QUASI-CARTAN COMPANIONS OF ELLIPTIC CLUSTER ALGEBRAS

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ABSTRACT

QUASI-CARTAN COMPANIONS OF ELLIPTIC CLUSTER ALGEBRAS

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There is an analogy between combinatorial aspects of cluster algebras and diagrams corresponding to skew-symmetrizable matrices. In this thesis, we study quasi-Cartan companions of skew-symmetric matrices in the mutation-class of exceptional elliptic diagrams. In particular, we establish the existence of semipositive admissible quasi-Cartan companions for these matrices and exhibit some other invariant properties.

Keywords: Cluster Algebra, Quasi-Cartan Companion, Admissible Companion, Exceptional Elliptic Diagrams

ÖZ

ELİPTİK CLUSTER CEBİRLERİNİN QUASİ-CARTAN EŞLENİKLERİ

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Cluster cebirlerinin kombinatorik yönleriyle anti-simetrize edilebilir matrisler arasında benzerlik mevcuttur. Bu çalışmada istisnai eliptik diyagramların mutasyon sınıflarındaki quasi-Cartan eşleniklerini çalışıyoruz. Bilhassa, bu diyagramların mutasyon sınıflarındaki elemanların admissible quasi-Cartan eşlenikleri olduğunu gösteriyoruz; ayrıca diğer bazı değişmezlerini buluyoruz.

Anahtar Kelimeler: Cluster Cebiri, Quasi-Cartan Eşleniği, Admissible Eşlenik, İstisnai Eliptik Diyagramlar

To My Mother Sevim Velioğlu and Father İsmail Velioğlu

(For whom I have written this thesis without whom I was not able to write.)

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Before anything else, here, when I thank someone, it is far more than an appreciation showing ceremony of a single person, instead it is from a man who is grateful for happiness and relief of his parents when he completed the process of writing a thesis.

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CHAPTER 1

INTRODUCTION

Cluster algebras are certain commutative rings, whose generators and relations recursively inter-related. They were first introduced-invented by Sergey Fomin and Andrei Zelevinsky in [5]. Shortly after the cluster algebras introduced, it has been recognized that the subject is related to diverse areas of mathematics such as algebraic geometry, representation theory, Teichmüller spaces and combinatorics of various kind. In this thesis, we will study the last one of the examples: Combinatorial aspects of cluster algebras. There is a strong correlation between diagrams corresponding to skew-symmetrizable matrices and cluster algebras. Even though we will not apply the structure theory of cluster algebras directly, in order to point out this correlation, we will give some basic definitions and properties of cluster algebras and give some examples to illustrate this relation. When we are making the definitions, giving properties and exhibiting examples we will be closely following [7]. Let us begin with a brief description of Cluster Algebras.

1.1 Brief Description of a Cluster Algebra

A cluster algebra is a commutative ring with unity and without zero-divisors, endowed with a distinguished set of generators called *cluster variables*. These cluster variables could be finite or infinite. However, there are subsets of cluster variables of equal cardinality which are called *clusters* and the cardinality of a cluster is called the *rank* of the cluster algebra. Moreover, all of the clusters could be recursively derived from a cluster that is called the initial cluster.

1.2 Definition of a Seed

Let n be a positive integer and let $\mathbb{Q}(x_1, ..., x_n)$ be the field generated by the $x_1, ..., x_n$. A *seed* is a pair (B, u) where $B = (B_{ij})$ is a skew-symmetrizable matrix (see Chapter 2) such that the diagram $\Gamma(B)$ (see Chapter 2) without loops or 2-cycles and the vertices of $\Gamma(B)$ is labeled by the set $\{1, 2, ..., n\}$ and $u = \{u_1, u_2, ..., u_n\}$ is a set of elements in $\mathbb{Q}(x_1, ..., x_n)$ such that $\{u_1, u_2, ..., u_n\}$ generates $\mathbb{Q}(x_1, ..., x_n)$ freely.

1.3 Definition of Seed Mutation

Let (B, u) be a seed and let k be a vertex of $\Gamma(B)$, then mutation μ_k of (B, u) at k is defined to be another seed (B', u') where B' is the matrix mutation (see Chapter 2) of B and $u' = \{u'_1, ..., u'_n\}$ is the set of elements in $\mathbb{Q}(x_1, ..., x_n)$ such that u and u' satisfy a special relation below which is called the *exchange relation*:

$$u_k u'_k = \prod_i u_i^{[B_{ik}]_+} + \prod_i u_i^{[-B_{ik}]_+}$$

Note that for a diagram Γ of a skew-symmetric matrix $B = (B_{ij})$, i.e. $\Gamma(B) = \Gamma$, the exchange relation takes the form:

$$u_k u'_k = \prod_{edges: i \to k} u_i + \prod_{edges: k \to j} u_j$$

where product over empty set defined to be 1.

It could be checked easily that the mutation operation on seeds is involutive, i.e. $\mu_k^2(B, u) = (B, u)$.

1.4 Cluster Algebras and Diagrams

Now fixing a skew-symmetrizable matrix B, and $(B, \{x_1, ..., x_n\})$ being the initial seed, a cluster associated to B is a set $u = \{u_1, ..., u_n\}$ such that there is a seed $(\tilde{B}, u)$ obtained from te initial seed $(B, \{x_1, ..., x_n\})$ by a sequence of mutations. The union of all such elements u is called cluster variables. Then cluster algebra A_B corresponding to the matrix B is defined as $\mathbb{Q}$-subalgebra of $\mathbb{Q}(x_1, ..., x_n)$ generated by cluster variables. Now if $(B, \{x_1, ..., x_n\})$ and $(\tilde{B}, \{u_1, ..., u_n\})$ are as above then the natural isomorphism $\mathbb{Q}(x_1, ..., x_n) \simeq \mathbb{Q}(u_1, ..., u_n)$ induces a $\mathbb{Q}$-algebra isomorphism $A_B \simeq A_{\tilde{B}}$. This induced isomorphism preserves clusters and cluster variables since it is induced by the natural isomorphism $\mathbb{Q}(x_1, ..., x_n) \simeq \mathbb{Q}(u_1, ..., u_n)$ which sends x_i to u_i. Two diagrams Γ, Γ' (see Definition 2.2) are said to be mutation-equivalent if one is obtained from the other by a sequence of diagram mutations (see Definition 2.4). Now if $B, \tilde{B}$ are mutation-equivalent as matrices (the diagrams $\Gamma(B)$ and $\Gamma(\tilde{B})$ are also mutation equivalent.) then the cluster algebras A_B and $A_{\tilde{B}}$ are isomorphic by above construction. Therefore, studying mutation-classes (equivalence class of a diagram taking mutation equivalence as our equivalence relation) of diagrams and matrices; and putting efforts to find some algebraic invariants on their mutation-classes may give some direct results on cluster algebras or may help to understand cluster algebras.

1.5 An Example

Consider the $\mathbb{Q}$-algebra generated by variables x_i for $i \in \mathbb{Z}$, which satisfy the recursive relation which is called exchange relation:

$$x_{i-1}x_{i+1} = 1 + x_i$$

Let us consider the variables x_i for $i \geq 3$,

$$x_3 = \frac{1 + x_2}{x_1}$$

$$x_4 = \frac{1 + x_3}{x_2} = \frac{1 + x_1 + x_2}{x_1 x_2}$$

$$x_5 = \frac{1 + x_4}{x_3} = \frac{x_1 x_2 + x_1 + x_2 + 1}{x_1 x_2} \frac{x_1}{1 + x_2} = \frac{1 + x_1}{x_2}$$

$$x_6 = \frac{1 + x_5}{x_4} = \frac{x_1 + x_2 + 1}{x_2} \frac{x_1 x_2}{1 + x_1 + x_2} = x_1$$

$$x_7 = \frac{1 + x_6}{x_5} = (1 + x_1)\frac{x_2}{1 + x_1} = x_2$$

Therefore, x_i's are 5-periodic. Thus cluster variables are $\{x_1, x_2, x_3, x_4, x_5\}$ and clusters are $\{x_i, x_{i+1}\}$ for $i = 1, 2, 3, 4$. Now since clusters are of cardinality 2. Then the rank of this cluster algebra is 2.

1.6 Main Aim of the Thesis: The Combinatorial Aspect of Diagrams and Mutations

In this thesis we mainly study the combinatorial aspects of diagrams, more precisely we will try to establish an algebraic invariant on mutation-class of each of the elliptic diagrams $E_6^{(1,1)}$, $E_7^{(1,1)}$ and $E_8^{(1,1)}$ (see Figure 4.1). We will exhibit the existence of an admissible quasi-Cartan companion (see Definition 2.17) for any element in the mutation class of each of the elliptic diagrams $E_6^{(1,1)}$, $E_7^{(1,1)}$ and $E_8^{(1,1)}$. Moreover, these admissibility structure of quasi-Cartan companions change via the companion-mutation to stay admissible while the diagrams in the mutation-class are changing under the diagram mutation. Then we will investigate reflection group relations arising from the quasi-Cartan companions of skew-symmetric matrices in each mutation class. Finally, we study the quasi-Cartan companions in a mutation-class of a diagram corresponding to a triangulation of a triangulable surface and exhibit admissibility structure of the quasi-Cartan companions of elements in its mutation-class. Meanwhile, it is well-known that $E_6^{(1,1)}$, $E_7^{(1,1)}$ and $E_8^{(1,1)}$ are diagrams that are not originated from a triangulation of a surface. As a result, we will show that there are admissible quasi-Cartan companions of skew-symmetric matrices in mutation-classes of a diagram obtained from a triangulation of a surface (see Chapter 6) and diagrams that are not obtained from triangulation of a surface.

CHAPTER 2

BASIC DEFINITIONS

2.1 Skew-symmetrizable matrices and their diagrams

Firstly, we give some definitions and statements from [1, 6]. Throughout the thesis, a matrix will always be a square integer matrix without further description.

Definition 2.1 *Let $B = (B_{ij})$ be an $n \times n$ integer matrix. Then the matrix B is called skew-symmetrizable if there exists a diagonal matrix D whose entries are positive such that DB is skew-symmetric.*

We could characterize skew-symmetrizable matrices as follows [6, Lemma 7.4]: B is skew-symmetrizable if and only if B is sign-skew-symmetric (i.e. for any i, j either $B_{ij} = B_{ji} = 0$ or $B_{ij}B_{ji} < 0$) and for all $k \geq 3$ and all $i_1, \ldots, i_k$, it satisfies

$$B_{i_1 i_2} B_{i_2 i_3} \cdots B_{i_k i_1} = (-1)^k B_{i_2 i_1} B_{i_3 i_2} \cdots B_{i_1 i_k} \,. \tag{2.1}$$

The above characterization is employed in the following construction which connects skew-symmetrizable matrices and graphs [6, Definition 7.3]:

Definition 2.2 *Let n be a positive integer and let $I = \{1, 2, ..., n\}$. The* diagram *of a skew-symmetrizable (integer) matrix $B = (B_{ij})_{i,j \in I}$ is the weighted directed graph $\Gamma(B)$ with the vertex set I such that there is a directed edge from i to j if and only if $B_{ij} > 0$, and this edge is assigned the weight $|B_{ij}B_{ji}|$. By a* diagram*, we mean the diagram of a skew-symmetrizable matrix.*

Now let us recall some basic notions (notations and conventions) on graphs (or diagrams) :

1. For a diagram Γ, by abusing the notation we denote by the same symbol Γ the underlying undirected graph of a diagram.
2. We occasionally denote an edge between vertices i and j by $\{i, j\}$. If an edge $e = \{i, j\}$ has weight which is equal to 1, then it will not be specified in the picture.
3. If all edges of a diagram Γ have weight 1, then we call Γ *simply-laced*.
4. By a *subdiagram* of Γ, we always mean a diagram Γ' obtained from Γ by taking an induced (full) directed subgraph on a subset of vertices and keeping all its edge weights the same as in Γ.
5. A vertex v of a diagram Γ is called *source* (*sink*) if each edge incident to v is oriented away (towards) v.

The property (2.1) puts a condition on weights of graphs obtained from skew- symmetrizable matrices as in the above manner. To be more precise, let Γ be as in the definition: a cycle C in Γ is an induced (full) subgraph whose vertices can be labeled by $\{1, 2, ..., r\}, r \geq 3$, such that there is an edge between i and j if and only if $|i - j| = 1$ or $\{i, j\} = \{1, r\}$. If the weights of the edges in C are $w_1, w_2, ..., w_r$, then the product $w_1 w_2 ... w_r$ is a perfect square (i.e. square of an integer) by (2.1).

Example 2.3 *We will determine the diagrams corresponding to the matrices given below.*

$$A = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -2 \\ -1 & 2 & 0 \end{bmatrix}$$

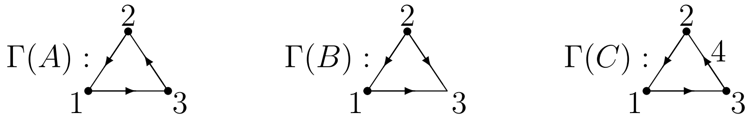


Figure 2.1: Diagrams corresponding to the matrices above

Definition 2.4 *For any vertex k in a diagram Γ, there is the associated mutation μ_k which changes Γ as follows:*

- *The orientations of all edges incident to k are reversed, their weights intact.*
- *For any vertices i and j which are connected in Γ via a two-edge oriented path going through k (see Figure 2.2), the direction of the edge $\{i, j\}$ in $\mu_k(\Gamma)$ and its weight c' are uniquely determined by the rule*

$$\pm\sqrt{c} \pm \sqrt{c'} = \sqrt{ab}\,, \tag{2.2}$$

  where the sign before $\sqrt{c}$ (resp., before $\sqrt{c'}$) is "$+$" if i, j, k form an oriented cycle in Γ (resp., in $\mu_k(\Gamma)$), and is "$-$" otherwise. Here either c or c' can be equal to 0, which means that the corresponding edge is absent.
- *The rest of the edges and their weights in Γ remain unchanged.*

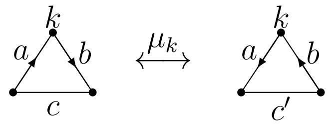


Figure 2.2: Diagram mutation

Definition 2.5 *Let $B = (B_{ij})$ be an $n \times n$ skew-symmetrizable matrix. Then $\mu_k(B)$ 'mutation of B at k' for $k = 1, ..., n$ is a matrix $B' = (B'_{ij})$ such that $B'_{ij} = -B_{ij}$ if $i = k$ or $j = k$; and $B'_{ij} = B_{ij} + \operatorname{sgn}(B_{ik})[B_{ik}B_{kj}]_+$ otherwise; where $[x]_+ = x$ if $x > 0$ and $[x]_+ = 0$ else.*

Basics of Mutation :

1. The operation is involutive, i.e. $\mu_k(\mu_k(\Gamma)) = \Gamma$, so it defines an equivalence relation on the set of all diagrams.
2. Two diagrams are called *mutation-equivalent* if they can be obtained from each other by applying a sequence of mutations.
3. The *mutation class* of a diagram Γ is the set of all diagrams which are mutation-equivalent to Γ.
4. If B is a skew-symmetrizable matrix, then $\Gamma(\mu_k(B)) = \mu_k(\Gamma(B))$.

An important class of diagrams that behave very nicely under mutations are finite type diagrams:

2.2 Diagrams of Skew-symmetrizable matrices of finite type

Definition 2.6 *A diagram* Γ *is said to be of* finite type *if for any diagram* Γ' *which is mutation-equivalent to* Γ*, weight of any edge of* Γ' *is equal to* $1, 2$ *or* 3*. A diagram is said to be of* infinite type *if it is not of finite type.*

Some Properties Of Finite Type Diagrams:

- Any subdiagram of a finite type diagram is also of finite type.
- A diagram which is mutation-equivalent to a diagram of finite type is of finite type.
- A diagram of finite type is of finite mutation type, i.e. its mutation class is finite.

Proposition 2.7 *Suppose that* Γ *is a diagram of finite type. Then any cycle in* Γ *is oriented.*

Proof. Firstly we will prove the proposition for a triangle (cycle of order 3). Suppose for contradiction there is a non-oriented triangle C in Γ. Let C have edges with weights a,b,c and assume without loss of generality the only vertex which is not a source or sink is the vertex say k at which edges with weights a and b are incident. Then mutating diagram at vertex k we obtain $c' = (\sqrt{a}\sqrt{b}+\sqrt{c})^2 \geq 4$ (since $(\sqrt{a}\sqrt{b}+\sqrt{c}) \geq 2$) which cannot be true since Γ is of finite type, hence any mutation equivalent diagram to Γ cannot have an edge of weight exceeding 3. Hence any triangle in Γ is oriented.

Now induction on number of vertices of the cycle will prove the proposition. Assume the claim of the proposition true for any cycle of order $n-1$ and let C be the cycle of order n. Now if the order is even, then there may be needed a mutation to obtain a vertex which is not a source or sink. Without loss of generality assume there is such a vertex say k. Mutating Γ at k and consider the subdiagram of C with the vertex k is omitted then we obtain a cycle of order $(n-1)$. Now by the induction assumption this new cycle is oriented. However, the newly arising edge in the diagram and original edges coming to the vertex and going out from vertex points the same orientation. Hence C itself is oriented.

Therefore, any cycle in a diagram Γ of finite type is oriented. □

Classification of diagrams of finite type carried out by Fomin and Zelevinsky in [6, Theorem 8.6]. We could state that result as:

Theorem 2.8 *A connected diagram is of finite type if and only if it is mutation-equivalent to an arbitrarily oriented Dynkin diagram (Fig. 2.4).*

2.3 Diagrams of Skew-symmetrizable matrices of finite mutation type

Definition 2.9 *A diagram Γ is said to be of finite mutation type (or mutation-finite) if there are only finitely many elements in the mutation class containing Γ.*

If Γ is a diagram of finite mutation type, then any diagram which is mutation-equivalent to Γ is also of finite mutation type by definition. Also any subdiagram of Γ is of finite mutation type,

Proposition 2.10 *A connected diagram Γ of order (i.e. the number of vertices) at least 3 is a diagram of finite mutation type if and only if the weight of any edge of any diagram in the mutation class of Γ is at most 4.*

Proof. If part is almost obvious. However, to be more precise, let Γ be a diagram of n vertices. Then number of diagrams in the mutation class of Γ is bounded by 2^{n^2} without considering weights (even if we allow any vertex can be connected to all of the vertices in the diagram including loops and 2-cycles). The number of edges in any diagram is bounded by n^2; again weights are not considered. Now the weight of any edge cannot exceed 4. Even if we allow for any edge to have all possible weights (0,1,2,3,4) then for any edge there are 5 possibilities. Hence 5^{n^2} bounds the number of edges with weights in a fixed element in the mutation class of Γ, so $2^{n^2}5^{n^2}$ bounds the number of elements in the mutation class of Γ. Therefore, Γ is mutation finite.

For the only if part we prove contrapositive of the statement. That is, if there is a diagram in the mutation class of Γ which has an edge whose weight exceeds 4 then mutation class of Γ has infinitely many elements. Without loss of generality assume that Γ has such an edge. Then we will show that for any three edges forming a triangle with weights $a \geq b \geq c \geq 0$ (we allow degenerate triangles, i.e. one of the edges of the triangle is missing) such that $b \geq 1$ and $a \geq 5$, there is a sequence of mutations (depending on orientation the length of this sequence will be 1 or 2) that increase sum of the weights of edges obtained via mutations. First assume the triangle is oriented. Then mutating Γ at the vertex where edges with weight a and b incident we obtain edges with weights a,b,$(\sqrt{ab} - \sqrt{c})^2$ sum of whose weights greater than the original triple since $(\sqrt{ab}-\sqrt{c})^2 = ab+c-2\sqrt{abc} > c$ since $2\sqrt{abc} = 2\sqrt{a}\sqrt{bc}$ and $2\sqrt{a} < a$ and $\sqrt{bc} \leq b$.

Let us now assume the triangle is not oriented. Then mutating at the source or sink if necessary, we may assume that there is a vertex such that the edge with weight a enters and the edge whose weight b goes out or the other way. Now we can make the mutation at such vertex to which the edges with weight a and b incident we obtain edges with weights $a, b, (\sqrt{ab}+\sqrt{c})^2$ whose sum obviously greater than $a+b+c$ which is the sum of the weights of original edges. Actually if we denote the resulting weight triple (a', b', c'), in the non-oriented case $(a', b', c') = (a, b, (\sqrt{ab} + \sqrt{c})^2)$ and in this case obviously $c' \geq b'$ and after the mutation we obtain an oriented triangle. Now apply the mutation at the vertex to which a' and c' are incident. Now if we can show the existence of an infinite sequence of mutations for the oriented case that increases total weight at each step we are done. But we know $(a', b', c') = (a, b, ab + c - 2\sqrt{abc})$ and we show $c' \geq b' = b$. For convenience, $c = 0$ case yields $c' = ab \geq b = b'$. Now suppose c is nonzero and $a \geq 9$ then $c' = ab + c - 2\sqrt{abc} = \sqrt{a}\sqrt{b}(\sqrt{a}\sqrt{b} - 2\sqrt{c}) + c \geq b' = b$ since $(\sqrt{a}\sqrt{b} - 2\sqrt{c}) \geq 1$. For the rest of the case, i.e. when $a \leq 8$ there are not too many choices for triple (a, b, c) since product of abc must be perfect square and the least value c' could get is 8 corresponding to the triple $(6, 3, 2)$ which is greater than or equal to a and hence $b' = b$. Now we have an oriented triangle to make the mutation at the vertex at which a' and c' incident will make the total weight greater and this process allow us to increase the total weight at each step of mutations. Therefore, the mutation class is infinite.

□

Lemma 2.11 *Let B be a skew-symmetric matrix and $\Gamma(B)$ be a diagram corresponding to B. Let $\Gamma' = \mu_k(\Gamma(B))$. Then $B' = \mu_k(B)$ is skew-symmetric such that $\Gamma' = \Gamma(B')$. Moreover, the weight of any edge of $\Gamma(B)$ where B a skew-symmetric matrix is perfect square.*

Proof. Let $B' = \mu_k(B)$. Consider the triangle $\{i, j, k\}$. First suppose it is non-oriented. If k is a source or sink in the triangle then $B'_{ij} = B_{ij}$, $B'_{ji} = B_{ji}$, $B'_{ik} = -B_{ik}$, $B'_{ki} = -B_{ki}$, $B'_{kj} = -B_{kj}$, $B'_{jk} = -B_{jk}$. Hence $B' = \mu_k(B)$ is skew-symmetric. Now suppose k is not a source or sink. Then, without loss of generality, we may assume B_{ik}, B_{kj} both positive. Only thing we have to worry is B'_{ij} and B'_{ji}. However, $B'_{ij} = B_{ij} + B_{ik}B_{kj}$ and

$B'_{ji} = B_{ji} - B_{ki}B_{jk} = -B_{ij} - B_{ik}B_{kj}$ since B is skew-symmetric.

Thus $B'_{ij} = -B'_{ji}$.

Now suppose the triangle is oriented. Then, without loss of generality, we may assume B_{ik}, B_{kj} both positive. Then $B'_{ij} = B_{ij} + B_{ik}B_{kj}$ and $B'_{ji} = B_{ji} - B_{ki}B_{jk} = -B_{ij} - B_{ik}B_{kj}$ since B is skew symmetric. Thus $B'_{ij} = -B'_{ji}$.

Last part is almost obvious. The weight of the edge $\{i, j\}$ is $|B_{ij}B_{ji}|$ and $B_{ij} = -B_{ji}$ by B being skew symmetric. Hence the weight of the edge $\{i, j\}$ is $(B_{ij})^2$.

□

2.4 Quasi-Cartan companions of skew-symmetrizable matrices

A description of finite type diagrams was obtained in [1] employing the notion of "quasi-Cartan matrices", which we will use in this thesis to describe the mutation classes of diagrams:

Definition 2.12 *Let A be an $n \times n$ integer matrix. Then the matrix A is called symmetrizable if there exists a diagonal matrix D whose entries are positive such that DA is symmetric. We say that A is a quasi-Cartan matrix if it is symmetrizable and all of its diagonal entries are equal to* 2*.*

The symmetrizable matrix A is sign-symmetric, i.e. $\operatorname{sgn}(A_{i,j}) = \operatorname{sgn}(A_{j,i})$. We say that A is (semi)positive if DA is positive (semi)definite, i.e. (resp. $x^T DAx \geq 0$) $x^T DAx > 0$ for all $x \neq 0$ (here x^T is the transpose of x which is a vector viewed as a column matrix). We say that u is a *radical* vector of A if $Au = 0$; we call u *sincere* if all of its coordinates are non-zero. We call A *indefinite* if it is not semipositive. A quasi-Cartan matrix is a *generalized Cartan matrix* if all of its non-zero off-diagonal entries are negative.

We are going to benefit from the following equivalence relation on quasi-Cartan matrices :

Definition 2.13 *Quasi-Cartan matrices A and A' are called* equivalent *if they have the same symmetrizer D, i.e D is a diagonal matrix whose entries are positive such that both $C = DA$ and $C' = DA'$ are symmetric, and $C' = E^T CE$ for some integer matrix E with determinant* ± 1.

A characteristic example of the equivalence for quasi-Cartan matrices is the matrices obtained via the sign change operation: To be more precise, the '*sign change* at (vertex) k' replaces A by A' obtained by multiplying the k-th row and column of A by -1.

Quasi-Cartan matrices and skew-symmetrizable matrices are inter-related via the following definition:

Definition 2.14 *Let B be a skew-symmetrizable matrix. A* quasi-Cartan companion *of B is a quasi-Cartan matrix A satisfying $|A_{ij}| = |B_{ij}|$ for all $i \neq j$. Also we say that A is a quasi-Cartan companion of a diagram Γ if it is a companion for a skew-symmetrizable matrix B whose diagram is equal to Γ.*

Example 2.15 *Let*

$$B = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

Then there are more than 1 quasi-Cartan companion for B. We exhibit two quasi-Cartan companions for B to illustrate:

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

$$A' = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

Here A and A' are quasi-Cartan companions of the matrix B. For future reference, note that two quasi-Cartan companions of the the same matrix need not be equivalent as in this case. In our case A is invertible but A' is not, thus they could not be equivalent companions.

Definition 2.16 The restriction of the companion A of a diagram Γ to a subdiagram Γ' *is the quasi Cartan matrix obtained from* A *by removing the rows and columns corresponding to the vertices of* Γ *which are not in* Γ'.

If B is skew-symmetric, then any quasi-Cartan companion of it is symmetric.

Let us note that for a diagram Γ, we may view a quasi-Cartan companion A as a sign assignment to the edges (of the underlying undirected graph) of Γ; more explicitly an edge $\{i, j\}$ is assigned the sign of the entry A_{ij} (which is the same as the sign of A_{ji} because A is sign-symmetric).

Now we will recall the notion 'admissibility'.

Definition 2.17 *[9, Definition 2.10] Suppose that* B *is a skew-symmetrizable matrix and let* A *be a quasi-Cartan companion of* B*. Then* A *is called* admissible *if it satisfies the following sign condition: for any cycle* C *in* Γ*, the product* $\prod_{\{i,j\}\in C}(-A_{ij})$ *over all edges of* C *is negative if* C *is oriented and positive if* C *is non-oriented.*

Basic Properties: The sign condition in the definition above could also be understood as follows: if C is (non)oriented, then there is exactly an (resp. even) odd number of edges $\{i, j\}$ such that $(A_{ij}) > 0$(recall that, since A is symmetrizable, we have $\mathrm{sgn}(A_{ij}) = \mathrm{sgn}(A_{ji})$). In this case we sometimes say A assigns odd(or even) $(+)$ to edges of C. Therefore, an admissible quasi-Cartan companion distinguishes the oriented cycles from non-oriented cycles in a diagram. Note also that A is admissible if and only if its restriction to any cycle is admissible. Hence if we restrict an admissible companion to a subdiagram then it stays admissible. We also note that changing sign at a vertex does not violate admissibility.

In general, for a diagram Γ, an admissible quasi-Cartan companion may not exist. It necesserily exists if Γ has *no* non-oriented cycles [1, Corollary 5.2]. Now the following theorem shows that if an admissible companion exists, then it will be unique up to sign changes.

Thanks to following theorem, we will have the uniqueness of admissible companions up to sign changes. For the proof of the Theorem 2.18 see [9, Theorem 2.11]

Theorem 2.18 *Let B be a skew-symmetrizable matrix. Suppose A and A' are admissible quasi-Cartan companions of B. Then A and A' could be recovered from each other by a sequence of simultaneous sign changes in rows and columns. In particular, A and A' are equivalent .*

Example 2.19 *Let*

$$B = \begin{bmatrix} 0 & 1 & -1 & 0 \\ -1 & 0 & 1 & -1 \\ 1 & -1 & 0 & 1 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & 1 & -1 \\ -1 & 1 & 2 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix}$$

$$A' = \begin{bmatrix} 2 & 1 & -1 & 0 \\ 1 & 2 & -1 & -1 \\ -1 & -1 & 2 & 1 \\ 0 & -1 & 1 & 2 \end{bmatrix}$$

Now A and A' are admissible quasi-Cartan companions of the same matrix B and one can be obtained from the other only by sequence of sign changes. In our case A' could be obtained from A by changing signs of first and third columns and rows and this corresponds changing signs at vertices 1 and 3 of $\Gamma(B)$.

The one of the main tools in this thesis will be the following mutation operation on quasi-Cartan companions which will be defined in complete analogy with the mutation operation on skew-symmetrizable matrices:

Definition 2.20 *Let Γ be a diagram and suppose A be a quasi-Cartan companion of Γ. Let k be a vertex in Γ. 'The mutation of A at k' is the quasi-Cartan matrix A' such that for any $i, j \neq k$: $A'_{ik} = \operatorname{sgn}(B_{ik})A_{ik}$, $A'_{kj} = -\operatorname{sgn}(B_{kj})A_{kj}$, $A'_{ij} = A_{ij} - \operatorname{sgn}(A_{ik}A_{kj})[B_{ik}B_{kj}]_+$. The quasi-Cartan matrix A' is equivalent to A. It is a quasi-Cartan companion of $\mu_k(\Gamma)$ if A is admissible [1, Proposition 3.2].*

Example 2.21 *Let $\Gamma = \Gamma(B)$ for,*

$$B = \begin{bmatrix} 0 & 2 & -1 \\ -1 & 0 & 1 \\ 0 & -2 & 1 \end{bmatrix}$$

Then,

$$A = \begin{bmatrix} 2 & -2 & 1 \\ -1 & 2 & -1 \\ 1 & -2 & 2 \end{bmatrix}$$

is an admissible quasi-Cartan companion for B. Then mutating B and A at the vertex 2 we will obtain, the matrices,

$$\mu_2(B) = \begin{bmatrix} 0 & -2 & 1 \\ 1 & 0 & -1 \\ -1 & 2 & 0 \end{bmatrix}$$

$$\mu_2(A) = \begin{bmatrix} 2 & -2 & -1 \\ -1 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix}$$

Thus $\mu_2(A)$ is a quasi-Cartan companion for $\mu_2(B)$.

However, instead of taking the above A as the quasi-Cartan companion which is admissible let us take a non-admissible companion A' s.t,

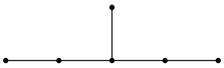
$$A' = \begin{bmatrix} 2 & -2 & -1 \\ -1 & 2 & -1 \\ -1 & -2 & -2 \end{bmatrix}$$

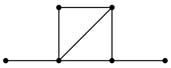
Then $\mu_2(B)$ is the same matrix but let us consider $\mu_2(A')$ where,

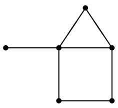
$$\mu_2(A') = \begin{bmatrix} 2 & -2 & -3 \\ -1 & 2 & 1 \\ -3 & 2 & 2 \end{bmatrix}$$

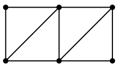
Now $\mu_2(A')$ is not a quasi-Cartan companion for $\mu_2(B)$.

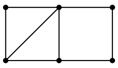
Example 2.22 *Even if A is admissible $\mu_k(A)$ may not be admissible : e.g. if A is an admissible quasi-Cartan companion of the diagram 'The Ears' in Figure 4.2 and k is the vertex x_3, then $\mu_k(A)$ is not admissible.*

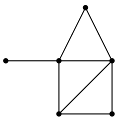
(1) 

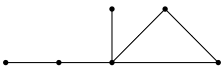
(2) 

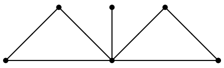
(3) 

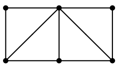
(4) 

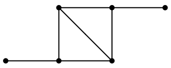
(5) 

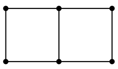
(6) 

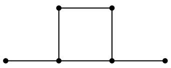
(7) 

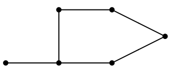
(8) 

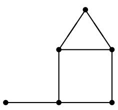
(9) 

(10) 

(11) 

(12) 

(13) 

(14) 

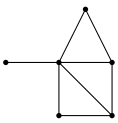
(15) 

Figure 2.3: Diagrams in the mutation class of E_6. Each cycle in the diagrams is oriented.

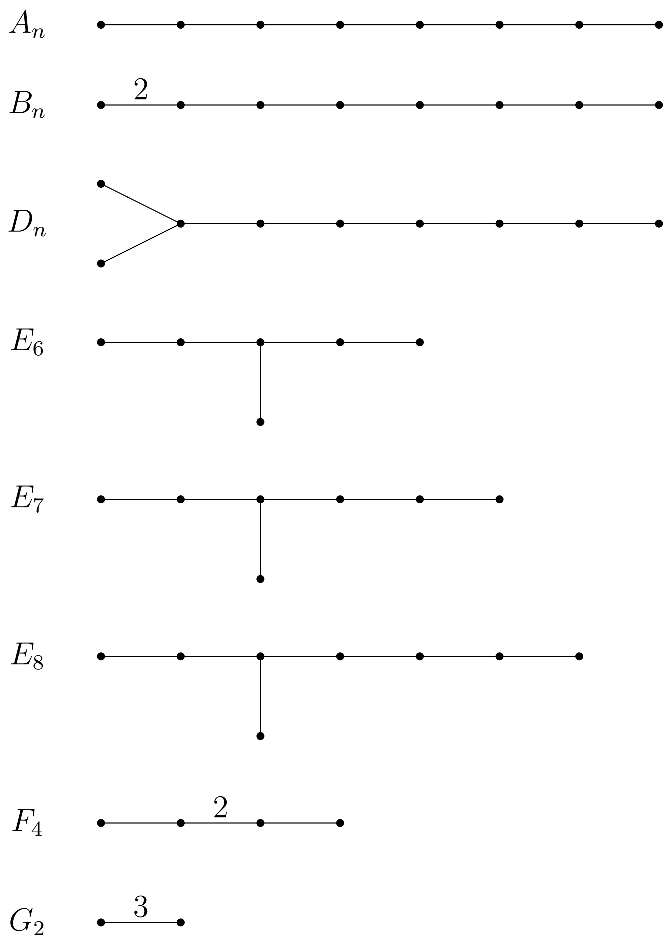


Figure 2.4: Dynkin diagrams are arbitrary orientations of the Dynkin graphs given above; all orientations of the same Dynkin graph are mutation-equivalent to each other (this definition of a Dynkin diagram has been introduced in [6].

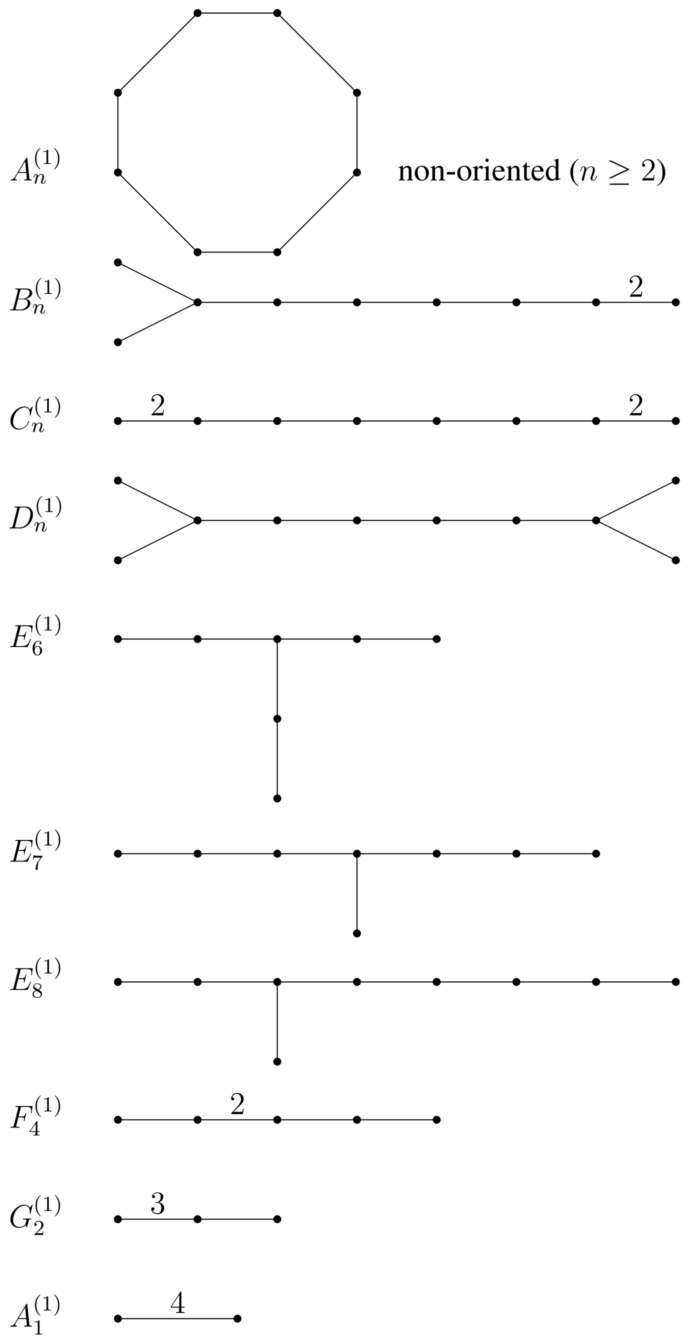


Figure 2.5: Extended Dynkin diagrams are orientations of the extended Dynkin graphs given above; the graphs apart from $A_n^{(1)}$ are assumed to be arbitrarily oriented; each $X_n^{(1)}$ has $n+1$ vertices

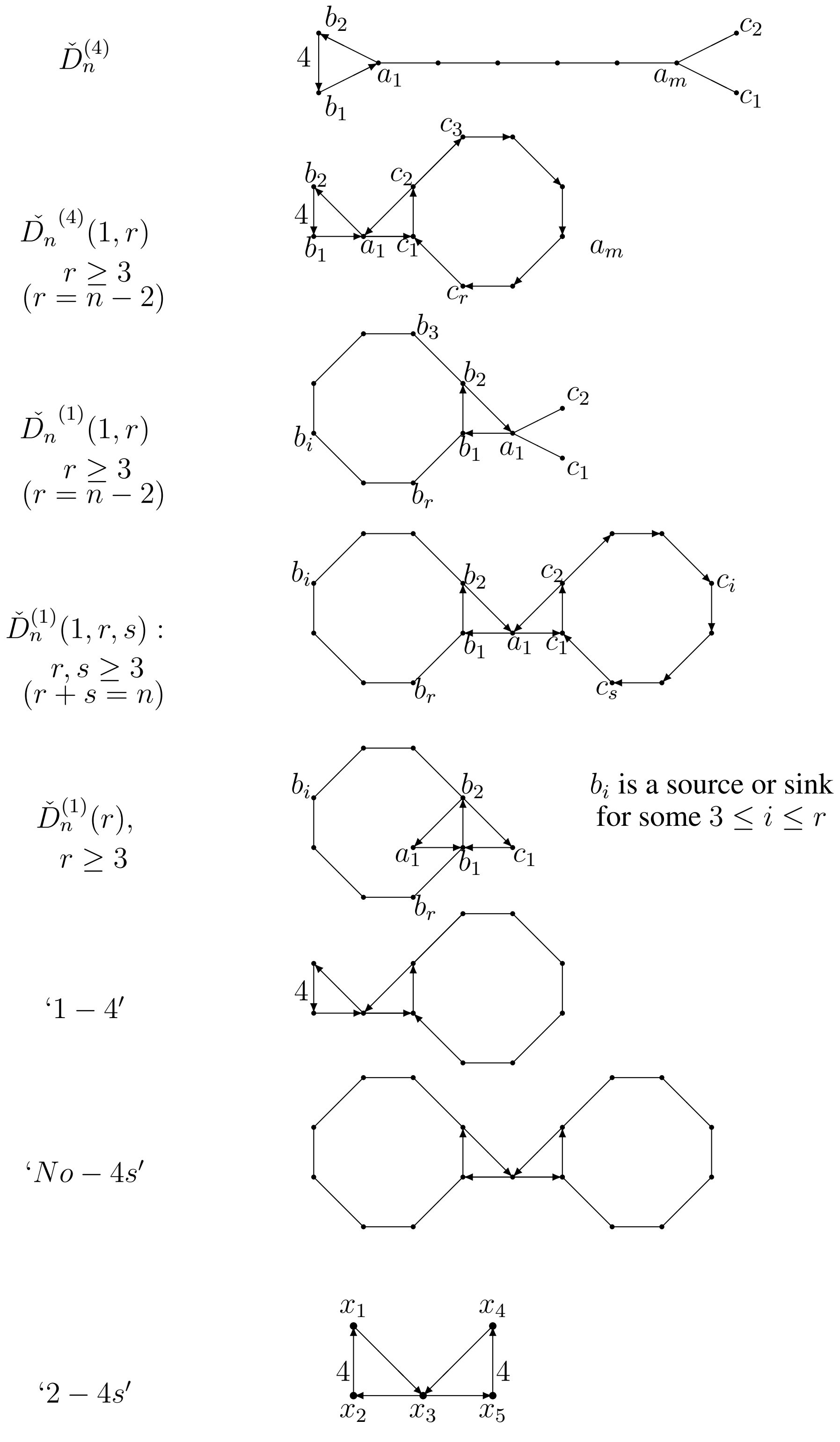


Figure 2.6: Diagrams that occur in the proof of Lemma 4.5. Note that cycles without a specified orientation are assumed to be non-oriented.

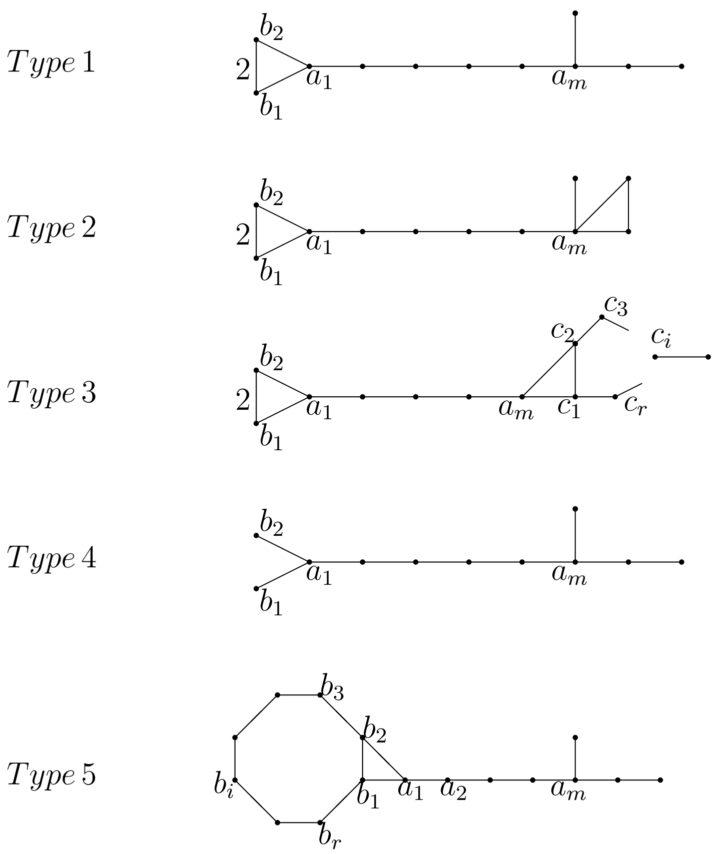


Figure 2.7: Some diagrams of infinite mutation type; each edge can be taken to be arbitrarily oriented. (Obtained in [8])

CHAPTER 3

PROPERTIES OF SEMIPOSITIVE QUASI-CARTAN COMPANIONS

In this chapter we will exhibit some properties of semipositive quasi-Cartan companions. Their most basic property that we will use is the following:

Proposition 3.1 *[9, Proposition 4.1] Suppose that A is a semipositive quasi-Cartan companion of a diagram Γ. Suppose also that u is a radical vector for the restriction of A to a subdiagram Σ, i.e. u is in the span of the standard basis vectors which correspond to the vertices in Σ and $x^T Au = 0$ for all x in the same span. Then u is a radical vector for A as well (i.e. $x^T Au = 0$ for* all *x).*

For the proof of above proposition see [9, Proposition 4.1]

Let us exhibit some other properties of semipositive quasi-Cartan companions by the following proposition:

Proposition 3.2 *Let Γ be a diagram and A be a semipositive quasi-Cartan companion of Γ. Then we have the following:*

(i) The weight of any edge of Γ cannot exceed 4.

(ii) The restriction of A to any edge of Γ of weight 4 is non-positive.

(iii) If e is any edge whose weight is 4, then any connected three-vertex diagram that contains e is a triangle whose edge weights are either $4, 1, 1$ or $4, 4, 4$ or $4, 2, 2$ or $4, 3, 3$.

(iv) If C is a non-simply-laced cycle, then the product $\prod_{\{i,j\}\in C}(-A_{ij})$ over all edges of C is negative (hence A assigns $(+)$ to odd number of edges of C).

(v) Suppose C is a simply-laced cycle such that for each edge of C the corresponding entry of A is -1. Let u be the vector whose coordinates are 1 in the vertices of C and 0 in the remaining vertices. Then u is a radical vector for A.

(vi) Suppose that C is a simply-laced cycle such that the product $\prod_{\{i,j\}\in C}(-A_{ij})$ over all edges of C is positive(i.e A assigns even number of $(+)$). If a vertex k is connected to C, then it is connected to at least two vertices in C.

(vii) Suppose that Γ is simply-laced and let C be a cycle in Γ such that the product $\prod_{\{i,j\}\in C}(-A_{ij})$ over all edges of C is positive. If a vertex is connected to C, then it is connected to exactly an even number of vertices in C.

(viii) Suppose that {i,j} is an edge of Γ of weight 4 and suppose $A_{ij} = A_{ji} = -2$ then the vector u whose i^{th} and j^{th} coordinate are 1 and the rest of the coordinates are zero, is a radical vector for A.

Proof. Let Γ be a diagram and let A be a quasi-Cartan companion of Γ that is semipositive.

(i): Assume Γ contains an edge whose weight exceeds 4. Then, without loss of generality, we may assume $\{1, 2\}$ is such an edge (i.e. $(A_{12})(A_{21}) \geq 5$). Then let D be the symmetrizer matrix for A whose first two diagonal entries are d_1 and d_2. Also let $C = DA$ and $C_{1,2}$ denote 2×2 principal minor corresponding to the vertices 1 and 2. Then $det(C_{1,2}) = d_1 d_2(4 - (A_{12})(A_{21})) < 0$ since $(A_{12})(A_{21}) \geq 5$. Thus, C could not be positive semi-definite and so A could not be semipositive by definition, which contradicts to semipositivity of A. Therefore, our assumption that Γ contains an edge whose weight exceeds 4 is wrong.

(ii): Assume Γ contains an edge whose weight equals to 4. Then, without loss of generality, we may assume $\{1, 2\}$ is such an edge (i.e. $(A_{12})(A_{21}) = 4$). Then let D be the symmetrizer matrix for A whose first two diagonal entries are d_1 and d_2. Also let $C = DA$ and $C_{1,2}$ denote 2×2 principal minor corresponding to the vertices 1 and 2. Now $det(C_{12}) = d_1 d_2(4 - (A_{12})(A_{21})) = 0$ since $(A_{12})(A_{21}) = 4$. Thus, C

could not be positive definite and so restriction of A to $1, 2$ whose weight 4 could not be positive by definition.

(iii): Let e be an edge of weight 4. We may assume without loss of generality that e is the edge corresponding to A_{12} and A_{21}. Also let us assume without loss of generality that the third vertex which forms a connected 3-vertex diagram together with 1 or 2, is the vertex 3(which corresponds 3rd column and row of A). Then, perfect square condition forces the edge weights to the following triples: $(4, 1, 0)$, $(4, 2, 0)$, $(4, 3, 0)$, $(4, 4, 0)$ which represent non-triangle connected 3-vertex digrams and $(4, 4, 1)$, $(4, 1, 1)$, $(4, 2, 2)$, $(4, 3, 3)$, $(4, 4, 4)$ which represent triangles with corresponding weights determined by the triples. Now let D be the symmetrizer matrix for A whose first two diagonal entries are d_1 , d_2 and d_3. Also let $C = DA$ and $C_{1,2,3}$ denote 3×3 principal minor corresponding to the entries d_1 , d_2 and d_3. Then,

$$det(C_{1,2,3}) = d_1d_2d_3[(4-A_{12}A_{21})+(A_{12}A_{23}A_{31}+A_{21}A_{32}A_{13}-2A_{23}A_{32}-2A_{13}A_{31})]$$

and since $A_{12}A_{21} = 4$ we have,

$$det(C_{1,2,3}) = d_1d_2d_3[(A_{12}A_{23}A_{31} + A_{21}A_{32}A_{13} - 2A_{23}A_{32} - 2A_{13}A_{31})]$$

Here we note that if entries corresponding to a pair of vertices are zero then this means exactly that one of $A_{13} = A_{31}$ and $A_{23} = A_{32}$ is 0. Then in either case $det(C_{1,2,3}) < 0$ contradicting semipositivity of A. Therefore, $(4, 1, 0)$, $(4, 2, 0)$, $(4, 3, 0)$, $(4, 4, 0)$ cannot occur as edge weights. Now for the case $(4, 4, 1)$, without loss of generality, we may asssume $A_{23}A_{32} = 4$ and $A_{13}A_{31} = 1$ and symmetrizability condition forces $A_{12}A_{23}A_{31} = A_{21}A_{32}A_{13} = \mp 4$. Now if $A_{12}A_{23}A_{31} = A_{21}A_{32}A_{13} = -4$ then $det(C_{1,2,3}) = -18d_1d_2d_3$, and if $A_{12}A_{23}A_{31} = A_{21}A_{32}A_{13} = 4$ then $det(C_{1,2,3}) = -2d_1d_2d_3$. In both cases $det(C_{1,2,3})$ is negative and so A cannot be semipositive. Therefore, $(4, 4, 1)$ cannot occur as edge weight triple. For the rest of the triples, there are always a sign choice for the entries of A as we are going to show now. First we note that determinants of all principal minors of $C_{1,2,3}$ of length 1 are positive being equal $2d_i$ for $i = 1, 2, 3$. Also determinants of the principal minors of length 2 of $C_{1,2,3}$ are equal to $d_id_j(4 - A_{ij}A_{ji})$ for $i \neq j$ and $i, j = 1, 2, 3$ and $A_{ij}A_{ji} \leq 4$ shows $det(C_{i,j}) \geq 0$. Rest is to arrange (if possible) sign of the entries to guarantee $det(C_{1,2,3}) \geq 0$. Now for $(4, 1, 1)$ symmetrizability condition forces $A_{12}A_{23}A_{31} = A_{21}A_{32}A_{13} = \pm 2$. Now choosing each entry positive makes

$A_{12}A_{23}A_{31} = A_{21}A_{32}A_{13} = 2$ and hence $det(C_{1,2,3}) = 0$ which does not violate semipositivity. Therefore, $(4,1,1)$ may occur. Now for $(4,2,2)$ symmetrizability condition forces $A_{12}A_{23}A_{31} = A_{21}A_{32}A_{13} = \pm 4$. Now choosing each entry positive makes $A_{12}A_{23}A_{31} = A_{21}A_{32}A_{13} = 4$ and hence $det(C_{1,2,3}) = 0$. Therefore, $(4,2,2)$ may occur. For $(4,3,3)$ and $(4,4,4)$ by the same token $det(C_{1,2,3}) = 0$.

(iv): Let C be a non-simply-laced cycle and assume $\prod_{\{i,j\}\in C}(-A_{ij})$ over all edges of C positive(i.e. there are even number of edges assigned $(+)$ by A). Now by changing signs at vertices we may further assume all edges assigned $(-)$ by A (see proof of part (vi) of this proposition). Actually this process gives a matrix equivalent to A but two equivalent companions are both semipositive or neither of them semipositive. Hence there is no harm to assume A assigns all edges $(-)$ (see proof of part (vi) of this proposition). Now assume without loss of generality that $A_{12} \leq -2$, and let $D = (d_1, .., d_n)$ be a symmetrizer for A and set $C = DA$ and let $x = (1, 1, .., 1)$. Now consider, $x^T Cx = d_1(2 + A_{12} + A_{1n}) + d_2(2 + A_{21} + A_{23}) + ... + d_i(2 + A_{i(i-1)} + A_{i(i+1)}) + ... + d_n(2 + A_{n1} + A_{n(n-1)}) < 0$ since $2 + A_{12} + A_{1n} < 0$ and rest of the expressions in other parantheses are less than or equal to zero by $A_{12} \leq -2$ and rest of the nonzero entries of A less than or equal to -1. This contradicts semipositiveness of A hence the product $\prod_{\{i,j\}\in C}(-A_{ij})$ over all edges of C must be negative.

(v): Let C be simply-laced cycle of length n such that for each edge corresponding entry of A is -1. We may assume without loss of generality that the cycle C is the subdiagram of Γ with vertices $i = 1, 2, ..., n$. By abusing the notation we let A to denote restriction of A to the cycle C for the time being, it means the diagonal entries are 2 ; $A_{i(i+1)} = Ai(i-1) = -1$ for $i = 2, .., n$ and $A_{12} = A_{n(n-1)} = A_{1n} = A_{n1} = -1$ and rest of the entries are 0. Then $Ax = 0$ for $x = (x_1, .., x_n)$ s.t. $x_i = 1$ for $i = 1, .., n$ since entries of any row of A add up to 0. Hence x is a radical vector for the restriction matrix of A to the cycle C . Now let $u = (u_1, ..., u_m)$ s.t $u_i = 1$ for $i = 1, .., n$ and $u_i = 0$ for $i > n$. Now by Proposition 3.1, such u is a radical vector for A.

(vi): Let C be simply-laced cycle of length n such that $\prod_{\{i,j\}\in C}(-A_{ij})$ over all edges of C positive(i.e. there are even number of edges assigned $(+)$ by A). Now changing signs of some vertices(i.e. multiplying corresponding rows and columns of A by

-1 or equivalently changing the signs of the entries corresponding edges that are incident on the vertices) if necessary we may assume C is as in the part (v). We could do this operation and get a matrix as in (v) since A assigns even number of $(+)$ signs to C. Indeed,we first make sign changes at vertices which are incident to edges both assigned $(+)$ by A. Now each sign change turns 2 $(+)$ signed edges into 2 $(-)$ signed edges. Hence number of positive signed edges are again even. Now all positive signed edges are incident to a vertex whose other incident edge is negative. Now if any positive signed edge is left, say there are one incident to vertex k. Now change sign of vertices starting from k consecutively for adjacent vertices up to the vertex say l where 2 edges incident to l are positive. Then make the sign change at l makes them negative and hence we have made two originally $(+)$ assigned edges negative and the other edges that are negative between k and l stays negative after 2 sign changes for each. If necessary, apply this process to any other such vertices. This process kills two positive edges and creates two negative edges after each process hence at the end all edges assigned $(-)$. The new matrix A' obtained via sign changes at vertices is equivalent to original companion A. Thus, without loss of generality we may assume $A' = A$. Now if a vertex $k > n$ in Γ is connected to exactly one vertex of C. Say that vertex is j in C.Therefore, $A_{kj} \neq 0$. Then the first n entries of k^{th} row of A is zero but the A_{kj}. Now letting u be as in part (v). Then k^{th} component of Au is nonzero being equal to $A_{kj} \neq 0$ contradicting u being radical vector for A. Therefore, if a vertex is connected to C then it must be connected to at least two of the vertices of C.

(vii): Let Γ be simply-laced and Let C be as in part (vi). Again if necessary we can make sign changes to assume C, A and u are as in part (v). Assume that a vertex $k > n$ is conected to an odd number say m of vertices of C then there are m nonzero entries in the first n entries of the k^{th} row of A and they are all 1 or -1 since Γ is simply-laced. Then if the number of positive entries even(odd) then number of negative entries is odd(even) since number of nonzero entries is odd. Thus k^{th} component of Au is nonzero contradicting u being a radical vector for A. Therefore, if a vertex is connected to C then it must be connected to even number of vertices of C.

(viii): Suppose {i,j} is such an edge. Then restriction of A to {i,j} is 2×2 matrix whose diagonal entries are 2 and off-diagonal entries are -2. Then two dimensional vector with coordinates 1 is a radical vector for the restriction and hence by Proposition 4.1. the vector whose i^{th} and j^{th} coordinate are 1 and the rest is zero, is a radical vector for A.

□

Next we exhibit some properties of semipositive admissible quasi-Cartan companions:

Proposition 3.3 *Let Γ be a diagram. Suppose that A is a semipositive admissible quasi-Cartan companion. Then we have the following:*

(i) If e is an edge of weight 4, then any three-vertex subdiagram that contains e is an oriented *triangle (see also part (iii) in the above proposition).*

(ii) Any non-oriented cycle C is simply-laced. Moreover, the restriction of A to C is not positive.

(iii) Any diagram in Figure 2.5 has an admissible quasi-Cartan companion of corank 1 with a sincere radical vector.

Proof. Let Γ be a diagram and suppose that A is a semipositive admissible quasi-Cartan companion.

(i): Let e be an edge of weight 4. Then by (iii) of Proposition 3.2 any three vertex diagram containing e is a triangle. Now by part (iv) of the same proposition odd number of edges must be assigned $(+)$ by A. Now since A is admissible the triangle must be oriented by the definition of admissibility.

(ii) Let C be a non-oriented cycle. By admissibility of A; even number of edges of C must be assigned $(+)$ by A. If C were non-simply laced, odd number of edges of C would be assigned positive by A by Proposition 3.2 part (iv) which would be a contradiction. Thus, C must be simply laced. Now since C is simply laced and even number of edges assigned $(+)$ by A and so by part (v) of the Proposition 3.2 (Also see part (vi) of the same proposition to see how A transforms into matrix that satisfies conditions in part (v)) there is a radical vector for the matrix A restricted to C hence the restriction of A to C could not be positive.

(iv): We will exhibit sample companion matrices of diagrams with arbitrary number of vertices for $n = 7$ which will be of general certainty. For each case, the vector u in the very next to that matrix will be its sincere radical vector.

$$A_7^{(1)} = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix} \qquad u = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$B_7^{(1)} = \begin{bmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & 2 \end{bmatrix} \qquad u = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \end{bmatrix}$$

$$C_7^{(1)} = \begin{bmatrix} 2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & 2 \end{bmatrix} \qquad u = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$D_7^{(1)} = \begin{bmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 2 \end{bmatrix} \qquad u = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \\ 2 \\ 2 \\ 1 \\ 1 \end{bmatrix}$$

$$E_6^{(1)} = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix} \qquad u = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 2 \\ 1 \\ 2 \\ 1 \end{bmatrix}$$

$$E_7^{(1)} = \begin{bmatrix} 2 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix} \qquad u = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 2 \\ 3 \\ 2 \\ 1 \end{bmatrix}$$

$$E_8^{(1)} = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix} \qquad u = \begin{bmatrix} 2 \\ 4 \\ 6 \\ 3 \\ 5 \\ 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}$$

$$F_4^{(1)} = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -2 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{bmatrix} \qquad u = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 2 \\ 1 \end{bmatrix}$$

$$G_2^{(1)} = \begin{bmatrix} 2 & -3 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \qquad u = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

Now to see the admissible quasi-Cartan companions of the diagrams in Figure 2.5 are of corank 1 it is enough to observe that when we are to construct a radical for vector for each, one row of that matrix such that only two nonzero entries exist, forces two entries of the radical vector to be certain integers or the multiples of them by same integer. Once we fix such initial data, all other entries are forced to take only one value. Therefore, there could not exist more than one radical vector for each. On the other hand we have exhibited a radical vector for each. Therefore, there are one and only one radical vector exists for each and thus the companions are of corank 1.

□

Lemma 3.4 *Let A be a quasi-Cartan companion of a diagram Γ and let u be a radi-*

cal vector for A. Also let A' be the matrix obtained by a sequence of sign changes at some set of vertices of Γ(i.e. multiplying columns and rows of A by -1 corresponding to the set of vertices) and let u' be the vector obtained by making the sign changes at coordinates corresponding to the same set of vertices. Then u' is a radical vector for A'.

Proof. Let A be a quasi-Cartan companion of some diagram Γ and let u be a radical vector for A. Let $V = \{i_1, .., i_m\}$ be the set of vertices at which we will make sign changes and obtain A'. Then $A'_{ij} = A_{ij}$ if $i \notin V$ and $j \notin V$, or i and $j \in V$. Also $A'_{ij} = -A_{ij}$ if exactly one of i and j is in V . Furthermore, we have $u'_i = u_i$ if $i \notin V$ and $u'_i = -u_i$ if $i \in V$. Now first let $k \in V$. Then the k^{th} row of the A' is the same as A except for the columns that appearing in V being minus of that of A. Thus, $(A'u')_k = (Au)_k = 0$. Now let $k \in V$. Then $A'_{kj} = A_{kj}$ if $j \in V$ and $A'_{kj} = -A_{kj}$ if $j \notin V$. Thus $(A'u')_k = (Au)_k = 0$. Thus, $A'u' = 0$. Therefore, u' is a radical vector for A'.

□

Lemma 3.5 *Let A be a semipositive quasi-Cartan companion of corank r and let u be a radical vector for A. Suppose i^{th} coordinate of u is 1. Now let A' be the matrix obtained by removing the i^{th} row and column of A. Then the corank of A' is equal to $(r - 1)$.*

Proof. Let A be a semipositive quasi-Cartan companion of corank r and let u be a radical vector for A whose i^{th} coordinate is 1. Now let A' be the matrix obtained by removing the i^{th} row and column of A. Suppose A' has corank greater than or equal to r. This is equivalent to saying that A' has at least m linearly independant radical vectors say $\{v_1, ..., v_m\}$ where $m \geq r$. Now by Proposition 3.1 $\{u_1, ..., u_m\}$, obtained from $\{v_1, ..., v_m\}$ by placing an extra coordinate which is zero appearing at i^{th} coordinate, are radical vectors for A. Moreover, $\{u_1, ..., u_m\}$ are linearly independant since $\{v_1, ..., v_m\}$ is. Now since corank of A is r, there could not be more than r linearly equivalent vectors. Thus, $m \leq r$. Now suppose $m = r$. Then there must be a linear combination of the vectors $\{u_1, ..., u_m\}$ which equals to u. This is impossible since i^{th} coordinate of u is nonzero however, i^{th} coordinate of each vector in the set $\{u_1, ..., u_m\}$ is zero. Therefore, the corank of A' is less than or equal to $(r-1)$. Now let $m \leq (r-2)$. Suppose $\{v_1, .., v_m\}$ are maximal linearly independant radical vectors for A'. Then there corresponds linearly independant radical vectors $\{u_1, ..., u_m\}$ of A and note that i^{th} coordinate of each vector in the set $\{u_1, ..., u_m\}$ is zero. Now suppose there is another radical vector $\tilde{u}$ apart from u whose i^{th} coordinate is also nonzero. Then there exist nonzero c and $\tilde{c}$ such that i^{th} coordinate of $cu + \tilde{c}\tilde{u}$ is zero and thus $cu + \tilde{c}\tilde{u}$ lies in the span of $\{u_1, ..., u_m\}$. Otherwise, there must exist another radical vector w whose i^{th} coordinate is zero and $\{u_1, ..., u_m, w\}$ linearly independant. However, this is impossible since in this case restriction of these set to A' must give linearly independant set of vectors but $\{v_1, .., v_m\}$ is maximal set already. Therefore, the set $\{u_1, ..., u_m, u, \tilde{u}\}$ is linearly dependant since we exhibited nontrivial relation and thus there could not be any other radical vector with nonzero i^{th} coordinate hence $\{u_1, ..., u_m, u\}$ is maximal linearly independant set of radical vectors for A. As a result, it must be of corank at most $r-1$ which gives contradiction. Therefore, A' is of corank $r-1$. □

Lemma 3.6 *Let A be a quasi-Cartan companion and let u be a radical vector whose i^{th} coordinate is 1. Let A' be the matrix obtained by removing the i^{th} row and column of the matrix A. Then if A' is semipositive then A is semipositive.*

Proof. We prove the lemma by proving contrapositive of the statement. Let A be an $n \times n$ quasi-Cartan companion which is not semipositive. We can further assume $i = n$ for the sake of simplicity (relabeling vertices this is always possible). Let $C = DA$ where D is the symmetrizer of A. Let $u = (u_1, .., u_n)$ where $u_n = 1$. Then $Cu = 0$ since D is diagonal matrix. Hence, if we denote columns of C as C_k; we have $C_1u_1 + .. + C_{n-1}u_{n-1} + C_n = 0$ which implies $C_n = -(C_1u_1 + .. + C_{n-1}u_{n-1})$ and $C_{nn} = -(C_{n1}u_1 + .. + C_{n(n-1)}u_{n-1}) = -(C_{1n}u_1 + .. + C_{(n-1)n}u_{n-1})$ Now since A is not semipositive, by definition there exists a vector $x = (x_1, .., x_n)$ s.t. $x^TCx < 0$. Now let $\tilde{u}$ and $\tilde{x}$ are the vectors obtained from u and x respectively by omitting the last coordinates of u and x. We also let $C' = D'A'$ where D' is the symmetrizer of A' obtained by D by removing last row and column of D. We denote columns of C' C'_k and abusing the notation even C' does not have the n^{th} row or column we set C'_n

as the vector obtained from C_n by removing the last coordinate of C_n. Therefore, we have $C'_n = -(C'_1 u_1 + .. + C'_{n-1} u_{n-1})$. Now our claim is the following:

Claim: $(\tilde{x} - x_n\tilde{u})^T C'(\tilde{x} - x_n\tilde{u}) < x^T C x < 0$.

To prove the claim, note that,

1. $$\begin{aligned}(\tilde{x}-x_n\tilde{u})^T C'(\tilde{x}-x_n\tilde{u}) &= \tilde{x}^T C'\tilde{x} - \tilde{x}^T C'(x_n\tilde{u}) - (x_n\tilde{u})^T C'\tilde{x} + (x_n\tilde{u})^T C'(x_n\tilde{u}) \\ &= \tilde{x}^T C'\tilde{x} - 2x_n(\tilde{x}^T C'\tilde{u}) + x_n^2(\tilde{u}^T C'\tilde{u})\end{aligned}$$

   Since,

2. $\tilde{x}^T C'\tilde{u} = \tilde{u}^T C'\tilde{x}$.

   Then,

3. $\tilde{x}^T C'\tilde{x} = \sum_{i,j}^{n-1} \tilde{x}^i C'_{ij}\tilde{x}_j = \sum_{i,j}^{n-1} x_i C_{ij} x_j$

   Thus,

4. $$\begin{aligned}-2x_n(\tilde{x}^T C'\tilde{u}) &= -2x_n(\tilde{x}^T(C'_1\tilde{u}_1 + .. + C'_{n-1}\tilde{u}_{n-1})) \\ &= 2x_n(\tilde{x}^T(-(C'_1 u_1 + .. + C'_{n-1}u_{n-1}))) \\ &= 2x_n(\tilde{x}^T C'_n) \\ &= \sum_i^{n-1} 2x_n C'_{in} x_i \\ &= \sum_i^{n-1} 2x_n C_{in} x_i\end{aligned}$$

5. $$\begin{aligned}x_n^2(\tilde{u}^T C'\tilde{u}) = x_n^2(\tilde{u}^T C'_n) &= x_n^2(C'_{in}u_i + ... + C'_{(n-1)n}u_{n-1}) \\ &= x_n^2(C_{in}u_i + ... + C_{(n-1)n}u_{n-1}) \\ &= x_n^2(-C_{nn}) \\ &= -x_n^2 C_{nn}\end{aligned}$$

   Now, combining (3),(4),(5) in (1) we have,

6. $$\begin{aligned}(\tilde{x} - x_n\tilde{u})^T C'(\tilde{x} - x_n\tilde{u}) &= \sum_{i,j}^{n-1} x_i C_{ij} x_j + \sum_i^{n-1} 2x_n C_{in} x_i - x_n^2 C_{nn} \\ &< \sum_{i,j}^{n-1} x_i C_{ij} x_j + \sum_i^{n-1} 2x_n C_{in} x_i + x_n^2 C_{nn} \\ &= x_T C x < 0\end{aligned}$$

Therefore, the claim is valid. Hence, A' is not semipositive. Thus we proved the contrapositive statement. As a result, if A' is semipositive then A must be semipositive.

□

Lemma 3.7 *Let C be a cycle (oriented or not). Let Ck be a diagram obtained by connecting a new vertex k to C and let A be a companion of Ck such that the product $\prod_{\{i,j\}\in C}(-A_{ij})$ is negative. Suppose that k is connected to an even number of vertices in C. Suppose also that k is connected to C in such a way that it is connected to two vertices which are not connected to each other in C (this condition excludes only the case when k is connected to exactly two vertices in C and those vertices are connected to each other). Then Ck necessarily has a cycle C' which contains k such that $\prod_{\{i,j\}\in C'}(-A_{ij})$ is positive.*

Proof. Let C be a cycle and let k be a new vertex that is connected to C in even number of vertices and not just to two adjacent vertices. Now let A be the companion of Ck such that $\prod_{\{i,j\}\in C}(-A_{ij})$ is negative(i.e. odd number of edges of C assigned $(+)$ by A). Now suppose for all cycles $C^{'}$ of Ck , $\prod_{\{i,j\}\in C'}(-A_{ij})$ is negative(i.e odd number of edges of $C^{'}$ assigned $(+)$ by A). Since we exclude the case that k is connected to exactly 2 vertices that are adjacent; Ck has exactly even number(Say $2m$) of cycles containing k. Now the total sum of the number of $(+)$ edges in these $2m$ cycles add up to be even since there are even number of cycles. However, we note that if an edge connecting k to C is assigned $(+)$ then it contributes twice to that total sum. Thus, if we discard the contribution of such connecting edges to the total sum, again the number of edges assigned $(+)$ is even but this number is exactly the number of $(+)$ edges of C. Therefore, the number of $(+)$ edges of C must be even which contradicts the assumption of the lemma. Thus, there must be a cycle of Ck such that even number of edges assigned $(+)$ that is there exist a cycle C' in Ck which contains k such that $\prod_{\{i,j\}\in C'}(-A_{ij})$ is positive.

□

CHAPTER 4

QUASI-CARTAN COMPANIONS OF ELLIPTIC DIAGRAMS

Exceptional elliptic diagrams $E_6^{(1,1)}$, $E_7^{(1,1)}$ and $E_8^{(1,1)}$ are diagrams that are not originated from a triangulation of a surface, nevertheless they are of finite mutation type by [2]. In this chapter we will establish our main result of the thesis (see Theorem 4.1 below). More precisely, we will prove the existence of an admissible quasi-Cartan companion for any element in mutation classes of $E_6^{(1,1)}$, $E_7^{(1,1)}$ and $E_8^{(1,1)}$, where admissibility is also preserved under mutation.

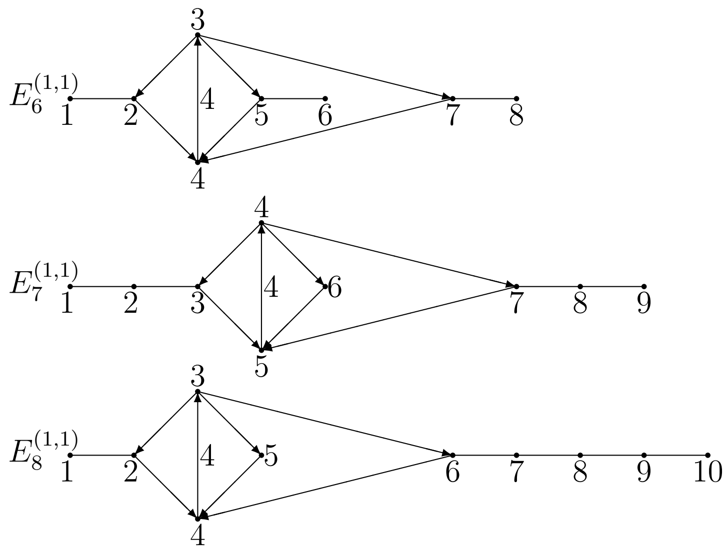


Figure 4.1: Exceptional elliptic diagrams of finite mutation type we will investigate.

Theorem 4.1 (The Main Theorem): *Let Γ be a diagram which is mutation equivalent to one of $E_6^{(1,1)}$, $E_7^{(1,1)}$ and $E_8^{(1,1)}$ (see Figure 4.1). Then Γ has a semipositive admissible quasi-Cartan companion of corank 2 whose invariant properties (admissibility, semipositivity, and corank) are preserved under mutation.*

The following result which is obtained by A. Seven in [8] is going to be one of the stepping stones to prove the Main Theorem.

Proposition 4.2 *Let Γ be a diagram which is mutation equivalent to one of $E_6^{(1,1)}$, $E_7^{(1,1)}$ and $E_8^{(1,1)}$. Then Γ contains a subdiagram which is mutation equivalent to E_6 (see Figure 2.4).*

Now we will prove the following proposition which we employ in the proof of the Theorem 4.1:

Proposition 4.3 *Each of $E_6^{(1,1)}$, $E_7^{(1,1)}$ and $E_8^{(1,1)}$ has an admissible semipositive quasi-Cartan companion of corank 2.*

Proof. For $E_n^{(1,1)}$, $n = 6, 7, 8$; we will denote the corresponding admissible quasi-Cartan companion as $A(E_n^{(1,1)})$

$$A(E_6^{(1,1)}) = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & 2 & -1 & 0 & -1 & 0 \\ 0 & -1 & 2 & 2 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix}$$

Now we note that labeling of the vertices as in the Figure 4.1. Now $A(E_6^{(1,1)})$ admissible since it assigns odd number(only 1 in our case) to three oriented triangles of $E_6^{(1,1)}$ by choosing the entries $(A(E_6^{(1,1)}))_{34}$ and $(A(E_6^{(1,1)}))_{43}$ positive and the rest of the entries negative. Now a straightforward calculation yields that $A(E_6^{(1,1)})$ has a radical vector and it must be of the form $(a, 2a, x, y, 2b, b, 2c, c)^T$ and $x + y = 3a = 3b = 3c$. Thus, x and y free and choosing x and y determines $a = b = c$. Therefore, $A(E_6^{(1,1)})$ is of corank 2. Actually $(1, 2, 3, 0, 2, 1, 2, 1)^T$, $(1, 2, 0, 3, 2, 1, 2, 1)^T$ generates the space of radical vectors for $A(E_6^{(1,1)})$. However, to apply the lemma 3.6, we note that taking $x = -y = 1$ generates the radical vector $u = (0, 0, 1, -1, 0, 0, 0, 0)^T$.

Now to see $A(E_6^{(1,1)})$ is semipositive since we have a radical vector u whose third coordinate is 1. Now we could apply the Lemma 3.6. Let $(A(E_6^{(1,1)}))'$ denote the matrix obtained by removing the 3^{rd} column and row of $A(E_6^{(1,1)})$. Then $(A(E_6^{(1,1)}))'$ corresponds to admissible quasi-Cartan companion of the subdiagram obtained by removing the vertex $\{3\}$ which is equal to $E_6^{(1)}$ (see Figure 2.5). Also we know $E_6^{(1)}$ is extended Dynkin and each of admissible quasi-Cartan companions of it is semipositive. Therefore, by the Lemma 3.6, $A(E_6^{(1,1)})$ is also semipositive. Actually to see $A(E_6^{(1,1)})$ is of corank 2 we could have applied Lemma 3.5 after showing there is a radical vector with coordinate corresponding to one of the vertices incident to edge e of weight 4 is 1, but we have shown it above by a direct proof.

$$A(E_7^{(1,1)}) = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & 2 & -1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & 2 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 2 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix}$$

Firstly we note that labeling of the vertices as in the Figure 4.1. $A(E_7^{(1,1)})$ is admissible since it assigns odd number(only 1 in our case) to three oriented triangles of $E_7^{(1,1)}$ by choosing $(A(E_7^{(1,1)}))_{45}$ and $(A(E_7^{(1,1)}))_{54}$ positive and the rest of the entries negative. Then $A(E_7^{(1,1)})$ has a radical vector and it must be of the form $(a, 2a, 3a, x, y, c, 3b, 2b, b)^T$ and $x + y = 4a = 4b = 2c$. Thus, x and y free and choosing x and y determines $2a = 2b = c$. Therefore, $A(E_7^{(1,1)})$ is of corank 2. Actually $(1, 2, 3, 4, 0, 2, 3, 2, 1)^T$, $(1, 2, 3, 0, 4, 2, 3, 2, 1)^T$ generates the space of radical vectors for $A(E_7^{(1,1)})$. However, to apply the Lemma 3.6, we note that taking $x = -y = 1$ generates the radical vector $u = (0, 0, 0, 1, -1, 0, 0, 0, 0)^T$.

Now to see $A(E_7^{(1,1)})$ is semipositive since we have a radical vector u whose fourth coordinate is 1 ; hence we could apply the Lemma 3.6. Now let $(A(E_7^{(1,1)}))'$ denote the matrix obtained by removing the 4^{th} column and row of $A(E_7^{(1,1)})$. Then $(A(E_7^{(1,1)}))'$ corresponds to admissible quasi-Cartan companion of the subdiagram obtained by removing the vertex $\{4\}$ which is equal to $E_7^{(1)}$ (see Figure 2.5). Also we know $E_7^{(1)}$ is extended Dynkin and each of admissible quasi-Cartan companions of it is semipositive. Therefore, by the Lemma $A(E_7^{(1,1)})$ is also semipositive. Similarly as above to see $A(E_7^{(1,1)})$ is of corank 2 we could apply Lemma 3.5 after showing there is a radical vector with coordinate corresponding to one of the vertices incident to edge e of weight 4 is 1.

$$A(E_8^{(1,1)}) = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & 2 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & 2 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix}$$

Similarly as in above cases, we note that labeling of the vertices as in the Figure 4.1. $A(E_8^{(1,1)})$ is admissible since it assigns odd number(only 1 in our case) to three oriented triangles of $E_8^{(1,1)}$ by choosing $A(E_8^{(1,1)})_{34}$ and $A(E_8^{(1,1)})_{43}$ positive and the rest of the entries negative. If $A_8^{(1,1)}$ has a radical vector it must be of the form $(a, 2a, x, y, c, 5b, 4b, 3b, 2b, b)^T$ and $x + y = 3a = 3b = 3c$. Thus, x and y free and choosing x and y determines $3a = 6b = 2c$. Therefore, $A_8^{(1,1)}$ is of corank 2. Actually $(2, 4, 6, 0, 3, 5, 4, 3, 2, 1)^T$, $(2, 4, 0, 6, 3, 5, 4, 3, 2, 1)^T$ generates the space of radical vectors for $A(E_8^{(1,1)})$. However, to apply the Lemma 3.6, we note that taking $x = -y = 1$ generates the radical vector $u = (0, 0, 1, -1, 0, 0, 0, 0, 0, 0)^T$.

Now to see $A(E_8^{(1,1)})$ is semipositive since we have a radical vector u whose third coordinate is 1 ; we could apply the Lemma 3.6. Now let $A(E_8^{(1,1)})'$ denote the matrix obtained by removing the 3^{rd} column and row of $A(E_8^{(1,1)})$. Then $A(E_8^{(1,1)})'$ corresponds to admissible quasi-Cartan companion of the subdiagram obtained by removing the vertex $\{3\}$ which is equal to $E_8^{(1)}$ (see Figure 2.5). Also we know $E_8^{(1)}$ is extended Dynkin and each of admissible quasi-Cartan companions of it is semipositive. Therefore, by the Lemma 3.6, $A(E_8^{(1,1)})$ is also semipositive. Again to see $A(E_8^{(1,1)})$ is of corank 2 we could have applied Lemma 3.5 after showing there is a radical vector with coordinate corresponding to one of the vertices incident to edge e of weight 4 is 1.

□

At the end of the chapter we will prove the "Main Theorem" by showing the following proposition:

Proposition 4.4 *Let Γ be a diagram that is mutation equivalent one of $E_6^{(1,1)}$, $E_7^{(1,1)}$ and $E_8^{(1,1)}$. Suppose Γ has an admissible quasi-Cartan companion A. Let k be a vertex of Γ and $\Gamma' = \mu_k(\Gamma)$ and $A' = \mu_k(A)$. Then A' is an admissible quasi-Cartan companion of Γ'.*

Now we need the following three lemmas 4.5, 4.6, 4.9 to prove the Proposition 4.4.

Lemma 4.5 *Let Γ' be a diagram corresponding to a skew-symmetric matrix and A' be the semipositive quasi-Cartan companion of Γ' of corank 2 and let $A'' = \mu_k(A')$. Suppose A' is not an admissible quasi-Cartan companion. Then A'' is not an admissible quasi-Cartan companion of $\Gamma = \mu_k(\Gamma')$ or the diagram $\Gamma = \mu_k(\Gamma')$ contains a subdiagram that belongs to Figure 2.6.*

Proof. Before starting the proof we make the following conventions : $A'|_C$ will occasionally denote restriction of A' to the subdiagram C. Moreover, we will occasionally say that two cycles are 'adjacent' if they share at least one edge. Also, for convenience, we will denote the subdiagram $\{C, k\}$ by Ck.

By assumption we have that A' is non-admissible. Thus, there is a cycle C such that $A'|_C$ is non-admissible namely sign condition is not satisfied.

First we will consider the case when k is on C. Let us first suppose k is a source or sink. Then since there is a source or sink in C, this amounts to saying that C is a non-oriented cycle. Moreover, since $A'|_C$ is non-admissible, by definition, exactly an odd number of edges of C assigned $(+)$ by A'. Since k is a source or sink in C, k is also a source or sink in $\mu_k(C)$. Thus $\mu_k(C)$ is also a non-oriented cycle. Furthermore, A'' assigns exactly an odd number of $(+)$ to the edges of $\mu_k(C)$. This is because of the following fact: By definition of companion mutation, if k is a source in C, mutation at k changes sign of both of the edges incident to k; and if k is a sink in C, then there is no change in the signs of the edges of C. Thus A'' is non-admissible on $\mu_k(C)$. Let us now suppose k is not a source or sink. In this case if C is a triangle, then A'' fails to be quasi-Cartan companion. If k is not a source or sink and C is not a triangle then the cycle $\mu_k(C) - \{k\}$ cannot satisfy the sign condition of admissibility by definition. Therefore, A'' is non-admissible.

We will continue the proof assuming k is not a vertex of C. From now on we may assume:

$(*)$ On any cycle containing k, A' satisfies the sign condition of admissibility.

Here we note that for the rest of the proof weight of any edge in Γ' is at most 4 by semipositivity of A' (see Proposition 3.2 (i)) and thus weight of any edge in Γ' is either 1 or 4 by the Lemma 2.11 about diagrams of skew-symmetric matrices.

Case 0: If k is connected to exactly 1 vertex in C then mutating Ck at k we have $\mu_k(C) = C$ with $A'' = A'$ restricted on C. Therefore, A'' is non-admissible.

Case 1: C is an oriented cycle.

A' must assign even number of $(+)$ signs to C to violate admissibility. We may assume without loss of generality that all edges of C is assigned $(-)$ by A' since otherwise changing the signs at some vertices of C will yield such assignment. Furthermore, note that by Proposition 3.2 part (iv) C must be simply-laced.

Subcase 1.1: Suppose k is connected to a vertex, say c of C by an edge of weight 4. By $(*)$ and Proposition 3.3 part (i), k is connected to vertices say c_1 and c_2 that are adjacent to c to form two oriented triangles $\{k, c, c_1\}$ and $\{k, c, c_2\}$. Moreover, k is not connected to any other vertex in C. However, if one of $\{k, c, c_1\}$ and $\{k, c, c_2\}$ is oriented then the other one must be non-oriented. This is because C is oriented. Thus, this possibility is overthrown.

Therefore, for the rest of Case 1, we will assume the subdiagram Ck to be simply-laced.

Subcase 1.2: k is connected to exactly 2 vertices say c_1 and c_2 of C; and c_1 and c_2 are adjacent.

Suppose first the triangle $\{k, c_1, c_2\}$ is oriented. Then exactly one of $\{k, c_1\}$ or $\{k, c_2\}$ is assigned $(+)$ by A'. Moreover, the effect of μ_k on Ck is to eliminate the edge $\{c_1, c_2\}$. Therefore, $\mu_k(Ck)$ is an oriented cycle. Then, either A'' assigns $(+)$ to both $\{k, c_1\}$ and $\{k, c_2\}$ or $(-)$ to both of them. Since all edges of C assumed to have $(-)$ at the beginning, A'' either assigns $(+)$ to exactly two edges of $\mu_k(Ck)$ or to no edges of $\mu_k(Ck)$. Thus, $A''|_{\mu_k(Ck)}$ is non-admissible. Then so is for A''.

Suppose now $\{k, c_1, c_2\}$ is non-oriented. If k is source or sink in Ck, then A'' is non-admissible on $\mu_k(C) = C$. Suppose k is not a source or sink in Ck. Then by $(*)$ A' restricted to $\{k, c_1, c_2\}$ is admissible. Furthermore, by assumption A' assigns $(-)$ to the edge $\{c_1, c_2\}$. Therefore, we have that $\{k, c_1\}$ and $\{k, c_2\}$ assigned both $(+)$ or $(-)$ by A'. Then the effect of μ_k is just to change the weight of the edge $\{c_1, c_2\}$ leaving it with same direction and same sign assignment made by A''. Therefore, A'' is non-admissible on $\mu_k(C)$ which is a non-oriented cycle.

Hence, for the rest of this case we will assume that if k is connected to C then k is connected to C in exactly 2 vertices, that are non-adjacent; or more than 2 vertices.

Now for the rest of this case we note that,

1. If k is connected to at least $n = 2$ non-adjacent vertices or more than $n = 3$ vertices of C there is exactly n cycles in Ck containing k.

2. If there is an oriented cycle Z in Ck containing k then for any cycle in Ck

containing k and adjacent to Z must be non-oriented.

3. The vertex k cannot be connected to an odd number of vertices of C by Proposition 3.2 (vii).

4. If k is connected to $n = 2m$ vertices of C , combining (2) and (3) there are at most m oriented cycles in Ck containing k.

5. The number of oriented cycles in Ck containing k must be even. Otherwise, Ck violates sign condition but this is impossible since by $(*)$ we assume A' is admissible on the cycles containing k.

6. If k is connected to exactly 2 non-adjacent vertices, then there could not be any oriented cycles in Ck containing k. Otherwise, there must be exactly 2 oriented adjacent cycles by (1) and (5) and which is impossible by (2).

7. Any vertex in Ck that is connected to a non-oriented cycle in Ck must be connected to at least 2 by Proposition 3.2 (vi) or even number of vertices by (vii) of a non-oriented cycle containing k. Using either one and combining with (2), any oriented cycle in Ck that contains k is a triangle and so is true for adjacent non-oriented cycles containing k in Ck except for the case that k is connected to exactly 2 non-adjacent vertices of C. In this case there are exactly 2 cycles that contain k and they must be non-oriented by (6) and Proposition 3.2 forces these cycles and C to be squares.

8. All edges of C can be assumed to be assigned $(-)$ by A'. Otherwise, we could make the sign changes on the vertices to have such assignment.

9. For any two adjacent non-oriented cycles containing k, we may assume that that A' assigns $(-)$ to all edges of them. Since if this is not the case, all edges of them that are incident to k must be assigned $(+)$ by A'. Changing the sign only at the vertex k solves the problem.

10. By Proposition 3.2 (v) and (8), there is a radical vector $\tilde{u}$ for $A'|_C$ whose coordinates are all 1's. Thus by Proposition 3.1, $\tilde{u}$ generates a radical vector u of Γ' which has 1's corresponding to vertices of C and 0's for the rest.

11. If there are two adjacent non-oriented cycles say C_1 and C_2 that contain k in Ck similarly to the last item, there are radical vectors $\tilde{u_1}$ and $\tilde{u_2}$ respectively of

$A'|_{Ck}$ which has all 1's corresponding to vertices of C_1 and C_2 respectively and the rest are zeros. Thus by Proposition 3.1, $\tilde{u_1}$ and $\tilde{u_2}$ generates radical vectors u_1 and u_2 of Γ' which has 1's corresponding to vertices of C_1 and C_2 and 0's for the rest.

12. If there are two adjacent non-oriented cycles say C_1 and C_2 that contains k then u_1 , u_2 and u as being above form linearly independant set of radical vectors for Γ' and thus corank exceeds 2 being at least 3, which is impossible in our case. This is because there are exactly 3 radical vectors u_1 , u_2 and u for Γ'. Now we consider the coordinates corresponding to common vertices shared by exactly two of C_1, C_2 and C. In this way we see that for such a coordinate, two of the vectors have 1 in that coordinate the other has 0.

13. Two non-oriented cycles C_1 and C_2 that contains k and assigned $(-)$ or $(+)$ to all of their edges that are incident to k(the edges that is common with C is assumed to have $(-)$) generate radical vectors u_1 and u_2. Moreover, $\{u_1, u_2, u\}$ is a linearly independant set of vectors hence corank of A' is greater than 3. By the same reasoning as (12), C_1, C_2 contain k but C does not, and similarly for the other vertices. Suppose the edges of C_1 and C_2 that are incident to k are all assigned $(-)$. Then by the same argument as in (12), u_1, u_2, u are linearly independant. In the latter case we can assume $(+)$ assigened edges are also $(-)$ by changing the sign at k.

Subcase 1.3: k is connected to exactly $n = (2m{+}1)2$ vertices for $m \geq 1$ or connected exactly to 2 vertices, that are adjacent.

Then combining (2),(4) and (5) , there are 2 adjacent non-oriented cycles(actually must be squares by (7) if k is connected to exactly two non-adjacent vertices of C and triangles if k is connected to C in more than 3 vertices) containing k. Thus, by (12) corank must be greater than 2 which is impossible.

Subcase 1.4: k is connected to exactly $n = (2m)2$ vertices of C for $m \geq 2$.

Suppose first, the number of oriented cycles containing k is less than 2m. Then there are 2 adjacent non-oriented cycles containing k. Therefore, by (12) corank is greater than 3.

Now suppose the number of non-oriented cycles is exactly 2m. Then there are at least 2m non-oriented cycles and m non-oriented cycles with all edges assigned $(-)$. Moreover, $m \geq 2$ by assumption. Thus, corank is greater than 3 by (13).

Subcase 1.5: k is connected to exactly 4 vertices of C.

If there is no oriented cycles containing k then corank is greater than 3.

If there are 2 oriented cycles in Ck containing k then there are exactly 2 non-oriented cycles contain k and have the different signed edges for their edges that is connected to k. Then the radical vectors obtained from these 2 cycles and the radical vectors obtained from C form a linearly dependant set which does not violate corank since they contribute corank at most 2 being linearly dependant(actually the exact contribution is 2). If we consider $\mu_k(Ck)$, then it will be one of the following 3 forms:

i) Two oriented triangles that has exactly one vertex common (the common vertex is k) and each of them has exactly one edge of weight 4 that is not incident to common vertex k. This happens when C is a square. (See '2-4s' in Figure 2.6)

ii) Two oriented triangles that has exactly one vertex (the common vertex is k), one is simply-laced and a non-oriented cycle attached to the edge that is not incident to common vertex k, and the other one of them has exactly one edge of weight 4 that is not incident to common vertex k. This happens when exactly one of the non-oriented cycles is triangle. (See '1-4' in Figure 2.6)

iii) Two oriented simply-laced triangles such that each of the edges in both traingles that is not incident to common vertex k is part of a non-oriented triangle. Moreover, such non-oriented cycles have no common vertices and none of them contains k. This happens when neither of non-oriented cycles is triangle. (See 'No-4s' in Figure 2.6)

Case 2: Suppose C is a non-oriented cycle.

In this case A' assigns odd number of (+) to the edges of C to violate admissibility. In this case;

1. If k is connected to C in exactly $n > 2$ vertices or exactly $n = 2$ non-adjacent vertices, then there are exactly n cycles in Ck containing k.

2. If k is connected to C in exactly $n > 2$ vertices or exactly 2 non-adjacent vertices. Then there are odd number of oriented cycles in Ck that contains k by sign condition and by $(*)$. In particular there always exists an oriented cycle in Ck that contains k under the same conditions.

3. If k is connected to C via even number $n = 2m$ for $m \geq 2$ of vertices or k is connected to C exactly in 2 nonadjacent vertices, then there exists a non-oriented cycle in Ck that contains k.

Subcase 2.1: k is connected to C say in the vertex c with an edge e of weight 4.

Similar to the oriented case, by Proposition 3.2, k is connected to the adjacent vertices of c say c_1 and c_2 and no other vertices. Then k is connected to exactly 3 vertices. The cycles $\{k, c, c_1\}$ and $\{k, c, c_2\}$ are oriented triangles by Proposition 3.3 (i) and by $(*)$. Moreover, this forces k to be a source or sink in $Ck - \{c\}$. Therefore, the cycle $Ck - \{c\}$ must be non-oriented. However, there must be exactly an odd number of oriented cycles in Ck containing k by (1) and (2) which gives a contradiction.

For the rest of the case 2, we may assume weight of any edge that connects k to C is less than 4 that is to say it is 1 since we are dealing with diagrams that is mutation equivalent to diagrams of skew symmetric matrices.

Subcase 2.2: Suppose k is connected to exactly one vertex in C.

Then $A''|_{\mu_k(C)} = A''|_C$ is again non-admissible. Therefore, A'' is non-admissible.

Thus, for the rest of the proof of Case 2, we may assume k is connected to at least 2 vertices of C.

Subcase 2.3: Suppose C has an edge e of weight 4.

Then C is a triangle by Proposition 3.2 (iii).

Then either k is connected to C in exactly two adjacent vertices of C that are incident to the edge of weight 4 or k is connected to all three vertices of C. First suppose that k is connected to exactly two vertices of C that are adjacent. Then k must be connected to two ends of e and hence there is an oriented triangle with one edge is e

with k as the third vertex. Then in $\mu_k(\Gamma')$, $\mu_k(C)$ is a non-oriented cycle with edges assigned same signs by A''. The only difference is its being simply-laced. Therefore, $A''|_C$ is non-admissible and so is for A''.

Now suppose k is connected all of the vertices of the triangle C. Then the triangle containing the vertex k and the edge e is oriented. Thus, the other 2 triangles containing k cannot be both oriented. Therefore, the other 2 triangles containing k must be both non-oriented by (2). Thus in $\mu_k(\Gamma')$, $\mu_k(C)$ is a non-oriented cycle with same signs assigned to its edges by A''. Hence $A''|_{\mu_k(C)}$ is non-admissible.

Thus for the rest of the Case 2, we may assume Ck is simply-laced.

Subcase 2.4: k is connected to exactly 2 adjacent vertices say c_1 and c_2 of C.

First assume that the cycle $\{k, c_1, c_2\}$ is non-oriented. Now if k is a source or sink then C is again a non-oriented cycle in $\mu_k(\Gamma')$ and A'' assigns the same signs to edges of C as A' assigns to C. Hence $A''|_C$ is non-admissible. Now if k is not a source or sink. Then the signs of the edges of $\mu_k(C)$ is the same as those of C after mutation at k. Moreover, $\mu_k(C)$ is a non-oriented cycle. Therefore, $A''|_{\mu_k(C)}$ is non-admissible.

Now suppose $\{k, c_1, c_2\}$ is oriented. Then μ_k destroys the edge $\{c_1, c_2\}$. The resulting cycle $\mu_k(Ck)$ is again non-oriented. This is because if there is a source or sink of C other than c_1 and c_2, trivially it remains what it was(source or sink). If only source and sink are c_1 and c_2. Then source stays as source and sink stays as sink after applying μ_k. Now if $\{c_1, c_2\}$ were $(-)$, for the other edges $\{k, c_1\}$, $\{k, c_2\}$ of the oriented triangle $\{k, c_1, c_2\}$, A' assigns exactly one $(-)$ and one $(+)$. After applying μ_k the signs of the edges $\{k, c_1\}$, $\{k, c_2\}$ are both $(+)$ or both $(-)$. Thus, $\mu_k(Ck)$ has the same number of $(+)$ edges $\bmod 2$. Now if $\{c_1, c_2\}$ were $(+)$, A' assigns $(-)$ or $(+)$ to both $\{k, c_1\}$, $\{k, c_2\}$ .Mutation at k changes sign of only one of the edge that is incident to k. Thus A'' assigns same number of $(+)$ to non-oriented cycle $\mu_k(Ck)$ as A' assigns to $C \bmod 2$. Therefore, A'' is non-admissible on $\mu_k(Ck)$.

Subcase 2.5: k is connected to exactly 2 vertices say c_1 and c_2 of C, that are non-adjacent.

Then there is one non-oriented cycle say C_1 containing k by (3) and one oriented cycle say C_2 containing k by (2). Note also that C_2 is an oriented square by Proposition 3.2 (vi) or (vii). Meanwhile C_1 has a source or sink other than the vertices that are adjacent to k, otherwise C would be oriented. Now $\mu_k(Ck)$ is of type $\check{D}_n^{(1)}(r)$ (see Figure 2.6).

Subcase 2.6: k is connected to exactly 3 vertices of C.

Then there are either 1 or 3 oriented cycles in Ck containing k. However, there could not be three pairwise adjacent oriented cycles in Ck containing k. Therefore, there are 2 non-oriented cycles C_1, C_2 and an oriented cycle C_3 containing k. Now by Proposition 3.2 (vi); C_1,C_2 and C_3 are triangles. Then the common vertex of C_2 and C_3 that is not a vertex of C_1, is connected to exactly 3 vertices of C_1 which contradicts Proposition 3.2 (vii).

Subcase 2.7: k is connected to exactly 4 vertices of C

Now by (3) there is at least one non-oriented cycle containing k and there exists at least one oriented cycle by (2) containing k. Actually there is exactly 1 or 3 oriented cycles in Ck containing k. Let C_1, C_2, C_3 and C_4 be the the cycles in Ck containing k.

Firstly suppose that there is exactly one such oriented cycle say C_4 is the one. Then each of the cycles C_1, C_2, C_3 and C_4 is adjacent to a non-oriented cycle hence they are all triangles by Proposition 3.2 (vi) (therefore C needs to be a square). We note that edges of C_1, C_2, C_3 may assumed to be assigned all $(-)$ by A'. We also note that in this case C_1, C_2, C_3 generates radical vectors for each one by Proposition 3.2 (v). These vectors are trivially linearly independant. To show this fact, it is enough to observe that two of the such non-oriented cycles must be non-adjacent and thus, these cycles have a vertex that is not included in the other 2 such non-oriented cycles. Therefore, corank exceeds 2 which is impossible.

Now suppose there are exactly 3 oriented cycles in Ck containing k say C_2, C_3 and C_4 are such cycles and C_1 is the non-oriented one. Without loss of generality, C_2, C_3 are assumed to be the cycles that are adjacent to C_1. Then by Proposition 3.2 (vi) or (vii) C_2 and C_3 are triangles. Let V and V' denote the number of vertices of C_1 and

C_4 respectively. Now if ,

1. $V > 3$ and $V' = 3$ then $\mu_k(Ck)$ is of type $\check{D}_n^{(1)}(1, r)$ (see Figure 2.6)
2. $V > 3$ and $V' > 3$ then $\mu_k(Ck)$ is of type $\check{D}_n^{(1)}(1, r, s)$ (see Figure 2.6)
3. $V = 3$ and $V' = 3$ then $\mu_k(Ck)$ is of type $\check{D}_n^{(4)}$ (see Figure 2.6)
4. $V = 3$ and $V' > 3$ then $\mu_k(Ck)$ is of type $\check{D}_n^{(4)}(1, r)$ (see Figure 2.6)

Subcase 2.8: k is connected to exactly $n = 2m + 1$ vertices of C where $m > 1$.

In this case there are at least 5 cycles in Ck containing k and one of these cycles say C_1 must be non-oriented. Now, there exist a vertex say c in C that is connected to C_1 only in the vertex k. Indeed, c could be taken as the vertex that is in C and incident to common edge of any 2 adjacent cycles in Ck that contain k and that are non-adjacent to C_1. This contradicts to Proposition 3.2 (vi).

Subcase 2.9: k is connected to exactly $n = 2m$ vertices of C where $m > 2$.

Now by Lemma 3.7, there is a non-oriented cycle Z in Ck containing k. Now, there exist a vertex say c in C that is connected to C_1 only in the vertex k. Indeed, c could be taken as the vertex that is in C and incident to common edge of any 2 adjacent cycles in Ck that contain k and that are non-adjacent to C_1. This contradicts to Proposition 3.2 (vi).

□

Lemma 4.6 *Let Γ be a diagram. Suppose Γ contains a diagram from Figure 2.6. Then Γ is mutation-equivalent to a diagram that contains one of the diagrams from the Figure 4.2 below as a subdiagram.*

Proof.

When we say, some vertex v is a source or sink in the cases below, our intention is that the vertex v is a source or sink in the cycle we want to transform. In any case we will

$TheEars$ $\quad$ $2-4s$

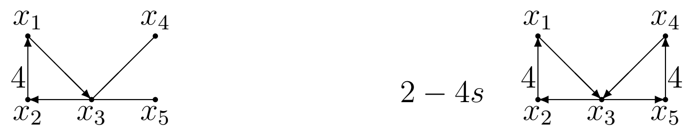


Figure 4.2: Edges whose orientation unspecified are assumed to be arbitrarily oriented.

not refer source or sink in the whole corresponding diagram even if they correspond to a source or sink in whole diagram by chance.

Case 1: Γ contains $\check{D}_n^4$.

Then applying the mutations μ_{a_2},μ_{a_3},..,μ_{a_m-1},μ_{a_m} in the written order from left to right, and then consider the subdiagram $\{b_1, b_2, a_1, c_1, c_2\}$ is exactly the 'The Ears'.

Case 2: Γ contains $\check{D}_n^{(4)}(1, r)$.

Then applying the mutations μ_{c_3},μ_{c_4},..,μ_{c_r-1},μ_{c_r} in the written order from left to right, and then consider the subdiagram determined by the vertices $\{b_1, b_2, a_1, c_1, c_2\}$ is exactly 'The Ears'.

Case 3: Γ contains $\check{D}_n^{(1)}(1, r)$.

In the non-oriented cycle C determined by $b_i's$, considering the set of vertices X that are either source or sink , if the b_1 is not a source for the non-oriented cycle then the smallest indexed vertex in X is sink and the greatest indexed one is source. If b_1 is a source vertex, then the least indexed vertex in the set X apart from b_1 is a sink. Note also that two source vertices or two sink vertices cannot occur in the cycle in a row. Consider the set $X \cup \{b_1, b_2\}$ (Of course b_1 could be source and b_2 could be sink.). The set of vertices $X \cup \{b_1, b_2\}$ determines a non-oriented cycle say Z obtained via applying mutations in middle vertices that are not in $X \cup \{b_1, b_2\}$. It is a subdiagram of $\check{D}_n^{(1)}(1, r)$ and the edges in Z are characterized as being one of the following three forms: (1) orienting from a source to sink or (2) from a source to b_1 or (3) from b_2 to a sink. Now proceeding in this manner inductively (if both the vertices $\{b_1, b_2\}$ are source or sink and if the resulting non-oriented cycle is not a triangle mutate at some vertex that is not in $\{b_1, b_2\}$ and proceed) to obtain a non-oriented triangle with one side is the edge $\{b_1, b_2\}$ say $\{x, b_1, b_2\}$ (Actually x must be in the set X.) Then if the vertex x is not a source or sink then we mutate the diagram at x and consider the subdiagram $\{b_1, b_2, a_1, c_1, c_2\}$ which is exactly the 'The Ears'. Now if x is a source or sink then, consider one of b_1 and b_2 which is not a sink or source of the non-oriented triangle. Say for the sake of simplicity it is b_1. Then, without loss of generality, we assume the edges $\{a_1, c_1\}$ and $\{a_1, c_2\}$ is oriented from a_1 to c_i. Otherwise, we mutate the diagram at one of or both of the vertices c_1 c_2 to reach that assumption. Then apply in the written order μ_{b_1},μ_{a_1} and consider the subdiagram $\{x, b_1, b_2, c_1, c_2\}$ which is exactly 'The Ears'.

Case 4: Γ contains $\check{D}_n^{(1)}(1, r, s)$.

Apply the same mutations for the oriented cycle as in Case 2 and then apply the same process for the non-oriented cycle and the triangle adjacent to it as in the Case 3, then consider the subdiagram $\{x, b_1, b_2, c_1, c_2\}$ where x is such a vertex as in the Case 3. Now $\{x, b_1, b_2, c_1, c_2\}$ is exactly the 'The Ears'.

Case 5: Γ contains $\check{D}_n^{(1)}(r)$.

Apply the same process as in the Case 3. Now if x taken as in that case , unlike the cases above now we have only one possibility for the character of x in the resulting non-oriented triangle $\{x, b_1, b_2\}$. In this case and in the resulting triangle, x must be a source or sink. This is because we have at least a vertex as source or sink in the non-oriented cycle determined by $\{b_3, .., b_r\}$. Suppose without loss of generality that b_1 is not a source or sink in $\{x, b_1, b_2\}$. Then apply μ_{b_1} and consider the subdiagram determined by the vertices $\{x, b_1, b_2, a_1, c_1\}$ which is exactly 'The Ears' .

Case 6: Γ contains '1-4' or 'No-4s'

Apply the same process as in the Case 3 to the unique non-oriented cycle of '1-4' or to both of the non-oriented cycles of 'No-4s' create '2-4s'.

Case 7: Γ contains '2-4s'

Trivially Γ contains '2-4s'.

□

With the following lemma below, we will obtain very efficient tools to show a diagram is mutation-infinite .

Lemma 4.7 *(Special Types of Mutation-Infinite Diagrams)*

- *Let Γ be a simply-laced diagram which contains a non-oriented cycle C and a vertex v such that v is adjacent exactly to an odd number of vertices in Γ. Then Γ is mutation-infinite.*
- *Let Γ be a simply-laced diagram which contains a cycle C and a vertex v such that v is adjacent exactly to $2m+1$ vertices in Γ for $m > 0$. Then Γ is mutation-infinite.*
- *Let Γ be a diagram which contains at least two non-oriented cycles C and no oriented cycles. Then Γ is mutation-infinite.*
- *A non-simply-laced non-oriented cycle C is mutation-infinite.*
- *A connected 3-vertex diagram V which contains an edge of weight 4 and that is not a triangle then V is mutation-infinite.*
  In particular if a diagram Γ contains a subdiagram that satisfies the conditions of one of the items above then Γ is mutation-infinite.

For the proof of the above statement see the proof [8, Proposition 2.1]

Lemma 4.8 *Let Γ be a diagram that is mutation-equivalent to the diagram E_6. Then;*

(i) For any two pairwise non-adjacent vertices i, j of Γ, there is another vertex, say k of Γ, such that k is adjacent to exactly one of the vertices i, j.

(ii) For any cycle C in Γ, there is a vertex, say k of Γ, such that k is adjacent to exactly one vertex in C.

Proof. By inspection on the diagrams in the Figure 2.3. □

We will repeatedly use the Lemmas 4.7 and 4.8 above without further mention in the proof of the Lemma 4.9 below to show a diagram is mutation-infinite.

Lemma 4.9 *Let Γ be a diagram. Suppose Γ contains a diagram from Figure 4.2 as a subdiagram and contains a subdiagram E that is mutation equivalent to E_6. Then Γ is of infinite mutation type. In particular Γ cannot be mutation equivalent to any of $E_6^{(1,1)}$, $E_7^{(1,1)}$ and $E_8^{(1,1)}$.*

Proof. Let us make a comment about the proof before commencing. Actually it is enough for our purposes to assume Γ to be in mutation class of one of $E_6^{(1,1)}$, $E_7^{(1,1)}$ and $E_8^{(1,1)}$. In such a case we could employ corank of corresponding skew-symmetric matrix say B for Γ. The corank of B is 2. We recall that in the subdiagrams if there is a non-oriented cycle or a special kind of subdiagram V that if a vertex is connected to V then that vertex must be connected to even number of vertices in V then this situation generates a radical vector for $\bmod 2$ reduction of B which contributes the corank 1 for each. Then our job would be far more simple and very many situations that is made complicated by non-oriented cycles or vertices that connects to even number of vertices in some subdiagram would be overcame. The proof would be substantially shorter and less technical. However, we have tried to give more general proof that works independantly of corank.

Throughout the proof, we will denote occasionally subdiagrams of the form 'The Ears' or '2-4s' by X for the sake of brevity. Also throughout the proof, we will occasionally denote the fact that x and y are adjacent to each other by '$x \sim y$'. Accordingly we denote x is not adjacent to y by '$x \nsim y$'.

Let Γ be such a diagram satisfying the conditions. First, we note that if Γ has at least 11 vertices then Γ is mutation-infinite by [2, Lemma 7.3]. Thus, we assume Γ has at most 10 vertices and thus E and X has at least 1 common vertex since E has exactly 6 and X has exactly 5 vertices. Note also that if any vertex e that belongs to E is adjacent to 'The Ears' or '2-4s' with an edge of weight 4, then the subdiagram determined by the vertices of 'The Ears' or '2-4s' and e generates a mutation-infinite subdiagram. Thus, Γ itself is mutation-infinite. Now suppose for the rest of the proof if a vertex e in E is adjacent to one of our diagrams in Figure 4.2 then weight of the

the edge that connects e to X is 1.

We will consider the lemma for 'The Ears' first.

Let Γ contain a subdiagram in the form 'The Ears'

Just before commencing the proof of this case, we note that there is at least 2 vertices of E that is not contained in X. This is because of the fact that E_6 has 6 vertices and so for any element in its mutation class; and the number of common vertices of X and E cannot exceed 4 since x_1 and x_2 cannot be both in E. Otherwise, there would be an edge of weight 4 in E which is impossible. Note also that, the vertices labeled by e of E are non-common with the vertices of subdiagram X in discussion unless otherwise stated.

First of all, we will consider the case, if there is a vertex e of E that is adjacent to exactly one of x_4 and x_5. We will show that if this is the case then Γ is mutation-infinite. We may assume without loss of generality that $e \sim x_4$. If e is adjacent to no other vertices of 'The Ears', then we have that the subdiagram $\{x_1, x_2, x_3, x_4, e\}$ is of Type 1 hence the subdiagram is mutation-infinite. Therefore, Γ is mutation-infinite. Now if $e \sim x_3$ but $e \nsim x_1$ and hence non-adjacent to x_2. Otherwise, we have a three-vertex subdiagram with one edge of weight 4 that is not a triangle therefore the situation generates infinite mutation-type subdiagram. For this situation in this case and for the rest the following will be assumed without further mentioning:

($*$) For any 3-vertex subdiagram containing an edge of weight 4 is an oriented triangle preventing the diagram from being trivially mutation-infinite.

Now the subdiagram $\{x_1, x_2, x_3, x_4, x_5, e\}$ is of Type 2 hence the subdiagram is infinite mutation type. Therefore, Γ is mutation-infinite. Now consider the situation where $e \sim x_1$ (and trivially $e \sim x_2$) and $e \nsim x_3$. Then there is at least 1 simply-laced non-oriented cycle (consider for instance $\{x_1, x_3, x_4, e\}$) to which x_5 is adjacent with a single edge of weight 1. Therefore, we have a mutation-infinite subdiagram (simply-laced) of Γ which makes Γ mutation-infinite. Now suppose $e \sim x_1$ (and trivially $e \sim x_2$) and $e \sim x_3$. Then we have at least 1 simply-laced non-oriented triangle (consider for instance $\{x_1, x_3, e\}$) to which x_5 is adjacent in a single edge of weight 1. Therefore, Γ is mutation-infinite. Now we have covered all the cases when there is a vertex of E that is adjacent to exactly one of x_4 and x_5. For each such case we have figured out that there is a mutation-infinite subdiagram of Γ which leaves Γ no opportunity other than being muatation-infinite. Therefore, we assume for the rest of the proof that:

($**$) There is no vertex in $E - X$ that is adjacent to exactly one of x_4 and x_5.

In particular this implies the fact that both x_4 and x_5 cannot be vertices of E, without Γ being mutation-infinite. This is because of the fact that x_4 and x_5 are vertices of E which are non-adjacent; and there is only one vertex that is adjacent to x_4 which is x_3. Then since $E \sim E_6$ and by the Lemma 4.8, there is a vertex e (regardless of x_3 being a vertex of E or not) that is adjacent to exactly one of x_4 and x_5. Thus it creates the exact copy of the situation we have just proved which generates a mutation-infinite subdiagram. Therefore, we will assume for the rest, at least one of x_4 and x_5 is not in E.

Since we assumed ($**$) for the rest, we will prove the lemma in the following two cases which are logical negations of each other :

1. There is no vertex e of $E - X$ that is adjacent to one of x_4 and x_5 (then Γ is mutation-infinite.)
2. There is at least one vertex e of $E - X$ that is adjacent to both of x_4 and x_5 (then Γ is mutation-infinite.)

Now we will prove the first statement.

Case 1: There is no vertex e of E that is adjacent to one of x_4 and x_5, (then Γ is mutation-infinite.)

Now we will prove the Case 1 in the subcases. We will start with:

1.1: There is a vertex e of $E - X$ s.t $e \sim x_3$.

1.1.1: $e \sim x_1$

Then there is a simply-laced non-oriented triangle, take for instance $\{x_1, x_3, e\}$, to which x_5 is adjacent with a single edge of weight 1. Therefore, Γ is mutation-infinite.

1.1.2: e is adjacent to no other vertices of X.

Then $\{x_1, x_2, x_3, x_4, x_5, e\}$ is a mutation-infinite subdiagram. Hence Γ is mutation-infinite.

Then for the rest of the Case 1, we may assume the following:

$(***)$ There is no vertex in $E - X$ that is adjacent to x_3.

Therefore, we will consider the following:

1.2: There is e in $E - X$ such that $e \sim x_1$

Now since $e \sim x_1$ and for any vertex $v \in E - X$, v could be adjacent to no vertices of $X - \{x_1, x_2\}$. we have the following three possibilities:

1. Common vertices of E and X are exactly $\{x_1, x_3, x_4\}$.
2. Common vertices of E and X are exactly $\{x_1, x_3\}$.
3. $\{x_1\}$ is the unique common vertex of E and X.

Now we will consider these 3 possibilities case by case:

1.2.1: Common vertices of E and X are exactly $\{x_1, x_3, x_4\}$.

Now valencies of x_3, x_4 are 2 and 1 in E respectively and x_3 must be adjacent to a vertex in E that is not contained in a cycle in E since x_1 could not be contained in a cycle in E, otherwise this cycle would be non-oriented. Trivially x_3 cannot be contained in any cycle in E since its valency is 2 and one of the vertices that x_3 is adjacent cannot be a part of any cycle in E. Furthermore, valency of x_1 is at least 2 in E being adjacent to x_3 and e. Using these data, by inspection in the diagrams in Figure 2.3, one may observe easily that E could be only (1) in Figure 2.3, Thus, we have a subdiagram of Type 4 which makes Γ mutation-infinite.

1.2.2: Common vertices of E and X are exactly $\{x_1, x_3\}$.

Here, valency of x_3 is 1 in E and x_3 must be adjacent to a vertex in E that is not contained in a cycle in E since x_1 could not be contained in a cycle in E, otherwise this cycle would be non-oriented. Trivially x_3 cannot be contained in any cycle in E since its valency is 1. Furthermore, valency of x_1 is at least 2 in E being adjacent to x_3 and e. Using these data, by inspection in the diagrams in Figure 2.3, one may observe easily that E could be only (1) or (7) in Figure 2.3, and there are two choices for x_1 in (1) and that determines the connection shape of E and X. For the first choice of connection we have a subdiagram of Type 4, and for the second type of connection we have two subdiagrams that are of Type 4 and Type 1 which makes Γ mutation-infinite. Now if E is the diagram (7). Then there is a unique choice. In this case we have a subdiagram of Type 2 that makes Γ mutation-infinite.

1.2.3: $\{x_1\}$ is the unique common vertex of E and X.

Now there are two possibilities:

1. There is another vertex say e' of E other than e that is connected to x_1. Now if $e \sim e'$ then $\{e, e', x_1\}$ is a triangle in E which would be non-oriented but it is impossible for E to contain non-oriented cycles. Then suppose $e \nsim e'$. Now there must be a vertex e'' in E that is connected to exactly one of e and e'. Suppose without loss of generality that $e \sim e''$ Now we have that $e'' \nsim x_1$ otherwise the triangle $\{e, x_1, e''\}$ would be non-oriented which is impossible. However, in this case subdiagram of Γ determined by $\{x_1, x_3, x_4, x_5, e, e', e''\}$ is of Type 4. Therefore, Γ is mutation-infinite.
2. There is no other vertices of E to which x_1 is adjacent. In this case, x_1 is not a part of the any cycle in E. Now by inspection(see Figure 2.3) all of the possible diagrams contains subdiagrams that are either of Type 1, Type 2, Type 3 or Type 4. Therefore, Γ is mutation-infinite.

Case 2: There is at least one vertex e of $E - X$ that is adjacent to both of x_4 and x_5.

Now let e be such a vertex in $E - X$ that is adjacent to both x_4 and x_5. In addition to this vertex e, suppose there is another vertex e' in $E - X$(trivially other than x_3) that is also adjacent to both x_4 and x_5. Then there are 32 possible subdiagrams of Γ determined by the vertices $\{x_1, x_2, x_3, x_4, x_5, e, e'\}$ which are mutation-infinite. Thus, Γ is mutation-infinite. We will not exhibit all of them. Nevertheless we will exhibit 4 typical examples of these 32 possibilities:

1. e and e' are only adjacent to x_4 and x_5 in X and non-adjacent to each other. Then there is a new vertex e'' in E which is adjacent to exactly one of e and e'. Then if e'' is not adjacent to any other vertex in X. Then the diagram has a subdiagram of Type 1. Thus, Γ is mutation-infinite. Else if e'' is adjacent to some other vertex in X then there is a simply-laced non-oriented cycle and a vertex that is adjacent to one of the vertices of the non-oriented cycle in discussion with odd number of edges of weight 1. Hence Γ is mutation-ininite.

2. Suppose e is adjacent to no other vertices and $e' \sim x_3$ and e' is adjacent to no other vertices. There are three cycles in the subdiagram $\{x_3, x_4, x_5, e, e'\}$ and whatever choice for orientations of the three cycles is made there is at least one non-oriented cycle which is simply-laced and x_1 is adjacent to that cycle with exactly one edge of weight 1. Now discard x_2 from the subdiagram. Now x_1 and the non-oriented cycle generates a simply-laced subdiagram. Therefore, Γ is mutation-infinite.

3. $e \sim x_1, x_2, e'$ and $e' \sim x_1, x_2, e$. Then e is adjacent to simply-laced cycle $\{e', x_3, x_4, x_5\}$ in exactly 3 vertices with edges of weight 1. Therefore, Γ is mutation-infinite.

4. $e \sim x_1, x_2, e'$ and $e' \sim x_1, x_2, x_3, e$. Then the subdiagram $\{e, e', x_1, x_3\}$ is simply-laced and has only two cycles $\{x_1, e, e'\}$ and $\{x_1, x_3, e'\}$. These 2 triangles are non-oriented. Therefore, Γ is mutation-infinite.

The remaining possibilities are handled in the same way; and in each possible case, we have checked that Γ is mutation-infinite only by the methods in these 4 typical examples above.

Therefore, for the rest of the case, we assume there is exactly one vertex e in $E - X$ that is adjacent to both x_4 and x_5. However, there is no vertex v in $E - X$ which is adjacent to one of the vertices x_4 and x_5 without being adjacent to both of x_4 and x_5 otherwise Γ is mutation-infinite. Therefore, we may assume all the vertices in $E - X$ other than e cannot be adjacent to any of them. In particular x_4 and x_5 is assumed not to be both in E. Therefore, without loss of generality, we may assume that x_5 is not in E throughout the proof of this case.

Firstly in the lemma below we will make some assumptions which will be valid throughout this case and give their proofs what if otherwise assumed :

Lemma 4.10 *1. We may assume by* ($* * *$) *of Case 1, there is no* $e' \neq e$ *in* $E - X$ *such that* $e' \sim x_3$*. Otherwise,* Γ *is mutation-infinite.*

2. *We may assume there cannot be two distinct vertices* e' *and* e'' *in* $E - X$ *such that* $e', e'' \sim x_1$ *(Note that unlike the above fact, one of* e', e'' *may be e.). Otherwise,* Γ *is mutation-infinite.*

Proof.

For each assumption above we will give proofs of them (i.e. if our assumptions in the lemma were wrong then Γ would be mutation-infinite.).

Pf.of (1): For $e' \neq e$ in $E - X$, $e' \nsim x_4, x_5$. Therefore, discarding e from the subdiagram we are in the same situation as in Case 1. Therefore, by ($* * *$) of Case 1, there is no such e'. Otherwise, Γ is mutation-infinite.

Pf.of (2): Otherwise, let e', e'' be such vertices that is adjacent to x_1. Then there are two cases. e could be one of the vertices e', e'' or not. First consider the case where e is not one of them. Then if they are not adjacent there is e''' that is adjacent to exactly one of e', e''. Suppose without loss of generality that $e''' \sim e''$. In this case if e''' is non-adjacent to any other vertex in the subdiagram we have subdiagram of Type 4. Now if $e = e'''$ or $e''' \sim e$ then e' is adjacent to simply-laced non-oriented cycle $\{x_1, x_3, e'', e, x_4\}$ or $\{x_1, x_3, e''', e'', e, x_4\}$ respectively via a single edge of weight 1. Now if e', e'' are adjacent to each other then e''' is adjacent to simply-laced non-oriented triangle $\{x_1, e', e''\}$ if $e''' \nsim x_1$ or and if $e''' \sim x_1$ we could consider x_3 is adjacent to $\{x_1, e', e''\}$ via a unique edge of weight 1. Now suppose $e = e''$. Then if e, e' non-adjacent then e' is adjacent to simply-laced non-oriented cycle $\{x_1, x_3, e, x_4\}$ or $\{x_1, x_3, e\}$ (where the first one of the cycles corresponding the situation $x_3 \nsim e$ besides the second one $x_3 \sim e$). Thus, e and e' must be adjacent, in which case x_4 or

x_5 is adjacent to the simply-laced non-oriented triangle $\{x_1, e', e\}$ via a unique edge of weight 1. Γ is mutation-infinite.

□

In the rest of the proof we will only be able to exhibit what possible diagrams in Figure 2.3 E could be in the light of the assumptions of this case, above lemma and working practice of earlier cases for the common vertices that will be assumed in each subcase. This is due to the fact that we need direct check in each subcase. Nevertheless we will show the exact type of some of the possible subdiagrams in Case 2.1. Note that there is at least one common vertex of E and X as in above cases. As it is explained in the beginning of the proof, if they do not share a common vertex then they would have total 11 vertices.

Case 2.1: x_4 is the unique common vertex of E and X.

Here we note that e could be adjacent to x_3 or x_1 (or both). Then we have 4 different possible subdiagrams in hand in the very beginning of each subcase. Now if there is e' (could be same as e) that is adjacent to x_1, then we have an oriented cycle with one edge of weight 4 and the vertices incident to that edge of weight 4 is only adjacent to e'. Here, we have the opportunity to discard x_3 from the subdiagram or else we employ the triangle $\{x_1, x_2, x_3\}$. Here we note that, since there is no other vertex of E that is adjacent to x_4 by assumption, valency of x_4 is 1 in E being adjacent only to e. Then possible subdiagrams E (see Figure 2.3) such that our diagram Γ could have as a subdiagram are (1),(2),(3),(6),(7),(8),(10),(12),(13),(14),(15). For each choice and for each various connection types the subdiagram determined by the vertices of X and E are one of Type 1,...,5. For instance if $E = (13)$ and e is non-adjacent to any other vertex. Then we have subdiagram of Type 1. At this point, if $E = (7)$ and $e \sim x_1$ we have a subdiagram of Type 2. Thus Γ is mutation-infinite.

Case 2.2: x_3 is the unique common vertex of E and X.

Then $e \sim x_3$. There is no other vertices in Γ that is adjacent to x_3 by the above lemma. Then the possible E's are (1),(2),(3),(6),(7),(8),(10),(12),(13),(14),(15). For each choice and for each various connection types by direct check, the subdiagram determined by the vertices of X and E are one of Type 1,...,5.

Case 2.3: x_1 is the unique common vertex of E and X.

Then x_1 could not be contained in any cycle in E. The possible E's are (1), (2), (3), (6), (7), (8), (10), (12), (13), (14), (15) by inspection. For each choice and for each various connection types by direct check, the subdiagram determined by the vertices of X and E are one of Type 1,...,5. Therefore, Γ is mutation-infinite.

Case 2.4: x_3, x_4 are exactly the common vertices of E and X.

Since there is no vertex in $E - X$ other than e could be adjacent to x_3. Now if $e \sim x_3$ we have a triangle in E. In this situation valency of x_3 and x_4 are 2 and they are contained in the same triangle in E. Therefore, the possible cases are (7),(8). Now if E is (7) or (8) then we have a subdiagram of Type 1 or Type 2. Now if $e \nsim x_3$. Then valency of x_3 is 1 and valency of x_4 is 2 in E. They are adjacent but not contained in any cycle in E. Then E could be (1) or (7). Then we have a subdiagram that fits one of Type 1 or Type 2. Therefore, Γ is mutation-infinite.

Case 2.5: x_1, x_4 are exactly the common vertices of E and X.

Now x_1 cannot be contained in a cycle in E and its valency must be 1 in E since x_3 is not a common vertex and there is at most 1 vertex can be joined to x_1(and in fact there must be at least 1 since E is connected). Also valency of x_4 is 1 in E. By inspection possible E's are (1),(2),(7),(10),(12). Then again by direct check the subdiagram determined by the vertices of X and E are one of Type 1,...,5. Therefore, Γ is mutation-infinite.

Case 2.6: x_1, x_3 are exactly the common vertices of E and X.

Valency of x_1 is at least 1 and at most 2 in E. Also we have that x_1 could not be contained in a cycle in E and the valency of x_3 in E is at least 1 and at most 2. However, e cannot be adjacent to x_1; and x_3 cannot be contained in any cycle in E. Also note that x_1 and x_3 are adjacent. Then possible E's are (1) and (7). Then by direct check, the subdiagram determined by the vertices of X and E are one of Type 1,...,5. Therefore, Γ is mutation-infinite.

Case 2.7: x_1, x_3, x_4 are exactly the common vertices of E and X.

In this case, valency of x_1 is at least 1 and at most 2 in E. We also have that x_1 could not be contained in a cycle in E and valency of x_3 in E is at least 2 and at most 3. This is because there is no vertex in $E-X$ other than e, could be adjacent to x_3. Note also that if x_3 is contained in a cycle(such a cycle must be the triangle $\{x_3, x_4, e\}$) in E then its valency is 3 in E and otherwise its valency is 2 in E. Moreover, the valency of x_4 in E is 2. Also taking adjacencies into account we obtain the fact that there is no diagram in Figure 4.2 E could be. Thus, there is no diagrams to investigate.

Let Γ contain a subdiagram in the form '2-4s'

We will prove this case considering the various possibilities of common vertices of E and X.

Before commencing the proof we will state a fact which will be valid in each possible common vertex combination.

$(*)$ If some vertex e of $E - X$ is adjacent to the middle vertex x_3 which is contained in the both of the triangles, then e must be adjacent to all other vertices of X. Otherwise, if e is adjacent to no other vertex of X then we obtain subdiagram in the form of X_6 and E is already in the mutation class of E_6 then Γ contains an X_6 and a subdiagram mutation-equivalent to E_6 hence it is mutation-infinite. Now if e is adjacent to only one triangle that is $e \sim x_1, x_2$ but $e \nsim x_4, x_5$, Then the triangle $\{x_1, x_3, e\}$ is simply-laced non-oriented cycle and x_4 is adjacent to this triangle with a unique edge of weight 1 therefore Γ is mutation-infinite. We note that the roles of x_1, x_2 and x_4, x_5 are the same and the situation symmetric in each. Hence, we will use the representatives x_1,x_4 in each triangles. And when we will consider exactly one vertex from one of the two triangles, this vertex will be x_1. Now we start to work in cases:

Case 1: x_1, x_3, x_4 are exactly the common vertices of E and X.

Now if there is $e \in X - E$ such that $e \sim x_3$, by $(*)$ we must have e is adjacent to all other vertices of X, however in this case the triangle $\{x_1, x_3, e\}$ is a non-oriented triangle in E which is impossible since E does not have non-oriented cycles. Thus there is no such e which is adjacent to x_3. Therefore, valency of x_3 in E is 2 being adjacent to only x_1, x_4. Note also that since E is connected there is at least one e in

$E - X$ that is adjacent to one of the common vertices and it cannot be adjacent to x_3, without loss of generality we may assume $e \sim x_1$ and this makes valency of x_1 in E is at least 2. We have one more datum that is each of x_1, x_3, x_4 cannot be contained in cycles in E otherwise we have a non-oriented cycle in E which is not possible. Then by inspection, there is only one possibility for E and that is E must be in the form of (1) in Figure 2.3 and thus Γ contains a subdiagram of Type 1 by direct check. Thus, Γ is mutation-infinite.

Case 2: x_1, x_3 are exactly the common vertices of E and X.

There is no vertex $e \in E - X$ that is adjacent to x_3 by the above argument in Case 1. Then valency of x_3 in E is 1 and valency of x_1 is at least 2. Also we have that each of x_1, x_3 cannot be contained in cycles in E. There are only two possibilities for E and that is E could be in the form of (1) or (7) in Figure 2.3 by inspection and to get rid of the possibility for a vertex in $E - X$ to be adjacent to x_4 or x_5 we discard them from the subdiagram and thus Γ contains a subdiagram of Type 1 or Type 2 respectively if E is of the form (1) or (7) by direct check. Therefore, Γ is mutation-infinite.

Case 3: x_1, x_4 are exactly the common vertices of E and X.

3.1: There is $e \in E - X$ such that $e \sim x_3$.

Then for such e, e is adjacent to all other vertices of X by ($*$). Now consider $\{x_1, x_2, x_4, x_5, e\}$ is a subdiagram of Γ of the form '2-4s', discarding x_3 from the subdiagram. Thus, we consider the vertex e as x_3 then mimic the proof of Case 1 which is already closed.

3.2: There is no $e \in E - X$ such that $e \sim x_3$.

Then by direct check on the diagrams in Figure 2.3 which has two vertices each of which is not contained in a cycle we have that Γ is mutation-infinite.

Case 4: x_1 is the unique common vertex of E and X.

4.1: There is $e \in E - X$ such that $e \sim x_3$.

Then for such e, e is adjacent to all other vertices of X by ($*$). Now consider $\{x_1, x_2, x_4, x_5, e\}$ is a subdiagram of Γ of the form '2-4s', discarding x_3 from the subdiagram. Thus, we consider the vertex e as x_3 then we are back in the Case 2 which is already closed.

4.2: There is no $e \in E - X$ such that $e \sim x_3$.

Discard x_4, x_5 and to get rid of the possibilities what if $e \in E - X$ is adjacent to one of them. Then by direct check on the diagrams in Figure 2.3 which has a vertex not contained in a cycle we have that Γ is mutation-infinite.

Case 5: x_3 is the unique common vertex of E and X.

There must exist $e \in E - X$ which is adjacent to x_3. Then for such e, e is adjacent to all other vertices of X by $(*)$. Now consider the subdiagram determined by the vertices $\{x_1, x_2, x_4, x_5, e\}$. Then this subdiagram of Γ is of the form '2-4s'. Thus, any other vertex say e' which is adjacent to e must be adjacent to x_1, x_2, x_4, x_5. Therefore, inductively, all vertices of E must be adjacent to each of x_1, x_2, x_4, x_5 and this is trivially true for x_3. We also have the fact that valency of any vertex of a diagram that is mutation equivalent to E_6 cannot exceed 4. Then there is a pair e, e' in $E - X$ s.t $e \sim x_3$ $e' \sim e$ and $e' \nsim x_3$ However, in this case consider the subdiagram determined by the vertices $\{x_1, x_3, e, e'\}$. Now this diagram does not contain any oriented cycles but contains exactly two non-oriented cycles. Hence Γ is mutation-infinite.

Thus we have proven the lemma for an arbitrary Γ that satisfies the conditions of the lemma. Now let Γ be a diagram which is mutation equivalent to any of $E_6^{(1,1)}$, $E_7^{(1,1)}$ and $E_8^{(1,1)}$ and suppose Γ satisfies conditions of the lemma then Γ would be mutation-infinite since any diagram that is mutation-equivalent to one of $E_6^{(1,1)}$, $E_7^{(1,1)}$ and $E_8^{(1,1)}$ contains a subdiagram that is mutation equivalent to E_6. This is because of the fact that each of $E_6^{(1,1)}$, $E_7^{(1,1)}$ and $E_8^{(1,1)}$ contains a subdiagram of the form of E_6. Then one may apply [8, Theorem 1.4] to get the result. Meanwhile, it is well-known that $E_6^{(1,1)}$, $E_7^{(1,1)}$ and $E_8^{(1,1)}$ are mutation-finite(see [2]), thus Γ has to be mutation-finite being mutation-equivalent to a mutation-finite diagram. Therefore, Γ cannot have any of 'The Ears' and '2-4s' as a subdiagram.

□

Now the proof of the Proposition 4.4 will follow as corollary:

Proof. (**of Proposition 4.4**) We will prove the contra-positive statement of the proposition. Let Γ be a diagram that is mutation equivalent one of $E_6^{(1,1)}$, $E_7^{(1,1)}$ and $E_8^{(1,1)}$. Suppose Γ has a quasi-Cartan companion A. Let k be a vertex of Γ and $\Gamma' = \mu_k(\Gamma)$ and $A' = \mu_k(A)$. Now by Lemma 4.5, if A' is non-admissible then A is non-admissible or Γ contains a diagram from Figure 2.6. However, by Lemma 4.6, if a diagram contains one of the diagrams in Figure 2.6 then it is mutation equivalent to a diagram that contains 'The Ears' or '2-4s' as a subdiagram. At this point, without loss of generality, we may assume Γ contains one of the 2 such diagrams as a subdiagram. Then by Proposition 4.2, Γ must contain a subdiagram mutation-equivalent to E_6. Now, by Lemma 4.9, Γ is mutation-infinite which is impossible since Γ is mutation equivalent to one of $E_6^{(1,1)}$, $E_7^{(1,1)}$ and $E_8^{(1,1)}$, each of which is known to be mutation-finite. Then Γ itself must also be mutation-finite. Therefore, the diagram Γ cannot contain a diagram in Figure 2.6. Thus A is non-admissible. □

Now we are in the position to prove the Main Theorem (Theorem 4.1) as a corollary in the light of lemmas and propositions above:

Proof. (**of Main Theorem**) Let Γ be a diagram that is mutation equivalent to one of $E_6^{(1,1)}$, $E_7^{(1,1)}$ and $E_8^{(1,1)}$. Note that by Proposition 4.3, each of $E_6^{(1,1)}$, $E_7^{(1,1)}$ and $E_8^{(1,1)}$ has a semipositive admissible quasi-Cartan companion of corank 2. Now if Γ is obtained from one of $E_6^{(1,1)}$, $E_7^{(1,1)}$ and $E_8^{(1,1)}$ by a finite sequence S of mutations and A is a matrix obtained from the corresponding quasi-Cartan companion by the same S. Then by Proposition 4.4, A is an admissible quasi-Cartan companion of Γ. We also know that equivalent quasi-Cartan matrices have the same corank and if one of them is semipositive then so is the other. We also have that finite sequence of mutations applied to an admissible quasi-Cartan matrix at each step, creates a matrix that is equivalent (as equivalence of quasi-Cartan matrices) to the original one. Therefore, Γ has a semipositive admissible quasi-Cartan companion of corank 2 . □

CHAPTER 5

REFLECTION GROUP RELATIONS

Even though most of the discussions in this chapter are valid for skew-symmetrizable and their symmetrizable quasi-Cartan companions, for the sake of simplicity we will work with skew-symmetric matrices and their symmetric quasi-Cartan companions. Throughout the chapter, B_0 will denote the skew-symmetric matrix whose diagram is one of $E_6^{(1,1)}$, $E_7^{(1,1)}$, $E_8^{(1,1)}$ and A_0 be an admissible quasi-Cartan companion corresponding to B_0. Now we know that from the previous chapter, if B is in the mutation class of B_0, then we obtain A, an admissible quasi-Cartan companion for B via the same set of mutations that changes B_0 into B.

Symmetric quasi-Cartan companions could be considered as the Gram matrices of some symmetric bilinear form. Also under a change of basis, a Gram matrix turn into a congruent matrix,i.e. if G is a Gram matrix and M be the change of basis matrix then the Gram matrix of the bilinear form with respect to the new basis is M^TGM. We also know that any two admissible companions of the same matrix is equivalent(as quasi-Cartan companions) which is a matrix congruence with $detM$ is 1 or -1. Similarly mutation-equivalent quasi-Cartan companions are equivalent quasi-Cartan matrices (hence congruent). Note also that mutation of an admissible quasi-Cartan companion is a quasi-Cartan companion of the corresponding skew-symmetric matrices.

Now let A_0 be the Gram matrix of some symmetric bilinear form. That is $A_{0_{ij}} = (e_i, e_j)$ where $\{e_1, .., e_n\}$ is the standard basis. For each e_i, we have a reflection s_{e_i} such that $s_{e_i}(e_j) = e_j - (e_j, e_i^\vee)e_i = e_j - A_{0_{ij}}e_i$. Now the group generated by the reflections is called the reflection group associated to $e_1, ..., e_n$. Throughout the chapter the symmetric bilinear form denoted by (.,.) will correspond to a symmetric quasi-Cartan companion. Note that in our setting 'standard basis' is used by no means to imply the basis elements are orthogonal. In fact they are not orthogonal under the symmetric bilinear form (.,.) determined by A_0.

Definition 5.1 *For any vector α, we say α is non-isotropic (w.r.t. a fixed symmetric bilinear form) if $(\alpha, \alpha) \neq 0$. Now for any non-isotropic vector α of a symmetric bilinear form (.,.), we define $\alpha^\vee = \frac{2\alpha}{(\alpha,\alpha)}$. Moreover, the linear operator s_α defined by $s_\alpha(\beta) = \beta - (\beta, \alpha^\vee)\alpha$ for any vector β, is called the reflection corresponding to α.*

Lemma 5.2 *For any non-isotropic vector α, the reflection s_α has order 2 and s_α fixes(pointwise) the hyperplane that is orthogonal to α and sends the elements of the line generated by α to its negative. Moreover, s_α is recovered by this data.*

Proof.
$$\begin{aligned} s_\alpha^2(\beta) &= s_\alpha(s_\alpha(\beta)) \\ &= s_\alpha(\beta) - (s_\alpha(\beta), \alpha^\vee)\alpha \\ &= \beta - 2(\beta, \alpha^\vee)\alpha + (\beta, \alpha^\vee)(\alpha, \alpha^\vee)\alpha \\ &= \beta - 2(\beta, \alpha^\vee)\alpha + 2(\beta, \alpha^\vee)\alpha \\ &= \beta \end{aligned}$$

Therefore, the order of s_α is 2. Now to see s_α fixes the hyperplane orthogonal to α; let β be a vector such that $(\alpha, \beta) = 0$. Then we have $(\beta, \alpha^\vee) = 0$. Thus, $s_\alpha(\beta) = \beta - (\beta, \alpha^\vee)\alpha = \beta$ which shows s_α fixes(pointwise) the hyperplane. Now consider $s_\alpha(t\alpha) = t\alpha - (t\alpha, \alpha^\vee)\alpha = t\alpha - 2t\alpha = -t\alpha$ which shows s_α sends the elements of the line generated by α to its negative. Moreover, trivially any transformation that satisfies these properties is equal to s_α.

□

Definition 5.3 *Let $\{e_1, .., e_n\}$ be the standard basis for the n-dimensional vector space $\mathbb{Z}^n$ equipped with symmetric bilinear form $(.,.)$ determined by a symmetric quasi-Cartan companion A_0. Then the reflections s_{e_i} will be called principal reflections for each $i = 1, ...n$.*

Definition 5.4 *(***Reflection Group***) The reflection group W generated by the principal reflections is called principal reflection group.*

Definition 5.5 *(**Canonical vector**) Let β be a vector such that, there is $w \in W$ and an e_k s.t $\beta = w(e_k)$ then β will be called a canonical vector.*

Lemma 5.6 *Let α be a non-isotropic vector and s_α is the corresponding reflection . Then s_α is an orthogonal transformation and if w be an element of W. Then $ws_\alpha w^{-1} = s_{w(\alpha)}$.*

Proof. First consider,

$$
\begin{aligned}
(s_\alpha(x), s_\alpha(y)) &= (x - (x, \alpha^\vee)\alpha, y - (y, \alpha^\vee)\alpha) \\
&= (x, y) - (\frac{2(x, \alpha)(y, \alpha)}{(\alpha, \alpha)}) - (\frac{2(x, \alpha)(y, \alpha)}{(\alpha, \alpha)}) + \frac{2(x, \alpha)}{(\alpha, \alpha)}\frac{2(y, \alpha)}{(\alpha, \alpha)}(\alpha, \alpha) \\
&= (x, y)
\end{aligned}
$$

which shows s_α is an orthogonal transformation. Also, $ws_\alpha w^{-1}(w(\alpha)) = -w(\alpha)$. Thus, $ws_\alpha w^{-1}$ sends elements in the line generated by $w(\alpha)$ to its negative. Now to see $ws_\alpha w^{-1}$ fixes hyperplane orthogonal to $w(\alpha)$ note that w is orthogonal transformation and $w(\gamma)$ is in the hyperplane orthogonal to $w(\alpha)$ if and only if γ is in the hyperplane orthogonal to α. Then $ws_\alpha w^{-1}(w(\gamma)) = ws_\alpha(\gamma) = w(\gamma)$. Therefore, $ws_\alpha w^{-1} = s_{w(\alpha)}$. □

Lemma 5.7 *If e_i is an element of the standard basis for V and w is an element of the principal reflection group W then $(w(e_i))^\vee = w(e_i)$. (i.e. for each canonical vector β we have $\beta^\vee = \beta$). In particular, each canonical vector β is non-isotropic.*

Proof. It is enough to check

$$
\begin{aligned}
(s_{e_j}(e_i))^\vee &= (e_i - (e_i, e_j)e_j)^\vee \\
&= \frac{2(e_i - (e_i, e_j)e_j)}{(e_i - (e_i, e_j)e_j, e_i - (e_i, e_j)e_j)} \\
&= \frac{2(e_i - (e_i, e_j)e_j)}{(e_i, e_i) - (e_i, e_j)^2 - (e_i, e_j)^2 + (e_i, e_j)^2(e_j, e_j)} \\
&= \frac{2(e_i - (e_i, e_j)e_j)}{(e_i, e_i)} \\
&= (e_i - (e_i, e_j)e_j).
\end{aligned}
$$

$= s_{e_j}(e_i).$

Therefore, the result follows. □

Definition 5.8 *(***Companion Basis***): A basis $\{\beta_1, ..., \beta_n\}$ of the lattice generated by standard basis $e_1, ..., e_n$ will be called companion basis for the skew-symmetric matrix B or its quasi-Cartan companion A if $A_{ij} = (\beta_i, \beta_j^\vee)$ where each β_i is a canonical vector.*

Definition 5.9 *Let $\{\beta_1, ..., \beta_n\}$ be a companion basis for a skew-symmetric matrix B or its quasi-Cartan companion A. Then the mutation $\mu_k(\beta_i)$ for $k = 1, ..., n$ is defined as follows:*

- $\mu_k(\beta_i) = \beta_i - A_{ik}\beta_k$ *if $i \neq k$ and $B_{ik} > 0$*
- $\mu_k(\beta_i) = \beta_i$ *if $i \neq k$ and $B_{ik} \leq 0$*
- $\mu_k(\beta_k) = -\beta_k$

Lemma 5.10 *Let A be a symmetric admissible quasi-Cartan companion and also $\{\beta_1, ..., \beta_n\}$ be a companion basis for A such that $A_{ij} = (\beta_i, \beta_j)$ then $A' = \mu_k(A)$ is a quasi-Cartan companion with companion basis $\{\beta_1', ..., \beta_n'\}$ where $\beta_i' = \mu_k(\beta_i)$ such that $A_{ij}' = (\beta_i', \beta_j')$. Moreover, $\{s_{\beta_1'}, ..., s_{\beta_n'}\}$ generates the same group as $\{s_{\beta_1}, ..., s_{\beta_n}\}$ does .*

Proof. Since A is admissible then A' is a quasi-Cartan companion. Now if $B_{ik} \leq 0$ then $(\beta_i', \beta_k') = (\beta_i, -\beta_k) = -A_{ik}$. Now since $A_{ik}' = \text{sgn}(B_{ik})A_{ik}$ we have $A_{ik}' = -A_{ik}$ therefore $A_{ik}' = -A_{ik} = (\beta_i', \beta_k')$. Now suppose $B_{ik} > 0$. Then $(\beta_i', \beta_k') = (\beta_i - A_{ik}\beta_k, -\beta_k) = -A_{ik} + A_{ik}A_{kk} = -A_{ik} + 2A_{ik} = A_{ik}$ and $A_{ik}' = \text{sgn}(B_{ik})A_{ik}$ hence $A_{ik}' = A_{ik}$ therefore $A_{ik}' = A_{ik} = (\beta_i', \beta_k')$. Note that since A and A' is symmetric there is no need to check A_{ki}'.

Now for the rest of the proof suppose $i \neq k$ and $j \neq k$.

Case 1: $B_{ik} \leq 0$ and $B_{jk} \leq 0$.

$(\beta'_i, \beta'_j) = (\beta_i, \beta_j) = A_{ij}$ and,

$A'_{ij} = A_{ij} - \text{sgn}(A_{ik}A_{kj})[B_{ik}B_{kj}]_+$.

Now since $B_{jk} \leq 0$ we have $B_{kj} \geq 0$ hence

$[B_{ik}B_{kj}]_+ = 0$. Therefore, $A'_{ij} = A_{ij} = (\beta'_i, \beta'_j)$.

Case 2: $B_{ik} > 0$ and $B_{jk} \leq 0$

$(\beta'_i, \beta'_j) = (\beta_i - A_{ik}\beta_k, \beta_j) = A_{ij} - A_{ik}A_{kj}$

Now if $B_{jk} = 0$ we have $A_{jk} = A_{kj} = 0$(or $B_{kj} = 0$) and,

$A'_{ij} = A_{ij} - \text{sgn}(A_{ik}A_{kj})[B_{ik}B_{kj}]_+ = A_{ij}$ therefore,

$(\beta'_i, \beta'_j) = A'_{ij}$

Now suppose $B_{jk} < 0$. Then we have $B_{kj} > 0$. Now if A_{ik},A_{kj} have the same signs then,

$A'_{ij} = A_{ij} - B_{ik}B_{kj} = A_{ij} - A_{ik}A_{kj}$ by the mutation formula, therefore,

$(\beta'_i, \beta'_j) = A'_{ij}$.

Also, if A_{ik},A_{kj} have opposite signs then,

$A'_{ij} = A_{ij} + B_{ik}B_{kj} = A_{ij} - A_{ik}A_{kj}$. Again we have

$(\beta'_i, \beta'_j) = A'_{ij}$.

Case 3: $B_{ik} > 0$ and $B_{jk} > 0$

$$\begin{aligned}(\beta'_i, \beta'_j) &= (\beta_i - A_{ik}\beta_k, \beta_j - A_{jk}\beta_k) \\ &= A_{ij} - A_{ik}A_{jk} - A_{ik}A_{kj} + A_{ik}A_{jk}A_{kk} = A_{ij} \text{ and,}\end{aligned}$$

$A'_{ij} = A_{ij} - \text{sgn}(A_{ik}A_{kj})[B_{ik}B_{kj}]_+ = A_{ij}$ therefore,

$(\beta'_i, \beta'_j) = A'_{ij}$.

Now we have to show $\{\beta'_1, ..., \beta'_n\}$ is actually a companion basis, i.e.

$A'_{ij} = (\beta'_i, (\beta'_j)^\vee)$ and each β'_i is a canonical vector corresponding to the symmetric bilinear form (.,.) and $\{\beta'_1, ..., \beta'_n\}$ is a linearly independant set of vectors. Linear independance is trivial since $\{\beta'_1, ..., \beta'_n\}$ is obtained by a single mutation from a linearly independant set $\{\beta_1, ..., \beta_n\}$. Then consider,

$$(\beta'_j)^\vee = \frac{2\beta'_j}{(\beta'_j, \beta'_j)} = \frac{2\beta'_j}{A'_{jj}} = \frac{2\beta'_j}{2} = \beta'_j$$

.

Thus the equality $A'_{ij} = (\beta'_i, (\beta'_j)^\vee)$ follows. Now to show each β'_i is a canonical vector, by definition of the mutation it is enough to consider the case $i \neq k$ and $B_{ik} > 0$. Then we have,

$$\beta'_i = \beta_i - A_{ik}\beta_k = \beta_i - (\beta_i, \beta_k)\beta_k = \beta_i - (\beta_i, \beta_k^\vee)\beta_k = s_{\beta_k}(\beta_i)$$

Thus β'_i is a canonical vector. Therefore, $\{\beta'_1, ..., \beta'_n\}$ is a companion basis for A'.

Now to show they generate the same group consider $s_{\beta'_i}$. Then if $i \neq k$ we have, $s_{\beta'_i} = s_{s_{\beta_k}(\beta_i)} = s_{\beta_k}s_{\beta_i}s_{\beta_k}$ for $B_{ik} > 0$ and $s_{\beta'_i} = s_{\beta_i}$ for $B_{ik} \leq 0$. Meanwhile for $i = k$ we have $s_{\beta'_k} = s_{-\beta_k} = s_{\beta_k}$. Thus, symmetrically for $B_{ik} > 0$ we also have,

$$s_{\beta_i} = s_{\beta'_k}s_{\beta'_i}s_{\beta'_k}$$

Also, in the cases $B_{ik} \leq 0$ and $i = k$ we have that $s_{\beta'_i} = s_{\beta_i}$. Therefore, the groups generated by $\{s_{\beta'_1}, ..., s_{\beta'_n}\}$ and $\{s_{\beta_1}, ..., s_{\beta_n}\}$ are equal.

□

Lemma 5.11 *Let B be the matrix which lies in the mutation class of B_0 where $\Gamma(B_0)$ is one of the elliptic diagrams $E_6^{(1,1)}$, $E_7^{(1,1)}$, $E_8^{(1,1)}$. Let A_0 be admissible (symmetric) quasi-Cartan companion of B_0 .Let $\{\beta_1, ..., \beta_n\}$ (obviously n is one of 6,7,8) be a companion basis for A. Then the reflections corresponding any companion basis that is obtained via the sequence of mutations from $\{e_1, ..., e_n\}$ generate W.*

Proof. This lemma follows as corollary of the lemma 5.10 above by using that lemma as the inductive step. The base of induction is to define the symmetric bilinear form as $(e_i, e_j) = A_{0_{ij}}$ where $\{e_1, ..., e_n\}$ is the standard basis. By the above discussions trivially $\{e_1, ..., e_n\}$ is a companion basis for A_0. Now since $\{s_{e_1}, ..., s_{e_n}\}$ generate W; by the above discussion in the lemma 5.10, the result follows. □

The following theorem exhibits properties on orders of reflections or certain compositions of reflections corresponding to canonical vectors making use of admissible companions and proves equality of reflection groups generated by distinct companion bases.

Theorem 5.12 *Let B be the matrix which lies in the mutation class of B_0 where $\Gamma(B_0)$ is one of the elliptic diagrams $E_6^{(1,1)}$, $E_7^{(1,1)}$, $E_8^{(1,1)}$. Let A_0 be admissible (symmetric) quasi-Cartan companion of B_0 and W be the principal reflection group corresponding to A_0. Suppose A is an admissible quasi-Cartan companion of B and let $\{\beta_1, ..., \beta_n\}$ (obviously n is one of 6,7,8) be a companion basis for A. Then the reflections $\{s_i = s_{\beta_i} : i = 1, ..., n\}$ satisfy the following relations:*

1. $s_i^2 = e$
2. *For vertices i, j in $\Gamma(B)$ $(s_i s_j)^{m_{ij}} = e$ for m_{ij} as in Lemma 5.13*
3. *For any oriented cycle $C = \{1, ..., d\}$ in $\Gamma(B)$ and any vertex i in C, taking $|A_{jk}|$ as $|A_{j'k'}|$ if $j = j' \pmod{d}$ for $j > d$; $k = k' \pmod{d}$ for $k > d$ and s_l as $s_{l'}$ if $l = l' \pmod{d}$ for $l > d$ where $1 \leq j', k', l' < d$. Then we have,*

   $(s_i s_{i+1} ... s_{i+d-1} s_{i+d-2} ... s_{i+1})$ has order m where m is determined by the following condition:
   Let $p = (|A_{i(i+1)}|...|A_{(i+d-2)(i+d-1)}| - |A_{(i+d-1)(i+d)}|)^2$, Now,

   (a) *if $p = 0$ then $m = 2$*
   (b) *if $p = 1$ then $m = 3$*

Moreover, the Coxeter group generated by the reflections $\{s_i = s_{\beta_i} : i = 1, ..., n\}$ is the same group as W.

We need the following lemma to prove the thorem.

Lemma 5.13 *Let $\{\beta_i, \beta_j\}$ (for $i \neq j$) be linearly independant canonical vectors defined by symmetric quasi-Cartan companion A_0. Then the order m_{ij} of $s_{\beta_i}s_{\beta_j}$ is determined as follows: Let $x = (\beta_i, \beta_j^\vee)(\beta_j, \beta_i^\vee) = (\beta_i, \beta_j)(\beta_j, \beta_i) = (\beta_i, \beta_j)^2$. Now if $x = 0$ then $m_{ij} = 2$; if $x = 1$ then $m_{ij} = 3$ and if $x = 4$ then $m_{ij} = \infty$.*

Proof. Let V be the lattice defined by non-degenerate symmetric bilinear form (.,.) corresponding to A. Firstly we note that by Lemma 5.7, β_i, β_j are non-isotropic vectors. Now consider $V' = span\{\beta_i, \beta_j\}$ in V and let V'' be the orthogonal complement of V'. Then s_{β_i} and s_{β_j} fix the elements of V'' pointwisely. Thus, $s_{\beta_i}s_{\beta_j}$ fixes V'' pointwisely. Therefore, to determine the order of $s_{\beta_i}s_{\beta_j}$ it is enough to find the order of $s_{\beta_i}s_{\beta_j}$ on V'. Now, without loss of generality we can take $i = 1$ and $j = 2$ for the sake of simplicity and we use the symbols s_1, s_2 for s_{β_1}, s_{β_2} respectively. We also note that $A_{12} = (\beta_1, \beta_2^\vee) = (\beta_2, \beta_1^\vee) = (\beta_1, \beta_2)$,

Case 1: $x = 0$

Then,

$s_1s_2(c_1\beta_1 + c_2\beta_2) = -c_1\beta_1 - c_2\beta_2$.

Then,

$(s_1s_2)^2(c_1\beta_1 + c_2\beta_2) = s_1s_2(-c_1\beta_1 - c_2\beta_2) = c_1\beta_1 + c_2\beta_2$

Thus, the order of s_1s_2 on V' is 2. Then $m_{ij} = 2$

Case 2: $x = 1$

Subcase 2.1: $(\beta_1, \beta_2) = 1$

Then,

$s_1s_2(c_1\beta_1 + c_2\beta_2) = c_2\beta_1 - (c_1 + c_2)\beta_2$.

Hence,

$(s_1s_2)^2(c_1\beta_1 + c_2\beta_2) = s_1s_2(c_2\beta_1 - (c_1 + c_2)\beta_2) = -(c_1 + c_2)\beta_1 + c_1\beta_2$

Therefore,

$(s_1s_2)^3(c_1\beta_1 + c_2\beta_2) = s_1s_2(-(c_1 + c_2)\beta_1 + c_1\beta_2) = c_1\beta_1 + c_2\beta_2$.

Subcase 2.2: $(\beta_1, \beta_2) = -1$

Then,

$s_1s_2(c_1\beta_1 + c_2\beta_2) = -c_2\beta_1 + (c_1 - c_2)\beta_2$.

Thus,

$(s_1s_2)^2(c_1\beta_1 + c_2\beta_2) = s_1s_2(-c_2\beta_1 + (c_1 - c_2)\beta_2) = (c_2 - c_1)\beta_1 - c_1\beta_2$
Therefore,
$(s_1s_2)^3(c_1\beta_1 + c_2\beta_2) = s_1s_2((c_2 - c_1)\beta_1 - c_1\beta_2) = c_1\beta_1 + c_2\beta_2.$
Thus the order of s_1s_2 on V' is 3. Therefore, $m_{ij} = 3$

Case 3: $(\beta_1, \beta_2) = \pm 2$.

Then when n goes large, absolute value of the coefficient of β_1 in $(s_1s_2)^n(\beta_2)$ increases strictly to diverge to ∞. Thus, $m_{ij} = \infty$

□

Proof. of the Theorem 5.12

We first consider,
(i): For any i, we have the reflection s_i that fixes all the vectors pointwisely in the hyperplane which is orthogonal to β_i and sends each vector in the line generated by β_i to its negative. Thus, $(s_i)^2 = e$ and obviously $(s_i) \neq e$ therefore the order of s_i is 2.

Now we will consider (ii). Any pair $\{\beta_i, \beta_j\}$ consists of linearly independant vectors as they are being a part of the companion basis. Then (ii) directly follows by the lemma where m_{ij}'s are as in the lemma 5.13.

Now we have to consider (iii). To show this consider the two cases $p = 0$ or $p = 1$. There is no other possibility as our diagrams are connected and correspond to admissible semipositive symmetric matrices. Thus, any three vertex diagram with an edge of weight 4 is triangle with possible weight triples as (4,1,1) and (4,4,4) however if a vertex connects to a triangle (4,4,4) then it creates a non-oriented cycle with an edge of weight 4 but it is impossible in our case. Now we have that $p = 0$ iff all edges of C have weight 1; and $p = 1$ iff C is triangle in the form (4,1,1). Then consider,

Case 1: $p = 0$

Then all edges of C have weight 1 and by abusing notation we denote restriction of A to C by A, then without loss of generality we may assume $A_{1d} = A_{d1} = 1 > 0$ and the rest of the off-diagonals of A are -1 for A to be admissible. Otherwise, we change signs at some vertices which creates equivalent admissible companion and companion basis corresponding to that companion. Also without loss of generality we may assume $i = 1$ for the sake of simplicity. Then,

$(s_2...s_d...s_3s_2) = s_{s_2s_3...s_{d-1}(\beta_d)}$

and so,

$(s_1s_2...s_d...s_3s_2) = s_1s_{s_2s_3...s_{d-1}(\beta_d)}$

Now since,

$s_2s_3...s_{d-1}(\beta_d) = (\beta_d + ... + \beta_2).$

We have,

$s_{s_2s_3...s_{d-1}(\beta_d)} = s_{(\beta_d+...+\beta_2)}.$

Now we note that $\beta_1, (\beta_d + ... + \beta_2)$ is a linearly independant pair of canonical vectors otherwise $\{\beta_1, ...\beta_n\}$ would not be a companion basis and we also have $(\beta_1, (\beta_d + ... + \beta_2)) = 0$. Now $x = (\beta_1, (\beta_d + ... + \beta_2))^2 = 0$ as it is defined in the Lemma 5.13. Now by the same lemma,

$(s_1s_2...s_d...s_3s_2) = s_1s_{s_2s_3...s_{d-1}(\beta_d)}$ has order 2.

Case 2: $p = 1$

Now C must be triangle(4,1,1) as it is discussed above. Then let $C = \{1, 2, 3\}$. Also by abusing notation we denote restriction of A to C by A Now without loss of generality assume $i = 1$. Then since $s_2s_3s_2 = s_{s_2(\beta_3)}$ and

$s_2(\beta_3) = \beta_3 - A_{23}\beta_2$. Then $\beta_1, \beta_3 - A_{23}\beta_2$ are linearly independant pair of canonical vectors. Moreover, $(\beta_1, \beta_3 - A_{23}\beta_2) = A_{13} - A_{12}A_{23}$. Now regardless of which edge has weight 4 and how the signs are chosen to satisfy admissibilty, this number is 1 in absolute sense. Hence, $x = (\beta_1, s_2(\beta_3))^2 = 1$ as in the form in the lemma 5.13. Therefore, $(s_1s_2s_3s_2)$ has order 3.

Now the group W is generated by $\{s_i = s_{\beta_i} : i = 1, ..., n\}$ follows immediately after

Lemma 5.11. Now by part (i) and (ii) $\{s_i = s_{\beta_i} : i = 1, ..., n\}$ satisfies the same relations in the group W as $\{s_{e_1}, ..., s_{e_n}\}$ does. Then for W, $\{s_i = s_{\beta_i} : i = 1, ..., n\}$ is another presentation.

□

Theorem 5.14 *Let B be the matrix which lies in the mutation class of B_0 where $\Gamma(B_0)$ is one of the elliptic diagrams $E_6^{(1,1)}$, $E_7^{(1,1)}$, $E_8^{(1,1)}$. Let A_0 be admissible (symmetric) quasi-Cartan companion of B_0. Suppose A is admissible quasi-Cartan companion of B. Let $\{\beta_1, ..., \beta_n\}$ (obviously n is one of 6,7,8) be a companion basis for A. Then the reflections $\{s_i = s_{\beta_i} : i = 1, ..., n\}$ satisfy the first two relations in [3, Table 4.1].(The other relations corresponding to rest of the subdiagrams cannot occur in our diagrams)*

Proof. For the first relation, consider $(s_1s_2s_3s_4s_3s_2) = s_1s_{s_2s_3(\beta_4)}$. And without loss of generality we may assume $A_{13} = A_{31} > 0$ and the rest of the off-diagonal entries are non-positive. Now $s_2s_3(\beta_4) = \beta_4 + \beta_3 + \beta_2$. Then $\beta_1, \beta_4 + \beta_3 + \beta_2$ is linearly independant pair of canonical vectors. And $(\beta_1, \beta_4 + \beta_3 + \beta_2) = 0$. Thus, $x = 0$ as it is in the lemma 5.13. Therefore, the order of $(s_1s_2s_3s_4s_3s_2)$ is 2 by the lemma.

For the second one, consider,

$(s_1s_2s_3s_2s_1s_4s_5...s_ns_{(n+1)}s_n...s_5s_4) = s_{s_1s_2(\beta_3)}s_{s_4s_5...s_n(\beta_{(n+1)})}$ and then without loss of generality we may assume $A_{3(n+1)} = A_{(n+1)3} > 0$ and the rest of the off-diagonal entries are non-positive. Then we have that,
$s_1s_2(\beta_3) = \beta_1 + \beta_2 + \beta_3$ and, $s_4s_5...s_n(\beta_{(n+1)}) = \beta_4 + \beta_5 + ... + \beta_{(n+1)}$ which are linearly independant. Now consider $(\beta_1 + \beta_2 + \beta_3, \beta_4 + \beta_5 + ... + \beta_{(n+1)}) = 0$. Thus, $x = 0$ as it is in the lemma 5.13. Therefore, the order of $(s_1s_2s_3s_2s_1s_4s_5...s_ns_{(n+1)}s_n...s_5s_4)$ is 2 by the lemma.

□

CHAPTER 6

MUTATION CLASS OF A DIAGRAM ORIGINATED FROM A TRIANGULABLE SURFACE

There are diagrams, consider for example exceptional diagrams, which are not originated from triangulation of surfaces. In fact, very small number of diagrams are coming from triangulations of surfaces compared to rest of the diagrams. We will consider one specific example that is seen in the Figure 6.1. For the basics of the triangulation-diagram correspondance and for the construction of diagrams corresponding to a triangulation of a surface, see [4].

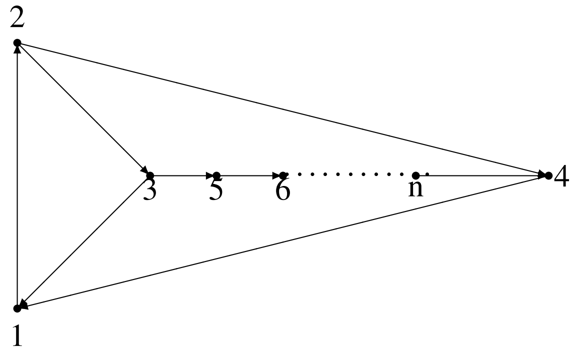


Figure 6.1: A diagram with n-vertices that is originated from a triangulation

The diagram in Figure 6.1 corresponds to the torus with exactly one boundary component and $n-3$ marked points on the boundary component when the diagram has exactly n vertices. In other words number of marked points of the triangulation is equal to the number of edges that connects the vertex 3 to vertex 4 in Figure 6.1, namely the number of edges on the middle line.

Now our aim is to observe that any element in the mutation class of the diagram has an admissible quasi-Cartan companion.

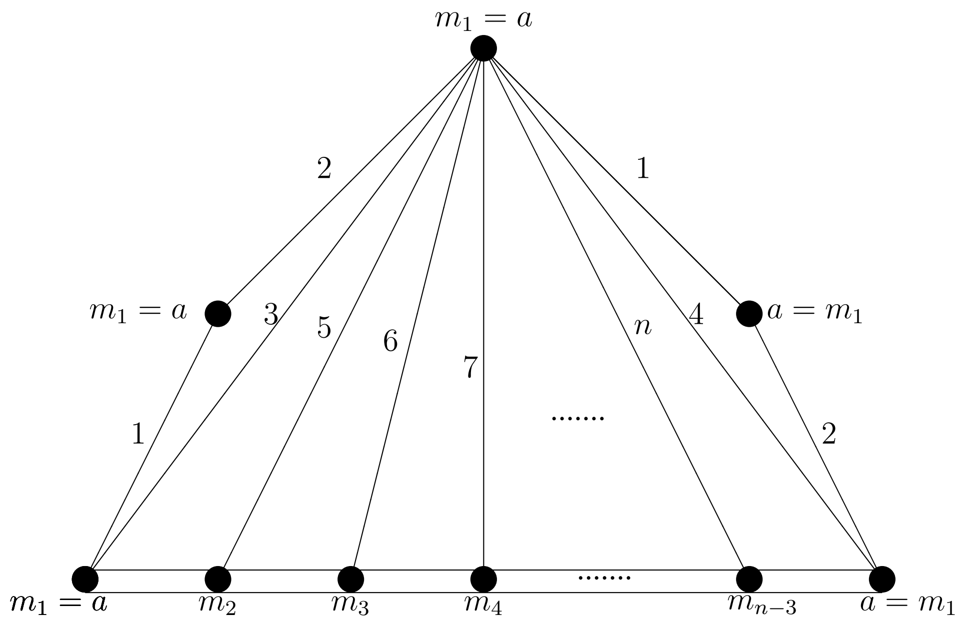


Figure 6.2: A triangulation of the surface of genus 1 with 1 boundary component and (n-3) marked points on the boundary component whose diagram corresponds to the diagram in Figure 6.1. The double line below in the triangulation stands for the boundary where m_i for $i = 1, ..., (n-3)$ denotes the (n-3) marked points in the boundary.

Theorem 6.1 *Any diagram which is mutation equivalent to the diagram in the Figure 6.1 has an admissible quasi-Cartan companion.*

To see this we have to examine the mutation class of the diagram and we will do this, examining the different triangulations obtained via sequence of flips that give the same surface. A flip at the edge k of a triangulation corresponds to mutation at the vertex k of its diagram pair.

Proof. First of all we make the following conventions:

1. Since sequence of any flips yield another triangulation of the same surface, there are two arcs that must be in the same role as arcs $1, 2$. However, then we relabel that vertices as $1, 2$ and thus we may assume arcs $1, 2$ stay as arcs of triangulation which occur at the boundary(we do not mean boundary of the surface) of the gluing scheme after any sequence of flips. In the first three cases this fact is obvious. For that reason, we will not provide an explanation. In the Case 4, we will provide a brief explanation of this fact for that case.

2. A proof given for 1 will be assumed valid for 2 since the roles of 1 and 2 are symmetric.

3. In a flipped triangulation diagram if we don't permit a certain type of arc that has one vertex common with arc labeled 1, then by symmetry we don't permit the arc with same properties for the arc 2 since the roles of 1 and 2 are symmetric. Therefore, allowing such kind of an arc for 2 would create the unwanted case for 2 as in 1 which we want to avoid.

Now there are 4 main cases for the flipped triangulations which are:

1. In the flipped triangulation there are arcs 3 and 4 where $1, 2, 3$ and $1, 2, 4$ form two ideal triangles.

2. In the flipped triangulation there is exactly one arc 3 such that $1, 2, 3$ is oriented clockwise.

3. In the flipped triangulation there is exactly one arc 3 such that $1, 2, 3$ is oriented counter-clockwise.

4. In the flipped triangulation there is no arc together with $1, 2$ that form a triangle.

Case 1: In the flipped triangulation there are arcs 3 and 4 where $1, 2, 3$ and $1, 2, 4$ form two ideal triangles.

Firstly these two ideal triangles can occur in only one way and arcs $1, 2, 3$ and $1, 2, 4$ in the triangulation must be triangles oriented clockwise. Thus, in the diagram pair

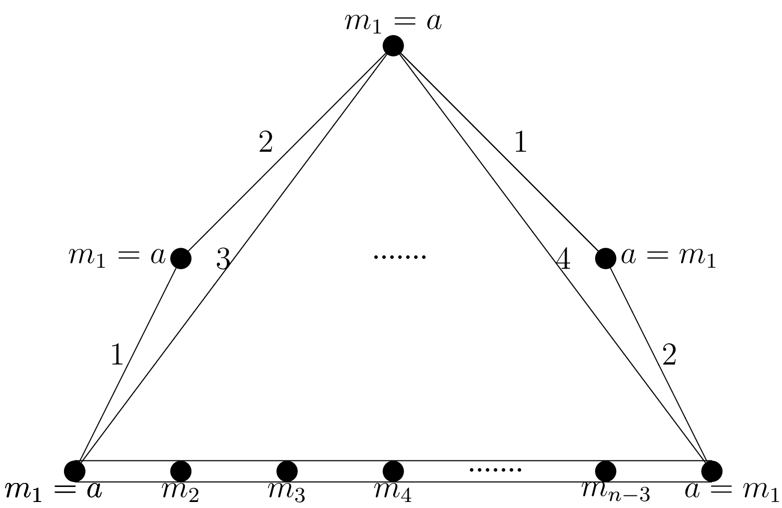


Figure 6.3: Triangulation in Case 1

of the flipped triangulation we have two oriented cycles $\{1, 2, 3\}$ and $\{1, 2, 4\}$ with an edge from 1 to 2 of weight 4. At this point, we note that regardless of the position of the rest of the arcs, 3 is connected to 4 via a path that contains only the vertices that corresponds to some subset of the arcs labeled $3, 4, 5, ..., n$. Observe also that the vertices $5, 6, ..., n$ cannot be adjacent to 1 or 2 since 3 and 4 are exactly the vertices that are adjacent to 1 or 2. Therefore, 1 and 2 are sink and source respectively and this creates 2 non-oriented cycles that is determined by the vertices $1, 3, 4$ together with the vertices of the path from 3 to 4 and; $2, 3, 4$ together with vertices of the same path. Also we have oriented triangles $\{1, 2, 3\}$ and $\{1, 2, 4\}$. These 2 non-oriented cycles and two oriented triangles are characteristic for the flipped triangulation that is considered in this case. Each of the oriented triangles $\{1, 2, 3\}$ and $\{1, 2, 4\}$ share exactly an edge with the each non-oriented cycles and these edges are different for each. Now these are the all cycles that contain 1 or 2. There could be some more cycles originated from the 'interior' arcs $3, ..., n$ but they must be oriented triangles and they cannot share sides. Hence for two such triangles only possibility is sharing a vertex. However, a non-characteristic oriented triangle either shares an edge with a characteristic non-oriented cycle and it must share the same edge with the other characteristic non-oriented cycle; or shares no edges with any other cycles.

Now an admissible structure on the quasi-Cartan companion for the diagram pair of the flipped triangulation that is our concern in this case could be constructed as follows: Let quasi-Cartan companion A assign $(+)$ to each edge of the two chracteristic oriented triangles and $(-)$ for the remaining edges of the characteristic non-oriented cycles. Therefore, A has already assigned $(-)$ to unique edge of any of non-characteristic oriented triangles that is common with characteristic ones, and let A assign one $(+)$ and one $(-)$ to the remaining two edges of each non-characteristic triangles that shares an edge with non-oriented cycles. At this point, the companion A is free while assigning signs to rest of the oriented triangles (that shares no edges with any other cycles). The companion A either assigns exactly 3 $(+)$ or exactly 1 $(+)$ to edges of the rest of the oriented triangles. Now characteristic oriented triangles have exactly 3 $(+)$ and non-oriented cycles have exactly 2 $(+)$. Also non-characteristic triangles(must be oriented) which shares an edge with a cycle have exactly 1 $(+)$. Moreover non-characteristic triangles which shares no edges with any other cycle have exactly 1 $(+)$ or exactly 3 $(+)$. These are all of the cycles our diagrams can have in Case 1. Thus admissibility follows.

Case 2: In the flipped triangulation there is exactly one arc 3 such that $1, 2, 3$ is oriented clockwise.

There are 2 subcases:

- There is an arc(loop) y connecting the free ends of arc 1 in the gluing scheme of the triangulation.
- There is no such y in the flipped tringulation.

Subcase 2.1: There is an arc(loop) y connecting the free ends of arc 1 in the gluing scheme of the triangulation.

There is an edge of weight 4 from 3 to 1. Furthermore, considering 3 as in the role of 1, 1 as in the role of 2, 3 as in the role of 4, y as in the role of the vertex 3; this is the same case as in Case 1. Thus, the admissibility follows.

Subcase 2.2: There is no such y in the flipped tringulation.

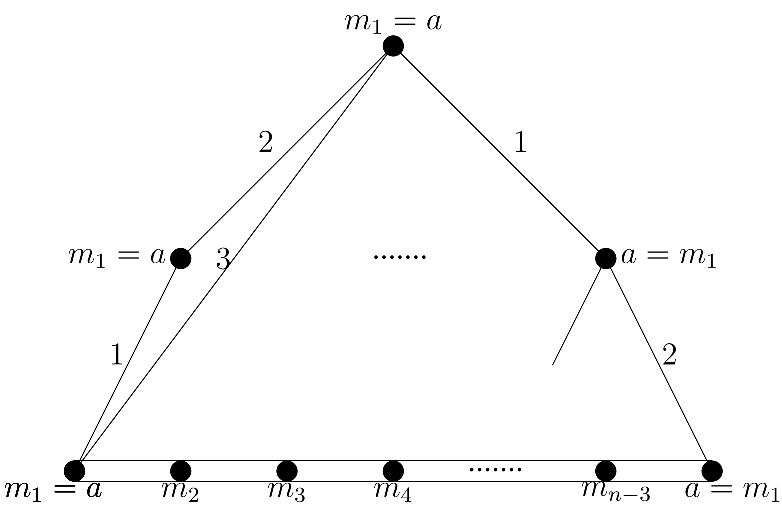


Figure 6.4: Triangulation in Case 2

Now if there is an arc that connects the lower end of the arc 1 in the left with lower end of 2 in the right in the gluing scheme, then either there must be an arc connceting free ends of 1 in the gluing scheme which is not possible by the assumption or there is an arc 4 that makes $1, 2, 4$ an ideal triangle in order this diagram to be a triangulation. However, the latter case is a subcase that is implicitly treated in Case 1. Hence, we could suppose there is no such arc. Then each of the remaining arcs start and end at different marked points at the boundary hence are non-loops.

Now $\{1, 2, 3\}$ is a simply-laced oriented triangle in the pair-diagram with an edge from 1 to 2. Now suppose without loss of generality that x is the arc connecting 'principal' marked point of triangulation indexed by a at the 'upper midpoint'(in the gluing scheme) to another marked point and belongs to an ideal triangle which contains 1. Thus, there is an edge from x to 1 in the pair diagram. Also suppose that y is the arc connecting 'principal' marked point of triangulation indexed by a at the right between $1, 2$ in the gluing scheme to another marked point and belongs to an ideal triangle which contains 2. Hence, there is an edge from y to 2 in the pair diagram.

In the pair diagram $3, x, y$ are connected via a path not containing $1, 2$. There is a vertex w that is connected to y via the path and the part of the ideal triangle $\{1, x, w\}$ and hence $\{1, x, w\}$ determines an oriented triangle with an edge from w to x in the diagram.

Now, the cycles determined by $1, 2$ together with the part of the path from w to y; 1 together with the part of the path from 3 to x; and 2 together with the part of the path from 3 to y are non-oriented. These 3 non-oriented cycles and 2 oriented triangles $\{1, 2, 3\}$ and $\{1, x, w\}$ are characteristic. Now each characteristic non-oriented cycle shares exactly one edge with each of the characteristic oriented triangles $\{1, 2, 3\}$ and $\{1, x, w\}$. For a characteristic non-oriented cycle and characteristic oriented triangles $\{1, 2, 3\}$ and $\{1, x, w\}$, there is no edge that is common to three of them. Now consider the possible remaining oriented triangles. Then the situation for non-characteristic oriented triangles is the same as in Case 1. At this point, if the quasi-Cartan companion of the diagram say A assigns signs to edges of the diagram as in Case 1, then the admissibility follows.

Case 3: In the flipped triangulation there is exactly one arc 3 such that $1, 2, 3$ is oriented counter-clockwise.

We could consider this case as the top $1, 2$ and 3 form an oriented cycle. Then there are two subcases:

- There is an arc y in the gluing scheme from lower end of left 1 to lower end of right 1 .
- There is no arc y in the gluing scheme from lower end of left 1 to lower end of right 1.

Subcase 3.1: There is an arc y in the gluing scheme from lower end of left 1 to lower end of right 1.

Here behaving 3 as 2 and 2 as 4; also y as 3 as in Case 1 then the situation is the same as in Case 1. Then admissibility follows.

Subcase 3.2: There is no arc y in the gluing scheme from lower end of left 1 to lower end of right 1.

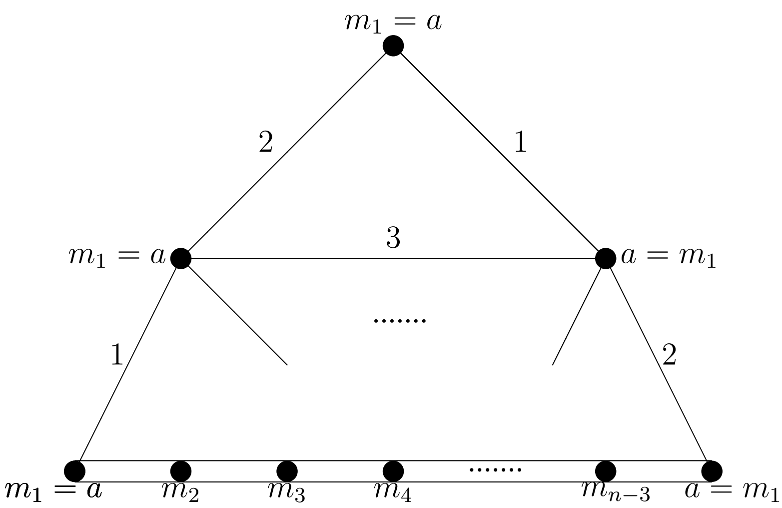


Figure 6.5: Triangulation in Case 3

Suppose there is an arc in the gluing scheme from lower end of left 1 to lower end of right 2. Then one of the diagonals must be inscribed as an arc of the triangulation which creates the one of the earlier cases. Therefore, we suppose there is no such arc.

Then there must exist two arcs x and y in the gluing scheme such that x is an arc from the principal marked point a at left in the middle to another marked point(not a) and y is an arc from a at right in the middle to a marked point which is not a. Then without loss of generality we may suppose x, y and 3 form an oriented cycle in the pair diagram, otherwise there are x' and y' that satisfies this and there is a sequence of edges from x to x' and y to y'. Now for such x, there is a path from 1 to x and a path from y to 2. Now, in the pair diagram $\{1, 2, 3\}$ and $\{x, y, 3\}$ are charecteristic oriented triangles and the cycle determined by $2, 3$ and the path connecting y to 2 is characteristic non-oriented cycle and similarly the cycle determined by $1, 3$ and the path connecting 1 to x is characteristic non-oriented cycle. Also the third characteristic non-oriented cycle is the one determined by the vertices discarding the vertex 3 . Here, between the characteristic cycles exact same properties hold as in above cases. For the other oriented triangles that are not characteristic, if they share a side with a non-oriented cycle they must share the same side with exactly two of the non-oriented cycles. Then admissibility follows if A assigns signs as in cases above.

Case 4: In the flipped triangulation there is no arc together with $1, 2$ that form an ideal triangle.

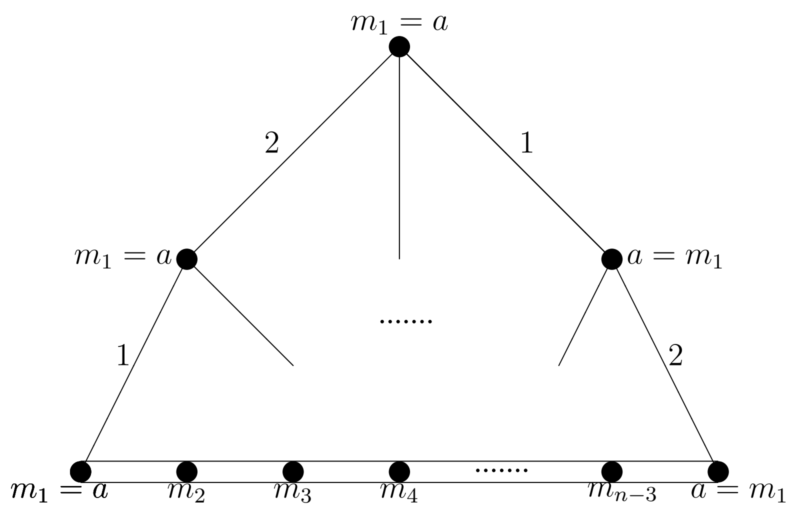


Figure 6.6: Triangulation in Case 4

We first note that if we mutate the pair diagram at the vertex 1 (or 2 symmetrically) then either we obtain the same triangulation type (relabeling the vertices) for the mutated diagram or we obtain the triangulation corresponding to one of the earlier cases.

Now, there must exist an arc x together with left-hand 1 in the gluing scheme which lies in an ideal triangle; an arc w together with left-hand 2 in the gluing scheme which lies in an ideal triangle; an arc y together with right-hand 1 in the gluing scheme which lies in an ideal triangle and an arc z together with right-hand 2 in the gluing scheme which lies in an ideal triangle. Moreover, it may be that $w = y$. However, without loss of generality, we may assume w and y are distinct. Since $w = y$ generates the same diagram up to path that connects w to y. And this path will not affect the characteristic cycles. Now in the pair diagram, $\{2, w, w'\}$ and $\{1, y, y'\}$ are the characteristic oriented triangles, where x', y' are the remaining arcs in the corresponding ideal triangles. Also, non-oriented cycles are the ones determined by $\{1, 2, y', w', z\}$; $\{1, x, w', w, y\}$; $\{2, w, y, y', z\}$. Now the characteristic cycles and non-characteristic oriented triangles share the same properties as in above cases. Thus, an admissible structure can be put on A.

Furthermore, we want to note that there is another way to see that there is an admissible quasi-Cartan companion for each element in the mutation class. First we show the existence of a semipositive admissible quasi-Cartan companion A of corank 2 of the original diagram. Then we mimic the proof of the lemma 4.5.

$$A = \begin{bmatrix} 2 & 2 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix}$$

Now, a straightforward calculation yields that kernel (radical vectors) of the linear operator that corresponds to A consists of the vectors of the form:
$(x, a-x, a, a, a, a, a, a, a, a)$ with x and a free hence A is of corank 2. At this point, if we let $x = 1$ then the first coordinate of the such radical vector is 1. Thus, if we erase first row and column of A and thus obtain a matrix A'. Now by the lemma 3.6 if A' is semipositive then so is A. However, A' is admissible quasi-Cartan companion for extended Dynkin diagram $A_n^{(1)}$ whose admissible quasi-Cartan companions are semipositive of corank 1. Therefore, A' is semipositive then so is A. Then if the elements in the mutation class of the original diagram do not contain any diagram from Figure 2.6, then any diagram in the mutation class has an admissible companion coming from the original one directly by the lemma 4.5. At this step we note the following: For each of the cases above the edges that are not contained in a cycle in the pair diagram, could only be incident to a vertex (not a vertex contained in any of the characteristic cycles) of a non-characteristic oriented triangle or incident to a

vertex that is incident to another edge which is not contained in a cycle. However, elements in the mutation class of the original diagram cannot have any such diagrams from Figure 2.6 by inspection on the elements of the mutation class that occur in Cases 1-2-3-4. Then the result follows.

□

REFERENCES

[1] M. Barot, C. Geiss, and A. Zelevinsky. Cluster algebras of finite type and positive symmetrizable matrices. *J. London Math. Soc. (2)*, 73(3):545–564, 2006.

[2] A. Felikson, M. Shapiro, and P. Tumarkin. Skew-symmetric cluster algebras of finite mutation type. *J. Eur. Math. Soc. (JEMS)*, 14(4):1135–1180, 2012.

[3] A. Felikson and P. Tumarkin. Coxeter groups and their quotients arising from cluster algebras. *International Mathematics Research Notices*, 2015.

[4] S. Fomin, M. Shapiro, and D. Thurston. Cluster algebras and triangulated surfaces. I. Cluster complexes. *Acta Math.*, 201(1):83–146, 2008.

[5] S. Fomin and A. Zelevinsky. Cluster algebras. I. Foundations. *J. Amer. Math. Soc.*, 15(2):497–529 (electronic), 2002.

[6] S. Fomin and A. Zelevinsky. Cluster algebras. II. Finite type classification. *Invent. Math.*, 154(1):63–121, 2003.

[7] B. Keller. Cluster algebras and derived categories. In *Derived categories in algebraic geometry*, EMS Ser. Congr. Rep., pages 123–183. Eur. Math. Soc., Zürich, 2012.

[8] A. I. Seven. Quivers of finite mutation type and skew-symmetric matrices. *Linear Algebra Appl.*, 433(6):1154–1169, 2010.

[9] A. I. Seven. Cluster algebras and semipositive symmetrizable matrices. *Trans. Amer. Math. Soc.*, 363(5):2733–2762, 2011.

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