LEADING TWIST LIGHT CONE DISTRUBUTION AMPLITUDES OF P-WAVE HEAVY QUARKONIA AND THEIR COUPLINGS TO PSEUDOSCALAR AND VECTOR MESONS

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ABSTRACT

LEADING TWIST LIGHT CONE DISTRUBUTION AMPLITUDES OF P-WAVE HEAVY QUARKONIA AND THEIR COUPLINGS TO PSEUDOSCALAR AND VECTOR MESONS

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The problem analyzed in this work is the calculation of the coupling of $L = 1$ axial-vector heavy quarkonia to vector and pseudo-scalar open flavor mesons. Calculations are based on light-cone QCD sum rules, with inputs from independently performed quark model calculations for the analysis of radially excited states. Those couplings are of interest for being inputs for effective theory calculations on the quark content and transitions of axial-vector $X_{c/b}$ mesons. Results involve profiles for the light-cone distribution amplitudes of charmonia and bottomonia, their respective leptonic decay constants and the mentioned coplings. Charmonium content of $X(3872)$ is also discussed in light of the results.

Keywords: light-cone, sum rule, distribution amplitude, QCD, hadron

ÖZ

P-DALGA AĞIR KUARKONYUMLARIN TVİST-2 IŞIK KONİSİ DAĞILIM GENLİKLERİ VE PSÖDOSKALER VE VEKTÖR MEZONLARA ÇİFTLENİMLERİ

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Bu çalışmada incelenen problem, $L = 1$ aksiyel-vektör ağır kuarkonyumların açık çeşnili psödo-skaler ve vektör mezonlara çiftlenimlerinin hesaplanmasıdır. Hesaplar ışık-konisi KRD toplam kuralları çerçevesindedir ve radyal uyarılmış durumlar için, bağımsız kuark model hesaplarından girdiler içermektedir. Bu çiftlenimlerle, aksiyel-vektör $X_{c/b}$ mezonlarının içeriği ve geçişleri üzerine etkin teori hesaplarında girdi olabilecekleri için ilgilenilmektedir. Sonuçlar, çarmonyum ve botomonyumların ışık-konisi dağılım genlikleri için profiller, ilgili durumların leptonik bozunma sabitleri ve bahsi geçen çiftlenimleri içermektedir. Ayrıca, $X(3872)$'nin çarmonyum içeriği de sonuçlar ışığında tartışılmıştır.

Anahtar Kelimeler: ışık-konisi, toplam kuralı, dağılım genliği, KRD, hadron

...to my dear wife Ceren

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TABLE OF CONTENTS

APPENDICES

LIST OF TABLES

TABLES

LIST OF FIGURES

FIGURES

LIST OF ABBREVIATIONS

h.c.	Hermitian conjugate
L	Orbital angular momentum quantum number
QED	Quantum electrodynamics
QCD	Quantum chromodynamics
OPE	Operator product expansion
LCDA	Light-cone distribution amplitude
PVA	Pseudo-scalar, vector, axial-vector
RQN	Radial quantum number
SVZ	Shifmann-Vainstein-Zakharov (relating to Shifmann-Vainstein-Zakharov sum rules)

CHAPTER 1

INTRODUCTION

Since the discovery of the first charmonium resonance in 1974 [1, 2], the charmonium spectrum has attracted attention in particle physics community. This dicovery has been followed by the discoveries of other heavy mesons including other charmonia and bottomonia, as well as developments in theoretical analyses [1]. Within this period, the quark model assignments (that is, quark - anti-quark pair for mesons and three (anti-)quarks for (anti-)baryons [3]) for the internal structure of hadrons have still been commonly used [2, 4]. However, hadrons which could be in partial conflict with quark model assignments have also been observed in the course of these developments [?,3,5]. There are various states which exhibit such behaviour and pictures trying to explain the observed phenomena. The most recent example is the discovery of the penta-quark state by LHCb [8]. Another example, which is also a motive for this work, is the famous $X(3872)$ [9–13]. $X(3872)$ has quantum numbers $J^{PC} = 1^{++}$ [13] and these quantum numbers are consistent with $L = 1$ and $S = 1$ (L: orbital angular momentum quantum number, S: total spin of the quark - anti-quark pair). There are models which describe $X(3872)$ as a $\bar{D}D^* + h.c.$ molecule [11], while there are other models discussing both molecular and charmonium contributions to the content of $X(3872)$ [10, 14]. One can also find discussions on the content of other exotica in the literature (see e.g. [15] for a sum rules analysis for $Z_c(3900)$).

In this work, the motivation is to describe a framework for examining how ground state and radially excited $L = 1$ axial-vector heavy quarkonia couple to lighter open-flavor vector and pseudo-scalar mesons. These couplings are essential in understanding the interactions of the mentioned mesons, as well as in determining the content of possible exotic states involving these mesons, such as $X(3872)$. The method which is at the core of the analysis is light-cone QCD sum rules, which is a variant of the sum rules technique [16–19], whose details will be discussed below. The necessary ingredients for using this technique are light-cone distribution amplitudes (LCDAs), on which detailed discussions can be found in [16–18, 20–39] and references therein. In order to calculate the LCDAs, this work makes use of 2-particle wavefunctions of a heavy quarkonium calculated in the framework of quark model [41].

In the following sections, concepts essential to this work will be summarized, relevant examples from the literature will be presented, and the idea applied in this work will be explained. Other chapters are organized as follows. In Chapter 2, theoretical background on the sum rules technique, light-cone distribution amplitudes and wavefunctions will be presented and relavant information on the quark model calculations which are used in this work will be reviewed. In Chapter 3, calculation of the leptonic decay constants and LCDAs of heavy quarkonia will be discussed and resulting leptonic decay constants and LCDAs will be presented. In Chapter 4, light-cone sum rules analysis for $L = 1$ heavy quarkonia will be discussed and numerical results for the pseudo-scalar - vector - axial-vector (PVA) couplings of heavy quarkonia will be presented. Comments on the charmonium content of $X(3872)$ will also be given in Chapter 4. Chapter 5 gives a summary of the work.

1.1 Hadrons

Until after the first decades of the 20th century the physics community knew only a few fundamental "particles", the electron, the proton, and the photon [42]. It was also observed by the community that the existence of and interactions between the electron and the proton involved certain conserved properties, today known as baryon and lepton number conservation [2] that is related to those particles' being "fundamentally distinct". With the discovery of the neutron by J. Chadwick in 1932 [2], the community was more enlightened about the

Table1.1: Quarks and respective masses [2]. The upper row has electric charge $e_q = +2/3$ the lower row has $e_q = -1/3$ [7].

u (up) ($2\,MeV$)	c (charm) ($1200\,MeV$)	t (top) ($174000\,MeV$)
d (down) ($5\,MeV$)	s (strange) ($1000\,MeV$)	b (bottom) ($4200\,MeV$)

content of the atomic nucleus. Further propositions for and discoveries of yet unknown particles (whether "elementary" or not) followed in the next decades. However, the notion of an "elementary particle" was still essential for describing the building blocks of matter.

The history of the notion of "elementary building blocks of matter" goes back to the famous Greek philoshopers of the presocratic school [43]. However, it is still central to physical sciences, and practically applicable in a wide variety of problems in modern physics and chemistry [44]. The first known hadrons were also thought as "elementary" until the introduction of "quarks" in 1964 [2]. Beginning from these years around 1964, the community began describing hadrons as composites of quarks. This description is referred to as the "quark model" in the literature [2, 45–50]. Combined with techniques firstly developed within non-relativistic quantum mechanics, quark model also provides means to calculate the hadron spectrum and interactions, which is also referred to as "potential models" [47, 51]. With adding the force-carriers between quarks, gluons [2], hadrons are now described as composites of quarks, anti-quarks and gluons in the modern wiev [42].

There are two sub-categories under the category "hadron": mesons and baryons [2, 7, 42, 50]. In quark model, mesons are defined as quark - anti-quark bound states, while baryons are bound states of three quarks (or three anti-quarks for anti-baryons) [2, 50]. However, this classification ignores the fact that those bound states can include gluons as well. Referring to the modern view, one can argue that this classification can be generalized as follows: mesons are bosonic bound states or resonances of quarks, anti-quarks and gluons, while baryons are fermionic bound states or resonances of the same objects. In the relevant sections below, it will be discussed that such a classification is also valid for exotic hadrons, objects involving more than one set of mesonic or baryonic quark content, and/or involving gluons or even no quarks at all.

Quarks have been proposed as particles having spin 1/2. The reason for this is that in order to build both bosonic and fermionic composites, one needs fermionic building blocks; otherwise no fermionic state (so no baryon) can be built. For this reason, (anti-)baryons have to include an odd number of (anti-)quarks, and the smallest possible number is 3.

Some properties of quarks can be derived directly from the properties of hadrons. As an example, consider the proton and the neutron. They have nearly the same mass [7] and are both fermions. Proton carries electric charge +1 and neutron is electrically neutral. Due to being fermions, both can be assigned at least three quarks. So one needs at least two types of quarks to construct a proton and a neutron, and their electric charges have to be fractional. The u and d-quarks are thus introduced, the former having electric charge +2/3 and the latter having −1/3 [2, 7, 42, 50] (see Table 1.1 for the types of quarks). Similarly, quarks are assigned baryon number 1/3 (so anti-quarks have baryon number −1/3). The fundamental reason for assigning three quarks to the proton and the neutron is discussed below.

As quarks are fermions, they are expected to obey Fermi exclusion principle. This means, no two quarks in a hadron can have the same set of quantum numbers; they have to differ in at least one quantum number [42]. The classic example is the Δ^{++} baryon [2, 42]. Δ^{++} has electric charge +2, and has spin 3/2 [7]. $\Delta^{++}(1232)$ is the lightest spin 3/2 baryon, and so the only possibility for its quark assignment is three u-quarks having the same spin projection with zero relative orbital angular momentum. However, this poses a problem: all quarks are of the same type, they have the same spin projection. So there has to be an additional quantum number which will have different values for the quarks. So, a new degree of freedom, called "color charge" has been introduced [2, 42, 50]. Color charge is a conserved charge and is related to the strong force binding sub-hadronic particles together [2, 42, 50].

However, this color degree of freedom was something not observable independently, or at least up to that time that degree of freedom could not be observed independently. This is known as the "confinement phenomenon", and the color charge has been postulated to be confined in hadrons [2, 42, 50]. In accordance with this postulate, it has also been postulated that all physically observable objects should be in color singlet configurations [2, 42, 50]. Relying on the early quark model prescription that baryons are composed of three quarks, one is led to introducing three different types of color (say, "red, blue and green") [2, 42].

There is also experimental evidence that the number of colors is 3. Let R be the ratio of hadron production rate in electron-positron collisions to the rate of muon - anti-muon production rate in the same events; that is:

$$R = \frac{\sigma(e^+e^- \to hadrons)}{\sigma(e^+e^- \to \mu^+\mu^-)}. \tag{1.1}$$

When the e^+e^- centre of momentum energy E_{cm} is much larger than the quark masses, one has:

$$R = \frac{\sigma(e^+e^- \to hadrons)}{\sigma(e^+e^- \to \mu^+\mu^-)} = N_c \sum_f e_f^2, \tag{1.2}$$

where N_c is the number of colors, and f denotes quark flavor and e_f denotes the electric charge of a quark of flavor f [2, 42]. For the light u, d, s quarks, for example, $R = \frac{2}{3}N_c$, which is valid when $E_{cm} > 2\,GeV$ [42]. When one considers the c-quark along with others, one obtains $R = \frac{10}{9}N_c$ (and this is valid when $E_{cm} > 3\,GeV$) etc. [42]. In Fig. 1.1, experimantal values at various energies have been presented [42]. Apart from the resonances (that is, ω, ρ, ψ etc.), the plateaus correspond to various values of R; the plateau of u, d, s quarks appears just after $1\,GeV$, c-quark is included above roughly $4\,GeV$, and b-quark is included above $10\,GeV$ [42], which correspond to values given by $R = N_c \sum_f e_f^2$. This plot is a classic example, explaining why $N_c = 3$ is favoured by the experiments.

As argued above, the number of colors is 3, so the corresponding symmetry group is $SU(3)$. So, the state vector of a quark carries a color index, which can take one of the three different values. The force carriers between quarks,

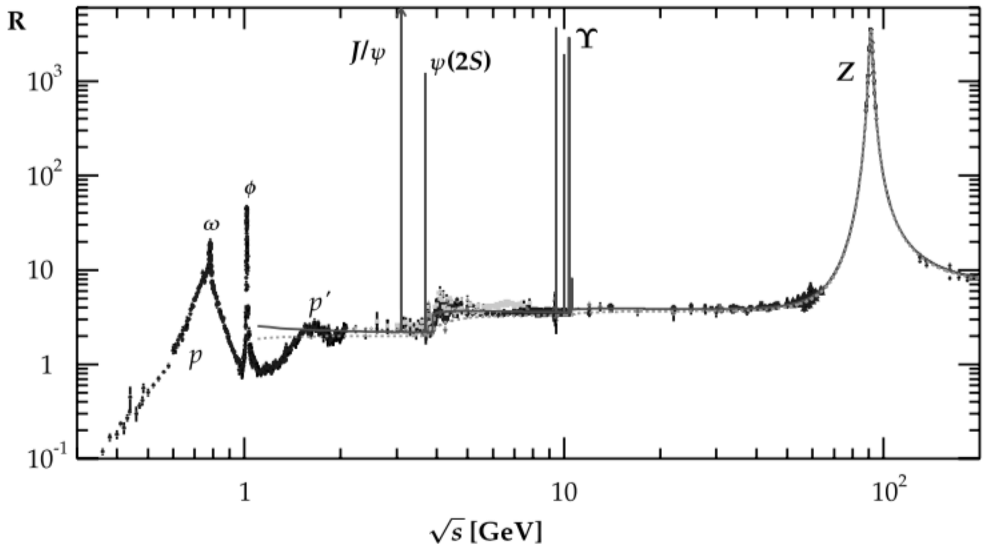

Figure 1.1: $R = \frac{\sigma(e^+e^- \to hadrons)}{\sigma(e^+e^- \to \mu^+\mu^-)}$ plotted against $\sqrt{s}$, where s is the invariant mass squared of e^+e^- [42].

gluons, also carry color charge [42]. Since color charge is conserved, and there is nothing dictating the interactions to take place only among objects having the same color, interactions can change the color of an object, conserving the overall color charge. Then, it appears that the state vector of a gluon should carry two color indices [42]. Mathematically speaking, this is equivalent to saying that quarks are in the fundamental representation of $SU(3)$, and gluons are in the adjoint representation, and the state vectors mentioned above can be thought as those representations [42]. The postulate that all hadrons should be in color singlet combinations severely restricts the spectrum of possible combinations. To understand the relation between this postulate and hadron contents, one shold consider possible combinations of $SU(3)$ representations. The conventions and reasoning can be found in [2].

The fundamental representation can be realized as the eight 3×3 matrices, known as Gell-Mann matrices:

$$
\lambda^1 \equiv \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \lambda^2 \equiv \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \lambda^3 \equiv \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix},
$$

$$
\lambda^4 \equiv \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \lambda^5 \equiv \begin{bmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix}, \lambda^6 \equiv \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},
$$

$$
\lambda^7 \equiv \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix}, \lambda^8 \equiv \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}, \tag{1.3}
$$

satisfying $\left[\lambda^a, \lambda^b\right] = 2if^{abc}\lambda^c$. The three column matrices:

$$
|red\rangle \equiv \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, |blue\rangle \equiv \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, |green\rangle \equiv \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{1.4}
$$

can be used for defining quark color states.

One can understand the fundamental representation as 3 basis vectors in the "color space", and the Gell-Mann matrices as the rotation matrices in the same space. Let 3 represent the number of quark colors, and $\bar{3}$ represent that of anti-quarks. Then, quark - anti-quark combinations can be written as $\bar{3}\otimes 3$ where $\otimes$ is the tensor product (or direct product) operation. That means, one can construct 9 combinations of 3 quarks and 3 anti-quarks. These nine combinations can be categorized as a "singlet", which is invariant under the application of rotations in color space (in other words, transforms to itself under transformations in color space), and an octet, whose members transform to one another or combinations of each other under the action of transformations in color space. One writes this relation as

$$
\bar{3} \otimes 3 = 1 \oplus 8, \tag{1.5}
$$

$\oplus$ being the direct sum operation, in group theory jargon. So, one observes that $\bar{3} \otimes 3$ combinations provide a singlet, and this state can be observed in nature.

These are the conventional mesons. One can also consider the combination of 2 quarks:

$$3 \otimes 3 = \bar{3} \oplus 6; \tag{1.6}$$

that is, combination of 2 quarks gives two groups, one behaves as if it were an anti-quark, and the other is the member of a sextet. This means, a 2-quark state, also called a "diquark", cannot be realized in isolation. However, if one combines the $\bar{3}$ group with a third quark, then one again obtains a singlet and additional groups, that is,

$$3 \otimes 3 \otimes 3 = 1 \oplus 8 \oplus 8 \oplus 10. \tag{1.7}$$

This singlet appearing in these combinations corresponds to conventional baryons. The reasoning for conventional anti-baryons is exactly the same, with $\bar{3} \leftrightarrow 3$.

One can also consider the combination of the singlets in, say, the $\bar{3}\otimes 3$ case. Then, one obtains a singlet and various other states (whose details are not important at this point), and this singlet corresponds to "molecules", meson-meson bound states. Similarly, the combination of a diquark and an anti-diquark is also $\bar{3}\otimes 3$ as demonstrated above, so this combination also gives a singlet, which corresponds to a "tetra-quark", etc. This is related to "exotic states", discussed below.

1.2 Exotic Hadrons

As argued above, the meson - baryon classification can (and most probably should) be done from a more general view, as the classification of bosonic and fermionic bound states or resonances of quarks and gluons, respectively. For example, more than one quark - anti-quark pair can be present in a meson. Or, more precisely, the probability for finding more than one fermion pair in a meson is non-zero. In terms of atomic physics, this is equivalent to arguing that in a hydrogen atom, for instance, the probability for finding an electron - positron pair, in addition to the already known proton and electron, is non-zero. In such a description for the hydrogen atom, chemical properties would be expected to remain unchanged, provided that the electron - positron pair does not change the quantum numbers that are relevant to the chemical properties. So one could argue that the state vector of the hydrogen atom is a superposition of an electron - proton vector and another vector involving the electron and proton plus the electron - positron pair.

In the literature, it is widely accepted that describing mesons as quark - anti-quark pairs and (anti-)baryons as bound states of three (anti-)quarks do not suffice to explain all properties and interactions of hadrons. This view is also supported by experimental observations. A notable example is the case of $X(3872)$, which is also a motive for this work. According to [10, 52], the ratio of the branching fractions for the decays $X \to J/\psi\pi\pi$ and $X \to J/\psi\pi\pi\pi$ is of $O(1)$, and in [10] the authors argue that DD^* bound state picture for this observation is favored. However, again according to [10] the production rate of X in proton - anti-proton collisions requires a charmonium component, whose contribution is roughly 1/20 [10]. So, for this example, one is led to consider a superposition state involving a pure charmonium component and a DD^* bound state [10, 14].

There are various types of exotic hadrons. Although this work deals with the heavy quarkonium spectrum, it is known that the discussion on exotics is equally (and even more) relevant to the light sectors as well [53–56]. One can list meson-meson, meson-baryon and baryon-baryon bound states, which are also reffered to as "molecules", states involving only gluons and called "glueballs", multi-quark - gluon bound states called "hybrids", and directly bound more than three quarks called "tetraquarks, pentaquarks,..." etc [3, 53, 55–59]. In this sense, all atomic nuclei are in fact hadronic molecules. It has also been argued in the literature that molecules and many-quark states are indeed equivalent, in the sense that they span the same Hilbert space to which the hadron state of interest belongs [60]. Taking this observation into account, it is possible to classify all hadron states as multi-quark states (those involving quarks (and/or anti-quarks) only), glueballs (those involving gluons only), and hybrids (those involving both quarks and gluons).

Particle Data Group review on "Non-$q\bar{q}$ candidates" involves an extensive literature on the status of exotic states as of 2015 [7][1]. Here, some notes on various exotic candidates have been presented, with the aim to demonstrate the relevance of model building studies to investigate the properties of exotic candidates.

1.2.1 Glueballs

According to [7], lightest glueballs are expected to have 0^{++} and 2^{++} J^{PC} quantum numbers, with masses expected around $1700\,MeV$ and $2300\,MeV$, respectively; and this expectation is supported by theoretical investigations such as lattice calculations and QCD sum rules. There are various articles discussing possible glueball candidates, and a number of f_0 mesons appear as 0^{++} glueball candidates, though which are mainly glueballs and which ones are mainly light $q\bar{q}$ states is still being investigated [7, 54, 61]. There are certain sets of J^{PC} numbers, such as 0^{--}, 0^{+-}, 1^{-+}, 2^{+-} which cannot be $q\bar{q}$ J^{PC} numbers (in the PDG review on "Non-$q\bar{q}$ candidates" these are named as "exotic glueballs"), and those are expected above $2\,GeV$, together with the 0^{-+} state [7].

1.2.2 Tetraquarks and meson molecules

f_0 mesons are also discussed in the context of tetraquark and molecule states, and $a_0(980)$ is cited as another candidate for both [7]. According to [9] and [60], there are tetraquark and/or molecule candidates in the heavy sectors as well; one of which is the $X(3872)$. According to [62], two photon couplings can provide information to distinguish scalar tetraquarks, molecules and conventional mesons. In [7], $f_1(1420)$ is also cited as a molecule candidate, with the expected content of $K\bar{K}\pi$, and $a_1(1414)$ as its isovector partner. Heavy molecule candidates cited in [7] are $D^*_{s0}(2317)^\pm$, $D_{s1}(2460)^\pm$, $X(3872)$, $Y(4260)$, $Y(4360)$, $Y(4660)$, $Z_c(3900)^\pm$, $Z_c(4430)^\pm$, $Z_b(10610)^\pm$ and $Z_b(10650)^\pm$. Charged mesons which are known to involve a heavy quarkonium pair (i.e. $Z_c^\pm(3900)$, $Z_c(4430)^\pm$, $Z_b^\pm(10610)$ and $Z_b^\pm(10650)$) clearly require a non-$q\bar{q}$ picture, while for the neutral states one should analyse decay properties, masses, quantum numbers etc.

[1] Particle Data Group review on "Non-$q\bar{q}$ candidates"

of the mesons in order to decide whether they are mainly conventional mesons or not.

1.2.3 Baryonia and Hybrid Mesons

Baryons can also form bound states, resulting in mesonic quantum numbers [7] or meson-baryon bound states can also be formed, as demonstrated in the recent work by LHCb [8]. One can think of the atomic nucleus as a molecule composed of baryons as well. However, there are not many candidates poposed for baryonia. The only state cited in [7] is $f_2(1565)$ as a 2^{++} proton - anti-proton molecule.

There are various hybrid candidates cited in [7]: $\pi_1(1400)$, $\pi_1(1600)$, $\pi_1(2015)$, having quantum numbers $J^{PC} = 1^{-+}$; $\pi(1800)$ with $J^{PC} = 0^{-+}$; $\eta_2(1870)$ with $J^{PC} = 2^{-+}$.

1.2.4 Other States

In principle, all quark-gluon bound states conforming to a certain set of quantum numbers, corresponding to the quantum numbers of a hadron, contributes to the state vector of the hadron [7]. Beyond the above mentioned possibilities, other multi-quark/gluon states are possible. An example, which has also recently been observed is the pentaquark state reported by LHCb [8].

1.3 Pseudoscalar - Vector - Axialvector Couplings

So far, hadron contents have been the core of discussion, and have led to the idea that one needs to consider multi-particle states as well as the minimalistic picture of quark model in order to have a deeper understanding of the hadron spectrum and interactions. Within the context of quantum mechanics, hadrons observed in experiments are expected to be eigenstates of some operator observable. From the QCD point of view, this observable is nothing but H^{QCD}, the hamiltonian of QCD [25, 26, 28]. Neither the multi-quark nor the hybrid states are themselves eigenstates of this hamiltonian [25, 26, 28]. Then, in the presence of interactions, there may be non-zero overlaps of different contributions. To make this point more clear, one can consider $X(3872)$ once again as an example. One can consider the matrix element $\langle \bar{D}D^*(molecule)|J_{int.}|c\bar{c}(L = 1)\rangle$ which measures the probability of transition of a pair of $\bar{D}$ and D^* mesons to a $L = 1$ charmonium as a result of the interaction $J_{int.}$ (here, $J_{int.}$ is the interaction part of the QCD hamiltonian). In order to construct a model for $X(3872)$ involving both pure charmonium and DD^* molecule, one needs to calculate this quantity.

Making use of this matrix element, one can define what are called the "PVA couplings":

$$\langle \bar{D}(q)D^*(p', \eta)|J_{int.}|c\bar{c}(p' + q, \epsilon)\rangle \equiv G_1\,(\epsilon \cdot q)\,(\eta^* \cdot q) + G_2\,(q \cdot p')\,(\eta^* \cdot \epsilon)\,, \quad (1.8)$$

where ϵ and η are the polarization vectors of $c\bar{c}$ and D^*, respectively. The numbers G_1 and G_2 are what are called PVA couplings. One can define the same probability amplitude for the coupling of axial-vector bottomonium to B and B^* mesons in exactly the same way.

This definition is motivated by all possible scalar products of the vectors q, p', ϵ, η. All hadrons are on mass-shell (by construction), so $\epsilon \cdot P = \eta \cdot p' = 0$. The vector meson state dictates the presence of η and ϵ in each term, since the state vector of a vector particle involves the polarization vector. Then, one is left with the terms in 1.8, and $\varepsilon_{\mu\nu\alpha\beta}q^\mu p'^\nu \epsilon^\alpha \eta^\beta$, but this term does not conserve parity while the interactions considered here are parity conserving. So, the most general combination of vectors satisfying the physical constraints is given in Eq. (1.8).

As the problem of calculating these numbers can itself be of interest, the motives of this work include commenting on and (if possible) calculating the charmonium content of $X(3872)$ and its bottom-system counterpart. Such couplings can be used as vertex factors in effective theories (see e.g. [14]), and the contents of interest can also be deduced from these theories. What the results of this work imply for the charmonium content of $X(3872)$ will be summarized in the last chapter.

CHAPTER 2

THEORETICAL BACKGROUND

In this chapter, the theoretical framework necessary to understand QCD sum rules, light-cone distribution amplitudes and light-cone wavefunctions will be reviewed.

For Section 2.2, main references to be followed are [17, 20–22]. For the fundamental concepts related to complex functions Ref. [63] has been used. [64] is the original article on SVZ sum rules and [16] is the original article on light-cone sum rules. The references [12,18,31–35,37–40] involve various applications of the technique. [23] discusses the advantages of light-cone sum rules over three point SVZ sum rules. [19] is another review on the technique, but is an older one with respect to [17]. [65] discusses calculating spectral densities for two independent momenta using Borel transforms.

For Section 2.3, main references to be followed are [24, 25, 28–30, 36, 66–69]. [25] is an extensive discussion on the physical motivations and theoretical properties of the light-cone wavefunctions and [28] is equivalent to a textbook on the same subjects. [26, 27] also discuss the properties and use of light-cone wavefunctions. [29] discusses light-cone quantization in detail. [30, 36] involve derivations of the light-cone distribution amplitudes for $L = 1$ mesons. [66–68] involve the calculation spin configurations of hadronic matrix elements.

2.1 Quantum Chromodynamics

Quantum chromodynamics (QCD) was first formulated by D.Gross and F.Wilczek and independently by H. D. Politzer in 1973 and led the inventors to the Nobel Prize in 2004 [42]. The phenomenon bringing the Nobel Prize to Gross, Wilczek and Politzer was "asymptotic freedom". The strength of attraction between color charge carriers has been postulated to be increasing as the seperation between them increases [42]. This is equivalent to saying that the coupling strength of the color field increases with increasing separation. This is in contrast to QED, whose strength decreases with increasing separation.

QCD is accepted as the underlying theory of sub-hadronic physics [2,4,17,70,71] and has the structure of a conventional field theory [17,70,72]. Just as in QED, perturbative calculations are also used in QCD. However, all properties and interactions of hadrons cannot be derived completely using only perturbative calculations. Even one of the fundamental properties of sub-hadronic interactions, the "confinement" phenomenon cannot be explained in a perturbative framework [42]. Experiments observe hadrons, and sub-hadronic degrees of freedom cannot be observed independently. This phenomenon is known as confinement (of sub-hadronic particles or equivalently, of color charge which is discussed below) [2,4,42,50,70]. For this reason, techniques which do not rely on perturbation theory are necessary [17].

One of the well known methods developed to meet this requirement is "QCD Sum Rules" [17,64], originally developed by Shifmann, Vainstein and Zakharov. Another method, which is more familiar from non-relativistic quantum mechanics is the quark model (also referred to as quark potential model) [47,51]. There are other methods for analyzing hadron structure and interactions discusssed extensively in the literature, such as lattice QCD (see e.g. [1]) and effective field theories (e.g. [14]) which are also in use for analyzing the hadron spectrum and interactions [14].

When one considers the evolution of electrodynamics, from simple concepts like "lines of force" to highly sophisticated quantum electrodynamics (QED), one is led to the idea that the theory involving this color degree of freedom would most probably be formulated in terms of the structure underlying QED. And history reveals, this is indeed the case. QCD is a $SU(3)$ gauge theory and involves the full machinery of quantum field theory, and even more [2, 17, 42, 50, 70].

Modern particle physics is based on quantum field theory, according to which the fundamental constituents of the universe are fields extended in all space and time, whose perturbations of quantized energy-momentum are perceived as "particles" [70, 73]. So, for example, in QED, electrons and positrons are the perturbations of the fermionic field, and photons are the perturbations of the bosonic field which are included in the following lagrangian [70, 73]:

$$L = \bar{\psi}(x)\left(i\gamma^{\mu}D_{\mu} - m\right)\psi(x) - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}, \tag{2.1}$$

where $A_{\mu}(x)$ and $\psi(x)$ are the bosonic and fermionic fields respectively, $\bar{\psi} = \psi^{\dagger}\gamma^{0}$, γ^{μ} are Dirac matrices satisfying $\{\gamma^{\mu}, \gamma^{\nu}\} = 2g^{\mu\nu}$, $g_{\mu\nu}$ is the metric tensor, $D_{\mu} \equiv \partial_{\mu} - ieA_{\mu}(x)$ is the covariant derivative, e is charge of the positron, $F_{\mu\nu} = \partial_{\mu}A_{\nu}(x) - \partial_{\nu}A_{\mu}(x)$ is the field strength tensor. Summation over repeated indices is implied.

The fields involved in this lagragian are supposed to extremize the action $S = \int d^4xL$. One varies S with respect to ψ, $\bar{\psi}$ and A_{μ} to calculate the Euler-Lagrange equations satisfied by these fields to calculate the dynamical evolution of the fields [70, 73].

The essence of the above Lagrangian is that is invariant under transformations of the form [42]:

$$\begin{aligned} \psi(x) &\rightarrow \psi^{'}(x) = \exp\left[ie\theta(x)\right]\psi(x), \\ A_{\mu}(x) &\rightarrow A_{\mu}^{'}(x) = A_{\mu}(x) + \frac{i}{e}\partial_{\mu}\theta(x), \end{aligned} \tag{2.2}$$

where $\theta(x)$ is some scalar function. This is indeed a gauge transformation, already known from classical electrodynamics [70,73]. In classical electrodynamics, the physics involved is independent from the gauge chosen for the vector potential A_{μ}, and the idea is intact in the QED case as well. The gauge transformation amounts to a local change in the phase of $\psi(x)$, the Lagrangian remains independent from the overall phase of the fermion field. Gauge invariance is also named as gauge symmetry.

The group of transformations leaving the QED Lagrangian gauge invariant (that is, the symmetry group) is known as $U(1)$, whose elements are scalar functions. In fact, this group structure is related directly to the conservation of electric charge; and since there is only one type of charge in QED, the relevant group is $U(1)$. However, in QCD case, one has three types of conserved charge, and together with the confinement phenomenon, this dictates the symmetry group be $SU(3)$ [42]. Then, one seeks for a Lagrangian which is invariant under local $SU(3)$ transformations. The following Lagrangian has this property:

$$L = \bar{\psi}^i(x)\left(i\gamma^\mu D_\mu - m\right)\psi^i(x) - \frac{1}{2}Tr\left(F^{\mu\nu}F_{\mu\nu}\right). \tag{2.3}$$

Here $i = 1, 2, 3$ is the color index for quarks, Tr denotes trace, and the gluon field, covariant derivative and field strength tensor are the following [42]:

$$\begin{aligned} A_\mu &\equiv A_\mu^a(x)\frac{\lambda^a}{2}, \\ D_\mu &\equiv \partial_\mu - ie_s A_\mu(x), \\ F_{\mu\nu} &\equiv \partial_\mu A_\nu - \partial_\nu A_\mu - ie_s\left[A_\mu, A_\nu\right], \end{aligned} \tag{2.4}$$

where e_s replaces the electronic charge, $a = 1, \ldots, 8$ and λ^a are the 3×3 traceless hermitian matrices satisfying:

$$\left[\frac{\lambda^a}{2}, \frac{\lambda^b}{2}\right] = if^{abc}\frac{\lambda^c}{2}, \tag{2.5}$$

f^{abc} being the structure constants of $SU(3)$ [42]. λ^a are also called Gell-Mann matrices [42]. With the above mentioned fields, the QCD Lagrangian is invariant under local gauge transformations of the form $\theta(x) = \frac{1}{2}\lambda^a\theta^a(x)$, with the following transformation properties [42]:

$$\begin{aligned} \psi(x) &\rightarrow \psi'(x) = \exp\left[ie_s\theta(x)\right]\psi(x), \\ A_\mu(x) &\rightarrow A'_\mu(x) = \exp\left[ie_s\theta(x)\right]\left(A_\mu(x) + \frac{i}{e_s}\partial_\mu\right)\exp\left[-ie_s\theta(x)\right]. \end{aligned} \tag{2.6}$$

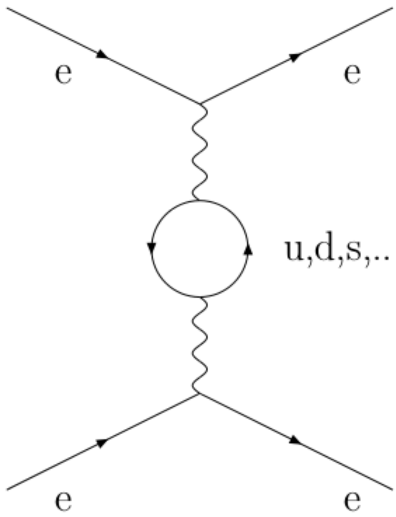

Figure 2.1: Virtual quark - anti-quark pair in the process $e^-e^- \to e^-e^-$, where the solid lines represent the electrons and the wavy line represents the photon exchanged between them. The loop corresponds to the pair, and $u, d, s, ...$ denote various quark flavors [17].

Ref. [42] is a valuable resource on QCD. One can also find many references related to various topics in QCD in Ref. [3].

2.2 QCD Sum Rules

2.2.1 SVZ Sum Rules: General Concepts and Reasoning

Having briefly introduced the ingredients of QCD, one can now proceed to QCD sum rules. This technique was originally developed by M. A. Shifman, A. I. Vainshtein and V. I. Zakharov in 1979 [64].

In order to understand the subject properly, one can use an example. In this chapter, the example presented in [17] will be used. This example is the creation of a quark - anti-quark pair in electron-electron scattering (see Fig. 2.1). Then, the object of interest is the two-point correlation function:

$$\Pi_{\mu\nu}(q) = i \int d^4x \exp(iq \cdot x)\langle 0|T\{j_\mu(x)j_\nu(0)\}|0\rangle = (q_\mu q_\nu - q^2 g_{\mu\nu})\Pi(q^2). \quad (2.7)$$

Here, q is the 4-momentum of the virtual photon exchanged by the electrons and is spacelike (i.e. $q^2 < 0$) and $j_\mu = \bar{\psi}^c \gamma_\mu \psi^c$ is the quark current, $c = 1, 2, 3$ is the color index for quark operators, and γ_μ are Dirac matrices satisfying $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}$, $g_{\mu\nu}$ being the metric tensor. T denotes time ordering:

$$T\{A(t_1), B(t_2)\} \equiv \theta(t_1 - t_2)A(t_1)B(t_2) + \theta(t_2 - t_1)B(t_2)A(t_1), \tag{2.8}$$

where A, B are two arbitrary operators and $\theta(t)$ is the Heaviside step function. The right hand side of 2.7 is dictated by the Lorentz structure and charge conservation: the only independent 4-vector is q_μ and the electromagnetic current is conserved, that is, $\partial_\mu j^\mu = 0$, whose Fourier transform reads $q_\mu j^\mu = 0$.

The task is now to calculate the function $\Pi(q^2)$.

The idea is to calculate this function in two limiting cases and matching the results relying on analytical continuation. The limiting cases can be named as the "QCD limit" where $-q^2$ is very large and the relevant degrees of freedom are quarks and gluons, and the "hadronic limit" where $0 < q^2$ and the relevant degrees of freedom are hadrons. Traditionally, these limits are referred to as the small and large distance, respectively, in the literature. The reason is that for very large $-q^2$, the exponential in the integrand oscillates rapidly except for small values of the separation x, so this limit is the "short distance" limit, and hence the other is the long distance limit.

To proceed further for the calculation of the correlation function, one can insert an identity operator, which is a sum over projectors onto free hadronic states involving various numbers of on-shell hadrons, between the current operators:

$$\begin{aligned}
\mathbb{1} = |0\rangle\langle 0| &+ \int \frac{d^4k}{(2\pi)^4}\theta(E^0)(2\pi)\delta(k^2 - m_h^2)\sum_h |h(k)\rangle\langle h(k)| \\
&+ \int \frac{d^4k d^4k'}{(2\pi)^8}\theta(E)\theta(E')(2\pi)^2\delta(k^2 - m_h^2)\delta(k'^2 - m_{h'}^2) \\
&\times \sum_{h,h'} |h(k)h'(k')\rangle\langle h(k)h'(k')| + \cdots
\end{aligned}$$

$$= |0\rangle\langle 0| + \int \frac{d\vec{k}}{(2\pi)^3 2\omega} \sum_h \left| h(\vec{k})\rangle\langle h(\vec{k}) \right|$$
$$+ \int \frac{d\vec{k}d\vec{k}'}{(2\pi)^6 4\omega\omega'} \sum_{h,h'} \left| h(\vec{k})h'(\vec{k}')\rangle\langle h(\vec{k})h'(\vec{k}') \right| + \cdots \qquad (2.9)$$

One obtains matrix elements of the form $\langle 0|j_\mu(x)|V(k,\eta)\rangle$, where k is the 4-momentum of the vector meson state appearing in the momentum integrations (when using the unitarity relation) and η is the polarization 4-vector of the same object (satisfying $\eta \cdot k = 0$). Using the translation property: $j_\mu(x) = e^{i\hat{x}\cdot\hat{P}} j_\mu(0) e^{-i\hat{x}\cdot\hat{P}}$, where $\hat{x}$ and $\hat{P}$ are position and momentum operators, one obtains

$$\langle 0|j_\mu(x)|V(k,\eta)\rangle = e^{-ik\cdot x}\langle 0|j_\mu(0)|V(k,\eta)\rangle, \qquad (2.10)$$

and the resulting matrix element defines the decay constant of the meson:

$$\langle 0|j_\mu(0)|V(k,\eta)\rangle \equiv f_V m_V \eta_\mu. \qquad (2.11)$$

The right hand side involves the polarization 4-vector of the meson, due to that the wavefunction of a vector particle is proportional to its polarization 4-vector. Making use of another important relation concerning vector particles, namely:

$$\sum_\eta \eta_\mu \eta_\nu^* = \frac{k_\mu k_\nu}{m_V^2} - g_{\mu\nu}, \qquad (2.12)$$

one arrives at:

$$\Pi_{\mu\nu}(q) = \frac{-f_V^2}{q^2 - m_V^2}\left(q_\mu q_\nu - m_V^2 g_{\mu\nu}\right) + \cdots, \qquad (2.13)$$

where excited and continuum states are denoted by $\cdots$. Owing to $Im\left(\frac{-1}{q^2-m_V^2}\right) = \pi\delta\left(q^2 - m_V^2\right)$, where Im denotes "imaginary part", one obtains:

$$\frac{1}{\pi} Im\Pi_{\mu\nu}(q) = f_V^2\left(q_\mu q_\nu - m_V^2 g_{\mu\nu}\right)\delta\left(q^2 - m_V^2\right) + \cdots. \qquad (2.14)$$

Since the excited states and continuum are more complicated, one may isolate the ground state contribution to the correlator and introduce a function (yet unknown) to incorporate the remaining contributions:

$$\frac{1}{\pi} Im\Pi(q^2) = f_V^2\delta\left(q^2 - m_V^2\right) + \rho(q^2)\theta\left(q^2 - s_0\right), \qquad (2.15)$$

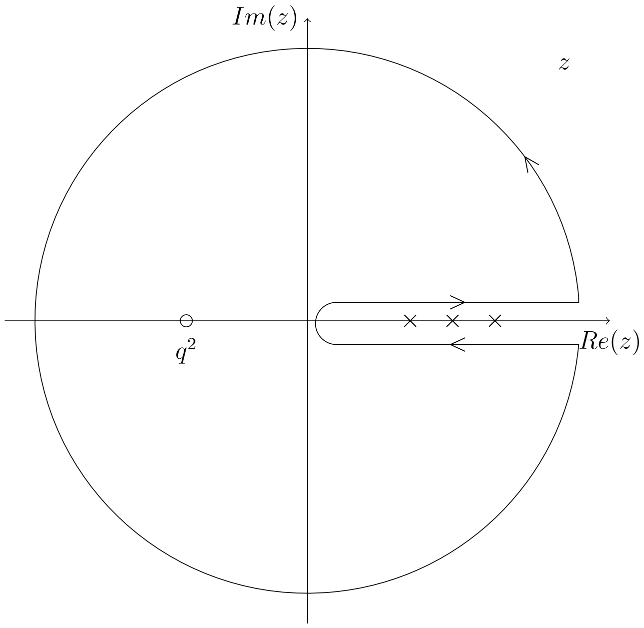

Figure 2.2: The contour to be used for the correlation function. The crosses represent hadronic thresholds, satisfying $q^2 = m^2_{hadron}$ [17].

where s_0 is the threshold over which hadronic continuum appears. It has been noted in [17] that for systems involving heavy quarks, there are heavy quarkonium resonances below the pair production threshold for the mesons created by the current j in the correlator, so one has to take the below threshold states into account and calculate $\sum_V Im\Pi(q^2)$ over the below-threshold states.

After this point, one exploits the analiticity properties of the correlation function, to relate the above form of the correlation function to $\Pi(q^2)$ at an arbitrary value of q^2.

Using Cauchy's integral formula, one can express an arbitrary function $f(z)$ as

$$f(z) = \frac{1}{2\pi i}\oint_C dw \frac{f(w)}{w-z}, \tag{2.16}$$

where C is some closed contour enclosing the point $w = z$ in the complex plane, and the function $f(w)$ is assumed to be analytic on and within this contour.

Now consider the function $\Pi(q^2)$ and the contour shown in Fig. 2.2. The motivation for choosing this contour is that one needs to match the $q^2 > 0$ and large $-q^2$ limits of $\Pi(q^2)$. On the positive real axis, there are poles (indicated by the crosses) of the correlator corresponding to hadron masses (in the example being discussed, masses of the vector bosons that can be created by the current j) and a branch cut where the continuum starts, so the portions parallel to the real axis are infinitesimally shifted away from the real axis. Let the radius of the circular part be R, which will go to infinity at the end of the calculation. Along this contour:

$$\Pi(q^2) = \frac{1}{2\pi i}\left\{\int_{|z|=R} dz\frac{\Pi(z)}{z-q^2} + \int_0^R dx\frac{\Pi(x+i\varepsilon)-\Pi(x-i\varepsilon)}{x-q^2}\right\}, \qquad (2.17)$$

where $\varepsilon > 0$. If the function $\Pi(z)$ tends to zero with any positive power of $1/q^2$, then the integral over the circle vanishes. When this condition is not satisfied, one can subtract the first few terms of the Taylor expansion of $\Pi(q^2)$ around $q^2 = 0$ from $\Pi(q^2)$. It has been noted in [17] that QCD calculations reveal that the function $\Pi(q^2)$ is indeed divergent, while subtracting only the first term of its Taylor expansion suffices to cure this problem.

Noting that $\Pi(q^2)$ is real for $q^2 < t_{min}$ where $t_{min} = \min\{m_V^2, s_0\}$, one can make use of "Schwarz reflection principle", which states that if a function $f(z)$ is real for real values of z, and is analytic in some region including the real axis, then $f^*(z) = f(z^*)$, where $*$ denotes complex conjugation [63]. Then, $\Pi(z+i\varepsilon) - \Pi(z-i\varepsilon) = 2iIm\Pi^{(i)}(q^2)$. Using this conclusion, one arrives at what is called a *dispersion relation*:

$$\Pi(q^2) = \frac{1}{\pi}\int_{t_{min}}^{\infty} ds\frac{Im\Pi(s)}{s-q^2-i\varepsilon}. \qquad (2.18)$$

Subtracting the first term of the Taylor expansion, one obtains:

$$\bar{\Pi}(q^2) \equiv \Pi(q^2) - \Pi(0) = \frac{q^2}{\pi}\int_{t_{min}}^{\infty} ds\frac{Im\Pi(s)}{s(s-q^2)}. \qquad (2.19)$$

Substituting the imaginary part 2.15 into the above expression, one obtains:

$$\Pi(q^2) = \frac{q^2 f_V^2}{m_V^2(m_V^2-q^2)} + q^2\int_{t_{min}}^{\infty} ds\frac{\rho(s)}{s(s-q^2)} + \Pi(0). \qquad (2.20)$$

It has been noted in [17] that $\Pi(0) = 0$ due to gauge invariance, but is kept in order to remind possible subtractions or polynomials of q^2 that can appear in calculations for different correlators.

The use of the above dispersion relation still requires further operations, in order to seperate the ground state more efficiently. The reason for this is that the "cleanest" information coming from the dispersion relation is on the ground state, since the technique proceeds without having the exact knowledge on the structure of the hadron of interest. So, one makes use of an operation called "Borel transform" to eliminate the unknown subtraction terms. This operation is defined as:

$$B_{Q^2 \to M^2} f(Q^2) \equiv \lim_{\substack{Q^2 \to \infty \\ n \to \infty}\, \to M^2} \frac{(Q^2)^{n+1}}{n!} \left(-\frac{d}{dQ^2} \right)^n f(Q^2), \tag{2.21}$$

where M^2 is called the Borel parameter. Since it is not a parameter introduced by the physics of the problem, numerical results are expected to be independent of M^2.

Above definition for the Borel transform is slightly different from the one used in [17], and is useful when one switches to Euclidean space (where $Q^2 \equiv -q^2$) at some step within the calculations. However, the general reasoning is exactly the same.

Some Borel transforms most commonly used in the calculations are the following:

$$\begin{aligned}
B_{M^2} \left(\frac{1}{(s+Q^2)^k} \right) &= \frac{1}{\Gamma(k)} \frac{\exp(-s/M^2)}{(M^2)^{k-1}}, \\
B_{M^2} \left((Q^2)^k \right) &= 0, \\
B_{M^2} \left[\exp(-tQ^2) \right] &= M^2 \delta(tM^2 - 1), \\
B_{M^2} \left[\frac{Q^2}{s+Q^2} \right] &= B_{M^2} \left[1 - \frac{s}{s+Q^2} \right] = -s \exp(-s/M^2),
\end{aligned} \tag{2.22}$$

where $\Gamma(k)$ is the gamma function.

The Borel transform leads to exponential suppression of the hadronic continuum:

$$\Pi(M^2) = f_V^2 \exp\left(-\frac{m_V^2}{M^2} \right) + \int_{s_0}^{\infty} ds \rho(s) \exp\left(-\frac{s}{M^2} \right). \tag{2.23}$$

After this point, the task is to calculate the correlation function in QCD. This means, one should express the current operators in terms of quark and gluon fields. In principle, all possible combinations of these fields corresponding to the same set of quantum numbers are possible. However, various combinations of quark and gluon fields will inevitably differ in the number of constituents. However, QCD calculation is supposed to be valid at the large $-q^2$ limit, and in this limit quarks are nearly free (thanks to asymptotic freedom). For this reason, one identifies the current operators with the quark model assingments for the quarks; in a sense, one applies a minimal prescription. For example, if the hadron of interest is the D meson, the quark model assignment is $c\bar{u}$; and since D meson is a pseudo-scalar meson, the corresponding current is $\bar{c}i\gamma_5 u$, where c and u are the spinor fields representing the charm and up quarks, respectively; and γ_μ are the Dirac matrices [2, 42, 70, 73].

At the first step of the calculation, one considers the time ordered product in the correlation function. According to Wick's theorem, the time ordered product can be written as a sum over all possible contractions of the quark fields [42, 70, 73]. Writing the quark operators and Dirac matrices in terms of respective color and spinor indices, one obtains:

$$\begin{aligned}
&\langle 0|T\left\{\bar{\psi}(x)\gamma_\mu\psi(x), \bar{\psi}(0)\gamma_\nu\psi(0)\right\}|0\rangle \\
=&\langle 0|T\left\{\bar{\psi}^i_A(x)\left(\gamma_\mu\right)_{AB}\psi(x)^i_B, \bar{\psi}^j_C(0)\left(\gamma_\nu\right)_{CD}\psi^j_D(0)\right\}|0\rangle \\
=&\left(\gamma_\mu\right)_{AB}\left(\gamma_\nu\right)_{CD} \times \langle 0|(:\bar{\psi}^i_A(x)\psi(x)^i_B\bar{\psi}^j_C(0)\psi^j_D(0): + :\bar{\psi}^i_A(x)\psi^j_D(0): \overbrace{\psi(x)^i_B\bar{\psi}^j_C(0)} \\
&+ :\psi(x)^i_B\bar{\psi}^j_C(0): \overbrace{\bar{\psi}^i_A(x)\psi^j_D(0)} + \overbrace{\psi(x)^i_B\bar{\psi}^j_C(0)}\,\overbrace{\bar{\psi}^i_A(x)\psi^j_D(0)})|0\rangle,
\end{aligned} \tag{2.24}$$

where :: denotes normal ordering, meaning that all creation operators are to be placed to the left of the annihilation operators in the quark fields, and $\overbrace{\psi(x)^i_B\bar{\psi}^j_C(0)}$ denotes contraction of the operators $\psi(x)^i_B\bar{\psi}^j_C(0)$ etc. $i, j = 1, 2, 3$ are color indices and $A, B, C, D = 1, 2, 3, 4$ are the spinor indices of the objects. Contractions are equal to propagators, which read in the free quark approximation:

$$iS^{free,ij}_{BC} = \overbrace{\psi(x)^i_B\bar{\psi}^j_C(0)} = \delta^{ij}\int\frac{d^4k}{(2\pi)^4}e^{-ik\cdot x}\frac{\left(k_\mu\gamma^\mu + m\right)_{BC}}{k^2 - m^2}, \tag{2.25}$$

where m is the quark mass, and S^{free} is the free quark propagator.

There are terms which involve operators that are not contracted. Those have the following structure: $\langle 0| : \psi(x)^i_B \bar{\psi}^j_C(0) : |0\rangle$. For those terms, one can expand the operators around $x = 0$ in the following way:

$$\psi(x) = \psi(0) + x^\alpha \partial_\alpha \psi(x)|_{x=0} + \cdots , \tag{2.26}$$

which is indeed a Taylor expansion. In the Fock-Schwinger gauge, defined by:

$$x^\alpha A^a_\alpha(x) = 0, \tag{2.27}$$

where $A^a_\alpha(x)$ is the gluon field, one has $x^\mu \partial_\mu = x^\mu D_\mu$, where D is the gauge convariant derivative. This relation provides the use of full equations of motion (not necessarily the equations of motion for the free fields) in the calculations. After performing the expansion, one obtains matrix elements of the form $\langle 0|\bar{\psi}(0)\psi(0)|0\rangle$ etc. where all operators are defined at the same spacetime point 0. These terms are called "condensates".

The above expansion is called an "operator product expansion" (OPE). OPE converts the bilocal (i.e. defined at two spacetime points - which are x and 0 in this example-) correlation function into a sum over the vacuum expectation values of local operators:

$$\Pi(q^2) = \sum_d C_d(q^2)\langle 0|O_d|0\rangle, \tag{2.28}$$

where C_d, called "Wilson coefficients" are ordinary functions of q^2 and O_d are local operators arising due to the expansion, involving various products of the fields, and d is the mass dimension of the operators. For example, fermion fields have mass dimension 3/2, and so the operator $\bar{\psi}\psi$ has $d = 3$, etc. Wilson coefficients scale with inverse powers of q^2; higher dimension terms involve higher powers of $1/q^2$. This allows one to truncate the expansion at a desired order, since it is guarranteed that increasing mass dimension means decreasing contribution to the correlation function in the large $-q^2$ limit.

Returning to the example presented in [17], one remembers that the task has been calculating the correlation function in QCD. This calculation yields two

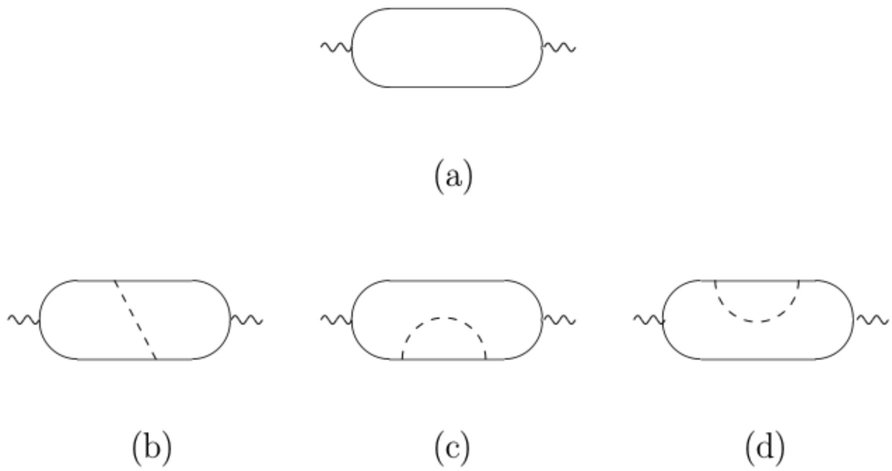

Figure 2.3: Some Feynmann diagrams relevant for the perturbative calculations: (a) the free quark loop, (b,c,d) perturbative QCD corrections [17].

contributions, one coming from Feynmann diagrams like the ones demonstrated in Fig. 2.3, and the other coming from the condensates.

The perturbative part gives up to the diagram (a) in Fig. 2.3 (details of this calculation are not directly useful in the problem studied in this work, so are omitted for brevity):

$$\Pi^0(q^2) = \frac{q^2}{\pi}\int ds \frac{Im\Pi(s)}{s(s-q^2)}, \tag{2.29}$$

where $Im\Pi(s) = \frac{1}{8\pi}v(3-v^2)\theta(s-4m^2)$ and $v = \sqrt{1-4m^2/s}$.

One can make further approximations for $\Pi^0(q^2)$, but the general flow of arguments is important here, so details of further calculations in [17] will not be discussed here. $\Pi(q^2)$ becomes:

$$\Pi(q^2) = \Pi^0(q^2) + \Pi^{cond.}(q^2), \tag{2.30}$$

where $\Pi^{cond.}(q^2)$ involves the contributions of the condensates. One performs a

Borel transform for $\Pi(q^2)$, and arrives at the following sum rule:

$$f_V^2 e^{-\frac{m_V^2}{M^2}} + \int\limits_{s_0}^{\infty} ds\rho(s) e^{-\frac{s}{M^2}}$$
$$= \frac{1}{\pi}\int_{4m^2}^{\infty}\frac{ds}{8\pi}\sqrt{1-\frac{4m^2}{s}}\left(2+\frac{4m^2}{s}\right)e^{-\frac{s}{M^2}} + \Pi^{cond.}(M^2). \tag{2.31}$$

For sufficiently large values of $-q^2$, one can make the "quark-hadron duality" approximation and write

$$\int\limits_{s_0}^{\infty} ds\rho(s) e^{\left(-\frac{s}{M^2}\right)} \simeq \frac{1}{\pi}\int\limits_{s_0^{pert.}}^{\infty} dsIm\Pi^{pert.}(s)e^{\left(-\frac{s}{M^2}\right)}, \tag{2.32}$$

where $s_0^{pert.}$ is expected to be close to the first excited vector meson mass squared.

Up to this point, the general flow of calculations in the framework of sum rules has been presented. From now on, attention will be focused on light-cone sum rules.

2.2.2 Light-Cone Sum Rules

The motivation for using light-cone sum rules is generally to calculate couplings of various hadronic states, and this information can be deduced from a correlation function like the following one:

$$F(q,p) = i\int d^4x \exp(iq\cdot x)\langle 0|T\left\{J(x)J^{'}(0)\right\}|H(p)\rangle, \tag{2.33}$$

where p is the 4-momentum of H. Here, the hadron H is on-shell by construction.

Experience from SVZ sum rules teaches that one will obtain terms involving $\frac{1}{(p-q)^2-m_{h'}^2}$ ($m_{h'}$ is the mass of the hadron created by the current J') and need the Borel transform with respect to $(p-q)^2$. This borel transform will thus be necessary for the QCD calculation as well. So the large $-(p-q)^2$ limit will also be of interest in this case. Owing to:

$$(p-q)^2 = p^2 + q^2 - 2p\cdot q, \tag{2.34}$$

it is observed that as long as p is finite, $p{\cdot}q$ can also be large as well (the relevance of this variable will be clear in a moment). This fact upsets the expansion around $x = 0$. However, and expansion around $x^2 = 0$ is still possible. One can see the discussion given in [17], which is reviewed here.

Let:

$$Q^2 \equiv -q^2, \;\; \nu \equiv q \cdot p, \;\; \xi \equiv 2\nu/Q^2. \tag{2.35}$$

The statement that $p \cdot q$ can be large is equivalent to:

$$|\nu| \sim |(p-q)^2| \sim Q^2 \gg \Lambda^2_{QCD}, \tag{2.36}$$

where Λ_{QCD} is the scale at which the QCD coupling e_s diverges. The region where Eq. 2.36 is valid corresponds to $\xi \sim 1$. Consider a reference frame where $|p^0|, |\vec{p}| \sim \mu$ (that is, where the hadron mass is negligible with respect to its 3-momentum) for some scale μ which satisfies $\mu^2 \ll Q^2, \nu$. Let also in this frame $\vec{q}_\perp = 0$, so $q = (q^0, 0, 0, q^3)$. This gives:

$$\nu = p \cdot q = p^0 q^0 - p^3 q^3 \sim \mu(q^0 - q^3) \Rightarrow q^0 - q^3 \sim \nu/\mu. \tag{2.37}$$

Using $q \cdot q = (q^0)^2 - (q^3)^2 = -Q^2$, one obtains $q^0 + q^3 \sim -Q^2\mu/\nu$. Together with Eq. 2.37, the conclusion that $q^0 \sim Q^2\xi/(4\mu) + 2\mu/\xi \sim\sim Q^2\xi/(4\mu)$ is achieved.

One can approximate $q \cdot x$ as

$$q \cdot x = q^0 x^0 - q^3 x^3 \simeq \frac{Q^2\xi}{4\mu} x^0 - \left(\sqrt{\frac{Q^4\xi^2}{16\mu^2} + Q^2}\right) x^3 \simeq \frac{Q^2\xi}{4\mu}(x^0 - x^3) - \frac{2\mu}{\xi} x^3. \tag{2.38}$$

The dominant contribution to the correlation function comes from the region where the exponential $e^{iq\cdot x}$ does not oscillate rapidly. This corresponds to the region where both terms at the end of Eq 2.38 are of $O(1)$. This gives:

$$(x^0)^2 \simeq (x^3 + 4\mu/(Q^2\xi))^2 \simeq (x^3)^2 + \cdots . \tag{2.39}$$

Then, one concludes that the dominant contribution comes from $x \cdot x = x^2 \sim 1/Q^2 \to 0$, and there is no short distance dominance owing to $x^0 \sim x^3 \sim$

$\xi/(2\mu) \gg 1/\sqrt{Q^2}$. For that reason, an expansion in terms of local operators around $x = 0$ cannot be truncated at a finite order.

The OPE in the SVZ technique is performed around $x = 0$. This time, one perfoms an expansion around $x^2 = 0$, that is, near the light-cone. Calculation of the hadronic sum and application of Borel transforms proceeds exactly in the same way with the SVZ technique. However, different matrix elements arise in the QCD calculation, which involve the on-shell hadron state and are not directly related to the condensates. As an example, one can consider matrix elements of the type $\langle M(p)|\bar{\psi}(x)\Gamma\phi(0)|0\rangle$ for a meson M having 4-momentum p.

Luckily, the terms in the expansion in local operators can still be arranged in inverse powers of Q^2. To see this, one again considers an OPE around $x = 0$, but without expecting to truncate this expansion at a finite order. Following from [17], the OPE has the form:

$$\bar{\psi}(x)\Gamma\phi(0) = \sum_{r=0}^{\infty} \frac{1}{r!}\bar{\psi}(0)\overleftarrow{D}_{\mu_1}\cdots\overleftarrow{D}_{\mu_r}\Gamma\phi(0)x^{\mu_1}\cdots x^{\mu_r}, \tag{2.40}$$

where the arrow to the left implies that derivatives act to the left. The matrix elements of the operators in the OPE have the following structure:

$$\begin{aligned}\langle M(p)|\bar{\psi}(0)\overleftarrow{D}_{\mu_1}\ldots\overleftarrow{D}_{\mu_r}\Gamma\phi(0)|0\rangle &= (-i)^r p_{\mu_1}\ldots p_{\mu_r}p_{(\Gamma)}M_r \\ &+(-i)^r g_{\mu_1\mu_2}p_{\mu_3}\ldots p_{\mu_r}p_{(\Gamma)}M_r^{'} + \ldots,\end{aligned} \tag{2.41}$$

where $p_{(\Gamma)}$ stands for the Lorentz indices dictated by Γ, and there are further terms involving more factors of the metric tensor $g_{\mu\nu}$. The factors M_r, $M_r^{'}$, which do not involve x-dependence, will be discussed below. So, the following structure is obtained:

$$\begin{aligned}&\langle M(p)|\bar{\psi}(0)\overleftarrow{D}_{\mu_1}\ldots\overleftarrow{D}_{\mu_r}\Gamma\phi(0)|0\rangle x^{\mu_1}\ldots x^{\mu_r} = \\ &(-i)^r\,(p\cdot x)^r\,p_{(\Gamma)}M_r + (-i)^r x^2\,(p\cdot x)^{r-2}\,p_{(\Gamma)}M_r^{'} + \cdots\,.\end{aligned} \tag{2.42}$$

Defining

$$M_r \equiv (constant) \times \int_0^1 du u^r \phi(u,\mu), \tag{2.43}$$

one obtains factors of the form $(-iup \cdot x)^r$. When summed over r, exponentials of the form $e^{iup \cdot x}$ are obtained. As an example, consider the matrix element discussed in [17]. The generic term of the expansion reads:

$$\langle \pi^0(p)|\bar{\psi}(0)\overleftarrow{D}_{\mu_1} \ldots \overleftarrow{D}_{\mu_r}\Gamma\phi(0)|0\rangle. \tag{2.44}$$

The expansion arising from these matrix elements gives:

$$\frac{1}{Q^2}\sum_{r=0}^{\infty}\xi^r M_r + \frac{4}{Q^4}\sum_{r=2}^{\infty}\frac{\xi^{r-2}}{r(r-1)}M_r^{'} + \ldots. \tag{2.45}$$

The operators appearing with the same power of Q^2 have a certain common property. The difference of their mass dimensions and spins are all equal. This difference is called "twist".

So, the hadron-to-vacuum matrix elements can be written in the following way:

$$\langle H(p)|\bar{\psi}(x)\Gamma\phi(0)|0\rangle|_{x^2=0} \equiv (factors)\int_0^1 du \exp(iup \cdot x)\phi(u,\mu), \tag{2.46}$$

where the "factors" are determined by the Lorentz structure of the current operator and quantum numbers of the hadron H, and involve so-called "leptonic decay constants". The variable u spans the whole range of quark seperation from $y \equiv ux \sim 0$ to $\sim x$. The functions $\phi(u,\mu)$ are called "light-cone distribution amplitues (LCDAs)".

Although they can be deduced from experiment as well, in the literature there are methods to calculate LCDAs. Next section is devoted to summarizing the main concepts relevant to calculating LCDAs by making use of wavefunctions obtained using quark model.

2.3 Light-Cone Wavefunctions

So far, no detailed discussion has been presented on how the quark-gluon structure of hadrons can be incorporated in the above techniques. For such a discussion, one needs a picture of hadrons in terms of QCD degrees of freedom.

This can be achieved by introducing wavefunctions. For instance, for a state involving a quark- anti-quark pair, one writes:

$$|H(p)\rangle \equiv \sum_{\lambda,\bar{\lambda}} \int \frac{d^3q}{(2\pi)^3} \Psi_{\lambda,\bar{\lambda}}(\vec{q}) |q_1(\vec{p}-\vec{q},\lambda)\bar{q}_2(\vec{q},\bar{\lambda})\rangle, \tag{2.47}$$

where $|q_1(\vec{p}-\vec{q},\lambda)\bar{q}_2(\vec{q},\bar{\lambda})\rangle$ is the quark - anti-quark state having a quark of 3-momentum $\vec{p}-\vec{q}$ and spin projection λ, and an anti-quark of 3-momentum $\vec{q}$ and spin projection $\bar{\lambda}$, and $\Psi_{\lambda,\bar{\lambda}}(\vec{q})$ is the coresponding wavefunction. The sum over spins is according to the quantum numbers of the hadron, that is, implying relevant spin and orbital angular momentum combinations. The information on orbital angular momentum is encoded in the wavefunction. Notice that 3-momenta of the quarks add up to the 3-momentum of the hadron, whereas their energies do not add up to the energy of the hadron due to binding. The conventions on defining the wavefunction may differ in various works (e. g. one may include energy denominators in the integration measure), and the definition and normalization of the Fock states may also be different, but the idea is always the same. One picks up a Fock state of definite 3-momentum and spin projections, and adds up such Fock states over all possible momenta and with the relevant orbital angular momentum quantum numbers.

This is how one defines a hadronic wavefunction (in momentum space) in standard Minkowski coordinates. However, the light-cone expansion motivates another set of coordinates, namely the light-cone coordinates, which is more suitable for use in the light-cone expansion.

Light-cone coordinates are defined as:

$$k^{\pm} = k^0 \pm k^3, \vec{k}_\perp \equiv (k^1, k^2) = \vec{k}_\perp. \tag{2.48}$$

One can define the Fock states having definite 3-momentum in the sense that k^+ and $\vec{k}_\perp$ components are conserved, and k^- components play the role of energy (it is indeed named as "light-cone energy" in the literature). The relation between wavefunctions in different coordinate systems is a complicated issue (see e. g. [74]). However, it is easier to relate wavefunctions when expressed in terms of relative momentum variables, at least for a conventional meson consisting of a quark and an anti-quark. In this case, wavefunctions calculated in standard Minkowski and light-cone coordinates can be related as:

$$\Psi(k^+, \vec{k}_\perp) \equiv N \times \sqrt{\left|\frac{\partial k^3}{\partial k^+}\right|} \Psi(\vec{k}_\perp, k^3), \tag{2.49}$$

where N is determined from the normalization relation for the wavefunction (normalization also depends on conventions, but the invariance of normalization among various coordinate systems leads to this relation). This relation arises owing to the fact that normalization of the wavefunction should be consistent in any corrdinate system, meaning that:

$$\int dk^3 d^2k_\perp |\Psi(\vec{k}_\perp, k^3)|^2 = \int dk^+ d^2k_\perp \left|\frac{\partial k^3}{\partial k^+}\right| |\Psi(k^+, \vec{k}_\perp)|^2. \tag{2.50}$$

One can also define the light-cone momentum fractions carried by the constituents of the hadron in the following way:

$$u_i \equiv \frac{k_i^+}{p^+}, \tag{2.51}$$

where k_i is the momentum of the i-th constituent and p is hadron momentum. By definition, $0 \leq u_i \leq 1$ and $\sum_i u_i = 1$. So, this variable can be identified with the u variable mentioned in Eq. 2.46.

Having this technology at hand, one can define field operators in terms of light-cone coordinates as well, and simply calculates the matrix elements appearing in the QCD calculations by taking care of relevant angular momentum combinations.

However, the issue of distinguishing between radial excitations of the same hadron is still open. Up to this point, the only information on radial excitations was that "the relevant information is encoded in the wavefunctions". If one constructs and solves a Schrödinger-like equation, this issue can be understood. Rather than this approach, in the literature, model functions motivated by the underlying physics have been preferred. Using model functions is also preferred in this work, and the model functions have been calculated in the framework of quark model [41].

2.4 Quark Model Calculations

The wavefunctions used in this work have been calculated using the Godfrey-Isgur hamiltonian [41]. For a more recent treatment on the charmonium spectrum one can consult Ref. [75].

The Hamiltonian used in [47] is the following:

$$H = H_0 + H_{ij}^{conf.} + H_{ij}^{hyp.} + H_{ij}^{so} \tag{2.52}$$

where H_0 is the kinetic term, $H_{ij}^{conf.}$ is the confinement term, $H_{ij}^{hyp.}$ is the hyperfine term and H_{ij}^{so} is the spin-orbit term. This hamiltonian has been constructed in meson rest frame so that its eigenvalues lead directly to relevant masses. This hamiltonian has been diagonalized using the eigenfunctions of the 3-dimensional simple harmonic oscillator as a basis [41]. This basis leads to the following wavefunctions:

$$\begin{aligned}
&\Psi^{QM}(\vec{\kappa}; n, L, S, J, J_z) \qquad (2.53)\\
&= \sum_{L_z,S_z} \mathtt{C} \times \langle L, L_z; S, S_z | L, S, J, J_z\rangle \times \chi_{S,S_z} \times Y_{L,L_z}(\theta_\kappa, \phi_\kappa)\\
&\times \sum_{m=0}^{N} h_{nm} \sqrt{2 \times \frac{2}{\nu^3} \frac{m!}{\Gamma(m+L+\frac{1}{2})\,[2(L+m)+1]}}\,(\nu\kappa)^L \exp\left[-\frac{\kappa^2}{2\nu^2}\right] \mathtt{L}_m^{l+1/2}(\frac{\kappa^2}{\nu^2}),
\end{aligned}$$

where $\vec{\kappa}$ is the relative momentum of the particles, $\mathtt{L}_m^{l+1/2}(\frac{\kappa^2}{\nu^2})$ are Laguerre polynomials, $Y_{L,L_z}(\theta_\kappa, \phi_\kappa)$ are spherical harmonics in momentum space. n is the radial quantum number, $\mathtt{C} = \frac{1}{\sqrt{3}}(R\bar{R} + B\bar{B} + G\bar{G})$ is the color part and χ_{S,S_z} is the spin part of the wavefunction. Functions having the above form constitute a complete orthonormal basis. h_{nm} are the elements of the n-th eigenvector of the hamiltonian given in Eq.(2.52). The parameter ν parametrizes the frequency of the oscillator. For numerical calculations, the expansion has been truncated at some finite dimension of the hamiltonian matrix.

Dressed c and b quark masses have been taken to be $m_c = 1628\,MeV$, $m_b = 4977\,MeV$ in those calculations. Other relevant parameters can be found in [47].

The calculated masses presented in Table 2.1 are used in numerical calculations in this work. In labeling the states, spectroscopic notation is used: for $^{2S+1}L_J$,

Table2.1: Masses of the first three levels of charmonia and bottomonia calculated in quark model, and experimentally observed states having the same quantum numbers. Notation: ${}^{2S+1}L_J$; S: total spin of the quark - anti-quark pair; L: relative orbital angular momentum; J: total spin of the meson.

Masses	$c\bar{c}$			$b\bar{b}$		
$M\,(GeV) \setminus n$	$n=1$	$n=2$	$n=3$	$n=1$	$n=2$	$n=3$
$M_{^3P_0}\,(\chi_{q0})$	3.37	3.88	4.30	9.81	10.2	10.7
$M_{^3P_1}\,(\chi_{q1})$	3.54	3.97	4.33	9.89	10.3	10.6
$M_{^1P_1}\,(h_q)$	3.53	3.96	4.37	9.88	10.3	10.6
$M_{^3P_2}\,(\chi_{q2})$	3.54	3.98	4.34	9.89	10.3	10.6
Masses (exp.)	$c\bar{c}$			$b\bar{b}$		
$M\,(GeV) \setminus n$	$n=1$	$n=2$	$n=3$	$n=1$	$n=2$	$n=3$
$M_{^3P_0}\,(\chi_{q0})$	3.41475	–	–	9.85944	10.2325	–
$M_{^3P_1}\,(\chi_{q1})$	3.51066	–	–	9.89278	10.25546	10512.1
$M_{^1P_1}\,(h_q)$	3.52538	–	–	9.8993	10.2598	–
$M_{^3P_2}\,(\chi_{q2})$	3.55620	3.9272	–	9.91221	10.26865	–

S is the total spin of the quark - anti-quark pair, L is relative orbital angular momentum and J is the total spin of the meson.

CHAPTER 3

TWIST-2 LIGHT-CONE DISTRIBUTION AMPLITUDES AND LEPTONIC DECAY CONSTANTS FOR P-WAVE CHARMONIA AND BOTTOMONIA

The remaining necessary ingredients for calculating the coupling parameters are the LCDAs for the ground states and the first two radial excitations of these mesons; and these LCDAs have been calculated using corresponding quark model wavefunctions. As a first step to test the reliability of the quark model wavefunctions, one may calculate the corresponding meson masses and compare these values with the existing experimental data [41]. As can be observed in Table 2.1, there is a good agreement between the calculated and experimental values. So, in this work, quark model wavefunctions calculated in the framework of [47] have been used.

3.1 Relevant matrix elements and LCDAs

In [30, 35], LCDAs of $L = 1$ mesons can be extracted certain hadronic matrix elements involving the follwing quantities:

$$\begin{aligned} z^\alpha &= x^\alpha - P^\alpha \frac{1}{m_{c\bar{c}}^2}\left(P\cdot x - \sqrt{(P\cdot x)^2 - x^2 m_{c\bar{c}}^2}\right) \\ &= x^\alpha - P^\alpha \frac{x^2}{2P\cdot z} + O(x^4), \end{aligned} \tag{3.1}$$

$$p_\alpha \equiv P_\alpha - m_{c\bar{c}}^2 \frac{z_\alpha}{2P\cdot z}, \tag{3.2}$$

$$\epsilon_{\perp\alpha}^{*(\sigma)} \equiv \epsilon_\alpha^{*(\sigma)} - \frac{\epsilon^{*(\sigma)}\cdot z}{p\cdot z}\left(p_\alpha - \frac{m_{c\bar{c}}^2}{2p\cdot z} z_\alpha\right), \tag{3.3}$$

where $z^2 = p^2 = 0$. One can write the polarization tensor of the tensor meson in terms of two polarization vectors for a massive vector particle [30]:

$$\epsilon^{\mu\nu}(m) = \langle 11; m', m'' | 11; 2, m\rangle \epsilon^{\mu}(m')\epsilon^{\nu}(m''), \tag{3.4}$$

where m, m', m'' denote spin projections along a certain (say, x^3) axis and $\langle 11; m', m'' | 11; 2, m\rangle$ are the corresponding Clebsh-Gordon coefficients for two spin-1 objects forming a spin-2 combination. One can decompose $\epsilon^{\mu\nu} z_\nu$ in longitidunal and transverse parts as follows [30]:

$$\epsilon_{\parallel}^{\mu\nu} z_\nu = \frac{\epsilon^{\alpha\nu} z_\alpha z_\nu}{p^\alpha z_\alpha}\left(p^\mu - z^\mu \frac{m_{c\bar{c}}^2}{2p\cdot z}\right),\ \epsilon_{\perp}^{\mu\nu} z_\nu = \epsilon^{\mu\nu} z_\nu \epsilon_{\parallel}^{\mu\nu} z_\nu, \tag{3.5}$$

which is cited for completeness.

The matrix elements, in terms of the above defined quantities are as follows [30]:

$$\begin{aligned}&\langle 0\,|\bar{q}(-z)\gamma^\mu q(z)|\, S(P)\rangle\,|_{z^2=0}\\ &= f_S P^\mu \int_0^1 du \phi_S(u),\end{aligned} \tag{3.6}$$

$$\begin{aligned}&\langle 0\,|\bar{q}(-z)\gamma^\mu \gamma_5 q(z)|\, A(P, \epsilon_{\lambda=0})\rangle\,|_{z^2=0}\\ &= i f_A M_A \epsilon^\mu \int_0^1 du \phi_{A\parallel}(u),\end{aligned} \tag{3.7}$$

$$\begin{aligned}&\frac{1}{2}\left\langle 0 \left|\bar{q}(-z)(\gamma^\mu z\cdot \hat{D} + \gamma^\nu z_\nu \hat{D}^\mu) q(z)\right| T(P, \epsilon_{\lambda=0})\right\rangle |_{z^2=0}\\ &= f_T M_T^2 \epsilon^{\mu\nu} z_\nu \int_0^1 du \xi \phi_{T\parallel}(u),\end{aligned} \tag{3.8}$$

$$\begin{aligned}&\langle 0\,|\bar{q}(-z)\sigma^{\mu\nu} z_\nu \epsilon_{\perp\mu}\gamma_5 q(z)|\, A(P, \epsilon_{\lambda=\pm1})\rangle\,|_{z^2=0}\\ &= f_A^{\perp} \int_0^1 du \phi_{A\perp}(u)\,(\epsilon\cdot\epsilon_\perp)\,(P\cdot z)\,,\end{aligned} \tag{3.9}$$

$$\langle 0\,|\bar{q}(-z)\sigma^{\mu\nu}z_\nu\epsilon_{\perp\mu\rho}z^\rho q(z)|\,T(P,\epsilon_{\lambda=\pm1})\rangle\,|_{z^2=0}$$
$$= if_T^\perp M_T\int_0^1 du\phi_{T\perp}(u)\left(\epsilon^{\mu\nu}z_\nu\epsilon_{\perp\mu\rho}z^\rho\right), \tag{3.10}$$

where z is the spacetime seperation between the quark and the anti-quark, ϵ^μ and $\epsilon^{\mu\nu}$ are the polarization vector and tensor of the relevant mesons, P is the four-momentum, M and f are the mass and decay constant of the relevant mesons, and $\hat{D}$ is the gauge covariant derivative.

It can be observed that, depending on the charge conjugation parities (C-parity) of the operators and meson states, LCDAs are either symmetric or anti-symmetric functions of u with respect to $u = 1/2$. C parities of the operators and meson states are as follows. Scalar ($\bar{\psi}\psi$) and axialvector $\bar{\psi}\gamma_\mu\gamma_5\psi$ operators have $C = +1$, pseudoscalar ($\bar{\psi}\gamma_5\psi$), vector ($\bar{\psi}\gamma_\mu\psi$) and tensor $\bar{\psi}\sigma_{\mu\nu}\psi$ operators have $C = -1$. For a meson state, C-parity is calculated as $(-1)^{L+S}$, where L is relative orbital angular momentum and S is the total spin of the quark - anti-quark pair. Remembering that all mesons of interest have $L = 1$, $S = 1$ scalar ($^3P_0$), $S = 1$ axialvector ($^3P_1$) and $S = 1$ tensor ($^3P_2$) meson states have $C = +1$, $S = 0$ axialvector ($^1P_1$) meson state has $C = -1$. As a result, the LCDAs $\phi_{^1P_{1\perp}}(u)$ and $\phi_{^3P_{1\parallel}}(u)$ are symmetric, $\phi_{^3P_0}(u)$, $\phi_{^3P_{1\perp}}(u)$, $\phi_{^1P_{1\parallel}}(u)$, $\phi_{^3P_{2\parallel}}(u)$ and $\phi_{^3P_{2\perp}}(u)$ are anti-symmetric with respect to $u = 1/2$. To understand how this conclusion derives, and remembering that the objects of interest are current operators having the form $\psi(x)\Gamma\psi'(y)$, one can consider a generic matrix element of the form:

$$\langle M|O(x,y)|0\rangle = \langle M|\left(C^\dagger C\right)O(x,y)\left(C^\dagger C\right)|0\rangle, \tag{3.11}$$

where C is the charge-conjugation operator and $C^\dagger$ is its hermitian conjugate satisfying $C^\dagger C = 1$. Since the C-parity of the vacuum state is $+1$, the matrix element in Eq. 3.11 becomes:

$$= \langle\bar{M}|\bar{O}(y,x)|0\rangle, \tag{3.12}$$

where $|\bar{M}\rangle = C_M|M\rangle$ and $\bar{O}(y,x) = C_O \times C^\dagger O(x,y)C$ are the charge conjugates, C_M and C_O are the C-parities of the meson M and the operator $O(x,y)$, respectively. Then the following equality is obtained:

$$C_M \times C_O \times \langle \bar{M}|\bar{O}(y,x)|0\rangle = \langle M|O(x,y)|0\rangle. \tag{3.13}$$

So, the C-parities of the meson state and the operator are multiplied. The interchange of x and y arises due to the q and $\bar{q}$ [70]. When the above matrix elements are considered, in which $x = -z$ and $y = z$, this change can be accompanied by interchanging u and $\bar{u} \equiv 1-u$ in the integrals. This determines the symmetry/anti-symmetry of the LCDAs with respect to $u = 1/2$.

In [30], a slightly different notation is used for the LCDAs, which are related to the notation used in this work in the following way: $\phi_{^1P_{1\perp}}(u) = \phi_{^1A_{1\perp}}(u)$, $\phi_{^3P_{1\|}}(u) = \phi_{^3A_{1\|}}(u)$, $\phi_{^3P_0}(u) = \phi_S(u)$, $\phi_{^3P_{1\perp}}(u) = \phi_{^3A_{1\perp}}(u)$, $\phi_{^1P_{1\|}}(u) = \phi_{^1A_{1\|}}(u)$, $\phi_{^3P_{2\|}}(u) = \phi_{T\|}(u)$, $\phi_{^3P_{2\perp}}(u) = \phi_{T\perp}(u)$.

From these matrix elements, the following expressions for the distribution amplitudes are obtained in [30]:

$$\phi_{^3P_0}(u) = \frac{\sqrt{2}}{f_{^3P_0}} \int \frac{d^2\kappa_\perp}{2(2\pi)^3} \frac{(\bar{u}-u)m_q}{\sqrt{u\bar{u}}} \varphi_{^3P_0}(m_q,u,\vec{\kappa}_\perp), \tag{3.14}$$

$$\phi_{^3P_1\|}(u) = \frac{2\sqrt{3}}{f_{^3P_1\|}} \int \frac{d^2\kappa_\perp}{2(2\pi)^3} \frac{\kappa_\perp^2}{\sqrt{u\bar{u}}M_0(m_q,u,\vec{\kappa}_\perp)} \varphi_{^3P_1\|}(m_q,u,\vec{\kappa}_\perp), \tag{3.15}$$

$$\phi_{^1P_1\|}(u) = \frac{\sqrt{6}}{f_{^1P_1\|}} \int \frac{d^2\kappa_\perp}{2(2\pi)^3} \frac{(\bar{u}-u)m_q}{\sqrt{u\bar{u}}} \varphi_{^1P_1\|}(m_q,u,\vec{\kappa}_\perp), \tag{3.16}$$

$$\begin{aligned}\phi_{^3P_2\|}(u) = {} & \frac{\sqrt{6}}{f_{^3P_2\|}} \int \frac{d^2\kappa_\perp}{2(2\pi)^3} \frac{(\bar{u}-u)}{\sqrt{u\bar{u}}} \varphi_{^3P_2\|}(m_q,u,\vec{\kappa}_\perp) \\ & \times \left[M_0(m_q,u,\vec{\kappa}_\perp) - m_q - \frac{\kappa_\perp^2}{M_0(m_q,u,\vec{\kappa}_\perp)+2m_q}\right],\end{aligned} \tag{3.17}$$

$$\phi_{^3P_1\perp}(u) = \frac{\sqrt{3}}{f_{^3P_1\perp}} \int \frac{d^2\kappa_\perp}{2(2\pi)^3} \frac{(\bar{u}-u)m_q}{\sqrt{u\bar{u}}} \varphi_{^3P_1\perp}(m_q,u,\vec{\kappa}_\perp), \tag{3.18}$$

$$\phi_{^1P_1\perp}(u) = \frac{\sqrt{6}}{f_{^1P_1\perp}} \int \frac{d^2\kappa_\perp}{2(2\pi)^3} \frac{\kappa_\perp^2}{\sqrt{u\bar{u}} M_0(m_q, u, \vec{\kappa}_\perp)} \varphi_{^1P_1\perp}(m_q, u, \vec{\kappa}_\perp), \quad (3.19)$$

$$\phi_{^3P_2\perp}(u) = \frac{\sqrt{6}}{f_{^3P_2\perp}} \int \frac{d^2\kappa_\perp}{2(2\pi)^3} \frac{(\bar{u} - u)}{\sqrt{u\bar{u}}} \times \left[m_q + \frac{2\kappa_\perp^2}{M_0(m_q, u, \vec{\kappa}_\perp) + 2m_q} \right] \varphi_{^3P_2\perp}(m_q, u, \vec{\kappa}_\perp), \quad (3.20)$$

where $\bar{u} = 1 - u$, m_q is the quark mass used in the framework in which the functions $\varphi(m_q, u, \vec{\kappa}_\perp)$ are calculated, $\vec{\kappa}$ is (half) the relative momentum of the particles and

$$M_0^2 = \frac{m_q^2 + \kappa_\perp^2}{u} + \frac{m_q^2 + \kappa_\perp^2}{\bar{u}} \quad (3.21)$$

where

$$\xi \equiv 1 - 2u. \quad (3.22)$$

The functions $\varphi(m_q, u, \vec{\kappa}_\perp)$ are the radial parts of the momentum-space wavefunctions of the quark - anti-quark pairs.

In this work, these expressions have been used to calculate the LCDAs of $c\bar{c}/b\bar{b}$ systems.

Here, the quark mass issue appears for the first time. As mentioned above, values for quark masses may be different in the context of sum rules and quark models. On the other hand, the above relations are independent of both sum rules and quark model calculations. So the only criterion for determining the correct values for the quark masses is the calculation of $\varphi(m_q, u, \vec{\kappa}_\perp)$.

Following [30], one may factor the radial and orbital parts of the wavefunction as:

$$\varphi_{L,L_z}(u, \kappa_\perp) = \varphi_p(u, \kappa_\perp) \kappa_{L_z}(u, \kappa_\perp) \quad (3.23)$$

where $\kappa_{L_z=\pm1} = \mp i(\kappa_1 \pm i\kappa_2)/\sqrt{2}$ and $\kappa_{L_z=0} = \kappa_3(u, \kappa_\perp)$; and the wavefunction is normalized as:

$$\int \frac{dud^2\kappa_\perp}{(2\pi)^3} \varphi_{L,L_z}(u, \kappa_\perp)\varphi^*_{L',L'_z}(u, \kappa_\perp) = \delta_{L,L'}\delta_{L_z,L'_z}. \tag{3.24}$$

One may calculate the wavefunction in terms of the standard Minkowski coordinates. If $K(|\vec{\kappa}|)$ is the radial part of the wavefunction calculated as such, one can relate the function φ_p and $K(|\vec{\kappa}|)$ in the following way:

$$\varphi_p(u, \kappa_\perp) = A \times \sqrt{\left|\frac{\partial \kappa_z}{\partial u}\right| \frac{K(|\vec{\kappa}|)}{|\vec{\kappa}|}}, \tag{3.25}$$

where $\vec{\kappa} = \vec{\kappa}(u, \vec{\kappa_\perp})$, $\kappa_z = \kappa_z(u, \vec{\kappa_\perp})$ and one calculates the normalization constant A using the normalization integral Eqn.3.24. This relation is obtained by demanding the normalization be consistent in all coordinate systems as in Eq. 2.50, and noticing that the wavefunctions in spherical polar coordinates are given as K functions times the spherical harmonics Y_{L,L_z}, which read $Y_{L,L_z} = \kappa_{L,L_z}/|\vec{\kappa}|$ for $L = 1$.

To proceed furhter for the leptonic decay constants, one can make use of the normalization conditions for the LCDAs [30, 35]:

$$\int_0^1 du\phi_{even}(u) = 1, \int_0^1 du(1-2u)\phi_{odd}(u) = 1. \tag{3.26}$$

One can also calculate the the leptonic decay constants by directly using the matrix elements (see e.g. [4]). However, as can be observed, part of these matrix elements are exactly equal to zero when the quark and the anti-quark meet at the same spacetime point. Such instances requires one to search for different matrix elements involving the same decay constants; so the procedure followed in [30] appears to be more practical.

If one neglects spin-orbit effects, the functions φ_p can be assumed to be the same for all states having the same n and L values, and the following equalities can be derived [30, 35]:

$$\sqrt{3}f_{^3P_0} = f_{^1P_1\parallel} = \sqrt{2}f_{^3P_1\perp} \equiv f_{odd}, \; \frac{f_{^3P_1\parallel}}{\sqrt{2}} = f_{^1P_1\perp} \equiv f_{even}. \tag{3.27}$$

$$\phi_{^3P_0} = \phi_{^1P_1\parallel} = \phi_{^3P_1\perp} \equiv \phi_{odd}, \ \phi_{^3P_1\parallel} = \phi_{^1P_1\perp} \equiv \phi_{even}. \tag{3.28}$$

In [30], the following values have been calculated for the decay constants. For charmonia:

$$\begin{aligned} f_{odd} &= 0.0884\,GeV, \ f_{even} = 0.109\,GeV; \\ f_{T\parallel} &= 0.124\,GeV, \ f_{T\perp} = 0.0978\,GeV; \end{aligned} \tag{3.29}$$

and for bottomonia:

$$\begin{aligned} f_{odd} &= 0.0674\,GeV, \ f_{even} = 0.0716\,GeV; \\ f_{T\parallel} &= 0.0750\,GeV, \ f_{T\perp} = 0.0692\,GeV. \end{aligned} \tag{3.30}$$

In [35], the decay constant $^1P_{1\perp}$ (corresponding to the h_c meson in [35]) is calculated as

$$f_{^1P_{1\perp}} = 0.192\,GeV. \tag{3.31}$$

These numerical values will be compared with the corresponding decay constants for the ground states, while f_{even} and f_{odd} are to be compared with the spin-weighted averages of the corresponding decay constants. Spin-weigthed avarages are as follows:

$$\begin{aligned} f_{odd} &= \frac{1 \times f_{^3P_0} + 3 \times f_{^1P_1\parallel} + 3 \times f_{^3P_1\perp}}{7}, \\ f_{even} &= \frac{3 \times f_{^3P_1\parallel} + 3 \times f_{^1P_1\perp}}{6}. \end{aligned} \tag{3.32}$$

There is no discussion on the radial excitations in [30]; however, as will be discussed below, the LCDA profiles of [30] correspond to ground state profiles in this work. This is the reason why decay constants given in [30] are also to be compared with those of ground states in this work.

Results obtained in this work and the above relations will be compared and discussed later on.

The only missing ingredient at this point is determining the radial part of the wavefunction. In [30], the following model function is used for all $L = 1$ states:

$$\varphi_p(u, k_\perp) = \frac{4\sqrt{2}}{\beta}\left(\frac{\pi}{\beta^2}\right)^{3/4}\sqrt{\frac{d\kappa_z}{du}}\exp\left(-\frac{|\vec{\kappa}|^2}{2\beta^2}\right). \tag{3.33}$$

Making use of wavefunctions calculated in the framework of quark model gives one the chance to calculate the LCDAs of the excited states in addition to that of the ground state. Furthermore, this approach can serve as a test for the validity of the approximation used in [30] which leads to the above mentioned model wavefunction for quarkonia.

The values obtained for the leptonic decay constants of $n = 1, 2, 3$ (n: radial quantum number) $c\bar{c}$ and $b\bar{b}$ systems are presented in Tables 3.1, 3.2 and 3.3. Calculations have been performed for two different cut-off values (quark mass and infinity) imposed on integrations over transverse relative momentum. Since the systems investigated are heavy quarkonia, quark dynamics is expected to be approximately non-relativistic, whereas relativistic effects may not be completely suppressed. With the motivation to understand the extent of the relativistic effects, two different cut-off values are used. Working with light-cone variables, one observes that the (+) components of the momentum vectors are already bounded (that is, $0 \leq u \leq 1$). So the only unbounded components are the transverse components, and this is why the cut-off is imposed on transverse momentum components. However, this does not completely define the "relativistic region". Remembering the definition of the u variable, Eq. 2.51,it is observed that $u \sim 1$ and $1 - u \sim 1$ correspond to relativistic momenta for finite $k_\perp$. So, the conclusion is that the domain in which relativistic effects are important, namely the "relativistic region", can be expressed as $k_\perp > m_q$, $\xi \equiv 1 - 2u \sim \pm 1$.

Calculated decay constants for $c\bar{c}$ and $b\bar{b}$ systems are presented in Tables 3.1, 3.2 and 3.3. The tables are organized so as to compare the results with Eq. 3.27 and Eq. 3.28.

The decay constants are roughly of the same order for $n = 1, 2, 3$ for each meson state. Numerical values given in [30] can be as small as 50 percent of the values obtained in this work. The conclusion here is that decay con-

Table3.1: Decay constants $f_{^1P_1}, f_{^3P_0}, f_{^3P_{1\perp}}$ for relevant charmonia and bottomonia.

$n \setminus f\ (GeV)$	$f_{^3P_0}$	$f_{^3P_1\perp}$	$f_{^1P_1}$	$\sqrt{3}f_{^3P_0}$	$\sqrt{2}f_{^3P_1\perp}$	f_{odd}
charmonia	$\Lambda = \infty$					
$n = 1$	0.109	0.0959	0.142	0.189	0.136	0.118
$n = 2$	0.0801	0.0881	0.129	0.139	0.125	0.105
$n = 3$	0.0755	0.0824	0.133	0.131	0.117	0.103
	$\Lambda = m_c$					
$n = 1$	0.0916	0.0875	0.127	0.159	0.124	0.105
$n = 2$	0.0588	0.0741	0.107	0.102	0.105	0.0860
$n = 3$	0.0459	0.0615	0.0946	0.0795	0.0870	0.0735
bottomonia	$\Lambda = \infty$					
$n = 1$	0.104	0.0802	0.119	0.180	0.113	0.100
$n = 2$	0.103	0.0832	0.124	0.178	0.118	0.104
$n = 3$	0.131	0.0834	0.143	0.227	0.118	0.116
	$\Lambda = m_b$					
$n = 1$	0.0972	0.0794	0.117	0.168	0.112	0.0981
$n = 2$	0.0976	0.0822	0.121	0.169	0.116	0.101
$n = 3$	0.118	0.0820	0.136	0.204	0.116	0.0951

stants are sensitive to the set of parameters used. The relations 3.27 and 3.28 appear to be partially satisfied, and the differences are interpreted as spin-orbit effects. $|f_{^1P_1} - \sqrt{2}f_{^3P_{1\perp}}|$ differences (which are smaller than 15 percent of min $\{f_{^1P_1}, \sqrt{2}f_{^3P_{1\perp}}\}$) are less than $|f_{^1P_1} - \sqrt{3}f_{^3P_0}|$ differences. Comparing $f_{^1P_{1\perp}}$ and $f_{^3P_1}/\sqrt{2}$ values, one observes that for $n = 1, 2$ the differences are less than 10 percent of min $\{f_{^1P_{1\perp}}, f_{^3P_1}/\sqrt{2}\}$, while for $n = 3$ the differences are greater, with the exception of $n = 3$, $\Lambda = m_c$ charmonium. These observations suggest that spin-orbit effects are not negligible, though one can at least determine the order of magnitude of the decay constants ignoring the spin-orbit effects.

One can also make the interpretation that the differences between the results for the decay constants for the ground states ($n = 1$) and those of [30] and [35] are also a measure of the spin-orbit coupling.

The value of the cut-off effects charmonia decay constants more significantly than those of bottomonia, while both values increase with increasing value of cut-off.

Table3.2: Decay constants $f_{^1P_1\perp}, f_{^3P_1}$ for relevant charmonia and bottomonia.

$n \setminus f\,(GeV)$	$f_{^3P_1}$	$f_{^1P_1\perp}$	$\frac{f_{^3P_1}}{\sqrt{2}}$	f_{even}	$f_{^3P_1}$	$f_{^1P_1\perp}$	$\frac{f_{^3P_1}}{\sqrt{2}}$	f_{even}
charmonia	$\Lambda = \infty$				$\Lambda = m_c$			
$n = 1$	0.264	0.199	0.187	0.232	0.185	0.133	0.131	0.159
$n = 2$	0.279	0.209	0.197	0.244	0.143	0.101	0.101	0.122
$n = 3$	0.290	0.246	0.205	0.268	0.0852	0.0595	0.0603	0.0724
bottomonia	$\Lambda = \infty$				$\Lambda = m_b$			
$n = 1$	0.182	0.138	0.129	0.160	0.173	0.126	0.122	0.146
$n = 2$	0.197	0.148	0.139	0.173	0.184	0.135	0.130	0.156
$n = 3$	0.204	0.182	0.144	0.193	0.187	0.153	0.132	0.170

Table3.3: Decay constants $f_{^3P_2}, f_{^3P_{2\perp}}$ for tensor charmonia and bottomonia.

$n \setminus f\,(GeV)$	$f_{^3P_2}$	$f_{^3P_2\perp}$	$f_{^3P_2}$	$f_{^3P_2\perp}$
charmonia	$\Lambda = \infty$		$\Lambda = m_c$	
$n = 1$	0.198	0.141	0.177	0.128
$n = 2$	0.229	0.142	0.189	0.118
$n = 3$	0.245	0.140	0.182	0.101
bottomonia	$\Lambda = \infty$		$\Lambda = m_b$	
$n = 1$	0.133	0.113	0.131	0.112
$n = 2$	0.148	0.121	0.146	0.119
$n = 3$	0.178	0.137	0.168	0.131

The observation that charmonia are affected more compared to bottomonia is expected, due to the fact that bottom quark is heavier than charm quark, and so relativistic effetcs are expected to be smaller for bottom quarks. However, for decay constants, the effect of relativistic momenta are still observable, as the values calculated for $\Lambda = \infty$ are greater than those calculated for $\Lambda = m_q$.

3.2 Fits for the LCDAs

It is possible to fit suitable functions to the LCDAs for practical applications. Finding suitable fit functions which reflect the general properties of the LCDAs also provides one a means to use the LCDAs without giving reference to any specific model.

In [35], the following functional forms have been used for the ground states:

$$\phi(\xi) = c(\beta)(1-\xi^2)\xi\exp[-\frac{\beta}{(1-\xi^2)}]$$
$$\psi(\xi) = -\int_{-1}^{\xi} dt\phi(t) = \frac{c(\beta)}{2}(1-\xi^2)^2 E_3[\frac{\beta}{(1-\xi^2)}] \tag{3.34}$$

where $\xi \equiv 1-2u$, $E_3[x] \equiv \int_1^\infty dt \frac{\exp[-xt]}{t^3}$, and the parameters c and β are to be calculated. These forms have been derived using the conventional SVZ sum rules. The idea in this work is to calculate the LCDAs using an independent approach and making use of these functions within sum rule calculations, and also calculating the LCDAs for the first two excited states. The first issue is handled by using quark model wavefunctions, and the second one is handled by motivating fit functions (making use of the above forms) which are suitable for the excited states. The fit functions used for the ground states are also slightly different from those derived in [35].

The fit functions used in this work have the following forms. For even LCDAs:

$$n=1: \psi(\xi) = a(1-\xi^2)^2\left(E_3[\frac{\beta}{(1-\xi^2)}] + b\exp[-\frac{\xi^2}{c}]\right) \tag{3.35}$$
$$n=2,\,3: \psi(\xi) = a\left\{\frac{1}{1+\frac{(\xi^2-\xi_0^2)^2}{\sigma^2}} + b\exp[-\frac{\xi^2}{c}]\right\}\exp[-\frac{\beta}{(1-\xi^2)}],$$

and for odd LCDAs:

$$n=1: \phi(\xi) = a\xi(1-\xi^2)\{\exp[-\frac{\beta}{(1-\xi^2)}] + b\exp[-\frac{\xi^2}{c}]\} \tag{3.36}$$
$$n=2,\,3: \phi(\xi) = -\frac{d}{d\xi}\left\{a\left[\frac{1}{1+\frac{(\xi^2-\xi_0^2)^2}{\sigma^2}} + b\exp[-\frac{\xi^2}{c}]\right]\exp[-\frac{\beta}{(1-\xi^2)}]\right\}.$$

Fits have been performed using a χ^2 minimization procedure in *Mathematica* software. Data sets have been obtained from the original LCDAs (those calculated directly from the quark model wavefunctions), and the above forms have been fitted to those data sets. Weight functions in the form $(1-\xi^2)^r$ have been used, where r is a non-zero number which can be large as 40. The motivation for using such weight functions is that the regions $\xi \sim \pm 1$ correspond to highly relativistic momenta and this form of a weight function suppresses the importance of these regions in determining the fit functions.

LCDAs calculated directly from the quark model wavefunctions and fits are presented in the Fig. A.1-A.7. Relevant fit parameters are presented in Tables A.1-A.7.

Certain patterns exhibited by the LCDAs have been observed. Ground state profiles are the same with those obtained in [30]. For even LCDAs, the ground state has a single global maximum at $\xi = 0$. The first excited state has a local minimum at $\xi = 0$ and two symmetric global maxima. The second excited state has a single global maximum at $\xi = 0$ in the shape of a bump. For odd LCDAs there is a node at $\xi = 0$ for all states. The ground state has one global maximum. The first excited state has two additional nodes, and four extrema, whose signs are alternating. The second excited state has six extrema and no additional nodes, and all extrema lying on the same side of the origin have the same sign. For all states, the extrema are closer to $\xi = 0$ for bottomonia as compared to charmonia, and the functions approach to zero as $e^{-\frac{\beta}{1-\xi^2}}$. For odd LCDAs, the global extrema approach to $\xi = \pm 1$ as n increases, meaning that the relativistic region becomes more significant for increasing n.

It has also been observed that charmonia are more sensitive to the cut-off compared to bottomonia. ${}^1P_{1\perp}$ and ${}^3P_{1\parallel}$ LCDAs for charmonia calculated using $\Lambda = m_c$ even do not exhibit the above mentioned pattern. Although charm quark is considered as a "heavy" quark with a mass above $1\,GeV$, this observation suggests that relativistic effects are still important for the charmonium system. The quark model calculations have been based on the "relativized" model of [47]. However, as any model based on writing effective non-causal potentials (such as those involving Coulomb $1/r$ or confinement kr terms), the ability of quark models in incorporating relativistic effects still appears to be a matter of debate.

CHAPTER 4

COUPLING OF AXIALVECTOR HEAVY QUAKONIA TO PSEUDOSCALAR AND VECTOR MESONS

In order to calculate the coupling of an axial-vector meson to psuedo-scalar and vector mesons within the context of light-cone sum rules, the object of interest is the two point correlator relating an on-shell axial-vector state to the hadronic vacuum via pseudo-scalar and vector current operators. The 1^{++} state is considered on-mass-shell, and the 0^- and 1^- currents represent the corresponding mesons to which the axial-vector couples ($\bar{D}$ and D^* mesons for the charmonia, $\bar{B}$ and B^* for the bottomonia case) at hadronic level. Using this correlation function, one can calculate the overlap of the 1^{++} state with a bound state of the 0^- and 1^- states, which involves the parameters measuring the strength of coupling of these states.

The calculations presented below involve the general flow calculations. Although the charmonium case is discussed in the flow, calculations for the bottomonium case are exactly the same, the only difference being the relevant parameters used.

4.1 Correlation function: Phenomenology

The object of interest in calculating the pseudoscalar - vector - axialvector couplings is the correlation function:

$$F_\nu^{*phen}(q,p,\epsilon) = -i \int d^4x \exp(iq\cdot x) \langle 0 \,| T\left\{J_5(x) J_\nu(0)\right\} |\, c\bar{c}(P,\epsilon)\rangle\,. \qquad (4.1)$$

Here, the operators $J_5(x)$ and $J_\nu(0)$ are composed of quark field operators which are solutions of QCD equations of motion. However, they can also be expressed in terms of the *free* solutions satisfying $(i\gamma^\mu\partial_\mu - m)\psi = 0$ (ψ representing c and/or u quark) such that $J_5 = \bar{c}i\gamma_5 u$ which creates a $\bar{D^0}$ (or destroys a D^0) and $J_\mu = \bar{u}\gamma_\mu c$ which creates a D^{*0} (or destroys a $\bar{D^{0*}}$) [70]:

$$F_\nu^{*phen}(q,p,\epsilon) = -i\int d^4x \exp(iq\cdot x)$$
$$\times \left\langle 0 \left| T\left\{ J_5(x)J_\nu(0)\exp\left[i\int d^4y J_{int}(y)\right]\right\}\right| c\bar{c}(P,\epsilon)\right\rangle . \quad (4.2)$$

Here, $J_{int}(y)$ represents the interaction terms in the QCD Hamiltonian. The interaction term appears due to making use of the free quark fields in the correlation function, and the effect of interactions are encoded in the J_{int} term. Discussion on why such correlation functions are suitable for calculating hadronic couplings can be found in [17, 18, 23, 37].

Considering only the first non-zero contribution to the correlation function, phenomenology gives the following expression:

$$F_\nu^*(q,P,\epsilon) = \int d^4x d^4y \exp(iq\cdot x)\left\langle 0\left|T\left\{J_5(x)J_\nu(0)J_{int}(y)\right\}\right| c\bar{c}(P,\epsilon)\right\rangle . \quad (4.3)$$

Here, contribution of all terms in the time ordered product are listed. The arrows indicate that the terms on the right hand side appear in calculating the matrix element which is written on the left hand side.

$$\langle 0\left|J_5(x)J_\nu(0)J_{int}(y)\right| c\bar{c}(P,\epsilon)\rangle \to \sum_{(\lambda)} \frac{1}{4EE'(E+E'-P^0)(q^0-E)},$$

$$\langle 0\left|J_5(x)J_{int}(y)J_\nu(0)\right| c\bar{c}(P,\epsilon)\rangle$$
$$\to \sum_{(\lambda)} \frac{(-1)}{4EE'(E-E'-P^0)}\left\{\frac{1}{(p')^0+E'} + \frac{1}{(q^0-E)}\right\},$$

$$\langle 0\left|J_{int}(y)J_5(x)J_\nu(0)\right| c\bar{c}(P,\epsilon)\rangle$$
$$\to \sum_{(\lambda)} \frac{(-1)}{4EE'(q^0+E)}\left\{\frac{1}{(p')^0+E'} - \frac{1}{(E+E'+P^0)}\right\},$$

$$\langle 0\left|J_{int}(y)J_\nu(0)J_5(x)\right| c\bar{c}(P,\epsilon)\rangle \to \sum_{(\lambda)} \frac{(-1)}{4EE'(E+E'+P^0)(q^0+E)},$$

$$\langle 0\,|J_\nu(0)J_5(x)J_{int}(y)|\,c\bar{c}(P,\epsilon)\rangle \to \sum_{(\lambda)} \frac{1}{4EE'(E+E'-P^0)((p')^0-E')},$$

$$\langle 0\,|J_\nu(0)J_{int}(y)J_5(x)|\,c\bar{c}(P,\epsilon)\rangle \to \sum_{(\lambda)} \frac{1}{4EE'(q^0+E)((p')^0-E')}, \tag{4.4}$$

where $E=\sqrt{\vec{q}^2+m_D^2}$, $E'=\sqrt{\vec{p}'^2+m_{D^*}^2}$, and $q^0+(p')^0=P^0$. All terms in the above expressions are multiplied by the hadronic matrix elements $\left\langle 0\,|J_5|\,\bar{D}(q)\right\rangle$, $\langle 0\,|J_\nu|\,D^*(p',\eta)\rangle$ and $\left\langle \bar{D}^0(q)D^{*0}(p',\eta^{(\lambda)})\,|J_{int}|\,c\bar{c}(P,\epsilon^{(\sigma)})\right\rangle$.

One makes use of the following relation in x^0 and y^0 integrations:

$$\lim_{t\to\pm\infty}\exp(i\alpha t) = \lim_{t\to\pm\infty}\lim_{\varepsilon\to 0}\exp[i(\alpha\pm i\varepsilon)t], \tag{4.5}$$

and the resolution of identity for the hadronic matrix elements:

$$\begin{aligned}
1 &= |0\,\rangle\langle\,0| + \int \frac{d^4k}{(2\pi)^4}\theta(E^0)(2\pi)\delta(k^2-m_h^2)\sum_h |h(k)\,\rangle\langle\,h(k)| \\
&+ \int \frac{d^4k d^4k'}{(2\pi)^8}\theta(E)\theta(E')(2\pi)^2\delta(k^2-m_h^2)\delta(k'^2-m_{h'}^2) \\
&\sum_{h,h'} |h(k)h'(k')\,\rangle\langle\,h(k)h'(k')| + \cdots \\
&= |0\,\rangle\langle\,0| + \int \frac{d\vec{k}}{(2\pi)^3 2\omega}\sum_h \left|h(\vec{k})\,\rangle\langle\,h(\vec{k})\right| \\
&+ \int \frac{d\vec{k}d\vec{k}'}{(2\pi)^6 4\omega\omega'}\sum_{h,h'}\left|h(\vec{k})h'(\vec{k}')\,\rangle\langle\,h(\vec{k})h'(\vec{k}')\right| + \cdots
\end{aligned} \tag{4.6}$$

Normalization of the states is relativistic ([70]):

$$\langle h(\vec{p},\lambda)|h(\vec{p}',\lambda')\rangle = 2\sqrt{\vec{p}^2+m^2}(2\pi)^3\delta^{(3)}(\vec{p}-\vec{p}')\delta_{\lambda,\lambda'}, \tag{4.7}$$

where λ,λ' denote the spins of the corresponding hadrons.

The matrix element defining the leptonic decay constant f_{H^*} of a vector meson H^* is ([17, 18, 42]):

$$\langle 0\,|\bar{q}\gamma_\mu Q|\,H^*(P,\eta)\rangle = m_{H^*}f_{H^*}\eta_\mu^{(H^*)}, \tag{4.8}$$

and that of a pseudo-scalar meson is:

$$\langle 0\,|\bar{q}\gamma_\mu\gamma_5 Q|\,H(P)\rangle = if_H P_\mu. \tag{4.9}$$

The Lorentz structure in the matrix elements 4.8 and 4.9 determine the Lorentz structure of the right hand sides. For 4.9, there is only one Lorentz vector at hand, P_μ, so the right hand side involves P_μ. For 4.8, there are two Lorentz vectors, η_ν and P_μ. The state vector of a vector particle carries information on its polarization state, and the vector carrying this information is η_ν. So, the state vector has to be proportional to η_ν. In principle, 4.8 can also involve P_μ as well; however, the coefficient of P_μ has to be proportional to η_ν, and the only possibility is contracting η_ν and P_μ, which is zero, by definition.

Using equations of motion, one multiplies 4.9 with $-iP^\mu$and obtains ([17,18,42]):

$$\langle 0 \left| \bar{q} i \gamma_5 Q \right| H(P) \rangle = \frac{f_H m_H^2}{m_Q + m_q}. \tag{4.10}$$

For vector particles, one has ([17, 18, 42]):

$$\sum_\lambda \eta_\mu^{*(\lambda)}(k') \eta_\nu^{(\lambda)}(k') = -g_{\mu\nu} + \frac{k'_\mu k'_\nu}{m^2}. \tag{4.11}$$

The coupling of $c\bar{c}$ to the $\bar{D}D^*$ molecule is defined by the matrix element:

$$\begin{aligned} &\left\langle \bar{D}^0(k) D^{*0}(k', \eta^{(\lambda)}) \left| J_{int} \right| c\bar{c}(k+k', \epsilon^{(\sigma)}) \right\rangle \\ &\equiv \left[G_1 (\epsilon^{(\sigma)} \cdot k)(\eta^{*(\lambda)} \cdot k) + G_2 (k' \cdot k)(\epsilon^{(\sigma)} \cdot \eta^{*(\lambda)}) \right] \equiv g(k, k', \epsilon, \eta), \end{aligned} \tag{4.12}$$

which has been discussed in Section 1.3, Eq. 1.8.

Remembering that the hadronic sum involves summations over relevant quantum numbers of the hadron states (here, the polarization vector of the vector meson), one needs to calculate the sum over η. Performing this summation, one obtains:

$$\sum_\lambda \eta_\nu^{(\lambda)} g(k, k', \epsilon, \eta) = \frac{(\epsilon^{(\sigma)} \cdot k)(k' \cdot k)}{m_{D^*}^2} (G_1 - G_2) k'_\nu - G_1 (\epsilon^{(\sigma)} \cdot k) k_\nu - G_2 (k' \cdot k) \epsilon_\nu^{(\sigma)}. \tag{4.13}$$

Then the correlation function becomes:

$$\begin{aligned} F_\nu^{*phen}(p', q, \epsilon) &= (subtractraction\, terms) - \frac{m_{\bar{D}}^2 m_{D^*} f_{\bar{D}} f_{D^*}}{m_c (m_D^2 - q^2)(m_{D^*}^2 - p'^2)} \\ &\times \left\{ \frac{(\epsilon^{(\sigma)} \cdot q)(p' \cdot q)}{m_{D^*}^2} (G_1 - G_2) p'_\nu - G_1 (\epsilon^{(\sigma)} \cdot q) q_\nu - G_2 (p' \cdot q) \epsilon_\nu^{(\sigma)} \right\}. \end{aligned} \tag{4.14}$$

and after the Wick rotation $k^0 \to -ik_E^0$:

$$F_{E,\nu}^{*phen}(p', q, \epsilon) = (subtraction\, terms) - \frac{m_{\bar{D}}^2 m_{D^*} f_{\bar{D}} f_{D^*}}{m_c(m_D^2 + q_E^2)(m_{D^*}^2 + p_E'^2)}$$
$$\times \left\{ \frac{(\epsilon_E^{(\sigma)} \cdot q_E)(p_E' \cdot q_E)}{m_{D^{*0}}^2}(G_1 - G_2)p_{E,\nu}' + G_1(\epsilon_E^{(\sigma)} \cdot q_E)q_{E,\nu} + G_2(p_E' \cdot q_E)\epsilon_{E,\nu}^{(\sigma)} \right\}. \tag{4.15}$$

4.2 Correlation function: QCD

On the QCD side, the correlation function can be expressed as (complex conjugate of the function presented in the previous section):

$$F_\nu^{QCD} = i \int d^4z \exp(-iq \cdot z)$$
$$\times \left\langle c\bar{c}(P, \epsilon) \middle| T \left\{ \bar{c}(z) i\gamma_5 u(z) \bar{u}(0) \gamma_\nu c(0) \exp \left[i \int d^4y J_{int}(y) \right] \right\} \middle| 0 \right\rangle. \tag{4.16}$$

On the QCD side, the zeroth order term in the expansion of $\exp\left[i \int d^4y J_{int}(y)\right]$ already gives a nonzero contribution. To demonstrate this, one may consider:

$$F_\nu^{QCD} = i \int d^4z \exp(-iq \cdot z) \left\langle c\bar{c}(P, \epsilon) \middle| T\{\bar{c}(z) i\gamma_5 u(z) \bar{u}(0) \gamma_\nu c(0)\} \middle| 0 \right\rangle. \tag{4.17}$$

Using the time ordering, one obtains:

$$F_\nu^{QCD} = i \int d^4z \exp(-iq \cdot z) \theta(z^0) \left\langle c\bar{c}(P, \epsilon) \middle| \bar{c}(z) i\gamma_5 u(z) \bar{u}(0) \gamma_\nu c(0) \middle| 0 \right\rangle$$
$$+ i \int d^4z \exp(-iq \cdot z) \theta(-z^0) \left\langle c\bar{c}(P, \epsilon) \middle| \bar{u}(0) \gamma_\nu c(0) \bar{c}(z) i\gamma_5 u(z) \middle| 0 \right\rangle. \tag{4.18}$$

In terms of the spinor components of the objects (capital Latin subscripts denote spinor indices, small latin superscripts denote color indices):

$$\bar{c}(z) i\gamma_5 u(z) \bar{u}(0) \gamma_\nu c(0) = \theta(z^0) \bar{c}_A^i(z) \left(i\gamma_5\right)_{AB} u_B^i(z) \bar{u}_C^j(0) \left(\gamma_\nu\right)_{CD} c_D^j(0)$$
$$+ \theta(-z^0) \bar{u}_C^j(0) \left(\gamma_\nu\right)_{CD} c_D^j(0) \bar{c}_A^i(z) \left(i\gamma_5\right)_{AB} u_B^i(z)$$
$$= \theta(z^0) \left(i\gamma_5\right)_{AB} \left(\gamma_\nu\right)_{CD} \bar{c}_A^i(z) c_D^j(0) u_B^i(z) \bar{u}_C^j(0)$$
$$+ \theta(-z^0) \left(i\gamma_5\right)_{AB} \left(\gamma_\nu\right)_{CD} c_D^j(0) \bar{c}_A^i(z) \bar{u}_C^j(0) u_B^i(z), \tag{4.19}$$

where the fact that operators of different flavors anti-commute has been used. So, the time ordered product becomes:

$$T\{\bar{c}(z)i\gamma_5 u(z)\bar{u}(0)\gamma_\nu c(0)\}$$

$$= (i\gamma_5)_{AB}(\gamma_\nu)_{CD}\left(\theta(z^0)\bar{c}^i_A(z)c^j_D(0)u^i_B(z)\bar{u}^j_C(0) + \theta(-z^0)c^j_D(0)\bar{c}^i_A(z)\bar{u}^j_C(0)u^i_B(z)\right). \tag{4.20}$$

Then, one observes the following:

$$T\{\bar{c}(z)i\gamma_5 u(z)\bar{u}(0)\gamma_\nu c(0)\} = (i\gamma_5)_{AB}(\gamma_\nu)_{CD}\, T\{\bar{c}^i_A(z)c^j_D(0)\}T\{u^i_B(z)\bar{u}^j_C(0)\}, \tag{4.21}$$

owing to the fact that $\theta(z^0)\theta(-z^0) = 0$ and $\theta(z^0)\theta(z^0) = \theta(z^0)$. The time ordered product of u-quark operators gives:

$$\langle 0 \left| T\{u^i_B(z)\bar{u}^j_C(0)\}\right| 0\rangle = iS^{(u)ij}_{BC} + \langle 0 \left|: u^i_B(z)\bar{u}^j_C(0) :\right| 0\rangle, \tag{4.22}$$

where the normal ordered product gives condensates involving u-quarks. The propagator:

$$iS^{(u)ij}_{BC} \simeq \delta^{ij}\frac{iz^\alpha(\gamma_\alpha)_{BC}}{2\pi^2 z^4} \tag{4.23}$$

for light quarks [17]. The condensate terms deserve some discussion. When expanded, the expression for $\langle 0 \left|: u^i_B(z)\bar{u}^j_C(0) :\right| 0\rangle$ brings terms in the form

$$\left\langle 0 \left|: u^i_B(0)\overleftarrow{D}_{\mu_1}\cdots\overleftarrow{D}_{\mu_r}\bar{u}^j_C(0) :\right| 0\right\rangle z^{\mu_1}\cdots z^{\mu_r}, \tag{4.24}$$

where the matrix elements are independent of z. Each factor of z^{μ_i} $(i = 1\ldots r)$ can be obtained in the following way:

$$z^{\mu_1}\cdots z^{\mu_r}e^{-iq\cdot z} = (i)^r\frac{\partial}{\partial q_{\mu_1}}\cdots\frac{\partial}{\partial q_{\mu_r}}e^{-iq\cdot z}. \tag{4.25}$$

When the matrix element involving c-quarks is written in terms of the LCDAs, the z-integral takes the following form:

$$\int d^4z e^{-iq\cdot z+uP\cdot z} = (2\pi)^4\delta^4(uP-q), \tag{4.26}$$

and the derivatives of this function with respect to q_μ are obtained. After performing the Borel transformation, this term does not contribute to the correlation function.

Returning to c-quarks, using Wick's theorem, one obtains:

$$T\{\bar{c}(z)\gamma_5\gamma_\alpha\gamma_\nu c(0)\} =: \bar{c}(z)\gamma_5\gamma_\alpha\gamma_\nu c(0) : + \overbrace{\bar{c}(z)\gamma_5\gamma_\alpha\gamma_\nu c(0)}, \tag{4.27}$$

where :: denotes normal ordering and $\overbrace{AB}$ denotes the contraction of operators A, B (here the c-quark operators). Since the matrix element without any c-quark operators will give zero (owing to presence of the $c\bar{c}$ state), the correlation function (without the condensate terms) becomes the following:

$$F_\nu^{QCD} = -i \int d^4z \exp(-iq\cdot z)\frac{z^\alpha}{2\pi^2 z^4}\left\langle c\bar{c}(P,\epsilon)\Big| : \bar{c}(z)\gamma_5\gamma_\alpha\gamma_\nu c(0) : \Big|0\right\rangle. \tag{4.28}$$

One has:

$$\gamma_\alpha\gamma_\nu = -i\sigma_{\alpha\nu} + g_{\alpha\nu}, \tag{4.29}$$

and according to [36], the relevant expression up to twist-2 should involve the σ term only (the g term contributions begin from twist-3).

One uses the definitions from [30, 36] for the matrix element (see Section 3.1). Then:

$$F_\nu^{QCD} = \frac{f_{c\bar{c}}^\perp}{2\pi^2}\int d^4z \int_0^1 du \exp(i(up-q)\cdot z)\frac{z^\alpha}{z^4}(\epsilon^*_{\perp\alpha}p_\nu - \epsilon^*_{\perp\nu}p_\alpha)\Phi_\perp(u), \tag{4.30}$$

where:

$$p\cdot z = P\cdot z - m_{c\bar{c}}^2\frac{z^2}{2P\cdot z},$$

$$\epsilon^*_\perp\cdot z = \epsilon^*\cdot z - \frac{\epsilon^*\cdot z}{p\cdot z}\left(p\cdot z - \frac{m_{c\bar{c}}^2}{2p\cdot z}z^2\right) = \epsilon^*\cdot z\frac{m_{c\bar{c}}^2 z^2}{2\,(p\cdot z)^2} = \epsilon^*\cdot z\frac{m_{c\bar{c}}^2 z^2}{2\left(P\cdot z - m_{c\bar{c}}^2\frac{z^2}{2P\cdot z}\right)^2}. \tag{4.31}$$

At $O\left(1/z^4\right)$, one obtains:

$$p\cdot z \simeq P\cdot z,\ \epsilon^*_\perp\cdot z\simeq 0,\ \exp(iup\cdot z)\simeq\exp(iuP\cdot z). \tag{4.32}$$

So one has:

$$F_\nu^{QCD} = -\frac{f_{c\bar{c}}^\perp}{2\pi^2}\int_0^1 du\Phi_\perp(u)\int d^4z\frac{\exp\{i\,(uP-q)\cdot z\}}{z^4}$$

$$\times\left\{-\left(\epsilon^*\cdot z\right)P_\nu + \frac{\left(\epsilon^*\cdot z\right)m_{c\bar{c}}^2}{(P\cdot z)}z_\nu - (P\cdot z)\,\epsilon^*_\nu\right\}. \tag{4.33}$$

In the previous section, the complex conjugate of this function was used. The complex conjugate is:

$$F_{\nu}^{*QCD} = -\frac{f_{c\bar{c}}^{\perp}}{2\pi^2}\int_0^1 du\Phi_{\perp}(u)\int d^4z\frac{\exp\left\{-i\left(uP-q\right)\cdot z\right\}}{z^4}$$

$$\times\left\{-\left(\epsilon\cdot z\right)P_{\nu}+m_{c\bar{c}}^2\frac{(\epsilon\cdot z)}{(P\cdot z)}z_{\nu}+\left(P\cdot z\right)\epsilon_{\nu}\right\}. \tag{4.34}$$

Now the correlation function can be examined term by term.

For factors involving z^{μ}, one may take derivatives with respect to $k^{\mu}\equiv(uP-q)^{\mu}$. For the $1/(P\cdot z)$ factor, one may define:

$$\varphi(u)\equiv-\int_1^u dv\Phi_{\perp}(v); \tag{4.35}$$

and integrate by parts with respect to u once to get:

$$\int_0^1 du\exp\left\{-i\left(uP-q\right)\cdot z\right\}\Phi_{\perp}(u)\frac{(\epsilon\cdot z)}{(P\cdot z)} = \int_0^1 du\varphi(u)\epsilon\cdot\frac{\partial}{\partial k}\exp\left\{-ik\cdot z\right\}. \tag{4.36}$$

Since $\int_0^1 du\Phi_{\perp}(u)=0$ [30,36], one obtains $\varphi(0)=0$ in the above line.

At this point, the correlation function becomes:

$$F_{\nu}^{*QCD} = \frac{-if_{c\bar{c}}^{\perp}}{2\pi^2}\int_0^1 du\{-P_{\nu}\Phi_{\perp}(u)\left(\epsilon\cdot\frac{\partial}{\partial k}\right)+\epsilon_{\nu}\Phi_{\perp}(u)\left(P\cdot\frac{\partial}{\partial k}\right)$$

$$+m_{c\bar{c}}^2\varphi(u)\left(\epsilon\cdot\frac{\partial}{\partial k}\right)\frac{\partial}{\partial k^{\nu}}\}\int d^4z\frac{1}{z^4}\exp\left\{-ik\cdot z\right\}. \tag{4.37}$$

Now one can perform a Wick rotation $z^0\equiv-iz_E^0$, $\vec{z}\equiv\vec{z}_E$ which leads to:

$$F_{E,\nu}^{*QCD} = -\frac{f_{c\bar{c}}^{\perp}}{2\pi^2}\int_0^1 du\{P_{E,\nu}\Phi_{\perp}(u)\left(\epsilon_E\cdot\frac{\partial}{\partial k_E}\right)-\epsilon_{E,\nu}\Phi_{\perp}(u)\left(P_E\cdot\frac{\partial}{\partial k_E}\right)$$

$$+m_{c\bar{c}}^2\varphi(u)\left(\epsilon_E\cdot\frac{\partial}{\partial k_E}\right)\frac{\partial}{\partial k_E^{\nu}}\}\int d^4z_E\frac{1}{z_E^4}\exp\left\{ik_E\cdot z_E\right\}. \tag{4.38}$$

One can proceed now by calculating the z-integral.

Making use of:

$$\frac{1}{(z_E^2)^n} = \frac{1}{\Gamma(n)}\int_0^{\infty}dt\exp(-tz_E^2)t^{n-1}, \tag{4.39}$$

one obtains:

$$\int d^4 z_E \frac{1}{z_E^4} \exp\{ik_E \cdot z_E\} = \int d^4 z_E \int_0^\infty t \exp\left\{-tz_E^2 + ik_E \cdot z_E\right\} dt. \qquad (4.40)$$

The Gaussian integral can be evaluated as:

$$\int d^4 z \exp\{-tz^2 + ik \cdot z\} = \int d^4 z \exp\{-t(z - \frac{ik}{2t})^2 - \frac{k^2}{4t}\} = e^{-\frac{k^2}{4t}} \frac{\pi^2}{t^2}, \qquad (4.41)$$

owing to the fact that in one dimension $\int_\infty^\infty dz e^{-tz^2} = \sqrt{\frac{\pi}{t}}$ and one can shift the integration variable z since the integration is over all space. So, one obtains:

$$\int d^4 z_E \int_0^\infty t \exp\left\{-tz_E^2 + ik_E \cdot z_E\right\} dt = \int_0^\infty dt \frac{\pi^2}{t} \exp\left\{-\frac{k_E^2}{4t}\right\}. \qquad (4.42)$$

Making a change of variables $v \equiv 1/t$ one obtains [1]:

$$\int_0^\infty dt \frac{\pi^2}{t} \exp\left\{-\frac{k_E^2}{4t}\right\} = \int_0^\infty dv \frac{\pi^2}{v} \exp\left\{-\frac{vk_E^2}{4}\right\}. \qquad (4.43)$$

One now plugs this result into the expression for the correlation function and obtains:

$$\begin{aligned} F_{E,\nu}^{*QCD} &= -\frac{f_{c\bar{c}}^\perp}{2\pi^2} \int_0^1 du\{P_{E,\nu} \Phi_\perp(u) \left(\epsilon_E \cdot \frac{\partial}{\partial k_E}\right) - \epsilon_{E,\nu} \Phi_\perp(u) \left(P_E \cdot \frac{\partial}{\partial k_E}\right) \\ &\quad - m_{c\bar{c}}^2 \varphi(u) \left(\epsilon_E \cdot \frac{\partial}{\partial k_E}\right) \frac{\partial}{\partial k_E^\nu}\} \int_0^\infty dv \frac{\pi)^2}{v} \exp\left\{-\frac{vk_E^2}{4}\right\} \\ &= f_{c\bar{c}}^\perp \int_0^1 du\{-P_{E,\nu} \frac{\Phi_\perp(u) (\epsilon_E \cdot q_E)}{(uP - q)_E^2} + \epsilon_{E,\nu} \left[\frac{-\Phi_\perp(u) (P_E \cdot (uP - q)_E) + m_{c\bar{c}}^2 \varphi(u)}{(uP - q)_E^2}\right] \\ &\quad + \frac{2m_{c\bar{c}}^2 \varphi(u) (\epsilon_E \cdot q_E)}{(uP - q)_E^4} (uP - q)_{E,\nu}\}. \end{aligned} \qquad (4.44)$$

Introducing $p_E^{'} \equiv P_E - q_E$ and $\bar{u} \equiv 1 - u$, one obtains for the scalar products $P_E \cdot (uP - q)_E = \left(m_{c\bar{c}}^2 + p_E^{'2} - q_E^2\right)/2$ and $(uP - q)_E^2 = u\bar{u}m_{c\bar{c}}^2 + up_E^{'2} + \bar{u}q_E^2$ and the correlation function becomes the following:

$$F_{E,\nu}^{*QCD}$$

$$= f_{c\bar{c}}^\perp \int_0^1 du\{p_{E,\nu}^{'} (\epsilon_E \cdot q_E) \left[-\frac{\Phi_\perp(u)}{\left(u\bar{u}m_{c\bar{c}}^2 + up_E^{'2} + \bar{u}q_E^2\right)} + \frac{2m_{c\bar{c}}^2 u\varphi(u)}{\left(u\bar{u}m_{c\bar{c}}^2 + up_E^{'2} + \bar{u}q_E^2\right)^2}\right]$$

[1] Although the resulting integral 4.43 is divergent, what are needed are its derivatives, which are convergent.

$$+q_{E,\nu}\left(\epsilon_E\cdot q_E\right)\left[\frac{\Phi_\perp(u)}{\left(u\bar{u}m_{c\bar{c}}^2+up_E'^2+\bar{u}q_E^2\right)}-\frac{2m_{c\bar{c}}^2\bar{u}\varphi(u)}{\left(u\bar{u}m_{c\bar{c}}^2+up_E'^2+\bar{u}q_E^2\right)^2}\right]$$

$$+\epsilon_{E,\nu}\frac{m_{c\bar{c}}^2\varphi(u)-\frac{1}{2}\Phi_\perp(u)\left((1-2u)m_{c\bar{c}}^2+p_E'^2-q_E^2\right)}{u\bar{u}m_{c\bar{c}}^2+up_E'^2+\bar{u}q_E^2}\}. \tag{4.45}$$

Remembering that:

$$F_{E,\nu}^{phen.*}(q,p',\epsilon)=-\frac{m_D^2m_{D^*}f_Df_{D^*}}{m_c(m_D^2+q_E^2)(m_{D^*}^2+p_E'^2)}$$

$$\times\left\{\frac{(\epsilon_E\cdot q_E)(p_E'\cdot q_E)}{m_{D^*}^2}(G_1-G_2)p_{E,\nu}'+G_1(\epsilon_E\cdot q_E)q_{E,\nu}+G_2(p_E'\cdot q_E)\epsilon_{E,\nu}\right\} \tag{4.46}$$

one has the following relations (factors multiplying $(\epsilon_E\cdot q_E)p_{E,\nu}'$, $(\epsilon_E\cdot q_E)q_{E,\nu}$, ϵ_E respectively):

$$F_{p'}^{QCD}=f_{c\bar{c}}^\perp\int_0^1du\left\{\frac{-\Phi_\perp(u)}{\left(u\bar{u}m_{c\bar{c}}^2+up_E'^2+\bar{u}q_E^2\right)}+\frac{2m_{c\bar{c}}^2u\varphi(u)}{\left(u\bar{u}m_{c\bar{c}}^2+up_E'^2+\bar{u}q_E^2\right)^2}\right\}$$

$$F_{p'}^{pheno.}=\frac{(G_1-G_2)m_D^2f_Df_{D^*}}{m_{D^*}m_c}\frac{\left(m_{c\bar{c}}^2+p_E'^2+q_E^2\right)/2}{(m_D^2+q_E^2)(m_{D^*}^2+p_E'^2)};$$

$$F_q^{QCD}=f_{c\bar{c}}^\perp\int_0^1du\left\{\frac{\Phi_\perp(u)}{\left(u\bar{u}m_{c\bar{c}}^2+up_E'^2+\bar{u}q_E^2\right)}-\frac{2m_{c\bar{c}}^2\bar{u}\varphi(u)}{\left(u\bar{u}m_{c\bar{c}}^2+up_E'^2+\bar{u}q_E^2\right)^2}\right\}$$

$$F_q^{pheno.}=-\frac{G_1m_{\bar{D}}^2m_{D^*}f_{\bar{D}}f_{D^*}}{m_c}\frac{1}{(m_D^2+q_E^2)(m_{D^*}^2+p_E'^2)};$$

$$F_\epsilon^{QCD}=f_{c\bar{c}}^\perp\int_0^1du\left\{\frac{m_{c\bar{c}}^2\varphi(u)-\frac{1}{2}\Phi_\perp(u)\left((1-2u)m_{c\bar{c}}^2+p_E'^2-q_E^2\right)}{u\bar{u}m_{c\bar{c}}^2+up_E'^2+\bar{u}q_E^2}\right\}$$

$$F_\epsilon^{pheno.}=\frac{G_2m_{\bar{D}}^2m_{D^*}f_{\bar{D}}f_{D^*}}{m_c}\frac{\left(m_{c\bar{c}}^2+p_E'^2+q_E^2\right)/2}{(m_D^2+q_E^2)(m_{D^*}^2+p_E'^2)}. \tag{4.47}$$

Since ϵ is fixed but arbitrary, $\epsilon\cdot q$ terms can be cancelled from the two sides.

4.3 Borel transform and couplings

One can write the dispersion relations and consider the contributions coming from them following the treatment in [65]. The correlation function calculated in QCD contains terms having the form $\int_0^1 du \frac{f(u)}{(u(p')^2_E+\bar{u}q^2_E+u\bar{u}m^2_{c\bar{c}})^n}$. Since the calculation will proceed with Euclidean momenta, it is convenient to supress the cumbersome notation p_E from now on: $p_E \to p$. Then, one defines:

$$\int_0^1 du \frac{f(u)}{(up'^2+\bar{u}q^2+u\bar{u}m^2_{c\bar{c}})^n} \equiv \int_0^\infty ds_1 \int_0^\infty ds_2 \frac{\rho_n(s_1,s_2)}{(s_1+p'^2)(s_2+q^2)}. \tag{4.48}$$

Now one performs a double Borel transform with respect to q^2 and $p2^2$:

$$B_{M_2^2}(p'^2)B_{M_1^2}(q^2)\int_0^1 du \frac{f(u)}{(up'^2+\bar{u}q^2+u\bar{u}m^2_{c\bar{c}})^n}$$

$$\equiv B_{M_2^2}(p'^2)B_{M_1^2}(q^2)\int_0^\infty ds_1 \int_0^\infty ds_2 \frac{\rho_n(s_1,s_2)}{(s_2+p'^2)(s_1+q^2)}$$

$$= \int_0^\infty ds_1 \int_0^\infty ds_2 \rho_n(s_1,s_2)\exp(-\frac{s_1}{M_1^2}-\frac{s_2}{M_2^2}) \tag{4.49}$$

To obtain the spectral density, one need to perform two more Borel transforms with respect to $1/M_1^2$ and $1/M_2^2$, and obtains:

$$B_{\tau_2}(\frac{1}{M_2^2})B_{\tau_1}(\frac{1}{M_1^2})\left\{\int_0^\infty ds_1 \int_0^\infty ds_2 \rho_n(s_1,s_2)\exp(-\frac{s_1}{M_1^2}-\frac{s_2}{M_2^2})\right\} = \frac{1}{\tau_1\tau_2}\rho_n(\frac{1}{\tau_1},\frac{1}{\tau_2}). \tag{4.50}$$

Having calculated the spectral density, one can write the correlation function in terms of a dispersion integral.

This is the general way of obtaining the spectral density when one has two independent momenta. However one may need a further simplification since the Borel transform of the above function also includes obviously the DA itself. When the pseudo-scalar and vector meson masses are close, one can assume $M_1^2 = M_2^2 = 2M^2$. The mass difference is $0.14\,GeV$ for D and D^* mesons, and is $0.05\,GeV$ for B and B^* mesons and so this approximation can be done. One defines:

$$s \equiv (s_1+s_2)/2;\ \beta \equiv s_1/(s_1+s_2)$$

$$\Rightarrow \int_0^\infty ds_1 \int_0^\infty ds_2 = \int_0^1 d\beta \int_0^\infty ds \begin{vmatrix} \partial s_1/\partial s & \partial s_2/\partial s \\ \partial s_1/\partial\beta & \partial s_2/\partial\beta \end{vmatrix} = \int_0^\infty ds \cdot s \int_0^1 d\beta. \tag{4.51}$$

Then, the calculation proceeds as follows:

$$\int_0^\infty ds_1 \int_0^\infty ds_2 \rho_n(s_1,s_2)\exp(-\frac{s_1}{M_1^2}-\frac{s_2}{M_2^2}) \equiv \int_0^\infty ds\cdot s\int_0^1 d\beta\rho_n(s,\beta)\exp(-\frac{s}{M^2})$$

$$= \int_0^\infty ds\cdot\rho_n(s)\exp(-\frac{s}{M^2}), \tag{4.52}$$

where $\rho_n(s)\equiv\int_0^1 d\beta\rho_n(s,\beta)$. Taking one more Borel transform here suffices to calculate the spectral density.

Consider again $\int_0^1 du\frac{f(u)}{(u(p')^2_E+\bar{u}q^2_E+u\bar{u}m^2_{c\bar{c}})^n}$:

$$B_{p'^2,M_2^2}B_{q^2,M_1^2}\left\{\int_0^1 du\frac{f(u)}{(up'^2+\bar{u}q_E^2+u\bar{u}m^2_{c\bar{c}})^n}\right\}$$

$$= B_{p'^2,M_2^2}\left\{\int_0^1 du\frac{f(u)}{\bar{u}^n\Gamma(n)M_1^{2(n-1)}}\exp\left[-\frac{\frac{u}{\bar{u}}p'^2+um^2_{c\bar{c}}}{M_1^2}\right]\right\}$$

$$= \int_0^1 du\frac{f(u)}{\bar{u}^n\Gamma(n)M_1^{2(n-1)}}\exp\left[-\frac{um^2_{c\bar{c}}}{M_1^2}\right]M_2^2\delta\left(\frac{uM_2^2}{\bar{u}M_1^2}-1\right)$$

$$= \frac{f(u_0)\exp\left[-\frac{u_0m^2_{c\bar{c}}}{M_1^2}\right]}{\bar{u}_0^n\Gamma(n)M_1^{2(n-1)}}\frac{M_2^2}{\frac{M_2^2}{\bar{u}_0^2M_1^2}} = \frac{f(u_0)\exp\left[-\frac{u_0m^2_{c\bar{c}}}{M_1^2}\right]}{\bar{u}_0^{n-2}\Gamma(n)M_1^{2(n-2)}}, \tag{4.53}$$

where $u_0 = M_1^2/(M_1^2+M_2^2)$. Choosing $M_1^2=M_2^2=2M^2$ gives:

$$B_{p'^2,M_2^2}B_{q^2,M_1^2}\left\{\int_0^1 du\frac{f(u)}{(up'^2+\bar{u}q^2+u\bar{u}m^2_{c\bar{c}})^n}\right\}$$

$$= \left\{\begin{array}{c} f(\frac{1}{2})\exp\left[-\frac{m^2_{c\bar{c}}}{4M^2}\right]M^2,\ n=1 \\ f(\frac{1}{2})\exp\left[-\frac{m^2_{c\bar{c}}}{4M^2}\right],\ n=2 \end{array}\right\}. \tag{4.54}$$

So, one proceeds as follows. Defining $F_1(M^2,m^2_{c\bar{c}})\equiv f(\frac{1}{2})\exp\left[-\frac{m^2_{c\bar{c}}}{4M^2}\right]M^2$ and $F_2(M^2,m^2_{c\bar{c}})\equiv f(\frac{1}{2})\exp\left[-\frac{m^2_{c\bar{c}}}{4M^2}\right]$, one observes that:

$$-4\frac{\partial F_1}{\partial m^2_{c\bar{c}}} = F_2 \Rightarrow -4\frac{\partial\rho_1}{\partial m^2_{c\bar{c}}} = \rho_2. \tag{4.55}$$

For $n = 2$:

$$f(\frac{1}{2})B_{M^{-2},\tau^{-1}}\left\{\exp\left[-\frac{m_{c\bar{c}}^2}{4M^2}\right]\right\} = \int\limits_0^\infty ds \cdot \rho_2(s)\frac{1}{\tau}\delta(\frac{s}{\tau}-1) = \rho_2(\tau)$$

$$\Rightarrow \rho_2(s) = f(\frac{1}{2})\frac{1}{s}\delta\left(\frac{m_{c\bar{c}}^2}{4s}-1\right). \tag{4.56}$$

So, for $n = 1$:

$$\rho_1(s) = f(\frac{1}{2})\theta\left(1-\frac{m_{c\bar{c}}^2}{4s}\right). \tag{4.57}$$

Here, it is convenient to summarize the procedure for finding the spectral densities:

$$B_{M^{-2},s^{-1}}\left\{\left[B_{p'^2,M_2^2}B_{q^2,M_1^2}F(q^2,p'^2)\right]_{M_1^2} = M_2^2 \equiv 2M^2\right\}$$

$$\equiv B_{q^2,p'^2\to M^{-2},s^{-1}}F(q^2,p'^2) = \rho(s), \tag{4.58}$$

where

$$\left[B_{p'^2,M_2^2}B_{q^2,M_1^2}F(q^2,p'^2)\right]_{M_1^2=M_2^2\equiv 2M^2} \equiv \int\limits_0^\infty ds \cdot \rho(s)\exp(-\frac{s}{M^2}). \tag{4.59}$$

In $J^{PC} = 1^{++}$ case, $s_0 = ((m_D+\alpha)^2 + (m_{D^*}+\alpha)^2)/2$ $(0\,GeV \leq \alpha \leq 0.5\,GeV)$, that is, slightly above one fourth of the $c\bar{c}/b\bar{b}$ mass squared. Threshold parameters are generally chosen to be slightly above the mass squared of the relevant hadrons, and are expected to be less than the mass squared of the first excited states. Since $s = (s_1+s_2)/2$, the above mentioned threshold is obtained. So the above relation becomes, in its final form (after continuum subtraction):

$$\left[B_{p'^2,M_2^2}B_{q^2,M_1^2}F(q^2,p'^2)\right]^{s_0}_{M_1^2=M_2^2\equiv 2M^2} = \int\limits_0^{s_0} ds \cdot \rho(s)\exp(-\frac{s}{M^2}), \tag{4.60}$$

where it is understood that the two sides are not exactly equal, but the Borel transforms supress the excited states and continuum and one is left with the ground state terms on both sides.

Consider once again the OPE and QCD calculations for the correlation function. The coefficients of the p' term were:

$$I:\ F_{p'}^{QCD} = f_{c\bar{c}}^{\perp}\int_0^1 du\left\{\frac{-\Phi_{\perp}(u)}{\left(u\bar{u}m_{c\bar{c}}^2+up_E'^2+\bar{u}q_E^2\right)} + \frac{2m_{c\bar{c}}^2u\varphi(u)}{\left(u\bar{u}m_{c\bar{c}}^2+up_E'^2+\bar{u}q_E^2\right)^2}\right\}$$

$$F_{p'}^{pheno.} = \frac{(G_1-G_2)m_D^2 f_D f_{D^*}}{m_{D^*}m_c}\frac{\left(m_{c\bar{c}}^2+p_E'^2+q_E^2\right)/2}{(m_D^2+q_E^2)(m_{D^*}^2+p_E'^2)}.$$

Then:

$$B_{q^2,p'^2\to M^{-2},s^{-1}}\int_0^1 du\left\{-\frac{\Phi_\perp(u)}{up_E'^2+\bar{u}q_E^2+u\bar{u}m_{c\bar{c}}^2}+\frac{2m_{c\bar{c}}^2\varphi(u)u}{\left(up_E'^2+\bar{u}q_E^2+u\bar{u}m_{c\bar{c}}^2\right)^2}\right\}$$

$$\equiv \rho_I(s) = -\Phi_\perp(\frac{1}{2})\theta\left(1-\frac{m_{c\bar{c}}^2}{4s}\right)+2m_{c\bar{c}}^2\varphi(\frac{1}{2})\frac{1}{2s}\delta\left(\frac{m_{c\bar{c}}^2}{4s}-1\right)$$

$$= \frac{m_{c\bar{c}}^2}{s}\varphi(\frac{1}{2})\delta\left(\frac{m_{c\bar{c}}^2}{4s}-1\right), \tag{4.61}$$

due to the fact that $\Phi_\perp(\frac{1}{2})=0$ for being asymmetric with respect to $u=1/2$. For the right hand side:

$$f_{c\bar{c}}^\perp\int_0^{s_0} ds\cdot\rho_I(s)\exp(-\frac{s}{M^2})$$

$$= B_{q^2,p'^2\to M^{-2},s^{-1}}\left\{\frac{(G_1-G_2)m_D^2 f_D f_{D^*}}{m_{D^*}m_c}\frac{\left(m_{c\bar{c}}^2+p_E'^2+q_E^2\right)/2}{(m_D^2+q_E^2)(m_{D^*}^2+p_E'^2)}\right\}$$

$$= B_{q^2,p'^2\to M^{-2},s^{-1}}\{\frac{(G_1-G_2)m_D^2 f_D f_{D^*}}{2m_{D^*}m_c}\{\frac{m_{c\bar{c}}^2}{(m_D^2+q_E^2)(m_{D^*}^2+p_E'^2)}$$

$$+\frac{1}{(m_D^2+q_E^2)}\left(1-\frac{m_{D^*}^2}{(m_{D^*}^2+p_E'^2)}\right)+\frac{1}{(m_{D^*}^2+p_E'^2)}\left(1-\frac{m_D^2}{(m_D^2+q_E^2)}\right)\}\}$$

$$= B_{q^2,p'^2\to M^{-2},s^{-1}}\left\{\frac{(G_1-G_2)m_D^2 f_D f_{D^*}}{2m_{D^*}m_c}\frac{m_{c\bar{c}}^2-m_D^2-m_{D^*}^2}{(m_D^2+q_E^2)(m_{D^*}^2+p_E'^2)}\right\}$$

$$\Rightarrow F_{p'}^{QCD} = f_{c\bar{c}}\varphi\left(\frac{1}{2}\right)m_{c\bar{c}}^2\exp[-\frac{m_{c\bar{c}}^2}{4M^2}]$$

$$= F_{p'}^{pheno.} = \frac{(G_1-G_2)m_D^2 f_D f_{D^*}}{2m_{D^*}m_c}\left(m_{c\bar{c}}^2-m_D^2-m_{D^*}^2\right)\exp[-\frac{m_D^2+m_{D^*}^2}{2M^2}], \tag{4.62}$$

$$\Rightarrow G_1-G_2 = 2\varphi\left(\frac{1}{2}\right)\frac{f_{c\bar{c}}^\perp}{f_D f_{D^*}}\frac{m_{c\bar{c}}^2 m_{D^*}m_c}{m_D^2\left(m_{c\bar{c}}^2-m_D^2-m_{D^*}^2\right)}\exp\left[\frac{2\left(m_D^2+m_{D^*}^2\right)-m_{c\bar{c}}^2}{4M^2}\right]. \tag{4.63}$$

The q terms were:

$$II:\ F_q^{QCD} = f_{c\bar{c}}^\perp\int_0^1 du\left\{\frac{\Phi_\perp(u)}{up'^2+\bar{u}q^2+u\bar{u}m_{c\bar{c}}^2}-\frac{2m_{c\bar{c}}^2\varphi(u)\bar{u}}{\left(up'^2+\bar{u}q^2+u\bar{u}m_{c\bar{c}}^2\right)^2}\right\}$$

$$F_q^{pheno.} = -\frac{G_1 m_D^2 m_{D^*} f_D f_{D^*}}{m_c} \frac{1}{(m_D^2+q^2)(m_{D^*}^2+p'^2)}.$$

Following the above steps, one obtains:

$$\rho_{II}(s) = -\frac{m_{c\bar{c}}^2}{s}\varphi(\frac{1}{2})\delta\left(\frac{m_{c\bar{c}}^2}{4s}-1\right). \tag{4.64}$$

$$\Rightarrow F_q^{QCD} = -f_{c\bar{c}}^{\perp}\varphi\left(\frac{1}{2}\right) m_{c\bar{c}}^2 \exp\left[-\frac{m_D^2+m_{D^*}^2}{2M^2}\right]$$

$$= F_q^{pheno.} = -\frac{G_1 m_D^2 m_{D^*} f_D f_{D^*}}{m_c} \exp\left[-\frac{m_{c\bar{c}}^2}{4M^2}\right], \tag{4.65}$$

$$\Rightarrow G_1 = \varphi\left(\frac{1}{2}\right)\frac{f_{c\bar{c}}^{\perp}}{f_D f_{D^*}}\frac{m_{c\bar{c}}^2 m_c}{m_D^2 m_{D^*}} \exp\left[\frac{2\left(m_D^2+m_{D^*}^2\right)-m_{c\bar{c}}^2}{4M^2}\right]. \tag{4.66}$$

The ϵ terms were:

$$F_\epsilon^{QCD} = f_{c\bar{c}}^{\perp}\int_0^1 du \left\{\frac{m_{c\bar{c}}^2\varphi(u) - \frac{1}{2}\Phi_\perp(u)\left((1-2u)m_{c\bar{c}}^2 + p'^2 - q^2\right)}{u\bar{u}m_{c\bar{c}}^2 + up'^2 + \bar{u}q^2}\right\}$$

$$F_\epsilon^{pheno.} = \frac{G_2 m_D^2 m_{D^*} f_D f_{D^*}}{m_c}\frac{\left(m_{c\bar{c}}^2 + p'^2 + q^2\right)/2}{(m_D^2+q^2)(m_{D^*}^2+p'^2)}.$$

Consider the following term on the left hand side:

$$B_{p'^2,M_2^2}B_{q^2,M_1^2}\left\{\int_0^1 du \frac{\Phi_\perp(u)\,(1-2u)\,m_{c\bar{c}}^2}{up'^2+\bar{u}q^2+u\bar{u}m_{c\bar{c}}^2}\right\}$$

$$\rightarrow \left[\Phi_\perp(u)\,(1-2u)\,m_{c\bar{c}}^2\theta\left(1-\frac{m_{c\bar{c}}^2}{4s}\right)\right]_{u=1/2} = 0. \tag{4.67}$$

So this term does not contribute to the result.

Consider now the other terms on the left hand side:

$$B_{all\,op.}\left\{\int_0^1 du \frac{m_{c\bar{c}}^2\varphi(u)}{up'^2+\bar{u}q^2+u\bar{u}m_{c\bar{c}}^2}\right\} = m_{c\bar{c}}^2\varphi(\frac{1}{2})\theta\left(1-\frac{m_{c\bar{c}}^2}{4s}\right). \tag{4.68}$$

$$\lim_{M_1^2,M_2^2\to 2M^2} B_{p'^2,M_2^2}B_{q^2,M_1^2}\left\{\frac{1}{2}\int_0^1 du \frac{\Phi_\perp(u)\,(p'^2-q^2)}{up'^2+\bar{u}q^2+u\bar{u}m_{c\bar{c}}^2}\right\}$$

$$= \lim_{M_1^2,M_2^2\to 2M^2} B_{p'^2,M_2^2}$$

$$\left\{\frac{1}{2}\int_0^1 du\Phi_\perp(u)\left[\frac{p'^2/\bar{u}}{\frac{u}{\bar{u}}p'^2+q^2+um_{c\bar{c}}^2} - \frac{1}{\bar{u}}\left(1-\frac{\frac{u}{\bar{u}}p'^2+um_{c\bar{c}}^2}{\frac{u}{\bar{u}}p'^2+q^2+um_{c\bar{c}}^2}\right)\right]\right\}$$

$$= \lim_{M_1^2, M_2^2 \to 2M^2} B_{p'^2, M_2^2}$$

$$\left\{ \frac{1}{2} \int_0^1 du \Phi_\perp(u) \exp\left[-\frac{u m_{c\bar{c}}^2}{M_1^2} \right] \left(\frac{u}{\bar{u}} m_{c\bar{c}}^2 - M_1^2 \bar{u} \frac{\partial}{\partial u} \right) \left[M_2^2 \delta \left(\frac{u M_2^2}{\bar{u} M_1^2} - 1 \right) \right] \right\};$$

since $\Phi_\perp(0) = \Phi_\perp(1) = 0$. This leads to the following:

$$= \lim_{M_1^2, M_2^2 \to 2M^2} B_{p'^2, M_2^2}$$

$$\left\{ \frac{1}{2} \int_0^1 du M_2^2 \delta \left(\frac{u M_2^2}{\bar{u} M_1^2} - 1 \right) \left(\frac{u}{\bar{u}} m_{c\bar{c}}^2 - M_1^2 \frac{\partial}{\partial u} \right) \left[\bar{u} \Phi_\perp(u) \exp\left[-\frac{u m_{c\bar{c}}^2}{M_1^2} \right] \right] \right\}$$

$$= -M^4 \Phi'_\perp(\frac{1}{2}) \exp\left[-\frac{m_{c\bar{c}}^2}{4M^2} \right]. \tag{4.69}$$

Then:

$$\tilde{\rho}_{III}(s) = B_{M^{-2}, s^{-1}} \left\{ -M^4 \Phi'_\perp(\frac{1}{2}) \exp\left[-\frac{m_{c\bar{c}}^2}{4M^2} \right] \right\}$$

$$= B_{M^{-2}, s^{-1}} \left\{ -M^4 \Phi'_\perp(\frac{1}{2}) \int_0^\infty dt \times t \times \exp\left[-\frac{m_{c\bar{c}}^2 + 4t}{4M^2} \right] \right\}$$

$$= -M^4 \Phi'_\perp(\frac{1}{2}) \int_0^\infty dt \times t \times \frac{1}{s} \delta \left[\frac{m_{c\bar{c}}^2 + 4t}{4s} - 1 \right] = -\Phi'_\perp(\frac{1}{2}) \left(s - \frac{m_{c\bar{c}}^2}{4} \right) \theta \left(s - \frac{m_{c\bar{c}}^2}{4} \right). \tag{4.70}$$

$$\Rightarrow F_\epsilon^{pheno.} = \frac{G_2 f_D f_{D^*} m_D^2 m_{D^*}}{2m_c} \exp\left[-\frac{m_D^2 + m_{D^*}^2}{2M^2} \right]$$

$$= F_\epsilon^{QCD} = 2 f_{c\bar{c}}^\perp \int_{m_{c\bar{c}}^2/4}^{s_0} ds \exp\left[-\frac{s}{M^2} \right] \left\{ m_{c\bar{c}}^2 \varphi(\frac{1}{2}) - \Phi'_\perp(\frac{1}{2}) \left(s - \frac{m_{c\bar{c}}^2}{4} \right) \right\}. \tag{4.71}$$

Finally, the third relation gives the following:

$$III:\ G_2 = \frac{f_{c\bar{c}}^\perp}{f_D f_{D^*}} \frac{4 m_c \exp\left[\frac{2\left(m_D^2 + m_{D^*}^2 \right)}{4M^2} \right]}{m_D^2 m_{D^*} \left(m_{c\bar{c}}^2 - m_D^2 - m_{D^*}^2 \right)}$$

$$\times \int_{m_{c\bar{c}}^2/4}^{s_0} ds \exp\left[-\frac{s}{M^2} \right] \left\{ m_{c\bar{c}}^2 \varphi(\frac{1}{2}) - \Phi'_\perp(\frac{1}{2}) \left(s - \frac{m_{c\bar{c}}^2}{4} \right) \right\}. \tag{4.72}$$

As a summary, all three relations for the couplings have been collected together at this point:

$$I:\ G_1-G_2=2\varphi\left(\tfrac{1}{2}\right)\frac{f^{\perp}_{c\bar{c}}}{f_D f_{D^*}}\frac{m^2_{c\bar{c}}m_{D^*}m_c}{m^2_D\left(m^2_{c\bar{c}}-m^2_D-m^2_{D^*}\right)}\exp\left[\frac{2\left(m^2_D+m^2_{D^*}\right)-m^2_{c\bar{c}}}{4M^2}\right];$$
$$II:\ G_1=\varphi\left(\tfrac{1}{2}\right)\frac{f^{\perp}_{c\bar{c}}}{f_D f_{D^*}}\frac{m^2_{c\bar{c}}m_c}{m^2_D m_{D^*}}\exp\left[\frac{2\left(m^2_D+m^2_{D^*}\right)-m^2_{c\bar{c}}}{4M^2}\right];$$
$$III:\ G_2=\frac{f^{\perp}_{c\bar{c}}}{f_D f_{D^*}}\frac{4m_c\exp\left[\frac{2\left(m^2_D+m^2_{D^*}\right)}{4M^2}\right]}{m^2_D m_{D^*}\left(m^2_{c\bar{c}}-m^2_D-m^2_{D^*}\right)}$$
$$\times\int_{m^2_{c\bar{c}}/4}^{s_0}ds\exp\left[-\tfrac{s}{M^2}\right]\left\{m^2_{c\bar{c}}\varphi(\tfrac{1}{2})-\Phi'_{\perp}(\tfrac{1}{2})\left(s-\tfrac{m^2_{c\bar{c}}}{4}\right)\right\}. \quad (4.73)$$

Before proceeding further, one can concentrate on the relations giving G_1, G_2 and G_1-G_2 one more time. The relations giving G_1 and G_1-G_2 are independent of s_0, and one observes that

$$\frac{G_1-G_2}{G_1}=\frac{2m^2_V}{m^2_{q\bar{q}}-m^2_P-m^2_V}, \quad (4.74)$$

that is, their ratio is also independent from the LCDA at twist-2 accuracy. It is an important question whether this result is valid only for this level of accuracy or is a general result, which cannot be addressed in this work. However, this relation is another sum rule, and being independent of the LCDAs (at least at this level of accuracy), it can be useful in interpreting the results.

At this point, one has the desired sum rules for 1^{++} heavy quarkonia. The parameters appearing in the above expressions can be categorized as follows: masses of c/b-quarks and $\bar{D}/\bar{B}$ and D^*/B^* mesons, leptonic decay constants of $\bar{D}/\bar{B}$ and D^*/B^* mesons are inputs from experiment, whose values can be found in [7]. $c\bar{c}/b\bar{b}$ masses have been calculated in quark model (in [41]), and are presented in Appendix 2.4. Leptonic decay constants of the quarkonium states, whose values can be found in Tables 3.1, 3.2 and 3.3, have been calculated using the relations derived in [30] and the quark model wavefunctions calculated in the framework of [41, 47].

The above mentioned experimental values are as follows. Leptonic decay constants of $\bar{D}/\bar{B}$ and D^*/B^* mesons [17]: $f_D = 180\,MeV$, $f_{D^*} = 270\,MeV$, $f_B = 170\,MeV$, $f_{B^*} = 195\,MeV$. Masses of c/b quarks, $\bar{D}/\bar{B}$ and D^*/B^*

mesons [7]: $m_c = 1.28\,GeV$, $m_b = 4.66\,GeV$, $m_D = 1.87\,GeV$, $m_B = 5.28\,GeV$, $m_{D^*} = 2.01\,GeV$, $m_{B^*} = 5.33\,GeV$.

4.4 Numerical Results

Up to this section, necessary ingredients for calculating the couplings of $L = 1$ axial-vector quarkonia to pseudo-scalar and vector mesons (PVA couplings) have been demonstrated and calculated. In addition to these, LCDAs of $L = 1$ charmonia and bottomonia have been calculated. In this chapter, numerical results for PVA couplings will be presented.

Before proceeding further for the couplings, masses and decay constants of the vector and pseudo-scalar mesons should be given.

Table4.1: Masses and decay constants of D, D^*, B, B^* mesons used in calculations.

Meson	Mass (GeV) [7]	Decay Constant (GeV) [17]
D	1.87	0.18
D^*	2.01	0.27
B	5.28	0.17
B^*	5.33	0.195

Results for the couplings can now be presented. The factors involving certain values of the derivative and first integral of the LCDAs have been calculated using the fitted functions presented in the previous section. The fit parameters for $n = 1, 2$ bottomonia are not affected by the cut-off. As a consequence, numerical values of the corresponding couplings are also not affected by the cut-off.

In calculating the couplings, the threshold s_0 has been chosen as $\frac{(m_P+\alpha)^2+(m_V+\alpha)^2}{2}$, where $0 \leq \alpha \leq 0.5\,GeV$.

As observed in equations 4.62, 4.65 and 4.71, the spectral densities leading to the first two equations involve delta functions, so the resulting $F_{p'}^{QCD}$ and F_q^{QCD} functions are independent of s_0, while F_ϵ^{QCD} depends on s_0 and so depends on

α. For this reason, in order to determine the value of α, F_ϵ^{QCD} has been used. The dependence on α is exponential, and G_2 is dependent on another parameter, namely the Borel parameter M^2 (which is also true for G_1). However, one can observe that $-\frac{\partial \ln[F_\epsilon^{pheno.}]}{\partial(1/M^2)} = \frac{m_P^2+m_V^2}{2}$, that is, independent of M^2 and α. So, defining the following function:

$$g^{QCD}(M^2,\alpha) \equiv -\frac{\partial \ln[F_\epsilon^{QCD}]}{\partial(1/M^2)}, \tag{4.75}$$

one can consider plotting $g^{QCD}(M^2,\alpha)$ and $\frac{m_P^2+m_V^2}{2}$ versus α and try to determine a sensible interval for α which lies within $0 \leq \alpha \leq 0.5\,GeV$.

One also expects the couplings to change slowly (or not to change at all) with M^2, since this is not a parameter dictated by the physics involved, but is a calculational tool to suppress the contributions of the hadronic continuum. So, the dependence of the couplings on the Borel parameter are also examined.

Choices for the values of α and M^2 and the resulting values for the couplings are discussed below.

As can be observed in the plots, the couplings do not change significantly for values of M^2 larger than the vector meson mass squared. For this reason, both for charmonia and for bottomonia, the Borel parameters are chosen to be around the vector meson mass squared; $5\,GeV^2$ for charmonia and $25\,GeV^2$ for bottomonia. This behaviour is not affected by the dependence on α. The cut-off dependence is obviously insignificant for bottomonia LCDAs, and as the numbers and plots are calculated one observes that the cut-off dependence for charmonia affects the numbers but not M^2 or α dependece behaviour.

α dependence of G_2 appears to be case dependent. For $n = 1$ charmonium, one can find a value of α such that independently calculated values for G_1, G_2 and $G_1 - G_2$ agree, and this value of α lies within the previously prescribed interval. The values of α corresponding to the two different cut-off cases have been used in the relevant plots and in calculating the couplings, which have been presented in Table 4.2. For $n = 2$ charmonium, such a value of α ($\sim 0.338\,GeV$) causes $s_0 = 3.89694\,GeV$, that is, slightly below $m_{c\bar{c}}^2/4$. The conclusion is that $G_2 = 0$ for $n = 2$ charmonium. This conclusion is also supported by the fact that for

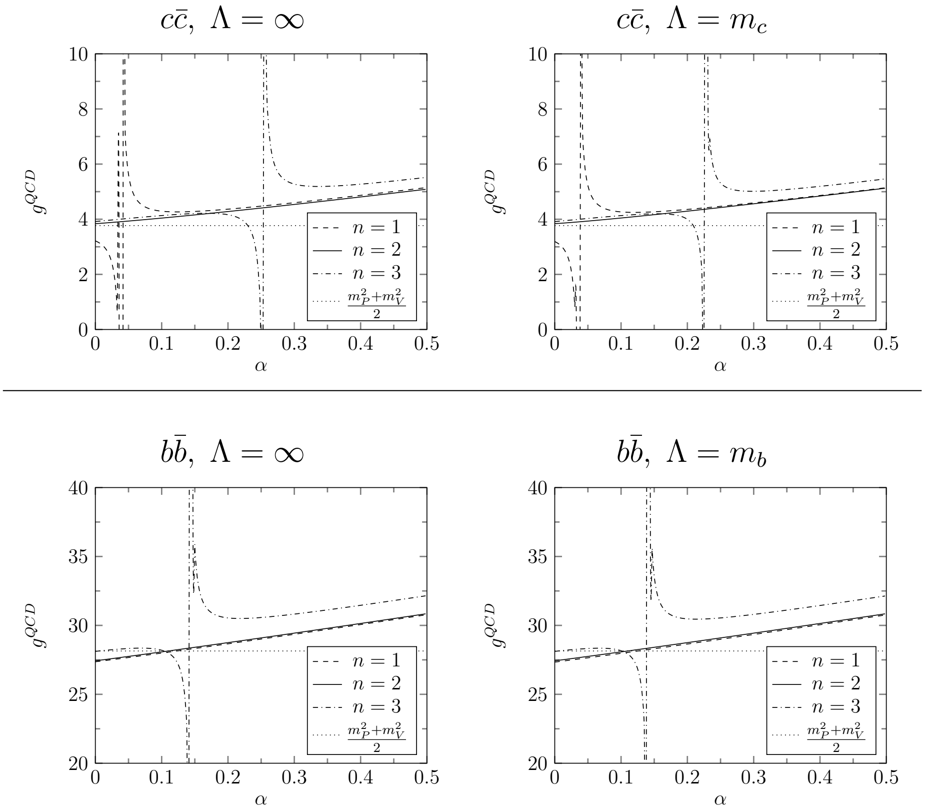

Figure 4.1: Plots of g^{QCD} versus α.

$m_{c\bar{c}} \sim 3.95\,GeV$, 4.74 implies $G_2 \sim 0$, and in this work $m_{c\bar{c}} = 3.97\,GeV$ ($c\bar{c}$ mass around this value is also suggested in other references in the literature, such as [10, 75]). $n = 3$ charmonium case is the same with $n = 1$ case, except that the sign and order of magnitude of G_2 is different. For $n = 1$ bottomonium, the value of α making G_1, G_2 and $G_1 - G_2$ agreed is negative. For $n = 2$ bottomonium, even the signs of G_1, G_2 and $G_1 - G_2$ cannot be consistent with the given parameters and the interval for α. This leads to the same conclusion with the $n = 2$ charmonium case. For $n = 3$ bottomonium, the case is the same with $n = 1, 3$ charmonium cases, except that the signs of G_2 for $n = 1$ and $n = 3$ bottomonia are the same.

In order to estimate the extent of errors in calculating the PVA couplings, it is possible to consider how much the fit functions deviate from the original ones. As can be observed from the plots, the deviations $|\frac{\phi - \phi^{fit}}{\min \phi, \phi^{fit}}|$ are not large in the region $|\xi| < 0.5$, while on the "tails", that is, in the regions beyond $|\xi| < 0.5$

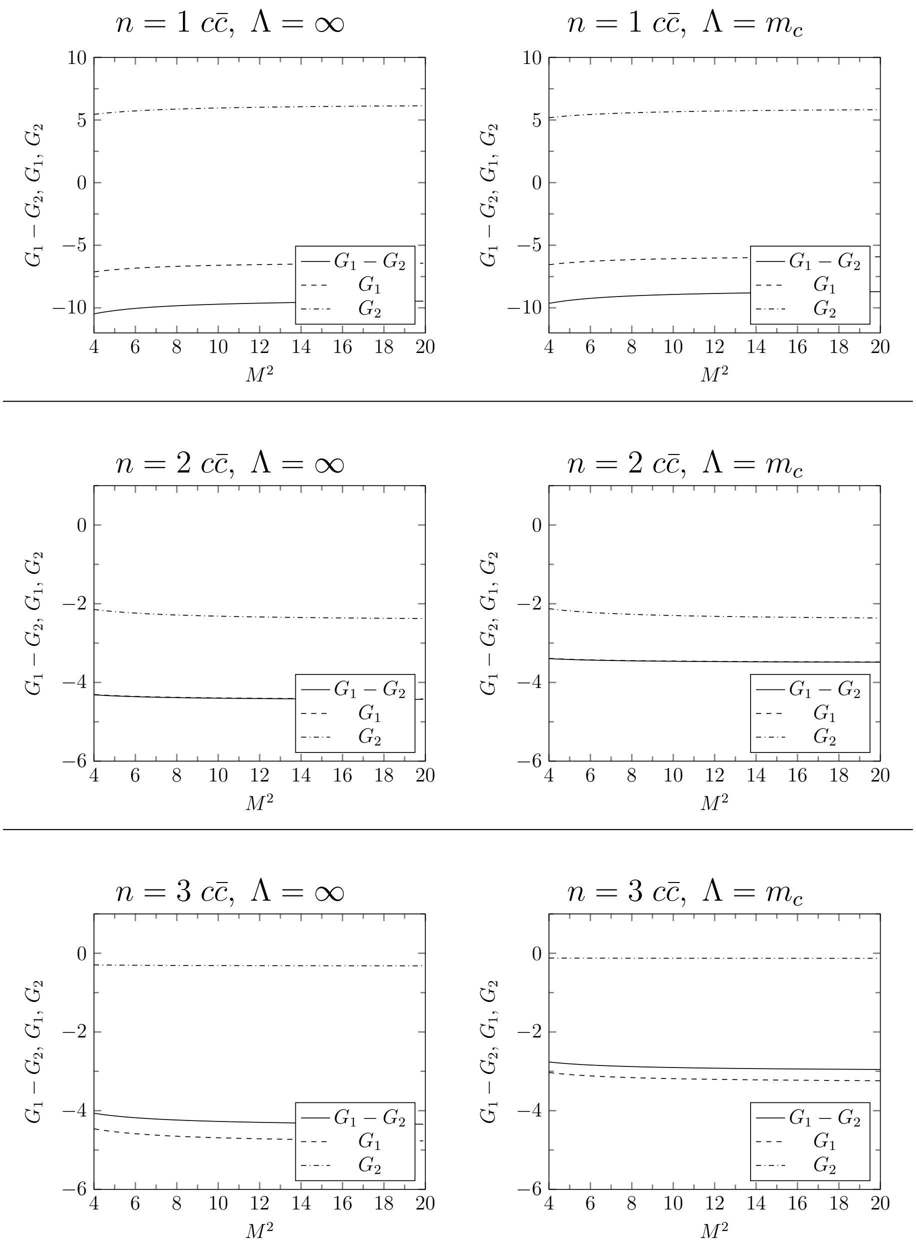

Figure 4.2: Plots for $G_1 - G_2$, G_1 and G_2 versus M^2 ($\alpha = 0.2\,GeV$).

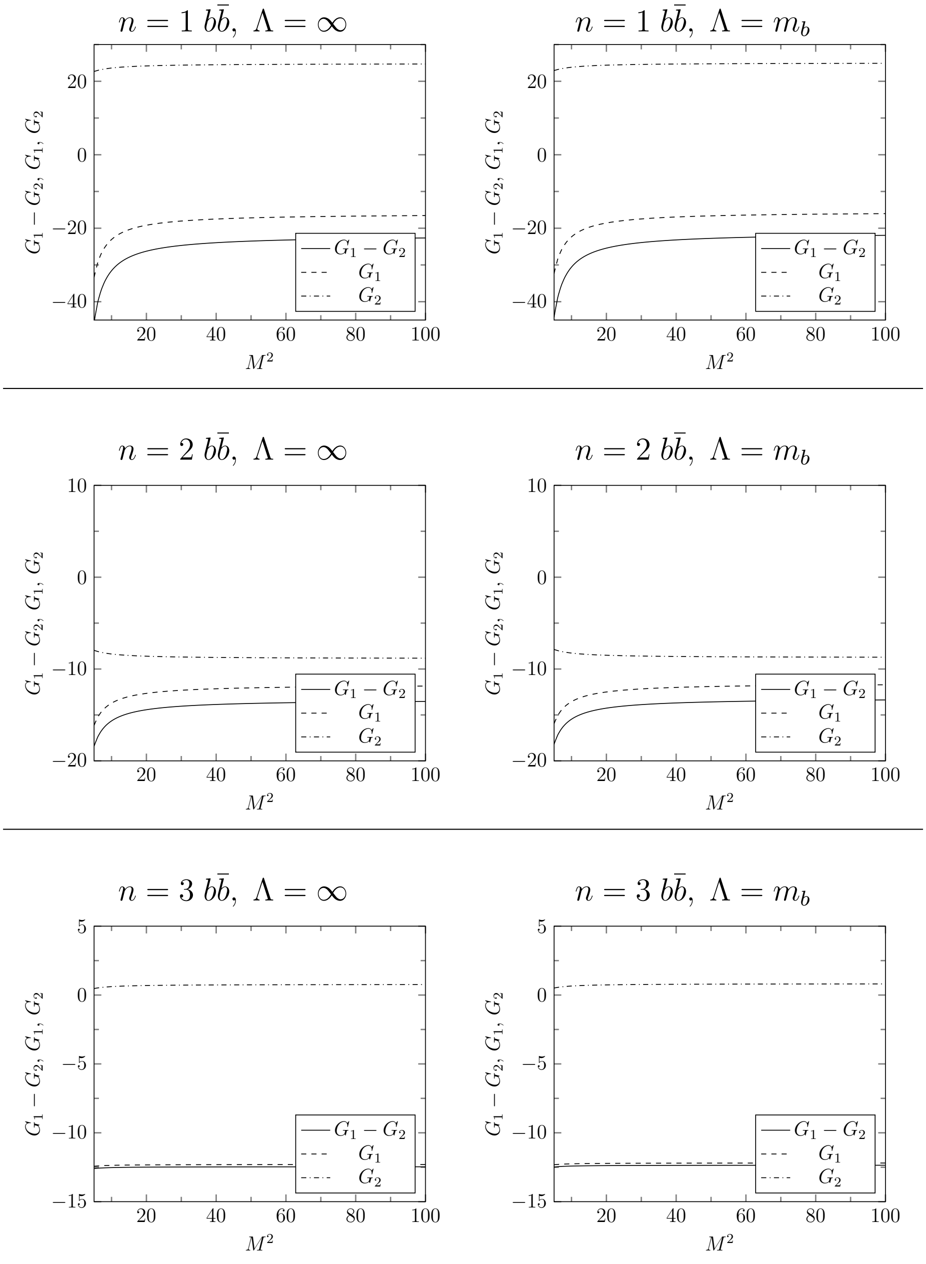

Figure 4.3: Plots for $G_1 - G_2$, G_1 and G_2 versus M^2 ($\alpha = 0.2\,GeV$).

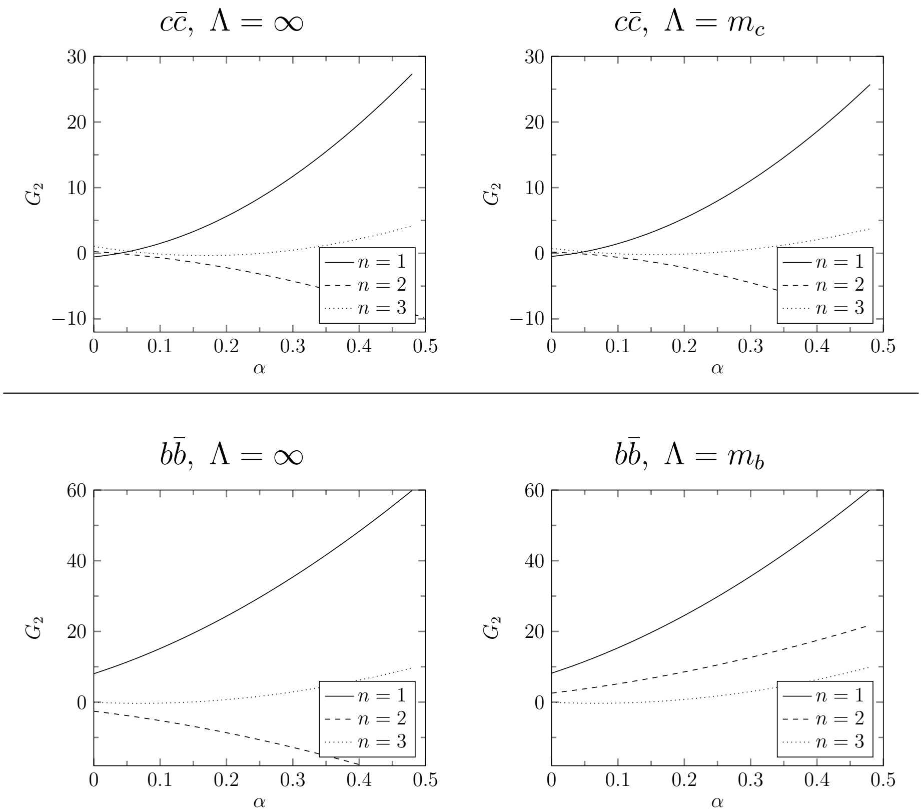

Figure 4.4: Plots of G_2 versus α, where $M^2 = 5\,GeV^2$ for charmonia and $M^2 = 25\,GeV^2$ for bottomonia.

Table4.2: PVA couplings. The values of M^2 and s_0 are $5\,GeV^2$ and $4.59\,GeV^2$ for all charmonia, $25\,GeV^2$ and $30.3\,GeV^2$ for all bottomonia.

$c\bar{c}\,(\Lambda = \infty)$			
	$G_1 - G_2\,(GeV^{-1})$	$G_1\,(GeV^{-1})$	$G_2\,(GeV^{-1})$
$n = 1$	-10.2	-6.95	5.62
$n = 2$	-4.34	-4.34	-2.20
$n = 3$	-4.13	-4.54	-0.30
$c\bar{c}(\Lambda = m_c)$			
	$G_1 - G_2\,(GeV^{-1})$	$G_1\,(GeV^{-1})$	$G_2\,(GeV^{-1})$
$n = 1$	-9.40	-6.39	5.34
$n = 2$	-3.42	-3.41	-2.18
$n = 3$	-2.81	-3.08	-0.12
$b\bar{b}\,(\Lambda = \infty)$			
	$G_1 - G_2\,(GeV^{-1})$	$G_1\,(GeV^{-1})$	$G_2\,(GeV^{-1})$
$n = 1$	-25.3	-18.5	24.3
$n = 2$	-14.2	-12.5	-8.66
$n = 3$	-12.5	-12.3	0.71
$b\bar{b}\,(\Lambda = m_b)$			
	$G_1 - G_2\,(GeV^{-1})$	$G_1\,(GeV^{-1})$	$G_2\,(GeV^{-1})$
$n = 1$	-25.3	-18.5	24.5
$n = 2$	-14.2	-12.5	-8.55
$n = 3$	-12.5	-12.3	0.75

can be large as the functions approach to zero exponentially in these regions. However, the contributions from these regions to the sum rules obtained in this work are not significant, owing to the fact that the first integrals and derivatives at $\xi = 0$ of the LCDAs will enter into the sum rules, where the integrals are less sensitive to the exact values of the original function, while the behaviour of the original and fit functions at $\xi = 0$ match rather well so that $|\frac{\phi-\phi^{fit}}{\min \phi,\phi^{fit}}|$ cannot be large. Other contributions to the errors are from the estimated errors in the other parameters, such as the vector and pseudoscalar decay constants and masses. The dependence of the couplings on the Borel parameter is another source of error. It can be observed from the plots that above $M^2 = 5\, GeV^2$ for charmonia and $M^2 = 25\, GeV^2$ for bottomonia, the couplings converge rapidly, and it appears that this source cannot have a significant contribution to the error in the couplings. The error in quarkonium masses and decay constants cannot be estimated here, though it can be asserted that their contribution to the overall error can be expected to have the same extent with that of the LCDAs.

According to [17], the decay constants of D, D^* and B, B^* mesons with corresponding errors are

$$\begin{aligned} f_D &= 180 \pm 30\, MeV,\ f_B = 170 \pm 30\, MeV, \\ f_{D^*} &= 270 \pm 35\, MeV,\ f_B = 195 \pm 35\, MeV. \end{aligned} \tag{4.76}$$

So, the fractional error in these numbers, $\Delta f/f$, is largest for the B meson, being around 18 percent.

According to PDG listings [7], the errors in the measured masses of D, D^* and B, B^* mesons is rather small, e.g. below 0.05 percent, so can safely be neglected for an estimation.

Since the decay constants of D, D^* and B, B^* are inversely proportional to the couplings, the fractional errors are to be summed to estimate the error in the couplings:

$$\frac{\Delta G}{G} = \frac{\Delta f_P}{f_P} + \frac{\Delta f_V}{f_V}. \tag{4.77}$$

The conclusion is that the error in the couplings due to the D, D^* and B, B^* mesons is around 35 percent, which is also the extent of the overall error.

4.5 Couplings and charmonium content of $X(3872)$

Calculated values for the couplings also give insights on the quarkonium content of $X_{c/b}$ mesons. To describe this insight, one can consider the expression for the coupling matrix element (Eq. 4.12) once again:

$$\left\langle \bar{D}^0(k) D^{*0}(k', \eta^{(\lambda)}) \left| J_{int} \right| c\bar{c}(k+k', \epsilon^{(\sigma)}) \right\rangle$$

$$\equiv \left[G_1(\epsilon^{(\sigma)} \cdot k)(\eta^{*(\lambda)} \cdot k) + G_2(k' \cdot k)(\epsilon^{(\sigma)} \cdot \eta^{*(\lambda)}) \right] \equiv g(k, k', \epsilon, \eta). \qquad (4.78)$$

The mass difference between lowest lying X states and sum of the vector and pseudo-scalar masses for the excited states is not large; so one can consider the molecule states in the nonrelativistic approximation and the rest frame of the $c\bar{c}$ system. In this system, $\epsilon \sim (0, \vec{\epsilon})$, while $k \sim (m_D, \vec{0})$ and similar for $k^{'}$. So, the first term is negligible and the second term is rather of interest. When charmonium states are considered, one observes that G_2 is negligible for $n = 2$. This means, the coupling of $n = 2$ charmonium to DD^* molecule is negligible. It has been asserted in [14] that one expects roughly 5 percent charmonium contribution in $X(3872)$, and this contribution is most probably from $n = 2$. However, the conclusion here disfavors this argument; the charmonium conitribution to $X(3872)$ will most probably be from $n = 1$ state.

To estimate this contribution, one can resort to [14]. To make use of the results obtained in [14], it is necessary to convert the normalization of states in this work to those of [14]. In order to make the normalizations consistent, one divides the matrix element by $\sqrt{m_{q\bar{q}} m_P m_V}$:

$$\left\langle \bar{P}(k) V(k', \eta^{(\lambda)}) \left| J_{int} \right| q\bar{q}(k+k', \epsilon^{(\sigma)}) \right\rangle$$

$$\rightarrow \frac{\left\langle \bar{P}(k) V(k', \eta^{(\lambda)}) \left| J_{int} \right| q\bar{q}(k+k', \epsilon^{(\sigma)}) \right\rangle}{\sqrt{m_{q\bar{q}} m_P m_V}}$$

$$= \frac{G_2 \Delta m^2}{2\sqrt{m_{q\bar{q}} m_P m_V}} \equiv d, \qquad (4.79)$$

where $\Delta m^2 \equiv m_{qq}^2 - m_P^2 - m_V^2$. The variable d is the coupling strength discussed in [14]. Conversion from GeV to fm is $1\, fm = 0.197\, GeV^{-1}$.

Due to the sensitivity of the sum rule involving G_2 on the parameters (and especially s_0), the other sum rules involving G_1 and $G1-G_2$ are used to calculate G_2 as $G_1 - (G_1 - G_2)$.

Table4.3: PVA couplings when states are normalized to 1. Value of M^2 is $5\,GeV^2$ for charmonia and $25\,GeV^2$ for bottomonia.

	$c\bar{c}\,(\Lambda=\infty)$		$c\bar{c}\,(\Lambda=m_c)$	
	$d\,(GeV^{-1/2})$	$d\,(fm^{1/2})$	$d\,(GeV^{-1/2})$	$d\,(fm^{1/2})$
$n=1$	2.42	1.08	2.25	1.00
$n=2$	0	0	0.01	0.01
$n=3$	-0.44	-0.20	-0.36	-0.16
	$b\bar{b}\,(\Lambda=\infty)$		$b\bar{b}\,(\Lambda=m_b)$	
	$d\,(GeV^{-1/2})$	$d\,(fm^{1/2})$	$d\,(GeV^{-1/2})$	$d\,(fm^{1/2})$
$n=1$	8.46	3.76	8.46	3.76
$n=2$	2.49	1.10	2.49	1.10
$n=3$	0.33	0.14	0.33	0.14

According to [14] and [76], the interval $0.1\,fm^{1/2} \leq |d| \leq 0.25\,fm^{1/2}$ correspond to 10-30 percent of charmonium contribution to $X(3872)$. This conclusion has been based on the following assumptions: (a) $c\bar{c}$ state has mass $3.906\,GeV$, (b) the coupling is of the same order for all radial excitations and (c) the dominant contribution comes from $n=2$. The charmonium contribution discussed in [14] is dependent on the coupling d. In this work, it appears that the coupling of to $n=2,3$ charmonium to the molecule state is rather small, while $n=1$ appears to have a larger coupling. With the parameters used in this work and according to the calculations in the framework of [14], $n=1$ charmonium contribution to $X(3872)$ is about 15 percent, $n=2$ contribution is negligible and $n=3$ contribution is around 1 percent [76].

CHAPTER 5

SUMMARY AND CONCLUSIONS

In this work, leptonic decay constants and LCDAs of $L = 1$ axial-vector heavy quarkonia and their couplings to vector and pseudo-scalar open flavor mesons have been calculated.

The decay constants are roughly of the same order for $n = 1, 2, 3$ for each meson state. The relations 3.27 and 3.28 are partially satisfied. The deviations from these relations are interpreted as effects of spin-orbit interations. However, for each n, the decay constants related in 3.27 and 3.28 are of the same order, and hence the order of magnitude of decay constants can still be deduced correctly without taking spin-orbit effects into account.

Functions fitted to the LCDAs and the observed patterns provide profiles for the LCDAs, which can be used independently from the details of the calculation performed in this work. Thus, the fit functions, fitted LCDA parameters and/or the profiles can be used by experimentalists. The extrema are closer to $\xi = 0$ for bottomonia as compared to charmonia for all states. All functions approach to zero as $e^{-\frac{1}{1-\xi^2}}$. Ground state profiles are the same with the profiles of the functions calculated in [30].

Couplings of $L = 1$ charmonia and bottomonia to D, D^* and B, B^* mesons have been calculated. The couplings converge to their limiting values (i.e. $M^2 \to \infty$) rapidly for values of M^2 larger than the vector meson mass squared and this behaviour is not affected by α nor by cut-off values used in calculating the LCDAs and decay constants. So, when calculating the values presented in Table

4.2, M^2 values are chosen around the vector meson mass squared; $5\,GeV^2$ for charmonia and $25\,GeV^2$ for bottomonia.

The charmonium contributions to $X(3872)$ have been deduced, using the results of this work and Ref. [14]. These contributions (about 15 percent from $n = 1$, negligible from $n = 2$ and about 1 percent from $n = 3$ [76]) are to be tested by experimental data, which is yet not available.

APPENDIX A

LCDA FIT PARAMETERS

Fit parameters and plots of the LCDAs calculated in this work have been presented in the tables and figures below.

TableA.1: $^3P_0$ fit parameters

$c\bar{c}$, $\Lambda = \infty$						
	a	ξ_0	σ^2	β	b	c
$n = 1$	5.29486	–	–	0.867659	1.36101	0.352628
$n = 2$	6.98919	0.257499	0.182999	1.50283	−0.560786	0.149576
$n = 3$	1.98475	0.51822	0.0928117	0.847995	0.482135	0.083937
$c\bar{c}$, $\Lambda = m_c$						
	a	ξ_0	σ^2	β	b	c
$n = 1$	5.87696	–	–	1.00251	1.2953	0.346534
$n = 2$	7.26777	0.255466	0.185855	1.52799	−0.567534	0.152166
$n = 3$	2.22227	0.538532	0.063798	0.736003	0.262874	0.0490782
$b\bar{b}$, $\Lambda = \infty$						
	a	ξ_0	σ^2	β	b	c
$n = 1$	14.9131	–	–	1.29523	2.47014	0.070175
$n = 2$	4.08343	0	0.018278	1.01745	−0.155892	0.0244758
$n = 3$	6.5	0.288561	0.0244281	1.70944	0.260454	0.0285309
$b\bar{b}$, $\Lambda = m_b$						
	a	ξ_0	σ^2	β	b	c
$n = 1$	7.88045	–	–	1.03344	5.01843	0.0756304
$n = 2$	4.15287	0	0.0173548	1.01512	−0.173547	0.0256279
$n = 3$	3.33759	0.280277	0.0230325	1.10439	0.283173	0.0283235

TableA.2: ${}^1P_{1\perp}$ fit parameters

$c\bar{c}$, $\Lambda = \infty$						
	a	ξ_0	σ^2	β	b	c
$n = 1$	1.78365	–	–	1.4891	0.636329	0.431836
$n = 2$	2.43681	0.284756	0.0966684	0.921282	0	–
$n = 3$	1.53215	0.487944	0.122605	0.734715	0.562724	0.122482
$c\bar{c}$, $\Lambda = m_c$						
	a	ξ_0	σ^2	β	b	c
$n = 1$	2.26917	–	–	2.14573	0.541302	0.383543
$n = 2$	1.40861	0.370795	0.0301022	0.303023	0	–
$n = 3$	2.15625	0.530031	0.0483686	0.23274	0.269402	1.01888
$b\bar{b}$, $\Lambda = \infty$						
	a	ξ_0	σ^2	β	b	c
$n = 1$	11.7196	–	–	1.74248	0.126342	0.0638176
$n = 2$	1.59483	0.12902	0.00748694	0.1	0	–
$n = 3$	17.2954	0.254288	0.0186876	2.57763	0.18766	0.0149827
$b\bar{b}$, $\Lambda = m_b$						
	a	ξ_0	σ^2	β	b	c
$n = 1$	13.3807	–	–	1.94856	0.120252	0.0635912
$n = 2$	1.61774	0.135557	0.00687409	0.1	0	–
$n = 3$	8.28066	0.24319	0.0150691	1.86152	0.242844	0.0159363

TableA.3: ${}^1P_{1\parallel}$ fit parameters

$c\bar{c}$, $\Lambda = \infty$						
	a	ξ_0	σ^2	β	b	c
$n = 1$	7.28898	–	–	4.6828	1.79244	0.273028
$n = 2$	8.65428	0.242683	0.167489	1.86131	−0.326097	0.128336
$n = 3$	2.8194	0.514286	0.0924345	1.13424	0.456194	0.0903501
$c\bar{c}$, $\Lambda = m_c$						
	a	ξ_0	σ^2	β	b	c
$n = 1$	53.6273	–	–	2.39208	0.155202	0.273479
$n = 2$	10.0745	0.269738	0.160284	1.92211	−0.426175	0.13256
$n = 3$	2.74678	0.516953	0.0704437	0.980324	0.294645	0.0616262
$b\bar{b}$, $\Lambda = \infty$						
	a	ξ_0	σ^2	β	b	c
$n = 1$	10.5205	–	–	1.50861	5.65674	0.0638543
$n = 2$	24.2512	0	0.0147382	2.63148	−0.199554	0.0189217
$n = 3$	3.28997	0.232939	0.0123631	0.987534	0.300752	0.0169844
$b\bar{b}$, $\Lambda = m_b$						
	a	ξ_0	σ^2	β	b	c
$n = 1$	9.97137	–	–	1.52337	6.0868	0.0639454
$n = 2$	8.82176	1.25×10^{-6}	0.0113222	1.63918	−0.18903	0.0186983
$n = 3$	18.256	0.255062	0.0130517	2.61225	0.240445	0.0151002

TableA.4: ${}^3P_{1\perp}$ fit parameters

$c\bar{c}$, $\Lambda = \infty$						
	a	ξ_0	σ^2	β	b	c
$n = 1$	47.933	–	–	2.33766	0.183997	0.263943
$n = 2$	9.78918	0.142262	0.180577	1.92684	−0.391433	0.12737
$n = 3$	2.62768	0.474425	0.07959	1.05711	0.437048	0.0611108
$c\bar{c}$, $\Lambda = m_c$						
	a	ξ_0	σ^2	β	b	c
$n = 1$	58.1733	–	–	2.42818	0.148617	0.258012
$n = 2$	12.3322	0.156543	0.184174	2.02338	−0.500837	0.135897
$n = 3$	2.71267	0.487316	0.062559	0.966985	0.318898	0.044592
$b\bar{b}$, $\Lambda = \infty$						
	a	ξ_0	σ^2	β	b	c
$n = 1$	460.545	–	–	3.96925	0.127859	0.0535103
$n = 2$	24.2512	0	0.0147382	2.63148	−0.199554	0.0189217
$n = 3$	1.33436	0.207932	0.009408	0.1	0.314583	0.0117902
$b\bar{b}$, $\Lambda = m_b$						
	a	ξ_0	σ^2	β	b	c
$n = 1$	460.778	–	–	3.98292	0.129095	0.515475
$n = 2$	24.2512	0	0.0147382	2.63148	−0.199554	0.0189217
$n = 3$	2.50176	0.21579	0.00891723	0.692022	0.312109	0.0119042

TableA.5: $^3P_{1\parallel}$ fit parameters

$c\bar{c}$, $\Lambda = \infty$						
$\Lambda = \infty$	a	ξ_0	σ^2	β	b	c
$n = 1$	1.89725	–	–	0.833424	0.531147	0.334782
$n = 2$	3.59963	0.000054	0.148262	0.571469	−0.551464	0.67177
$n = 3$	2.32785	0.469035	0.109016	0.999494	0.406331	0.0780892
$c\bar{c}$, $\Lambda = m_c$						
	a	ξ_0	σ^2	β	b	c
$n = 1$	11.121	–	–	1.91255	0.0836092	0.343286
$n = 2$	5.84685	1×10^{-8}	0.152987	0.33432	−0.847095	0.529515
$n = 3$	1.34405	0.487261	0.0170547	0.05	0.0564819	0.00851709
$b\bar{b}$, $\Lambda = \infty$						
	a	ξ_0	σ^2	β	b	c
$n = 1$	15.1067	–	–	1.99935	0.104585	0.0630898
$n = 2$	1.62749	0.129254	0.00695058	0.1	0	–
$n = 3$	4.52742	0.216833	0.0131697	1.26096	0.233993	0.0111891
$b\bar{b}$, $\Lambda = m_b$						
	a	ξ_0	σ^2	β	b	c
$n = 1$	8.43243	–	–	1.68704	0.200742	0.0638126
$n = 2$	1.63952	0.133876	0.00654186	0.1	0	–
$n = 3$	33.7026	0.252602	0.0148385	3.14863	0.181075	0.00958596

TableA.6: ${}^3P_{2\perp}$ fit parameters

$c\bar{c}$, $\Lambda = \infty$						
	a	ξ_0	σ^2	β	b	c
$n=1$	18.9386	–	–	1.53348	0.588362	0.187505
$n=2$	9.78268	0.000301	0.159954	1.92414	−0.331021	0.125204
$n=3$	2.98352	0.455036	0.0789049	1.21573	0.49737	0.0706649
$c\bar{c}$, $\Lambda = m_c$						
	a	ξ_0	σ^2	β	b	c
$n=1$	23.5577	–	–	1.70536	0.469824	0.191909
$n=2$	6.95313	0.2615	0.122794	1.72871	−0.251262	0.0784634
$n=3$	3.28322	0.4731	0.0684879	1.16459	0.309103	0.0461067
$b\bar{b}$, $\Lambda = \infty$						
	a	ξ_0	σ^2	β	b	c
$n=1$	5.03627	–	–	1.36603	12.6441	0.066995
$n=2$	54.0733	0	0.0147382	3.38888	−0.199327	0.0189271
$n=3$	4.27052	0.223597	0.0109257	1.21376	0.305811	0.0155296
$b\bar{b}$, $\Lambda = m_b$						
	a	ξ_0	σ^2	β	b	c
$n=1$	4.60379	–	–	1.34201	14.0572	0.0666158
$n=2$	22.905	1.04×10^{-6}	0.0121405	2.55386	−0.184869	0.0175767
$n=3$	7.66472	0.231103	0.0112419	1.75192	0.282943	0.014799

TableA.7: ${}^3P_{2\parallel}$ fit parameters

$c\bar{c}$, $\Lambda = \infty$						
	a	ξ_0	σ^2	β	b	c
$n = 1$	11.3608	–	–	1.21471	0.697849	0.272678
$n = 2$	4.44771	0.232039	0.158319	1.41475	−0.119605	0.0808185
$n = 3$	2.3202	0.461568	0.0998141	1.01046	0.418644	0.0858612
$c\bar{c}$, $\Lambda = m_c$						
	a	ξ_0	σ^2	β	b	c
$n = 1$	14.4417	–	–	1.40581	0.551405	0.278178
$n = 2$	8.83337	0.173308	0.229571	1.85411	−0.360409	0.141981
$n = 3$	2.51931	0.476137	0.0902674	0.976052	0.253069	0.0561185
$b\bar{b}$, $\Lambda = \infty$						
	a	ξ_0	σ^2	β	b	c
$n = 1$	5.24954	–	–	1.11938	10.5179	0.069816
$n = 2$	26.8594	0	0.0147382	2.73992	−0.168346	0.0189217
$n = 3$	3.78742	0.222412	0.0131263	1.13605	0.259637	0.0157952
$b\bar{b}$, $\Lambda = m_b$						
	a	ξ_0	σ^2	β	b	c
$n = 1$	4.7464	–	–	1.09225	11.8753	0.0695242
$n = 2$	26.8594	0	0.0147382	2.73992	−0.168346	0.0189217
$n = 3$	2.28	0.222311	0.0105893	0.673188	0.326787	0.0180464

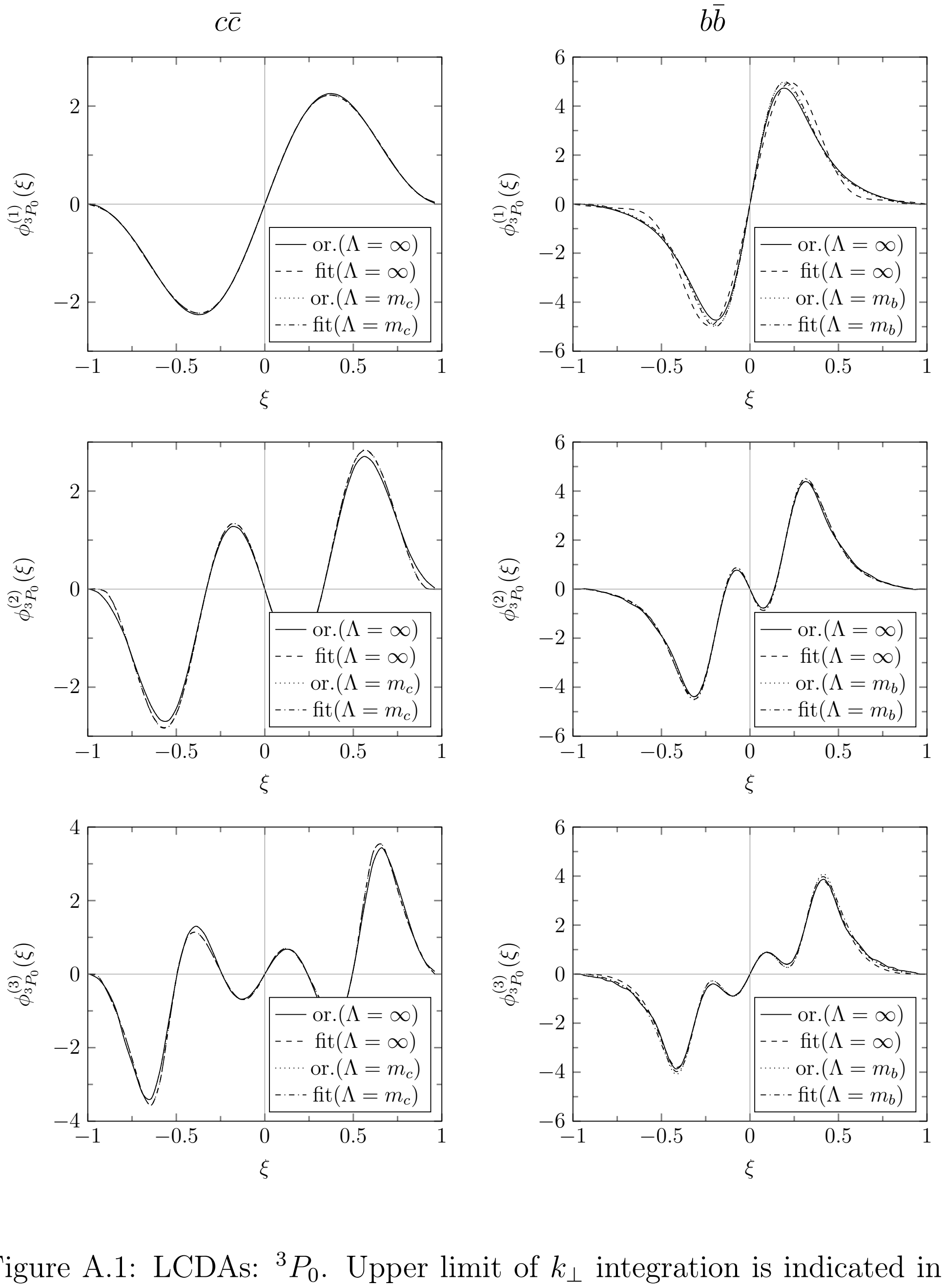

Figure A.1: LCDAs: $^3P_0$. Upper limit of $k_\perp$ integration is indicated in parantheses. "or." refers to the original function and "fit" refers to the fitted function. The radial quantum number n is indicated in parantheses as superscript: $\phi^{(n)}(u)$.

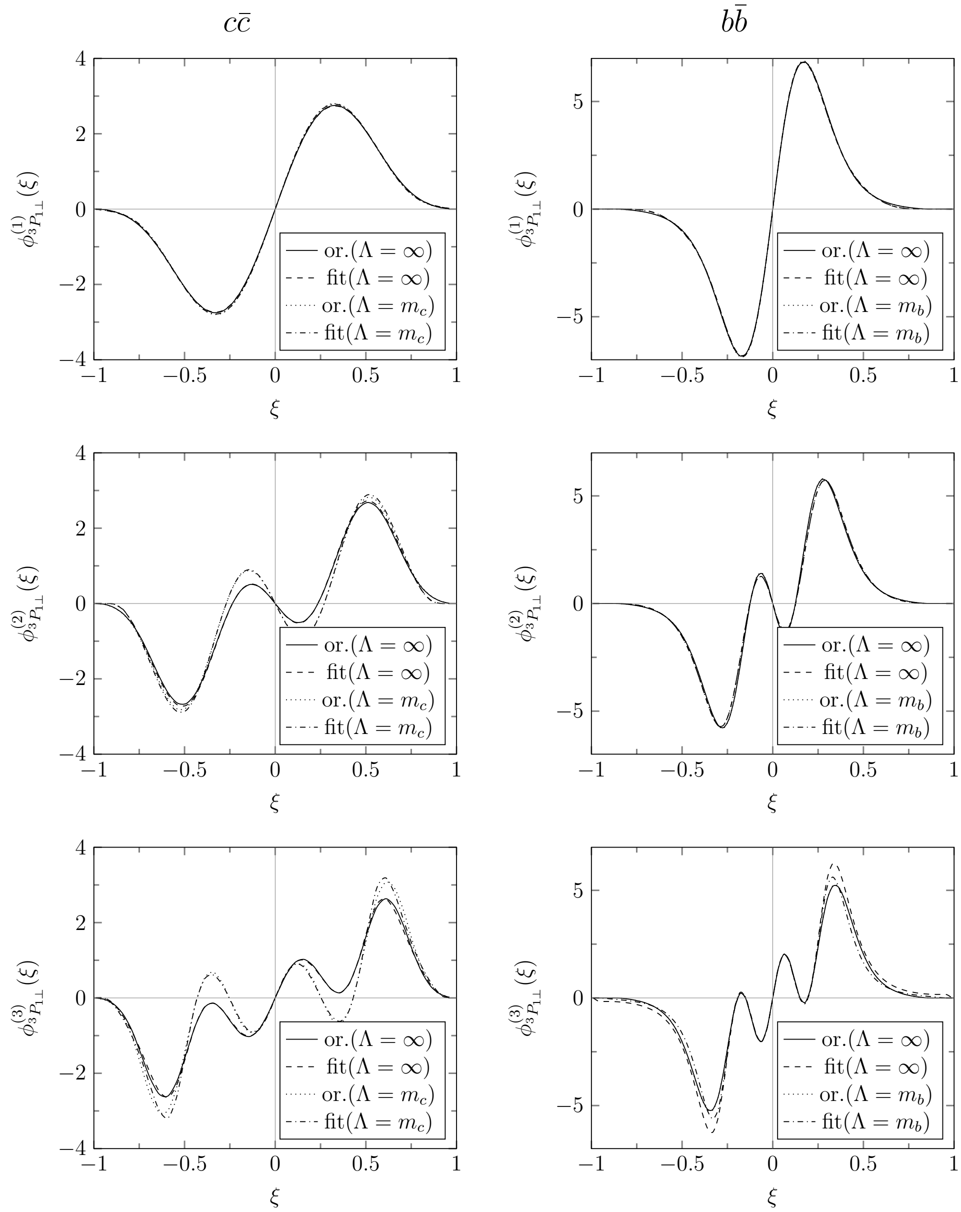

Figure A.2: LCDA plots as in Fig. A.1, but for $^3P_{1\perp}$ states.

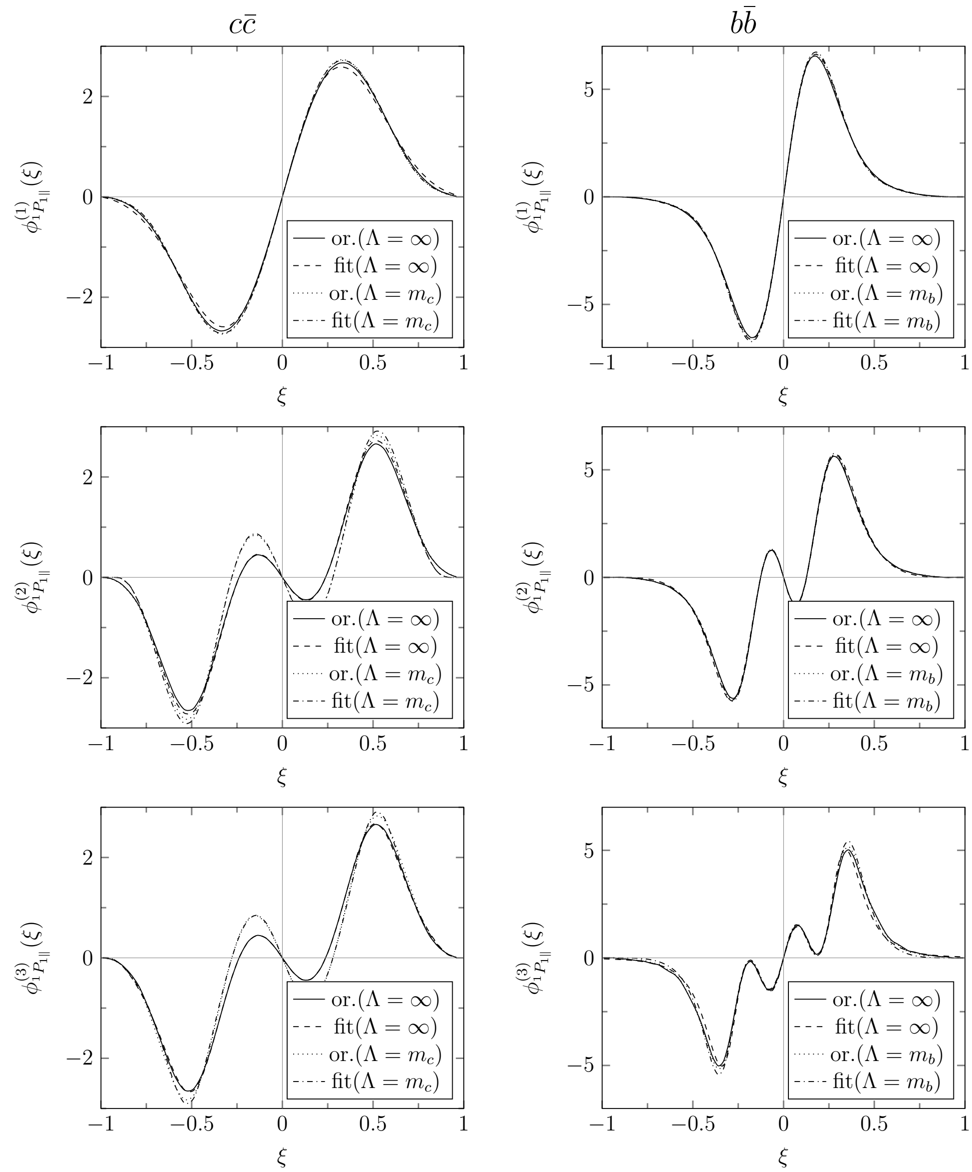

Figure A.3: LCDA plots as in Fig. A.1, but for ${}^1P_{1\parallel}$ states.

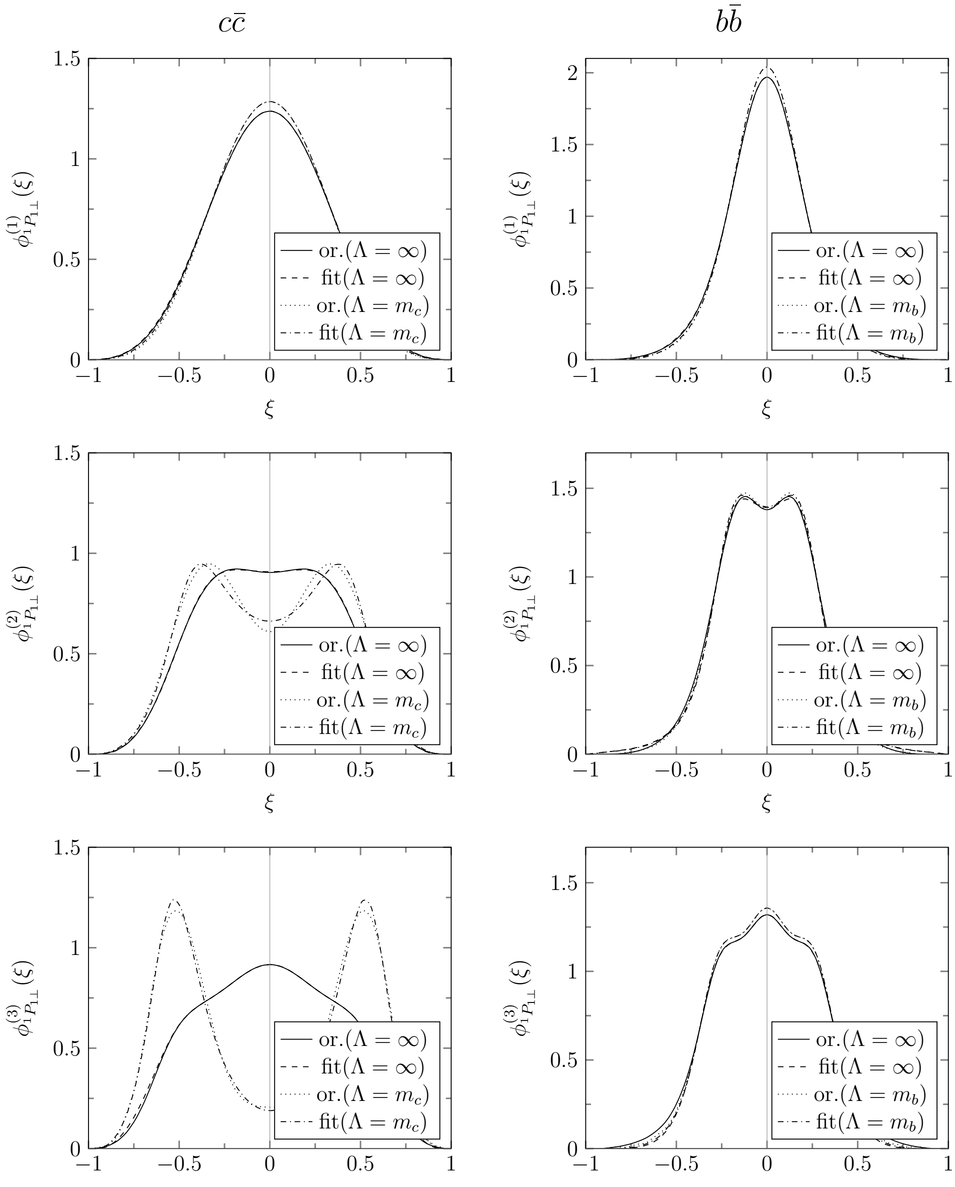

Figure A.4: LCDA plots as in Fig. A.1, but for $^1P_{1\perp}$ states.

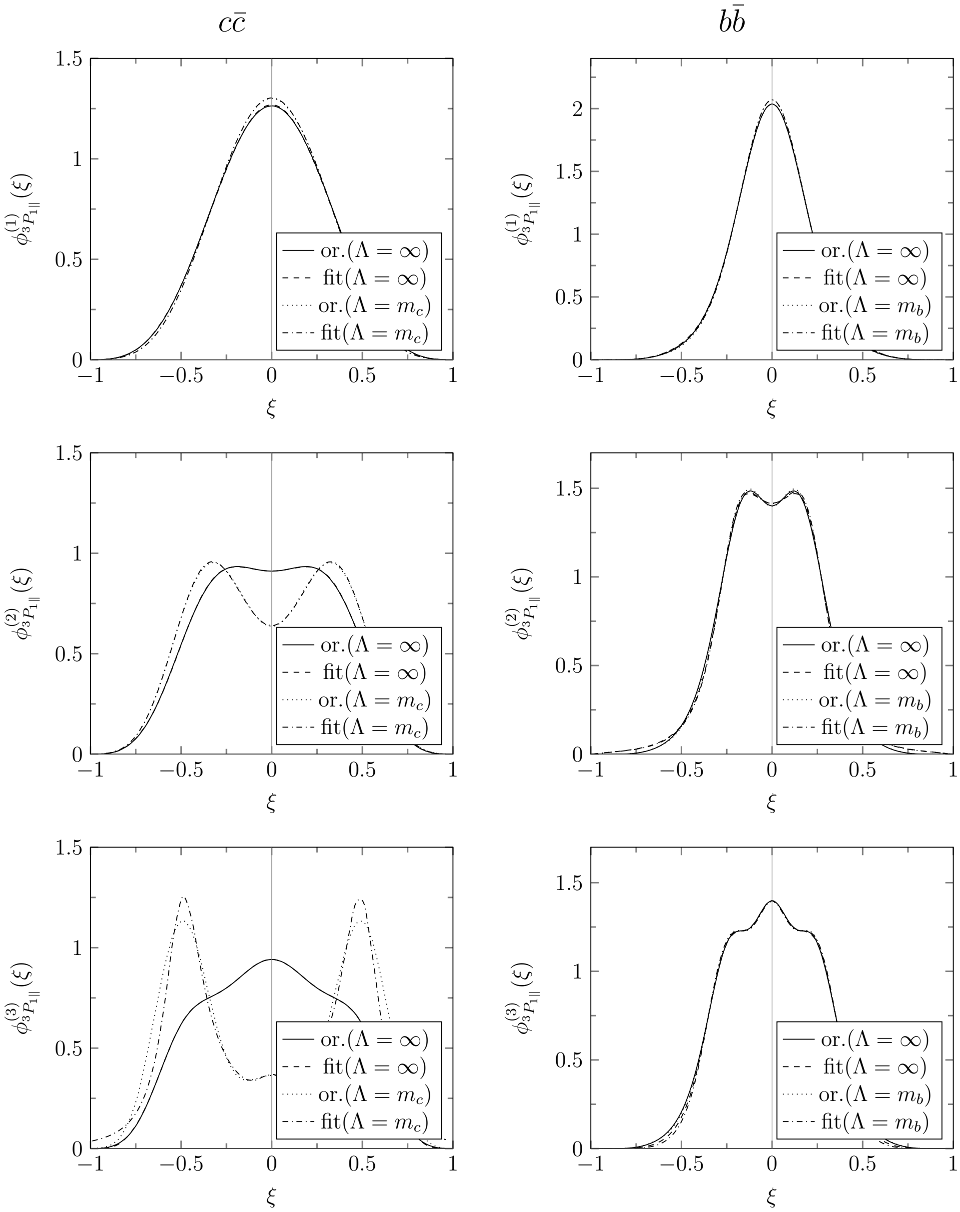

Figure A.5: LCDA plots as in Fig. A.1, but for $^3P_{1\parallel}$ states.

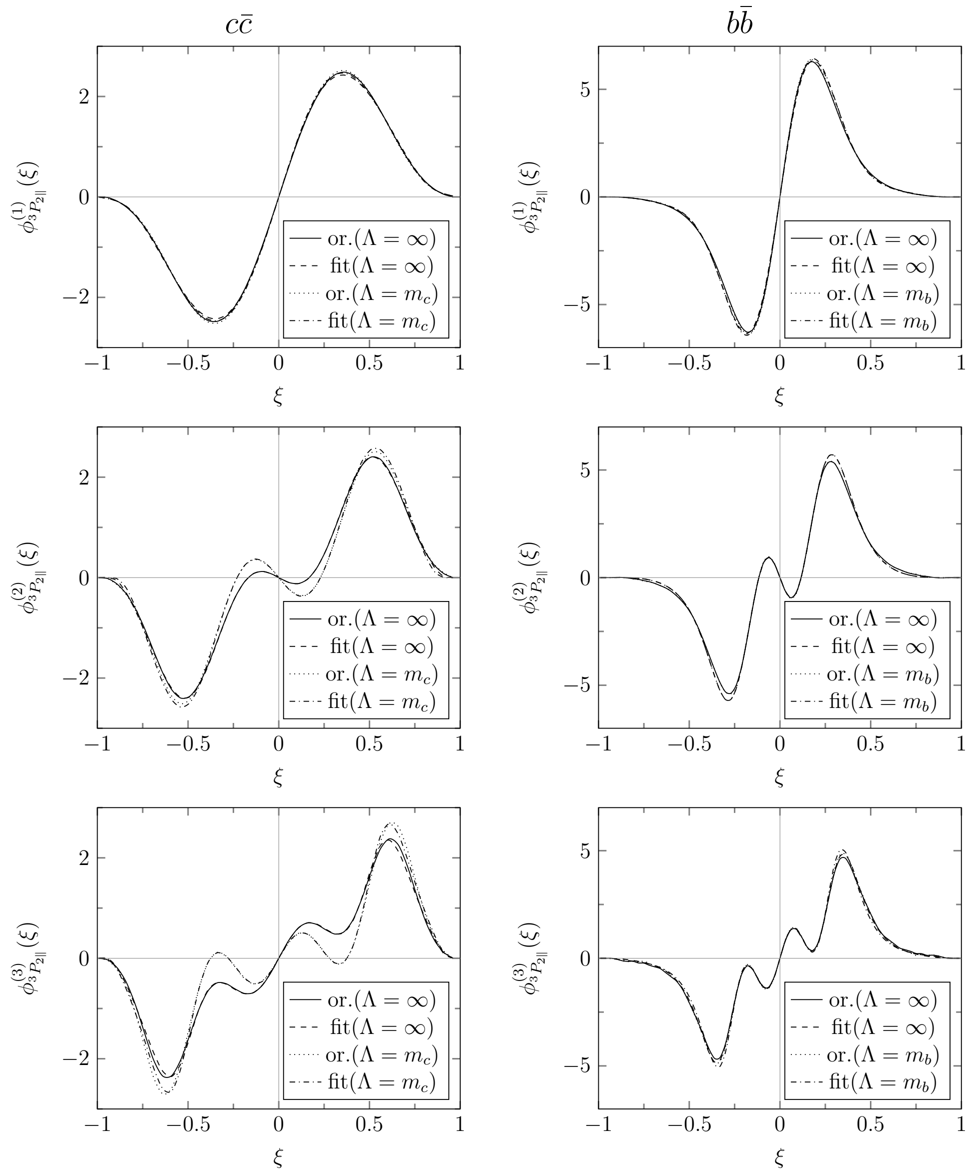

Figure A.6: LCDA plots as in Fig. A.1, but for $^3P_{2\parallel}$ states.

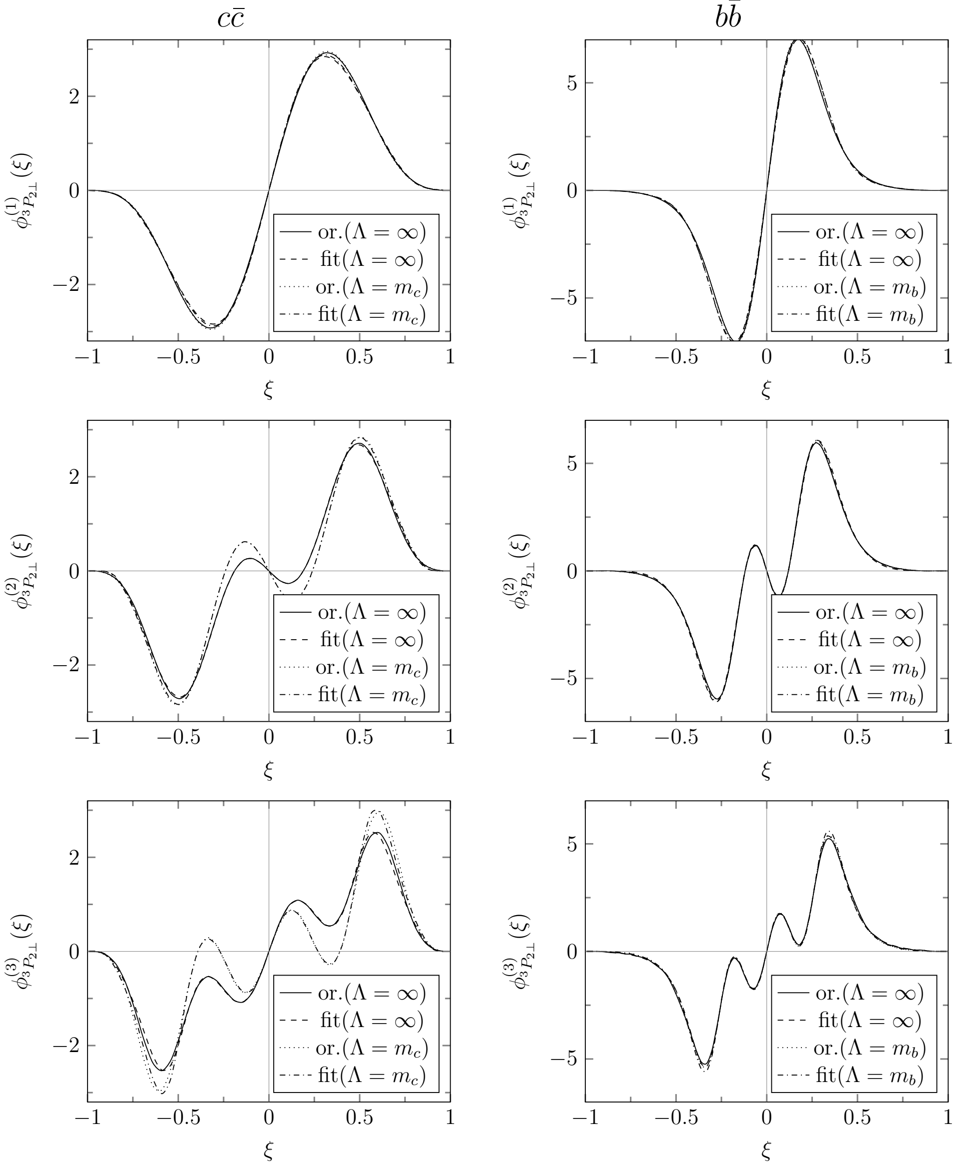

Figure A.7: LCDA plots as in Fig. A.1, but for $^3P_{2\perp}$ states.

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