}\right)^{-4} - \left(1 + \frac{j}{30}\right)^{-4}, \quad j \in \mathbb{Z}^+. \tag{6.5}$$

6.1 Ruin-Related Quantities Comparison between Takaful and Conventional Insurance

For the simulations in this section we make use the same parameters from Kim and Drekic [20] in order to obtain comparable results from our study (i.e. $v = 10$, $g = 0$, $l_1 = 0$, $l_2 = 20$, $l_3 = 50$, $c = 5$, $d = 1$, $\kappa = 0.01$, and $\nu = 0.75$). However, there are some uncommon parameters that do not belong to both our risk model and KD risk model. In particular, while the Takaful fund risk model does not utilize any interest rate when its external fund is in debt position, Kim and Drekic [20] assign an interest rate $\kappa_2 = 0.02$ in their risk model. Additionally, the only source of dividend payments in their risk model is the underwriting surplus, where Kim and Drekic [20] define their amount to be random instead of the excess from the threshold level l_3. Further, they consider a degenerate distribution with all the probability mass on 2 (i.e., $c_1 = c_2 = 2$, so that $b_2 = 1$) for the distribution of the random dividend payments.

Takaful business model employed in this section is waqf model, which is the only business model that can assign a positive value to v or g. Therefore, following our convention for waqf model, we set $l_5 = 0$ and use $\acute{\kappa} = (1 - x)\kappa$ as the growth rate of the external fund since the portion $x \in (0, 1)$ of the investment gain is distributed as dividends to the operator/shareholders. Specifically, we take the optimum value $x = 0.5$ so that the investment gain is shared equally between the shareholders and the Takaful fund.

Some ruin-related quantities of both Takaful insurance and conventional insurance for $\tau = 25, 50, 75$ with changing values of l_4 (denoted by β in KD risk model) are given in Table 6.1 and Table 6.2 respectively. The numerical results in the tables are obtained by using the first inter-claim time distribution (a) given in Equation 6.1 and the claim size distribution given in Equation 6.5. $\Psi^{(KD)}(v, g, \tau)$ and $\Upsilon^{(KD)}(v, g, \tau)$ are the expression of finite-time ruin probability and expected total discounted dividends from underwriting surplus respectively associated with the risk model introduced by Kim and Drekic [20].

Table 6.1: Ruin-related quantities of Takaful insurance employing waqf model with $l_1 = 0$, $l_2 = 20$, $l_3 = 50$, $l_5 = 0$, $c = 5$, $d = 1$, $\nu = 0.75$, $\kappa = 0.01$, and $x = 0.5$.

l_4	$\Psi(10, 0, 25)$	$\Psi(10, 0, 50)$	$\Psi(10, 0, 75)$	$\Gamma(10, 0, 75)$	$\Upsilon(10, 0, 75)$
0	0.175706	0.202137	0.213999	0.0247213	0.253017
-4	0.143771	0.168367	0.179699	0.0248108	0.254967
-8	0.118909	0.141663	0.152407	0.0248297	0.256363
-12	0.0992321	0.120196	0.130329	0.0248340	0.257386
-16	0.0834483	0.102708	0.112225	0.0248349	0.258150
-20	0.0706449	0.0883024	0.0972144	0.0248351	0.258731

Table 6.2: Ruin-related quantities of conventional insurance with $l_1 = 0$, $l_2 = 20$, $l_3 = 50$, $c = 5$, $d = 1$, $\nu = 0.75$, $\kappa_1 = \kappa = 0.01$, and $\kappa_2 = 0.02$.

l_4	$\Psi^{(KD)}(10, 0, 25)$	$\Psi^{(KD)}(10, 0, 50)$	$\Psi^{(KD)}(10, 0, 75)$	$\Upsilon^{(KD)}(10, 0, 75)$
0	0.174830	0.196614	0.204672	0.248444
-4	0.144086	0.164662	0.172498	0.250317
-8	0.119948	0.139303	0.146891	0.251686
-12	0.100726	0.118899	0.126230	0.252692
-16	0.0852287	0.102274	0.109353	0.253445
-20	0.0726360	0.0886259	0.0954679	0.254016

As the minimum possible level of the external fund l_4 is reduced (i.e., the maximum loan amount is increased), the finite-time ruin probabilities decrease monotonically for all $\tau \leq 75$ for both Takaful and conventional insurance under the specific setting considered here. Having more funds available has greater benefit to Takaful insurance because the loan from the Takaful operator has no interest rate. Hence, the ruin probabilities of Takaful insurance decrease more rapidly when l_4 is reduced. We observe that for $\tau = 25, 50$, the finite-time ruin probabilities of Takaful insurance is lower than the ruin probabilities of conventional insurance when $l_4 = 20$, even though they were higher when $l_4 = 0$. The ruin probability comparison is better illustrated in Figure 6.1. Table 6.1 and Table 6.2 also show that the expected total discounted dividends from underwriting surplus generated by Takaful insurance is higher than conventional insurance since it is defined to be the excess of fund in the surplus process from l_3. Higher dividend payments in Takaful insurance case can result in also higher ruin probabilities if we ignore the effect of the interest rate in KD risk model, which can be observed when $l_4 = 0$. However, under waqf model the dividends from underwriting surplus is not distributed to the operator/shareholders, which is the case in conventional insurance. Alternatively, the Takaful operator gets its compensation from half of the investment gain generated in every term plus regular fixed fees from the participants. Although under the specific setting, the expected total discounted dividends from investment gain is relatively small compared to the expected total discounted dividends from underwriting surplus. For all cases in Table 6.1, the operator can be expected to get higher total dividends compared to the conventional insurer if it charges regular fee payment of 0.07 from the participants, which is equal to 1.4% of the contribution rate.

Similar results shown in Table 6.3 and Table 6.4 are obtained when we use different distributions for the inter-claim time, which are given in Equations 6.1, 6.2, 6.3, and 6.4. In other words, for lower τ and lower l_4 the finite-time ruin probabilities of Takaful insurance are lower than its conventional counterpart. However, due to the distribution of investment gain and higher underwriting surplus, the finite-time probabilities of Takaful insurance becomes higher then conventional insurance for higher τ. Since the dividend payments from underwriting surplus is not distributed to the shareholders, the conventional insurer seems to be more profitable than the operator employing waqf model if we consider only investment gain as the source of dividends. Accordingly, the operator can get extra income from charging regular fixed fee payments from the participants. For all cases in Table 6.3, the operator can be expected to get higher total dividends compared to the conventional insurer if it charges regular fee payment of 0.06 from the participants, which is equal to 1.2% of the contribution rate.

Table 6.3: Ruin-related quantities of Takaful insurance employing waqf model with $l_1 = 0$, $l_2 = 20$, $l_3 = 50$, $l_4 = -10$, $l_5 = 0$, $c = 5$, $d = 1$, $\nu = 0.75$, $\kappa = 0.01$, and $x = 0.5$.

a_j	$\Psi(10,0,25)$	$\Psi(10,0,50)$	$\Psi(10,0,75)$	$\Gamma(10,0,75)$	$\Upsilon(10,0,75)$
(a)	0.108514	0.130364	0.140804	0.0248326	0.256913
(b)	0.0737698	0.0919374	0.100546	0.0279442	0.244161
(c)	0.0578656	0.0744973	0.0824136	0.0299859	0.238418
(d)	0.193628	0.224262	0.239616	0.0195488	0.242705

Table 6.4: Ruin-related quantities of conventional insurance with $l_1 = 0$, $l_2 = 20$, $l_3 = 50$, $l_4 = -10$, $c = 5$, $d = 1$, $\nu = 0.75$, $\kappa_1 = \kappa = 0.01$, and $\kappa_2 = 0.02$.

a_j	$\Psi^{(KD)}(10,0,25)$	$\Psi^{(KD)}(10,0,50)$	$\Psi^{(KD)}(10,0,75)$	$\Upsilon^{(KD)}(10,0,75)$
(a)	0.109811	0.128569	0.136029	0.252225
(b)	0.0739737	0.0893741	0.0955145	0.249026
(c)	0.0577812	0.0719240	0.0775949	0.247518
(d)	0.198521	0.225533	0.241374	0.227710

We observe that the variance of (a), (b), (c), and (d) are 22.96, 8.25, 4.24, and 64.89 respectively. For both Takaful and conventional insurance under this setting, higher variance of the inter-claim time distribution results in higher ruin probabilities. Moreover, Table 6.1 and Table 6.3 show inverse relationship between the finite-time ruin probabilities and the expected total discounted dividends from investment gain in our risk model. In other words, the lower the probability of ruin occurrence, the higher the expected total discounted dividends from investment gain. On the other hand, Table 6.3 and Table 6.4 do not show any relation between the ruin probabilities and the expected total discounted dividends from underwriting surplus for both Takaful and conventional insurance. The expected total discounted dividends from underwriting surplus is highest for (a) in both Takaful and conventional insurance risk model , lowest for (d) in conventional insurance risk model, but lowest for (c) in Takaful risk model.

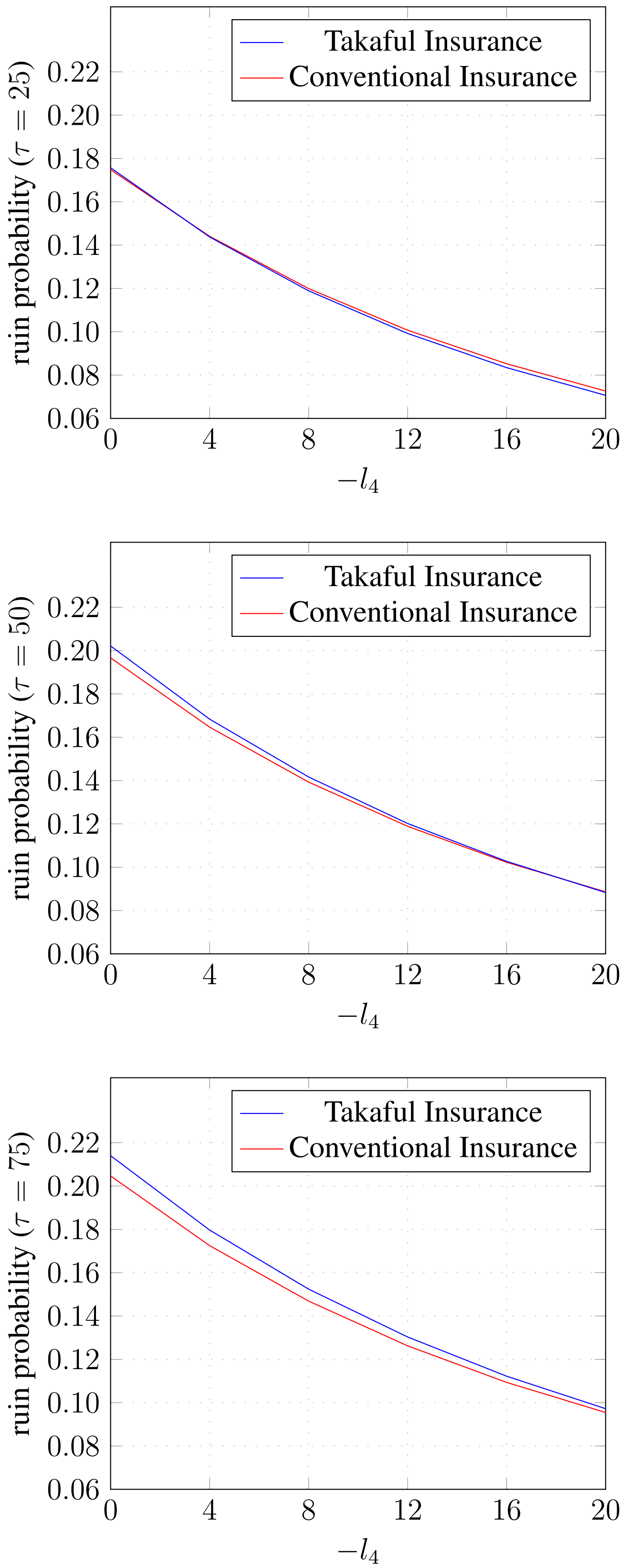


Figure 6.1: Ruin probability comparison between Takaful insurance employing waqf model and conventional insurance, based on the settings in Table 6.1 and Table 6.2.

6.2 Sensitivity Analyses

In this section we investigate the behavior of finite-time ruin probabilities ($\Psi(v, g, \tau)$), expected total discounted dividends from investment gain ($\Gamma(v, g, \tau)$), expected total discounted dividends from underwriting surplus ($\Upsilon(v, g, \tau)$), and expected total discounted fee payments ($\Phi(v, g, \tau)$) toward changing values of some parameters in our risk model. For all simulations in this section we employ non-waqf model with $v = g = 0$, $\acute{c} = 0.2$, $l_1 = 0$, $\kappa = 0.01$, and $\nu = (1+\kappa)^{-1}$. Further, the inter-claim time distribution (b) given in Equation 6.2 and the claim size distribution given in Equation 6.5 are utilized for the simulations in this section.

Table 6.5: Ruin-related quantities of Takaful insurance employing non-waqf model with $l_1 = 0$, $l_2 = 10$, $l_3 = 40$, $l_4 = -10$, $l_5 = 0$, $c = 5$, $\acute{c} = 0.2$, $\kappa = 0.01$, and $\nu = 1/1.01$.

Finite-Time Ruin Probabilities

d	$\Psi(0,0,15)$	$\Psi(0,0,30)$	$\Psi(0,0,45)$	$\Psi(0,0,60)$	$\Psi(0,0,75)$
1	0.104112	0.128140	0.142272	0.151434	0.157710
2	0.103311	0.119241	0.125635	0.128860	0.130670
3	0.103281	0.117225	0.121422	0.123153	0.123991
4	0.103281	0.117153	0.121129	0.122663	0.123361

Expected Total Discounted Dividends from Investment Gain

d	$\Gamma(0,0,15)$	$\Gamma(0,0,30)$	$\Gamma(0,0,45)$	$\Gamma(0,0,60)$	$\Gamma(0,0,75)$
1	0.647710	2.63181	5.53250	9.01661	12.8447
2	1.28210	5.22575	11.0658	18.1657	26.0480
3	1.86642	7.53018	15.9694	26.3066	37.8469
4	2.36099	9.26592	19.3831	31.6859	45.3745

Expected Total Discounted Dividends from Underwriting Surplus

d	$\Upsilon(0,0,15)$	$\Upsilon(0,0,30)$	$\Upsilon(0,0,45)$	$\Upsilon(0,0,60)$	$\Upsilon(0,0,75)$
1	6.26198	30.2256	50.7717	68.1244	82.8837
2	0.904899	13.1713	25.2814	35.6415	44.5133
3	0	1.73000	5.46695	9.05194	12.1894
4	0	0	0.00321832	0.0248885	0.0574872

Expected Total Discounted Fee Payments

d	$\Phi(0,0,15)$	$\Phi(0,0,30)$	$\Phi(0,0,45)$	$\Phi(0,0,60)$	$\Phi(0,0,75)$
1	2.61396	4.74589	6.54309	8.07028	9.37381
2	2.61410	4.75634	6.57964	8.14169	9.48331
3	2.61410	4.75823	6.58791	8.15881	9.50996
4	2.61410	4.75826	6.58830	8.15989	9.51190

First of all, the table above shows that the expected total discounted dividend from investment gain and the expected total discounted dividend from underwriting surplus increase rapidly with τ, whilst the expected total discounted fee payments seems to have constant growth. Notice that from the definition of expected total discounted fee payments, its amount for specific τ is proportional to the amount of one single fee payment $\acute{c}$. In other words, by increasing the amount of $\acute{c}$, we can increase the expected total discounted fee payments without affecting the other ruin-related quantities. Therefore under some settings, expected total discounted fee payments can be higher than expected total discounted dividends from investment gain or underwriting surplus for lower τ, but its amount eventually becomes the lowest for higher τ due to its slower growth rate.

From Table 6.5 we observe that that by raising the amount of d from 1 to 4 for $\tau \leq 75$, the finite-time ruin probabilities and the expected total discounted dividends from underwriting surplus decrease, whereas the expected total discounted dividends from investment gain and the expected fee payments increase. The reason is that higher amount of deposit d implies more fund is kept at the external fund and less fund remains in the surplus process. Therefore, the probability that the surplus process reaches the threshold level l_3 for underwriting surplus distribution is lower, and the probability of generating more investment return is higher. Further, lower dividends from underwriting surplus results in more total fund in the surplus process and the external fund that can be used to cover future claim payments, which then generates lower finite-time ruin probabilities. Consequently, lower ruin probabilities imply the probability that the Takaful system survives until later time points is higher, thus the expected total discounted fee payments from the participants increases.

In contrast, increasing the minimum acceptable threshold level l_1 results in more fund to be kept at the surplus process as reserve, hence less fund is transferred to the external fund to be invested. Therefore, less significant contrasting behavior can be observed when we increase the level of l_1 from 0 to 10 under the setting of $l_2 = 10$, $l_3 = 40$, $l_4 = -10$, $l_5 = 0$, $c = 5$, $d = 3$, $\acute{c} = 0.2$, $\kappa = 0.01$, and $\nu = 1/1.01$. For $\tau = 75$, the finite-time ruin probability increases from 0.123991 to 0.124074, the expected total discounted dividends from investment gain decreases from 37.8469 to 37.2793, the expected total discounted dividends from underwriting surplus increases from 12.1894 to 12.8467, and expected total discounted fee payments decreases from 9.50996 to 9.50968.

Contrasting behavior with Table 6.5 can also be observed in Table 6.6, where we increase the threshold level l_2 as the trigger level for deposit/investment activities. Higher l_2 implies more time is required for the surplus process to start making deposit to the external fund, hence investment activities begin at later time points, resulting in lower expected total discounted dividends from investment gain. On the other hand, by raising l_2, the surplus process requires less time to reach the threshold level l_3, which then generates higher expected total discounted dividends from underwriting surplus. Accordingly, less total negative fund in the surplus process and the external fund gives negative impact to the finite-time ruin probabilities and the expected total discounted fee payments.

Table 6.6: Ruin-related quantities of Takaful insurance employing non-waqf model with $l_1 = 0$, $l_3 = 40$, $l_4 = -10$, $l_5 = 0$, $c = 5$, $d = 3$, $\acute{c} = 0.2$, $\kappa = 0.01$, and $\nu = 1/1.01$.

Finite-Time Ruin Probabilities

l_2	$\Psi(0,0,15)$	$\Psi(0,0,30)$	$\Psi(0,0,45)$	$\Psi(0,0,60)$	$\Psi(0,0,75)$
10	0.103281	0.117225	0.121422	0.123153	0.123991
20	0.103281	0.117551	0.122105	0.124041	0.124995
30	0.103420	0.119169	0.124630	0.127048	0.128273
40	0.105043	0.125865	0.134321	0.138483	0.140775

Expected Total Discounted Dividends from Investment Gain

l_2	$\Gamma(0,0,15)$	$\Gamma(0,0,30)$	$\Gamma(0,0,45)$	$\Gamma(0,0,60)$	$\Gamma(0,0,75)$
10	1.86642	7.53018	15.9694	26.3066	37.8469
20	1.19588	6.04950	13.7014	23.2532	34.0164
30	0.671054	4.62850	11.2678	19.7176	29.3297
40	0.288434	3.00483	7.88101	14.2021	21.4490

Expected Total Discounted Dividends from Underwriting Surplus

l_2	$\Upsilon(0,0,15)$	$\Upsilon(0,0,30)$	$\Upsilon(0,0,45)$	$\Upsilon(0,0,60)$	$\Upsilon(0,0,75)$
10	0	1.73000	5.46695	9.05194	12.1894
20	0.0334652	4.38145	9.52301	13.9736	17.7921
30	1.43225	9.45646	16.4421	22.3972	27.5056
40	6.40044	20.1815	31.7473	41.5948	50.0303

Expected Total Discounted Fee Payments

l_2	$\Phi(0,0,15)$	$\Phi(0,0,30)$	$\Phi(0,0,45)$	$\Phi(0,0,60)$	$\Phi(0,0,75)$
10	2.61410	4.75823	6.58791	8.15881	9.50996
20	2.61410	4.75796	6.58660	8.15609	9.50578
30	2.61408	4.75594	6.58027	8.14480	9.48965
40	2.61350	4.74546	6.55276	8.09840	9.42481

The expected total discounted dividends comparison for $\tau = 1, 2, ..., 75$ based on the settings in Table 6.5 and Table 6.6 is better illustrated in Figure 6.2 and Figure 6.3 respectively. From these figures, we observe that the expected total discounted dividends from underwriting surplus is more sensitive than expected total discounted dividends from investment gain, whereas expected total discounted fee payments is the least sensitive. Moreover, we can also see that the graph of expected total discounted dividend from investment gain is concave up, whilst the the other graphs are concave down. Therefore under some settings, expected total discounted dividends from investment gain can be lower than expected total discounted dividends from underwriting surplus or fee payments for lower τ, but its amount eventually becomes the highest for higher τ due to exponential growth of the investment activities.

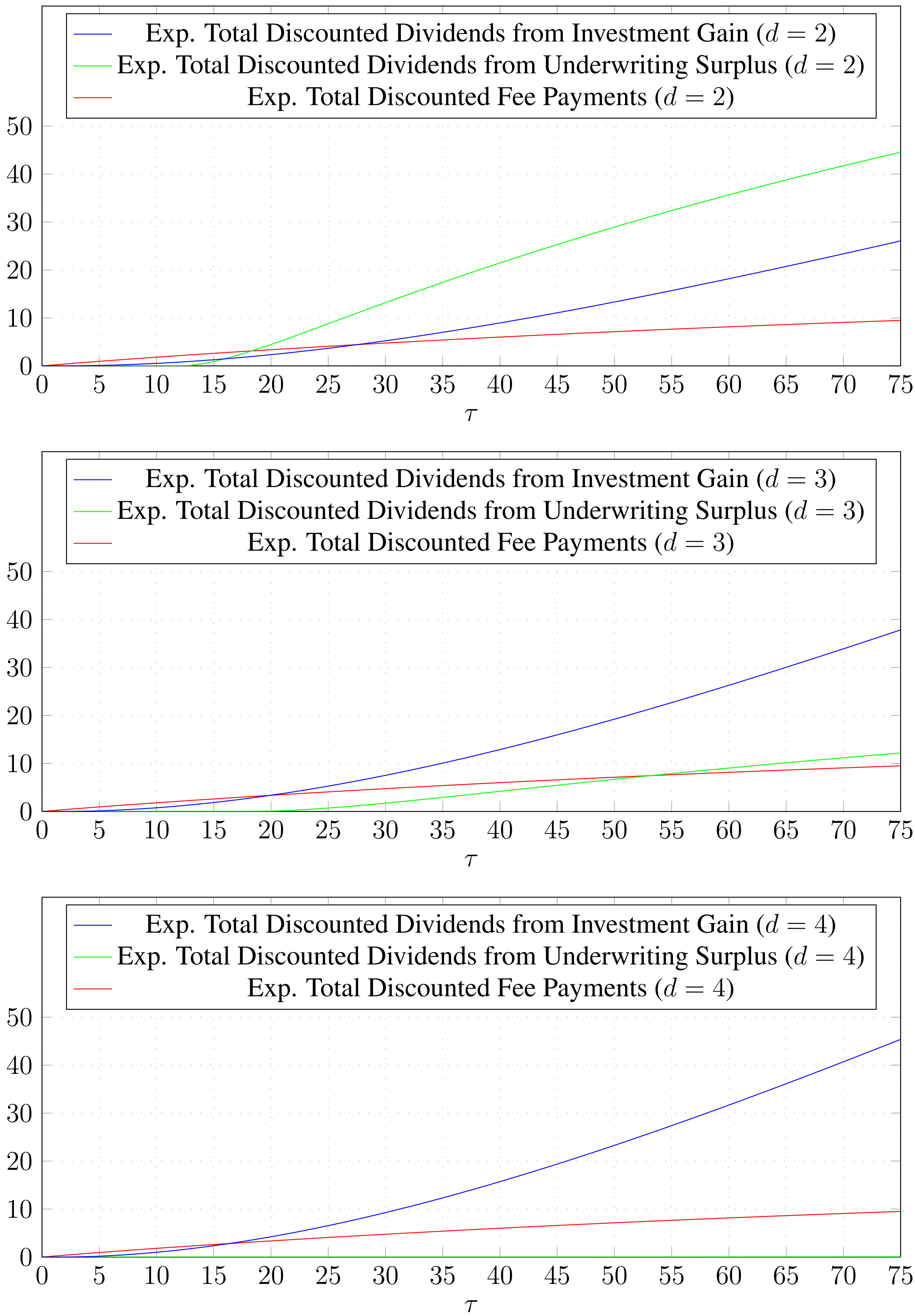


Figure 6.2: Expected total discounted dividends comparison based on the setting in Table 6.5.

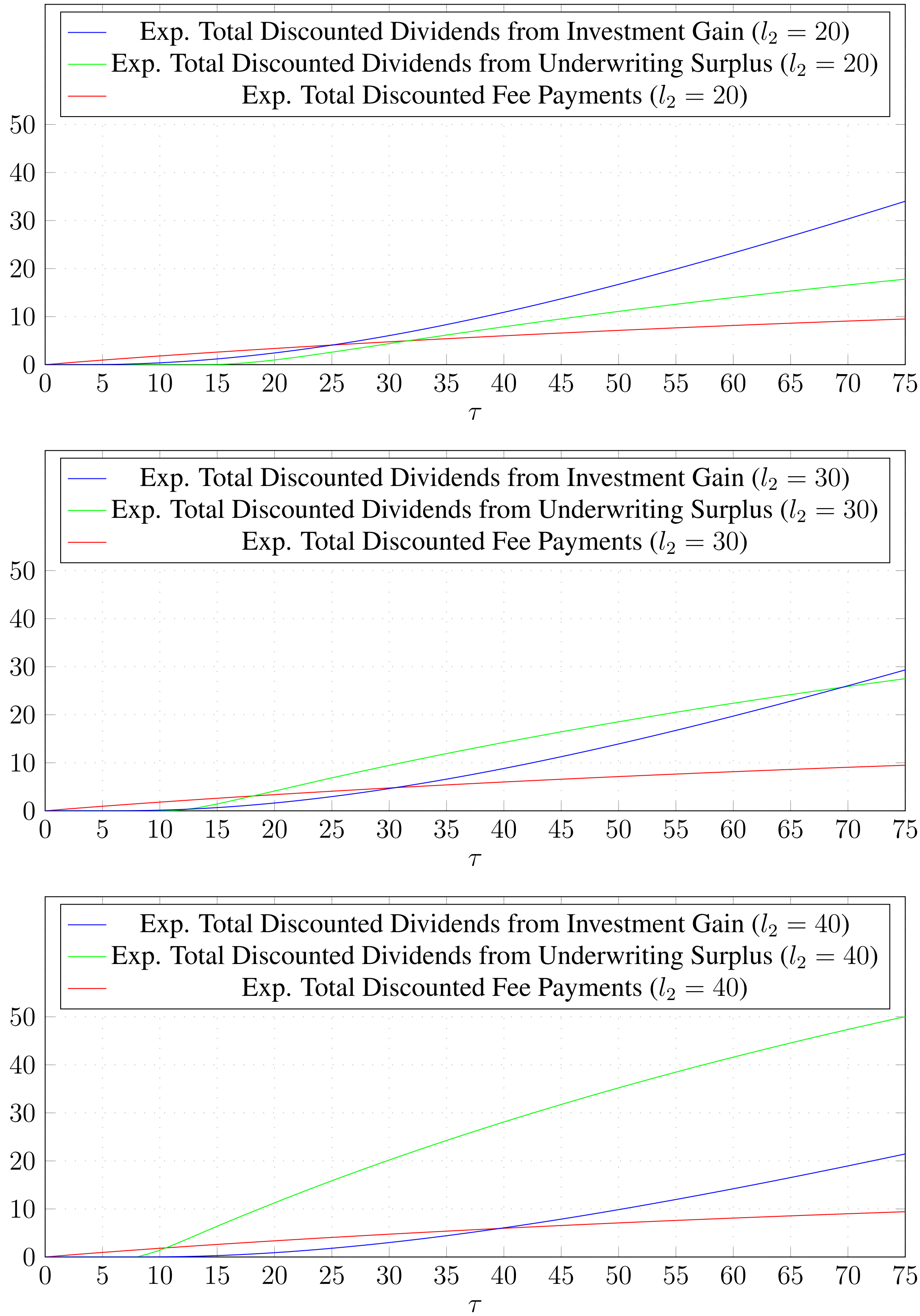


Figure 6.3: Expected total discounted dividends comparison based on the setting in Table 6.6.

Table 6.7: Ruin-related quantities of Takaful insurance employing non-waqf model with $l_1 = 0$, $l_2 = 10$, $l_4 = -10$, $l_5 = 0$, $c = 5$, $d = 3$, $\acute{c} = 0.2$, $\kappa = 0.01$, and $\nu = 1/1.01$.

Finite-Time Ruin Probabilities

l_3	$\Psi(0,0,15)$	$\Psi(0,0,30)$	$\Psi(0,0,45)$	$\Psi(0,0,60)$	$\Psi(0,0,75)$
40	0.103281	0.117225	0.121422	0.123153	0.123991
30	0.103293	0.117909	0.122646	0.124667	0.125664
20	0.105224	0.123302	0.129604	0.132407	0.133832
10	0.129712	0.162924	0.176460	0.183289	0.187148

Expected Total Discounted Dividends from Investment Gain

l_3	$\Gamma(0,0,15)$	$\Gamma(0,0,30)$	$\Gamma(0,0,45)$	$\Gamma(0,0,60)$	$\Gamma(0,0,75)$
40	1.86642	7.53018	15.9694	26.3066	37.8469
30	1.86641	7.50359	15.8123	25.9079	37.1207
20	1.85417	7.27306	15.0793	24.4573	34.8040
10	1.63329	5.91837	11.8223	18.7424	26.2553

Expected Total Discounted Dividends from Underwriting Surplus

l_3	$\Upsilon(0,0,15)$	$\Upsilon(0,0,30)$	$\Upsilon(0,0,45)$	$\Upsilon(0,0,60)$	$\Upsilon(0,0,75)$
40	0	1.73000	5.46695	9.05194	12.1894
30	0.392530	5.71499	11.0427	15.6144	19.5362
20	4.76734	13.5250	20.8947	27.1891	32.5915
10	16.4527	31.2825	43.7049	54.2646	63.2973

Expected Total Discounted Fee Payments

l_3	$\Phi(0,0,15)$	$\Phi(0,0,30)$	$\Phi(0,0,45)$	$\Phi(0,0,60)$	$\Phi(0,0,75)$
40	2.61410	4.75823	6.58791	8.15881	9.50996
30	2.61410	4.75753	6.58523	8.15368	9.50237
20	2.61312	4.74757	6.56239	8.11767	9.45410
10	2.58851	4.64475	6.36962	7.83744	9.09358

Using Table 6.7 we investigate the effect of reducing the threshold level l_3 as the trigger level for distribution of dividends from underwriting surplus. Lower level of l_3 implies more fund is distributed as dividends from underwriting surplus, which has negative impact on the availability of fund in the surplus process. For this reason, the finite-time ruin probabilities for lower l_3 are higher, which means the probability of surviving later time points is smaller. Accordingly, it reduces the expected total discounted dividends from investment gain and expected total discounted fee payments, which can be observed clearly as we set l_3 to be equal to $l_2 = 10$.

Table 6.8: Ruin-related quantities of Takaful insurance employing non-waqf model with $l_1 = 0$, $l_2 = 10$, $l_3 = 40$, $l_4 = -10$, $c = 5$, $d = 3$, $\acute{c} = 0.2$, $\kappa = 0.01$, and $\nu = 1/1.01$.

Finite-Time Ruin Probabilities

l_5	$\Psi(0,0,15)$	$\Psi(0,0,30)$	$\Psi(0,0,45)$	$\Psi(0,0,60)$	$\Psi(0,0,75)$
0	0.103281	0.117225	0.121422	0.123153	0.123991
50	0.102993	0.116143	0.120047	0.121668	0.122460
100	0.102993	0.115802	0.119139	0.120462	0.121111
150	0.102993	0.115802	0.119007	0.120112	0.120623

Expected Total Discounted Dividends from Investment Gain

l_5	$\Gamma(0,0,15)$	$\Gamma(0,0,30)$	$\Gamma(0,0,45)$	$\Gamma(0,0,60)$	$\Gamma(0,0,75)$
0	1.86642	7.53018	15.9694	26.3066	37.8469
50	0	4.51263	13.0845	23.6124	35.3346
100	0	0	6.86074	17.8350	30.1516
150	0	0	0.507320	10.3205	23.2667

Expected Total Discounted Dividends from Underwriting Surplus

l_5	$\Upsilon(0,0,15)$	$\Upsilon(0,0,30)$	$\Upsilon(0,0,45)$	$\Upsilon(0,0,60)$	$\Upsilon(0,0,75)$
0	0	1.73000	5.46695	9.05194	12.1894
50	0	1.73001	5.46795	9.05631	12.1982
100	0	1.73001	5.46800	9.05731	12.2017
150	0	1.73001	5.46800	9.05732	12.2020

Expected Total Discounted Fee Payments

l_5	$\Phi(0,0,15)$	$\Phi(0,0,30)$	$\Phi(0,0,45)$	$\Phi(0,0,60)$	$\Phi(0,0,75)$
0	2.61410	4.75823	6.58791	8.15881	9.50996
50	2.61422	4.76004	6.5923	8.16576	9.51924
100	2.61422	4.76026	6.59381	8.16917	9.52462
150	2.61422	4.76026	6.59389	8.16967	9.52577

In addition, from Table 6.8 we investigate the effect of increasing the threshold level l_5, which is the threshold level for investment gain distribution. Raising the level of l_5 results in less expected total discounted dividend from investment gain because more fund is kept at the external fund instead of distributed as dividends, which means more fund is available for future claim payments. Hence, from the table we can observe positive effect to the other ruin-related quantities as we increase the level of l_5. However, significant impact is only realized by the expected total discounted dividends from investment gain, which is illustrated in Figure 6.4.

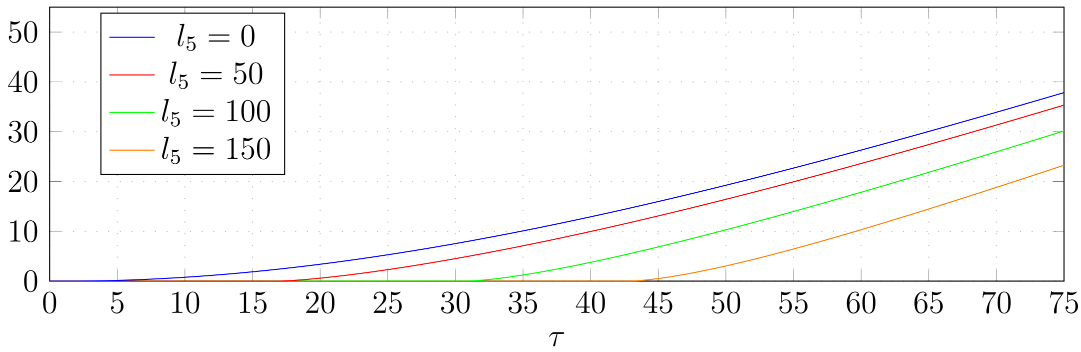


Figure 6.4: Comparison of expected total discounted dividends from investment gain based on the setting in Table 6.8.

Lastly, we analyze the change in size of the ruin-related quantities when we increase the amount of the contribution rate c from 5 to 6, under the setting of $l_1 = 0$, $l_2 = 10$, $l_3 = 40$, $l_4 = -10$, $l_5 = 0$, $d = 3$, $\acute{c} = 0.2$, $\kappa = 0.01$, and $\nu = 1/1.01$. For $\tau = 75$, there is a significant reduction of the finite-time ruin probability from 0.123991 to 0.0930544 and a significant increase of expected total discounted dividend from underwriting surplus from 12.1894 to 47.4903 due larger cash inflow from the participants to the Takaful fund. Moreover, for $\tau = 75$, the expected total discounted dividends from investment gain increases from 37.8469 to 41.4282, and the expected total discounted fee payments increases from 9.50996 to 9.77116 since the Takaful system is expected to survive until later time points.

From all the simulations in this section, we observe that the expected total discounted dividends from investment gain and fee payments are most likely to increase when the finite-time ruin probabilities decreases since their amount do not directly affected by the claim distributions. In other words, their amount depend mostly on the capability of the Takaful system to survive until later time points. To sum up, Table 6.9 shows the effect of each parameter in our risk model toward the ruin-related quantities based on the simulations performed in Section 6.1 and Section 6.2.

Table 6.9: Parameter influence on the ruin-related quantities.

Increasing Parameter	$\Psi(v, g, \tau)$	$\Gamma(v, g, \tau)$	$\Upsilon(v, g, \tau)$	$\Phi(v, g, \tau)$
d	$\downarrow$	$\Uparrow$	$\downarrow$	$\Uparrow$
l_1	$\Uparrow$	$\downarrow$	$\Uparrow$	$\downarrow$
l_2	$\Uparrow$	$\downarrow$	$\Uparrow$	$\downarrow$
l_3	$\downarrow$	$\Uparrow$	$\downarrow$	$\Uparrow$
l_4	$\Uparrow$	$\downarrow$	$\downarrow$	$\downarrow$
l_5	$\downarrow$	$\downarrow$	$\Uparrow$	$\Uparrow$
c	$\downarrow$	$\Uparrow$	$\Uparrow$	$\Uparrow$
$\acute{c}$	$-$	$-$	$-$	$\Uparrow$

6.3 Expected Total Discounted Dividends Comparison

In this section we investigate the profitability of each Takaful business models described in Chapter 2 by comparing the expected total discounted dividends paid to the operator/shareholders. The parameters used in this section are $l_1 = 0$, $l_2 = 10$, $l_3 = 40$, $l_4 = -10$, $l_5 = 0$, $c = 5$, $d = 3$, $\kappa = 0.01$, $\nu = (1 + \kappa)^{-1}$, and $\upsilon = 0$. For non-waqf model g is equal to 0, whereas for waqf model g is equal to 10, which is the amount of the initial waqf fund provided by the shareholders. The share x of investment gain, the share y of underwriting surplus, and the amount of individual fee payment $\acute{c}$ which are entitled to the operator are defined differently for each business model. In particular, we set $x = 50\%$ for the operator employing mudarabah model or waqf model, and we set $\acute{c} = 0.2$ for the operator employing wakalah model. On the other hand, for the operator employing hybrid model, it receives half the investment gain share of the operator employing mudarabah model and half fee payments of the operator employing wakalah model (i.e. $x = 25\%$ and $\acute{c} = 0.1$). Further, we also set $x = 25\%$ and $y = 75\%$ for the operator employing modified mudarabah model, and we set $\acute{c} = 0.1$ for the operator employing waqf model. Lastly, the inter-claim time distribution (b) given in Equation 6.2 and the claim size distribution given in Equation 6.5 are utilized for the simulations in this section.

The expected total discounted dividends for the operator/shareholders of each Takaful business model are illustrated in Figure 6.5. To begin with, we note that the initial waqf fund injection by the shareholders of waqf model obviously has positive impact to the ruin probability. The fact that half of the investment gain remains in the external fund of waqf model further helps the availability of fund for claim coverage. In fact our simulation shows that for $\tau = 75$, the finite-time ruin probability of non-waqf model is equal to 0.123991, whilst the finite-time ruin probability of waqf model is equal to 0.0759064.

Due to the steady nature of the expected total discounted fee payments, we observe that the operator employing wakalah model has a more stable expected income starting at early time points. However at later stages the expected total discounted dividends for the operator is lower than the other Takaful business models. Therefore wakalah model is more suitable for an operator that underwrites short term Takaful contracts, such as the case in most general Takaful business. On the other hand, lower expected total discounted dividends can be observed in mudarabah model at early time points with exponential growth over time, which means it is more acceptable for long term contracts, such as the case in family Takaful. Under this setting, modified version of mudarabah model is expected to offer more dividends than mudarabah model for $\tau = 42, 43, ..., 67$, but is expected to offer less at later time points because the graph of the expected total discounted dividends from underwriting surplus is concave down, whereas the graph of the expected total discounted dividends from investment gain is concave up. Hybrid model can be utilized to combine mudarabah model and wakalah model in order to help an operator at early time points in collecting enough fund for investment activities. Therefore, hybrid model as shown in Figure 6.5 is expected to generate more dividends than mudarabah model at early stages, and more dividends than wakalah model at later time points.

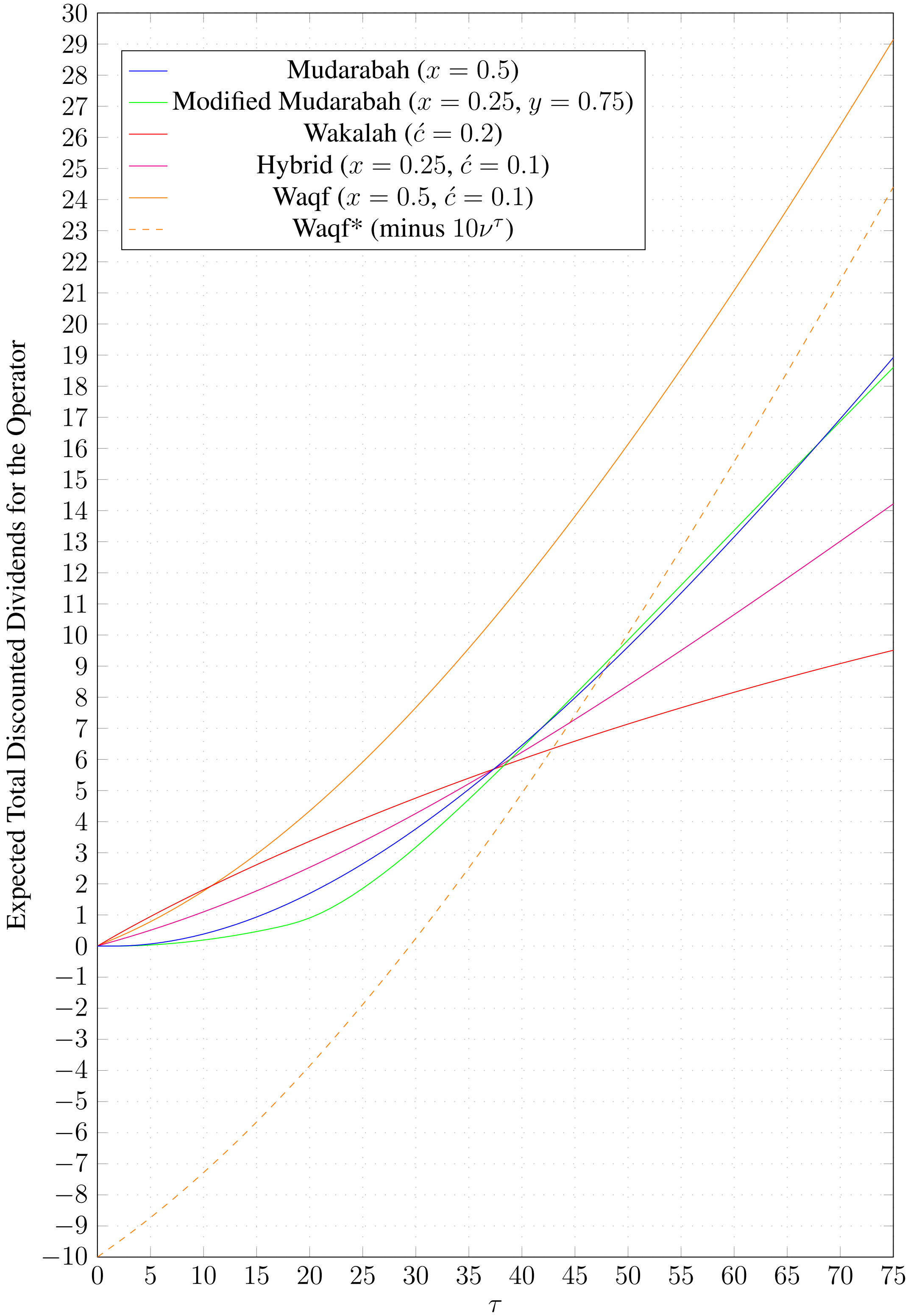


Figure 6.5: Expected total discounted dividends for different Takaful business models with $l_1 = 0$, $l_2 = 10$, $l_3 = 40$, $l_4 = -10$, $l_5 = 0$, $c = 5$, $d = 3$, $\kappa = 0.01$, $\nu = 1/1.01$, $v = 0$, $g = 0$ for non-waqf model, and $g = 10$ for waqf model.

Waqf model seems to offer more dividends compared to the other business models due to the initial waqf fund injection from the shareholders, which boosted the investment activities of the Takaful fund. However, the shareholders relinquish their ownership of the initial waqf capital when they creates the waqf fund, thus it requires time to generate enough dividends in order to get the amount of the initial waqf capital back. In Figure 6.5, we plot the graph of the expected total discounted dividends for the operator employing waqf model minus the present value of the initial waqf fund whose amount remains at constant $g = 10$ for all τ.

CHAPTER 7

CONCLUSION

The comparison between numerical results from our simulations and the results from Kim and Drekic [20], we observe that if the maximum loan amount is high enough, Takaful insurance (particularly with waqf model) has lower finite-time ruin probabilities than conventional insurance since its loan undertakings bear no interest rate. Furthermore, over time Takaful insurance is expected to generate higher dividend payments since the dividends can also be generated from investment gain in addition to underwriting surplus, which is the only source of dividends in KD risk model. However, higher amount of dividends payments from underwriting surplus and investment gain results in also higher ruin probabilities for Takaful insurance when we increase the time parameter. We also observe that under specific settings, the expected total discounted dividends from investment gain is much lower than the expected total discounted dividends from underwriting surplus. And since the operator employing waqf model is not entitled to the underwriting surplus, it can keep its competitiveness relying on the regular fee payments from the participants.

Considerable excess of fund in the surplus process and the external fund at the end of every term is not necessarily all distributed as dividends since some reserve fund is required to cover future claim payments. From our study we find that by changing the values of some parameters in our risk model, we can adjust the amount of reserve to be kept at the Takaful fund. Furthermore, we can set the amount of reserve allocated to the surplus process and the external fund separately so that we can adjust the amount of ruin probabilities and expected total discounted dividends associated with our risk model. For instance, more reserve allocated to the external fund results in higher expected total discounted dividends from investment gain but lower expected total discounted dividends from underwriting surplus. In contrast, more reserve allocated to the surplus process results in higher expected total discounted dividends from underwriting surplus, which also increases the ruin probabilities since underwriting surplus essentially reduces the amount of fund available for claim coverage. Consequently, high ruin probabilities have negative effect on the expected total discounted fee payments, whose amount depends only on the capability of the Takaful fund to survive later time points.

Another important observation from our findings is the significant positive impact of higher contribution rate on the ruin probabilities and expected total discounted dividends from underwriting surplus. Since the participants are entitled to the whole underwriting surplus (or its portion in modified mudarabah model), Takaful operators can consider this findings to keep high contribution rate attractive to the participants by increasing their awareness of dividend payments they are entitled to. Moreover, high contribution rate also increases the expected total discounted dividends from investment gain and expected total discounted fee payments. This information can be considered by the operator to adjust the contribution to fee ratio, i.e. the operator can increase the contribution rate and reduce the amount of individual fee payment in order to decrease the probability of ruin occurrence.

Our study shows that waqf model is better then the other business models in terms of ruin probabilities and expected total discounted dividends due to financial help from the initial waqf capital provided by the shareholders. Another reason is that waqf model is the only Takaful business model in which the investment gain is shared between the shareholders and the Takaful fund itself. However, since the shareholders lost their ownership of the initial waqf fund, it may take some time to generate dividends that cover the amount of the initial waqf capital. From our findings, we observe that wakalah model offers a steady expected total dividend income for the operator as the fee payments does not directly depend on the inter-claim time nor the claim size distributions. Therefore, it can be the most profitable business model for early time points, but eventually becomes the least profitable at later stages. On the other hand, we find that mudarabah model is less profitable at early time points since a number of termly contributions is required to achieve enough amount of fund to generate decent dividend payments from investment gain. Therefore, as the level of external fund increases over time, mudarabah model is expected to generate more profit then the other non-waqf business models. Modified version of mudarabah model can help the operator to extract more profit at early time points under some settings by gaining some shares of the underwriting surplus. However, we observe that expected total discounted dividends from underwriting surplus eventually becomes lower than expected total discounted dividend from investment gain. Moreover, there are severe critics from many scholars arguing the ownership of the underwriting surplus being solely the participants. Alternatively, a Takaful operator can combine mudarabah model and wakalah model into a hybrid model which is more profitable than mudarabah model at early time points and more profitable than wakalah model at later stages.

Our study can help an operator employing a specific business model to find optimum dividend payments and to keep moderate ruin probabilities at the same time. For example, transferring more fund for investment activities increases expected total discounted dividends from investment gain while reducing expected underwriting surplus and ruin probabilities. This can be done by increasing deposit amount, setting lower trigger for investment activities, or setting higher trigger for underwriting surplus distribution. To avoid usury, Takaful insurance strictly forbids any activity in fixed return investment, thus the investment rate in our model which is assumed to be constant can be replaced in order to better portray Takaful insurance scheme. Therefore as extension for future research, we consider employing a stochastic investment rate to improve our Takaful insurance risk model.

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