PRICING AND HEDGING OF QUOTIENT OPTIONS IN ISTANBUL STOCK EXCHANGE

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ABSTRACT

PRICING AND HEDGING OF QUOTIENT OPTIONS IN ISTANBUL STOCK EXCHANGE

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Multi-asset options are derivatives written on more than one underlying asset. As a special case of multi-asset options, quotient options are written on the ratio of two underlying assets. They may be used to replace pair trading. We review the literature on quotient options within the Black-Scholes-Merton framework and pair trading. We study the performance of the delta hedging algorithm given by the BSM framework when it is applied to the quotient options traded in Borsa Istanbul. We also compare the market prices of the same quotient options to the prices suggested by the BSM model.

Keywords : option pricing, delta-hedging, Black-Scholes-Merton formula, quotient options, BIST30

ÖZ

BORSA İSTANBUL'DA ORAN OPSİYONLARININ FİYATLAMASI VE KORUNUMU

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Birden fazla dayanağa sahip opsiyonlara çoklu opsiyon denir. Çoklu opsiyonların özel bir çeşidi olan oran opsiyonları dayanakların birbirlerine bölünmesiyle bulunur. Oran opsiyonlar ikili yatırım stratejisinin yerine kullanılabilir. Bu çalışmada, oran opsiyonların avantajları, fiyatlaması ve riskten korunum performansları Black-Scholes-Merton (BSM) çatısı altında incelenmiştir. Oran opsiyonları ve ikili yatırım üzerine kaynak taraması yapılmıştır. Oran opsiyonların Borsa İstanbul verileri kullanılarak fiyatlaması ve korunumu uygulamaları BSM çatısı altında yapılmıştır. Fiyatlama için BSM'den gelen formül kullanışmıştır. Ayrıca market fiyatlarıyla BSM modelinin fiyatları karşılaştırılmıştır.

Anahtar Kelimeler: opsiyon fiyatlaması, riske karşı korunum, Black-Scholes-Merton formülü, oran opsiyonları, BIST30

To My Family

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LIST OF ABBREVIATIONS

$\mathbb{R}$	Real Numbers
BSM	Black-Scholes-Merton Model

CHAPTER 1

INTRODUCTION

A financial derivative is a financial product(contract), whose value is derived from its underlying (or underlying assets). In this study, a type of financial derivatives, quotient options, is reviewed and applied to real-life data of Istanbul Stock Exchange.

The topic of our thesis, *quotient options*, is a class of multi-asset options, defined as follows: given two prices $S^{(1)}$ and $S^{(2)}$, a quotient option call option with maturity T and strike K pays $\left(\frac{S_T^{(1)}}{S_T^{(2)}} - K\right)^+$; here $S^{(i)}$ can be the prices of stocks, commodities, futures etc. A model, based on the Black Scholes Merton (BSM) framework (including explicit pricing formulas), to price these types of options is already in existence, see for instance, [6], [10] or [26]. Because the quotient option involves two underlying assets, the BSM model involves exponentials of two correlated Brownian motions (BM); the correlation coefficient ρ between the BMs is one of the parameters of the models to be estimated from data. The goal of this thesis is to apply the BSM framework to the pricing and hedging of quotient warrants written on assets traded in the Istanbul Stock Exchange.

The idea to design such an option originates from the strategy of pair trading, which consists of trading two stocks whose prices are highly correlated, once the prices of the stocks move away from each other the investor takes positions that would bring in a profit when the stock prices converge back to each other. Pair trading by its nature may involve many transactions on two underlying stocks; instead of a pair trade, to reduce the number of transactions the investor may prefer to buy /sell a quotient option written on the ratio of the stocks. Chapter 2 covers pair trading and quotient options in further detail.

In Chapter 3 we review the BSM framework for quotient options. The treatment of these options within the BSM framework is well known and a number of approaches are available in the current literature. We review two of them: [26] and [6]. In particular, we review how these references derive explicit formulas for the price of the call warrants; the derivatives of these prices give us the deltas to be used in the hedging algorithm.

The main results of our thesis are given Chapter 4 where we apply the formulas of the BSM model reviewed in Chapter 3. For $S^{(1)}$, $S^{(2)}$ we take two separate pairs:

(Sabanci[1] and Koc[2] Holdings) and (Garanti Bankasi [3] and Akbank [4]. Is Bankasi is a market maker in Borsa Istanbul selling and buying quotient options on these pairs. Chapter 4 does two things:

1. it applies the hedging algorithm implied by the BSM model to quotient options on the above pairs over two initial dates (08.03.2016, 30.12.2015), two separate maturities (30 and 60 days), four strikes ($K = 1.5, 1.1, 1, 0.9$) and two types of options (puts and calls). For this, the model has to be fit to market data, which we do by using historical volatility and correlation estimation directly from the price data of the underlyings.

2. it compares the prices of the quotient options observed in the market to those implied by the BSM model.

The hedging algorithm given by the BSM model is in continuous time; in real life we obviously hedge in discrete time. Our price data is daily so we perform the hedge daily. This and model error lead to a hedging error (which is identically 0 for a perfectly hedged portfolio). A precise definition of the hedging error is given Chapter 4 and this error (normalized by the initial price of the option) will be our primary performance measure in our evaluation of the BSM framework. The interest rate is assumed constant and is estimated from an average of the benchmark interest rate [5] over a single month preceding the hedge period.

Further comments on our results and about future work is given in the Conclusion.

[1] see https://www.google.com/finance?q=IST%3ASAHOL
[2] see https://www.google.com/finance?q=IST%3AKCHOL
[3] see https://www.google.com/finance?q=IST%3AGARAN
[4] see https://www.google.com/finance?q=IST%3AAKBNK
[5] see, e.g, http://www.bloomberght.com/tahvil/gosterge-faiz

CHAPTER 2

QUOTIENT OPTIONS AND PAIR TRADING

Firstly we review the concept of pair trading, which is one of the motivations for trading in quotient options. Pair trading consists of buying two highly correlated stocks whose difference or ratio follow a mean reverting process. When the ratio process wanders away from the mean the trader expects it to return to the mean in a short while and takes positions to benefit from such a reversion. A more detailed review is given in the next section. Books on the basics and applications of the subject include [11], [24] and [25]. As for real-life applications, one can refer to [22] for Brazilian financial market, and to [13] for Dhaka Stock Exchange. Another interesting application on energy futures markets can be found in [18].

Among the standard references on the Black-Scholes-Merton model (the model used in this thesis) are [3], [15] and [20]. This thesis is concerned with the application of the BSM framework to quotient options. There are a number of works deriving explicit pricing formulas for a range of exotic options under the BSM framework, see, for example, [26, 6]; both of these works cover quotient options. An early paper on the pricing of quotient options under the BSM frameworks is [10]. In this work we will mostly follow the exposition in [26], a review of the derivations in [26, 6] for quotient options is given in the next chapter. Another reference of interest on exotic options is [9]. Finally, we would like to mention [2] an application is given of quotient options written on the ratio of futures contract.

2.1 Pair Trading

2.1.1 Introduction to Pair Trading

As we have already indicated in the precious section, pair trading consists of buying two highly correlated stocks whose difference or ratio follow a mean reverting process. When the ratio process wanders away from the mean the trader expects it to return to the mean in a short while and takes positions to benefit from such a reversion. The goal in pair trading is to have "market neutral" investment strategy, i.e., to obtain investment results regardless of the direction of the market. A definition along these lines is also given in [11]:

“Pairs trading: a non-directional, relative-value investment strategy that seeks to identify two companies with similar characteristics whose equity securities are currently trading at a price relationship that is outside their historical trading range. This investment strategy entails buying the undervalued security while short selling the overvalued security, thereby maintaining market neutrality.”

However, there is major downside to pair trading; transaction costs. Suppose that we execute pair trading strategy with two assets. At the beginning, we pay two commissions for each of the short and long positions. At the end, to close the positions, we again pay commissions. Meaning that four times we are faced with transaction costs so as to execute pair trading. If the pair trading involves a longer horizon more transactions may need to be executed for each mean reversion, adding to transaction costs. For a discussion of sensitivity of profitability of pair trading to transaction costs see [5], also [12]. A less costly alternative is to purchase a quotient option that is compatible with the direction of the mean reversion, which we discuss in the next section.

2.2 Quotient Options

2.2.1 Foundations of Quotient Options

To lay the foundations of quotient options, one must define the necessary background of these exotic options. Starting with exotic options, we now step by step define classes of options, to narrow our focus on solely quotient options. Since we assert that quotient option is a special case, now we have the following definitions.

Definition 2.1. An option with a different structure of payoff and underlyings is considered an exotic option.

Definition 2.2. An option written on more than one underlying asset is defined as a multi-asset option, whose value depends on the overall performance of the underlying assets. If assets are from different type from each other, multi-asset options are sometimes also called cross-asset options.

What makes multi-asset options unique and difficult to price from other exotic options is its number of underlyings. With multiple underlying, the dimension of the problem and the computational complexity of the problem increase; this is the main challenge with multi-asset options. These options are mostly traded over-the-counter. One may have different reason to enter into exotic option positions, such as its different payoff calculations giving advantages in some cases, or to hedge ongoing position through its features of underlying assets. For more on exotic option see [26] and [6], on their hedging refer to [23], and also see [9] for market applications. Now we move on to define what is central in quotient options.

Definition 2.3. Correlation option is a multi-asset option written on correlated assets.

In some sources, such as [26], it is asserted that correlation options and multi-asset options refer to the same concept. It does hold true under some circumstances, however, multi-asset option does not need necessarily to be written on correlated assets.

Definition 2.4. Quotient option is a multi-asset option, written on the ratio of assets S_1 and S_2. Let κ be a binary operator taking 1 for call option and -1 for put option, K be the strike price and T be the time to expiration. European quotient option's payoff at expiration is given by

$$C = \left[\kappa \frac{S_2(T)}{S_1(T)} - \kappa K\right]^+. \tag{2.1}$$

2.2.2 Correlation Between Assets

Correlated assets tend to move together regardless of the market direction. When it is said that a pair of assets are correlated, we infer that assets' returns are correlated, not the prices. Meaning that ρ is the correlation coefficient between the returns of the two assets, in terms of quotient option, returns of the two underlying assets. Before explaining this concept from financial perspective, we first define two continuous random variables X and Y, let $f(x)$ and $f(y)$ be their corresponding density functions, and $g(x, y)$ as their joint density function of X and Y. Means of continuous random variables are given by the following

$$\mathbb{E}(X) = \int_{-\infty}^{\infty} x f(x) dx, \tag{2.2}$$

$$\mathbb{E}(Y) = \int_{-\infty}^{\infty} y f(y) dy. \tag{2.3}$$

And the variances

$$Var(X) = \mathbb{E}[(x - \mathbb{E}(X))^2] = \int_{-\infty}^{\infty} (x - \mathbb{E}(X))^2 f(x) dx, \tag{2.4}$$

$$Var(Y) = \mathbb{E}[(y - \mathbb{E}(Y))^2] = \int_{-\infty}^{\infty} (y - \mathbb{E}(Y))^2 f(y) dy. \tag{2.5}$$

Mean is the average. Thinking in terms of asset prices, it indicates on average what the asset value takes on. Since we are calculating the returns of the underlying assets, we can obtain the returns on average. Variance measures how spread the data is. Taking square root of variance, we obtain standard deviation. In terms of our case, we calculate the standard deviation σ_i for asset i , meaning that σ indicates to the standard deviation of asset's daily return over a period. This is also called historical volatility, or simply volatility. One more concept is needed to calculate correlation, covariance.

$$Cov(X, Y) = \mathbb{E}[X - \mathbb{E}(X)][Y - \mathbb{E}(Y)], \tag{2.6}$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [x - \mathbb{E}(X)][y - \mathbb{E}(Y)] g(x, y) dx dy. \tag{2.7}$$

Covariance indicates the relationship of two random variables. In our case, for instance, positive covariance is an indicator of assets' returns moving together, negative covariance is also indicating the direction of two assets', moving opposite. Now we define correlation, which is given by

$$\rho = \frac{Cov(X, Y)}{\sigma_x \sigma_y}. \tag{2.8}$$

As can be seen, covariance and correlation refer to the movements, indicating the direction. What seperates correlation from covariance is that correlation also tells us the magnitude or say the degree of movements. If we were to assume strong positive correlation between two assets, we would be able to assert that two assets are moving the same way, but now we are given a measure to answer how strong the relationship is. Another example is that the correlation between gasoline prices and oil production in total is negative due to supply and demand. Meaning that with more production, gasoline prices are inversely moving. Correlation takes values within $[-1, 1]$. If correlation is 1, it is positive perfect correlation, if -1, it is negative perfect correlation.

In financial markets, assets having perfect correlation is theoretical. Therefore, we assert that correlated assets move within these bounds. Since we are dealing with a sample not the population, we utilize sample data formulas, instead of (2.2), (2.3), (2.4), (2.5), and (2.6). Let us present the formula to calculate volatility, also see [15].

Let S_t be the price of the asset at time t, $n + 1$ number of observations, however we calculate from the sample data so we use $n - 1$, and τ lenght of time interval in years, we assume there are 252 trading days in a year. Define daily return u_i for $i = 1, 2, 3, ..., n$ as

$$u_i = \ln\left(\frac{S_i}{S_{i-1}}\right). \tag{2.9}$$

With $\bar{u}$ as the mean of daily asset returns, unbiased estimate s of the standard deviation of the daily asset returns are given by

$$s = \sqrt{\frac{\sum_{i=1}^{n}(u_i - \bar{u})^2}{n - 1}}. \tag{2.10}$$

Variable s is an estimate of $\sigma\sqrt{\tau}$, so that volatility is given by

$$\hat{\sigma} = \frac{s}{\sqrt{\tau}}. \tag{2.11}$$

Determining an appropriate value of n is challenging. A rule of thumb is suggested by [15], if we value an option with time to expiration one year, use one year daily data to compute σ, and then annualize it. Now we present formula for sample covariance in terms of asset's prices. We are given two assets $S_i^{(1)}$ and $S_i^{(2)}$ for $i = 1, 2, ..., n$ with $\sigma_{S^{(1)}}$ and $\sigma_{S^{(2)}}$ as volatilities of asset returns per annum, setting u_i and v_i of assets $S^{(1)}$ and $S^{(2)}$, respectively, as the percentage changes (or returns) is given by the following. Notice one can also use the form in (2.9).

$$u_i = \frac{S_i^{(1)} - S_{i-1}^{(1)}}{S_{i-1}^{(1)}}, \tag{2.12}$$

$$v_i = \frac{S_i^{(2)} - S_{i-1}^{(2)}}{S_{i-1}^{(2)}}. \tag{2.13}$$

The reason that they can be used interchangeably is that if one takes logarithm of simple returns, logarithmic return is found. While $\bar{u}$ and $\bar{v}$ are mean values of asset returns. Again we take $n-1$ as the sample size and divide by it, so the covariance formula is given by

$$Cov(S^{(1)}, S^{(2)}) = \frac{1}{n-1}\sum_{i=1}^{n}(u_i - \bar{u})(v_i - \bar{v}). \tag{2.14}$$

Therefore, using (2.14), the formula of correlation between two assets' returns is obtained

$$\rho = \frac{Cov(S^{(1)}, S^{(2)})}{\sigma_{S^{(1)}}\sigma_{S^{(2)}}}. \tag{2.15}$$

Back to the previous example, using the method above, the correlation coefficient of assets' returns is found as 0.8131. Since we are interested in the returns, not the prices themselves. Let us now give a graphical example of assets' returns.

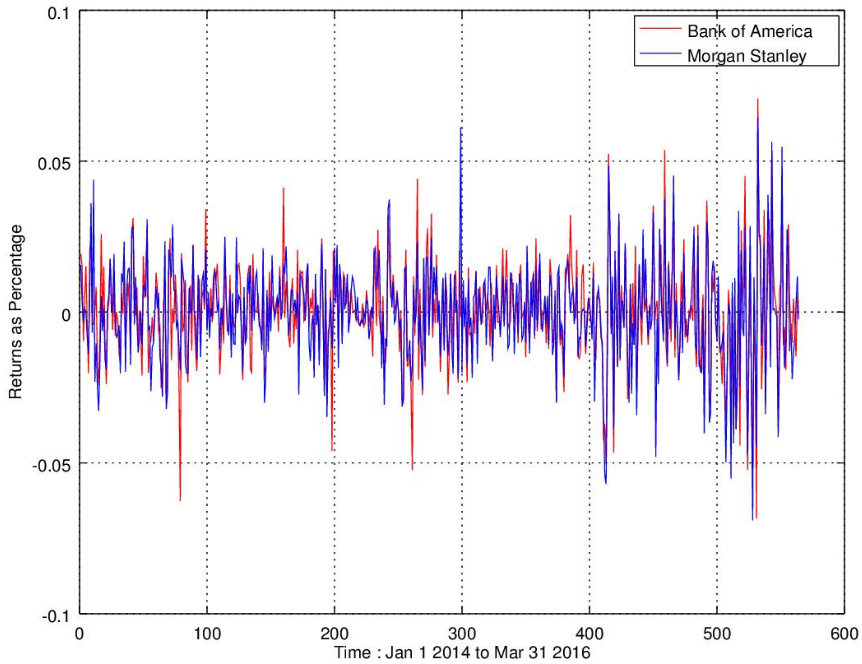

Figure 2.1: Daily Returns of Bank of America and Morgan Stanley

It can be seen in Figure 2.1 that daily returns present similar movements, meaning that a quotient option as well as pair trading can be applied for these two assets. Considering the payoff structure given (2.1), we also look for the graphs of price ratios and return ratios of Bank of America and Morgan Stanley from New York Stock Exchange. Below graph could give us insight into the option to be written. For instance, setting $K = 2$ would have created many opportunities, however, if we were to be more precise: in the last hundred trading days setting strike price as 2 dollars would be applicable for a call option with time to maturity is about of a quarter.

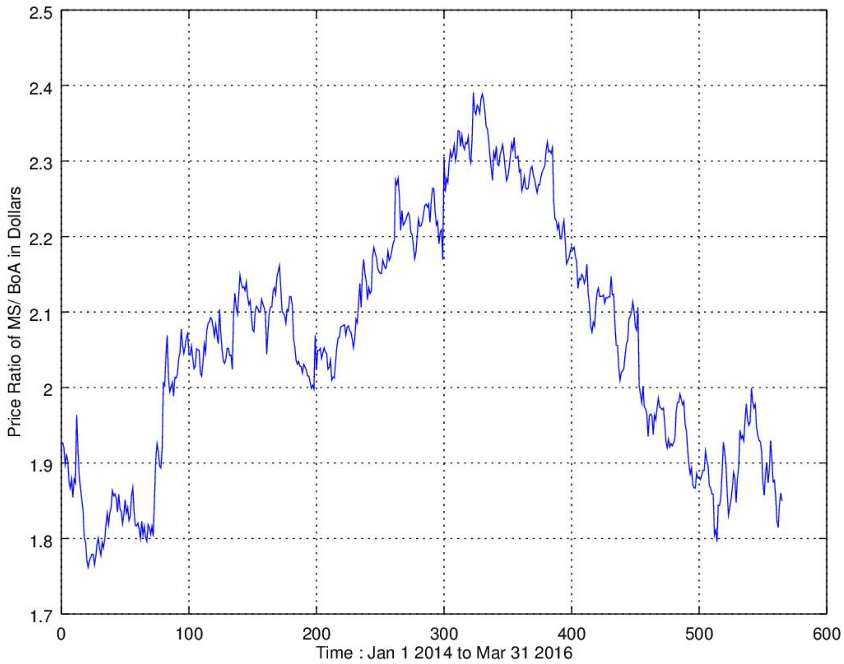

Figure 2.2: Price Ratio of Bank of America and Morgan Stanley

2.2.3 Modeling Multi-Asset Options

An asset's behavior can be described by a stochastic differential equation. Using that equation, we reach a one-dimensional partial differential equation (PDE). The multiple asset case is no different. Now that we are given two assets, we have a set of stochastic differential equations, one equation for each of the assets. After that, utilizing Ito's Lemma Multidimensional, what we reach is a multidimensional PDE governing the dynamics of two-assets, and solved as a terminal-boundary value problem. Suggested by [17], we should reduce the multidimensional PDE to one-dimensional PDE by introducing suitable composed variables into the governing PDE. Now let us introduce what is used to price quotient options. First we define geometric Brownian motion, and the CEV model is given.

Let Z be a Brownian motion, μ be the percentage drift, and σ be the percentage volatility. The solution to the following SDE

$$dS_t = \mu S_t dt + \sigma S_t dZ_t. \tag{2.16}$$

is called a geometric Brownian motion with $\mu, \sigma \in \mathbb{R}$, and written as

$$S_t = S_0 e^{(\mu - \frac{\sigma^2}{2})t + \sigma Z_t}. \tag{2.17}$$

In a risk-neutral setting, μ can be replaced by r. However, it can also be replaced by $r - g$ with g as the dividend rate, see [26], such as

$$dS = (r - g)Sdt + \sigma SdZ. \tag{2.18}$$

This process is quite common, and suggested in most cases, see [17]. For a detailed discussion of geometric Brownian motion, see [19], and for a practical approach refer to [15]. There is another process used in modeling multi-asset options, which is the Constant Elasticity of Variance (CEV) model. CEV is put forward as

$$dS_t = \mu S_t dt + \sigma S_t^{\gamma} dZ_t. \tag{2.19}$$

Each cases of the CEV model is studied, see [8] for the case of $\gamma < 0$, and thorough literature review is given in [16], also suggested for pricing quotient options in [2] in futures markets.

CHAPTER 3

QUOTIENT OPTIONS IN CONTINUOUS-TIME

3.1 General Framework

In this part, we present the Black-Scholes-Merton model. The PDE for the standard call option is the Black-Scholes-Merton framework will be referred to when analyzing quotient options. Let P be the price of the security, r the interest rate, D as the dividend rate, σ the volatility of the security, T is the expiration date and the option value is given by C. The governing PDE of Black-Scholes-Merton is given by

$$\frac{\partial C}{\partial t} + (r - D)P\frac{\partial C}{\partial P} + \frac{1}{2}\sigma^2 P^2 \frac{\partial^2 C}{\partial P^2} - rC = 0. \tag{3.1}$$

which is subject to final condition $C(P,T) = \max(P - K, 0)$

$$C(P,t) = Pe^{-D(T-t)}\mathcal{N}(d_1) - Ke^{-r(T-t)}\mathcal{N}(d_2), \tag{3.2}$$

$$d_1 = \frac{\ln(\frac{P}{K}) + (r - D + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}, \tag{3.3}$$

$$d_2 = d_1 - \sigma\sqrt{T-t}. \tag{3.4}$$

Here $\mathcal{N}$ denotes the cumulative distribution function of standard normal distribution. For more information on this model see [3], [20] and [21].

3.2 Pricing Quotient Options

3.2.1 A Model For Pricing Quotient Options

A pricing formula for quotient options is already derived by [6], or [26]. Here we set out to study the formula step by step. The price processes are assumed to be

$$dS_1 = \mu_1 S_1 dt + \sigma_1 S_1 dZ_1, \tag{3.5}$$

$$dS_2 = \mu_2 S_2 dt + \sigma_2 S_2 dZ_2, \tag{3.6}$$

where $Z = (Z_1, Z_2)$ is a two dimensional Brownian motion with cross variation $\langle dZ_1, dZ_2 \rangle = \rho dt$. Let $C(S_1, S_2, T)$ be the value of the option written on the ratio of two assets at expiration. Now set a portfolio consisting of a long position on quotient option, and two short positions in the underlying assets with amounts Δ_1 and Δ_2, such as

$$\Pi = C - \Delta_1 S_1 - \Delta_2 S_2. \tag{3.7}$$

And the increment of the portfolio is given by

$$d\Pi = dC - \Delta_1 dS_1 - \Delta_2 dS_2. \tag{3.8}$$

In (3.8), we have random variables. Therefore, we apply Itô's Lemma Multidimensional. The purpose here is to get rid of the randomness coming from the Brownian motion, and be able to use deterministic calculus after we reach a PDE governing the dynamics of the option. We have

$$dC = \frac{\partial C}{\partial S_1} dS_1 + \frac{1}{2}\frac{\partial^2 C}{\partial S_1^2}(dS_1)^2 + \frac{1}{2}\frac{\partial^2 C}{\partial S_2^2}(dS_2)^2 + \frac{\partial^2 C}{\partial S_1 \partial S_2}(dS_1 dS_2) + \frac{\partial C}{\partial t} dt. \tag{3.9}$$

Using (3.5) and (3.6)

$$\begin{aligned} dC = \frac{\partial C}{\partial t} dt + \frac{\partial C}{\partial S_1}(\mu_1 S_1 dt + \sigma_1 S_1 dZ_1) + \frac{1}{2}\frac{\partial^2 C}{\partial S_1^2}(\mu_1 S_1 dt + \sigma_1 S_1 dZ_1)^2 \\ + \frac{\partial C}{\partial S_2}(\mu_2 S_2 dt + \sigma_2 S_2 dZ_2) + \frac{1}{2}\frac{\partial^2 C}{\partial S_2^2}(\mu_2 S_2 dt + \sigma_2 S_2 dZ_2)^2 \\ + \frac{\partial^2 C}{\partial S_1 \partial S_2}(\mu_1 S_1 dt + \sigma_1 S_1 dZ_1)(\mu_2 S_2 dt + \sigma_2 S_2 dZ_2), \end{aligned} \tag{3.10}$$

$$
\begin{aligned}
dC = \frac{\partial C}{\partial t}dt + \frac{\partial C}{\partial S_1}\mu_1 S_1 dt + \frac{\partial C}{\partial S_1}\sigma_1 S_1 dZ_1 + \frac{1}{2}\frac{\partial^2 C}{\partial S_1^2}\mu_1^2 S_1^2 dt^2 + \frac{\partial^2 C}{\partial S_1^2}\mu_1\sigma_1 S_1^2 dt dZ_1 \\
+ \frac{1}{2}\frac{\partial^2 C}{\partial S_1^2}\sigma_1^2 S_1^2 dZ_1^2 + \frac{\partial C}{\partial S_2}\mu_2 S_2 dt + \frac{\partial C}{\partial S_2}\sigma_2 S_2 dZ_2 + \frac{1}{2}\frac{\partial^2 C}{\partial S_2^2}\mu_2^2 S_2^2 dt^2 \\
+ \frac{\partial^2 C}{\partial S_2^2}\mu_2\sigma_2 S_2^2 dt dZ_2 + \frac{1}{2}\frac{\partial^2 C}{\partial S_2^2}\sigma_2^2 S_2^2 dZ_2^2 + \frac{\partial^2 C}{\partial S_1 \partial S_2}\mu_1\mu_2 S_1 S_2 dt^2 \\
+ \frac{\partial^2 C}{\partial S_1 \partial S_2}\mu_1\sigma_2 S_1 S_2 dt dZ_2 + \frac{\partial^2 C}{\partial S_1 \partial S_2}\sigma_1\mu_2 S_1 S_2 dt dZ_1 \\
+ \frac{\partial^2 C}{\partial S_1 \partial S_2}\sigma_1\sigma_2 S_1 S_2 dZ_1 dZ_2, \quad (3.11)
\end{aligned}
$$

Using $dZ_i^2 = 0$, $dZ_1 dZ_2 = \rho dt$, $dt^2 = 0$ and $dt dZ_i = 0$ for $i = 1, 2$, and with dS_1 and dS_2 is already defined; now rearranging the terms

$$
\begin{aligned}
dC = \left[\frac{\partial C}{\partial t} + \frac{1}{2}\frac{\partial^2 C}{\partial S_1^2}\sigma_1^2 S_1^2 + \frac{1}{2}\frac{\partial^2 C}{\partial S_2^2}\sigma_2^2 S_2^2 + \rho\sigma_1\sigma_2 S_1 S_2 \frac{\partial^2 C}{\partial S_1 \partial S_2}\right] dt \\
+ \frac{\partial C}{\partial S_1}dS_1 + \frac{\partial C}{\partial S_2}dS_2. \quad (3.12)
\end{aligned}
$$

Set Δ_1 and Δ_2 as $\frac{\partial C}{\partial S_1}$ and $\frac{\partial C}{\partial S_2}$ to eliminate risk, so that the portfolio is riskless. Substituting (3.12) into (3.8), we reach increment of the portfolio value

$$
d\Pi = \left[\frac{\partial C}{\partial t} + \frac{1}{2}\frac{\partial^2 C}{\partial S_1^2}\sigma_1^2 S_1^2 + \frac{1}{2}\frac{\partial^2 C}{\partial S_2^2}\sigma_2^2 S_2^2 + \rho\sigma_1\sigma_2 S_1 S_2 \frac{\partial^2 C}{\partial S_1 \partial S_2}\right] dt. \quad (3.13)
$$

Since the portfolio is riskless, should have a return of interest rate r. Now

$$
d\Pi = r\Pi = r\left(C - \frac{\partial C}{\partial S_1}S_1 - \frac{\partial C}{\partial S_2}S_2\right) dt. \quad (3.14)
$$

From above, we find

$$
\begin{aligned}
\frac{\partial C}{\partial t} + \frac{1}{2}\frac{\partial^2 C}{\partial S_1^2}\sigma_1^2 S_1^2 + \frac{1}{2}\frac{\partial^2 C}{\partial S_2^2}\sigma_2^2 S_2^2 + \rho\sigma_1\sigma_2 S_1 S_2 \frac{\partial^2 C}{\partial S_1 \partial S_2} \\
+ rS_1 \frac{\partial C}{\partial S_1} + rS_2 \frac{\partial C}{\partial S_2} - rC = 0. \quad (3.15)
\end{aligned}
$$

Now we can introduce dividends, even though in our study we assume that the dividend rates are zero. Let g_i be the dividend rate of asset i for $i = 1, 2$. Subject to final condition $C(S_1, S_2, T) = \max\left(\frac{S_2}{S_1} - K, 0\right)$, we solve the following governing PDE

$$\frac{\partial C}{\partial t} + \frac{1}{2}\frac{\partial^2 C}{\partial S_1^2}\sigma_1^2 S_1^2 + \frac{1}{2}\frac{\partial^2 C}{\partial S_2^2}\sigma_2^2 S_2^2 + \rho\sigma_1\sigma_2 S_1 S_2 \frac{\partial^2 C}{\partial S_1 \partial S_2} + (r - g_1)S_1\frac{\partial C}{\partial S_1} + (r - g_2)S_2\frac{\partial C}{\partial S_2} - rC = 0. \quad (3.16)$$

Let time to expiration be $\tau = T - t$, (3.16) can be written as

$$\frac{\partial C}{\partial \tau} = \frac{1}{2}\frac{\partial^2 C}{\partial S_1^2}\sigma_1^2 S_1^2 + \frac{1}{2}\frac{\partial^2 C}{\partial S_2^2}\sigma_2^2 S_2^2 + \rho\sigma_1\sigma_2 S_1 S_2 \frac{\partial^2 C}{\partial S_1 \partial S_2} + (r - g_1)S_1\frac{\partial C}{\partial S_1} + (r - g_2)S_2\frac{\partial C}{\partial S_2} - rC. \quad (3.17)$$

We assume the exact solution is $C(S_1, S_2, t) = e^{-r\tau}\phi(z, \tau)$ with $z = \frac{S_2}{S_1}$. Our purpose is to reduce (3.17) to 1-D, by introducing composed variables. However, to achieve that, let us solve (3.17) by taking the partial derivatives of exact solution:

$$\frac{\partial C}{\partial t} = -e^{-r(T-t)}\phi\left(\frac{S_2}{S_1}, T - t\right) + rC, \quad (3.18)$$

$$\frac{\partial C}{\partial S_1} = \frac{\partial \phi}{\partial z}\frac{\partial}{\partial S_1}\left(\frac{S_2}{S_1}\right) = -\frac{\partial \phi}{\partial z}S_2 S_1^{-2}e^{-r\tau}, \quad (3.19)$$

$$\frac{\partial^2 C}{\partial S_1^2} = \frac{\partial}{\partial S_1}\left(-\frac{\partial \phi}{\partial z}S_2 S_1^{-2}e^{-r\tau}\right), \quad (3.20)$$

$$= S_2 S_1^{-2}\frac{\partial^2 \phi}{\partial z^2}S_2 S_1^{-2}e^{-r\tau} + 2\frac{\partial \phi}{\partial z}S_2 S_1^{-3}e^{-r\tau}, \quad (3.21)$$

$$\frac{\partial C}{\partial S_2} = \frac{\partial \phi}{\partial z}\frac{\partial}{\partial S_2}\left(\frac{S_2}{S_1}\right) = \frac{\partial \phi}{\partial z}S_1^{-1}e^{-r\tau}, \quad (3.22)$$

$$\frac{\partial^2 C}{\partial S_2^2} = \frac{\partial}{\partial S_2}\left(\frac{\partial \phi}{\partial z}S_1^{-2}e^{-r\tau}\right) = \left(\frac{\partial^2 \phi}{\partial z^2}S_1^{-2}e^{-r\tau}\right), \quad (3.23)$$

$$\frac{\partial^2 C}{\partial S_1 \partial S_2} = \frac{\partial C}{\partial S_1}\left(\frac{\partial \phi}{\partial z}S_1^{-1}e^{-r\tau}\right), \quad (3.24)$$

$$= -\frac{\partial^2 \phi}{\partial z^2}S_2 S_1^{-3}e^{-r\tau} - \frac{\partial \phi}{\partial z}S_1^{-2}e^{-r\tau}. \quad (3.25)$$

Putting the above partial derivatives into (3.17), with z as the ratio of the assets

$$\frac{\partial C}{\partial \tau} = \frac{1}{2}\sigma_1^2 z^2 \frac{\partial^2 \phi}{\partial z^2} + \sigma_1^2 z \frac{\partial \phi}{\partial z} + \frac{1}{2}\sigma_2^2 z^2 \frac{\partial^2 \phi}{\partial z^2} - \rho\sigma_1\sigma_2 z^2 \frac{\partial^2 \phi}{\partial z^2} - \rho\sigma_1\sigma_2 z \frac{\partial \phi}{\partial z} - (r - g_1) z \frac{\partial \phi}{\partial z} + (r - g_2) z \frac{\partial \phi}{\partial z}. \quad (3.26)$$

Rearranging the terms

$$\frac{\partial C}{\partial \tau} = \frac{1}{2}(\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2) z^2 \frac{\partial^2 \phi}{\partial z^2} + (\sigma_1(\sigma_1 - \rho\sigma_2) + (g_1 - g_2)) z \frac{\partial \phi}{\partial z}. \quad (3.27)$$

If we let $\alpha_1 = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}$ and $\alpha_2 = \sigma_1(\sigma_1 - \rho\sigma_2) + (g_1 - g_2)$, (3.17) is reduced to the following form

$$\frac{\partial \phi}{\partial \tau} = \frac{\alpha_1^2}{2} z^2 \frac{\partial^2 \phi}{\partial z^2} + \alpha_2 z \frac{\partial \phi}{\partial z}. \quad (3.28)$$

Here, α_1 is also called the aggregate volatility, α_a see [6]. Now comparing with Black-Scholes-Merton (3.1), it can be seen that (3.28) is a special case of (3.1), letting $D = -\alpha_2$, $r = 0$, and $\sigma = \alpha_1$

$$\phi(z, \tau) = z e^{\alpha_2 \tau} N(d_1) - K N(d_2), \quad (3.29)$$

$$d_1 = \frac{\ln(\frac{z}{K}) + (\alpha_2 + \frac{\alpha_1^2}{2})\tau}{\alpha_1 \sqrt{\tau}}, \quad (3.30)$$

$$d_2 = d_1 - \alpha_1 \sqrt{\tau}. \quad (3.31)$$

Since $c(S_1, S_2, t) = e^{-r\tau}\phi(z, \tau)$ with the other variables put into the equation, pricing formula is

$$e^{-r\tau} \left[\frac{S_2}{S_1} e^{\sigma_1(\sigma_1 - \rho\sigma_2)\tau + (g_1 - g_2)\tau} \mathcal{N}(d_1) - K\mathcal{N}(d_2) \right]. \quad (3.32)$$

Let us introduce a variable κ taking the value of 1 for call options and -1 for put options. Writing (3.32) explicitly, we obtain the final form of a pricing formula for quotient options written on stocks with dividends g_1 and g_2

$$\kappa\left[\frac{S_2}{S_1}e^{\sigma_1(\sigma_1-\rho\sigma_2)\tau+(g_1-g_2-r)\tau}\mathcal{N}(\kappa d_1)-Ke^{-r\tau}\mathcal{N}(\kappa d_2)\right], \tag{3.33}$$

$$d_1=\frac{\ln(\frac{S_2}{S_1K})+(\sigma_1^2-\rho\sigma_1\sigma_2+g_1-g_2+\frac{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2}{2})\tau}{\sqrt{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2}\sqrt{\tau}}, \tag{3.34}$$

$$d_2=d_1-\sqrt{\tau}\sqrt{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2}. \tag{3.35}$$

3.2.2 An alternative approach using Two-Asset Binary Options

We now review an approach to pricing quotient options developed in [6], which reduces the task of pricing quotient options to that of pricing two-asset binary options.

Definition 3.1. Binary option is a case of exotic options, in which the payoff can take only two possible outcomes.

Let us have a binary option with time to maturity T written on S_1 and S_2, with K as the strike price and r is the interest rate. Since a binary option can take only two outcomes for each asset, in case of two assets, there are four possible outcomes. Given $(a_1,a_2,a_3)\in(\pm)$, $(p,q)\in\mathbb{R}$; H and K are some positive constants, K also serves as the strike price. The payoffs are given by the following equations:

$$C_1(S_1,S_2,T)=S_1^pS_2^q\mathbb{I}_{(a_1S_1>a_1H)}\mathbb{I}_{(a_2S_2>a_2K)}, \tag{3.36}$$

$$C_2(S_1,S_2,T)=S_1^pS_2^q\mathbb{I}_{(a_3S_1>a_3S_2)}, \tag{3.37}$$

$$C_3(S_1,S_2,T)=S_1^pS_2^q\mathbb{I}_{(a_1S_1>a_1H)}\mathbb{I}_{(a_3S_1>a_3S_2)}, \tag{3.38}$$

$$C_4(S_1,S_2,T)=S_1^pS_2^q\mathbb{I}_{(a_2S_2>a_2K)}\mathbb{I}_{(a_3S_1>a_3S_2)}. \tag{3.39}$$

The indicator functions are exercise conditions of the quotient options, meaning that under which conditions the option is exercised is indicated by these functions. Among these, C_2 is appropriate to model the behavior ofquotient options, and its value at time t is given by $C_2(S_1,S_2,t)=S_1^pS_2^qe^{\mu\tau}N(a_3d)$ with τ being the time to expiration. Asset are assumed to follow Geometric Brownian Motion (2.17), with volatility σ_1 and σ_2, while σ is the aggregate volatility. The payoff of a European quotient call option is given by

$$C(S_1,S_2,T)=\left(\frac{S_1}{S_2}-K\right)\mathbb{I}_{(S_1>S_2K)}. \tag{3.40}$$

Dividing by K, and setting $z=KS_2$, which also follows Geometric Brownian Motion due to S_2. Here aim is to substitute $z=KS_2$ into the two-asset binary formula. We obtain

$$C(S_1, S_2, T) = K\left(\frac{S_1}{z} - 1\right)\mathbb{I}_{(S_1>z)}. \quad (3.41)$$

Now we define $C_a = \frac{S_1}{z}\mathbb{I}_{(x>z)}$, and setting $p = 1$ and $q = -1$ in (3.37), and using its value function we find

$$C_a(S_1, S_2, t) = \frac{S_1}{z}e^{\mu\tau}\mathcal{N}(d(S_1, z, \tau)), \quad (3.42)$$

$$\mu = -r + \sigma_2(\sigma_2 - \rho\sigma_1), \quad (3.43)$$

$$\sigma = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}, \quad (3.44)$$

$$d(S_1, z, t) = \frac{1}{\sigma\sqrt{\tau}}\left(\ln(\frac{S_1}{z}) + \left(\frac{\sigma_2^2}{2} - \frac{\sigma_1^2}{2} + \sigma^2\right)\tau\right). \quad (3.45)$$

Now define $C_b = \mathbb{I}_{(x>z)}$, and setting $p = q = 0$ in (3.37), we obtain

$$C_b(S_1, z, t) = e^{-r\tau}\mathcal{N}(d'(S_1, z, \tau)), \quad (3.46)$$

$$d'(S_1, x, \tau) = \frac{1}{\sigma\sqrt{\tau}}\left(\ln\left(\frac{S_1}{KS_2}\right) + \frac{1}{2}(\sigma_2^2 - \sigma_1^2)\tau\right). \quad (3.47)$$

With (3.42) and (3.46), from the equation $C(S_1, S_2, t) = K(C_a - C_b)(S_1, S_2, t)$, we obtain

$$C(S_1, S_2, t) = \frac{S_1}{S_2}e^{\mu\tau}\mathcal{N}(d(S_1, KS_2, \tau)) - Ke^{-r\tau}\mathcal{N}(d'(S_1, KS_2, \tau)). \quad (3.48)$$

3.3 Hedging and Sensitivities in Continuous-time

Hedging requires a parameter known as delta, which equals the derivative of the prices with respect to the price of the underlying; thus, obviously, delta is the sensitivity of the option price to changes in the price of the underlying. Aside from the delta, sensitivities to change in time, volatility are often of interest. As is well known, these derivatives are collectively refered to as the “Greeks of the option.” In this section, we give formulas for the greeks for the pricing of quotient options under the BSM framework. Since we have a correlation option, there comes parameters concerning the correlation. We will in particular use the delta in the hedging application of the next chapter.

Recall that $\mathcal{N}$ denotes the cumulative distribution function of standard normal distribution, and let $\mathcal{N}'$ denote the derivative of the cumulative distribution function, both are given by

$$\mathcal{N}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}} dt, \tag{3.49}$$

$$\mathcal{N}'(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}. \tag{3.50}$$

Using (3.5), (3.6) and (3.50), we obtain an identity we will use in calculating the sensitivities

$$\frac{\mathcal{N}'(d_1)}{\mathcal{N}'(d_2)} = \frac{\frac{1}{\sqrt{2\pi}}}{\frac{1}{\sqrt{2\pi}}} e^{\frac{d_2^2 - d_1^2}{2}}, \tag{3.51}$$

$$= e^{\frac{(d_1 - \sqrt{\tau} - d_1)(d_1 - \alpha_1\sqrt{\tau} + d_1)}{2}}. \tag{3.52}$$

Therefore, using the above relation, we reach

$$\mathcal{N}'(d_1) S_2 e^{\sigma_1(\sigma_1 - \rho\sigma_2)\tau + (g_1 - g_2)\tau} = S_1 K \mathcal{N}'(d_2). \tag{3.53}$$

Definition 3.2. Delta(Δ) of an option measures the rate of change of the option's value with respect to changes in the underlying asset's price. Meaning that λ amount of change in delta changes the price of an option by the amount of $\lambda\Delta$, obtained by taking the partial derivative of payoff with respect to asset's price, $\frac{\partial C}{\partial S}$.

Since we have two assets, we also have two deltas. Taking the partial derivative of (3.33) with respect to S_1 and S_2, letting $\eta = \sigma_1(\sigma_1 - \rho\sigma_2)\tau + (g_1 - g_2 - r)\tau$, we obtain

$$\frac{\partial C}{\partial S_1} = -\kappa \frac{S_2}{S_1^2} e^{\eta} \mathcal{N}(\kappa d_1) + \kappa \frac{S_2}{S_1} e^{\eta} \mathcal{N}'(d_1) \frac{\partial d_1}{\partial S_1} - \kappa K e^{-r\tau} \mathcal{N}'(d_2) \frac{\partial d_2}{\partial S_1}, \tag{3.54}$$

$$\frac{\partial C}{\partial S_2} = \kappa \frac{1}{S_1} e^{\eta} \mathcal{N}(\kappa d_1) + \kappa \frac{S_2}{S_1} e^{\eta} \mathcal{N}'(d_1) \frac{\partial d_1}{\partial S_2} - \kappa K e^{-r\tau} \mathcal{N}'(d_2) \frac{\partial d_2}{\partial S_2}, \tag{3.55}$$

$$\frac{\partial d_2}{\partial S_1} = \frac{\partial d_1}{\partial S_1} - \frac{\partial}{\partial S_1} \alpha_1 \sqrt{\tau}, \tag{3.56}$$

$$\frac{\partial d_2}{\partial S_2} = \frac{\partial d_1}{\partial S_2} - \frac{\partial}{\partial S_2} \alpha_1 \sqrt{\tau}, \tag{3.57}$$

$$K \mathcal{N}'(d_2) = \frac{S_2}{S_1} \mathcal{N}'(d_1) e^{\sigma_1(\sigma_1 - \rho\sigma_2)\tau + (g_1 - g_2)\tau}. \tag{3.58}$$

Partial derivatives of d_1 and d_2 with respect to any variable cancel each other out, and we are left with partial derivative of $\alpha_1\sqrt{\tau}$ with respect to variable in question. This fact is used in all calculations regarding the sensitivities. Given the binary operator κ 1 for call options and -1 for put options, Δ_1 and Δ_2 of S_1 and S_2 are given by

$$\Delta_1 = -\kappa \frac{S_2}{S_1^2} e^{\sigma_1(\sigma_1 - \rho\sigma_2)\tau + (g_1 - g_2 - r)\tau} \mathcal{N}(\kappa d_1), \tag{3.59}$$

$$\Delta_2 = \frac{\kappa}{S_1} e^{\sigma_1(\sigma_1 - \rho\sigma_2)\tau + (g_1 - g_2 - r)\tau} \mathcal{N}(\kappa d_1). \tag{3.60}$$

Definition 3.3. Theta(Θ) of an option measures the rate of change of the option's value with respect to changes in the time to maturity, obtained by taking the partial derivative of the payoff with respect to time to maturity, $\frac{\partial C}{\partial \tau}$.

Theta of an option is also known as the time decay. Taking the partial derivative of (3.33) with respect to τ

$$\begin{aligned} \frac{\partial C}{\partial \tau} = \kappa \frac{S_2}{S_1}(\sigma_1(\sigma_1 - \rho\sigma_2) + (g_1 - g_2 - r)) e^{\eta} \mathcal{N}'(\kappa d_1) + \kappa \frac{S_2}{S_1} e^{\eta} \mathcal{N}'(d_1) \frac{\partial d_1}{\partial \tau} \\ + \kappa r K e^{-r\tau} \mathcal{N}(\kappa d_2) - \kappa K e^{-r\tau} \mathcal{N}'(d_2) \frac{\partial d_2}{\partial \tau}. \end{aligned} \tag{3.61}$$

Rearranging the terms and using (3.53), we obtain the theta as

$$\begin{aligned} \Theta = \kappa \frac{S_2}{S_1}(\sigma_1(\sigma_1 - \rho\sigma_2) + (g_1 - g_2 - r)) e^{\sigma_1(\sigma_1 - \rho\sigma_2)\tau + (g_1 - g_2 - r)\tau} \mathcal{N}(\kappa d_1) \\ + \kappa \frac{S_2}{S_1} e^{\sigma_1(\sigma_1 - \rho\sigma_2)\tau + (g_1 - g_2 - r)\tau} \frac{\alpha_1}{2\sqrt{\tau}} \mathcal{N}'(d_1) + \kappa r K e^{-r\tau} \mathcal{N}(\kappa d_2). \end{aligned} \tag{3.62}$$

Theta is given in terms of year, therefore we divide it by 252 to find the time decay for a trading day.

Definition 3.4. Vega($\mathcal{V}$) of an option measures the rate of change of the option's value with respect to changes in volatility of the underlying, given by $\frac{\partial C}{\partial \sigma}$.

Since we have two assets, we also have two vegas. Taking the partial derivative of (3.33) with respect to volatilities σ_1 and σ_2, we obtain

$$\begin{aligned} \frac{\partial C}{\partial \sigma_1} =& \kappa \frac{S_2}{S_1}(2\sigma_1\tau - \rho\sigma_2\tau) e^{\eta} \mathcal{N}(\kappa d_1) + \kappa \frac{S_2}{S_1} e^{\eta} \mathcal{N}'(d_1) \frac{\partial d_1}{\partial \sigma_1} \\ & - \kappa K e^{-r\tau} \mathcal{N}'(d_2) \frac{\partial d_2}{\partial \sigma_1}, \end{aligned} \tag{3.63}$$

$$\begin{aligned} \frac{\partial C}{\partial \sigma_2} =& \kappa \frac{S_2}{S_1}(-\rho\sigma_1\tau) e^{\eta} \mathcal{N}(\kappa d_1) + \kappa \frac{S_2}{S_1} e^{\eta} \mathcal{N}'(d_1) \frac{\partial d_1}{\partial \sigma_2} \\ & - \kappa K e^{-r\tau} \mathcal{N}'(d_2) \frac{\partial d_2}{\partial \sigma_2}. \end{aligned} \tag{3.64}$$

Rearranging the terms and using the identity (3.53), we obtain vegas

$$\mathcal{V}_1 = \kappa \frac{S_2}{S_1} e^{\eta} \left[(2\sigma_1 \tau - \rho\sigma_2 \tau)\mathcal{N}(\kappa d_1) + \frac{\sigma_1 - \rho\sigma_2}{\alpha_1} \sqrt{\tau} \mathcal{N}'(d_1) \right], \tag{3.65}$$

$$\mathcal{V}_2 = \kappa \frac{S_2}{S_1} e^{\eta} \left[-\rho\sigma_1 \tau \mathcal{N}(\kappa d_1) + \frac{\sigma_2 - \rho\sigma_1}{\alpha_1} \sqrt{\tau} \mathcal{N}'(d_1) \right]. \tag{3.66}$$

Vega is given in the absolute value. If the absolute value is high, we assert that the underlying's volatility has a high effect on the option value.

Definition 3.5. Rho(ρ) measures the rate of change of the option's value with respect to the interest rate.

We assumed that the interest rate for borrowing and lending is the same, and denoted by r. Rho of the quotient option is given by

$$\frac{\partial C}{\partial r} = \kappa \frac{S_2}{S_1} (-\tau) e^{\eta} \mathcal{N}(\kappa d_1) + \kappa \tau K e^{-r\tau} \mathcal{N}(\kappa d_2). \tag{3.67}$$

From now on, we derive the higher-order sensitivities, starting with a second-order measure and moving on to cross-sensitivities used for multi-asset options. We now define gamma of an option, in our case we are given two gammas for each of the assets.

Definition 3.6. Gamma(Γ) of an option measures the rate of change in the delta with respect to the changes in the underlying price, given by $\frac{\partial^2 C}{\partial S^2}$.

Taking the second-order partial derivative of (3.33) with respect to assets, and utilizing deltas:

$$\frac{\partial^2 C}{\partial S_1^2} = \frac{\partial}{\partial S_1} \Delta_1, \tag{3.68}$$

$$\frac{\partial^2 C}{\partial S_2^2} = \frac{\partial}{\partial S_2} \Delta_2. \tag{3.69}$$

Using the identity and rearranging the terms, we obtain two gammas

$$\Gamma_1 = \kappa \frac{S_2}{S_1^3} e^{\eta} \left[2\mathcal{N}(\kappa d_1) + \frac{\mathcal{N}'(d_1)}{\alpha_1 \sqrt{\tau}} \right], \tag{3.70}$$

$$\Gamma_2 = \frac{\kappa}{S_1 S_2} \frac{e^{\eta}}{\alpha_1 \sqrt{\tau}} \mathcal{N}'(d_1). \tag{3.71}$$

From the hedger's view, a small gamma is preferred, so that the frequency of rebalancing can be reduced.

Definition 3.7. Chi measures the rate of change in the value of an option with respect to correlation coefficient.

Taking the partial derivative of (3.33) with respect to ρ, we obtain

$$\frac{\partial C}{\partial \rho} = \kappa \frac{S_2}{S_1}(-\sigma_1\sigma_2\tau)e^{\eta}\mathcal{N}(\kappa d_1) + \kappa \frac{S_1}{S_2}e^{\eta}\mathcal{N}'(d_1)\frac{\partial d_1}{\partial \rho} - \kappa K e^{-r\tau}\mathcal{N}'(d_2)\frac{\partial d_2}{\partial \rho}. \quad (3.72)$$

Rearranging the terms, and using (3.53), we obtain

$$\frac{\partial C}{\partial \rho} = \kappa \frac{S_2}{S_1} e^{\sigma_1(\sigma_1 - \rho\sigma_2)\tau + (g_1 - g_2 - r)\tau}\left[-\sigma_1\sigma_2\tau\mathcal{N}(\kappa d_1) - \frac{\sigma_1\sigma_2\sqrt{\tau}}{\alpha_1}\mathcal{N}'(d_1)\right]. \quad (3.73)$$

Definition 3.8. Cross-gamma measures the rate of change of delta in one underlying in response to a change in the level of another underlying, given by $\frac{\partial^2 C}{\partial S_1 \partial S_2}$.

Taking partial derivative of deltas, regardless the order of assets; we form

$$\frac{\partial^2 C}{\partial S_1 \partial S_2} = \frac{\partial}{\partial S_1}\Delta_2 = \frac{\partial}{\partial S_1}\left[\frac{\kappa}{S_1} e^{\sigma_1(\sigma_1 - \rho\sigma_2)\tau + (g_1 - g_2 - r)\tau}\mathcal{N}(\kappa d_1)\right], \quad (3.74)$$

$$\Gamma_{cross} = -\frac{\kappa}{S_1^2}e^{\eta}\left[\mathcal{N}(\kappa d_1) + \frac{\mathcal{N}'(d_1)}{\alpha_1\sqrt{\tau}}\right]. \quad (3.75)$$

CHAPTER 4

APPLICATIONS OF QUOTIENT OPTIONS

In this chapter, we apply the BSM formulas for the quotient options given in the previous chapters to several quotient options traded in BIST30. All prices below are in Turkish Lira. As explained in the Introduction, the quotient options traded in BIST 30 are written on the pairs (Koc, Sabanci) and (Garanti, Akbank). We begin this chapter by making several observations on the price movements of these underlyings.

4.1 Analysis of The Underlyings

In this part, we analyze the underlyings of our quotient options.

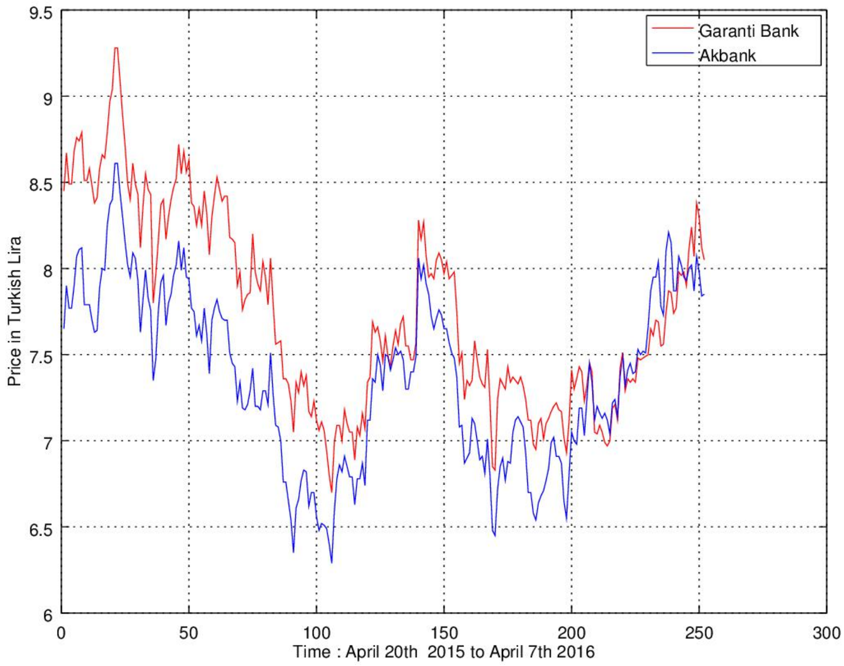

Figure 4.1: Price Movements of Garanti and Akbank

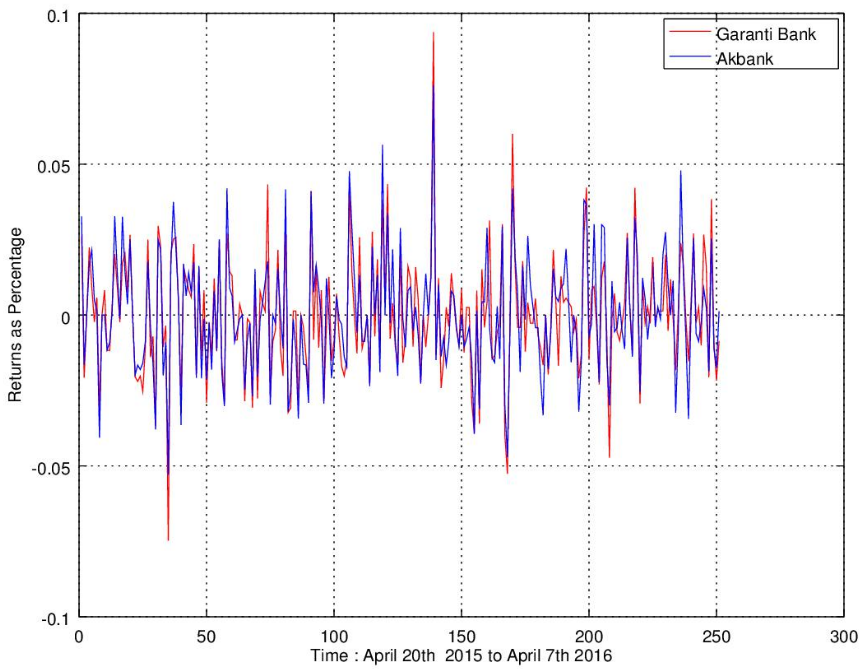

Figure 4.2: Returns of Garanti and Akbank

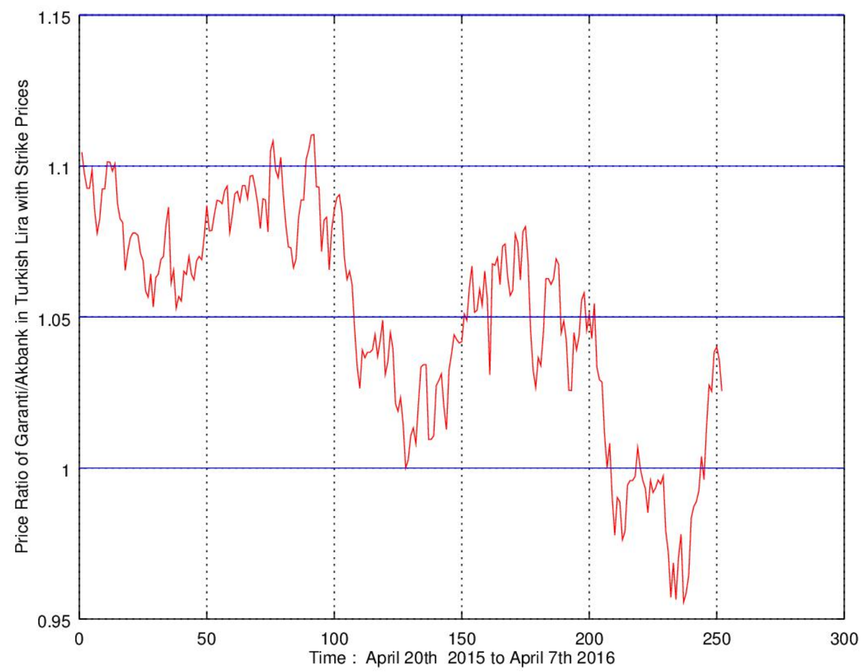

Figure 4.3: Garanti and Akbank Ratio With Different Strikes

In Figure 4.1, there is a clear tendency for these two assets to have similar movements. Especially, at sharp turning points the behavior of assets are quick to take a similar trend. However, we are to form this through the suggestion of correlated assets' returns, not the prices, therefore movements of returns are more important.

In Figure 4.2, returns of these two assets exhibit correlation, which are given in Table 4.14 for different periods of time. Meaning that the pair is a good candidate for pair trading and quotient options. Returns swing between 0.05 and -0.05 with occasional outliers. Since quotient options are written on the ratio of prices, it would give an idea to see the movement of the ratio, to be able to think in terms of quotient options; different strikes are set in Figure 4.3.

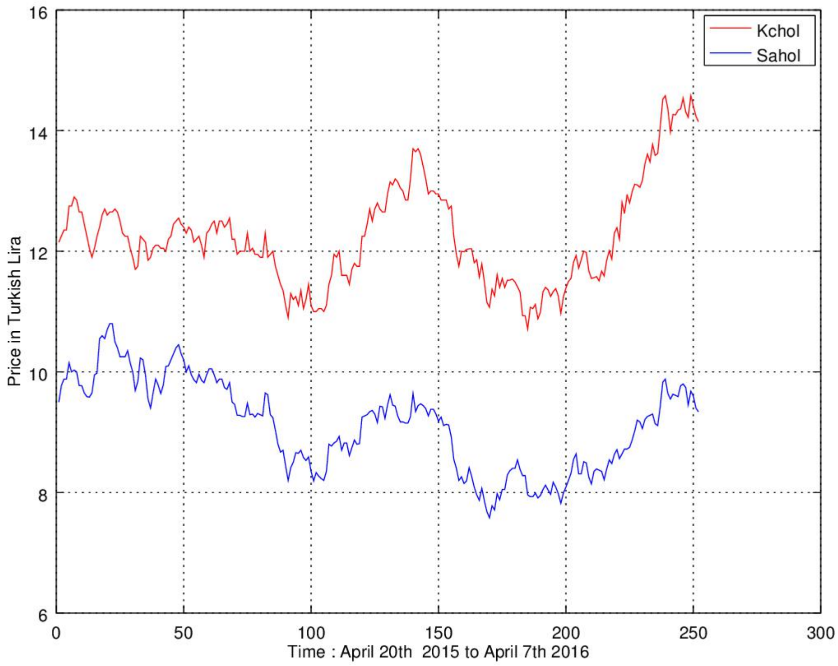

Figure 4.4: Price Movements of Koc and Sabanci

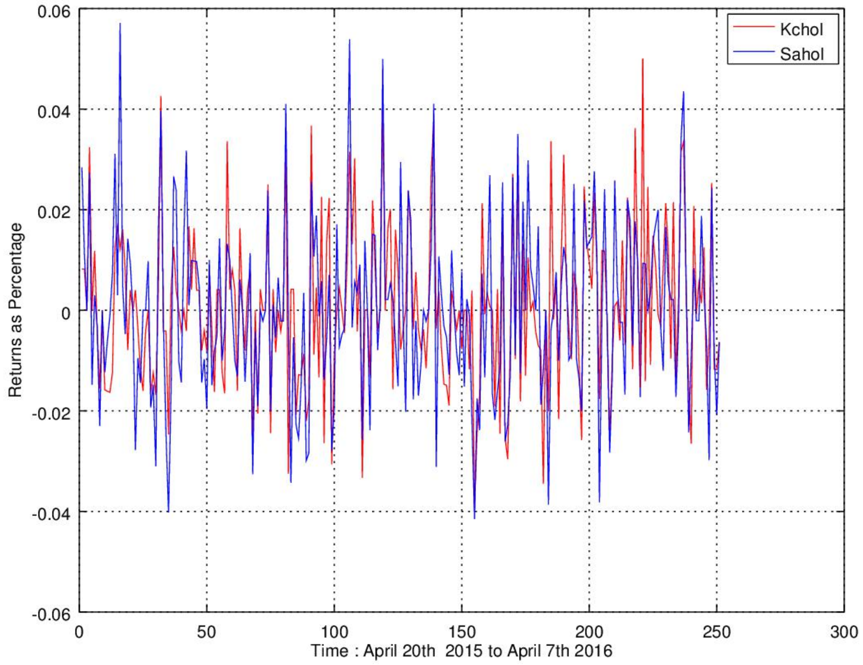

Figure 4.5: Returns of Koc and Sabanci

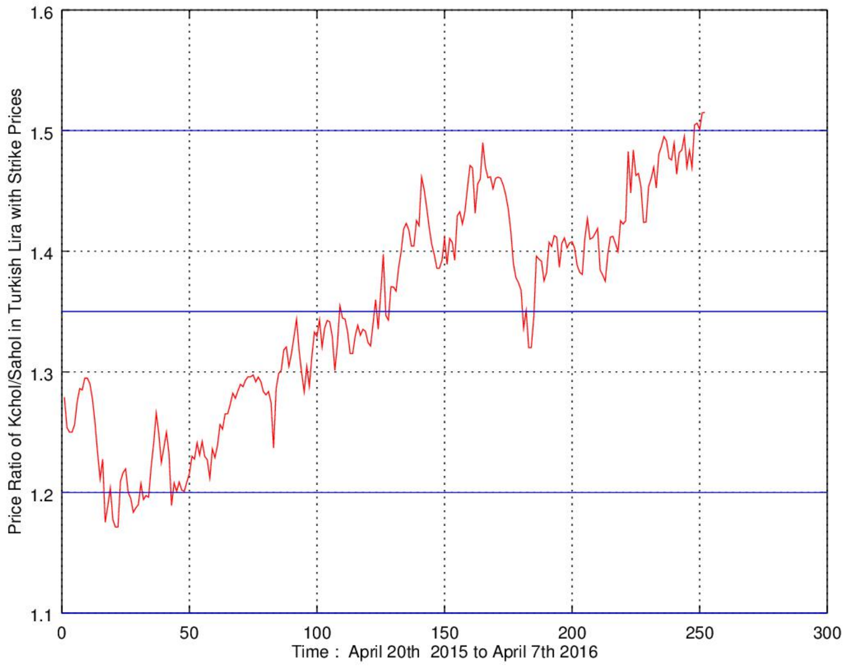

Figure 4.6: Koc and Sabanci Ratio with Different Strikes

In Figure 4.4, for the most part, prices are moving together. Clearly a good pair for a quotient option to be underwritten. However, the difference between the prices is more significant compared to Garanti and Akbank, meaning that a higher strike price should be set, and is by the firm as 1.5. Looking at the returns of these two, we see a high amount of tendency for comoving, which suggests that a pair trade may be applicable. Again in Figure 4.6, we see that 1.5 is quite higher for an option to be in-the-money. However, in terms of a put option, it would be advantageous. Since in the last 252 trading days, the ratio never exceeded 1.5. Therefore it is expected for the option to have a higher price, which already has in Table 4.13.

4.2 Hedging

In this part, we explore the hedging performances. We have 8 hedging scenarios for each a table of the last hedge error and average hedge error is given, and a graphic of hedges. Four calls and four puts with four different strike prices as 1.5, 1.1, 1, 0.9, and two different starting dates and two different maturities: 30-days and 60-days; from these we have excluded those options which are deeply out of the money, which effectively makes their price 0. In our case, 19 hedges are excluded.

Let ϕ_t be the positions in S_1, S_2 and bond at time t, and V_t be the value of this portfolio at time t. Hedge error is given by the formula

$$H_t = H_{t-1}(1+r) + V_t(\phi_t) - V_t(\phi_{t-1}). \tag{4.1}$$

We are issuing 1000 options, and we track the error by $\frac{H_t}{C_0 1000}$, therefore we base the hedge error on the first call price. We will refer to this ratio as the "normalized hedge error".

For the hedge scenarios, we use the following values

Table 4.1: Values for Hedge Scenarios 1

For $T = 30$, Starting: Dec 30th 2015			
Underlyings	**Volatility**	**Correlation Coefficient**	**Interest Rate**
Garan	0.3623	0.8378	10.54
Akbank	0.33725		
Kchol	0.2794	0.7767	
Sahol	0.31339		

Table 4.2: Values for Hedge Scenarios 2

For $T = 30$, Starting: March 8th 2016			
Underlyings	**Volatility**	**Correlation Coefficient**	**Interest Rate**
Garan	0.2639	0.8470	10.76
Akbank	0.2632		
Kchol	0.2614	0.62089	
Sahol	0.2521		

Table 4.3: Values for Hedge Scenarios 3

For $T = 60$, Starting: Dec 30th 2015			
Underlyings	**Volatility**	**Correlation Coefficient**	**Interest Rate**
Garan	0.3683	0.8790	10.302
Akbank	0.3366		
Kchol	0.2605	0.7309	
Sahol	0.2939		

Table 4.4: Values for Hedge Scenarios 4

For $T = 60$, Starting: March 8th 2016			
Underlyings	**Volatility**	**Correlation Coefficient**	**Interest Rate**
Garan	0.2714	0.8340	10.802
Akbank	0.2814		
Kchol	0.292	0.0769	
Sahol	0.2885		

We firstly present the average error and last hedge errors in hedging. Tables also give the starting points, and in any of it maturity is fixed at either 30 or 60 days. In graphs, dashed lines represent the hedging errors of put contracts.

Table 4.5: Hedge 1: $T = 30$, Starting March 8th 2016

Underlyings	**Type**	**Strike Price**	**Last Normalized Hedge Error**	**Average Normalized Hedge Error**
Garan/Akbank	Call	1	-0.594	-0.284
Garan/Akbank	Call	0.9	-0.049	-0.035
Garan/Akbank	Put	1.5	-0.009	-0.003
Garan/Akbank	Put	1.1	0.008	0.009
Garan/Akbank	Put	1	-0.188	-0.071

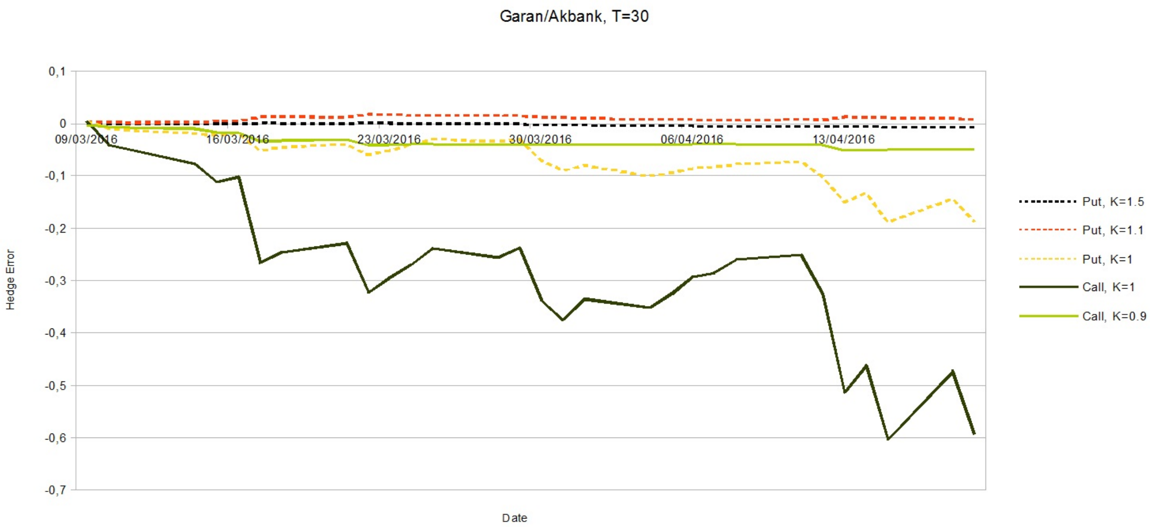

Figure 4.7: Hedge 1: $T = 30$, Starting March 8th 2016

Table 4.6: Hedge 2: $T = 30$, Starting Dec 30th 2015

Underlyings	**Type**	**Strike Price**	**Last Normalized Hedge Error**	**Average Normalized Hedge Error**
Garan/Akbank	Call	1.1	0.288	0.211
Garan/Akbank	Call	1	0.080	0.029
Garan/Akbank	Call	0.9	-0.021	-0.012
Garan/Akbank	Put	1.5	-0.009	-0.004
Garan/Akbank	Put	1.1	0.068	0.048
Garan/Akbank	Put	1	0.569	0.234

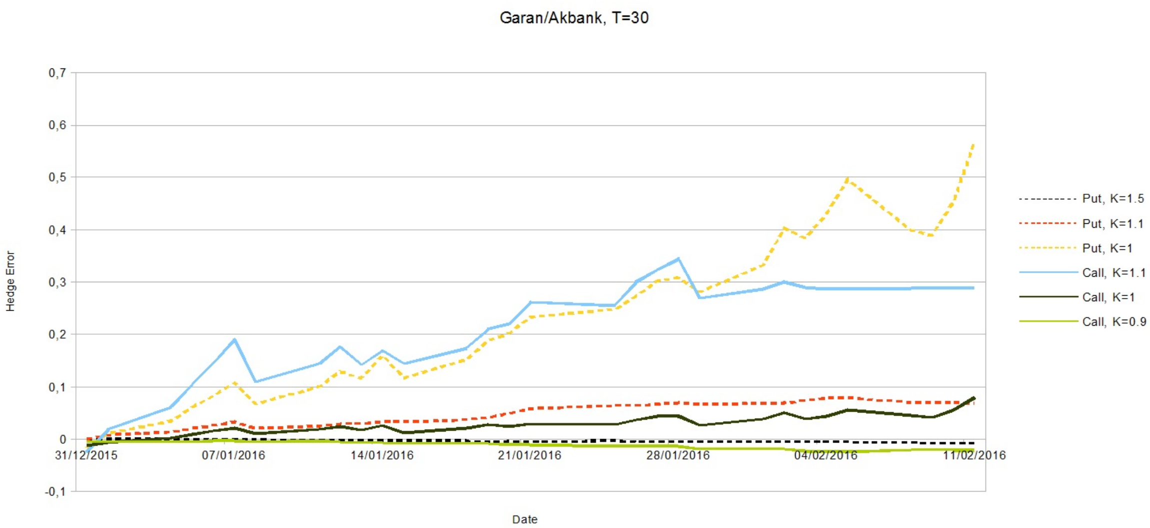

Figure 4.8: Hedge 2: $T = 30$, Starting Dec 30th 2015

Table 4.7: Hedge 3: $T = 30$, Starting March 8th 2016

Underlyings	Type	Strike Price	Last Normalized Hedge Error	Average Normalized Hedge Error
Kchol/Sahol	Call	1.5	0.549	0.199
Kchol/Sahol	Call	1.1	-0.018	-0.007
Kchol/Sahol	Call	1	-0.017	-0.007
Kchol/Sahol	Call	0.9	-0.016	-0.007
Kchol/Sahol	Put	1.5	0.221	0.074

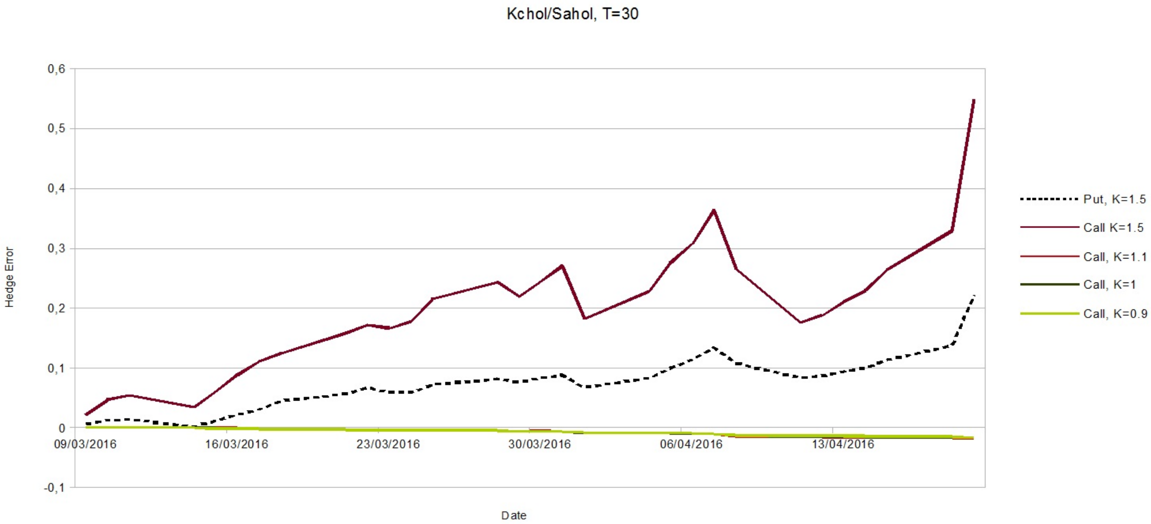

Figure 4.9: Hedge 3: $T = 30$, Starting March 8th 2016

Table 4.8: Hedge 4: T=30, Starting Dec 30th 2015

Underlyings	Type	Strike Price	Last Normalized Hedge Error	Average Normalized Hedge Error
Kchol/Sahol	Call	1.5	0.566	0.029
Kchol/Sahol	Call	1.1	-0.015	-0.006
Kchol/Sahol	Call	1	-0.013	-0.005
Kchol/Sahol	Call	0.9	-0.013	-0.005
Kchol/Sahol	Put	1.5	0.003	-0.006

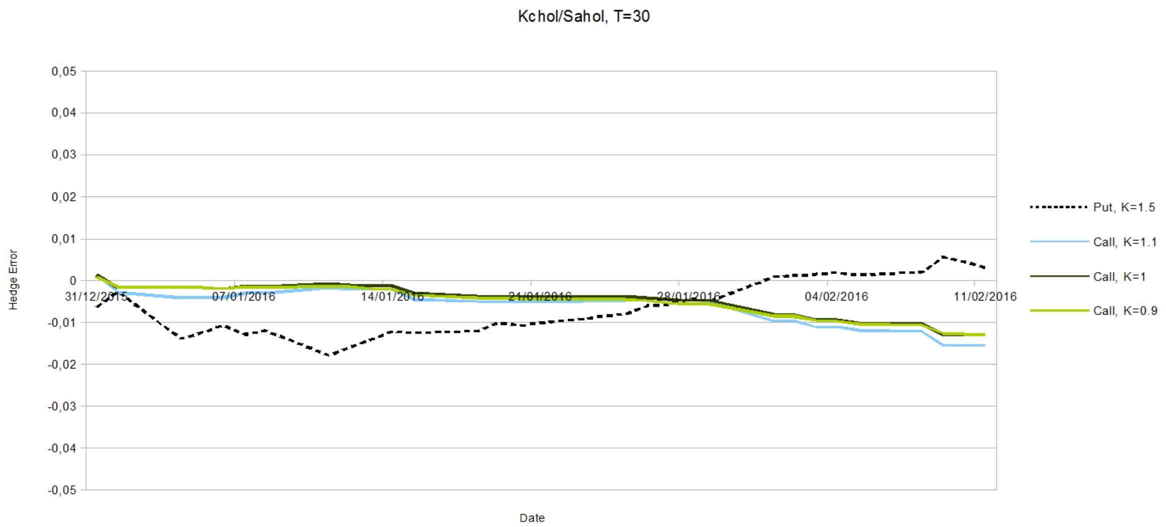

Figure 4.10: Hedge 4: $T = 30$, Starting Dec 30th 2015

Table 4.9: Hedge 5: $T = 60$, Starting March 8th 2016

Underlyings	Type	Strike Price	Last Normalized Hedge Error	Average Normalized Hedge Error
Garan/Akbank	Call	1.1	-0.348	-0.256
Garan/Akbank	Call	1	-0.087	-0.059
Garan/Akbank	Call	0.9	-0.033	-0.034
Garan/Akbank	Put	1.5	-0.025	-0.011
Garan/Akbank	Put	1.1	-0.032	-0.011
Garan/Akbank	Put	1	-0.026	-0.025
Garan/Akbank	Put	0.9	-0.174	-0.183

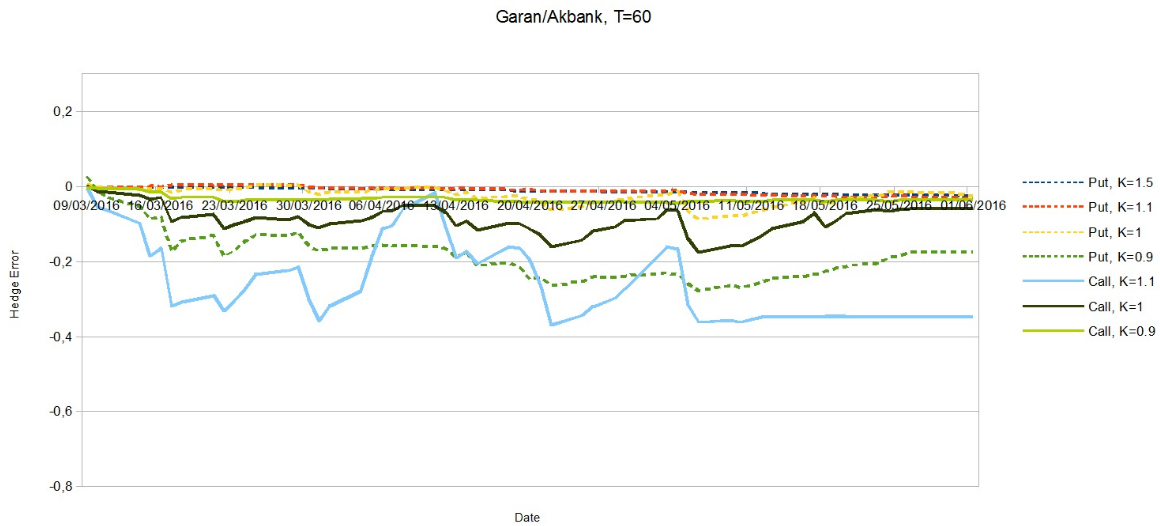

Figure 4.11: Hedge 5: $T = 60$, Starting March 8th 2016

Table 4.10: Hedge 6: $T = 60$, Starting Dec 30th 2015

Underlyings	Type	Strike Price	Last Normalized Hedge Error	Average Normalized Hedge Error
Garan/Akbank	Call	1.1	-0.003	-0.009
Garan/Akbank	Call	1	0.020	-0.007
Garan/Akbank	Call	0.9	-0.055	-0.027
Garan/Akbank	Put	1.5	-0.016	-0.008
Garan/Akbank	Put	1.1	0.049	0.024
Garan/Akbank	Put	1	0.41	0.13
Garan/Akbank	Put	0.9	0.367	0.208

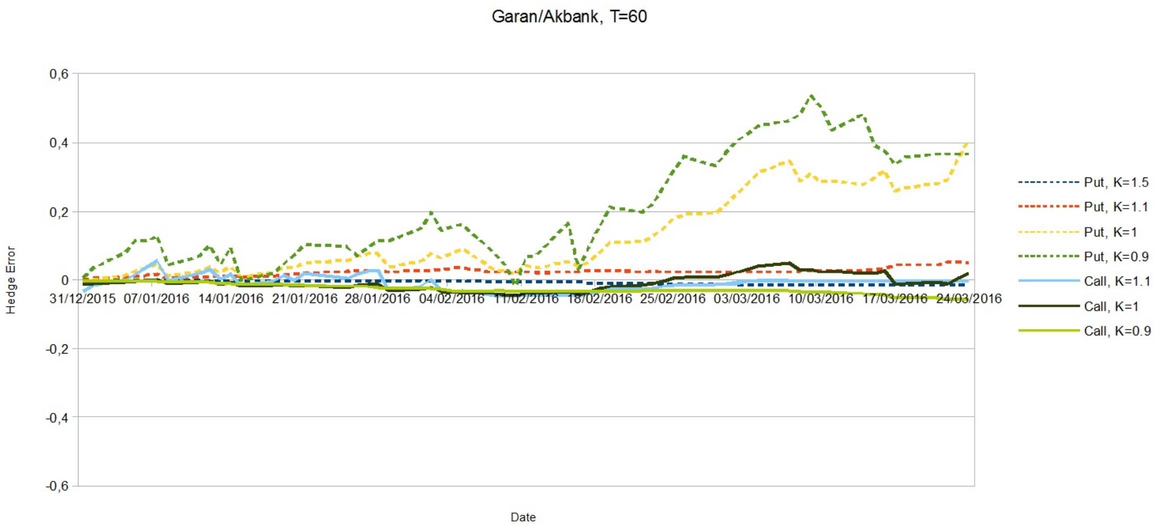

Figure 4.12: Hedge 6: $T = 60$, Starting Dec 30th 2015

Table 4.11: Hedge 7: $T = 60$, Starting March 8th 2016

Underlyings	Type	Strike Price	Last Normalized Hedge Error	Average Normalized Hedge Error
Kchol/Sahol	Call	1.5	0.715	0.315
Kchol/Sahol	Call	1.1	0.038	0.020
Kchol/Sahol	Call	1	0.018	0.008
Kchol/Sahol	Call	0.9	0.009	0.003
Kchol/Sahol	Put	1.5	0.434	0.186
Kchol/Sahol	Put	1.1	0.627	0.471

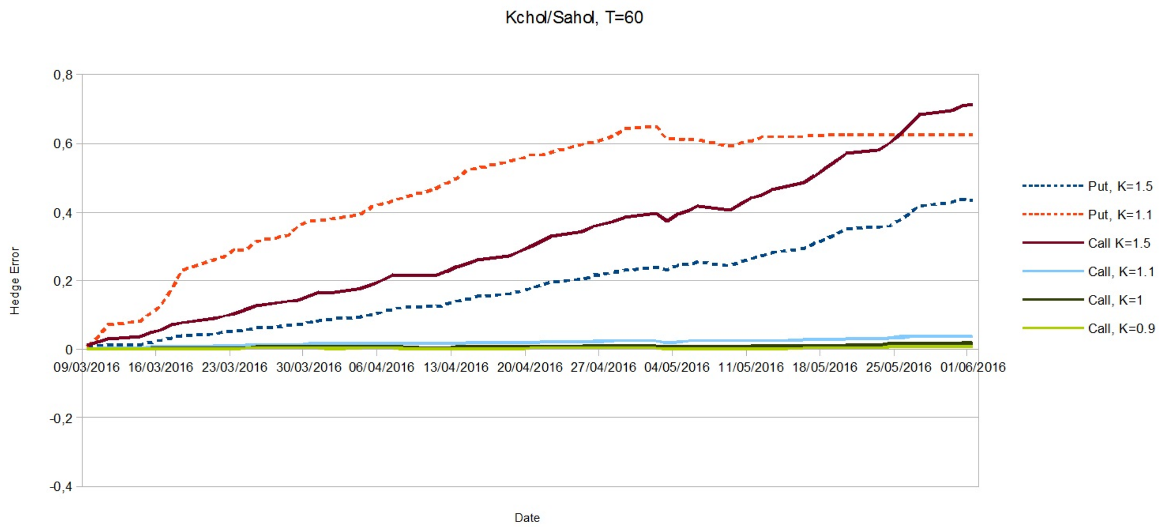

Figure 4.13: Hedge 7: $T = 60$, Starting March 8th 2016

Table 4.12: Hedge 8: $T = 60$, Starting Dec 30th 2015

Underlyings	Type	Strike Price	Last Normalized Hedge Error	Average Normalized Hedge Error
Kchol/Sahol	Call	1.5	0.105	-0.019
Kchol/Sahol	Call	1.1	-0.016	-0.009
Kchol/Sahol	Call	1	-0.017	-0.009
Kchol/Sahol	Call	0.9	-0.018	-0.009
Kchol/Sahol	Put	1.5	-0.041	-0.027

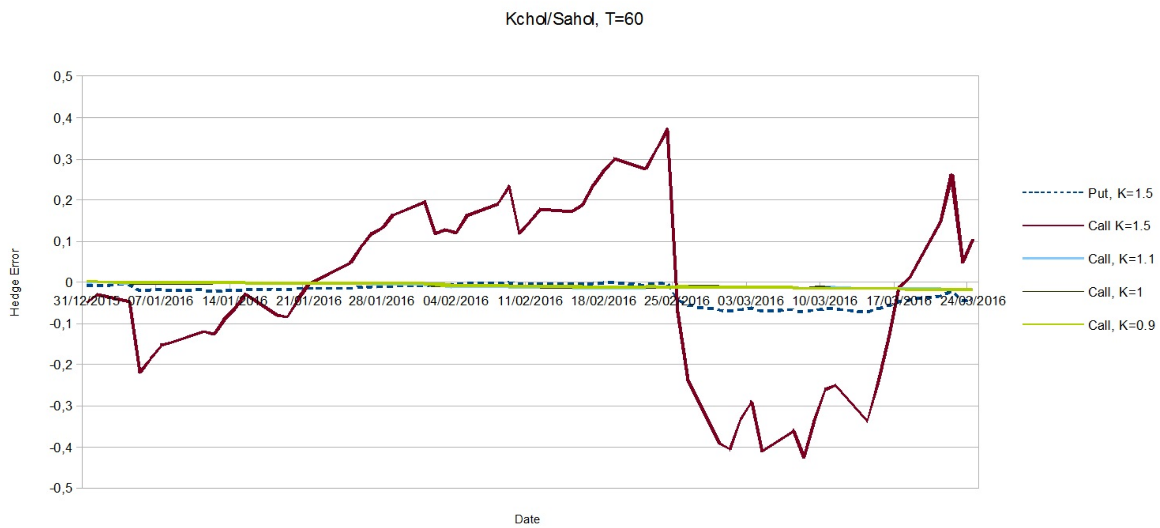

Figure 4.14: Hedge 8: $T = 60$, Starting Dec 30th 2015

4.2.1 Comments on the results

Our overall impression is that the BSM has limited success in hedging quotient options over the periods and products covered above. A successful hedge corresponds to a hedging error close to 0 throughout the hedging period. In each of the tables given above, we see a huge variability in the hedging error; sometimes the hedge ends reasonably close to successful (surprisingly, most hedges of strikes close to 1 seem to give reasonable hedges).

There may be many reasons for a hedge to perform poorly. One main reason for a hedge to fail is very volatile price movements, since delta-hedging works better with small changes. Another determinant is the length of maturity, for most it produced lower average hedge error given 30-days period. Let's say an option matures in money, if the premium we get is higher, we are likely to close this hedge with lower amounts of loss. For most of the cases, the performance of hedging is decided by how high or low the strike price is set, given the spread between the ratio of two assets and K. For instance, in Hedge 1, call with 1.5 strike price did not perform well even though it ended with almost 100 percent profit for the hedger. In the same example, calls with 0.9 strike price moves very closely around zero, indicating a very well hedging performance, if not a perfect one. Since we do not buy or sell fractional amounts of assets, a perfect hedge becomes not possible. However, calls with 1.1 strike price ended with a lost almost the double what we invest.

Analyzing Hedge 3, given the ratio of Kchol/Sahol, puts with lower strike prices performed poorly (although ending with profit for the hedger). The strike set by the firm is 1.5, and puts with 1.5 strike performed better. Calls with 0.9, 1 and 1.1 strike prices move around zero, suggesting a well-performing hedge. Let us compare Hedge 1 and Hedge 5 which has longer time to maturity. Hedge 5 outperforms Hedge 1 almost each cases, very lower average hedge errors. Maturity affects the option price, and in-the-money option could end up out-of-the-money. To sum up, among other contributors to the performances, the choice of strike price seem to have a very important role.

4.3 Pricing

We apply BSM formula to four different option contracts traded in the market. The data given in Table 4.13 taken from Trademaster platform of Is Investment in April 25th 2016, with expiration in April 29th. Closing prices of 25th is used. Two call and put quotient options are written on Garanti Bank, Akbank, Koc Holding, Sabanci Holding, which are all listed in Istanbul Stock Exchange (Borsa Istanbul). Prices of Garanti and Akbank are 8.31 and 8.35, of Koc Holding(Kchol) and Sabanci Holding(Sahol) are 14.85 and 10.05.

Table 4.13: Prices: The Market Data

Underlyings	Type	Strike Price	Bid	Ask
Garan/Akbank	Call	1	0.01	0.02
Garan/Akbank	Put	1	0.01	0.02
Kchol/Sahol	Call	1.5	0.01	0.02
Kchol/Sahol	Put	1.5	0.04	0.05

Bid price refers to the price the market maker is willing to buy at that price, and ask price is the price the market maker is willing to sell at that price.

Table 4.14: Volatilities and Correlations of Garanti and Akbank

	3-years	180-days	90-days	60-days	30-days	10-days
Garan Volatility	0.3643	0.3033	0.2536	0.2650	0.2711	0.2690
Akbank Volatility	0.35116	0.30377	0.27988	0.27789	0.28492	0.28057
Correlation Coefficient	0.8966	0.8521	0.8028	0.8046	0.7921	0.7516

Table 4.15: Volatilities and Correlations of Kchol and Sahol

	3-years	180-days	90-days	60-days	30-days	10-days
Kchol Volatility	0.2779	0.2618	0.2513	0.2491	0.2352	0.1992
Schol Volatility	0.31804	0.27741	0.25753	0.25183	0.28367	0.21988
Correlation Coefficient	0.7104	0.7187	0.6730	0.6706	0.7894	0.8568

Correlations and volatilities presented in Table 4.14 and Table 4.15 are done using the methods given in Chapter 2. Time to expiration is $\frac{4}{252}$, and the interest rate is again set as 0.1. We use the prices mentioned in the beginning. By BSM Red. 1-D, we refer to (3.33), and by Buchen Binary to (3.48).

Table 4.16: Prices for April 25th

Garan/Akbank	Volatility& Correlation for	Bid	Ask	BSM Red. 1-D	Buchen Binary
Call	180-days	0.01	0.02	0.0050858	0.0050567
	90-days			0.0053019	0.0052722
	60-days			0.0053179	0.0052882
	30-days			0.0057329	0.0057021
	10-days			0.0063450	0.0063129
Put	180-days	0.01	0.02	0.0097076	0.0096784
	90-days			0.0098339	0.0098042
	60-days			0.0098900	0.0098603
	30-days			0.010281	0.010250
	10-days			0.010869	0.010837

Table 4.17: Prices for April 25th

Kchol/Sahol	Volatility& Correlation for	Bid	Ask	BSM Red. 1-D	Buchen Binary
Call	180-days	0.01	0.02	0.006687	0.00656346
	90-days			0.00686957	0.00678017
	60-days			0.00670018	0.0066329
	30-days			0.00496206	0.00490298
	10-days			0.00165859	0.00162519
Put	180-days	0.04	0.05	0.0284596	0.0285333
	90-days			0.0284596	0.0286738
	60-days			0.0285527	0.0284854
	30-days			0.0266635	0.0266044
	10-days			0.0237578	0.0237244

As for Garan/Akbank call option, we produce close prices, however; it suggests that the call option is overvalued. For the put options, the prices are very close or just above the market price. Meaning that put option on Garan/Akbank is undervalued. For Kchol/Sahol, the call and put seems to be overpriced. The reason that prices are calculated using different time periods is to have an idea how much change it has on the prices. However, not much of a significance is found, except 10-days prices for Kchol/Sahol call.

Now let us price again the same four different contracts with five different days(time to maturity). The expiration date is April 29th 2016(now we are pricing the same contracts used in the previous example, only earlier), and the first contract we have is on February 8th. Again strike prices are set as 1 for Garan/Akbank, and 1.5 for Kchol/Sahol. Interest rate is 10.48 percent so we set $r = 0.10$ like the above example. 90 days volatilities and correlations are calculated starting from the day before it starts to 90 days backward. Volatilities for Garanti and Akbank are 0.328, 0.32302 and correlation is 0.8597. Volatilities for Kchol and Sahol are 0.2837, 0.25746 and correlation is 0.7162.

On February 8th, prices for Garanti and Akbank are 7.05 and 7.12, for Kchol and Sahol are 11.67 and 8.25.

Table 4.18: Prices for February 8th

Underlyings	Type	Bid	Ask	BSM Red 1D	Buchen Binary
Garan/Akbank	Call	0.06	0.07	0.0356386	0.0313811
Garan/Akbank	Put	0.06	0.07	0.0344478	0.0274991
Kchol/Sahol	Call	0.07	0.08	0.0412074	0.0369499
Kchol/Sahol	Put	0.15	0.16	0.111352	0.104403

Now we go one day further, on February 9th, prices for garanti and akbank are 7.04 and 7.20, for kchol and sahol are 11.55 and 8.14

Table 4.19: Prices for February 9th

Underlyings	Type	Bid	Ask	BSM Red 1D	Buchen Binary
Garan/Akbank	Call	0.05	0.06	0.0297346	0.0254694
Garan/Akbank	Put	0.06	0.07	0.0355584	0.0286971
Kchol/Sahol	Call	0.07	0.08	0.0473853	0.0431201
Kchol/Sahol	Put	0.14	0.15	0.108313	0.101451

On February 10th, prices for Garanti and Akbank are 7.09 and 7.16, for Kchol and Sahol are 11.56 and 8.35.

Table 4.20: Prices for February 10th

Underlyings	Type	Bid	Ask	BSM Red 1D	Buchen Binary
Garan/Akbank	Call	0.06	0.07	0.0351941	0.0310769
Garan/Akbank	Put	0.05	0.06	0.024765	0.0184446
Kchol/Sahol	Call	0.07	0.08	0.0408069	0.0366897
Kchol/Sahol	Put	0.16	0.17	0.131122	0.124802

On February 11th, prices for Garanti and Akbank are 7.05 and 7.13, for Kchol and Sahol are 11.58 and 8.39.

Table 4.21: Prices for February 11th

Underlyings	Type	Bid	Ask	BSM Red 1D	Buchen Binary
Garan/Akbank	Call	0.06	0.07	0.034265	0.0302055
Garan/Akbank	Put	0.05	0.06	0.0233444	0.0172182
Kchol/Sahol	Call	0.07	0.08	0.0413295	0.03727
Kchol/Sahol	Put	0.17	0.18	0.133908	0.127782

On February 12th, prices for Garanti and Akbank are 6.99 and 7.16, for Kchol and Sahol are 11.51 and 8.37.

Table 4.22: Prices for February 12th

Underlyings	Type	Bid	Ask	BSM Red 1D	Buchen Binary
Garan/Akbank	Call	0.05	0.06	0.0283863	0.0243323
Garan/Akbank	Put	0.04	0.05	0.0217623	0.0158484
Kchol/Sahol	Call	0.07	0.08	0.0476742	0.0436202
Kchol/Sahol	Put	0.17	0.18	0.13736	0.131446

In all of the above pricing, we see that the options are overpriced, meaning that the fair value is lower than the market value: it is not advisable to enter into any positions of these contracts.

4.4 Sensitivities

Table 4.23: Sensitivities: Results

Sensitivity Parameter	**Garan/Akbank Call**	**Garan/Akbank Put**	**Kchol/Sahol Call**	**Kchol/Sahol Put**
Delta 1	-0.0501512	0.0688868	-0.0426022	0.104244
Delta 2	0.0503926	-0.0692184	0.0288318	-0.0705488
Theta	0.0011	-0.0011	0.0016	-0.0016
Vega 1	0.0244126	0.0252966	0.029685	0.0330861
Vega 2	0.00688933	0.00633154	0.0229484	0.0212443
Rho	-0.00010222	-0.000172798	-0.000109027	-0.000455369
Gamma 1	0.273403	-0.27789	0.201403	-0.21367
Gamma 2	0.263913	-0.263913	0.0883623	-0.0883623
Chi	-0.0209983	0.0211745	-0.0204545	0.0210908
Cross Gamma	-0.268684	0.270938	-0.133434	0.137585

We use the contracts prevailing in 25th of April. Sensitivities measure the effects of a change in one variable while keeping others constant. Deltas give us the answer that if x amount of change in Garanti' price, we assert that $0.05x$ change in the option's value occurs. Notice that for calls delta 1 is negative whereas the reverse is true for puts. The reason for that comes from the structure of the option. Meaning that the ratio is affected negatively if the price of denominator rises, therefore a negative delta for call options. For puts, the expectation is that the ratio will be smaller than the strike, so it is in the interest of a trader for a raise in the denominator, forcing the put option to end up in-the-money. Vegas are given in absolute values, if it has a small value, we assert that volatility has a small impact on the option's value. In our cases, the absolute values are quite small, therefore the expected effect of a change in the underlyings' volatilities are rather small. Rho seems insignificant in our cases. Given the gammas, one thing should be noticed, the gamma of Kchol is, compared to Sahol, very small. This means that the rate of change of delta of Kchol is slow. The correlation plays a part in the pricing of quotient options, and the sensitivity of it is measured by Chi, also called as Correlation-delta. It is negative for call options, indicating that higher the positive change in the correlation adversely affect the price. A similar idea appears in pair trading where traders wait for the relaxing moments of correlation.

CHAPTER 5

CONCLUSION

The goal of this thesis was the application of the BSM framework to a number of quotient options traded in BIST30. Our main tool in this assessment was the hedging performance of the model.

We analyzed graphically the pairs used in Istanbul Stock Exchange. Correlations are found to determine if the pairs are good candidate for quotient options. We saw from our graphical analysis and empirical computations that indeed the pairs studied (Koc/Sabanci) and (Garanti/Akbank) do exhibit high correlations. We applied the BSM hedging algorithm to quotient options written on the above pairs traded in Borsa Istanbul. This involed the investigation of 45 hedge scenarios, to see the hedging performances of the contracts in real-life. To the best of our understanding and based on our results the BSM has a limited capability in hedging the quotient options of the types and periods covered in this thesis. We also compared the market prices and those suggested by the BSM faremwork. We observed that overall the market prices to be greater than those suggested by BSM.

Future work may focus on: 1) using implied volatilities in the hedging algorithm (rather than historical as done in this thesis) and 2) try alternative volatility models that allow also hedging the volatility.

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