ANISOTROPIC SOLUTIONS FOR GENERALISED HOLOGRAPHY

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ABSTRACT

ANISOTROPIC SOLUTIONS FOR GENERALISED HOLOGRAPHY

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We first find that the four dimensional cosmological Einstein-Yang-Mills theory with $SU(2)$ gauge group admits Lifshitz spacetime as a base solution for the dynamical exponent $z > 1$. Motivated by this, we next demonstrate numerically that the field equations admit black hole solutions which behave regularly on the horizon and at spatial infinity for different horizon topologies. In the second part, using an off-shell Killing spinor analysis we perform a systematic investigation of the supersymmetric background and black hole solutions of the $\mathcal{N} = (1, 1)$ Cosmological New Massive Gravity model. We find new solutions with a time-like Killing vector that are absent in the $\mathcal{N} = 1$ case. An example of such a solution is a Lifshitz spacetime. The solutions described in this thesis, can be used as backgrounds for holography beyond AdS/CFT.

Keywords: Lifshitz spacetime, Einstein-Yang-Mills, Non-Abelian Groups, Gauge field, Black Hole, Holography, Non-Relativistic Spacetime

ÖZ

LİFSHİTZ UZAY-ZAMANI ÇÖZÜMLERİ

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lk olarak, dört boyutlu, $SU(2)$ ayar gruplu kozmolojik Einstein-Yang-Mills teorisinin dinamik katsayı $z > 1$ için Lifshitz uzay zamanını temel çözüm olarak aldığı bulundu. Bundan yola çıkarak, nümerik olarak olay ufkunda ve uzaysal sonsuzda düzenli davranan farklı olay ufku topolojilerine sahip kara delik çözümleri bulundu. İkinci kısımda off-shell Killing spinör analizi kullanılarak $\mathcal{N} = (1, 1)$ Kozmolojik New Massive Gravity kuramı sistematik bir şekilde incelendi. $\mathcal{N} = 1$ durumundan farklı olarak zamansal Killing vektörü içeren Lifshitz uzay-zamanı çözümü bulundu. Bu tezde betimlenen çözümler AdS/CFT den öte holografi analizleri için arkaplan olarak kullanılabilir.

Anahtar Kelimeler: Lifshitz Uzay-Zamanı, Einstein-Yang-Mills, Abelyen Olmayan Gruplar, Ayar Alanları, Karadelikler, Holografi, Rölativistik Olmayan Uzay-Zamanlar

"THERE is a pleasure in the pathless woods,
There is a rapture on the lonely shore,
There is society, where none intrudes,
By the deep sea, and music in its roar:
I love not man the less, but Nature more,
From these our interviews, in which I steal
From all I may be, or have been before,
To mingle with the Universe, and feel
What I can ne'er express, yet cannot all conceal."

Lord Byron

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CHAPTER 1

INTRODUCTION

The holographic principle has its roots dating back to the works of t'Hooft and Susskind [1, 2]. In its most general form, it basically states that the degrees of freedom of a gravitational system grows with the surface area that closes on the system, not the volume. This sounds rather counterintuitive. If we think in terms of the entropy of the $d + 1$ dimensional bulk and d dimensional boundary theories, the bulk will have an higher entropy for large enough temperature T. However, there is an important point to consider: In a gravitational system black holes will begin to pop up [3] at large enough energies. Thus we have a natural cut-off for the gravitational system dictated by the black holes. The usual thermodynamical computation will break down for large temperatures and entropy will be proportional to the black hole entropy, i.e. the surface area. Therefore, the thermal entropy and black hole entropy is connected through the holographic principle.

These arguments are quite general and it is expected that holographic principle should be applicable for any quantum theory of gravity. However, it is hard to find examples in which the dictionary and the computational framework between bulk/boundary physics is lucid and applicable. The most studied and well-known example is the Anti de Sitter (AdS) /Conformal Field Theory (CFT) correspondence [4], which states an equivalence between a CFT living on the conformal boundary of a string theory on asymptotically locally AdS (AlAdS) spacetime. The correspondence relates different regimes of two theories, i.e. if one of them is perturbative the other is strongly coupled. Therefore the observables that are hard/impossible to compute on one side is manageable on the other side of the duality. From the viewpoint of the CFT side, the observables we can compute are the correlation functions of gauge invariant operators and their symmetry relations. Classical symmetries of a quantum field theory action will give rise to Noether charges and if these symmetries are local then the correlation functions satisfy kinematical relations that are called *Ward identities*. However, the classical symmetries of the systems is not always preserved at the quantum level and broken by quantum corrections. A well-known example is the vanishing trace of the energy momentum tensor of a classical conformal invariant theory, which may be broken by quantum effects that introduces a scale and a non-zero trace for the correlation function of the energy momentum tensor. The Ward identities of these broken symmetries are called *anomalous*.

In order to compute the correlation functions of a given CFT, we first write down the generating function

$$Z[g_{(0)}, \phi_{(0)}] = \int [D\varphi^A] \exp\left(- \int d^d x \sqrt{g_{(0)}} [\mathcal{L}_{\text{CFT}}(\varphi^A; g_{(0)}) + \phi_{(0)} O(\varphi^A) \right), \quad (1.1)$$

where $g_{(0)}$ is the background metric, $\phi_{(0)}$ is the source for the composite operator O and φ^A denotes all fields in the theory. Then, it is straightforward to generate connected diagrams for the operator $O(\varphi^A)$, e.g. the two point correlation function is

$$\langle O(x) O(0) \rangle = \frac{\delta^2 W}{\delta\phi_{(0)}(x) \delta\phi_{(0)}(0)} \bigg|_{\phi_{(0)}=0}, \quad (1.2)$$

where $W = \log Z$ and the background metric is flat $g_{(0)ij} = \delta_{ij}$. We need to introduce counter-terms to the action in (1.1) in order to obtain finite results for physical observables. This, in turn, might break the classical symmetry of the action and make the corresponding Ward identities anomalous.

The AdS/CFT correspondence states that: Instead of employing the generating function (1.1) for the computation of correlation functions in the regime where the theory is strongly coupled and perturbation is not possible, the following equivalence is valid at the leading order [4, 5]

$$S_{\text{on-shell}}[f_{(0)}] = -W[f_{(0)}], \quad (1.3)$$

where $S_{\text{on-shell}}[f_0]$ is the on-shell value of the supergravity action with f_0 representing the boundary values of all the bulk fields. Therefore, the two point correlation function (1.2) reads

$$\langle O(x)O(0)\rangle = -\frac{\delta^2 S_{\text{on-shell}}}{\delta\phi_{(0)}(x)\delta\phi_{(0)}(0)}\bigg|_{\phi_{(0)}=0} . \tag{1.4}$$

The dictionary of the AdS/CFT correspondence can be summarised as follows [5]:

- The bulk fields of the gravity theory are in correspondence with the gauge invariant operators of the CFT. The information about the field theory is contained in the bulk theory.
- The operators of the field theory are coupled to the sources ($\phi_{(0)}$'s in (1.1)) which are identified with the leading boundary behaviour of the bulk fields.
- The generating function of the CFT is identified with the string partition function.

Before we move on with the generalisation of these ideas, let us expound the definitions given in the dictionary and try to pin down what is meant by the leading boundary behaviour and how to expand the metric as a function of distance to the boundary.

The asymptotically locally AdS spacetimes have a well defined boundary in the sense that at a point where the boundary is located, there is a conformal structure instead of a metric, i.e. a set of boundary metrics that are related by conformal transformations (see Sec. 2.1.3). Following the AdS/CFT dictionary, given a conformal structure at the boundary we must be able to determine an Einstein space of constant curvature. Luckily, this hard problem is studied by Fefferman, Graham and Lee [6, 7]. It was shown that for the class of AlAdS metrics it is possible to expand the bulk metric as follows

$$ds^2 = \frac{d\rho^2}{4\rho^2} + \frac{1}{\rho}g_{ij}(x,\rho)dx^i dx^j, \tag{1.5}$$

where

$$g_{ij}(x,\rho) = g_{(0)ij} + \rho g_{(2)ij} + \cdots + \rho^{d/2}(g_{(d)ij} + (\log\rho)\, h_{(d)ij}) + \cdots , \tag{1.6}$$

with ρ as a radial coordinate and x denoting the boundary coordinates.

To make the meaning of the expansion coefficients clear, let us consider cosmological Einstein gravity coupled to a scalar sector with a generic potential [5][1]

$$S = \int d^{d+1}x\sqrt{g}\left[-\frac{1}{2\kappa^2}R + \frac{1}{2}g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi + V(\Phi)\right], \tag{1.7}$$

with the potential of the form

$$V(\Phi) = \frac{\Lambda}{\kappa^2} + \frac{1}{2}m^2\Phi^2 + g\Phi^3 + \cdots , \tag{1.8}$$

where Λ is the cosmological constant, m^2 is the mass of the scalar field and is related to the scaling dimension of the CFT operator as $m^2 = (\Delta - d)\Delta$.

The boundary expansion of the metric (1.5) holds for any field in AlAdS space, scalar, spinor or tensor. Therefore it is possible to expand the scalar field as

$$\Phi(x,\rho) = \rho^{\Delta-d/2}\phi(x,\rho), \tag{1.9}$$

$$\phi(x,\rho) = \phi_{(0)} + \rho\phi_{(2)} + \cdots \rho^{\Delta-d/2}(\phi_{(2\Delta-d)} + (\log\rho)\psi_{(2\Delta-d)}) + \cdots \tag{1.10}$$

Plugging these expansions (1.5), (1.9) to the field equations coming from the action (1.7), we find the relations between the expansion coefficients. It turns out that, all of the coefficients except $\phi_{(2\Delta-d)}$ and the traceless transverse part of $g_{(d)ij}$ are determined in terms of $g_{(0)ij}$ and $\phi_{(0)}$, which are the aforementioned sources that couple to the dual operators [8, 9]. The determined parts of $g_{(d)ij}$ is related to the Ward identities and anomalies. Also the coefficient in front of the logarithmic terms are responsible for the gravitational and matter part of the conformal anomaly [10].

Employing these expansions we can now compute the correlation functions by differentiating the on-shell value of the action with respect the sources (1.4). However, as almost every computation of field theory is plagued with infinities, so is this one. As a result of the infinite volume of the AdS spacetime, the $S_{\text{on-shell}}$ value is divergent, which can be renormalized by introducing a cut-off and adding suitable counter-terms. This whole mechanism is called *holographic renormalisation* and studied extensively with different techniques for various cases.

Having discussed the basic concepts of AdS/CFT correspondence, let us turn our attention to the main subject of the thesis, i.e. non-relativistic spacetimes. Originally

[1] It is possible to extend the matter type to gauge fields, spinors... For the sake of simplicity let us stick to the scalar case.

formulated on $AdS_5 \times S^5$, the concept of holography thrived on different spacetimes that are solutions of string theory. If we are to believe in t'Hooft and Susskind's arguments, it is reasonable that these bulk geometries are dual to field theories that live on the boundary. Motivated by this, lots of phenomenological ("bottom up") models are proposed and most of them does not even have a string theory completion. Some of these different models support solutions that have anisotropy between coordinates and time, which are believed to be the dual geometries for condensed matter systems like strongly correlated electrons. Lifshitz spacetime is one of the geometries that exhibit these non-relativistic symmetries and will be the main topic of this thesis. Along the way we will also discuss the properties of Schrödinger spacetimes, which are intimately related to Lifshitz spacetimes.

Unlike AdS/CFT correspondence, the dictionary of non-relativistic (generalised) holography is not well established. First of all, there is no well defined boundary like the one we have discussed in AdS spacetimes. Moreover, as we will expound in chapter 2, the null geodesics with spatial momenta experience a potential which impedes them, so they can not reach the boundary. On the other hand, infalling observers encounter infinite tidal forces as they go deep in the bulk, which signals the geodesic incompleteness of Lifshitz spacetimes. Therefore, the communication between bulk and boundary is not complete and there is no analogue of Fefferman-Graham expansion from which one can read off the boundary behaviour of fields and deduce the correlation functions.

Fortunately, there are other ways to attack the problem. One of the techniques used is based on the fact that $z = 2$ Lifshitz spacetimes are related to AlAdS spacetimes by Scherk-Schwarz reduction [11, 12]. By computing 5-dimensional vacuum expectation values (vevs) on AlAdS using well known techniques, 4-dimensional vevs can be analysed and counterterms for holographic renormalisation can be obtained. The most important result of [11, 12] was to show that the boundary geometry of Lifshitz spacetime is described by an extension of Newton-Cartan (NC) geometry with a torsion tensor, called torsional Newton-Cartan (TNC) geometry. The inclusion of TNC changes the behaviour of the field theories, conserved currents and symmetries drastically. Contrary to the relativistic case where we have a pseudo-Riemannian manifold endowed with a non-degenerate metric $g_{\mu\nu}$, NC/TNC has much more structure. The real problem is the particle number symmetry, which is a central charge for Galilean algebra, i.e. commutes with all of the operators, but appears on the right handside of the commutator of translations and Galilean boosts (see Sec. 2.2.3). Therefore, it has to be related with the spacetime symmetries. Because of this fact, the geometry and the connections can not be determined only in terms of the d dimensional metric g_{ij} but we need extra data, a 1-form, positive semi-definite $d-1$ rank tensor and a $U(1)$ connection related to the particle number. This extra baggage makes the coupling of field theories to curved NC backgrounds difficult [13, 14]. The other aspect of the geometry of TNC is the fact that it can be realised from the gauging of the Schrödinger algebra and imposing curvature constraints [15, 16], just as in the case of General Relativity when it is obtained by gauging the Poincaré algebra (see Sec. 4.2.4). Although the reduction technique for the holographic analysis employed in [11, 12] is robust and reliable, it is not really possible to extend that holographic analysis to other configurations with different types of matter. One should be able to obtain the theory to be examined, from the higher dimensional ones with AdS backgrounds through TsT (T-duality, shift, T-duality)+Scherk-Schwarz transformations and obviously that is not possible for all cases.

Another approach to holographic renormalization that is in principle applicable to all models with a Lifshitz background employs the Hamiltonian formalism [5, 17]. First, let us quickly review the procedure for AdS. One starts by writing the Hamiltonian by employing Arnowitt-Deser-Misner (ADM) formalism through which the corresponding momenta of fields can be obtained. However, as AdS is a non-compact spacetime, these momenta diverge. At finite radius one can cast momenta as functionals of the bulk fields at hypersurface defined by radius r, but we also know that momenta are r-derivatives of the bulk fields. Then in the spirit of Hamilton-Jacobi theory in classical mechanics, functional partial differential equations can be defined for the momenta. Although it is not easy to solve these functional PDEs, it is suitable for the asymptotic analysis. At that point AdS analysis follows through the expansion of the metric and matter fields (1.5), (1.9), however as we have discussed it is not easy to define fall-off conditions for asymptotically non-AdS backgrounds. One can extract behaviour by studying the linearised equations but the authors of [18, 19] instead construct a recursion procedure based on the covariant expansion of the Hamilton-Jacobi solution in eigenfunctions of suitable derivative operator. The analysis is carried out for a general Einstein-Proca-dilaton model and in principle should be applicable to different models.

The moral of the story is, holography of Lifshitz and Schrödinger spacetimes is a valuable playground for studying the idea of t'Hooft and Susskind, i.e. the dual nature of gravity and field theories should extend outside of the AdS/CFT correspondence. However, the technical difficulties and lack of tools hinder the holographic study of these spacetimes. In this thesis we won't be dealing with the Lifshitz holography, instead we first steer to the solutions with Lifshitz backgrounds supported by non-abelian matter. Then in the second part, the supersymmetric solutions of three dimensional higher curvature theory will be investigated. Both of these solutions exhibit properties that make them interesting test grounds for the Lifshitz holography.

The plan of the thesis is as follows: We will set out with a discussion of AdS spacetime and its properties, which will be useful to compare with non-relativistic spacetimes. Later in Chapter 1 we will investigate the symmetries and the causal structure of non-relativistic spacetimes including Lifshitz and Schrödinger. In Chapter 2 we will show that it is possible to support Lifshitz spacetime with a non-abelian matter and present some numerical solutions. The final chapter will be on the classification of solutions of 3-dimensional $\mathcal{N} = 2$ quadratic curvature supergravity, which admits Lifshitz spacetime as a solution.

CHAPTER 2

NON-RELATIVISTIC SPACETIMES

2.1 AdS and its properties

2.1.1 AdS from an embedding and global coordinates

We will start with the properties of AdS spacetime in order to gain insight for symmetries and causal properties which will guide us through the discussion of non-relativistic (NR) spacetimes. It is enlightening to see the features of AdS that are crucial for holography and not shared by NR spacetimes. Conversely on the NR side, partial breaking of the spacetime symmetries bring in extra structure, e.g. conserved particle number and anisotropic scaling.

AdS$_n$ spacetimes are the maximally symmetric solution of the Einstein equations with cosmological constant derived from the action

$$S = \frac{1}{\kappa}\int d^n x\sqrt{|g|}(R + 2\Lambda), \tag{2.1}$$

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \Lambda g_{\mu\nu}. \tag{2.2}$$

From (2.2), it follows that

$$R_{\mu\nu} = \frac{2\Lambda}{2-n}g_{\mu\nu} \tag{2.3}$$

which renders the solution as Einstein space. Demanding the Riemann tensor of the solution to be

$$R_{\mu\nu\tau\rho} = \frac{2\Lambda}{(n-1)(n-2)}(g_{\mu\tau}g_{\nu\rho} - g_{\mu\rho}g_{\nu\tau}), \tag{2.4}$$

makes AdS_n a maximally symmetric space. These spacetimes can be defined as an embedding of particular quadratic surface in flat spacetimes. Let us consider $\mathcal{M}$ as an m-dimensional submanifold of an n-dimensional manifold $\mathcal{N}$ with the metric g_N. Then the embedding $f : \mathcal{M} \to \mathcal{N}$ will induce a pullback of the metric $g_M = f^{\star} g_N$, which is given by [20]

$$g_{M\mu\nu} = g_{N\alpha\beta} \frac{\partial f^{\alpha}}{\partial x^{\mu}} \frac{\partial f^{\beta}}{\partial x^{\nu}}. \tag{2.5}$$

As an example consider the well known embedding of a sphere S^n in Euclidean spacetime $ds^2 = \sum_{i=0}^{n} dx_i^2$ with the quadratic given as

$$X_0^2 + X_1^2 + \cdots + X_n^2 = L^2. \tag{2.6}$$

In this form it is obvious that the quadratic (2.6) respects the symmetries of the embedding space i.e. $X^i \to X^j \Lambda_j^i$ with $\Lambda_j^i \in \mathrm{SO}(n+1)$. In order to find a set of global coordinates, we need to solve the constraint (2.6). For $n = 3$

$$\begin{aligned} X_0 &= r \sin\theta \cos\phi, \\ X_1 &= r \sin\theta \sin\phi, \\ X_2 &= r \cos\theta, \end{aligned}$$

employing (2.5) with $g_{N\alpha\beta} = \delta_{\alpha\beta}$, we find the familiar result

$$ds^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{2.7}$$

The defining quadratic for Lorentzian[1] AdS_n differs from (2.6) with a signature change. Instead of all positive signs, we will now have a hyperboloid with two sheets (note that we have also changed the sign of L^2)

$$-X_0^2 - X_n^2 + \sum_{i=1}^{n-1} X_i^2 = -L^2. \tag{2.8}$$

This time it is not possible to take Euclidean or Minkowski signature for manifest spacetime symmetries, we should embed (2.8) in $ds^2 = -dX_0^2 - dX_n^2 + \sum_{i=1}^{n-1} dX_i^2$, so that the AdS_n will be homogeneous and have $\mathrm{SO}(2, n-1)$ symmetry. The following

[1] The Euclidean AdS_n is defined through the quadratic $-X_0^2 + \sum_{i=1}^{n} X_i^2 = -L^2$ with the embedding space with Minkowski metric $ds^2 = -dX_0^2 + \sum_{i=1}^{n} dX_i^2$.

set of coordinates is a solution to the constraint (2.8)

$$\begin{aligned}X_0 &= L\cosh\rho\cos\tau,\\ X_n &= L\cosh\rho\sin\tau,\\ X_i &= L\,\Omega_i\sinh\rho,\end{aligned}$$

with $\sum_{i=1}^{n-1}\Omega_i^2 = 1$. Then the induced metric (2.5) will read

$$ds^2 = L^2(-\cosh^2\rho d\tau^2 + d\rho^2 + \sinh^2\rho d\Omega_i^2). \tag{2.9}$$

Note that the metric $d\Omega_i^2$ possesses the symmetries of SO$(n-1)$. Therefore AdS$_n$ has the manifest symmetry of $SO(2)\times SO(n-1)$ in these coordinates. Taking coordinate ranges $\rho\in(0,\infty)$ and $\tau\in[0,2\pi)$, the hyperboloid is fully covered once, hence we have global coordinates on AdS$_n$. The timelike Killing vector has norm squared $\cosh^2\rho$ which is non-vanishing and well defined everywhere. However for physical applications, AdS$_n$ in global coordinates is not quite suitable because of the closed timelike curves which will spoil causality. Near $\rho\simeq 0$ the global metric (2.9) goes like [21]

$$ds^2 \simeq L^2(-d\tau^2 + d\rho^2 + \rho^2 d\Omega_i^2), \tag{2.10}$$

which has the topology of $S^1\times\mathbb{R}^n$ with timelike S^1, that signals closed timelike curves (CTC). In order to obtain a causal spacetime we will consider the universal cover of global AdS$_n$. In essence, the covering space of a manifold $\mathcal{M}$ is a "larger" manifold that locally looks like discrete copies of $\mathcal{M}$. The covering space $\mathcal{N}$ of the manifold $\mathcal{M}$ is defined with a projection $p:\mathcal{N}\to\mathcal{M}$ such that [22]

- the projection p is onto and continuous,
- every point in $x\in\mathcal{M}$ has a neighbourhood U such that $p^{-1}(U)$ is homeomorphic to $U\times\Lambda$, where Λ is a discrete space.

Through this projection we can define lifts of paths on $\mathcal{M}$, which is the process of unwrapping loops, hence ending up in a manifold with a simpler fundamental group[2]

[2] Remember, the fundamental group basically counts the number of topologically distinct loops on a manifold.

Finally, the universal covering is the manifold for which the fundamental group is trivial. For the case of AdS$_n$ we will unwrap the CTC by taking $-\infty < \tau < \infty$, ending in a causal spacetime.

There is another incarnation of AdS$_n$ which is used extensively. Defining $(n-2)$ dimensional Lorentz vector x^i and a radial coordinate $u > 0$, the following redefinition of embedding coordinates

$$\begin{aligned}X_0 =& \frac{u}{2}\left(1 + \frac{1}{u^2}(L^2 + x^2 - t^2)\right), \quad X_i = \frac{Lx^i}{u}\\ X_{n-1} =& \frac{u}{2}\left(1 - \frac{1}{u^2}(L^2 - x^2 + t^2)\right), \quad X_n = \frac{Lt}{2},\end{aligned} \tag{2.11}$$

brings the metric to the form

$$ds^2 = \frac{L^2}{u^2}[du^2 - dt^2 + dx_i dx^i]. \tag{2.12}$$

In this set of coordinates $SO(1,1) \times SO(1, n-2)$ is manifest and it has $u =$ constant Minkowski slices, because of the manifest symmetry this set of coordinates is called *Poincaré* coordinates. The $SO(1,1)$ symmetry acts as [21]

$$(u, t, x^i) \to (au, at, ax^i), \quad a > 0. \tag{2.13}$$

Unlike AdS in global coordinates, the Poincaré coordinates have a timelike Killing vector that has a vanishing norm at $u \to \infty$, which is called the horizon of the Poincaré coordinates.

Having discussed the construction and the properties of AdS in different coordinates, we are now in a position to explore the behaviour of particle trajectories in AdS.

2.1.2 Particle motion in AdS

In order to analyse particle trajectories in AdS spacetime we can start from the free particle Lagrangian (square root action) and study the equations of motion for the timelike, spacelike or null cases. By introducing an auxiliary field (gauge degree of freedom), "the nasty-looking" square root action can be simplified even more [23]. However, it is wiser to approach from the embedding definition of AdS, which we have discussed in the previous section. Let us use that knowledge and define the particle motion on AdS as an action with the constraint (2.8)

$$S = \int d\tau \left[\frac{1}{2}\eta_{\mu\nu}\dot{X}^\mu(\tau)\dot{X}^\nu(\tau) + \Lambda\left(\eta_{\mu\nu}X^\mu(\tau)X^\nu(\tau) + L^2\right)\right], \qquad (2.14)$$
$$\eta_{\mu\nu} \equiv \mathrm{diag}(-1, 1, \cdots, -1),$$

where we have parametrised the curves with τ and employed the Lagrange multiplier Λ to ensure that the particle stays on the surface defined by the quadratic (2.8). The equations of motion follow as

$$\ddot{X}^\mu = 2\Lambda X^\mu, \qquad X^2 + L^2 = 0. \qquad (2.15)$$

Now to eliminate Λ, take second derivative of the constraint equation

$$X_\mu \ddot{X}^\mu = -\dot{X}_\mu \dot{X}^\mu. \qquad (2.16)$$

Plugging (2.16) in (2.15), we find the Lagrange multiplier $\Lambda = \dot{X}_\mu \dot{X}^\mu/(2L^2)$. A priori, the norm $\dot{X}_\mu \dot{X}^\mu$ is an arbitrary function of τ. However, it is possible to set the norm to a constant value. First, observe that the antisymmetric tensor $k_{\mu\nu} = X_\mu \dot{X}_\nu - X_\nu \dot{X}_\mu$ is conserved on shell, i.e. $\dot{k}_{\mu\nu} = 0$ and its norm is

$$k_{\mu\nu}k^{\mu\nu} = -2L^2 \dot{X}^\mu \dot{X}_\mu. \qquad (2.17)$$

Since $k_{\mu\nu}$ is conserved, (2.17) allows us to set $\dot{X}^\mu \dot{X}_\mu =$ constant. Moreover, the field equations (2.15) are invariant under rescaling of the parameter τ so that we can set $\eta_{\mu\nu}\dot{X}^\mu \dot{X}^\nu = \sigma$, where σ can take values $\pm 1, 0$ depending on whether the curve we consider is spacelike, timelike or null [3].

After eliminating Λ, the particles on AdS satisfy the equation

$$\ddot{X}^\mu - \frac{\sigma}{L^2}X^\mu = 0. \qquad (2.18)$$

It is obvious that the null geodesics are straight lines on the embedding space [24]. On the other hand timelike ones ($\sigma = -1$) are

$$X^\mu(\tau) = c^\mu \cos(\tau/L) + s^\mu \sin(\tau/L), \qquad (2.19)$$

with $c_\mu c^\mu = s_\mu s^\mu = -L^2$ and $c_\mu s^\mu = 0$, coming from the constraint equation (2.8). In the previous section we have discussed about taking the universal cover of AdS to get

[3] It is easy to show that the field equations (2.15) are reparametrization invariant with $\tau = \tau(\lambda)$ if $\dfrac{d^2\lambda}{d\tau^2} = 0$.

rid of CTCs yet we now see that timelike particles in AdS have periodic trajectories. Basically, AdS spacetime behaves like a box. This state of affairs is summarised by Gibbons in [25] as

“Many physicists are unhappy with the CTCs in AdS$_{p+2}$ *and seek to assuage their feelings of guilt by claiming to pass to the universal covering spacetime AdS*$_{p+2}$*. In this way they feel that they have exorcised the demon of “acausality”. However therapeutic uttering these words may be, nothing is actually gained in this way. Consider for example the behaviour of test particles. Every timelike geodesic on AdS*$_{p+2}$ *is a closed curve of the same durations equal to* $2\pi L$*, which Heraclitus would have called the “Great Year”.”*

2.1.3 Conformal Infinity

Developed by Penrose [26, 27], the concept of conformal infinity is not limited to AdS spacetimes. The need for such concept originated from the study of ”isolated systems” like binary stars and their gravitational wave characterisation ([28] and references therein). Initially people have studied the problem of gravitational radiation by classifying the behaviour of Riemann tensor and Bianchi identities at large distances. It was Penrose who noticed that, by making a conformal transformation $g = \omega^2 \tilde{g}$, one can bring infinity to a finite coordinate value, which translates into attaching boundary points to the physical spacetime $\tilde{g}$ and ending up in a larger manifold with a metric g. Since conformal transformations leave the light cones invariant, the causal structure of $\tilde{g}$ will be the same after the transformation[4]. One of the well-known examples is the 4-dimensional Minkowski spacetime which is diffeomorphic to the inner part of Einstein universe or $\mathbb{R} \times S^3$ [29].

To make the discussion clear, let us define *asymptotically simple spacetime* (conformally compact manifold) as a Lorentzian manifold $(\tilde{\mathcal{M}}, \tilde{g}_{ab})$, with conformally related partner $(\mathcal{M}, g_{ab})$ having the following properties [28]

- $\tilde{\mathcal{M}}$ is an open submanifold of $\mathcal{M}$ with smooth boundary $\partial\tilde{\mathcal{M}} = X$,
- there exists a smooth scalar field Ω on $\mathcal{M}$ such that $g_{ab} = \Omega^2 \tilde{g}_{ab}$ on $\tilde{\mathcal{M}}$ and, so

[4] In Euclidean signature conformal transformations maps $S^n \to S^n$.

that $\Omega = 0$, $d\Omega \neq 0$ on X,

- every null geodesic in $\tilde{\mathcal{M}}$ acquires a past and future endpoint on X.

Moreover a spacetime is *asymptotically flat* if it also satisfies $\tilde{R}_{ab} = 0$ near the boundary. The Minkowski spacetime is an example of both simple and asymptotically flat spacetime. The AdS/dS spacetimes are asymptotically simple but not asymptotically flat.

The couple $(\mathcal{M}, \Omega)$ defines a *conformal structure* at the boundary, i.e given a function Ω that has the given properties above, any other function $\Omega\, e^{\alpha}$ is also acceptable if α is a function with no zeros or poles at the boundary. Therefore we have a set of boundary metrics that are related by conformal transformations. The non-degenerate boundary metric $g|_X \equiv g_0$ is a representative of the conformal class of metrics on X.

Computing the curvature tensor of $\tilde{g}_{\mu\nu}$ we find [30]

$$R_{\kappa\lambda\mu\nu}[\tilde{g}] = |d\Omega|_g^2(\tilde{g}_{\kappa\mu}\tilde{g}_{\nu\lambda} - \tilde{g}_{\kappa\nu}\tilde{g}_{\lambda\mu}) + \mathcal{O}(\Omega^{-3}), \tag{2.20}$$

where $|d\Omega|_g^2 \equiv g^{\mu\nu}\partial_\mu\Omega\,\partial_\nu\Omega$. From the properties of Ω it is apparent that as we approach boundary, i.e. as $\Omega \to 0$, the curvature tensor (2.20) will begin to look like AdS, as the first term is of the order Ω^{-4}. The important point here is that, we have not used the equation of motion to obtain (2.20). Requiring (2.2) to hold fixes $|d\Omega|_g^2 = -2\Lambda/(n-1)(n-2)$. Then *asymptotically locally AdS (AlAdS)* spacetimes are defined as solutions of Einstein's equations for which the Riemann tensor approaches (2.20) asymptotically. On the other hand, *asymptotically AdS (AAdS)* spacetimes are exactly of the form (2.4) without deviations.

Being conformally compact, AdS spacetime in global coordinates (2.21) can be conformally extended. First, to put global AdS (2.9) in a manageable form consider a coordinate transformation $\tan\theta = \sinh\rho, \quad \theta \in [0, \pi/2)$,

$$ds^2 = \frac{L^2}{\cos^2\theta}(-d\tau^2 + d\theta^2 + \sin^2\theta\ d\Omega_{n-1}^2). \tag{2.21}$$

Then, multiplying by $\cos^2\theta/L^2$ we have a conformal completion of AdS which is actually a part of the Einstein universe,

$$ds^2 = -d\tau^2 + d\theta^2 + \sin^2\theta\ d\Omega_{n-1}^2. \tag{2.22}$$

We are characterising as "part of" since, normally the full Einstein universe covers $\theta \in [0, \pi)$ but for AdS $\theta \in [0, \pi/2)$. The conformal factor blows up at $\theta = \pi/2$, therefore there is a spherical boundary at that point. Note that, the conformal completion doesn't uniquely define the larger spacetime which we extend the physical spacetime into. By performing a different transformation we will end up in different spacetimes.

2.1.4 Conformal symmetries and AdS

The action of the generator of a symmetry with the parameter ϵ^A can be defined as a linear operator $\delta(\epsilon)$ acting on fields in a chosen representation. The parameters ϵ^A depend on coordinates if symmetry is local, otherwise they are only constants for global symmetries. The most general form of symmetry operation is defined as [31]

$$\delta(\epsilon) = \epsilon^A T_A \tag{2.23}$$

where T_A is an operator that acts on fields. The operators T_A can be taken as a matrix representation of a Lie algebra, i.e. $T_A \phi^i = -(t_A)^i_j \phi^j$ with commutator $[t_A, t_B] = f_{AB}{}^C t_C$, and ϕ^i are the fields that transform in the chosen matrix representation.

Following the definition of conformal transformation discussed in the previous section, i.e. $g_{ab} = \Omega^2 \tilde{g}_{ab}$, we can derive the equations that are satisfied by the infinitesimal generators of the transformations which are also called *conformal Killing vectors*.

Let us define the infinitesimal form of the transformations at first order as $\tilde{x}_\mu = x_\mu + \xi_\mu(x)$ and let $\Omega(x) = 1 + \omega(x)$, and study the conformal transformation in flat spacetimes. Plugging in, we find at first order

$$\partial_\mu \xi_\nu + \partial_\nu \xi_\mu = \omega(x)\, \eta_{\mu\nu}. \tag{2.24}$$

Taking the trace to derive the equation for generators, we find

$$\partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \frac{2}{D} \partial_\alpha \xi^\alpha \eta_{\mu\nu} = 0. \tag{2.25}$$

First of all, note that something special happens in two dimensions obviously. Setting $D = 2$ in (2.25) and writing all possible equations we find

$$\partial_1 \xi_1 = \partial_2 \xi_2, \tag{2.26}$$

$$\partial_2 \xi_1 = -\,\partial_1 \xi_2, \tag{2.27}$$

which are the Cauchy-Riemann conditions from complex analysis. This remarkable thing tells us that it is possible to define a holomorphic function $\xi(z) \equiv \xi_1 + i\xi_2$ which makes the generators of conformal transformations at $D = 2$ holomorphic maps i.e. $\partial_{\bar{z}}\xi = 0$. Since $\xi(z)$ is holomorphic, it will have a Laurent expansion, which signals at $D = 2$ we have infinite dimensional algebra, namely the *Witt* algebra.

Before going into the case $D > 2$, let us remember the Killing vector equation and its solution for the flat spacetime

$$\partial_\mu \xi_\nu + \partial_\nu \xi_\mu = 0. \tag{2.28}$$

Hitting it with one more derivative and summing up the equations with permuted indices we find $\partial_\alpha \partial_\mu \xi_\nu = 0$. This tells us that the generator of symmetries for flat spacetimes, i.e. the Poincaré group is linear in coordinates x_μ. The solution is given by

$$\xi^\mu(x) = a^\mu + \lambda^{\mu\nu} x_\nu, \quad \text{where} \quad \lambda^{\mu\nu} = -\lambda^{\nu\mu}, \tag{2.29}$$

where a^μ are D parameters for translations and $\lambda^{\mu\nu}$ are $D(D-1)/2$ parameters for Lorentz transformations. Together they form $D(D+1)/2$ parameter Poincaré algebra.

Performing the same trick for the equation (2.25) this time, we find $\partial_\alpha \partial_\rho \partial_\mu \xi_\nu = 0$, indicating generators for the conformal group are at most quadratic in x_μ. The solutions are well known

$$\xi^\mu(x) = a^\mu + \lambda^{\mu\nu} x_\nu + \lambda_D x^\mu + (x^2 \lambda_K^\mu x^2 - 2x^\mu x_\alpha \lambda_K^\alpha). \tag{2.30}$$

Again, we have the Poincaré part, so it is a subalgebra of the conformal algebra. The λ_D is the parameter for dilatations and the last part λ_K is the parameter for special conformal transformations. Adding these extra parameters to Poincaré, we have the $(D+1)D(D+2)/2$ dimensional conformal algebra which is actually isomorphic to the $SO(2, D)$ algebra.

The most general conformal transformation acting on fields of a conformal theory is given by

$$\delta(\epsilon) = a^\mu P_\mu + \frac{1}{2}\lambda^{\mu\nu} M_{[\mu\nu]} + \lambda_D D + \lambda_K^\mu K_\mu. \tag{2.31}$$

The non-vanishing commutators of the conformal algebra are

$$
\begin{aligned}
[M_{\mu\nu}, M_{\rho\sigma}] &= 4\eta_{[\mu[\rho}M_{\sigma]\nu]}, \quad [P_\mu, M_{\nu\rho}] = 2\eta_{\mu[\nu}P_{\rho]}, \\
[K_\mu, M_{\nu\rho}] &= 2\eta_{\mu[\nu}K_{\rho]}, \quad [P_\mu, K_\nu] = 2(\eta_{\mu\nu}D + M_{\mu\nu}) \\
[D, P_\mu] &= P_\mu, \quad [D, K_\mu] = -K_\mu.
\end{aligned} \tag{2.32}
$$

Several observations are in order. First, the generators P_μ, K_μ are vectors and D is a scalar under the Lorentz group $SO(1, D-1)$. The last two commutators indicate that P_μ, K_μ are ladder operators similar to the simple harmonic oscillator in quantum mechanics with the hamiltonian operator D in this case.

Secondly, there is one special discrete transformation, that is not connected to the identity, called *inversion*, which is extensively used in discussions of CFTs and holography. The inversion is defined as $I : x_\mu \to \tilde{x}_\mu = x_\mu/x^2$, acting twice we have the identity map $I^2 = \mathbb{I}$. By combining translations with inversion, $IP_\mu I$, we can reproduce finite special conformal transformations K_μ

$$
IP^\mu I = K^\mu = \frac{x^\mu + a^\mu x^2}{1 + a^2x^2 + 2a.x}. \tag{2.33}
$$

Adding this discrete transformation to the $SO(2, D)$ generators, we end up in $O(2, D)$.

Finally, to see the connection with AdS, let us assemble the generators in the following fashion with $\hat{\mu}, \hat{\nu} = 0, \cdots, D,\ D+1$

$$
M^{\hat{\mu}\hat{\nu}} = \begin{bmatrix} M^{\mu\nu} & \frac{1}{2}(P^\mu - K^\mu) & \frac{1}{2}(P^\mu + K^\mu) \\ -\frac{1}{2}(P^\mu - K^\mu) & 0 & -D \\ -\frac{1}{2}(P^\mu + K^\mu) & D & 0 \end{bmatrix}. \tag{2.34}
$$

The new generators will satisfy the $SO(2, D)$ algebra

$$
[M_{\hat{\mu}\hat{\nu}}, M_{\hat{\rho}\hat{\sigma}}] = 4\hat{\eta}_{[\hat{\mu}[\hat{\rho}}M_{\hat{\sigma}]\hat{\nu}]} \tag{2.35}
$$

where $\hat{\eta} = \mathrm{diag}(-1, 1, 1, 1 \cdots - 1)$. Previously we have defined AdS$_{D+1}$ as an embedding (2.8) with an embedding metric having the same signature. Therefore the AdS$_{D+1}$ has the isometry group $O(2, D)$.

2.2 Non-Relativistic Spacetimes

In this section we will focus on the main theme of the thesis, the non-relativistic (NR) spacetimes. However, before going into details about NR spacetimes, let us discuss the physical importance of the scale invariance and start the discussion from the viewpoint of Renormalization Group. The discussion will be conceptual and may not be crucial for the rest of the thesis, but it is really easy to get smothered by the amount of jargon used in papers. Therefore, we believe it is a good idea to cover the origins and motivation to introduce anisotropic scale invariant spacetimes.

2.2.1 Regularization, renormalization basics

Assume we are studying a QFT given by a Lagrangian involving coupling constants like masses and charges. From computations we have done in QFT courses, we know that even the easiest observables we can compute are generally plagued by infinities. In order to overcome this problem one can first impose a *frequency cut-off* Λ for which, we simply ignore all waves above Λ in a Fourier expansion of a field in the theory we consider. That is called *regularization*[5]. If we think in terms of Euclidean signature, i.e. Wick rotating the time component $t \to it$, the frequency cut-off will turn into the *distance cut-off*. Both terms are used interchangeably depending on the area of research (high energy, condensed matter).

Take $\lambda\phi^4$ theory as an example from Zee [32]. It is easy to show that λ^2 correction to the meson-meson scattering amplitude is

$$\mathcal{M} = \frac{1}{2}(-\lambda)^2 i^2 \int \frac{d^4k}{(2\pi)^4}\left(\frac{1}{k^2 - m^2 + i\epsilon}\right)\left(\frac{1}{(K-k)^2 - m^2 + i\epsilon}\right), \tag{2.36}$$

where $K \equiv k_1 + k_2$ is the sum of the momenta of inital particles. It is obvious that for large values of k, the integrand is logarithmic divergent ($d^4k/k^4 \sim dk/k$). By introducing a cut-off Λ (2.36) reads

$$\mathcal{M} = -i\lambda + iC\lambda^2\left[\log\left(\frac{\Lambda^2}{s}\right) + \log\left(\frac{\Lambda^2}{t}\right) + \log\left(\frac{\Lambda^2}{u}\right)\right] + \mathcal{O}(\lambda^3), \tag{2.37}$$

where it was assumed that $m^2 \ll K^2$ so that we have no m^2 in (2.37), C is a numerical constant and the kinematic variables $s \equiv K^2 = (k_1 + k_2)^2$, $t \equiv (k_1 - k_3)^2$, $u \equiv$

[5] There are other types of regularization schemes which are suitable for different cases.

$(k_1 - k_4)^2$ are used (k_3, k_4 are momenta for final particles). By introducing a scale Λ we effectively have increased the number of parameters in our theory which may seem inconvenient at first. However, if we try to do an experiment and measure the interaction we will see that the results will depend on the energy or how hard we smash them into each other. So there is no harm seeing an energy scale around. The parameters $\lambda,\ \Lambda,\ s,\ t,\ u$ are called *bare* values that appear in the Lagrangian. There is also a *physical* value of these parameters $\lambda_P,\ \Lambda_P,\ s_0,\ t_0,\ u_0$, which correspond to the values that are used, found in the experiment. In renormalization, our goal is to write all observables in terms of physical quantities.

Now assume you have measured a physical coupling constant λ_P at a given momentum $s_0,\ t_0,\ u_0$, and we search for the bare coupling constant that gives this physical value for a given Λ. If we can somehow find these different bare values corresponding to physical measurements as $\Lambda \to 0$, we have a *renormalizable theory* in hand, otherwise the theory is *non-renormalizable*. Following the lines of [32] the meson-meson scattering amplitude reads

$$\mathcal{M} = -i\lambda_P + iC\lambda_P^2 \left[\log\left(\frac{s_0}{s}\right) + \log\left(\frac{t_0}{t}\right) + \log\left(\frac{u_0}{u}\right)\right] + \mathcal{O}(\lambda_P^3). \qquad (2.38)$$

Note that the dependence on energy scale cut-off has disappeared as should be the case for a renormalizable theory.

It is also possible that the zero bare coupling constants can have nonzero physical counterparts. A massless particle can gain mass through these *quantum corrections*. In that case one has to start renormalization from a more general action that respects symmetries and degrees of freedom. If this number of extra terms is finite then we have a renormalizable theory, otherwise not.

2.2.2 Renormalization Group

In the previous section we have discussed the difference between bare coupling constants and how to define physical ones through renormalization. The example of meson-meson scattering amplitude (2.38) tells us the well-known fact that the amplitude, in other words coupling constant λ_P, depends on the momentum of particles involved in the interaction. Taking $s_0 = t_0 = u_0 = \mu^2$ to simplify things, (2.38) reads

[32]

$$\mathcal{M} = -i\lambda_P(\mu) + iC\lambda_P^2(\mu)\left[\log\left(\frac{\mu^2}{s}\right) + \log\left(\frac{\mu^2}{t}\right) + \log\left(\frac{\mu^2}{u}\right)\right] + \mathcal{O}(\lambda_P(\mu)^3). \tag{2.39}$$

The simplest and meaningful question to ask here is the behaviour of the coupling constant λ_P at different momentum values $\mu' \sim \mu$. The relation is given by subtraction

$$\lambda_P(\mu') = \lambda_P(\mu) + 3C\lambda_P(\mu)^2 \log\left(\frac{\mu'^2}{\mu^2}\right) + \mathcal{O}(\lambda_P^3), \tag{2.40}$$

which can be cast in the form of a differential equation

$$\mu\frac{d\lambda_P}{d\mu} = 6C\lambda_P^2 + \mathcal{O}(\lambda_P^3). \tag{2.41}$$

This equation, giving us the behaviour of the coupling constant at different momentum values, is called the *renormalization group (RG) flow*.

Motivated by this example, given a QFT we can imagine n coupling constants living in n-dimensional space, each of them satisfying a flow equation

$$\mu\frac{dg_i}{d\mu} = \beta_i(g_1, \cdots g_i), \tag{2.42}$$

where $\beta_i(g)$ is called the *beta function* of the theory. Then depending on the sign of the right hand side, the couplings either grow or shrink as we go to bigger values on the length scale (i.e. decreasing energy).

We can play this game for both renormalizable/non-renormalizable theories. The difference is, in non-renormalizable theories, the couplings will scale with the positive power of energy, i.e. they will grow infinitely as we go through short distance scales (ultraviolet). On the other hand couplings in renormalizable ones will scale with the zeroth power of energy,

The asymptotic behaviour of RG flows are controlled by the fixed points where the beta function approaches to zero $\beta_i(g_*) = 0$. The points that are at large distance scales are called *infrared fixed points* and the ones at smaller are called *ultraviolet fixed points*. These points correspond to scale invariant QFTs, meaning the theory will transform simply under scaling. For Lorentz invariant field theories we may have the conformal symmetry and a conformal field theory (CFT) at these points. The

CFTs are special in the sense that, all QFTs can be thought of as a deformation from the fixed point. In other words, they are like the building blocks for QFTs.

Starting from the most general action with the desired number of degrees of freedom and symmetries

$$S = S_0 + \int d^D x \sum_i g_i \theta_i(x), \quad \text{where } S_0 = \int d^D x (\partial\phi)^2, \tag{2.43}$$

with $\sum_i g_i \theta_i(x)$ indicating all possible interactions. We can now change the distance scale and look for the behaviour of the β function. The operators θ_i can be classified into three different species according to their behaviour near fixed points as we look into larger distance scales

- relevant: flow away from fixed point,
- irrelevant: flow to fixed point,
- marginal: preserves scaling symmetry.

We can classify theories according to their behaviour of operators when we change the energy scale. Although we might have two (microscopically) different systems in hand, at long distances these two different systems might have the same type of relevant operators, i.e. the physical observables are identical. Therefore, they have the same scaling behaviour or they are in the same *universality class*.

There is a special phenomenon at the infrared fixed points of the renormalization group called *2nd-order phase transition*. The phase transitions can be crudely defined as the non-analycity of some thermodynamic quantity. The transition between phases of water is a first order phase transition in which the internal energy changes discontinuously. There is a point, called *critical point*, where the boundary between liquid and gas form of water disappears. Basically you can not differentiate between liquid and gas phases, they seem to exist at the same time and the system becomes scale invariant in some sense.

Another canonical example of critical point is the Curie temperature in ferromagnets. Above the Curie temperature ferromagnets, like iron, do not display magnetization. However, if we place a ferromagnet in an external magnetic field and crank down the system below the Curie temperature, the spins will align themselves in the same direction as the external field and material is magnetised. At the Curie temperature, the state of being magnetised in the up or down direction dies out and the system is again scale invariant in the sense that examining crystal lattice we see spin-up or spin down regions. If we look closely (changing distance scale), we see that these regions have subregions that are opposite in spin variable, i.e. there is no distinguished length scale.

The observables at the critical points have a remarkably simple form because of the scaling symmetry. Let us define an operator $\sigma(x)$, which is 1 or 0 depending on the spin state of an atom at the point x. Then the correlation of two spins is the expectation value $\langle\sigma(x)\sigma(y)\rangle$, called *correlation function*. Away from the critical point, the correlation functions roughly decay exponentially depending on the distance on points $|x-y|$ and a length scale L called *correlation length*,

$$\langle\sigma(x)\sigma(y)\rangle \sim \exp(-|x-y|/L). \tag{2.44}$$

As we advance towards the critical point, the scale invariance of the theory kicks in and dictates a special form for the correlation functions. The 2-point correlation functions have to decay with a power law

$$\langle\sigma(x)\sigma(y)\rangle \sim \frac{1}{|x-y|^{\xi}}. \tag{2.45}$$

where ξ is called the *critical exponent*. Similarly, the higher order correlation functions, *n-point functions*, satisfy power laws at the critical point, with the forms determined by the scale transformations.

In this short section, we have tried to convey some ideas of the renormalization group ideas, phase transitions, fixed points and the scale invariance at fixed points. Our discussion was nowhere rigorous and just focused on the ideas, concepts that are crucial to motivate the study of NR spacetimes with anisotropic scaling.

2.2.3 Lifshitz and Schrödinger backgrounds

In the previous section we have argued that, at fixed points of the RG flow, theories exhibit scale invariance. For Lorentz invariant theories scaling has to act the same way on both space and time,

$$t \to \lambda t, \quad x \to \lambda x. \tag{2.46}$$

However, the scaling behaviour does not have to be isotropic for non-relativistic theories. Instead we could have the *dynamical scaling*, commonly seen in condensed matter systems

$$t \to \lambda^z t, \quad x^i \to \lambda x^i, \; i = 1, \cdots d \text{ and } \; z \neq 1, \tag{2.47}$$

where z is called the *dynamical critical exponent*[6]. The simplest example with non-relativistic scaling $z = 2$, is the good old Schrödinger equation derived from the Lagrangian $\psi^\dagger(\partial_t - \nabla^2)\psi$. Another canonical example arising in critical points of the phase diagrams of certain materials is the Lifshitz field theory

$$\mathcal{L} = \int d^2x \, dt \, \left[(\partial_t \phi)^2 - \kappa (\nabla^2 \phi)^2 \right]. \tag{2.48}$$

This model is studied in the context of strongly correlated electron systems [33, 34] and it lies in the universality class of some important systems.

Following the logic of AdS/CFT correspondence, we should then look for spacetimes which are dual to theories living at fixed points enjoying the anisotropic scale invariance. Besides scale invariance with dilatation generator D, one should also demand the invariance under space and time translations P_i, H, spatial rotations M_{ij}, P and T symmetry. Then these set of generators will satisfy the following algebra of commutators

$$[D, M_{ij}] = 0, \quad [D, P_i] = P_i, \quad [D, H] = zH,$$
$$[M_{ij}, M_{kl}] = 4\eta_{[i[k} M_{l]j]}, \quad [P_i, M_{jk}] = 2\eta_{i[j} P_{k]}. \tag{2.49}$$

(2.50)

By choosing a Maurer-Cartan basis for this solvable group [35]

$$e^r = \frac{dr}{r}, \quad e^i = \frac{dx^i}{r}, \quad e^t = \frac{dt}{r^z}, \tag{2.51}$$

one is lead to the following metric for Lifshitz spacetimes

$$ds^2 = L^2 \left(-\frac{dt^2}{r^{2z}} + \frac{dr^2}{r^2} + \frac{dx_i dx^i}{r^2} \right). \tag{2.52}$$

[6] We can even have anisotropy between spatial components.

which was first constructed in [36]. The Killing vectors of (2.52) satisfying the algebra (2.49) are

$$M_{ij} = -(x_i\partial_j - x_j\partial_i), \quad P_i = -\partial_i, \quad H = -\partial_t, \quad D = -(zt\partial_t + x_i\partial_i + r\partial_r). \tag{2.53}$$

The non-relativistic nature of (2.52) can also be seen from the near boundary behaviour of lightcones. Taking a factor of $1/r^2$ out, (2.52) reads

$$ds^2 = \frac{L^2}{r^2}\left(-\frac{dt^2}{r^{2(z-1)}} + dr^2 + dx_i dx^i\right). \tag{2.54}$$

Now, consider a radial slice at $r = r_*$ with the induced metric

$$ds^2 = \frac{L^2}{r_*^2}\left(-c(r_*)^2 dt^2 + dx_i dx^i\right), \tag{2.55}$$

where $c(r_*) = r_*^{(1-z)}$. Therefore as we approach the boundary, i.e. as $r_* \to 0$ for $z > 1$, the effective speed of light will diverge, forcing lightcones to open up and flatten at the boundary. For $z < 1$ the boundary lightcone closes up leading to the so called *Carroll* theories.

Apart from the Lifshitz symmetries, one could naively demand a simpler non-relativistic symmetry. Instead of adding scale invariance, we can expect a non-relativistic theory that is invariant under rotations, space and time translations and Galilean boosts $x_i \to x_i + v_i t,\ t \to t$ with $i = 1, \cdots, d$. Denoting the Galilean boosts with K_i we have the following commutators

$$\begin{aligned} &[M_{ij}, M_{kl}] = 4\eta_{[i[k}M_{l]j]}, \quad [P_i, M_{jk}] = 2\eta_{i[j}P_{k]}, \\ &[M_{ij}, K_k] = 2\delta_{k[i}K_{j]}, \quad [P_j, K_i] = 0, \quad [H, K_i] = P_i. \end{aligned} \tag{2.56}$$

However this set of commutators is not actually enough for physically relevant systems. This can be seen by the fact that the Lagrangians are changed by a total derivative under the Galilean boosts [37].

As a fundamental example for the action of Galilean boosts, consider the change of action for a free particle under the infinitesimal transformation $\delta x^i = v^i t$,

$$S = \frac{1}{2}\int_{t_1}^{t_2} M\dot{x}_i^2 dt \quad \to \quad \delta S = \frac{d}{dt}\left[M x_i v^i\right]_{t_1}^{t_2}. \tag{2.57}$$

So the Noether current of this transformation will be

$$\begin{aligned} J_G &= -M\dot{x}_i v^i t + M x_i v^i, \\ &= -p_i v^i t + M x_i v^i. \end{aligned} \tag{2.58}$$

On the other hand the Noether current for translations $\delta x^i = a^i$ is

$$J_P = -M\dot{x}_j a^j = -p_j a^j. \tag{2.59}$$

The Poisson bracket of charges of these currents will reveal the structure of the symmetry algebra

$$\begin{aligned} \{Q_G, Q_P\} &= \int d^{D-1}x \left(\frac{\delta Q_G}{\delta x^i} \frac{\delta Q_P}{\delta p^i} - \frac{\delta Q_G}{\delta p^i} \frac{\delta Q_P}{\delta x^i} \right) \\ &= -M v_i a^i, \end{aligned} \tag{2.60}$$

which basically points out the fact that, the non-commutativity of boosts and translations are related to the mass of the particle. Then with this hindsight, the central extension of Galilean algebra reads

$$[P_j, K_i] = 0 \quad \to \quad [P_j, K_i] = -\delta_{ij} N. \tag{2.61}$$

which is called the *Bargmann algebra*. For a single particle $N = M$, and in general N counts the number of particles with a certain mass. There is even more to the story when $d = 1$ or $d = 2$ (meaning we have 1 or 2 spatial dimension and one time dimension). For the latter case, one can introduce 3 central charges M, K, E (we have also defined $M_{12} = J,\ i.j = 1, 2$.)[38, 39, 40]

$$\begin{aligned} [J, K_i] = \epsilon_{ij} K_j, &\quad [J, P_i] = \epsilon_{ij} P_j \\ [J, H] = E, &\quad [K_i, P_j] = \delta_{ij} M \\ [K_i, K_j] = \epsilon_{ij} K, &\quad [K_i, H] = P_i, \quad [P_i, H] = [P_1, P_2] = 0. \end{aligned} \tag{2.62}$$

For this algebra to make sense in group level one should set $E = 0$ [38]. This can be seen from the action of finite rotations on H (assuming $E = e.\mathcal{I}$)

$$\begin{aligned} e^{\theta J} H e^{-\theta J} &= (1 + \theta J + \cdots) H (1 - \theta J + \cdots), \\ &= H - \theta [H, J] \\ &= H - e\theta\mathcal{I}. \end{aligned} \tag{2.63}$$

In order to have the same values for energy at $\theta = 0$ and $\theta = 2\pi$ one should set $E = 0$. Before leaving the discussion of the $d = 2$ algebra, let us make a final comment about the possible physical applications. By defining $X_i = K_i/M$, we can modify the commutators $[X_i, P_j] = \delta_{ij}$ and $[X_i, X_j] = \epsilon_{ij} K/M^2$ and in this new phase space variables the coordinates are non-commutative. The non-commutative Galilean invariant models have been studied [39, 41, 42] (see the references therein) based on this extended algebra.

The Bargmann algebra that is conformally extended to include dilatations of the form (2.47) is called Schrödinger algebra and is denoted by $\mathtt{sch_z}(d)$ [43, 44]. First performed in [45], the idea of geometrizing the Schrödinger algebra for $z = 2$ is motivated by the symmetry properties of fermions at unitarity [46, 47]. Being a strongly coupled system, fermions interact with a short ranged potential that is fine-tuned to support a zero-energy bound state and is scale invariant in the limit of zero range potential. Then following AdS/CFT correspondence once again, one should be able to investigate the system with its gravity dual that realizes the Schrödinger symmetry geometrically.

There is a special $z = 2$ value which makes $\mathtt{sch_2}(d)$ a subset of conformal algebra $O(2, d{+}2)$, therefore introducing a special conformal transformation. This fact can be deduced from the massless Klein-Gordon equation in $d + 2$ dimensional Minkowski spacetime [45]

$$\Box\phi \equiv -\partial_t^2\phi + \sum_{i=1}^{d+1} \partial_i^2\phi = 0. \tag{2.64}$$

Introducing light-cone coordinates

$$x^{\pm} = \frac{x^0 \pm x^{d+1}}{\sqrt{2}}, \tag{2.65}$$

(2.64) reads

$$\left(-2\frac{\partial}{\partial x^-}\frac{\partial}{\partial x^+} + \sum_{i=1}^{d} \partial_i^2\right)\phi = 0. \tag{2.66}$$

Identifying $\partial/\partial x^- \equiv -im$, we have the following Schrödinger equation with x^+ acting as time coordinate

$$\left(2im\frac{\partial}{\partial x^+} + \sum_{i=1}^{d} \partial_i^2\right)\phi = 0. \tag{2.67}$$

Several comments are in order. First the equation (2.67) has a scaling symmetry (2.47) with $z = 2$, i.e. $\mathtt{sch}_2(d)$. Moreover, massless Klein-Gordon equation is conformal invariant (no scale) which makes $\mathtt{sch}_2(d)$ a subgroup of $O(2, d+2)$. Therefore the $\mathtt{sch}_2(d)$ algebra can be embedded into the conformal algebra (2.32) with the following identifications [45][7]

$$
\begin{aligned}
&M = \tilde{P}^+,\ H = \tilde{P}^-,\ P^i = \tilde{P}^i,\ M^{ij} = \tilde{M}^{ij},\\
&K^i = \tilde{M}^{i+},\ D = \tilde{D} + \tilde{M}^{+-},\ C = \frac{\tilde{K}^+}{2},
\end{aligned}
\tag{2.68}
$$

where $\tilde{P}^+ \equiv (\tilde{P}^0 + \tilde{P}^{d+1})/\sqrt{2}$. The non-zero commutators of $\mathtt{sch}_2(d)$ are

$$
\begin{aligned}
&[M_{ij}, M_{kl}] = 4\eta_{[i[k}M_{l]j]}, \quad [P_i, M_{jk}] = 2\eta_{i[j}P_{k]},\\
&[M_{ij}, K_k] = 2\delta_{k[i}K_{j]}, \quad [P_j, K_i] = -\delta_{ij}M, \quad [H, K_i] = P_i,
\end{aligned}
\tag{2.69}
$$

$$
[D, P_i] = -P_i, \quad [D, K_i] = K_i, \quad [D, H] = -2H, \tag{2.70}
$$

$$
[D, C] = 2C, \quad [H, C] = D. \tag{2.71}
$$

The central charge M of the Bargmann algebra again commutes with all of the generators in $\mathtt{sch}_2(d)$. Moreover, the triplet C, H, D forms an $\mathtt{sl}(2, \mathbb{R})$ algebra. Unlike AdS or Lifshitz case, the symmetries of $\mathtt{sch}_2(d)$ is realised by the following $d+2$ dimensional metric

$$
ds^2 = -\frac{2(dx^+)^2}{r^4} + \frac{-2dx^+dx^- + dx^idx^i + dr^2}{r^2} \tag{2.72}
$$

The generators (2.68) generates the following isometries of the metric [45]

$$
\begin{aligned}
&P^i : x^i \to x^i + a^i,\ H : x^+ \to x^+ + a,\ M : x^- \to x^- + a,\\
&K^i : x^i \to x^i - a^i x^+,\ x^- \to x^- - a^i x^i,\\
&D : x^i \to (1-a)x^i,\ r \to (1-a)r,\ x^+ \to (1-a)^2 x^+,\ x^- \to x^-,\\
&C : r \to (1 - ax^+)r,\ x^i \to (1 - ax^+)x^i,\ x^+ \to (1 - ax^+)x^+,\\
&\quad x^- \to x^- - \frac{a}{2}(x^i x^i + r^2).
\end{aligned}
\tag{2.73}
$$

From the action of generators on coordinates (2.73), we see that the null coordinate x^- is related with the particle number. In non-relativistic theories the mass has generally discrete spectrum. By compactifying the coordinate x^- we can achieve discrete spectrum, yet this approach has its drawbacks [48].

[7] Note that we have denoted the generators of conformal algebra with tilde signs $\tilde{M}_{ij}$.

Unlike the $z = 2$ case, for a generic exponent $z \neq 1, 2$, it is not possible to extend the Bargmann algebra to include special conformal generator C to the $\mathtt{sch_z}(d)$. Yet, it is possible include the dilatations with the following change of commutators in (2.71)

$$[D, H] = -zH, \quad [D, M] = (z-2)M, \quad [D, K_i] = (z-1)K_i, \tag{2.74}$$

where we also have set $C = 0$. It is apparent that for generic values of z, the particle number M is not a central extension. The metric corresponding to $z \neq 1, 2$ is

$$ds^2 = -\frac{2(dx^+)^2}{r^{2z}} + \frac{-2dx^+dx^- + dx^i dx^i + dr^2}{r^2}. \tag{2.75}$$

In this section we have discussed spacetimes that are possible candidates of gravity duals of non-relativistic theories. Although our focus in this thesis will be mainly on Lifshitz spacetimes, we also have discussed the important example of Schrödinger spacetimes. Both of these spacetimes possess symmetries that are crucial for non-relativistic scale invariant field theories. The main difference being the particle number generator M. Theories exhibiting Lifshitz symmetries will have particle production (since particle number is not conserved). On the other hand Schrödinger spacetimes with dynamical exponent $z = 2$ are endowed with a conserved particle number. Our next step will be to check the particle motion in these spacetimes.

2.3 Particle Motion in Lifshitz and Schrödinger spacetimes

In order to discuss geodesic motion of particles in Lifshitz spacetimes let us first construct the conserved quantities along a geodesic [49]. If ξ^μ is a Killing vector and γ is a geodesic with a tangent vector u^μ, then $\xi_\mu u^\mu$ is conserved along the geodesic γ. This fact can be easily seen from the following equality

$$u^\mu \nabla_\mu (\xi_\nu u^\nu) = u^\mu u^\nu \nabla_\mu \xi_\nu + \xi_\nu u^\mu \nabla_\mu u^\nu = 0, \tag{2.76}$$

where the first term in the equality is zero by Killing equation and the second term vanishes by the geodesic equation. The timelike and spatial Killing vectors of Lifshitz spacetime will then amount to the following conserved quantities

$$E \equiv \left(\frac{r}{L}\right)^{-2z} \dot{t}, \quad P_i \equiv \left(\frac{L}{r}\right)^2 \dot{x}_i, \tag{2.77}$$

where we have parametrized the geodesic tangent vector with τ with the components $u(\tau)^\mu = (t(\tau), r(\tau), x_i(\tau))$ and dot denotes the derivative with respect to τ. The norm of the tangent vector is either timelike, spacelike or null $u_\mu u^\mu = -\kappa$, $\kappa = \pm 1, 0$, respectively. Replacing the conserved quantities in the norm of the tangent vector, we have the following equation for radial behaviour of geodesics

$$\left(\frac{L}{r}\right)^{2(z+1)} \dot{r}^2 = E^2 - V_{\text{eff}}, \tag{2.78}$$

where the effective potential for geodesics is [50]

$$V_{\text{eff}} = \left(\frac{L}{r}\right)^{2z} \kappa + \left(\frac{L}{r}\right)^{2(z-1)} \mathbf{P}^2. \tag{2.79}$$

As a warm-up, consider $z = 1$ case, which corresponds to the AdS in Poincaré patch (2.12). The spatial momenta term in the effective potential becomes constant and depending on the value of κ we have different behaviours. For timelike geodesics $\kappa = 1$ (2.78) reads

$$\dot{r}^2 = \frac{r^4}{L^4}(E^2 - \mathbf{P}^2) - \frac{r^2}{L^2} \tag{2.80}$$

with $E^2 - \mathbf{P}^2 \equiv M^2 > 0$. The radial geodesics have turning points, i.e. $\dot{r} = 0$ at $r_{\text{max}} = \pm L/M$, meaning they can not reach the boundary situated at $r = 0$. For null geodesics $\kappa = 0$, V_{eff} is just a constant. Therefore null geodesics in AdS can reach the boundary and turn back in a finite time $r(t) = \pm(M/E)t$.

The behaviour of geodesics in Lifshitz spacetimes depends heavily on V_{eff}. For timelike ones the transverse momentum term in V_{eff} diverges as $r \to 0$, but the first term diverges faster, making timelike geodesics turn around some minimum value like their counterparts in AdS. The null geodesics can reach the boundary in a finite time if their transverse momenta $\mathbf{P} = 0$, i.e. they are radial. The non-radial geodesics will experience a potential forcing them to turn back at some minimum value. This has important consequences on the holography of Lifshitz spacetimes: An observer at the boundary of spacetime will not be able to receive the signals with transverse momenta making the reconstruction of the bulk from the boundary impossible [50, 51, 52].

The other side, $r \to \infty$ of the Poincaré like coordinate system (2.52) is incomplete in the sense that any timelike geodesic crosses the horizon in finite time, just like in AdS. However in AdS we already know that this is a relic of coordinate system (Poincaré patch). It is possible to extend the spacetime by choosing a global coordinate patch. To answer the question whether there is a similar extension for Lifshitz spacetime, one can look for the tidal forces, i.e. how the separation between family of geodesics behave as we approach $r \to \infty$.

Consider a congruence of geodesics with parameter τ and a tangent vector V^μ with the separation vector ξ^μ, then the geodesic deviation is given by

$$\frac{D^2\xi^\alpha}{D\tau^2} = -K^\alpha{}_\beta\,\xi^\beta, \tag{2.81}$$

where $K^\alpha{}_\mu = R^\alpha{}_{\beta\mu\nu}V^\beta V^\nu$. To simplify the discussion assume we have two parallel and radial geodesics. Plugging in (2.52) the transverse components yield

$$\frac{D^2\xi^i}{D\tau^2} = -\xi^i\left(\frac{\dot{r}}{r}\right)^2 + \frac{1}{L^2}\left(\frac{L}{r}\right)^{2z} z\xi^i\dot{t}^2. \tag{2.82}$$

Employing previous definitions for energy and norm of tangent vector of geodesic, we have the following [50]

$$\frac{D^2\xi^y}{D\tau^2} = \xi^y\frac{E^2}{L^2}\left[(1-z)\left(\frac{r}{L}\right)^{2z} - \frac{\kappa}{E^2}\right]. \tag{2.83}$$

For $z = 1$, observers reach $r = \infty$ in finite proper time without any singularity, signalling geodesic completeness of AdS. The tidal force becomes divergent as $r \to \infty$ when $z > 1$, so Lifshitz spacetimes have singularities, falling observers experience infinite tidal forces, i.e. they are geodesically incomplete.

Although we won't discuss in detail, things are more peculiar for Schrödinger spacetimes. First of all, neither timelike nor null geodesics reach the boundary of the spacetime [53]. However tidal forces are more forgiving, behaving like $(z-1)r^{4-2z}$. Therefore they are divergent between $1 < z < 2$ and finite for $z \geq 2$, making the global coordinate extension of Schrödinger spacetimes possible [53].

In this chapter we have summarised the symmetry and causal properties of various spacetimes. Our starting point was maximally symmetric AdS spacetime which has quite unique properties e.g. conformal completion, behaviour of geodesics that we have discussed. We then consider our main objective, non-relativistic spacetimes. By deforming the conformal algebra we introduced anisotropic scaling for NR spacetimes. The Lifshitz algebra is simple, besides the usual generators of rotations and translations, it accommodates the dilatation generator. On the other hand, the algebra of Schrödinger spacetimes has more structure. Addition of Galilean boosts introduces a central charge M related with particle number. Finally, we have checked the geodesics and causal properties of Lifshitz spacetimes. In fact, the geodesics do not exhibit the same properties as their AdS counterparts, making the definition of holographic dictionary quite difficult.

CHAPTER 3

EINSTEIN-YANG-MILLS-LIFSHITZ SOLUTIONS

In this chapter we will try to answer whether it is possible to support Lifshitz spacetime with non-abelian matter. In order to tame the problem into a more palpable form, we will make use of the symmetries of the spacetime and gauge fields.

In the search for solutions one could bluntly insert unknown functions to field equations and try to find a solution to those coupled, highly nonlinear equations. Moreover Einstein-Yang-Mills (EYM) system inherits other difficulties to tackle. On one hand we have diffeomorphism invariance of gravity, on the other gauge invariance of YM theory introducing extra degrees of freedom. Remedy of this mess will be the symmetries we impose on the solution. First, to keep things as simple as possible we will assume static fields, i.e. there is no change in time. The Achilles' heel of this problem will be the planar symmetry. By imposing planar symmetry both on metric and the gauge field we will greatly simplify the problem. This is easier said than done, especially for the gauge field. One must conceive how symmetries on the spacetime manifold manifest themselves on gauge fields. This process will be carried out in detail in this thesis. After fixing the gauge for planar symmetric $SU(2)$ gauge field, we will look for the configurations that support Lifshitz spacetime as a background. This chapter is based on the work that is published in [4].

The outline of this chapter is as follows: Sec. 3.1 starts with the problem of finding the most general form of gauge fields that respect the given spacetime symmetry. After presenting the procedure, an easy $U(1)$ example will be given Sec. 3.1.1. Then the spherically symmetric and planar symmetric $SU(2)$ ansatz will be discussed. Having dealt with symmetric gauge fields, we then describe in Sec. 3.2 how to use those to reduce the fields equations into a simpler form, invoking Palais' Symmetric Criticality theorems. Starting from Sec. 3.3, we begin constructing the EYM solutions. In Sec. 3.3.1 we will show that it is possible to find a planar symmetric configuration with a Lifshitz background. Later, in Sec. 3.5 we will find series and numerical black hole solutions and investigate their thermal properties in Sec. 3.6.

3.1 Gauge fields and Symmetry

In this section we will briefly review the interplay between the symmetries of spacetime and gauge fields. In what follows we will make use of the construction given in [55, 56], although the problem had been worked out by mathematicians long before [57, 58]. Important examples are 't Hooft-Polyakov monopole [59, 60] and Witten's multi-instanton ansatz [61], which we will make use in the search for a EYM solution that exhibits $SO(3)$ symmetry. By introducing a systematic way, we will be able to generate different type of Ansätze for different symmetry groups. Furthermore there are far-reaching and interesting applications of this construction, like reduction of the Yang-Mills action to a gauge theory with Higgs fields in lower dimension if the metric of the spacetime manifold has the same symmetries as the gauge fields. To keep the discussion short, we will avoid calculational details and give a simple example of $U(1)$ gauge field with planar symmetry at the end of this section.

Let σ be a mapping of D dimensional manifold $\mathcal{M}$ onto itself

$$\sigma : \mathcal{M} \to \mathcal{M}; \qquad x \longmapsto \bar{x} \tag{3.1}$$

In this general form σ could define an action of the symmetry group over the spacetime manifold $\mathcal{M}$. In order to use Lie derivative in construction, we will focus on the infinitesimal version of σ defined by the vector field $\xi^\mu(x)$.

$$\bar{x}^\mu = x^\mu + \epsilon\xi^\mu(x) \tag{3.2}$$

As well known, under this action, change in the objects living on manifold $\mathcal{M}$ can be computed by the Lie derivative which is given by

$$\mathcal{L}_\xi T^{\rho\sigma\cdots}_{\mu\nu\cdots} = \xi^\lambda\partial_\lambda T^{\rho\sigma\cdots}_{\mu\nu\cdots} + (\partial_\mu\xi^\lambda)T^{\rho\sigma\cdots}_{\lambda\nu\cdots} + (\partial_\nu\xi^\lambda)T^{\rho\sigma\cdots}_{\mu\lambda\cdots} + \cdots - (\partial_\lambda\xi^\rho)T^{\lambda\sigma\cdots}_{\mu\nu\cdots} - \cdots . \tag{3.3}$$

As in the case of Killing vectors of a metric, we say a scalar, vector or tensor possesses the symmetry generated by $\xi^\mu(x)$ when its Lie derivative along $\xi^\mu(x)$ vanishes

$$\mathcal{L}_\xi V_\mu = 0. \tag{3.4}$$

The gauge fields are different objects, with one leg on the manifold $\mathcal{M}$ and the other on the group manifold, thus they behave differently. Now, consider the gauge group G with generators T^a satisfying $[T^a, T^b] = g^{abc}T^c$ (where g^{abc} are called structure constants of the gauge group, not of the spacetime symmetry group) and let the normalization be given by $\text{Tr}(T^aT^b) = 2\delta^{ab}$. Under a gauge transformation with the group element $g(x) \in G$ the gauge field $A_\mu(x) = A_\mu(x)^a T^a$ transforms as

$$A^g_\mu(x) = gA_\mu(x)g^{-1} + (\partial_\mu g)g^{-1}. \tag{3.5}$$

Expanding the group element g around identity will yield in the first order

$$\begin{aligned}
A^g_\mu(x) &= (1 + W^a(x)T^a)A_\mu(x)(1 - W^b(x)T^b) \\
&\qquad + \left[\partial_\mu(1 + W^c(x)T^c)\right](1 - W^d(x)T^d), \\
&= A_\mu(x) + (\partial_\mu W(x) + [A_\mu(x), W(x)]), \\
&= A_\mu(x) + D_\mu W(x),
\end{aligned} \tag{3.6}$$

where $D_\mu \equiv \partial_\mu + [A_\mu, \quad]$ and $W(x) = W^a(x)T^a$ is in the Lie algebra of G. The right hand side of the symmetry expression for gauge fields will not be zero. Since we have found that the gauge fields are equivalent up to a total divergence of a Lie algebra valued function, (3.4) will turn into

$$\mathcal{L}_\xi A_\mu = D_\mu W(x). \tag{3.7}$$

This is something expected, somehow the extra structure we imposed must appear. We will further restrict the function $W(x)$ by assuming that both sides of (3.7) transforms in the same way. Before that, let us massage the left side of (3.7). Using the explicit form of the Lie derivative, (3.7) can be cast in the form

$$\mathcal{L}_\xi A_\mu = \xi^\rho F_{\rho\mu} + D_\mu(\xi^\rho A_\rho), \qquad \text{where} \quad F_{\rho\mu} = \partial_\rho A_\mu - \partial_\mu A_\rho + [A_\rho, A_\mu]. \tag{3.8}$$

With (3.7) we can relate the field strength and the function $W(x)$ as

$$\xi^\rho F_{\rho\mu} = D_\mu \phi; \qquad \phi(x) \equiv W(x) - \xi^\rho A_\rho. \tag{3.9}$$

In this form, the transformation of $W(x)$ is more apparent. Under gauge transformations, the gauge field and field strength transforms as

$$\begin{aligned} A_\mu(x) &\Longrightarrow gA_\mu(x)g^{-1} + (\partial_\mu g)g^{-1}, \\ F_{\mu\nu} &\Longrightarrow g^{-1}F_{\mu\nu}g. \end{aligned} \tag{3.10}$$

In order to have a gauge covariance, ϕ must transform as $\phi \Longrightarrow g^{-1}\phi g$, which in turn implies that

$$W(x) \Longrightarrow g^{-1}W(x)g + \xi^\alpha g^{-1}\partial_\alpha g. \tag{3.11}$$

An interesting analogy with the Riemannian geometry is possible here. The commutator of derivatives is equal to the Riemann tensor

$$[\nabla_\mu, \nabla_\nu]\xi_\alpha = R_{\mu\nu\alpha}{}^\rho \xi_\rho. \tag{3.12}$$

When ξ^μ are Killing vectors, from elementary properties of Killing vectors it is easy to show that

$$R_{\nu\mu\alpha}{}^\rho \xi_\rho = \nabla_\alpha \nabla_\nu \xi_\mu. \tag{3.13}$$

To make the connection with the first equation in (3.9) obvious, let us supress the two indices ν, μ

$$R_{..\alpha}{}^\rho \xi_\rho = \nabla_\alpha(\nabla_. \xi_.). \tag{3.14}$$

Now, both (3.14) and (3.9) expresses the same property, i.e. the projection of curvatures along the generator of coordinate transformation is equal to the gradient of some quantity. In the case of Riemannian geometry that “some quantity” is the derivative of the generator again. On the other hand, for non-Abelian gauge theories that quantity is not specified. In some sense Riemannian geometry has more structure [56].

Another simple observation we can make is the case of one symmetry generator $\xi^\mu(x)$. Having one vector field we can always choose a frame so that $\xi^\mu(x) = (1, 0, 0, \cdots)$. Then our symmetry equation will simplify to

$$\partial_0 A_\mu = \partial_\mu W - A_\mu W + W A_\mu. \tag{3.15}$$

The goal is to make $W = 0$ by choosing a suitable g, so that A_μ will be independent of x^0. Consider (3.11), the choice

$$g(x) = (\text{Const}) \exp\left[-\int_0^{x_0} W(y, \cdots) dy \right] \tag{3.16}$$

sets $W = 0$. In this gauge we see that A_μ is independent of x^0. However when there are several symmetries present, life is not that easy. One can not simultaneously make all components of several generators zero in one frame.

Let us consider several symmetries with ξ^μ_m where $1 \leq m \leq N$. In this case our symmetry equation will modify into

$$\mathcal{L}_{\xi_m} A_\mu = D_\mu W_m, \tag{3.17}$$

with ξ's satisfying an algebra i.e.

$$[\xi_m, \xi_n]^\mu \equiv \xi^\rho_m \partial_\rho \xi^\mu_n - \xi^\rho_n \partial_\rho \xi^\mu_m = f_{mnp} \xi^\mu_p. \tag{3.18}$$

Notice that, we keep the structure constants of the algebra of spacetime symmetry generators and algebra of gauge group generators different for now. In order to use this algebra, let us consider the commutator of Lie derivatives

$$(\mathcal{L}_{\xi_m} \mathcal{L}_{\xi_n} - \mathcal{L}_{\xi_n} \mathcal{L}_{\xi_m}) A_\mu = \mathcal{L}_{\xi_m}(D_\mu W_n) - \mathcal{L}_{\xi_n}(D_\mu W_m). \tag{3.19}$$

After a little manipulation, (3.19) reduces to

$$\mathcal{L}_\eta A_\mu = D_\mu(\mathcal{L}_{\xi_m} W_n - \mathcal{L}_{\xi_n} W_m + [W_m, W_n]) = D_\mu W_\eta. \tag{3.20}$$

Assuming $D_\mu W_\eta \neq 0$, we require for consistency that

$$\mathcal{L}_{\xi_m} W_n - \mathcal{L}_{\xi_n} W_m + [W_m, W_n] - f_{mnp} W_p = 0. \tag{3.21}$$

First, solving the above consistency condition with a given set of coordinate transformations and employing the equation (3.7) for A_μ, we will be able to get rid of the extra degrees of freedom in symmetric gauge fields. Note that, there is a special case here. If the Lie derivatives of W_f and W_g are constant, i.e. $\mathcal{L}_{\xi_m} W_n = \mathcal{L}_{\xi_n} W_m = 0$, then the gauge functions satisfy

$$[W_m, W_n] = f_{mnp} W_p = W_\eta. \tag{3.22}$$

Remember that f_{mnp} is the structure constant of the spacetime symmetry group. Therefore, we have basically embedded the spacetime symmetries into the gauge group, making the compensating gauge transformations global.

Before moving into the examples let us discuss about the reduction of Yang-Mills actions into the Yang-Mills-Higgs system. After some manipulations, the consistency conditions (3.21) and the definition of gauge covariant scalar (3.9) will lead to

$$\xi_m^\mu D_\mu \phi_n - \xi_n^\mu D_\mu \phi_m + [\phi_m, \phi_n] - \phi_k = \xi_m^\mu \xi_n^\nu F_{\nu\mu} \tag{3.23}$$

where $[\xi_m, \xi_n] \equiv \xi_k$ and ϕ_k is the related scalar with ξ_k. Finally using the projection (3.9)

$$\xi_n^\mu \xi_m^\nu F_{\nu\mu} = [\phi_m, \phi_n] - \phi_k. \tag{3.24}$$

This powerful equality states that certain projections of curvature are directly related to the scalars. Following the lines of [56], one can decompose the Yang-Mills action as follows: Assume the gauge fields are invariant under coordinate transformations that are the elements of some submanifold $\mathcal{M}'$. Then, we can separate the indices μ, ν parallel to $\mathcal{M}'$ and its complement $\mathcal{M}''$, i.e. $\mathcal{M} = \mathcal{M}' \cup \mathcal{M}''$. The Yang-Mills action will decompose as

$$\begin{aligned}
\mathcal{L} =& \frac{1}{2} \operatorname{Tr} F^{\mu\nu} F_{\mu\nu}, \\
=& \frac{1}{2} \operatorname{Tr} F^{||\,||} F_{||\,||} + \operatorname{Tr} F^{||\,\perp} F_{||\,\perp} + \frac{1}{2} \operatorname{Tr} F^{\perp\,\perp} F_{\perp\,\perp}, \\
=& \frac{1}{2} \operatorname{Tr} F^{\perp\,\perp} F_{\perp\,\perp} + (D_\perp \phi)(D^\perp \phi) - \operatorname{Tr} \frac{1}{2} \left([\phi, \phi] - \phi \right)^2,
\end{aligned} \tag{3.25}$$

where in the final line we have used (3.24), (3.9). The first term in (3.25) is the pure Yang-Mills in submanifold $\mathcal{M}'$, the second term is a kinetic term for Higgs field and the final one is the potential for the Higgs field. This type of splits has been studied along the equivalence of solutions and obtaining Weinberg-Salam model from a pure Yang-Mills theory [56, 61, 62].

3.1.1 An easy example

In this section we will discuss a simple, well-known application of the procedure we have introduced. The problem is simple: *What is the form of the vector potential of a uniform magnetic field in the z direction ?* Answer is well-known and repeatedly used in electromagnetic theory and quantum mechanics courses. Stated in more "rigorous" way we need to find a $U(1)$ gauge field, invariant under $E(2)$, that is rotations about the z axis and translations along x or y. Generators of $E(2)$ are simple to guess

$$P_Y = -\frac{\partial}{\partial y}, \qquad P_X = \frac{\partial}{\partial x}, \qquad M = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}, \tag{3.26}$$

with P_X, P_Y describing the translations and M is the rotation about the z axis. Written in polar coordinates

$$P_Y = -\sin\varphi\frac{\partial}{\partial\rho} - \frac{\cos\varphi}{\rho}\frac{\partial}{\partial\varphi}, \qquad P_X = \cos\varphi\frac{\partial}{\partial\rho} - \frac{\sin\varphi}{\rho}\frac{\partial}{\partial\varphi}, \qquad M = \frac{\partial}{\partial\varphi}. \tag{3.27}$$

In addition, we will make use of the commutation relations of generators

$$[P_X, P_Y] = 0, \quad [P_X, M] = -P_Y, \quad [P_Y, M] = P_X. \tag{3.28}$$

Let us turn back to (3.21) and make use of the machinery we have developed in the previous section. Since $U(1)$ is an Abelian gauge group, (3.21) will simplify into

$$\mathcal{L}_{\xi_m} W_n - \mathcal{L}_{\xi_n} W_m = f_{mnp} W_p. \tag{3.29}$$

From the transformation properties of W's we see that for the rotation generator M

$$W_M \Longrightarrow g^{-1} W_M g + g^{-1} M^\mu \partial_\mu g, \quad M^\mu = (0, 1). \tag{3.30}$$

Here we have made our first choice and set $W_M = 0$ (Note that, we have fixed only the φ part of $g(\rho, \varphi)$). The choice made here is quite general, for most of the problems the rotation generator part of W is set to zero. In this example that choice makes the gauge field A_μ independent of φ, i.e.

$$\mathcal{L}_M A_\mu = 0. \tag{3.31}$$

With this choice we can assume the following form of A_μ

$$A = A_\rho(\rho) d\rho + A_\varphi(\rho) d\varphi. \tag{3.32}$$

One could push further and make use of (3.5) and make $A_\rho(\rho) = 0$ by fixing the residual ρ dependence on $g(\rho, \varphi)$. Then (3.32) reduces to having one component only:

$$A = A_\varphi(\rho) d\varphi. \tag{3.33}$$

The remaining equations for P_X, P_Y read

$$\mathcal{L}_M W_{P_X} = W_{P_Y} \tag{3.34}$$

$$\mathcal{L}_M W_{P_Y} = -W_{P_X}. \tag{3.35}$$

These will generate the first order coupled equations with solution

$$W_{P_Y} = f(\rho)\cos\varphi \tag{3.36}$$

$$W_{P_X} = f(\rho)\sin\varphi. \tag{3.37}$$

The explicit forms of W_{P_Y}, W_{P_X} can be used to extract differential equations from

$$\mathcal{L}_{P_Y} A_\mu = \partial_\mu W_{P_Y}, \qquad \mathcal{L}_{P_X} A_\mu = \partial_\mu W_{P_X}. \tag{3.38}$$

(3.38) amounts to two simple differential equations

$$f(\rho) - \partial_\rho A_\varphi(\rho) + \frac{A_\varphi(\rho)}{\rho} = 0, \qquad \frac{df}{d\rho} = \frac{A_\varphi(\rho)}{\rho^2}, \tag{3.39}$$

with a solution

$$A(\rho) = c_1 + c_2\rho^2. \tag{3.40}$$

We cast this in a more familiar form by turning to Cartesian coordinates

$$A = B\rho^2 d\phi = B(xdy - ydx), \tag{3.41}$$

where B is a constant. In this simple problem we have actually used a sledgehammer to crack a nut, but this is a nice, simple example to practice on this machinery. The power of this can be seen on non-abelian groups with complicated symmetries. In the next section we will discuss about $SO(3)$ invariant $SU(2)$ gauge field ansatz given by Witten [61].

3.1.2 Spherically and Planar Symmetric $SU(2)$ Ansatze

Witten gave a seminal ansatz for the $SU(2)$ gauge field with rotational symmetry in [61]. We will make use of this ansatz to simplify EYM equations, however we will not expound it in here. Derivation with the construction we have introduced can be found in [55] or [56]. [55] uses the coset techniques which is rather complicated, whereas [56] directly solves the differential equations as we did in the previous section.

The following static $SU(2)$ connections are invariant under $SO(3)$ and the connected part of $SO(2,1)$ [55, 61]

$$A = q(r)T^3 dt + p(r)T^3 dr + \big(w(r)T^1 + u(r)T^2\big)d\theta \\ + \big(w(r)\Omega_k(\theta)\, T^2 - u(r)\Omega_k(\theta)\, T^1 + \tilde{\Omega}_k(\theta)T^3\big)d\phi, \tag{3.42}$$

for $k = 1, -1$, where $\Omega_1(\theta) \equiv \sin\theta$, $\Omega_{-1}(\theta) \equiv \sinh\theta$, $\tilde{\Omega}_1(\theta) \equiv \cos\theta$, $\tilde{\Omega}_{-1}(\theta) \equiv \cosh\theta$. The values of $k = \pm 1$ control whether the ansatz is symmetric under spherical rotations or hyperbolic rotations.

This expression still has a $U(1)$ gauge freedom [63], which can be used to set $u(r) = 0$. Next, to simplify the discussion, we will only consider the gauge field strengths with vanishing electric part, i.e. $q(r) = 0$ which is rather restrictive. In fact it was shown in [64] that (assuming appropriate asymptotic) the Reissner-Nordström solution is the only static black hole with non-zero YM electric field. However, all of this was for asymptotically flat backgrounds which obviously does not necessarily apply to Lifshitz spacetimes. Nevertheless, for the sake of simplicity, we shall restrict ourselves to the purely magnetic case in this work. The extension of this work is done in [65], where the authors coupled an extra $U(1)$ field and found exact solutions with colourful vacua in different dimensions.

Taking these considerations into account, we have the simplified version of the ansatz (3.42)

$$A = \begin{cases} w(r)T^1 d\theta + (w(r)\Omega_k(\theta)\, T^2 + \tilde{\Omega}_k(\theta)\, T^3)d\phi; & \text{for } k = \pm 1 \\ w(r)T^1 d\theta + w(r)T^2 d\phi; & \text{for } k = 0 \end{cases} . \tag{3.43}$$

The form of the Lifshitz metric (2.52) we have given in the previous chapter is planar symmetric in the spatial part. This form will be the actual background solution which we will dress with blackholes. The non-abelian gauge field configuration respecting the symmetry of the plane is a subgroup of the Poincaré group and studied extensively in [66, 67]. Additionally, we shall again restrict ourselves to the static and purely magnetic case which leads to the $SU(2)$ gauge connection

$$A_\mu dx^\mu = w(r)T^1 dx_1 + w(r)T^2 dx_2. \tag{3.44}$$

3.2 Palais' Principle of Symmetric Criticality

Having dealt with the procedure for finding group invariant gauge fields and presented the ones we are going to use, let us now discuss how to implement these symmetric fields and further simplify the field equations. The Palais' Principle of Symmetric Criticality (PSC) asserts that given a symmetry, the field equations restricted to fields that are invariant under symmetry action are equivalent to the field equations that are obtained through a symmetry reduced Lagrangian [68]. The PSC has been employed in a variety of cases in gravitational theories [69, 70, 71, 72] involving higher curvature ones and so on.

However one should be careful before implementing PSC, it may not be possible to impose the symmetry on the fields and retrieve correct, reduced field equations. The theory may have non-trivial boundary terms in the restricted variational principle [73, 74] which spoils equivalence. The conditions on the applicability of PSC on gravitational theories depend on the group action regardless of the Lagrangian, spacetime manifold [75, 76].

Now let us give a simple example which will also be useful in the EYM model. The most general spherically symmetric metric in four dimensions in Schwarzschild gauge is given by the following

$$ds^2 = -S(r,t)^2\mu(r,t)dt^2 + \frac{dr^2}{\mu(r,t)} + r^2 d\Omega^2, \tag{3.45}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ is the usual metric on the 2-sphere S^2. This form of the metric is used by Weyl [69] to derive Schwarzschild solution in a very classy way. Plugging (4.179) in cosmological Einstein action amounts to

$$\begin{aligned} I_E &= \int d^4x\sqrt{-g}(R-2\Lambda) \\ &= \int dr\,dt\,d\Omega \Bigg\{ S(r,t)\left[2 - 2r^2\Lambda_0 - 2\mu(r,t) - r^2\mu(r,t)'' - 4r\mu(r,t)'\right] \\ &- \left[4r\mu(r,t)S(r,t)' + 3r^2S(r,t)'\mu(r,t)' + 2r^2\mu(r,t)S(r,t)''\right] \\ &+ \frac{r^2\dot{S}(r,t)\dot{\mu}(r,t)}{(S(r,t)\mu(r,t))^2} + \frac{2r^2S(r,t)\dot{\mu}(r,t)^2}{S(r,t)\mu(r,t)^3} - \frac{r^2\ddot{\mu}(r,t)}{S(r,t)\mu(r,t)^2}\Bigg\} \end{aligned} \tag{3.46}$$

where dot denotes t derivatives and prime indicates derivative with respect to the coordinate r. One can immediately see the consequences of Birkhoff's theorem, i.e.

the last three terms can be cast as a time total derivative boundary term. Therefore there are no time derivative terms in the integrated action. The main objective is to get rid of the second radial derivative terms with integration by parts. After performing necessary integrations, we have the following

$$I_E = \int dr\, d\Omega\, 2S(\mu + r\mu' - 1 + r^2\Lambda) + \text{Boundary Terms.} \tag{3.47}$$

The Schwarzschild-de-Sitter solution easily follows from Euler-Lagrange equations for $\mu(r), s(r)$

$$\mu(r) = 1 - \frac{r^2\Lambda}{3} + \frac{c_1}{r}, \quad S(r) = c_2. \tag{3.48}$$

We have seen that, by employing the PSC theorem and reducing the Lagrangian before varying it, we can extract the simplest form of field equations. This procedure will be really valuable when we couple non-linear Yang-Mills fields to the Einstein sector.

3.3 Lifshitz configurations

In this section we will construct the Lifshitz background supported by the planar symmetric $SU(2)$ gauge field (3.44). As well known, Einstein gravity with a negative cosmological constant does not admit anisotropic backgrounds as a solution. One can consider matter couplings to engineer these backgrounds. One of the first matter configuration considered is the string theory motivated p-form fields in [36], which is conjectured to be the gravitational dual of $2+1$ dimensional field theories modelling quantum critical behaviour in strongly correlated electron systems. In [77] different types of anisotropic backgrounds (including spatial anisotropy) were constructed, with massive gauge fields, $U(1)$ fields with dilaton-like couplings [78] which are themselves better studied models for gravity duals.

Another possibility is to employ higher order derivative terms to source the metric. Once we depart from Einstein gravity and add higher curvature corrections, the amended theories begin to accommodate these backgrounds as a solution [79, 80, 81, 82]. The downside of these theories is that they are not wieldy for the applications of holography, e.g. they don't possess a well defined variational principle which makes the definition quantities, procedures (like conserved charges, holographic renormalisation) somewhat sketchy. Moreover, higher curvature theories may have unitarity, ghost problems which are important for relativistic field theories, though it may not be crucial for the applications of non-relativistic holography.

The matter Lagrangians with non-abelian gauge fields have been employed in holographic superconductor models [83, 84], with AdS/Schwarzschild black hole backgrounds. In [85] Lifshitz scaling on these models were considered. However, in these examples non-abelian matter was used as an extra degree of freedom not to support the background spacetime. In this part of the thesis we will attack the problem of finding the non-abelian matter configuration that will support Lifshitz background. After solving that problem, the next task we undertake is the dressing up of this background solution with black holes.

3.3.1 Colored Lifshitz Vacua

We will consider four dimensional cosmological EYM theory with the gauge group $SU(2)$ described by the action

$$S = \int d^4x\, \sqrt{-g}\Big((R - 2\Lambda) - \frac{1}{2g_{\text{YM}}^2}\mathrm{Tr}\, F_{\mu\nu}F^{\mu\nu}\Big), \tag{3.49}$$

where Λ is the cosmological and g_{YM}^2 is the gauge coupling constant in dimensions of $1/\text{length}^2$.

Note that we have kept the coupling g_{YM}^2 and cosmological constant Λ explicit, in order to determine their dependence on the dynamical critical exponent z, which basically controls the geometry. This is actually quite different from the AdS and asymptotically flat cases [86, 87] in which they are both solutions of the cosmological Einstein and YM field is used only as a hair parameter, not for supporting the geometry. Therefore with hindsight we expect that at conformal limit, i.e. $z = 1$, we expect $g_{\text{YM}}^2 \to \infty$ and two sectors will decouple. Because of the decoupling, it is not possible to recover the results of [86] for a crosscheck.

Einstein field equations following from the action (3.49) read

$$R_{\mu\nu} - \Lambda g_{\mu\nu} = \frac{1}{g_{\text{YM}}^2}T_{\mu\nu}, \tag{3.50}$$

with the traceless YM stress-energy tensor defined as

$$T_{\mu\nu} \equiv \mathrm{Tr}\,(F_{\mu}{}^{\alpha}F_{\nu\alpha} - \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}), \tag{3.51}$$

and the YM field equations

$$D_{\mu}F^{\mu\nu} = 0. \tag{3.52}$$

We can immediately determine the cosmological constant by exploiting the traceless nature of the stress-energy tensor. Taking the trace of (3.50) with respect to the metric (2.52) yields[1]

$$\Lambda = -\frac{3+2z+z^2}{2L^2}. \tag{3.53}$$

The next step is now to bring all ingredients into play and find the field equations of EYM with a non-abelian gauge field configuration respecting the symmetry of the plane. Besides the symmetric gauge field (3.44), we will also employ a plane symmetric metric ansatz that is similar to the spherical one (4.179) we have discussed in PSC theorem. For our purposes it is convenient to consider the following form

$$ds^2 = L^2\Big(-S(r)^2\mu(r)dt^2 + \frac{dr^2}{\mu(r)} + r^2 d\mathbf{x}^2\Big). \tag{3.54}$$

As we have discussed in the previous section, we just plug in the gauge field (3.44) and metric (3.54) into the action (3.49)

$$\begin{aligned} I_{EYM} = \int dr\, 2L^2 S(r)\ \left[\mu(r) + r\mu(r)' + r^2\Lambda\right] \\ + \frac{S(r)}{g^2_{\mathrm{YM}}}\left(\frac{w(r)^4}{2L^2r^2} + \mu(r)w(r)'^2\right). \end{aligned} \tag{3.55}$$

Invoking PSC and varying reduced action we end up with the following field equations

$$S^{-1}S' = \frac{1}{2L^2g^2_{\mathrm{YM}}}\frac{(w')^2}{r}, \tag{3.56}$$

$$(\mu w')' = \frac{w^3}{r^2} - \frac{1}{2L^2g^2_{\mathrm{YM}}}\frac{\mu(w')^3}{r}, \tag{3.57}$$

$$r\mu' + \mu + L^2r^2\Lambda = -\frac{1}{2g^2_{\mathrm{YM}}L^2}\Big(\frac{w^4}{2r^2} + \mu(w')^2\Big), \tag{3.58}$$

[1] Note that in this chapter we will use the transformed form of the metric (2.52) with $r \to 1/r$, which basically maps the boundary from $r = 0$ to $r \to \infty$.

with prime denoting the ordinary derivative with respect to r. The choice $S(r) = r^{z-1}$, $\mu(r) = r^2$ will reproduce Lifshitz spacetimes. Taking this into account and using (3.53) as the cosmological constant value, it is straightforward to show that the Lifshitz spacetime (2.52) is a solution for all $z > 1$ provided that the gauge field and the coupling constant are chosen as

$$w(r) = \pm\sqrt{z+1}\,r, \quad g^2_{\text{YM}} = \frac{1}{2L^2}\frac{(z+1)}{(z-1)}. \tag{3.59}$$

There are various remarks that need to be made at this point. First of all, there is a sign ambiguity in the gauge field which can be deduced from the invariance of the field equations (3.56), (3.57), (3.58) under $w(r) \to -w(r)$. That actually corresponds to a gauge transformation [88]. Hence, in what follows we will choose the positive sign gauge field. We need $z > 1$ in order to have real gauge fields, which signals the "critical slowing down" of the possible dual field theories [36]. As we have pointed out earlier, the conformal limit is special in the sense that the YM part decouples from the gravity action and, as well-known, the AdS spacetime is a solution of (3.50) without matter fields, provided $\Lambda = -3/L^2$. On the other hand the decoupled gauge field is also a solution to the pure YM part, which is in some sense the AdS analogue of the flatspace solution given in [66, 67].

The five dimensional extension to our four dimensional solution was later done in [65] and it was shown that the solution can also be extended to $D = 5$ with the following couplings

$$\Lambda = -\frac{(D-2)[(z^2+(D-2)z+(D-1)]}{4L^2}, \quad g^2_{\text{YM}} = \frac{z+D-3}{(D-2)(z-1)L^2}, \tag{3.60}$$

and the gauge field

$$A = \sum_{a=1}^{3} T^a A^a = w(r)(T^1 dx^1 + T^2 dx^2 + T^3 dx^3), \tag{3.61}$$

where T^a are the Pauli matrices. Having found the non-abelian configuration that supports Lifshitz asymptotics, in the next section we will dress up this background geometry to obtain black hole solutions.

3.4 Heavenly Bodies, Hair, YM solitons

The concept of a "black hole" has its roots even in Newtonian Era. Laplace has discussed existence of such objects and called them *dark stars*. He argued that *"the attractive force of a heavenly body could be so large, that light could not flow out of it"*[29]. The real deal started with the Einstein's theory of gravitation; shortly after its construction Schwarzschild [89] came up with a solution. However this solution was plagued with singularities at the centre $r = 0$ and $r = r_g$, which were not fully understood for almost thirty years. The structure of spacetime was revealed with the pioneering works of Finkelstein, Fronsdal, Kruskal and Szekeres [90, 91, 92, 93].

In the middle of 1960s black hole studies gained an impetus with Kerr's rotating solution [94]. At this era the black hole *"no-hair"* conjecture proposed by Israel, Penrose and Wheeler had a paramount importance on the black hole theory. The conjecture states that:

> *All stationary, asymptotically flat, four dimensional electro-vacuum black hole spacetimes are characterised by their mass, angular momentum and electric charge.*

This conjecture has a striking similarity with a statistical system that is in thermal equilibrium. In statistical physics a system in equilibrium can be described by a small set of variables E, V, T, N (energy, volume, temperature, number of particles) although we have large number of accessible states for each particles. Likewise a static black hole is described by mass, angular momentum and charge, regardless of its microstates if any. Moreover, the black hole mass variation formula [95] and the area increase theorem [29] are analogous to energy variation and second law of thermodynamics (entropy increase). These strong resemblances led people to believe in a generalised version of the no hair conjecture which simply states that all black hole equilibrium configurations are characterised by mass, angular momentum and global gauge field charges. By global charges we mean quantities which are measured from infinity, far from the event horizon.

The proof of this conjecture, as stated above, took some serious effort and required strong assumptions such as stationarity, asymptotic flatness, Einstein equations. If we relax some of the conditions such as adding a cosmological constant, "topological black holes" [96, 97] starts to appear with non-spherical event horizon. That contradicts with Israel's theorem [98, 99] used in the proof of the no-hair conjecture, which states that staticity guarantees spherical symmetry of the horizon. This alone shows that the no-hair conjecture in its original form is too restrictive.

Another way to relax the conditions is to add matter. One of the first examples is the Bocharova-Bronnikov-Melnikov-Bekenstein (BBMB) blackhole [100, 101, 102] which is basically the extremal Reissner-Nordstrom black hole with a conformally coupled scalar field. For our purposes, the related example is the asymptotically flat EYM blackholes [103]. However both examples are highly unstable which saves the no-hair conjecture for stable black holes. On the other hand, asymptotically AdS EYM blackholes with $SU(2)$ gauge fields are actually stable and they require a new parameter to describe the geometry outside of the horizon [86, 104, 105]. Therefore in addition to mass, charge and angular momentum we have another parameter, extending the no-hair conjecture. In some sense the conjecture still holds, i.e. we have finite number of parameters describing the black hole.

Besides the no-hair conjecture there were also a number of reasons that made research on solutions with self gravitating fields a no man's land until 1989.

i) Lichnerowicz theorem [106] proves that there are no gravitational solitons. Likewise Einstein-Maxwell system does not admit solitons [107].

ii) Deser's simple argument [108] proves that there are no static solutions to Yang-Mills equations in four dimensional spacetime. Also he further proved that three dimensional Einstein-Yang-Mills equations do not admit solitons.

These arguments are plausible enough to conjecture that there are no EYM solitons. When Bartnik-McKinnon [103] found globally regular solutions to $SU(2)$ EYM theory, it was a big surprise. Considered on their own both theories are not capable of supporting solitons, but taken together the non-linearities of the gravitational and gauge fields seem to balance themselves.

3.4.1 Einstein Yang-Mills Equations

The importance of black hole solutions in holography comes from the principle that they describe the finite temperature behaviour of those dual field theories. With AdS asymptotics there are beautiful examples of analytic blackholes in various theories and dimensions. Passing from AdS to Lifshitz spacetime we break some of the symmetry and this is no good for finding exact solutions. The analytic black holes with Lifshitz background are actually quite rare. Things get harder especially with a matter coupling [77, 78] for generic dynamical exponent z, [109, 110, 111, 112] for fixed value of z.

Recall that, another possibility to support Lifshitz spacetimes is the curvature corrections. Although this approach opens up the way for large families of analytic black holes in different dimensions [82, 80, 79, 81], again the holographic discussion of these solutions are tricky. On the other hand, it is still possible to learn a great deal of properties from the numerical solutions, which were explored for theories with massive gauge fields and p-forms with generic z values and for different horizon topologies [113, 114, 115, 116, 117]. Motivated by our background solution, let us now gather the ingredients together and investigate the numerical solutions of EYM Lifshitz system thoroughly.

We will start with a more general form of the metric (3.54)

$$ds^2 = L^2\Big(- S(r)^2\mu(r)dt^2 + \frac{dr^2}{\mu(r)} + r^2 d\Omega_k^2\Big), \tag{3.62}$$

where we control the spatial part of the metric with the parameter k. It is clear that, $k = 0$ corresponds to the planar symmetric case, in which we have employed in the background solution, $k = 1$ yields the spherically symmetric metric, and $k = -1$ option is invariant under hyperbolic rotations. We also need to consider the general form of the gauge field, i.e. spherical symmetric ansatz (3.43) for the cases $k = \pm 1$.

Utilising all these, the equations (3.56), (3.57), (3.58) can be put into a more general form covering major topologies [87]

$$S^{-1}S' = \frac{1}{2L^2 g_{\text{YM}}^2}\frac{(w')^2}{r}, \tag{3.63}$$

$$(\mu w')' = \frac{w(w^2-k)}{r^2} - \frac{1}{2L^2 g_{\text{YM}}^2}\frac{\mu (w')^3}{r}, \tag{3.64}$$

$$r\mu' + \mu + L^2 r^2 \Lambda - k = -\frac{1}{2g_{\text{YM}}^2 L^2}\Big(\frac{(w^2-k)^2}{2r^2} + \mu (w')^2\Big). \tag{3.65}$$

Although this form seems compact and useful for the constraints on the fields, for the sake of numerical study it is a bit impractical for the reasons we have described in the appendix. What we need is the first order equations and it can be done by simply redefining the metric and the gauge field functions. It should be done in such a way that the Lifshitz vacuum (2.52) can be explicitly recovered at large radius. One can achieve this with simple redefinitions

$$\begin{aligned} w(r) \equiv& \sqrt{z+1}\, r h(r), \quad \mu(r) \equiv \frac{r^2}{g(r)^2}, \\ S(r) \equiv& r^{z-1} f(r) g(r), \quad w'(r) \equiv \sqrt{z+1}\, j(r). \end{aligned} \tag{3.66}$$

Provided $f(r)$, $g(r)$, $h(r)$, $j(r)$ go to one in the large r limit, i.e. when $r \gg 1$, we recover the Lifshitz background solution for the EYM system with $k = 0$. To make this fact quite clear and point out the differences with asymptotically AdS solutions of EYM let us write down the metric $k = 0$ explicitly

$$ds^2 = L^2\left(-r^{2z} f(r)^2 dt^2 + \frac{dr^2}{r^2} g(r)^2 + r^2 d\mathbf{x}^2\right). \tag{3.67}$$

For comparison, the form of the metric used in the AdS analysis is [86]

$$\begin{aligned} ds^2 =& L^2\Big(- S(r)^2 \mu(r) dt^2 + \frac{dr^2}{\mu(r)} + r^2 d\Omega_k^2\Big), \\ & \text{where} \quad \mu(r) = 1 - \frac{2m(r)}{r} - \frac{\Lambda r^2}{3}, \end{aligned} \tag{3.68}$$

from which you can define the mass of the black hole. Similar to the Lifshitz case $m(r) \to 1$ as we go further from the event horizon. This choice also reflects the fact that the gauge field is a hair parameter in this case, without it the solution is Schwarzschild-AdS. On the other hand, for our case we recover the plain background for the large radius.

The set of transformations (3.66) will yield the following first order system of equations

$$
\begin{aligned}
rf(r)' = &-f(r)\Big((z-1) - \frac{j(r)^2}{2}(z-1) + \frac{g(r)^2h(r)^4}{4}(z^2-1) \\
&\qquad - \frac{g(r)^2}{4}(3+2z+z^2) + \frac{3}{2}\Big) \\
&- kf(r)g(r)^2\Big\{\frac{k}{4r^4}\frac{(z-1)}{(z+1)} - \frac{h(r)^2}{2r^2}(z-1) - \frac{1}{2r^2}\Big\}, && (3.69) \\
rj(r)' = &\; j(r) + g(r)^2h(r)^3(z+1) - \frac{g(r)^2j(r)}{2}(z^2+2z+3) \\
&\qquad + \frac{g(r)^2h(r)^4j(r)}{2}(z^2-1) \\
&- k\Big\{g(r)^2\Big(\frac{h(r)^2j(r)}{r^2}(z-1) - k\frac{j(r)}{2r^4}\frac{(z-1)}{(z+1)} + \frac{j(r)}{r^2} + \frac{h(r)}{r^2}\Big)\Big\}, && (3.70) \\
rg(r)' = &\; \frac{g(r)j(r)^2}{2}(z-1) + \frac{g(r)^3h(r)^4}{4}(z^2-1) - g(r)^3(3+2z+z^2) + \frac{3g(r)}{2} \\
&+ kg(r)^3\Big\{\frac{k}{4r^4}\frac{(z-1)}{(z+1)} - \frac{h(r)^2}{2r^2}(z-1) - \frac{1}{2r^2}\Big\}, && (3.71) \\
rh(r)' = &\; j(r) - h(r). && (3.72)
\end{aligned}
$$

There are important points to discuss here. First, the equations are highly nonlinear which is a common characteristic of the EYM system. This very fact makes the analytic study impossible, and despite our efforts, we couldn't find an exact solution with non-trivial gauge field functions. There are ways to overcome this difficulty. In [65] another $U(1)$ field is coupled to the EYM system and by using $SU(2)$ only to support the Lifshitz spacetime, the new $U(1)$ field is used to construct black holes. Since one does not touch the $SU(2)$ sector, the equations are easy to control. In this work we will only consider $SU(2)$ fields by attacking the set of equations (3.69), (3.70), (3.71), (3.72) numerically.

Secondly, spherical and hyperbolic cases $k = \pm 1$ involve terms with $1/r^2$ and $1/r^4$ which decays at long distances. Motivated by this fact, in the large r limit, we will replace the spherical and hyperbolic parts by a flat one [113, 114], which forces all of the unknown functions that appear in the numerical solution to have the same asymptotic behavior, $f(r) = g(r) = h(r) = j(r) = 1$.

Note that the last three equations form a closed system which will be studied separately. The first equation will be considered afterwards. In addition, the right hand side of the first equation is linear in the function $f(r)$, i.e. we can scale the function $f(r)$ freely. This freedom is essentially a gauge choice, corresponding to the rescaling of the time coordinate [36]. This property will be critical to get the correct asymptotics after the numerical integration. We will choose the initial value of $f(r)$ such that at large distances all functions will go to unity.

In the next section we will lay the foundations for the numerical study. More specifically, we will expand the functions $f(r),\ g(r),\ h(r),\ j(r)$ at large r and separately at the horizon, for all possible values of the parameter k but for a fixed value of z. The asymptotic form of the solutions will yield the behaviour of functions at large r and the possible shooting parameter at the horizon which is of paramount importance.

3.5 Series and Numerical Solutions

In this section, we will look for the series solutions of equations (3.69), (3.70), (3.71), (3.72) by first expanding them at large radius to see how they behave. Later we will expand functions at the horizon and try to extract a shooting parameter for the numerical analysis. After the series expansions, we will move on to the numerical study for various cases.

3.5.1 Series solution for the large radius

After the usual transformation $r = 1/x$ in field equations (3.69), (3.70), (3.71), (3.72), we can proceed with the small x expansions

$$f(r) = \sum_{n=0}^{\infty} \tilde{f}_n x^n, \quad g(r) = \sum_{n=0}^{\infty} \tilde{g}_n x^n, \quad h(r) = \sum_{n=0}^{\infty} \tilde{h}_n x^n, \quad j(r) = \sum_{n=0}^{\infty} \tilde{j}_n x^n. \tag{3.73}$$

Imposing the Lifshitz asymptotics, i.e. $\tilde{f}_0 = \tilde{g}_0 = \tilde{h}_0 = \tilde{j}_0 = 1$, we found that the behaviour of solutions is rather interesting for different values of z. When z is odd there is a solution for all cases $k = 0, \pm 1$. However, when z is even only the planar geometry $k = 0$ survives. This is obtained after plugging in the expansions (3.73) in transformed equations and working order by order in x. Rather than giving the details, we prefer simply to state the results for different cases[2]:

For $z = 2$ and $k = 0$, we have

$$f(r) = 1 - \frac{9h_L}{2r^4} - \frac{1557}{176}\frac{h_L^2}{r^8} + \mathcal{O}(1/r^{16}) + \cdots\,, \tag{3.74}$$

$$g(r) = 1 + \frac{6h_L}{r^4} + \frac{1143}{22}\frac{h_L^2}{r^8} + \mathcal{O}(1/r^{16}) + \cdots\,, \tag{3.75}$$

$$h(r) = 1 + \frac{h_L}{r^4} + \frac{405}{44}\frac{h_L^2}{r^8} + \mathcal{O}(1/r^{16}) + \cdots\,, \tag{3.76}$$

$$j(r) = 1 - \frac{3h_L}{r^4} - \frac{2835}{44}\frac{h_L^2}{r^8} + \mathcal{O}(1/r^{16}) + \cdots\,. \tag{3.77}$$

However, for $z = 3$ and with generic k, the solution is

$$f(r) = 1 + \frac{k}{2r^2} + \frac{127}{1352}\frac{k^2}{r^4} + \mathcal{O}(1/r^5) + \cdots\,, \tag{3.78}$$

$$g(r) = 1 + \frac{23}{676}\frac{k^2}{r^4} + \frac{12h_L}{r^5} + \mathcal{O}(1/r^6) + \cdots\,, \tag{3.79}$$

$$h(r) = 1 - \frac{3}{338}\frac{k^2}{r^4} + \frac{h_L}{r^5} + \mathcal{O}(1/r^6) + \cdots\,, \tag{3.80}$$

$$j(r) = 1 + \frac{9}{338}\frac{k^2}{r^4} - \frac{4h_L}{r^5} + \mathcal{O}(1/r^6) + \cdots\,. \tag{3.81}$$

Note that, in both solutions there is only one parameter h_L that characterises the system at large r. As a side remark, the difference between even and odd is an artefact from the expansion (3.73) we have considered. Note that we have not considered fractional powers which can resolve the issue for even z. Another approach could be to use a different choice of coordinates to investigate the solutions for large r. For the sake of completeness, in numerical part of the calculations we will fix $z = 3$ to cover all possible cases of topologies.

3.5.2 Series solution about the event horizon

Having found the large r expansion and its dependence on a single parameter, let us consider the functions on the presumed horizon. The numerical integration will start from the horizon so the initial values and the bounds on the functions is essential to proceed. Besides the existence of horizon we will assume that the g_{tt} and g_{rr}

[2] To keep the following discussion simple, we only present our findings for the $z = 2$ and $z = 3$ cases. The generic behaviour of the solutions are captured by the $z = 2$ choice for even $z = 4, 6, 8, \cdots$ or by the $z = 3$ choice for odd $z = 5, 7, 9, \cdots$.

components of the metric (3.62) must have a simple zero and a simple pole [113, 114] at the finite horizon $r = R_0$. This condition basically guarantees non-extremal black holes, which leads to the following horizon expansions of the functions

$$f(r) = \sqrt{r - R_0} \sum_{n=0}^{\infty} f_n (r - R_0)^n, \tag{3.82}$$

$$g(r) = \frac{1}{\sqrt{r - R_0}} \sum_{n=0}^{\infty} g_n (r - R_0)^n. \tag{3.83}$$

Before employing these expansions let us focus on the bounds of the gauge field functions at the horizon. We already know the large r values for the functions. On the other hand, the horizon values will define a shooting parameter for the system, i.e. we can choose different values. However there must be some kind of upper/lower value above/below which the system is not well-defined physically. To discuss these constraints, consider the general form of the field equations (3.63), (3.64), (3.65) we have discussed in the previous section. Start with the equation (3.64) on the horizon

$$\mu'(R_0) w'(R_0) = \frac{w(R_0)(w^2(R_0) - k)}{R_0^2}, \tag{3.84}$$

where we have used $\mu(R_0) = 0$ on the horizon. Now use (3.65) to replace $\mu'(R_0)$

$$\mu'(R_0) = \frac{k - L^2 R_0^2 \Lambda}{R_0} - \frac{1}{2L^2 g_{\text{YM}}^2 R_0} \left(\frac{(w^2(R_0) - k)^2}{2R_0^2} \right). \tag{3.85}$$

These two equations imply that the gauge field function $w(r)$ and its derivative is related at the horizon as

$$w'(R_0) = \frac{w(R_0)(w^2(R_0) - k)}{\left(kR_0 - \frac{1}{2g_{\text{YM}}^2 L^2} \frac{(w^2(R_0) - k)^2}{2R_0} - L^2 R_0^3 \Lambda \right)}, \tag{3.86}$$

employing the identification $w(R_0) = \sqrt{z+1} R_0 h_0$ and $w'(R_0) = \sqrt{z+1} j_0$

$$j(R_0) = j_0 = \frac{2h_0 R_0 \left(h_0^2 R_0^2 (z+1) - k \right)}{2kR_0 + R_0^3 \left(z^2 + 2z + 3 \right) - \frac{(z-1)\left(k - h_0^2 R_0^2 (z+1)\right)^2}{R_0 (z+1)}} \quad \text{for} \quad z > 1, \tag{3.87}$$

where $h_0 \equiv h(R_0)$ and $j_0 \equiv j(R_0)$. The equality (3.87) relates the expansion coefficients on the horizon for different values of k. At this point, it is worth emphasising the meaning of h_0. For that, consider a non-coordinate basis for the one-forms [36]

$$\theta_t = L r^z f(r) dt, \quad \theta_{x_i} = L r dx^i, \quad \theta_r = L \frac{g(r)}{r} dr, \quad i = 1, 2 \tag{3.88}$$

in which the planar metric (3.62) takes the form $ds^2 = \eta^{\mu\nu} d\theta_\mu d\theta_\nu$ with $\eta^{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$. Taking this into account, the gauge connection can be written as

$$A = \frac{\sqrt{z+1}}{L} h(r)(T^1\theta_1 + T^2\theta_2). \tag{3.89}$$

This relation suggests that, up to some normalization, h_0 can be considered as the strength of the gauge field at the horizon. Going back to equation (3.87), we see that the planar case is somewhat special. For $k = 0$, the horizon radius cancels out, and j_0 depends only on the gauge field strength, h_0 and the dynamical exponent z.

The bound for the strength of the gauge field follows from the regularity condition of the derivative of the function $\mu(r)$ at the horizon

$$\left.\frac{d\mu}{dr}\right|_{r=R_0} > 0. \tag{3.90}$$

Then, with the help of (3.65), one finds that

$$k - \frac{1}{2g_{\mathrm{YM}}^2 L^2} \frac{(w^2(R_0) - k)^2}{2R_0^2} - L^2 R_0^2 \Lambda > 0. \tag{3.91}$$

In terms of $w(R_0) = \sqrt{z+1}\, h_0$, this inequality further simplifies to

$$\frac{R_0^2(z+1)\Big(2k + R_0^2(3 + 2z + z^2)\Big)}{(z-1)} > (k - R_0^2(z+1)h_0^2)^2. \tag{3.92}$$

This inequality effectively reduces the possible values of h_0 that can be chosen as a shooting parameter. Again the planar case $k = 0$ differs from others, where h_0 becomes independent of the horizon radius and is solely bounded by the dynamical critical exponent z, meaning that once a suitable h_0 is found for a given z, it will always be a solution for black holes with different radii. There is even more to the story for the hyperbolic case $k = -1$. The event horizon radius is bounded by the value of the cosmological constant as

$$|\Lambda| > \frac{1}{L^2 R_0^2}(1 + \frac{1}{4g_{\mathrm{YM}}^2 R_0^2 L^2}). \tag{3.93}$$

For a crosscheck, the above relations and bounds can be obtained from near horizon expansions. If the functions $h(r), j(r)$ are finite on the horizon, they can be expanded as

$$h(r) = \sum_{n=0}^{\infty} h_n (r - R_0)^n, \tag{3.94}$$

$$j(r) = \sum_{n=0}^{\infty} j_n (r - R_0)^n. \tag{3.95}$$

Inserting the expansions (3.73), (3.94), (3.95) into (3.69), (3.70), (3.71), (3.72), we find that for fixed z, solutions are characterised by two free parameters h_0, the strength of the gauge field at the horizon, and R_0, the horizon radius.

As a simple example, for $z = 2$ and $k = 0$, one gets

$$g_0 \rightarrow \frac{\sqrt{2R_0}}{\sqrt{11 - 3h_0^4}}, \tag{3.96}$$

$$j_0 \rightarrow \frac{6h_0^3}{11 - 3h_0^4}, \tag{3.97}$$

$$h_1 \rightarrow \frac{h_0\,(3h_0^4 + 6h_0^2 - 11)}{(11 - 3h_0^4)\,R_0}, \tag{3.98}$$

$$g_1 \rightarrow \frac{\sqrt{2}\,(18h_0^8 + 27h_0^6 - 99h_0^4 + 121)}{(11 - 3h_0^4)^{\,5/2}\sqrt{R_0}}, \tag{3.99}$$

$$f_1 \rightarrow \frac{f_0\,(-27h_0^8 + 9h_0^6 + 165h_0^4 - 242)}{(11 - 3h_0^4)^{\,2}R_0}. \tag{3.100}$$

At the last equality f_0 appears to be a free parameter, however from the previous section we already know that $f(r)$ can be scaled without affecting the equation of motion. Therefore it is just an overall normalization factor, not a free parameter. The bound we have found on h_0 guarantees real values for g_0, h_0, e.g. for $z = 2$, $k = 0$ the strength of the gauge field must be $h_0^4 < 11/3$ which agrees with (3.92). Finally the value of j_0 (3.87) is also recaptured here.

To sum up, in this section we have established the procedure for numerical computation, by finding how the gauge field strength h_0 depends on the value of $R_0,$ k and z. Now fixing one of the event horizon radius R_0 for a given topology, we numerically integrate the functions and force them to converge to unity at infinity by fine tuning the initial value h_0.

3.5.3 Numerical solutions

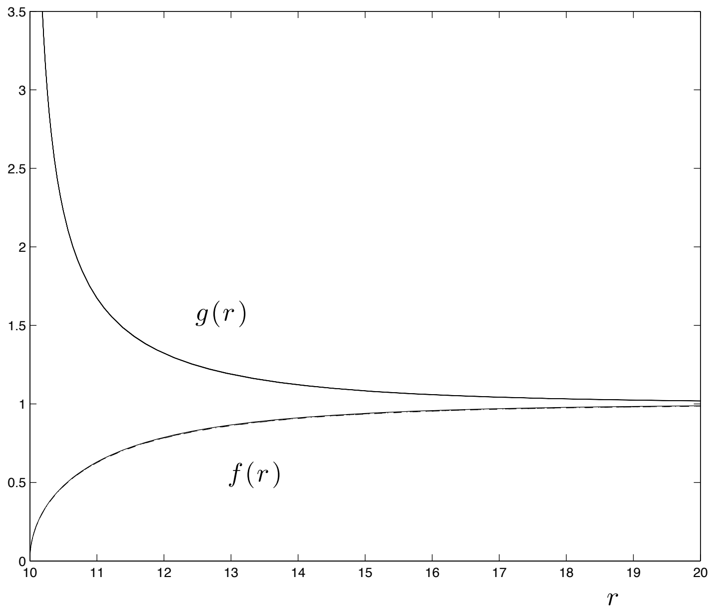
In this section we will perform the numerical integration of the field equations by fine-tuning the initial conditions specified at the horizon. Our program of choice will be the MATLAB's differential equation solver ode45 in default settings, which uses Runga-Kutta method with variable step-size and the relative tolerance value is 10^{-3}

(0.1 accuracy).

Before going into details, let us summarize our findings and compare them to the EYM solutions with different asymptotics. Spherically symmetric, asymptotically AdS solutions are found in continuous open intervals [86], i.e. $0 < \omega_h < \omega_h^c$ where ω_h is the shooting parameter and ω_h^c is the upper bound for the value of gauge field above which there is no solution. In contrast, our solutions have a unique critical value of shooting parameter h_0 within the allowed region (3.92), at which we capture the desired asymptotics. Moreover, AdS solutions exhibit nodes for sufficiently small values of $|\Lambda|$ [86], whereas our solutions are nodeless.

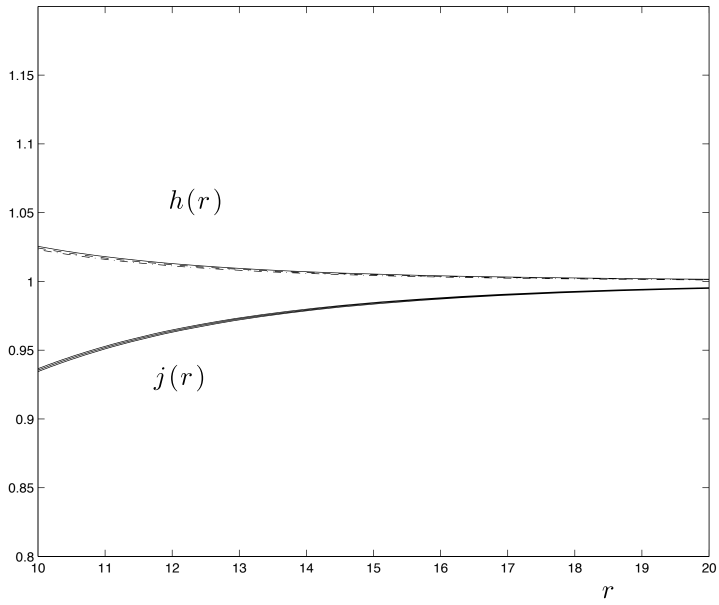
We start the analysis with larger black holes, and fix $z = 3$ in order to cover all possible topologies and to compare results. We see that the solutions behave similarly regardless of the topology of the event horizon. In figures 3.1 and 3.2 we set $R_0 = 10$ and plotted the functions for all values of k. However, since their behaviours are the same for all k, they coalesce on top of each other and seem as one graph. For the sake

Figure 3.1 The figure plots the metric functions $f(r)$ and $g(r)$ as a function of radius r. This is an example of a large black hole with $R_0 = 10$, where the plots overlap for all values of k.



of clarity, let us present the initial value of the gauge field function, i.e. the shooting

Figure 3.2 The figure shows the gauge field functions $h(r)$ and $j(r)$ as a function of radius r with $R_0 = 10$. The initial values of functions for different topologies are very close to each other. Graphs for different topologies merge into one.



parameter

$$h_0 = \begin{cases} 1.025530137, & \text{for} \quad k = 1, \\ 1.023139854, & \text{for} \quad k = -1, \\ 1.024335678, & \text{for} \quad k = 0, \end{cases} \tag{3.101}$$

with the cut-off value $r_{max} = 1000$. By fine tuning the shooting parameter h_0 to desired order, we can extend the numerical integration to a larger distance. Simply setting $h_0 = 1.025530137219$, for $k = 1$ and $R_0 = 10$ the asymptotics is extended to $r_{max} = 10.000$. Note that in all calculations we need the value of j_0, which follows from the equality (3.87).

We next focus on the smaller black holes by fixing $R_0 = 0.5$. The functions now differ in behaviour according to their horizon topology. The metric function $f(r)$ reaches a maximum value before converging to unity (Figures 3.3, 3.4 and 3.5), whereas in planar and hyperbolic cases functions monotonically converge to one. The radial component of the metric function $g(r)$ first dips then approaches to one for $k = 1$. From the graphs we also see that the spherical and hyperbolic solutions decay slower than the planar ones. The shooting parameters we have used for the small black holes

Figure 3.3 A small black hole with $R_0 = 0.5$. Figure shows the metric function $f(r)$ for different cases $k = 1, -1, 0$. The solid line corresponds to $k = 1$, the dashed line to $k = 0$ and dot-dashed line represents $k = -1$, respectively.

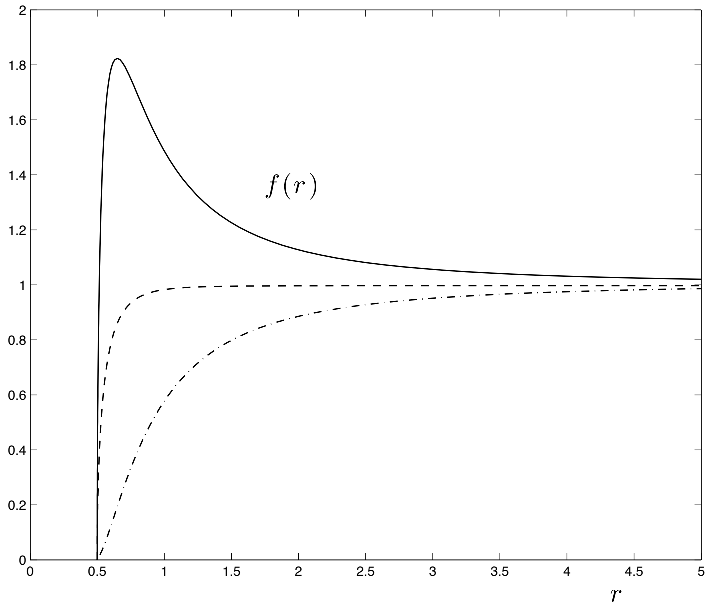


Figure 3.4 The figure illustrates the metric function $g(r)$ with a small radius $R_0 = 0.5$. The solid line indicates $k = 1$, while the $k = 0$ and $k = 1$ cases are represented by dashed and dot-dashed lines, respectively.

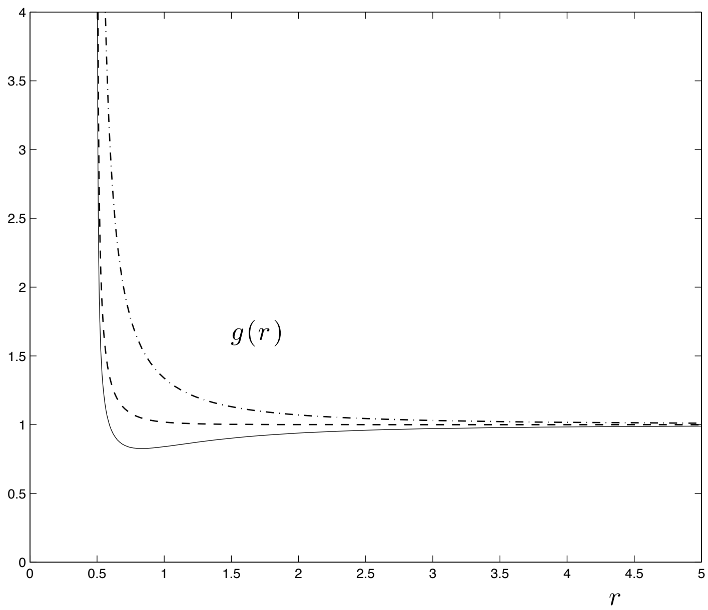
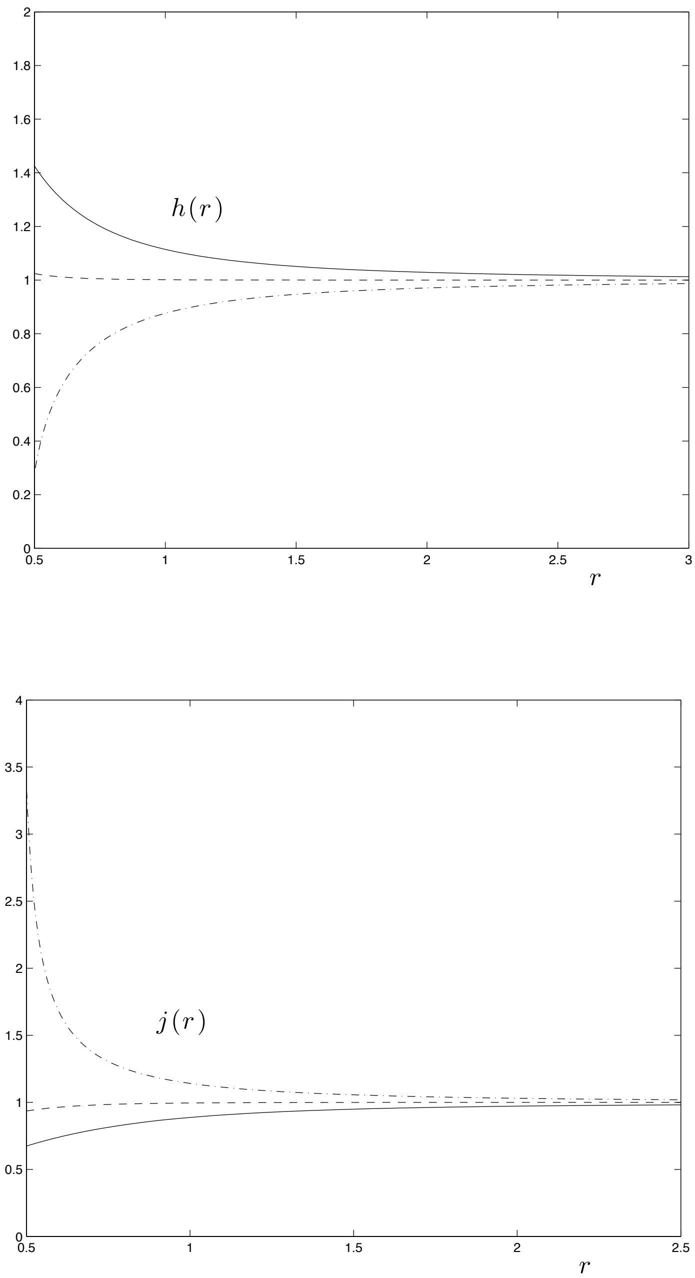


Figure 3.5 The gauge field function $h(r)$ is displayed on the top and $j(r)$ at the bottom, both as functions of r. In both graphs $R_0 = 0.5$. The solid line indicates $k = 1$, while the $k = 0$ and $k = 1$ cases are represented by dashed and dot-dashed lines, respectively.



with radius $R_0 = 0.5$ are as follows

$$h_0 = \begin{cases} 1.425617169, & \text{for} \quad k = 1, \\ 0.278652475, & \text{for} \quad k = -1, \\ 1.024335678, & \text{for} \quad k = 0. \end{cases} \tag{3.102}$$

The next task we undertake is to compare the analytic bound (3.92) with the values of h_0 we have found in numerical analysis for different radii. In the previous section we have shown that the planar black holes have a unique value of h_0 that is compatible with all radii R_0. Behaviours change if we consider spherical or hyperbolic event horizons. The spherical ones demand stronger gauge fields as the radius gets smaller, in contrast the hyperbolic ones can support weaker gauge fields for smaller radii. In [113, 114] the same behaviour was observed with abelian fields. Plotting analytic bound (3.92) along with the numerical values of h_0 with respect to different R_0, we see the bound is saturated as the radius gets smaller (Figure 3.6). Finally we observe that the lower limit on the horizon radius of hyperbolic black holes (3.93) is compatible with the numerical results for $z = 3$, i.e. from Figure 3.7 it is obvious that numerical integration stops and there is no solution below $R_0 \sim 0.48$.

3.6 Thermal behavior

Finally let us focus on the thermal behaviour, a property that we can compute and discuss from the numerical data we obtain from the solutions. Since we don't have the exact solution, we will resort to the Wick rotation technique.

For convenience let us display the metric (3.67) after a Wick rotation $\tau = it$

$$ds^2 = L^2 \left(r^{2z} f(r)^2 d\tau^2 + \frac{dr^2}{r^2} g(r)^2 + r^2 d\Omega^2 \right). \tag{3.103}$$

Now, assume a point outside but close to the horizon, i.e. $r = R_0 + \epsilon$ where $\varepsilon > 0$. Since we don't have the exact solution, consider the near horizon expansions of functions $f(r)$ and $g(r)$ for non-extremal black holes (3.83)

$$f(R_0 + \varepsilon) = \sqrt{\varepsilon}\, f_0, \tag{3.104}$$

$$g(R_0 + \varepsilon) = \frac{1}{\sqrt{\varepsilon}}\, g_0, \tag{3.105}$$

Figure 3.6 The inequality (3.92) as a function of R_0 is plotted with a solid line for $k = 1$. The dashed line corresponds to the numerical values of h_0 as a function of R_0 for spherically symmetric black holes.

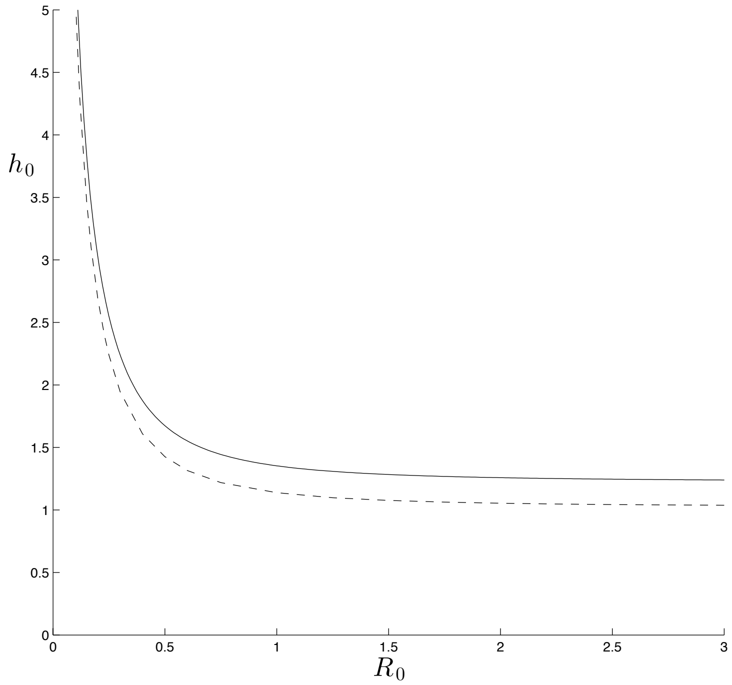
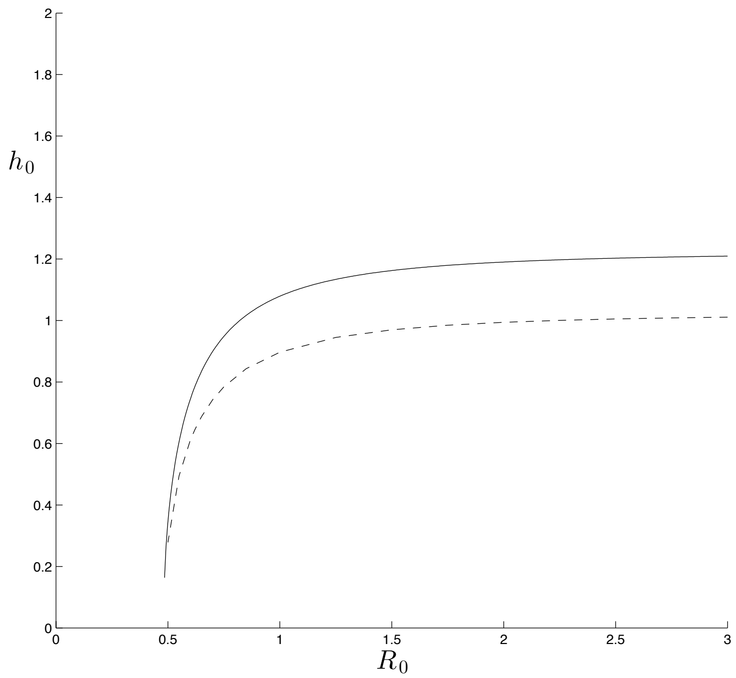


Figure 3.7 The inequality (3.92) as a function of R_0 is plotted with a solid line for $k = -1$. The dashed line corresponds to the numerical values of h_0 as a function of R_0 for hyperbolically symmetric black holes. The lower bound (3.93) on the horizon radius is apparent.



and plugging these in (3.103)

$$ds^2 = L^2 \left[(R_0+\varepsilon)^{2z} f_0^2 \varepsilon d\tau^2 + \frac{d\varepsilon^2}{(R_0+\varepsilon)^2}\frac{g_0^2}{\varepsilon} + (R_0+\varepsilon)^2 d\Omega^2 \right], \tag{3.106}$$

$$\sim L^2 \left[R_0^{2z} f_0^2 \varepsilon d\tau^2 + \frac{g_0^2}{R_0^2 \varepsilon} d\varepsilon^2 + R_0^2 d\Omega^2 \right]. \tag{3.107}$$

The metric near the horizon (3.107) can now be written as a product manifold of some 2-dimensional manifold times S^2 of radius R_0. After the transformations, $\rho = 2g_0\sqrt{\varepsilon}$ and $\tau = 2g_0\phi/R_0^{z+1} f_0$, the two dimensional metric

$$ds_2^2 = L^2 \left[R_0^{2z} f_0^2 \varepsilon d\tau^2 + \frac{g_0^2}{R_0^2 \varepsilon} d\varepsilon^2 \right], \tag{3.108}$$

will transform into a metric looking like a plane in polar coordinates

$$ds_2^2 = \frac{L^2}{R_0^2}\left[\rho^2 d\phi^2 + d\rho^2\right] \tag{3.109}$$

with angle ϕ that is not restricted to the range $(0, 2\pi)$. But we know from the appendix that Euclidean time is related to the periodic 2π polar angle ϕ which in turn makes τ periodic with $4\pi g_0 \phi/R_0^{z+1} f_0$ and finally the temperature will be the inverse period [115]

$$T = \frac{f_0 R_0^{z+1}}{4\pi g_0}. \tag{3.110}$$

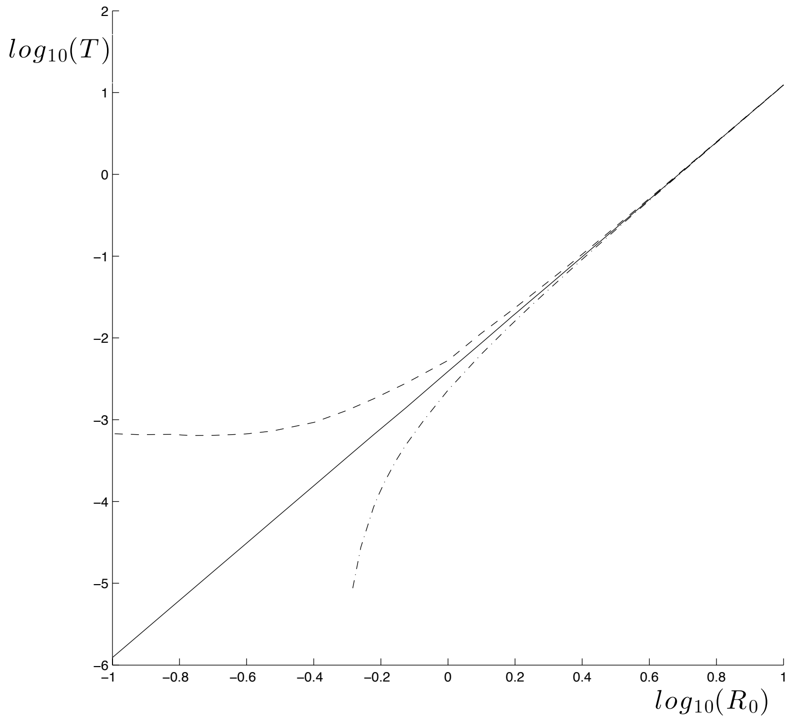
The coefficient g_0 is determined from the series solution near the horizon

$$g_0 = \frac{\sqrt{2(z+1)} R_0^{3/2}}{(2h_0^2 k R_0^2 (z-1) + h_0^4 R_0^2 (1-z)(z+1) - k^2 \frac{(z-1)}{(z+1)} + 2k + R_0^2(3+2z+z^2))^{1/2}}. \tag{3.111}$$

The other coefficient f_0 was actually a normalization constant for the numerical solutions, so it depends on the shooting parameter h_0. By fixing $z = 3$, we can now compare the temperature of black holes with different sizes and topology. The algorithm of finding temperature is simple, first we will find solutions with different R_0 values, then since we know the shooting parameter h_0 and f_0 from normalization, the equation (3.110) will directly lead us to temperature. Following this algorithm we have plotted Figure 3.8.

Similar to their behaviour in section 3.5.3, the large black holes with different topologies agree on the higher temperature. On the other hand, the smaller ones are cooler and differ in their cooling rates, e.g. hyperbolic ones have a higher cooling rate than the planar ones. The thermal behaviour of these black holes is opposite to their AdS counterparts, where the Hawking temperature increases with the ever decreasing radius causing thermal instability. Moreover, it is clear that the EYM black holes do not exhibit Hawking-Page transition. A similar thermal behaviour is observed for the Lifshitz black holes supported by abelian p-forms [113, 114] which indicates that the black holes become extremal, i.e. they have zero Hawking temperature in the vanishing black hole size.

Figure 3.8 Temperature versus horizon radius for $z = 3$. The different topologies are represented by a solid line $k = 0$, by a dashed line $k = 1$ and a dot-dashed line $k = -1$.



CHAPTER 4

$\mathcal{N} = (1,1)$ COSMOLOGICAL NEW MASSIVE SUPERGRAVITY SOLUTIONS

In this chapter we will focus on the solutions of a higher curvature supergravity (SUGRA) model, namely, the three dimensional $\mathcal{N} = (1,1)$ New Massive Gravity [5]. Apart from thr maximally symmetric AdS backgrounds, we will see that the theory supports plethora of backgrounds including Lifshitz spacetimes. Introducing supersymmetry will actually help us finding solutions, since the Killing spinor equation will be first order and constrains the metric functions. However, in order to get to the solution phase, we need to establish serious machinery forged for Clifford algebra. After we have the right tools, it will be easy to manipulate identities and extract the information we need.

The organisation of this section is as follows: In section 4.1, we give a brief summary of the Clifford algebra and spinor properties. We will try to summarise all of the machinery that is needed to perform off-shell Killing spinor analysis. Section 4.2 starts by presenting a general formalism for the conformal construction. The subsection 4.2.4 of section 4.2 is devoted to the construction of Einstein gravity from a conformal gauge theory. In section 4.3 we will give the origin and details of the theory we are going to study. Section 4.4 basically deals with the definition and useful identities for off-shell Killing spinors. In sections 4.5 and 4.6 we will investigate and classify solutions with null and timelike Killing vectors, respectively.

4.1 Spinors and Clifford Algebra

Before we move on to the construction of three dimensional $\mathcal{N}=2$ SUGRA theories, we will first cover the basics of Clifford algebras and spinors. Incorporating fermions into the theories is always a formidable task, even for flat spacetimes. The local SUSY transformations demand spacetime to be curved, making the discussion even harder. Along the way, we will see that most of the identities and properties related to the spinors depend on the dimension of spacetime. Therefore, to keep the discussion more general we do not fix the dimension at first. In what follows we will basically summarise the conventions and definitions of [31], so more detailed discussion is in the book. Instead of just referring to the source, we will try to put all the ingredients we need in a nutshell.

4.1.1 Clifford Algebra

As well known, while working on a relativistic wave equation Dirac came up with matrices satisfying the following anti-commutation relations

$$\gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\mu=2\eta^{\mu\nu}\mathbb{I}. \tag{4.1}$$

These matrices are the elements of *Clifford algebra*. For general D, the gamma matrices are complex, and the spinor they act on is also complex, called the *Dirac spinor*. However, in some dimensions the real representation of gamma matrices can be chosen, and the spinors they act on is also real, which are called *Majorana spinors*. By tensor multiplying the Pauli matrices we can construct gamma matrices, resulting in $2^{[D/2]}$ dimensional representations[1].

The basis of the Clifford algebra consists of the identity, the D matrices γ^μ and possible combinations of gamma matrices. From (4.1) we see, it is not possible to take symmetric combinations, which will always reduce to simpler expressions. Therefore we take the anti-symmetric ones, and define

$$\gamma^{\mu_1\mu_2\cdots\mu_r}\equiv\gamma^{[\mu_1}\cdots\gamma^{\mu_r]},\quad \gamma^{\mu\nu}=\tfrac{1}{2}(\gamma^\mu\gamma^\nu-\gamma^\nu\gamma^\mu), \tag{4.2}$$

[1] Here $[D/2]$ refers to the integer part of $D/2$.

where $D \geq r$. There are different types of gamma matrix contractions that are frequently used in SUGRA theories, summarized as

$$\gamma^{\mu_1\cdots\mu_r\nu_1\cdots\nu_s}\gamma_{\nu_s\cdots\nu_1} = \frac{(D-r)!}{(D-r-s)!}\gamma^{\mu_1\cdots\mu_r}, \tag{4.3}$$

$$\gamma_\rho\gamma^{\mu_1\cdots\mu_r}\gamma^\rho = (-1)^r(D-2r)\gamma^{\mu_1\mu_2\cdots\mu_r}, \tag{4.4}$$

$$\gamma^{\mu\nu\rho}\gamma_{\sigma\tau} = \gamma^{\mu\nu\rho}{}_{\sigma\tau} + 6\gamma^{[\mu\nu}{}_{[\tau}\delta^{\rho]}{}_{\sigma} + 6\gamma^{[\mu}\delta^{\nu}{}_{[\tau}\delta^{\rho]}{}_{\sigma]}. \tag{4.5}$$

Although it can not be expressed in a general form, the last equality can be computed for different cases. The basic idea is, multiplication of two Clifford matrices can be written as a totally antisymmetric matrix with all indices plus other lower order matrices with possible index pairings.

4.1.2 Basis of Clifford Algebra

The basis of Clifford algebra is spanned by different sets depending on the dimension of spacetime. For even $D = 2m$, the following 2^D matrices is a basis

$$\{\Gamma^A = \mathbb{I}, \gamma^\mu, \gamma^{\mu_1\mu_2}, \cdots, \gamma^{\mu_1\cdots\mu_D}\}, \tag{4.6}$$

with index vales $\mu_1 < \mu_2 < \cdots < \mu_r$. On top of these, one can also define a highest rank element for even dimensions

$$\gamma_* \equiv (-i)^{m+1}\gamma_0\gamma_1\cdots\gamma_{D-1}, \quad \text{or} \quad \gamma_{\mu_1\mu_2\cdots\mu_D} = i^{m+1}\epsilon_{\mu_1\mu_2\cdots\mu_D}\gamma_*, \tag{4.7}$$

which is hermitian and satisfies $\gamma_*^2 = 1$. The highest rank element is used in the study of chiral fermions by defining projection operators $P_L = \frac{1}{2}(\mathbb{I}+\gamma_*)$ and $P_R = \frac{1}{2}(\mathbb{I}-\gamma_*)$.

In odd dimensions $D = 2m+1$ the basis (4.6) changes considerably. The highest rank element can be included to the generators and two sets of $2m+1$ generating elements are defined as follows

$$\gamma^\mu_\pm = (\gamma^0, \gamma^1, \cdots \gamma^{(2m-1)}, \gamma^{2m} = \gamma_*). \tag{4.8}$$

Because of this, the highest element in (4.6) is some factor times unity, meaning that there is no chiral projector in odd dimensions. Moreover, the rank r and rank $D-r$ branches are related by

$$\gamma_\pm^{\mu_1\mu_2\cdots\mu_r} = \pm i^{m+1}\frac{1}{(D-r)!}\epsilon^{\mu_1\cdots\mu_D}\gamma_{\pm\mu_D\cdots\mu_{r+1}}, \tag{4.9}$$

effectively reducing the number of matrices in $D = 2m+1$ dimensional Clifford algebra to m. We will be dealing with $D = 3$ theories so that the structure of Clifford algebra will be quite simple i.e. we have at most rank one, 2×2 matrices, which will be chosen as Pauli matrices.

Finally, defining a basis for Clifford algebra we can expand any matrix M in the basis $\{\Gamma^A\}$ satisfying the orthogonality relation $\mathrm{Tr}(\Gamma^A \Gamma_B) = 2^m \delta^A_B$,

$$M = \sum_A m_A \Gamma^A, \quad \text{where} \quad m_A = \frac{1}{2^m}\,\mathrm{Tr}(M\Gamma_A). \tag{4.10}$$

4.1.3 Symmetry Properties

The symmetry properties of gamma matrices are rather odd. In order to define the notion of symmetric/antisymmetric matrices, we will introduce a unitary matrix C, called the charge conjugation matrix. One can then define the multiplication of $C\Gamma^A$ as symmetric or anti-symmetric as

$$(C\Gamma^r)^T = -t_r C\Gamma^r, \quad t_r = \pm 1. \tag{4.11}$$

We don't actually need all t_r's to find the symmetry properties of different ranks. From the rank $r = 0$ and $r = 1$ relations

$$C^T = -t_0 C, \quad \text{and} \quad \gamma^{\mu T} = t_0 t_1 C \gamma^\mu C^{-1}, \tag{4.12}$$

we can extract any other ranks. Consider the second rank gamma matrix $(C\gamma^{\mu\nu})^T$

$$(C\gamma^{\mu\nu})^T = \frac{1}{2}\left[(C\gamma^\mu\gamma^\nu)^T - (C\gamma^\nu\gamma^\mu)^T\right], \tag{4.13}$$

$$= \frac{1}{2}\left[t_1 t_0 (\gamma^{\nu T} C^T \gamma^\mu - \gamma^{\mu T} C^T \gamma^\nu)\right], \tag{4.14}$$

$$= t_0 C\gamma^{\mu\nu} = -t_2 C\gamma^{\mu\nu}, \tag{4.15}$$

so $t_2 = -t_0$ and $t_3 = -t_1$. By following similar arguments, it can be shown that $t_{r+4} = t_r$. The values of t_0 and t_1 depend on the dimension we study and given in a table in [31]. For $D = 3$, $t_{1,2} = -1$ and $t_{0,3} = +1$.

We are working in hermitian representations with $\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0$. Therefore, the complex conjugation and charge conjugation matrix are related as follows

$$\gamma^{\mu *} = -t_0 t_1 B \gamma^\mu B^{-1}, \quad \text{where} \quad B = i t_0 C \gamma^0 \tag{4.16}$$

and B also satisfies $B^*B = -t_1\mathbb{I}$.

In order to produce scalars, we need a conjugate to the spinors in general. We will define the *Majorana conjugate* as follows

$$\bar{\lambda} \equiv \lambda^T C. \tag{4.17}$$

Employing this definition and (4.11) the spinor bilinear $\bar{\lambda}\gamma_{\mu_1\cdots\mu_r}\chi$ can be flipped as

$$(\bar{\lambda}\gamma_{\mu_1\cdots\mu_r}\chi)^T = \chi^T \gamma_{\mu_1\cdots\mu_r}{}^T C^T \lambda, \tag{4.18}$$

$$= t_r \chi^T C \gamma_{\mu_1\cdots\mu_r}\lambda, \tag{4.19}$$

$$= t_r \bar{\chi}\gamma_{\mu_1\cdots\mu_r}\lambda. \tag{4.20}$$

Note that, the components of the spinors are Grassmann numbers, so flipping components will bring up an extra minus sign. The flipping relations will play an important role in the discussion of the Dirac spinor we use in this work.

4.1.4 Working with Indices

It is possible to introduce indices and contraction rules for spinors. We will denote the basic spinor with the index down $\lambda \to \lambda_\alpha$, where $\alpha = 1, \cdots 2^{[D/2]}$. The Majarona conjugate is spinor with an index up $\bar{\lambda} \to \lambda^\alpha$. The relation between the spinors with up and down indices follows from (4.17)

$$\lambda^\alpha = \mathcal{C}^{\alpha\beta}\lambda_\beta. \tag{4.21}$$

so $\mathcal{C}^{\alpha\beta}$ are the components of C^T. Unlike spacetime indices, the order of the contraction of spinor indices is important. We will use the convention NW-SE, i.e. indices up must be contracted with the indices on their right and vice versa. Therefore, for lowering of indices, introduce $\mathcal{C}_{\alpha\beta}$

$$\lambda_\alpha = \lambda^\beta \mathcal{C}_{\beta\alpha}, \tag{4.22}$$

which must obey $\mathcal{C}^{\alpha\beta}\mathcal{C}_{\gamma\beta} = \delta_\gamma{}^\alpha$ and $\mathcal{C}_{\beta\alpha}\mathcal{C}^{\beta\gamma} = \delta_\alpha{}^\gamma$. The bilinears we have introduced in the previous section can also be expressed with indices

$$\bar{\lambda}\gamma_\mu\chi = \lambda^\alpha(\gamma_\mu)_\alpha{}^\beta\chi_\beta. \tag{4.23}$$

Note that, γ matrices will be expressed as one index down and one index up.

4.1.5 Reality Conditions

In order to check whether a quantity is real or hermitian, we should define a complex conjugation. Directly taking the complex conjugate of a contracted quantity e.g a scalar, will most probably include taking the complex conjugates of the gamma matrices and C, which will complicate computations. So we will introduce an operation, which will simplify to taking a complex conjugate when acting on scalars. Let us define *charge conjugate* of a spinor as

$$\lambda^C \equiv B^{-1}\lambda^*. \tag{4.24}$$

The Majorana conjugate spinor is then

$$\begin{aligned}\bar{\lambda}^C =&(B^{-1}\lambda^*)^T C,\\ &= \lambda^\dagger (B^{-1})^T C,\\ &= it_0\lambda^\dagger (C^{-1})^T(\gamma^0)^T C,\\ &= it_0^2\lambda^\dagger (C^{-1})(\gamma^0)^T C,\\ &= -it_0t_1(i\lambda^\dagger\gamma^0),\end{aligned} \tag{4.25}$$

where we have used $(B^{-1})^T = -it_0(C^{-1})^T((\gamma^0)^{-1})^T$. The charge conjugate of matrices is defined as $M^C \equiv B^{-1}M^*B$, which makes the gamma matrix transformation to be

$$(\gamma_\mu)^C = B^{-1}\gamma_\mu^* B = B^{-1}(-t_0t_1B\gamma_\mu B^{-1})B = -t_0t_1\gamma_\mu, \tag{4.26}$$

where we have employed (4.16). The complex conjugation of bilinears can be constructed as follows

$$(\bar{\lambda}M\chi)^* \equiv (\bar{\lambda}M\chi)^C = (-t_0t_1)\bar{\lambda}^C M^C\chi^C. \tag{4.27}$$

Armed with this definition, we can impose reality conditions on spinors

$$\lambda = \lambda^C = B^{-1}\lambda^*. \tag{4.28}$$

The spinors that satisfy this condition will be called *Majorana spinors*. By simply taking the complex conjugate of (4.28), we see that the reality condition is not compatible in all dimensions

$$\lambda^* = B\lambda \to \lambda = B^*\lambda^* \to \lambda = B^*B\lambda. \tag{4.29}$$

Thanks to the equality $B^*B = -t_1\mathbb{I}$, we find that Majorona condition is only possible for dimensions with $t_1 = -1$. For $t_1 = -1$ there is a branching with $t_0 = \pm 1$. The first case i.e. $t_1 = -1$ and $t_0 = 1$ is available in $D = 2, 3, 4, \bmod 8$. We are able to choose a representation where all γ-matrices are real and from (4.16) it is possible to choose $B = \mathbb{I}$ up to a phase, setting $C = i\gamma^0$. The other possibility $t_1 = -1$ and $t_0 = -1$ is relevant for $D = 8, 9$ in which γ matrices can now be chosen as purely imaginary.

4.1.6 Dirac Spinors in $3D\ \mathcal{N} = 2$

The SUGRA theory we are going to study will be $\mathcal{N} = 2$, i.e. we have 4 supercharges (4 spinor components). In previous sections, we have discussed that in three dimensions spinors have 2 components. Therefore, in order to have 4 supercharges we either choose Majorana spinors and label them with indices or we choose Dirac spinors and deal with complex entries. In [119], Dirac spinors were used to construct the theory, so we will proceed with Dirac spinors.

However, choosing Dirac spinors over Majorana ones brings in extra structure, which can be observed from decomposing a Dirac spinor into two Majorana spinors. Let us assume a Dirac spinor that is given as

$$\epsilon = \epsilon_1 + i\epsilon_2, \tag{4.30}$$

where ϵ_1 and ϵ_2 are 2 dimensional Majorana spinors, i.e. $\epsilon_i^* = \epsilon_i$ with $i = 1, 2$. Note that, we have chosen $B = \mathbb{I}$ and $C = i\gamma^0$. So that the Dirac conjugation, i.e. $\bar{\epsilon} = i\epsilon^\dagger\gamma^0$, on a Majorana spinor is basically a Majorana conjugate (4.17). Considering this we have the following spinors and conjugates

$$\begin{aligned}\epsilon^* =& \epsilon_1 - i\epsilon_2\\ \bar{\epsilon} =& \bar{\epsilon}_1 - i\bar{\epsilon}_2\\ \bar{\epsilon^*} =& \bar{\epsilon}_1 + i\bar{\epsilon}_2 \equiv \tilde{\epsilon}.\end{aligned} \tag{4.31}$$

Given these spinors and conjugates we will check the flipping relations, similar to the ones in the previous section. From now on we will assume our spinors are commuting for the reasons that are described in Sec. 4.4. Keeping this in mind, let us look for different bilinears we can construct from (4.31). Consider

$$\bar{\epsilon}\Gamma\chi = (\bar{\epsilon}_1 - i\bar{\epsilon}_2)\Gamma(\chi_1 + i\chi_2), \tag{4.32}$$

$$= \bar{\epsilon}_1\Gamma\chi_1 + i\bar{\epsilon}_1\Gamma\chi_2 - i\bar{\epsilon}_2\Gamma\chi_1 + \bar{\epsilon}_2\Gamma\chi_2, \tag{4.33}$$

$$= -\, t_r \left[(\bar{\chi}_1 + i\bar{\chi}_2)\Gamma\epsilon_1 - i(\bar{\chi}_1 + i\bar{\chi}_2)\Gamma\epsilon_2\right], \tag{4.34}$$

$$= -\, t_r(\tilde{\chi}\Gamma\epsilon^*), \tag{4.35}$$

where t_r is the rank of the gamma matrix. In three dimensions, the possible ranks are $t_0 = 1$ and $t_1 = -1$. Following the same strategy, i.e. decomposing into Majorana and using the flipping rules, we obtain [119]

$$\bar{\lambda}\Gamma^r\chi^* = -t_r\bar{\chi}\Gamma^r\lambda^*, \tag{4.36}$$

$$\tilde{\lambda}\Gamma^r\chi = -t_r\tilde{\chi}\Gamma^r\lambda. \tag{4.37}$$

These equalities (4.35), (4.36), (4.37) will be the backbone of our off-shell Killing spinor analysis.

4.1.7 Fierz Identity

Previously, we have seen that the gamma matrices form a complete basis $\{\Gamma^A\}$ and any matrix can be expanded in terms of the basis as in (4.10). Now armed with the index structure for spinors and matrices, let us proceed and derive the basic Fierz identity. Consider the product $\delta_\alpha{}^\beta\delta_\gamma{}^\delta$, treat the indices α, δ as spectators and β, γ as the matrix indices for M in (4.10), which now reads

$$\delta_\alpha{}^\beta\delta_\gamma{}^\delta = \sum_A (m_A)_\alpha{}^\delta(\Gamma^A)_\gamma{}^\beta, \tag{4.38}$$

employing the definition of m_A in (4.10)

$$\delta_\alpha{}^\beta\delta_\gamma{}^\delta = \frac{1}{2^m}\sum_A (\Gamma_A)_\alpha{}^\delta(\Gamma^A)_\gamma{}^\beta, \tag{4.39}$$

The equation (4.39) is the basic *Fierz identity*, which we can utilize to manipulate the bilinears we construct.

As an example, we consider the three dimensional Fierz identity for the commuting Dirac spinors that is used in this chapter. With that goal in mind, multiply (4.39) with the commuting Dirac spinors $\epsilon_1^\alpha \epsilon_{2\beta}\chi^\gamma$

$$\epsilon_1^\beta \epsilon_{2\beta}\chi^\delta = \tfrac{1}{2}\left[\epsilon_1^\delta \chi^\beta \epsilon_{2\beta} + \epsilon_1^\alpha (\gamma_\mu)_\alpha{}^\delta (\gamma^\mu)_\gamma{}^\beta \epsilon_{2\beta}\chi^\gamma\right], \tag{4.40}$$

$$(\bar\epsilon_1\epsilon_2)\bar\chi = \tfrac{1}{2}(\bar\chi\epsilon_2)\bar\epsilon_1 + \tfrac{1}{2}(\bar\chi\gamma^\mu\epsilon_2)\bar\epsilon_1\gamma_\mu, \tag{4.41}$$

multiplying with ϵ_3

$$(\bar\epsilon_1\epsilon_2)\bar\chi\epsilon_3 = \tfrac{1}{2}(\bar\chi\epsilon_2)\bar\epsilon_1\epsilon_3 + \tfrac{1}{2}(\bar\chi\gamma^\mu\epsilon_2)\bar\epsilon_1\gamma_\mu, \tag{4.42}$$

then renaming the spinors for convenience. The Fierz identity in three dimensions amounts to

$$(\bar\epsilon_1\gamma^\mu\epsilon_2)\gamma_\mu\epsilon_3 = 2(\bar\epsilon_1\epsilon_3)\epsilon_2 - (\bar\epsilon_1\epsilon_2)\epsilon_3. \tag{4.43}$$

4.2 Conformal construction

In this section we discuss the procedure of conformal construction which is easy to understand once you have the right tools. However, it gets too arduous to implement this once we consider higher curvature corrections. Therefore starting from defining tools and technology, we will try to convey the idea of conformal construction, but will not expound the whole procedure with higher curvature terms. In this section we will follow the book by van Proeyen and Freedman [31].

4.2.1 Gauge theories, symmetries

We have actually discussed a special example of transformations and parameters in chapter 1, Sec. 2.1.4, where we have derived the parameters of conformal algebra and given the non-zero commutators of generators. Here, we will take this issue in a more general approach without restricting the algebra.

We will define an infinitesimal transformation as a linear operation on fields, with a parameter ϵ^A. For constant parameter we have a *global (rigid) transformation* and transformations with parameters that depend on spacetime points are called *local transformations*. With the operator T_A acting on fields, the infinitesimal transformation reads

$$\delta(\epsilon) = \epsilon^A T_A \tag{4.44}$$

with index A running over possible symmetries. We will deal with symmetries that can be realised as matrix transformations, i.e. $T_A\phi^i = -(t_A)^i{}_j\phi^j$ with ϕ^i's are the fields that transform in a given representation and the matrix generators satisfy

$$[t_A, t_B] = f_{AB}{}^C t_C. \tag{4.45}$$

It is easy to show that the commutation also holds for the infinitesimal transformations defined through (4.44)

$$[\delta(\epsilon_1), \delta(\epsilon_2)] = \delta(\epsilon_3^C = \epsilon_2^B \epsilon_1^A f_{AB}{}^C). \tag{4.46}$$

Note that, up to now we have taken the parameters as bosonic objects. For supersymmetry we need fermionic ones, so we will change T_A with Majorana spinors Q_α and the parameters as Majorana spinor conjugates $\bar{\epsilon}^\alpha$

$$\delta(\epsilon) = \bar{\epsilon}^\alpha Q_\alpha. \tag{4.47}$$

This time the generators will satisfy an anti-commutation algebra

$$\{Q_\alpha, Q_\beta\} = f_{\alpha\beta}{}^C T_C. \tag{4.48}$$

The structure of supersymmetric theory is apparent here. By acting two times with the elements of the fermionic algebra, we have a sum over bosonic ones. Another important result of (4.48) is the commutator of infinitesimal transformations

$$[\delta(\epsilon_1), \delta(\epsilon_2)] = \bar{\epsilon}_2^\alpha \bar{\epsilon}_1^\beta (Q_\alpha Q_\beta + Q_\beta Q_\alpha) = \bar{\epsilon}_2^\alpha \bar{\epsilon}_1^\beta f_{\alpha\beta}{}^C T_C = \delta(\epsilon_3^C = \bar{\epsilon}_2^\alpha \bar{\epsilon}_1^\beta f_{\alpha\beta}{}^C), \tag{4.49}$$

which confirms that equation (4.46) is valid even for fermionic symmetries.

4.2.2 Gauge fields

While constructing field theories we are guided by the symmetry principles of the problem. For a theory to enjoy local symmetries we need to introduce gauge fields denoted by $B(x)^A$, where the index A covers the localized symmetries. The transformation of gauge fields is given by

$$\delta(\epsilon)B(x)_\mu^A = \partial_\mu \epsilon(x)^A + \epsilon(x)^C B(x)_\mu^B f_{BC}{}^A. \tag{4.50}$$

Under this transformation it is easy to show that the commutator (4.46) is satisfied.

For an illustration of how fields transform under Poincaré plus supersymmetry transformations and to point out peculiarities, let us consider the following algebra with the structure constants and parameters[2]

$$
\begin{aligned}
[M_{ab}, M_{cd}] =& 4\eta_{[a[c}M_{d]b]}, \quad f_{[ab][cd]}{}^{[ef]} = 8\eta_{[c[b}\delta^{[e}_{a]}\delta^{f]}_{d]}, \\
[P_a, M_{bc}] =& 2\eta_{a[b}P_{c]}, \quad f_{a,[bc]}{}^{d} = 2\eta_{a[b}\delta^{d}_{c]} \\
\{Q_\alpha, Q_\beta\} =& -\frac{1}{2}(\gamma^a)_{\alpha\beta}P_a \quad f_{\alpha\beta}{}^{a} = -\frac{1}{2}(\gamma^a)_{\alpha\beta} \\
[M_{ab}, Q] =& -\frac{1}{2}\gamma_{ab}Q, \quad f_{[ab],\alpha}{}^{\beta} = -\frac{1}{2}(\gamma_{ab})_\alpha{}^\beta
\end{aligned}
\tag{4.51}
$$

with the gauge fields and parameters as

$$\omega_\mu^{ab}, \quad \lambda^{ab}, \quad \text{Lorentz Transformations,} \tag{4.52}$$

$$e_\mu^a, \quad \xi^a, \quad \text{Translations,} \tag{4.53}$$

$$\bar{\epsilon}^\alpha, \quad \bar{\psi}_\mu^\alpha, \quad \text{Supersymmetry.} \tag{4.54}$$

As an example, consider the transformation of the vielbein e_μ^a under SUSY. Starting from the general form of transformation (4.50), we pick the free index $A = a$ as Lorentz index. However, the first term, i.e. derivative of parameter, can not have a Lorentz index for SUSY transformation, it should have a spinor index. From this we deduce that first term is absent

$$\delta(\epsilon)e_\mu^a = \epsilon^C B_\mu^B f_{BC}{}^a. \tag{4.55}$$

The only non-zero structure constant comes from the commutator of SUSY $f_{\alpha\beta}{}^a = -1/2(\gamma^a)_{\alpha\beta}$, plugging this in and using the fact that gravitino is the gauge field of SUSY transformations

$$\delta(\epsilon)e_\mu^a = \epsilon^\alpha B_\mu^\beta f_{\beta\alpha}{}^a = \bar{\epsilon}^\alpha \bar{\psi}_\mu^\beta \left(-\frac{1}{2}(\gamma^a)_{\alpha\beta}\right) = \frac{1}{2}\bar{\epsilon}\gamma^a\psi_\mu, \tag{4.56}$$

where we have also lowered the index on gravitino, which brings in an extra minus sign. This algorithm of reading of transformations from the algebra has limitations which can easily be seen from the translations of the gravitino, $\delta_P\psi$. Writing down the general form, we see again that the derivative does not contribute. Moreover, there is also no contribution from the second part, meaning $\delta_P\psi = 0$. However, it is a well known fact that SUSY commutator on gravitino is a translation and it is non-zero. In the next section we will handle this anomaly of translations more carefully.

[2] In this chapter we have used Greek indices $\mu, \nu, \rho, \cdots$ as coordinate (curved) indices and latin ones $a, b, c, \cdots$ as local frame indices.

4.2.3 Covariant Derivatives

In order to construct Lagrangians with kinetic terms that exhibit local symmetries, we need to introduce a notion of covariant derivative in which the transformation of the derivative of a field does not contribute a term with the derivative of a parameter. In short, we need to subtract the terms that has derivatives of transformation parameters. Assuming the fields in theory transform as $\delta(\epsilon)\phi^i(x) = \epsilon^A(T_A\phi^i(x))$, the following is the definition for a covariant derivative

$$D_\mu\phi^i \equiv \left(\partial_\mu - \delta(B_\mu)\right)\phi^i = \left(\partial_\mu - B_\mu^A T_A\right)\phi^i, \tag{4.57}$$

where $\delta(B_\mu)$ refers to the inclusion of all gauge transformations with B_μ^A as a gauge parameter. It is easy to show that if the symmetry algebra is closed off-shell, then the gauge transformations commute with covariant differentiation, i.e. $\delta(D_\mu\phi^i) = \epsilon^A D_\mu(T_A\phi^i)$. The gauge field strength or curvature is defined as usual

$$[D_\mu, D_\nu] = -\delta(R_{\mu\nu}), \quad R_{\mu\nu}^A = 2\partial_{[\mu}B_{\nu]}^A + B_\nu^C B_\mu^B f_{BC}{}^A, \tag{4.58}$$

again $\delta(R_{\mu\nu})$ is a gauge transformation with curvature as a gauge parameter. An important example is the curvature for translations $R_{\mu\nu}(P^a)$. Following the definition (4.58)

$$R_{\mu\nu}(P^a) = 2\,\partial_{[\mu}e_{\nu]}^a + e_\mu^d\omega_\nu^{bc}f_{d[bc]}{}^a + \psi_\nu^\alpha\psi_\mu^\beta f_{\beta\alpha}{}^a, \tag{4.59}$$

$$= 2\,\partial_{[\mu}e_{\nu]}^a + 2\,\omega_{[\mu}^{ab}e_{\nu]b} - \tfrac{1}{2}\bar{\psi}_\mu\gamma^a\psi_\nu. \tag{4.60}$$

The last term is the familiar torsion term of SUGRA and from the vielbein postulate, i.e. $\nabla_{[\mu}e_{\nu]}^a = \partial_{[\mu}e_{\nu]}^a + \omega_{[\mu}^{ab}e_{\nu]b} - \Gamma_{[\mu\nu]}^\sigma e_\sigma^a = 0$, which is equivalent to the first Cartan structure equation. Therefore by imposing $R_{\mu\nu}(P^a) = 0$, we effectively find the spin connection ω_μ^{ab} in terms of the other gauge fields in symmetry group we study. This is the first constraint we see, along the way we will encounter more to express the spin connection as a composite field. Before moving on to the principles of covariant differentiation [31], which will make computations a lot easier, let us point out to an ambiguity in the definition (4.57). Consider the covariant derivative of a field with a symmetry group including translations. According to our formula

$$D_\mu\phi = \partial_\mu\phi - e_\mu^a\partial_a\phi = 0, \tag{4.61}$$

signalling to a peculiarity in translations, similar to the case in the previous section. Thus, from now on, we will exclude translations in the sum over group indices. Armed with the covariant differentiation and a proper curvature definition we now have a chance to construct Lagrangians with kinetic terms.

The guiding principles for covariant derivative are the following [31]

- The covariant derivative D_a of a covariant object is a covariant object.
- The principle above implies that the derivate of a covariant quantity does not contain the derivative of a parameter.
- The transformation of a covariant object is covariant if the algebra is closed off-shell. The result is that, there is no explicit gauge field around after the transformation of a covariant object.

Note that, in the first principle covariant derivative is written in flat coordinates D_a and this is crucial for the completeness of principles (the proof is in Appendix 11A.1 [31]).

Let us apply these principles to the simple example of a chiral multiplet that can be embedded in SUGRA. Without going into details, assume that the complex scalar Z in the multiplet transforms under SUSY as $\frac{1}{\sqrt{2}}\bar{\epsilon}P_L\chi$ (where χ is a Majorana spinor) and under the YM gauge group as $\delta_{YM}Z = -\theta^A t_A Z$. The transformation of the gravitino is $\delta\psi_\mu = \partial_\mu\epsilon + \frac{1}{4}\omega_\mu^{ab}\gamma_{ab}\epsilon$, and finally assume that the gauge field transforms under SUSY as $\delta A_\mu^A = -\frac{1}{2}\bar{\epsilon}\gamma_\mu\lambda^A$, where λ^A is a Majorana spinor. Following the definition of the covariant derivative, we will use the gauge fields as parameters, resulting in the covariant derivative of Z as

$$D_a Z = \epsilon_a^\mu\left(\partial_\mu Z - \frac{1}{\sqrt{2}}\bar{\psi}_\mu P_L\chi + A_\mu^A t_A Z\right). \tag{4.62}$$

The transformation (4.62) can be decomposed as follows: First, δ will hit the vielbein $\delta e_a^\mu = -\lambda_a{}^b e_b^\mu$, which contributes a term $-\lambda_a{}^b D_b Z$. Then, we find that $\partial_\mu\delta Z$ will generate $\frac{1}{\sqrt{2}}\partial_\mu(\bar{\epsilon}P_L\chi)$ under SUSY, however, the second principle demands that there is no derivative on ϵ and the first principle requires that covariant objects will have covariant derivatives. From these arguments we deduce $\partial_\mu\delta Z \rightarrow \frac{1}{\sqrt{2}}\bar{\epsilon}D_\mu P_L\chi$. By the same arguments under YM transformations $\partial_\mu \delta Z \to -\theta^A t_A D_\mu Z$. The transformation of the second term $-\frac{1}{\sqrt{2}}\delta(\bar{\psi}_\mu P_L \chi)$ does not produce anything since, $\delta\psi_\mu$ has the derivative of a parameter and explicit gauge fields (principle 2-3). Moreover, no matter what $\delta P_L\chi$ is, it will always have an explicit gauge field in front so that, that piece also does not contribute. With the same reasoning, the last term produces a single term $-\frac{1}{2}(\bar{\epsilon}\gamma_\mu\lambda^A)t_A Z$. In its final form one has

$$\delta D_a Z = -\lambda_a{}^b D_b Z + \tfrac{1}{\sqrt{2}}\bar{\epsilon} D_a P_L \chi - \theta^A t_A D_a Z - \tfrac{1}{2}(\bar{\epsilon}\gamma_a \lambda^A) t_A Z. \tag{4.63}$$

The principles allow us to compute the transformation of a covariant quantity, without going into the actual calculations, which will be straightforward but arduous. This procedure is indeed outstanding when one considers, the conformal construction of higher derivative SUGRA theories.

4.2.4 Gauging Conformal Algebra and Einstein Gravity

Let us now examine the machinery of this conformal construction in a simple, purely bosonic example from [31]. We will acquire the Einstein action from a gauge multiplet that is coupled to a scalar field. By a gauge multiplet, we mean the gauge fields of the conformal algebra (2.32) we have discussed in chapter 1. The following set is the gauge fields and parameters of (2.32)

$$\omega_\mu^{ab}, \quad \lambda^{ab}, \quad \text{Lorentz Transformations}, \tag{4.64}$$

$$e_\mu^a, \quad \xi^a, \quad \text{Translations}, \tag{4.65}$$

$$b_\mu, \quad \lambda_D, \quad \text{Dilatation}, \tag{4.66}$$

$$f_\mu^a, \quad \lambda_K^a, \quad \text{Superconformal Transformations}. \tag{4.67}$$

The transformation of gauge fields follow from (4.50) easily. The important point is to throw away the translations in the summation. Let us consider several examples. The first one is the transformation of the vielbein δe_μ^a

$$\begin{aligned}\delta e_\mu^a =& \epsilon^C B_\mu^B f_{BC}{}^a \\ =& \lambda^{db} e_\mu^c f_{c[db]}{}^a + \lambda_D e_\mu^b \delta_b^a, \\ \delta e_\mu^a =& -\lambda^{ab} e_{\mu b} - \lambda_D e_\mu^a.\end{aligned} \tag{4.68}$$

Note that we have not taken the derivative of the parameter part, since it is a translation $\partial\xi^a$. Then, on the second line the non-zero commutators $[M,P]=P$ and $[D,P]=P$ are employed. On the other hand, the gauge field of special conformal transformations transform as

$$\delta f_\mu^a = \partial_\mu \lambda_K^a + \epsilon^C B_\mu^B f_{BC}{}^a, \tag{4.69}$$

and this time the second part has four non-zero terms. Then using the structure constants of commutators

$$\delta f_\mu^a = \partial_\mu \lambda_K^a - b_\mu \lambda_K^a + \omega_\mu^{ab} \lambda_{Kb} - \lambda^{ab} f_{\mu b} + \lambda_D f_\mu^a. \tag{4.70}$$

Following the same logic, the remaining gauge field transformations are

$$\begin{aligned}\delta\omega_\mu^{ab} =& \partial_\mu \lambda^{ab} + 2\omega_{\mu c}{}^{[a}\lambda^{b]c} - 4\lambda_K^{[a} e_\mu^{b]} \\ \delta b_\mu =& \partial_\mu \lambda_D + \lambda_K^a e_{\mu a}.\end{aligned} \tag{4.71}$$

This set of gauge fields is too crowded for Einstein theory. The metric or the vielbein is the only physical field that is relevant. Therefore, somehow we need to write the other fields in terms of vielbeins or gauge away the irrelevant ones. The way to do that is the curvature constraints that is similar to the one we have employed in the previous section $R_{\mu\nu}(P^a) = 0$. But this time, the curvature of translations has an extra term from the dilatation

$$\begin{aligned}R_{\mu\nu}(P^a) =& 2\partial_{[\mu} e_{\nu]}^a + 2\omega_{[\mu}{}^{ab} e_{\nu]b} + B_\nu^D B_\mu^b f_{bD}{}^a + B_\nu^b B_\mu^D f_{Db}{}^a \\ =& 2\partial_{[\mu} e_{\nu]}^a + 2\omega_{[\mu}{}^{ab} e_{\nu]b} + 2b_{[\mu} e_{\nu]}^a = 0.\end{aligned} \tag{4.72}$$

Solving for ω_μ^{ab} is simple: Just multiply (4.72) with a vielbein $e^{\nu b}$ and consider the sum of permutations, then ω_μ^{ab} reads

$$\begin{aligned}\omega_\mu{}^{ab}(e,b) =& 2e^{\nu[a}\partial_{[\mu} e_{\nu]}^{b]} - e^{\nu[a} e^{b]\sigma} e_{\mu c}(\partial_\nu e_\sigma^c) + b_\nu e_\mu^{[a} e^{|\nu|b]} \\ =& \omega_\mu{}^{ab}(e) + b_\nu e_\mu^{[a} e^{|\nu|b]}.\end{aligned} \tag{4.73}$$

The curvature of Lorentz transformations is related to the special conformal gauge field

$$\begin{aligned}R_{\mu\nu}{}^{ab}(M) =& R_{\mu\nu}{}^{ab}(e) + 8 f_{[\mu}^{[a} e_{\nu]}^{b]} \\ \text{where} \quad R_{\mu\nu}{}^{ab} \equiv& 2\partial_{[\mu}\omega_{\nu]}^{ab} + 2\omega_{[\mu}{}^a{}_{|c|}\omega_{\nu]}{}^{cb}.\end{aligned} \tag{4.74}$$

Multiplying by the inverse vielbein and taking the trace we find the gauge field f_μ^a in terms of curvatures that are function of vielbeins

$$2(D-2)f_\mu^a = \frac{1}{2(D-1)}e_\mu^a R - R_\mu^a. \qquad (4.75)$$

With the help of these constraints and curvatures, we have managed to write the non-physical fields in terms of e_μ^a and b_μ. Of course, the dilatation gauge field is still an extra degree of freedom which can be gauged away by fixing the K transformations. Note that, under $\delta_K b_\mu = 2\lambda_{K\mu}$, so by a clever choice b_μ can be set to zero.

After setting the stage for the construction of the action, let us consider a scalar field ϕ that transforms as

$$\delta\phi = w\lambda_D\phi, \qquad (4.76)$$

where w is the Weyl weight of the scalar ϕ. In order to generate f_μ^a's, which is equal to the curvature scalar, consider the conformal D'Lambertian

$$\Box^C\phi = \eta^{ab}D_a D_b\phi. \qquad (4.77)$$

The explicit form of the above expression is a mess to compute if we continue bluntly. However, the principles of covariant differentiation will guide us through this mess. Start with the first derivative of ϕ which is straightforward

$$D_a\phi = e_a^\mu(\partial_\mu - wb_\mu)\phi, \qquad (4.78)$$

where we have used the dilatation gauge field as a gauge parameter. Then the transformation of this will be

$$\delta D_a\phi = \delta e_a^\mu D_\mu\phi + e_a^\mu(\delta\partial_\mu\phi) - e_a^\mu w(\delta b_\mu)\phi - e_a^\mu w b_\mu\delta\phi. \qquad (4.79)$$

The first term is trivial from (4.68), which is $-\lambda_a{}^b D_b\phi + \lambda_D D_a\phi$. The second one has a derivative of ϕ, from the principles we know it should be a covariantized derivative, i.e. $w\lambda_D(D_a\phi)$. The third term has the transformation of dilatation field which has a derivative term that we throw away and the rest will contribute as $-2\lambda_{aK}w\phi$. The last term has an open gauge field so it does not contribute. The final result is

$$\delta D_a\phi = -\lambda_a{}^b D_b\phi + (w+1)\lambda_D(D_a\phi) - 2\lambda_{aK}w\phi. \qquad (4.80)$$

From this transformation we can immediately write down the second derivative, using the basic rule, i.e by taking the gauge fields as the parameters, we have

$$\Box^C \phi = e^{\mu a} \left(\partial_\mu D_a \phi - (w+1) b_\mu (D_a \phi) + \omega_{\mu ab} D^b \phi + 2w f_{\mu a} \phi \right). \tag{4.81}$$

We finally have generated the gauge field f_μ^a which will bring in a curvature term. Before writing down the action, we need to fix the Weyl weight w, by demanding a conformal invariant action, i.e. we check for the $\delta \Box^C \phi$

$$\delta \Box^C \phi = (w+2) \lambda_D \Box^C \phi + (2D - 4w - 4) \lambda_K^a D_a \phi. \tag{4.82}$$

So by choosing $w = D/2 - 1$, the following action is invariant under conformal transformations

$$I = -\frac{1}{2} \int d^D x \, e \, \phi \Box^C \phi, \tag{4.83}$$

where e is the determinant of the metric. After gauge fixing $b_\mu = 0$ and replacing $e_a^\mu f_\mu^a = -R/\left(4(D-1)\right)$, the action (4.83) reads

$$\begin{aligned} I = &-\frac{1}{2} \int d^D x e \phi \Box^C \phi = -\frac{1}{2} \int d^D x e \phi \left[D^a (e_a^\nu \partial_\nu \phi) - \frac{R(D-2)\phi}{4(D-1)} \right] \\ = &\int dx^D e \left[\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \frac{(D-2)}{8(D-1)} R \phi^2 \right]. \end{aligned} \tag{4.84}$$

With a final gauge fixing on ϕ we can retrieve Einstein gravity.

In this simple example we have seen that by coupling a gauge multiplet to a scalar field, and with suitable constraints and gauge choices, it is possible to generate Einstein gravity. The supersymmetric version has the same logic, albeit with more gauge fields and symmetries to handle.

4.3 3-Dimensional Supergravity Theories

Despite having a simpler fermionic algebra, three dimensional supergravity theories offer a rich structure. The theories we are going to consider will have extended supersymmetry i.e. $\mathcal{N} = 2$ with 4 real supercharges. There are several ways to obtain different 3-dimensional $\mathcal{N} = 2$ theories. One of the obvious ways is to consider the dimensional reduction of the four dimensional $\mathcal{N} = 1$ SUGRA action. The Poincaré versions of four dimensional $\mathcal{N}=1$ and three dimensional $\mathcal{N}=2$ SUGRA are very similar. However it is not possible to recover all possible three dimensional $\mathcal{N}=2$ theories with dimensional reduction, and especially those theories that possess Chern-Simons terms which are unique to three dimensions. One should use more systematic ways, such as the superconformal method or superspace techniques. Besides being systematic, these tools also provide the off-shell formulations of such SUGRA theories which are also valuable for their own sake.

When it comes to AdS, there are several disguises of supergravity theories in $3D$, which were found by Achuarro and Townsend [120]. Basically three dimensional $\mathcal{N}$ extended theories have AdS supergroups $OSp(p|2;\mathbb{R})\otimes OSp(q|2;\mathbb{R})$, where $p,q\in\mathbb{Z}^+$ and $\mathcal{N}=p+q$ with $p\geq q$. The $(0,0)$ theory corresponds to the good old cosmological Einstein theory without supersymmetry. The $(1,0)$ theory is the basic $\mathcal{N}=1$ theory that first appeared in [121, 122], with the action

$$e^{-1}\mathcal{L}_{\rm R} = R-2S^2-\bar{\psi}_\mu\gamma^{\mu\nu\rho}D_\nu(\omega)\psi_\rho\,, \tag{4.85}$$

$$e^{-1}\mathcal{L}_{\rm C} = S+\tfrac{1}{8}\bar{\psi}_\mu\gamma^{\mu\nu}\psi_\nu\,, \tag{4.86}$$

$$\mathcal{L} = \mathcal{L}_{\rm R}+\mathcal{L}_{\rm C} \tag{4.87}$$

where the first term corresponds to the Poincaré part and the second is the cosmological constant. This Lagrangian is invariant under the following transformation rules

$$\delta e_\mu{}^a = \tfrac{1}{2}\bar{\epsilon}\gamma^a\psi_\mu\,, \tag{4.88}$$

$$\begin{aligned}\delta\psi_\mu &= D_\mu(\omega)\epsilon+\tfrac{1}{2}S\gamma_\mu\epsilon\,,\\ \delta S &= \tfrac{1}{4}\bar{\epsilon}\gamma^{\mu\nu}\psi_{\mu\nu}(\omega)-\tfrac{1}{4}S\bar{\epsilon}\gamma^\mu\psi_\mu\,,\end{aligned} \tag{4.89}$$

where

$$D_\mu(\omega)\epsilon = \Big(\partial_\mu+\tfrac{1}{4}\omega_\mu{}^{ab}\gamma_{ab}\Big)\epsilon,\quad \psi_{\mu\nu}=D_{[\mu}(\omega)\psi_{\nu]}\,. \tag{4.90}$$

Here the off-shell multiplet consists of a graviton, a real scalar and a Majorana gravitino, with the off-shell degrees of freedom 3+1=4. In $\mathcal{N}=2$, we have different choices. The first one is the $(1,1)$ theory where the Poincaré part is [123, 124]

$$e^{-1}\mathcal{L}_{EH}=R+2V^2-2\,|S|^2-\left(\bar{\psi}_\mu\,\gamma^{\mu\nu\rho}D_\nu(\widehat{\omega})\psi_\rho+h.c.\right)\,, \tag{4.91}$$

and the cosmological part is

$$e^{-1}\mathcal{L}_C=S-\frac{1}{4}\tilde{\psi}_\mu\,\gamma^{\mu\nu}\,\psi_\nu+h.c., \tag{4.92}$$

with two auxiliary fields, a complex scalar S and a real vector V_μ (3+2+3=8). The theory is invariant under supersymmetry transformations

$$\begin{aligned}
\delta e_\mu{}^a &= \tfrac{1}{2}\bar{\epsilon}\gamma^a\psi_\mu + h.c.,\\
\delta\psi_\mu &= D_\mu(\widehat{\omega})\,\epsilon - \tfrac{1}{2}\mathrm{i}V_\nu\,\gamma^\nu\gamma_\mu\,\epsilon - \tfrac{1}{2}S\gamma_\mu\,(B\epsilon)^*\,,\\
\delta V_\mu &= \tfrac{1}{8}\mathrm{i}\bar{\epsilon}\,\gamma^{\nu\rho}\gamma_\mu\Big(\psi_{\nu\rho} - \mathrm{i}V_\sigma\gamma^\sigma\gamma_\nu\,\psi_\rho - S\gamma_\nu\,(B\psi_\rho)^*\Big) + h.c.,\\
\delta S &= -\tfrac{1}{4}\tilde{\epsilon}\,\gamma^{\mu\nu}\Big(\psi_{\mu\nu} - \mathrm{i}V_\sigma\,\gamma^\sigma\gamma_\mu\psi_\nu - S\gamma_\mu\,(B\psi_\nu)^*\Big)\,,
\end{aligned} \tag{4.93}$$

where

$$D_\mu(\widehat{\omega})\epsilon = (\partial_\mu + \tfrac{1}{4}\widehat{\omega}_\mu{}^{ab}\,\gamma_{ab})\epsilon\,, \qquad \psi_{\mu\nu} = 2D_{[\mu}(\widehat{\omega})\psi_{\nu]}\,. \tag{4.94}$$

The $(1,1)$ theory can be obtained as a dimensional reduction from $\mathcal{N}=1$, $D=4$ AdS supergravity. The other choice $(2,0)$ is the novel one. It is given by the Poincaré and cosmological parts [119, 125, 126]

$$e^{-1}\mathcal{L}_{EH} = R - 2G^2 - 8D^2 - 8\epsilon^{\mu\nu\rho}\,C_\mu\,\partial_\nu V_\rho, \tag{4.95}$$

$$e^{-1}\mathcal{L}_C = 2D - \epsilon^{\mu\nu\rho}\,C_\mu\,G_{\nu\rho} - \left(\frac{1}{8}\bar{\psi}_\mu\,\gamma^{\mu\nu}\,\psi_\nu + h.c.\right)\,, \tag{4.96}$$

where the gravitino ψ_μ is a Dirac vector spinor, V_μ is a $U(1)_R$ symmetry gauge field, C_μ is a vector gauge field and D is an auxiliary real scalar. We have also defined

$$G_\mu := \epsilon_{\mu\nu\rho}G^{\nu\rho}\,, \qquad G^2 := G_\mu G^\mu\,. \tag{4.97}$$

The local supersymmetry transformation rules that leaves the $(2,0)$ theory invariant are

$$\begin{aligned}
\delta e_\mu{}^a &= \tfrac{1}{2}\,\bar{\epsilon}\,\gamma^a\,\psi_\mu + h.c.,\\
\delta\psi_\mu &= \big(\partial_\mu + \tfrac{1}{4}\widehat{\omega}_\mu{}^{ab}\,\gamma_{ab} - \mathrm{i}V_\mu\big)\,\epsilon - \tfrac{1}{2}\mathrm{i}\,\gamma_\mu\gamma\cdot\widehat{G}\epsilon - \gamma_\mu\,D\epsilon,\\
\delta C_\mu &= -\tfrac{1}{4}\mathrm{i}\,\bar{\epsilon}\,\psi_\mu + h.c.,\\
\delta V_\mu &= -\tfrac{1}{2}\mathrm{i}\bar{\epsilon}\,\gamma^\nu\widehat{\psi}_{\mu\nu} + \tfrac{1}{8}\mathrm{i}\bar{\epsilon}\,\gamma_\mu\gamma\cdot\widehat{\psi} - \tfrac{1}{2}\bar{\epsilon}\,\gamma\cdot\widehat{G}\,\psi_\mu + \mathrm{i}D\bar{\epsilon}\,\psi_\mu + h.c.,\\
\delta D &= -\tfrac{1}{16}\bar{\epsilon}\,\gamma\cdot\widehat{\psi} + h.c.,
\end{aligned} \tag{4.98}$$

where the $U(1)_R$ covariant gravitino field strength is given by

$$\widehat{\psi}_{\mu\nu} = 2\left(\partial_{[\mu} + \tfrac{1}{4}\widehat{\omega}_{[\mu|}{}^{ab}\,\gamma_{ab} - \mathrm{i}V_{[\mu}\right)\psi_{\nu]} - \mathrm{i}\gamma_{[\mu}\gamma\cdot\widehat{G}\psi_{\nu]} - 2D\,\gamma_{[\mu}\psi_{\nu]}\,. \tag{4.99}$$

The Chern Simons term makes this theory unique to three dimensions, i.e. it can not be obtained from dimensional reduction like the $(1,1)$ theory.

Similar to the case in which the four dimensional $\mathcal{N} = 1$ theory has different conformal compensators coupled to a Weyl multiplet. The $(1,1)$ and $(2,0)$ actions are constructed from different compensating multiplets and they are off-shell supersymmetric theories, which means that one does not need to impose the field equations for invariance under supersymmetry transformations. One can consider the scalar multiplet for the $(1,1)$ theory and the vector multiplet for the $(2,0)$ theory. There is even a third multiplet called the linear complex multiplet which is dual to the scalar multiplet for the $(1,1)$ theory.

These actions can be extended to involve higher derivative terms. However, performing these extensions in an off-shell setting is not an easy task at all. Obviously one can not simply add an R^2 or $R_{\mu\nu}R^{\mu\nu}$ term. The superconformal construction should be extended to involve higher order derivatives [119].

Luckily for three dimensions we have a special higher curvature extension i.e. Chern-Simons terms

$$e^{-1}\mathcal{L}_{\rm CS} = \quad -\frac{1}{4\mu}\,\varepsilon^{\mu\nu\rho}\left(R_{\mu\nu}{}^{ab}\omega_{\rho ab} + \tfrac{2}{3}\omega_{\mu}^{ab}\omega_{\nu b}{}^{c}\omega_{\rho ca}\right) + \frac{1}{\mu}\varepsilon^{\mu\nu\rho}F_{\mu\nu}V_{\rho}\,, \tag{4.100}$$

and the fermionic CS piece, up to quartic fermion terms, is given by

$$e^{-1}\mathcal{L}_{\rm CS} = \quad \frac{1}{4\mu}\,\varepsilon^{\mu\nu\rho}\left(R_{\rho\tau} - \tfrac{1}{4}Rg_{\rho\tau}\right)\bar{\psi}_{\mu}\gamma^{\tau}\psi_{\nu} - \frac{1}{\mu}\,\bar{R}^{\mu}\gamma_{\nu}\gamma_{\mu}R^{\nu} \tag{4.101}$$

with $R^{\mu} = \varepsilon^{\mu\nu\rho}(D_{\nu} - iA_{\nu})\psi_{\rho}$. These terms are really special since they are invariant under supersymmetry transformations without the need for extra terms, so one can simply couple CS terms to the actions we have discussed. By coupling these higher derivative terms, the auxiliary fields begin to propagate. In [127], the authors simply take the $(1,1)$ cosmological Einstein theory coupled to Chern-Simons terms (4.100). Then, they consider the off-shell Killing spinors (in Lorentzian signature) and find the spacelike-squashed, timelike-stretched AdS_3 for the spacelike and timelike norms of auxiliary field in the $(1,1)$ theory.

Motivated by the off-shell Killing spinor and supersymmetric background analysis of [127, 128, 129], we will try to do the same to the $(1,1)$ higher curvature theory given by the action below

$$\begin{aligned}\mathcal{L}_{\rm CNMG} &= \sigma(R+2V^2-2|S|^2)+4MA \qquad (4.102)\\ &+\frac{1}{m^2}\Big[R_{\mu\nu}R^{\mu\nu}-\tfrac{3}{8}R^2-R_{\mu\nu}V^\mu V^\nu-F_{\mu\nu}F^{\mu\nu}+\tfrac{1}{4}R(V^2-B^2)\\ &\quad +\tfrac{1}{6}|S|^2(A^2-4B^2)-\tfrac{1}{2}V^2(3A^2+4B^2)-2V^\mu B\partial_\mu A\Big]\,,\end{aligned}$$

where $\sigma=\pm1$ controls the sign of Einstein term and the complex auxiliary scalar is decomposed as $S\equiv A+iB$. The following are the field equations derived from the variations of the fields A, B, V_μ and $g_{\mu\nu}$ respectively

$$\begin{aligned}0 &= 4M-4\sigma A+\frac{1}{m^2}\left[\frac{2}{3}A^3-B^2A-3V^2A+2\left(\nabla\cdot V\right)B+2V^\mu\partial_\mu B\right]\,,\\ 0 &= 4\sigma B+\frac{1}{m^2}\left[\frac{1}{2}RB+A^2B+\frac{8}{3}B^3+4V^2B+2V^\mu\partial_\mu A\right]\,,\\ 0 &= 4\sigma V_\mu-\frac{1}{m^2}\left[2R_{\mu\nu}V^\nu+4\nabla_\nu F_\mu{}^\nu+V_\mu\left(3A^2+4B^2-\frac{R}{2}\right)+2B\partial_\mu A\right]\,,\\ 0 &= \sigma\Big(R_{\mu\nu}+2V_\mu V_\nu-\frac{1}{2}g_{\mu\nu}[R+2V^2-2(A^2+B^2)]\Big)-2g_{\mu\nu}MA\\ &+\frac{1}{m^2}\Bigg[\Box R_{\mu\nu}-\frac{1}{4}\nabla_\mu\nabla_\nu R+\frac{9}{4}RR_{\mu\nu}-4R^\rho_\mu R_{\nu\rho}-2F_\mu{}^\rho F_{\nu\rho}\\ &+\frac{1}{4}RV_\mu V_\nu-2R^\rho_{(\mu}V_{\nu)}V_\rho-\frac{1}{2}\Box(V_\mu V_\nu)+\nabla_\rho\nabla_{(\mu}(V_{\nu)}V^\rho)\\ &+\frac{1}{4}R_{\mu\nu}(V^2-B^2)-\frac{1}{4}\nabla_\mu\nabla_\nu(V^2-B^2)-\frac{1}{2}V_\mu V_\nu(3A^2+4B^2)\\ &-2BV_{(\mu}\partial_{\nu)}A-\frac{1}{2}g_{\mu\nu}\Big(\frac{13}{8}R^2+\frac{1}{2}\Box R-3R^2_{\rho\sigma}-R_{\rho\sigma}V^\rho V^\sigma\\ &+\nabla_\rho\nabla_\sigma(V^\rho V^\sigma)-F^2_{\rho\sigma}+\frac{1}{4}R(V^2-B^2)-\frac{1}{2}\Box(V^2-B^2)\\ &+\frac{1}{6}(A^2+B^2)(A^2-4B^2)-\frac{1}{2}V^2(3A^2+4B^2)-2BV^\rho\partial_\rho A\Big)\Bigg]\,. \qquad (4.103)\end{aligned}$$

The $m\to\infty$ limit is the cosmological Poincaré supergravity, i.e. A, B and V_μ can be eliminated algebraically. In the next section, we will show that the Minkowski or AdS_3 is a maximally supersymmetric background. By imposing projection conditions on ϵ, we can generate more solutions, with less supersymmetry.

Without higher derivative terms, the auxiliary fields can be eliminated from the theory, resulting in an on-shell supergravity theory with the field content $(e_\mu{}^a, \psi_\mu)$. However, once those terms are added, the 'auxiliary' fields become dynamical and contribute to the supersymmetric backgrounds allowed by the CNMG Lagrangian (4.102).

4.4 Killing Spinors

Killing spinors are one of the powerful tools in the realm of the classical solutions of the supersymmetric theories. The supergravity theories are invariant under supersymmetry transformations with arbitrary spinors $\epsilon(x)$ as transformation parameters. Killing spinors are the subset of these arbitrary spinors, for which the classical solutions of the theory remain invariant when transformed with one of the elements of this subset. Therefore, Killing spinors do not have to preserve all of the supersymmetry. In order to find these spinors we have to make an important assumption: the classical solution we are going to look for, whether it is a background, a black hole a pp-wave or whatever it is, it should have vanishing fermions, which means that

$$\delta(\epsilon)\text{boson} = \epsilon\text{fermion} = 0 \qquad \delta(\epsilon)\text{fermion} = \epsilon\text{boson} = 0. \tag{4.104}$$

The first equation is trivially satisfied since all fermions vanish, but from the second one we have a condition for the spinor $\epsilon(x)$ and that is the Killing spinor equation. The theory we are looking for may have $\mathcal{N}$ number of supersymmetries, but as we have said, the solutions may not preserve all of the supersymmetries of the theory. The supergravity solutions with Killing spinors are called Bogomol'nyi, Prasad and Sommerfield (BPS) solutions, e.g. 1/2 BPS means a solution with a Killing spinor preserving half of the supersymmetry of a theory.

From (4.104) and the gravitino transformation, we see that $\delta\psi_\mu = 0 = D_\mu\epsilon(x)$, where D_μ is the covariant derivative for the theory or background we consider (for Minkowski background we simply have ∂_μ). The integrability condition will then follow from the commutator of $[D_\mu, D_\nu]\epsilon(x) = \frac{1}{2}R_{\mu\nu}{}^{ab}\gamma_{ab}\epsilon(x)$. This condition will largely constraint the geometry Killing spinor is living in. Finally, it can be easily seen from the Killing spinor bilinears, $K^\mu = \bar{\epsilon}\gamma^\mu\epsilon$, one can construct Killing vectors K_μ, such that $\nabla_{(\mu}K_{\nu)} = 0$. The Killing vectors constructed from spinors provide valuable information about the solutions we are after. Since the Killing spinor equations are first order in derivatives, it is easier to solve and obtain constraints on metric functions.

4.4.1 Off-Shell Killing Spinors

Although we have discussed Killing spinors based on solutions of a theory, we can extend this definition to the off-shell Killing spinors. The Killing spinor equation follows from (4.104) which is the result of gravitino transformation. If we are given a theory with auxiliary fields, e.g. the $(1,1)$ or $(2,0)$ theories, the off-shell Killing spinors can be studied by simply setting $\delta\psi_\mu = 0$ from (4.98), (4.93). Thus the most general form of the metric and auxiliary fields can be found assuming that there exists at least one Killing spinor. From the off-shell Killing spinor bilinears, one can construct null or timelike Killing vectors and look for the possible forms of pp-wave solutions metrics with a timelike Killing vector. Once the equations of motion is imposed, the solutions are found.

4.4.2 Maximally Supersymmetric Backgrounds

Let us start the analysis of off-shell Killing spinors by considering the maximally supersymmetric backgrounds i.e. Killing spinors without any projection conditions on them. We start with the Killing spinor equation obtained through the gravitino transformation (4.93)

$$\begin{aligned}\mathcal{D}_\mu\epsilon =& D_\mu(\widehat{\omega})\epsilon - \frac{1}{2}\mathrm{i}V_\nu\,\gamma^\nu\gamma_\mu\,\epsilon - \frac{1}{2}S\gamma_\mu\epsilon^* = 0\,,\\ =& D_\mu(\widehat{\omega})\epsilon - \frac{i}{2}V_\mu\epsilon + \frac{i}{2}\gamma_\mu{}^\rho V_\rho\epsilon - \frac{1}{2}A\gamma_\mu\epsilon^* - \frac{i}{2}B\gamma_\mu\epsilon^*,\end{aligned} \tag{4.105}$$

where we have used $\gamma_m\gamma_n = \gamma_{\mu\nu} + g_{\mu\nu}$ and defined $S = A + iB$. To extract a geometric quantity out of this first order equation, we will take one more derivative and check the commutator or integrability condition. The second derivative will read

$$
\begin{aligned}
\mathcal{D}_\mu\mathcal{D}_\nu\epsilon = & D_\mu D_\nu\epsilon - \frac{i}{2}(\nabla_\mu V_\nu)\epsilon + \frac{i}{2}\gamma_\nu{}^\rho(\nabla_\mu V_\rho)\epsilon + \frac{i}{2}\gamma_\nu{}^\rho V_\rho D_\mu\epsilon \\
& - \frac{1}{2}(\nabla_\mu A)\gamma_\nu\epsilon^* - \frac{1}{2}A\gamma_\nu D_\mu\epsilon^* - \frac{i}{2}(\nabla_\mu B)\gamma_\nu\epsilon^* - \frac{i}{2}B\gamma_\nu D_\mu\epsilon^* \\
& - \frac{i}{2}V_\mu D_\nu\epsilon - \frac{1}{4}V_\mu V_\nu\epsilon + \frac{1}{4}\gamma_\nu{}^\rho V_\mu V_\rho\epsilon + \frac{i}{4}AV_\mu\gamma_\nu\epsilon^* - \frac{1}{4}BV_\mu\gamma_\nu\epsilon^* \\
& + \frac{i}{2}\gamma_\mu{}^\rho V_\rho D_\nu\epsilon + \frac{1}{4}\gamma_\mu{}^\rho V_\rho V_\nu\epsilon - \frac{1}{4}\gamma_\mu{}^\rho\gamma_\nu{}^\sigma V_\rho V_\sigma\epsilon - \frac{i}{4}AV_\rho\gamma_\mu{}^\rho\gamma_\nu\epsilon^* + \frac{1}{4}BV_\rho\gamma_\mu{}^\rho\gamma_\nu\epsilon^* \\
& - \frac{1}{2}A\gamma_\mu D_\nu\epsilon^* - \frac{i}{4}AV_\nu\gamma_\mu\epsilon^* + \frac{i}{4}AV_\rho\gamma_\mu\gamma_\nu{}^\rho\epsilon + \frac{1}{4}A^2\gamma_\mu\gamma_\nu\epsilon - \frac{i}{4}AB\gamma_\mu\gamma_\nu\epsilon \\
& - \frac{i}{2}B\gamma_\mu D_\nu\epsilon^* + \frac{1}{4}BV_\nu\gamma_\mu\epsilon^* - \frac{1}{4}BV_\rho\gamma_\mu\gamma_\nu{}^\rho\epsilon^* + \frac{i}{4}AB\gamma_\mu\gamma_\nu\epsilon + \frac{1}{4}B^2\gamma_\mu\gamma_\nu\epsilon.
\end{aligned} \tag{4.106}
$$

Note that, when D_μ hits on vectors it becomes ∇_μ and the derivative of the gamma matrices is zero $\nabla_\mu\gamma_\nu = 0$. Antisymmetrizing on μ, ν

$$
\begin{aligned}
[\mathcal{D}_\mu, \mathcal{D}_\nu]\epsilon = & \frac{1}{4}R_{\mu\nu}{}^{\rho\sigma}\gamma_{\rho\sigma}\epsilon - \frac{i}{2}F_{\mu\nu}\epsilon + i\gamma_{[\nu}{}^\rho(\nabla_{\mu]}V_\rho)\epsilon - (\nabla_{[\mu}A)\gamma_{\nu]}\epsilon^* - i(\nabla_{[\mu}B)\gamma_{\nu]}\epsilon^* \\
& + iAV_{[\mu}\gamma_{\nu]}\epsilon^* - BV_{[\mu}\gamma_{\nu]}\epsilon^* - \frac{1}{2}\gamma_{[\mu}{}^\rho\gamma_{\nu]}^\sigma V_\rho V_\sigma\epsilon - \frac{i}{2}AV_\rho\gamma_{[\mu}{}^\rho\gamma_{\nu]}\epsilon^* \\
& + \frac{1}{2}BV_\rho\gamma_{[\mu}{}^\rho\gamma_{\nu]}\epsilon^* + \frac{i}{2}AV_\rho\gamma_{[\mu}\gamma_{\nu]}{}^\rho - \frac{1}{2}BV_\rho\gamma_{[\mu}\gamma_{\nu]}{}^\rho\epsilon^* + \frac{1}{2}(A^2+B^2)\gamma_{\mu\nu}\epsilon.
\end{aligned} \tag{4.107}
$$

We will now employ the product rules for gamma matrices similar to the one (4.5), $\gamma_\mu{}^\rho\gamma_\nu = \gamma_\mu{}^\rho{}_\nu + \gamma_\mu\delta_\nu{}^\rho - \gamma^\rho g_{\mu\nu}$. Applying this equality on (4.107)

$$
\begin{aligned}
[\mathcal{D}_\mu, \mathcal{D}_\nu]\epsilon = & \frac{1}{4}R_{\mu\nu}{}^{\rho\sigma}\gamma_{\rho\sigma}\epsilon - \frac{i}{2}F_{\mu\nu}\epsilon + i\gamma_{[\nu}{}^\rho(\nabla_{\mu]}V_\rho)\epsilon - (\nabla_{[\mu}A)\gamma_{\nu]}\epsilon^* - i(\nabla_{[\mu}B)\gamma_{\nu]}\epsilon^* \\
& + iAV_{[\mu}\gamma_{\nu]}\epsilon^* - BV_{[\mu}\gamma_{\nu]}\epsilon^* - V^\rho V_{[\nu}\gamma_{\mu]\rho}\epsilon + \frac{1}{2}V^2\gamma_{\mu\nu}\epsilon + iA\gamma_{\mu\nu}{}^\rho V_\rho\epsilon^* \\
& - B\gamma_{\mu\nu}{}^\rho V_\rho\epsilon^* + \frac{1}{2}(A^2+B^2)\gamma_{\mu\nu}\epsilon.
\end{aligned} \tag{4.108}
$$

Collecting terms that has the same rank gamma matrices and keeping in mind that $\gamma_{\mu\nu\rho} = -\epsilon_{\mu\nu\rho}$ in three dimensions, one has

$$
\begin{aligned}
[\mathcal{D}_\mu, \mathcal{D}_\nu]\epsilon = & \frac{1}{4}\Big(R_{\mu\nu}{}^{\rho\sigma} + 2\delta^\rho_\mu\delta^\sigma_\nu(A^2+B^2) + 2\delta^\rho_\mu\delta^\sigma_\nu V^2 - 4\mathrm{i}\delta^\sigma_{[\nu}\nabla_{\mu]}V^\rho - 4\delta^\sigma_{[\nu}V_{\mu]}V^\rho\Big)\gamma_{\rho\sigma}\epsilon \\
& - \delta^\sigma_{[\nu}\Big(\partial_{\mu]}A + BV_{\mu]}\Big)\gamma_\sigma\epsilon^* - \mathrm{i}\delta^\sigma_{[\nu}\Big(\partial_{\mu]}B - AV_{\mu]}\Big)\gamma_\sigma\epsilon^* - \frac{1}{2}\mathrm{i}F_{\mu\nu}\epsilon \\
& + \mathrm{i}\epsilon_{\mu\nu\rho}V^\rho(A + \mathrm{i}B)\epsilon^* = 0.
\end{aligned} \tag{4.109}
$$

Before moving on to the cases with broken supersymmetry, let us try to simplify the integrability condition and capture the AdS and Minkowski backgrounds from

(4.109) by multiplying it with γ^μ. After rearranging terms and utilising gamma matrix contractions (4.3), $\gamma^\rho\gamma_{\rho\mu} = 2\gamma_\mu$, $\gamma^\rho\gamma_{\rho\mu\nu} = \gamma_{\mu\nu}$, one gets

$$
\begin{aligned}
\gamma^\mu[\mathcal{D}_\mu, \mathcal{D}_\nu]\epsilon = &\Big(\frac{1}{2}R_{\mu\nu} - \frac{i}{2}F_{\mu\nu} - \frac{i}{2}(\nabla_\alpha V^\alpha)g_{\mu\nu} + (A^2 + B^2)g_{\mu\nu} \\
&\qquad - \frac{1}{2}V_\mu V_\nu + \frac{1}{2}V^2 g_{\mu\nu}\Big)\gamma^\mu\epsilon \\
&- \frac{i}{2}(\nabla_\mu V_\rho)\gamma_\nu{}^{\mu\rho}\epsilon + \left(-\frac{1}{2}(\nabla_\mu A) - \frac{i}{2}\nabla_\mu B + \frac{1}{2}BV_\mu - \frac{i}{2}AV_\mu\right)\gamma^\mu{}_\nu\epsilon^* \\
&+ \left((\nabla_\nu A) + i(\nabla_\nu B) - iAV_\nu + BV_\nu\right)\epsilon^* = 0.
\end{aligned}
\tag{4.110}
$$

From the first line

$$
R_{\mu\nu} = iF_{\mu\nu} + i(\nabla_\alpha V^\alpha)g_{\mu\nu} - 2(A^2 + B^2)g_{\mu\nu} + V_\mu V_\nu - V^2 g_{\mu\nu}, \tag{4.111}
$$

we see that AdS is a solution for $F_{\mu\nu} = 0, \quad V_\mu = \text{constant}, \quad \partial_\mu A = \partial_\nu B = 0$ and $-1/\sqrt{A^2 + B^2}$ equals to the AdS_3 radius.

4.4.3 Off-Shell Killing Spinor Analysis

After finding the maximally supersymmetric backgrounds, let us proceed to the construction of bilinears out of Killing spinors. These bilinears will constrain the functions that appear in the background solutions. In what follows, we will assume that we have at least one unbroken supersymmetry and follow the discussion of [127].

Consider the flipping relations of bilinears we have found (4.36), (4.37) and let us assume that the spinor appearing in the equalities is the same Killing spinor ϵ. Then for rank zero matrices[3] , i.e. identity matrix, these relations imply

$$
\bar{\epsilon}\epsilon^* = \tilde{\epsilon}\epsilon = 0\,. \tag{4.112}
$$

From the remaning bilinear (4.35) we can define a scalar as follows

$$
\begin{aligned}
\bar{\epsilon}\epsilon = &- \tilde{\epsilon}\epsilon = \bar{\epsilon}_1\epsilon_1 + \bar{\epsilon}_2\epsilon_2 + i(\bar{\epsilon}_1\epsilon_2 - \bar{\epsilon}_2\epsilon_1), \\
=&2i\bar{\epsilon}_1\epsilon_2 = if, \quad \text{where} \quad f \equiv 2\bar{\epsilon}_1\epsilon_2,
\end{aligned}
\tag{4.113}
$$

where f is a real function since ϵ_1, ϵ_2 are Majorana spinors and for Majorana spinors $\bar{\epsilon}\epsilon = -\bar{\epsilon}\epsilon = 0$. The bilinears that include rank-1 are more interesting. A close scrutiny

[3] Note that, here we use the rank of the matrix as the number of spacetime indices on a gamma matrix.

of (4.35), (4.36) shows that

$$
\begin{aligned}
K^\mu \equiv \bar{\epsilon}\gamma^\mu\epsilon = \tilde{\epsilon}\gamma^\mu\epsilon^* =&\bar{\epsilon}_1\gamma^\mu\epsilon_1 + \bar{\epsilon}_2\gamma^\mu\epsilon_2 + i(\bar{\epsilon}_1\gamma^\mu\epsilon_2 - \bar{\epsilon}_2\gamma^\mu\epsilon_1),\\
K^\mu =& \bar{\epsilon}_1\gamma^\mu\epsilon_1 + \bar{\epsilon}_2\gamma^\mu\epsilon_2. \qquad (4.114)\\
L^\mu \equiv \bar{\epsilon}\gamma^\mu\epsilon^* =&\bar{\epsilon}_1\gamma^\mu\epsilon_1 - \bar{\epsilon}_2\gamma^\mu\epsilon_2 - i(2\bar{\epsilon}_1\gamma^\mu\epsilon_2),\\
L^\mu =&S^\mu + iT^\mu,\\
\text{where} \quad S^\mu \equiv&\bar{\epsilon}_1\gamma^\mu\epsilon_1 - \bar{\epsilon}_2\gamma^\mu\epsilon_2, \quad T^\mu \equiv -2\bar{\epsilon}_1\gamma^\mu\epsilon_2, \qquad (4.115)
\end{aligned}
$$

where $K_\mu\,(L_\mu)$ is a real (complex) vector. The norm of the vectors can be obtained from the three dimensional Fierz identity (4.43), assuming all spinors are the same

$$(\bar{\epsilon}\gamma^\mu\epsilon)\gamma_\mu\epsilon = (\bar{\epsilon}\epsilon)\epsilon. \qquad (4.116)$$

Multiply this with $\bar{\epsilon}$ and use the definition (4.113)

$$(\bar{\epsilon}\gamma^\mu\epsilon)\bar{\epsilon}\gamma_\mu\epsilon = (\bar{\epsilon}\epsilon)^2 = K_\mu K^\mu = -f^2. \qquad (4.117)$$

The function f is real, therefore the vector K^μ can be either timelike or null. On the other hand, the norm of $T_\mu T^\mu$ is spacelike

$$T_\mu T^\mu = 4\bar{\epsilon}_1\left[(\bar{\epsilon}_1\gamma^\mu\epsilon_2)\gamma_\mu\epsilon\right] = 4\bar{\epsilon}_1\left[2(\bar{\epsilon}_1\epsilon_2)\epsilon_1 - (\bar{\epsilon}_1\epsilon_2)\epsilon_2\right] = 4(\bar{\epsilon}_1\epsilon_2)^2 = f^2. \qquad (4.118)$$

Although we won't use the following equalities, let us state them for convenience

$$
\begin{aligned}
S_\mu S^\mu =&\bar{\epsilon}_1\left[(\bar{\epsilon}_1\gamma^\mu\epsilon_1)\gamma_\mu\epsilon_1\right] - 2\bar{\epsilon}_1\left[(\bar{\epsilon}_2\gamma^\mu\epsilon_2)\gamma_\mu\epsilon_1\right] + \bar{\epsilon}_2\left[(\bar{\epsilon}_2\gamma^\mu\epsilon_2)\gamma_\mu\epsilon_2\right],\\
=&-2\bar{\epsilon}_1\left[(\bar{\epsilon}_2\gamma^\mu\epsilon_2)\gamma_\mu\epsilon_1\right].\\
=&-2\bar{\epsilon}_1\left[2(\bar{\epsilon}_2\epsilon_1)\epsilon_2 - (\bar{\epsilon}_2\epsilon_2)\epsilon_1\right],\\
S_\mu S^\mu =&4(\bar{\epsilon}_1\epsilon_2)^2 = f^2, \qquad (4.119)\\
S^\mu T_\mu =&-2\bar{\epsilon}_1\left[(\bar{\epsilon}_1\gamma^\mu\epsilon_2)\gamma_\mu\epsilon_1\right] + 2\bar{\epsilon}_2\left[(\bar{\epsilon}_1\gamma^\mu\epsilon_2)\gamma_\mu\epsilon_2\right],\\
S^\mu T_\mu =&0. \qquad (4.120)
\end{aligned}
$$

After finding all of the vectors constructed from the bilinears, let us check what kind of differential equations they satisfy. It is obvious from the previous discussions that the bilinear K^μ must satisfy the Killing vector equation. For the sake of completeness, we first write down the Killing spinor equation and its conjugates

$$
\begin{aligned}
D_\mu(\widehat{\omega})\epsilon =&\frac{i}{2}V_\nu\gamma^\nu\gamma_\mu\epsilon + \frac{1}{2}S\gamma_\mu\epsilon^*, \qquad (4.121)\\
D_\mu(\widehat{\omega})\bar{\epsilon} =&-\frac{i}{2}V_\nu\bar{\epsilon}\gamma_\mu\gamma^\nu - \frac{1}{2}S^*\tilde{\epsilon}\gamma_\mu, \qquad (4.122)\\
D_\mu(\widehat{\omega})\epsilon^* =&-\frac{i}{2}V_\nu\gamma^\nu\gamma_\mu\epsilon^* + \frac{1}{2}S^*\gamma_\mu\epsilon. \qquad (4.123)
\end{aligned}
$$

The derivative of K_μ reads

$$\begin{aligned}
\nabla_\mu K_\nu = \nabla_\mu(\bar{\epsilon}\gamma_\nu\epsilon) =&(D_\mu(\widehat{\omega})\bar{\epsilon})\gamma_\nu\epsilon + \bar{\epsilon}\gamma_\nu(D_\mu(\widehat{\omega})\epsilon),\\
=&-\frac{i}{2}V_\alpha\bar{\epsilon}\gamma_\mu\gamma^\alpha\gamma_\nu\epsilon - \frac{1}{2}S^*\tilde{\epsilon}\gamma_\mu\gamma_\nu\epsilon + \frac{i}{2}V_\alpha\bar{\epsilon}\gamma_\nu\gamma^\alpha\gamma_\mu\epsilon + \frac{1}{2}S\bar{\epsilon}\gamma_\nu\gamma_\mu\epsilon^*,\\
=&iV_\alpha\bar{\epsilon}\gamma_{[\nu}\gamma^\alpha\gamma_{\mu]}\epsilon - \frac{1}{2}S^*\tilde{\epsilon}\gamma_{\mu\nu}\epsilon + \frac{1}{2}S\bar{\epsilon}\gamma_{\nu\mu}\epsilon^*. \qquad (4.124)
\end{aligned}$$

Symmetrising in μ, ν we find $\nabla_{(\mu}K_{\nu)} = 0$. Therefore, depending on the value of f, K^μ is a timelike or null Killing vector. Let us cast (4.124) in a more suggestive form for future manipulations, by considering gamma matrix identities from Sec. 4.1:[4]

$$\begin{aligned}
\nabla_{[\mu}K_{\nu]} = \partial_{[\mu}K_{\nu]} =&iV_\alpha\bar{\epsilon}\gamma_{[\nu}\gamma^\alpha\gamma_{\mu]}\epsilon - \frac{1}{2}S^*\tilde{\epsilon}\gamma_{\mu\nu}\epsilon + \frac{1}{2}S\bar{\epsilon}\gamma_{\nu\mu}\epsilon^*,\\
=&iV_\alpha\left[\delta^\alpha_\nu\,\bar{\epsilon}\gamma_\mu\epsilon + \bar{\epsilon}\gamma_\nu{}^\alpha\gamma_\mu\epsilon - \mu\leftrightarrow\nu\right] - \frac{1}{2}S^*\tilde{\epsilon}\gamma_{\mu\nu}\epsilon + \frac{1}{2}S\bar{\epsilon}\gamma_{\nu\mu}\epsilon^*,\\
=&iV_\alpha\left[\delta^\alpha_\nu\,\bar{\epsilon}\gamma_\mu\epsilon + \bar{\epsilon}\gamma_\nu{}^\alpha{}_\mu\epsilon + \delta^a_\mu\,\bar{\epsilon}\gamma_\nu\epsilon - \delta_{\mu\nu}\,\bar{\epsilon}\gamma^a\epsilon - \mu\leftrightarrow\nu\right]\\
&\quad - \frac{1}{2}S^*\tilde{\epsilon}\gamma_{\mu\nu}\epsilon + \frac{1}{2}S\bar{\epsilon}\gamma_{\nu\mu}\epsilon^*,\\
=&iV_\alpha\bar{\epsilon}\gamma_\nu{}^\alpha{}_\mu\epsilon + \frac{1}{2}S^*\,\varepsilon^{\alpha\mu\nu}\tilde{\epsilon}\gamma_\alpha\epsilon - \frac{1}{2}S\varepsilon^{\alpha\nu\mu}\tilde{\epsilon}\gamma_\alpha\epsilon^*,\\
=&-iV_\alpha\varepsilon_\nu{}^\alpha{}_\mu\,\bar{\epsilon}\epsilon + \frac{1}{2}\varepsilon^\alpha{}_{\mu\nu}\left[S^*(\tilde{\epsilon}\gamma_\alpha\epsilon) + S(\tilde{e}\gamma_\alpha\epsilon^*)\right],\\
=&\varepsilon_{\mu\nu}{}^\alpha\left[V_\alpha f + \frac{1}{2}(S^*L^{*\alpha} + SL^\alpha)\right]. \qquad (4.125)
\end{aligned}$$

As similar identity holds for L_μ

$$\begin{aligned}
\nabla_\mu L_\nu =&\nabla_\mu[\bar{\epsilon}\gamma_\nu\epsilon^*] = (D_\mu(\widehat{\omega})\bar{\epsilon})\gamma_\nu\epsilon^* + \bar{\epsilon}\gamma_\nu(D_\mu(\widehat{\omega})\epsilon^*),\\
=&-\frac{i}{2}V_\alpha\bar{\epsilon}\gamma_\mu\gamma^\alpha\gamma_\nu\epsilon^* - \frac{1}{2}S^*\tilde{\epsilon}\gamma_\mu\gamma_\nu\epsilon^* - \frac{i}{2}V_\alpha\bar{\epsilon}\gamma_\nu\gamma^\alpha\gamma_\mu\epsilon^* + \frac{1}{2}S^*\bar{\epsilon}\gamma_\nu\gamma_\mu\epsilon,\\
=&-iV_\alpha\left[\delta^\alpha_\mu\bar{\epsilon}\gamma_\nu\epsilon^* + \delta^\alpha_\nu\bar{\epsilon}\gamma_\mu\epsilon^* - g_{\mu\nu}\bar{\epsilon}\gamma^\alpha\epsilon^*\right]\\
&-\frac{1}{2}S^*\left[\tilde{\epsilon}\gamma_{\mu\nu}\epsilon^* + g_{\mu\nu}\tilde{\epsilon}\epsilon^*\right] + \frac{1}{2}S^*\bar{\epsilon}\gamma_{\nu\mu}\epsilon + \frac{1}{2}S^*g_{\mu\nu}\bar{\epsilon}\epsilon.\\
\nabla_\mu L_\nu =&-iV_\mu L_\nu - iV_\nu L_\mu + ig_{\mu\nu}V_\alpha L^\alpha + S^*\varepsilon_{\mu\nu}{}^\rho K_\rho + ifS^*g_{\mu\nu}. \qquad (4.126)
\end{aligned}$$

[4] In deriving (4.125) the duality relation between gamma matrices $\gamma^\mu = \frac{1}{2}\varepsilon^{\mu\nu\rho}\gamma_{\rho\nu}$, and the highest rank element $\gamma_\pm^{\mu\nu\rho} = \pm i^2\varepsilon^{\mu\nu\rho}$ for odd dimensional Clifford algebra is used. Finally three index version of gamma product rule is employed $\gamma^{\nu\alpha}{}_\mu + 2\gamma^{[\nu}\delta^{\alpha]}_\mu$. Moreover, $\nabla_{[\mu}K_{\nu]} = \partial_{[\mu}K_{\nu]}$ i.e. we set the torsion to zero, since we have imposed vanishing fermions as a Killing spinor condition.

Likewise the derivative of $i\partial_\mu f = \partial_\mu(\bar{\epsilon}\epsilon)$ is

$$
\begin{aligned}
i\partial_\mu f =& \partial_\mu(\bar{\epsilon}\epsilon) = D_\mu(\widehat{\omega})(\bar{\epsilon}\epsilon) = (D_\mu(\widehat{\omega})\bar{\epsilon})\epsilon + \bar{\epsilon}D_\mu(\widehat{\omega})\epsilon, \\
=& -\frac{i}{2}V_\nu\bar{\epsilon}\gamma_\mu\gamma^\nu\epsilon - \frac{1}{2}S^*\tilde{\epsilon}\gamma_\mu\epsilon + \frac{i}{2}V_\nu\bar{\epsilon}\gamma^\nu\gamma_\mu\epsilon + \frac{1}{2}S\bar{\epsilon}\gamma_\mu\epsilon^*, \\
=& -\frac{1}{2}S^*L_\mu^* + \frac{1}{2}SL_\mu - iV_\nu\bar{\epsilon}\gamma_\mu{}^\nu\epsilon. \\
\partial_\mu f =& -\frac{i}{2}\left(SL_\mu - S^*L_\mu^*\right) + \varepsilon_{\mu\nu}{}^\rho V^\nu K_\rho.
\end{aligned}
\tag{4.127}
$$

Also the Levi-Civita contraction of K_μ and L_μ is as follows

$$
\begin{aligned}
\varepsilon^{\mu\nu\rho}K_\nu L_\rho =& \varepsilon^{\mu\nu\rho}(\bar{\epsilon}\gamma_\nu\epsilon)(\bar{\epsilon}\gamma_\rho\epsilon^*), \\
=& \frac{1}{2}\varepsilon^{\mu\nu\rho}\left[(\bar{\epsilon}\gamma_\rho\gamma_\nu\epsilon)(\bar{\epsilon}\epsilon^*) + (\bar{\epsilon}\gamma_\rho\gamma^\sigma\gamma_\nu\epsilon)(\bar{\epsilon}\gamma_\sigma\epsilon^*)\right], \\
=& \frac{1}{2}\varepsilon^{\mu\nu\rho}\left[-\bar{\epsilon}\gamma_\rho\gamma_\nu\gamma^\sigma\epsilon + 2\delta_\nu^\sigma\bar{\epsilon}\gamma_\rho\epsilon\right]\bar{\epsilon}\gamma_\sigma\epsilon^*, \\
=& \frac{1}{2}\varepsilon^{\mu\nu\rho}\left[-\bar{\epsilon}\gamma_{\rho\nu}\gamma^\sigma\epsilon + 2\delta_\nu^\sigma\bar{\epsilon}\gamma_\rho\epsilon]\right]\bar{\epsilon}\gamma_\sigma\epsilon^*, \\
=& \frac{1}{2}\varepsilon^{\mu\nu\rho}\varepsilon_{\rho\nu}{}^\sigma(\bar{\epsilon}\epsilon)L_\sigma = ifL^\mu.
\end{aligned}
\tag{4.128}
$$

Finally, another useful Levi-Civita contraction is

$$
\varepsilon^{\mu\nu\rho}L_\mu L_\nu^* = \varepsilon^{\mu\nu\rho}(S_\mu + iT_\mu)(S_\beta - iT_\nu) = 2iS_\mu T_\nu\varepsilon^{\mu\nu\rho}, \tag{4.129}
$$

plugging in the definitions of S_μ and T_μ (4.115)

$$
\varepsilon^{\mu\nu\rho}L_\mu L_\nu^* = 4i\left[\bar{\epsilon}_1\gamma_\mu\epsilon_1 - \bar{\epsilon}_2\gamma_\mu\epsilon_2\right](\bar{\epsilon}_1\gamma_\nu\epsilon_2)\varepsilon^{\mu\nu\rho}. \tag{4.130}
$$

To put this bilinear into a suitable form, we turn back to Fierz identity (4.43) and first multiply it with γ_ν

$$
(\bar{\epsilon}_1\gamma^\mu\epsilon_2)\gamma_\nu\gamma_\mu\epsilon_3 = 2(\bar{\epsilon}_1\epsilon_3)\gamma_\nu\epsilon_2 - (\bar{\epsilon}_1\epsilon_2)\gamma_\nu\epsilon_3, \tag{4.131}
$$

and contract with $\bar{\epsilon}_4$

$$
(\bar{\epsilon}_1\gamma^\mu\epsilon_2)\bar{\epsilon}_4\gamma_{\nu\mu}\epsilon_3 + (\bar{\epsilon}_1\gamma_\nu\epsilon_2)\bar{\epsilon}_4\epsilon_3 = 2(\bar{\epsilon}_1\epsilon_3)\bar{\epsilon}_4\gamma_\nu\epsilon_2 - (\bar{\epsilon}_1\epsilon_2)\bar{\epsilon}_4\gamma_\nu\epsilon_3. \tag{4.132}
$$

To generate our first identity, take $3, 4 \to 1$ (keeping in mind that $\bar{\epsilon}\epsilon = 0$)

$$
(\bar{\epsilon}_1\gamma^\mu\epsilon_2)\bar{\epsilon}_1\gamma_{\nu\mu}\epsilon_1 = -(\bar{\epsilon}_1\epsilon_2)\bar{\epsilon}_1\gamma_\nu\epsilon_1, \tag{4.133}
$$

and for the second identity $3, 4 \to 2$

$$
(\bar{\epsilon}_1\gamma^\mu\epsilon_2)\bar{\epsilon}_2\gamma_{\nu\mu}\epsilon_2 = (\bar{\epsilon}_1\epsilon_2)\bar{\epsilon}_2\gamma_\nu\epsilon_2. \tag{4.134}
$$

Now, we are in a position to express equation (4.130) in a more compact way. First replace $\varepsilon^{\mu\nu\rho}\gamma_\rho = -\gamma_{\mu\nu}$

$$\varepsilon^{\mu\nu\rho} L_\mu L_\nu^* = 4i\left[(\bar{\epsilon_1}\gamma_\mu\epsilon_2)\bar{\epsilon_1}\gamma^{\rho\mu}\epsilon_1 - (\bar{\epsilon_1}\gamma_\mu\epsilon_2)\bar{\epsilon_2}\gamma^{\rho\mu}\epsilon_2\right], \tag{4.135}$$

then employ the Fierz identities we have derived in (4.133) and (4.134) to get

$$\varepsilon^{\mu\nu\rho} L_\mu L_\nu^* = 4i\left[(\bar{\epsilon}_1\gamma^\rho\epsilon_1) + (\bar{\epsilon}_2\gamma^\rho\epsilon_2)\right](\bar{\epsilon}_1\epsilon_2) = 2if\, K^\rho. \tag{4.136}$$

By computing various identities (4.125), (4.126), (4.127) (4.128) we have set the stage for the analysis of Killing spinors according to their norms. We will start from the null case, after a through analysis, timelike case will be considered.

4.5 The Null Killing Vector

We first start with the case, where the function f introduced in eq. (4.113) is zero everywhere, i.e. $f = 0$, which makes K_μ a null Killing vector. Previously, we have considered Dirac spinors as an object constructed out of Majorana spinors. It also was shown that the norm $\bar{\epsilon}\epsilon = 2i\bar{\epsilon}_1\epsilon_2 \equiv if$ with ϵ_1 and ϵ_2 as different Majorana spinors. Therefore, the $f = 0$ case is only possible for Dirac spinors that is proportional to a real spinor ϵ_0 up to a phase factor

$$\epsilon = e^{-\mathrm{i}\frac{\theta}{2}}\,\epsilon_0\,. \tag{4.137}$$

This observation will lead to a simplification of the following vectors we have defined previously

$$K^\mu = \tilde{\epsilon}\gamma^\mu\epsilon^* = \bar{\epsilon}_0\gamma^\mu\epsilon_0, \tag{4.138}$$

$$L^\mu = \bar{\epsilon}\gamma^\mu\epsilon^* = e^{i\theta}\bar{\epsilon}_0\gamma^\mu\epsilon_0, \tag{4.139}$$

$$L^\mu = e^{i\theta}K_\mu. \tag{4.140}$$

The equations (4.125), (4.127), then imply

$$\partial_{[\mu}K_{\nu]} - \varepsilon_{\mu\nu\rho}K^\rho\,\mathrm{Re}(Se^{i\theta}) = 0, \tag{4.141}$$

$$\mathrm{Im}(Se^{i\theta})K_\mu + \varepsilon_{\mu\nu}{}^{\rho}V^\nu K_\rho = 0. \tag{4.142}$$

Exploiting (4.140), the derivative L_μ (4.126) will read

$$\begin{aligned} \nabla_\mu L_\nu =& \nabla_\mu (e^{i\theta} K_\nu), \\ i(\partial_\mu \theta) K_\nu + \nabla_\mu K_\nu =& - iV_\nu K_\mu - iV_\mu K_\nu + ig_{\mu\nu} K_\beta V^\beta + e^{-i\theta} \varepsilon_{\mu\nu\rho} K^\rho S^*. \end{aligned} \tag{4.143}$$

Antisymmetrize the latter in $\mu\nu$, then plug it in (4.141) to find

$$\begin{aligned} i(\partial_{[\mu}\theta) K_{\nu]} + \varepsilon_{\mu\nu\rho} K^\rho \mathrm{Re}(Se^{i\theta}) =& \left[\mathrm{Re}(Se^{i\theta}) - i\mathrm{Im}(Se^{i\theta})\right] \varepsilon_{\mu\nu\rho} K^\rho, \\ i(\partial_{[\mu}\theta) K_{\nu]} =& - i\mathrm{Im}(Se^{i\theta}) \varepsilon_{\mu\nu\rho} K^\rho. \end{aligned} \tag{4.144}$$

Using the previous result on the imaginary part of $Se^{i\theta}$ (4.142), we can derive a constraint on the auxiliary field V_μ

$$\begin{aligned} i(\partial_{[\mu}\theta) K_{\nu]} =& i\varepsilon_{\mu\nu\rho} \varepsilon^\rho{}_{\alpha\beta} V^\alpha K^\beta. \\ =& - i2V_{[\mu} K_{\nu]}, \end{aligned} \tag{4.145}$$

Assuming that this holds for all K_ν, we find

$$V_\mu = -\frac{1}{2}\partial_\mu \theta. \tag{4.146}$$

On the other hand, we can learn a great deal on the geometry and the metric by contracting the equation (4.141) with K^ν

$$K^\mu \nabla_\mu K_\nu = 0\,. \tag{4.147}$$

By Frobenius' theorem, we find that K is hypersurface orthogonal and thus tangent to affinely parametrized geodesics. One can, adopt coordinates (u, v, x), with v being an affine parameter along the geodesics, i.e.

$$K^\mu\, \partial_\mu = \frac{\partial}{\partial v}\,. \tag{4.148}$$

Then it is possible to choose an *adapted coordinate system* such that the functions appearing in the metric does not depend on v [130]

$$ds^2 = h_{ij}(x,u)\, dx^i\, dx^j + 2P(x,u)\, du\, dv\,, \tag{4.149}$$

where $x^i = (x, u)$. Without loss of generality, this metric can be cast in the following form by a coordinate transformation [129, 131]

$$ds^2 = dx^2 + 2P(x,u)\, du\, dv + Q(x,u)\, du^2\,, \tag{4.150}$$

with $\sqrt{|g|} = P$. With this choice of metric and coordinates, let us reconsider equation (4.125)

$$\partial_x K_u = \varepsilon_{xuv}(S^* e^{-i\theta} + S e^{i\theta}), \tag{4.151}$$

which is the only non-zero combination. The Levi-Civita pseudo tensor is defined as $\varepsilon_{\mu\nu\rho} = \sqrt{-g}\epsilon_{\mu\nu\rho}$ in terms of the tensor density $\epsilon_{\mu\nu\rho}$, where we use the convention $\epsilon_{xuv} = 1$. The equation (4.151) will then take the form [127]

$$S e^{\mathrm{i}\theta} + S^* e^{-\mathrm{i}\theta} \;=\; \partial_x \log P(x,u)\,. \tag{4.152}$$

Note that until now we have not used any field equation, everything was off-shell and derived from the transformation properties of the gravitino. In the next subsection, we will make use of this off-shell analysis and investigate the solutions of CNMG with the null Killing vector.

4.5.1 The pp-wave solution with $A = \pm 1/l, B = 0$

We start by setting S to be a constant; to be more precise we set $A = 1/l$ and $B = 0$ in order to simplify field equations and capture AdS asymptotics when $Q(x,u) = 0$ in (4.150). Each choice of signs of $1/l$, covers different part of AdS and two charts together cover the whole AdS [129].

Using (4.127) we obtain

$$\epsilon_{\mu\nu\rho} V^\nu K^\rho = -\frac{1}{l} K_\mu \sin\theta(u,x)\,. \tag{4.153}$$

The non-zero u component of this equation reads

$$\frac{1}{l} K_u \sin\theta(u,x) = P(u,x) V_x\,. \tag{4.154}$$

For nowhere vanishing $P(u,x)$, this equation can be used to integrate (4.146). After a simple $\int d\theta/\sin\theta$ integration we find $\theta(u,x)$ to be

$$\begin{aligned} c(u)e^{-2x/l} &= \tan\frac{\theta(u,x)}{2}, \\ \theta(u,x) &= \arctan\left(\frac{2\,c(u)\,e^{-2x/l}}{1-c(u)^2\,e^{-4x/l}}\right), \end{aligned} \tag{4.155}$$

for an arbitrary function $c(u)$. With our choices, the equality (4.152) simplifies to

$$\frac{2}{l}\cos\theta(u,x) = \partial_x \log P(u,x)\,, \tag{4.156}$$

which, upon using (4.155) and the trigonometric relation $\cos\arctan(x) = 1/\sqrt{1+x^2}$, yields

$$P(u,x) = P(u)[\,e^{2x/l} + e^{-2x/l}c^2(u)]\,, \tag{4.157}$$

where $P(u)$ is an arbitrary function of u. The function $P(u)$ can be set to unity without loss of generality [129, 130]. This point is actually the farthest we can reach, without invoking field equations. To proceed, we substitute into the (4.150) vector field equation (4.103) and find $c(u) = 0$. Before moving onto the metric field equations, let us present the final form of the metric

$$ds^2 = dx^2 + 2\,e^{2x/l}\,du\,dv + Q(u,x)\,du^2\,, \tag{4.158}$$

and the limit $l \to \infty$ gives rise to the pp-wave in a Minkowski background.

The final task we undertake is to find the function $Q(u,x)$. For notational simplicity, we set $l = 1$ and feed the metric (4.158) into the metric field equations. The result is a linear fourth order ordinary differential equation with constant coefficients

$$(2+4\sigma m^2)\,Q' - (9+2\sigma m^2)\,Q'' + 8Q''' - 2Q'''' = 0\,, \tag{4.159}$$

where the prime denotes a derivative with respect to x. The most general solution of this differential equation is well known and given by

$$Q(x,u) = e^{(1-\sqrt{\frac{1}{2}-\sigma m^2})x}C_1(u) + e^{(1+\sqrt{\frac{1}{2}-\sigma m^2})x}C_2(u) + e^{2x}C_3(u) + C_4(u) \tag{4.160}$$

where the functions $C_i(u)\,, i = 1,\cdots,4$, are arbitrary functions of u. The solution we have found is the same as the one in [128] which is the $\mathcal{N} = 1$ version of the theory we study. Therefore we expect that 1/4 of the supersymmetry is conserved, which is a property we will show in the next section. The non-supersymmetric version [132] also matches to the solution we have found.

The solution $Q(u,x)$ we have found is not in the most desirable form. It actually has redundant pieces that can be gauged away by a set of suitable coordinate transformations. To that end, consider the following coordinate transformations [131]

$$x = \widetilde{x} - \tfrac{1}{2}\log a'\,, \qquad u = a(\widetilde{u})\,, \qquad v = \widetilde{v} - \tfrac{1}{4}e^{-2\widetilde{x}}\,\frac{a''}{a'} + b(\widetilde{u})\,, \tag{4.161}$$

where $a(\widetilde{u})$ and $b(\widetilde{u})$ are arbitrary functions of $\widetilde{u}$ and here prime denotes a derivative with respect to $\widetilde{u}$. The next step is to choose the arbitrary functions $a(\widetilde{u})$ and $b(\widetilde{u})$ such that, the differential equations

$$\left(\frac{a''}{a'}\right)' - \frac{1}{2}\left(\frac{a''}{a'}\right)^2 - 2(a')^2\,\widetilde{C}_4(\widetilde{u}) = 0\,, \qquad b' + \frac{1}{2}a'\,\widetilde{C}_3(\widetilde{u}) = 0\,, \tag{4.162}$$

are satisfied, i.e. the functions $\widetilde{C}_3$ and $\widetilde{C}_4$ can be set to zero. Thus one is naturally led to set $C_3 = C_4 = 0$ without loss of generality. In addition to this, after the transformations we get the new set of functions as

$$\widetilde{C}_1(\widetilde{u}) = C_1(a(\widetilde{u}))[a'(\widetilde{u})]^{\frac{1}{2}(3+\sqrt{\frac{1}{2}-\sigma m^2})},\ \widetilde{C}_2(\widetilde{u}) = C_2(a(\widetilde{u}))[a'(\widetilde{u})]^{\frac{1}{2}(3-\sqrt{\frac{1}{2}-\sigma m^2})}.$$

So far we have not fixed the values of the parameters σm^2. With a quick glance, it is easy to see that for the $\sigma m^2 = \frac{1}{2}$ case, the function with coefficient C_1 degenerates with C_2, whereas for the $\sigma m^2 = -\frac{1}{2}$ case the function with coefficient C_1 degenerates with C_4, while the function with coefficient C_2 degenerates with C_3. Following the theory of differential equations, we generate linearly independent solutions for these special values of the parameters by simply multiplying xe^{2x}, etc,

$$\begin{aligned}
\sigma m^2 = \tfrac{1}{2} \quad &: \qquad Q(u,x) = e^x\,D_1(u) + x\,e^x\,D_2(u) + e^{2x}\,D_3(u) + D_4(u)\,,\\
\sigma m^2 = -\tfrac{1}{2} \quad &: \qquad Q(u,x) = x\,e^{2x}\,D_1(u) + x\,D_2(u) + e^{2x}\,D_3(u) + D_4(u)\,.
\end{aligned}$$

Here $D_i(u)\,,\ i = 1,\ldots,4$, are arbitrary functions of u. Setting $D_3 = D_4 = 0$, we are led to the following cases:

$$\begin{aligned}
\sigma m^2 \neq \pm\tfrac{1}{2} \quad &: \quad ds^2 = dx^2 + 2\,e^{2x}\,du\,dv\\
&\qquad\qquad + \Big(e^{(1-\sqrt{\frac{1}{2}-\sigma m^2})x}\,D_1(u) + e^{(1+\sqrt{\frac{1}{2}-\sigma m^2})x}\,D_2(u)\Big)\,du^2\,,\\
\sigma m^2 = \tfrac{1}{2} \quad &: \quad ds^2 = dx^2 + 2\,e^{2x}\,du\,dv + \Big(e^x\,D_1(u) + x\,e^x\,D_2(u)\Big)\,du^2\,,\\
\sigma m^2 = -\tfrac{1}{2} \quad &: \quad ds^2 = dx^2 + 2\,e^{2x}\,du\,dv + \Big(x\,e^{2x}\,D_1(u) + x\,D_2(u)\Big)\,du^2\,.
\end{aligned}$$

This is the most general solution we can find for the null case. The next question that comes to mind is the amount of supersymmetry that these solutions preserve. In the next section we will attack this problem by working out the Killing spinor equation (4.105).

4.5.2 Killing Spinor Analysis

The following set of orthonormal frames is the most suitable for the construction of the Killing spinors of the pp-wave metric (4.158) [131]

$$e^0 = e^{\frac{2x}{l}-\beta}\,dv, \qquad e^1 = e^{\beta}du + e^{\frac{2x}{l}-\beta}\,dv, \qquad e^2 = dx\,, \tag{4.163}$$

where $Q(u,x) = e^{2\beta(u,x)}$. A simple calculation shows that the spin-connections are

$$\begin{aligned}
\omega_{01} &= -\dot{\beta}\,du - \Big(\beta' - \frac{1}{l}\Big)dx\,,\\
\omega_{02} &= -\Big(\beta' - \frac{1}{l}\Big)e^{\beta}\,du - \frac{1}{l}\,e^{\frac{2x}{l}-\beta}\,dv\,,\\
\omega_{12} &= \beta'\,e^{\beta}\,du + \frac{1}{l}\,e^{\frac{2x}{l}-\beta}\,dv\,,
\end{aligned} \tag{4.164}$$

where

$$\dot{\beta} \equiv \frac{\partial\beta}{\partial u}\,, \qquad \beta' \equiv \frac{\partial\beta}{\partial x}\,. \tag{4.165}$$

Plugging in our choices to the Killing spinor equation (4.105)[5]

$$0 = d\epsilon + \frac{1}{4}\,\omega_{ab}\,\gamma^{ab}\epsilon + \frac{1}{2l}\,\gamma_a\,e^a\,\epsilon^*\,. \tag{4.166}$$

Until now we have not made a choice for the γ matrices. As we have discussed previously, in three dimensions the representation is real and 2×2 dimensional. Hence a natural choice is

$$\gamma_0 = \mathrm{i}\sigma_2\,, \qquad \gamma_1 = \sigma_1\,, \qquad \gamma_2 = \sigma_3\,, \tag{4.167}$$

where σ_i's are the standard Pauli matrices. With this choice the Killing spinor equation reads

$$\begin{aligned}
0 = d\epsilon \;&+\; \tfrac{1}{2}\Big(\dot{\beta}\,\sigma_3\,\epsilon - e^{\beta}\beta'(\sigma_1 + \mathrm{i}\sigma_2)\,\epsilon + \frac{1}{l}e^{\beta}\sigma_1\,(\epsilon + \epsilon^*)\Big)du\\
&-\; \frac{1}{2l}\,e^{\frac{2x}{l}-\beta}\,(\sigma_1 + \mathrm{i}\sigma_2)\,(\epsilon - \epsilon^*)\,dv\\
&+\; \frac{1}{2}\,\Big(\beta'\sigma_3\,\epsilon - \frac{1}{l}\,\sigma_3\,(\epsilon - \epsilon^*)\Big)\,dx\,.
\end{aligned} \tag{4.168}$$

In order to read the components of the spinor explicitly, we decompose the Dirac spinor into two Majorana spinors as $\epsilon = \xi + \mathrm{i}\zeta$

$$\epsilon = \begin{pmatrix} \xi_1 + \mathrm{i}\zeta_1 \\ \xi_2 + \mathrm{i}\zeta_2 \end{pmatrix}\,, \tag{4.169}$$

[5] Note that, in this section we have used $A = -1/l$ which does not effect the results.

and obtain the following equations for the components

$$
\begin{aligned}
0 &= d\xi_1 + \frac{1}{2}\dot{\beta}\,\xi_1\,du - e^{\beta}(\beta' - \frac{1}{l})\,\xi_2\,du + \frac{1}{2}\,\xi_1\,\beta'\,dx\,, \\
0 &= d\xi_2 + \frac{1}{l}\,e^{\beta}\,\xi_1\,du - \frac{1}{2}\,\dot{\beta}\,\xi_2\,du - \frac{1}{2}\,\beta'\,\xi_2\,dx\,, \\
0 &= d\zeta_1 + \frac{1}{2}\,\dot{\beta}\,\zeta_1\,du - e^{\beta}\,\beta'\,\zeta_2\,du - \frac{2}{l}\,e^{\frac{2x}{l}-\beta}\,\zeta_2\,dv + \frac{1}{2}\,(\beta' - \frac{2}{l})\,\zeta_1\,dx\,, \\
0 &= d\zeta_2 - \frac{1}{2}\,\dot{\beta}\,\zeta_2\,du - \frac{1}{2}\,(\beta' - \frac{2}{l})\,\zeta_2\,dx\,.
\end{aligned}
\tag{4.170}
$$

By choosing $\xi_1 = \xi_2 = 0$ the first two equations are uniquely solved. For the last two equations, we have a non-trivial solution with a generic function $\beta(u, x)$

$$
\zeta_1 = e^{-\frac{1}{2}\beta + \frac{x}{l}}\,, \qquad\qquad \zeta_2 = 0\,. \tag{4.171}
$$

A close scrutiny shows that there exists an additional solution for the special case that $\beta = x$

$$
\zeta_1 = (u + 2v)e^{\frac{1}{2}x}\,, \qquad\qquad \zeta_2 = e^{-\frac{1}{2}x}\,, \tag{4.172}
$$

which corresponds to the first case given in eq. (4.163) with a parameter $\sigma m^2 \neq \pm 1/2$ and the functions $D_1(u) = 0$, $D_2(u) = 1$. However, from the Killing spinor equations we have to choose $\sigma m^2 = -1/2$ which is in conflict with the solution in the previous section. Therefore, the only solution for the pp-wave Killing spinor equation reads

$$
\xi_1 = \xi_2 = \zeta_2 = 0\,, \qquad\qquad \zeta_1 = e^{-\frac{1}{2}\beta + \frac{x}{l}}\,. \tag{4.173}
$$

Since we are left with only a one-component spinor, we deduce that the pp-wave solutions all preserve $1/4$ of the supersymmetries. The $l \to \infty$ limit, i.e. where we recover Minkowski pp-wave solutions, the equations for ξ and ζ degenerate, making the number of Killing spinors the same for both AdS and Minkowski pp-wave solutions.

As a final remark, let us investigate the maximally supersymmetric AdS spacetime and see how the supersymmetry enhances. For the choice $D_1 = D_2 = 0$, the metric reduces to AdS_3 in a Poincaré patch

$$
ds^2 = dx^2 + 2e^{2x/l}\,du\,dv = dx^2 + e^{2x/l}\,(-dt^2 + d\phi^2)\,. \tag{4.174}
$$

With a similar choice of basis

$$
e^0 = e^{x/l}\,dt\,, \qquad e^1 = e^{x/l}\,d\phi\,, \qquad e^2 = dx\,. \tag{4.175}
$$

the spin connections are

$$\omega_{02} = -\frac{1}{l}\, e^{x/l}\, dt\,, \qquad \omega_{12} = \frac{1}{l}\, e^{x/l}\, d\phi\,. \tag{4.176}$$

The Killing spinor equation then reads

$$d\epsilon - \frac{1}{2l}\, e^{x/l}\Big(\sigma_1\epsilon - \mathrm{i}\sigma_2\epsilon^*\Big)dt - \frac{1}{2l}\, e^{x/l}\Big(\mathrm{i}\sigma_2\epsilon - \sigma_1\epsilon^*\Big)d\phi + \frac{1}{2l}\,\sigma_3\,\epsilon^* dx = 0\,. \tag{4.177}$$

Repeating the same decomposition as before, $\epsilon = \xi + \mathrm{i}\zeta$, produces the following set of equations

$$\begin{aligned}
0 &= d\xi_1 + \tfrac{1}{2l}\,\xi_1 dx\,,\\
0 &= d\xi_2 - \tfrac{1}{l}\, e^{x/l}\,\xi_1 dt + \tfrac{1}{l}\, e^{x/l}\,\xi_1 d\phi - \tfrac{1}{2l}\,\xi_2 dx\,,\\
0 &= d\zeta_1 - \tfrac{1}{l}\, e^{x/l}\,\zeta_2\, dt - \tfrac{1}{l}\, e^{x/l}\,\zeta_2\, d\phi - \tfrac{1}{2l}\,\zeta_1 dx\,,\\
0 &= d\zeta_2 + \tfrac{1}{2l}\,\zeta_2\, dx\,.
\end{aligned} \tag{4.178}$$

Now, the good thing about these equations as opposed to the previous ones is the decoupling of the ξ and ζ terms, which increases the number of independent solutions. It is easy to show that the following is a solution

1. $\xi_1 = 0, \quad \chi_2 = e^{\frac{x}{2l}}, \quad \zeta_1 = \zeta_2 = 0,$
2. $\xi_1 = e^{-\frac{x}{2l}}, \quad \chi_2 = \frac{1}{l}\, e^{\frac{x}{2l}}(t-\phi), \quad \zeta_1 = \zeta_2 = 0,$
3. $\xi_1 = \chi_2 = 0, \quad \zeta_1 = e^{\frac{x}{2l}}, \quad \zeta_2 = 0,$
4. $\xi_1 = \chi_2 = 0, \quad \zeta_1 = \frac{1}{l}\, e^{\frac{x}{2l}}(t+\phi), \quad \zeta_2 = e^{-\frac{x}{2l}},$

Therefore, the AdS_3 spacetime is maximally symmetric, i.e. there are 4 non-trivial spinor components, or to put it in a more fancy way AdS_3 enjoys supersymmetry enhancement with four Killing spinors.

4.6 The Timelike Killing Vector

In this section, we will consider the solutions with at least one timelike Killing vector i.e. $f \neq 0$. With a strategy similar to the null case, we start by decomposing the off-shell Killing spinor bilinears and try to come up with constraints on the metric and the auxiliary fields, which in turn will help us to track down the families of background solutions.

We start by introducing a coordinate t such that $K^\mu \partial_\mu = \partial_t$. Our metric ansatze that is compatible with this choice is the following [127]

$$ds^2 = -e^{2\varphi(x,y)}\,(dt + B_\alpha(x,y)\,dx^\alpha)^2 + e^{2\lambda(x,y)}(dx^2 + dy^2)\,, \tag{4.179}$$

where $\lambda(x,y)$ and $\varphi(x,y)$ are arbitrary functions and $B_\alpha\,(\alpha = x,y)$ is a vector with two components. We also require that the functions appearing in the metric (4.180) do not depend on t. A natural choice for the dreibein and its inverse components reads

$$e^t{}_0 = f^{-1}\,, \qquad e^t{}_i = -f^2 W_i\,, \qquad e^\alpha{}_0 = 0\,, \qquad e^\alpha{}_i = e^{-\lambda}\delta^\alpha_i\,, \tag{4.180}$$

$$e^0{}_t = f\,, \qquad e^i{}_t = 0\,, \qquad e^0{}_\alpha = f^{-1}e^\lambda W_\alpha\,, \qquad e^i{}_\alpha = e^\lambda \delta^i_\alpha\,, \tag{4.181}$$

where we have defined $f \equiv e^\varphi$ and $W_\alpha = e^{2\varphi-\lambda}B_\alpha$. During the analysis we will jump from curved indices to flat ones back and forth, so we define $\mu = (t,\alpha)$ for the curved indices and $a = (0,i)$ for the flat ones, respectively. Rather than giving the details, we prefer simply to state the components of the spin connection ω_{abc} in the flat basis,

$$\begin{aligned}
\omega_{00i} &= -e^{-\lambda}\,f^{-1}\partial_i f\ , \\
\omega_{0ij} &= -\omega_{ij0} \;=\; f\,e^{-2\lambda}\,\partial_{[i}\left(W_{j]}e^\sigma f^{-2}\right)\ , \\
\omega_{ijk} &= 2e^{-\lambda}\,\delta_{i[j}\partial_{k]}\lambda\ .
\end{aligned} \tag{4.182}$$

Before delving into spinor bilinears, let us remember the Lorentz covariant derivative and its action on dreibeins. We define $D_\mu e^a_\nu \equiv \partial_\mu e^a_\nu + \omega_\mu{}^{ab}e_{\nu b}$, anti-symmetrizing in $\mu\nu$ we have $D_{[\mu}e^a_{\nu]} = 0 = R_{\mu\nu}(P^a)$ where $R_{\mu\nu}(P^a)$, is the curvature of translations we have defined previously. The covariant derivative of e^a_μ is, $\nabla_\mu e^a_\nu = D_\mu e^a_\nu + \Gamma^r_{\mu\nu}e^a_\rho = 0$, so that the Lorentz covariant derivative and the usual covariant derivative are related by $\nabla_\mu K_\nu = e^a_\nu D_\mu K_a$.

Now, consider the (4.127) projected to flat indices

$$
\begin{aligned}
e_i^\mu \partial_\mu f =& -\frac{i}{2}\left(SL_i - S^* L_i^*\right) + e_i^\mu \varepsilon_{\mu\nu}{}^\rho V^\nu K_\rho, \\
e^{-\lambda}\partial_i f =& e_i^t\, \varepsilon_{t\alpha\beta}\, V^\alpha K^\beta + e_i^\alpha\, \varepsilon_{\alpha\nu\rho} V^\nu K^\rho - \frac{i}{2}\left(SL_i - S^* L_i^*\right), \\
e^{-\lambda}\partial_i f =& \varepsilon_{ij0} f V^j - \frac{i}{2}\left(SL_i - S^* L_i^*\right), \\
\partial_i f =& e^\lambda \left[\varepsilon_{ij} f V^j - \frac{i}{2}\left(SL_i - S^* L_i^*\right)\right],
\end{aligned} \tag{4.183}
$$

where we have used $K^\rho = (1, 0, 0)$ and $\varepsilon_{ij0} = \varepsilon_{ij}$. The second equation which will help us to relate the components of the vector L_μ is the $e_i^\mu e_0^\nu$ projection of (4.125)

$$
e_i^\mu e_0^\nu\, \partial_{[\mu} K_{\nu]} = D_{[i} K_{0]} = e_i^\mu e_0^\nu\, \varepsilon_{\mu\nu}{}^\alpha \left[V_\alpha f + \frac{1}{2}(S^* L_\alpha^* + S L_\alpha)\right]. \tag{4.184}
$$

The derivative terms on the left hand side read

$$
D_i K_0 = e_i^\mu \partial_\mu K_0 + \omega_{i00} K^0 = -e^{-\lambda}\partial_i f, \tag{4.185}
$$

$$
D_0 K_i = e_0^\mu \partial_\mu K_i + \omega_{0i0} K^0 = e^{-l}\partial_i f, \tag{4.186}
$$

where $K^0 = e_\mu^0 K^\mu = f$ and $K^i = e_\mu^i K^\mu = 0$ are used. Plugging those in (4.184)

$$
\begin{aligned}
-e^{-\lambda}\,\partial_i f =& \varepsilon_{i0j}\left[V^j f + \frac{1}{2}(S^* L^{*j} + S L^j)\right], \\
\partial_i f =& e^\lambda \varepsilon_{ij}\left[V^j f + \frac{1}{2}(S^* L^{*j} + S L^j)\right].
\end{aligned} \tag{4.187}
$$

Comparing (4.183), (4.187) we see that the components of L_a in the flat basis are related as

$$
L_1 = i L_2. \tag{4.188}
$$

With this relation and the Levi-Civita contraction of L_μ's (4.136), it is easy to show that

$$
\begin{aligned}
e_\mu^0 \varepsilon^{\mu\nu\rho} L_\nu L_\rho^* =& 2 i f e_\mu^0 K^\mu, \\
\varepsilon^{0ij} L_i L_j^* = 2 i f^2 =& 2 i |L_1|^2, \\
f^2 = |L_2|^2 = |L_1|^2.&
\end{aligned} \tag{4.189}
$$

Therefore, we can choose L_a to be

$$
L_a = e^{\varphi + i c} \begin{pmatrix} 0 \\ 1 \\ -i \end{pmatrix}, \tag{4.190}
$$

where c is a time independent real function. We also deduce the fact that $L_0 = 0$ from the orthogonality of $K^\mu L_\mu = 0$. Let us now try to find the components of the auxiliary vector V_μ. To that end, it is sufficient to invert (4.187) by multiplying ε^{ij} with it

$$V^k = f^{-1}\left[-e^{-\lambda}\varepsilon^{ki}(\partial_i f) + \frac{1}{2}(S^* L^{*k} + SL^k)\right]. \tag{4.191}$$

From the components V_1 and V_2 one can define

$$V_+ \equiv V_1 + iV_2 = 2ie^{-\lambda}f^{-1}\partial_{\bar{z}}f - f^{-1}SL_1, \tag{4.192}$$

where $z = x + iy$, $\bar{z} = x - iy$ is the usual complex variables, and $\partial_z = \frac{1}{2}(\partial_x - i\partial_y)$, $\partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y)$ are the holomorphic and anti-holomorphic derivatives. The definition (4.192) boils down to a first order differential equation between the auxiliary fields and the metric components, by using the explicit form of L_1 (4.190)

$$\begin{aligned} V_+ =& 2ie^{-\lambda}e^{-\varphi}(\partial_{\bar{z}}e^{\varphi}) - e^{-\varphi}Se^{\varphi+ic}, \\ =& 2ie^{-\lambda}(\partial_{\bar{z}}\varphi) - ie^{-\lambda}\partial_{\bar{z}}(\varphi + \lambda - ic), \\ =& ie^{-\lambda}\partial_{\bar{z}}(\varphi - \lambda + ic). \end{aligned} \tag{4.193}$$

There is one last spinor bilinear identity (4.126) that we haven't utilised. It is suitable to project this into the flat basis i, j

$$D_iL_j = -iV_iL_j - iV_jL_i + i\delta_{ij}L_aV^a + S^*\varepsilon_{ij0}K^0 + ifS^*\delta_{ij}. \tag{4.194}$$

Let us first focus on the left hand side of the equation (4.194) and employ spin connections (4.182) we have computed before

$$D_iL_j = e^{-\sigma}\left[\partial_iL_j + \delta_{ij}(\partial_k\sigma)L^k - L_i(\partial_j\sigma)\right]. \tag{4.195}$$

For completeness, let us display all of the components of (4.195)

$$D_1L_1 = e^{-\sigma}\left[\partial_1L_1 - iL_1(\partial_2\sigma)\right], \tag{4.196}$$

$$D_2L_2 = e^{-\sigma}\left[\partial_2L_1 + iL_1(\partial_1\sigma)\right], \tag{4.197}$$

$$D_1L_2 = e^{-\sigma}\left[\partial_1L_2 - L_1(\partial_2\sigma)\right], \tag{4.198}$$

$$D_2L_2 = e^{-\sigma}\left[\partial_2L_2 + L_1(\partial_1\sigma)\right], \tag{4.199}$$

where we have used the equality of $L_1 = iL_2$. The useful components will be again holomorphic and anti holomorphic combinations as follows

$$
\begin{aligned}
D_z L_1 =& \frac{1}{2}(D_1 - iD_2)L_1, \\
=& \frac{1}{2}\left[e^{-\sigma}(\partial_1 L_1 - iL_1(\partial_2\sigma) - i\partial_2 L_1 + L_1(\partial_1\sigma)\right], \\
=& \frac{1}{2}\left[e^{-\sigma}\{(\partial_1 - i\partial_2)L_1 + L_1(\partial_1 - i\partial_2)\sigma\}\right], \\
D_z L_1 =& e^{-\sigma}(\partial_z L_1 + L_1\sigma\partial_z\sigma) = e^{-2\sigma}\partial_z(L_1 e^{\sigma}), \qquad (4.200) \\
D_{\bar z} L_1 =& \frac{1}{2}(D_1 + iD_2)L_1, \\
=& \frac{1}{2}\left[e^{-\sigma}(\partial_1 L_1 - iL_1(\partial_2\sigma) + i\partial_2 L_1 - L_1(\partial_1\sigma)\right], \\
=& \frac{1}{2}\left[e^{-\sigma}\{(\partial_1 + i\partial_2)L_1 - L_1(\partial_1 + i\partial_2)\sigma\}\right], \\
D_{\bar z} L_1 =& \partial_{\bar z}(e^{-\sigma}L_1). \qquad (4.201)
\end{aligned}
$$

Having dealt with the left hand side of (4.194), we now concentrate on the right hand side for the components of holomorphic and anti-holomorphic combinations

$$
\begin{aligned}
D_1 L_1 =& -iV_1 L_1 + iV_2 L_2^* + ifS^*, \\
=& L_1(V_2 - iV_1) + ifS^*, \\
=& -iV_+ L_1 + ifS^*, \qquad (4.202) \\
D_2 L_1 =& -iV_2 L_1 - iV_1 L_2 - fS^*, \\
=& -V_+ L_1 - fS^*. \qquad (4.203)
\end{aligned}
$$

It is now easy to see that $D_{\bar z} L_1 = -iV_+ L_1$ and $D_z L_1 = ifS^*$. Plugging in the explicit expression for V_+ (4.193), the anti-holomorphic derivative amounts to

$$
\begin{aligned}
D_{\bar z} L_1 =& -iL_1 V_+, \\
=& -ie^{\varphi + ic}\left[2if^{-1}e^{-l}\partial_{\bar z} f - f^{-1}SL_1\right], \\
=& 2e^{ic-\lambda}(\partial_{\bar z} f) + ife^{2ic}S. \qquad (4.204)
\end{aligned}
$$

In order to get a constraint on S, replace the left hand side with (4.200)

$$
\partial_{\bar z}(e^{-\lambda}L_1) = 2e^{ic-\lambda}(\partial_{\bar z} f) + ife^{2ic}S. \qquad (4.205)
$$

Now, the left hand side can also be written as

$$
\partial_{\bar z}(fe^{ic-\lambda}) = -e^{2(ic-\lambda)}\partial_{\bar z}(fe^{\lambda - ic}) + 2(\partial_{\bar z} f)e^{ic-\lambda}, \qquad (4.206)
$$

combining with right hand side

$$
\begin{aligned}
\partial_{\bar{z}}(fe^{\lambda-ic}) =& -ife^{2\lambda}S,\\
\partial_{\bar{z}}(e^{\varphi+\lambda-ic}) =& -ie^{2\lambda+\varphi}S,\\
e^{\varphi+\lambda-ic}\partial_{\bar{z}}(\varphi+\lambda-ic) =& -ie^{2\lambda+\varphi}S,\\
S =& ie^{-\lambda-ic}\partial_{\bar{z}}(\varphi+\lambda-ic).
\end{aligned} \tag{4.207}
$$

There are still two unconstrained functions we need to discuss, the metric function $B_\alpha(x, y)$ and the V_0 component of the auxiliary vector field. From the i, j component of (4.125) it is easy to see that

$$
\begin{aligned}
\nabla_{[i}K_{j]} =& \varepsilon_{ij}{}^{0}V_0 f,\\
e^{-\sigma}\partial_i K_j + \omega_{[ij]0}K^0 =& \varepsilon_{ij}{}^{0}V_0 f,\\
V_0 =& -\frac{1}{2}\varepsilon^{ij}\omega_{[ij]0}.
\end{aligned} \tag{4.208}
$$

The explicit form of (4.182) will help to bridge $B_\alpha(x, y)$ and V_0 as

$$
\begin{aligned}
\omega_{ij0} =& -fe^{-2\lambda}\partial_{[i}(W_{j]}e^{\sigma}f^{-2}),\\
=& fe^{-2\lambda}\partial_{[i}B_{j]},\\
\varepsilon^{ij}\partial_i B_j =& 2e^{2\lambda-\varphi}V_0.
\end{aligned} \tag{4.209}
$$

Let us now restate all of the ingredients that will be used in the search of solutions with a timelike vector [127]

$$V_0 \ = \ -\frac{1}{2}\,\epsilon^{ij}\,\omega_{ij0}\ , \tag{4.210}$$

$$V_1 + iV_2 \ = \ \mathrm{i}e^{-\lambda}\,\partial_{\bar{z}}\,(\varphi-\lambda+ic)\ , \tag{4.211}$$

$$S \ = \ \mathrm{i}e^{-\lambda-ic}\,\partial_{\bar{z}}\,(\varphi+\lambda-ic)\ , \tag{4.212}$$

$$\epsilon^{ij}\partial_i B_j \ = \ 2V_0\,e^{2\lambda-\varphi}\ . \tag{4.213}$$

There remains now to make an ansatz for the vector field V_μ, so that we can solve eqs. (4.210)–(4.213) and determine the metric functions λ and φ. In the next section, following the same logic in [127], we will look for background solutions.

4.6.1 Classification of Supersymmetric Background Solutions

In order to solve the set of conditions on auxiliary fields and the metric functions, we will take the vector field and the complex scalar to be

$$S = \Lambda \,, \qquad V_a = \text{const} \,, \qquad V_2 = 0 \,, \qquad c = 0 \,. \tag{4.214}$$

These choices will lead to the following components of the spin connection given in (4.182) in a flat basis

$$\begin{aligned} \omega_{002} &= -(\Lambda + V_1) \,, \qquad \omega_{112} = \Lambda - V_1 \,, \\ \omega_{120} &= \omega_{201} = -\omega_{012} = V_0 \,. \end{aligned} \tag{4.215}$$

After setting $V_2 = c = 0$, we can solve for λ and φ by decomposing eqs. (4.211) and (4.212) into real and imaginary parts. The function B_y can be set to zero by a gauge choice. These simple manipulations will amount to the following differential equations for φ, λ and B_x

$$e^{-\lambda}\partial_y\varphi = -(V_1 + \Lambda), \tag{4.216}$$

$$e^{-\lambda}\partial_y\lambda = V_1 - \Lambda, \tag{4.217}$$

$$\partial_y B_x = -2V_0\, e^{2\lambda-\varphi}, \tag{4.218}$$

with $\partial_x\varphi = \partial_x\lambda = 0$.

Up until this point, we have not used the equations of motion, we have just considered the set of fields that close the off-shell supersymmetry algebra. From the transformation of gravitino, we have defined a Killing spinor equation and from the bilinear constructed out of Killing spinors, a Killing vector is defined. Remarkably, we have reduced the problem of finding metric functions into a first order differential equations. The solutions of eqs. (4.216)–(4.218) will not be the complete solution. At the end of the day, we have to use the field equations in order to fix the couplings and find the final form of the solution.

Before we move on with the solutions, let us summarise the results of this section. The solutions will bifurcate depending on the value of the vector component V_1 and V_0. For the sake of clarity, we have tabulated all supersymmetric background solutions allowed by the theory (4.102) in Table 4.1.

Table 4.1: Classification of supersymmetric background solutions of the $\mathcal{N} = (1,1)$ CNMG. The solutions are categorized with respect to the values of the components of the auxiliary vector V_a.

	V^2	V_0	V_1	Equation
Round AdS_3	0	0	0	4.221
$AdS_2 \times \mathbb{R}$	> 0	0	Λ	4.224
Null-Warped AdS_3	0	$\pm\Lambda$	Λ	4.227
Spacelike Squashed AdS_3	> 0	$< \Lambda$	Λ	4.231
Timelike Streched AdS_3	< 0	$> \Lambda$	Λ	4.233
AdS_3 pp-wave	0	V_0	εV_0	4.240
Lifshitz	> 0	0	$\neq 0$ and $\neq \Lambda$	4.245

4.6.1.1 The case $V_1 = 0$

Let us start with the basic case, i.e. $V_1 = 0$. The supersymmetry constraint equations (4.216)–(4.218) yield

$$\lambda = -\log(\Lambda y), \qquad \varphi = \log(\frac{1}{\Lambda y}), \qquad B_x = \frac{2V_0}{\Lambda}\log(\Lambda y). \tag{4.219}$$

Then it is simpler to invoke vector equation (4.103), which implies $V_0 = 0$ for $\Lambda \neq 0$. Finally, in order to fix the coupling M, we employ the scalar equation

$$M = -\frac{\Lambda^3}{6m^2} + \Lambda\sigma. \tag{4.220}$$

Thus, the metric reads

$$ds^2 = \frac{l^2}{y^2}\left(-dt^2 + dx^2 + dy^2\right), \tag{4.221}$$

which describes the round AdS_3 spacetime with $l = -1/\Lambda$, see Table 4.1.

4.6.1.2 The case $V_1 = \Lambda \neq 0$

Setting $V_1 = \Lambda$, we obtain

$$\lambda = 0, \qquad \varphi = -2\Lambda y, \qquad B_x = -\frac{V_0}{\Lambda}e^{2\Lambda y}. \tag{4.222}$$

This time the vector and the scalar field equation lead to the different choices of parameters which we investigate in subclasses A, B and C.

A. $V_0 = 0$**,** $\Lambda = -2\sqrt{\frac{m^2\sigma}{7}}$**,** $M = \frac{7\Lambda^3}{12m^2} + \Lambda\sigma$

This set of parameters will lead to the metric

$$ds^2 = -e^{4\Lambda y}dt^2 + dx^2 + dy^2\,. \tag{4.223}$$

A simple coordinate transformation $y = \frac{\log r}{2\Lambda}$, $x = \frac{x'}{2\Lambda}$ will bring the metric into a more recognisable form

$$ds^2 = \frac{l^2}{4}(-r^2dt^2 + \frac{dr^2}{r^2} + dx^2), \tag{4.224}$$

which is $AdS_2 \times \mathbb{R}$. Although appeared in different coordinates, this background is also given in the bosonic version of NMG, [133, 134].

B. $V_0 = \pm\Lambda$**,** $\Lambda = -\sqrt{\frac{-2m^2\sigma}{7}}$**,** $M = -\frac{\Lambda^3}{6m^2} + \Lambda\sigma$

The second set of parameters leads to the metric

$$ds^2 = -e^{4\Lambda y}dt^2 \pm 2e^{2\Lambda y}dtdx + dy^2\,. \tag{4.225}$$

After a coordinate transformation

$$y = l\log u, \qquad t = lx^-, \qquad x = \pm\frac{lx^+}{2}, \tag{4.226}$$

the metric (4.225) can be cast in a more familiar form [135]

$$ds^2 = l^2\left[\frac{du^2 + dx^+dx^-}{u^2} - \left(\frac{dx^-}{u^2}\right)^2\right], \tag{4.227}$$

which is null warped AdS_3.

C. $V_0 = \pm\sqrt{\frac{7\Lambda^2 - 4m^2\sigma}{21}}$**,** $M = -\frac{\Lambda^3}{3m^2} + \frac{8\Lambda\sigma}{7}$

In this subclass, we first fix the value of V_0 from the vector equation then the field equation for the metric will fix the value of the parameter M. Using these values for the parameters, we have the following form for the metric

$$ds^2 = \frac{V^2}{\Lambda^2}\Big(dx + \frac{V_0\Lambda}{V^2}\,e^{2\Lambda y}\,dt\Big)^2 - \frac{\Lambda^2}{V^2}\,e^{4\Lambda y}\,dt^2 + dy^2\,. \tag{4.228}$$

After making a coordinate transformation $\frac{V_0\Lambda}{V^2}e^{2\Lambda y} = \frac{1}{z}$, the metric reads

$$ds^2 = \frac{V^2}{\Lambda^2}\Big(dx + \frac{dt}{z}\Big)^2 - \frac{1}{z^2}\frac{V^2}{\Lambda^2}\frac{dt^2}{\nu^2} + \frac{dy^2}{4\Lambda^2 z^2}\,, \tag{4.229}$$

where $\nu^2 = 1 - \frac{V^2}{\Lambda^2} < 1$.

However, these solutions do not cover the whole story for this subclass: provided that $V^2 > 0$, which implies $7\Lambda^2 + 2m^2\sigma > 0$, we have $1 > \nu^2 > 0$. After a coordinate transformation

$$x = \frac{x'\nu}{2V}\,, \qquad\qquad t = \frac{t'\nu}{2V}\,, \tag{4.230}$$

the metric (4.228) can be cast into the following form

$$ds^2 \;\; = \;\; \frac{l^2}{4}\Big[\frac{-dt^2 + dz^2}{z^2} + \nu^2\Big(dx + \frac{dt}{z}\Big)^2\Big]\,, \tag{4.231}$$

which is the metric of spacelike squashed AdS_3 with squashing parameter ν^2.

The second branch is $V^2 < 0$, i.e. $7\Lambda^2 + 2m^2\sigma < 0$, performing a coordinate transformation

$$x = \frac{x'}{2}\sqrt{\frac{-\nu^2}{V^2}}\,, \qquad\qquad t = \frac{t'}{2}\sqrt{\frac{-\nu^2}{V^2}}\,, \tag{4.232}$$

after which the metric (4.228) can be written in the following form

$$ds^2 = \frac{l^2}{4}\Big[\frac{dt^2 + dz^2}{z^2} - \nu^2\Big(dx + \frac{dt}{z}\Big)^2\Big]\,, \tag{4.233}$$

where $\nu^2 > 1$. The metric (4.233) is one of the manifestations of the timelike stretched AdS_3 background.

4.6.1.3 The case $V_1 \neq \Lambda$ and $V_1 \neq 0$

This class of solutions have $V_1 \neq \Lambda$ and $V_1 \neq 0$. Following the same steps as before, we compute the metric functions as follows

$$\lambda = -\log(z), \qquad \varphi = \log(z^\alpha), \qquad B_x = -\frac{V_0}{V_1}z^{-(1+\alpha)}, \tag{4.234}$$

where

$$z \equiv (\Lambda - V_1)y, \qquad\qquad \alpha \equiv \frac{V_1 + \Lambda}{V_1 - \Lambda}\,. \tag{4.235}$$

Using the components of the vector equation, we find

$$V_0(V_0^2 - V_1^2)(\Lambda - V_1) = 0\,. \tag{4.236}$$

From (4.236) it is straightforward to see that this subclass has two different branches, i.e. $V_0 = 0$ and $V_1 = \varepsilon V_0$ with $\varepsilon^2 = 1$.

A. $V_1 = \varepsilon V_0\,, \varepsilon = \pm 1$**,** $\quad V_0 = -\varepsilon\Lambda \pm \sqrt{\dfrac{\Lambda^2 - 2m^2\sigma}{2}}$

Plugging in the values of parameters, the vector equation leads to

$$2V_0^2 + 4\varepsilon V_0 + \Lambda^2 + 2m^2\sigma = 0\,. \tag{4.237}$$

After solving the field equation for A, we find parameter M to be

$$M = \frac{-\Lambda^3}{6m^2} + \Lambda\sigma\,. \tag{4.238}$$

Finally plugging in the metric functions, the metric reads

$$ds^2 = -z^{2\alpha}(-dt + 2\varepsilon z^{-1-\alpha}dx)dt + \frac{1}{(V_1 - \Lambda)^2}\frac{dz^2}{z^2}\,.$$

After a coordinate transformation [127]

$$z = u^{\frac{(\Lambda - V_1)}{\Lambda}}\,, \qquad t = lx^-\,, \qquad x = \frac{\varepsilon l x^+}{2}\,, \tag{4.239}$$

this metric can be recast as follows

$$ds^2 = l^2\left[\frac{du^2 + dx^+dx^-}{u^2} - u^{2(\frac{\Lambda - V_1}{\Lambda})}\left(\frac{dx^-}{u^2}\right)^2\right]\,. \tag{4.240}$$

This is the metric of a AdS_3 pp-wave. Note that the limit $V_1 \to \Lambda$ will lead us to the minus null warped AdS_3 metric of (4.227), as expected.

B. $V_0 = 0$**,** $\quad V_1 = \dfrac{\alpha+1}{\alpha-1}$**,** $\quad M = \frac{\Lambda(9V_1^2 - 2\Lambda^2)}{12m^2} + \Lambda\sigma$

The final spacetime we consider appears for $V_0 = 0$. Rather than solving the vector equation for V_1 as we did in the previous cases, we set $V_1 = \dfrac{\alpha+1}{\alpha-1}$ using (4.235). The field equations further imply that

$$(1 - 14\alpha - 7\alpha^2)\Lambda^2 + 4m^2(-1+\alpha)^2\sigma = 0, \tag{4.241}$$

whose solution is given by

$$\Lambda = -\sqrt{\frac{4m^2\sigma(\alpha-1)^2}{(1 - 14\alpha - 7\alpha^2)}}\,. \tag{4.242}$$

Here, we would like to restrict our attention to $\alpha < 0$, as α will be the minus of the Lifshitz exponent, thus giving rise to spacetimes with positive Lifshitz exponent

(1) $\alpha < \frac{1}{7}(-7 - 2\sqrt{14})$ then $m^2\sigma > 0$,

(2) $\frac{1}{7}(-7 - 2\sqrt{14}) < \alpha < 0$ then $m^2\sigma < 0$,

Provided that the vector field components are chosen as discussed, we obtain the **Lifshitz** metric

$$ds^2 = l^2\Big[- y^{2\alpha}dt^2 + \frac{1}{y^2}(dx^2 + dy^2)\Big] , \tag{4.243}$$

where l is the Lifshitz radius which is defined as

$$l^2 = \frac{1}{(V_1 - \Lambda)^2} . \tag{4.244}$$

We have redefined t as $t \to (V_1 - \Lambda)^{2\alpha+2}t$. Note that in the limit $V_1 \to 0$ one obtains the round AdS_3 metric given in eq. (4.221). Taking $y = 1/r$ gives the metric in the standard form

$$ds^2 = l^2\Big(- r^{-2\alpha}dt^2 + r^2dx^2 + \frac{1}{r^2}dr^2\Big) , \tag{4.245}$$

where l^2 and V_1 are given in terms of α and Λ as [6]

$$l^2 = \Big(\frac{\alpha - 1}{2\Lambda}\Big)^2. \tag{4.246}$$

As shown in [127], all the supersymmetric backgrounds that we have found in this section except the AdS_3 metric preserve $1/4$ of the supersymmetries.

4.6.2 Killing Spinor Analysis for the Lifshitz Solution

In order to construct the Killing spinor explicitly for the Lifshitz solution, we introduce the following orthonormal frame for the metric

$$e^0 = lr^{-\alpha}dt , \qquad e^1 = lrdx , \qquad e^2 = -\frac{l}{r}dr , \tag{4.247}$$

where the minus sign in e^2 is due to transformation $r = 1/y$ in (4.245). It follows that the components of the spin-connection are given by

$$\omega_{02} = -\alpha r^{-\alpha}dt , \qquad \omega_{12} = -rdx . \tag{4.248}$$

[6] Note that the standard Lifshitz exponent z in the literature is given by $z = -\alpha$.

For the timelike case, the Killing spinor equation is given by

$$d\epsilon + \frac{1}{4}\omega_{ab}\gamma^{ab}\epsilon - \frac{1}{2}\mathrm{i}\gamma^b\gamma_a V_b e^a\epsilon - \frac{1}{2}(A+\mathrm{i}B)\gamma_a e^a\epsilon^* = 0\,. \qquad (4.249)$$

By the following choice of the γ matrices

$$\gamma_0 = \mathrm{i}\sigma_2\,, \qquad \gamma_1 = \sigma_1\,, \qquad \gamma_2 = \sigma_3\,, \qquad (4.250)$$

where σ_i are the standard Pauli matrices, the Killing spinor equation reads

$$\begin{aligned} 0 &= d\epsilon + \frac{1}{4r}\Big((\alpha+1)\sigma_2\epsilon + (\alpha-1)\sigma_3\epsilon^*\Big)dr \\ &\quad -\frac{1}{4}r\Big(-2\mathrm{i}\sigma_2\epsilon + \mathrm{i}(\alpha+1)\epsilon + (\alpha-1)\sigma_1\epsilon^*\Big)dx \\ &\quad +\frac{1}{4}r^{-\alpha}\Big(-2\alpha\sigma_1\epsilon + \mathrm{i}(\alpha+1)\sigma_3\epsilon - \mathrm{i}(\alpha-1)\sigma_2\epsilon^*\Big)dt\,, \end{aligned} \qquad (4.251)$$

where we have used the relations

$$V_1 l = \frac{\alpha+1}{2}\,, \qquad Al = \frac{\alpha-1}{2}\,. \qquad (4.252)$$

Decomposing the Dirac spinor into two Majorana spinor as $\epsilon = \xi + \mathrm{i}\zeta$, the Killing spinor equation gives rise to the following four equations

$$\begin{aligned} 0 &= d\xi_1 + \frac{1}{4}r^{-\alpha}\Big(-(\alpha+1)\zeta_1 + (1-3\alpha)\xi_2\Big)dt + \frac{1}{4}r\Big((\alpha+1)\zeta_1 - (\alpha-3)\xi_2\Big)dx \\ &\quad +\frac{1}{4r}\Big((\alpha+1)\zeta_2 + (\alpha-1)\xi_1\Big)dr\,, \\ 0 &= d\xi_2 + \frac{1}{4}(\alpha+1)r^{-\alpha}(\zeta_2-\xi_1)dt + \frac{1}{4}r(\alpha+1)(\zeta_2-\xi_1)dx \\ &\quad +\frac{1}{4r}\Big((1-\alpha)\xi_2 - (\alpha+1)\zeta_1\Big)dr\,, \\ 0 &= d\zeta_1 + \frac{1}{4}r^{-\alpha}(\alpha+1)(-\zeta_2+\xi_1)dt + \frac{1}{4}r(\alpha+1)(\zeta_2-\xi_1)dx \\ &\quad +\frac{1}{4r}\Big((1-\alpha)\zeta_1 - (\alpha+1)\xi_2\Big)dr\,, \\ 0 &= d\zeta_2 + \frac{1}{4}r^{-\alpha}\Big((1-3\alpha)\zeta_1 - (\alpha+1)\xi_2\Big)dt + \frac{1}{4}r\Big((\alpha-3)\zeta_1 - (\alpha+1)\xi_2\Big)dx \\ &\quad +\frac{1}{4r}\Big((\alpha-1)\zeta_2 + (\alpha+1)\xi_1\Big)\,. \end{aligned} \qquad (4.253)$$

Setting $\xi_1 = \zeta_2$ we observe that we can consistently set $\xi_2 = \zeta_1 = 0$ by their equations. Noticing that ξ_2 and ζ_1 equations remain the same under the identification $\xi_1 = \zeta_2$ and $\xi_2 = -\zeta_1$, whereas there is a remaining $-r^{\alpha}(\alpha-1)\xi_2 dt$ term for ξ_1 and ζ_2 equations under this identification, we impose $\xi_1 = \zeta_2$ and $\xi_2 = -\zeta_1$ and set $\xi_2 = \zeta_1 = 0$. Note that the first condition corresponds to $\sigma_2\epsilon = \epsilon$ and the second constraint corresponds to $\sigma_1\epsilon^* = -\mathrm{i}\epsilon$. Imposing this constraint the Killing spinor equations reduce to

$$0 = d\xi_1 + \frac{1}{2r}\alpha\xi_1 dr\,, \qquad 0 = d\zeta_2 + \frac{1}{2r}\alpha\zeta_2 dr\,. \tag{4.254}$$

These equations imply one Killing spinor with

$$\xi_2 = \zeta_1 = 0, \qquad \xi_1 = \zeta_2 = r^{-\alpha/2} \tag{4.255}$$

Therefore, we conclude that $1/4$ of the supersymmetries are preserved for the Lifshitz solution. Note that the case $\alpha = -1$ has a supersymmetry enhancement with four Killing spinors, which corresponds to the maximally symmetric AdS_3.

CHAPTER 5

COMMENTS AND DISCUSSIONS

In this work anisotropic solutions for different matter coupled models have been studied. Our main focus was on the solutions with a Lifshitz background. The "anisotropy" stems from the scaling difference between time and radial components of the metric which is the reason for naming these solutions as non-relativistic spacetimes. However, they are pseudo-Riemannian metrics constructed using a fully covariant procedure. These spacetimes are important playgrounds for the extension of holography beyond AdS and as we have seen in Chapter 2 they do not share the unique properties of AdS. Inspecting particle geodesics and tidal forces, we have seen that the bulk-boundary communication is problematic and the Lifshitz spacetime is geodesically incomplete. The non-relativistic algebra which exhibits a conserved particle number also differs significantly from the AdS one. Because of this central charge and the distinct nature of space and time, the construction of the non-relativistic field theory on a Newton-Cartan background is quite peculiar.

In the second part, we have investigated a possible non-abelian matter configuration that supports Lifshitz background. As we have discussed, there should be matter couplings in order to break the symmetries of Einstein space and generate solutions with non-relativistic symmetries. The most studied models include Proca and scalar fields that are easy to control and investigate holographically. Other than those, the higher curvature models have attracted lots of attention. Since, it is easier to find black hole solutions on top of these backgrounds. The downside of these theories is that, they are not wieldy for the applications of holography.

One might think that it would have been harder to find the correct non-abelian configuration. However, after gauge fixing the planar ansatz using the procedure defined in Sec. 3.1, it is just a matter of solving a simple differential equation. On the other hand, dressing this background with a black hole is a formidable task. As a common characteristic of Einstein-Yang-Mills (EYM) system, there is no exact solution and we had to resort to numerical methods to find black holes with different horizon topologies. We have also analysed the thermal behaviour of the numerical solutions by computing the Hawking temperature for all types of black holes. We have found that there is a rapid decay in temperature as the black hole radius gets smaller, and moreover black holes do not display Hawking-Page transition. In this respect, the EYM black holes and the abelian counterparts [113, 114] have quite similar characteristics, but they both differ considerably from their conformal cousins and some of the Lifshitz black hole solutions to string theory [136].

Along these lines, it is reasonable to ask whether the Schrödinger spacetime (2.75) is supported by a non-abelian matter configuration (preferably $SU(2)$). However, before writing down the field equations, one should look for an $SU(2)$ ansatz that respects the part of the symmetries of spacetime we consider. In Lifshitz problem we have worked with a planar symmetric metric (2.52), so the ansatz was the planar symmetric $SU(2)$ and given in [67, 66]. On the other hand, the Schrödinger spacetime (2.75) has a planar symmetry and a null Killing direction, therefore we expect to use a different ansatz for that problem. The procedure we have reviewed can be used to find the most general $SU(2)$ ansatz that respects null+planar symmetries of (2.75). Another related problem is the uplift of the Lifshitz solution we have found. It was shown that the four-dimensional EYM theory with $SU(2)$ gauge group can be obtained by a dimensional reduction of 11-dimensional SUGRA model [137]. Therefore, the Lifshitz solutions must have an uplift that may be important for understanding the higher dimensional origins of these spacetimes. Having a Lifshitz solution in hand, a holographic study on the EYM action can be performed. Employing the uplift we've discussed, it may be possible to extract information about the field theory on the boundary with a reduction akin to [11]. Otherwise, the method of [19] must be applicable to this case, since it is quite general.

In the last part of the thesis we have investigated the supersymmetric backgrounds of the 3-dimensional $\mathcal{N} = (1,1)$ Cosmological New Massive Gravity (CNMG) model with the Lagrangian (4.102). At the beginning of the chapter we have given numerous tools and identities for the study of supersymmetry. We have also given the procedure for defining covariant derivatives and the algorithm for gauging the Poincare algebra. The conformal construction of supergravity (SUGRA) theories is itself a separate topic so we have shortly discussed the multiplets that are used in constructing the CNMG model and other incarnations of $\mathcal{N} = 2$ SUGRA theories.

Our weapon of choice to attack the problem was off-shell Killing spinor analysis, which proved to be a very efficient one. We have seen that the background solutions are classified according to the norm of the Killing vector constructed out of Killing spinors. In the first case, when the Killing vector is null, the $\mathcal{N} = (1, 1)$ analysis reduces to that of the $\mathcal{N} = 1$ CNMG model. Since, the matter fields, i.e. the auxiliary massive vector V_μ and the auxiliary pseudo-scalar B vanishes. Therefore, the solution boils down to same pp-wave type that is found in $\mathcal{N} = 1$, which preserves $1/4$ of the supersymmetries. We have also shown that in the AdS_3 limit, there is a supersymmetry enhancement.

The timelike case was much more richer. In particular, we did consider a special class of solutions in which the pseudo-scalar B vanishes. Then, all the supersymmetric solutions can be classified in terms of the components V_a of the massive vector in the flat basis. A subclass of these solutions, with different parameters, are also solutions of the supersymmetric TMG model, as we have tabulated in the section. In addition to these solutions, we found that the $\mathcal{N} = (1, 1)$ CNMG model possesses a Lifshitz solution. All these background solutions preserve $1/4$ of the supersymmetries. Note that the bosonic NMG also supports Lifshitz solutions and even has an exact black hole solution for $z = 3$ [80]. However, despite our efforts we were not able to find a supersymmetric Lifshitz black hole solution.

A possible extension to what we have done in here can also be applied to the $\mathcal{N} = (2, 0)$ CNMG model. This time the model accommodates two auxiliary vectors and a real scalar as well as the graviton and the gravitino. Given that the $\mathcal{N} = (2, 0)$ theory with matter couplings has new supersymmetric solutions, we also expect that the $\mathcal{N} = (2, 0)$ CNMG model exhibits different supersymmetric solutions. Therefore, it would be interesting to see what the consequences of the different field content is for the supersymmetric solutions of the model.

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APPENDIX A

SHOOTING METHOD

In this section we will try to give a pedagogical explanation of the shooting method. Our discussion will not be rigorous or even complete. However, it will throw light on certain transformations we made on the field equations (3.69), (3.70), (3.71), (3.72), how numerical study follows from these equations and the significance of the shooting parameter h_0.

Consider a second order boundary value problem as follows

$$\frac{d^2y(x)}{dx^2} = F(x, y(x), \frac{dy(x)}{dx}) \tag{A.1}$$

with the boundary values

$$y(a) = y_a, \quad y(b) = y_b. \tag{A.2}$$

which is actually quite similar to the problem we have. Assuming that, one already knows the value of the gauge or metric function at the horizon and at infinity, the question is to find what happens in between. In the shooting method, one tries to reduce the boundary value problem to an initial value problem, which one can solve with the known methods like Euler, Runge-Kutta and so on. By the initial value problem, one means that one has the value of function at one point $y(a) = y_a$ and its derivative $dy(a)/dx = Y_a$. However, for our problem we only have the information about the boundary values, i.e. $y(a) = y_a$, $y(b) = y_b$.

What we do is, we assign a reasonable value for the derivative of the function at $dy(a)/dx = Y_a$, then try to see with the numerical methods, whether we can reach the point $x = b$ with the value $y(b) = p$. Obviously, we can't hit the value y_b at the first trial, then with a second choice of the value of derivative $dy(a)/dx = Z_a$ we may hit the value at $y(b) = q$. Now, by interpolating with the values p and q we can approach to the actual y_b value as close as we like. Because of this trial and error of hitting a boundary value, this numerical method is called the *shooting method*.

The reason we have reduced the second order equations to first order ones is to apply the numerical methods of Euler or Runge-Kutta. After a definition of the first derivative of $y(x)$ with $dy(x)/dx = g(x)$, we reduce the second order problem to a system of first order differential equations

$$\frac{dy(x)}{dx} = g(x) = f_1(x, y, g), \qquad \frac{dg(x)}{dx} = f_2(x, y, g), \tag{A.3}$$

with the initial values $y(a) = y_a$ and $g(a) = Y_a$. After this, we can attack the problem with the usual methods we have pointed out in the main body.

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