LOCALIZATION TECHNIQUES IN COMPUTATION OF EQUIVARIANT J-GROUPS AND EQUIVARIANT CROSS SECTIONS OF STIEFEL MANIFOLDS

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ABSTRACT

LOCALIZATION TECHNIQUES IN COMPUTATION OF EQUIVARIANT J-GROUPS AND EQUIVARIANT CROSS SECTIONS OF STIEFEL MANIFOLDS

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Let G be a finite group and X be a finite G-connected G-CW complex. The main purpose of this dissertation is to find means for computing the equivariant J-groups, $JO_G(X)$, and then to obtain solutions for the equivariant cross section problem of Stiefel manifolds.

We give an alternative method for computing $JO_G(X)$. We do our computations for the cases: X is a free G-space, X is a trivial G-space, and X is a one point set. We find the orders of elements of $JO_G(\mathbb{F}P^k)$ for various projective spaces, and then we use the results to obtain a partial solution for the equivariant cross section problem of Stiefel manifolds.

Without using Atiyah-Segal completion theorem, we prove two methods for computing $JO(X)$. Then we show how to use these methods to find the orders of elements of $JO(X)$, and also to find $JO(X)$. Our illustrative example is $\mathbb{C}P^k$, the complex projective space.

Key words: Localization, Equivariant J-groups, Stiefel manifolds, Cross sections, Adams operations, Bott classes.

ÖZ

EKUVARYANT J-GRUPLARININ HESAPLANMASINDA LOKALİZASYON TEKNİKLERİ VE STIEFEL MANİFOLDLARININ EKUVARYANT KESİTLERİ

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G sonlu bir grup ve X de G-bağlantılı bir G-CW kompleksi olsun. Bu tezin ana amacı ekuvaryant J-grupları $JO_G(X)$' in hesaplanması ve Stiefel manifoldlarının ekuvaryant kesitlerinin varlık problemine çözüm bulmaktır.

$JO_G(X)$' in hesabı için değişik bir yöntem bulduk. Hesaplarımızı X' in serbest, aşikar, tek nokta G-uzayı olduğu durumlar için yaptık. Çeşitli projektif uzaylar için $JO_G(\mathbb{F}P^k)$' nın elementlerinin mertebesini hesapladık ve bunları Stiefel manifoldlarının ekuvaryant kesitlerinin varlığı problemine kısmi bir çözüm bulmak için kullandık.

$JO(X)$' in hesaplanması için Atiyah-Segal tamlama teoremine dayanmayan iki yöntem bulduk. Bu yöntemleri kullanarak önce $JO(X)$' in elemanlarının mertebesini, sonra da $JO(X)$' in nasıl hesaplanacağını gösterdik. Bunları örnek uzay olarak seçtiğimiz kompleks projektif uzay $\mathbb{C}P^k$ için hesapladık.

Anahtar Sözcükler: Lokalizasyon, Ekuvaryant J-grupları, Stiefel manifoldları, Kesitler, Adams operasyonları, Bott sınıfları.

TO MY FAMILY

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Chapter 1

INTRODUCTION

Equivariant J-groups emerged from the following basic question in equivariant homotopy theory: Let X be a G-space, usually, a finite G-connected G-CW complex where G is a finite group, and let ξ, η be two spherical G-fibrations over X. Are ξ and η G-fibre homotopy equivalent? To show that ξ and η are not G-fibre homotopy equivalent, one usually tries to find suitable invariants such as equivariant characteristic classes. On the other hand, to show that ξ and η are G-fibre homotopy equivalent, one usually tries to use some geometrical constructions. Our main purpose in this dissertation is to study the above question for spherical G-fibrations which come out from G-vector bundles over X.

Let $\mathbb{F}$ be one of the classical division algebras over $\mathbb{R}$, namely the real numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$, or the quaternionic numbers $\mathbb{H}$. Let $Sph_G(X)$ be the Grothendieck group of all stable G-fibre homotopy classes of spherical G-fibrations over X, and let $K\mathbb{F}_G(X)$ be the Grothendieck group of all stable isomorphism classes of $\mathbb{F}$ G-vector bundles over X. The J-homomorphism $J\mathbb{F}_G : K\mathbb{F}_G(X) \to Sph_G(X)$ is the map induced by associating to an $\mathbb{F}$ G-vector bundle E its underlying spherical G-fibration $S(E)$. We are interested in computing the image of $J\mathbb{F}_G$, which we denote by $J\mathbb{F}_G(X)$. In this direction, we prove a useful formula for computing $J\mathbb{F}_G(X)$ and compute the orders of elements of $J\mathbb{F}_G(X)$ for various projective spaces.

If $r : K\mathbb{F}_G(X) \to K\mathbb{R}_G(X)$ is the realification homomorphism, $\mathbb{F}$ being $\mathbb{C}$ or $\mathbb{H}$, then we show that the induced map $\tilde{r} : J\mathbb{F}_G(X) \to J\mathbb{R}_G(X)$ is a monomorphism. Therefore, since the problem is to determine whether $S(E) = S(F)$ in $J\mathbb{F}_G(X)$ or not for two $\mathbb{F}$ G-vector bundles E and F over X, we only need to consider the groups $J\mathbb{R}_G(X)$. From now on, we will use the standard notation $JO_G(X)$ for $J\mathbb{R}_G(X)$, and $KO_G(X)$ for $K\mathbb{R}_G(X)$.

One of the most important applications of the equivariant J-groups is the equivariant cross section problem of Stiefel manifolds. Let M be an $\mathbb{F}$ G-module, the Stiefel manifold of $\mathbb{F}$ k-frames in M is denoted by $V\mathbb{F}_k(M)$. Let $p : V\mathbb{F}_k(M) \to S(M)$ be the projection map which sends a frame $(u_1, \ldots, u_k)$ to u_1. $V\mathbb{F}_k(M)$ inherits a natural G-action from M and p is a G-map. The equivariant cross section problem of Stiefel manifolds is to determine the largest value of k for which p has a G-section. This problem is reduced to a problem about the groups $JO_G(X)$ as follows:

Let $\mathbb{F}P^{k-1}$ be the $\mathbb{F}$ projective space with trivial G-action, $\xi_{k-1}(\mathbb{F})$ be the canonical Hopf $\mathbb{F}$ line bundle over $\mathbb{F}P^{k-1}$, and $\xi_{k-1}(\mathbb{F}) \otimes_{\mathbb{F}} \mathbf{M}$ (or $\overline{\xi_{k-1}(\mathbb{F})} \otimes_{\mathbb{F}} \mathbf{M}$, where $\overline{\xi_{k-1}(\mathbb{F})}$ is the conjugate bundle to $\xi_{k-1}(\mathbb{F})$, if $\mathbb{F} = \mathbb{H}$) be the $\mathbb{F}$ G-vector bundle over $\mathbb{F}P^{k-1}$ associated to the tensor product of the $\mathbb{F}$ G-vector bundle $\xi_{k-1}(\mathbb{F})$, with the trivial G-action, and the trivial $\mathbb{F}$ G-vector bundle $\mathbf{M} = \mathbb{F}P^{k-1} \times M$. A necessary condition for p to have a G-section is that $r(\xi_{k-1}(\mathbb{F}) \otimes_{\mathbb{F}} \mathbf{M} - \mathbf{M})$ vanishes in $JO_G(\mathbb{F}P^{k-1})$. Further, under some mild restrictions on G and M this condition can be shown to be sufficient.

There are two important technical tools in studying G-fibre homotopy equivalence, namely the Adams operations ψ^k and the Bott cannibalistic characteristic classes θ_k. A famous conjecture due to Adams states that:

If E is a real G-vector bundle over X and k is prime to the order of G, then there exist stable G-maps $f : S(E) \to S(\psi^k(E))$ such that f^H has a degree which divides a power of k for each $H \leq G$.

The non-equivariant Adams' conjecture, that is the case $G = \{e\}$, is proved to be true by Quillen in 1970. In general, it is well-known now that Adams' conjecture holds for all finite groups.

The connection between G-fibre homotopy equivalence and Bott classes is given in the following easily proved resut:

If E and F are two $Spin(G)$-bundles over X such that $S(E)$ is G-fibre homotopy equivalent to $S(F)$, then there is a unit $x \in KO_G(X)$ such that $\theta_k(E)x = \theta_k(F)\psi^k(x)$ for all $k \geq 1$.

Let $TO_G(X) = \{E - F \in KO_G(X) : S(E \oplus \mathbf{V})$ is G-fibre homotopy equivalent to $S(F \oplus \mathbf{V})$ for some real G-module $V\}$, it is easy to see that $TO_G(X) = \ker JO_G$. Thus the computation of $JO_G(X) \cong KO_G(X)/TO_G(X)$ is equivalent to the computation of $TO_G(X)$.

To compute $TO(X)$, that is the case $G = \{e\}$, Adams in 1963 introduced two computable subgroups $WO(X)$ and $VO(X)$ of $KO(X)$:

$$WO(X) = \bigcap_f \widetilde{KSO}(X)_f$$

where the intersection runs over all functions $f : \mathbb{N} \to \mathbb{N}$ and $\widetilde{KSO}(X)_f = < k^{f(k)}(\psi^k - 1)(u) : u \in \widetilde{KSO}(X) \text{ and } k \in \mathbb{N} >$.

$$\begin{aligned} VO(X) \;\; = \{ \;\; & x \in \widetilde{KSO}(X) : \theta_k(x) = \frac{\psi^k(1+u)}{1+u} \text{ in } 1 + \widetilde{KSO}(X) \otimes \mathbb{Q}_k \\ & \text{for all } k \in \mathbb{N} \text{ and some } u \in \widetilde{KSO}(X)\} \end{aligned}$$

where $\widetilde{KSO}(X)$ is the subgroup of $KO(X)$ of orientable elements of virtual dimension zero, and $\mathbb{Q}_k = \{n/k^m : n, m \in \mathbb{Z}\}$.

Adams proved that $TO(X) \subseteq VO(X)$, $VO(X) = WO(X)$, and $WO(X) \subseteq TO(X)$ for $X = \mathbb{R}P^n$, $X = \mathbb{C}P^n$, and $X = S^m$ with $m \not\equiv 0 \bmod 8$. The non-equivariant Adams' conjecture is equivalent to the statment that $WO(X) \subseteq TO(X)$ for any finite CW-complex X. Using the fact that

$$TO(\mathbb{C}P^m) = WO(\mathbb{C}P^m) = VO(\mathbb{C}P^m),$$

Adams and Walker in 1965 completed the solution of the non-equivariant cross section problem of complex Stiefel manifolds. Using the new-proved Adams' conjecture, Sigrist and Suter in 1973 completed the solution of the non-equivariant cross section problem of quaternionic Stiefel manifolds.

The above two formulae of $WO(X)$ and $VO(X)$ have the difficulties that in the first, we need to take the intersection over all functions $f : \mathbb{N} \to \mathbb{N}$, and in the second we need to show the equality $\theta_k(x) = \psi^k(1+u)/(1+u)$ for all $k \in \mathbb{N}$. Now, $JO(X) \cong \mathbb{Z} \oplus \widetilde{JO}(X)$ where $\widetilde{JO}(X) = \widetilde{KO}(X)/TO(X)$. Also, from Atiyah we know that $\widetilde{JO}(X)$ is a finite abelian group. So, the localization $\widetilde{JO}(X)_{(p)}$ of $\widetilde{JO}(X)$ at a prime p is isomorphic to the p-summand of $\widetilde{JO}(X)$, in the prime factorization of $\widetilde{JO}(X)$. Since $\widetilde{JO}(X)_{(p)} \cong \widetilde{KO}(X)_{(p)}/TO(X)_{(p)}$, it is natural to ask about formulae for $WO(X)_{(p)}$ and $VO(X)_{(p)}$. Using Atiyah-Tall paper about λ-rings and some techniques of tom Dieck and Hauschild, we prove that:

$$WO(X)_{(p)} = (\psi^{k_p} - 1)(\widetilde{KSO}(X)_{(p)}),$$

and

$$\begin{aligned} VO(X)_{(p)} = \;\; \{ \;\; & \frac{x}{m} \in \widetilde{KSO}(X)_{(p)} : \theta_{k_p}(x) = \frac{\psi^{k_p}(1+u)}{1+u} \text{ in } 1 + \widetilde{KSO}(X)_p \\ & \text{for some } u \in \widetilde{KSO}(X)_p\} \end{aligned}$$

where $\widetilde{KSO}(X)_p$ denotes the p-adic completion of $\widetilde{KSO}(X)$, and k_p is an odd generator of the group of units in $(\mathbb{Z}/p^2\mathbb{Z})^*$. To show the significance of the above two formulae of $TO(X)_{(p)}$, we use them first to find the J-orders of elements of $\widetilde{KO}(\mathbb{C}P^m)$ and then we show how to use them to find the group $\widetilde{JO}(\mathbb{C}P^m)$ itself.

Following Adams, to compute $TO_G(X)$ for $G \neq \{e\}$, one usually tries to find a formula for $TO_G(X)$ which involves one or both of ψ^k, θ_k. More precisely, one tries to find an equivariant analogue of $WO(X)$ and VO(X). But since equivariant KO_G-theory essentially involves the real representation theory of G, the situation is more complicated and one first tries to treat the case X is a point and then works with the localization $TO_G(X)_{(p)}$, instead of $TO_G(X)$ itself. Also, as for many problems in representation theory, one usually treats the p-group case and then climbs up to the general group case.

In the case $X = \{*\}$ is a one point set, we write $JO_G(*)$ instead of $JO_G(X)$. Since $KO_G(*) \cong RO(G)$, the problem of computing $JO_G(*)$ can be reduced to the famous problem raised by Adams in 1963, about the existence of G-maps of a given degree between the associated spheres of two real representations of G. $TO_G(*)$ correspondes to the subgroup of $RO(G)$ which is usually denoted by $RO_h(G)$. Let $RO_0(G) = \{V - W \in RO(G) :\ \dim V^H = \dim W^H$ for all subgroups H of $G\}$, then $RO_h(G) \subseteq RO_0(G)$. Let $RO(G)_{\Gamma_G} = RO(G)/RO_0(G)$ (the index Γ_G will be discussed later), and let $jo(G) = RO_0(G)/RO_h(G)$. Then $RO(G)_{\Gamma_G}$ is a free abelian group and $jo(G)$ is a finite abelian group. Using some results of Atiyah-Tall, Snaith, Lee-Wasserman, and tom Dieck we prove that:

(i) If G is a finite group, then $JO_G(*) \cong RO(G)_{\Gamma_G} \oplus jo(G)$.

(ii) If G is a finite p-group then $RO_0(G) = (1 - \psi^{k_p})(RO(G))$, and $RO_h(G) = (1 - \psi^{k_p})^2(RO(G))$.

Using the above two facts, we give extensive explicit computations for $JO_G(*)$ for finite abelian groups. In general, the groups $JO_G(*)$ are understood for compact topological groups.

To compute $TO_G(X)$ when $G \neq \{e\}$ and X is not a one point set, tom Dieck and Hauschild introduced an intermediate J-group where G-fibre homotopy equivalence is replaced by a weaker condition. We say that two G-vector bundles E and F are stably p-equivalent, written $E \overset{(p)}{\sim} F$, if there are p-equivalences $S(E \oplus \mathbf{V}) \to S(F \oplus \mathbf{V})$ and $S(F \oplus \mathbf{V}) \to S(E \oplus \mathbf{V})$ for some real

G-module V (a fibrewise G-map $f : S(E) \to S(F)$ is p-equivalence if it has degree prime to p on all fixed sets of each fibre). Let

$$TO_G^{(p)}(X) = \{\frac{E-F}{m} \in KO_G(X)_{(p)} : E \overset{(p)}{\sim} F\},$$

then $TO_G^{(p)}(X)$ is a well-defined subgroup of $KO_G(X)_{(p)}$.

Using Atiyah-Segal completion theorem and some facts about λ-rings, tom Dieck and Hauschild proved that: If G is a finite p-group, and $JO_G^{(p)}(X) = KO_G(X)_{(p)}/TO_G^{(p)}(X)$. Then

$$TO_G^{(p)}(X) = (1-\psi^{k_p})(\widetilde{KSO}_G(X)_{(p)})$$

and

$$JO_G(X)_{(p)} \cong JO_G^{(p)}(X) \oplus jo(G)_{(p)}.$$

Later on McClure reduced the finite group case to the finite p-group case. So, the computation of $JO_G(X)_{(p)}$ is reduced to the computation of $JO_G^{(p)}(X)$ and $jo(G)_{(p)}$, where G is a finite p-group. We have discussed the groups $jo(G)$, so now we concentrate on the groups $JO_G^{(p)}(X)$.

From one point of view, one can look at the formula $TO_G^{(p)}(X) = (1 - \psi^{k_p})(\widetilde{KSO}_G(X)_{(p)})$ as the localization at p of an equivariant analogue of $WO(X)$. We embarked working on this dissertation with the hope that we can find an equivariant analogue of $VO(X)$. In this direction, we prove that:

$$TO_G^{(p)}(X) = \{ \frac{x}{m} \in \widetilde{KSO}_G(X)_{(p)} : \theta_{k_p}(x) = \frac{1+u}{\psi^k(1+u)} \text{ in } 1 + \widetilde{KSO}_G(X)_p \text{ for some } u \in \widetilde{KSO}_G(X)_p\}.$$

We use the above formula to give an alternative proof of the main theorem of Önder about the equivariant cross section problem of complex Stiefel manifolds for the case $G = \mathbb{Z}/2\mathbb{Z}$.

There are two extreme kinds of G-spaces, free G-spaces and trivial G-spaces. If X is a free G-space, we show that $J\mathbb{F}_G(X) \cong J\mathbb{F}(X/G)$. Thus for free group actions, the computation of the equivariant J-groups reduces to the non-equivariant ones. An important class of spaces which arise from free group action on spheres is the lens spaces $L^n(m) = S^{2n+1}/(\mathbb{Z}/m\mathbb{Z})$. It would perhaps be interesting to use methods of equivariant J-groups to compute the

J-groups of lens spaces, instead of the usual methods of non-equivariant J-groups, to avoid some difficulties in computing the groups $JO(L^n(m))$. That is we suggest to compute $JO_{\mathbb{Z}/m\mathbb{Z}}(S^{2n+1})$ instead of $JO(L^n(m))$!.

Recall that to solve the equivariant cross section problem of Stiefel manifolds, it is important to understand the groups $JO_G(\mathbb{F}P^k)$ where $\mathbb{F}P^k$ is considered as a trivial G-space. Also, from McClure, we only need to consider the case G is a finite p-group. Therefore, in the remainder of this introduction, we assume that X is a trivial G-space and G is a finite p-group with no type $\mathbb{H}$ irreducibles (the reason for the last assumption will be discussed later).

Let $Irr(G, \mathbb{R})$ denote the set of all irreducible real representations of G. It follows from Namboodiri that ψ^{k_p} permutes the irreducibles in $Irr(G, \mathbb{R})$, preserving their types. Let $\theta_1, \dots, \theta_s$ denote the type $\mathbb{R}$ orbits, and $\theta_{s+1}, \dots, \theta_{s+t}$ denote the type $\mathbb{C}$ orbits. For $i = 1, \dots, s+t$, let $\theta_i = \{V_{i,j} : j = 1, \dots, d_i\}$ for some $d_i \in \mathbb{N}$. Let $RO(G)_{\theta_i}$ be the subgroup of $RO(G)$ generated by elements of θ_i. If $x \in \widetilde{KO}_G(X)$, then $x = \sum_{i=1}^{s+t} x_i + \sum_{i=1}^{s+t} w_i$ where $x_i \in \widetilde{KO}(X) \otimes RO(G)_{\theta_i}$ for $i = 1, \dots, s$, $x_i \in \widetilde{KU}(X) \otimes RO(G)_{\theta_i}$ for $i = s+1, \dots, s+t$, and $w_i \in RO(G)_{\theta_i} \cap IO(G)$ where $IO(G)$ is the ideal of $RO(G)$ of elements of virtual dimension zero.

Let l_i be the order of x_i in $\widetilde{KO}(X) \otimes RO(G)_{\theta_i}/(1-\psi^{k_p})$ (resp. in $\widetilde{KU}(X) \otimes RO(G)_{\theta_i}/(1-\psi^{k_p}))$ for $i = 1, \dots, s$ (resp. for $i = s+1, \dots, s+t$). Where the denominator $(1-\psi^{k_p})$ indicates that we factor out the image of $(1-\psi^{k_p})$. Let m_i be the order of w_i in $RO(G)_{\theta_i}/(1-\psi^{k_p})^2$. Then the order of x in $JO_G(X)_{(p)}$ is the least common multiple of $\{l_i, p^{\nu_p(m_i)} : i = 1, \dots, s+t\}$ where $\nu_p(m_i)$ denotes the exponent of m_i in its prime factorization. Let τ_i denote the inverse of the permutation determined by the action of ψ^{k_p} on θ_i and let $w_i = \sum_{j=1}^{d_i} n_i V_{i,j}$. If $N_j = n_{\tau_i(j)} + 2n_{\tau_i^2(j)} + \cdots + (d_i - 1)n_{\tau_i^{d_i-1}(j)}$ and q is any prime number, then we prove that:

$$\nu_q(m_i) = \max\{\nu_q(d_i) - \nu_q(N_j) : j = 1, \dots, d_i, \text{ and } N_j \neq 0\}.$$

For $X = \mathbb{R}P^m$ or S^n with $n \equiv 1$ or $2 \bmod 8$, we prove that

$$JO_G^{(p)}(X) \cong \widetilde{JO}_G^{(p)}(X) \oplus (RO(G)_{\Gamma_G})_{(p)}.$$

Where

$$\widetilde{JO}_G^{(p)}(\mathbb{R}P^m) \cong \begin{cases} 0 & \text{if } p \neq 2 \\ \mathbb{Z}/2^{f(m)}\mathbb{Z} \otimes RO(G)_{\mathbb{R},\Gamma_G} \oplus \mathbb{Z}/2^{[m/2]}\mathbb{Z} \otimes RO(G)_{\mathbb{C},\Gamma_G} & \text{if } p = 2 \end{cases}$$

where $f(m)$ is the number of integers q with $q \equiv 0, 1, 2,$ or $4 \bmod 8$ and $0 < q \leq m$.

If $n \equiv 1 \bmod 8$, then

$$\widetilde{JO}_G^{(p)}(S^n) \cong \begin{cases} 0 & \text{if} \quad p \neq 2 \\ \mathbb{Z}/2\mathbb{Z} \otimes RO(G)_{\mathbb{R},\Gamma_G} & \text{if} \quad p = 2. \end{cases}$$

If $n \equiv 2 \bmod 8$, then

$$\widetilde{JO}_G^{(p)}(S^n) \cong \begin{cases} (\widetilde{KU}(S^n) \otimes RO(G)_{\mathbb{C}})_{(p)}/(1-\psi^{k_p}) & \text{if} \quad p \neq 2 \\ \mathbb{Z}/2\mathbb{Z} \otimes RO(G)_{\mathbb{R},\Gamma_G} \oplus (\widetilde{KU}(S^n) \otimes RO(G)_{\mathbb{C}})_{(2)}/(1-\psi^3) & \text{if} \quad p = 2. \end{cases}$$

Also, we find the J-orders of elements of $JO_G^{(p)}(S^{4n})$ and give another form of the results of Önder about the J-orders of elements of $JO_G^{(p)}(\mathbb{C}P^m)$.

The equivariant cross section problem of real (resp. complex) Stiefel manifolds is partially solved by Namboodiri (resp. Önder). In order to obtain a solution for the equivariant cross section problem of quaternionic Stiefel manifolds, we first compute Adams operations on $\widetilde{KO}(\mathbb{H}P^m)$, $\widetilde{KU}(\mathbb{H}P^m)$, and then find the J-orders of elements of $JO_G(\mathbb{H}P^m)$.

The arrangement of this dissertation is as follows: In Chapter 2, we present some basic background materials about equivariant vector bundles, $K\mathbb{F}_G$-theories, λ-rings, and the equivariant cross section problem of Stiefel manifolds. Also, we give simlpe formulae for computing Adams operations, Bott classes, and symmetric functions.

The backbone of this dissertation is Chapter 3, in which we present our suggestive formula for $TO_G^{(p)}(X)$ and then discuss the cases: X is a point, X is free G-space, and X is a trivial G-space.

In Chapter 4, we apply the ideas of Chapter 3 to find $JO_G(\mathbb{R}P^m)$ and $JO_G(S^m)$ and then to find the J-orders of elements of $JO_G(\mathbb{C}P^m)$ and $JO_G(\mathbb{H}P^m)$. Also, we give a partial solution for the equivariant cross section problem of quaternionic Stiefel manifolds.

Chapter 5 is of independent interest, in which we give two formulae of $TO(X)_{(p)}$ and then compute the J-orders of elements of $KO(\mathbb{C}P^m)$ and suggest a method for computing $JO(\mathbb{C}P^m)$.

In Chapter 6, we give an outline for further study about equivariant J-groups and equivariant cross section problem of Stiefel manifolds. These include suggestions to study $JO_G(X)$ for any compact topological group G, not necessarily finite, and to study $JO_G(L^n(m))$.

Chapter 2

EQUIVARIANT TOPOLOGICAL K-THEORY

2.1 Introduction

In this preliminary chapter, we present basic background material about equivariant topological K-theories. Since we have no occasion to consider algebraic K-theory, from now on, by equivariant K-theory we mean equivariant topological K-theory. In the literature authors usually consider complex equivariant K-theory and then briefly mention the results which hold for real and quaternionic K-theories. Also, they usually restrict themselves to finite groups and compact G-spaces. For our purposes, we shall consider the three theories, real, comlpex, and quaternionic, simultaneously. Also, we shall state the results in the most general form, up to our knowledge.

Our philosophy in writing this chapter is to make it a practical chapter, so if we state that two specific objects (groups, rings) are isomorphic we will give an explicit isomorphism without any proofs.

Since equivariant vector bundles are the backbone of the whole subject, we discuss them in Section 1. Then in Section 2, we construct the cofunctor $K\mathbb{F}_G(X)$ and state some of its basic properties. If $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, then $K\mathbb{F}_G(X)$ is a special λ-ring. Therefore, it will be very helpful to remember some basic facts concerning λ-rings. So in Section 3, we discuss λ-rings and take $K\mathbb{F}_G(X)$ as an illustrative example. To lead the reader to the main purpose of this dissertation, namely the computation of the groups $JO_G(X)$, we consider in

Section 4 one of the most important applications of equivariant J-groups, the equivariant cross section problem of Stiefel manifolds.

Throughout this chapter, G will be a topological group, X a connected G-space, and $\mathbb{F}$ one of the classical division algebras over $\mathbb{R}$, namely the real numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$, or the quaternionic numbers $\mathbb{H}$. Due to the non-commutative multiplicative structure of $\mathbb{H}$, many times, we will be forced to exclude the case $\mathbb{F} = \mathbb{H}$.

2.2 Equivariant vector bundles

In this section, we present some basic definitions, examples, and properties of equivariant vector bundles. Main references for ordinary vector bundles are Husemoller [31], Atiyah [6], and Karoubi [36]. Equivariant vector bundles are mainly discussed by Atiyah [6], and Segal [55], also one can look at Petrie-Randall [53] and tom Dieck [24]. For the classification of equivariant vector bundles, we have Bierstone [15], Lashof [41], Wasserman [64], and tom Dieck [24].

Definition 2.2.1 *An $\mathbb{F}$ G-vector bundle over a G-space X is a G-space E together with a G-map $p : E \to X$ such that*

(i) *$p : E \to X$ is an $\mathbb{F}$-vector bundle over X.*

(ii) *For any $g \in G$ and $x \in X$, the group action $g : E_x \to E_{gx}$ is $\mathbb{F}$-linear.*

Examples.

(1) If V is an $\mathbb{F}$ G-module, then $\mathbf{V} = X \times V$ is an $\mathbb{F}$ G-vector bundle over X, it is the trivial $\mathbb{F}$ G-vector bundle over X associated to V.

(2) If E and F are two $\mathbb{F}$ G-vector bundles over X, then the Whitney sum $E \oplus F$ is an $\mathbb{F}$ G-vector bundle over X, with $g(v \oplus w) = gv \oplus gw$ for each $g \in G$ and $v \oplus w \in E \oplus F$. Also, if $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ then the tensor product $E \otimes F$ is an $\mathbb{F}$ G-vector bundle over X, with $g(v \otimes w) = gv \otimes gw$ for each $g \in G$ and $v \otimes w \in E \otimes F$. Note that since $\mathbb{H}$ is not commutative, we do not have a definition for the product of two $\mathbb{H}$ G-vector bundles over X. Let $Vect\mathbb{F}_G(X)$ denote the set of all isomorphism classes of $\mathbb{F}$ G-vector bundles over X. Then $(Vect\mathbb{F}_G(X), \oplus, \otimes)$ is a semiring for $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ and $(Vect\mathbb{H}_G(X), \oplus)$ is a semigroup.

Convention. Unless otherwise indicated, from now on, G-vector bundles and G-modules will be real, complex, or quaternionic. Also, if we speak about the product of two G-vector bundles (resp. two G-modules) then we implicitly assume that $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

In the following three examples, we assume that G is a compact Lie group.

(3) (X is a homogeneous G-space, Segal [55], and tom Dieck [24], p. 67).

Let H be a closed subgroup of G. Then $X = G/H$ is a G-space with $g_1(g_2H) = g_1g_2H$ for each g_1, $g_2 \in G$. Let $p : E \to X$ be a G-vector bundle over X and let $V = p^{-1}(eH)$ be the fiber over eH considered as an H-module. $G \times_H V$ is G-space with $g_1(g_2 \times_H v) = g_1 g_2 \times_H v$ for each g_1, $g_2 \in G$ and $v \in V$. Then the canonical map $G \times_H V \to E$, given by $g \times_H v \mapsto gv$, is an isomorphism of G-vector bundles over X. On the other hand, if V is an H-module then the canonical map $G \times_H V \to G/H$, given by $g \times_H v \mapsto gH$, is a G-vector bundle over G/H. Thus any G-vector bundle over a homogeneous space $X = G/H$ has the form $G \times_H V$ for some H-module V.

(4) (X is a free G-space, Atiyah [6], p. 36, Segal [55], and Dieck [24], p. 68).

Let X be a free G-space and X/G be the associated orbit space. If E is a G-vector bundle over X, then E/G is a vector bundle over X/G. On the other hand, if $p : F \to X/G$ is a vector bundle over X/G then $q^*(F)$ is a G-vector bundle over X where $q : X \to X/G$ is the quotient map and $q^*(F) = \{(v, x) \in F \times X : p(v) = q(x)\}$. ($q^*(F)$ is a free G-space with $g(v, x) = (v, gx)$ for each $g \in G$ and $(v, x) \in q^*(F)$). Now, it is easy to see that $q^*(F)/G \cong F$ and $q^*(E/G) \cong E$. So, $Vect\mathbb{F}_G(X) \cong Vect\mathbb{F}(X/G)$ (as semirings for $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ and as semigroups for $\mathbb{F} = \mathbb{H}$).

(5) (X is a trivial G-space, Atiyah [6], p. 37, Segal [55], Becker [14], and tom Dieck [24] p. 68).

Let $Irr(G, \mathbb{F})$ denote the set of all irreducible $\mathbb{F}$ representations of G. For $V \in Irr(G, \mathbb{F})$, let $F_V = Hom_G({}_{\mathbb{F}}V, {}_{\mathbb{F}}V)$ then $F_V \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ if $\mathbb{F} = \mathbb{R}$, and $F_V = \mathbb{C}$ if $\mathbb{F} = \mathbb{C}$. So, if M is a G-module over $\mathbb{F}$ and $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ then

$$M \cong \bigoplus_{V \in Irr(G,\mathbb{F})} Hom_G({}_{\mathbb{F}}V\ ,_{\mathbb{F}} M) \otimes_{F_V} V \tag{2.1}$$

V is a left F_V-module via the evaluation map $\phi \otimes v \mapsto \phi(v)$, $Hom_G({}_{\mathbb{F}}V\ ,_{\mathbb{F}} M)$ is a right F_V-module via $\phi \otimes \psi = \phi \circ \psi$. Now, let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ and E be an $\mathbb{F}$ G-vector bundle over a trivial G-space X. Then $Hom_G(\mathbf{V},\ E)$ is a right F_V-vector bundle over X. So from (2.1), we get

$$E \cong \bigoplus_{V \in Irr(G,\mathbb{F})} Hom_G(\mathbf{V},\ E) \otimes_{F_V} \mathbf{V} \tag{2.2}$$

Basic Properties.

The following properties of G-vector bundles can be found, in some form, in Atiyah [6], Segal [55], and Husemoller [31]. Sometimes , in these references, they are stated in a form weaker than the one we present here, for example they are stated for finite G-CW complexes, or for non-equivariant vector bundles.

We will try to present these properties in the most general form, that is the proofs of the original statements can be used with little modifications for the proofs of the new ones.

(1) Let E and F be two $\mathbb{F}$ G-vector bundles over X. Let $HOM(E, F)$ be the $\mathbb{F}$-module of all $\mathbb{F}$ G-vector bundle homomorphism from E to F. If $f \in HOM(E, F)$, then f is an isomorphism if and only if for each $x \in X$, $f_x : E_x \to F_x$ is an $\mathbb{F}$-isomorphism.

In the following properties, we assume that G is a compact Lie group.

(2) If X is a paracompact G-space. Then any G-vector bundle over X has an invariant metric. Hence if E is a G-vector bundle over X, then there is a G-vector bundle F such that $E \oplus F \cong \mathbf{V}$ for some G-module V.

(3) Let X be a paracompact G-space, and let $f_0, f_1 : X \to Y$ be two G-homotopic G-maps. If E is a G-vector bundle over Y, then $f_0^*E \cong f_1^*E$ as G-vector bundles.

(4) $Vect\mathbb{F}_G$ defines a cofunctor from the category of G-spaces and G-maps to the category of Semirings (Semigroups if $\mathbb{F} = \mathbb{H}$). In fact, $Vect\mathbb{F}_G$ defines a cofunctor from the category of paracompact G-spaces and G-homotopic G-maps to the category of Semirings (Semigroups if $\mathbb{F} = \mathbb{H}$).

(5) If Y is a closed G-contractible subspace of a compact G-space X. Then the quotient map $q : X \to X/Y$ induces an isomorphism

$$q^* : Vect\mathbb{F}_G(X/Y) \to Vect\mathbb{F}_G(X).$$

Definition 2.2.2 *A G-section of a G-vector bundle $p : E \to X$ is G-map $s : X \to E$ such that $p \circ s = id|_X$.*

Let $\Gamma(E)$ (resp. $\Gamma_G(E)$) denote the set of all sections (resp. G-sections) of a G-vector bundle $p : E \to X$. Then we have:

(1) $\Gamma(E)$ is an $\mathbb{F}$-module. Further, with the compact open topology, $\Gamma(E)$ is a G-space with $(gs)(x) = gs(g^{-1}x)$ for each $g \in G$, $s \in \Gamma(E)$, and $x \in X$.

(2) $\Gamma_G(E)$ is an $\mathbb{F}$-submodule of $\Gamma(E)$. Further, the G-fixed point set $\Gamma(E)^G$ of $\Gamma(E)$ is equal to $\Gamma_G(E)$.

(3) If $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ and E, F are two G-vector bundles over X. Then

$$HOM(E, F) \cong \Gamma_G(Hom(E, F))$$

the isomorphism is given by sending $f \in HOM(E,F)$ to s_f where

$$s_f : X \to Hom(E,F)$$

is given by $s_f(x) = f|_x$ for each $x \in X$.

We close this section by discussing the classification of equivariant bundles. Let G be a compact Lie group and H be a topological group.

Definition 2.2.3 *A G-equivariant H-principal bundle over a G-space X is a G-space E together with a G-map $p : E \to X$ such that:*

(i) $p : E \to X$ is an H-principal bundle.

(ii) For $v \in E$, $g \in G$, and $h \in H$, the relation $(gv)h = g(vh)$ holds.

Let $B(G,H)$ denote the classifying space of numerable (in the sense of tom Dieck [24]) G-equivariant H-principal bundles and let $E(G,H)$ be the universal G-equivariant H-principal bundle over $B(G,H)$. For the construction of $E(G,H)$ and $B(G,H)$ see tom Dieck [24] p. 58.

Let X be a G-space and E be a numerable G-equivariant H-principal bundle over X, then there is a G-map $f_E : X \to B(G,H)$ such that

$$f_E^* E(G,H) \cong E$$

as G-equivariant H-principal bundles.

Theorem 2.2.4 *(G-homotopy classification theorem of numerable G-equivariant H-principal bundles).*

The assignment $E \to f_E$, defines a ono-to-one correspondence between the set of isomorphism classes of numerable G-equivariant H-principal bundles over a G-space X and the set $[X, B(G,H)]_G$ of all G-homotopy classes of G-maps from X to $B(G,H)$.

If V is a G-module, let V^∞ be the direct sum of infinitely many copies of V. Let $W_{\mathbb{F}}(G) = \bigoplus_{V \in Irr(G,\mathbb{F})} V^\infty$. For an integer k, let $V_k(W_{\mathbb{F}}(G))$ be the G-Stiefel manifold of G-equivariant k-frames in $W_{\mathbb{F}}(G)$ and let $G_k(W_{\mathbb{F}}(G))$ be the G-Grassmann manifold of G-equivariant k-subspaces of $W_{\mathbb{F}}(G)$. Then $V_k(W_{\mathbb{F}}(G)) \to G_k(W_{\mathbb{F}}(G))$, $(u_1, \ldots, u_k) \mapsto$ the space spanned by $\{u_1, \ldots, u_k\}$

is a universal G-equivariant $U_{\mathbb{F}}(k)$-principal bundle. (Here, as in Husemoller [31], $U_{\mathbb{F}}(k)$ denotes $O(k)$ for $\mathbb{F} = \mathbb{R}$, $U(k)$ for $\mathbb{F} = \mathbb{C}$, and $Sp(k)$ for $\mathbb{F} = \mathbb{H}$). Now, if $E \to X$ is a k-dimensional $\mathbb{F}$ G-vector bundle over a paracompact G-space X, then there is a G-map $h_E : X \to G_k(W_{\mathbb{F}}(G))$ such that

$$h_E^*(V_k(W_{\mathbb{F}}(G))) \times_{U_{\mathbb{F}}(k)} \mathbb{F}^k \cong E \quad \text{as } \mathbb{F}\, G - \text{vector bundles.}$$

Theorem 2.2.5 *(G-homotopy classification theorem of k-dimensional $\mathbb{F}$ G-vector bundles over a paracompact G-space X).*

The assignment $E \to h_E$ defines a ono-to-one correspondence between the set $Vect\mathbb{F}_{G,k}(X)$ of isomorphism classes of k-dimensional $\mathbb{F}$ G-vector bundles over a paracompact G-space X and the set $[X,\, G_k(W_{\mathbb{F}}(G))]_G$ of all G-homotopy classes of G-maps from X to $G_k(W_{\mathbb{F}}(G))$.

2.3 $K\mathbb{F}_G(X)$

In this section, we define the ring $K\mathbb{F}_G(X)$ (only a group for $\mathbb{F} = \mathbb{H}$) where X is a compact or locally compact G-space and G is a compact topological group. Then we present some of the most important features of $K\mathbb{F}_G(X)$, such as Thom isomorphism theorem and Bott periodicity theorem.

Let X be any G-space. Recall that $Vect\mathbb{F}_G(X)$ denotes the semiring (semigroup for $\mathbb{F} = \mathbb{H}$) of all isomorphism classes of $\mathbb{F}$ G-vector bundles over X. Let $K\mathbb{F}_G^0(X)$ be the ring (group for $\mathbb{F} = \mathbb{H}$) completion of $Vect\mathbb{F}_G(X)$. Elements of $K\mathbb{F}_G^0(X)$ are formal differences $E - F$ of isomorphism classes of $\mathbb{F}$ G-vector bundles over X modulo the equivalence relation $E_1 - F_1 = E_2 - F_2$ if and only if $E_1 \oplus F_2 \oplus F \cong E_2 \oplus F_1 \oplus F$ for some $\mathbb{F}$ G-vector bundle F over X. $K\mathbb{F}_G^0(X)$ is a group with

$$(E_1 - F_1) + (E_2 - F_2) = E_1 \oplus E_2 - F_1 \oplus F_2$$

for $E_1 - F_1,\, E_2 - F_2 \in K\mathbb{F}_G^0(X)$, $0 = X \times 0$ (the 0-bundle over X). Also, if $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ then $K\mathbb{F}_G^0(X)$ is a commutative ring with

$$(E_1 - F_1)(E_2 - F_2) = E_1 \otimes E_2 \oplus F_1 \otimes F_2 - E_1 \otimes F_2 - F_1 \otimes E_2,$$

$1 = X \times \mathbb{F}$ (the trivial 1-dimensional $\mathbb{F}$ G vector bundle over X, where $\mathbb{F}$ is the trivial 1-dimensional $\mathbb{F}$ G-module).

Now, let X be a G-space with a base-point x_0 (i.e., $x_0 \in X^G$). Then

$$K\mathbb{F}_G^0(X) \xrightarrow{i_{x_0}^*} K\mathbb{F}_G^0(\{x_0\}) \longrightarrow 0$$

splits, where $i_{x_0} : \{x_0\} \to X$ is the inclusion map. So,

$$K\mathbb{F}_G^0(X) \cong \text{ ker } i_{x_0}^* \oplus K\mathbb{F}_G^0(\{x_0\}).$$

Let $K\tilde{\mathbb{F}}_G^0(X) = \text{ ker } i_{x_0}^*$ and identify $R\mathbb{F}(G)$ with $K\mathbb{F}_G^0(\{x_0\})$ by sending an $\mathbb{F}$ G-module V to $\{x_0\} \times V$. Then we obtain,

$$K\mathbb{F}_G^0(X) \cong K\tilde{\mathbb{F}}_G^0(X) \oplus R\mathbb{F}(G).$$

Under the above identification $E - F \in K\mathbb{F}_G^0(X)$ corresponds to

$$(E \oplus F_{x_0} - F \oplus E_{x_0}) + (E_{x_0} - F_{x_0}) \in K\tilde{\mathbb{F}}_G^0(X) \oplus R\mathbb{F}(G).$$

Next, we define $K\mathbb{F}_G(X)$ where X is a compact or locally compact G-space. If X is compact, let $X^+ = X \cup \{*\}$ be the disjoint union of X and a point. If X is locally compact but not compact, let X^+ be the one point compactification of X. Now, X^+ is a G-space a with base-point. Define $K\mathbb{F}_G(X) = K\tilde{\mathbb{F}}_G^0(X^+)$, frequently, we denote $K\mathbb{F}_G(X)$ by $KO_G(X)$ for $\mathbb{F} = \mathbb{R}$, $KU_G(X)$ for $\mathbb{F} = \mathbb{C}$, and $KSp_G(X)$ for $\mathbb{F} = \mathbb{H}$.

Remark 1. If X is compact then $K\mathbb{F}_G(X) \cong K\mathbb{F}_G^0(X)$. If X is locally compact but not compact then $K\mathbb{F}_G(X) \ncong K\mathbb{F}_G^0(X)$, for example let $X = \mathbb{R}$ (with the usual topology) then $KU(X) = K\tilde{U}^0(S^1) = 0$, while $KU^0(X) \cong \mathbb{Z}$.

Examples.

(1) If X is a point, then $K\mathbb{F}_G(X) \cong R\mathbb{F}(G)$ (the character ring of G). On the other hand, if $G = \{e\}$ then $K\mathbb{F}_G(X) \cong K\mathbb{F}(X)$ (the non-equivariant $K\mathbb{F}$-theory of X).

(2) Let H be a closed subgroup of G, then $K\mathbb{F}_G(G/H) \cong R\mathbb{F}(H)$ (see Example 3, §2.2).

(3) If X is a free G-space, then $K\mathbb{F}_G(X) \cong K\mathbb{F}(X/G)$ (see Example 4, §2.2).

(4) If X is a trivial G-space then, from Example 5 §2.2,

$$KU_G(X) \cong KU(X) \otimes RU(G) \qquad \text{as rings, and}$$

$$KO_G(X) \cong KO(X) \otimes RO(G)_{\mathbb{R}} \oplus KU(X) \otimes RO(G)_{\mathbb{C}} \oplus KSp(X) \otimes RO(G)_{\mathbb{H}}$$

as groups, where $RO(G)_{\mathbb{F}}$ is the subgroup of $RO(G)$ generated by type $\mathbb{F}$ irreducible real G-modules.

Basic Properties.

(1) If X is a trivial G-space and G is finite with $Irr(G,\mathbb{R}) = \{V_1,\dots,V_n\}$. Then from (2.2)

$$KO_G(X) \cong \bigoplus_{i=1}^{n} KF_{V_i}(X) \qquad (2.3)$$

where $F_{V_i} = Hom_G({}_{\mathbb{R}}V_i, {}_{\mathbb{R}}V_i) \in \{\mathbb{R},\mathbb{C},\mathbb{H}\}$. Let $\widetilde{K\mathbb{F}}_G(X)$, with a wide tilde over $K\mathbb{F}$ to distinguish it from $K\tilde{\mathbb{F}}_G(X) = \ker i^*_{x_0}$, be the subgroup of $K\mathbb{F}_G(X)$ of elements of virtual dimension zero. If $y \in \widetilde{KO}_G(X)$ and $M = m_0V_1+\cdots+m_nV_n$ is a real G-module then the image of $y \otimes_{\mathbb{R}} \mathbf{M}$ under the identification in (2.3) is

$$\sum_{i=1}^{n} m_i y \otimes_{\mathbb{R}} F_{V_i}.$$

In particular, if y has finite order in $\widetilde{KO}(X)$ and t_i is the order of $y \otimes_{\mathbb{R}} F_{V_i}$ in $KF_{V_i}(X)$ then the order of $y \otimes_{\mathbb{R}} \mathbf{M}$ in $KO_G(X)$ is

$$lcm\left\{\frac{t_0}{(t_0,m_0)},\dots,\frac{t_n}{(t_n,m_n)}\right\}.$$

(2) Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. If $x \in K\mathbb{F}_G(X)$ is a unit then $\dim x = \pm 1$. Further, if $G = \{e\}$ and $\dim x = \pm 1$ then x is a unit in $K\mathbb{F}(X)$.

Proof of (2). Let $x \in K\mathbb{F}_G(X) \cong \widetilde{K\mathbb{F}}_G(X) \oplus \mathbb{Z}$ be a unit. If $\dim x = n$, then $x = n + y$ for some $y \in \widetilde{K\mathbb{F}}_G(X)$. Suppose $x^{-1} = m + z$ for some $z \in \widetilde{K\mathbb{F}}_G(X)$ then $xx^{-1} = nm+nz+my+yz = 1$. So $nm = 1$, namely $n = \pm 1$. On the other hand, if $G = \{e\}$ and $\dim x = 1$, then $x = 1 + y$ for some $y \in \widetilde{K\mathbb{F}}(X)$ (dim $x = -1$ is similar). Since $\widetilde{K\mathbb{F}}(X)$ is nilpotent then $\sum_{i=1}^{\infty}(-1)^i y^i \in \widetilde{K\mathbb{F}}(X)$. Now, $x^{-1} = 1 + \sum_{i=1}^{\infty}(-1)^i y^i \in K\mathbb{F}(X)$.

(3) $K\mathbb{F}_G$ defines a cofunctor from the category of compact (resp. locally compact) G-spaces and G-homotopic (resp. G-homotopic proper) G-maps to the category of Rings (Groups if $\mathbb{F} = \mathbb{H}$) (see Property 4, §2.2).

Remark 2. If $f : X \to Y$ is a G-map between two locally compact G-spaces, then in general we can not extend f to $f^+ : X^+ \to Y^+$. But if f is proper, then f can be extended to $f^+ : X^+ \to Y^+$, for this reason $K\mathbb{F}_G$ is only a cofunctor for proper G-maps on the category of locally compact G-spaces.

(4) Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, X be a compact or locally compact G-space and A be a closed G-subspace of X. Let $K\mathbb{F}_G(X,A) = K\tilde{\mathbb{F}}_G(X^+/A^+)$, and for each $n \geq 1$ let $K\mathbb{F}_G^{-n}(X) = K\tilde{\mathbb{F}}_G(S^nX^+)$ (here as usual $S^nX^+ = S^n \wedge X^+$). Using the fact

that,

$$K\mathbb{F}_G(X,A) \to K\mathbb{F}_G(X) \to K\mathbb{F}_G(A)$$

is exact, we get the long exact sequence

$$\cdots \to K\mathbb{F}_G^{-n}(X,A) \to K\mathbb{F}_G^{-n}(X) \to K\mathbb{F}_G^{-n}(A) \to \cdots$$

$$\cdots \to K\mathbb{F}_G^{-n+1}(X,A) \to \cdots K\mathbb{F}_G(X,A) \to K\mathbb{F}_G(X) \to K\mathbb{F}_G(A).$$

For the rest of this section, we assume that X a is locally compact G-space where G is a compact topological group.

Theorem 2.3.1 *(Thom isomorphism theorem for complex G-vector bundles, Atiyah [6], p. 103, and Segal [55]).*

If E is a complex G-vector bundle over X. Then the Thom homomorphism $\phi : KU_G(X) \to KU_G(E)$ given by $x \mapsto x.\lambda_E$ is an isomorphism, where λ_E is the equivariant complex Thom class of E.

Corollary 2.3.2 *(Bott periodicity theorem in equivariant complex K-theory)*

There is a natural isomorphism $KU_G^{-n}(X) \to KU_G^{-n-2}(X)$ for each $n \in \mathbb{Z}$.

Definition 2.3.3 *Let E be a real $8n$-dimensional G-vector bundle over X. A $Spin(G)$-structure on E consists of a principal $Spin(8n)$-bundle $\tilde{E}$ and an action of G on $\tilde{E}$ through principal bundle maps, together with an equivalence $\tilde{E} \times_{Spin(8n)} \mathbb{R}^{8n} \to E$. The G-structure on $\tilde{E} \times_{Spin(8n)} \mathbb{R}^{8n}$ is given by*

$$g(v \times_{Spin(8n)} w) = gv \times_{Spin(8n)} w.$$

Theorem 2.3.4 *(Thom isomorphism theorem for real G-vector bundles with $Spin(G)$-structure, Atiyah [4]).*

If E is a real G-vector bundle over X with $Spin(G)$-structure, then the Thom homomorphism $\phi : KO_G(X) \to KO_G(E)$ given by $x \mapsto x.\mu_E$ is an isomorphism, where μ_E is the equivariant real Thom class of E.

Corollary 2.3.5 *(Bott periodicity theorem in equivariant real K-theory)*

There is a natural isomorphism $KO_G^{-n}(X) \to KO_G^{-n-8}(X)$ for each $n \in \mathbb{Z}$.

Using Property 4 and the above two corollaries, it is easy to see that $K\mathbb{F}_G$ (for $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$) gives a generalized cohomology theory. For a good reference on general cohomology theory and K-theory see Hilton [30].

2.4 λ-Rings

In this section, we assume that $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Adams operations are the most important technical tools in studying the rings $K\mathbb{F}_G(X)$ Since these operations are defined in any λ-ring and since $K\mathbb{F}_G(X)$ is a λ-ring, it will be very helpful to remember the basic properties of these rings. λ-rings are discussed by Atiyah-Tall [12], tom Dieck [25] , Ch. 3, and Knutson [38], Ch. 1.

In this section, we first present some basic properties of λ-rings and show that $K\mathbb{F}_G(X)$ is a special λ-ring. Then, we discuss Adams operations and suggest a formula for computing these operations. Bott cannibalistic classes will play a central role in computing the groups $JO_G(X)$. These classes are defined in any special λ-ring, so we discuss them and show how to compute them in some useful cases. Finally, we state Atiyah-Tall-Segal exponential isomorphism theorem for λ-rings [12], [11] and Atiyah-Segal completion theorem for equivariant real and complex K-theories [10].

Definition 2.4.1 *A λ-ring is a commutative ring R with unity 1 and a countable set of maps $\lambda^n : R \to R$ such that for all $x, y \in R$,*

(i) $\lambda^0(x) = 1$

(ii) $\lambda^1(x) = x$

(iii) $\lambda^n(x+y) = \sum_{r=0}^{n} \lambda^r(x)\lambda^{n-r}(y)$.

Let $1 + R[[t]]^+$ be the multiplicative group of formal power series in t with constant term 1. Then $\lambda_t : R \to 1 + R[[t]]^+$ such that $\lambda_t(x) = \sum_{n\geq 0} \lambda^n(x)t^n$ is an exponential map, i.e., $\lambda_t(x+y) = \lambda_t(x)\lambda_t(y)$. If $x \in R$ and $\lambda_t(x)$ is a polynomial of degree n in t, then we say that x is of dimension n. Let $P(R)$ denote the set of all finite demensional elements in R, then $P(R)$ is a λ-semiring.

Example 1. $K\mathbb{F}_G(X)$ is a λ-ring with

$$\lambda^n(E-F) = \sum_{i+j=k} (-1)^j \wedge^i(E) S^j(F) \qquad \text{(Atiyah [6], p. 119)}$$

where $\wedge^i(E)$ is the ith exterior power of E and $S^j(F)$ is the jth symmetric power of F. Also, $P(K\mathbb{F}_G(X)) = Vect\mathbb{F}_G(X)$. Here as usual, X is a compact or locally compact G-space, G is a compact topological group, and E, F are G-vector bundles over X.

A λ-ring R is augmented if there is a λ-homomorphism $\epsilon : R \to \mathbb{Z}$, where $\mathbb{Z}$ is λ-ring with

$$\lambda^k(m) = \begin{cases} 0 & \text{if } k > m \\ \binom{m}{k} & \text{if } k \leq m \end{cases}$$

and

$$\lambda^k(-m) = (-1)\binom{k+m-1}{k}$$

for each $k, m \in \mathbb{N}$.

$K\mathbb{F}_G(X)$ is an augmented λ-ring with $\epsilon(E-F) = \dim E - \dim F$. Further, $\widetilde{K\mathbb{F}}_G(X) = \ker \epsilon$ is a λ-ideal of $K\mathbb{F}_G(X)$.

To define the notion of special λ-rings, we need to recall some facts about symmetric functions. Good references on symmetric functions are Husemoller [31], Ch. 13, and Knutson [38], Ch. 1. For each $n, q \in \mathbb{N}$, let

$$S_{q,n}(x_1, \ldots, x_q) = \begin{cases} \sum_{i_1 < \ldots < i_n} x_{i_1} \cdots x_{i_n} & \text{if } n \leq q \\ 0 & \text{if } n > q \end{cases}$$

be the nth elementary symmetric function in the indeterminates $x_1, \ldots, x_q$. The nth Newton polynomial in the indeterminates $x_1, \ldots, x_q$ is the unique polynomial $Q_{q,n}$ which satisfies

$$Q_{q,n}(S_{q,1}, \ldots, S_{q,n}) = T_{q,n}(x_1, \ldots, x_q)$$

where

$$T_{q,n}(x_1, \ldots, x_q) = \sum_{i=1}^{q} x_i^n$$

is the nth symmetric power polynomial in $x_1, \ldots, x_q$. For example, $Q_{2,2} = x_1^2 - 2x_2$, $Q_{1,2} = x_1^2$, and $Q_{2,1} = x_1$. Actually, we only need to find $Q_{q,n}$ for $n = q$ because if $q > n$ then $Q_{q,n}(x_1, \ldots, x_q) = Q_{n,n}(x_1, \ldots, x_n)$ and if $q < n$ then $Q_{q,n}(x_1, \ldots, x_q) = Q_{n,n}(x_1, \ldots, x_q, 0, \ldots, 0)$. So $Q_{q,n}$ is independent of q for $n \geq q$. Therefore, it is customary to write Q_n instead of $Q_{q,n}$.

Now, let $\sigma_{r,n}$ be the nth elementary symmetric polynomial in the indeterminates $y_1, \ldots, y_r$. Then for each $n, m \in \mathbb{N}$, define the universal symmetric polynomials

$$P_n(S_{q,1}, \ldots, S_{q,n}, \ldots, \sigma_{r,1}, \ldots, \sigma_{r,n}) = \text{ the coefficient of } t^n \text{ in } \prod_{i,j}(1 + x_i y_j t)$$

and

$$P_{n,m}(S_{q,1},\ldots,S_{q,nm}) = \text{ the coefficient of } t^n \text{ in } \prod_{i_1<\cdots<i_m}(1+x_{i_1}\cdots x_{i_m}t).$$

Actually, we only need to find P_n for $n = r = q$ and $P_{n,m} = P_{m,n}$ for some $q \geq nm$. For example,

$$P_2(S_{2,1},S_{2,2},\sigma_{2,1},\sigma_{2,2}) = S_{2,1}^2\sigma_{2,2} + S_{2,2}\sigma_{2,1}^2 - 2S_{2,2}\sigma_{2,2},$$

$$P_3(S_{3,1},S_{3,2},S_{3,3},\sigma_{3,1},\sigma_{3,2},\sigma_{3,3}) = (S_{3,1}S_{3,2} - 3S_{3,3})\sigma_{3,1}\sigma_{3,2}$$
$$+(-3S_{3,3} - 3S_{3,1}S_{3,2} + S_{3,1}^3)\sigma_{3,3} + S_{3,3}\sigma_{3,1}^3,$$
$$P_{1,2}(S_{q,1},S_{q,2}) = S_{q,2},$$

and

$$P_{2,3}(S_{q,1},\ldots,S_{q,6}) = S_{q,6} - S_{q,5}S_{q,1} + S_{q,4}S_{q,2}.$$

Definition 2.4.2 *A λ-ring R is a special λ-ring if*

(i) $\lambda_t(1) = 1 + t$

(ii) for all $x, y \in R$ and $n \in \mathbb{N}$, $\lambda^n(xy) = P_n(\lambda^1(x),\ldots,\lambda^n(x),\lambda^1(y),\ldots,\lambda^n(y))$

(iii) for all $x \in R$ and $n, m \in \mathbb{N}$, $\lambda^m(\lambda^n(x)) = P_{n,m}(\lambda^1(x),\ldots,\lambda^{nm}(x))$.

Example 2. $K\mathbb{F}_G(X)$ is a special λ-ring. This follows directly from the G-splitting principle and the fact that a λ-ring R is special if and only if for any $a \in P(R)$ there exists a λ-homomorphism $f : R \to R'$ such that $f(a)$ is a sum of one dimensional elements in R'.

Now, we are ready to discuss Adams operations on special λ-rings. From Atiyah-Tall [12], we know that any natural operation on a special λ-ring R is a polynomial in the λ-operations. The Grothendieck operations γ^i are useful in studying the immersion theory of manifolds (see Atiyah[5]), they are defined by the relation

$$\gamma_t(x) = \sum_{j\geq 0}\gamma^j(x)t^j = \lambda_{t/(1-t)}(x)$$

or equivalently,

$$\gamma^n(x) = \lambda^n(x+n-1);$$

$x \in R$ and $n \in \mathbb{N}$. The most important operations in $K\mathbb{F}_G$-theory are the Adams operations, they are defined by

$$\psi_t(x) = \sum_{n\geq 1} \psi^n(x)t^n = -t\frac{d}{dt}\log\,\lambda_{-t}(x)$$

or equivalently,

$$\psi^n(x) = Q_{n,n}(\lambda_1(x), \ldots, \lambda^n(x)).$$

From Husemoller [31] (Proposition 1.8, p. 177), if $f(t) = 1 + x_1 t + \cdots + x_q t^q \in 1 + R[[t]]^+$, then

$$\sum_{n\geq 0}(-1)^n Q_{q,n}(x_1, \ldots, x_q) = -t\frac{d}{dt}\log\,f(x) \tag{2.4}$$

By differentiating the right hand side of (2.4) and compairing coefficients, we obtain

$$Q_{q,n}(x_1, \ldots, x_q) = (-1)^{n+1} \sum_{i+j=n} ix_i a_j$$

where $a_0 = 1$ and $a_j = -\sum_{r+s=j} a_r x_s$

So, we have the following 'computable' formula for ψ^n :

$$\psi^n(x) = (-1)^{n+1} \sum_{i+j=n} i\lambda^i(x)a_j$$

where $a_0 = 1$ and $a_j = -\sum_{r+s=j} a_r \lambda_s(x)$.

Further, ψ^k enjoys the following useful properties on $K\mathbb{F}_G(X)$:

(1) $\psi^k \circ \psi^l = \psi^{kl}$.

(2) If E is a 1-dimensional G-vector bundle over X, then $\psi^k(E) = E^k$.

(3) $c \circ \psi^k = \psi^k \circ c$, where c is the complexification homomorphism.

(4) $r \circ \psi^k = \psi^k \circ r$, where r is the realification homomorphism.

All the above properties are well-known, except possibly (4). For a proof of (4) in the non-equivariant case see Karoubi [36], Proposition 7.40 p. 265. Using the fact that a k-dimensional complex G-vector bundle E over a G-space X is isomorphic to $h_E^*(V_k(W_\mathbb{C}(G))) \times_{U(k)} \mathbb{C}^k$ for some G-map $h_E : X \to G_k(W_\mathbb{C}(G))$, the proof of (4) for $G \neq \{e\}$ will be the same as Karoubi's proof for the case $G = \{e\}$.

Before we discuss Bott exponential maps θ_k and θ_k^{or}, we briefly recall some facts about localization and completion. Let H be a finitely generated abelian group. For a prime p let $H_{(p)} = \{h/m : h \in H, m \in \mathbb{Z} \text{ with } (p,m) = 1\}$ denote the localization of H at p, then $H_{(p)}$ is canonically isomorphic to $\mathbb{Z}_{(p)} \otimes H$. Also, let $H_p = \lim_{\leftarrow n} H/p^n H$ denote the p-adic completion of H then H_p is canonically isomorphic to $\mathbb{Z}_p \otimes H$. For a rational number q, $\nu_p(q)$ will denote the exponent of p in the prime factorization of q.

Lemma 2.4.3 (i) *Let H be a finite abelian group of order N. Then the following groups are canonically isomorphic:*

$$H_{(p)} \cong H_p \cong H(p)$$

where $H(p) = \{h \in H : h$ has order a power of $p\}$. Consequently, if $h \in H$ has order m then the order of $h/1$ in $H_{(p)}$ = the order of $1 \otimes h$ in $H_p = p^{\nu_p(m)}$.

(ii) *If H is a finitely generated abelian gruop, then $H_{(p)}$ is canonically embedded in H_p.*

Proof. (i) If $N = p^n$ then $H = H(p)$, so in this case we only need to show that $H_{(p)} \cong H$ and $H_p \cong H$. Let $\phi_{(p)} : H \to H_{(p)}$ be the canonical map which sends h to $h/1$. $\phi_{(p)}$ is injective because, if $h/1 = 0$ in $H_{(p)}$ then $sh = 0$ in H for some $s \in \mathbb{N}$ with $(p,s) = 1$, hence h=0. To show that $\phi_{(p)}$ is onto, let $h/s \in H_{(p)}$. Choose $d, k \in \mathbb{Z}$ such that $kp^r + ds = 1$ where p^r is the order of h in H. Then $h/s = dh/1 = \phi_{(p)}(dh)$. Similarly, let $\phi_p : H \to H_p$ be the canonical map which sends h to $1 \otimes h$. ϕ_p is injective because

$$\bigcap_{i=1}^{\infty} p^i H = \{e\}$$

(see [9] for this and other back ground material on completions and localizations). To show that ϕ_p is onto, let $(h_i + p^i H) \in H_p$. Then $h_i - h_j \in p^i H$ for all $j \geq i$. So, $h_j = h_n$ for all $j \geq n$, because $p^n H = \{e\}$. Hence, $\phi_p(h_n) = (h_n + p^i H) = (h_i + p^i H)$. Thus if H is a finite p-group then $H_{(p)} \cong H_p \cong H(p)$. Now let $N = p_1^{n_1} \cdots p_t^{n_t}$ be the prime factorization of N. Then $H \cong H(p_1) \oplus \cdots \oplus H(p_t)$. Using the fact that localization and completion are exact functors on the category of finintely generated abelian groups, we obtain that

$$H_{(p)} \cong H(p_1)_{(p)} \oplus \cdots \oplus H(p_t)_{(p)} \text{ and } H_p \cong H(p_1)_p \oplus \cdots \oplus H(p_t)_p.$$

If $p_i = p$ then $H(p_i)_{(p)} \cong H(p_i)_p \cong H(p)$. If $p_i \neq p$ then one can easily show that $H(p_i)_{(p)} \cong H(p_i)_p = 0$. Hence $H_{(p)} \cong H_p \cong H(p)$ for any finite abelian group H. The other part of (i) is now easily established.

(ii) If $(p, m) = 1$ then $(m+p^i\mathbb{Z})$ is a unit in $\mathbb{Z}_p$, say $(m+p^i\mathbb{Z})^{-1} = (m_i+p^i\mathbb{Z})$. So $\phi : \mathbb{Z}_{(p)} \to \mathbb{Z}_p$ such that $\phi(n/m) = (nm_i+p^i\mathbb{Z})$ is a well-defined homomorphism. ϕ is injective, because if $\phi(n/m) = 0$ in $\mathbb{Z}_p$ then $nm_i \in p^i\mathbb{Z}$ for each $i \geq 1$. But $mm_i - 1 \in p^i\mathbb{Z}$ for each $i \geq 1$. So $m_i \notin p\mathbb{Z}$ for all i. Hence

$$n \in \bigcap_{i=1}^{\infty} p^i\mathbb{Z},$$

namely $n = 0$. Thus $\mathbb{Z}_{(p)}$ can be canonically embedded in $\mathbb{Z}_p$. Now, (ii) follows directly from (i). This completes the proof.

Let R be a special augmented λ-ring with augmentation $\epsilon : R \to \mathbb{Z}$, and let $\tilde{R} = \ker \epsilon$. Let $R(n)$ be the subgroup of R generated by all monomials $\gamma^{n_1}(a_1)\cdots\gamma^{n_r}(a_r)$ with $\epsilon(a_i) = 0$ for all i, and $\sum n_i \geq n$. Then

$$R(0) \supseteq R(1) \supseteq \cdots$$

is the γ-filtration of R. From now on, we assume that

(i) R is finitely generated as an abelian group by $x_1 = 1, x_2, \ldots, x_m$ which are finite dimensional with $\epsilon(x_r) = \dim x_r$.

(ii) The γ-topology on $\tilde{R}$, the topology defined by the γ-filtration, is finer than the p-adic toplology.

Example 3. $K\mathbb{F}_G(X)$ is a special λ-ring which satisfies the above two conditions. (tom Dieck [25], Proposition 3.8.6, p. 45).

For each $k \in \mathbb{N}$, define $\theta_k : P(R) \to R$ by

$$\theta_k(x) = \prod_{\substack{u^k-1=0 \\ u\neq 1}} \lambda_{-u}(x)$$

where $x \in P(R)$ and the product is taken over all kth roots of unity except 1. If $(k, p) = 1$, then θ_k can be extended to R with values in $R \otimes \mathbb{Z}_p$ by $\theta_k(x - y) = \theta_k(x)\theta_k(y)^{-1}$ for $x, y \in P(R)$. For each $k \in \mathbb{N}$ with $(k, p) = 1$, $\rho_k : \tilde{R} \otimes \mathbb{Z}_p \to 1 + \tilde{R} \otimes \mathbb{Z}_p$ (here $1 + \tilde{R} \otimes \mathbb{Z}_p$ is the multiplicative group of elements $1 + x$ with $x \in \tilde{R} \otimes \mathbb{Z}_p$) is given by

$$\rho_k(x) = \prod_{\substack{u^k-1=0 \\ u\neq 1}} \gamma_{u/(u-1)}(x).$$

The main algebraic theorem of Atiyah-Tall [12] states that:
If p is an odd prime, and h is a generator of $\Gamma = \lim_{\overleftarrow{n}} \Gamma_{p^n}$ where Γ_{p^n} is the Galois group of $\mathbb{Q}(w)$ over $\mathbb{Q}$, and w is a primitive p^nth root of unity. Then ρ_h induces an isomorphism

$$\rho_{h,\Gamma} : (\tilde{R} \otimes \mathbb{Z}_p)_\Gamma \to (1 + \tilde{R} \otimes \mathbb{Z}_p)_\Gamma$$

where the index Γ indicates that we factor out the image of $1 - \psi^h$.

Let $x \in P(R)$ be an element of dimension n. x is said to be orientable if $\lambda^r(x) = \lambda^{n-r}(x)$ for each $r = 0, \dots, n$. R is said to be orientable if all elements of $P(R)$ are orientable.

Example 4. If $KSO_G(X)$ is the subgroup of $KO_G(X)$ generated by orientable G-vector bundles, then $KSO_G(X)$ is an orientable special λ-ring.

Suppose that R is orientable, and let $P(R(2))$ be the semiring of $P(R)$ generated by even dimensional elements. Let k be an odd integer, and J be a set of kth roots of unity $u \neq 1$ which contains from each pair u, u^{-1} exactly one element. The operation $\theta_k^{or} : P(R(2)) \to R$ is defined by

$$\theta_k^{or}(x) = k^m \prod_{u \in J} \lambda_{-u}(x)(1-u)^{-2m}.$$

With a few calculations, we get

$$\theta_k^{or}(x) = \prod_{u \in J} \lambda_{-u}(x)(-u^{-1})^m. \tag{2.5}$$

As with θ_k, if $(k, p) = 1$, then we can extend θ_k^{or} to $R(2)$ with values in $R(2) \otimes \mathbb{Z}_p$.

For a 2-dimensional $z \in R(2)$, the following useful identity is proved in tom Dieck [25], p. 52 : If $k = 2q + 1$ is an odd integer, then

$$\theta_k^{or}(z) = 1 + \psi^1(z) + \cdots + \psi^q(z). \tag{2.6}$$

Similarly, $\rho_k^{or} : \tilde{R} \otimes \mathbb{Z}_p \to 1 + \tilde{R} \otimes \mathbb{Z}_p$ is given by

$$\rho_k^{or}(x) = \prod_{u \in J} \gamma_{u/(u-1)}(x).$$

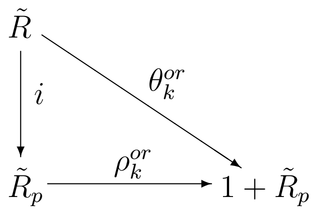
Now, we show how to define θ_{2k}^{or} and ρ_{2k}^{or} for each $k \in \mathbb{N}$. If $p \neq 2$ then $1/2 \in \mathbb{Z}_p$, So we can define $\theta_{2k}^{or} : R \to 1 + R_p$. by

$$\theta_{2k}^{or}(x - y) = \Big(\prod_{\substack{u^{2k}-1=0 \\ u \neq 1}} \lambda_{-u}(x)\Big(\prod_{\substack{u^{2k}-1=0 \\ u \neq 1}} \lambda_{-u}(y)\Big)^{-1}\Big)^{1/2}.$$

Similarly, we can define

$$\rho_{2k}^{or}(x) = \Big(\prod_{\substack{u^{2k}-1=0 \\ u \neq 1}} \gamma_{\frac{u}{u-1}}(x)\Big)^{1/2}.$$

If $(p, k) = 1$ (k may be 2) then the following diagram is commutative:



where i is defined by $i(x) = 1 \otimes x$, it is not necessarily injective. From Atiyah-Tall [12], we know that the induced map

$$\rho_{k_p,\Gamma}^{or} : (\tilde{R}_p)_\Gamma \to (1 + \tilde{R}_p)_\Gamma \tag{2.7}$$

is an isomorphism. In fact, from the main theorem of Atiyah-Segal [11] we know that there is a natural isomorphism between $\tilde{R}_p$ and $1 + \tilde{R}_p$.

Example 5. If $R = KSO_G(X)$ and E is an $8n$-dimensional real vector bundle with $Spin(G)$-structure, then one can define the Bott class $\tilde{\theta}_k(E) = \phi^{-1}(\mu_E)$. From tom Dieck [25], Proposition 3.15.2, we get $\theta_k^{or}(E) = \tilde{\theta}_k(E)$. Similarly, if $R = KU_G(X)$ and E is a complex G-vector bundle over X then $\theta_k(E) = \phi^{-1}(\lambda_E)$.

Finally, we state Atiyah-Segal completion theorem.

Theorem 2.4.4 *(Atiyah-Segal completion theorem for equivariant complex K-theory)*

Let X be a finite G-CW complex, where G is a finite p-group. Then there is an isomorphism

$$KU_G(X)_p \to KU(X_G)$$

where $X_G = EG \times_G X$.

Theorem 2.4.5 *(Atiyah-Segal completion theorem for equivariant real K-theory)*

Let X be a finite G-CW complex, where G is a finite p-group. Then there is an isomorphism

$$KSO_G(X)_p \to KSO(X_G).$$

2.5 The equivariant cross section problem of Stiefel manifolds

One of the most important applications of the equivariant J-groups is the equivariant cross section problem of Stiefel manifolds, for the topology of Stiefel manifolds see James [33]. The solution for the non-equivariant case was completed by Adams [2], 1961 (the real part), Atiyah-Todd, Adams-Walker [13],[3], 1963-1965 (the complex part), and Sigrist-Suter [56], 1973 (the quaternionic part). The real equivariant case was discussed by Becker [14], 1971, Kakutani [35], and Namboodiri [46], 1982. The complex equivariant case is discussed by Önder [51],[50], 1995-1998. However, many important cases, in the equivariant case, are still open.

The main purpose of this section is to lead the reader to the important role of the equivariant J-groups in solving the equivariant cross section problem of Stiefel manifolds.

Throughout this section, M will be an n-dimensional $\mathbb{F}$ G-module, where G is a compact topological group, and $\mathbb{F} = \mathbb{R}, \mathbb{C},$ or $\mathbb{H}$. Also, we assume that M has an $\mathbb{F}$ G-invariant inner product $(\,|\,) : M \times M \to \mathbb{F}$, so that $S(M) = \{x \in M : (x|x) = 1\}$ is a G-space.

A G-equivariant $\mathbb{F}$ vector field on $S(M)$ is a G-map $s : S(M) \to S(M)$ such that $(x|s(x)) = 0$ for all $x \in S(M)$. The G-equivariant $\mathbb{F}$ vector field problem on M reads as follows:
What is the maximal number of linearly independent G-equivariant $\mathbb{F}$ vector fields on $S(M)$? We call this number the $\mathbb{F}$ G-field number of M.

For each $k \leq n$, let

$$V\mathbb{F}_k(M) = \{(x^{(1)}, \ldots, x^{(k)}) \in M^k : (x^{(i)}|x^{(j)}) = \delta_{ij}\}$$

be the Stiefel manifold of $\mathbb{F}$ k-frames in M. $V\mathbb{F}_k(M)$ is a G-space with

$$g(x^{(1)}, \ldots, x^{(k)}) = (gx^{(1)}, \ldots, gx^{(k)}).$$

Consider the following G-fibration:

$$p : V\mathbb{F}_k(M) \to S(M)$$
$$(x^{(1)}, \ldots, x^{(k)}) \mapsto x^{(1)}.$$

Then we have:

Proposition 2.5.1 *$S(M)$ has $\mathbb{F}$ $(k-1)$ G-fields if and only if p has a G-equivariant cross section.*

Proof. Let $s_1, \ldots, s_{k-1}$ be $(k-1)$ linearly independent G-equivariant $\mathbb{F}$ vector fields on $S(M)$. Define $s : S(M) \to V\mathbb{F}_k(M)$ by

$$s(x) = (x, s_1(x), \ldots, s_{k-1}(x)).$$

Then s is a G-section of p. Conversely, if $s : S(M) \to V\mathbb{F}_k(M)$ is a G-section of p then $pr_i \circ s$, $i = 2, \ldots, k$ are $(k-1)$ linearly independent G-equivariant $\mathbb{F}$ vector fields on $S(M)$, where pr_i denotes the ith projection map. This completes the proof.

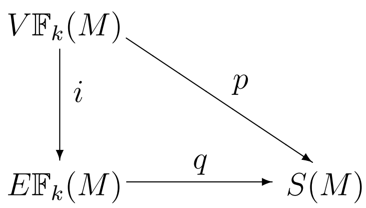
For each $n \in \mathbb{N}$, $\mathbb{F}^n$ is an $\mathbb{F}$-module, so $S(\mathbb{F})$ acts on $S(\mathbb{F}^n)$. Similarly, $S(\mathbb{F})$ acts on $S(M)$. Let $E\mathbb{F}_k(M) = \{f : S(\mathbb{F}^k) \to S(M) : f \text{ is } S(\mathbb{F}) - \text{equivariant}\}$. $E\mathbb{F}_k(M)$ is a G-space with $(gf)(x) = gf(x)$ for $f \in E\mathbb{F}_k(M)$, $g \in G$ and $x \in S(\mathbb{F}^n)$. Let $q : E\mathbb{F}_k(M) \to S(M)$ be the map defined by $q(f) = f(e^{(1)} = (1, 0, \ldots, 0))$. Clearly, q is a G-map.

Proposition 2.5.2 *If $V\mathbb{F}_k(M) \xrightarrow{p} S(M)$ has a G-section, then*

$$E\mathbb{F}_k(M) \xrightarrow{q} S(M)$$

has a G-section.

Proof. Consider the following commutative diagram:



where $i(x^{(1)}, \ldots, x^{(k)}) : S(\mathbb{F}^n) \to S(M)$ is given by $(\alpha_1, \ldots, \alpha_n) \mapsto \sum_{i=1}^{k} \alpha_i x^{(i)}$. i is an inclusion, in fact we can regard $V\mathbb{F}_k(M)$ as the set $\{f : \mathbb{F}^k \to M : f$ is a norm preserving $\mathbb{F} - \text{linear map}\}$. So, $V\mathbb{F}_k(M)$ can be viewed as a subset of $E\mathbb{F}_k(M)$ in a proper way. Now, if $s : S(M) \to V\mathbb{F}_k(M)$ is a G-section of p. Then $i \circ s$ is a G-section of q.

For any real number r, let $\lceil r \rceil$ denote the smallest integer larger than or equal to r. For each $k \leq n$, consider the following condition C_k on M :
C_k : If $H \leq G$ with $\dim_{\mathbb{F}} M^H > 0$, then $\dim_{\mathbb{F}} M^H \geq 2k - 2 + \lceil 2/\dim_{\mathbb{R}} \mathbb{F} \rceil$.

Proposition 2.5.3 *If $q : E\mathbb{F}_k(M) \to S(M)$ has a G-section and M satisfies condition C_k, then $V\mathbb{F}_k(M) \xrightarrow{p} S(M)$ has a G-section.*

Proof. For $\mathbb{F} = \mathbb{R}$ this is Theorem 3.1 of Namboodiri [46]. For $\mathbb{F} = \mathbb{C}$ this is Lemma 3.1 of Önder [51]. The proof for $\mathbb{F} = \mathbb{H}$ is exactly the same, we only need to use the fact that $i : V\mathbb{H}_k(\mathbb{H}^n) \to E\mathbb{H}_k(\mathbb{H}^n)$ is an $(8n - 8k + 1)$-equivalence (see Lemma 2.11 of [43]).

Now, we present the main definition of this dissertation which will be used throughout the remaining chapters.

Definition 2.5.4 *(Main Definition)*

Let X be a G-space and ξ, η be two spherical G-fibrations over X. We say that ξ and η are stably G-fiber homotopy equivalent, shortly, $\xi \wedge X \times S^V \simeq_G \eta \wedge X \times S^V$, if there is a real G-module V such that $\xi \wedge X \times S^V$ is G-fiber homotopy equivalent to $\eta \wedge X \times S^V$,i.e., there is a fiberwise G-map

$$f : \xi \wedge X \times S^V \to \eta \wedge X \times S^V$$

which has a fiberwise G-homotopy inverse, where S^V is the one-point compactification of V with base-point at ∞.

Now, let us see the relation between G-fiber homotopy equivalence and the equivariant cross section problem of Stiefel manifolds. Let $\mathbb{F}P^k$ be the $\mathbb{F}$ projective space with trivial G-action. Let $\xi_k(\mathbb{F})$ be the canonical Hopf $\mathbb{F}$ line bundle (resp. the conjugate of the Hopf $\mathbb{H}$ line bundle) over $\mathbb{F}P^k$ for $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ (resp. over $\mathbb{H}P^k$). $\{S(M), E\mathbb{F}_k(M)\}_G$ will denote the space of G-maps from $S(M)$ to $E\mathbb{F}_k(M)$ and $\{S(\mathbf{M}), S(\xi_k(\mathbb{F}) \otimes_{\mathbb{F}} \mathbf{M})\}_G$ will denote the space of fiberwise G-maps from $S(\mathbf{M})$ to $S(\xi_k(\mathbb{F}) \otimes_{\mathbb{F}} \mathbf{M})$ where $\mathbf{M} = \mathbb{F}P^k \times M$ is the trivial $\mathbb{F}$ G-vector bundle over $\mathbb{F}P^k$ with fiber M.

Lemma 2.5.5 *There exists a homeomorphism*

$$\phi : \{S(M), E\mathbb{F}_k(M)\}_G \to \{S(\boldsymbol{M}), S(\xi_k(\mathbb{F}) \otimes_{\mathbb{F}} \boldsymbol{M})\}_G$$

with the following property: A G-map $s : S(M) \to E\mathbb{F}_k(M)$ is such that $q \circ s$ has G-degree 1 if and only if $\phi(s) : S(\boldsymbol{M}) \to S(\xi_k(\mathbb{F}) \otimes_{\mathbb{F}} \boldsymbol{M})$ has fiberwise G-degree 1.

Proof. Exactly the same as the proof of Lemma 3.4 of [46].

Theorem 2.5.6 *(i) If $p : V\mathbb{F}_k(M) \to S(M)$ has a G-section, then*

$$S(\xi_{k-1}(\mathbb{F}) \otimes_{\mathbb{F}} \boldsymbol{M}) \simeq_G S(\boldsymbol{M}).$$

(ii) If M satisfies condition C_k and $S(\xi_{k-1}(\mathbb{F}) \otimes_{\mathbb{F}} \boldsymbol{M}) \simeq_G S(\boldsymbol{M})$, then

$$p : V\mathbb{F}_k(M) \longrightarrow S(M)$$

has G-section.

Proof. (i) Follows from Proposition 2.5.2, Lemma 2.5.5, and the equivariant Dold's theorem mod-k (see Chapter 3).
(ii) Follows from Lemma 2.5.5 and Proposition 2.5.3.

From the above theorem, the G-field number of M is the largest k such that M satisfies condition C_k and $S(\xi_{k-1}(\mathbb{F}) \otimes_{\mathbb{F}} \mathbf{M}) \simeq_G S(\mathbf{M})$. So, the equivariant cross section problem of Stiefel manifolds has been reduced to a question about G-fiber homotopy equivalence.

Chapter 3

EQUIVARIANT J-GROUPS

3.1 Introduction

An important problem in equivariant homotopy theory is the following:
Given two spherical G-fibrations ξ, η over a G-space X. Are ξ and η G-fibre homotopy equivalent? To show that ξ and η are not G-fibre homotopy equivalent, one usually tries to find suitable invariants such as equivariant characteristic classes. On the other hand, to show that ξ and η are stably G-fibre homotopy equivalent, one usually tries to use some geometrical constructions. The aim of this chapter is to discuss the above problem for spherical G-fibrations which come out from G-vector bundles over a finite G-CW complex X, G being a finite group. In the next chapter, we shall use the ideas of this chapter to discuss the equivariant J-groups of various projective spaces.

In Section 2, we construct the Grothendieck group $Sph_G(X)$ of all stable equivalence classes of spherical G-fibrations over a G-space X and then consider the equivariant J-groups, $J\mathbb{F}_G(X)$, as subgroups of $Sph_G(X)$. Also, we give reasons for only considering real equivariant J-groups, $JO_G(X)$. We show that if X is a free G-space then the computation of the equivariant J-groups reduces to the computation of the non-equivariant ones. Section 2 is closed by reducing the equivariant cross section problem of Stiefel manifolds to a question about the equivariant J-groups.

In Section 3, we give a comprehensive survey about the case X is a point and then we give explicit computations for $JO_G(\text{one point})$ for finite abelian p-groups G.

In Section 4, we discuss tom Dieck and Hauschild method for computing $JO_G^{(p)}(X)$, the main part of $JO_G(X)_{(p)}$, for finite p-groups. Then we present McClure reduction of the problem from finite group case to p-group case. Also, in this section, we prove our main theoretical result in this dissertation, an alternative method for computing $JO_G^{(p)}(X)$. Finally, in Section 5 we consider the case X is a trivial G-space.

3.2 $Sph_G(X)$ and $JO_G(X)$

Let $Sph_G(X)$ be the group completion of the semigroup of all stable G-fibre homotopy classes of spherical G-fibrations over X, with addition given by fibrewise join. Define

$$J\mathbb{F}_G : K\mathbb{F}_G(X) \longrightarrow Sph_G(X)$$

$$E - F \mapsto S(E) - S(F).$$

We denote the image of $J\mathbb{F}_G$ by $J\mathbb{F}_G(X)$. It is easy to see that ker $J\mathbb{F}_G = T\mathbb{F}_G(X) = \{E - F \in \widetilde{K\mathbb{F}}_G(X) : S(E \oplus \mathbf{V}) \simeq_G S(F \oplus \mathbf{V})$ for some $\mathbb{F}$ G-module $V\}$. So, $J\mathbb{F}_G(X) \cong K\mathbb{F}_G(X)/T\mathbb{F}_G(X)$. In fact, this last identification of $J\mathbb{F}_G(X)$ is its usuall definition (see Adams [1], for the non-equivariant case, and Petrie-Randall [53], Ch. 1 §§ 9 and 13 for the equivariant case).

Proposition 3.2.1 *$J\mathbb{F}_G$ is a cofunctor from the category of locally compact G-spaces and G-homotopic proper G-maps to the category of finitely generated abelian groups.*

Proof. Let $f : X \to Y$ be a proper G-map. If $E - F \in T\mathbb{F}_G(Y)$ then $f^*E - f^*F \in T\mathbb{F}_G(X)$, because if $S(E \oplus Y \times V) \simeq_G S(F \oplus Y \times V)$ for some $\mathbb{F}$ G-module V then it is easy to see that $S(f^*E \oplus X \times V) \simeq_G S(f^*F \oplus X \times V)$. So f^* is defined from $J\mathbb{F}_G(Y)$ to $J\mathbb{F}_G(X)$, then the result follows.

Let $r : KU_G(X) \to KO_G(X)$, $c : KO_G(X) \to KU_G(X)$, and $q : KU_G(X) \to KSp_G(X)$ be the realification, complexification, and quaternionization homomorphisms, respectively.

Lemma 3.2.2 *With the above notation, we have the following inclusions:*

(i) $r(TU_G(X)) \subseteq TO_G(X)$.

(ii) $c(TO_G(X)) \subseteq TU_G(X)$.

(iii) $q(TU_G(X)) \subseteq TSp_G(X)$.

Proof. (i) Let $E - F \in TU_G(X)$, then $S(E \oplus \mathbf{V}) \simeq_G S(F \oplus \mathbf{V})$ for some complex G-module V. $S(rE)$ is isomorphic to $S(E)$ as spherical G-fibrations. So $S(rE \oplus r\mathbf{V}) \simeq_G S(rF \oplus r\mathbf{V})$, namely $rE - rF \in TO_G(X)$.
(ii) Let $E - F \in TO_G(X)$, then $S(E \oplus \mathbf{V}) \simeq_G S(F \oplus \mathbf{V})$ for some real G-module V. $S(cE)$ is isomorphic to the fibrewise join $S(E) * S(E)$ as spherical G-fibrations. So $S(cE \oplus c\mathbf{V}) \simeq_G S(cF \oplus c\mathbf{V})$, namely $cE - cF \in TU_G(X)$.
(iii) is similar to (ii).

Proposition 3.2.3 *We have homomorphisms $\tilde{r} : J\mathbb{F}_G(X) \to JO_G(X)$ for $\mathbb{F} = \mathbb{C}$ or $\mathbb{H}$. Similarly, we have $\tilde{c}$, $\tilde{q}$.*

Proof. Follows directly from the above lemma.

Proposition 3.2.4 *$\tilde{r} : J\mathbb{F}_G(X) \to JO_G(X)$ is a monomorphism for $\mathbb{F} = \mathbb{C}$ or $\mathbb{H}$.*

Proof. We prove this proposition for $\mathbb{F} = \mathbb{C}$, the case $\mathbb{F} = \mathbb{H}$ is similar. Let $\tilde{r}(E - F + TU_G(X)) = 0$ in $JO_G(X)$ then $rE - rF \in TO_G(X)$. So, $S(rE \oplus \mathbf{V}) \simeq_G S(rF \oplus \mathbf{V})$ for some real G-module V. $S(E \oplus c\mathbf{V})$ is isomorphic to $S(rE) * S(\mathbf{V}) * S(\mathbf{V})$. Hence, $S(E \oplus c\mathbf{V}) \simeq_G S(F \oplus c\mathbf{V})$, namely $E - F \in TU_G(X)$. This completes the proof.

For $x \in K\mathbb{F}_G(X)$, the $J\mathbb{F}_G$-order of x is the order of $x + T\mathbb{F}_G(X)$ in $J\mathbb{F}_G(X)$.

Theorem 3.2.5 *(i) If E is a complex G-vector bundle over X with finite JU_G-order, then the JO_G-order of rE is equal to the JU_G-order of E.*
(ii) If F is a real G-vector bundle over X with finite JO_G-order n, then the JU_G-order of cF is equal to $n/(2, n)$.

Proof. (i) Suppose the order of E in $JU_G(X)$ is m and the order of rE in $JO_G(X)$ is n. Then $m(E - \dim E) \in TU_G(X)$. So, $m(rE - 2(\dim E)) \in TO_G(X)$, hence $n|m$. On the other hand, $n(rE - 2(\dim E)) \in TO_G(X)$ implies that $n(E - \dim E) \in TU_G(X)$, because $\tilde{r}$ is injective. Hence $m|n$.
(ii) From (i) the JU_G-order of cF is the JO_G-order of $rcF = 2F$. The result follows.

Remark 1. Let E, F be two $\mathbb{F}$ G-vector bundles over X. The main purpose of the groups $J\mathbb{F}_G(X)$ is to see whether $S(E) = S(F)$ in $Sph_G(X)$ or not. From Proposition 3.2.4, $S(E) = S(F)$ in $Sph_G(X)$ if and only if $rE - rF \in TO_G(X)$. Therefore, we only need to consider the equivariant real J-groups $JO_G(X)$.

In the following theorem, we show that if X is a free G-space then the computation of the equivariant J-groups $J\mathbb{F}_G(X)$ reduces to the non-equivariant ones.

Theorem 3.2.6 *If X is a free G-space, then $J\mathbb{F}_G(X) \cong J\mathbb{F}(X/G)$.*

Proof. We show that $JO_G(X) \cong JO(X/G)$. (The cases $\mathbb{F} = \mathbb{C}$ or $\mathbb{H}$ are exactly the same). From Example 3 §2.3, we know that $\phi : KO_G(X) \to KO(X/G)$ given by $\phi(E_1 - E_2) = E_1/G - E_2/G$ is an isomorphism. Also, $\phi^{-1}(F_1 - F_2) = q^*F_1 - q^*F_2$ where $q : X \to X/G$ is the quotient map. To show that $JO_G(X) \cong JO(X/G)$, it is enough to show that $\phi(TO_G(X)) = TO(X/G)$, because then $\tilde{\phi} : JO_G(X) \to JO(X/G)$ is an isomorphism. To show that $\phi(TO_G(X)) \subseteq TO(X/G)$, we only need to show that if $S(E_1) \simeq_G S(E_2)$, where E_1, E_2 are two G-vector bundles over X, then $S(E_1/G) \simeq S(E_2/G)$. Let

$$f_1 : S(E_1) \to S(E_2) \text{ and } f_2 : S(E_2) \to S(E_1)$$

be fibrewise G-maps such that

$$f_2 \circ f_1 \overset{H_1}{\simeq_G} id|_{S(E_1)} \text{ and } f_1 \circ f_2 \overset{H_2}{\simeq_G} id|_{S(E_2)}$$

where H_1, H_2 are fibrewise G-homotopies. Define,

$$\tilde{f}_1 : S(E_1/G) \to S(E_2/G) \text{ and } \tilde{H}_1 : S(E_1/G) \times I \to S(E_1/G)$$

such that $\tilde{f}_1(Ge_1) = Gf(e_1)$ and $\tilde{H}_1(Ge_1, t) = GH_1(e_1, t)$. Similarly, define $\tilde{f}_2$ and $\tilde{H}_2$. It is easy to see that

$$\tilde{f}_2 \circ \tilde{f}_1 \overset{\tilde{H}_1}{\simeq} id|_{S(E_1)} \text{ and } \tilde{f}_1 \circ \tilde{f}_2 \overset{\tilde{H}_2}{\simeq} id|_{S(E_2)}.$$

Hence, $S(E_1) \simeq_G S(E_2)$ implies that $S(E_1/G) \simeq S(E_2/G)$.

On the other hand, to show that $\phi(TO_G(X)) \subseteq TO(X/G)$ we only need to show that if $S(F_1) \simeq S(F_2)$, where F_1, F_2 are two vector bundles over X/G, then $S(q^*(F_1)) \simeq_G S(q^*(F_2))$. Let d_1, d_2, D_1, D_2 be the maps and homotopies giving the equivalence $S(F_1) \simeq S(F_2)$. Define

$$\bar{d}_1 : S(q^*(F_1)) \to S(q^*(F_2))$$

such that $\bar{d}_1((x, \alpha_1)) = (x, d_1(\alpha_1))$. $\bar{d}_1$ is a G-map because

$$\bar{d}_1(g(x, \alpha_1)) = \bar{d}_1(gx, \alpha_1) = (gx, d_1(\alpha_1)) = g\bar{d}_1(x, \alpha_1).$$

Similarly, define $\bar{d}_2, \bar{D}_1, \bar{D}_2$. Then $\bar{d}_1, \bar{d}_2$ are G-maps and $\bar{D}_1, \bar{D}_2$ are fibrewise G-homotopies. Hence, $S(F_1) \simeq S(F_2)$ implies that $S(q^*(F_1)) \simeq_G S(q^*(F_2))$. Consequently, $\phi(TO_G(X)) = TO(X/G)$ as asserted. This completes the proof of the theorem.

Example 1. (Lens Spaces)

Let $L^n(m) = S^{2n+1}/(\mathbb{Z}/m\mathbb{Z})$ be the orbit space of the standard free action of $\mathbb{Z}/m\mathbb{Z}$ on S^{2n+1} (consider $\mathbb{Z}/m\mathbb{Z} = \{e^{2\pi ik/m} : k = 0, \ldots, m-1\}$). From the above theorem, we have $JO_{\mathbb{Z}/m\mathbb{Z}}(S^{2n+1}) \cong JO(L^n(m))$. J-groups of lens spaces are discussed by Kobayashi-Murakami-Sugawara [39], Dibağ-Kirdar [21], and Kirdar [37]. However, many important cases are still open and the computations of these groups using usual methods of non-equivariant J-groups involve many difficulties. Therefore, it may be a good problem, if one tries to use methods of equivariant J-groups to compute $JO_{\mathbb{Z}/m\mathbb{Z}}(S^{2n+1})$ instead of $JO(L^n(m))$?!.

Let M be an $\mathbb{F}$ G-module. From Theorem 2.5.6, if $V\mathbb{F}_k(M) \xrightarrow{p} S(M)$ has a G-section then $S(\xi_{k-1}(\mathbb{F}) \otimes \mathbf{M}) \simeq_G S(\mathbf{M})$. So, a necessary condition for $V\mathbb{F}_k(M) \xrightarrow{p} S(M)$ to have a G-section is that $r(\xi_{k-1}(\mathbb{F}) \otimes \mathbf{M} - \mathbf{M})$ vanishes in $JO_G(\mathbb{F}P^{k-1})$, where $\mathbb{F}P^{k-1}$ is considered as a trivial G-space. Further, if $G = \{e\}$ then this condition is also sufficient. Thus the non-equivariant cross section problem of Stiefel manifolds reduces to the computation of the orders of $\xi_{k-1}(\mathbb{F})$ in $JO(\mathbb{F}P^{k-1})$. For $G \neq \{e\}$, the situation is more complicated due to the fact that stably G-fibre homotopy equivalence does not, in general, imply G-fibre homotopy equivalence. So, one tries to define a variant of $J\mathbb{F}_G(X)$ so that stably G-fibre homotopy equivalence, with some conditions, implies G-fibre homotopy equivalence.

Let

$$\begin{aligned} Gap(M) = \ & \{V : V \text{ is a real } G-\text{module such that if } K < H < G \text{ and} \\ & \quad \dim_{\mathbb{R}} V^K > \dim_{\mathbb{R}} V^H \text{ then } \dim_{\mathbb{R}} M^K > \dim_{\mathbb{R}} M^H\}. \end{aligned}$$

Let

$$\mathcal{U}(M) = \bigoplus_{W_i \in Gap(M) \cap Irr(G,\mathbb{R})} W_i^{\infty}.$$

Define

$$KO_G(X, \mathcal{U}(M)) = \{E - F \in KO_G(X) : E_x, F_x \subseteq \mathcal{U}(M, G_x) \text{ for some } x \in X\}$$

where, the inclusion is understood as inclusion up to isomorphism, and $\mathcal{U}(M, G_x) = \mathcal{U}(M)$ but as a G_x-space,

$$TO_G(X,\mathcal{U}(M)) = \{E - F \in KO_G(X,\mathcal{U}(M)) : S(E \oplus V) \simeq_G S(F \oplus V) \text{ for some } V \in Gap(M)\}$$

and let

$$JO_G(X,\mathcal{U}(M)) = KO_G(X,\mathcal{U}(M))/TO_G(X,\mathcal{U}(M)).$$

Let

$$d_{\mathbb{F}} = \begin{cases} 0 & \text{if } \mathbb{F} = \mathbb{R} \\ 1 & \text{if } \mathbb{F} = \mathbb{C} \text{ or } \mathbb{H} \end{cases}$$

Then we have:

Theorem 3.2.7 *Let M be an $\mathbb{F}$ G-module. If*

$$dim_{\mathbb{R}} M^G \geq \max\{3, (dim_{\mathbb{R}} \mathbb{F})(2k - d_{\mathbb{F}})\} \tag{3.1}$$

and $r(\xi_{k-1}(\mathbb{F}) \otimes \boldsymbol{M} - \boldsymbol{M})$ vanishes in $JO_G(\mathbb{F}P^{k-1},\mathcal{U}(M))$, then

$$V\mathbb{F}_k(M) \xrightarrow{p} S(M)$$

has a cross G-section.

Proof For $\mathbb{F} = \mathbb{R}$ this is Theorem 5.2 of [46]. For $\mathbb{F} = \mathbb{C}$ this is Theorem 2.2 of [50]. So, let us prove the case $\mathbb{F} = \mathbb{H}$. We shall imitate the proofs of the real and complex cases. The main tool in proving this theorem is Theorem 4.2 of [46]. If $r(\xi_{k-1}(\mathbb{H}) \otimes \mathbf{M} - \mathbf{M})$ vanishes in $JO_G(\mathbb{H}P^{k-1},\mathcal{U}(M))$ then

$$S(r(\xi_{k-1}(\mathbb{H}) \otimes \mathbf{M}) \oplus \mathbf{V}) \simeq_G S(r\mathbf{M} \oplus \mathbf{V})$$

for some $V \in Gap(M)$. Now, let $X = \mathbb{H}P^{k-1}$ with the trivial G-action, $N = rM$ and $\eta = r(\xi_{k-1}(\mathbb{H}) \otimes \mathbf{M})$ then condition (3.1) implies that conditions (1), (2), (3), and (4) of Theorem 4.2 of [46] are satisfied. So, we are done if we prove that $\dim_{\mathbb{R}} M^K - \dim_{\mathbb{R}} M^H > 4k - 3$ whenever $V \in Gap(M)$, and $H, K \in Iso(M)$ with $K < H$ and $\dim_{\mathbb{R}} V^K > \dim_{\mathbb{R}} V^H$. By restriction to H-fixed point sets,

$$S(r(\xi_{k-1}(\mathbb{H}) \otimes \mathbf{M}) \oplus \mathbf{V}) \simeq_G S(r\mathbf{M} \oplus \mathbf{V})$$

implies that

$$S(\dim_{\mathbb{H}} M^H \xi_{k-1}(\mathbb{H}) \oplus \dim_{\mathbb{H}} V^H) \simeq_G S(\dim_{\mathbb{H}} M^H \oplus \dim_{\mathbb{H}} V^H).$$

So, $\dim_{\mathbb{H}} M^H(r(\xi_{k-1}(\mathbb{H})-1)) = 0$ in $JO(\mathbb{H}P^{k-1})$. Thus, $\dim_{\mathbb{H}} M^H$ divides c_k where c_k is the quaternionic James number defined in [56]. Hence, $\nu_2(\dim_{\mathbb{H}} M^H) \geq \nu_2(c_k)$. Similarly, $\nu_2(\dim_{\mathbb{H}} M^K) \geq \nu_2(c_k)$. Since $\dim_{\mathbb{H}} M^K - \dim_{\mathbb{H}} M^H > 0$ then $\nu_2(\dim_{\mathbb{H}} M^H - \dim_{\mathbb{H}} M^K) \geq \nu_2(c_k)$. But

$$\nu_2(c_k) = \max\{2s - \nu_2(s), 2k-1 : 1 \leq s \leq k-1\}.$$

So, $\dim_{\mathbb{H}} M^K - \dim_{\mathbb{H}} M^H \geq 2k-1$ or equivalently, $\dim_{\mathbb{R}} M^K - \dim_{\mathbb{R}} M^H \geq 8k-4$. Clearly, $8k-4 > 4k-3$ for each $k \geq 1$, this completes the proof.

From the above theorem, the $\mathbb{F}$ G-field number of M is the largest k such that M satisfies condition (3.1) and $r(\xi_{k-1}(\mathbb{F}) \otimes \mathbf{M} - \mathbf{M})$ vanishes in $JO_G(\mathbb{F}P^{k-1}, \mathcal{U}(M))$. Therefore, it is of great importance to find means for computing $JO_G(X, \mathcal{U}(M))$?.

3.3 $JO_G(X)$; X is a point

In this section, we discuss the groups $JO_G(X)$ where X is a point and G is a compact Lie group, for simplicity, we write these groups by $JO_G(*)$. First, we need to recall a collection of definitions and simple facts concerning the representations of compact Lie groups. Let $\mathrm{Irr}(G, \mathbb{F})$ denote the set of all irreducible $\mathbb{F}$ representations of G ($\mathbb{F}$ being $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$). Since every $\mathbb{F}$ representation of G is semisimple ([31], Ch. 12, Theorem 6.5). Then $R\mathbb{F}(G)$ is a free abelian group with basis $\mathrm{Irr}(G, \mathbb{F})$. Let $\mathrm{Irr}(G, \mathbb{F})_{\mathbb{R}}$, $\mathrm{Irr}(G, \mathbb{F})_{\mathbb{C}}$, and $\mathrm{Irr}(G, \mathbb{F})_{\mathbb{H}}$ denote the subset of $\mathrm{Irr}(G, \mathbb{F})$ of elements of type $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$, respectively. Then $\mathrm{Irr}(G, \mathbb{F})$ is the disjoint union of $\mathrm{Irr}(G, \mathbb{F})_{\mathbb{R}}$, $\mathrm{Irr}(G, \mathbb{F})_{\mathbb{C}}$, and $\mathrm{Irr}(G, \mathbb{F})_{\mathbb{H}}$ ([19], Ch. 2, Theorem 6.3). So,

$$R\mathbb{F}(G) = R\mathbb{F}(G)_{\mathbb{R}} \oplus R\mathbb{F}(G)_{\mathbb{C}} \oplus R\mathbb{F}(G)_{\mathbb{H}}$$

where $R\mathbb{F}(G)_{\mathbb{R}}$, $R\mathbb{F}(G)_{\mathbb{C}}$, and $R\mathbb{F}(G)_{\mathbb{H}}$ are the free abelian groups with basis $\mathrm{Irr}(G, \mathbb{F})_{\mathbb{R}}$, $\mathrm{Irr}(G, \mathbb{F})_{\mathbb{C}}$, and $\mathrm{Irr}(G, \mathbb{F})_{\mathbb{H}}$, respectively.

Let $R\mathbb{F}_0(G) = \{V - W \in R\mathbb{F}(G) : \dim V^H = \dim W^H$ for all subgroups H of $G\}$. If $\mathcal{P}$ is a collection of prime numbers and V, W are two $\mathbb{F}$ representations of G, then V, W are said to be $J_{\mathcal{P}}$-equivalent if there are G-maps $f : S(V) \to S(W)$, $h : S(W) \to S(V)$ such that $\nu_p(deg\, f) = \nu_p(deg\, h) = 0$ for all $p \notin \mathcal{P}$. We define $T\mathbb{F}_{\mathcal{P}}(G) = \{V - W \in R\mathbb{F}(G) : V \oplus U$ is $J_{\mathcal{P}} -$

equivalent to $W \oplus U$ for some $\mathbb{F}\ G-$ module $U\}$, and

$$J\mathbb{F}_{\mathcal{P}}(G) = R\mathbb{F}_{\mathcal{P}}(G)/T\mathbb{F}_{\mathcal{P}}(G).$$

For the rest of this section, we assume that $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, unless otherwise indicated.

Let G be a finite group of order N. Then $\mathbb{F}\ G$-modules are realizable over the field $\mathbb{Q}(u)$ where $u = e^{2\pi i/N}$. Let Γ_G (or Γ_N) be the Galois group of $\mathbb{Q}(u)$ over $\mathbb{Q}$ then $\Gamma_N = \{\alpha^k : \mathbb{Q}(u) \to \mathbb{Q}(u) : (k, N) = 1,\ \alpha^k|_{\mathbb{Q}} = id$ and $\alpha^k(u) = e^{2\pi ik/N}\} \cong (\mathbb{Z}/N\mathbb{Z})^*$ (the group of units of $\mathbb{Z}/N\mathbb{Z}$), the isomorphism is given by $\alpha^k \leftrightarrow k$. Γ_G acts on $R\mathbb{F}(G)$ via its action on character values, in fact $\alpha^k(x) = \psi^k(x)$ where ψ^k are the Adams operations on $R\mathbb{F}(G)$ ([12], Proposition 3.1). Actually, Γ_G acts on $\mathrm{Irr}(G, \mathbb{F})$. Let $\mathbb{Z}[\Gamma_G]$ be the integral group ring of Γ_G and $I(\Gamma_G)$ its augmentation ideal. Let $W\mathbb{F}(G) = I(\Gamma_G)R\mathbb{F}(G)$ and $R\mathbb{F}(G)_{\Gamma_G} = R\mathbb{F}(G)/W\mathbb{F}(G)$.

Now, we are ready to discuss the groups $JO_G(*)$. Clearly, $JO_G(*) \cong RO(G)/RO_h(G)$ where $RO_h(G) = \{V - W \in RO(G) : S(V \oplus U)$ is $G-$ homotopy equivalent to $S(W \oplus U)$ for some real $G-$module $U\}$. So, given two (real) representations V, W of a compact Lie group G, when does there exist a G-map $f : S(V) \to S(W)$ which has a G-homotopy inverse? This question was raised by Adams in a more general form in 1963, when he asked: Given two (orthogonal) representations V, W of a compact Lie group G, When does there exist a G-map $f : S(V) \to S(W)$ of degree $k \in \mathbb{N}$? Adams' question has been answered in many stages as follows:

(1) (Atiyah-Tall [12], 1968).

Let G be a finite p-group ($p \neq 2$). Let $\mathcal{P}$ be the set of all prime numbers except p. Then $J\mathbb{F}_{\mathcal{P}}(G) = R\mathbb{F}(G)_{\Gamma_G}$ ($\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$), i.e., for any two $\mathbb{F}\ G$-modules V, W there is a (stable) G-map $f : S(V) \to S(W)$ of degree prime to p if and only if $V - W \in W\mathbb{F}(G)$.

(2) (Snaith [57], 1970).

Snaith extended Atiyah-Tall (1) to include the case $p = 2$ for complex representations.

(3) (Lee-Wasserman [42], 1975).

Lee and Wasserman discussed the Adams' question in great details. They considered the question for any compact Lie group G and showed that: If V, W

are two representaions of a compact Lie group G, then there exists a stable G-map : $S(V) \to S(W)$ of degree k if and only if there exists a stable G_p-map : $S(V) \to S(W)$ of degree k for every prime p dividing the order of G/G_0, where G_0 is the component of G containing 0 and G_p is a p-sylow subgroup of G, in some sense. So, if G is a finite group then the Adams' question reduces to the analogous question for p-groups. Among the important results of Lee and Wasserman are :

(i) They extended Atiyah-Tall (1) to include the case $p = 2$ for real representations (Theorem 3.18, p. 47).

(ii) For any compact Lie group G and any set of primes $\mathcal{P}$, $\tilde{r} : J\mathbb{F}_{\mathcal{P}}(G) \to JO_P(G)$ is a monomorphism where r is the realification homomorphism and $\mathbb{F} = \mathbb{C}$ or $\mathbb{H}$. So, we only need to consider the Adams' question for real representations (Proposition 1.2, p. 9).

(iii) (Main result, Theorem 3.20, p. 48).

Let V, W be two (real) representaions of a compact Lie group G, and $\mathcal{P}$ be a collection of primes. Then V is stably $J_{\mathcal{P}}$-equivalent to W if and only if the following conditions are satisfied:

(a) $\omega_1(V) = \omega_1(W)$, where $\omega_1(V) \in Hom(G, \mathbb{Z}_2)$ is defined by $\omega_1(V)(g) = det(g : V \to V)$.

(b) $\dim V^H = \dim W^H$ for all finite cyclic subgroups $H \leq G$ such that $H/(H \cap G_0)$ is a p-group, $p \notin \mathcal{P}$.

(iv) If G is a compact connected Lie group then for any collection $\mathcal{P}$ of primes $TO_{\mathcal{P}}(G) = \{0\}$ (Corollary 2.18, p. 20).

(v) If G is a finite group, then $RO_0(G) = WO(G)$ (Proposition 3.17, p. 43).

(vi) Let G be a compact Lie group and $\mathcal{P}$ be a collection of primes. If $2 \notin \mathcal{P}$ then $JO_{\mathcal{P}}(G)$ is a free abelian group, if $2 \in \mathcal{P}$ then $JO_{\mathcal{P}}(G)$ is a direct sum of a free abelian group and $Hom(G, \mathbb{Z}_2)$ (Corollary 3.14, p. 38).

(4) (tom Dieck and T.Petrie [22], [23], [25], and [26], 1978).

Let G be a finite group and $jo(G) = RO_0(G)/RO_h(G)$.

(i) $I(\Gamma_G)^2 RO(G) \subseteq RO_h(G)$. Further, if G is a finite p-group then $I(\Gamma_G)^2 RO(G) = RO_h(G)$. ([22] and [26] Theorem 1 and 5, respectively).

(ii) Let $V \in \operatorname{Irr}(G,\mathbb{R})$ and $o(V) = \Gamma_G V$ be the orbit of V under the Γ_G-action on $\operatorname{Irr}(G,\mathbb{R})$. Since Γ_G is abelian then the isotropy group $(\Gamma_G)_W$ of $W \in o(V)$ is independent of W, so we denote this isotropy group by $\Gamma_{o(V)}$. Now, let $\{\mathcal{V}_1, \ldots, \mathcal{V}_n\}$ be a set of representatives for the Γ_G-action on $\operatorname{Irr}(G,\mathbb{R})$. Then

$$jo(G) \cong \bigoplus_{i=1}^{n} \Gamma_G/\Gamma_{o(\mathcal{V}_i)}. \qquad ([22], p.\,185)$$

(iii) For an arbitrary finite group G, let $E(G)$ denote the set of hyperelementary subgroups of G. Then

$$res : jo(G) \longrightarrow \prod_{H \in E(G)} jo(H)$$

is injective ([22], Proposition 5.1), a subgroup H of a finite group G is called hyperelemetrary if there exists an exact sequence

$$0 \longrightarrow S \longrightarrow H \longrightarrow P \longrightarrow 0$$

with a finite p-group P (for some prime p) and a cyclic group S of order prime to p.

Remark 1. The analogous results of (i),(ii), and (iii) hold for complex G-modules.

(iv) Two real G-modules V and W are said to be locally J-equivalent, shortly $V \sim_{loc} W$, if for each subgroup H of G there exists a G-module U and G-maps $f : S(V \oplus U) \to S(W \oplus U)$, $g : S(W \oplus U) \to S(V \oplus U)$ such that f^H and g^H have degree one. Let $TO_G^{loc} = \{V - W \in RO(G) : V \sim_{loc} W\}$. Then $TO_G^{loc} = RO_0(G)$ for any finite group G ([25], Theorem 10.1.2).

(v) Let $\alpha = V - W \in RO(G)$ and let $\omega_G^{\alpha} = \{S^V, S^W\}$ be the stable G-homotopy group of pointed stable G-maps $S^V \to S^W$, where S^V denotes the one-point compactification of V. Then we have a bilinear pairing $\omega_G^{\alpha} \times \omega_G^{\beta} \longrightarrow \omega_G^{\alpha+\beta}$. In particular, ω_G^{α} is a module over ω_G^0, the stable equivariant homotopy ring of spheres in dimension zero. Let G be a compact Lie group, then from [26] we have:

(a) If $\alpha \in RO_0(G)$ then ω_G^{α} is a projective module of rank one over ω_G^0.

(b) $\alpha \in RO_h(G)$ if and only if ω_G^{α} is a free ω_G^0-module.

(c) $jo(G) \longrightarrow Pic(\omega_G^0)$, given by $\alpha + RO_h(G) \mapsto \omega_G^{\alpha}$, is an injective homomorphism, where $Pic(\omega_G^0)$ is the Picard group of ω_G^0.

Now, let us use the above results to study $JO_G(*)$.

(I) If G is a compact connected Lie group, then by (3)-(iv), $TO_{\mathcal{P}}(G) = 0$ for any set of primes $\mathcal{P}$, in particular $RO_h(G) = 0$. In fact, one can use (3)-(iii) and Proposition 2.25 of Lee-Wasserman to show that $RO_0(G) = 0$. Hence, we have:

Theorem 3.3.1 *Let G be a compact connected Lie group. Then*

$$JO_G(*) \cong RO(G).$$

(II) Let G be a finite group. Then
$RO(G)_{\Gamma_G} = RO(G)/RO_0(G) = JO_G^{loc}(*) = RO(G)/TO_G^{loc}$ ((3)-(v) and (4)-(iv)).

Now, $RO(G)_{\Gamma_G}$ is a free abelian group, in fact if $\{\mathcal{V}_1, \ldots, \mathcal{V}_s\}$ is a set of representatives for the Γ_G-action on $\mathrm{Irr}(G, \mathbb{R})$, then

$$WO(G) = < \mathcal{V}_i - \alpha(\mathcal{V}_i) : i = 1, \ldots, s \text{ and } \alpha \in \Gamma_G >$$

and hence $RO(G)_{\Gamma_G}$ is a free abelian group with basis

$$\{\mathcal{V}_i + WO(G) : i = 1, \ldots, s\}.$$

Theorem 3.3.2 *If G is a finite group, then $JO_G(*) \cong RO(G)_{\Gamma_G} \oplus jo(G)$.*

Proof. Since $RO(G)_{\Gamma_G}$ is a free abelian group then the following sequence splits:

$$0 \longrightarrow RO_0(G)/RO_h(G) \xrightarrow{\tilde{i}} RO(G)/RO_h(G) \xrightarrow{\tilde{q}} RO(G)/RO_0(G) \longrightarrow 0$$

The result follows.

From the above theorem, to find $JO_G(*)$ explicitly, we need to find the rank of $RO(G)_{\Gamma_G}$ and to compute $jo(G)$. So, we close this section by computing the rank of $RO(G)_{\Gamma_G}$ for any finite abelian group G and calculating the groups $jo(G)$ for any finite abelian p-group G. In the rest of this section, we assume that $G = \mathbb{Z}/n_1\mathbb{Z} \times \cdots \times \mathbb{Z}/n_r\mathbb{Z} = \{(j_1, \ldots, j_r) : j_i \in \{0, \ldots, n_i - 1\}\}$ is a finite abelian group.

$\mathrm{Irr}(G, \mathbb{C}) = \{V_{(j_1,\ldots,j_r)} : (j_1, \ldots, j_r) \in G$ where $V_{(j_1,\ldots,j_r)} = \mathbb{C}$ with G-action given by $(k_1, \ldots, k_r)z = e^{2\pi i j_1 k_1/n_1} \cdots e^{2\pi i j_r k_r/n_r} z$ for each $z \in V_{(j_1,\ldots,j_r)}$ and $(k_1, \ldots, k_r) \in G\}$.

$$\mathrm{Irr}(G, \mathbb{C})_{\mathbb{R}} = \{V_{(j_1,\ldots,j_r)} \in \mathrm{Irr}(G, \mathbb{C}) : (j_1, \ldots, j_r)^{-1} = (j_1, \ldots, j_r)\},$$

$$\mathrm{Irr}(G,\mathbb{C})_{\mathbb{C}} = \{V_{(j_1,\ldots,j_r)} \in \mathrm{Irr}(G,\mathbb{C}) : (j_1,\ldots,j_r) \neq (j_1,\ldots,j_r)\},$$

and $\mathrm{Irr}(G,\mathbb{C})_{\mathbb{H}} = \phi$. $R(G) \cong \mathbb{Z}[G]$ (the group ring of G over $\mathbb{Z}$), the isomorphism is given by $V_{(j_1,\ldots,j_r)} \leftrightarrow (j_1,\ldots,j_r)$. Also, for each $k \in \mathbb{N}$

$$\psi^k(V_{(j_1,\ldots,j_r)}) = V_{(j_1,\ldots,j_r)^k}.$$

For each $(j_1,\ldots,j_r) \in G$, let

$$[(j_1,\ldots,j_r)] = \{(j_1,\ldots,j_r),(j_1,\ldots,j_r)^{-1} = (n_1 - j_1,\ldots,n_r - j_r)\}.$$

If $(j_1,\ldots,j_r)^{-1} \neq (j_1,\ldots,j_r)$ let $\mathcal{V}_{[(j_1,\ldots,j_r)]} = rV_{(j_1,\ldots,j_r)}$ (note that $(j_1,\ldots,j_r)^{-1} \neq (j_1,\ldots,j_r)$ implies that $V_{(j_1,\ldots,j_r)} \in \mathrm{Irr}(G,\mathbb{C})_{\mathbb{C}}$, so $V_{(j_1,\ldots,j_r)^{-1}} = \overline{V}_{(j_1,\ldots,j_r)}$ and hence $rV_{(j_1,\ldots,j_r)} = rV_{(j_1,\ldots,j_r)^{-1}}$). If $(j_1,\ldots,j_r)^{-1} = (j_1,\ldots,j_r)$ and $(k_1,\ldots,k_r) \in G$ define

$$L^{(j_1,\ldots,j_r)}_{(k_1,\ldots,k_r)} = \sum_{j_i \neq 0} k_i$$

and let $\mathcal{V}_{[(j_1,\ldots,j_r)]} = \mathbb{R}$ with a G-action given by

$$(k_1,\ldots,k_r)x = (-1)^{L^{(j_1,\ldots,j_r)}_{(k_1,\ldots,k_r)}} x$$

for $x \in \mathbb{R}$, it is easy to see that $c\mathcal{V}_{[(j_1,\ldots,j_r)]} = V_{(j_1,\ldots,j_r)}$. Then

$$\mathrm{Irr}(G,\mathbb{R})_{\mathbb{R}} = \{\, V_{[(j_1,\ldots,j_r)]} : (j_1,\ldots,j_r)^{-1} = (j_1,\ldots,j_r)\},$$

$$\mathrm{Irr}(G,\mathbb{R})_{\mathbb{C}} = \{\mathcal{V}_{[(j_1,\ldots,j_r)]} : (j_1,\ldots,j_r)^{-1} \neq (j_1,\ldots,j_r)\},$$

and $\mathrm{Irr}(G,\mathbb{R})_{\mathbb{H}} = \phi$. Using the fact that ψ^k commutes with the realification and complexification homomorphisms for each $k \in \mathbb{N}$, we obtain

$$\psi^k(\mathcal{V}_{[(j_1,\ldots,j_r)]}) = \mathcal{V}_{[(j_1,\ldots,j_r)^k]}$$

if $(j_1,\ldots,j_r)^{-1} = (j_1,\ldots,j_r)$ or $(j_1,\ldots,j_r)^k \neq (j_1,\ldots,j_r)^{-k}$, but if $(j_1,\ldots,j_r)^{-1} \neq (j_1,\ldots,j_r)$ and $(j_1,\ldots,j_r)^k = (j_1,\ldots,j_r)^{-k}$ then

$$\psi^k(\mathcal{V}_{[(j_1,\ldots,j_r)]}) = 2\mathcal{V}_{[(j_1,\ldots,j_r)^k]}.$$

In particular, if $k \in \Gamma_G$ then $\psi^k(\mathcal{V}_{[(j_1,\ldots,j_r)]}) = \mathcal{V}_{[(j_1,\ldots,j_r)^k]}$ for all $(j_1,\ldots,j_r) \in G$.

Now, we are ready to compute rank $RO(G)_{\Gamma_G} = |\mathrm{Irr}(G,\mathbb{R})/\Gamma_G| =$ the number of representatives of the Γ_G-action on $\mathrm{Irr}(G,\mathbb{R})$.

Proposition 3.3.3 $|\mathit{Irr}(G,\mathbb{C})/\Gamma_G| = |\mathit{Irr}(G,\mathbb{R})/\Gamma_G|$.

Proof. Recall that $\Gamma_G = \{k \in \{1, \ldots, |G|-1\} : (k, |G|) = 1\}$. Also, for $k \in \Gamma_G$, ψ^k acts on $\mathrm{Irr}(G, \mathbb{R})_{\mathbb{F}}$, $\mathrm{Irr}(G, \mathbb{C})_{\mathbb{F}}$ for $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. If $|G| > 1$, then $|G| - 1 \in \Gamma_G$ and $\psi^{|G|-1}(V_{(j_1,\ldots,j_r)}) = V_{(j_1,\ldots,j_r)^{-1}}$. Now, it is easy to see that $\phi : \mathrm{Irr}(G, \mathbb{C})/\Gamma_G \longrightarrow \mathrm{Irr}(G, \mathbb{R})/\Gamma_G$ such that $V_{(j_1,\ldots,j_r)}/\Gamma_G \mapsto \mathcal{V}_{[(j_1,\ldots,j_r)]}/\Gamma_G$ is a one-to-one correspondence.

Γ_G acts on G by sending $(s, g) \mapsto g^s$ for $s \in \Gamma_G$ and $g \in G$.

Proposition 3.3.4 *$|Irr(G, \mathbb{C})/\Gamma_G| = (\sum_{g \in G} |\Gamma_g|)/|\Gamma_G|$ where Γ_g is the isotropy group of G at g.*

Proof. From the above discussion about $\mathrm{Irr}(G, \mathbb{C})$, it follows that

$$|\mathrm{Irr}(G, \mathbb{C})/\Gamma_G| = |G/\Gamma_G|.$$

Now, the result follows from basic facts of group theory.

Corollary 3.3.5 *rank $RO(G)_{\Gamma_G} = (\sum_{g \in G} |\Gamma_g|)/|\Gamma_G|$.*

Lemma 3.3.6 *Let G be a cyclic group of order $n = p_1^{r_1} \cdots p_s^{r_s}$. Then for $j, m \in \{1, \ldots, n-1\}$, $\psi^k(V_j) = V_m$ for some $k \in \Gamma_G$ if and only if for any $i = 1, \ldots, s$ either $\nu_{p_i}(j) = \nu_{p_i}(m) < r_i$ or $\nu_{p_i}(j)$, $\nu_{p_i}(m) \geq r_i$.*

Proof. Let $\psi^k(V_j) = V_m$ then $kj - m = tn$ for some $t \in \mathbb{N}$. Let $i \in \{1, \ldots, s\}$ if $\nu_{p_i}(j) \geq r_i$ then $\nu_{p_i}(m) \geq r_i$ because $(k, p_i) = 1$. If $\nu_{p_i}(j) < r_i$ then $\nu_{p_i}(m) = \nu_{p_i}(j)$ because if $\nu_{p_i}(m) \neq \nu_{p_i}(j)$ then $\nu_{p_i}(kj - m) = \min\{\nu_{p_i}(j), \nu_{p_i}(m)\} < r_i$ which is a contradiction. This shows the first implication. The other implication is similar.

Theorem 3.3.7 *Let G be a cyclic group of order $p_1^{r_1} \cdots p_s^{r_s}$. Then*

$$|Irr(G, \mathbb{C})/\Gamma_G| = (r_1 + 1) \cdots (r_s + 1).$$

Proof. Follows directly from the above lemma.

Corollary 3.3.8 *If G is a cyclic group of order $n = p_1^{r_1} \cdots p_s^{r_s}$. Then*

$$\textit{rank } RO(G)_{\Gamma_G} = (r_1 + 1) \cdots (r_s + 1).$$

Proposition 3.3.9 *Let $G = \mathbb{Z}/n_1\mathbb{Z} \times \cdots \times \mathbb{Z}/n_r\mathbb{Z}$. Then*

$$|Irr(G, \mathbb{C})/\Gamma_G| \geq \prod_{i=1}^{r} |Irr(\mathbb{Z}/n_i\mathbb{Z}, \mathbb{C})/\Gamma_{\mathbb{Z}/n_i\mathbb{Z}}|.$$

Proof. Let $(j_1,\ldots,j_r)$, $(k_1,\ldots,k_r)\in G$. and $k\in\Gamma_G$ such that $\psi^k(V_{(j_1,\ldots,j_r)})=V_{(k_1,\ldots,k_r)}$. Then $\psi^k(V_{j_i})=V_{k_i}$ for each $i=1,\ldots,r$. Since $(k \bmod n_i, n_i)=1$, then

$$\phi:\mathrm{Irr}(G,\mathbb{C})/\Gamma_G\to\prod_{i=1}^{r}\mathrm{Irr}(\mathbb{Z}/n_i\mathbb{Z},\mathbb{C})/\Gamma_{\mathbb{Z}/n_i\mathbb{Z}}$$

given by $\phi(V_{(j_1,\ldots,j_r)}/\Gamma_G)=(V_{j_1}/\Gamma_{\mathbb{Z}/n_1\mathbb{Z}},\ldots,V_{j_r}/\Gamma_{\mathbb{Z}/n_r\mathbb{Z}})$ is well-defined. Clearly ϕ is onto, the result follows.

Corollary 3.3.10 *Let $G=\mathbb{Z}/n_1\mathbb{Z}\times\cdots\times\mathbb{Z}/n_r\mathbb{Z}$. Then*

$$\textit{rank } RO(G)_{\Gamma_G}\geq\prod_{i=1}^{r}\textit{rank } RO(\mathbb{Z}/n_i\mathbb{Z})_{\Gamma_{\mathbb{Z}/n_i\mathbb{Z}}}.$$

Remark 2. In general, rank $RO(G)_{\Gamma_G}\neq\prod_{i=1}^{r}$ rank $RO(\mathbb{Z}/n_i\mathbb{Z})_{\Gamma_{\mathbb{Z}/n_i\mathbb{Z}}}$. For instance, using Corollary 3.3.5, rank $RO(\mathbb{Z}/3\mathbb{Z}\times\mathbb{Z}/6\mathbb{Z})_{\Gamma_{\mathbb{Z}/3\mathbb{Z}\times\mathbb{Z}/6\mathbb{Z}}}=10$ while, (rank $RO(\mathbb{Z}/3\mathbb{Z})_{\Gamma_{\mathbb{Z}/3\mathbb{Z}}}$)(rank $RO(\mathbb{Z}/6\mathbb{Z})_{\Gamma_{\mathbb{Z}/6\mathbb{Z}}}$) $=8$.

Now, we compute $jo(G)$ where G is a finite abelian p-group.
First, let p be an odd prime and $G=\mathbb{Z}/p^n\mathbb{Z}$ be a cyclic p-group of order p^n for some $n\in\mathbb{N}$. Then $\{\mathcal{V}_0, rV_1, rV_p, rV_{p^2}\ldots, rV_{p^{n-1}}\}$ is a set of representatives for the Γ_G-action on $\mathrm{Irr}(G,\mathbb{R})$. So,

$$jo(G)\cong\Gamma_G/\Gamma_{o(\mathcal{V}_0)}\oplus\Gamma_G/\Gamma_{o(rV_1)}\oplus\cdots\oplus\Gamma_G/\Gamma_{o(rV_{p^{n-1}})}.$$

Γ_G is cyclic of order $p^n(p-1)$. So $\Gamma_G/\Gamma_{o(\mathcal{V}_0)}$ is cyclic of order 1 and for $i=0,\ldots,n-1$ $\Gamma_G/\Gamma_{o(rV_{p^i})}$ is cyclic of order $|o(rV_{p^i})|=|o(V_{p^i})|/2$.

$$o(V_{p^i})=\{V_j:\nu_p(j)=i\}.$$

So, $|o(V_{p^i})|=$ the number of integers $1\leq r<p^n$ such that $\nu_p(r)=i$, namely $\mid o(V_{p^i})\mid=\mid(\mathbb{Z}_{p^{n-i}})^*\mid=p^{n-i-1}(p-1)$. Hence, we have:

Theorem 3.3.11 *If G is a cyclic p-group of order p^n, p is odd, then*

$$jo(G)\cong\bigoplus_{i=0}^{n-1}\mathbb{Z}/(p^{n-i-1}(p-1)/2)\mathbb{Z}.$$

Now, Let $G=\mathbb{Z}/\mathbb{Z}_{p^{n_1}}\times\cdots\times\mathbb{Z}/\mathbb{Z}_{p^{n_r}}$ be a finite abelian p-group. Let

$$\{\mathcal{V}_{(0,\ldots,0)}, rV_{(j_1,\ldots,j_r)}: rV_{(j_1,\ldots,j_r)}\in S\subseteq\mathrm{Irr}(G,\mathbb{R})\}$$

be a set of representatives for the Γ_G-action on $\mathrm{Irr}(G,\mathbb{R})$. Let

$$|o(V_{(j_1,\ldots,j_r)}|=2d_{(j_1,\ldots,j_r)}.$$

Γ_G is cyclic of order $p^{n_1+\cdots+n_r-1}(p-1)$. So $\Gamma_G/\Gamma_{o(rV_{(j_1,\ldots,j_r)})}$ is cyclic of order $d_{(j_1,\ldots,j_r)}$. Hence

$$jo(G) \cong \bigoplus_{rV_{(j_1,\ldots,j_r)} \in S} \mathbb{Z}/d_{(j_1,\ldots,j_r)}\mathbb{Z}.$$

Now, consider the case $p=2$. Let $G = \mathbb{Z}/2^n\mathbb{Z}$, then

$$\{\mathcal{V}_0, \mathcal{V}_{2^{n-1}}, rV_1, rV_2, rV_{2^2}, \ldots, rV_{2^{n-2}}\}$$

is a set of representatives for the Γ_G-action on $\mathrm{Irr}(G,\mathbb{R})$. $o(\mathcal{V}_0) = \{\mathcal{V}_0\}$, $o(\mathcal{V}_{2^{n-1}}) = \{\mathcal{V}_{2^{n-1}}\}$ and for $i = 1, \ldots, n-2$

$$|o(rV_{2^i})| = |o(V_{2^i})|/2 = 2^{n-i-2}.$$

If $n \geq 3$ then Γ_G is the direct sum of two cyclic groups, one consisting of ± 1 and the other is cyclic of order $2^{|G|-2}$ [32], Ch. 4. Since $\{\pm 1\} \subseteq \Gamma_{o(V_{2^i})}$, then $\Gamma_G/\Gamma_{o(V_{2^i})}$ is cyclic of order 2^{n-i-2}. Hence we have:

Theorem 3.3.12 *If G is a cyclic group of order 2^n. Then*

$$jo(G) \cong \bigoplus_{i=0}^{n-2} \mathbb{Z}/2^{n-i-2}\mathbb{Z}.$$

Let $G = \mathbb{Z}/2^{n_1}\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/2^{n_r}\mathbb{Z}$ be a finite abelian 2-group. Let

$$\{\mathcal{V}_{(0,\ldots,0)}, \mathcal{V}_{[(j_1,\ldots,j_r)]} : \mathcal{V}_{[(j_1,\ldots,j_r)]} \in S \subseteq \mathrm{Irr}(G,\mathbb{R})\}$$

be a set of representatives for the Γ_G-action on $\mathrm{Irr}(G,\mathbb{R})$. Since $\{\pm 1\} \subseteq \Gamma_{o(\mathcal{V}_{[(j_1,\ldots,j_r)]})}$ then $\Gamma_G/\Gamma_{o(\mathcal{V}_{[(j_1,\ldots,j_r)]})}$ is cyclic of order $|\Gamma_{o(\mathcal{V}_{[(j_1,\ldots,j_r)]})}| = a_{(j_1,\ldots,j_r)}$, say. Consequently,

$$jo(G) \cong \bigoplus_{\mathcal{V}_{[(j_1,\ldots,j_r)]} \in S} \mathbb{Z}/a_{(j_1,\ldots,j_r)}\mathbb{Z}.$$

3.4 The main theoretical result, an alternative formula of $TO_G^{(p)}(X)$

If $G = \{e\}$, then by Adams [1] and Quillen [54], it is shown that $TO(X) = WO(X) = VO(X)$, see Chapter 1 for more details:

$$WO(X) = \bigcap_f \widetilde{KSO}(X)_f \tag{3.2}$$

where the intersection runs over all functions $f : \mathbb{N} \to \mathbb{N}$ and $\widetilde{KSO}(X)_f = < k^{f(k)}(\psi^k - 1)(u) : u \in \widetilde{KSO}(X) \text{ and } k \in \mathbb{N} >$.

$$\begin{aligned} VO(X) \;\; = \{ \;\; & x \in \widetilde{KSO}(X) : \theta_k(x) = \frac{\psi^k(1+u)}{1+u} \text{ in } 1 + \widetilde{KSO}(X) \otimes \mathbb{Q}_k \\ & \text{for all } k \in \mathbb{N} \text{ and some } u \in \widetilde{KSO}(X)\} \end{aligned} \qquad (3.3)$$

It is easy to find $JO(S^n)$ and $JO(\mathbb{R}P^n)$ by using $WO(X)$ (see Adams [1]). On the other hand, it is easier to use $VO(X)$ to find the order of $\xi_n(\mathbb{C})$ in $JO(\mathbb{C}P^n)$ (see [43], Ch. 4). Therefore, to treat $JO(X)$, it is important to know both subgroups $VO(X)$ and $WO(X)$.

For a finite p-group G, an equivariant analogue of $WO(X)$ has been given by tom Dieck and Hauschild [25], Ch. 11. They defined an intermediate computable subgroup $TO_G^{loc}(X)$ of $KO_G(X)$ which holds the main part of $TO_G(X)$. The localization of $TO_G^{loc}(X)$ at p is given by

$$TO_G^{loc}(X)_{(p)} = (1 - \psi^{k_p})(KO_G(X)_{(p)})$$

where k_p is a generator of the group of units in $\mathbb{Z}/p^2\mathbb{Z}$. Later on, McClure [45] reduced the general finite group case to the p-group case.

We embarked working on this dissertation with the hope that we can find an analogue of $VO(X)$ in the equivariant case. To present our suggestive formula for an equivariant analogue of $VO(X)$, we first need to recall some results concerning G-fibre homotopy equivalence of sphere bundles.

(1) (Brouwer degree theorem, see Spanier [58], p. 398).

For $n \geq 1$, two maps $f, g : S^n \to S^n$ are homotopic if and only if $\deg f = \deg g$. Consequently, a map $f : S^n \to S^n$ is a homotopy equivalence if and only if $\deg f = \pm 1$.

(2) (Non-equivariant Dold's theorem mod-k, Adams [1], 1963).

Let ξ, η be two sphere-bundles over a finite CW complex X. If there is a fibrewise map $f : \xi \to \eta$ of degree $\pm k$ on each fibre, then there exists a non-negative integer n such that the Whitney multiples $k^n\xi, k^n\eta$ are fibre homotopy equivalent.

A converse of this theorem is given by Dibağ [20], 1982 and Tanaka [59], 1983.

(3) (Equivariant Dold's theorem mod-k, Hauschild-Waner [29], 1983).

Let ξ, η be two spherical G-fibrations over a finite G-CW complex X, and assume that corresponding fibres are stably equivalent with respect to the action of the appropriate (isotropy) subgroups, i.e., for each $x \in X$, ξ_x is stably G_x-fibre homotopy equivalent to η_x. Let $f : \xi \to \eta$ be a fiberwise G-map of degree N in the Burnside ring $A(G)$ of G, and assume that N divides a power of k in $A(G)$. Then there exists an integer $n \geq 0$ and a stable fiberwise G-equivalence $g : k^n\xi \to k^n\eta$.

A converse of this theorem is given by Önder [51], 1995.

The most important tool in computing $JO_G(X)$ is the Adams' conjecture which appeared in the first part of his famous papers on the groups $JO(X)$, [1].

(4) (Non-equivariant Adams' conjecture, Quillen [54], 1970).

Let E be a real vector bundle over a finite CW-complex X. If $k \in \mathbb{N}$, then there is a stable map $\phi : S(\psi^k(E)) \to S(E)$ of degree a power of k.

(5) (Equivariant Adams' conjecture, Hauschild-Waner [29], 1983).

Let G be a cyclic group of order prime to $k \in \mathbb{N}$, X be a finite G-connected G-CW complex, and $x \in KO_G(X)$. Then there is an integer $n \geq 0$ such that

$$JO_G(sk^n(\psi^k - 1)(x)) = 0$$

where JO_G is the homomorphism given in §3.2, and s is the smallest positive integer such that $k^s \equiv 1 \mod |G|$.

Versions of varying generality of this theorem are given by tom Dieck [25], McClure [45], and Fiedorowicz-Hauschild-May [28]. In general it is well-known now that equivariant Adams' conjecture holds for all finite groups.

The second important tool in computing $JO_G(X)$ is the Bott classes.

(6) (Non-equivariant case, Adams [1], 1963).

Let E, F be two $Spin(8n)$-bundles over X. If $S(E) \simeq S(F)$, then there exists an element $x \in \widetilde{KO}(X)$ such that

$$\theta_k(E) = \theta_k(F)\frac{\psi^k(1+x)}{1+x}$$

for all $k \geq 1$.

(7) (Equivariant case, Becker [14], 1971).

Let E, F be two $Spin(G)$-bundles over a G-space X. If $S(E) \simeq_G S(F)$, then there is a unit $x \in KO_G(X)$ such that

$$\theta_k(E) = \theta_k(F)\frac{\psi^k(x)}{x}$$

for all $k \geq 1$.

Now, we are ready to present tom Dieck-Hauschild and McClure description of the main part of $JO_G(X)_{(p)}$, namely $JO_G^{(p)}(X)$. Throughout the remainder of this chapter, G will be a finite group, X a finite G-connected G-CW complex, p a prime number, $k_2 = 3$, and k_p is an odd generator of $(\mathbb{Z}/p^2\mathbb{Z})^*$ for $p \neq 2$.

Definition 3.4.1 *A fibrewise G-map $f : \xi \to \eta$, between spherical G-fibrations over X, is called a p-equivalence if it has degree prime to p on all fixed sets of each fibre. We say that ξ and η are stably p-equivalent, written $\xi \overset{(p)}{\sim} \eta$, if there are p-equivalences $\xi \wedge S^V \to \eta \wedge S^V$ and $\eta \wedge S^V \to \xi \wedge S^V$ for some real G-module V.*

Let

$$TO_G^{(p)} = \{\frac{E-F}{m} \in KO_G(X)_{(p)} : S(E) \overset{(p)}{\sim} S(F)\}.$$

Then $TO_G^{(p)}(X)$ is a well-defined subgroup of $KO_G(X)_{(p)}$. Let

$$JO_G^{(p)}(X) = KO_G(X)_{(p)}/TO_G^{(p)}(X).$$

Definition 3.4.2 *Let E, F be two G-vector bundles over X. E and F are said to be stably locally homotopy equivalent, written $E \sim_{loc} F$, if for any $H \leq G$ there exists a G-module V and fibrewise G-maps $f : S(E \oplus \boldsymbol{V}) \to S(F \oplus \boldsymbol{V})$ and $g : S(F \oplus \boldsymbol{V}) \to S(E \oplus \boldsymbol{V})$ such that f^H and g^H are ordinary fibre homotopy equivalences.*

Let

$$TO_G^{loc}(X) = \{E - F \in KO_G(X) : E \sim_{loc} F\}.$$

Then $TO_G^{loc}(X)$ is a well-defined subgroup of $KO_G(X)$. Let

$$JO_G^{loc}(X) = KO_G(X)/TO_G^{loc}(X).$$

Theorem 3.4.3 *(Proposition 11.4.2 of [25])*
$TO_G(X)_{(p)} \subseteq TO_G^{loc}(X)_{(p)} \subseteq TO_G^{(p)}(X)$. Further, if G is a finite p-group then $TO_G^{loc}(X)_{(p)} = TO_G^{(p)}(X)$.

The relation between $JO_G^{(p)}(X)$ and $JO_G(X)_{(p)}$ is given by the following theorem which can be considered as another form of the equivariant Dold's theorem mod-k (3).

Theorem 3.4.4 *(Theorem 1.1 of [45])*
Let X be a finite G-connected G-CW complex and E, F be two G-vector bundles over X. Then the following conditions are equivalent:

(i) $E = F$ *in* $JO_G(X)_{(p)}$.

(ii) $E = F$ *in* $JO_G^{(p)}(X)$ *and* $E_x = F_x$ *in* $JO_{G_x}(\{x\})_{(p)}$ *for all* $x \in X$.
Further, $JO_G(X)_{(p)} \cong JO_G^{(p)}(X) \oplus jo(G)$.

Remark 1. If $G = \{e\}$, then from the above theorem

$$JO(X)_{(p)} = JO^{(p)}(X), \quad \text{namely } TO(X)_{(p)} = TO^{(p)}(X).$$

A finite group G is p-regular if it has no non-trivial p-group quotient. Let C_p be a set containing exactly one represetative for each conjugacy class of p-regular cyclic subgroups of G (note that a cyclic group is p-regular if and only if its order is prime to p). For $H \in C_p$, pick a p-sylow subgroup $P(H)$ of the group NH/H, where NH denotes the normalizer of H. If $E \to X$ is a G-vector bundle over X and $H \in C_p$, then $E^H \to X^H$ is a $P(H)$-vector bundle. If $S(E) \overset{(p)}{\sim} S(F)$ then clearly $S(E^H) \overset{(p)}{\sim} S(F^H)$.

The following theorem of McClure reduces the general finite group case to the p-group case.

Theorem 3.4.5 *(Theorem 2.2 of [45])*
Let E, F be G-vector bundles over X. If $E^H = F^H$ in $JO_{P(H)}^{(p)}(X^H)$ for all $H \in C_p$, then $E = F$ in $JO_G^{(p)}(X)$.

Corollary 3.4.6 *If $x \in KO_G(X)$ has order p^n in $JO_G^{(p)}(X)$, then*

$$n = \max\{n^H : H \in C_p\}$$

where p^{n^H} is the order of x^H in $JO_{P(H)}^{(p)}(X^H)$.

Theorem 3.4.7 *Let $|G| = p_1^{n_1} \cdots p_r^{n_r}$. If $z \in RO(G)$ has order m in $jo(G)$ and m_{p_i} is the order of z in $jo(G_{p_i})$ for some p_i-sylow subgroup G_{p_i} of G, then*

$$m = lcm\{m_{p_i} : i = 1, \ldots, r\}.$$

Proof. Follows from Lee-Wasserman §3.3 (3).

Corollary 3.4.8 *Let* $|G| = p_1^{n_1} \cdots p_r^{n_r}$. *If* $x \in KO_G(X)$ *has order* p^n *in* $JO_G(X)_{(p)}$. *Then*

$$n = \max\{n^H, \nu_p(m_{p_i}) : H \in C_p \text{ and } i = 1, \ldots, r\}$$

where n^H, m_{p_i} *are defined in Corollary 3.4.6 and Theorem 3.4.7.*

Proof. Follows directly from Corollary 3.4.6, Theorem 3.4.7, and Theorem 3.4.4.

From the above corollary the problem of computing $JO_G(X)_{(p)}$ where G is a finite group reduces to the problem of computing $TO_G^{(p)}(X)$ where G is a p-group. So, in the rest of this chapter, we assume that G is finite p-group.

Using the notion of λ-rings §2.4, Atiyah-Segal completion theorem (Theorem 2.4.5), and tom Dieck-Hauschild [25], Ch. 11, we obtain the following commutative diagram:

$$\begin{array}{ccccccc}
 & 0 & & & & & \\
 & \downarrow & & & & & \\
 & \widetilde{KSO}_G(X)_{(p),\Gamma} & & \xrightarrow{\quad\tilde{q}\quad} & & & \widetilde{KSO}_G(X)_{(p)}/TO_G^{(p)}(X) \\
 & \downarrow{\scriptstyle i_\Gamma} & & & & & \downarrow{\scriptstyle \tilde{\theta}_{k_p,\Gamma}} \\
0 \longrightarrow & \widetilde{KSO}_G(X)_{p,\Gamma} & \xrightarrow{\rho^{or}_{k_p,\Gamma}} & (1 + \widetilde{KSO}_G(X)_p)_\Gamma & \xrightarrow{\alpha_\Gamma} & & (1 + \widetilde{KSO}(X_G))_\Gamma
\end{array}$$

where the index Γ indicates that we factor out the image of $1 - \psi^k$,

$$i : \widetilde{KSO}_G(X)_{(p)} \to \widetilde{KSO}_G(X)_p$$

$$x/m \mapsto x \otimes (1/m)$$

$\rho^{or}_{k_p} : \widetilde{KSO}_G(X)_p \to 1 + \widetilde{KSO}_G(X)_p$ is the exponential map given in § 2.4, $\alpha : \widetilde{KSO}_G(X)_p \to \widetilde{KSO}(X_G)$ is the Atiyah-Segal isomorphism, and $\tilde{\theta}_{k_p,\Gamma}$ is defined so as to make the diagram commutative.

Theorem 3.4.9 *(see Theorem 11.4.1 of [25])*

$$TO_G^{(p)}(X) = (1 - \psi^{k_p})(\widetilde{KSO}_G(X)_{(p)}) \qquad \textit{(Formula I)}$$

Remark 2. The formula $TO_G^{(p)}(X) = (1-\psi^{k_p})(\widetilde{KSO}_G(X)_{(p)})$ given in the above theorem may be thought of as an analogue of $WO_G(X)_{(p)}$, without being able to define $WO_G(X)$ properly.

Next, we use the above diagram to obtain a formula of $TO_G^{(p)}(X)$ which may be considered as an analogue of $VO_G(X)_{(p)}$, again without being able to define $VO_G(X)$ properly.

Theorem 3.4.10 *(An alternative formula of $TO_G^{(p)}(X)$).*

$$TO_G^{(p)}(X) = \{\; x/m \in \widetilde{KSO}_G(X)_{(p)} : \theta^{or}_{k_p}(x) = \frac{1+u}{\psi^{k_p}(1+u)} \text{ in } 1+\widetilde{KSO}_G(X)_p \text{ for some } u \in \widetilde{KSO}_G(X)_p\} \qquad \text{(Formula II)}$$

Proof. Clearly, the right hand side of the above equality is a well-defined subgroup of $\widetilde{KSO}(X)_{(p)}$. i_Γ is injective implies that

$$i(\widetilde{KSO}_G(X)_{(p)}) \cap (\psi^{k_p}-1)(\widetilde{KSO}_G(X)_p) = i((\psi^{k_p}-1)(\widetilde{KSO}_G(X)_{(p)})). \quad (3.4)$$

From (2.7) $\rho^{or}_{k_p,\Gamma}$ is an isomorphism, so

$$\rho^{or}_{k_p}(\psi^{k_p}-1)(\widetilde{KSO}_G(X)_p) = (\psi^{k_p}-1)(1+\widetilde{KSO}_G(X)_p). \quad (3.5)$$

Now let $x/m \in TO_G^{(p)}(X)$, then by Formula I of $TO_G^{(p)}(X)$,

$$\frac{x}{m} \in (\psi^{k_p}-1)(\widetilde{KSO}_G(X)_{(p)}).$$

Hence from (3.4) and (3.5)

$$\theta^{or}_{k_p}(x/m) = \rho^{or}_{k_p}(i(x/m)) = \frac{1+u}{\psi^{k_p}(1+u)} \text{ in } 1+\widetilde{KSO}_G(X)_p$$

for some $u \in \widetilde{KSO}(X)_p$, this shows the first implication. On the other hand, if

$$\theta^{or}_{k_p}(x/m) = \frac{1+u}{\psi^{k_p}(1+u)} \text{ in } 1+\widetilde{KSO}_G(X)_p$$

for some $u \in \widetilde{KSO}(X)_p$ Then

$$\rho^{or}_{k_p}(i(x/m)) = \frac{1+u}{\psi^{k_p}(1+u)}.$$

Again by using (3.4) and (3.5), we get $x/m \in (1-\psi^{k_p})(\widetilde{KSO}_G(X)_{(p)})$. This completes the proof.

3.5 $JO_G(X)$; X is a trivial G-space

In this section X will be a trivial G-space, where G is a finite group. If $k \in \mathbb{N}$ with $(k, |G|) = 1$, then from Namboodiri [46], Lemma 6.1, ψ^k permutes the irreducibles in $\mathrm{Irr}(G, \mathbb{R})$, preserving their types. Let $\theta_{k,1}, \ldots, \theta_{k,l}$ denote the type $\mathbb{R}$ orbits, $\theta_{k,l+1}, \ldots, \theta_{k,l+m}$ denote the type $\mathbb{C}$ orbits, and $\theta_{k,l+m+1}, \ldots, \theta_{k,l+m+n}$ denote the type $\mathbb{H}$ orbits. For $i = 1, \ldots, l+m+n$, let $\theta_{k,i} = \{V_{i,j} : j = 1, \ldots, d_i\}$ for some $d_i \in \mathbb{N}$.

Now, from Example 4 § 2.3,

$$\begin{aligned} KO_G(X) \cong KO(X) \otimes RO(G)_{\theta_{k,1}} \oplus \cdots \oplus KO(X) \otimes RO(G)_{\theta_{k,l}} \oplus \\ KU(X) \otimes RO(G)_{\theta_{k,l+1}} \oplus \cdots \oplus KU(X) \otimes RO(G)_{\theta_{k,l+m}} \oplus \\ KSp(X) \otimes RO(G)_{\theta_{k,l+m+1}} \oplus \cdots \oplus KSp(X) \otimes RO(G)_{\theta_{k,l+m+n}} \end{aligned} \tag{3.6}$$

where $RO(G)_{\theta_{k,i}}$ is the subgroup of $RO(G)$ generated by elements of $\theta_{k,i}$. Let

$$x = \sum_{i=1}^{l+m+n} \sum_{j=1}^{d_i} x_{i,j} \otimes V_{i,j}$$

be an element of the right hand side of (3.6), where $x_{i,j} \in KO(X)$ for $i = 1, \ldots, l$, $x_{i,j} \in KU(X)$ for $i = l+1, \ldots, l+m$, and $x_{i,j} \in KSp(X)$ for $i = l+m+1, \ldots, l+m+n$, then the image of x in $KO_G(X)$ under the identification in (3.6) is

$$\begin{aligned} \sum_{i=1}^{l} \sum_{j=1}^{d_i} x_{i,j} \otimes_{\mathbb{R}} \mathbf{V}_{i,j} + \sum_{i=1}^{l+m} \sum_{j=1}^{d_i} r(x_{i,j} \otimes_{\mathbb{C}} \mathbf{V}_{i,j}) \\ + \sum_{i=1}^{l+m+n} \sum_{j=1}^{d_i} r(x^*_{i,j} \otimes_{\mathbb{H}} \mathbf{V}_{i,j}) \end{aligned} \tag{3.7}$$

where $x^*_{i,j}$ is the conjugate bundle to $x_{i,j}$.

For $i = 1, \ldots, l$, let $KO_{G,\theta_{k,i}}(X)$ denote the image of $KO(X) \otimes RO(G)_{\theta_{k,i}}$ in $KO_G(X)$ under the identification in (3.6). Similarly, define $KO_{G,\theta_{k,i}}(X)$ for $i = l+1, \ldots, l+m+n$. Then we have:

$$KO_G(X) = \bigoplus_{i=1}^{l+m+n} KO_{G,\theta_{k,i}}(X) \tag{3.8}$$

now, ψ^k preserves the summands $KO_{G,\theta_{k,i}}(X)$ for $i = 1, \ldots, l$. Also, from §2.4, $r \circ \psi^k = \psi^k \circ r$. So ψ^k preserves the summands $KO_{G,\theta_{k,i}}(X)$ for $i =$

$l+1,\dots,l+m$. Thus for $i=1,\dots,l+m$, we can define the quotient group $JO_{G,\theta_{k,i}}(X) = KO_{G,\theta_{k,i}}(X)/(1-\psi^k)$ where the denominator $1-\psi^k$ indicates that we factor out the image of $1-\psi^k$. The situation is different with the quaternionic summands, $KO_{G,\theta_{k,i}}(X)$, $i=l+m+1,\dots,l+m+n$, because Adams operations are not defined on $KSp(X)$. So, we can not say that ψ^k will preserve the summands $KO_{G,\theta_{k,i}}(X)$ for $i=l+m+1,\dots,l+m+n$, but, if ψ^k preserves a summand $KO_{G,\theta_{k,i}}(X)$ for some $i=l+m+1,\dots,l+m+n$, then we can define $JO_{G,\theta_{k,i}}(X) = KO_{G,\theta_{k,i}}(X)/(1-\psi^k)$.

Recall that $KO_G(X) \cong \widetilde{KO}_G(X) \oplus \mathbb{Z}$. Also, it is easy to see that

$$\widetilde{KO}_G(X) = \bigoplus_{i=1}^{l+m+n} \widetilde{KO}_{G,\theta_{k,i}}(X) \oplus \mathbf{IO}(G) \tag{3.9}$$

where $\widetilde{KO}_{G,\theta_{k,i}}(X)$ is the image of $\widetilde{KO}(X) \otimes RO(G)_{\theta_{k,i}}$ in $KO_G(X)$, for $i=1,\dots,l$. $\widetilde{KO}_{G,\theta_{k,i}}(X)$ are similarly defined for $i=l+1,\dots,l+m+n$, and $\mathbf{IO}(G) = \{\mathbf{V}-\mathbf{W} : V-W \in RO(G) \text{ and } \dim V = \dim W\}$,i.e., $\mathbf{IO}(G)$ is the image of $IO(G) = \{V-W \in RO(G) : \dim V = \dim W\}$ in $KO_G(X)$.

We know that ψ^k preserves all summands in (3.9) except possibly the quaternionic summands. So, we define $\widetilde{JO}_{G,\theta_{k,i}}(X) = \widetilde{KO}_{G,\theta_{k,i}}(X)/(1-\psi^k)$ if $i=1,\dots,l+m$, or if ψ^k preserves the summand $\widetilde{KO}_{G,\theta_{k,i}}(X)$ for some $i=l+m+1,\dots,l+m+n$. If G is a finite p-group, then from the above discussion, we directly obtain:

Theorem 3.5.1 *If ψ^{k_p} preserves the quaternionic summands in (3.8), then*

$$JO_G^{(p)}(X) \cong \bigoplus_{i=1}^{l+m+n} JO_{G,\theta_{k_p,i}}(X)_{(p)}.$$

In particular, if G has no type $\mathbb{H}$ irreducibles, then

$$JO_G^{(p)}(X) \cong \bigoplus_{i=1}^{l+m} JO_{G,\theta_{k_p,i}}(X)_{(p)}.$$

Theorem 3.5.2 *If ψ^{k_p} preserves the quaternionic summands in (3.8), then*

$$JO_G^{(p)}(X) \cong \bigoplus_{i=1}^{l+m+n} \widetilde{JO}_{G,\theta_{k_p,i}}(X)_{(p)} \oplus (RO(G)_{\Gamma_G})_{(p)}$$

In particular, if G has no type $\mathbb{H}$ irreducibles, then

$$JO_G^{(p)}(X) \cong \bigoplus_{i=1}^{l+m} \widetilde{JO}_{G,\theta_{k_p,i}}(X)_{(p)} \oplus (RO(G)_{\Gamma_G})_{(p)}$$

Corollary 3.5.3 *If ψ^{k_p} preserves the quaternionic summands in (3.8), then*

$$JO_G(X)_{(p)} \cong \bigoplus_{i=1}^{l+m+n} \widetilde{JO}_{G,\theta_{k_p,i}}(X)_{(p)} \oplus JO_G(*)_{(p)}$$

In particular, if G has no type $\mathbb{H}$ irreducibles, then

$$JO_G(X)_{(p)} \cong \bigoplus_{i=1}^{l+m} \widetilde{JO}_{G,\theta_{k_p,i}}(X)_{(p)} \oplus JO_G(*)_{(p)}.$$

For simplicity, from now on we suppose that G is a finite p-group with no type $\mathbb{H}$ irreducibles. Note that if $x \in KO_G(X)$ has finite order in $JO_G^{(p)}(X)$, then $x \in \widetilde{KO}_G(X)$. For $x \in \widetilde{KO}_G(X)$, the $J^{(p)}$-order of x is the order of $x + TO_G^{(p)}(X)$ in $JO_G^{(p)}(X)$ and the $J_{(p)}$-order of x is the order of $x + TO_G(X)_{(p)}$ in $JO_G(X)_{(p)}$.

Theorem 3.5.4 *Let $x = \sum_{i=1}^{l+m} x_i + \boldsymbol{z} \in \widetilde{KO}_G(X)$, where $x_i \in \widetilde{KO}_{G,\theta_{k,i}}(X)$ for $i = 1, \ldots, l+m$, and $z \in IO(G)$.*
(i) If x has finite order in $JO_G^{(p)}(X)$, then x_i has finite order in $\widetilde{JO}_{G,\theta_{k_p,i}}(X)_{(p)}$ for each $i = 1, \ldots, l+m$, and dim $z^H = 0$ for each $H \leq G$.
(ii) If n_i is the order of x_i in $\widetilde{JO}_{G,\theta_{k_p,i}}(X)_{(p)}$, and n_z is the order of z in $jo(G)_{(p)}$, then the $J^{(p)}$-order of x is lcm$\{n_i : i = 1, \ldots, l+m\}$ and the $J_{(p)}$-order of x is lcm$\{n_i, n_z : i = 1, \ldots, l+m\}$.

Proof. The proof is obvious.

From the above theorem, to find the $J_{(p)}$-order of $x = \sum_{i=1}^{l+m} x_i + \mathbf{z} \in \widetilde{KO}_G(X)$, we only need to consider the following three cases:

Case 1. $x \in \widetilde{KO}_{G,\theta_{k_p}}(X)$ where θ_{k_p} is an orbit of real type of the ψ^{k_p}-action on $\mathrm{Irr}(G, \mathbb{R})$. Let $\theta_{k_p} = \{V_1, \ldots, V_d\}$, then $x = \sum_{i=1}^{d} a_i \otimes_{\mathbb{R}} \mathbf{V}_i$ for some $a_i \in \widetilde{KO}(X)$. To find the $J_{(p)}$-order of x, we need to find the smallest p^v such that

$$p^v x = (1 - \psi^{k_p})(\sum_{i=1}^{d} a'_i \otimes_{\mathbb{R}} \mathbf{V}_i)$$

for some $a'_i \in \widetilde{KO}(X)_{(p)}$. (See Chapter 4 for explicit calculations).

Case 2. $x \in \widetilde{KO}_{G,\theta_{k_p}}(X)$ where θ_{k_p} is an orbit of complex type of the ψ^{k_p}-action on $\mathrm{Irr}(G, \mathbb{R})$. Let $\theta_{k_p} = \{V_1, \ldots, V_d\}$, then $x = \sum_{i=1}^{d} r(b_i \otimes_{\mathbb{C}} \mathbf{V}_i)$ for some

$b_i \in \widetilde{KU}(X)$. To find the $J_{(p)}$-order of x, we need to find the smallest p^v such that

$$p^v \sum_{i=1}^{d} b_i \otimes_{\mathbb{C}} \mathbf{V}_i = (1-\psi^{k_p})(\sum_{i=1}^{d} b'_i \otimes_{\mathbb{C}} \mathbf{V}_i)$$

for some $b'_i \in \widetilde{KU}(X)_{(p)}$. (See Chapter 4 for explicit calculations).

Case 3. x is the image of $z \in IO(G)$ in $\widetilde{KO}_G(X)$ (i.e., $x = \mathbf{z}$). To find the $J_{(p)}$-order of x, we need to find the order of $z \in jo(G)_{(p)}$. So, we close this chapter by computing the order of z in $jo(G)$ where G is any finite p-group.

Recall that by §3.4 that, $jo(G) = RO_0(G)/RO_h(G)$,

$$RO_0(G) = (1-\psi^{k_p})(RO(G)) \text{ and } RO_h(G) = (1-\psi^{k_p})^2(RO(G)).$$

Since ψ^{k_p} permutes the irreducibles in $\mathrm{Irr}(G,\mathbb{R})$, preserving their types, we only need to consider the case $z = \sum_{i=1}^{d} n_i V_i \in RO(G)_{\theta_{k_p}}$, where $\theta_{k_p} = \{V_1, \ldots, V_d\}$ is an orbit (real, complex, or quaterionic) of the ψ^{k_p}-action on $\mathrm{Irr}(G,\mathbb{R})$. Let τ be the inverse of the permutation determined by the action of ψ^{k_p} on $V_1, \ldots, V_d$.

Now, suppose z has finite order m in $jo(G)$. Then, m is the smallest positive integer such that

$$mz = (1-\psi^{k_p})^2(y) \tag{3.10}$$

for some $y = \sum_{i=1}^{d} m_i V_i \in RO(G)_{\theta_{k_p}}$. It is easy to see that

$$(1-\psi^{k_p})^2(y) = (m_1 - 2m_{\tau(1)} + m_{\tau^2(1)})V_1 + \cdots + (m_d - 2m_{\tau(d)} + m_{\tau^2(d)})V_d.$$

Comparing coefficients of variuos V_i in (3.10), we get the following system of equations:

$$\begin{aligned}
mn_1 &= m_1 - 2m_{\tau(1)} + m_{\tau^2(1)} \\
mn_2 &= m_2 - 2m_{\tau(2)} + m_{\tau^2(2)} \\
&\vdots \\
mn_d &= m_d - 2m_{\tau(d)} + m_{\tau^2(d)}
\end{aligned}$$

Adding all these equations, we get $m(n_1 + \cdots + n_d) = 0$. So, a necessary condition for z to have a finite order in $jo(G)$ is that $n_1 + \cdots + n_d = 0$, of course this condition is also sufficient.

The above system of equations has the following solutions

$$m_i = \frac{-m(n_i + 2n_{\tau(i)} + 3n_{\tau^2(i)} + \cdots + dn_{\tau^{d-1}(i)})}{d}$$

using the fact that $n_i + n_{\tau(i)} + n_{\tau^2(i)} + \cdots + n_{\tau^{d-1}(i)} = 0$, we get

$$m_i = \frac{-mN_i}{d}$$

where $N_i = n_{\tau(i)} + 2n_{\tau^2(i)} + \cdots + (d-1)n_{\tau^{d-1}(i)}$. So, for any prime q

$$\nu_q(m) = \max\{\nu_q(d) - \nu_q(N_i) : i = 1, \ldots d, \text{ and } N_i \neq 0\}.$$

So far, we have proved:

Theorem 3.5.5 *Let G be a finite p-group and $z = \sum_{i=1}^{d} n_i V_i \in RO(G)_{\theta_{k_p}}$.*
(i) z has finite order in $jo(G)$ if and only if $\sum_{i=1}^{d} n_i = 0$.
(ii) If z has order m in $jo(G)$ and q is any prime number, then

$$\nu_q(m) = \max\{\nu_q(d) - \nu_q(N_i) : i = 1, \ldots d, \textit{ and } N_i \neq 0\}$$

where $N_i = n_{\tau(i)} + 2n_{\tau^2(i)} + \cdots + (d-1)n_{\tau^{d-1}(i)}$.

So from now on, to find the $J_{(p)}$-orders of elements of $\widetilde{KO}_G(X)$, we only need to consider the first two cases. In other words, we only need to find the $J^{(p)}$-orders of elements of $\widetilde{KO}_G(X)$.

Chapter 4

EQUIVARIANT J-GROUPS OF PROJECTIVE SPACES

4.1 Introduction

Throughout this chapter G will be a finite group with no type $\mathbb{H}$ irreducibles, and $\mathbb{F}P^k$ will be considered as a trivial G-space. Recall that if M is an $\mathbb{F}$ G-module such that $\dim_{\mathbb{R}} M^G \geq \max\{3, (\dim_{\mathbb{R}} \mathbb{F})(2k-d_{\mathbb{F}})\}$ and $r(\xi_{k-1}(\mathbb{F})\otimes_{\mathbb{F}}\mathbf{M}-\mathbf{M})$ vanishes in $JO_G(\mathbb{F}P^{k-1},\mathcal{U}(M))$, then $V\mathbb{F}_k(M) \xrightarrow{p} S(M)$ has a G-section. In many cases, $r(\xi_{k-1}(\mathbb{F}) \otimes_{\mathbb{F}} \mathbf{M} - \mathbf{M})$ vanishes in $JO_G(\mathbb{F}P^{k-1},\mathcal{U}(M))_{(p)}$ if and only if it vanishes in $JO_G(\mathbb{F}P^{k-1})_{(p)}$. Therefore, it is of great importance to understand the groups $JO_G(\mathbb{F}P^{k-1})$. Önder [52], [50] has used Formula I of $TO_G^{(p)}(X)$ to compute the J-orders of elements of $\widetilde{KO}_G(\mathbb{C}P^m)$ and then to give a solution for the equivariant cross section problem of complex Stiefel manifolds. In this chapter, we use the ideas of the previous chapter to compute the equivariant J-groups of various projective spaces.

As we have seen in the last chapter, we only need to consider the case G is a finite p-group. So, in the remainder of this chapter, we assume that G is a finite p-group. In Section 2, we discuss the groups $JO_G(\mathbb{R}P^m)$ and $JO_G(S^m)$. In Section 3, we use Formula I of $TO_G^{(p)}(X)$ to give a formula for the J-orders of elements of $\widetilde{KO}_G(\mathbb{C}P^m)$. This formula may be considered as another form of the formula given in [52]. Then we show how the computations of the J-orders of elements of $\widetilde{KO}_G(\mathbb{C}P^m)$ can be used to obtain a partial solution for the equivariant cross section problem of complex Stiefel manifolds. To show the significance of our new formula of $TO_G^{(p)}(X)$, we use it to give an alternative proof of Theorem 1.1 [51] for the case $G = \mathbb{Z}/2\mathbb{Z}$. Then we discuss the advantages and disadvantages of both formulae of $TO_G^{(p)}(X)$.

Finally, in Section 4, we consider the quaternionic projective space $\mathbb{H}P^m$. We compute Adams operations on $\widetilde{KO}(\mathbb{H}P^m)$, $\widetilde{KU}(\mathbb{H}P^m)$, and then we compute the J-orders of elements of $\widetilde{KO}_G(\mathbb{H}P^m)$. Also, we obtain a partial solution for the equivariant cross section problem of quaternionic Stiefel manifolds.

4.2 $JO_G(\mathbb{R}P^m)$ and $JO_G(S^m)$

Let G be a finite p-group with no type $\mathbb{H}$ irreducibles. If X is a trivial G-space, then

$$KO_G(X) \cong \widetilde{KO}(X) \otimes RO(G)_{\mathbb{R}} \oplus \widetilde{KU}(X) \otimes RO(G)_{\mathbb{C}} \oplus RO(G).$$

So,

$$JO_G^{(p)}(X) \cong \widetilde{JO}_G^{(p)}(X) \oplus (RO(G)_{\Gamma_G})_{(p)}$$

where

$$\widetilde{JO}_G^{(p)}(X) = (\widetilde{KO}(X) \otimes RO(G)_{\mathbb{R}})_{(p)}/(1-\psi^{k_p}) \oplus (\widetilde{KU}(X) \otimes RO(G)_{\mathbb{C}})_{(p)}/(1-\psi^{k_p}).$$

Now, suppose $\psi^{k_p} = id$ on $\widetilde{KO}(X)$. Then

$$(1-\psi^{k_p})((\widetilde{KO}(X) \otimes RO(G)_{\mathbb{R}})_{(p)}) = (\widetilde{KO}(X) \otimes (1-\psi^{k_p})(RO(G)_{\mathbb{R}}))_{(p)}.$$

Since $(RO(G)_{\mathbb{R}}/(1-\psi^{k_p})(RO(G)_{\mathbb{R}})$ is a free abelian group, the following sequence splits:

$$0 \longrightarrow (1-\psi^{k_p})(RO(G)_{\mathbb{R}}) \xrightarrow{i} RO(G)_{\mathbb{R}} \xrightarrow{q} RO(G)_{\mathbb{R}}/(1-\psi^{k_p})(RO(G)_{\mathbb{R}}) \longrightarrow 0.$$

Hence the following sequence is exact

$$0 \longrightarrow \widetilde{KO}(X) \otimes (1-\psi^{k_p})(RO(G)_{\mathbb{R}}) \xrightarrow{id \otimes i} \widetilde{KO}(X) \otimes RO(G)_{\mathbb{R}} \xrightarrow{id \otimes q}$$
$$\widetilde{KO}(X) \otimes (RO(G)_{\mathbb{R}}/(1-\psi^{k_p})(RO(G)_{\mathbb{R}})) \longrightarrow 0.$$

Consequently,

$$(\widetilde{KO}(X) \otimes RO(G)_{\mathbb{R}})_{(p)}/(1-\psi^{k_p}) \cong (\widetilde{KO}(X) \otimes RO(G)_{\mathbb{R},\Gamma_G})_{(p)}.$$

Similarly, if $\psi^{k_p} = id$ on $\widetilde{KU}(X)$, then

$$(\widetilde{KU}(X) \otimes (RO(G)_{\mathbb{C}})_{(p)}/(1-\psi^{k_p}) \cong (\widetilde{KU}(X) \otimes RO(G)_{\mathbb{C},\Gamma_G})_{(p)}.$$

So, we have:

Theorem 4.2.1 *If X is a trivial G-space such that $\psi^{k_p} = id$ on $\widetilde{KO}(X)$ and $\widetilde{KU}(X)$, then*

$$\widetilde{JO}_G^{(p)}(X) \cong (\widetilde{KO}(X) \otimes (RO(G)_{\mathbb{R},\Gamma_G})_{(p)} \oplus (\widetilde{KU}(X) \otimes (RO(G)_{\mathbb{C},\Gamma_G})_{(p)}.$$

Corollary 4.2.2 *If we consider $\mathbb{R}P^m$ as a trivial G-space, then*

$$\widetilde{JO}_G^{(p)}(\mathbb{R}P^m) \cong \begin{cases} 0 & \text{if } p \neq 2 \\ \mathbb{Z}/2^{f(m)}\mathbb{Z} \otimes RO(G)_{\mathbb{R},\Gamma_G} \oplus \mathbb{Z}/2^{[m/2]}\mathbb{Z} \otimes RO(G)_{\mathbb{C},\Gamma_G} & \text{if } p = 2 \end{cases}$$

where $f(m)$ is the number of integers q with $q \equiv 0, 1, 2,$ or 4 mod 8 and $0 < q \leq m$.

Proof. Follows from the above theorem and the fact that, $RO(G)_{\mathbb{R},\Gamma_G}$, $RO(G)_{\mathbb{C},\Gamma_G}$ are free abelian groups, $\widetilde{KO}(\mathbb{R}P^m)$ is cyclic of order $2^{f(m)}$ with $\psi^{k_p} = id$, and $\widetilde{KU}(\mathbb{R}P^m)$ is cyclic of order $2^{[m/2]}$ with $\psi^{k_p} = id$. For the computations of $\widetilde{KO}(\mathbb{R}P^m)$ and $\widetilde{KU}(\mathbb{R}P^m)$, see Adams [2].

Now, let us study $JO_G^{(p)}(S^m)$ where S^m is a trivial G-space.

Lemma 4.2.3 *(i) If $k \in \mathbb{N}$, then $\psi^k : \widetilde{KO}(S^{4m}) \to \widetilde{KO}(S^{4m})$ is given by $\psi^k(x) = k^{2m}x$, and $\psi^k : \widetilde{KU}(S^{2m}) \to \widetilde{KU}(S^{2m})$ is given by $\psi^k(x) = k^m x$.*
(ii) If $m \equiv 1$ or 2 mod 8, then $\psi^k : \widetilde{KO}(S^m) \to \widetilde{KO}(S^m)$ is given by

$$\psi^k(x) = \begin{cases} 0 & \text{if } k \text{ is even} \\ x & \text{if } k \text{ is odd} \end{cases}$$

Proof. (i) is proved in Adams [2].

(ii) Let $f : \mathbb{R}P^m \to S^m$ be a map of degree 1. Then $f^* : \widetilde{KO}(S^m) \to \widetilde{KO}(\mathbb{R}P^m)$ is a monomorphism (see the proof of Theorem 7.4 of Adams [2]). Now, the result follows from the computations of ψ^k on $\widetilde{KO}(\mathbb{R}P^m)$ and the naturality of Adams operations.

Corollary 4.2.4 *(i) If $m \equiv 1$ mod 8, then*

$$\widetilde{JO}_G^{(p)}(S^m) \cong \begin{cases} 0 & \text{if } p \neq 2 \\ \mathbb{Z}/2\mathbb{Z} \otimes RO(G)_{\mathbb{R},\Gamma_G} & \text{if } p = 2 \end{cases}$$

(ii) If $m \equiv 2$ mod 8, then

$$\widetilde{JO}_G^{(p)}(S^m) \cong \begin{cases} (\widetilde{KU}(S^m) \otimes RO(G)_{\mathbb{C}})_{(p)}/(1 - \psi^{k_p}) & \text{if } p \neq 2 \\ \mathbb{Z}/2\mathbb{Z} \otimes RO(G)_{\mathbb{R},\Gamma_G} \oplus (\widetilde{KU}(S^m) \otimes RO(G)_{\mathbb{C}})_{(2)}/(1 - \psi^3) & \text{if } p = 2 \end{cases}$$

Proof. Follows from Theorem 4.2.1 and Lemma 4.2.3.

Finally, let us find the $J_{(p)}$-orders of elements of $\widetilde{KO}_G(S^{4m})$. Recall that $\widetilde{KU}(S^{4m}) \cong <\beta_{4m}>$ is infinite cyclic with $\psi^k(\beta_{4m}) = k^{2m}\beta_{4m}$. Using, the fact that

$$c : \widetilde{KO}(S^{4m}) \to \widetilde{KU}(S^{4m})$$

is an isomorphism, we directly get

$$\widetilde{KO}(S^{4m}) =< \gamma_{4m} >$$

is infinite cyclic with $\psi^k(\gamma_{4m}) = k^{2m}\gamma_{4m}$, where $\gamma_{4m} = c^{-1}(\beta_{4m})$.

As we have shown in Chapter 3, to find the $J_{(p)}$-order of $x \in \widetilde{KO}_G(S^{4m})$, we only need to consider the following two cases:

Case 1. Let $\theta_{k_p} = \{V_1, \ldots, V_d\}$ be an orbit of real type of ψ^{k_p}-action on $\mathrm{Irr}(G, \mathbb{R})$. If $x \in \widetilde{KO}_{G,\theta_{k_p}}(S^{4m})$, then $x = \sum_{i=1}^{d} a_i\gamma_{4m} \otimes_{\mathbb{R}} \mathbf{V}_i$ for some $a_i \in \mathbb{Z}$. To find the order of x in $\widetilde{JO}_G^{(p)}(S^{4m})$, we need to find the smallest v such that

$$p^v x = (1 - \psi^{k_p})(\sum_{i=1}^{d} a_i'\gamma_{4m} \otimes_{\mathbb{R}} \mathbf{V}_i) \tag{4.1}$$

for some $a_i' \in \mathbb{Z}_{(p)}$. Let τ be the inverse of the permutation determined by the ψ^{k_p}-action on $V_1, \ldots, V_d$. Comparing coefficients of various V_i in (4.1), we get

$$p^v a_n \gamma_{4m} = a_n'\gamma_{4m} - a_{\tau(n)}' k_p^{2m}\gamma_{4m}$$

in $\widetilde{KO}(S^{4m})_{(p)}$ for $n = 1, \ldots, d$. Hence

$$\begin{aligned} p^v a_1 &= a_1' - k_p^{2m} a_{\tau(1)}' \\ p^v a_2 &= a_2' - k_p^{2m} a_{\tau(2)}' \\ &\vdots \\ p^v a_d &= a_d' - k_p^{2m} a_{\tau(d)}' \end{aligned}$$

The above system has the following solutions:

$$a_n' = \frac{p^v S_n}{(1 - k_p^{2md})}$$

where

$$S_n = \sum_{i=0}^{d-1} a_{\tau^i(n)} k_p^{2mi}.$$

Using the fact that $a_n' \in \mathbb{Z}_{(p)}$ for $n = 1, \ldots, d$, we get

$$v \geq \max\{\nu_p(1 - k_p^{2md}) - \nu_p(S_n) : n = 1, \ldots, d, \text{ and } S_n \neq 0\}.$$

Hence, the order of x in $JO_G(S^{4m})_{(p)}$ is

$$p^{\max\{\nu_p(1-k_p^{2md}) - \nu_p(S_n) : n=1,\ldots,d, \text{ and } S_n \neq 0\}}.$$

Case 2. Let $\theta_{k_p} = \{V_1, \ldots, V_d\}$ be an orbit of complex type of ψ^{k_p}-action on $\mathrm{Irr}(G, \mathbb{R})$. If $x \in \widetilde{KO}_{G,\theta_{k_p}}(S^{4m})$, then $x = \sum_{i=1}^{d} r(b_i\beta_{4m} \otimes_{\mathbb{C}} \mathbf{V}_i)$ for some $b_i \in \mathbb{Z}$. To find the order of x in $\widetilde{JO}_G(S^{4m})_{(p)}$, we need to find the smallest v such that

$$p^v \sum_{i=1}^{d} b_i\beta_{4m} \otimes_{\mathbb{C}} \mathbf{V}_i = (1 - \psi^{k_p})(\sum_{i=1}^{d} b'_i\beta_{4m} \otimes_{\mathbb{C}} \mathbf{V}_i) \tag{4.2}$$

for some $b'_i \in \mathbb{Z}_{(p)}$. As in case 1, we obtain that the $J_{(p)}$-order of x is

$$p^{\max\left\{\nu_p(1-k_p^{2md})-\nu_p(S'_n):n=1,\ldots,d, \text{ and } S'_n\neq 0\right\}},$$

where

$$S'_n = \sum_{i=0}^{d-1} b_{\tau^i(n)} k_p^{2mi}.$$

4.3 $JO_G(\mathbb{C}P^m)$ and the equivariant cross section problem of complex Stiefel manifolds

Let G be a finite p-group with no type $\mathbb{H}$ irreducibles, and consider $\mathbb{C}P^m$ as a trivial G-space. Önder [52] has used Formula I of $TO_G^{(p)}(X)$ to compute the $J_{(p)}$-orders of elements of $\widetilde{KO}_G(\mathbb{C}P^m)$. In this section, we first use Formula I of $TO_G^{(p)}(X)$ to give another form of Önder's results, and then we show how to use the computations of the $J_{(p)}$-orders of elements of $\widetilde{KO}_G(\mathbb{C}P^m)$ to obtain a partial solution for the equivariant cross section problem of complex Stiefel mainfolds. Finally, we use Formula II of $TO_G^{(p)}(X)$ to give an alternative proof of Theorem 1.1 in [51] for the case $G = \mathbb{Z}/2\mathbb{Z}$.

For simplicity, we assume that $m = 2t$ (the case m is odd is similar). Recall that $\widetilde{KO}(\mathbb{C}P^m) = \mathbb{Z}[y](\text{mod } y^{t+1})$ where $y = r\xi_m(\mathbb{C}) - 2$, and $\widetilde{KU}(\mathbb{C}P^m) = \mathbb{Z}[v](\text{mod } v^{m+1})$ where $v = \xi_m(\mathbb{C}) - 1$. As we have seen in Chapter 3, to find the $J_{(p)}$-order of $x \in \widetilde{KO}_G(\mathbb{C}P^m)$, we only need to consider the following two cases:

Case1. $x = \sum_{n=1}^{d} a^n(y) \otimes_{\mathbb{R}} \mathbf{V}_n$, where $V_1, \ldots, V_d$ are irreducibles comprising an orbit of real type of the ψ^{k_p}-action on $\mathrm{Irr}(G, \mathbb{R})$, and

$$a^n(y) = \sum_{s=1}^{t} a_s^n y^s; \quad a_s^n \in \mathbb{Z}$$

for $n = 1, \ldots, d$. Now, let m_1 be the order of x in $JO(\mathbb{C}P^m)_{(p)}$, then $\nu_p(m_1)$ is the smallest v such that

$$p^v x = (1 - \psi^{k_p})(\sum_{n=1}^{d} b^n(y) \otimes_{\mathbb{R}} \mathbf{V}_n) \tag{4.3}$$

where $b^n(y) = \sum_{s=1}^{t} b_s^n y^s, b_s^n \in \mathbb{Z}_{(p)}$ for $n = 1, \ldots, d$.

Let τ be the inverse of the permutation determined by the ψ^{k_p}-action on $V_1, \ldots, V_d$. Comparing coefficients of variuos V_i in (4.3), we get

$$p^v a^n(y) = b^n(y) - \psi^{k_p}(b^{\tau(n)}(y)),\ n = 1, \ldots, d. \tag{4.4}$$

Let $k_p = 2q + 1$ then from [3] Theorem 2.2, and [43] Lemma 3.6,

$$\psi^{k_p}(y) = y(\sum_{j=0}^{q} \frac{k_p}{2j+1} \binom{q+j}{2j} y^j)^2 = y(\sum_{j=0}^{q} b_j y^j)^2, \tag{4.5}$$

where

$$b_j = \frac{k_p}{2j+1} \binom{q+j}{2j},\ j = 0, \ldots, q.$$

So, for $r = 2, \ldots, t$

$$\psi^{k_p}(y^r) = (\psi^{k_p}(y))^r = \sum_{\substack{j=0 \\ j \le t-r}}^{2rq} C_{j,r} y^{r+j}$$

where,

$$C_{j,r} = \sum_{\substack{i_1 + \cdots + i_{2r} = j \\ i_1, \ldots, i_{2r} \in \{0, \ldots, q\}}} b_{i_1} b_{i_2} \ldots b_{i_{2r}}. \tag{4.6}$$

The coefficient of y^r in $\psi^{k_p}(b^{\tau(n)}(y))$ is

$$\sum_{s=j}^{r-1} C_{r-s,s} b_s^{\tau(n)} + b_r^{\tau(n)} k_p^{2r}$$

where $r \in \{(j-1) + 2(j-1)q + 1, \min\{j + 2jq, t\}\}$. So, we have

$$p^v a_r^n = b_r^n - \sum_{s=j}^{r-1} C_{r-s,s} b_s^{\tau(n)} - b_r^{\tau(n)} k_p^{2r}$$

$r = 1, \ldots, t$, and $n = 1, \ldots, d$. If $r = 1$, then we have the following system of equations:

$$\begin{aligned} p^v a_1^1 &= b_1^1 - b_1^{\tau(1)} k_p^2 \\ p^v a_1^2 &= b_1^2 - b_1^{\tau(2)} k_p^2 \\ &\vdots \\ p^v a_1^d &= b_1^d - b_1^{\tau(d)} k_p^2. \end{aligned}$$

The above system has the following solutions

$$b_1^n = \frac{p^v M_{1,n}}{(1-k_p^{2d})}$$

where

$$M_{1,n} = \sum_{i=1}^{d} a_1^{\tau^{i-1}(n)} k_p^{2(i-1)}.$$

Similarly, for a fixed $1 \le r \le t$, we have the following system of equations:

$$p^v a_r^n = b_r^n - \sum_{s=j}^{r-1} C_{r-s,s} b_s^{\tau(n)} - b_r^{\tau(n)} k_p^{2r} \qquad \text{for } n = 1, \ldots, d.$$

Using induction, the above system has the following solutions:

$$b_r^n = \frac{p^v M_{r,n}}{(1-k_p^{2d}) \cdots (1-k_p^{2dr})}$$

where

$$M_{r,n} = A_{r,n}(1-k_p^{2d}) \cdots (1-k_p^{2d(r-1)}) +$$

$$\sum_{i=1}^{d} (\sum_{s=j}^{r-1} C_{r-s,s} M_{s,\tau^i(n)} (1-k_p^{2d(s+1)}) \cdots (1-k_p^{2d(r-1)}) k_p^{2r(i-1)});$$

$$A_{r,n} = \sum_{i=1}^{d} a_r^{\tau^{i-1}(n)} k_p^{2r(i-1)}.$$

Using the fact that $b_r^n \in \mathbb{Z}_{(p)}$ for $r = 1, \ldots, t$, $n = 1, \ldots, d$, we get

$$\begin{aligned} \nu_p(m_1) = \max\{&\nu_p((1-k_p^{2d}) \cdots (1-k_p^{2dr})) - \nu_p(M_{r,n}) : \\ &r = 1, \ldots, t, n = 1, \ldots, d \text{ and } M_{r,n} \neq 0\}. \end{aligned} \tag{4.7}$$

Case2. $x = r(\sum_{n=1}^{d} a^n(v) \otimes_{\mathbb{C}} \mathbf{V}_n)$, where $V_1, \ldots, V_d$ are irreducibles comprising an orbit of complex type of the ψ^{k_p}-action on $\mathrm{Irr}(G, \mathbb{R})$, and $a^n(v) = \sum_{s=1}^{m} a_s^n v^s$, $a_s^n \in \mathbb{Z}$ for $n = 1, \ldots, d$. Now, let m_2 be the order of x in $JO(\mathbb{C}P^m)_{(p)}$, then $\nu_p(m_2)$ is the smallest v such that

$$p^v \sum_{n=1}^{d} a^n(v) \otimes_{\mathbb{C}} \mathbf{V}_n = (1-\psi^{k_p})(\sum_{n=1}^{d} b^n(y) \otimes_{\mathbb{C}} \mathbf{V}_n) \tag{4.8}$$

where $b^n(v) = \sum_{s=1}^{m} b_s^n v^s$, $b_s^n \in \mathbb{Z}_{(p)}$ for $n = 1, \ldots, d$.

Let τ be the inverse of the permutation determined by the ψ^{k_p}-action on $V_1, \ldots, V_d$. Comparing coefficients of variuos V_i in (4.8), we get

$$p^v a^n(v) = b^n(v) - \psi^{k_p}(b^{\tau(n)}(v)),\ n = 1, \ldots, d. \tag{4.9}$$

The coefficient of v^r in $\psi^{k_p}(b^{\tau(n)}(v))$ is

$$\sum_{s=j}^{r-1} b_s^{\tau(n)} D_{r,s} + b_r^{\tau(n)} k_p^r$$

where $D_{r,s} = h_{i_1} \cdots h_{i_s}$, $i_1 + \cdots + i_s = r$, $i_1, \ldots, i_s \in \{1, \ldots, k_p\}$, $h_i = \binom{k_p}{i}$, and $r \in \{k_p j - k_p + 1, k_p j\}$. From (4.9), we have

$$p^v a_r^n = b_r^n - \sum_{s=j}^{r-1} D_{r,s} b_s^{\tau(n)} - b_r^{\tau(n)} k_p^r$$

$r = 1, \ldots, m$, and $n = 1, \ldots, d$. If $r = 1$, then we have the following system of equations:

$$\begin{aligned} p^v a_1^1 &= b_1^1 - b_1^{\tau(1)} k_p \\ p^v a_1^2 &= b_1^2 - b_1^{\tau(2)} k_p \\ &\vdots \\ p^v a_1^d &= b_1^d - b_1^{\tau(d)} k_p. \end{aligned}$$

The above system has the following solutions

$$b_1^n = \frac{p^v N_{1,n}}{(1 - k_p^d)}$$

where

$$N_{1,n} = \sum_{i=1}^{d} a_1^{\tau^{i-1}(n)} k_p^{(i-1)}.$$

Similarly, for a fixed $1 \leq r \leq m$, we have the following system of equations:

$$p^v a_r^n = b_r^n - \sum_{s=j}^{r-1} D_{r,s} b_s^{\tau(n)} - b_r^{\tau(n)} k_p^r \qquad \text{for } n = 1, \ldots, d.$$

Using induction, the above system has the following solutions:

$$b_r^n = \frac{p^v N_{r,n}}{(1 - k_p^d) \cdots (1 - k_p^{dr})}$$

where

$$N_{r,n} = B_{r,n}(1 - k_p^d) \cdots (1 - k_p^{d(r-1)}) +$$
$$\sum_{i=1}^{d} (\sum_{s=j}^{r-1} D_{r,s} N_{s,\tau^i(n)} (1 - k_p^{d(s+1)}) \cdots (1 - k_p^{dr}) k_p^{r(i-1)});$$

$$B_{r,n} = \sum_{i=1}^{d} a_r^{\tau^{i-1}(n)} k_p^{r(i-1)}.$$

Using the fact that $b_r^n \in \mathbb{Z}_{(p)}$ for $r = 1, \ldots, m$, $n = 1, \ldots, d$, we get

$$\nu_p(m_2) = \max\{\nu_p((1-k_p^d)\cdots(1-k_p^{dr})) - \nu_p(N_{r,n}) : \\ r = 1, \ldots, m, n = 1, \ldots, d \text{ and } N_{r,n} \neq 0\}. \tag{4.10}$$

Now, let us use the above computations for the $J_{(p)}$-orders of elements $\widetilde{KU}(\mathbb{C}P^m)$ to discuss the equivariant cross section problem of complex Stiefel manifolds.

Let M be a complex G-module. From Theorem 2.5.6, if $V\mathbb{C}_{m+1}(M) \xrightarrow{p} S(M)$ has a G-section then $S(\xi_m(\mathbb{C}) \otimes_{\mathbb{C}} \mathbf{M}) \simeq_G S(\mathbf{M})$. So, a necessary condition for $V\mathbb{C}_{m+1}(M) \xrightarrow{p} S(M)$ to have a G-section is that $r(\xi_m(\mathbb{C}) \otimes_{\mathbb{C}} \mathbf{M} - \mathbf{M})$ vanishes in $JO_G(\mathbb{C}P^m)$. To use the above computations, we first need to find the image of $r(\xi_m(\mathbb{C}) \otimes_{\mathbb{C}} \mathbf{M} - \mathbf{M})$ under the identification in (3.6). Unfortunately, it is not clear how to do that for general complex G-modules, because r is not a ring homomorphism and it is difficult to recognize the image of $r(\xi_m(\mathbb{C}) \otimes_{\mathbb{C}} \mathbf{M})$ under the identification in (3.6). To avoid this difficulty, we shall assume that $M = c\tilde{M}$ for some real G-module $\tilde{M}$. Then

$$r(\xi_m(\mathbb{C}) \otimes_{\mathbb{C}} \mathbf{M} - \mathbf{M}) = y \otimes_{\mathbb{R}} \tilde{M}.$$

For simplicity, let $\tilde{M} = a_1V_1 + \cdots + a_dV_d$, where $V_1, \ldots, V_d$ are irreducibles comprising an orbit of real type of the ψ^{k_p}-action on $\mathrm{Irr}(G, \mathbb{R})$, and $a_i \in \mathbb{Z}^+$ for each $i = 1, \ldots, d$. Then

$$y \otimes_{\mathbb{R}} \tilde{M} = \sum_{n=1}^{d} a^n(y) \otimes_{\mathbb{R}} \mathbf{V}_n$$

where $a^n(y) = \sum_{s=1}^{t} a_s^n y^s$;

$$a_s^n = \begin{cases} a_n & \text{if} \quad s = 1 \\ 0 & \text{if} \quad s \neq 1. \end{cases}$$

Now, if $y \otimes_{\mathbb{R}} \tilde{M}$ vanishes in $JO_G(\mathbb{C}P^m)$ then the $J_{(p)}$-order of $y \otimes_{\mathbb{R}} \tilde{M}$ is zero. So, from (4.7)

$$\nu_p((1-k_p^{2d})\cdots(1-k_p^{2dr})) \leq \nu_p(M_{r,n})$$

for $r = 1, \ldots, t, n = 1, \ldots, d$ and $M_{r,n} \neq 0$. With these symbols we have:

Theorem 4.3.1 *If $V\mathbb{C}_{m+1}(M) \xrightarrow{p} S(M)$ has a G-section then*

$$\nu_p(M_{r,n}) \geq \nu_p((1-k_p^{2d})\cdots(1-k_p^{2dr}))$$

for $r = 1, \ldots, t, n = 1, \ldots, d$ and $M_{r,n} \neq 0$. Further, similar implications hold for primes other than p, by using Corollary 3.4.6.

From Theorem 3.2.7, if $\dim_{\mathbb{R}} M^G \geq \max\{3, 4k-2\}$ then a sufficient condition on M so that $V\mathbb{C}_{m+1}(M) \xrightarrow{p} S(M)$ has a G-section is that $r(\xi_m(\mathbb{C}) \otimes_{\mathbb{C}} \mathbf{M} - \mathbf{M})$ vanishes in $JO_G(\mathbb{C}P^m, \mathcal{U}(M))$. So, we face the problem of computing $JO_G(\mathbb{C}P^m, \mathcal{U}(M))$. Note that if $\mathcal{U}(M) = G\mathbb{R}^\infty$, for example $G = \mathbb{Z}/p\mathbb{Z}$, then $JO_G(X, \mathcal{U}(M)) = JO_G(X)$. So, in this case the conditions given in the above theorem are also sufficient provided that $\dim_{\mathbb{R}} M^G \geq \max\{3, 4k-2\}$.

Remark 1. It is still an interesting problem to consider the equivariant cross section problem of complex Stiefel manifolds in the following cases:

(1) M is a complex G-module which is not the complexification of a real G-module.

(2) G is not a cyclic p-group.

In the following example, we use Formula II of $TO_G^{(p)}(X)$ to give an alternative proof of Theorem 1.1 in [51] for the case $G = \mathbb{Z}/2\mathbb{Z}$.

Example1. Let M be a complex $\mathbb{Z}/2\mathbb{Z}$-module, say $M = aV_0 + bV_1$, where $\mathrm{Irr}(\mathbb{Z}/2\mathbb{Z}, \mathbb{C}) = \{V_0, V_1\}$. We need to find sufficient conditions on M, so that $r(\xi_m(\mathbb{C}) \otimes \mathbf{M} - \mathbf{M})$ vanishes in $JO^{(2)}_{\mathbb{Z}/2\mathbb{Z}}(\mathbb{C}P^m)$, $\mathbb{C}P^m$ being a trivial $\mathbb{Z}/2\mathbb{Z}$-space. It is easy to see that $r(\xi_m(\mathbb{C}) \otimes \mathbf{M} - \mathbf{M}) = ay \otimes \mathcal{V}_0 + by \otimes \mathcal{V}_1$ where $\mathrm{Irr}(\mathbb{Z}/2\mathbb{Z}, \mathbb{R}) = \{\mathcal{V}_0, \mathcal{V}_1\}$, $y = r\xi_m(\mathbb{C}) - 2$. According to Formula II, $r(\xi_m(\mathbb{C}) \otimes \mathbf{M} - \mathbf{M})$ vanishes in $JO^{(2)}_{\mathbb{Z}/2\mathbb{Z}}(\mathbb{C}P^m)$ if and only if

$$\theta_3^{or}(ay \otimes \mathcal{V}_0 + by \otimes \mathcal{V}_1) = \frac{1+u}{\psi^3(1+u)} \quad \text{in } 1 + \widetilde{KSO}_{\mathbb{Z}/2\mathbb{Z}}(\mathbb{C}P^m)_2 \tag{4.11}$$

for some $u \in \widetilde{KSO}_{\mathbb{Z}/2\mathbb{Z}}(\mathbb{C}P^m)_2$. To solve (4.11), we first need to compute $\theta_3^{or}(ay \otimes \mathcal{V}_0 + by \otimes \mathcal{V}_1) = \theta_3^{or}(y \otimes \mathcal{V}_0)^a \theta_3^{or}(y \otimes \mathcal{V}_1)^b$.

$$\theta_3^{or}(y \otimes \mathcal{V}_0) = \theta_3^{or}(r\xi_m(\mathbb{C}) \otimes \mathcal{V}_0 - 2\mathcal{V}_0)$$

$$= \theta_3^{or}(r\xi_m(\mathbb{C}) \otimes \mathcal{V}_0)\theta_3^{or}(2\mathcal{V}_0)^{-1}.$$

Note that, $r\xi_m(\mathbb{C}) \otimes \mathcal{V}_0$ is an orientable 2-dimensional real $\mathbb{Z}/2\mathbb{Z}$-vector bundle over $\mathbb{C}P^m$ because $\lambda^2(r\xi_m(\mathbb{C}) \otimes \mathcal{V}_0) = P_2(r\xi_m(\mathbb{C}), 1, \mathcal{V}_0, 0) = 1$. Simlarly, $2\mathcal{V}_0$ is an orientable 2-dimensional real $\mathbb{Z}/2\mathbb{Z}$-vector bundle over $\mathbb{C}P^m$.

From (2.6), $\theta_3^{or}(r\xi_m(\mathbb{C})\otimes\mathcal{V}_0) = 1+\psi^1(r\xi_m(\mathbb{C})\otimes\mathcal{V}_0) = 1+r\xi_m(\mathbb{C})\otimes\mathcal{V}_0$, $\theta_3^{or}(2\mathcal{V}_0) = 1+2\mathcal{V}_0$. Hence,

$$\theta_3^{or}(y\otimes\mathcal{V}_0) = (1+r\xi_m(\mathbb{C})\otimes\mathcal{V}_0)(1/3\mathcal{V}_0) = 1+1/3y\otimes\mathcal{V}_0.$$

Similarly, we get

$$\theta_3^{or}(y\otimes\mathcal{V}_1) = (1+r\xi_m(\mathbb{C})\otimes\mathcal{V}_1)(1+2\mathcal{V}_1)^{-1}$$

$$= (1+r\xi_m(\mathbb{C})\otimes\mathcal{V}_1)(-1/3\mathcal{V}_0+2/3\mathcal{V}_1) = 1+1/3y\otimes(2\mathcal{V}_0-\mathcal{V}_1).$$

So,

$$\theta_3^{or}(ay\otimes\mathcal{V}_0+by\otimes\mathcal{V}_1) = (1+1/3y\otimes\mathcal{V}_0)^a(1+1/3y\otimes(2\mathcal{V}_0-\mathcal{V}_1))^b$$

$$= (1+1/3y)^a\otimes\mathcal{V}_0(1+1/3y)^b\otimes\mathcal{W},$$

for some $\mathcal{W}\in RO(G)$. If $\nu_2(a)\geq\nu_2(b_m(y))$, then

$$\theta_3^{or}(ay) = (1+1/3y)^a = \frac{1+b^0(y)}{\psi^3(1+b^0(y))}$$

for some $b^0(y)\in\widetilde{KSO}(\mathbb{C}P^m)_2$. Similarly, if $\nu_2(b)\geq\nu_2(b_m(y))$ then

$$\theta_3^{or}(by) = \frac{1+b^1(y)}{\psi^3(1+b^1(y))}$$

for some $b^1(y)\in\widetilde{KSO}(\mathbb{C}P^m)_2$. So, we get

$$\theta_3^{or}(ay\otimes\mathcal{V}_0+by\otimes\mathcal{V}_1) = \frac{1+u}{\psi^3(1+u)}$$

provided that $\nu_2(a)\geq\nu_2(b_m(y))$, and $\nu_2(b)\geq\nu_2(b_m(y))$. Thus a sufficient condition on M so that $r(\xi_m(\mathbb{C})\otimes\mathbf{M}-\mathbf{M})$ vanishes in $JO^{(2)}_{\mathbb{Z}/2\mathbb{Z}}(\mathbb{C}P^m)$ is that $\nu_2(a)\geq\nu_2(b_m(y))$, and $\nu_2(b)\geq\nu_2(b_m(y))$.

Remark 2. It was easy to use Formula I of $TO_G^{(p)}(X)$ to find the orders of elements of $JO_G^{(p)}(\mathbb{C}P^m)$ because of the following facts:

(1) ψ^{k_p} permutes the irreducibles in $\mathrm{Irr}(G,\mathbb{R})$, preserving their type.

(2) ψ^{k_p} is a ring homomorphism, so it is easy to compute $\psi^{k_p}(x)$ for each $x\in KO_G(X)$.

As, we have seen in the above example, to use Formula II of $TO_G^{(p)}(X)$ we face two difficulties:

(1) We need to compute $\theta^{or}_{k_p}(x)$, for this purpose it is possible, at least in principle, to use formula (2.5). However, the calculations are lenghty and tediuos.

(2) We need to calculate $(1+u)/\psi^{k_p}(1+u)$ which involves a question about the multiplicative structure of $RO(G)$.

However, in some favourable cases, one may hope to obtain direct information from the equation

$$\theta^{or}_{k_p}(x) = \frac{1+u}{\psi^k(1+u)}$$

to avoid the linear-algebra calculations in using Formula I (see [43] for such cases where $G = \{e\}$ and $X = \mathbb{C}P^m$ or $\mathbb{H}P^m$).

4.4 $JO_G(\mathbb{H}P^m)$ and the equivariant cross section problem of quaternionic Stiefel manifolds

Let G be a finite group with no type $\mathbb{H}$ irreducibles and consider $\mathbb{H}P^m$ as a trivial G-space. In order to compute the $J_{(p)}$-orders of elements of $\widetilde{KO}_G(\mathbb{H}P^m)$, we first need to compute Adams operations on $\widetilde{KO}(\mathbb{H}P^m)$ and $\widetilde{KU}(\mathbb{H}P^m)$.

From [56] and [43], if $g : \mathbb{C}P^{2m+1} \to \mathbb{H}P^m$ is the canonical projection and $h : \mathbb{C}P^{2m} \to \mathbb{H}P^m$ is the restriction of g to $\mathbb{C}P^{2m}$ then the induced map

$$h^* : KO(\mathbb{H}P^m) \to KO(\mathbb{C}P^{2m}) = \mathbb{Z}[y](\bmod y^{m+1})$$

is a monomorphism. Its image is the free abelian group with basis

$$\{1, e_i y^i : i = 1, \ldots, m\},$$

where

$$e_i = \begin{cases} 1 & \text{if } i \text{ is even} \\ 2 & \text{if } i \text{ is odd.} \end{cases}$$

Let $u_i = h^{*^{-1}}(e_i y^i)$, then

$$\widetilde{KO}(\mathbb{H}P^m) = < u_1, \ldots, u_m >$$

with $u_i u_j = e_{i,j} u_{i+j}$, where

$$e_{i,j} = \begin{cases} 4 & \text{if} \quad i \text{ and } j \text{ are odd} \\ 1 & \quad\quad \text{otherwise.} \end{cases}$$

With these symbols, we have:

Theorem 4.4.1 *If $k = 2q + 1$ is an odd integer, then*

$$\psi^k : \widetilde{KO}(\mathbb{H}P^m) \to \widetilde{KO}(\mathbb{H}P^m)$$

is given by

$$\psi^k(u_r) = \sum_{\substack{j=0 \\ j \leq m-r}}^{2rq} \frac{e_r}{e_{r+j}} C_{j,r} u_{r+j}$$

where $C_{j,r}$ is given by (4.6).

Proof. By naturality of Adams operations,

$$h^*(\psi^k(u_r)) = \psi^k(h^*(u_r)) = \psi^k(e_r y^r) = e_r \psi^k(y^r) = \sum_{\substack{j=0 \\ j \leq m-r}}^{2rq} \frac{e_r}{e_{r+j}} C_{j,r} e_{r+j} y^{r+j}.$$

With a little calculations, we get

$$h^*(\sum_{\substack{j=0 \\ j \leq m-r}}^{2rq} \frac{e_r}{e_{r+j}} C_{j,r} u_{r+j}) = \sum_{\substack{j=0 \\ j \leq m-r}}^{2rq} \frac{e_r}{e_{r+j}} C_{j,r} e_{r+j} y^{r+j}.$$

Now, the result follows by the injectivity of h^*.

From [56] if $w = c\xi_m(\mathbb{H}) - 2 \in \widetilde{KU}(\mathbb{H}P^m)$, then $\widetilde{KU}(\mathbb{H}P^m) = \mathbb{Z}[w](\text{mod } w^{m+1})$.

Theorem 4.4.2 *If $k = 2q + 1$ is an odd integer, then*

$$\psi^k : \widetilde{KU}(\mathbb{H}P^m) \to \widetilde{KU}(\mathbb{H}P^m)$$

is given by

$$\psi^k(w^r) = \sum_{\substack{j=0 \\ j \leq m-r}}^{2rq} C_{j,r} w^r$$

where $C_{j,r}$ is given by (4.6).

Proof. Let $c : \widetilde{KO}(\mathbb{H}P^m) \to \widetilde{KU}(\mathbb{H}P^m)$ be the complexification homomorphism. From [56] Proposition 1.6, c is a monomorphism with

$$c(\widetilde{KO}(\mathbb{H}P^m)) = < 1, e_i w : i = 1, \ldots, m > .$$

Also, from [56] Proposition 1.4, $g^*: KU(\mathbb{H}P^m) \to KU(\mathbb{C}P^{2m+1})$ is a monomorphism with $g^*(w) = v + \bar{v}$ where $v = \xi_{2m+1}(\mathbb{C}) - 1$.

$$g^*(c(u_i)) = g^*(c(h^{*^{-1}}(e_i y^i))) = e_i(v+\bar{v})^i = g^*(e_i w^i).$$

Hence, $c(u_i) = e_i w^i$. Now, we show that

$$g^*(\psi^k(w^r)) = g^*(\sum_{\substack{j=0 \\ j \le m-r}}^{2rq} C_{j,r} w^r).$$

$$g^*(\psi^k(w^r)) = g^*(\frac{1}{e_r}\psi^k(c(u_r))) = g^*(\frac{1}{e_r}c(\psi^k(u_r)))$$
$$= g^*(\frac{1}{e_r}c(\sum_{\substack{j=0 \\ j \le m-r}}^{2rq} \frac{e_r}{e_{r+j}} C_{j,r} u_{r+j})) = g^*(\sum_{\substack{j=0 \\ j \le m-r}}^{2rq} C_{j,r} w^r).$$

g^* is injective, so the result follows.

Now, we are ready to compute the $J_{(p)}$-orders of elements of $\widetilde{KO}(\mathbb{H}P^m)$. As usuall, we only need to consider the following two cases:

Case1. $x = \sum_{n=1}^{d} a^n(u_1, \ldots, u_m) \otimes_{\mathbb{R}} \mathbf{V}_n$, where

$$a^n(u_1, \ldots, u_m) = a_1^n u_1 + \cdots + a_m^n u_m \in \widetilde{KO}(\mathbb{H}P^m),$$

and $V_1, \ldots, V_d$ are irreducibles comprising an orbit of real type of the ψ^{k_p}-action on $\mathrm{Irr}(G, \mathbb{R})$. Now, let n_1 be the order of x in $JO(\mathbb{H}P^m)_{(p)}$, then $\nu_p(n_1)$ is the smallest v such that

$$p^v x = (1 - \psi^{k_p})(\sum_{n=1}^{d} b^n(u_1, \ldots, u_m) \otimes_{\mathbb{R}} \mathbf{V}_n) \tag{4.12}$$

where $b^n(u_1, \ldots, u_m) = \sum_{s=1}^{m} b_s^n u_s$; $b_s^n \in \mathbb{Z}_{(p)}$ for $n = 1, \ldots, d$.

Let τ be the inverse of the permutation determined by the ψ^{k_p}-action on $V_1, \ldots, V_d$. Comparing coefficients of variuos V_i in (4.12), we get

$$p^v a^n(u_1, \ldots, u_m) = b^n(u_1, \ldots, u_m) - \psi^{k_p}(b^{\tau(n)}(u_1, \ldots, u_m)),\ n = 1, \ldots, d.$$

Let $k_p = 2q + 1$, then from Theorem 4.4.1, the coefficient of u_r in

$$\psi^{k_p}(b^{\tau(n)}(u_1, \ldots, u_m))$$

is

$$\sum_{s=j}^{r-1} \frac{e_s}{e_r} C_{r-s,s} b_s^{\tau(n)} + b_r^{\tau(n)} k_p^{2r}$$

where $r \in \{(j-1)+2(j-1)q+1, min\{j+2jq, t\}\}$. So, we have

$$p^v a_r^n = b_r^n - \sum_{s=j}^{r-1} \frac{e_s}{e_r} C_{r-s,s} b_s^{\tau(n)} - b_r^{\tau(n)} k_p^{2r}$$

$r = 1, \ldots, t$, and $n = 1, \ldots, d$. If $r = 1$, then we have the following system of equations:

$$\begin{array}{rcl} p^v a_1^1 & = & b_1^1 - b_1^{\tau(1)} k_p^2 \\ p^v a_1^2 & = & b_1^2 - b_1^{\tau(2)} k_p^2 \\ & \vdots & \\ p^v a_1^d & = & b_1^d - b_1^{\tau(d)} k_p^2. \end{array}$$

The above system has the following solutions

$$b_1^n = \frac{p^v H_{1,n}}{(1-k_p^{2d})}$$

where

$$H_{1,n} = \sum_{i=1}^{d} a_1^{\tau^{i-1}(n)} k_p^{2(i-1)}.$$

Similarly, for a fixed $1 \le r \le m$, we have the following system of equations:

$$p^v a_r^n = b_r^n - \sum_{s=j}^{r-1} \frac{e_s}{e_r} C_{r-s,s} b_s^{\tau(n)} - b_r^{\tau(n)} k_p^{2r} \qquad \text{for } n = 1, \ldots, d.$$

Using induction, the above system has the following solutions:

$$b_r^n = \frac{p^v H_{r,n}}{(1-k_p^{2d}) \cdots (1-k_p^{2dr})}$$

where

$$H_{r,n} = A'_{r,n}(1-k_p^{2d}) \cdots (1-k_p^{2d(r-1)}) +$$

$$\sum_{i=1}^{d} (\sum_{s=j}^{r-1} \frac{e_s}{e_r} C_{r-s,s} H_{s,\tau^i(n)} (1-k_p^{2d(s+1)}) \cdots (1-k_p^{2d(r-1)}) k_p^{2r(i-1)});$$

$$A'_{r,n} = \sum_{i=1}^{d} a_r^{\tau^{i-1}(n)} k_p^{2r(i-1)}.$$

Using the fact that $b_r^n \in \mathbb{Z}_{(p)}$ for $r = 1, \ldots, t$, $n = 1, \ldots, d$, we get

$$\begin{aligned} \nu_p(n_1) = \max\{\nu_p((1-k_p^{2d}) \cdots (1-k_p^{2dr})) - \nu_p(H_{r,n}) : \\ r = 1, \ldots, m, n = 1, \ldots, d \text{ and } H_{r,n} \neq 0\} \end{aligned} \qquad (4.13)$$

Case2. $x = r(\sum_{n=1}^{d} a^n(w) \otimes_{\mathbb{C}} \mathbf{V}_n)$, where $a^n(w) = \sum_{s=1}^{m} a_s^n w^s \in \widetilde{KU}(\mathbb{H}P^m)$ for $n = 1, \ldots, d$, and $V_1, \ldots, V_d$ are irreducibles comprising an orbit of complex type of the ψ^{k_p}-action on $\mathrm{Irr}(G, \mathbb{R})$. Now, let n_2 be the order of x in $JO(\mathbb{H}P^m)_{(p)}$, then $\nu_p(n_2)$ is the smallest v such that

$$p^v \sum_{n-1}^{d} a^n(w) \otimes_{\mathbb{C}} \mathbf{V}_n = (1 - \psi^{k_p})(\sum_{n=1}^{d} b^n(w) \otimes_{\mathbb{C}} \mathbf{V}_n) \tag{4.14}$$

where $b^n(w) = \sum_{s=1}^{m} b_s^n w^s; b_s^n \in \mathbb{Z}_{(p)}$ for $n = 1, \ldots, d$.

Let τ be the inverse of the permutation determined by the ψ^{k_p}-action on $V_1, \ldots, V_d$. Comparing coefficients of variuos V_i in (4.14), we get

$$p^v a^n(w) = b^n(w) - \psi^{k_p}(b^{\tau(n)}(w)),\ n = 1, \ldots, d. \tag{4.15}$$

From Theorem 4.4.2, the coefficient of w^r in $\psi^{k_p}(b^{\tau(n)}(w))$ is

$$\sum_{s=j}^{r-1} b_s^{\tau(n)} C_{r,s} + b_r^{\tau(n)} k_p^r.$$

Exactly as in Case 1 § 4.3, we get

$$\nu_p(n_2) = \max\{\nu_p((1 - k_p^{2d}) \cdots (1 - k_p^{2dr})) - \nu_p(F_{r,n}) : \tag{4.16}$$

$$r = 1, \ldots, m, n = 1, \ldots, d \text{ and } F_{r,n} \neq 0\} \tag{4.17}$$

Where

$$F_{1,n} = \sum_{i=1}^{d} a_1^{\tau^{i-1}(n)} k_p^{2(i-1)}$$

and for $2 \leq r \leq m,\ 1 \leq n \leq d$

$$F_{r,n} = B'_{r,n}(1 - k_p^{2d}) \cdots (1 - k_p^{2d(r-1)}) +$$

$$\sum_{i=1}^{d} (\sum_{s=j}^{r-1} C_{r-s,s} F_{s,\tau^i(n)} (1 - k_p^{2d(s+1)}) \cdots (1 - k_p^{2d(r-1)}) k_p^{2r(i-1)});$$

$$B'_{r,n} = \sum_{i=1}^{d} a_r^{\tau^{i-1}(n)} k_p^{2r(i-1)}.$$

Finally, we consider the equivariant cross section problem of quaternionic Stiefel manifolds. Let M be a quaternionic G-module. For the same reasons given in §4.3, to treat complex G-modules, let $M = q\tilde{M}$ for some real G-module $\tilde{M}$. Then

$$r(\xi_m(\mathbb{H}) \otimes_{\mathbb{H}} \mathbf{M} - \mathbf{M}) = 2u_1 \otimes_{\mathbb{R}} \tilde{M}$$

(see [43] Proposition 3.10).

For simplicity, let $\tilde{M} = a_1V_1 + \cdots + a_dV_d$, where $V_1, \ldots, V_d$ are irreducibles comprising an orbit of real type of the ψ^{k_p}-action on $\mathrm{Irr}(G, \mathbb{R})$, and $a_i \in \mathbb{Z}^+$ for each $i = 1, \ldots, d$. Then

$$2u_1 \otimes_{\mathbb{R}} \tilde{M} = \sum_{n=1}^{d} a^n(u_1, \ldots, u_m) \otimes_{\mathbb{R}} \mathbf{V}_n$$

where $a^n(u_1, \ldots, u_m) = a_1^n u_1 + \cdots + a_m^n u_m$;

$$a_s^n = \begin{cases} 2a_n & \text{if} \quad s = 1 \\ 0 & \text{if} \quad s \neq 1. \end{cases}$$

Now, if $V\mathbb{H}_{m+1}(M) \xrightarrow{p} S(M)$ has a G-section, then $2u_1 \otimes_{\mathbb{R}} \tilde{M}$ vanishes in $JO_G(\mathbb{H}P^m)$. Hence from (4.13)

$$\nu_p((1 - k_p^{2d}) \cdots (1 - k_p^{2dr})) \leq \nu_p(H_{r,n})$$

for $r = 1, \ldots, m, n = 1, \ldots, d$ and $H_{r,n} \neq 0$. So, we have:

Theorem 4.4.3 *If $V\mathbb{H}_{m+1}(M) \xrightarrow{p} S(M)$ has a G-section, then*

$$\nu_p(H_{r,n}) \geq \nu_p((1 - k_p^{2d}) \cdots (1 - k_p^{2dr}))$$

for $r = 1, \ldots, m, n = 1, \ldots, d$ and $H_{r,n} \neq 0$. Further, similar implications hold for primes other than p, by using Corollary 3.4.6.

From Theorem 3.2.7, if $\dim_{\mathbb{R}} M^G \geq max\{3, 8k-4\}$ then a sufficient condition on M so that $V\mathbb{H}_{m+1}(M) \xrightarrow{p} S(M)$ has a G-section is that $r(\xi_m(\mathbb{H}) \otimes_{\mathbb{H}} \mathbf{M} - \mathbf{M})$ vanishes in $JO_G(\mathbb{H}P^m, \mathcal{U}(M))$. If $\mathcal{U}(M) = G\mathbb{R}^\infty$, then $JO_G(\mathbb{H}P^m, \mathcal{U}(M)) = JO_G(\mathbb{H}P^m)$. So, in this case, the conditions given in the above theorem are also sufficient provided that $\dim_{\mathbb{R}} M^G \geq \max\{3, 8k - 4\}$.

Chapter 5

NON-EQUIVARIANT J-GROUPS

5.1 Introduction

Let $G = \{e\}$ and let X be a connected finite dimensional CW-complex. Denote $JO(X)$ instead of $JO_G(X)$. Then $JO(X) \cong \mathbb{Z} \oplus \widetilde{JO}(X)$ where $\widetilde{JO}(X) = \widetilde{KO}(X)/TO(X)$. By Adams [1], $\widetilde{JO}(X)$ is isomorphic to the group of all stable fibre homotopy classes of real vector bundles over X as defined in Atiyah [7]. Also, from Atiyah [7] $\widetilde{JO}(X)$ is a finite group. So, if p is a prime number then $\widetilde{JO}(X)_{(p)}$ is isomorphic to the p-summand of $\widetilde{JO}(X)$, in the prime factorization of $\widetilde{JO}(X)$.

Localization is an exact functor on the category of finitely generated abelian groups. So, $\widetilde{JO}(X)_{(p)} \cong \widetilde{KO}(X)_{(p)}/TO(X)_{(p)}$. Our purpose in this chapter is to use Atiyah-Tall [12] to obtain two computable formulae of $TO(X)_{(p)}$. Those two localized formulae of $TO(X)_{(p)}$ can be directy obtained from the two formulae of $TO_G^{(p)}(X)$ given in Chapter 3, by taking X as a trivial $\mathbb{Z}/p\mathbb{Z}$-space and using the fact that Adams operation ψ^{k_p} permutes the irreducibles in $Irr(\mathbb{Z}/p\mathbb{Z}, \mathbb{R})$, preserving their types. However, our derivation of those two formulae of $TO(X)_{(p)}$ avoids the use of Atiyah-Segal completion theorem, which was used to derive the two formulae of $TO_G^{(p)}(X)$. To show the significance of those two localized formulae of $TO(X)$, we first use them to find the $J-$orders of elements of $\widetilde{KO}(\mathbb{C}P^m)$ and then we show how to use them to find the group $\widetilde{JO}(\mathbb{C}P^m)$ itself.

In Section 2, using some facts from §2.4, we obtain the following commutative diagram:

$$
\begin{array}{ccccccc}
 & & 0 & & & & \\
 & & \downarrow & & & & \\
 & & \widetilde{KSO}(X)_{(p),\Gamma} & \xrightarrow{\tilde{q}} & \widetilde{KSO}(X)_{(p)}/TO(X)_{(p)} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle i_\Gamma} & & \downarrow{\scriptstyle \tilde{\theta}^{or}_{k_p}} & & \\
0 & \longrightarrow & \widetilde{KSO}(X)_{p,\Gamma} & \xrightarrow{\rho^{or}_{k_p,\Gamma}} & 1+\widetilde{KSO}(X)_{p,\Gamma} & \longrightarrow & 0.
\end{array}
$$

From the above diagram we obtain the following two formulae of $TO(X)_{(p)}$:

$$TO(X)_{(p)} = (\psi^{k_p} - 1)(\widetilde{KSO}(X)_{(p)}). \qquad \text{(Formula I)}$$

$$TO(X)_{(p)} = \{ \ \frac{x}{m} \in \widetilde{KSO}(X)_{(p)} : \theta^{or}_{k_p}(x) = \frac{\psi^{k_p}(1+u)}{1+u} \text{ in } 1+\widetilde{KSO}(X)_p \text{ for some } u \in \widetilde{KSO}(X)_p \}. \qquad \text{(Formula II)}$$

Let $y_m = r\xi_m(\mathbb{C}) - 2$ where $\xi_m(\mathbb{C})$ is the complex Hopf line bundle over $\mathbb{C}P^m$. Then $\widetilde{KO}(\mathbb{C}P^m) =< y_m, y_m^2, \ldots, y_m^s >$, as an abelian group, for some $s \in \mathbb{N}$. In Section 3, we first prove a useful formula for $\theta_p(y_m^n)$ for $n = 1, \ldots, s$. Then we apply Formulae I and II of $TO(X)_{(p)}$ to find $b_m(P_m(y; m_1, \ldots, m_s))$, the J-order of $P_m(y; m_1, \ldots, m_s) = m_1 y + m_2 y^2 + \cdots + m_s y^s \in \widetilde{KO}(\mathbb{C}P^m)$. Önder [49] has induced the formula $TO(X)_{(2)} = (\psi^3 - 1)(\widetilde{KO}(X)_{(2)})$ from the formula of $TO_{\mathbb{Z}/2\mathbb{Z}}(X)_{(2)}$ by taking X as a trivial $\mathbb{Z}/2\mathbb{Z}$-space, then he applied this formula of $TO(X)_{(2)}$ to give an alternative computation of the 2-primary factors of $b_m(y)$.

Our results in this section may be interpreted as a special case of the results of §4.3 where $G = \mathbb{Z}/p\mathbb{Z}$, however, we obtain sharper results in giving a simple formula for the 2 and 3 primary factors of the J-orders of the canonical generators of $\widetilde{JO}(\mathbb{C}P^m)$. This simple formula was conjectured in [48] where we used the formula of $VO(X)$ given in (3.3) to obtain a formula for $b_m(P_m(y; m_1, \ldots, m_s))$ which may be considered as an alternative to the formula given here. The reader may be confused, why do we need two formulae for $b_m(P_m(y; m_1, \ldots, m_s))$? The problem here is that both formulae of $b_m(P_m(y; m_1, \ldots, m_s))$ has its own advantages and disadvantages, that is, even if we give two formulae for $b_m(P_m(y; m_1, \ldots, m_s))$, those two formulae involve many computational problems so we present both of them and at the same time hope to obtain a simpler one. Finally, in Section 4, we show how

Formulae I and II of $TO(X)_{(p)}$ can be used to find the group $\widetilde{JO}(X)$. Our illustrative example is $\mathbb{C}P^4$.

5.2 Two computable forumlae of $TO(X)_{(p)}$

Let $KSO(d)(X)$ be the group obtained by symmetrization of the semigroup $VectSO(d)(X)$ of all isomorphic classes of real vector bundles over X with structural group $SO(dn)$ for $n = 1, 2, \dots$. $KSO(d)(X)$ is monomorphically embedded in $KO(X)$ as the subgroup of classes x such that $\omega_1(x) = 0$ and $\dim(x) = dn$ for some $n \in \mathbb{N}$, i.e.,

$$KSO(d)(X) = \{E - F \in KO(X) :\ \dim(E - F) = dn \text{ and } E, F \text{ are orientable}\}.$$

Let $\widetilde{KSO}(d)(X) = \{E - F \in KSO(d)(X) :\ \dim E = \ \dim F\}$. It is easy to see that $KSO(d)(X) = d\mathbb{Z} \oplus \widetilde{KSO}(d)(X)$ and $\widetilde{KSO}(d)(X) = \widetilde{KSO}(1)(X)$ for each $d \geq 1$. So, for simplicity, we write $\widetilde{KSO}(X)$ instead of $\widetilde{KSO}(d)(X)$. From § 2.4, $\widetilde{KSO}(X)$ is an orientable $\gamma-$ring and $\widetilde{KSO}(X)_p$ is an orientable p-adic $\gamma-$ring, so we have:

Lemma 5.2.1 *(An analogue of Proposition* 5.3 *of [12]) If* $(p, k) = 1$, k *may be* 2, *then the following diagram is commutative:*

$$\begin{array}{ccc} \widetilde{KSO}(X) & & \\ \Big\downarrow i & \searrow \theta_k^{or} & \\ \widetilde{KSO}(X)_p & \xrightarrow{\rho_k^{or}} & 1 + \widetilde{KSO}(X)_p \end{array}$$

where $i : \widetilde{KSO}(X) \to \widetilde{KSO}(X)_p$ *is given by* $i(x) = 1 \otimes x$, *it is not necessarily injective.*

Now, we give our main theorem in this section.

Theorem 5.2.2 *Let* p *be a prime number and* k_p *be a generator of the group of units* $(\mathbb{Z}/p^2\mathbb{Z})^*$ *in* $\mathbb{Z}/p^2\mathbb{Z}$ *(*k_p *may be* 2*). Then the following diagram is commutative:*

$$\begin{array}{ccccccc}
 & & 0 & & & & \\
 & & \downarrow & & & & \\
 & & \widetilde{KSO}(X)_{(p),\Gamma} & \xrightarrow{\tilde{q}} & \widetilde{KSO}(X)_{(p)}/TO(X)_{(p)} & \longrightarrow & 0 \\
 & & \downarrow i_\Gamma & & \downarrow \tilde{\theta}^{or}_{k_p} & & \\
0 & \longrightarrow & \widetilde{KSO}(X)_{p,\Gamma} & \xrightarrow{\rho^{or}_{k_p,\Gamma}} & 1+\widetilde{KSO}(X)_{p,\Gamma} & \longrightarrow & 0
\end{array}$$

where the index Γ indicates that we factor out the image of $(\psi^{k_p}-1)$ and $\tilde{q}$ is the quotient map.

Proof. First, we show that rows and columns are well-defined and exact.
(a) Using Lemma 2.4.3, the fact that localization and completion are exact functors on the category of finitely generated abelian groups, and the naturality of Adams operations, we have the following identifications:

$$\widetilde{KSO}(X)_{(p)}/(\psi^{k_p}-1)(\widetilde{KSO}(X)_{(p)}) = \widetilde{KSO}(X)_{(p)}/((\psi^{k_p}-1)(\widetilde{KSO}(X)))_{(p)} =$$

$$(\widetilde{KSO}(X)/(\psi^{k_p}-1)(\widetilde{KSO}(X)))_{(p)} \subseteq \widetilde{KSO}(X)_p/((\psi^{k_p}-1)(\widetilde{KSO}(X)))_p =$$

$$\widetilde{KSO}(X)_p/(\psi^{k_p}-1)(\widetilde{KSO}(X)_p).$$

Hence, $i : \widetilde{KSO}(X)_{(p)} \to \widetilde{KSO}(X)_p$ defined by $i(x/m) = (m)^{-1} \otimes x$ induces a monomorphism $\widetilde{i} : \widetilde{KSO}(X)_{(p),\Gamma} \to \widetilde{KSO}(X)_{p,\Gamma}$.
(b) By Theorem 4.5 of Atiyah-Tall [12], $\rho^{or}_{k_p}$ induces an isomorphism

$$\rho^{or}_{k_p,\Gamma} : \widetilde{KSO}(X)_{p,\Gamma} \to 1 + \widetilde{KSO}(X)_{p,\Gamma}.$$

(c) To show that $(\psi^{k_p}-1)(KSO(X)_{(p)}) \subseteq TO(X)_{(p)}$. Let

$$\frac{E-F}{m} \in \widetilde{KSO}(X)_{(p)},$$

then

$$(\psi^{k_p}-1)(\frac{E-F}{m}) = (\frac{\psi^{k_p}E-E}{m}) - (\frac{\psi^{k_p}F-F}{m}).$$

By Quillen [54] (Adams' conjecture), there is a fiberwise map of degree a power of k_p between $\psi^{k_p}E$ and E. So, by Dold's Theorem mod-k [1]-I

$$k_p^n(\psi^{k_p}E-E) \in TO(X)$$

for some integer n. Since $(p, k_p) = 1$ then

$$(\frac{\psi^{k_p}E-E}{m}) = \frac{k_p^n(\psi^{k_p}E-E)}{k_p^n m} \in TO(X)_{(p)}.$$

Similarly,

$$\frac{\psi^{k_p}F - F}{m} \in TO(X)_{(p)}$$

and hence

$$(\psi^{k_p} - 1)(\frac{E - F}{m}) \in TO(X)_{(p)}.$$

Thus, we have an epimorphism $\tilde{q} : \widetilde{KSO}(X)_{(p),\Gamma} \to \widetilde{KSO}(X)_{(p)}/TO(X)_{(p)}$.

(d) It is easy to see that $\theta^{or}_{k_p} : \widetilde{KSO}(X)_{(p)} \to 1+\widetilde{KSO}(X)_p$ given by $\theta^{or}_{k_p}(x/m) = (\theta^{or}_{k_p}(x))^{1/m}$ is an exponential map. Let

$$\frac{E - F}{m} \in TO(X)_{(p)}$$

then $nS(E)$ is stably fibre homotopy equivalent to $nS(F)$ for some n with $(p, n) = 1$. So by [1]-(II), Corollary 5.8,

$$\theta^{or}_{k_p}(E - F)^n = \frac{\psi^{k_p}(1 + u)}{1 + u} \text{ in } 1 + \widetilde{KSO}(X)_p$$

for some $u \in \widetilde{KSO}(X)$. Since $(p, n) = 1$, then

$$(1 + u)^{1/nm} = 1 + w \text{ in } 1 + \widetilde{KSO}(X)_p$$

for some $w \in \widetilde{KSO}(X)_p$. Hence

$$\begin{aligned}\theta^{or}_{k_p}(\frac{E - F}{m}) &= \theta^{or}_{k_p}(E - F)^{1/m} = (\theta^{or}_{k_p}(E - F)^n)^{1/nm} \\ &= \frac{\psi^{k_p}(1 + u)^{1/nm}}{(1 + u)^{1/nm}} = \frac{\psi^{k_p}(1 + w)}{1 + w}.\end{aligned}$$

Thus $\theta^{or}_{k_p}$ induces a homomorphism

$$\tilde{\theta}^{or}_{k_p} : \widetilde{KSO}(X)_{(p)}/TO(X)_{(p)} \to 1 + \widetilde{KSO}(X)_{p,\Gamma}.$$

Finally, we show the commutativity of our diagram.

Let $x/m \in \widetilde{KSO}(X)_{(p)}$. Then

$$\begin{aligned}\tilde{\theta}^{or}_{k_p} \circ \tilde{q}(x/m + (\psi^{k_p} - 1)(\widetilde{KSO}(X)_{(p)})) &= \tilde{\theta}^{or}_{k_p}(x/m + TO(X)_{(p)}) \\ &= \theta^{or}_{k_p}(x)^{1/m} + (\psi^{k_p} - 1)(1 + \widetilde{KSO}(X)_p).\end{aligned}$$

On the other hand,

$$\begin{aligned}&(\rho^{or}_{k_p,\Gamma} \circ i_\Gamma)(x/m + (\psi^{k_p} - 1)(\widetilde{KSO}(X)_{(p)})) \\ &= \rho^{or}_{k_p,\Gamma}(i(x/m) + (\psi^{k_p} - 1)(\widetilde{KSO}(X)_p)) \\ &= \rho^{or}_{k_p}(i(x/m)) + (\psi^{k_p} - 1)(1 + \widetilde{KSO}(X)_p)).\end{aligned}$$

Now, the result follows from Lemma 5.2.1. This completes the proof of the theorem.

Corollary 5.2.3 *(Formula I of $TO(X)_{(p)}$)*

$$TO(X)_{(p)} = (\psi^{k_p} - 1)(\widetilde{KSO}(X)_{(p)}).$$

Proof. Since $\tilde{\theta}^{or}_{k_p} \circ \tilde{q} = \rho^{or}_{k_p,\Gamma} \circ i_\Gamma$, then $\tilde{q}$ is injective and hence an isomorphism. So, $TO(X)_{(p)} = (\psi^{k_p} - 1)(\widetilde{KSO}(X)_{(p)})$.

Corollary 5.2.4 *(Formula II of $TO(X)_{(p)}$)*

$$TO(X)_{(p)} = \{\tfrac{x}{m} \in \widetilde{KSO}(X)_{(p)} : \theta^{or}_{k_p}(x) = \frac{\psi^{k_p}(1+u)}{1+u} \text{ in } 1 + \widetilde{KSO}(X)_p \text{ for some } u \in \widetilde{KSO}(X)_p\}.$$

Proof. Clearly, the right hand side of the above equality is a well-defined subgroup of $\widetilde{KSO}(X)_{(p)}$. i_Γ is injective implies that

$$i(\widetilde{KSO}(X)_{(p)}) \cap (\psi^{k_p} - 1)(\widetilde{KSO}(X)_p) = i((\psi^{k_p} - 1)(\widetilde{KSO}(X)_{(p)})) \quad (5.1)$$

the fact that $\rho^{or}_{k_p,\Gamma}$ is an isomorphism implies that

$$\rho^{or}_{k_p}(\psi^{k_p} - 1)(\widetilde{KSO}(X)_p) = (\psi^{k_p} - 1)(1 + \widetilde{KSO}(X)_p). \quad (5.2)$$

Now let $x/m \in TO(X)_{(p)}$, then by Formula I of $TO(X)_{(p)}$

$$x/m \in (\psi^{k_p} - 1)\widetilde{KSO}(X)_{(p)}.$$

Hence using (5.1) and (5.2), we have

$$\theta^{or}_{k_p}(x/m) = \rho^{or}_{k_p}(i(x/m)) = \frac{\psi^{k_p}(1+u)}{1+u} \text{ in } 1 + \widetilde{KSO}(X)_p$$

for some $u \in \widetilde{KSO}(X)_p$, this shows the first implication. The other implication is similar. This completes the proof.

If X is a finite $CW-$complex then $\widetilde{JO}(X)$ is a finite abelian group. So, by Lemma 2.4.3, the p-primary factor of the order of $x + TO(X) \in \widetilde{JO}(X)$ is the order of $x + TO(X)_{(p)} \in \widetilde{JO}(X)_{(p)}$. To find the order of $x + TO(X)_{(p)} \in \widetilde{JO}(X)_{(p)}$, we need to find the smallest p^m such that $p^m x \in TO(X)_{(p)}$. So if we use Formula I of $TO(X)_{(p)}$, given in Corollary 5.2.3, then we need to find the smallest p^m which solves

$$p^m x = (\psi^{k_p} - 1)(u) \in \widetilde{KSO}(X)_{(p)}$$

for some $u \in \widetilde{KSO}(X)_{(p)}$. But if we use Formula II of $TO(X)_{(p)}$, given in Corollay 5.2.4, then we need to find the smallest p^m which solves

$$\theta_{k_p}^{or}(x)^{p^m} = \frac{\psi^{k_p}(1+u)}{1+u} \text{ in } 1+\widetilde{KSO}(X)_p$$

for some $u \in \widetilde{KSO}(X)_p$.

Finally, we show how to use Formulae I and II of $TO(X)_{(p)}$ to find $\widetilde{JO}(X)_{(p)}$.

Lemma 5.2.5 *Let G be a finitely generated abelian group. Let H be a subgroup of G such that $(G/H)_{(p)}$ is finite (p a prime number). Then* $(G/H)_{(p)} =< g_1 + H_{(p)}, \ldots, g_n + H_{(p)} >$.

Proof. G/H is finite because if G/H has an infinite summand then so is $(G/H)_{(p)}$. To prove that $(G/H)_{(p)} =< g_1+H_{(p)}, \ldots, g_n+H_{(p)} >$, we only need to show that $g_i/m+H_{(p)} \in< g_1+H_{(p)}, \ldots, g_n+H_{(p)} >$ for each $i = 1, \ldots, n$ and all $m \in \mathbb{Z}$ with $(p, m) = 1$. Let p^r be the order of $g_i/m+H_{(p)}$ in $(G/H)_{(p)}$. Choose $d, k \in \mathbb{Z}$ such that $kp^r + dm = 1$. Then $(1 - dm)(g_i/m) = p^r k(g_i m) \in H_{(p)}$. Hence $g_i/m + H_{(p)} = dg_i + H_{(p)}$, namely $g_i/m \in< g_i + H_{(p)} >$. This completes the proof of the lemma.

Now, since $\widetilde{JO}(X)$ is a finite abelian group then Lemma 5.2.5 applies and

$$\widetilde{JO}(X)_{(p)} =< \alpha_1 = y_1 + TO(X)_{(p)}, \ldots, \alpha_n = y_n + TO(X)_{(p)} >$$

where

$$\widetilde{KO}(X) =< y_1, \ldots, y_n > .$$

Using Formulae I and II of $TO(X)_{(p)}$, one may try to find relations between $\alpha_1, \ldots, \alpha_n$ and then give $\widetilde{JO}(X)_{(p)}$ explicitly.

In the remainder of this chapter, we shall apply the above ideas to $\mathbb{C}P^m$, the complex projective space.

5.3 J-orders of elements of $\widetilde{KO}(\mathbb{C}P^m)$

Let L be a non-trivial complex line bundle over $\mathbb{C}P^m$. Using the theory of characteristic classes, it is easy to see that $L \cong \xi_m(\mathbb{C})^n$ or $L \cong \overline{\xi_m(\mathbb{C})}^n$ for some $n \in \mathbb{N}$ where $\overline{\xi_m(\mathbb{C})}$ denotes the conjugate bundle to $\xi_m(\mathbb{C})$. In the next lemma we find the image of $\xi_m(\mathbb{C})^n$ under the realification homomorphism $r : KU(\mathbb{C}P^m) \to KO(\mathbb{C}P^m)$. Clearly, $r(\overline{\xi_m(\mathbb{C})}^n) = r(\xi_m(\mathbb{C})^n)$.

For each $r, s \in \mathbb{N}$ with $r > s$, let $b_{r,s}$ be the coefficient of $(\xi_m(\mathbb{C})^s + \overline{\xi_m(\mathbb{C})}^s)$ in $(\xi_m(\mathbb{C}) + \overline{\xi_m(\mathbb{C})} - 2)^r$. Using the fact that $\xi_m(\mathbb{C})\overline{\xi_m(\mathbb{C})}= 1$, it is easy to see that:

$$b_{r,s} = (-2)^{r-s}\binom{r}{s} + \sum_{\substack{j=s+1 \\ j=s+2s_j}}^{r} (-2)^{r-j}\binom{r}{j}\binom{j}{s_j} \tag{5.3}$$

Further, $b_{r,0}$ is the constant term of $(\xi_m(\mathbb{C}) + \overline{\xi_m(\mathbb{C})} - 2)^r$.

For each $n \in \mathbb{N}$ and $s = 1, \ldots, n$, define $d_{n,s}$ by the recurrence relation $d_{n,n} = 1$ and for $s = n-1, n-2, \ldots, 1$,

$$d_{n,s} = -(d_{n,s+1}b_{s+1,s} + d_{n,s+2}b_{s+2,s} + \cdots + d_{n,n}b_{n,s}). \tag{5.4}$$

Lemma 5.3.1 *Let $n, m \in \mathbb{N}$,*

(i) *If $m = 2t$, then $r(\xi_m(\mathbb{C})^n) = (d_{n,1}y_m + d_{n,2}y_m^2 + \cdots + d_{n,n}y_m^n + 2)(mod\ y_m^{t+1})$.*

(ii) *If $m = 4t+3$, then*

$$r(\xi_m(\mathbb{C})^n) = (d_{n,1}y_m + d_{n,2}y_m^2 + \cdots + d_{n,n}y_m^n + 2)(mod\ y_m^{2t+1}).$$

(iii) *If $m = 4t+1$, then*

$$r(\xi_m(\mathbb{C})^n) = (d_{n,1}y_m + d_{n,2}y_m^2 + \cdots + d_{n,n}y_m^n + 2)(mod\ y_m^{2t+2}).$$

(iv) $d_{n,1} = n^2$.

Proof. Let $c : KO(\mathbb{C}P^m) \to KU(\mathbb{C}P^m)$ be the complexification homomorphism.

(i) $cr(\xi_m(\mathbb{C})^n) = \xi_m(\mathbb{C})^n + \overline{\xi_m(\mathbb{C})}^n$. On the other hand, from (5.3),(5.4) and the fact that $y^n = 0$ for $n > t$ we have

$$c((d_{n,1}y_m + d_{n,2}y_m^2 + \cdots + d_{n,n}y_m^n + 2)(mod\ y_m^{t+1})) = \xi_m(\mathbb{C})^n + \overline{\xi_m(\mathbb{C})}^n.$$

Using the fact that c is a monomorphism for $m = 2t$, we get

$$r(\xi_m(\mathbb{C})^n) = (d_{n,1}y_m + d_{n,2}y_m^2 + \cdots + d_{n,n}y_m^n + 2)(mod\ y_m^{t+1}).$$

(ii) The same as (i), since c is a monomorphism for $m = 4t+3$.

(iii) Let $i : \mathbb{C}P^{4t+1} \to \mathbb{C}P^{4t+2}$ be the inclusion map. Then $i^* : KO(\mathbb{C}P^{4t+2}) \to KO(\mathbb{C}P^{4t+1})$ is an epimorphism and maps $r(\xi_{4t+2}(\mathbb{C})^n)$ to $r(\xi_{4t+1}(\mathbb{C})^n)$. Hence

$$r(\xi_{4t+1}(\mathbb{C})^n) = i^*(r(\xi_{4t+2}(\mathbb{C})^n))$$

$$= i^*((d_{n,1}y_{4t+2} + d_{n,2}y_{4t+2}^2 + \cdots + d_{n,n}y_{4t+2}^n + 2)(\text{mod } y_{4t+2}^{2t+2}))$$
$$= (d_{n,1}y_{4t+1} + d_{n,2}y_{4t+1}^2 + \cdots + d_{n,n}y_{4t+1}^n + 2)(\text{mod } y_{4t+1}^{2t+2}).$$

(iv) $d_{n,1}$ is the constant term of

$$\frac{cr(\xi_m(\mathbb{C})^2) - 2}{\xi_m(\mathbb{C}) + \overline{\xi_m(\mathbb{C})} - 2} = \frac{\xi_m(\mathbb{C})^n + \overline{\xi_m(\mathbb{C})}^n - 2}{\xi_m(\mathbb{C}) + \overline{\xi_m(\mathbb{C})} - 2} =$$

$(\xi_m(\mathbb{C})^{n-1}+\xi_m(\mathbb{C})^{n-2}+\cdots+\xi_m(\mathbb{C})+1)((\overline{\xi_m(\mathbb{C})}^{n-1}+\overline{\xi_m(\mathbb{C})}^{n-2}+\cdots+\overline{\xi_m(\mathbb{C})}+1).$

Hence,

$d_{n,1} = n + 2(n-1) + 2(n-2) + \cdots + 2 = n + 2((n-1) + (n-2) + \cdots + 1)$

$= n + 2(\frac{n(n-1)}{2}) = n^2$. This completes the proof of the lemma.

Now, we use the above lemma to find $\theta_p(y_m^n) \in 1 + \widetilde{KO}(\mathbb{C}P^m) \otimes \mathbb{Q}_p$.

Theorem 5.3.2 *Let $m = 2t$ for some $t \geq 1$. Then*

(i) $\theta_2(y_m) = (1 + \frac{y_m}{4})^{\frac{1}{2}}$ *and for any odd integer $p = 2q + 1$,*

$$\theta_p(y_m) = \sum_{\substack{j=0 \\ j \leq t}}^{q} \frac{1}{2j+1}\binom{q+j}{2j} y_m^j.$$

(ii) *For $2 \leq n \leq t$, let $p = 2$ or $2q + 1$. Then $\theta_p(y_m^n) =$*

$$\left(\frac{\psi^p(d_{n,1} + \cdots + d_{n,n}y_m^{n-1})}{(d_{n,1} + \cdots + d_{n,n}y_m^{n-1})}\right)^{\frac{1}{2}} \frac{1}{\theta_p(y_m)^{d_{n,1}-1}\theta_p(y_m^2)^{d_{n,2}} \ldots \theta_p(y_m^{n-1})^{d_{n,n-1}}}.$$

Proof. (i) This is Lemma 5.4 of [43].

(ii) Let $\eta = 2\xi_m(\mathbb{C})^n + 2\overline{\xi_m(\mathbb{C})}^n$. η is a 4-dimensional complex vector bundle over $\mathbb{C}P^m$, also $\wedge^4(\eta) = \wedge^2(2\xi_m(\mathbb{C})^n) \wedge^2 (2\overline{\xi_m(\mathbb{C})}^n) = 1$. Hence,

$$c\theta_p(r\eta) = \theta_p(\eta) = \left(\frac{\xi_m(\mathbb{C})^{np} + \overline{\xi_m(\mathbb{C})}^{np} - 2}{\xi_m(\mathbb{C})^n + \overline{\xi_m(\mathbb{C})}^n - 2}\right)^2 = \left(\frac{\psi^p(\xi_m(\mathbb{C})^n + \overline{\xi_m(\mathbb{C})}^n - 2)}{\xi_m(\mathbb{C})^n + \overline{\xi_m(\mathbb{C})}^n - 2}\right)^2.$$

From Lemma 5.3.1,

$$\xi_m(\mathbb{C})^n + \overline{\xi_m(\mathbb{C})}^n - 2 = cr(\xi_m(\mathbb{C})^n) - 2 = c(d_{n,1}y_m + \cdots + d_{n,n}y_m^n).$$

Also,

$r\eta = 2r(cr(\xi_m(\mathbb{C})^n)) = 2rc(r(\xi_m(\mathbb{C})^n)) = 4r(\xi_m(\mathbb{C})^n) = 4d_{n,1}y_m+\cdots+4d_{n,n}y_m^n+8.$

Thus,

$$c\theta_p(r\eta) = c((p\theta_p(y_m)^{d_{n,1}}\theta_p(y_m^2)^{d_{n,2}}\dots\theta_p(y_m^{n-1})^{d_{n,n-1}}\theta_p(y_m^n))^4)$$

and

$$\begin{aligned}\theta_p(\eta) &= \left(\frac{c(\psi^p(y_m))c(\psi^p(d_{n,1}+d_{n,2}y_m+\cdots+d_{n,n}y_m^{n-1}))}{c(y_m)c(d_{n,1}+d_{n,2}y_m+\cdots+d_{n,n}y_m^{n-1})}\right)^2 \\ &= \left(c(p^2\theta_p(y_m)^2)c(\frac{\psi^p(d_{n,1}+d_{n,2}y_m+\cdots+d_{n,n}y_m^{n-1})}{(d_{n,1}+d_{n,2}y_m+\cdots+d_{n,n}y_m^{n-1})})\right)^2 \\ &= c\left((p\theta_p(y_m))^4(\frac{\psi^p(d_{n,1}+d_{n,2}y_m+\cdots+d_{n,n}y_m^{n-1})}{(d_{n,1}+d_{n,2}y_m+\cdots+d_{n,n}y_m^{n-1})})^2\right).\end{aligned}$$

Now, c is a monomorphism implies that

$$\begin{aligned}&p\theta_p(y_m)^{d_{n,1}}\theta_p(y_m^2)^{d_{n,2}}\dots\theta_p(y_m^{n-1})^{d_{n,n-1}}\theta_p(y_m^n) \\ &= p\theta_p(y_m)\left(\frac{\psi^p(d_{n,1}+d_{n,2}y_m+\cdots+d_{n,n}y_m^{n-1})}{(d_{n,1}+d_{n,2}y_m+\cdots+d_{n,n}y_m^{n-1})}\right)^{\frac{1}{2}}.\end{aligned}$$

The result follows.

Remark 1. Using Lemma 5.3.1 (ii) and (iii), we obtain a similar formula for $\theta_p(y_m^n)$ where m is an odd integer.

Corollary 5.3.3 *$\theta_p \circ \psi^n = \psi^n \circ \theta_p$ on $\widetilde{KO}(\mathbb{C}P^m)$ for all $m, n, p \in \mathbb{N}$.*

Proof. For any space X, we know that $\psi^r \circ \psi^s = \psi^s \circ \psi^r$ and $\theta_{rs}(x) = \theta_r(x)\psi^r(\theta_s(x))$ for all $r, s \in \mathbb{N}$ and $x \in \widetilde{KSO}(X)$. Now, the result follows from the above theorem.

Recall that, $b_m(P_m(y_m; m_1, \dots, m_s))$ denotes the J-order of $P_m(y_m; m_1, \dots, m_s) \in \widetilde{KO}(\mathbb{C}P^m)$. By using the same method of Proposition 5.7. of [43], one can easily show that $b_{4t+3}(P_{4t+3}(y_{4t+3}; m_1, \dots, m_{2t+1})) = b_{4t+2}(P_{4t+2}(y_{4t+2}; m_1, \dots, m_{2t+1}))$. Also, $b_{4t+1}(P_{4t+1}(y_{4t+1}; m_1, \dots, m_{2t}, m_{2t+1}))$

$$= \begin{cases} b_{4t}(P_{4t}(y_{4t}; m_1, \dots, m_{2t})) & \text{if } m_{2t+1} = 0 \\ \operatorname{lcm}\{b_{4t}(P_{4t}(y_{4t}; m_1, \dots, m_{2t})), 2\} & \text{if } m_{2t+1} = 1. \end{cases}$$

Therefore, in the rest of this paper we shall assume that $m = 2t$ for some $t \geq 1$, unless othewise indicated.

In order to find $b_m(P_m(y; m_1, \dots, m_t))$, the following two lemmas will be useful.

Lemma 5.3.4 *Let k_p be a generator of $(\mathbb{Z}/p^2\mathbb{Z})^*$ (remember that we always take $k_2 = 3$). If $n \in \mathbb{N}$, then*

(i) $\nu_2(3^{2n} - 1) = 3 + \nu_2(n)$.

(ii) *For an odd prime p,*

$$\nu_p(k_p^{2n} - 1) = \begin{cases} 0 & \text{if } \; 2n \not\equiv 0 \; mod(p-1) \\ 1 + \nu_p(n) & \text{if } \; 2n \equiv 0 \; mod(p-1). \end{cases}$$

Proof. (i) is well-known.

(ii) Let $\nu_p(k_p^{2n} - 1) = s$, then $k_p^{2n} \equiv 1 \bmod p^s$. If $s \geq 1$ then $(\mathbb{Z}/p^s\mathbb{Z})^*$ is cyclic of order $p^{s-1}(p-1)$ with generator k_p ([32], Theorem 2, p. 43). So, $2n = p^{s-1}(p-1)d$ for some $d \in \mathbb{N}$ with $(d,p) = 1$ ([32], Lemma 3, p. 42). Hence, $s = 1 + \nu_p(n)$.

Lemma 5.3.5 *Let k_p be a generator of $(\mathbb{Z}/p^2\mathbb{Z})^*$, and $r, s \in \mathbb{N}$ with $r \geq s$. Then*

(i) $\nu_2(\prod_{i=s}^r (3^{2i} - 1)) = 3(r - s + 1) + \sum_{i=s}^r \nu_2(i)$.

(ii) *For an odd prime p, $\nu_p(\prod_{i=s}^r (k_p^{2i} - 1))$*

$$= \begin{cases} [\frac{2r}{p-1}] + \sum_{i=1}^{[\frac{2r}{p-1}]} \nu_p(i) - [\frac{2(s-1)}{p-1}] - \sum_{i=1}^{[\frac{2(s-1)}{p-1}]} \nu_p(i) & \text{if } \; p \leq 2r+1 \\ 0 & \text{if } \; p > 2r+1. \end{cases}$$

Proof. (i) $\nu_2(\prod_{i=s}^r (3^{2i} - 1)) = \sum_{i=s}^r \nu_2(3^{2i} - 1) = \sum_{i=s}^r (3 + \nu_2(i)) = 3(r - s + 1) + \sum_{i=s}^r \nu_2(i)$.

(ii) If $p > 2r + 1$, then $\nu_p(k_p^{2i} - 1) = 0$ for each $i = s, \ldots, r$. Hence

$$\nu_p(\prod_{i=s}^r (k_p^{2i} - 1)) = 0.$$

If $p \leq 2r + 1$, then $p - 1 = 2d$ for some $d \in \{1, \ldots, r\}$.

$$\nu_p(\prod_{i=s}^r (k_p^{2i} - 1)) = \sum_{i=s}^r \nu_p(k_p^{2i} - 1) =$$

$$\sum_{\substack{i=s \\ 2i \equiv 0 \bmod (p-1)}}^r (1 + \nu_p(i)) = \sum_{i=1}^{[\frac{2r}{p-1}]} (1 + \nu_p(2di)) - \sum_{i=1}^{[\frac{2(s-1)}{p-1}]} (1 + \nu_p(2di)).$$

But $\nu_p(2di) = \nu_p(i)$. So

$$\nu_p(\prod_{i=s}^{r}(k_p^{2i}-1)) = \sum_{i=1}^{[\frac{2r}{p-1}]}(1+\nu_p(i)) - \sum_{i=1}^{[\frac{2(s-1)}{p-1}]}(1+\nu_p(i)) =$$

$$[\frac{2r}{p-1}] + \sum_{i=1}^{[\frac{2r}{p-1}]}\nu_p(i) - [\frac{2(s-1)}{p-1}] - \sum_{i=1}^{[\frac{2(s-1)}{p-1}]}\nu_p(i).$$

This completes the proof.

Now, let k_p be an odd generator of $(\mathbb{Z}/p^2\mathbb{Z})^*$, say $k_p = 2q+1$ (take $k_2 = 3$).

Remark 2. We take k_p to be odd only to reduce the work, k_p is even works equally well.

According to Formula I of $TO(X)_{(p)}$, $\nu_p(b_m(P_m(y; m_1, \ldots, m_t)))$ is the smallest non-negative integer v such that

$$p^v P_m(y; m_1, \ldots, m_t) = (1-\psi^{k_p})(u) \quad \text{in } \widetilde{KO}(\mathbb{C}P^m)_{(p)} \tag{5.5}$$

for some $u \in \widetilde{KO}(\mathbb{C}P^m)_{(p)}$.

Recall that from §4.3

$$\psi^{k_p}(y^r) = (\psi^{k_p}(y))^r = \sum_{\substack{j=0 \\ j\le t-r}}^{2rq} C_{j,r}y^{r+j} \tag{5.6}$$

where

$$C_{j,r} = \sum_{\substack{i_1+\cdots+i_{2r}=j \\ i_1,\ldots,i_{2r}\in\{0,\ldots,q\}}} b_{i_1}b_{i_2}\ldots b_{i_{2r}}.$$

Let $u \in \widetilde{KO}(\mathbb{C}P^m)_{(p)}$, then $u = a_1y + \cdots + a_ty^t$ for some $a_i \in \mathbb{Z}_{(p)}$. Using (5.6), it is easy to see that the coefficient of y^r in $(1-\psi^{k_p})(u)$ is

$$-\sum_{i=j}^{r-1} C_{r-i,i}a_i + (1-k_p^{2r})a_r,$$

where $r \in \{(j-1)+2(j-1)q+1, \min\{j+2jq, t\}\}$, $r = 1, \ldots, t$.
So, from (5.5), we need to find the smallest v which solves the following system of equations in $\mathbb{Z}_{(p)}$:

$$-\sum_{i=j}^{r-1} C_{r-i,i}a_i + (1-k_p^{2r})a_r = p^v m_r$$

where $r = 1, \ldots, t$.
The above system has the following solutions:

$$a_r = \frac{p^v M_{k_p,r}(m_1, \ldots, m_t)}{(1 - k_p^2) \ldots (1 - k_p^{2r})},$$

where $M_{k_p,1}(m_1, \ldots, m_t) = m_1$ and for $r = 2, \ldots, t$,

$$M_{k_p,r}(m_1, \ldots, m_t) = \sum_{i=j}^{r-1} C_{r-i,i}(1 - k_p^{2(i+1)}) \ldots (1 - k_p^{2(r-1)}) M_{k_p,i}(m_1, \ldots, m_t)$$

$$+ m_r(1 - k_p^2) \ldots (1 - k_p^{2(r-1)}).$$

Now,$a_r \in \mathbb{Z}_{(p)}$ implies that

$$v \geq \max\{\nu_p(\prod_{i=1}^{r}(1 - k_p^{2i})) - \nu_p(M_{k_p,r}(m_1, \ldots, m_t)), 0 : r = 1, \ldots, t$$

$$\text{and } M_{k_p,r}(m_1, \ldots, m_t) \neq 0\}.$$

Hence, we have:

Theorem 5.3.6 $\nu_p(b_m(P_m(y; m_1, \ldots, m_t))) =$

$$\max\{\nu_p(\prod_{i=1}^{r}(1 - k_p^{2i})) - \nu_p(M_{k_p,r}(m_1, \ldots, m_t)), 0 : r = 1, \ldots, t$$

$$\textit{and } M_{k_p,r}(m_1, \ldots, m_t) \neq 0\}.$$

Now, let us use Formula II.

Let $\theta_{k_p}(P_m(y; m_1, \ldots, m_t)) = 1 + \alpha_1(m_1, \ldots, m_t)y + \cdots + \alpha_t(m_1, \ldots, m_t)y^t$ for some $\alpha_i(m_1, \ldots, m_t) \in \mathbb{Z}_p$ (see [48], Theorem 2.2). $\nu_p(b_m(P_m(y; m_1, \ldots, m_t)))$ is the smallest non-negative integer v such that

$$\theta_{k_p}(P_m(y; m_1, \ldots, m_t))^{p^v} = \frac{\psi^{k_p}(1 + u)}{1 + u} \quad \text{in } 1 + \widetilde{KO}(\mathbb{C}P^m)_p \tag{5.7}$$

for some $u \in \widetilde{KO}(\mathbb{C}P^m)_p$.
Let $u = b_1 y + \cdots + b_t y^t$ for some $b_i \in \mathbb{Z}_p$, with the above symbols, the coefficient of y^r in $\psi^{k_p}(u)$ is

$$\sum_{i=j}^{r-1} C_{r-i,i} b_i + b_r k_p^{2r}.$$

To avoid excessive notation, we write $\theta_{k_p}(P_m(y; m_1, \ldots, m_t))^{p^v} = 1 + \alpha_1 y + \cdots + \alpha_t y^t$,where α_i now inolves quantities containing p in some way.

From (5.7), we have $1+\psi^{k_p}(u) = 1 + d_1 y + \cdots + d_t y^t$ where

$$d_n = \sum_{\substack{i+s=n \\ b_0=\alpha_0=1}} b_i \alpha_s.$$

Thus

$$\sum_{i=j}^{r-1} C_{r-i,i} b_i + b_r k_p^{2r} = b_r + \sum_{\substack{s>0 \\ i+s=r}} b_i \alpha_s,$$

which implies that

$$b_r = \frac{\sum_{\substack{s>0 \\ i+s=r}} b_i \alpha_s - \sum_{i=j}^{r-1} C_{r-i,i} b_i}{(k_p^{2r}-1)} = \frac{L_{k_p,r}(m_1,\ldots,m_t)}{(k_p^2-1)\ldots(k_p^{2r}-1)}$$

where

$$L_{k_p,1}(m_1,\ldots,m_t) = \alpha_1$$

and for $r = 2, \ldots, t$,

$$L_{k_p,r}(m_1,\ldots,m_t) = \sum_{i=1}^{r-1} L_{k_p,r-i}(m_1,\ldots,m_t)\alpha_i (k_p^{2(r-i+1)}-1)\ldots(k_p^{2(r-1)}-1) -$$

$$\sum_{i=j}^{r-1} C_{r-i,i} L_{k_p,i}(m_1,\ldots,m_t)(k_p^{2(i+1)}-1)\ldots(k_p^{2(r-1)}-1) + \alpha_r (k_p^2-1)\ldots(k_p^{2(r-1)}-1).$$

Now, $b_i \in \mathbb{Z}_p$ for $i = 1, \ldots, t$ implies that

$$\nu_p(L_{k_p,r}(m_1,\ldots,m_t)) \geq \nu_p((k_p^2-1)\ldots(k_p^{2r}-1)).$$

So, we have:

Theorem 5.3.7 *$\nu_p(b_m(P_m(y; m_1,\ldots,m_t)))$ is the smallest v such that*

$$\nu_p(L_{k_p,r}(m_1,\ldots,m_t)) \geq \nu_p((k_p^2-1)\ldots(k_p^{2r}-1)) \text{ for each } r = 1,\ldots,t.$$

Using Lemma 3.2 and any one of the above two theorems, we directly obtain:

Corollary 5.3.8 *If $p > 2t+1$, then $\nu_p(b_m(P_m(y; m_1,\ldots,m_t))) = 0$. Consequently,*

$$\widetilde{JO}(\mathbb{C}P^m) \cong \bigoplus_{\text{for all primes } p \leq m+1} \widetilde{JO}(\mathbb{C}P^m)_{(p)}$$

From Theorem 5.3.6, to find $b_m(P_m(y; m_1, \ldots, m_t))$ we only need to find $\nu_p(M_{k_p,r}(m_1, \ldots, m_t))$ for $r = 1, \ldots, t$. Therefore, it may be a good problem if one tries to obtain a general formula for $\nu_p(M_{k_p,r}(m_1, \ldots, m_t))$ in terms of $r, k_p, m_1, \ldots, m_t$?

Next, we find $\nu_p(M_{k_p,r}(0, \ldots, m_r = 1, 0, \ldots, 0))$ for $p = 2, 3$ and then we obtain simple formulae for the 2 and 3 primary factors of the J-orders of the canonical generators of $\widetilde{JO}(\mathbb{C}P^m)$. These simple formulae have been already conjectured in [48].

For $n = 1, \ldots, t$, the J-order of $y^n + TO(\mathbb{C}P^m)$ is

$$b_m(P_m(y; 0, \ldots, m_n = 1, 0, \ldots, 0)).$$

Let $M_{k_p,r} = M_{k_p,r}(0, \ldots, m_n = 1, 0, \ldots, 0)/((1 - k_p^2) \cdots (1 - k_p^{2(n-1)}))$. Then $M_{k_p,r} = 0$ for $r < n$, $M_{k_p,n} = 1$ and for $r = n+1, \ldots, t$,

$$M_{k_p,r} = \sum_{\substack{i=j \\ i \geq n}}^{r-1} C_{r-i,i}(1 - k_p^{2(i+1)}) \ldots (1 - k_p^{2(r-1)}) M_{k_p,i}$$

where $r \in \{(j-1) + 2(j-1)q + 1, \, \min\{j + 2jq, t\}\}$.
Hence, from Theorem 5.3.6, we have

$$\nu_p(b_m(y^n)) = \max\{\nu_p(\prod_{i=n}^{r}(1 - k_p^{2i})) - \nu_p(M_{k_p,r}) : r = n, \ldots, t\}.$$

Proposition 5.3.9 *If $p = 2$ or 3 then*

$$\nu_p(M_{k_p,r}) = \sum_{s=1}^{[\frac{2(r-1)}{p-1}]} \nu_p(s) - \sum_{s=1}^{[\frac{2(n-1)}{p-1}]} \nu_p(s),$$

for each $r = n, \ldots, t$.

Proof. We prove this proposition for $p = 2$ (the case $p = 3$ is similar). Recall that $k_2 = 3$. So we need to show that $\nu_2(M_{3,r}) = r - n + \sum_{s=n}^{r-1} \nu_2(s)$ for $r = n+1, \ldots, t$, where

$$M_{3,r} = \sum_{\substack{i=j \\ i \geq n}}^{r-1} 3^{3i-r} \binom{2i}{r-i} (1 - 3^{2(i+1)}) \cdots (1 - 3^{2(r-1)}) M_{3,i}.$$

By induction on r, if $r = n+1$ then $\nu_2(M_{3,r}) = \nu_2\binom{2n}{1} = 1 + \nu_2(n)$. So let $n + 1 < r \leq t$, we claim that

$$\nu_2(3^{3i-r}\binom{2i}{r-i}(1 - 3^{2(i+1)}) \cdots (1 - 3^{2(r-1)}) M_{3,i}) > \nu_2(3^{2i-r}\binom{2(r-1)}{1} M_{3,r-1})$$

for each $\max\{j,n\} \le i < r-1$. Suppose that $i > n$ (the case $i = n$ is similar) then by induction hypothesis and Lemma 5.3.5,

$$\nu_2(3^{3i-r}\binom{2i}{r-i}(1-3^{2(i+1)})\cdots(1-3^{2(r-1)})M_{3,i})$$

$$= \nu_2\binom{2i}{r-i} + 3(r-i-1) + i - n - \nu_2(i) + \sum_{s=n}^{r-1}\nu_2(s).$$

On the other hand,

$$\nu_2(3^{2i-r}\binom{2(r-1)}{1}M_{3,r-1}) = r - n + \sum_{s=n}^{r-1}\nu_2(s).$$

So, we need to show that $\nu_2\binom{2i}{r-i} + 2(r-i-1) > \nu_2(2i)$. But this follows directly from the fact that $\nu_2\binom{2i}{r-i} = \nu_2(2i) - \nu_2(r-i)$ if $\nu_2(2i) \ge r-i-1$. This completes the proof of our claim. Hence, $\nu_2(M_{3,r}) = \nu_2(3^{2i-r}\binom{2(r-1)}{1}M_{3,r-1}) = r - n + \sum_{s=n}^{r-1}\nu_2(s)$. This completes the proof.

Unfortunately, the above proof can not be used for $p \neq 2, 3$.

Theorem 5.3.10 *If $p = 2$ or 3 and $1 \le n \le t$, then*

$$\nu_p(b_m(y^n)) = \max\left\{ s - [\frac{2(n-1)}{p-1}] + \nu_p(s) : [\frac{2n}{p-1}] \le s \le [\frac{2t}{p-1}] \right\}.$$

Proof. Let $p = 2$, then

$$\nu_2(b_m(y^n)) = \max\{\nu_2(\prod_{i=n}^{r}(1-3^{2i})) - \nu_2(M_{3,r}) : r = n, \ldots, t\}$$

$$= \max\{2r - 2n + 2 + \nu_2(2r) : r = n, \ldots, t\}$$

$$= \max\{s - 2(n-1) + \nu_2(s) : 2n \le s \le 2t\}.$$

The case $p = 3$ is similar.

Remark 3. (1) If Proposition 5.3.9 is true for some values of p, other than $2, 3$, then Theorem 5.3.10 will be also true for those values of p.

(2) By using the computer, we can prove Proposition 5.3.9 for $t \le 10$. Unfortunately, due to the large value of $M_{k_p,r}$, for higher values of t, using the computer leads to the integer-overflow problem.

5.4 $\widetilde{JO}(\mathbb{C}P^4)$

If $\widetilde{KO}(X) =< y_1, \ldots, y_n >$, then

$$\widetilde{JO}(X)_{(p)} =< \alpha_{1,p} = y_1 + TO(X)_{(p)}, \ldots, \alpha_{n,p} = y_n + TO(X)_{(p)} > .$$

So, to find $\widetilde{JO}(X)_{(p)}$, we need to find all relations between $\alpha_{1,p}, \ldots, \alpha_{n,p}$, i.e., we need to find 'sufficient' solutions for the equation:

$$c_1\alpha_{1,p} + \cdots + c_n\alpha_{n,p} = 0 \text{ in } \widetilde{JO}(X)_{(p)},\ c_1, \ldots, c_n \in \mathbb{Z}. \tag{5.8}$$

$c_1\alpha_{1,p} + \cdots + c_n\alpha_{n,p} = 0$ in $\widetilde{JO}(X)_{(p)}$, implies that $c_1y_1 + \cdots + c_ny_n \in TO(X)_{(p)}$. Now using Formulae I and II of $TO(X)_{(p)}$, one may try to find 'sufficient' solutions for (5.8). Our illustrative example in this section is $\mathbb{C}P^4$.

$\widetilde{KO}(\mathbb{C}P^4) = \{a_1y + a_2y^2 : a_1, a_2 \in \mathbb{Z},\ y^3 = 0\}$. So, $\widetilde{JO}(\mathbb{C}P^4)_{(p)} =< \alpha_{1,p} = y + TO(\mathbb{C}P^4)_{(p)}, \alpha_{2,p} = y^2 + TO(\mathbb{C}P^4)_{(p)} >=< \alpha_{1,p} > + < \alpha_{2,p} >$. To find relations between $\alpha_{1,p}$ and $\alpha_{2,p}$, we need to solve $c_1\alpha_{1,p} + c_2\alpha_{2,p} = 0$ in $\widetilde{JO}(\mathbb{C}P^4)_{(p)}$.

$\widetilde{JO}(\mathbb{C}P^4)_{(2)} =< \alpha_{1,2} = y + TO(\mathbb{C}P^4)_{(2)} > + < \alpha_{2,2} = y^2 + TO(\mathbb{C}P^4)_{(2)} >$. $< \alpha_{1,2} >$ is cyclic of order 64 and $< \alpha_{2,2} >$ is cyclic of order 16. Also, $2\alpha_{2,2} = 40\alpha_{1,2}$. Hence $\widetilde{JO}(\mathbb{C}P^4)_{(2)} \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/64\mathbb{Z}$, similarly, $\widetilde{JO}(\mathbb{C}P^4)_{(3)} \cong \mathbb{Z}/9\mathbb{Z}$ and $\widetilde{JO}(\mathbb{C}P^4)_{(5)} \cong \mathbb{Z}/5\mathbb{Z}$. Thus, by Corollary 5.3.8, we have:

Theorem 5.4.1 $\widetilde{JO}(\mathbb{C}P^4) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/64\mathbb{Z} \oplus \mathbb{Z}/9\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z}$.

Chapter 6

CONCLUSION

To compute $JO_G(X) = KO_G(X)/TO_G(X)$, we need to find effective means for computing $TO_G(X)$. For $G = \{e\}$, Adams defined two computable subgroups $VO(X)$ and $WO(X)$ of $KO(X)$. He proved that $VO(X) = WO(X)$, $VO(X) \subseteq TO(X)$, and conjectured that $TO(X) \subseteq WO(X)$. Later on Quillen proved Adams' conjecture.

A necessary and sufficient condition for $V\mathbb{F}_k(\mathbb{F}^n) \xrightarrow{p} S(\mathbb{F}^n)$ to have a cross section is that n is a multiple of the order of $\xi_{k-1}(\mathbb{F})$ in $JO(\mathbb{F}P^{k-1})$. Using the fact that $TO(X) = VO(X) = WO(X)$, it was easy to find the J-orders of $\xi_{k-1}(\mathbb{F})$ and then to obtain a complete solution for the non-equivariant cross section problem of Stiefel manifolds.

Although we have two computable formulae of $TO(X)$, in many cases, it is difficult to use these formulae to compute $JO(X)$. Therefore, one prefers to work with the localization of $TO(X)$ at a prime p instead of working with $TO(X)$. Using only Atiyah-Tall paper about λ-rings, we gave a simple proof for two computable formulae of $TO(X)_{(p)}$. These two localized formulae of $TO(X)$ may be thought of as the localization of $VO(X)$ and $WO(X)$ at p.

To show the significance of those two localized formulae of $TO(X)$, we first used them to compute the J-orders of elements of $KO(\mathbb{C}P^k)$ and then we showed how to use them to compute the group $JO(\mathbb{C}P^k)$ itself. Also, we gave a simple formula for the 2 and 3 primary factors of the J-orders of the canonical generator of $JO(\mathbb{C}P^k)$. We conjecture that this simple formula is liable to be true for other p-primary factors.

For a finite p-group G, an equivariant analogue of $WO(X)$ has been given by tom Dieck and Hauschild. We have proved what may be considered an equivariant analogue of $VO(X)$. McClure reduced the computations of the equivariant J-groups from finite group case to finite p-group case. So, now we have two methods for computing $JO_G(X)$ for any finite group G and any finite

G-connected G-CW complex X.

If M is an $\mathbb{F}$ G-module, then a necessary condition for $V\mathbb{F}_k(M) \xrightarrow{p} S(M)$ to have a G-section is that $r(\xi_{k-1}(\mathbb{F}) \otimes \mathbf{M} - \mathbf{M})$ vanishes in $JO_G(\mathbb{F}P^{k-1})$. We gave complete calculations for the J-orders of elements of $JO_G(\mathbb{F}P^{k-1})$ for all projective spaces. So, we obtain a necessary condition for $V\mathbb{F}_k(M) \xrightarrow{p} S(M)$ to have a G-section. If M satisfies the condition $\dim_{\mathbb{R}} M^G \geq \max\{3, (\dim_{\mathbb{R}} \mathbb{F})(2k - d_{\mathbb{F}})\}$ then a sufficient condition for $V\mathbb{F}_k(M) \xrightarrow{p} S(M)$ to have a G-section is that $r(\xi_{k-1}(\mathbb{F}) \otimes \mathbf{M} - \mathbf{M})$ vanishes in $JO_G(\mathbb{F}P^{k-1}, \mathcal{U}(M))$.

For $\mathbb{F} = \mathbb{R}$, it is shown by Namboodiri that $JO_G(\mathbb{R}P^{k-1}, \mathcal{U}(M))$ can be computed in the same way as the groups $JO_G(\mathbb{R}P^{k-1})$ are computed. Also, if $\mathcal{U}(M)) = G\mathbb{R}^\infty$ then $JO_G(X, \mathcal{U}(M)) = JO_G(X)$. So, we can obtain a solution for the equivariant cross section problem of Stiefel manifolds for a wide class of G-modules. However, many important cases are still open.

After the above survey about the equivariant J-groups and the equivariant cross section problem of Stiefel manifolds, we hope that the importance of the following problems will be apparent to the reader:

(1) The groups $JO_{\mathcal{P}}(G)$ were first discussed for finite p-groups by Atiyah-Tall, Snaith, tom Dieck, and Petrie. Then Lee and Wasserman reduced the problem from general compact Lie group case to the p-group case. So, it is reasonable to expect that a method similar to that of Lee and Wasserman may lead to the computations of $JO_G(X)$ for any compact Lie group G. Of course the main difficulty is the Adams' conjecture which is proved only for finite groups and it is not clear how to extend it for general compact Lie groups.

Directly after one obtains a method for computing $JO_G(X)$ where G is a compact Lie group, he can discuss the equivariant cross section problem of Stiefel manifolds for any compact Lie group G. Therefore, we think that this problem is important and fruitful.

(2) A sufficient condition for an $\mathbb{F}$ G-module M to have a $(k-1)$ $\mathbb{F}$ G-fields is that $r(\xi_{k-1}(\mathbb{F}) \otimes \mathbf{M} - \mathbf{M})$ vanishes in $JO_G(\mathbb{F}P^{k-1}, \mathcal{U}(M))$. So, we have the following general problem:
Let M be a real G-module and X be a G-space. Find means for computing $JO_G(X, \mathcal{U}(M)$?

Of course, one should first begin with finite p-groups. It is reasonable to conjecture that the following sequence is exact:

$$0 \to KO_G(X,\mathcal{U}(M))_{(p)} \xrightarrow{1-\psi^{kp}} KO_G(X,\mathcal{U}(M))_{(p)} \xrightarrow{q} JO_G^{(p)}(X,\mathcal{U}(M))_{(p)} \to 0$$

where q is the quotient map.

(3) The equivariant cross section problem of Stiefel manifolds has been discussed with the assumption that

$$\dim_{\mathbb{R}} M^G \geq \max\{3, (\dim_{\mathbb{R}} \mathbb{F})(2k - d_{\mathbb{F}})\}.$$

But if M is a free $\mathbb{F}$ G-module then $M^G = \{0\}$. So, free G-modules do not satisfy the above condition. If M is a free real G-module, the real G-field number of M is computed by Becker. So, it seems a good problem, but a difficult one, to compute the complex (resp. quaternionic) G-field number of complex (resp. quaternionic) free G-modules.

(4) We have shown that if X is a free G-space then $JO_G(X) \cong JO(X/G)$. We suggest to compute $JO_{\mathbb{Z}/m\mathbb{Z}}(S^{2n+1})$ instead of $JO(L^n(m))$ and also to compute the J-orders of elements of $JO_G(L^n(m))$ where $L^n(m)$ is considered as a trivial G-space.

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