ON PRODUCTS OF BLOCKS OF CONSECUTIVE INTEGERS

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ABSTRACT

ON PRODUCTS OF BLOCKS OF CONSECUTIVE INTEGERS

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In this thesis, an old conjecture of Erdős and Graham concerning integer squares obtained from products of disjoint blocks of consecutive integers is revisited. From arithmetic geometry point of view, the conjecture concerns the structure of integral points on certain projective hypersurfaces. These hypersurfaces are analyzed geometrically. The relation between the Erdős-Graham conjecture and some well-known conjectures in diophantine geometry and in number theory are explained. As for the computational aspect of the problem, an efficient algorithm for computer search is developed and in certain computationally challenging cases new numerical examples are obtained.

Keywords: Diophantine geometry, elementary number theory, polynomial parametrizations

ÖZ

ARDIŞIK TAMSAYI BLOKLARININ ÇARPIMLARI ÜZERİNE

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Bu tezde, ayrık tamsayı bloklarının çarpımlarından elde edilen tam kare sayılarla ilgili Erdős ve Graham'a ait bir varsayım incelenmiştir. Bu varsayımın sayılar teorisinde ve diophantine geometride bilinen bazı tahminlerle ilişkisi açıklanmıştır. Çözümlerin üzerinde bulunduğu projektif hiperyüzeyin geometrik ve bilgisayarlı hesaplamalar açısından analizi yapılmıştır. Bunlara ek olarak, bilgisayarlı hesaplamalar açısından zorlu olan ve daha önce örneği verilmemiş bazı özel durumlar için yeni örnekler elde edilmiştir.

Anahtar Kelimeler: Diophantine geometri, elemanter sayılar teorisi, polinom parametrizasyonları

To the memory of Mutlu Yıldız

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INTRODUCTION

It was shown by Erdős and Selfridge [8] that a product of consecutive integers is never a perfect power. In a subsequent article [7], it was asked by Erdős and Graham whether the same holds for products of disjoint blocks of consecutive integers. In fact, they conjectured that only finitely many integer squares could be obtained from products of disjoint blocks of four or more consecutive integers. Since then, considerable work has been done concerning the products of disjoint blocks of consecutive integers and some counterexamples to the conjecture were presented in [22, 17, 3, 4]. Recently, Bennett and Luijk [4] showed that infinitely many perfect integer squares could be obtained from products of five or more disjoint blocks of five consecutive integers by using a special univariate polynomial parametrization. We formulate the problem and list the known conjectures and facts as follows:

1.1 Formulation of the problem

Given integer parameters $r \geq 2$, $k \geq 4$, and variables x_i for $i = 1, 2, \ldots, r$, we define each block starting at x_i of length k with $b_k(x_i) := (x_i)(x_i + 1) \cdots (x_i + (k-1))$. We denote the products of blocks of length k with $[x_1, x_2, \ldots, x_r]_k := b_k(x_1)b_k(x_2) \cdots b_k(x_r)$. In this notation, the problem is to find (non-trivial) integer solutions of the equation:

$$[x_1, x_2, \ldots, x_r]_k = b_k(x_1)b_k(x_2) \cdots b_k(x_r) = y^2,\ x_i+(k-1) < x_{i+1},\ i = 1, 2, \ldots, r-1. \tag{1.1}$$

In this study, $(x_1, x_2, \ldots, x_r, y)$ positive integer tuple solutions of (1.1) will be investigated.

1.2 Literature review

In this section, we list some results and conjectures about the problem collected from various papers. As a supplementary note, some elementary number theoretical results on product of consecutive integers are given.

1.2.1 Results and conjectures about the problem

- For $(r = 4; k = 4)$ and $(r \geq 6; k = 4)$, equation (1.1) has infinitely many solutions [22].
- For $(r = 3; k = 4)$ and $(r = 5; k = 4)$, equation (1.1) has infinitely many solutions [3].
- For $(r \geq 5; k = 5)$, equation (1.1) has infinitely many solutions [4].
- For $(r \geq 1; k \geq 4)$, equation (1.1) may have at most finitely many solutions in positive integers conjectured in [7].
- For $(r = 2; k = 4)$, equation (1.1) may have at most finitely many solutions conjectured in [17, 3].
- For $(r = 2, 3; k = 5)$, equation (1.1) may have at most finitely many solutions conjectured in [4].
- For $(r = 4; k = 5)$, equation (1.1) may have infinitely many solutions conjectured in [4].
- For $(r \gg k; k \geq 4)$, equation (1.1) may have infinitely many solutions conjectured in [22].
- For $(r \geq 2; k \geq 6)$, almost nothing known, see the discussion in [4]. However, to the best our knowledge, we have shown the first numerical examples for $k = 6$ and $k = 7$.

- For $(1 \leq r < k-1; k \geq 4)$, equation (1.1) may have only finitely many solutions. Moreover, we have observed that product of square-free parts of disjoint blocks of length k are less likely to match each other infinitely often when the number of distinct blocks r is sufficiently less than block length k.

1.2.2 Results on products of consecutive integers

In this section, we present various results about products of consecutive integers obtained by elementary techniques.

Theorem 1. *Let $k \geq 2$ be given integer. Then, the equation*

$$x(x+1)\cdots(x+k-1) = y^n \text{ with } n \geq 2 \text{ and } xy \neq 0$$

has no integer solutions.

Proof. See, [8]. ☐

Erdős and Graham in [7], mentioned the following fact without giving a proof:

Theorem 2. *For any fixed $b > 1, r > 1 \in \mathbb{N}$, there exists only finitely many positive integer sequences $0 < m_1 < m_2 < \ldots < m_r$ with $m_{i+1} - m_i < b$ for $i = 1, 2, \cdots, r-1$ such that $m_1 m_2 \ldots m_r$ is a perfect integer square.*

Proof. The cases $r = 2$ and $r > 2$ are considered separately.
For the case $r = 2$, first of all, for any $0 < s < b \in \mathbb{Z}$, $0 < x \in \mathbb{Z}$, $y \in \mathbb{Z}$ integer solution of $x(x+s) = (x+\frac{s}{2})^2 - (\frac{s}{2})^2 = y^2$ gives a rational solution to the equation $1 = U^2 + V^2$ with $U = \frac{Y}{X}, 0 < V = \frac{Z}{X}, 0 < X = x + \frac{s}{2}, Y = y, 0 < Z = \frac{s}{2}$. Also, (U, V) is a rational solution to $1 = U^2 + V^2$ if and only if $(U, V) \in \{(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}), t \in \mathbb{Q}\} \cup \{(-1, 0)\}$. Using this rational parametrization, one can obtain $y = Y = X\frac{1-t^2}{1+t^2}$ with $t \in \mathbb{Q}$, or, $y = Y = (x+\frac{s}{2})\frac{n^2-m^2}{n^2+m^2}$ with $mn \neq 0 \in \mathbb{Z},\ \gcd(m,n) = 1$ and $\frac{s}{2} = Z = X\frac{2t}{1+t^2}$, or, $\frac{s}{2} = Z = (x+\frac{s}{2})\frac{2mn}{n^2+m^2}$ with $mn \neq 0 \in \mathbb{Z},\ \gcd(m,n) = 1$, or, $x = \frac{s}{2}\frac{(n-m)^2}{2mn}$ with $mn \neq 0 \in \mathbb{Z},\ \gcd(m,n) = 1$. Hence, $x = \frac{s}{2}\frac{(n-m)^2}{2mn}$ and $y = \frac{s}{2}\frac{n^2-m^2}{2mn}$ with $mn \neq 0 \in \mathbb{Z},\ \gcd(m,n) = 1$. In the case $m < n$, $x(n^2 - m^2) = y(n-m)^2$ implies $x(n+m) = y(n-m)$, which

also implies $x = n - m, y = n + m, s = \frac{4mn}{n-m}$. Since $s = \frac{4mn}{n-m}$ is an integer, and $mn \neq 0 \in \mathbb{Z}$, $\gcd(m, n) = 1$, implies $(n - m) \mid 4$, such that $s = n^2 - 4n$ if $n - m = 4$, $s = 2n^2 - 4n$ if $n - m = 2$, $s = 4n^2 - 4n$, if $n - m = 1$. Since $0 < s < b$, only finitely many choices possible for s, n, m and also for x, y. In the case $n < m$, $x(n^2 - m^2) = y(n - m)^2$ implies $x(n + m) = (-1)y(m - n)$, which also implies $x = m - n, y = -n - m, s = \frac{4mn}{m-n}$. Since $s = \frac{4mn}{m-n}$ is an integer, and $mn \neq 0 \in \mathbb{Z}$, $\gcd(m, n) = 1$, implies $(m - n) \mid 4$, such that $s = m^2 - 4m$ if $m - n = 4$, $s = 2m^2 - 4m$ if $m - n = 2$, $s = 4m^2 - 4m$, if $m - n = 1$. Since $0 < s < b$, again just only finitely many choices possible for s, n, m and also for x, y, so result follows for the case $r = 2$.

For the case $r > 2$, it is clear that $m_1 m_2 \ldots m_r$ is a perfect square if and only if m_1 satisfies one of the following $(b - 1)^{r-1}$ equations inductively defined below. When $r > 2$, each equation given in (1.2) defines a smooth algebraic curve of genus $g > 0$, by Siegel's theorem stated in [15] on integral points of a smooth algebraic curve of genus g, these equations each has only finitely many (x, y) integer tuple solutions. Also, such possible equations is also bounded by $(b - 1)^{r-1}$, so result follows for

$r > 2$ as well.

$$x = y^2 \longrightarrow \begin{cases} x(x+1) = y^2 \longrightarrow \begin{cases} x(x+1)(x+2) = y^2 \longrightarrow \cdots \\ \vdots \\ x(x+1)(x+b) = y^2 \longrightarrow \cdots \end{cases} \\ \\ x(x+2) = y^2 \longrightarrow \begin{cases} x(x+2)(x+3) = y^2 \longrightarrow \cdots \\ \vdots \\ x(x+2)(x+b+1) = y^2 \longrightarrow \cdots \end{cases} \\ \\ \vdots \\ x(x+b-1) = y^2 \longrightarrow \begin{cases} x(x+b-1)(x+b) = y^2 \longrightarrow \cdots \\ \vdots \\ x(x+b-1)(x+2b-2) = y^2 \longrightarrow \cdots \end{cases} \end{cases} \tag{1.2}$$

□

Remark 3. *Theorem could be extended to perfect powers as well.*

Proof. For all cases, i.e. $r > 1$, Siegel's theorem [15] could be applied, hence the finiteness result follows immediately. □

Theorem 4. *[16, 18] Let $\alpha_1, \alpha_2 \neq 1$ be two positive rational numbers and β_1, β_2 be two positive integers. Assume that $\Lambda := \beta_2 \log(\alpha_2) - \beta_1 \log(\alpha_1) \neq 0$ and $h(\frac{m}{n}) :=$ $\log(\max\{|m|, |n|\})$.*
Then, $\log(|\Lambda|) \geq -22 \left(\max\{\log(\frac{\beta_1}{h(\alpha_2)} + \frac{\beta_2}{h(\alpha_1)}) + 0.06, 21\}\right)^2 h(\alpha_1)h(\alpha_2)$.

Proof. See Remark 4 in [18]. □

Lemma 5. $\log(1-w) = -\Sigma_{n=1}^{\infty}\frac{1}{n}w^n$*, when* $|w| < 1$.

Proof. Taylor expansion of $\log(z)$ around $z = 1$, is $\Sigma_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(z-1)^n$. By taking into convergence interval $\log(z) = \Sigma_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(z-1)^n$, when $|z-1| < 1$. By taking $z - 1 = -w$, $\log(1-w) = -\Sigma_{n=1}^{\infty} \frac{1}{n} w^n$, when $|w| < 1$. □

Lemma 6. $\Sigma_{n=1}^{\infty} \frac{1}{n} w^n \leq 2w$, *when* $0 \leq w \leq \frac{1}{2}$.

Proof. Let $f(w) := 2w - \Sigma_{n=1}^{\infty} \frac{1}{n} w^n$. Since $f(0) = 0$, it is enough to show that $f(w)$ is non-decreasing to prove the lemma. $f'(w) = 2 - \Sigma_{n=0}^{\infty} w^n = 2 - \frac{1}{1-w} \geq 0$, as $0 \leq w \leq \frac{1}{2}$. □

Proposition 7. $|\log(1+x)| \leq 2|x|$, *when* $|x| \leq \frac{1}{2}$.

Proof. Case $(0 \leq x \leq \frac{1}{2})$: It is clear that $|\log(1+x)| = \log(1+x) \leq 2|x| = 2x$.

Case $(-\frac{1}{2} \leq x \leq 0)$: By a change of variable $-x = w$, and using Lemma 5, $|\log(1+x)| = |\log(1-w)| = \Sigma_{n=1}^{\infty} \frac{1}{n} w^n$, when $0 \leq w \leq \frac{1}{2}$. By also using Lemma 6, $\Sigma_{n=1}^{\infty} \frac{1}{n} w^n \leq 2w = 2|x|$. □

Evertse in [9], mentioned the following fact without giving a proof:

Theorem 8. *Let and $a, b, c \geq 1$ be integers. There is a computable number ζ, depending only on a, b, c such that the equation*

$$ax^n - by^n = c \text{ with } x \geq 1, y \geq 1, (x, y) \neq (1, 1)$$

has no integer solutions if $n > \zeta$.

Proof. Case $(x \geq 2, y = 1)$:
Since $ax^n - by^n = c$, then $ax^n = b + c$. This implies $a2^n \leq b + c$.

Case $(x = 1, y \geq 2)$:
Since $ax^n - by^n = c$, then $a - c = by^n$. This implies $b2^n \leq a - c$.

Case $(x, y \geq 2)$ and $(a = b)$:
Since $ax^n - ay^n = c$, then $x > y$.

Also, $ax^n - ay^n = a(x-y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1}) = c$ which implies $n2^{n-1} < ny^{n-1} < c$.

Case $(x \geq y \geq 2)$ and $(a \neq b)$:
Since $ax^n - by^n = c$, we have $1 - \frac{b}{a}(\frac{y}{x})^n = \frac{c}{ax^n}$ and $1 - \frac{c}{ax^n} = \frac{b}{a}(\frac{y}{x})^n$.
By taking logarithms, $\log(1 - \frac{c}{ax^n}) = \log(\frac{b}{a}) + n\log(\frac{y}{x}) = \log(\frac{b}{a}) - n\log(\frac{x}{y})$.
By using Proposition 7, $\frac{2c}{ax^n} \geq |\log(1 - \frac{c}{ax^n})| = |\log(\frac{b}{a}) - n\log(\frac{x}{y})|$.
By using Theorem 4,
$\log(\frac{2c}{ax^n}) \geq -22\left(\max\{\log(\frac{1}{\log(x)} + \frac{n}{\log(\max\{a,b\})}) + 0.06, 21\}\right)^2 \log(x)\log(\max\{a,b\})$.
Then, $n \leq 22\left(\max\{\log(\frac{1}{\log(x)} + \frac{n}{\log(\max\{a,b\})}) + 0.06, 21\}\right)^2 \log(\max\{a,b\}) + \frac{\log(\frac{2c}{a})}{\log(x)}$.
Therefore, $n \leq 22\left(\max\{\log(\frac{n+1}{\log(2)}) + 0.06, 21\}\right)^2 \log(\max\{a,b\}) + \frac{\log(\frac{2c}{a})}{\log(2)}$.

Case $(y \geq x \geq 2)$ and $(a \neq b)$:
Since $ax^n - by^n = c$, we have $\frac{a}{b}(\frac{x}{y})^n - 1 = \frac{c}{by^n}$ and $1 + \frac{c}{by^n} = \frac{a}{b}(\frac{x}{y})^n$.
By taking logarithms, $\log(1 + \frac{c}{by^n}) = \log(\frac{a}{b}) + n\log(\frac{x}{y}) = \log(\frac{a}{b}) - n\log(\frac{y}{x})$.
By using Proposition 7, $\frac{2c}{by^n} \geq |\log(1 + \frac{c}{by^n})| = |\log(\frac{a}{b}) - n\log(\frac{y}{x})|$.
By using Theorem 4,
$\log(\frac{2c}{by^n}) \geq -22\left(\max\{\log(\frac{1}{\log(y)} + \frac{n}{\log(\max\{a,b\})}) + 0.06, 21\}\right)^2 \log(y)\log(\max\{a,b\})$.
Then, $n \leq 22\left(\max\{\log(\frac{1}{\log(y)} + \frac{n}{\log(\max\{a,b\})}) + 0.06, 21\}\right)^2 \log(\max\{a,b\}) + \frac{\log(\frac{2c}{b})}{\log(y)}$.
Therefore, $n \leq 22\left(\max\{\log(\frac{n+1}{\log(2)}) + 0.06, 21\}\right)^2 \log(\max\{a,b\}) + \frac{\log(\frac{2c}{b})}{\log(2)}$. □

Corollary 9. *The equation*

$$x(x+1) = 2y^n \text{ with } n \geq 6726 \text{ and } x, y \geq 1$$

has no integer solutions.

Proof. For some u, v co-prime integers, $x = v^n, x+1 = 2u^n$. We have $2u^n - v^n = 1$. Using above theorem, $n \leq 22\left(\max\{\log(\frac{n+1}{\log(2)}) + 0.06, 21\}\right)^2 \log(2) + 1$. Then $n \leq 6725$ □

Theorem 10. *The diophantine equation* $x(x+1)\cdots(x+k-1) = p^{e_p}q^{e_q}$*,* $k \geq 2$*,* $p, q \in \{2,3,5,7,11,13\}$ *with* $e_p, e_q \in \mathbb{N}$*, has only positive* x *solutions* $x = 1,2,3,4,7,8$

when $k = 2$, has only positive x solutions $x = 1, 2$ when $k = 3$, has only positive x solution $x = 1$ when $k = 4$, and has no positive x integer solution when $k > 4$.

Proof. Using the method explained in [13], we could find solutions of the equations of type $p^{m_1}q^{n_1} - p^{m_2}q^{n_2} = 1$. The method first given by Størmer in [21].
Case $k = 2$: Since two consecutive integers always contain an even number, $x(x + 1) = 2^{e_2}q^{e_q}$.
Fundamental units of 5.3^2 Pell equations, $a^2 - 2^{\epsilon_2}q^{\epsilon_q}b^2 = 1$ where $q \in \{3, 5, 7, 11, 13\}$ and $\epsilon_2, \epsilon_q \in \{0, 1, 2\}$ are found. For each fundamental unit the equation $a^2 - 1 = 4x(x + 1)$ is solved for x. Then $x = 1, 2, 3, 4, 7, 8$ could be obtained.

Case $k = 3$: Since three consecutive integers always divisible by $3! = 6$,
$x(x + 1)(x + 2) = 2^{e_2}3^{e_3}$.
Fundamental units of 3^2 Pell equations, $a^2 - 2^{\epsilon_2}3^{\epsilon_3}b^2 = 1$ where $\epsilon_2, \epsilon_3 \in \{0, 1, 2\}$ are found. For each fundamental unit the equation $a^2 - 1 = 4t(t + 1)$ is solved for t. Then $t = 1, 2, 3, 8$ could be obtained. From these solutions $x = 1, 2$ could be obtained.

Case $k = 4$:Since four consecutive integers always divisible by $4! = 24$,
$x(x + 1)(x + 2)(x + 3) = 2^{e_2}3^{e_3}$.
Using the solutions found in Case $k = 3$, $x = 1$ could be obtained for the Case $k = 4$.

Case $k > 4$: Since five or more consecutive integers always divisible by $5! = 120$, $x(x + 1)\cdots(x + k - 1) = p^{e_p}q^{e_q}$, is not possible for any positive x value as left side always contains at least three distinct prime numbers $2, 3, 5$.

□

Lemma 11. *Let k be any positive integer. Then, the following equality holds:*

$$\Pi_{x=1}^{n} f(x)f(x+1)\cdots f(x+k) = \left(\Pi_{t=1}^{k} f(t)^{t}\right)\left(\Pi_{t=k+1}^{n} f(t)^{k+1}\right)\left(\Pi_{t=1}^{k} f(n+t)^{k+1-t}\right).$$

Proof. We prove the lemma by induction.
Case $k = 1$:

$$\Pi_{x=1}^{n} f(x)f(x+1) = f(1)\left(\Pi_{t=2}^{n} f(t)^2\right) f(n+1)$$

holds. Assume that the result holds for the Case $k = m$.

Case $k = m + 1$:

$\Pi_{x=1}^{n} f(x) \cdots f(x+m+1) = \left(\Pi_{x=1}^{n} f(x) \cdots f(x+m)\right)\left(\Pi_{x=1}^{n} f(x+m+1)\right) =$
$\left(\Pi_{t=1}^{m} f(t)^t\right)\left(\Pi_{t=m+1}^{n} f(t)^{m+1}\right)\left(\Pi_{t=1}^{m} f(n+t)^{m+1-t}\right)\left(\Pi_{x=1}^{n} f(x+m+1)\right) =$
$\left(\Pi_{t=1}^{m+1} f(t)^t\right)\left(\Pi_{t=m+2}^{n} f(t)^{m+2}\right)\left(\Pi_{t=1}^{m} f(n+t)^{m+1-t}\right)\left(\Pi_{x=n-m}^{n} f(x+m+1)\right) =$
$\left(\Pi_{t=1}^{m+1} f(t)^t\right)\left(\Pi_{t=m+2}^{n} f(t)^{m+2}\right)\left(\Pi_{t=1}^{m+1} f(n+t)^{m+2-t}\right).$

Since result holds for Case $k = m + 1$ as well, by induction result holds for arbitrary positive integer k. □

Remark 12. *Let $f(x) = ax^2 + bx + c$. Then, integer solution of the diophantine equation*

$$\Pi_{x=1}^{n} f(x)f(x+1) = y^2$$

corrresponds to integer solution of the diophantine equation

$$(a+b+c)(a(n+1)^2 + b(n+1) + c) = w^2.$$

Proof. Using Lemma 11, we have the following identity:

$$\Pi_{x=1}^{n} f(x)f(x+1) = f(1)\left(\Pi_{t=2}^{n} f(t)^2\right) f(n+1).$$

Moreover, by setting $f(x) = ax^2 + bx + c$, the equation transforms into

$$\Pi_{x=1}^{n} f(x)f(x+1) = (a+b+c)\left(\Pi_{t=2}^{n} f(t)^2\right)(a(n+1)^2 + b(n+1) + c) = y^2.$$

Hence, result follows. □

Theorem 13. *The diophantine equation*

$$\Pi_{x=1}^{n} x(x+1)(x+2) = y^3$$

has no integer solutions.

Proof. Using Lemma 11, we have the following identity:

$$\Pi_{x=1}^{n} f(x)f(x+1)f(x+2) = \left(\Pi_{t=1}^{2} f(t)^t\right)\left(\Pi_{t=3}^{n} f(t)^3\right)\left(\Pi_{t=1}^{2} f(n+t)^{3-t}\right).$$

Moreover, by setting $f(x) = x$, the equation transforms into

$$\Pi_{x=1}^{n} x(x+1)(x+2) = \left(\Pi_{t=1}^{2} t^t\right)\left(\Pi_{t=3}^{n} t^3\right)\left(\Pi_{t=1}^{2} (n+t)^{3-t}\right) = y^3.$$

To have integer solutions, $4(n+1)^2(n+2) = w^3$ must be satisfied for some integers n and w. Since $n+1$ and $n+2$ are coprime integers, the problem reduces to the following two cases:
Case $(n+2 = 2u^3, n+1 = v^3)$: The equation $n+2-(n+1) = 2u^3 - v^3 = 1$ is obtained. If we set $x = \frac{1}{u}, y = \frac{v}{u}$, then the equation transforms into $x^3 + y^3 = 2$. It is well known that, this equation isomorphic to the elliptic curve $Y^2 = X^3 - 27$ in [6]. To show this fact, we set $x = p+q, y = p-q$, then the equation transforms into $p^3 + 3pq^2 = 1$. If we further substitute $p = \frac{3}{X}, q^2 = \frac{X^3-27}{(3X)^2}, Y = 3qX$, then the equation transforms into $Y^2 = X^3 - 27$. Since this elliptic curve has rank 0, we only have torsion points which is just $(X, Y) = (3, 0)$. This torsion point corresponds to $(p, q) = (1, 0)$, $(x, y) = (1, 1)$ and $(u, v) = (1, 1)$. Then $n+2 = 2$ implies $n = 0$. Therefore, no integer solutions could be obtained.
Case $(n+2 = v^3, n+1 = 4u^3)$: The equation $n+2-(n+1) = v^3 - 4u^3 = 1$ is obtained. If we set $x = \frac{1}{u}, y = \frac{v}{u}$, then the equation transforms into $x^3+y^3 = 4$. Then, we again set $x = p+q, y = p-q$, then the equation transforms into $p^3 + 3pq^2 = 2$. If we further substitute $p = \frac{3}{X}, q^2 = \frac{2X^3-27}{(3X)^2}, Y = 3qX$, then the equation transforms into $Y^2 = 2X^3-27$. Moreover, we set $X' = 2X, Y' = 2Y$ and transform the equation into $Y'^2 = X'^3 - 108$. Since this elliptic curve has rank 0 and has no rational torsion points, no integer solutions could be obtained.

□

Theorem 14. *The diophantine equation*

$$\Pi_{x=1}^{n} x(x+1)(x+2)(x+3) = y^2$$

has infinitely many integer solutions.

Proof. Using Lemma 11, we have the following identity:

$$\Pi_{x=1}^{n} f(x)f(x+1)f(x+2)f(x+3) = \left(\Pi_{t=1}^{3} f(t)^t\right)\left(\Pi_{t=4}^{n} f(t)^4\right)\left(\Pi_{t=1}^{3} f(n+t)^{4-t}\right).$$

Moreover, by setting $f(x) = x$, the equation transforms into

$$\Pi_{x=1}^{n} x(x+1)(x+2)(x+3) = \left(\Pi_{t=1}^{3} t^t\right)\left(\Pi_{t=4}^{n} t^4\right)\left(\Pi_{t=1}^{3} (n+t)^{4-t}\right) = y^2.$$

To have integer solutions, $3(n+1)(n+3) = w^2$ must be satisfied for some integers n and w. Solution set is given by recurrence relations $n_{t+1} = 2n_t + w_t + 2$ and

$w_{t+1} = 3n_t + 2w_t + 6$ with initial condition $n_0 = 5, w_0 = 12$. Hence infinitely many integer solutions in n and y could be obtained from these solutions.

□

Remark 15. *We could obtain a square from product of four disjoint blocks of k consecutive integers by using any integer solution of the diophantine equation*

$$\Pi_{x=1}^{n} x(x+k)(x+2k)(x+3k) = y^2.$$

Proof. Let

$A = \left(\Pi_{t=1}^{k} t\right)\left(\Pi_{t=k+1}^{2k} t^2\right)\left(\Pi_{t=2k+1}^{3k} t^3\right), B = \left(\Pi_{t=n+1}^{n+k} t^3\right)\left(\Pi_{t=n+k+1}^{n+2k} t^2\right)\left(\Pi_{t=n+2k+1}^{n+3k} t\right).$

Then, we have $\Pi_{x=1}^{n} x(x+k)(x+2k)(x+3k) = A\left(\Pi_{t=3k+1}^{n} t^4\right) B = y^2$. To have integer solutions, $\left(\Pi_{t=1}^{k} t\right)\left(\Pi_{t=2k+1}^{3k} t\right)\left(\Pi_{t=n+1}^{n+k} t\right)\left(\Pi_{t=n+2k+1}^{n+3k} t\right) = w^2$ must be satisfied for some integers n and w. Hence, a square from product of four disjoint blocks of k consecutive integers could be obtained.

□

1.3 Results in this thesis

In this thesis, we present a new method generating parametrized integer solutions on some special affine hypersurfaces by producing special univariate polynomial parametrizations. Using some of these parametrizations, we verify all known counterexamples to the conjecture, except the one given for three disjoint blocks of four consecutive integers.

A full list of these parametrizations is given in the Appendix. These results, constitute the main contribution of the thesis to the literature on the problem described in 1.1.

Moreover, to the best of our knowledge, we produce the first examples of integer squares obtained from product of disjoint blocks of consecutive integers where each block has length six or seven.

The rest of the thesis is structured as follows: In Chapter 2, we investigate the geometry of algebraic varieties used to study the products of blocks of consecutive integers.

We also discuss the relation of the problem with some well-known conjectures in diophantine geometry. In Chapter 3, we give our main method and results. Finally, the thesis ends with some concluding remarks in Chapter 4

CHAPTER 2

GEOMETRY OF PRODUCTS OF BLOCKS OF CONSECUTIVE INTEGERS

We work with the affine variety X in $\mathbb{A}^{r+1}$ with affine coordinates $(x_1, x_2, \ldots, x_r, y)$ defined by the following equation:

$$\begin{aligned} y^2 &= b_k(x_1)b_k(x_2)\cdots b_k(x_r), \quad r, k \in \mathbb{N}_{>0} \\ b_k(x_i) &:= (x_i)(x_i+1)\cdots(x_i+(k-1)). \end{aligned} \tag{2.1}$$

2.1 Compactifying X

We compactify X, using different methods by separating $r = 1$ case and $r \geq 2$ case, as described in the following subsections.

2.1.1 Case $r = 1$

By using the compactification explained in [19] for hyperelliptic algebraic curves, we can compactify X smoothly by glueing two copies of X. This construction is presented as follows:

If $k = 2m$, then

$$\begin{aligned} X:\ y^2 &= x_1(x_1+1)\dots(x_1+(2m-1)), \\ X':\ y'^2 &= (1+x'_1)\dots(1+(2m-1)x'_1), \\ x'_1 &= \frac{1}{x_1}, \\ y' &= \frac{y}{x_1^m}. \end{aligned} \tag{2.2}$$

If $k = 2m+1$, then

$$\begin{aligned} X:\ y^2 &= x_1(x_1+1)\dots(x_1+2m), \\ X':\ y'^2 &= x'_1(1+x'_1)\dots(1+2mx'_1), \\ x'_1 &= \frac{1}{x_1}, \\ y' &= \frac{y}{x_1^{m+1}}. \end{aligned} \tag{2.3}$$

2.1.2 Case $r \geq 2$

We assume that rk is an even positive integer and we let B be the divisor in $\mathbb{P}^r$ defined by

$$\Pi_{i\in\{1,2,\dots,r\}} x_i(x_i+x_0)\cdots(x_i+(k-1)x_0) = 0.$$

We take $\mathcal{L} = \mathcal{O}_{\mathbb{P}^r}(\frac{rk}{2})$ so that $\mathcal{L}^{\otimes 2} = \mathcal{O}_{\mathbb{P}^r}(B)$. We have a section $s \in \Gamma(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(B))$, vanishing exactly along B. If we denote by L the total space of $\mathcal{L}$, then we have the bundle projection $\pi : L \longrightarrow \mathbb{P}^r$. Let $t \in \Gamma(L, \pi^*(\mathcal{L}))$ be the tautological section, then $\pi^* s = t^2$ in L gives a double cover of $\mathbb{P}^r$ which ramifies only on B. Since $B \neq 0$ and is reduced (as B consists of rk hyperplane divisors), the equation $\pi^* s = t^2$ defines an irreducible normal analytic space in L [2, p. 42].
Let $t_{m,n} = \frac{x_n}{x_m}$, $(n,m) \in \{0,1,\dots,r\} \times \{0,1,\dots,r\}$.

$$\begin{aligned} &x_0 \neq 0, \\ &\quad s_0 := \Pi_{i\in\{\hat{0},1,\dots,r\}} t_{0,i}(t_{0,i}+1)\cdots(t_{0,i}+(k-1)), \\ &j \neq 0, x_j \neq 0, \\ &\quad s_j := (1+t_{j,0})\cdots(1+(k-1)t_{j,0})\Pi_{i\in\{\hat{0},1,\dots,\hat{j},\dots,r\}} t_{j,i}(t_{j,i}+t_{j,0})\cdots(t_{j,i}+(k-1)t_{j,0}). \end{aligned} \tag{2.4}$$

Observe that, $x_m^{rk} s_m = x_n^{rk} s_n$, and X could be defined as $s_0 = t^2$ in L. Moreover, to construct the normal variety $\pi^* s = t^2$ in L, we need to glue these sections appropriately, as follows:

$$x_m x_n \neq 0,\ s_m = (\tfrac{x_n}{x_m})^{rk} s_n = (t_{m,n})^{rk} s_n,$$
$$(t_{m,0}, t_{m,1}, \ldots, \widehat{t_{m,m}}, \ldots, t_{m,r}, t) \sim (t_{n,0}, t_{n,1}, \ldots, \widehat{t_{n,n}}, \ldots, t_{n,r}, \tfrac{t}{(t_{m,n})^{\frac{rk}{2}}})$$

Therefore, $\{\pi^* s = t^2\} = \bigcup_{0 \leq i \leq r} \{s_i = t^2\} = \widetilde{X} \subset L$ is a normal ramified double cover of $\mathbb{P}^r$ compactifying X.

2.2 Notes on birational geometry

We will need and freely use the following general facts in the computations that will appear in the following section.

- If X is a normal projective variety, then the dualizing sheaf $w_X \cong \mathcal{O}_X(K_X)$. [14, Prop. 5.75]
- Let X, Y be projective varieties of the same dimension and $f : X \longrightarrow Y$ be a generically finite map. Then [14, Prop. 5.77]
  - we have a trace map, $\mathrm{Tr}_{X/Y} : f_*(w_X) \longrightarrow w_Y$.
  - if X, Y are normal varieties and f is a birational morphism, then $\mathrm{Tr}_{X/Y}$ is an isomorphism over the points where f^{-1} is an isomorphism.
- Let X be a projective variety, $D \subset X$ be an effective Cartier divisor. Then we have the adjunction formula which relates the dualizing sheaves w_D, w_X with $w_D \cong w_X(D) \otimes \mathcal{O}_D$.[14, Prop. 5.73]

2.3 X is a variety of general type

We show that X is variety of general type, using different methods again by separating $r = 1$ case with $r \geq 2$ case, described in the following subsections.

2.3.1 Case $r = 1$

After compactification of X, we have $\overline{X} := X \cup X'$ and the following ramified double cover map between smooth complex compact algebraic curves:

$$\begin{aligned} \pi : \overline{X} &\longrightarrow \mathbb{P}^1 \\ (x_1, y) &\mapsto x_1, \\ (x_1', y') &\mapsto x_1'. \end{aligned} \tag{2.5}$$

If $k = 2m$, then

$$\pi^{-1}(t) = \begin{cases} (x_1 = -t, y = 0), & t \in \{0, 1, 2, \ldots, 2m-1\}, \\ (x_1' = 0, y' = \pm 1), & t = \infty. \end{cases}$$

If $k = 2m+1$, then

$$\pi^{-1}(t) = \begin{cases} (x_1 = -t, y = 0), & t \in \{0, 1, 2, \ldots, 2m\}, \\ (x_1' = 0, y' = 0), & t = \infty. \end{cases}$$

Finally, we obtain the geometric genus of X by using Riemann-Hurwitz formula as follows:

$$g(X) = g(\overline{X}) = \left\lfloor \frac{k-1}{2} \right\rfloor = \begin{cases} m-1, & k = 2m, \\ m, & k = 2m+1. \end{cases}$$

If $k > 4$, then $g(X) \geq 2$ and we can conclude that the algebraic curve X is a variety of general type.

2.3.2 Case $r \geq 2$

Using recipes given in [2, p. 42, 182, 183], with the assumption that rk is an even positive integer, we have the following maps and results:

$$
\begin{aligned}
\pi : L &\longrightarrow \mathbb{P}^r \\
\sigma : \overline{X} &\longrightarrow \widetilde{X} \\
p := \pi|_{\widetilde{X}} &\circ \sigma \\
p : \overline{X} &\longrightarrow \mathbb{P}^r \\
\mathcal{L}^{\otimes 2} &= \mathcal{O}_{\mathbb{P}^r}(B) \\
\mathcal{L} &= \mathcal{O}_{\mathbb{P}^r}(\frac{rk}{2}) \\
p_*\mathcal{O}_{\overline{X}} &= \mathcal{O}_{\mathbb{P}^r} \oplus \mathcal{L}^{-1} && (2.6) \\
K_{\overline{X}} &= p^*(K_{\mathbb{P}^r} \otimes \mathcal{L}) && (2.7) \\
P_d &= h^0(\overline{X}, K^d_{\overline{X}}) && (2.8) \\
&= h^0(\mathbb{P}^r, p_*K^d_{\overline{X}}) = h^0(\mathbb{P}^r, p_*(p^*(K_{\mathbb{P}^r} \otimes \mathcal{L})^d)) && (2.9) \\
&= h^0(\mathbb{P}^r, (K_{\mathbb{P}^r} \otimes \mathcal{L})^d \otimes (\mathcal{O}_{\mathbb{P}^r} \oplus \mathcal{L}^{-1})) && (2.10) \\
&= h^0(\mathbb{P}^r, (K_{\mathbb{P}^r} \otimes \mathcal{L})^d) + h^0(\mathbb{P}^r, K^d_{\mathbb{P}^r} \otimes \mathcal{L}^{d-1}) && (2.11) \\
&= h^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(d(-(r+1) + \frac{rk}{2}))) + h^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(-(r+1)d + \frac{rk}{2}(d-1))) \\
&= \binom{d(-(r+1) + \frac{rk}{2}) + r}{r} + \binom{-(r+1)d + \frac{rk}{2}(d-1) + r}{r} [11], && (2.12)
\end{aligned}
$$

Whenever $k > 2 + \frac{2}{r}$ (i.e., $k > 3; r = 2, \quad k > 2; r > 2$), we can conclude that the algebraic hypersurface X is a variety of general type, as d-th plurigenera P_d becomes a polynomial of degree r in the variable d.

2.3.3 Relations with diophantine geometry

We have shown that the variety X is of general type. This fact enables us to extract some conjectural information about the rational points on X using the following conjectures stated in [12], in the order of increasing precision.

Conjecture 16 (Bombieri-Lang Conjecture I)**.** *Given a variety of general type X defined over an number field k, then k-rational points on X is not Zariski dense in X.*

Conjecture 17 (Bombieri-Lang Conjecture II)**.** *Given a variety of general type X defined over an number field k, then there is a dense Zariski open set $U \subset X$ such that for all number fields k'/k, k'-rational points on U is finite.*

Conjecture 18 (Bombieri-Lang Conjecture III)**.** *Given a variety of general type X defined over an number field k, and $U := X \setminus \{$union of all subvarieties of X that are not of general type$\}$, then for all number fields k'/k, k'-rational points on Zariski dense open set U is finite.*

Remark 19. *Counterexamples to the Erdős-Graham conjecture producing infinitely many integer solutions lie on a rational algebraic curve. These solutions do not contradict with Bombieri-Lang conjectures.*

2.3.4 Birational Automorphism Group of X

First of all, let use denote the Symmetric group of $r!$ elements with S_r. An element σ of S_r, maps the set $\{1, 2, \ldots, r\}$ into $\{1, 2, \ldots, r\}$ such that set-wise the equality $\{1, 2, \ldots, r\} = \{\sigma(1), \sigma(2), \ldots, \sigma(r)\}$ holds. Let us also denote the multiplicative Cyclic group of m elements with C_m. Also, there exists an element τ in C_m, such that set-wise the equality $\{\tau, \tau^2, \ldots, \tau^m = 1_{C_m}\} = C_m$ holds.

Now, we are able to give the following obvious automorphisms of $X \subset \mathbb{A}^{r+1}$, over an algebraically closed field with characteristic 0 or $p \nmid m$ as follows:

$$\begin{aligned} Aut_\sigma : X &\longrightarrow X \\ (x_1, \ldots, x_r, y) &\mapsto (x_{\sigma(1)}, \ldots, x_{\sigma(r)}, y), \end{aligned} \tag{2.13}$$

$$\begin{aligned} Aut_\tau : X &\longrightarrow X \\ (x_1, \ldots, x_r, y) &\mapsto (x_1, \ldots, x_r, \tau y), \end{aligned} \tag{2.14}$$

$$\begin{aligned} Aut_{\sigma_1} \circ Aut_{\sigma_2} &= Aut_{\sigma_1 \circ \sigma_2} \qquad & Aut_{\tau_1} \circ Aut_{\tau_2} = Aut_{\tau_2} \circ Aut_{\tau_1} = Aut_{\tau_1 \tau_2} \\ \{Aut_{\sigma_i} \mid \sigma_i \in S_r\} &\cong S_r \qquad & \{Aut_{\tau_i} \mid \tau_i \in C_2\} \cong C_2 \end{aligned}$$

Therefore, we have $S_r \times C_2 \subset Aut(X)$ and $|S_r \times C_2| = 2r! \le |Aut(X)|$.

Theorem 20. *There exists a constant λ_n depending only on the dimension n of the projective variety X such that the birational automorphism group of any projective variety X of general type has at most $\lambda_n vol(X, K_X)$ elements.*

Proof. See [10]. □

Corollary 21. $2r! \le |Aut(X)| \le \lambda_r vol(X, K_X)$

Proof. Above theorem implies the result immediately. □

Theorem 22. *If $r = 1$, then $2 \le |Aut(X)| \le 42(2 \left\lfloor \frac{k-1}{2} \right\rfloor - 2)$*

Proof. Since $r = 1$ case gives us an algebraic curve X, we applied the well-known upper bound $\lambda_1 \text{vol}(X, K_X) = 42(2g - 2)$ for curves, where g is the geometric genus of the curve. □

Theorem 23. *If $r = 2$, then $4 \le |Aut(X)| < 42^2 2(k - 3)^2$*

Proof. Since $r = 2$ case gives us an algebraic surface X, we applied the well-known upper bound $\lambda_2 \text{vol}(X, K_X) = 42^2 c_1^2$ given in [23] for surfaces, where c_1^2 is computed using the formula in [2, p. 183]. □

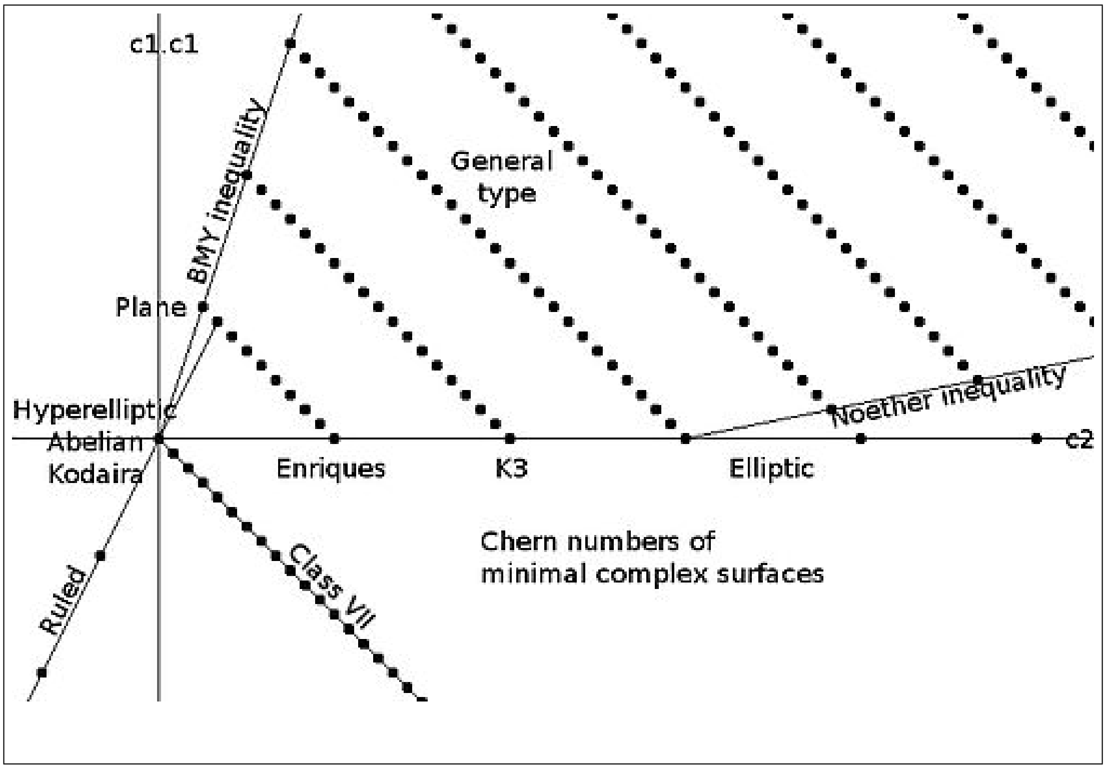

Figure 2.1: Enriques–Kodaira classification of compact complex surfaces, Wikipedia

2.4 Connections with some well-know conjectures in number theory

In this section, we give strong numerical evidence on possible connections of product of consecutive integers with some well-known conjectures in number theory. From now on, we assume that $k, m, n \in \mathbb{N}$, $k \geq 2, m \geq 2, n \geq 2$ with $\frac{1}{k} + \frac{1}{m} + \frac{1}{n} < 1$ holds and $b_k(x) := x(x+1)(x+2)\cdots(x+k-1)$ throughout the section.

Table 2.1: $1 \leq x^k, y^m, z^n \leq 10^{10}$ with $\gcd(x^k, y^m, z^n) = 1$

x	k	y	m	z	n	$x^k + y^m = z^n$
1	M	2	3	3	2	$(1)^M + (2)^3 = (3)^2$
2	5	7	2	3	4	$(2)^5 + (7)^2 = (3)^4$
2	7	17	3	71	2	$(2)^7 + (17)^3 = (71)^2$
13	2	7	3	2	9	$(13)^2 + (7)^3 = (2)^9$
3	5	11	4	122	2	$(3)^5 + (11)^4 = (2.61)^2$

Conjecture 24 (Fermat-Catalan Conjecture). *$x^k + y^m = z^n$ with $\gcd(x, y, z) = 1$, $|xyz| > 0$, has only finitely many integer solutions.*

Conjecture 25 (Beal's Conjecture). *$x^k + y^m = z^n$ with $\gcd(x, y, z) = 1$, $|xyz| > 0$, has any integer solution, then $2 \in \{k, m, n\}$.*

Table 2.2: $1 \leq b_k(x), b_m(y), b_n(z) \leq 10^{10}$ with $\gcd(b_k(x), b_m(y), b_n(z)) = \min(k!, m!)$

x	k	y	m	z	n	$b_k(x) + b_m(y) = b_n(z)$
1	5	1	5	15	2	$(2^3.3.5) + (2^3.3.5) = (2^4.3.5)$
1	5	1	6	4	4	$(2^3.3.5) + (2^4.3^2.5) = (2^3.3.5.7)$
1	5	2	5	4	4	$(2^3.3.5) + (2^4.3^2.5) = (2^3.3.5.7)$
1	5	4	5	18	3	$(2^3.3.5) + (2^6.3.5.7) = (2^3.3^2.5.19)$
1	6	3	6	144	2	$(2^4.3^2.5) + (2^6.3^2.5.7) = (2^4.3^2.5.29)$
1	9	1	10	6	7	$(2^7.3^4.5.7) + (2^8.3^4.5^2.7) = (2^7.3^4.5.7.11)$
1	9	2	9	6	7	$(2^7.3^4.5.7) + (2^8.3^4.5^2.7) = (2^7.3^4.5.7.11)$
1	11	1	11	4	9	$(2^8.3^4.5^2.7.11) + (2^8.3^4.5^2.7.11) = (2^9.3^4.5^2.7.11)$
1	11	1	12	9	8	$(2^8.3^4.5^2.7.11) + (2^{10}.3^5.5^2.7.11) = (2^8.3^4.5^2.7.11.13)$
1	11	2	11	9	8	$(2^8.3^4.5^2.7.11) + (2^{10}.3^5.5^2.7.11) = (2^8.3^4.5^2.7.11.13)$

Conjecture 26. *$b_k(x) + b_m(y) = b_n(z)$ with $\gcd(b_k(x), b_m(y), b_n(z)) = \min(k!, m!)$, $x, y, z > 0$, has only finitely many integer solutions.*

Conjecture 27. *$b_k(x) + b_m(y) = b_n(z)$ with $\gcd(b_k(x), b_m(y), b_n(z)) = \min(k!, m!)$, $x, y, z > 0$, has any integer solution, then $2 \in \{k, m, n\}$.*

Table 2.3: $1 \leq b_k(x), b_m(y), z^n \leq 10^{10}$ with $\gcd(b_k(x), b_m(y), z^n) = \min(k!, m!)$

x	k	y	m	z	n	$b_k(x) + b_m(y) = z^n$
1	3	22	2	2	9	$(2.3) + (2.11.23) = (2)^9$
1	4	1	5	12	2	$(2^3.3) + (2^3.3.5) = (2^2.3)^2$
801	3	40393	2	2	31	$(2.3^2.11.73.89.401) + (2.19.31.1063.1303) = (2)^{31}$

Conjecture 28. $b_k(x) + b_m(y) = z^n$ *with* $\gcd(b_k(x), b_m(y), z^n) = \min(k!, m!)$, $x, y, z > 0$*, has only finitely many integer solutions.*

Conjecture 29. $b_k(x) + b_m(y) = z^n$ *with* $\gcd(b_k(x), b_m(y), z^n) = \min(k!, m!)$, $x, y, z > 0$*, has any integer solution, then* $2 \in \{k, m, n\}$.

Table 2.4: $1 \leq b_k(x), y^m, b_n(z) \leq 10^{10}$ with $\gcd(b_k(x), y^m, b_n(z)) = \min(k!, n!)$

x	k	y	m	z	n	$b_k(x) + y^m = b_n(z)$
1	7	420	2	3	7	$(2^4.3^2.5.7) + (2^2.3.5.7)^2 = (2^6.3^4.5.7)$
1	7	2940	2	7	7	$(2^4.3^2.5.7) + (2^2.3.5.7^2)^2 = (2^6.3^3.5.7.11.13)$
37	3	2	11	238	2	$(2.3.13.19.37) + (2)^{11} = (2.7.17.239)$
1	11	55440	2	3	11	$(2^8.3^4.5^2.7.11) + (2^4.3^2.5.7.11)^2 = (2^9.3^5.5^2.7.11.13)$

Conjecture 30. $b_k(x) + y^m = b_n(z)$ *with* $\gcd(b_k(x), y, b_n(z)) = \min(k!, n!)$, $x, y, z > 0$*, has only finitely many integer solutions.*

Conjecture 31. $b_k(x) + y^m = b_n(z)$ *with* $\gcd(b_k(x), y, b_n(z)) = \min(k!, n!)$, $x, y, z > 0$*, has any integer solution, then* $2 \in \{k, m, n\}$.

Table 2.5: $1 \leq x^k, y^m, b_n(z) \leq 10^{10}$ with $\gcd(x^k, y^m, b_n(z)) = 1$

x	k	y	m	z	n	$x^k + y^m = b_n(z)$
1	M	1	M	1	2	$(1)^M + (1)^M = (2)$
1	M	11	3	36	2	$(1)^M + (11)^3 = (2^2.3^2.37)$
19	5	83	5	62781	2	$(19)^5 + (83)^5 = (2.3.17.1231.31391)$
7	8	299	3	5700	2	$(7)^8 + (13.23)^3 = (2^2.3.5^2.19.5701)$
323	3	5	12	16668	2	$(17.19)^3 + (5)^{12} = (2^2.3^2.79.211.463)$

Conjecture 32. *$x^k + y^m = b_n(z)$ with $\gcd(x, y, b_n(z)) = 1$, $x, y, z > 0$, has only finitely many integer solutions.*

Conjecture 33. *$x^k + y^m = b_n(z)$ with $\gcd(x, y, b_n(z)) = 1$, $x, y, z > 0$, has any integer solution, then $2 \in \{k, m, n\}$.*

Theorem 34. *$x^2 + y^2 = b_4(z)$, has infinitely many integer solutions.*

Proof. Let $b_4(z) = z(z+1)(z+2)(z+3) = z^4+6z^3+11z^2+6z$, and set $y = z^2+bz+c$, then $b_4(z) - y^2 = (6 - 2b)z^3 + (11 - 2c - b^2)z^2 + (6 - 2cb)z - c^2 = x^2$. If we also set $b = 3$ then, $y = z^2 + 3z + c$ and $b_4(z) - y^2 = 2(1 - c)z^2 + 6(1 - c)z - c^2 = x^2$. Moreover, the quadratic equation $2(1 - c)z^2 + 6(1 - c)z - c^2 = x^2$ has infinitely many integer solutions (x, z) for infinitely many integer parameter $c < 1$, and this completes the proof.

□

Remark 35. *Above theorem shows that the conjecture is very sensitive to the inequality condition $\frac{1}{k} + \frac{1}{m} + \frac{1}{n} < 1$.*

Table 2.6: $1 \le b_k(x), y^m, z^n \le 10^{10}$ with $\gcd(b_k(x), y^m, z^n) = 1$

x	k	y	m	z	n	$b_k(x) + y^m = z^n$
2	3	1	M	5	2	$(2^3.3) + (1)^M = (5)^2$
1	5	1	M	11	2	$(2^3.3.5) + (1)^M = (11)^2$
4	3	1	M	11	2	$(2^3.3.5) + (1)^M = (11)^2$
18	2	1	M	7	3	$(2.3^2.19) + (1)^M = (7)^3$
1	6	41	2	7	4	$(2^4.3^2.5) + (41)^2 = (7)^4$
2	5	41	2	7	4	$(2^4.3^2.5) + (41)^2 = (7)^4$
3	5	11	4	131	2	$(2^3.3^2.5.7) + (11)^4 = (131)^2$
15	3	337	2	7	6	$(2^4.3.5.17) + (337)^2 = (7)^6$
64	2	7	4	3	8	$(2^6.5.13) + (7)^4 = (3)^8$
1	7	1	M	71	2	$(2^4.3^2.5.7) + (1)^M = (71)^2$
2	6	1	M	71	2	$(2^4.3^2.5.7) + (1)^M = (71)^2$
4	5	89	2	11	4	$(2^6.3.5.7) + (89)^2 = (11)^4$
4	5	1679	2	41	4	$(2^6.3.5.7) + (23.73)^2 = (41)^4$
103	2	17	3	5	6	$(2^3.13.103) + (17)^3 = (5)^6$
22	3	59	2	5	6	$(2^4.3.11.23) + (59)^2 = (5)^6$
5	5	13	4	209	2	$(2^4.3^3.5.7) + (13)^4 = (11.19)^2$
3	6	5039	2	71	4	$(2^6.3^2.5.7) + (5039)^2 = (71)^4$
7	5	19	4	431	2	$(2^4.3^2.5.7.11) + (19)^4 = (431)^2$
4	6	929	2	31	4	$(2^6.3^3.5.7) + (929)^2 = (31)^4$
40	3	11	4	17	4	$(2^4.3.5.7.41) + (11)^4 = (17)^4$
44	3	163	2	7	6	$(2^3.3^2.5.11.23) + (163)^2 = (7)^6$
9	5	23	4	659	2	$(2^3.3^3.5.11.13) + (23)^4 = (659)^2$
55	3	1	M	419	2	$(2^3.3.5.7.11.19) + (1)^M = (419)^2$
3	7	1261	2	11	6	$(2^6.3^4.5.7) + (13.97)^2 = (11)^6$
10	5	199	2	23	4	$(2^4.3.5.7.11.13) + (199)^2 = (23)^4$
10	5	17	4	569	2	$(2^4.3.5.7.11.13) + (17)^4 = (569)^2$
10	5	41	4	1751	2	$(2^4.3.5.7.11.13) + (41)^4 = (17.103)^2$
6	6	13	4	601	2	$(2^5.3^3.5.7.11) + (13)^4 = (601)^2$
70	3	1189	2	11	6	$(2^4.3^2.5.7.71) + (29.41)^2 = (11)^6$
11	5	589	2	29	4	$(2^3.3^2.5.7.11.13) + (19.31)^2 = (29)^4$
72	3	41	2	5	8	$(2^4.3^2.37.73) + (41)^2 = (5)^8$

Table 2.7: Table 2.6 continued

12	5	19	4	809	2	$(2^7.3^2.5.7.13) + (19)^4 = (809)^2$
88	3	7	4	29	4	$(2^4.3^2.5.11.89) + (7)^4 = (29)^4$
13	5	41	4	1889	2	$(2^5.3.5.7.13.17) + (41)^4 = (1889)^2$
90	3	1009	2	11	6	$(2^3.3^2.5.7.13.23) + (1009)^2 = (11)^6$
968	2	13	6	7	8	$(2^3.3.11^2.17.19) + (13)^6 = (7)^8$
968	2	169	3	7	8	$(2^3.3.11^2.17.19) + (13^2)^3 = (7)^8$
8	6	23	4	1231	2	$(2^6.3^3.5.11.13) + (23)^4 = (1231)^2$
8	6	19289	2	139	4	$(2^6.3^3.5.11.13) + (19289)^2 = (139)^4$
9	6	1121	2	43	4	$(2^4.3^3.5.7.11.13) + (19.59)^2 = (43)^4$
9	6	61	4	4001	2	$(2^4.3^3.5.7.11.13) + (61)^4 = (4001)^2$
136	3	5	8	1721	2	$(2^4.3.17.23.137) + (5)^8 = (1721)^2$
138	3	14549	2	11	8	$(2^3.3.5.7.23.139) + (14549)^2 = (11)^8$
18	5	17	4	1801	2	$(2^4.3^3.5.7.11.19) + (17)^4 = (1801)^2$
152	3	5	8	1993	2	$(2^4.3^2.7.11.17.19) + (5)^8 = (1993)^2$
154	3	12013	2	23	6	$(2^3.3.5.7.11.13.31) + (41.293)^2 = (23)^6$
21	5	13	6	3347	2	$(2^4.3^2.5^2.7.11.23) + (13)^6 = (3347)^2$
21	5	19	6	7309	2	$(2^4.3^2.5^2.7.11.23) + (19)^6 = (7309)^2$
22	5	78911	2	281	4	$(2^5.3.5^2.11.13.23) + (7.11273)^2 = (281)^4$
12	6	29	6	24571	2	$(2^7.3^2.5.7.13.17) + (29)^6 = (24571)^2$
24	5	569	2	59	4	$(2^6.3^4.5^2.7.13) + (569)^2 = (59)^4$
24	5	36319	2	191	4	$(2^6.3^4.5^2.7.13) + (36319)^2 = (191)^4$
13	6	83	4	7799	2	$(2^6.3^3.5.7.13.17) + (83)^4 = (11.709)^2$
238	3	28321	2	13	8	$(2^5.3.5.7.17.239) + (127.223)^2 = (13)^8$
238	3	13	8	28799	2	$(2^5.3.5.7.17.239) + (13)^8 = (31.929)^2$
26	5	23	4	4169	2	$(2^4.3^4.5.7.13.29) + (23)^4 = (11.379)^2$
8	7	1	M	4159	2	$(2^7.3^3.5.7.11.13) + (1)^M = (4159)^2$
260	3	11413	2	23	6	$(2^3.3^2.5.13.29.131) + (101.113)^2 = (23)^6$
3	9	439	2	67	4	$(2^7.3^4.5^2.7.11) + (439)^2 = (67)^4$
5	8	439	2	67	4	$(2^7.3^4.5^2.7.11) + (439)^2 = (67)^4$
15	6	89	4	9521	2	$(2^7.3^3.5^2.17.19) + (89)^4 = (9521)^2$
29	5	9169	2	103	4	$(2^6.3^2.5.11.29.31) + (53.173)^2 = (103)^4$
354	3	1519	2	19	6	$(2^3.3.5.59.71.89) + (7^2.31)^2 = (19)^6$
32	5	53	4	7289	2	$(2^8.3^3.5.7.11.17) + (53)^4 = (37.197)^2$
10	7	7409	2	103	4	$(2^8.3^2.5^2.7.11.13) + (31.239)^2 = (103)^4$
34	5	18919	2	143	4	$(2^4.3^2.5.7.17.19.37) + (18919)^2 = (11.13)^4$
4	9	7831	2	109	4	$(2^9.3^4.5^2.7.11) + (41.191)^2 = (109)^4$

Table 2.8: Table 2.6 continued

434	3	8107	2	23	6	$(2^3.3.5.7.29.31.109)+(11^2.67)^2=(23)^6$
37	5	8551	2	113	4	$(2^4.3.5.13.19.37.41)+(17.503)^2=(113)^4$
9674	2	379	3	23	6	$(2.3^2.5^2.7.43.691)+(379)^3=(23)^6$
40	5	871	2	107	4	$(2^6.3.5.7.11.41.43)+(13.67)^2=(107)^4$
41	5	14341	2	137	4	$(2^3.3^3.5.7.11.41.43)+(14341)^2=(137)^4$
14832	2	17	6	5	12	$(2^4.3^2.7.13.103.163)+(17)^6=(5)^{12}$
14832	2	289	3	5	12	$(2^4.3^2.7.13.103.163)+(17^2)^3=(5)^{12}$
50	5	22721	2	173	4	$(2^4.3^4.5^2.13.17.53)+(22721)^2=(173)^4$
776	3	20455	2	31	6	$(2^4.3.7.37.97.389)+(5.4091)^2=(31)^6$
53	5	31681	2	197	4	$(2^4.3^4.5.7.11.19.53)+(13.2437)^2=(197)^4$
26	6	103	4	25351	2	$(2^4.3^4.5.7.13.29.31)+(103)^4=(101.251)^2$
834	3	44537	2	37	6	$(2^3.3.5.11.19.139.167)+(44537)^2=(37)^6$
55	5	2609	2	157	4	$(2^4.3.5.7.11.19.29.59)+(2609)^2=(157)^4$
55	5	131	4	29921	2	$(2^4.3.5.7.11.19.29.59)+(131)^4=(29921)^2$
55	5	167	4	37129	2	$(2^4.3.5.7.11.19.29.59)+(167)^4=(107.347)^2$
6	9	41791	2	223	4	$(2^8.3^4.5.7^2.11.13)+(23^2.79)^2=(223)^4$
903	3	42743	2	37	6	$(2^3.3.5.7.43.113.181)+(42743)^2=(37)^6$
912	3	32821	2	35	6	$(2^5.3.11.19.83.457)+(23.1427)^2=(5.7)^6$
934	3	41813	2	37	6	$(2^4.3^2.5.11.13.17.467)+(41813)^2=(37)^6$
17	7	139	4	40111	2	$(2^4.3^3.5.7.11.17.19.23)+(139)^4=(40111)^2$
64	5	167	4	45041	2	$(2^9.3.5.11.13.17.67)+(167)^4=(73.617)^2$
1240	3	53281	2	41	6	$(2^4.3^3.5.17.23.31.73)+(53281)^2=(41)^6$
75	5	167	4	59011	2	$(2^3.3^2.5^2.7.11.13.19.79)+(167)^4=(59011)^2$
1604	3	24811	2	41	6	$(2^3.3.5.11.73.107.401)+(43.577)^2=(41)^6$
82	5	247	4	88889	2	$(2^4.3.5.7.17.41.43.83)+(13.19)^4=(103.863)^2$
1614	3	23191	2	41	6	$(2^5.3.5.17.19.101.269)+(7.3313)^2=(41)^6$
1664	3	11591	2	41	6	$(2^8.3^2.5.7^2.13.17.37)+(67.173)^2=(41)^6$
84	5	7001	2	41	6	$(2^6.3^2.5.7.11.17.29.43)+(7001)^2=(41)^6$
1826	3	2263	2	5	14	$(2^3.3^2.7.11.29.83.457)+(31.73)^2=(5)^{14}$
1908	3	4339	2	17	8	$(2^3.3^2.5.23.53.83.191)+(4339)^2=(17)^8$

Conjecture 36. $b_k(x)+y^m=z^n$ *with* $\gcd(b_k(x),y,z)=1$, $x,y,z>0$, $k\neq 4$ *or* $y\neq 1$, *has only finitely many integer solutions.*

Conjecture 37. $b_k(x)+y^m=z^n$ *with* $\gcd(b_k(x),y,z)=1$, $x,y,z>0$, $k\neq 4$ *or* $y\neq 1$, *has any integer solution, then* $2\in\{k,m,n\}$.

Remark 38. *The condition* $k\neq 4$ *or* $y\neq 1$ *is necessary for the conjecture as there exists a trivial equality* $b_4(x)+1=(x^2+3x+1)^2$.

Theorem 39 (Darmon and Granville)**.** *If A, B, C, k, m, n are fixed positive integers with $\frac{1}{k} + \frac{1}{m} + \frac{1}{n} < 1$, then*

$$Ax^k + By^m = Cz^n$$

has at most finitely many solutions in coprime nonzero integers x, y, z.

Proof. See [5]. □

Remark 40. *If k, m, n are fixed positive integers with $\frac{1}{k} + \frac{1}{m} + \frac{1}{n} < 1$, then*

$$x^k + y^m = z^n$$

has at least the following list of solutions in coprime nonzero integers x, y, z.

$$1^k + 2^3 = 3^2,$$
$$2^5 + 7^2 = 3^4,$$
$$3^5 + 11^4 = 122^2,$$
$$2^7 + 17^3 = 71^2,$$
$$7^3 + 13^2 = 2^9,$$
$$43^8 + 96222^3 = 30042907^2,$$
$$33^8 + 1549034^2 = 15613^3,$$
$$17^7 + 76271^3 = 21063928^2,$$
$$1414^3 + 2213459^2 = 65^7,$$
$$9262^3 + 15312283^2 = 113^7.$$

Conjecture 41. *Above list contains all the solutions.*

Conjecture 42. *If k, m, n are fixed positive integers with $\frac{1}{k} + \frac{1}{m} + \frac{1}{n} < 1$, then any (x, y, z) positive integer tuple solutions to the below equations satisfy exactly only one of them.*

$$x^k + y^m = z^n, \quad \gcd(x, y, z) = 1, \tag{2.15}$$
$$x^k + y^m = b_n(z), \quad \gcd(x, y, b_n(z)) = 1, \tag{2.16}$$
$$b_k(x) + y^m = z^n, \quad \gcd(b_k(x), y, z) = 1, \tag{2.17}$$
$$b_k(x) + y^m = b_n(z), \quad \gcd(b_k(x), y, b_n(z)) = \min(k!, n!), \tag{2.18}$$
$$b_k(x) + b_m(y) = z^n, \gcd(b_k(x), b_m(y), z) = \min(k!, m!), \tag{2.19}$$
$$b_k(x) + b_m(y) = b_n(z), \gcd(b_k(x), b_m(y), b_n(z)) = \min(k!, m!). \tag{2.20}$$

CHAPTER 3

MAIN METHOD AND RESULTS

In this chapter,we will give details of the method we developed and also give the main results. In [4], Bennett and Luijk have given a special univariate polynomial parametrization which is the linear translate of the parametrization written bold in Table 3.1 to attack the problem for the $k = 5$ case. The method we present here, is designed to find such polynomial parametrizations for arbitrary r, k. Using this method, we constructed new families of infinitely many solutions in the following cases: $k = 4, r \geq 4$ (Theorem 46) and $k = 5, r \geq 5$ (Theorem 47). Moreover, it has been observed that one might find infinitely many such parametrizations for the case $k = 4$. Details of the method is given below.

3.1 The Method

Our algorithmic method provides two different options divided into two algorithms and a sketch of these algorithms is given below in the corresponding subsections.

3.1.1 Algorithm I (Linear parametrization)

Algorithm starts with a linear polynomial $x_1(x)$ whose coefficients are bounded by an integer parameter B. Then, it takes a linear polynomial $x_2(x)$ from the list $L_B^1 = \{ax_1(x) + b \mid 0 < a < H, -k < b < 0, a, b \in \mathbb{Z}\}$ of polynomials and checks degree of the square-free part of $[x_1(x), x_2(x)]_k$. If the degree is less than three, then it records the result and passes to the next element in the list L_B^1, and applies the same steps. If the degree is greater than two, then a new list L_B^2, which consists of scaled and translated non-eliminated factors of square-free part of $[x_1(x), x_2(x)]_k$, is created. Iterate routines until degree of square-free part of the product of all blocks is less than three. Furthermore, a more detailed algorithm for generating at most three disjoint blocks is given in the Appendix.

3.1.2 Algorithm II (Quadratic and linear parametrization)

Algorithm starts with a quadratic polynomial $x_1(x)$ whose coefficients are bounded by an integer parameter B. If number of irreducible quadratic factors of $b_k(x_1(x))$ over $\mathbb{Z}[x]$ is less than two, then algorithm tries to eliminate remaining linear factors of the square-free part of $b_k(x_1(x))$ by applying Method 1. If number of irreducible quadratic factors of $b_k(x_1(x))$ over $\mathbb{Z}[x]$ is greater than one, then algorithm takes a quadratic polynomial $x_2(x)$ from the list $Q_B^1 = \{ax_1^*(x) + b \mid 0 < a < H, -k < b < 0, a, b \in \mathbb{Z}\}$ where $x_1^*(x)$ is any irreducible quadratic factor of $b_k(x_1(x))$. Then, it checks number of irreducible quadratic factors of $[x_1(x), x_2(x)]_k$. If the number is less than two, then algorithm tries to eliminate remaining linear factors of the square-free part of $[x_1(x), x_2(x)]_k$ by applying Method 1. If the number is greater than one, then a new list Q_B^2, which consists of scaled and translated non-eliminated irreducible quadratic factors of $[x_1(x), x_2(x)]_k$, is created. Iterate routines until finding suitable quadratic polynomials. If suitable quadratic polynomials are found, then algorithm tries to eliminate remaining linear factors by applying the Method 1 until degree of square-free part of product of all blocks is less than three.

3.1.3 Notes about algorithms

Disjointness of the polynomial blocks is checked when a new polynomial block is added. Number of linear and quadratic polynomial blocks is bounded by parameters l and q, respectively. Hence, algorithms search for only finitely many combinations for given positive integer parameters B, k, l and q. On the other hand, the number of elements in the lists $L_B^1, \dots, L_B^l$ and $Q_B^1, \dots, Q_B^q$ generally grows with increasing number of polynomial blocks. For optimization purposes, these lists could be ordered and trimmed, so that each trimmed list could have the same cardinality.

3.1.4 Notes about polynomials

The following facts have been taken into consideration in constructing algorithms.

Theorem 43. *Let $p(x) \in \mathbb{Z}[x]$, if $\deg p(x) \geq 3$, then $p(x)$ or $q(x) = p(x) + 1$ can not be written as a product of linear polynomials over $\mathbb{Z}$.*

Proof. If $\deg p(x) = 2$, then the polynomial $p(x) = (x - (r-1))(x - (r+1))$ could be chosen for any integer r, such that $q(x) = p(x)+1 = (x-r)^2$ means $p(x)$ and $q(x)$ could be written as a product of linear polynomials over $\mathbb{Z}$. Moreover, this quadratic polynomial is unique up to the integer parameter r which is also proved below.

Firstly, we suppose that $\deg p(x) \geq 3$ and the contrary holds, then $p(x)$ and $q(x) = p(x)+1$ must have both integer roots. Now, for any integer $m \geq 3$, let the integer roots of $p(x)$ be given as $r_1, r_2, \dots, r_m$, and integer roots of $q(x)$ be given as $s_1, s_2, \dots, s_m$. Since $p(x)$ and $q(x)$ could be written as $p(x) = a_0x^m + a_1x^{m-1} + \cdots + a_{m-1}x + a_m$ and $q(x) = a_0x^m + a_1x^{m-1} + \cdots + a_{m-1}x + a_m + 1$ for some integers $a_0 \neq 0, a_1, \dots, a_m$.

Secondly, reading the product of integer roots from coefficients of their equations, $(-1)^m a_m/a_0 \in \mathbb{Z}$ and $(-1)^m(a_m + 1)/a_0 \in \mathbb{Z}$, we obtain $|a_0| = 1$. Reading the sum of integer roots from coefficients of their equations, $-a_1/a_0 \in \mathbb{Z}$, we obtain $r_1+r_2+\cdots+r_m = s_1+s_2+\cdots+s_m$ which is also equivalent to saying $\sum_{t=1}^{m}(r_t - s_t) = 0$.

Thirdly, $p(r_i) = q(s_j) = 0$ and $r_i \neq s_j$, for $i, j \in \{1, 2, \dots, m\}$. Since $p(r_i) = q(s_j) = 0$, we obtain $a_0(r_i^m - s_j^m) + a_1(r_i^{m-1} - s_j^{m-1}) + \cdots + a_{m-1}(r_i - s_j) = 1$. Therefore, for all $i, j \in \{1, 2, \dots, m\}$, $|r_i - s_j| = 1$ holds which also means ($|s_i - s_j| = 2$ or $|s_i - s_j| = 0$) and ($|r_i - r_j| = 2$ or $|r_i - r_j| = 0$). In order for these equations to hold, either r_i's or s_j's must be fixed. From now on, without loss of generality

r_i's fixed as $r_i = r$ for some integer r, such that $|r_i - r_j| = 0$ is trivially satisfied in this case. Also, using $|r_i - s_j| = 1$, we obtain that for any j, $s_j = r-1$ or $s_j = r+1$.

Finally, if the degree m is odd, $r_1 + r_2 + \cdots + r_m = mr = s_1 + s_2 + \cdots + s_m = mr + k_+(1) + k_-(-1)$ is not possible, since $k_+ + k_- = m$ is an odd integer, and this contradicts with our assumption. If the degree m is even, we again get a contradiction, since $r_1 r_2 \cdots r_m \pm 1 = r^m \pm 1 = s_1 s_2 \cdots s_m = (r-1)^{k_-}(r+1)^{k_+}$ if and only if $m = 2$, and $k_+ = 1, k_- = 1$. In addition to that, $p(x) = (x-(r-1))(x-(r+1))$ is the unique quadratic polynomial up to integer parameter r satisfying the condition. Therefore, excluding the degree two case, $p(x)$ or $q(x) = p(x)+1$ can not be written as a product of linear polynomials over $\mathbb{Z}$ when $\deg p(x) \geq 3$. □

Theorem 44. *Let $p(x) \in \mathbb{Z}[x]$, if $\deg p(x) \geq 2$, then $p(x)$, $q(x) = p(x) + 1$ or $w(x) = p(x) + 2$ can not be written as a product of linear polynomials over $\mathbb{Z}$.*

Proof. For the proof of the theorem, if $\deg p(x) \geq 3$, then using Theorem 43, $p(x)$ or $q(x) = p(x) + 1$ can not be written as a product of linear polynomials over $\mathbb{Z}$, so the theorem is proved in this case.

Firstly, we suppose that $\deg p(x) = 2$ and the contrary holds, then $p(x)$, $q(x) = p(x) + 1$ and $w(x) = p(x) + 2$ must have both integer roots. Let the integer roots of $p(x)$ be given as r_1, r_2, integer roots of $q(x)$ be given as s_1, s_2 and integer roots of $w(x)$ be given as t_1, t_2. Since $p(x)$, $q(x)$ and $w(x)$ could be written as $p(x) = a_0x^2 + a_1x + a_2$, $q(x) = a_0x^2 + a_1x + a_2 + 1$ and $w(x) = a_0x^2 + a_1x + a_2 + 2$ for some integers $a_0 \neq 0, a_1, a_2$.

Secondly, reading the product of integer roots from coefficients of their equations, $a_2/a_0 \in \mathbb{Z}$ and $(a_2+1)/a_0 \in \mathbb{Z}$, we obtain $|a_0| = 1$. Reading the sum of integer roots from coefficients of their equations, $-a_1/a_0 \in \mathbb{Z}$, we obtain $r_1 + r_2 = s_1 + s_2 = t_1 + t_2$ which is also equivalent to saying $\sum_{j=1}^{2}(r_j - s_j) = 0, \sum_{j=1}^{2}(r_j - t_j) = 0$ and $\sum_{j=1}^{2}(s_j - t_j) = 0$.

Thirdly, $p(r_i) = q(s_j) = 0$ with $r_i \neq s_j$, $p(r_i) = w(t_k) = 0$ with $r_i \neq t_k$ and $q(s_j) = w(t_k) = 0$ with $s_j \neq t_k$ for $i, j, k \in \{1, 2\}$. Since $p(r_i) = q(s_j) = 0$, we obtain $a_0(r_i^2 - s_j^2) + a_1(r_i - s_j) = 1$. Therefore, for all $i, j \in \{1, 2\}$, $|r_i - s_j| = 1$ holds which also means ($|s_1 - s_2| = 2$ or $|s_1 - s_2| = 0$) and ($|r_1 - r_2| = 2$ or $|r_1 - r_2| = 0$). In order for these equations to hold, either r_i's or s_j's must be fixed. From now on, without loss of generality r_i's fixed as $r_1 = r$, $r_2 = r$ for some integer r, such that $|r_1 - r_2| = 0$ is trivially satisfied in this case. Also, using $|r_i - s_j| = 1$, we obtain without loss of generality $s_1 = r - 1$, $s_2 = r + 1$.

Finally, using $q(s_j) = w(t_k) = 0$, we obtain $a_0(s_j^2 - t_k^2) + a_1(s_j - t_k) = 1$ and for all $j, k \in \{1, 2\}$, $|s_j - t_k| = 1$ holds. Hence, ($|s_1 - s_2| = 2$ or $|s_1 - s_2| = 0$) and ($|t_1 - t_2| = 2$ or $|t_1 - t_2| = 0$). Moreover, $s_1 = r - 1$, $s_2 = r + 1$, and $|s_j - t_k| = 1$ implies that $t_1 = r$, $t_2 = r$ which is not possible, since $r_i \neq t_k$ for all $i, k \in \{1, 2\}$ and this contradicts with our assumption. □

Corollary 45. *Let $n \geq 1$ for some integer n, $\vec{x} = (x_1, x_2, \cdots, x_n)$ and $p(\vec{x}) \in \mathbb{Z}[x_1, x_2, \cdots, x_n]$, if $\deg p(\vec{x}) \geq 3$, then $p(\vec{x})$ or $q(\vec{x}) = p(\vec{x}) + 1$ can not be written as a product of linear polynomials over $\mathbb{Z}$.*

Proof. For the proof of the corollary, assume on the contrary that $\deg p(\vec{x}) \geq 3$ and both $p(\vec{x})$ and $q(\vec{x}) = p(\vec{x}) + 1$ could be written as a product of linear polynomials over $\mathbb{Z}$. Therefore, there exists linear polynomials $l_n(t) \in \mathbb{Z}[t]$, such that we can form polynomials $\bar{p}(t) = p(l_1(t), l_2(t), \cdots, l_n(t))$ and $\bar{q}(t) = \bar{p}(t) + 1$ in $\mathbb{Z}[t]$ with $\deg p(\vec{x}) = \deg \bar{p}(t) = \deg \bar{q}(t) \geq 3$ and both $\bar{p}(t)$ and $\bar{q}(t)$ could be written as a product of linear polynomials over $\mathbb{Z}$, which contradicts the Theorem 46. □

3.2 Polynomial parametrizations

We have implemented the algorithms discussed briefly in the previous section in Sage [20]. Then, we have produced parametrized family of integer solutions to equation (1.1) by using this implementation. In Table 3.1, we present a brief list of selected parametrizations. A full list of polynomial parametrizations (which are distinct up to affine coordinate changes) can be found in the Appendix. Furthermore, if

$t = z + 13/4$ is substituted into the parametrization written bold in Table 3.1, one obtains $4z^2 + z - 5, 8z^2 + 2z - 4, 4z - 4, 4z + 1$ which is given in [4].

Table 3.1: A brief list of polynomial parametrizations

k	**r**	$\boldsymbol{x_1(t), x_2(t), \ldots, x_r(t)}$	square-free part of $\boldsymbol{[x_1(t), x_2(t), \ldots, x_r(t)]_k}$
4	4	$4t^2 - 25t + 36, 8t^2 - 50t + 74, 4t - 16, 4t - 12$	1
4	4	$12t - 48, 36t - 141, 8t - 32, 72t - 281$	$576t^2 - 4336t + 8149$
4	6	$14t + 18, 42t + 57, 21t + 27, 18t + 24, 126t + 175, 63t + 88$	1
4	6	$3t - 50, 6t - 101, 2t - 33, 6t - 94, 9t - 138, 18t - 275$	$324t^2 - 10404t + 83325$
5	4	$2t - 50, 6t - 150, 3t - 74, 6t - 142$	$216t^2 - 10368t + 124266$
5	**4**	$\boldsymbol{4t^2 - 25t + 34, 8t^2 - 50t + 74, 4t - 17, 4t - 12}$	$\boldsymbol{8t^2 - 50t + 70}$
5	6	$12t^2 - 43t + 35, 24t^2 - 86t + 74, 12t - 28, 4t - 8, 3t - 7, 6t - 14$	3

3.2.1 Use of Table 3.1

Suppose that we try to find a parametrized family of solutions to the equation $[x_1, x_2, \ldots, x_{15}]_5 = y^2$. Firstly, we compute square-free part$([1, 12, 24, 38, 285]_5) = 22$. Secondly, we see from the Table 3.1 that square-free part$([12u^2 - 43u + 35, 24u^2 - 86u + 74, 12u - 28, 4u - 8, 3u - 7, 6u - 14]_5) = 3$ and square-free part$([2t - 50, 6t - 150, 3t - 74, 6t - 142]_5) = 216t^2 - 10368t + 124266$. In addition, it can be shown that the quadratic Diophantine equation $66v^2 = 216t^2 - 10368t + 124266$ has infinitely many (v, t) positive integer tuple solutions by using Dario Alpern's generic two integer variable equation solver in [1]. Finally, if we concatenate all these blocks, we get a parametrized family of solutions as $[1, 12, 24, 38, 285, 2t - 50, 6t - 150, 3t - 74, 6t - 142, 12u^2 - 43u + 35, 24u^2 - 86u + 74, 12u - 28, 4u - 8, 3u - 7, 6u - 14]_5 = y(t, u)^2$.

3.2.2 Verifying square-free part of $[x_1(t), x_2(t), \ldots, x_r(t)]_k$

To begin with, we create a function called `VerifyList(k, L)` in the following lines of Sage code:

```
def VerifyList(k,L):
   Uni.<x>=PolynomialRing(ZZ);
```

```
block=lambda m,k: prod([m+i for i in range(k)]);
product=1;
for polynomial in L:
   product*=block(Uni(polynomial),k);
return prod([p^(e%2) for (p,e) in list(product.factor())])
```

Then, we select a parametrization from Table 3.1. We could assume that the polynomial parametrization $12t-48, 36t-141, 8t-32, 72t-281$ with block length $k = 4$ is selected. If we call `VerifyList`$(4, [\texttt{12} * \texttt{x} - \texttt{48}, \texttt{36} * \texttt{x} - \texttt{141}, \texttt{8} * \texttt{x} - \texttt{32}, \texttt{72} * \texttt{x} - \texttt{281}])$ in Sage, then we get the output as $576x^2 - 4336x + 8149$ which completes the verification.

3.3 New families of solutions

By using the polynomial parametrizations given in the previous section, we construct new families of solutions to equation (1.1) in the proofs of Theorem 46 and Theorem 47, as follows:

Theorem 46. *For* $(r \geq 3; k = 4)$*, equation (1.1) has infinitely many solutions.*

Proof. Case $r = 3$ is proved by by Bauer and Bennett using bivariate polynomial parametrizations in [3]. Moreover, in cases $(r = 4 \text{ or } r \geq 6)$ and $r = 5$, a unified proof using just univariate polynomials is presented below alternative to the ones given in [22, 3], respectively.

By using the parametrizations given in Table 3.1 together with Dario Alpern's generic two integer variable equation solver in [1], we get the following parametrized solutions:
Case $r = 4$: $[4t^2 - 25t + 36, 8t^2 - 50t + 74, 4t - 16, 4t - 12]_4$
Case $r = 5$: $[6, 12t - 48, 36t - 141, 8t - 32, 72t - 281]_4$
Case $r = 6$: $[14t + 18, 42t + 57, 21t + 27, 18t + 24, 126t + 175, 63t + 88]_4$
Case $r = 7$: $[6, 3t - 50, 6t - 101, 2t - 33, 6t - 94, 9t - 138, 18t - 275]_4$

As there is no fixed block in the parametrization of the case $r = 4$, we can duplicate it to obtain new solutions for cases $k = 4, r = 4t, t \geq 1$. In addition to that, by concatenating above blocks with disjoint copies of the blocks associated to the case $r = 4$ appropriately, infinitely many solutions can be obtained for each $r \geq 4, k = 4$ case, that is, $\cup_{t \geq 0}\{4t + 4, 4t + 5, 4t + 6, 4t + 7\} = \{r \in \mathbb{N} \mid r \geq 4\}$.

$\square$

Theorem 47. *For* $(r \geq 5; k = 5)$*, equation (1.1) has infinitely many solutions.*

Proof. In [4], this theorem is proved by Bennett and Luijk, but an alternative proof using new parametrizations is presented here.

By using the parametrizations given in Table 3.1 together with Dario Alpern's generic two integer variable equation solver in [1], we get the following parametrized solutions:

Case $r = 5$: $[6, 2t - 50, 6t - 150, 3t - 74, 6t - 142]_5$

Case $r = 6$: $[38, 285, 2t - 50, 6t - 150, 3t - 74, 6t - 142]_5$

Case $r = 7$: $[3, 36, 74, 2t - 50, 6t - 150, 3t - 74, 6t - 142]_5$

Case $r = 8$: $[13, 30, 45, 90, 2t - 50, 6t - 150, 3t - 74, 6t - 142]_5$

Case $r = 9$: $[1, 12, 24, 12t^2 - 43t + 35, 24t^2 - 86t + 74, 12t - 28, 4t - 8, 3t - 7, 6t - 14]_5$

Case $r = 10$: $[1, 16, 62, 152, 12t^2 - 43t + 35, 24t^2 - 86t + 74, 12t - 28, 4t - 8, 3t - 7, 6t - 14]_5$

Case $r = 11$: $[6, 38, 285, 2t - 50, 6t - 150, 3t - 74, 6t - 142, 2u - 50, 6u - 150, 3u - 74, 6u - 142]_5$

Case $r = 12$: $[12t^2 - 43t + 35, 24t^2 - 86t + 74, 12t - 28, 4t - 8, 3t - 7, 6t - 14, 12u^2 - 43u + 35, 24u^2 - 86u + 74, 12u - 28, 4u - 8, 3u - 7, 6u - 14]_5$

Case $r = 13$: $[6, 13, 30, 45, 90, 2t - 50, 6t - 150, 3t - 74, 6t - 142, 2u - 50, 6u - 150, 3u - 74, 6u - 142]_5$

Case $r = 14$: $[13, 30, 38, 45, 90, 285, 2t - 50, 6t - 150, 3t - 74, 6t - 142, 2u - 50, 6u - 150, 3u - 74, 6u - 142]_5$

Case $r = 15$: $[1, 12, 24, 38, 285, 2t - 50, 6t - 150, 3t - 74, 6t - 142, 12u^2 - 43u + 35, 24u^2 - 86u + 74, 12u - 28, 4u - 8, 3u - 7, 6u - 14]_5$

Case $r = 16$: $[1, 16, 38, 62, 152, 285, 2t - 50, 6t - 150, 3t - 74, 6t - 142, 12u^2 - 43u +$

$35, 24u^2 - 86u + 74, 12u - 28, 4u - 8, 3u - 7, 6u - 14]_5$

As there is no fixed block in the parametrization of the case $r = 12$, we can duplicate it to obtain new solutions for cases $k = 5, r = 12t, t \geq 1$. In addition to that, by concatenating above blocks with disjoint copies of the blocks associated to the case $r = 12$ appropriately, infinitely many solutions can be obtained for each $r \geq 12, k = 5$ case, that is, $\cup_{t\geq 0}\{12t + 5, 12t + 6, 12t + 7, 12t + 8, 12t + 9, 12t + 10, 12t+11, 12t+12, 12t+13, 12t+14, 12t+15, 12t+16\} = \{r \in \mathbb{N} \mid r \geq 5\}$. $\square$

3.4 Numerical results

We produce examples of block products for $k = 6$ and $k = 7$, as follows:

$[1, 7, 13, 19, 30, 166, 275, 830]_6, [1, 7, 13, 20, 42, 91, 340]_6,$
$[1, 7, 13, 20, 42, 154, 470, 527]_6, [1, 7, 13, 21, 28, 49, 114, 527]_6,$
$[1, 7, 13, 21, 28, 170, 341]_6, [1, 7, 13, 22, 28, 169, 341]_6,$
$[1, 7, 13, 22, 31, 54, 172, 341]_6, [1, 7, 13, 22, 171, 284, 341, 492]_6,$
$[1, 7, 13, 23, 29, 52, 115, 527]_6, [1, 7, 13, 26, 44, 56, 91, 117]_6,$
$[1, 7, 13, 27, 44, 165, 185, 663]_6, [1, 7, 13, 28, 39, 81, 245, 285]_6,$
$[1, 7, 13, 29, 61, 242, 285]_6, [1, 7, 13, 29, 84, 128, 262, 525]_6,$
$[1, 7, 13, 29, 90, 111, 185, 338]_6, [1, 7, 13, 29, 245, 290, 528, 581]_6,$
$[1, 7, 13, 32, 46, 61, 183]_6, [1, 7, 13, 34, 82, 245, 258, 524]_6,$
$[1, 7, 13, 34, 120, 140, 183, 779]_6, [1, 7, 13, 36, 45, 203, 256, 512]_6,$
$[1, 7, 13, 37, 46, 170, 203, 515]_6, [1, 7, 13, 41, 75, 85, 492, 710]_6,$
$[1, 8, 15, 22, 40, 152, 471, 527]_7, [1, 8, 15, 26, 42, 114, 152, 470]_7,$
$[1, 8, 15, 27, 54, 115, 169, 340]_7, [1, 8, 15, 28, 38, 82, 284, 492]_7,$
$[1, 8, 15, 29, 49, 56, 242, 527]_7, [1, 8, 15, 32, 46, 60, 182]_7,$
$[1, 8, 15, 33, 46, 60, 183]_7$, are all perfect squares.

Additionally, we produce examples for some cases mentioned in [4], as follows:

$[4, 13, 48]_5, [5, 13, 48]_5,$
$[1, 14, 24, 48]_5, [1, 15, 62, 152]_5, [1, 44, 62, 90]_5, [2, 13, 30, 62]_5, [6, 13, 31, 62]_5,$

$[6, 17, 62, 152]_5, [6, 54, 84, 168]_5, [9, 15, 55, 116]_5, [10, 18, 31, 152]_5, [12, 17, 56, 116]_5,$
$[13, 20, 91, 186]_5, [15, 44, 90, 152]_5, [17, 24, 33, 74]_5, [18, 32, 62, 152]_5, [23, 46, 152, 186]_5,$
$[28, 44, 54, 92]_5, [28, 55, 117, 152]_5, [30, 64, 132, 152]_5, [35, 45, 152, 184]_5,$
are all perfect squares.

CHAPTER 4

CONCLUSION

In this thesis, we have investigated both geometric and arithmetic aspects of the problem introduced in 1. We have produced polynomial parametrizations for cases $k = 4$ and $k = 5$. It has been observed that number of distinct univariate polynomial parametrizations (two parameterizations are distinct if and only if they produce different families of integer solutions) grows as the number of blocks r increases. On the other hand, it is not known whether number of all distinct univariate polynomial parametrizations for fixed parameters r, k with $k \geq 4$ and $r \geq k-1$ is finite or integer solutions obtained by them lie on a Zariski closed subvariety since the equation (1.1) defines a variety of general type in these cases. In a personal communication, Erhan Gürel from METU, suggested extending each block of length k left or right to obtain a parametrization for $k+1$ case. However, extending current parametrizations have not produced a suitable one for $k = 6$. He also asks whether there always exists a parametrization just consisting of linear polynomials. Moreover, we have illustrated connections with some well-known conjectures in number theory. Finally, we have shown the first examples of block products for $k = 6$ and $k = 7$ in Section 3.4, and their pattern gives some support to the conjecture of Ulas in [22].

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APPENDIX A

ALGORITHMS

Table A.1: Adding linear polynomial block

procedure LINBLOCK($k, L, p, sfreeq$)
  $flag = 1, q = product([p + j \mid j$ in $[0, \ldots, k - 1]])$
  sq =square-free part of $q * sfreeq$
  sq' =divide sq by degree zero integer-factor of sq
  qL = list of degree one polynomial factors of sq'
  if degree of $sq < 3$ **then**
    $flag = 0$
    write $L + [p]$
  end if
  return $qL, sq, flag$
end procedure

Table A.2: Good linear factors

```
procedure GOODLINF(k, L)
    count = 0, L_u = [ ], L_b = [ ], result = [ ]
    for ax + b in L do
        for c in [1, ..., 2^k] do
            for shift in [1 - k, ..., k - 1] do
                S = [cax + cb + shift + j | j in [0, ..., k - 1]]
                count = 0
                for ux + v in S do
                    for mx + n in L do
                        if u * n == m * v then
                            count = count + 1
                        end if
                    end for
                end for
                if count = 2 then
                    L_b = L_b + [cax + cb + shift]
                end if
                if count >= 3 then
                    L_u = L_u + [cax + cb + shift]
                end if
            end for
        end for
    end for
    Eliminate duplicate elements in the lists L_u and L_b
    result = L_u + L_b
    return first ten elements of the list result starting from L_u
end procedure
```

Table A.3: Main algorithm generating at most three disjoint polynomial blocks

```
procedure MAIN(k, B)
    L = [ ], P = [ ]
    for u in [1, ..., B] do
        for v in [−B, ..., B] do
            L = L + [ux + v]
        end for
    end for
    for p_1 in L do
        sfreeq_1 = product([p_1 + j | j in [0, ..., k − 1]])
        sfreeq'_1 =divide sfreeq_1 by degree zero integer-factor of sfreeq_1
        qL_1 = list of degree one polynomial factors of sfreeq'_1
        for p_2 in GOODLINF(k, qL_1) do
            P = [p_1]
            if p_2 is disjoint from P then
                qL_2, sfreeq_2, flag =LINBLOCK(k, P, p_2, sfreeq_1)
                if flag = 1 then
                    for p_3 in GOODLINF(k, qL_2) do
                        P = [p_1, p_2]
                        if p_3 is disjoint from P then
                            qL_3, sfreeq_3, flag =LINBLOCK(k, P, p_3, sfreeq_2)
                        end if
                    end for
                end if
            end if
        end for
    end for
    return 0
end procedure
```

APPENDIX B

POLYNOMIAL PARAMETRIZATIONS

Table B.1: Polynomial parametrizations I

k	r	$x_1(x), x_2(x), \ldots, x_r(x)^*$	square-free part of the product $b_k(x_1)b_k(x_2)\cdots b_k(x_r)$
4	3	$3x-48, 6x-94, 2x-33$	$36x^2-1140x+9009$
4	3	$3x-48, 6x-94, 2x-32$	$36x^2-1068x+7917$
4	3	$27x-48, 54x-94, 18x-33$	$324x^2-1140x+1001$
4	3	$36x-48, 72x-94, 24x-33$	$576x^2-1520x+1001$
4	3	$3x-48, 6x-95, 2x-33$	$36x^2-1164x+9405$
4	3	$48x-48, 96x-95, 32x-32$	$9216x^2-17472x+8265$
4	4	$5x-48, 10x-96, 20x-186, 4x-38$	$400x^2-7160x+32025$
4	4	$10x-50, 20x-97, 8x-40, 40x-197$	$1600x^2-15280x+36445$
4	4	$12x-48, 36x-141, 8x-32, 72x-281$	$576x^2-4336x+8149$
4	4	$25x-50, 50x-97, 20x-41, 100x-196$	$10000x^2-39800x+39565$
4	4	$27x-48, 81x-141, 18x-32, 162x-281$	$2916x^2-9756x+8149$
4	4	$33x-48, 99x-141, 198x-281, 22x-32$	$4356x^2-11924x+8149$
4	4	$35x-50, 70x-97, 28x-41, 140x-196$	$19600x^2-55720x+39565$
4	4	$36x-48, 108x-141, 216x-281, 24x-33$	$5184x^2-13872x+9273$
4	4	$42x-49, 84x-94, 56x-64, 24x-29$	$28224x^2-64848x+37149$
4	4	$42x-49, 84x-94, 56x-64, 24x-28$	$28224x^2-60144x+32025$
4	4	$45x-48, 135x-141, 30x-32, 270x-281$	$8100x^2-16260x+8149$
4	4	$4x^2-25x+36, 8x^2-50x+74, 4x-16, 4x-12$	1
4	4	$4x^2-23x+30, 8x^2-46x+62, 4x-15, 4x-11$	1
4	4	$4x^2-25x+36, 8x^2-50x+74, 4x-13, 4x-17$	$16x^2-104x+153$
4	5	$3x-45, 9x-132, 27x-393, 54x-785, 2x-31$	$324x^2-9732x+73005$
4	5	$5x-50, 10x-97, 6x-60, 20x-196, 12x-119$	$3600x^2-70440x+344505$
4	5	$5x-50, 10x-97, 30x-294, 60x-586, 4x-40$	$3600x^2-68280x+323565$
4	5	$10x-42, 30x-126, 6x-25, 30x-117, 15x-58$	$225x^2-1770x+3472$
4	5	$12x-48, 36x-141, 108x-420, 216x-838, 8x-33$	$5184x^2-41424x+82665$
4	5	$14x-49, 42x-144, 21x-72, 12x-43, 84x-289$	$7056x^2-49560x+86989$
4	5	$15x-48, 45x-141, 10x-33, 135x-420, 270x-838$	$8100x^2-51780x+82665$
4	5	$18x-48, 54x-141, 162x-420, 12x-32, 324x-839$	$11664x^2-58392x+72993$
4	5	$21x-17, 42x-35, 18x-15, 36x-29, 28x-22$	$7056x^2-10472x+3857$
4	5	$25x-50, 50x-97, 20x-41, 150x-294, 300x-586$	$90000x^2-359400x+358545$
4	5	$27x-48, 81x-141, 18x-32, 243x-420, 486x-839$	$26244x^2-87588x+72993$
4	5	$27x-33, 81x-96, 9x-12, 6x-9, 162x-190$	$324x^2-860x+561$
4	5	$30x-42, 90x-126, 18x-27, 10x-15, 45x-65$	$225x^2-590x+384$
4	5	$30x-42, 90x-126, 18x-27, 10x-16, 45x-65$	$225x^2-680x+512$
4	5	$33x-48, 99x-141, 297x-420, 594x-839, 22x-32$	$39204x^2-107052x+72993$
4	5	$33x-5, 66x-11, 22x-3, 198x-12, 18x-3$	$9801x^2-1584x+55$
4	5	$39x-48, 78x-93, 52x-65, 312x-378, 24x-31$	$97344x^2-242736x+151125$
4	5	$39x-48, 78x-93, 52x-65, 312x-378, 24x-30$	$10816x^2-25168x+14625$
4	5	$39x-48, 117x-141, 351x-420, 26x-32, 702x-839$	$54756x^2-126516x+72993$
4	5	$42x-49, 126x-144, 63x-72, 252x-289, 36x-42$	$63504x^2-141624x+78897$
4	5	$42x-48, 126x-141, 28x-33, 378x-420, 756x-838$	$63504x^2-144984x+82665$
4	5	$42x-17, 126x-51, 18x-9, 63x-28, 14x-8$	$441x^2-434x+104$
4	5	$44x-28, 132x-78, 33x-22, 36x-24, 396x-243$	$156816x^2-195624x+60973$
4	5	$45x-50, 90x-97, 270x-294, 540x-587, 36x-40$	$291600x^2-616680x+325785$
4	5	$50x-46, 150x-132, 30x-29, 75x-72, 150x-145$	$22500x^2-41100x+18733$
4	5	$4x^2-25x+36, 8x^2-50x+74, 4x-13, 6x-24, 12x-46$	$144x^2-840x+1161$
4	5	$4x^2-25x+36, 8x^2-50x+74, 4x-13, 6x-24, 12x-47$	$144x^2-888x+1269$
4	5	$4x^2-25x+36, 8x^2-50x+74, 4x-16, 4x-11, x-4$	x^2-5x+4
4	5	$12x^2-49x+47, 24x^2-98x+93, 4x-9, 12x-27, 6x-15$	$144x^2-600x+589$
4	5	$12x^2-44x+37, 36x^2-132x+111, 3x-6, 6x-15, 2x-5$	$36x^2-132x+113$

Table B.2: Polynomial parametrizations II

k	r	$x_1(x), x_2(x), \ldots, x_r(x)$	square-free part of the product $b_k(x_1)b_k(x_2)\cdots b_k(x_r)$
4	6	$14x+18, 42x+57, 21x+27, 18x+24, 126x+175, 63x+88$	1
4	6	$21x-9, 42x-18, 14x-7, 18x-9, 126x-52, 63x-28$	1
4	6	$21x+16, 42x+32, 14x+11, 18x+15, 126x+116, 63x+56$	1
4	6	$28x-27, 84x-78, 21x-21, 36x-36, 252x-245, 126x-122$	1
4	6	$28x+21, 84x+66, 42x+33, 36x+27, 126x+98, 252x+200$	1
4	6	$28x+32, 84x+99, 42x+48, 36x+42, 126x+151, 252x+301$	1
4	6	$28x+46, 84x+141, 42x+69, 36x+60, 126x+214, 252x+427$	1
4	6	$35x+14, 105x+48, 21x+9, 15x+6, 35x+18, 105x+54$	1
4	6	$40x+16, 120x+54, 45x+18, 72x+30, 360x+153, 180x+75$	1
4	6	$40x+21, 120x+63, 45x+24, 72x+39, 360x+204, 180x+102$	1
4	6	$42x-40, 84x-77, 28x-28, 36x-36, 252x-245, 126x-122$	1
4	6	$42x+23, 84x+49, 28x+14, 36x+18, 126x+67, 252x+133$	1
4	6	$42x+33, 84x+66, 28x+21, 36x+27, 126x+98, 252x+200$	1
4	6	$3x-50, 6x-101, 2x-33, 6x-94, 9x-138, 18x-275$	$324x^2-10404x+83325$
4	6	$3x-48, 6x-95, 18x-288, 2x-32, 27x-432, 54x-862$	$324x^2-9852x+74733$
4	6	$3x-45, 9x-132, 27x-393, 81x-1176, 162x-2350, 2x-31$	$324x^2-9716x+72757$
4	6	$3x-45, 9x-132, 27x-393, 81x-1176, 162x-2350, 2x-30$	$324x^2-9068x+63369$
4	6	$5x-50, 10x-97, 6x-60, 12x-118, 30x-294, 60x-587$	$3600x^2-69720x+337525$
4	6	$5x-50, 10x-97, 30x-294, 90x-879, 180x-1757, 4x-41$	$3600x^2-72040x+360185$
4	6	$5x-50, 10x-97, 30x-294, 90x-879, 180x-1757, 4x-40$	$3600x^2-68440x+325045$
4	6	$11x-47, 33x-138, 66x-275, 12x-51, 22x-94, 44x-187$	$17424x^2-144408x+299145$
4	6	$12x-43, 36x-126, 9x-33, 36x-131, 54x-198, 108x-395$	$11664x^2-83160x+148125$
4	6	$14x-49, 42x-144, 21x-71, 18x-63, 126x-430, 63x-217$	$441x^2-2940x+4896$
4	6	$14x-35, 42x-102, 21x-51, 12x-30, 126x-306, 252x-611$	$7056x^2-32984x+38493$
4	6	$15x-49, 30x-96, 60x-186, 12x-38, 20x-61, 60x-177$	$225x^2-1395x+2156$
4	6	$15x-29, 30x-58, 10x-20, 6x-13, 45x-78, 90x-154$	$8100x^2-31140x+29445$
4	6	$16x-48, 48x-141, 12x-36, 48x-136, 72x-201, 144x-400$	$20736x^2-117216x+165549$
4	6	$18x-48, 36x-94, 108x-273, 12x-32, 162x-408, 324x-815$	$11664x^2-57528x+70905$
4	6	$20x+41, 60x+123, 15x+30, 180x+384, 36x+74, 18x+35$	$8100x^2+33120x+33775$
4	6	$21x-36, 42x-71, 126x-216, 14x-25, 189x-324, 378x-646$	$15876x^2-55356x+48225$
4	6	$21x-30, 42x-58, 126x-165, 14x-21, 189x-246, 378x-490$	$15876x^2-44268x+30681$
4	6	$21x-17, 42x-35, 18x-15, 54x-42, 108x-82, 28x-23$	$63504x^2-98616x+38157$
4	6	$21x+14, 42x+32, 6x+3, 14x+11, 7x+3, 14x+6$	$196x^2+308x+117$
4	6	$22x-39, 66x-114, 33x-57, 44x-78, 24x-43, 264x-453$	$69696x^2-244464x+214269$
4	6	$24x-48, 48x-95, 144x-288, 16x-32, 216x-432, 432x-863$	$20736x^2-79008x+75081$
4	6	$25x-46, 50x-88, 30x-54, 90x-159, 450x-786, 225x-395$	$5625x^2-20200x+18124$
4	6	$25x-45, 50x-88, 30x-54, 90x-159, 450x-786, 225x-395$	$5625x^2-19300x+16548$
4	6	$27x-24, 54x-47, 162x-144, 18x-16, 243x-216, 486x-431$	$26244x^2-42228x+16809$
4	6	$28x-45, 84x-135, 36x-57, 252x-388, 63x-98, 126x-193$	$15876x^2-49392x+38391$
4	6	$30x-50, 60x-100, 20x-33, 60x-94, 90x-138, 180x-274$	$32400x^2-101160x+78861$
4	6	$30x-49, 60x-97, 120x-197, 24x-39, 120x-186, 40x-62$	$14400x^2-44880x+34869$
4	6	$30x-48, 60x-95, 12x-19, 20x-33, 18x-27, 36x-52$	$3600x^2-10840x+8085$
4	6	$32x-48, 96x-138, 36x-54, 72x-107, 96x-143, 288x-429$	$82944x^2-241344x+175497$
4	6	$33x-45, 66x-87, 22x-30, 44x-57, 264x-342, 24x-33$	$7744x^2-20592x+13673$
4	6	$33x-45, 66x-88, 198x-255, 22x-30, 297x-381, 594x-761$	$39204x^2-98340x+61641$
4	6	$33x-45, 66x-88, 198x-255, 22x-31, 297x-381, 594x-760$	$39204x^2-105204x+70401$
4	6	$35x-50, 70x-98, 42x-60, 126x-177, 630x-876, 315x-440$	$11025x^2-30170x+20633$
4	6	$35x-46, 70x-88, 42x-54, 126x-159, 630x-786, 315x-395$	$11025x^2-28280x+18124$
4	6	$35x-19, 105x-57, 105x-51, 21x-12, 14x-8, 30x-17$	$44100x^2-40740x+8925$
4	6	$35x-19, 105x-57, 105x-51, 42x-22, 15x-9, 10x-7$	$44100x^2-50820x+13965$
4	6	$36x-44, 108x-129, 27x-33, 108x-124, 162x-183, 324x-365$	$104976x^2-241704x+139065$
4	6	$39x-48, 117x-138, 234x-275, 78x-94, 468x-552, 52x-64$	$24336x^2-58136x+34713$
4	6	$39x-48, 78x-93, 52x-65, 36x-45, 312x-379, 72x-89$	$876096x^2-2147184x+1315509$
4	6	$39x-48, 78x-93, 52x-65, 36x-45, 156x-189, 468x-561$	$13689x^2-33111x+20020$
4	6	$39x-48, 78x-93, 52x-65, 468x-567, 936x-1132, 24x-30$	$97344x^2-226928x+132093$
4	6	$39x-48, 78x-93, 52x-65, 468x-567, 936x-1133, 24x-30$	$97344x^2-227344x+132561$
4	6	$39x-42, 78x-83, 234x-252, 26x-29, 351x-378, 702x-754$	$54756x^2-119652x+65337$
4	6	$39x-42, 78x-83, 234x-252, 26x-28, 351x-378, 702x-755$	$54756x^2-111540x+56625$
4	6	$39x-29, 117x-87, 234x-172, 52x-39, 18x-15, 36x-30$	$24336x^2-36920x+13949$
4	6	$42x-47, 84x-94, 28x-31, 252x-264, 504x-527, 24x-27$	$28224x^2-61264x+33201$
4	6	$42x-35, 126x-102, 63x-51, 378x-306, 756x-611, 36x-31$	$571536x^2-954072x+397761$
4	6	$44x-47, 132x-138, 33x-36, 132x-143, 198x-216, 396x-431$	$156816x^2-333432x+177141$
4	6	$44x-39, 132x-114, 66x-57, 88x-78, 528x-452, 48x-43$	$278784x^2-486816x+212377$
4	6	$45x-50, 90x-98, 54x-60, 162x-177, 810x-876, 405x-440$	$18225x^2-38790x+20633$

Table B.3: Polynomial parametrizations III

k	r	$x_1(x), x_2(x), \ldots, x_r(x)$	square-free part of the product $b_k(x_1)b_k(x_2)\cdots b_k(x_r)$
4	6	$45x-42, 90x-82, 270x-237, 30x-29, 405x-354, 810x-706$	$72900x^2-133740x+61161$
4	6	$49x-49, 147x-144, 294x-287, 84x-84, 98x-95, 196x-189$	$345744x^2-671496x+325941$
4	6	$49x-49, 147x-144, 294x-287, 98x-95, 294x-282, 42x-42$	$86436x^2-162876x+76713$
4	6	$49x-45, 147x-132, 294x-262, 98x-90, 294x-267, 42x-38$	$1764x^2-3060x+1325$
4	6	$50x-45, 100x-86, 60x-54, 180x-159, 450x-396, 900x-788$	$90000x^2-153400x+65321$
4	6	$50x-18, 150x-54, 75x-27, 450x-162, 45x-16, 90x-28$	$8100x^2-5148x+805$
4	6	$12x^2-41x+32, 24x^2-82x+63, 12x-17, 4x-10, 3x-7, 6x-15$	1
4	6	$12x^2-31x+17, 24x^2-62x+33, 4x-5, 12x-22, 6x-6, 3x-5$	1
4	6	$36x^2-3x-3, 72x^2-6x-4, 12x-2, 4x-1, 6x+1, 3x-1$	1
4	6	$48x^2-34x+3, 96x^2-68x+5, 24x-5, 8x-6, 6x-4, 12x-9$	1
4	6	$48x^2+14x-2, 96x^2+28x-5, 8x-2, 12x-3, 24x+7, 6x-1$	1
4	6	$4x^2-25x+36, 8x^2-50x+74, 4x-13, 6x-24, 18x-69, 36x-137$	$144x^2-872x+1233$
4	6	$4x^2-25x+36, 8x^2-50x+74, 4x-13, 6x-24, 18x-69, 36x-136$	$144x^2-856x+1197$
4	6	$12x^2-31x+17, 24x^2-62x+33, 12x-22, 9x-15, 4x-6, 18x-29$	$324x^2-792x+435$
4	6	$36x^2-20x, 108x^2-60x+6, 6x-3, 9x-5, 27x-12, 54x-23$	$2916x^2-1404x+69$
4	6	$36x^2-20x, 108x^2-60x+6, 6x-3, 9x-5, 27x-12, 54x-22$	$2916x^2-1188x+57$
4	6	$36x^2-20x, 108x^2-60x+6, 18x-9, 18x-4, 9x-6, 3x-3$	$9x^2-10x+1$
4	6	$12x^2-43x+35, 24x^2-86x+74, 12x-28, 4x-8, 3x-7, 6x-14$	$9x^2-33x+30$
4	6	$12x^2-29x+14, 24x^2-58x+32, 4x-8, 3x-6, 12x-12, 6x-8$	$9x^2-15x+6$
4	7	$12x^2-31x+17, 24x^2-62x+33, 12x-22, 9x-15, 27x-42, 4x-6, 54x-83$	$324x^2-768x+415$
4	7	$12x^2-31x+17, 24x^2-62x+33, 12x-22, 9x-15, 27x-42, 4x-6, 54x-82$	$324x^2-744x+395$
4	7	$36x^2+20x, 108x^2+60x+6, 18x+6, 9x+4, 9x, 18x, 3x-1$	$81x^2+36x-21$
4	8	$30x+31, 60x+62, 36x+38, 180x+204, 20x+20, 18x+18, 45x+51, 90x+102$	$225x^2+535x+318$
5	4	$2x-50, 6x-150, 3x-74, 6x-142$	$216x^2-10368x+124266$
5	4	$4x^2-25x+34, 8x^2-50x+74, 4x-17, 4x-12$	$8x^2-50x+70$
5	6	$10x-42, 30x-126, 15x-62, 30x-118, 12x-50, 24x-98$	$1728x^2-13824x+27645$
5	6	$10x-42, 30x-126, 15x-62, 30x-118, 6x-26, 3x-15$	$3x^2-29x+70$
5	6	$21x+15, 42x+31, 14x+10, 18x+15, 126x+116, 63x+56$	$2268x^2+4068x+1767$
5	6	$30x-32, 90x-96, 45x-47, 90x-88, 18x-20, 9x-11$	$243x^2-486x+231$
5	6	$42x+14, 84x+34, 24x+8, 168x+74, 28x+10, 56x+21$	$49392x^2+39102x+7728$
5	6	$42x+42, 84x+87, 36x+36, 252x+258, 126x+130, 63x+66$	$111132x^2+237699x+127092$
5	6	$12x^2-43x+35, 24x^2-86x+74, 12x-28, 4x-8, 3x-7, 6x-14$	3
5	6	$12x^2-41x+31, 24x^2-82x+62, 12x-18, 4x-10, 3x-8, 6x-16$	3
5	6	$12x^2-31x+16, 24x^2-62x+32, 4x-6, 12x-22, 3x-5, 6x-6$	3
5	6	$12x^2-29x+14, 24x^2-58x+32, 4x-8, 3x-6, 12x-12, 6x-8$	3
5	6	$12x^2-49x+46, 24x^2-98x+92, 4x-10, 12x-28, 6x-13, 12x-22$	$48x^2-208x+217$
5	6	$12x^2-43x+35, 24x^2-86x+74, 4x-9, 3x-7, 6x-14, 12x-28$	$12x^2-39x+27$
5	6	$12x^2-43x+35, 24x^2-86x+74, 12x-28, 4x-8, 6x-14, 3x-8$	$3x^2-11x+8$
5	6	$12x^2-41x+31, 24x^2-82x+62, 12x-18, 4x-10, 3x-8, 6x-15$	$108x^2-486x+528$
5	6	$12x^2-41x+31, 24x^2-82x+62, 12x-18, 3x-7, 12x-28, 4x-8$	$96x^2-320x+250$
5	6	$24x^2-37x+10, 48x^2-74x+20, 6x-7, 24x-16, 8x-7, 12x-10$	$48x^2-74x+23$
5	8	$10x+20, 20x+42, 12x+24, 24x+50, 15x+32, 60x+132, 120x+264, 40x+85$	$1200x^2+5170x+5568$
5	8	$20x-35, 60x-96, 30x-48, 10x-17, 12x-22, 6x-11, 15x-22, 30x-39$	$3600x^2-10380x+7326$
5	8	$21x-7, 42x-16, 42x-10, 14x-7, 84x-36, 12x-6, 28x-11, 84x-30$	$24696x^2-18522x+3451$
5	8	$21x+3, 42x+6, 28x+4, 84x+13, 168x+32, 24x+4, 56x+8, 168x+26$	$8064x^2+3744x+406$
5	8	$30x+31, 60x+62, 36x+38, 180x+204, 20x+20, 18x+18, 45x+51, 90x+102$	$450x^2+1015x+572$
5	8	$35x-18, 140x-64, 60x-28, 420x-186, 84x-38, 42x-21, 105x-50, 70x-36$	$102900x^2-94570x+21714$
5	8	$35x-18, 140x-64, 60x-28, 420x-186, 84x-38, 42x-21, 70x-36, 210x-100$	$463050x^2-416745x+93702$
5	8	$35x-18, 140x-64, 60x-28, 420x-186, 84x-38, 42x-21, 105x-53, 210x-108$	$308700x^2-292530x+68908$
5	8	$40x+6, 120x+21, 48x+8, 60x+9, 160x+32, 96x+18, 240x+48, 480x+95$	$115200x^2+45360x+4462$
5	8	$42x+20, 126x+66, 252x+133, 36x+16, 28x+12, 252x+116, 63x+27, 126x+59$	$666792x^2+685314x+175497$
5	8	$45x-24, 90x-48, 72x-36, 360x-170, 720x-336, 144x-70, 240x-115, 720x-342$	$194400x^2-193590x+48081$
5	8	$12x^2-43x+35, 24x^2-86x+74, 6x-11, 4x-8, 3x-9, 2x-6, 12x-28, 6x-17$	$36x^2-144x+119$
5	8	$12x^2-43x+35, 24x^2-86x+74, 6x-11, 4x-8, 3x-9, 2x-6, 6x-16, 12x-29$	$144x^2-516x+406$
5	8	$12x^2-35x+22, 24x^2-70x+48, 12x-18, 4x-7, 12x-12, 2x-4, 3x-4, 6x-6$	$72x^2-186x+92$
5	8	$12x^2-35x+22, 24x^2-70x+48, 12x-18, 4x-7, 2x-4, 3x-4, 6x-6, 12x-13$	$288x^2-864x+598$
5	8	$12x^2-35x+22, 24x^2-70x+48, 12x-18, 12x-24, 12x-12, 6x-11, 4x-6, 6x-6$	$54x^2-81x+21$
5	8	$12x^2-35x+22, 24x^2-70x+48, 12x-18, 12x-24, 12x-12, 6x-12, 4x-6, 6x-6$	$6x^2-14x+4$
5	8	$12x^2-31x+16, 24x^2-62x+32, 4x-5, 6x-5, 12x-6, 12x-22, 3x-6, 2x-5$	$72x^2-210x+75$
5	8	$12x^2-29x+14, 24x^2-58x+32, 12x-12, 6x-10, 4x-9, 2x-4, 3x-4, 6x-5$	$144x^2-348x+54$
5	8	$12x^2-19x+4, 24x^2-38x+12, 4x-4, 3x-4, 18x-16, 9x-11, 12x-16, 18x-24$	$648x^2-1296x+598$
5	8	$12x^2+13x, 24x^2+26x+4, 12x, 6x+1, 12x+12, 6x+6, 4x+2, 12x+5$	$216x^2+450x+150$

CURRICULUM VITAE

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EDUCATION

Degree	Institution	Year of Graduation
B.S.	Industrial Engineering, ITU	2009
B.S.	Mathematics Engineering, ITU	2008
High School	Dede Korkut Anatolian High School, Istanbul	2003

Language skills

Turkish (native), English (fluently), German (basic)

Coding skills

C, C++, Perl, LaTeX

Work experience

September 2010 - Present Research Assistant, Dept of Mathematics, METU

Papers submitted to international journals

- B. Yildiz, *A Note On a Problem of Erdős and Graham*, 2016.