NONAUTONOMOUS TRANSCRITICAL AND PITCHFORK BIFURCATIONS IN IMPULSIVE/HYBRID SYSTEMS

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ABSTRACT

NONAUTONOMOUS TRANSCRITICAL AND PITCHFORK BIFURCATIONS IN IMPULSIVE/HYBRID SYSTEMS

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The main purpose of this thesis is to study nonautonomous transcritical and pitchfork bifurcations in continuous and discontinuous dynamical systems. Two classes of discontinuity, impulsive differential equations and differential equations with an alternating piecewise constant argument of generalized type, are addressed. Moreover, the Bernoulli equation in impulsive as well as hybrid systems is introduced. For the former one, the corresponding jump equation is chosen so that after Bernoulli transformation the original system is reduced to a linear non-homogeneous system. For the latter, this is achieved by constructing a special type of transformation. Sufficient conditions are obtained for the existence of bounded solutions of the Bernoulli equations. Next, it is shown that different types of convergence analysis, such as pullback and forward remain as a fruitful idea in impulsive and hybrid systems. Furthermore, bifurcation scenarios are obtained depending on the sign of Lyapunov exponent by using these convergence analysis. Attraction and transition approaches are used to study bifurcation patterns in impulsive systems which cannot be solved explicitly. In other words, qualitative change in the attractor/reppeller pair is observed as a parameter goes though bifurcation value. Besides, finite-time analogues of nonautonomous transcritical and pitchfork bifurcations are investigated in impulsive systems. Illustrative examples with numerical simulations are provided to demonstrate the theoretical results.

Keywords: Nonautonomous Bifurcation, Discontinuous Dynamical Systems, Attractive Solution, Repulsive Solution, Picewise Constant Argument, Impulsive Differential Equation, Finite-time Dynamics

ÖZ

İMPALSİF/HİBRİD SİSTEMLERDE OTONOM OLMAYAN TRANSKRİTİK VE DİRGEN ÇATALLANMA

Kashkynbayev, Ardak
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Bu tezin asıl amacı sürekli ve süreksiz dinamik sistemlerde otonom olmayan transkritik ve dirgen çatallanmaların çalışılmasıdır. Bu tezde iki grup süreksizlik ele alınmıştır: impalsif diferensiyel denklemler ve değişken genel tipteki parçalı sabit argümanlı diferansiyel denklemler. İmpalsif ve hibrit sistemlerde Bernoulli denklemleri tanımlanmıştır. Bu sistemlerin ilki için, sıçrama denklemi, Bernoulli dönüşümünden sonra sistem homojen olmayan lineer sisteme dönüşecek şekilde seçilmiştir. İkinci sistem için bu, özel dönüşüm oluşturarak temin edilmiştir. Bernoulli denklemlerinin sınırlı çözümlerinin varlığı için yeterli şartlar elde edilmiştir. Geri çekme ve ileri gibi farklı yakınsaklık kavramlarının impalsif ve hibrit sistemlerde de yararlı olduğu gösterilmiştir. Üstelik bu yakınsaklık analizleri sonucu Lyapunov üssün işaretine bağlı olarak farklı çatallanma elde edilmiştir. Doğrudan çözülemeyen impalsif sistemlerde çekicilik ve geçiş yaklaşımları kullanılarak çatallanma modelleri çalışılmıştır. Başka bir ifadeyle, parametre çatallanma değerini geçerken çekici ve itici ikilisinin nitelikli değişime uğradığı gözlemlenmiştir. Buna ek olarak, otonom olmayan transkritik ve dirgen çatallanmasının impalsif sistemlerde sonlu zaman benzerleri incelenmiştir. Teorik sonuçların doğruluğu sayısal benzetim örnekleriyle gösterilmiştir.

Anahtar Kelimeler: Otonom Olmayan Çatallanma, Süreksiz Dinamik Sistemler, Çe-

kici Çözüm, İtici Çözüm, Parçalı Sabit Argüman, İmpalsif Diferansyel Denklem, Sonlu zaman Dinamikleri

To my beloved family

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CHAPTER 1

INTRODUCTION AND PRELIMINARIES

Generalizations of ordinary differential equations and difference equations as an abstract rule are called dynamical systems. Historically, this terminology was first used in the book of Birkhoff [42]. Although mathematical modeling of real world processes by means of differential equations goes back to Newton it has only after Poincaré and Lyapunov, often accepted as the founders of the theory of dynamical systems, these problems started to be considered from qualitative point of view. Poincaré premised topological and geometrical approach to analyze the dynamic behavior of solutions instead of traditional methods known before [106, 107, 108]. On the other hand, Lyapunov in his thesis was concerned with asymptotic behavior of solutions in the neighborhood of a fixed point [91]. Thus, both of the mentioned scientists have indisputable contributions on the vast developing theory of dynamical systems as we understand and accept it today.

Poincaré used an originally French word bifurcation to explain the splitting of asymptotic behavior in a dynamical system [108]. Since then, bifurcation has been regarded as the topological change in the qualitative nature of the states as parameter varies over a specified space. These parameters often regarded as influence of an environment to a system. In autonomous dynamical systems, the bifurcation theory is well developed and it is concerned with a qualitative change of an equilibrium point or a periodic solution as parameter ranges [50, 72, 73, 83, 127]. For instance, a stable equilibrium persists as stable to small fluctuations in a certain parameter range and as parameter crosses a critical threshold, called as a bifurcation value, the equilibrium becomes unstable or even does not exist at all. Another example is existence of a periodic solution around an equilibrium point and its disappearance in the vicinity of the bifurcation value. Hence, bifurcation is by no means exceptional but a typical property of dynamical systems such as in biochemical reactions, structural mechanics, cardiac arrhythmias in malfunctioning hearts and in many other models of biology [50, 69, 83].

Discontinuous dynamical systems has its origin started with academic work of Krylov and Bogolyubov [81], Introduction to Nonlinear Mechanics, where the authors studied a model of a clock. This study suggested that differential equations with pulse action can be considered for nonlinear mechanics. Later, Pavlidis introduced terminology of impulsive differential equations in his studies [98, 99, 100]. However, Samoilenko and Perestyuk in [121] developed the theory of impulsive differential equations in a more systematic way and parallel to the that of theory in ordinary differential equations. Similar ideas were used in the book of Bainov and Simenov [37]. The theory of impulsive differential equations at non-fixed moments of impulses improved significantly after Akhmet introduced the notion of so-called B-topology [1, 32]. This idea enables to handle more complex systems in a coherent and fruitful way. To be concrete, systems with impulses at variable moment of time was hard problem to overcame over decades. There were certain attempts to solve this issue in the past, however, today it seems that B-topology is an appropriate and suitable tool to address these kind of problems.

On the other hand, Myshkis in [96] accentuated on delay arguments that have interval of constancy. Nevertheless, the foundation of the theory of differential equations with piecewise constant arguments is due to Cooke and Wiener [54, 125]. As a prototype to interval of constancy Cooke and his coauthors considered greatest integer functions, i.e. $q(t) = [t]$ type functions. A great step towards a characterization of the theory was achieved by Akhmet, who emerged parallel developments similar to ordinary differential equations [2, 7, 9, 14, 22]. Before Akhmet, reduction to discrete equations was the main tool to treat differential equations with piecewise constant arguments. However, this method allows one to solve equations which start at integer values only. This reason was the main obstacle to make complete qualitative analysis of a solution such as stability and bifurcation of an equilibrium. Akhmet sailed through this issue by introducing an equivalent integral equations which permit not only to consider arbitrary piecewise constant functions as arguments but also to examine qualitative properties of a solution such as existence and uniqueness of solutions and existence of periodic and almost periodic solutions [10, 11, 12, 13, 21, 23]. Method of Lyapunov function and Lyapunov-Razumikhin technique are studied to analyze stability of differential equations with piecewise argument of generalized type in papers [15, 19]. In other words, there is no restriction on the distance between the switching moments of the argument. Moreover, the method proposed by Akhmet is less restrictive since it does not require additional assumptions to reduce an equation to a discrete system. Thus, the theory was significantly improved and it becomes possible to handle more complex problems.

If a mathematical model explicitly involves time-dependent vector field then it is the main object of nonautonomous dynamical systems to describe its nature. These models are usually given in terms of evolutionary equations which may be ordinary differential, delay or difference equations. We encounter with several limitations if one assumes that an environment which surrounds system is not variable in time. The main reason for this is that conditions in real world is often very different from ones in labs where models are generated. For instance, seasonal effects on different time scales or changes in nutrient supply should be taken into account when modeling predator-prey systems. Another example is to analyze possible impacts on a model after stimulating chemicals or dosing drags. Hence, there are several reasons to consider evolutionary equations with vector fields which explicitly depend on time. Statistical confirmation of this reason is often obvious since data which is obtained from a measurement may contain time-dependent parameters. There exist two approaches to study nonautonomous dynamical systems. The first one is the concept of process or two parameter semi-flow, studied by Dafermos and Hale [59, 71]. Another approach is skew product flows which has its origins in ergodic theory was under investigation by Sell [123, 124]. In this thesis, we treat process formulation to study nonautonomous dynamical systems. The classical notion of exponential dichotomy in a linear nonautonomous differential equations was introduced by Perron [102, 103] and has been under intensive research in [55, 60, 93, 117, 118, 119, 120]. For a systematic development of nonautonomous dynamical systems in recent years we refer to [77, 112].

1.1 Attraction and Bifurcation

There are strong ties between concepts of attraction and repulsion of an invariant set and bifurcation. It is possible to predicate the base of attraction on the studies of Lyapunov [91, 92] while the origin of the bifurcation theory starts with Poincaré [108]. A significant accomplishment in the development of bifurcation theory was achieved by Andronov and Pontryagin when they regarded so-called structural stability [35]. And, a new highlight was attained after Pliss introduced the center manifold theory which allows to reduce dimension of a system to a lower one [105]. A rigors application of the center manifold theory can be found in the book of Carr [45]. On the other hand, in the literature we encounter the term attractor in the book of Coddington and Levinson [51]. The axiom A attractor was presented by Smale in [126]. Ruelle and Takens introduced strange attractors when they studied the turbulent behavior in fluids [116], where the word strange was used to point out that the limit set has a fractal structure [66]. However, it was Conley to introduce a local attractor and established a connection between Morse decompositions and attractor-repeller pairs [53]. It is only in the last two decades that attraction has been intensively studied in nonautonomous dynamical systems. There are basically two types of attraction in nonautonomous systems: forward and pullback. The former one involves a moving invariant set and deals with attraction in Lyapunov asymptotic stability sense. The latter one involves fixed invariant set which starts progressively earlier in time. The notion of pullback attractors were adopted from random dynamical systems [57, 58] and were called as cocycle attractors in some papers [78, 80]. Apparently, first time in the literature pullback attractor was introduced in [76] to emphasize the difference from the forward attraction. Let us briefly give bifurcation scenarios in one dimensional autonomous systems and its generalizations to nonautonomous case.

1.1.1 The Transcritical Bifurcation

The normal form of the transcritical bifurcation in one-dimension system is as follows.

$$x' = \mu_1 x - \mu_2 x^2,$$

where $\mu_2 > 0$. It is easy to see that $x = 0$ and $x = \frac{\mu_1}{\mu_2}$ are the equilibrium points. As μ_1 varies, the stability of the equilibria change. To be precise, whenever $\mu_1 < 0$ the origin is stable and the equilibrium point $x = \frac{\mu_1}{\mu_2}$ is unstable whereas for $\mu_1 > 0$ the origin and the equilibrium point $x = \frac{\mu_1}{\mu_2}$ interchange their roles, i.e. the origin is unstable and the equilibrium point $x = \frac{\mu_1}{\mu_2}$ becomes stable (see Figure 1.1 for details). Therefore, $\mu_1 = 0$ regarded as the bifurcation value.

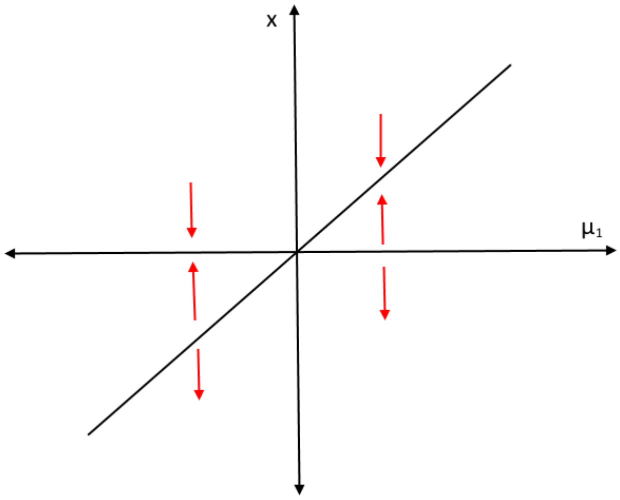

Figure 1.1: The transcritical bifurcation in one-dimensional system.

1.1.2 The Pitchfork Bifurcation

The normal form of the supercritical pitchfork bifurcation is as follows.

$$x' = \mu_1 x - \mu_2 x^3,$$

where $\mu_2 > 0$. One can confirm that if $\mu_1 < 0$ then $x = 0$ is the only equilibrium point and it is stable. On the other hand, if $\mu_1 > 0$ there are three equilibrium points $x = 0$, and $x = \pm\sqrt{\frac{\mu_1}{\mu_2}}$ with the origin no more stable and equilibrium points $x = \pm\sqrt{\frac{\mu_1}{\mu_2}}$ are stable. In other words, the trivial solution is repulsive in the open interval $\left(-\sqrt{\frac{\mu_1}{\mu_2}}, \sqrt{\frac{\mu_1}{\mu_2}}\right)$ for $\mu_1 > 0$ and repulsion shrinks to zero as $\mu_1 \searrow 0$, on the other hand, the trivial solution becomes attractive for $\mu_1 < 0$. Moreover, this trivial attraction undergo qualitative change and become nontrivial as $\mu_1 \nearrow 0$. Hence, $\mu_1 = 0$ is the bifurcation value (see Figure 1.2 for details).

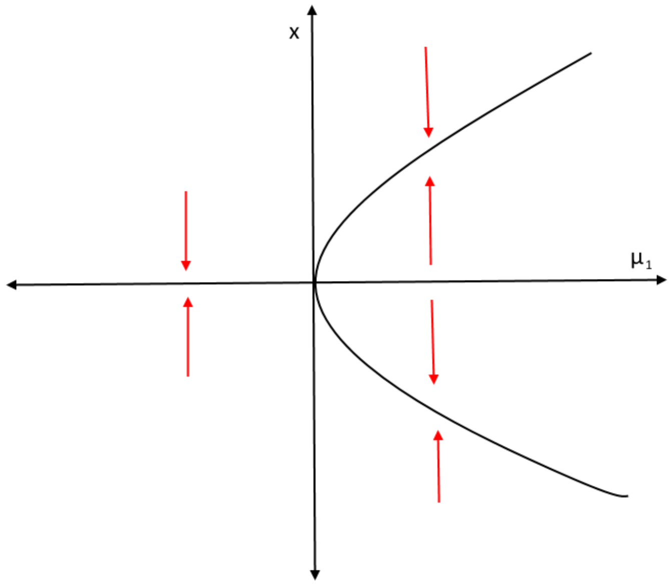

Figure 1.2: The pitchfork bifurcation in one-dimensional system.

1.1.3 The Saddle-node Bifurcation

The normal form of the saddle-node or as often called a fold bifurcation is as follows.

$$x' = \mu_1 - \mu_2 x^2,$$

where $\mu_2 > 0$. There is no an equilibrium point for $\mu_1 < 0$ and two equilibrium points, $x = \pm\sqrt{-\frac{\mu_1}{\mu_2}}$, a stable equilibrium point, $x = -\sqrt{-\frac{\mu_1}{\mu_2}}$, and an unstable equilibrium point, $x = \sqrt{-\frac{\mu_1}{\mu_2}}$, for $\mu_1 > 0$ (see Figure 1.3 for details). Thus, $\mu_1 = 0$ is the bifurcation value.

1.2 Nonautonomous Bifurcation

Despite of the fact that above systems are quite simple they have played important role in the development of the bifurcation theory. However, it may not be appropriate

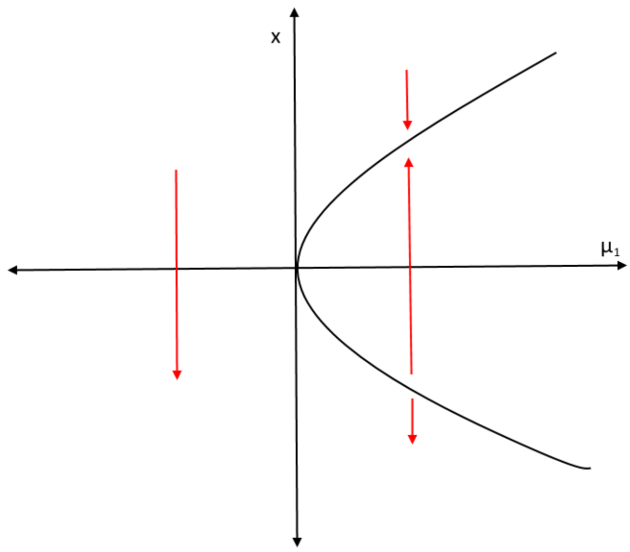

Figure 1.3: The saddle-node bifurcation in one-dimensional system.

to follow the same route as in autonomous dynamical systems to construct the bifurcation theory for nonautonomous dynamical systems. One of the reasons is that there may not exist an equilibrium point or a periodic solution of nonautonomous systems. Hence, most of the time the notion of equilibria is replaced with bounded trajectories. Another reason is that eigenvalues of a linearized system do not give proper information about asymptotic behavior of a solution. Thus, in scalar nonautonomous dynamical systems Lyapunov exponent seem to be an adequate tool to overcome this issue. From dynamic bifurcation viewpoint there are several approaches extending this theory to nonautonomous case. The mathematical foundations of nonautonomous bifurcation theory started with the paper of Langa et al. [86], where the authors proposed to study this theory by means of pullback and forward convergence and considered the following equations as a nonautonomous counterparts of the pitchfork and saddle-node bifurcations, respectively.

$$x' = \mu_1 x - \mu_2(t)x^3,$$

and

$$x' = \mu_1 - \mu_2(t)x^2,$$

where $0 < \mu_2(t) < \mu$. Next, Langa et al. in [88] considered nonautonomous transcritical and saddle-node bifurcations and obtained sufficient conditions on Taylor coefficients of the right hand side. Namely, the authors considered the following equations.

$$x' = \lambda\mu_1(t)x - \mu_2(t)x^2$$

and

$$x' = \lambda\mu_1(t) - \mu_2(t)x^2.$$

In [63, 74] the authors obtained results for nonautnomous saddle-node and transcritical bifurcations by using method of averaging. Nùñez and Obaya followed skew product approach in one-dimensional dynamical systems and studied bifurcation patterns depending on variation of the number and attraction properties of minimal sets [97]. One another approach is bifurcation of control sets based on Conley index theory was carried out by Colonius and his coauthors [52]. Finally, there are studies which describe bifurcation in nonautonomous dynamical systems by means of attractor and repeller pair. This approach deals with transitions of nonautonomous attractor which undergo qualitative change and become nontrivial when parameter pass through critical value [76, 79, 113]. The book of Rasmussen gives enlightening information about relation of attraction/repulsion and bifurcation concepts in nonautonomous systems [112].

1.3 The Bernoulli Equations

The origin of the Bernoulli differential equations go far beyond Poincaré and Lyapunov and apparently was first studied by Jacob Bernoulli in 1695 [41]. Although these equations have already became a classical subject in the theory of differential equations it has not been studied, for the best of our knowledge, in the discontinuous systems yet. One of the main reasons why this subject is attractive is that by means of the Bernoulli transformation they are reduced to a linear non-homogeneous equation; and hence can be solved explicitly. Despite its simplicity recent applications in nonautonomous bifurcation theory showed that a detailed insight into the discontinuous Bernoulli equations is necessary. We develop this simple idea to both impulsive and hybrid systems and carry out nonautonomous bifurcation analysis as well as analyze bounded solutions of these equations.

1.4 Principles of Impulsive Differential Equations

There are certain cases when continuous dynamical system fail to meet the needs of real world problems. Consider, for instance, the population dynamics of some species after epidemics or harvesting. One would expect a remarkable change in the population density of that species. Moreover, it is known fact that there is a very quick change of momentum when a pendulum of a clock crosses its equilibrium state [34]. In addition to these, we can give as an example a rapid alter in the velocity of an oscillating string while it is struck by a hammer [81]. If one wants to be a generous, it is a necessary that in all of the above cases the mathematical models involve discontinuity. One of the most common way to study discontinuity is by means of impulsive differential equations. Although there are huge amount of literature devoted to impulsive differential equations and its applications, ones in [1, 37, 38, 121] are the most commonly accepted as the fundamental work in this field.

There are two ways to involve discontinuity into a mathematical model. One is at prescribed moments of impulse and another one is at nonprescribed moments of impulse. Let us start describe with the first case and consider the following system.

$$\begin{aligned} x' &= f(t,x), \\ \Delta x|_{t=\theta_k} &= J_k(x), \end{aligned} \tag{1.1}$$

where $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, $\{\theta_k\}$ is a sequence of real numbers with the set of indexes $\mathbb{A}$ which is either finite or infinite, $J : \mathbb{A} \times \mathbb{R}^n \to \mathbb{R}^n$, $\Delta x|_{t=\theta_k} := x(\theta_k+) - x(\theta_k)$, and $x(\theta_k+) = \lim_{t\to\theta_k^+} x(t)$. Let us give details of the system (1.1). When $t \neq \theta_k$, a phase portrait of system (1.1) is characterized by differential equation counterpart of (1.1); it has jump at $t = \theta_k$ and satisfies difference equation counterpart of (1.1), i.e. $x(\theta_k+) - x(\theta_k) = J_k(x(\theta_k))$.

The second case is more sophisticated since impulse actions occur at nonprescribed moments, i.e. we consider systems of the form

$$\begin{aligned} x' &= f(t,x), \\ \Delta x|_{t=\theta_k(x)} &= J_k(x), \end{aligned} \tag{1.2}$$

where $\theta_k(x)$ are surfaces of discontinuity. One can easily see that impulse time in (1.2) by its own nature depend on solutions. Consequently, jump moments can be very different. Remarkable results concerning system (1.2) can be found in the accomplished book of Akhmet [1], where the author shed light on various topics like stability, periodic solutions of nonlinear systems, differentiability properties of nonautonomous systems and chaos.

In this thesis, however, we will only address systems with pulse action at prescribed moments. It worth nothing to say that both of the systems described above are nonautonomous and one cannot construct the theory based on autonomous systems. Thus, let us give a brief information on the theory of impulsive differential equations needed throughout of this thesis following the books [1, 121].

Let $\theta = \{\theta_k\}$, $i \in \mathbb{A} \subseteq \mathbb{Z}$, be a strictly increasing sequence of real numbers and a time interval $I \subseteq \mathbb{R}$.

Definition 1 *[1] A function $x : I \to \mathbb{R}^n$ is said to be from a class of functions $PC(I, \theta)$ if x is left continuous and x is continuous except at points from θ, where it has discontinuities of the first kind.*

In other words, $x \in PC(I, \theta)$ implies that the right limit exists and at points from θ one has $x(\theta+) = \lim_{t \to \theta+} x(t)$. Moreover, x is left continuous, i.e. $x(\theta-) = \lim_{t \to \theta-} x(t) = x(\theta)$.

Definition 2 *[1] A function $x : I \to \mathbb{R}^n$ is said to be from a class of functions $PC^1(I, \theta)$ if $x \in PC(I, \theta)$ and $x' \in PC(I, \theta)$, where the left derivative is considered at points from the set of θ.*

Let us state the existence and uniqueness theorems for (1.1). Consider an open connected set $O \subset \mathbb{R}^n$, and I be a open interval.

Theorem 1 *[1] Let $f : I \times O \to \mathbb{R}^n$ be a continuous and $\prod_k O \subseteq O$, $k \in \mathbb{A}$. Then, there is a $\Delta > 0$ such that for any $(t_0, x_0) \in I \times O$ there exists a solution $x(t, t_0, x_0)$ of* (1.1) *on $(t_0 - \Delta, t_0 + \Delta)$.*

We assume the following conditions, which will be useful in the next theorem.

(A0) The ordinary differential equation counterpart of (1.1) has a solution $x(t, t_0, x_0)$ which is unique in any interval of existence;

(A1) Its maximal interval of existence is an open set;

(A2) As t tends to cluster point of the interval any limit point of the set $(t, x(t))$ is a boundary point of $I \times O$.

Theorem 2 *[1] If conditions (A0)-(A2) are fulfilled, then each solution of* (1.1) *has a maximal interval of existence which satisfy one of the following alternatives:*

(i) it is an open interval (a, b) with any cluster point of the set $(t, x(t))$ as $t \to a$ or $t \to b$ belongs to the boundary of $I \times O$;

(ii) it is a half-open interval $(a, b]$, where $b \in \theta$ and any cluster point of the set $(t, x(t))$ as $t \to a$ belongs to the boundary of $I \times O$;

(iii) it is a half-open interval $(a, b]$, where both $a, b \in \theta$, the limit $x(a+)$ exists and it is interior point of O;

(iv) it is an open interval (a, b) with any cluster point of the set $(t, x(t))$ as $t \to b$ belongs to the boundary of $I \times O$, and the limit $x(a+)$ exists and it is interior point of O.

In order to have a uniqueness theorem we shall need the following conditions.

(A3) The function f is locally Lipschitzian;

(A4) There is at most one y which satisfy $x = y + J_k(y),\ k \in \mathbb{A}$, for every solution of (1.1).

Theorem 3 *[1] If the conditions (A3) and (A4) are fulfilled, then there exist a unique solution $x(t, t_0, x_0)$ of* (1.1).

One of the main features of impulsive differential equations is that it is possible to reduce (1.1) to an equivalent integral equation. This property is crucial in order to see relation to the theory of ordinary differential equations.

Theorem 4 *[1] A function $x \in PC(I,\theta)$ with $x(t_0) = x_0$ is a solution of* (1.1) *if and only if it satisfy the following integral equation.*

$$x(t) = \begin{cases} x_0 + \int_{t_0}^{t} f(s,x(s))ds + \sum_{t_0 \le \theta_k < t} J_k(x(\theta_k+)), \text{ if } t_0 \le t, \\ x_0 + \int_{t_0}^{t} f(s,x(s))ds - \sum_{t \le \theta_k < t_0} J_k(x(\theta_k+)), \text{ if } t_0 > t. \end{cases}$$

One of the main auxiliary lemmas in this thesis is the Gronwall-Belman inequality for piecewise continuous functions. We make use of the following lemma in Chapter 3.

Lemma 1 *Assume that $y, z \in PC(I,\theta)$ with $t_0, t \in I$, $y(t) \ge 0$, $z(t) > 0$, $\alpha_k \ge 0$, $k \in \mathbb{A}$, and c is nonnegative real number. Moreover, the following inequality is fulfilled.*

$$y(t) \le c + \int_{t_0}^{t} z(\tau)y(\tau)d\tau + \sum_{t_0 \le \theta_k < t} \alpha_k y(\theta_k) \; for \; t \ge t_0.$$

Then, the following inequality holds true.

$$y(t) \le c e^{\int_{t_0}^{t} z(\tau)d\tau} \prod_{t_0 \le \theta_k < t} (1 + \alpha_k) \; for \; t \ge t_0.$$

If, in addition to above, we have that $\alpha_k < 1$, $k \in \mathbb{A}$, and the following inequality is fulfilled.

$$y(t) \le c + \int_{t_0}^{t} z(\tau)y(\tau)d\tau - \sum_{t \le \theta_k < t_0} \alpha_k y(\theta_k) \; for \; t < t_0.$$

Then, the following inequality holds true.

$$y(t) \le c e^{-\int_{t_0}^{t} z(\tau)d\tau} \prod_{t \le \theta_k < t_0} (1 + \alpha_k)^{-1} \; for \; t < t_0.$$

Up to this point we have given the general description of an impulsive system. In our thesis, however, we consider impulsive system of the following.

$$\begin{aligned} x' &= A(t)x + f(t,x), \\ \Delta x|_{t=\theta_k} &= B_k x + J_k(x), \end{aligned} \tag{1.3}$$

where entries of $A(t)$ are from $PC(\mathbb{R},\theta)$ and B_k, $k \in \mathbb{A}$, are $n \times n$ real-valued matrices and B_k satisfy $\det(I + B_k) \ne 0$, $f : I \times O \to \mathbb{R}^n$ is a piecewise continuous function and $J_k : \mathbb{A} \times O \to \mathbb{R}^n$. Let $\Phi(t, s)$ be a fundamental matrix of the following linear impulsive system.

$$\begin{aligned} x' &= A(t)x, \\ \Delta x|_{t=\theta_k} &= B_k x. \end{aligned} \tag{1.4}$$

Lemma 2 *A function $x(t, t_0, x_0) \in PC^1(I, \theta)$ is a solution of* (1.3) *if and only if it satisfies the following integral equation.*

$$x(t) = \Phi(t, t_0)x_0 + \int_{t_0}^{t} \Phi(t, \tau)f(\tau, x(\tau))d\tau + \sum_{t_0 \le \theta_k < t} \Phi(t, \theta_k+)J_k(x(\theta_k)).$$

Let us denote by $i(I)$ the number of elements of θ in the interval I and consider the intervals $I_h = [t_0 - h, t_0 + h]$ and the open set $O_h = \{x \in O : \|x - x_0\| < H\}$ for the fixed $(t_0, x_0) \in I \times O$. Moreover, let $\delta_- = i[t_0 - h, t_0)$ and $\delta_+ = i[t_0, t_0 + h)$. In order to have the uniqueness of solutions for (1.3) we assume the following conditions.

- **(B0)** The matrix function $A(t)$ is bounded, i.e. $\sup_{t \in I} \|A(t)\| = A_0 < \infty$;
- **(B1)** The function f is Lipschitzian, i.e. there exists positive number L_f such that $\|f(t, x) - f(t, y)\| \le L_f \|x - y\|$ for all $x, y \in O$ and $t \in I$;
- **(B2)** The function J_k is Lipschitzian, i.e. there exists positive number L_J such that $\|J_k(x) - J_k(y)\| \le L_J \|x - y\|$ for all $x, y \in O$ and $(t, k) \in I \times \mathbb{A}$;
- **(B3)** $\sup_{I \times O} \|f(t, x)\| + \sup_{I \times \mathbb{A}} \|J_k(x)\| = A_1 < \infty$;
- **(B4)** $(A_0 + A_1)\, h + A_1 \max\{\delta_-, \delta_+\} < H$;
- **(B5)** $(A_0 + L_f)\, h + L_J \max\{\delta_-, \delta_+\} < 1$.

Theorem 5 *If conditions (B0)-(B5) are fulfilled, then* (1.3) *possesses a unique solution on I_h.*

Next, we study one of the main features of (1.3) which is stability.

Definition 3 *[1] The solution $x(t)$ of* (1.1) *is stable if for any $\epsilon > 0$, and $t_0 \in I$, there is $\delta(t_0, \epsilon) > 0$ such that for any other solution $y(t)$ of* (1.1) *with $\|x_0 - y_0\| < \delta(t_0, \epsilon)$ implies that $\|x(t) - y(t)\| < \epsilon$ for all $t \geq t_0$.*

A solution $x(t)$ of (1.1) is said to be asymptotically stable if $x(t)$ is stable and there is $\eta(t_0)$ such that $\|x(t) - y(t)\| \to 0$ as $t \to \infty$ whenever $\|x_0 - y_0\| < \eta(t_0)$.

Finally, we consider stability analysis of the trivial solution of (1.3). For this purpose we assume that $f(t, 0) = 0$ for all $t \in I$ and $J_k(0) = 0$ for all $k \in \mathbb{A}$. From now on we make use of the following assumption.

(B6) There exist real numbers $N > 1$ and $\xi > 0$ such that the fundamental matrix of (1.4) satisfy $\|\Phi(t, s)\| \leq Ne^{-\xi(t-s)}$ for all $t \geq s \geq t_0$.

Theorem 6 *If (B1)-(B2) and (B6) hold, then the trivial solution of* (1.3) *is asymptotically stable for the sufficiently small values of L_f and L_J.*

1.5 Principles of Differential Equations with Piecewise Constant Argument

One another way to study discontinuity in mathematical models is to consider differential equations with piecewise constant arguments. The need to study these equations raised from real world application problems which include but not limited to the damped as well as undamped loading systems based on a piecewise constant voltage, population dynamics, neural networks, the Froude pendulum and the Geneva mechanism [2, 4, 16, 17, 18, 39, 40, 68, 94, 95, 128]. Thus, despite the fact that differential equations with piecewise constant argument is a relatively new subject there is vast ongoing research in this field. Nevertheless, there are very few literature which treat this theory in a systematic manner. The book of Akhmet, Nonlinear Hybrid Continuous/Discrete Time Models, though, contains the basics of how this theory should be constructed. There are two types of arguments function, β and γ, that is under investigation in [2]. Let us give the description of the system. In [2], it was proposed to study the following equations with delayed time arguments.

$$x'(t) = f(t, x(t), x(\beta(t))), \tag{1.5}$$

where $x \in \mathbb{R}^n$, $t \in \mathbb{R}$, $f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous real-valued function and $\beta(t) = \theta_i$, $i \in \mathbb{Z}$, if $\theta_i \leq t < \theta_{i+1}$, θ_i is strictly increasing real-valued sequence such that $|\theta_i| \to \infty$ as $|i| \to \infty$. The argument function $\beta(t)$ is illustrated in Figure 1.4. One can see that by choosing $\theta_i = i$, $i \in \mathbb{Z}$, the greatest integer function, $[t]$, becomes a particular example of the argument function $\beta(t)$. It should be stressed out that the sequence θ_i is not necessarily an integer or a multiple of integer. Hence, equation (1.5) is an obvious generalization of any differential equations with delayed piecewise constant argument.

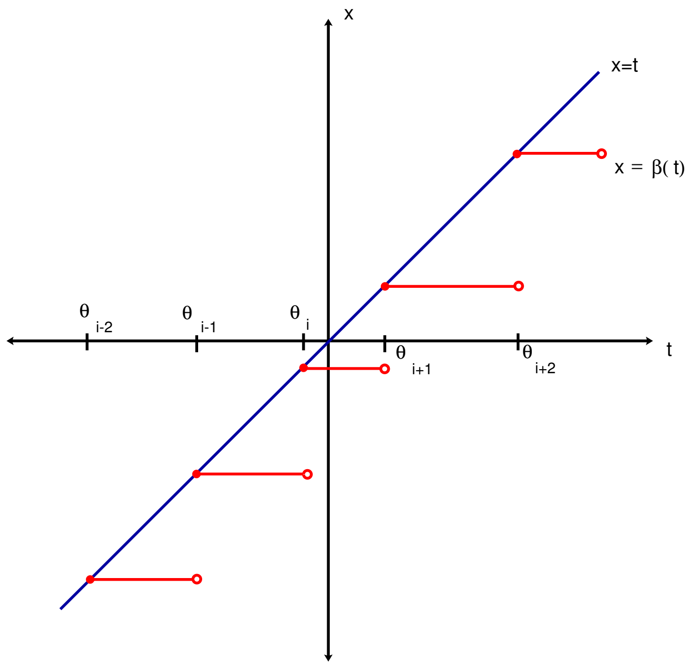

Figure 1.4: The deviated argument function $\beta(t)$. Clearly, it is of retarded type.

However, in the most adequate real world application problems there may be advanced arguments as well as retarded arguments. These equations are called as mixed type. Thus, deviating argument can change its nature during motion. To characterize these equations let us consider the following system.

$$x'(t) = f(t, x(t), x(\gamma(t))), \tag{1.6}$$

where $x \in \mathbb{R}^n$, $t \in \mathbb{R}$, $f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous real-valued function and $\gamma(t) = \zeta_i$, $i \in \mathbb{Z}$, if $\theta_i \leq t < \theta_{i+1}$, real-valued sequences θ_i, ζ_i are strictly increasing such that $\theta_i \leq \zeta_i \leq \theta_{i+1}$ and $|\theta_i| \to \infty$ as $|i| \to \infty$. As it is illustrated in Figure 1.5, the argument function $\gamma(t)$ is both retarded and advanced type. To be more concrete, the equation (1.6) is retarded whenever $\zeta_i < t \leq \theta_{i+1}$, i.e. $\gamma(t) < t$, and the equation (1.6) is advanced whenever $\theta_i \leq t < \zeta_i$, i.e. $\gamma(t) > t$. Therefore, deviated argument

$\gamma(t)$ is more general than argument $\beta(t)$ and the equation (1.6) is called differential equations with alternating piecewise constant argument of generalized type. In this thesis, we deal with the equation of mixed type described above in Chapter 4.

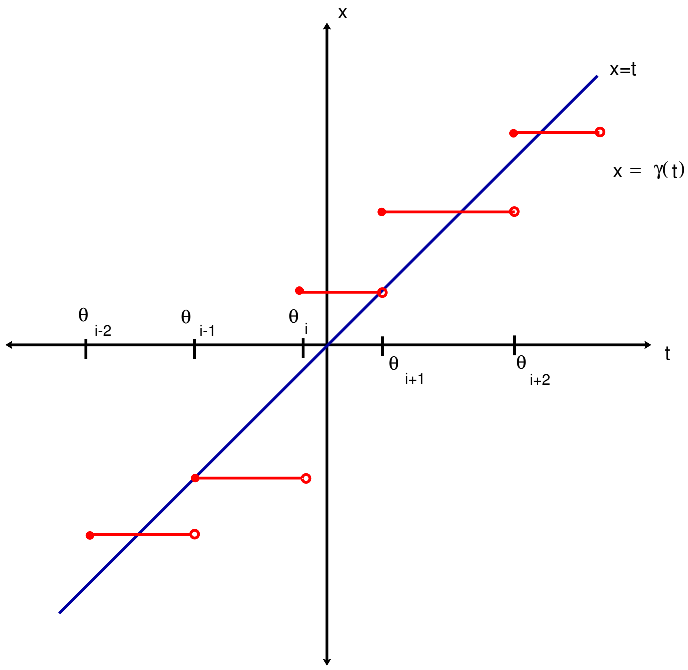

Figure 1.5: The deviated argument function $\gamma(t)$. One can see that argument is of mixed type.

Let us consider the following two equations which are in a maximal correspondence with this thesis.

$$u'(t) = A(t)u(t) + B(t)u(\gamma(t)), \tag{1.7}$$

and

$$u'(t) = A(t)u(t) + B(t)u(\gamma(t)) + g(t), \tag{1.8}$$

where $u \in \mathbb{R}^n$, $t \in \mathbb{R}$, $A, B : \mathbb{R} \to \mathbb{R}$ continuous $n \times n$ real-valued matrices,$f : \mathbb{R} \to \mathbb{R}^n$ continuous function and strictly increasing real-valued sequences θ_i and ζ_i, $i \in \mathbb{Z}$, are such that $\theta_i \leq \zeta_i \leq \theta_{i+1}$ and satisfy $\gamma(t) = \zeta_i$ for $\theta_i \leq t < \theta_{i+1}$, $i \in \mathbb{Z}$. In this thesis, the solutions of (1.7) and (1.8) are assumed to be continuous. Naturally, the right-hand sides of the equations (1.7) and (1.8) are discontinuous since the deviated function $\gamma(t)$ is discontinuous at the moments $t = \theta_i$, $i \in \mathbb{Z}$. In other words, the solutions of equations are assumed to be continuous and continuously differentiable within the intervals $[\theta_i, \theta_{i+1})$ for each $i \in \mathbb{Z}$.

Definition 4 *[2] A continuous function, $u(t)$, on $\mathbb{R}$ is a solution of* (1.7) *or* (1.8) *if the followings are satisfied.*

(i) the derivative $u'(t)$ exists at each point $t \in \mathbb{R}$ except, possibly, the points θ_i, $i \in \mathbb{Z}$, where the one-sided derivatives exist;

(ii) $u(t)$ satisfies (1.7) *and* (1.8) *on each interval (θ_i, θ_{i+1}) and at the points θ_i the right derivative of $u(t)$ fulfills* (1.7) *or* (1.8) *for each $i \in \mathbb{Z}$.*

Let $\Psi(t, s)$ be the fundamental matrix of the following linear system associated with (1.7) and (1.8).

$$v'(t) = A(t)v(t).$$

Next, we introduce a matrix function, $R_i(t)$, $i \in \mathbb{Z}$, which will be useful in what follows [2].

$$R_i(t) = \Psi(t, \zeta_i) + \int_{\zeta_i}^{t} \Psi(t, \tau)B(\tau)d\tau.$$

One would expect to have an initial function or an interval of initial values since we deal with differential equation with delay argument. However, the following regularity condition allows one to consider the initial value at a single point.

(R) $\det[R_i(t)] \neq 0$ for each fixed $i \in \mathbb{Z}$ and $\theta_i \leq t \leq \theta_{i+1}$.

To stress out the details of this idea let us consider the system (1.7) with values (t_0, u_0) be given such that $\theta_j \leq t_0 \leq \theta_{j+1}$ for a fixed $j \in \mathbb{Z}$ and $t_0 \neq \zeta_j$. Then, for $\theta_j \leq t \leq \theta_{j+1}$ the following functional differential equation is satisfied.

$$u'(t) = A(t)u(t) + B(t)u(\zeta_j).$$

Issuing from the theory of functional differential equations [67, 70, 82], in order to construct a solution one need the pair (t_0, u_0) and $(\zeta_j, u(\zeta_j))$. However, since the relation $u_0 = R_j(t_0)u(\zeta_j)$ and Condition (R) hold, and hence, $R_j(t_0)$ is an invertible matrix, to define a solution it is enough to have $u(t_0) = u_0$. Indeed, we attain at the following theorem.

Theorem 7 *[2] For every $(t_0, u_0) \in \mathbb{R} \times \mathbb{R}^n$ there exists a unique solution $u(t) = u(t, t_0, u_0)$ of* (1.7) *such that $u(t_0) = u_0$ if and only if the regularity condition (R) is valid.*

The last theorem emphasizes the similarity between the definition of the initial value problem of differential equation with piecewise constant argument of generalized type and that of ordinary differential equations. In particular, it implies that the set of solutions of (1.7) is $n-$dimensional linear space. Therefore, there exists fundamental matrix of (1.7) $U(t) = U(t,t_0), U(t_0,t_0) = I$, for a fixed t_0 such that

$$U'(t) = A(t)U(t) + B(t)U(\gamma(t)).$$

For a fixed $j \in \mathbb{Z}$ we assume that, without loss of generality, $\theta_j \leq t_0 \leq \zeta_j$, and define the fundamental matrix for the increasing t and arbitrary $k > j$ is as follows.

$$U(t) = R_k(t)\left(\prod_{l=j+1}^{k} R_l^{-1}(\theta_l)\, R_{l-1}(\theta_l)\right) R_j^{-1}(t_0),$$

where $\theta_k \leq t \leq \theta_{k+1}$. In the same manner, one can confirm that the fundamental matrix for the increasing t and arbitrary $m < j$ as follows.

$$U(t) = R_m(t)\left(\prod_{l=m}^{j-1} R_l^{-1}(\theta_{l+1})\, R_{l+1}(\theta_{l+1})\right) R_j^{-1}(t_0),$$

where $\theta_m \leq t \leq \theta_{m+1}$.

It can be verified that $U(t,s) = U(t)U^{-1}(s)$, for all $t,s \in \mathbb{R}$. Moreover, a solution of (1.7) with $u(t_0) = u_0$ satisfies, for all $t \in \mathbb{R}$, the following equation.

$$u(t) = U(t,t_0)u_0.$$

The uniqueness theorem for (1.8) is presented in the following theorem.

Theorem 8 *[2] If the regularity condition (R) holds for* (1.8)*, then there exists a unique solution* $u(t,t_0,u_0)$*, defined on* $\mathbb{R}$*, of* (1.8) *with* $u(t_0) = u_0$ *which satisfy the following equations.*

$$\begin{aligned} u(t,t_0,u_0) &= U(t,t_0)u_0 + U(t,t_0)\int_{t_0}^{\zeta_j} \Psi(t_0,\tau)g(\tau)d\tau \\ &+ \sum_{l=j}^{k-1} U(t,\theta_{l+1})\int_{\zeta_l}^{\zeta_{l+1}} \Psi(\theta_{l+1},\tau)g(\tau)d\tau + \int_{\zeta_k}^{t} \Psi(t,\tau)g(\tau)d\tau, \end{aligned}$$

with $\theta_j \le t_0 \le \theta_{j+1}, t \in [\theta_k, \theta_{k+1}], k > j$ *and*

$$\begin{aligned} u(t,t_0,u_0) &= U(t,t_0)u_0 + U(t,t_0)\int\limits_{t_0}^{\zeta_j} \Psi(t_0,\tau)g(\tau)d\tau \\ &+ \sum_{l=j}^{k} U(t,\theta_{l+1}) \int\limits_{\zeta_l}^{\zeta_{l+1}} \Psi(\theta_{l+1},\tau)g(\tau)d\tau + \int\limits_{\zeta_k}^{t} \Psi(t,\tau)g(\tau)d\tau, \end{aligned}$$

with $\theta_j \le t_0 \le \theta_{j+1}, t \in [\theta_k, \theta_{k+1}], k < j$.

To see the above equations in a more compact form let us denote $\widehat{[a,b]} = [a,b]$ if $a \le b$, and equal to $[b,a]$, otherwise for $a, b \in \mathbb{R}$. Moreover, let us define the following piecewise continuous matrix.

$$\Sigma(t,s) = \begin{cases} U(\theta_j,t_0)\Psi(t_0,s), & \text{if } t \in \widehat{[t_0,\zeta_j]}, \\ U(t,\theta_{l+1})\Psi(\theta_{l+1},s), & \text{if } t \in [\zeta_l,\zeta_{l+1}], \\ \Psi(t,s), & \text{if } t \in \widehat{[\zeta_k,t]}. \end{cases}$$

Then, we arrive at the following integral equation for (1.8).

$$u(t,t_0,u_0) = U(t,t_0)u_0 + \int\limits_{t_0}^{t} \Sigma(t,\tau)g(\tau)d\tau, \tag{1.9}$$

where $\Sigma(t,s)$ is called the Cauchy matrix and (1.9) is called the Cauchy representation formula. In this way, one can see the similarities and establish connection with the theory of ordinary differential equations. The similar results for quasilinear system are obtained in [2, 12]. Another approach to construct an integral equation is studied in [104].

1.6 Organization of the Thesis

The remaining part of this dissertation is organized as follows.

In Chapter 2, we consider nonautonomous transcritical and pitchfork bifurcations in continuous as well as discontinuous systems. The notions of so-called pullback attractor and forward attractor are implemented to analyze asymptotic behavior of systems. In the first part of the chapter, we study pitchfork bifurcation patterns based on pullback convergence which depend on the properties of the system in the past. Conditions which ensure transcritical bifurcation are obtained. In the second part of the chapter, we not only generalize the results obtained in the first part but we also attain less restrictive conditions to ensure nonautonomous bifurcation patters. Moreover, we introduce the Bernoulli equations in impulsive systems. The corresponding jump equation is constructed in special way that the whole system is reduced to a linear non-homogeneous system under the Bernoulli transformation. Both pullback and forward asymptotic behavior of the original system is analyzed based on reduced system. In addition to these, conditions to have bounded solutions for the Bernoulli equations are achieved. Appropriate numerical simulations which illustrate theoretical results are provided.

In Chapter 3, we study nonautonomous transcritical and pitchfork bifurcations in impulsive systems which are not explicitly solvable. That is, by any means of substitution it is not possible to obtain a solution. Bifurcation scenarios in this chapter are attained in terms of qualitative change in the attractor reppeller pair. Besides, we establish a new results in asymptotic behavior of linearized systems depending on entire time. In the remaining part of the chapter finite-time analogues of nonautonomous transcritical and pitchfork bifurcations are presented in impulsive systems. Illustrative examples which support the theoretical results are depicted.

Chapter 4 concerned with nonautonomous transcritical and pitchfork bifurcations in differential equations with alternating piecewise constant argument. The Bernoulli equation is presented for the hybrid systems. We construct special type of transformation so that original nonlinear system is converted to a linear non-homogeneous system. We premise that bifurcation scenarios depend on the sign of Lyapunov exponents. Besides, future and past asymptotic properties of bounded solutions are discussed. Appropriate examples with numerical simulations are given to illustrate the theoretical results.

Finally, in conclusion part we summarize the results of this thesis and give concluding remarks. Moreover, we discuss how this theory could be further developed.

CHAPTER 2

NONAUTONOMOUS TRANSCRITICAL AND PITCHFORK BIFURCATIONS IN AN IMPULSIVE BERNOULLI EQUATIONS

2.1 Nonautonomous Transcritical and Pitchfork Bifurcations in Impulsive Systems

In this chapter we discuss impulsive generalizations of the nonautonomous pitchfork and transcritical bifurcations. Scalar differential equations with fixed moments of impulses are considered. By means of certain systems we show that how the idea of pullback attracting sets remains a fruitful concept in the impulsive systems. Basics of the theory are provided.

2.1.1 Introduction

Asymptotic behavior of a solution near a fixed point and analysis of bifurcation is of a great importance in the qualitative theory of differential equations. In autonomous ordinary differential equations this theory is well developed. As in the autonomous systems, nonautonomous bifurcation is understood as a qualitative change in the structure and stability of the invariant sets of the system. However, to implement this concept in nonautonomous systems, locally defined notions of attractive and repulsive solutions are needed. There are currently qualitative studies which are devoted to nonautonomous bifurcation theory by treating pullback attractors [43, 44, 75, 77, 80, 86, 88, 112]. In the classical theory of stability one is interested in the asymptotic behavior of a solution as $t \to \infty$ for a fixed t_0, which is called forward attraction. On the other hand, the theory of pullback attraction deals with the asymptotic behavior of the solution as $t_0 \to -\infty$ for a fixed t [36, 43, 46, 49, 56, 57, 58, 80, 85, 87, 89, 122]. These two types of attraction give the same convergence analysis for autonomous dynamical systems. The approach of pullback attraction is required for the discussion of bifurcation analysis in nonautonomous differential equations by defining corresponding types of stability.

Modeling problems in the states of dynamical systems with time-dependent vector fields leads to nonautonomous problems. Moreover, these models may depend on some parameters which are accepted as the influence of an environment. In this case, it is an interesting issue to analyze qualitative changes when these parameters are varied. The main object of nonautonomous bifurcation theory is concerned in describing these changes. In addition to these, there may be abrupt changes at prescribed times in the real world evolutionary processes. These progressions are portrayed as impulsive phenomena [1, 37, 62, 84, 121], which are in no way, shape or form however regular in modeling in mechanics, electronics, biology, neural networks, medicine, and in social sciences [1, 4, 20, 31]. Hence, an impulsive differential equation is recognized as one of the central apparatuses to better comprehend the function of discontinuity in this present reality issues. Extending nonautonomous bifurcation theory to impulsive systems is a contemporary problem.

There are qualitative studies on asymptotic behavior of impulsive systems [1, 5, 29, 37, 84, 121]. There are also many studied which deal with bifurcation theory either in autonomous differential equations [1, 6, 30] or periodic equations with fixed moments of impulses [61, 64, 65]. However, differential equations with fixed moments of impulses are naturally nonautonomous differential equations. Consequently, one cannot construct the theory similar to autonomous systems of ordinary differential equations. Thus, in order to achieve results on fixed moments, it is crucial to extend idea of pullback attraction to impulsive systems for nonautonomous differential equations. Although the theory of impulsive differential equations is very developed nowadays, there are no results concerning analogues in [36, 43, 46, 57, 58, 77, 80, 85, 89, 122]. This appear to be due to the absence of papers concerning concrete systems analyzing the existence of nonautonomous bifurcations. It is hoped that present chapter fill this gap. The main novelty of the current study is to construct nonautonomous bifurcation theory for impulsive systems with appropriate definitions of pullback attracting sets. This is the very first step towards the bifurcation of nonautonomous differential equations with impulses.

2.1.2 Preliminaries

In this section, we introduce concepts of attractive and repulsive solutions, which are used to analyze asymptotic behavior of impulsive nonautonomous systems. This chapter is concerned with systems of the type

$$\begin{aligned} x' &= f(t,x), \\ \Delta x|_{t=\theta_i} &= J_i(x), \end{aligned} \tag{2.1}$$

where $\Delta x|_{t=\theta_i} := x(\theta_i+) - x(\theta_i)$, $x(\theta_i+) = \lim_{t\to\theta_i^+} x(t)$. The system (2.1) is defined on the set $\Omega = \mathbb{R} \times \mathbb{Z} \times G$ where $G \subseteq \mathbb{R}^n$. $\theta = \{\theta_i\}$ is a nonempty sequence with the set of indexes $\mathbb{Z}$, set of integers, such that $|\theta_i| \to \infty$ as $|i| \to \infty$. Let $\phi(t, t_0, x_0)$ be solution of (2.1). In this chapter, we deal with scalar impulsive differential equations such that $\phi(t, t_0, x_0)$ is non-continuable. Solutions are unique both forwards and backwards in time.

We say that the function $\phi : \mathbb{R} \to \mathbb{R}^n$ is from the set $PC(\mathbb{R}, \theta)$, where $\theta = \{\theta_i\}$ is an infinite set such that $|\theta_i| \to \infty$ as $|i| \to \infty$, if:

- ϕ is left continuous on $\mathbb{R}$;
- it is continuous everywhere except possibly points of θ where it has discontinuities of the first kind.

One cannot follow the same way in developing the theory for impulsive differential equations as for autonomous systems because there are certain problems. Namely, there may not be any equilibrium point at all. That is, it is hard to find a point x_0 which satisfies both $f(x_0, t) = 0$ for all $t \in \mathbb{R}$ and $J_i(x_0) = 0$ for all $i \in \mathbb{Z}$. Therefore, the notion of equilibrium point is replaced with a bounded solution or a complete trajectory, which is a particular examples of invariant sets. We investigate appearances and disappearances of complete trajectories that are stable and unstable in the pullback sense.

A time varying family of set $\mathfrak{A}(t)$ is invariant if $x_0 \in \mathfrak{A}(t_0)$ implies that $\phi(t, t_0, x_0) \in \mathfrak{A}(t)$. In order to study nonautonomous bifurcation with impulses we should define corresponding concepts of stability. In this chapter, we use Hausdorff semi-distance between sets X and Y as $d(X, Y) = \sup_{x \in X} \inf_{y \in Y} d(x, y)$.

2.1.2.1 Attraction and Stability

Asymptotic properties of continuous dynamics and dynamics with discontinuity are the same. Therefore, we shall use notion of pullback attracting sets without any change from [36, 43, 46, 49, 56, 57, 58, 78, 80, 85, 87, 89, 112, 122] and references therein. In autonomous system, to ensure that an invariant set $\mathfrak{A}$ is attracting it is enough have the existence of a neighborhood N of $\mathfrak{A}$ such that

$$d(\phi(t, 0, x_0), \mathfrak{A}) \to 0 \text{ as } t \to \infty.$$

In autonomous system asymptotic behavior of dynamics relies on upon given $t - t_0$ rather than the initial time only. Hence, the idea of attraction for autonomous systems is identical to the presence of a neighborhood N of $\mathfrak{A}$ for each fixed $t \in \mathbb{R}$,

$$d(\phi(t, t_0, x_0), \mathfrak{A}) \to 0 \text{ as } t_0 \to -\infty \text{ for all } x_0 \in N.$$

This is the principle thought under the pullback attraction [78, 122]. That is, we are interested in asymptotic behavior as $t_0 \to -\infty$ for fixed t, which makes it possible to analyze time-dependent sets.

Definition 5 *[78] An invariant set $\mathfrak{A}(t)$ is called pullback attracting if for every $t \in \mathbb{R}$*

$$\lim_{t_0 \to -\infty} d(\phi(t, t_0, x_0), \mathfrak{A}(t)) = 0.$$

Having given meanings of pullback attraction one needs to characterize related ideas of stability, instability and asymptotic stability in order to investigate asymptotic analysis in the pullback sense. Next, we start with defining stability in the pullback sense.

Definition 6 *[86] An invariant set $\mathfrak{A}(t)$ is pullback stable if for every $t \in \mathbb{R}$ and $\epsilon > 0$ there exists a $\delta(t) > 0$ such that for any $t_0 < t, x_0 \in N(\mathfrak{A}(t_0), \delta(t))$ implies that $\phi(t, t_0, x_0) \in N(\mathfrak{A}(t), \epsilon)$.*

An invariant set $\mathfrak{A}(t)$ is said to be pullback asymptotically stable if it is pullback stable and pullback attracting. As we are busy with scalar impulsive systems, one can verify that pullback attraction implies pullback stability for a bounded trajectories. Next, we state the following lemma which will be useful in what follows.

Lemma 3 *Let $y(t)$ be a locally pullback attracting complete trajectory of a scalar impulsive system. Then, $y(t)$ is also pullback stable.*

The proof of this lemma, given by Langa et al. in [88], for continuous case is the same for impulsive systems. Thus, the last lemma allows us to concentrate on only pullback attraction properties of a complete trajectory instead of carrying out pullback stability.

As one would expect pullback instability is characterized through the converse of pullback stability. That is, an invariant set $\mathfrak{A}(t)$ is called pullback unstable if it is not pullback stable, i.e. if there exists a $t \in \mathbb{R}$ and $\epsilon > 0$ such that for each $\delta > 0$, there exists a $t_0 < t$ and $x_0 \in N(\mathfrak{A}(t_0), \delta)$ such that $d(\phi(t, t_0, x_0), \mathfrak{A}(t)) > \epsilon$. However, the notion of unstable set, which Crauel defined for the random dynamical systems, seems to be more natural instrument in discontinuous dynamics point of view.

Definition 7 *[56] The unstable set, $U_{\mathfrak{A}(t)}$, of an invariant set $\mathfrak{A}(t)$ is defined as*

$$U_{\mathfrak{A}(t)} = \{u : \lim_{t \to -\infty} d(\phi(t, t_0, u), \mathfrak{A}(t)) = 0\}.$$

We say that $\mathfrak{A}(t)$ is asymptotically unstable if the relation $U_{\mathfrak{A}(t)} \neq \mathfrak{A}(t)$ is fulfilled for some t.

If $\mathfrak{A}(t)$ is invariant then one can see that $\mathfrak{A}(t) \subset U_{\mathfrak{A}(t)}$ is satisfied. Thus, from the last definition we have that $\mathfrak{A}(t)$ is strict subset of $U_{\mathfrak{A}(t)}$. In the sequel, we need the following result.

Proposition 9 *[86] If $\mathfrak{A}(t)$ is asymptotically unstable then it is also locally pullback unstable and cannot be locally pullback attracting.*

This result proven by Langa et al. in [86] is valid for impulsive systems. Thus, we omit the proof and refer to [86]. Note that the idea of the asymptotic instability is a definition of time-reversed forward attraction. Alternatively, it is conceivable to define instability as a time-reversed version of pullback attraction.

Definition 8 *[88] An invariant set $\mathfrak{A}(t)$ is pullback repelling if it is pullback attracting for time-reversed system, i.e., if for every $t \in \mathbb{R}$ and every $x_0 \in \mathbb{R}^n$,*

$$\lim_{t_0 \to \infty} d(\phi(t, t_0, x_0), \mathfrak{A}(t)) = 0.$$

2.1.3 The Pitchfork Bifurcation

In this section, we consider the following system

$$x' = p(t)x - q(t)x^3, \tag{2.2a}$$

$$\Delta x|_{t=\theta_i} = -x + \frac{x}{\sqrt{c_i + d_i x^2}}, \tag{2.2b}$$

where $p, q \in PC(\mathbb{R}, \theta)$. Assume that there exist constants A, B, C and D such that

$$|p(t)| < A < \infty \text{ and } 0 < c_i \leq C < \infty, \tag{2.3}$$

and

$$0 < b_0 \leq q(t) < B < \infty \;\text{ and }\; 0 < d_i \leq D < \infty, \tag{2.4}$$

for $i \in \mathbb{Z}$ and $t \in \mathbb{R}$. We suppose that there exist positive numbers $\underline{\theta}$ and $\overline{\theta}$ such that

$$\underline{\theta} \leq \theta_{i+1} - \theta_i \leq \overline{\theta}. \tag{2.5}$$

Moreover, there exists the limit

$$\lim_{t-s \to \infty} \frac{2\int_s^t p(u)du - \sum_{s \leq \theta_i < t} \ln c_i}{t - s} = \gamma. \tag{2.6}$$

By means of substitution $y = \dfrac{1}{x^2}$, the system (2.2) is converted to the impulsive linear non-homogeneous system

$$\begin{aligned} \dot{y} &= -2p(t)y + 2qt), \\ \Delta y|_{t=\theta_i} &= (c_i - 1)y + d_i. \end{aligned} \tag{2.7}$$

In what follows, we discuss the system (2.7) to analyze the system (2.2). Since $c_i \neq 0$, the transition matrix of the associated homogeneous part of (2.7) is, [1, 37, 121],

$$Y(t,s) = e^{-2\int_s^t p(u)du} \prod_{s\le\theta_i<t} c_i = e^{-\frac{2\int_s^t p(u)du - \sum_{s\le\theta_i<t}\ln c_i}{t-s}(t-s)}, \;\; t \ge s. \tag{2.8}$$

Lemma 4 *If $\alpha > \gamma > \beta > 0$, then there exist positive numbers M and m such that*

$$me^{-\alpha(t-s)} \le Y(t,s) \le Me^{-\beta(t-s)}, \;\; t \ge s. \tag{2.9}$$

Proof. By relation (2.6), there exists T such that if $t-s \ge T$, then
$\beta < \dfrac{2\int_s^t p(u)du - \sum_{s\le\theta_i<t}\ln c_i}{t-s} < \alpha.$

Consequently, by means of (2.3) and (2.5), it is true that

$$M = \sup_{0\le t-s\le T} e^{-2\int_s^t p(u)du} \prod_{s\le\theta_i<t} c_i$$

and

$$m = \inf_{0\le t-s\le T} e^{-2\int_s^t p(u)du} \prod_{s\le\theta_i<t} c_i.$$

Hence,

$$\begin{aligned} me^{-\alpha(t-s)} \le Y(t,s) &= e^{-2\int_s^T p(u)du + \sum_{s\le\theta_i<T}\ln c_i} e^{-2\int_T^t p(u)du + \sum_{T\le\theta_i<t}\ln c_i} \\ &\le Me^{-\beta(t-s)}, \end{aligned}$$

for $t \ge s$. The lemma is proved. □

Theorem 10 *Assume that (2.3), (2.4) and (2.6) hold for the system (2.2). Then, for $\gamma < 0$ the trivial solution is globally asymptotically pullback stable, and for $\gamma > 0$ the trivial solution is asymptotically unstable and complete trajectories $\pm\nu(t,\gamma)$ are locally asymptotically pullback stable and satisfy the following relation.*

$$\nu^2(t,\gamma) = \frac{1}{2\int_{-\infty}^t Y(t,s)q(s)ds + \sum_{\theta_i<t} Y(t,\theta_i+)d_i}.$$

Proof. By substitution $y = \dfrac{1}{x^2}$, we see that the solution of the system (2.2) satisfy the following equation, [1, 37, 121],

$$y(t,t_0,y_0) = Y(t,t_0)y_0 + 2\int_{t_0}^{t} Y(t,s)q(s)ds + \sum_{t_0 \le \theta_i < t} Y(t,\theta_i+)d_i. \qquad (2.10)$$

By means of (2.6), one can see that asymptotic behavior of $y(t,t_0,y_0)$ depends on the sign of γ.

Consider the case $\gamma < 0$. From (2.10) it follows that $y(t,t_0,y_0) \to \infty$ as $t_0 \to -\infty$. Thus, $x(t,t_0,x_0) \to 0$ both as $t_0 \to -\infty$ and as $t \to \infty$. Hence, all solutions are attracted both forwards and pullback to the trivial solution.

If $\gamma > 0$, then from (2.10) it follows that $y(t,t_0,y_0) \to 0$ as $t \to \infty$ implying that all solutions are unbounded as $t \to \infty$. However, as $t_0 \to -\infty$ we have

$$\lim_{t_0 \to -\infty} y(t,t_0,y_0) = 2\int_{-\infty}^{t} Y(t,s)q(s)ds + \sum_{\theta_i < t} Y(t,\theta_i+)d_i. \qquad (2.11)$$

The last equation reply that

$$\lim_{t_0 \to -\infty} x^2(t,t_0,x_0) = \nu^2(t,\gamma) = \frac{1}{2\int_{-\infty}^{t} Y(t,s)q(s)ds + \sum_{\theta_i < t} Y(t,\theta_i+)d_i} \qquad (2.12)$$

where $s, \theta_i \in (-\infty, t]$. By means of (2.5) and Lemma 4, one can show that

$$\begin{aligned} 0 < \frac{2mb_0}{\alpha} &< 2\int_{-\infty}^{t} Y(t,s)q(s)ds + \sum_{\theta_i < t} Y(t,\theta_i+)d_i \\ &< \frac{2BM}{\beta} + DM \sum_{\theta_i < t} e^{-\beta(t-\theta_i)} \\ &\le \frac{2BM}{\beta} + DM \sum_{i=0}^{\infty} e^{-i\beta\underline{\theta}} \\ &= \frac{2BM}{\beta} + DM \frac{1}{1 - e^{\beta\underline{\theta}}} < \infty. \end{aligned} \qquad (2.13)$$

Thus, $\nu^2(t,\gamma)$ is bounded both from above and from below. To see that $\nu(t,\gamma)$ is a complete trajectory, it would be enough to check that $\eta(t) = \dfrac{1}{\nu^2(t,\gamma)}$ satisfies (2.7). Indeed,

$$\begin{aligned} \dot{\eta} &= -4p(t)\int_{-\infty}^{t} Y(t,s)q(s)ds + 2Y(t,t)q(t) - 2p(t)\sum_{\theta_i < t} Y(t,\theta_i+)d_i \\ &= -2p(t)\left\{2\int_{-\infty}^{t} Y(t,s)q(s)ds + \sum_{\theta_i < t} Y(t,\theta_i+)d_i\right\} + 2q(t) \\ &= -2p(t)\eta + 2q(t). \end{aligned} \qquad (2.14)$$

To show that $\eta(t)$ satisfies the equation jumps, we note for fixed j it is true that $Y(\theta_j+, s) - Y(\theta_j, s) = (c_j - 1)Y(\theta_j, s)$; so that $Y(\theta_j+, s) = c_j Y(\theta_j, s)$. Then,

$$
\begin{aligned}
&\Delta\eta(t)|_{t=\theta_j} = \eta(\theta_j+) - \eta(\theta_j) \\
&= 2\int_{-\infty}^{\theta_j+} Y(\theta_j+, s)q(s)ds + \sum_{\theta_i<\theta_j+} Y(\theta_j+, \theta_j+)d_j \\
&-2\int_{-\infty}^{\theta_j} Y(\theta_j, s)q(s)ds - \sum_{\theta_i<\theta_j} Y(\theta_j, \theta_j+)d_j \\
&= 2c_j \int_{-\infty}^{\theta_j} Y(\theta_j, s)q(s)ds - 2\int_{-\infty}^{\theta_j} Y(\theta_j, s)q(s)ds + d_j \\
&+ \sum_{\theta_i<\theta_j} c_j Y(\theta_j, \theta_j+)d_j - \sum_{\theta_i<\theta_j} Y(\theta_j, \theta_j+)d_j \\
&= (c_j - 1)\left\{ 2\int_{-\infty}^{\theta_j} Y(\theta_j, s)q(s)ds + \sum_{\theta_i<\theta_j} Y(\theta_j, \theta_j+)d_j \right\} + d_j \\
&= (c_j - 1)\eta(\theta_j) + d_j.
\end{aligned}
\tag{2.15}
$$

The above analysis show that $\nu(t, \gamma)$ is pullback attracting. Thus, Lemma 3 implies that $\nu(t, \gamma)$ is pullback stable. Moreover, for $\gamma > 0$ all trajectories with $x_0 > 0$ are pullback attracted to $\nu(t, \gamma)$ and all trajectories with $x_0 < 0$ are pullback attracted to $-\nu(t, \gamma)$ as it is illustrated in Figure 2.1. By means of (2.10), it follows that

$$
\begin{aligned}
&x^2(t, t_0, x_0) = \\
&\frac{1}{y(t, t_0, y_0)} = \frac{1}{Y(t, t_0)x_0^{-2} + 2\int_{t_0}^{t} Y(t, s)q(s)ds + \sum_{t_0\le\theta_i<t} Y(t, \theta_i+)d_i} \\
&= \frac{1}{Y(t, t_0)(x_0^{-2} - \nu^{-2}(t_0)) + 2\int_{-\infty}^{t} Y(t, s)q(s)ds + \sum_{\theta_i<t} Y(t, \theta_i+)d_i}.
\end{aligned}
\tag{2.16}
$$

If $|x_0| < \nu(t_0)$ so that $x^{-2} - \nu^{-2}(t_0) > 0$, then $x(t)$ converge to 0 as $t \to -\infty$ implying that the origin is asymptotically unstable. This finalizes the proof of the theorem. $\square$

Remark 1 *In the similar manner, it can be easily shown that the results of Theorem 10 hold for the following system.*

$$
\begin{aligned}
&x' = p(t)x - q(t)x^3, \\
&\Delta x|_{t=\theta_i} = -x - \tfrac{x}{\sqrt{c_i + d_i x^2}}.
\end{aligned}
$$

Example 1 *Let* $p(t) \equiv a$*,* $c_i \equiv c$*, and* $\theta_i = ih$ *for the system (2.2) with* $h > 0$*. That*

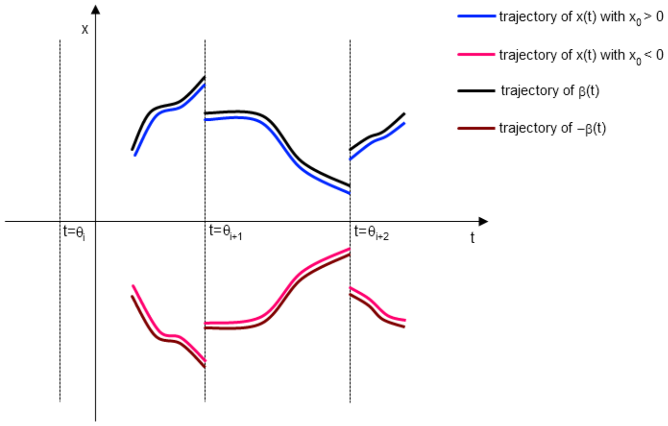

Figure 2.1: Asymptotic behavior of the system (2.2).

is,

$$\begin{aligned} x' &= ax - q(t)x^3, \\ \Delta x|_{t=ih} &= -x + \frac{x}{\sqrt{c+d_i x^2}}. \end{aligned} \tag{2.17}$$

Then $\gamma = 2a - \frac{1}{h}\ln c$. *By means of* $y = \frac{1}{x^2}$, *the system (2.17) is reduced to the linear impulsive system*

$$\begin{aligned} \dot{y} &= -2ay + 2q(t), \\ \Delta y|_{t=ih} &= (c-1)y + d_i. \end{aligned} \tag{2.18}$$

Asymptotic behavior of (2.18) depends on the sign of $2a - \frac{1}{h}\ln c = \gamma$, *and results of Theorem 10 are true for the system (2.17). If, in particular,* $c = 1$ *and* $d_i = 0$, *then there is no equation of jumps in the system (2.17). Moreover,* $\gamma = 2a$ *so that asymptotic behavior of (2.18) depends on the sign of* a. *Thus, results of Theorem 10 are generalization of the results obtained in the studies of Langa et al. in [86] and Caraballo and Langa in [43].*

2.1.4 The Transcritical Bifurcation

Consider the impulsive system

$$x' = p(t)x - q(t)x^2, \tag{2.19a}$$

$$\Delta x|_{t=\theta_i} = -x + \frac{x}{c_i + d_i x}, \tag{2.19b}$$

where $c_i > 0, d_i \in \mathbb{R}, i \in \mathbb{Z}, p, q \in PC(\mathbb{R}, \theta)$. Differently from the previous section, we do not impose any condition on the function p. However, as in the previous section, we suppose that there exist positive numbers $\underline{\theta}$ and $\overline{\theta}$ such that $\underline{\theta} \leq \theta_{i+1} - \theta_i \leq \overline{\theta}$, and there exists the limit

$$\lim_{t-s\to\infty} \frac{\int_s^t p(u)du - \sum_{s\leq\theta_i<t} \ln c_i}{t-s} = \gamma. \tag{2.20}$$

The function q and the numbers d_i are asymptotically positive as $t \to -\infty$ and $\theta_i \to -\infty$, respectively. In other words, there exist constants $\underline{b}$ and $\underline{d}$ such that

$$q(t) \geq \underline{b} > 0 \text{ for all } t \leq T^-, \text{ and } d_i \geq \underline{d} > 0 \text{ for all } \theta_i \leq T^-. \tag{2.21}$$

By means of substitution $x = \dfrac{1}{y}$, the system (2.19) is reduced to the following impulsive linear non-homogeneous differential equation.

$$\begin{aligned} \dot{y} &= -p(t)y + q(t), \\ \Delta y|_{t=\theta_i} &= (c_i - 1)y + d_i. \end{aligned} \tag{2.22}$$

The transition matrix of the associated homogeneous part of the system (2.22) is, [1],

$$Y(t,s) = e^{-\int_s^t p(u)du} \prod_{s\leq\theta_i<t} c_i = e^{-\frac{\int_s^t p(u)du - \sum_{s\leq\theta_i<t}\ln c_i}{t-s})(t-s)}, \quad t \geq s. \tag{2.23}$$

Assume that there exists a $\gamma_0 > 0$ such that

$$0 < m_\gamma \leq x_\gamma(t) = \frac{1}{\int_{-\infty}^t Y(t,s)q(s)ds + \sum_{\theta_i<t} Y(t,\theta_i+)d_i} \leq M_\gamma \tag{2.24}$$

for all $t \in \mathbb{R}, \quad i \in \mathbb{Z}, \quad 0 < \gamma < \gamma_0$, and

$$\liminf_{t_0\to-\infty} \frac{Y(t,t_0)}{\int_{t_0}^t Y(t,s)q(s)ds + \sum_{t_0\leq\theta_i<t} Y(t,\theta_i+)d_i} \geq m_\gamma > 0 \tag{2.25}$$

for all $-\gamma_0 < \gamma < 0$.

Theorem 11 *Assume that the above conditions hold for equation (2.19). Then, for* $-\gamma_0 < \gamma < 0$ *the origin is locally pullback attracting in* $\mathbb{R}$*; and for* $0 < \gamma < \gamma_0$ *the origin is asymptotically unstable and the trajectory* $x_\gamma(t)$ *is locally pullback attracting.*

Proof. By introducing transformation $x = \dfrac{1}{y}$ for equation (2.19), we see that the solution of the impulsive system (2.22) satisfy the following equation, [1, 37, 121],

$$y(t, t_0, y_0) = Y(t, t_0)y_0 + \int_{t_0}^{t} Y(t, s)q(s)ds + \sum_{t_0 \le \theta_i < t} Y(t, \theta_i+)d_i. \tag{2.26}$$

Transforming backwards we have

$$x(t, t_0, x_0) = \frac{1}{Y(t, t_0)x_0^{-1} + \int_{t_0}^{t} Y(t, s)q(s)ds + \sum_{t_0 \le \theta_i < t} Y(t, \theta_i+)d_i}. \tag{2.27}$$

By means of (2.20), one can see that asymptotic behavior of (2.27) depends on the sign of γ.

Consider the case $\gamma < 0$. From equation (2.27) and relation (2.20), it follows that $x(t, t_0, x_0) \to 0$ as $t_0 \to -\infty$ for any $x_0 \neq 0$ as long as $x(\xi, t_0, x_0)$ exists for all $\xi \in [t_0, t]$.

For $x_0 > 0$, it is sufficient to show that

$$Y(\xi, t_0)x_0^{-1} + \int_{t_0}^{\xi} Y(\xi, s)q(s)ds + \sum_{t_0 \le \theta_i < \xi} Y(\xi, \theta_i+)d_i > 0 \tag{2.28}$$

for $\xi \in [t_0, t]$. By means of (2.21), inequality (2.28) is satisfied provided that

$$Y(\xi, t_0)x_0^{-1} + \int_{T^-}^{\xi} Y(\xi, s)q(s)ds + \sum_{T^- \le \theta_i < \xi} Y(\xi, \theta_i+)d_i > 0 \tag{2.29}$$

for $\xi \in [T^-, t]$. Because of assumption (2.20), for t_0 small enough $Y(\xi, t_0)$ is bounded below on $(-\infty, T^-]$. Thus, (2.28) is satisfied if

$$x_0 < \frac{\inf_{t_0 \le T^-} Y(\xi, t_0)}{\sup_{\xi \in [T^-, t]} \left| \int_{T^-}^{\xi} Y(\xi, s)q(s)ds + \sum_{T^- \le \theta_i < \xi} Y(\xi, \theta_i+)d_i \right|}. \tag{2.30}$$

For $x_0 < 0$ the argument requires condition (2.25), which implies that there exists a μ_t such that

$$\frac{Y(\xi, t_0)}{\int_{t_0}^{\xi} Y(\xi, s)q(s)ds + \sum_{t_0 \le \theta_i < \xi} Y(\xi, \theta_i+)d_i} \ge \frac{m_\gamma}{2} \tag{2.31}$$

for all $t_0 \le \mu_t$. Now, it is sufficient to show that

$$Y(\xi, t_0)x_0^{-1} + \int_{t_0}^{\xi} Y(\xi, s)q(s)ds + \sum_{t_0 \le \theta_i < \xi} Y(\xi, \theta_i+)d_i < 0 \tag{2.32}$$

for $\xi \in [t_0, t]$. Denote $I(t_0, \xi) = \int_{t_0}^{\xi} Y(\xi, s)q(s)ds + \sum_{t_0 \le \theta_i < \xi} Y(\xi, \theta_i+)d_i$.

If $I(t_0, \xi) < 0$, then (2.32) is satisfied. If $I(t_0, \xi) > 0$, then we require

$$|x_0| < \frac{Y(\xi, t_0)}{\int_{t_0}^{\xi} Y(\xi, s)q(s)ds + \sum_{t_0 \le \theta_i < \xi} Y(\xi, \theta_i+)d_i},$$

which has the right-hand side of this expression is bounded below by $\frac{m_\gamma}{2}$ using (2.31). Therefore, for each t there exists a μ_t such that if $t_0 \le \mu_t$ and $|x_0|$ is sufficiently small, the solution exists on $[t_0, t]$ and, hence, the origin is locally pullback attracting.

Consider the case when $\gamma > 0$. If $x_0 > 0$, then as $t_0 \to -\infty$ (2.27) implies that

$$\lim_{t_0 \to -\infty} x(t, t_0, x_0) = x_\gamma(t) = \frac{1}{\int_{-\infty}^{t} Y(t, s)q(s)ds + \sum_{\theta_i < t} Y(t, \theta_i+)d_i} \tag{2.33}$$

as long as solution exists in the interval $[t_0, t]$. To ensure the existence, it is sufficient to have

$$Y(\xi, t_0)x_0^{-1} + \int_{t_0}^{\xi} Y(\xi, s)q(s)ds + \sum_{t_0 \le \theta_i < \xi} Y(\xi, \theta_i+)d_i > 0 \tag{2.34}$$

for $\xi \in [t_0, t]$. Let us show that (2.34) holds if we require $x_0 < (1 + \omega_t)x_\gamma(t_0)$ for some $\omega_t > 0$. Indeed,

$$\begin{aligned}
&Y(\xi, t_0)x_0^{-1} + \int_{t_0}^{\xi} Y(\xi, s)q(s)ds + \sum_{t_0 \le \theta_i < \xi} Y(\xi, \theta_i+)d_i \\
&> \frac{1}{1 + \omega_t} \left\{ \int_{-\infty}^{t_0} Y(\xi, s)q(s)ds + \sum_{\theta_i < t_0} Y(\xi, \theta_i+)d_i \right\} \\
&+ \int_{t_0}^{\xi} Y(\xi, s)q(s)ds + \sum_{t_0 \le \theta_i < \xi} Y(\xi, \theta_i+)d_i \\
&= \int_{-\infty}^{\xi} Y(\xi, s)q(s)ds + \sum_{\theta_i < \xi} Y(\xi, \theta_i+)d_i \\
&- \frac{\omega_t}{1 + \omega_t} \left\{ \int_{-\infty}^{t_0} Y(\xi, s)q(s)ds + \sum_{\theta_i < t_0} Y(\xi, \theta_i+)d_i \right\} > 0
\end{aligned} \tag{2.35}$$

for all $t_0 \le \xi \le t$. By (2.21), it suffices to show that last expression holds for $\xi \in [T^-, t]$. Choosing $\delta(t) = \omega_t m_\gamma$ it follows that $x_\gamma(t)$ is locally pullback attracting.

Assumption (2.24) implies that $0 < \int_{-\infty}^{t} Y(t,s)q(s)ds + \sum_{\theta_i<t} Y(t,\theta_i+)d_i < \infty$. Therefore, from equation (2.27) and relation (2.20), it follows that $x(t,t_0,x_0) \to 0$ as $t \to -\infty$, which implies that the origin is asymptotically unstable.

If $x_0 < 0$, then in order to solution $x(\xi,t_0,x_0)$ not to blow up in a finite time we need

$$Y(\xi,t_0)x_0^{-1} + \int_{t_0}^{\xi} Y(\xi,s)q(s)ds + \sum_{t_0\le\theta_i<\xi} Y(\xi,\theta_i+)d_i < 0,$$

for all $\xi \in \big[t_0,t\big]$. The last relation is satisfied if $I(\xi,t_0) = \int_{t_0}^{\xi} Y(\xi,s)q(s)ds + \sum_{t_0\le\theta_i<\xi} Y(\xi,\theta_i+)d_i < 0$. If $I(\xi,t_0) > 0$ we choose

$$\big|x_0\big| < \frac{Y(\xi,t_0)}{\int_{t_0}^{\xi} Y(\xi,s)q(s)ds + \sum_{t_0\le\theta_i<\xi} Y(\xi,\theta_i+)d_i},$$

which is bounded from below as it was proven in the case $\gamma < 0$.

This finalizes the proof of the theorem. □

Next, we want to formulate an impulsive extension of the system (2.19), which is related to the forward attraction. We assume that the function $q(t)$ and the numbers d_i are asymptotically positive as $t \to \infty$ and $\theta_i \to \infty$, respectively, and the balance condition (2.24) is valid. That is,

$$q(t) \ge \bar{b} > 0 \text{ for all } t \ge T^+, \text{ and } d_i \ge \bar{d} > 0 \text{ for all } \theta_i \ge T^+. \tag{2.36}$$

$$0 < m_\gamma \le x_\gamma(t) = \frac{1}{\int_{-\infty}^{t} Y(t,s)q(s)ds + \sum_{\theta_i<t} Y(t,\theta_i+)d_i} \le M_\gamma \tag{2.37}$$

for all $t \in \mathbb{R}$, $\quad 0 < \gamma < \gamma_0$.

Theorem 12 *Assume above conditions hold for equation (2.19). Then, for $-\gamma_0 < \gamma < 0$ the origin is locally forward attracting, and for $0 < \gamma < \gamma_0$ the trajectory $x_\gamma(t)$ is locally forward attracting. In addition, if*

$$0 < m_\gamma \le x_\gamma(t) = \frac{1}{\int_{t}^{\infty} Y(t,s)q(s)ds + \sum_{\theta_i<t} Y(t,\theta_i+)d_i} \le M_\gamma \tag{2.38}$$

for all $t \in \mathbb{R}, \gamma < 0$, then for $-\gamma_0 < \gamma < 0$ the trajectory $x_\gamma(t)$ is both asymptotically unstable and locally pullback repelling.

Proof. If $\gamma < 0$, the origin is locally forward attracting when x_0 is sufficiently small, since condition (2.36) implies that

$$\inf_{t\geq t_0}\left\{\int_{t_0}^{t} Y(t,s)q(s)ds + \sum_{t_0\leq\theta_i<t} Y(t,\theta_i+)d_i\right\} > -\infty. \tag{2.39}$$

If $\gamma > 0$, the trajectory $x_\gamma(t)$ is locally forward attracting, which easily follows from the following relation.

$$\left(\frac{1}{x(t)} - \frac{1}{x_\gamma(t)}\right) = Y(t,t_0)\left(\frac{1}{x_0} - \frac{1}{x_\gamma(t_0)}\right). \tag{2.40}$$

Therefore,

$$|x(t) - x_\gamma(t)| = \frac{x_\gamma(t)x(t)}{x_\gamma(t_0)x_0} e^{\left(\frac{-\int_{t_0}^{t} p(u)du + \sum_{t_0\leq\theta_i<t}\ln c_i}{t-t_0}\right)(t-t_0)} |x_0 - x_\gamma(t_0)|. \tag{2.41}$$

Using the balance condition (2.37) with $x_0 > 0$ implies that

$$\begin{aligned} x(t) &= \frac{1}{Y(t,t_0)x_0^{-1} + \int_{t_0}^{t} Y(t,s)q(s)ds + \sum_{t_0\leq\theta_i<t} Y(t,\theta_i+)d_i} \\ &\leq M_\gamma \frac{\int_{-\infty}^{t} Y(t,s)q(s)ds + \sum_{\theta_i<t} Y(t,\theta_i+)d_i}{Y(t,t_0)x_0^{-1} + \int_{t_0}^{t} Y(t,s)q(s)ds + \sum_{t_0\leq\theta_i<t} Y(t,\theta_i+)d_i} \\ &= M_\gamma \frac{Y(t,t_0)x_\gamma^{-1}(t_0) + \int_{t_0}^{t} Y(t,s)q(s)ds + \sum_{t_0\leq\theta_i<t} Y(t,\theta_i+)d_i}{Y(t,t_0)x_0^{-1} + \int_{t_0}^{t} Y(t,s)q(s)ds + \sum_{t_0\leq\theta_i<t} Y(t,\theta_i+)d_i}. \end{aligned} \tag{2.42}$$

Condition (2.36) guarantees that for t sufficiently large the integral and the sum in the numerator and denominator are positive. So, from the last expression it follows that

$$\limsup_{t\to\infty} x(t) \leq M_\gamma max\left\{1, \frac{x_0}{x_\gamma(t_0)}\right\}.$$

Therefore, any solution with $x_0 > 0$ is bounded as $t \to \infty$. Hence, from (2.41) it follows that $x_\gamma(t)$ is forward attracting as long as solutions do not blow up in a finite time. Next, we show that solution exists for $x_0 < (1+\omega_{t_0})x_\gamma(t_0)$.

$$\begin{aligned} &Y(t,t_0)x_0^{-1} + \int_{t_0}^{t} Y(t,s)q(s)ds + \sum_{t_0\leq\theta_i<t} Y(t,\theta_i+)d_i \\ &> \frac{1}{1+\omega_{t_0}}\left\{\int_{-\infty}^{t_0} Y(t,s)q(s)ds + \sum_{\theta_i<t_0} Y(t,\theta_i+)d_i\right\} \\ &+ \int_{t_0}^{t} Y(t,s)q(s)ds + \sum_{t_0\leq\theta_i<t} Y(t,\theta_i+)d_i \\ &= \int_{-\infty}^{t} Y(t,s)q(s)ds + \sum_{\theta_i<t} Y(t,\theta_i)d_i \\ &- \frac{\omega_{t_0}}{1+\omega_{t_0}}\left\{\int_{-\infty}^{t_0} Y(t,s)q(s)ds + \sum_{\theta_i<t_0} Y(t,\theta_i+)d_i\right\}. \end{aligned} \tag{2.43}$$

The last expression is positive for sufficiently small ω_{t_0} because of the assumption (2.36). Therefore, $x_\gamma(t)$ is locally forward attracting.

Under the final assumption (2.38), the results follow by making the transformations

$$\gamma \mapsto -\gamma, \quad x \mapsto -x, \;\; \theta \mapsto -\theta \;\; and \quad t \mapsto -t.$$

This finalizes the proof. □

Example 2 *Let $p(t) \equiv a$, $c_i \equiv c$, and $\theta_i = ih$ for the system (2.19) with $h > 0$. That is,*

$$\begin{aligned} x' &= ax - q(t)x^2, \\ \Delta x|_{t=ih} &= -x + \tfrac{x}{c+d_i x}. \end{aligned} \tag{2.44}$$

Then $\gamma = a - \frac{1}{h}\ln c$. By means of $y = \frac{1}{x}$, the system (2.17) is converted to the linear impulsive system

$$\begin{aligned} \dot{y} &= -ay + q(t), \\ \Delta y|_{t=ih} &= (c-1)y + d_i. \end{aligned} \tag{2.45}$$

Asymptotic behavior of (2.45) depends on the sign of γ, and results of Theorem 11 and Theorem 12 are true for the system (2.44). If $c = 1$ and $d_i = 0$, then $\gamma = a$ and there is no equation of jumps in the system (2.44).

2.2 An Impulsive Bernoulli Equations: The Transcritical and The Pitchfork Bifurcations

In this section, we study existence of the bounded solutions and asymptotic behavior of an impulsive Bernoulli equation. Moreover, we generalize nonautonomous pitchfork and transcritical bifurcation scenarios are investigated in the previous section. Illustrative examples with numerical simulations are given to support our theoretical results.

2.2.1 Introduction and Preliminaries

The Bernoulli equations constitute an important class of nonlinear differential equations. In this section we shall introduce a new type of impulsive equations. We say that an impulsive equation is of the Bernoulli type if it is reducible to an equation, which is linear and non-homogeneous in both its components, differential and impulsive. Thus, it is essentially nonlinear not only in its differential equation, but impulsive part also. It is important to note that the equation, which is under discussion in this section, is obtained not by a simple adding of an impulsive expression to the differential Bernoulli equation. Moreover, to the best of our knowledge, there is no study which deal with a discontinuous Bernoulli equations at all.

It is only in the recent decades, there has been intensive developments on time-dependent differential equations. Local theory of dynamical systems is concerned with asymptotic behavior of a fixed point or a periodic solution. However, in nonautonomous dynamical systems it is usually hard to find a fixed point or a periodic solution. Indeed, in many case they fail even to exist. Therefore, the notion of fixed points are generically endure as bounded solutions in the theory of time varying dynamical systems. There are abstract formulation of a nonautonomous dynamical systems as new concept of nonautonomous attractors which are called pullback attractors [46, 47, 77, 89, 112, 122]. We investigate appearances and disappearances of bounded solutions that are stable and unstable in the pullback and forward sense. In particular, it was possible to study bifurcation analysis in nonautonomous systems with pullback attractors [43, 75, 86, 88]. In previous section, we have studied nonautonomous transcritical and pitchfork bifurcations in impulsive systems. In the present section, we introduce a new and the most general impulsive Bernoulli equation, and discuss bifurcation analysis of these equations. The main equation under investigation is the following impulsive system,

$$\begin{aligned} &x' = p(t)x - q(t)x^n, \\ &\Delta x|_{t=\theta_i} = -x + \frac{x}{(c_i+d_ix^{n-1})^{\frac{1}{n-1}}}, \end{aligned} \tag{2.46}$$

where the functions $p, q : \mathbb{R} \to \mathbb{R}$ are continuous, the sequence of real numbers $\{\theta_i\}$, $i \in \mathbb{N}$, is such that there exist two real numbers $\underline{\theta}$ and $\overline{\theta}$ satisfying $\underline{\theta} \leq \theta_{i+1} - \theta_i \leq \overline{\theta}$, $\Delta x|_{t=\theta_i} := x(\theta_i+) - x(\theta_i)$ and $x(\theta_i+) = \lim_{t\to\theta_i^+} x(t)$. The system (2.46) is nonlinear not only in its differential equation part but in its impulsive part also. It consists of the Bernoulli equation and nonlinear impulsive one such that under the Bernoulli transformation, $y = x^{1-n}$, it is reduced to the following linear impulsive nonhomogenous equation,

$$\dot{y} = (1-n)p(t)y + (n-1)q(t),$$
$$\Delta y|_{t=\theta_i} = (c_i - 1)y + d_i.$$

This is the reason why we call (2.46) to be the impulsive Bernoulli equation. Moreover, the results obtained for the system (2.46) are also interpreted for the following continuous Bernoulli equation,

$$x' = p(t)x - q(t)x^n, \tag{2.47}$$

where the functions $p, q : \mathbb{R} \to \mathbb{R}$ are continuous. Thus, a new results are accomplished for (2.47). Let $x(t, t_0, x_0)$ be solution of (2.46) or (2.47). In this section, we deal with scalar differential equations such that $x(t, t_0, x_0)$ is continuable on $\mathbb{R}$. Solutions are unique both forwards and backwards in time. In the previous section we have considered the system (2.46) for $n = 2$ and $n = 3$. However, we did not state forward asymptotic analysis for the case $n = 3$. In this section, we state results for forward and pullback asymptotic analysis and n is allowed to be an arbitrary natural number. Moreover, we obtain conditions for (2.46) and (2.47) to have nontrivial bounded solutions on $\mathbb{R}$. Pullback asymptotic analysis of eq. (2.47) with $p(t) = const.$ and $n = 3$ has been carried out by Caraballo $\&$ Langa in [43] and Langa et al. in [86]. In [88], the authors considered (2.47) for $n = 2$ and $p(t) = \lambda a(t)$, where different bifurcation analysis are studied depending on the sign of the λ. In this section, we want to emphasize that we obtain different bifurcation scenarios for (2.46) which depend on $\gamma = \limsup\limits_{t-s\to\infty} \frac{\int_s^t (1-n)p(u)du + \sum_{s\leq\theta_i\leq t} \ln c_i}{t-s}$ and for (2.47) depend on $\gamma = \limsup\limits_{t-s\to\infty} \frac{\int_s^t (1-n)p(u)du}{t-s}$. Thus, the discontinuous system (2.46) satisfies the bifurcation conditions for the wider class of functions $p(t)$ than for continuous system (2.47). In other words, the bifurcation is cost by change of the exponents of a solution. This approach is premised for the first time in the literature in our papers [24, 26]. We continue with this idea in the present section and significantly improve the results obtained in the previous section. A theory of nonautonomous bifurcations in a Banach space is treated in terms of exponential dichotomy in a series of remarkable papers [109, 110, 111].

2.2.2 Bounded Solutions

In this section, we study the existence of a bounded solution of (2.46). It is easy to see that $x = 0$ is the trivial bounded solution of (2.46). In what follows, we are interested in the solutions which are bounded and different from zero. For this purpose, we shall need the following conditions.

(C1) There exist positive real numbers m and M such that $0 < m \leq q(t) \leq M$ for all $t \in \mathbb{R}$;

(C2) There exists positive real number L such that $0 \leq d_i \leq L$ for all $i \in \mathbb{Z}$.

By means of the transformation $y = x^{1-n}$, (2.46) is reduced to the following linear impulsive system,

$$\begin{aligned} \dot{y} &= (1-n)p(t)y + (n-1)q(t), \\ \Delta y|_{t=\theta_i} &= (c_i - 1)y + d_i. \end{aligned} \tag{2.48}$$

Let $\Psi(t, s)$ be the fundamental matrix of (2.48). One can find that

$$\Psi(t, s) = e^{\int_s^t (1-n)p(u)du} \prod_{s \leq \theta_i \leq t} c_i = e^{\int_s^t (1-n)p(u)du + \sum_{s \leq \theta_i \leq t} \ln c_i}.$$

Denote $\gamma = \limsup\limits_{t-s \to \infty} \frac{\int_s^t (1-n)p(u)du + \sum_{s \leq \theta_i \leq t} \ln c_i}{t-s}$. One can guarantee that there exist two positive numbers k and K such that

$$ke^{\gamma(t-s)} \leq ||\Psi(t, s)|| \leq Ke^{\gamma(t-s)}, \ \ s \leq t. \tag{2.49}$$

Lemma 5 *If (C1)-(C2) are satisfied, then (2.46) admits a nontrivial bounded solutions $\widetilde{x}(t)$ on $\mathbb{R}$ which satisfy the following equations*

$$\widetilde{x}^{n-1}(t) = \frac{1}{\int_{-\infty}^t \Psi(t, s)(n-1)q(s)ds + \sum_{\theta_i < t} \Psi(t, \theta_i+)d_i}, \ \ \textit{if } \gamma < 0,$$

$$\widetilde{x}^{n-1}(t) = -\frac{1}{\int_t^{\infty} \Psi(t, s)(n-1)q(s)ds - \sum_{t \leq \theta_i < \infty} \Psi(t, \theta_i+)d_i}, \ \ \textit{if } \gamma > 0.$$

Proof. Consider $\gamma < 0$. It suffices to show that $\widetilde{y}(t) = \int_{-\infty}^t \Psi(t, s)(n-1)q(s)ds + \sum_{\theta_i < t} \Psi(t, \theta_i+)d_i$ is a bounded solution of (2.48). Let us verify that $\widetilde{y}(t)$ satisfies

(2.48).

$$\begin{aligned}\dot{\widetilde{y}}(t) &= (1-n)p(t)\int_{-\infty}^{t}\Psi(t,s)(n-1)q(s)ds \\ &+ (n-1)\Psi(t,t)q(t) + (1-n)p(t)\sum_{\theta_i<t}\Psi(t,\theta_i+)d_i \\ &= (1-n)p(t)\left\{\int_{-\infty}^{t}\Psi(t,s)(n-1)q(s)ds + \sum_{\theta_i<t}\Psi(t,\theta_i+)d_i\right\} \\ &+ (n-1)q(t) \\ &= (1-n)p(t)\widetilde{y}(t) + (n-1)q(t).\end{aligned}$$

To show that $\widetilde{y}(t)$ satisfies the equation jumps, we note for fixed j it is true that $\Psi(\theta_j+,s) - \Psi(\theta_j,s) = (c_j-1)\Psi(\theta_j,s)$. Thus, $\Psi(\theta_j+,s) = c_j\Psi(\theta_j,s)$.

$$\begin{aligned}&\Delta\widetilde{y}(t)|_{t=\theta_j} = \widetilde{y}(\theta_j+) - \widetilde{y}(\theta_j) \\ &= \int_{-\infty}^{\theta_j+}\Psi(\theta_j+,s)(n-1)q(s)ds + \sum_{\theta_i<\theta_j+}\Psi(\theta_j+,\theta_j+)d_j \\ &- \int_{-\infty}^{\theta_j}\Psi(\theta_j,s)(n-1)q(s)ds - \sum_{\theta_i<\theta_j}\Psi(\theta_j,\theta_j+)d_j \\ &= c_j\int_{-\infty}^{\theta_j}\Psi(\theta_j,s)(n-1)q(s)ds - \int_{-\infty}^{\theta_j}\Psi(\theta_j,s)(n-1)q(s)ds + d_j \\ &+ \sum_{\theta_i<\theta_j}c_j\Psi(\theta_j,\theta_j+)d_j - \sum_{\theta_i<\theta_j}\Psi(\theta_j,\theta_j+)d_j \\ &= (c_j-1)\left\{\int_{-\infty}^{\theta_j}\Psi(\theta_j,s)(n-1)q(s)ds + \sum_{\theta_i<\theta_j}\Psi(\theta_j,\theta_j+)d_j\right\} + d_j \\ &= (c_j-1)\widetilde{y}(\theta_j) + d_j.\end{aligned}$$

Next, we show that $\widetilde{y}(t)$ is bounded and separated from zero.

$$\begin{aligned}&0 < \frac{mk(n-1)}{-\gamma} \le ||\widetilde{y}(t)|| \le \frac{MK(n-1)}{-\gamma} + LK\sum_{\theta_i<t}e^{\gamma(t-\theta_i)} \\ &\le \frac{MK(n-1)}{-\gamma} + LK\sum_{i=0}^{\infty}e^{i\gamma\underline{\theta}} = K\left(\frac{M(n-1)}{-\gamma} + \frac{L}{1-e^{\gamma\underline{\theta}}}\right) < \infty.\end{aligned}$$

Now consider $\gamma > 0$. Similarly, it suffices to show that $\widetilde{y}(t) = -\int_t^{\infty}\Psi(t,s)(n-1)q(s)ds - \sum_{t\le\theta_i<\infty}\Psi(t,\theta_i+)d_i$ is a bounded solution of (2.48). Let us verify that

$\widetilde{y}(t)$ satisfies (2.48).

$$\begin{aligned}
\dot{\widetilde{y}}(t) &= -(1-n)p(t)\int_t^{\infty}\Psi(t,s)(n-1)q(s)ds + (n-1)\Psi(t,t)q(t) \\
&- (1-n)p(t)\sum_{t\le\theta_i<\infty}\Psi(t,\theta_i+)d_i \\
&= (1-n)p(t)\left\{-\int_t^{\infty}\Psi(t,s)(n-1)q(s)ds - \sum_{t\le\theta_i<\infty}\Psi(t,\theta_i+)d_i\right\} \\
&+ (n-1)q(t) \\
&= (1-n)p(t)\widetilde{y}(t) + (n-1)q(t).
\end{aligned}$$

To show that $\widetilde{y}(t)$ satisfies the equation of jumps, we note for fixed j, it is true that $\Psi(\theta_j+,s)-\Psi(\theta_j,s) = (c_j-1)\Psi(\theta_j,s)$. Thus, $\Psi(\theta_j+,s) = c_j\Psi(\theta_j,s)$.

$$\begin{aligned}
&\Delta\widetilde{y}(t)|_{t=\theta_j} = \widetilde{y}(\theta_j+) - \widetilde{y}(\theta_j) \\
&= -\int_{\theta_j+}^{\infty}\Psi(\theta_j+,s)(n-1)q(s)ds - \sum_{\theta_j+\le\theta_i}\Psi(\theta_j+,\theta_j+)d_j \\
&+ \int_{\theta_j}^{\infty}\Psi(\theta_j,s)(n-1)q(s)ds + \sum_{\theta_j\le\theta_i}\Psi(\theta_j,\theta_j+)d_j \\
&= -c_j\int_{\theta_j}^{\infty}\Psi(\theta_j,s)(n-1)q(s)ds + \int_{\theta_j}^{\infty}\Psi(\theta_j,s)(n-1)q(s)ds + d_j \\
&- \sum_{\theta_j\le\theta_i}c_j\Psi(\theta_j,\theta_j+)d_j + \sum_{\theta_j\le\theta_i}\Psi(\theta_j,\theta_j+)d_j \\
&= (c_j-1)\left\{-\int_{\theta_j}^{\infty}\Psi(\theta_j,s)(n-1)q(s)ds - \sum_{\theta_j\le\theta_i}\Psi(\theta_j,\theta_j+)d_j\right\} + d_j \\
&= (c_j-1)\widetilde{y}(\theta_j) + d_j.
\end{aligned}$$

Next, we show that $\widetilde{y}(t)$ is bounded and separated from zero.

$$\begin{aligned}
&0 < \frac{mk(n-1)}{\gamma} \le ||\widetilde{y}(t)|| \le \frac{MK(n-1)}{\gamma} + LK\sum_{t\le\theta_i}e^{\gamma(t-\theta_i)} \\
&\le \frac{MK(n-1)}{\gamma} + LK\sum_{i=0}^{\infty}e^{-i\gamma\bar{\theta}} = K\left(\frac{M(n-1)}{\gamma} + \frac{L}{1-e^{\gamma\bar{\theta}}}\right) < \infty.
\end{aligned}$$

Therefore, $\widetilde{y}(t)$ is bounded and by (C1) it is separated from zero. The lemma is proved. □

Finally, let us show that $\widetilde{y}(t)$ is a unique solution of (2.48). Assume on the contrary, that there exists bounded solution $y_1(t)$ different from $\widetilde{y}(t)$. Then, $w_0(t) =: \widetilde{y}(t) - y_1(t)$ is a bounded solution of the following linear impulsive system

$$\begin{aligned}
\dot{w} &= (1-n)p(t)y, \\
\Delta w|_{t=\theta_i} &= (c_i-1)w.
\end{aligned} \tag{2.50}$$

But (2.50) admits only the trivial solution bounded on $\mathbb{R}$. Therefore, $\widetilde{y}(t) = y_1(t)$ and it implies that all bounded solutions which is different from zero have to satisfy the following equations,
$\widetilde{x}^{1-n}(t) = \int_{-\infty}^{t} \Psi(t,s)(n-1)q(s)ds + \sum_{\theta_i<t} \Psi(t,\theta_i+)d_i,$ if $\gamma < 0$,
$\widetilde{x}^{1-n}(t) = -\int_{t}^{\infty} \Psi(t,s)(n-1)q(s)ds - \sum_{t\le\theta_i<\infty} \Psi(t,\theta_i+)d_i,$ if $\gamma > 0$.

Thus, one can see that if n is even then, there is unique nontrivial bounded solution. If n is odd then, there are two nontrivial bounded solutions. In what follows, we have different bifurcation scenarios depending on the parity of n. In the next sections we deal with pitchfork and transcritical bifurcations respectively.

2.2.3 The Pitchfork Bifurcation

Consider (2.46) for $n = 2m+1$. That is,

$$\begin{aligned} x' &= p(t)x - q(t)x^{2m+1}, \\ \Delta x|_{t=\theta_i} &= -x + \frac{x}{(c_i+d_i x^{2m})^{\frac{1}{2m}}}, \end{aligned} \tag{2.51}$$

where $p, q \in PC(\mathbb{R}, \theta)$, $m \in \mathbb{N}$ and $c_i, d_i \in \mathbb{R}^+$ for all $i \in \mathbb{Z}$.

Theorem 13 *Suppose that (C1)-(C2) are fulfilled for (2.51). Then, for $\gamma > 0$ the trivial solution is asymptotically pullback and forward stable whereas the nontrivial bounded solutions $\widetilde{x}(t)$ are asymptotically unstable, and for $\gamma < 0$ the trivial solution is asymptotically unstable and the nontrivial bounded solutions are asymptotically pullback and forward stable.*

Proof. One can find that the solution of (2.51) satisfy the following equation, [1, 84, 121],

$$x^{2m}(t,t_0,x_0) = \frac{1}{\Psi(t,t_0)x_0^{-2m} + 2\int_{t_0}^{t} \Psi(t,s)mq(s)ds + \sum_{t_0\le\theta_i<t} \Psi(t,\theta_i+)d_i}. \tag{2.52}$$

In the previous section we have shown that (2.51) admits the trivial solution and two bounded solutions which satisfy the following equations

$$\widetilde{x}(t)=\begin{cases}\pm\left(\dfrac{1}{2\int_{-\infty}^{t}\Psi(t,s)mq(s)ds+\sum_{\theta_i<t}\Psi(t,\theta_i+)d_i}\right)^{\frac{1}{2m}}, & \text{if } \gamma<0\\ \pm\left(-\dfrac{1}{2\int_{t}^{\infty}\Psi(t,s)mq(s)ds+\sum_{t\le\theta_i<\infty}\Psi(t,\theta_i+)d_i}\right)^{\frac{1}{2m}}, & \text{if } \gamma>0\end{cases}.$$

One can see that asymptotic behavior of (2.51) depends on γ. We start with the case $\gamma > 0$. From (2.52) it follows that $x^{2m}(t, t_0, x_0) \to 0$ as $t_0 \to -\infty$ as well as $t \to \infty$. So, $x(t, t_0, x_0) \to 0$ as $t_0 \to -\infty$ and as $t \to \infty$, replying that all solutions are attracted both forwards and pullback to the point $\{0\}$.

To show that the nontrivial bounded solutions $\widetilde{x}(t)$ are asymptotically unstable notice that

$$x^{-2m}(t) - \widetilde{x}^{-2m}(t) = \Psi(t, t_0)\left(x_0^{-2m} - \widetilde{x}^{-2m}(t_0)\right). \tag{2.53}$$

From the last expression it follows that $x(t)$ converges to $\widetilde{x}(t)$ as $t \to -\infty$ whenever $\|x_0\| < \|\widetilde{x}(t_0)\|$.

If $\gamma < 0$, we notice that the expression (2.53) holds. Thus, one can see that $x(t)$ converges to $\widetilde{x}(t)$ both forward and pullback whenever $\|x_0\| < \|\widetilde{x}(t_0)\|$. To show that the origin is asymptotically unstable we rewrite the expression (2.53) as follows.

$$x^{2m}(t) = \frac{1}{\Psi(t, t_0)\left(x_0^{-2m} - \widetilde{x}^{-2m}(t_0)\right) + \widetilde{x}^{-2m}(t)},$$

which implies that $x(t)$ converges to 0 as $t \to -\infty$ whenever $\|x_0\| < \|\widetilde{x}(t_0)\|$. The theorem is proved. □

Remark 2 *In the similar manner, it can be easily shown that the results of Theorem 13 hold for the following system.*

$$\begin{aligned} x' &= p(t)x - q(t)x^{2m+1}, \\ \Delta x|_{t=\theta_i} &= -x - \tfrac{x}{(c_i + d_i x^{2m})^{\frac{1}{2m}}}. \end{aligned}$$

We obtain the similar results for the following equation.

$$x' = p(t)x - q(t)x^{2m+1}, \tag{2.54}$$

where p, q are continuous functions. For this particular case we have that $\gamma = \limsup\limits_{t-s\to\infty} \frac{\int_s^t (1-n)p(u)du}{t-s}$. If (C1) satisfied, one can show that the nontrivial bounded solutions satisfy the following equations,

$$\widetilde{x}(t) = \begin{cases} \pm \left(\dfrac{1}{2\int_{-\infty}^{t} \Psi(t,s) m q(s) ds} \right)^{\frac{1}{2m}}, & \text{if } \gamma < 0 \\ \pm \left(-\dfrac{1}{2\int_{t}^{\infty} \Psi(t,s) m q(s) ds} \right)^{\frac{1}{2m}}, & \text{if } \gamma > 0 \end{cases}.$$

Theorem 14 *Suppose that (C1) is fulfilled for (2.54). Then, for $\gamma > 0$ the trivial solution is asymptotically pullback and forward stable whereas the nontrivial bounded solutions $\widetilde{x}(t)$ are asymptotically unstable, and for $\gamma < 0$ the trivial solution is asymptotically unstable and the nontrivial bounded solutions are asymptotically pullback and forward stable.*

We omit the proof since it is the similar to that of Theorem 13.

2.2.4 The Transcritical Bifurcation

In this section we consider (2.46) for $n = 2m$. That is,

$$\begin{aligned} x' &= p(t)x - q(t)x^{2m}, \\ \Delta x|_{t=\theta_i} &= -x + \tfrac{x}{(c_i + d_i x^{2m-1})^{\frac{1}{2m-1}}}, \end{aligned} \tag{2.55}$$

where $c_i, d_i \in \mathbb{R}^+, i \in \mathbb{Z}, p, q \in PC(\mathbb{R}, \theta)$.

Theorem 15 *Suppose that (C1)-(C2) are fulfilled for (2.55). Then, for $\gamma > 0$ the trivial solution is asymptotically forward and pullback stable, and for $\gamma < 0$ the trivial solution is asymptotically unstable and the nontrivial bounded solution is forward and pullback stable.*

Proof. One can find that the solution of (2.55) satisfy the following equation, [1, 84, 121],

$$x^{2m-1}(t, t_0, x_0) = \frac{1}{\Psi(t,t_0)x_0^{-2m+1} + \int\limits_{t_0}^{t} \Psi(t,s)(2m-1)q(s)ds + \sum\limits_{t_0 \le \theta_i < t} \Psi(t, \theta_i+)d_i}. \tag{2.56}$$

In previous section we have shown that (2.55) admits the trivial solution and the non-trivial bounded solution which satisfy the following equations

$$\widetilde{x}(t) = \begin{cases} \left(\dfrac{1}{\int_{-\infty}^{t} \Psi(t,s)(2m-1)q(s)ds + \sum_{\theta_i<t} \Psi(t,\theta_i+)d_i}\right)^{\frac{1}{2m-1}}, & \text{if } \gamma < 0 \\ \left(-\dfrac{1}{\int_{t}^{\infty} \Psi(t,s)(2m-1)q(s)ds + \sum_{t\le\theta_i<\infty} \Psi(t,\theta_i+)d_i}\right)^{\frac{1}{2m-1}}, & \text{if } \gamma > 0 \end{cases}.$$

As in the previous section, it is clear that asymptotic behavior of (2.55) depends on the sign of γ. Consider the case $\gamma > 0$. From the equation (2.56) it follows that $x(t,t_0,x_0) \to 0$ as $t_0 \to -\infty$ and as $t \to \infty$ as long as $x(\xi,t_0,x_0)$ exists for all $\xi \in [t_0,t]$. If $x_0 > 0$, observe that

$$\Psi(\xi,t_0)x_0^{-2m+1} + \int_{t_0}^{\xi} \Psi(\xi,s)(2m-1)q(s)ds + \sum_{t_0\le\theta_i<\xi} \Psi(\xi,\theta_i+)d_i > 0,$$

for $\xi \in [t_0,t]$. Thus, $x(\xi,t_0,x_0)$ exists for all $\xi \in [t_0,t]$ and does not blow up as $t_0 \to -\infty$ and as $t \to \infty$.

If $x_0 < 0$, to ensure the existence of the solution $x(\xi,t_0,x_0)$ it is sufficient to show that

$$\Psi(\xi,t_0)x_0^{-2m+1} + \int_{t_0}^{\xi} \Psi(\xi,s)(2m-1)q(s)ds + \sum_{t_0\le\theta_i<\xi} \Psi(\xi,\theta_i+)d_i < 0,$$

for $\xi \in [t_0,t]$. Since $\int_{t_0}^{\xi} \Psi(\xi,s)(2m-1)q(s)ds + \sum_{t_0\le\theta_i<\xi} \Psi(\xi,\theta_i+)d_i > 0$, we require

$$|x_0| < \left(\frac{\Psi(\xi,t_0)}{\int_{t_0}^{\xi} \Psi(\xi,s)(2m-1)q(s)ds + \sum_{t_0\le\theta_i<\xi} \Psi(\xi,\theta_i+)d_i}\right)^{\frac{1}{2m-1}}.$$

However, we need to show that right-hand side of the last inequality is bounded from below. One can find that

$$\begin{aligned} &\frac{\Psi(\xi,t_0)}{\int_{t_0}^{\xi} \Psi(\xi,s)(2m-1)q(s)ds + \sum_{t_0\le\theta_i<\xi} \Psi(\xi,\theta_i+)d_i} \\ =\ &\frac{1}{-\widetilde{x}^{-2m+1}(t_0) + \Psi^{-1}(\xi,t_0)\widetilde{x}^{-2m+1}(t)}. \end{aligned}$$

It is easy to see that the last expression is bounded from below since $\widetilde{x}(t)$ is bounded and $\Psi^{-1}(\xi,t_0)$ is bounded for small enough t_0 or for large enough ξ.

Finally, we consider the case $\gamma < 0$. To show that the trivial solution is asymptotically unstable notice that

$$x^{2m-1}(t) = \frac{1}{\Psi(t,t_0)\left(x_0^{-2m+1} - \widetilde{x}^{-2m+1}(t_0)\right) + \widetilde{x}^{-2m+1}(t)}. \tag{2.57}$$

From the last expression it follows that $x(t)$ converges to 0 as $t \to -\infty$ for all $0 < x_0 < \widetilde{x}(t_0)$.

It remains to show that $\widetilde{x}(t)$ is forward and pullback stable. If $x_0 > 0$, then it is clear that

$$\Psi(\xi,t_0)x_0^{-2m+1} + \int_{t_0}^{\xi} \Psi(\xi,s)(2m-1)q(s)ds + \sum_{t_0 \le \theta_i < \xi} \Psi(\xi,\theta_i+)d_i > 0,$$

for $\xi \in [t_0,t]$. Thus, the solution $x(\xi,t_0,x_0)$ exists for all $\xi \in [t_0,t]$ and (2.57) implies that $\widetilde{x}(t)$ is forward and pullback stable for all $0 < x_0 < \widetilde{x}(t_0)$.

If $x_0 < 0$, then to ensure the existence of the solution $x(\xi,t_0,x_0)$ it is sufficient to show that

$$\Psi(\xi,t_0)x_0^{-2m+1} + \int_{t_0}^{\xi} \Psi(\xi,s)(2m-1)q(s)ds + \sum_{t_0 \le \theta_i < \xi} \Psi(\xi,\theta_i+)d_i < 0,$$

for $\xi \in [t_0,t]$. Since $\int_{t_0}^{\xi} \Psi(\xi,s)(2m-1)q(s)ds + \sum_{t_0 \le \theta_i < \xi} \Psi(\xi,\theta_i+)d_i > 0$, we require

$$|x_0| < \left(\frac{\Psi(\xi,t_0)}{\int_{t_0}^{\xi} \Psi(\xi,s)(2m-1)q(s)ds + \sum_{t_0 \le \theta_i < \xi} \Psi(\xi,\theta_i+)d_i} \right)^{\frac{1}{2m-1}}.$$

The right-hand side of the last inequality is bounded from below because the following relations is holds.

$$\frac{\Psi(\xi,t_0)}{\int_{t_0}^{\xi} \Psi(\xi,s)(2m-1)q(s)ds + \sum_{t_0 \le \theta_i < \xi} \Psi(\xi,\theta_i+)d_i} = \frac{1}{-\widetilde{x}^{-2m+1}(t_0) + \Psi^{-1}(\xi,t_0)\widetilde{x}^{-2m+1}(t)}.$$

The theorem is proved. □

Remark 3 *In the previous section, we have considered (2.55) for $m = 1$ and required asymptotic positivity for $q(t)$ and d_i instead of the conditions (C1)-(C2). Namely, we assumed that there exist positive constants $\bar{q}$ and $\bar{d}$ such that $q(t) \ge \bar{q}$ for all $t \le T$,* *and $d_i \geq \bar{d}$ for all $\theta_i \leq T$. Moreover, to ensure the existence of solution we required the balance conditions. That is to say we assumed that there exists $\gamma_0 > 0$ such that $0 < m < \widetilde{x}(t) < M$, for all $0 < \gamma < \gamma_0$, and $\liminf\limits_{t_0 \to -\infty} \frac{\Psi(t,t_0)}{\int_{t_0}^t \Psi(t,s)(2m-1)q(s)ds + \sum_{t_0 \leq \theta_i < t} \Psi(t,\theta_i+)d_i} \geq m > 0$ for all $-\gamma_0 < \gamma < 0$ hold. However, in the present section we show that the conditions (C1)-(C2) are enough to ensure the existence of the solution.*

Finally, we state the similar results for the following equation.

$$x' = p(t)x - q(t)x^{2m}, \tag{2.58}$$

where p, q are continuous functions. For this particular case we have that $\gamma = \limsup\limits_{t-s \to \infty} \frac{\int_s^t (1-n)p(u)du}{t-s}$. If (C1) satisfied, one can show that the nontrivial bounded solutions satisfy the following equations,

$$\widetilde{x}(t) = \begin{cases} \left(\dfrac{1}{\int_{-\infty}^t \Psi(t,s)(2m-1)q(s)ds} \right)^{\frac{1}{2m-1}}, & \text{if } \gamma < 0 \\ \left(-\dfrac{1}{\int_t^{\infty} \Psi(t,s)(2m-1)q(s)ds} \right)^{\frac{1}{2m-1}}, & \text{if } \gamma > 0 \end{cases}.$$

Theorem 16 *Suppose that (C1) is fulfilled for (2.58). Then, for $\gamma > 0$ the trivial solution is asymptotically forward and pullback stable, and for $\gamma < 0$ the trivial solution is asymptotically unstable and the nontrivial bounded solution is forward and pullback stable.*

We omit the proof since it is the similar to that of Theorem 15.

2.2.5 Illustrative Examples

In this section, to illustrate theoretical results of Theorem 13 we consider two examples.

Example 3 *Let us consider the following system,*

$$\begin{aligned} x' &= (6 + 2.5\sin(t^2))x - (18 + 3.5\cos(1 + \tfrac{t^2}{5}))x^7, \\ \Delta x|_{t=i} &= -x + \frac{x}{\left(\frac{i^2}{i^2+3} + \frac{10x^6}{1+i^2}\right)^{\frac{1}{6}}}, \end{aligned} \tag{2.59}$$

where $p(t) = 6 + 2.5\sin(t^2)$, $q(t) = 18 + 3.5\cos(1 + \frac{t^2}{5})$, $\theta_i = i, i \in \mathbb{N}$, $c_i = \frac{i^2}{i^2+3}$, $d_i = \frac{10}{1+i^2}$ *and* $n = 7$. *We check that all conditions of Theorem 13 are satisfied with* $m = 14.5$, $M = 21.5$ *and* $L = 5$. *However, we do not verify all consequences of Theorem 13. One can see that* $\gamma = \limsup_{t-s\to\infty} \frac{\int_s^t (-36-15\sin(u^2))du + \sum_{s\le i\le t} \ln \frac{i^2}{i^2+3}}{t-s} < 0$. *Thus, Theorem 13 guarantees that (2.59) has nontrivial bounded solutions which satisfy equations* $\widetilde{x}(t) = \pm \left(\frac{1}{6\int_{\infty}^{t} \Psi(t,s)(18+3.5\cos(1+\frac{s^2}{5}))ds + \sum_{s\le i<\xi} \Psi(\xi,i+)\frac{10}{1+i^2}} \right)^{\frac{1}{6}}$, *where* $\Psi(t,s) = e^{-6\int_s^t (6+2.5\sin(s^2))ds} \prod_{s\le i\le t} \frac{i^2}{i^2+3}$. *Figure 2.2 reveals that all solutions starting near the origin diverge from the origin and converge to the nontrivial bounded solutions* $\pm\widetilde{x}(t)$. *Therefore, the origin is asymptotically unstable and the bounded solutions are forward and pullback stable as expressed in the numerical simulations. Moreover, from the simulations it is seen that the nontrivial bounded solution satisfy the inequality* $0.6 \le ||\widetilde{x}(t)|| \le 1$.

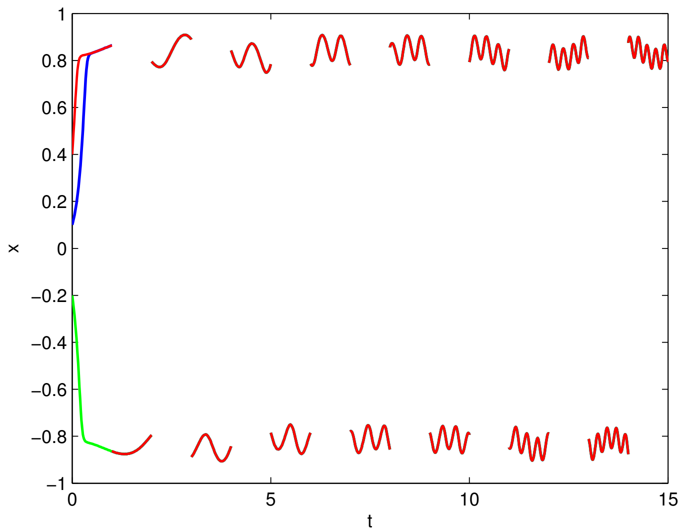

Figure 2.2: Asymptotic behavior of (2.59) for $t \in [0, 15]$. In the figure, the green color corresponds to the solution with initial value $x_0 = -0.2$, the blue color corresponds to the solution with initial value $x_0 = 0.1$ and the red color corresponds to the solution with initial value $x_0 = 0.4$. One can see that all solutions which start in the neighborhood of the origin diverge from the origin and converge to the nontrivial bounded solutions $\pm\widetilde{x}(t)$, which cannot seen through the simulations.

Example 4 *We consider the following system,*

$$
\begin{aligned}
x' &= -(1.01 + \sin(5 + \tfrac{t^3}{5}))x - (0.21 + 0.2\cos(1 + \tfrac{t^2}{5}))x^7, \\
\Delta x|_{t=i} &= -x + \frac{x}{\left(\frac{i^2+3}{i^2} + \frac{10x^6}{1+i^2}\right)^{\frac{1}{6}}},
\end{aligned} \tag{2.60}
$$

where $p(t) = -1.01 - \sin(5 + \frac{t^3}{5})$, $q(t) = 0.21 + 0.2\cos(1 + \frac{t^2}{5})$, $\theta_i = i, i \in \mathbb{N}$, $c_i = \frac{i^2+3}{i^2}$, $d_i = \frac{10}{1+i^2}$ *and* $n = 7$. *We check that all conditions of Theorem 13 are satisfied with* $m = 0.01$, $M = 0.41$ *and* $L = 5$. *However, we do not verify all consequences of Theorem 13. Note that for this example* $\gamma = \limsup_{t-s\to\infty} \frac{\int_s^t (6.06+6\sin(5+\frac{u^3}{5}))du + \sum_{s\le i\le t} \ln \frac{i^2+3}{i^2}}{t-s} > 0$. *Figure 2.3 reveals that all solutions starting near the origin eventually converge to the origin. Thus, the origin is forward and pullback stable as expressed in the numerical simulation.*

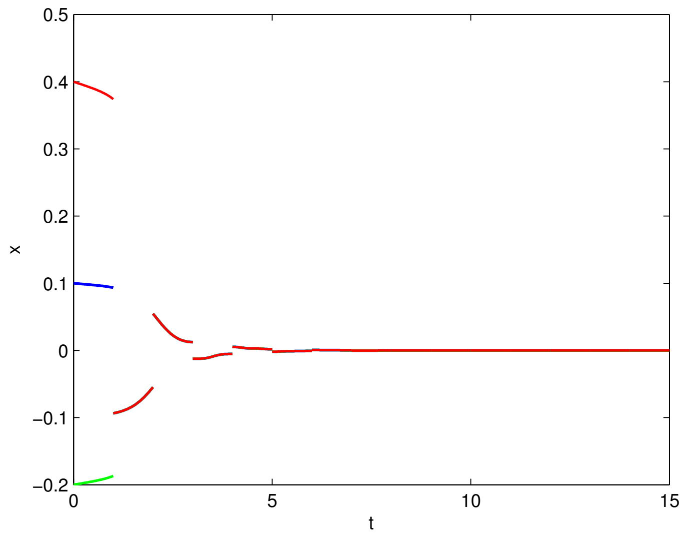

Figure 2.3: Asymptotic behavior of (2.60) for $t \in [0, 15]$. In the figure, the green color corresponds to the solution with initial value $x_0 = -0.2$, the blue color corresponds to the solution with initial value $x_0 = 0.1$ and the red color corresponds to the solution with initial value $x_0 = 0.4$. One can see that all solutions which start in the neighborhood of the origin eventually converge to the origin.

2.3 Discussion

The pitchfork and the transcritical bifurcations are considered for nonautonomous impulsive differential equations. Explicitly solvable models with the specific equations of jumps have been considered. This allowed us to categorize one-dimensional bifurcations in impulsive systems.

This theory could be developed in many ways. One can consider impulsive analogues for the pitchfork bifurcation and corresponding impulsive analogue for the transcritical bifurcation without finding explicit solution similarly to that done in [112]. Nonautonomous saddle-node bifurcation remains unconsidered even for one-dimensional impulsive systems. Finally, general theory of bifurcation in higher-dimensional systems with impulses has to developed.

In the present section, it is the first time the impulsive Bernoulli equation has been studied. It is clearly seen that in the second part of the chapter we have obtained results that are more general than those in the first section. This chapter provides new sufficient conditions guaranteeing the existence of the nontrivial bounded solutions. Moreover, both forward and pullback asymptotic behavior of the trivial and the nontrivial bounded solutions are studied. Different nonautonomous bifurcation scenarios depending on the asymptotic behavior of these solutions are obtained.

CHAPTER 3

NONAUTONOMOUS TRANSCRITICAL AND PITCHFORK BIFURCATIONS IN SCALAR NON-SOLVABLE IMPULSIVE DIFFERENTIAL EQUATIONS

3.1 Nonautonomous Transcritical and Pitchfork Bifurcations in Impulsive Systems

In this section, we consider impulsive analogues of nonautonomous transcritical and pitchfork bifurcations in the systems which cannot be solved explicitly. We extend the theorem on asymptotic properties of the quasilinear impulsive systems which was considered by Samoilenko and Perestyuk [121] to the entire time.

3.1.1 Introduction

In the previous chapter we studied nonautonomous bifurcations which are solvable by means of Bernoulli transformation. In particular, the concept of the pullback and forward attracting sets was used to analyze nonautonomous bifurcations which depend on the properties of the system in the past and future respectively. However, most of nonlinear discontinuous models are not of a Bernoulli type. In this chapter, we are concerned with the most general type of equations, which are naturally cannot be solved. Thus, we cannot expect to find a bounded or a periodic solution as in the previous chapter. Instead, we deal with asymptotic behavior of an equilibrium point. Thus, we need different approach than one in the previous chapter in order to describe bifurcation analysis.

As it was observed in pitchfork bifurcation in one-dimensional case in Introduction part of the thesis, a system undergo bifurcation if a particular solution of a differential equation gain or loss attractivity when parameter varies. Therefore, there is a strong relation between notions of attractiveness/repulsiveness of a solution and bifurcation theory. In this chapter we implement this idea to study various bifurcation scenarios and focus on systems with discontinuity. There are qualitative studies on asymptotic properties of the quasilinear impulsive systems of differential equations [1, 5, 29, 37, 84, 121]. In Chapter 2, we studied impulsive extensions of the nonautonomous pitchfork and transcritical bifurcation in the systems which are explicitly solvable models. The main novelty of this chapter is to study analogues of nonautonomous pitchfork and transcritical bifurcations in scalar impulsive systems which depend on properties of the system on entire time, i.e. past and future time. Moreover, we extend the results on stability of quasilinear systems based on first order approximation obtained by Samoilenko and Perestyuk [121] to entire time.

3.1.2 Preliminaries

We denote by $\mathbb{R}$ the set of all real numbers, $\mathbb{Z}$ the set of integers and write $\mathbb{R}_k^- := [-\infty, k)$ and $\mathbb{R}_k^+ := [k, \infty)$ for a given $k \in \mathbb{R}$. In this section we introduce concepts of attractive and repulsive solutions, which are used to analyze asymptotic behavior of impulsive systems. This section is concerned with systems of the type

$$\begin{aligned} x' &= f(t, x), \\ \Delta x|_{t=\theta_i} &= J_i(x), \end{aligned} \tag{3.1}$$

where $\Delta x|_{t=\theta_i} := x(\theta_i+) - x(\theta_i)$, $x(\theta_i+) = \lim_{t\to\theta_i^+} x(t)$. The system (3.1) is defined on the set $\Omega = I \times \mathbb{A} \times G$ where $G \subseteq \mathbb{R}^n$, I is the interval of the form $I = \mathbb{R}$, $I = \mathbb{R}_k^-$ or $I = \mathbb{R}_k^+$, respectively. θ is a nonempty sequence with the set of indexes $\mathbb{A}$ such that $|\theta_i| \to \infty$ as $|i| \to \infty$. Let $\phi(t, t_0, x_0)$ be solution of (3.1) which is continuable and unique on I.

Denote $PC(\mathbb{R}, \theta)$ space of piecewise left continuous functions with discontinuity of the first kind at points in θ. In this chapter, the Euclidean norm $||\cdot||$ and Hausdorff semi-distance between nonempty set X and Y as $d(X, Y) = \sup_{x\in X} \inf_{y\in Y} d(x, y)$ are used. For arbitrary ϵ-neighborhood of some point $x_0 \in \mathbb{R}^n$ we write $B_\epsilon(x_0) =$

$\{x \in \mathbb{R}^n : ||x - x_0|| < \epsilon\}$ and for arbitrary nonempty set $X \subset \mathbb{R}^n$ we define $\phi(t, t_0, X) := \bigcup_{x_0 \in X} \phi(t, t_0, x_0)$. A graph of function $g : A \to B$ is defined as $graphg = \{(a, b) \in A \times B : g(a) = b\}$.

A set $\mathcal{N} \subset I \times \mathbb{R}^n$ is called nanautonomous set if the set $\mathcal{N}(t) := \{x \in \mathbb{R}^n : (t, x) \in \mathcal{N}\}$, called as $t-$fibers, is not empty for all $t \in I$. $\mathcal{N}$ is said to be compact if all $t-$fibers are compact and $\mathcal{N}$ is said to be invariant if it satisfies $\phi(t, t_0, \mathcal{N}(t_0)) = \mathcal{N}(t)$ for all $t, t_0 \in I$.

Asymptotic properties of discontinuous dynamics and continuous one are the same. In what follows, we use definitions of attractivity and repulsivity without any changes form [112, 113].

Definition 9 *[112, 113] Let $\psi : \mathbb{R} \to \mathbb{R}^n$ be a solution of the system (3.1).*

- *A compact and invariant nonautonomous set $\mathscr{G}$ is all-time attractor if there exists an $\epsilon > 0$ such that*

$$\lim_{t\to\infty} \sup_{t_0 \in \mathbb{R}} d(\phi(t + t_0, t_0, B_\epsilon(\mathscr{G}(t_0))), \mathscr{G}(t + t_0)) = 0.$$

  All-time attraction radius of $\mathscr{G}$, denoted by $\mathcal{A}^{\pm}_{\mathscr{G}}$, is the supremum of all positive ϵ which satisfy the above relation;

- *If $graph\psi$ is an all-time attractor then $\psi(t)$ is called all-time attractive;*

- *A compact and invariant nonautonomous set $\mathscr{H}$ is all-time repeller if there exists an $\epsilon > 0$ such that*

$$\lim_{t\to\infty} \sup_{t_0 \in \mathbb{R}} d(\phi(t_0 - t, t_0, B_\epsilon(\mathscr{H}(t_0))), \mathscr{H}(t_0 - t)) = 0.$$

  All-time repulsion radius of $\mathscr{H}$, denoted by $\mathcal{R}^{\pm}_{\mathscr{H}}$, is the supremum of all positive ϵ which satisfy the above relation;

- *If $graph\psi$ is an all-time repeller then ψ is called all-time repulsive.*

In Chapter 2 we studied nonautonomous bifurcation patterns in the pullback and forward sense. In what follows to examine asymptotic analysis of the systems that depend in the past we define past attractivity and repulsivity. One can confirm that a past attractor is a local pullback attractor, defined in the previous chapter [48].

Definition 10 *[112, 113] Let $\psi : \mathbb{R}_k^- \to \mathbb{R}^n$ be a solution of the system (3.1).*

- *A compact and invariant nonautonomous set $\mathscr{G}$ is past attractor if there exists an $\epsilon > 0$ such that*

$$\lim_{t\to\infty} d(\phi(t_0, t_0 - t, B_\epsilon(\mathscr{G}(t_0 - t))), \mathscr{G}(t_0)) = 0 \text{ for all } t_0 \in \mathbb{R}_k^-.$$

  The past attraction radius of $\mathscr{G}$, denoted by $\mathcal{A}_{\mathscr{G}}^-$, is the supremum of all positive ϵ which satisfy the above relation;

- *If $graph\psi$ a past attractor then $\psi(t)$ is called past attractive;*

- *A compact and invariant nonautonomous set $\mathscr{H}$ is past repeller if there exists an $\epsilon > 0$ such that*

$$\lim_{t\to\infty} d(\phi(t_0 - t, t_0, B_\epsilon(\mathscr{H}(t_0))), \mathscr{H}(t_0 - t)) = 0 \text{ for all } t_0 \in \mathbb{R}_k^-.$$

  Past repulsion radius of $\mathscr{H}$, denoted by $\mathcal{R}_{\mathscr{H}}^-$, is the supremum of all positive ϵ such that there exists a $\widehat{k} \in \mathbb{R}_k^-$ with

$$\lim_{t\to\infty} d(\phi(t_0 - t, t_0, B_\epsilon(\mathscr{H}(t_0))), \mathscr{H}(t_0 - t)) = 0 \text{ for all } t_0 \le \widehat{k};$$

- *If $graph\psi$ is a past repeller then $\psi(t)$ is called past repulsive.*

Definition 11 *[112, 113] Let $\psi : \mathbb{R}_k^+ \to \mathbb{R}^n$ be a solution of the system (3.1).*

- *A compact and invariant nonautonomous set $\mathscr{G}$ is future attractor if there exists an $\epsilon > 0$ such that*

$$\lim_{t\to\infty} d(\phi(t + t_0, t_0, B_\epsilon(\mathscr{G}(t_0))), \mathscr{G}(t + t_0)) = 0 \text{ for all } t_0 \in \mathbb{R}_k^+.$$

  Future attraction radius of $\mathscr{G}$, denoted by $\mathcal{A}_{\mathscr{G}}^+$, is the supremum of all positive ϵ such that there exists a $\widehat{k} \in \mathbb{R}_k^+$ with

$$\lim_{t\to\infty} d(\phi(t + t_0, t_0, B_\epsilon(\mathscr{G}(t_0))), \mathscr{G}(t + t_0)) = 0 \text{ for all } t_0 \ge \widehat{k};$$

- *If $graph\psi$ is a future attractor then $\psi(t)$ is called future attractive;*

- *A compact and invariant nonautonomous set $\mathscr{H}$ is called future repeller if there exists an $\epsilon > 0$ with*

$$\lim_{t\to\infty} d(\phi(t_0, t_0 + t, B_\epsilon(\mathscr{H}(t + t_0))), \mathscr{H}(t_0)) = 0 \ \textit{for all} \ \ t_0 \in \mathbb{R}_k^+.$$

  Future repulsion radius of $\mathscr{H}$, denoted by $\mathcal{R}^+_{\mathscr{H}}$, is the supremum of all positive ϵ with the above relation;

- *If $graph\psi$ is a future repeller then $\psi(t)$ is called future repulsive.*

From definitions given above it follows that if a solution is future attractive then it is uniformly asymptotically stable. Moreover, every all-time attractor/repeller is both a past attractor/repeller and a future attractor/repeller.

3.1.3 Attraction and Repulsion in a Quasilinear Impulsive Systems

By means of definitions given above we analyze attraction and repulsion in quasilinear systems which is important in the stability analysis of solutions of nonlinear impulsive systems with fixed moments of impulses. We consider the system with interval I of the form $\mathbb{R}$, $\mathbb{R}_k^-$ or $\mathbb{R}_k^+$, respectively, and let

$$\begin{aligned} x' &= A(t)x + F(t, x), \\ \Delta x|_{t=\theta_i} &= B_i x + I_i(x), \end{aligned} \tag{3.2}$$

where $A \in PC(I, \theta)$, matrices B_i satisfy $det(B_i + I) \neq 0$, $F : I \times G \to \mathbb{R}^n$ and $I : \mathbb{A} \times G \to \mathbb{R}^n$. An infinite sequence θ_i satisfies $|\theta_i| \to \infty$ as $|i| \to \infty$. It is assumed that there exist positive constants $\underline{\theta}$ and $\overline{\theta}$ such that $\underline{\theta} \leq \theta_{i+1} - \theta_i \leq \overline{\theta}$. Denote $\phi(t, t_0, x_0)$ as the solution of (3.2) and $\Psi(t, s)$ as the fundamental matrix of the following system

$$\begin{aligned} x' &= A(t)x, \\ \Delta x|_{t=\theta_i} &= B_i x. \end{aligned} \tag{3.3}$$

Theorem 17 *If there exist $\alpha < 0$, $K \geq 1$ and $\delta > 0$ such that*

$$||\Psi(t, s)|| \leq K e^{\alpha(t-s)} \ \textit{for all} \ \ t \geq s,$$

and the functions $F(t,x)$ and $I_i(x)$ are Lipschitzian, i.e. there exists a positive number l such that

$$||F(t,x)|| \leq l||x||, \ \ ||I_i(x)|| \leq l||x|| \tag{3.4}$$

for all $t \in I, i \in \mathbb{A}$ and $||x|| < h, h > 0$. Then,

$$||\phi(t,t_0,x_0)|| \leq \delta e^{(\alpha+Kl+\frac{1}{\underline{\theta}}\ln(1+Kl))(t-t_0)} \text{ for all } t,t_0 \in I \text{ with } t \geq t_0,$$

i.e. for sufficiently small values of l, the origin is all-time attractive.

Proof. An equivalent integral equation of the system (3.2) can be written as [1, 121]:

$$\phi(t,t_0,x_0) = \Psi(t,t_0)x_0 + \int_{t_0}^{t} \Psi(t,s)F(s,\phi(s,t_0,x_0))ds + \sum_{t_0 \leq \theta_i < t} \Psi(t,\theta_i)I_i(\phi(\theta_i,t_0,x_0))$$

for all $t \geq t_0$. By using inequalities in (3.4) we get

$$||\phi(t,t_0,x_0)|| \leq Ke^{\alpha(t-t_0)}||x_0|| + \int_{t_0}^{t} Ke^{\alpha(t-s)}l||\phi(s,t_0,x_0)||ds + \sum_{t_0 \leq \theta_i < t} Ke^{\alpha(t-\theta_i)}l||\phi(\theta_i,t_0,x_0)||$$

for all $t \geq t_0$ is fulfilled. The last expression can be rewritten as

$$e^{-\alpha t}||\phi(t,t_0,x_0)|| \leq Ke^{-\alpha t_0}||x_0|| + \int_{t_0}^{t} Kle^{-\alpha s}||\phi(s,t_0,x_0)||ds + \sum_{t_0 \leq \theta_i < t} Kle^{-\alpha\theta_i}||\phi(\theta_i,t_0,x_0)||$$

for all $t \geq t_0$. Hence, by Gronwall-Bellman lemma for piecewise continuous functions ([1, 121]) it follows that

$$||\phi(t,t_0,x_0)|| \leq Ke^{(\alpha+Kl)(t-t_0)}(1+Kl)^{i[t_0,t)}||x_0|| \text{ for all } t \geq t_0. \tag{3.5}$$

By means of of the inequality $\theta_{i+1} - \theta_i \geq \underline{\theta}$ one can see that

$$||\phi(t,t_0,x_0)|| \leq Ke^{(\alpha+Kl+\frac{1}{\underline{\theta}}\ln(1+Kl))(t-t_0)}||x_0|| \text{ for all } t \geq t_0.$$

If l is small egough that

$$\alpha + Kl + \frac{1}{\underline{\theta}}\ln(1+Kl) < 0 \text{ for } \alpha < 0,$$

then the required result follows by choosing $\delta = Kh$. □

Theorem 18 *If there exist $\alpha > 0$, $K \geq 1$ and $\delta > 0$ such that*

$$||\Psi(t,s)|| \leq Ke^{\alpha(t-s)} \text{ for all } t \leq s,$$

and the functions $F(t,x)$ and $I_i(x)$ are Lipschitzian, i.e. there exists a positive number l such that

$$||F(t,x)|| \leq l||x||, \;\; ||I_i(x)|| \leq l||x|| \tag{3.6}$$

for all $t \in I, i \in \mathbb{A}$ and $||x|| < h, h > 0$. Then,

$$||\phi(t,t_0,x_0)|| \leq \delta e^{(\alpha - Kl + \frac{1}{\underline{\theta}}\ln(1-Kl))(t-t_0)} \text{ for all } t, t_0 \in I \text{ with } t \leq t_0,$$

i.e., for sufficiently small values of l, the origin is all-time repulsive.

Proof. An equivalent integral equation of the system (3.2) can be written as [1, 121]:

$$\phi(t,t_0,x_0) = \Psi(t,t_0)x_0 + \int_{t_0}^{t} \Psi(t,s)F(s,\phi(s,t_0,x_0))ds - \sum_{t \leq \theta_i < t_0} \Psi(t,\theta_i)I_i(\phi(\theta_i,t_0,x_0))$$

for all $t \leq t_0$. By using inequalities in (3.4) we get

$$||\phi(t,t_0,x_0)|| \leq Ke^{\alpha(t-t_0)}||x_0|| + \int_{t_0}^{t} Ke^{\alpha(t-s)}l||\phi(s,t_0,x_0)||ds + \sum_{t \leq \theta_i < t_0} Ke^{\alpha(t-\theta_i)}l||\phi(\theta_i,t_0,x_0)||$$

for all $t \leq t_0$ is fulfilled. The last expression can be rewritten as

$$e^{-\alpha t}||\phi(t,t_0,x_0)|| \leq Ke^{-\alpha t_0}||x_0|| + \int_{t_0}^{t} Kle^{-\alpha s}||\phi(s,t_0,x_0)||ds + \sum_{t \leq \theta_i < t_0} Kle^{-\alpha\theta_i}||\phi(\theta_i,t_0,x_0)||$$

for all $t \leq t_0$. Gronwall-Bellman lemma for piecewise continuous functions ([1, 121]) can be applied since l can be chosen such that $Kl < 1$. Thus,

$$||\phi(t,t_0,x_0)|| \leq Ke^{(\alpha-Kl)(t-t_0)}(1-Kl)^{-i[t,t_0)}||x_0|| \text{ for all } t \leq t_0. \tag{3.7}$$

By means of of the inequality $\theta_{i+1} - \theta_i \geq \underline{\theta}$ one can see that

$$||\phi(t,t_0,x_0)|| \leq Ke^{(\alpha - Kl + \frac{1}{\underline{\theta}}\ln(1-Kl))(t-t_0)}||x_0|| \text{ for all } t \leq t_0.$$

If l is small enough that

$$\alpha - Kl + \frac{1}{\underline{\theta}}\ln(1 - Kl) > 0 \ \text{ for } \ \alpha > 0,$$

then the required results follow if we choose $\delta = Kh$ since $||x_0|| < h$. $\square$

3.1.4 The Transcritical Bifurcation

In this section we study impulsive analogue of the nonautonomous transcritical bifurcation. Let $x_- < 0 < x_+$ and $\mu_- < \mu_+$ be real numbers and I be interval of the form $\mathbb{R}$, $\mathbb{R}_k^-$ or $\mathbb{R}_k^+$, respectively. Consider the system

$$\begin{aligned} x' &= p(t,\mu)x + q(t,\mu)x^2 + r(t,x,\mu), \\ \Delta x|_{t=\theta_i} &= c_i(\mu)x + d_i(\mu)x^2 + e_i(x,\mu), \end{aligned} \tag{3.8}$$

with piecewise continuous functions $p : I \times (\mu_-, \mu_+) \to \mathbb{R}, q : I \times (\mu_-, \mu_+) \to \mathbb{R}$ and $r : I \times (x_-, x_+) \times (\mu_-, \mu_+) \to \mathbb{R}$ satisfying $r(t, 0, \mu) = 0$. $c : \mathbb{A} \times (\mu_-, \mu_+) \to \mathbb{R}, d : \mathbb{A} \times (\mu_-, \mu_+) \to \mathbb{R}$ and $e : \mathbb{A} \times (x_-, x_+) \times (\mu_-, \mu_+) \to \mathbb{R}$ with $c_i(\mu) \neq -1$ and $e_i(0, \mu) = 0$. An infinite sequence θ_i satisfies $|\theta_i| \to \infty$ as $|i| \to \infty$. It is assumed that there exist positive constants $\underline{\theta}$ and $\overline{\theta}$ such that $\underline{\theta} \leq \theta_{i+1} - \theta_i \leq \overline{\theta}$. Let $\Psi_\mu(t, s)$ be the fundamental matrix of the following linear system.

$$\begin{aligned} x' &= p(t,\mu)x, \\ \Delta x|_{t=\theta_i} &= c_i(\mu)x. \end{aligned} \tag{3.9}$$

Assume that there exist $\mu_0 \in (\mu_-, \mu_+)$ such that are two functions $\alpha_1, \alpha_2 : (\mu_-, \mu_+) \to \mathbb{R}$ which are either both monotone increasing or both monotone decreasing satisfying $\lim_{\mu\to\mu_0} \alpha_1(\mu) = \lim_{\mu\to\mu_0} \alpha_2(\mu) = 0$ and $K \geq 1$ such that

$$\|\Psi_\mu(t,s)\| \leq Ke^{\alpha_1(\mu)(t-s)} \ \text{ for all } \ \mu \in (\mu_-, \mu_+) \ \text{ and } \ t, s \in I \ \text{ with } \ t \geq s,$$

$$\|\Psi_\mu(t,s)\| \leq Ke^{\alpha_2(\mu)(t-s)} \ \text{ for all } \ \mu \in (\mu_-, \mu_+) \ \text{ and } \ t, s \in I \ \text{ with } \ t \leq s.$$

The functions q and d_i satisfy one of the following conditions.

$$\begin{aligned} 0 &< \liminf_{\mu\to\mu_0} \inf_{t\in I} q(t,\mu) \leq \limsup_{\mu\to\mu_0} \sup_{t\in I} q(t,\mu) < \infty, \\ 0 &< \liminf_{\mu\to\mu_0} \inf_{i\in\mathbb{A}} d_i(\mu) \leq \limsup_{\mu\to\mu_0} \sup_{i\in\mathbb{A}} d_i(\mu) < \infty \end{aligned} \tag{3.10}$$

or

$$-\infty < \liminf_{\mu\to\mu_0} \inf_{t\in I} q(t,\mu) \le \limsup_{\mu\to\mu_0} \sup_{t\in I} q(t,\mu) < 0,$$
$$-\infty < \liminf_{\mu\to\mu_0} \inf_{i\in\mathbb{A}} d_i(\mu) \le \limsup_{\mu\to\mu_0} \sup_{i\in\mathbb{A}} d_i(\mu) < 0. \tag{3.11}$$

The functions r and e_i satisfy the following conditions.

$$\lim_{x\to 0} \sup_{\mu\in(\mu_0-|x|,\mu_0+|x|)} \sup_{t\in I} \frac{|r(t,x,\mu)|}{|x|^2} = 0,$$
$$\lim_{x\to 0} \sup_{\mu\in(\mu_0-|x|,\mu_0+|x|)} \sup_{i\in\mathbb{A}} \frac{|e_i(x,\mu)|}{|x|^2} = 0 \tag{3.12}$$

and there exists sufficiently small $l > 0$ such that

$$|r(t,x,\mu)| < l|x|, \quad |e_i(x,\mu)| < l|x|, \tag{3.13}$$

for all $\mu \in (\mu_-, \mu_+), t \in I, i \in \mathbb{A}$ and $x \in (x_-, x_+)$.

Theorem 19 *Assume that above conditions hold for the system (3.8). Then there exist $\widehat{\mu}_- < 0 < \widehat{\mu}_+$ such that*

- *If the functions α_1 and α_2 are monotone increasing, the origin is all-time attractive for $\mu \in (\widehat{\mu}_-, \mu_0)$ and all-time repulsive for $\mu \in (\mu_0, \widehat{\mu}_+)$. The following relations hold true.*

$$\lim_{\mu\to\mu_0^-} \mathcal{A}_0^\mu = 0 \;\; \textit{and} \;\; \lim_{\mu\to\mu_0^+} \mathcal{R}_0^\mu = 0; \tag{3.14}$$

- *If the functions α_1 and α_2 are monotone decreasing, the origin is all-time repulsive for $\mu \in (\widehat{\mu}_-, \mu_0)$ and all-time attractive for $\mu \in (\mu_0, \widehat{\mu}_+)$. The following relations hold true.*

$$\lim_{\mu\to\mu_0^+} \mathcal{A}_0^\mu = 0 \;\; \textit{and} \;\; \lim_{\mu\to\mu_0^-} \mathcal{R}_0^\mu = 0; \tag{3.15}$$

Hence, in both of the above cases the system (3.8) possesses an all-time bifurcation.

Proof. Let ϕ_μ be the general solution of the system (3.8). We may consider the case (3.10) since the functions α_1 and α_2 are monotone increasing. Choose $\widehat{\mu}_- < \mu_0 < \widehat{\mu}_+$ such that

$$0 < \inf_{\mu\in(\widehat{\mu}_-,\widehat{\mu}_+),t\in I} q(t,\mu) \le \sup_{\mu\in(\widehat{\mu}_-,\widehat{\mu}_+),t\in I} q(t,\mu) < \infty,$$
$$0 < \inf_{\mu\in(\widehat{\mu}_-,\widehat{\mu}_+),i\in\mathbb{A}} d_i(\mu) \le \sup_{\mu\in(\widehat{\mu}_-,\widehat{\mu}_+),i\in\mathbb{A}} d_i(\mu) < \infty. \tag{3.16}$$

By means of (3.13) and (3.16) one can see that Theorem 17 and Theorem 18 can be applied. Thus, we get attraction and repulsion of the origin as it was required to show in the theorem. It remains to show that relations (3.14) and (3.15) hold. Let us assume on the contrary that $\gamma = \limsup_{\mu \to \mu_0^-} \mathcal{A}_0^\mu > 0$. By means of (3.12) and (3.16) one can show that there exist $\widetilde{\mu} \in (\widehat{\mu}_-, \mu_0), x_0 \in (0, \gamma)$ and $J \in (0, \frac{x_0}{4K})$ such that

$$q(t,\mu)x^2 + r(t,x,\mu) > J \text{ and } d_i(\mu)x^2 + e_i(x,\mu) > J \tag{3.17}$$

for all $t \in I, i \in \mathbb{A}, \mu \in (\widetilde{\mu}_-, \mu_0)$ and $x_0 \in \left[\frac{x_0}{2K^2}, x_0\right]$. Next, fix $\widehat{\mu} \in (\widetilde{\mu}_-, \mu_0)$ such that $\mathcal{A}_0^{\widehat{\mu}} > x_0$ and $\alpha_2(\widehat{\mu}) \geq \alpha := -\frac{2KJ}{x_0} > -\frac{1}{2}$ so that $\phi_{\widehat{\mu}}(t, t_0, x_0)$ is attracted to the origin. Set $\psi(t) = \phi_{\widehat{\mu}}(t, t_0, x_0)$. Then, there exists $\tau \in I, \tau > t_0$, such that $\psi(\tau) \leq \frac{x_0}{2K^2}$. We choose minimal τ which satisfy this property. In other words, $\psi(\tau) > \frac{x_0}{2K^2}$ for all $t \in [t_0, \tau)$. Moreover, choose $t_1 \in [t_0, \tau)$ such that

$$\psi(t_1) = \frac{x_0}{2K} \text{ and } \psi(t) \in \left(\frac{x_0}{2K^2}, x_0\right] \text{ for all } t \in [t_1, \tau).$$

We write integral equation of the system (3.8) at $t = \tau$ for fixed $\widehat{\mu}$ which start at point t_1.

$$\begin{aligned}
\psi(\tau) = \Psi_{\widehat{\mu}}(\tau, t_1)\psi(t_1) + \int_{t_1}^{\tau} \Psi_{\widehat{\mu}}(\tau, s) \left(b(s, \widehat{\mu})(\psi(s))^2 + r(s, \psi(s), \widehat{\mu})\right) ds \\
+ \sum_{t_1 \leq \theta_i < \tau} \Psi_{\widehat{\mu}}(\tau, \theta_i) \left(d_i(\widehat{\mu})(\psi(\theta_i))^2 + e_i(\psi(\theta_i), \widehat{\mu})\right) \\
> \frac{x_0}{2K^2} e^{\alpha(\tau - t_1)} + \frac{J}{K} \int_{t_1}^{\tau} e^{\alpha(\tau - s)} ds \\
= e^{\alpha(\tau - t_1)} \left(\frac{x_0}{2K^2} + \frac{J}{K\alpha}\right) - \frac{J}{K\alpha} = \frac{x_0}{2K^2}
\end{aligned}$$

which is a contradiction and we arrive at $\lim_{\mu \to \mu_0^-} \mathcal{A}_0^\mu = 0$. In the similar fashion, one can show that $\lim_{\mu \to \mu_0^+} \mathcal{R}_0^\mu = 0$. We omit the proof of the second part since it be verified in the similar manner. This finalizes the proof theorem. □

3.1.5 The Pitchfork Bifurcation

In this section we study impulsive analogue of the nonautonomous pitchfork bifurcation. Let $x_- < 0 < x_+$ and $\mu_- < \mu_+$ be real numbers and I be interval of the form $\mathbb{R}$, $\mathbb{R}_k^-$ or $\mathbb{R}_k^+$, respectively. Consider the system

$$\begin{aligned}
x' &= p(t,\mu)x + q(t,\mu)x^3 + r(t,x,\mu), \\
\Delta x|_{t=\theta_i} &= c_i(\mu)x + d_i(\mu)x^3 + e_i(x,\mu),
\end{aligned} \tag{3.18}$$

with piecewise continuous functions $p : I \times (\mu_-, \mu_+) \to \mathbb{R}, q : I \times (\mu_-, \mu_+) \to \mathbb{R}$ and $r : I \times (x_-, x_+) \times (\mu_-, \mu_+) \to \mathbb{R}$ satisfying $r(t, 0, \mu) = 0$. $c : \mathbb{A} \times (\mu_-, \mu_+) \to \mathbb{R}, d : \mathbb{A} \times (\mu_-, \mu_+) \to \mathbb{R}$ and $e : \mathbb{A} \times (x_-, x_+) \times (\mu_-, \mu_+) \to \mathbb{R}$ with $c_i(\mu) \neq -1$ and $e_i(0, \mu) = 0$. An infinite sequence θ_i satisfies $|\theta_i| \to \infty$ as $|i| \to \infty$. It is assumed that there exist positive constants $\underline{\theta}$ and $\overline{\theta}$ such that $\underline{\theta} \leq \theta_{i+1} - \theta_i \leq \overline{\theta}$. Let $\Psi_\mu(t, s)$ be the fundamental matrix of the following linear system.

$$\begin{aligned} x' &= p(t, \mu)x, \\ \Delta x|_{t=\theta_i} &= c_i(\mu)x. \end{aligned} \tag{3.19}$$

Assume that there exist $\mu_0 \in (\mu_-, \mu_+)$ such that are two functions $\alpha_1, \alpha_2 : (\mu_-, \mu_+) \to \mathbb{R}$ which are either both monotone increasing or both monotone decreasing satisfying $\lim_{\mu \to \mu_0} \alpha_1(\mu) = \lim_{\mu \to \mu_0} \alpha_2(\mu) = 0$ and $K \geq 1$ such that

$$\|\Psi_\mu(t, s)\| \leq K e^{\alpha_1(\mu)(t-s)} \text{ for all } \mu \in (\mu_-, \mu_+) \text{ and } t, s \in I \text{ with } t \geq s,$$

$$\|\Psi_\mu(t, s)\| \leq K e^{\alpha_2(\mu)(t-s)} \text{ for all } \mu \in (\mu_-, \mu_+) \text{ and } t, s \in I \text{ with } t \leq s.$$

The functions q and d_i satisfy one of the following conditions.

$$\begin{aligned} 0 < \liminf_{\mu \to \mu_0} \inf_{t \in I} q(t, \mu) \leq \limsup_{\mu \to \mu_0} \sup_{t \in I} q(t, \mu) < \infty, \\ 0 < \liminf_{\mu \to \mu_0} \inf_{i \in \mathbb{A}} d_i(\mu) \leq \limsup_{\mu \to \mu_0} \sup_{i \in \mathbb{A}} d_i(\mu) < \infty \end{aligned} \tag{3.20}$$

or

$$\begin{aligned} -\infty < \liminf_{\mu \to \mu_0} \inf_{t \in I} q(t, \mu) \leq \limsup_{\mu \to \mu_0} \sup_{t \in I} q(t, \mu) < 0, \\ -\infty < \liminf_{\mu \to \mu_0} \inf_{i \in \mathbb{A}} d_i(\mu) \leq \limsup_{\mu \to \mu_0} \sup_{i \in \mathbb{A}} d_i(\mu) < 0. \end{aligned} \tag{3.21}$$

The functions r and r_i satisfy the following conditions.

$$\begin{aligned} \lim_{x \to 0} \sup_{\mu \in (\mu_0 - x^2, \mu_0 + x^2)} \sup_{t \in I} \frac{|r(t, x, \mu)|}{|x|^3} = 0, \\ \lim_{x \to 0} \sup_{\mu \in (\mu_0 - x^2, \mu_0 + x^2)} \sup_{i \in \mathbb{A}} \frac{|e_i(x, \mu)|}{|x|^3} = 0 \end{aligned} \tag{3.22}$$

and there exists sufficiently small $l > 0$ such that

$$|r(t, x, \mu)| < l|x|, \quad |e_i(x, \mu)| < l|x|, \tag{3.23}$$

for all $\mu \in (\mu_-, \mu_+), t \in I, i \in \mathbb{A}$ and $x \in (x_-, x_+)$.

Theorem 20 *Assume that above conditions hold for the system (3.18). Then there exist $\widehat{\mu}_- < 0 < \widehat{\mu}_+$ such that*

- *If the functions α_1 and α_2 are monotone increasing, the origin is all-time attractive for $\mu \in (\widehat{\mu}_-, \mu_0)$ and all-time repulsive for $\mu \in (\mu_0, \widehat{\mu}_+)$. The following relations hold true.*

$$\lim_{\mu \to \mu_0^-} \mathcal{A}_0^\mu = 0 \text{ and } \lim_{\mu \to \mu_0^+} \mathcal{R}_0^\mu = 0;$$

- *If the functions α_1 and α_2 are monotone decreasing, the origin is all-time repulsive for $\mu \in (\widehat{\mu}_-, \mu_0)$ and all-time attractive for $\mu \in (\mu_0, \widehat{\mu}_+)$. The following relations hold true.*

$$\lim_{\mu \to \mu_0^+} \mathcal{A}_0^\mu = 0 \text{ and } \lim_{\mu \to \mu_0^-} \mathcal{R}_0^\mu = 0;$$

Hence, in both of the above cases the system (3.18) possesses an all-time bifurcation.

The proof of the theorem is similar to that of Theorem 19.

3.2 Finite-Time Nonautonomous Bifurcations in Impulsive Systems

The purpose of this section is to investigate nonautonomous bifurcation in impulsive differential equations in the finite-time interval. The impulsive finite-time analogues of transcritical and pitchfork bifurcation are provided. An illustrative example is given with numerical simulations which supports theoretical results.

3.2.1 Introduction

There are qualitative papers devoted to nonautonomous bifurcation theory in continuous dynamical systems studied in the last twenty years [75, 77, 86, 88, 112, 113, 114]. In considering application problems which arise in the real world such as ocean or atmosphere dynamics [101], transport problems in fluid or any model in biological applications [33, 115] one come across with finite dynamics. There are several reasons why bounded set dynamics is of a great importance. The first and the most simple reason is that data obtained from measurements and observations is often given in a compact time interval. Another reason may be the interest in transient behavior of solutions regardless of time interval in which differential equation is defined. Therefore, there is increasing interest in the behavior of the system on bounded time interval coming from application point of view.

The main novelty of this section is to provide suitable and efficient concepts of finite-time bifurcation in the context of nonautonomous differential equations with impulses.

3.2.2 Preliminaries

We denote by $\mathbb{R}$ the set of all real numbers, $\mathbb{Z}$ the set of integers. In this section we introduce concepts of attractive and repulsive solutions, which are used to analyze asymptotic behavior of impulsive nonautonomous systems. This section is concerned with systems of the type

$$\begin{aligned} x' &= f(t,x), \\ \Delta x|_{t=\theta_i} &= J_i(x), \end{aligned} \tag{3.24}$$

where $\Delta x|_{t=\theta_i} := x(\theta_i+) - x(\theta_i)$, $x(\theta_i+) = \lim_{t\to\theta_i^+} x(t)$. The system (3.24) is defined on the set $\Omega = I \times \mathbb{A} \times G$ where $G \subseteq \mathbb{R}^n$, $I \subset \mathbb{R}$ is a finite compact time interval which contains only a finite number of impulse points θ_i with the set of indexes $\mathbb{A}$. Let $\phi(t, t_0, x_0)$ be general solution of (3.24) which is unique and noncontinuable.

Asymptotic properties of continuous dynamics and dynamics with discontinuous are the same. In what follows we use definitions of attractivity and repulsivity without any changes form [112, 114].

Definition 12 *[112, 114] Let $t_0 \in I$ and $T > 0$ is such that $t_0 + T \in I$.*

- *A compact and invariant nonautonomous set $\mathscr{G}$ is called $(t_0, T) - attractor$ if*

$$\limsup_{\epsilon\to 0^+} \frac{1}{\epsilon} d(\phi(t_0+T, t_0, B_\epsilon(\mathscr{G}(t_0))), \mathscr{G}(t_0+T)) < 1;$$

- *A solution* $\psi : [t_0, t_0 + T] \to \mathbb{R}^n$ *of* (3.24) *is called* $(t_0, T) - attractive$ *if* $graph\psi$ *is a* $(t_0, T) - attractor$.
- *A compact and invariant nonautonomous set* $\mathscr{H}$ *is called* $(t_0, T) - repeller$ *if*

$$\limsup_{\epsilon \to 0^+} \frac{1}{\epsilon} d(\phi(t_0 + T, t_0, B_\epsilon(\mathscr{H}(t_0 + T))), \mathscr{H}(t_0)) < 1.$$

- *A solution* $\psi : [t_0, t_0 + T] \to \mathbb{R}^n$ *of* (3.24) *is called* $(t_0, T) - repulsive$ *if* $graph\psi$ *is a* $(t_0, T) - repeller$.

The $(t_0, T) - attractor$ and $(t_0, T) - repeller$ satisfy the duality principle, i.e. under time reversal their roles are changed.

Example 5 *Let* $I := [t_0, t_0 + T]$ *be an interval containing a finite number of impulse points* θ_i *such that* $t_0 \leq \theta_1 < \theta_2 < ... < \theta_m \leq t_0 + T$ *for some* $t_0 \in \mathbb{R}$, $T > 0$ *and* $m \in \mathbb{N}$. *Consider the system*

$$\begin{aligned} x' &= a(t)x, \\ \Delta x|_{t=\theta_i} &= b_i x, \end{aligned} \tag{3.25}$$

with piecewise continuous function $a : I \to \mathbb{R}$ *and there exist constants* $b, B \in \mathbb{R}$ *such that* $-1 < b \leq b_i \leq B$. *Let* $\Psi : I \times I \to \mathbb{R}^n$ *be the transition matrix of the system* (3.24).

If $t_0 < \theta_1$, *then* $a(t)$ *is continuous on* $[t_0, \theta_1]$ *since* $a(t) \in PC(\mathbb{R}, \theta)$. *So, we have that* $\Psi(\theta_1, t_0) = e^{\int_{t_0}^{\theta_1} a(s)ds}$. *At* $t = \theta_1$ *solution makes a jump and we have that* $x(\theta_1+) = (1 + b_1)x(\theta_1)$. *Next,* $a(t)$ *is continuous on* $(\theta_1, \theta_2]$ *implies that* $\Psi(\theta_2, \theta_1) = e^{\int_{\theta_1}^{\theta_2} a(s)ds}(1 + b_1)$. *Proceeding in this way one can show that*

$$\Psi(t_0 + T, t_0) = \Psi(\theta_1, t_0)\Psi(\theta_2, \theta_1) \cdots \Psi(t_0 + T, \theta_m) = e^{\int_{t_0}^{t_0+T} a(s)ds} \prod_{i=1}^{m} (1 + b_i)$$

since $1 + b_i$ *is nonsingular matrix and commutes with any other matrix because* $1 + b_i > 0$.

If $t_0 = \theta_1$, *then the solution starts with a jump and we have that* $x(\theta_1+) = (1 + b_1)x(\theta_1)$. *Next,* $a(t)$ *is continuous on* $(\theta_1, \theta_2]$ *implies that* $\Psi(\theta_2, \theta_1) = e^{\int_{\theta_1}^{\theta_2} a(s)ds}(1 + b_1)$. *Discussing in the way one can show that*

$$\Psi(t_0 + T, t_0) = \Psi(\theta_2, \theta_1) \cdots \Psi(t_0 + T, \theta_m) = e^{\int_{t_0}^{t_0+T} a(s)ds} \prod_{i=1}^{m} (1 + b_i).$$

We want to point out that the basics of linear impulsive systems are fruitfully discussed in the books [1, 37, 121]. As a result, we have that

$$\begin{aligned}\Psi(t_0+T,t_0)=e^{\int_{t_0}^{t_0+T}a(s)ds}\prod_{i=1}^{m}(1+b_i) &\leq e^{\int_{t_0}^{t_0+T}a(s)ds}\prod_{i=1}^{m}(1+B)\\ &= e^{\int_{t_0}^{t_0+T}a(s)ds+m\ln(1+B)}.\end{aligned}$$

Therefore, any invariant and compact nonautonomous set is a $(t_0,T)-attractor$ *if*

$$\int_{t_0}^{t_0+T}a(s)ds+m\ln(1+B)<0.$$

By the same way one can say that any invariant and compact nonautonomous set is a $(t_0,T)-repeller$ *if* $\int_{t_0}^{t_0+T}a(s)ds+m\ln(1+b)>0$.

Definition 13 *[112, 114] The radius of* $(t_0,T)-attraction$ *of a* $(t_0,T)-attractor A$ *is defined by*
$\mathcal{A}_{\mathscr{G}}^{(t_0,T)}:=\sup\{\epsilon>0:d(\phi(t_0+T,t_0,B_{\widehat{\epsilon}}(\mathscr{G}(t_0))),\mathscr{G}(t_0+T))<\widehat{\epsilon}$ *for all* $\widehat{\epsilon}\in(0,\epsilon)\}$,
and the radius of $(t_0,T)-repulsion$ *of a* $(t_0,T)-repeller R$ *is defined by*
$\mathcal{R}_{\mathscr{H}}^{(t_0,T)}:=\sup\{\epsilon>0:d(\phi(t_0+T,t_0,B_{\widehat{\epsilon}}(\mathscr{H}(t_0+T))),\mathscr{H}(t_0))<\widehat{\epsilon}$ *for all* $\widehat{\epsilon}\in(0,\epsilon)\}$.

In Example 5 every invariant and compact set $S\subset[t_0,t_0+T]\times\mathbb{R}$ of the system (3.25) is

- $(t_0,T)-attractor$ with $\mathcal{A}_S^{(t_0,T)}=\infty$ if $\int_{t_0}^{t_0+T}a(s)ds+m\ln(1+B)<0$,
- $(t_0,T)-repeller$ with $\mathcal{R}_S^{(t_0,T)}=\infty$ if $\int_{t_0}^{t_0+T}a(s)ds+m\ln(1+b)>0$.

Definition 14 *[112, 114] We consider the impulsive system* (3.24)*, which depends on a parameter* μ*. The system* (3.24) *possesses a supercritical* $(t_0,T)-bifurcation$ *at* μ_0*,* $\mu_0\in(\mu_-,\mu_+)$*, if there exist a* $\widehat{\mu}>\mu_0$ *and a piecewise continuous function* $\psi:[t_0,t_0+T]\times(\mu_0,\widehat{\mu})\to\mathbb{R}^n$ *such that one of the following alternatives hold.*

- $\psi(\cdot,\mu)$ *is a* $(t_0,T)-attractive$ *solution of the system* (3.24) *for all* $\mu\in(\mu_0,\widehat{\mu})$*, and*

$$\lim_{\mu\to\mu_0^+}\mathcal{A}_{\psi(\cdot,\mu)}^{(t_0,T)}=0.$$

- $\psi(\cdot,\mu)$ *is a* $(t_0,T)-repulsive$ *solution of the system* (3.24) *for all* $\mu\in(\mu_0,\widehat{\mu})$, *and*

$$\lim_{\mu\to\mu_0^+}\mathcal{R}^{(t_0,T)}_{\psi(\cdot,\mu)}=0.$$

If in the above definition the limit $\mu\to\mu_0^+$ is replaced with $\mu\to\mu_0^-$ then we have subcritical $(t_0,T)-bifurcation.$

3.2.3 Attraction and Repulsion in a Quasilinear Impulsive System

In this section we study linear non-homogeneous impulsive systems in finite-time with definitions provided in the previous section. We show that these definition remain as a fruitful concept in the stability analysis of solutions of nonlinear systems with fixed moments of impulses. Let us consider the impulsive system in a compact interval $I:=[t_0,t_0+T]$ with m impulse points θ_i for some $t_0\in\mathbb{R}$, $T>0$ and $m\in\mathbb{N}$,

$$\begin{aligned} x' &= A(t)x+F(t,x),\\ \Delta x|_{t=\theta_i} &= B_ix+J_i(x), \end{aligned}\tag{3.26}$$

where $A\in PC(I,\theta)$, matrices B_i satisfy $\det(B_i+I)\neq 0$, $F:I\times G\to\mathbb{R}^n$ and $J:\mathbb{A}\times G\to\mathbb{R}^n$. Denote $\phi(t,t_0,x_0)$ as the solution of (3.26) and $\Psi:I\times I\to\mathbb{R}^{n\times n}$ as the transition matrix of the following linear system

$$\begin{aligned} x' &= A(t)x,\\ \Delta x|_{t=\theta_i} &= B_ix. \end{aligned}\tag{3.27}$$

Define $M_+:=\sup\{||\Psi(t,s)||:t_0\le s\le t\le t_0+T\}$ and
$M_-:=\sup\{||\Psi(t,s)||:t_0\le t\le s\le t_0+T\}$. Assume that the following conditions hold for the system (3.26):

(C1) $||\Psi(t_0+T,t_0)||<1$;

(C2) The functions $F(t,x)$ and $J_i(x)$ are Lipschitzian i.e., there exist positive number l such that $||F(t,x)||\le l||x||,\ \ ||J_i(x)||\le l||x||$ for all $t\in I, i\in\mathbb{A}$ and $||x||<h, h>0.$

Then one has the following theorem.

Theorem 21 *The origin is (t_0,T)-attractive for sufficiently small values of l, i.e.,*

$$||\phi(t_0+T,t_0,x_0)|| \leq \delta e^{M_+Tl+m\ln(1+M_+l)+\ln||\Psi(t_0+T,t_0)||}.$$

Now consider the following condition

(C3) $||\Psi(t_0,t_0+T)|| < 1.$

Theorem 22 *Assume that conditions (C2) and (C3) are true for the system* (3.26)*, then the origin is (t_0,T)-repulsive for sufficiently small values of l, i.e.,*

$$||\phi(t_0,t_0+T,x_0)|| \leq \delta e^{M_-Tl+m\ln(1+M_-l)+\ln||\Psi(t_0,t_0+T)||}.$$

Proof. An equivalent integral equation of the system (3.26) can be written as [1, 121]:

$$\begin{aligned}\phi(t,t_0,x_0) = \Psi(t,t_0)x_0 &+ \int_{t_0}^{t}\Psi(t,s)F(s,\phi(s,t_0,x_0))ds \\ &+ \sum_{t_0\leq\theta_i<t}\Psi(t,\theta_i)I_i(\phi(\theta_i,t_0,x_0)\end{aligned}$$

for all $t \in I$. Therefore, we get

$$\begin{aligned}||\phi(t,t_0,x_0)|| \leq ||\Psi(t,t_0)||||x_0|| &+ M_+l\int_{t_0}^{t}||\phi(s,t_0,x_0)||ds \\ &+ M_+l\sum_{t_0\leq\theta_i<t}||\phi(\theta_i,t_0,x_0)||\end{aligned}$$

for all $t \in I$ is fulfilled. Hence, by Gronwall-Bellman lemma for piecewise continuous functions [1, 121] it follows that

$$\begin{aligned}||\phi(t_0+T,t_0,x_0)|| &\leq ||\Psi(t_0+T,t_0)||e^{M_+lT}(1+M_+l)^{i[t_0,t_0+T)}||x_0|| \\ &\leq ||x_0||e^{\ln||\Psi(t_0+T,t_0)||+M_+lT+m\ln(1+M_+l)}\end{aligned}$$

where $i[t_0,t_0+T)$ is the number of elements of the sequence θ_i in the interval $[t_0,t_0+T)$. Since in this section $i[t_0,t_0+T)=m$, one can see that the required result follows by choosing $\delta = Kh$ for l small enough that $\ln||\Psi(t_0+T,t_0)|| + M_+lT + m\ln(1+M_+l) < 0$. We skip the prove of Theorem 22 since it can be proven analogously. □

3.2.4 Bifurcation Analysis

In this section we state and prove finite-time nonautonomous transcritical and pitchfork bifurcation results for impulsive systems. In what follows, the auxiliary theorems obtained in the previous section for higher dimensions will be used in the scalar case.

3.2.4.1 The Transcritical Bifurcation

In this subsection we study impulsive analogue of the nonautonomous transcritical bifurcation in finite-time. Let $x_- < 0 < x_+$ and $\mu_- < \mu_+$ be real numbers and let $I := [t_0, t_0 + T]$ with m impulse points θ_i. Consider the system

$$\begin{aligned} x' &= p(t,\mu)x + q(t,\mu)x^2 + r(t,x,\mu), \\ \Delta x|_{t=\theta_i} &= a_i(\mu)x + b_i(\mu)x^2 + c_i(x,\mu), \end{aligned} \tag{3.28}$$

with piecewise continuous functions $p : I \times (\mu_-, \mu_+) \to \mathbb{R}, q : I \times (\mu_-, \mu_+) \to \mathbb{R}$ and $r : I \times (x_-, x_+) \times (\mu_-, \mu_+) \to \mathbb{R}$ satisfying $r(t, 0, \mu) = 0$. $a : \mathbb{A} \times (\mu_-, \mu_+) \to \mathbb{R}, b : \mathbb{A} \times (\mu_-, \mu_+) \to \mathbb{R}$ and $c : \mathbb{A} \times (x_-, x_+) \times (\mu_-, \mu_+) \to \mathbb{R}$ with $a_i(\mu) \neq -1$ and $c_i(0, \mu) = 0$. Let $\Psi_\mu(t, s)$ be the fundamental matrix of the associated homogeneous part of the system

$$\begin{aligned} x' &= p(t,\mu)x, \\ \Delta x|_{t=\theta_i} &= a_i(\mu)x. \end{aligned} \tag{3.29}$$

Assume that there exists $\mu_0 \in (\mu_-, \mu_+)$ such that the following conditions hold.

(T1) $\Psi_\mu(t_0 + T, t_0) < 1$ for all $\mu \in (\mu_-, \mu_0)$ and $\Psi_\mu(t_0 + T, t_0) > 1$ for all $\mu \in (\mu_0, \mu_+)$;

or

(T1*) $\Psi_\mu(t_0 + T, t_0) > 1$ for all $\mu \in (\mu_-, \mu_0)$ and $\Psi_\mu(t_0 + T, t_0) < 1$ for all $\mu \in (\mu_0, \mu_+)$.

The functions q and b_i satisfy one of the following conditions.

(T2) $\liminf_{\mu\to\mu_0} \inf_{t\in I} q(t, \mu) > 0$ and $\liminf_{\mu\to\mu_0} \inf_{i\in\mathbb{A}} b_i(\mu) > 0$;

or

(T2*) $\limsup_{\mu\to\mu_0} \sup_{t\in I} q(t, \mu) < 0$ and $\limsup_{\mu\to\mu_0} \sup_{i\in\mathbb{A}} b_i(\mu) < 0$.

The functions r and c_i satisfy the following conditions.

(T3) $\lim_{x\to 0} \sup_{\mu\in(\mu_0-|x|,\mu_0+|x|)} \sup_{t\in I} \frac{|r(t,x,\mu)|}{|x|^2} = 0$;

(T4) $\lim_{x\to 0}\sup_{\mu\in(\mu_0-|x|,\mu_0+|x|)}\sup_{i\in\mathbb{A}}\frac{|c_i(x,\mu)|}{|x|^2}=0$;

(T5) There exists sufficiently small $l>0$ such that $|r(t,x,\mu)|<l|x|$, $|c_i(x,\mu)|<l|x|$ for all $\mu\in(\mu_-,\mu_+), t\in I, i\in\mathbb{A}$ and $x\in(x_-,x_+)$.

Theorem 23 *Assume that above conditions hold for the system* (3.28)*. Then there exist* $\widehat{\mu}_-<0<\widehat{\mu}_+$ *such that if (T1) is satisfied, then the origin is* $(t_0,T)-attractive$ *for* $\mu\in(\widehat{\mu}_-,\mu_0)$ *and* $(t_0,T)-repulsive$ *for* $\mu\in(\mu_0,\widehat{\mu}_+)$*. The following relations hold true.*

$$\lim_{\mu\to\mu_0^-}\mathcal{A}_0^\mu=0 \ \textit{and} \ \lim_{\mu\to\mu_0^+}\mathcal{R}_0^\mu=0. \tag{3.30}$$

In case (T1) is satisfied, the origin is* $(t_0,T)-repulsive$ *for* $\mu\in(\widehat{\mu}_-,\mu_0)$ *and* $(t_0,T)-attractive$ *for* $\mu\in(\mu_0,\widehat{\mu}_+)$*. The following relations hold true.*

$$\lim_{\mu\to\mu_0^+}\mathcal{A}_0^\mu=0 \ \textit{and} \ \lim_{\mu\to\mu_0^-}\mathcal{R}_0^\mu=0. \tag{3.31}$$

Hence, in both of the above cases the system (3.28) *possesses a* $(t_0,T)-bifurcation.$

Proof. We give the first part of the proof since second part can be proven in the similar manner. That is, (T1) is assumed. Let ϕ_μ be the general solution of the system (3.28). We may consider the case (T2). Choose $\widehat{\mu}_-<\mu_0<\widehat{\mu}_+$ such that

$$0<\inf_{\mu\in(\widehat{\mu}_-,\widehat{\mu}_+),t\in I}q(t,\mu) \ \text{and} \ 0<\inf_{\mu\in(\widehat{\mu}_-,\widehat{\mu}_+),i\in\mathbb{A}}b_i(\mu). \tag{3.32}$$

By means of (T4) and (3.32) one can see that Theorem 21 and Theorem 22 can be applied. Thus, we get attractivity and repulsivity of the origin as it was required to show in the theorem. Set

$$K:=\inf\{\Psi_\mu(t,s):t,s\in I,\mu\in(\widehat{\mu}_-,\mu_0)\}\in(0,1).$$

In order to show relations (3.30) and (3.31) hold we assume to the contrary that

$$\gamma=\limsup_{\mu\to\mu_0^-}\mathcal{A}_0^\mu>0.$$

By means of (T3) and (3.32) one can show that there exist $\widetilde{\mu}\in(\widehat{\mu}_-,\mu_0), x_0\in(0,K\gamma)$ and $L>0$ such that

$$q(t,\mu)x^2+r(t,x,\mu)>J \ \text{and} \ b_i(\mu)x^2+c_i(x,\mu)>J \tag{3.33}$$

for all $t \in I, i \in \mathbb{A}, \mu \in (\widetilde{\mu}_-, \mu_0)$ and $x_0 \in \left[Kx_0, \frac{x_0}{K}\right]$. Next, fix $\widehat{\mu} \in (\widetilde{\mu}_-, \mu_0)$ such that $\mathcal{A}_0^{\widehat{\mu}} > x_0$ and

$$\Psi_\mu(t_0 + T, t_0) \geq 1 - \frac{KJT}{x_0}. \tag{3.34}$$

Denote $\psi(t) = \phi_{\widehat{\mu}}(t, t_0, x_0)$. Since $\mathcal{A}_0^{\widehat{\mu}} > x_0$, we have

$$\psi(t_0 + T) < x_0. \tag{3.35}$$

Moreover, from the definition of K and by means of (3.33), we get

$$\psi(t_0 + T) \geq Kx_0 \text{ for all } t \in [0, T]. \tag{3.36}$$

There are two cases to be considered.

Case 1. There exist a $\tau \in (0, T]$ such that

$$\psi(t_0 + \tau) = \frac{x_0}{K}.$$

We take maximal τ which satisfy this relation. By means of (3.35), one can see that $\psi(t_0 + T) \leq \frac{x_0}{K}$ for all $t \in [\tau, T]$. Next, we consider the integral equation of the system (3.28) at $t_0 + T$ for fixed $\widehat{\mu}$ which start at point $t = t_0 + \tau$.

$$\begin{array}{rcl} \psi(t_0 + T) & = & \Psi_{\widehat{\mu}}(t_0 + T, t_0 + \tau)\dfrac{x_0}{K} \\ & + & \displaystyle\int_{t_0+\tau}^{t_0+T} \Psi_{\widehat{\mu}}(t_0 + T, s)\left(q(s, \widehat{\mu})(\psi(s))^2 + r(s, \psi(s), \widehat{\mu})\right) ds \\ & + & \displaystyle\sum_{t_0+\tau \leq \theta_i < t_0+T} \Psi_{\widehat{\mu}}(t_0 + T, \theta_i)\left(b_i(\widehat{\mu})(\psi(\theta_i))^2 + c_i(\psi(\theta_i), \widehat{\mu})\right) \\ & \geq & x_0 + KJ(T - \tau) + \displaystyle\sum_{t_0+\tau \leq \theta_i < t_0+T} KJ \\ & > & x_0. \end{array}$$

This is contraction for (3.35).

Case 2. For all $t \in (0, T]$, we have

$$\psi(t_0 + \tau) < \frac{x_0}{K}.$$

Next, from the integral equation of the system (3.28) at $t = t_0 + T$ for fixed $\widehat{\mu}$ which start at point t_0 we have

$$\begin{aligned}
\psi(t_0+T) &= \Psi_{\widehat{\mu}}(t_0+T, t_0+\tau)x_0 \\
&+ \int_{t_0}^{t_0+T} \Psi_{\widehat{\mu}}(t_0+T, s)\left(q(s,\widehat{\mu})(\psi(s))^2 + r(s,\psi(s),\widehat{\mu})\right) ds \\
&+ \sum_{t_0 \le \theta_i < t_0+T} \Psi_{\widehat{\mu}}(t_0+T, \theta_i)\left(b_i(\widehat{\mu})(\psi(\theta_i))^2 + c_i(\psi(\theta_i), \widehat{\mu})\right) \\
&\ge \left(1 - \frac{KJT}{x_0}\right) x_0 + KJT + KJm \\
&> x_0
\end{aligned}$$

where the last inequality follows by means of (3.33) and (3.34). We again arrive at contradiction for (3.35). Hence, we have that $\lim_{\mu \to \mu_0^-} \mathcal{A}_0^\mu = 0$. One can show following the similar route that $\lim_{\mu \to \mu_0^+} \mathcal{R}_0^\mu = 0$ and considering the condition (T2*). This finalizes the proof of the theorem. □

3.2.4.2 The Pitchfork Bifurcation

In this subsection we study impulsive analogue of the nonautonomous pitchfork bifurcation. Let $x_- < 0 < x_+$ and $\mu_- < \mu_+$ be real numbers and let $I := [t_0, t_0 + T]$ with m impulse points θ_i. Consider the system

$$\begin{aligned}
x' &= p(t,\mu)x + q(t,\mu)x^3 + r(t,x,\mu), \\
\Delta x|_{t=\theta_i} &= a_i(\mu)x + b_i(\mu)x^3 + c_i(x,\mu),
\end{aligned} \tag{3.37}$$

with piecewise continuous functions $p : I \times (\mu_-, \mu_+) \to \mathbb{R}, q : I \times (\mu_-, \mu_+) \to \mathbb{R}$ and $r : I \times (x_-, x_+) \times (\mu_-, \mu_+) \to \mathbb{R}$ satisfying $r(t, 0, \mu) = 0$. $a : \mathbb{A} \times (\mu_-, \mu_+) \to \mathbb{R}, b : \mathbb{A} \times (\mu_-, \mu_+) \to \mathbb{R}$ and $c : \mathbb{A} \times (x_-, x_+) \times (\mu_-, \mu_+) \to \mathbb{R}$ with $a_i(\mu) \neq -1$ and $c_i(0, \mu) = 0$. Let $\Psi_\mu(t, s)$ be the fundamental matrix of the linear system

$$\begin{aligned}
x' &= p(t,\mu)x, \\
\Delta x|_{t=\theta_i} &= a_i(\mu)x.
\end{aligned}$$

Assume that there exists $\mu_0 \in (\mu_-, \mu_+)$ such that following conditions hold.

(P1) $\Psi_\mu(t_0 + T, t_0) < 1$ for all $\mu \in (\mu_-, \mu_0)$ and $\Psi_\mu(t_0 + T, t_0) > 1$ for all $\mu \in (\mu_0, \mu_+)$;

or

(P1*) $\Psi_\mu(t_0+T, t_0) > 1$ for all $\mu \in (\mu_-, \mu_0)$ and $\Psi_\mu(t_0+T, t_0) < 1$ for all $\mu \in (\mu_0, \mu_+)$.

The functions q and b_i satisfy one of the following conditions.

(P2) $\liminf_{\mu\to\mu_0} \inf_{t\in I} q(t,\mu) > 0$ and $\liminf_{\mu\to\mu_0} \inf_{i\in\mathbb{A}} b_i(\mu) > 0$;
or

(P2*) $\limsup_{\mu\to\mu_0} \sup_{t\in I} q(t,\mu) < 0$ and $\limsup_{\mu\to\mu_0} \sup_{i\in\mathbb{A}} b_i(\mu) < 0$.

The functions r and c_i satisfy the following conditions.

(P3) $\lim_{x\to 0} \sup_{\mu\in(\mu_0-x^2,\mu_0+x^2)} \sup_{t\in I} \frac{|r(t,x,\mu)|}{|x|^3} = 0$;

(P4) $\lim_{x\to 0} \sup_{\mu\in(\mu_0-x^2,\mu_0+x^2)} \sup_{i\in\mathbb{A}} \frac{|c_i(x,\mu)|}{|x|^3} = 0$;

(P5) There exists sufficiently small $l > 0$ such that $|r(t,x,\mu)| < l|x|$, $|c_i(x,\mu)| < l|x|$ for all $\mu \in (\mu_-, \mu_+), t \in I, i \in \mathbb{A}$ and $x \in (x_-, x_+)$.

Theorem 24 *Assume that above conditions hold for the system* (3.37). *Then there exist* $\widehat{\mu}_- < 0 < \widehat{\mu}_+$ *such that if the conditions (P1) and (P2) are satisfied, then the origin is* $(t_0,T)-attractive$ *for* $\mu \in (\widehat{\mu}_-, \mu_0)$ *and* $(t_0,T)-repulsive$ *for* $\mu \in (\mu_0, \widehat{\mu}_+)$. *Moreover, it is true that* $\lim_{\mu\to\mu_0^-} \mathcal{A}_0^\mu = 0$.

If the conditions (P1) and (P2) are satisfied, then the origin is* $(t_0,T)-repulsive$ *for* $\mu \in (\widehat{\mu}_-, \mu_0)$ *and* $(t_0,T)-attractive$ *for* $\mu \in (\mu_0, \widehat{\mu}_+)$. *Moreover, it is true that* $\lim_{\mu\to\mu_0^+} \mathcal{R}_0^\mu = 0$.

If the conditions (P1) and (P2) are satisfied, the origin is* $(t_0,T)-attractive$ *for* $\mu \in (\widehat{\mu}_-, \mu_0)$ *and* $(t_0,T)-repulsive$ *for* $\mu \in (\mu_0, \widehat{\mu}_+)$. *Moreover, it is true that* $\lim_{\mu\to\mu_0^+} \mathcal{A}_0^\mu = 0$.

In case the conditions (P1) and (P2*) hold, the origin is* $(t_0,T)-repulsive$ *for* $\mu \in (\widehat{\mu}_-, \mu_0)$ *and* $(t_0,T)-attractive$ *for* $\mu \in (\mu_0, \widehat{\mu}_+)$. *Moreover, it is true that* $\lim_{\mu\to\mu_0^-} \mathcal{R}_0^\mu = 0$. *Hence, in all of the above cases the system* (3.37) *possesses a* $(t_0,T)-bifurcation$

The proof of the theorem is similar to that of Theorem 23.

3.2.5 An Example

In this section, we give an example illustrating our theoretical results by means of simulations. Consider the following system with $I := [0, 10]$ and impulse moments $\theta_i = i, 1 \leq i \leq 9$.

$$x' = \left(6\mu + \tfrac{5}{2}\mu\sin(\tfrac{t^3}{4})\right)x - \left(4\mu + \tfrac{7}{2}\mu\sin(\tfrac{t^5}{3}) + 2\right)x^2 + \left(\mu + \tfrac{1}{2}\mu\cos^2(t^3)\right)x^3,$$
$$\Delta x|_{t=i} = (1.5i\mu + 5\mu)\,x - (2i\mu + 5\mu + 3)\,x^2 + i\mu x^3,$$

where we have taken $p(t, \mu) = 6\mu + 2.5\mu\sin(t^3/4)$, $q(t, \mu) = 4\mu + 3.5\mu\sin(t^5/3) + 2$, $r(t, x, \mu) = (\mu + 0.5\mu\cos^2(t^3))\,x^3$ $a_i(\mu) = 1.5i\mu + 5\mu$, $b_i(\mu) = 2i\mu + 5\mu + 3$ and $d_i(x, \mu) = i\mu x^3$. One can verify that this system satisfies the conditions of Theorem 23. Simulation results support our theoretical discussion and reveal that all solutions starting in the neighborhood of the origin converge to the origin if $\mu < 0$, whereas for $\mu > 0$ all solutions starting in the neighborhood of the origin diverge from the origin.

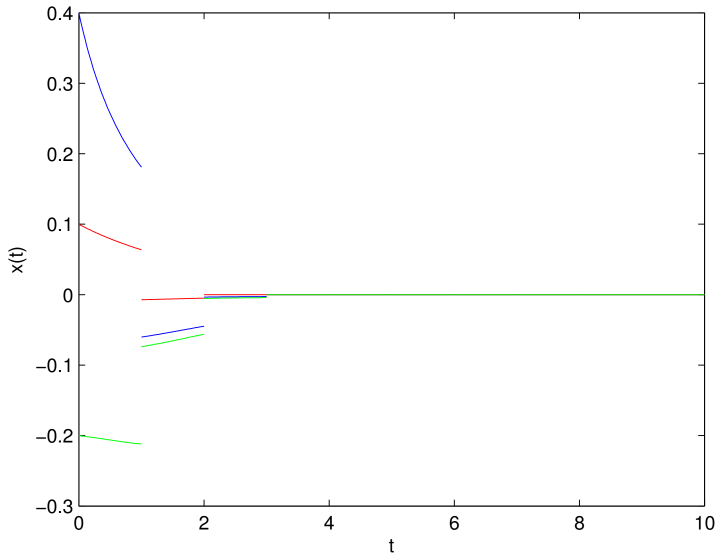

Figure 3.1: Asymptotic behavior of the solution for $\mu = -0.1$, where blue color represents the solution corresponding to $x_0 = 0.4$; red color represents the solution corresponding to $x_0 = 0.1$; and green color represents the solution corresponding to $x_0 = -0.2$.

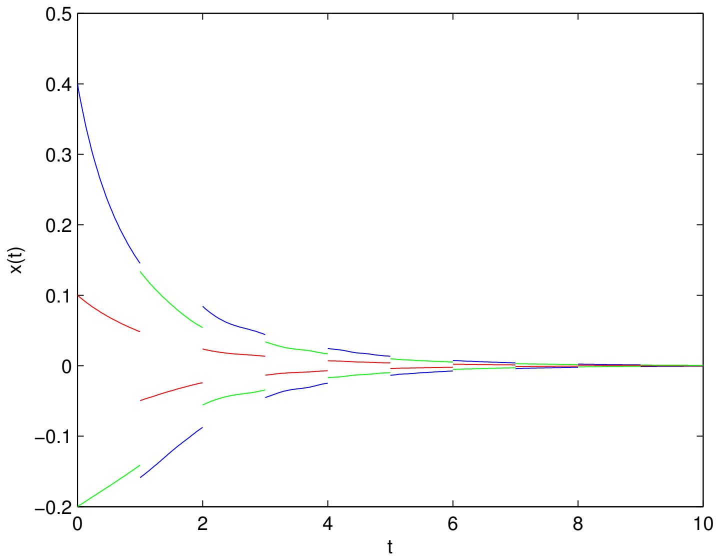

Figure 3.2: Asymptotic behavior of the solution for $\mu = -0.05$, where blue color represents the solution corresponding to $x_0 = 0.4$; red color represents the solution corresponding to $x_0 = 0.1$; and green color represents the solution corresponding to $x_0 = -0.2$.

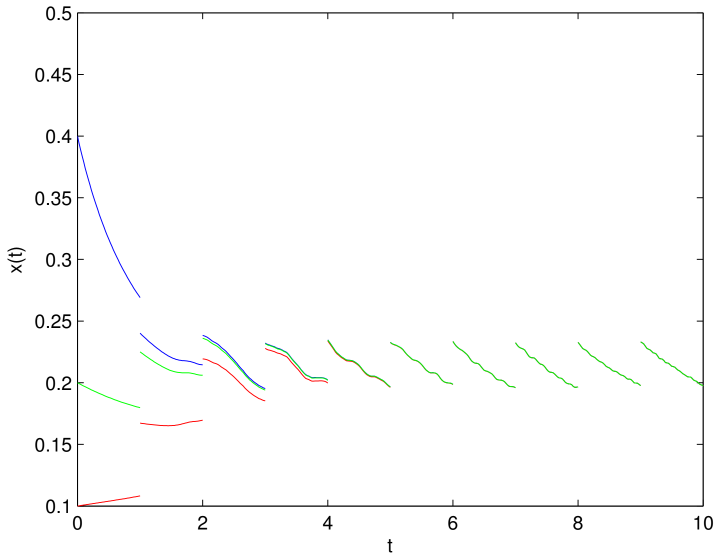

Figure 3.3: Asymptotic behavior of the solution for $\mu = 0.05$, where blue color represents the solution corresponding to $x_0 = 0.4$; red color represents the solution corresponding to $x_0 = 0.1$; and green color represents the solution corresponding to $x_0 = 0.2$.

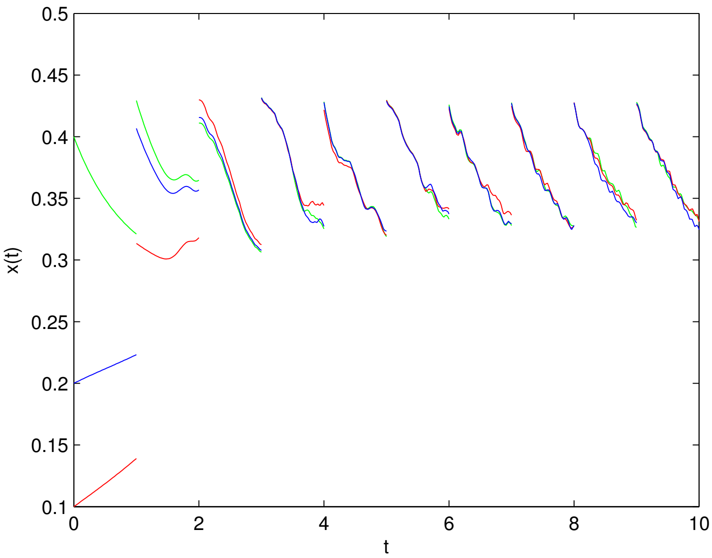

Figure 3.4: Asymptotic behavior of the solution for $\mu = 0.1$, where blue color represents the solution corresponding to $x_0 = 0.4$; red color represents the solution corresponding to $x_0 = 0.1$; and green color represents the solution corresponding to $x_0 = 0.2$.

From the simulation results, it is seen that $\lim_{\mu \to 0^-} \mathcal{A}_0^\mu = 0$ since solutions in Figure 3.2 converge more rapidly to the origin than those in Figure 3.1; and $\lim_{\mu \to 0^+} \mathcal{R}_0^\mu = 0$ since solutions in Figure 3.4 diverge more rapidly from the origin than those in Figure 3.3. Thus, the origin is $(0, 10) - attractive$ for $\mu \in (-0.1, 0)$ and $(0, 10) - repulsive$ for $\mu \in (0, 0.1)$. We conclude that this example possesses $(0, 10)-$ transcritical bifurcation.

3.3 Discussion

In this chapter, it is the first time nonautonomous transcritical and pitchfork bifurcations are studied for the most general impulsive systems with fixed time. Bifurcation patterns are obtained through loss of attraction and gain of repulsion. Furthermore, we discussed finite-time bifurcation scenarios in the scalar discontinuous systems which are applicable for the real world problems. We show that attractive and repulsive solutions play a great role in the theory of bifurcations. Numerical simulations suggest us that, as in Chapter 2, asymptotic behavior of the bounded solutions should also be examined for these type of equations. Finally, we note that this theory can be extended for higher dimensional systems.

CHAPTER 4

NONAUTONOMOUS TRANSCRITICAL AND PITCHFORK BIFURCATIONS IN BERNOULLI EQUATIONS WITH PIECEWISE CONSTANT ARGUMENT OF GENERALIZED TYPE

In this chapter, we study existence of the bounded solutions and asymptotic behavior of the Bernoulli equations with piecewise constant argument. Nonautonomous pitchfork and transcritical bifurcation scenarios are investigated. An example with numerical simulation is given to illustrate our results.

4.1 Introduction and Preliminaries

Cooke, Shah and Wiener were pioneers to initiate the theory of differential equations with piecewise constant argument [54, 125]. Since then, these equations have been under intensive investigations. The main idea of differential equation with piecewise constant argument is representing a hybrid of continuous and discrete dynamical systems and combining the properties of both the differential and difference equations. The concept of differential equations with piecewise constant argument has been generalized by introducing arbitrary piecewise constant functions as arguments in [2, 7, 8, 14].

The Bernoulli equations are important class of nonlinear systems in the classical theory of differential equation. Even though these equations have nonlinearities in continuous case it is possible to obtain the exact solution by certain transformations. In

Chapter 2, we investigated the Bernoulli equations with impulses. To the best of our knowledge, there is no study which deal with the Bernoulli equations with piecewise constant argument.

It is only in the recent decades, there has been intensive developments on time-dependent differential equations. Local theory of dynamical systems is concerned with asymptotic behavior of an equilibrium or a periodic solution. However, in nonautonomous dynamical systems it is usually hard to find an equilibrium point or a periodic solution. Indeed, in many case they fail even to exist. Therefore, equilibria generically persist as bounded solutions in the theory of time varying dynamical systems. There are abstract formulation of a nonautonomous dynamical systems as new concept of nonautonomous attractors which are called pullback attractors [46, 47, 77, 89, 112, 122]. We investigate appearances and disappearances of bounded solutions that are stable and unstable in the pullback and forward sense. In particular, it was possible to study bifurcation analysis in nonautonomous systems with pullback attractors [43, 75, 86, 88]. In Chapter 2, we have defined an impulsive Bernoulli equation and studied nonautonomous transcritical and pitchfork bifurcations analysis depending on Lyapunov exponents [24, 26, 27, 28]. In the current chapter we continue with scalar systems in hybrid dynamics and obtain results depending on the sign of these exponents. A theory of nonautonomous bifurcations in Banach space is treated in terms of exponential dichotomy in series of remarkable papers [109, 110, 111].

Let $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{R}$ be the sets of all integers, natural and real numbers, respectively. Fix two real-valued sequences θ_i, ζ_i, $i \in \mathbb{Z}$, such that $\theta_i < \theta_{i+1}$, $\theta_i \leq \zeta_i \leq \theta_{i+1}$ for all $i \in \mathbb{Z}$, $|\theta_i| \to \infty$ as $|i| \to \infty$. The main subject under investigation in this chapter is the following Bernoulli equation with piecewise constant argument of generalized type.

$$x'(t) = p(t)x(t) - q(t)x^n(t)x^n(\gamma(t)), \tag{4.1}$$

where $x \in \mathbb{R}$, $t \in \mathbb{R}$, $\gamma(t) = \zeta_i$, if $t \in [\theta_i, \theta_{i+1})$, $i \in \mathbb{Z}$, the functions $p, q : \mathbb{R} \to \mathbb{R}$ are continuous. Let $\phi(t, \gamma(t), t_0, x_0)$ be a solution of (4.1). In this chapter, we treat only scalar differential equations such that $\phi(t, \gamma(t), t_0, x_0)$ is continuable on $\mathbb{R}$. Solutions are unique both forwards and backwards in time.

In this chapter, we consider differential equation with both retarded and advanced piecewise constant argument of generalized type. That is, the argument function, $\gamma(t)$, is of a mixed type. The nonlinear term of the equation (4.1) is taken so that after substitution $y(t) + y(\gamma(t)) = 2x^{1-n}(t)x^{-n}(\gamma(t))$ the system (4.1) is converted to a linear nonhomogeneous system. The main novelty of this chapter is that the Bernoulli equation with piecewise constant argument (4.1) is considered for the first time in the literature. The remaining part of this chapter is organized as follows. In Section 4.2, we study bounded solutions of the Bernoulli equation (4.1). In Section 4.3 and Section 4.4, the pitchfork and the transcritical bifurcations are investigated respectively along with asymptotic properties of the bounded solutions.

In order to study nonautonomous bifurcation with piecewise constant argument we should define corresponding concepts of stability. In this chapter, we use Hausdorff semi-distance between sets X and Y as $d(X, Y) = \sup_{x\in X} \inf_{y\in Y} d(x, y)$.

4.1.1 Attraction and Stability

Asymptotic properties of continuous dynamics and hybrid dynamics are the same. Therefore, we shall use notion of pullback attracting sets without any change from [36, 43, 46, 49, 56, 57, 58, 78, 80, 87, 89, 112, 122].

Definition 15 *[78] An invariant set $\mathfrak{A}(t)$ is called pullback attracting if for every $t \in \mathbb{R}$*

$$\lim_{t_0\to-\infty} d(\phi(t, t_0, x_0), \mathfrak{A}(t)) = 0.$$

Having given meanings of pullback attraction one needs to characterize related ideas of stability, instability and asymptotic stability in order to investigate asymptotic analysis in the pullback sense. Next, we start with defining stability in the pullback sense.

Definition 16 *[86] An invariant set $\mathfrak{A}(t)$ is pullback stable if for every $t \in \mathbb{R}$ and $\epsilon > 0$ there exists a $\delta(t) > 0$ such that for any $t_0 < t, x_0 \in N(\mathfrak{A}(t_0), \delta(t))$ implies that $\phi(t, t_0, x_0) \in N(\mathfrak{A}(t), \epsilon)$.*

An invariant set $\mathfrak{A}(t)$ is said to be pullback asymptotically stable if it is pullback stable and pullback attracting. As we are busy with scalar impulsive systems, one can verify that pullback attraction implies pullback stability for a bounded trajectories. Next, we state the following lemma which will be useful in what follows.

Lemma 6 *Let $y(t)$ be a locally pullback attracting complete trajectory of a scalar impulsive system. Then, $y(t)$ is also pullback stable.*

The proof of this lemma, given by Langa et al. in [88], for continuous case is the same for impulsive systems. Thus, the last lemma allows us to concentrate on only pullback attraction properties of a complete trajectory instead of carrying out pullback stability.

As one would expect pullback instability is characterized through the converse of pullback stability. That is, an invariant set $\mathfrak{A}(t)$ is called pullback unstable if it is not pullback stable, i.e. if there exists a $t \in \mathbb{R}$ and $\epsilon > 0$ such that for each $\delta > 0$, there exists a $t_0 < t$ and $x_0 \in N(\mathfrak{A}(t_0), \delta)$ such that $d(\phi(t, t_0, x_0), \mathfrak{A}(t)) > \epsilon$. However, the notion of unstable set, which Crauel defined for the random dynamical systems, seems to be more natural instrument in discontinuous dynamics point of view.

Definition 17 *[56] The unstable set, $U_{\mathfrak{A}(t)}$, of an invariant set $\mathfrak{A}(t)$ is defined as*

$$U_{\mathfrak{A}(t)} = \{u : \lim_{t \to -\infty} d(\phi(t, t_0, u), \mathfrak{A}(t)) = 0\}.$$

We say that $\mathfrak{A}(t)$ is asymptotically unstable if the relation $U_{\mathfrak{A}(t)} \neq \mathfrak{A}(t)$ is fulfilled for some t.

If $\mathfrak{A}(t)$ is invariant then one can see that $\mathfrak{A}(t) \subset U_{\mathfrak{A}(t)}$ is satisfied. Thus, from the last definition we have that $\mathfrak{A}(t)$ is strict subset of $U_{\mathfrak{A}(t)}$. In the sequel, we need the following result.

Proposition 25 *[86] If $\mathfrak{A}(t)$ is asymptotically unstable then it is also locally pullback unstable and cannot be locally pullback attracting.*

This result proven by Langa et al. in [86] is valid for impulsive systems. Thus, we omit the proof and refer to [86]. Note that the idea of the asymptotic instability is a definition of time-reversed forward attraction. Alternatively, it is conceivable to define instability as a time-reversed version of pullback attraction.

Definition 18 *[88] An invariant set $\mathfrak{A}(t)$ is pullback repelling if it is pullback attracting for time-reversed system, i.e., if for every $t \in \mathbb{R}$ and every $x_0 \in \mathbb{R}^n$,*

$$\lim_{t_0 \to \infty} d(\phi(t, t_0, x_0), \mathfrak{A}(t)) = 0.$$

4.2 Bounded Solutions

In this section, we study existence of a bounded solution of (4.1). It is easy to see that $x = 0$ is a bounded solution of (4.1). We are interested in nonzero bounded solutions. For this purpose, we shall need the following conditions.

(C1) There exist positive numbers m and M such that $0 < m \leq q(t) \leq M$ for all $t \in \mathbb{R}$;

(C2) There exist positive numbers $\underline{\theta}$ and $\overline{\theta}$ such that $\underline{\theta} \leq \theta_{i+1} - \theta_i \leq \overline{\theta}$.

By means of the substitution $y(t) + y(\gamma(t)) = 2x^{1-n}(t)x^{-n}(\gamma(t))$ the system (4.1) is reduced to the following non-homogeneous linear system

$$y'(t) = (1-n)p(t)y(t) + (1-n)p(t)y(\gamma(t)) + 2(n-1)q(t). \tag{4.2}$$

One can see that $\Psi(t, s) = e^{\int_s^t (1-n)p(u)du}$ is the fundamental solution of the following linear system

$$z'(t) = (1-n)p(t)z(t).$$

In what follows, we introduce a function $R_i(t)$, $i \in \mathbb{Z}$, [2, 14],

$$R_i(t) = \Psi(t, \zeta_i) + \int_{\zeta_i}^{t} \Psi(t, s)(1-n)p(s)ds = 2e^{\int_{\zeta_i}^t (1-n)p(u)du} - 1.$$

We shall need the following modified regularity condition.

(C3) For every fixed $i \in \mathbb{Z}$, $R_i(t) > 0$, $\forall t \in [\theta_i, \theta_{i+1}]$.

Let $Y(t,s)$ be the fundamental matrix of the following linear system

$$y'(t) = (1-n)p(t)y(t) + (1-n)p(t)y(\gamma(t)). \tag{4.3}$$

Assume that $\theta_i < t_0 < \zeta_i$ for a fixed $i \in \mathbb{Z}$. One can confirm that [2, 8, 14],

$$Y(t,t_0) = R_j(t)\left(\prod_{k=j}^{i+1} R_k^{-1}(\theta_k)R_{k-1}(\theta_k)\right)R_i^{-1}(t_0),$$

for $t \in [\theta_j, \theta_{j+1}]$ and arbitrary $j > i$.

Denote $\alpha = \limsup\limits_{t-s\to\infty} \dfrac{\ln\|Y(t,s)\|}{t-s}$. One can guarantee that there exist two positive numbers k and K such that

$$ke^{\alpha(t-s)} \le \|Y(t,s)\| \le Ke^{\alpha(t-s)}, \;\; s \le t. \tag{4.4}$$

In what follows, it will be useful to define the following piecewise continuous matrix.

$$\Sigma(t,s) = \begin{cases} Y(\theta_i,t_0)\Psi(t_0,s), & \text{if } t \in [t_0 \hat{,} \zeta_i], \\ Y(t,\theta_{k+1})\Psi(\theta_{k+1},s), & \text{if } t \in [\zeta_k, \zeta_{k+1}], \\ \Psi(t,s), & \text{if } t \in [\zeta_j \hat{,} t], \end{cases} \tag{4.5}$$

where $t_0 \in [\theta_i, \theta_{i+1}]$, $t \in [\theta_j, \theta_{j+1}]$, $i < j$, $[a \hat{,} b] = [a,b]$ if $a \le b$, and equal to $[b,a]$, otherwise for $a, b \in \mathbb{R}$.

Lemma 7 *If (C1)-(C3) are satisfied, then (4.1) possesses nonzero bounded solutions $\widetilde{x}(t)$ on $\mathbb{R}$ which satisfy the following equations*

$$\widetilde{x}^{n-1}(t) = \begin{cases} \dfrac{2\left(\int_{-\infty}^{\zeta_j} \Sigma(\zeta_j,s)2(n-1)q(s)ds\right)^{\frac{n}{2n-1}}}{\int_{-\infty}^{t} \Sigma(t,s)2(n-1)q(s)ds + \int_{-\infty}^{\zeta_j} \Sigma(\zeta_j,s)2(n-1)q(s)ds}, & \text{if } \alpha < 0, \\ -\dfrac{2\left(\int_{\zeta_j}^{\infty} \Sigma(\zeta_j,s)2(n-1)q(s)ds\right)^{\frac{n}{2n-1}}}{\int_{t}^{\infty} \Sigma(t,s)2(n-1)q(s)ds + \int_{\zeta_j}^{\infty} \Sigma(\zeta_j,s)2(n-1)q(s)ds}, & \text{if } \alpha > 0, \end{cases}$$

where $t \in [\theta_j, \theta_{j+1}]$.

Proof. Consider $\alpha < 0$. First of all, let us show that

$$\begin{aligned} \widetilde{y}(t) &= \int_{-\infty}^{t} \Sigma(t,s)2(n-1)q(s)ds \\ &= \sum_{-\infty}^{k=j} Y(t,\theta_k) \int_{\zeta_{k-1}}^{\zeta_k} \Psi(\theta_k,s)2(n-1)q(s)ds + \int_{\zeta_j}^{t} \Psi(t,s)2(1-n)q(s)ds \end{aligned}$$

is a bounded solution of (4.2) for $t \in \big[\theta_j, \theta_{j+1}\big]$. Indeed,

$$\begin{aligned} \widetilde{y}'(t) &= \sum_{-\infty}^{k=j} \big[2(1-n)p(t)Y(t,\theta_k) \\ &+ 2(1-n)p(t)Y(\gamma(t),\theta_k)\big] \int_{\zeta_{k-1}}^{\zeta_k} \Psi(\theta_k,s)2(n-1)q(s)ds \\ &+ \int_{\zeta_j}^{t} 2(1-n)p(t)\Psi(t,s)2(1-n)q(s)ds + \Psi(t,t)2(n-1)q(t) \\ &= 2(1-n)p(t)\widetilde{y}(t) + 2(1-n)p(t)\widetilde{y}(\gamma(t)) + 2(n-1)q(t). \end{aligned}$$

Thus, $\widetilde{y}(t)$ satisfies (4.2). It is easy to see that $\widetilde{y}(t)$ is continuous in any interval $\big(\theta_i, \theta_{i+1}\big)$, $i \in \mathbb{Z}$. We show that $\widetilde{y}(t)$ is also continuous at points θ_i. Fix any $i \in \mathbb{Z}$.

$$\begin{aligned} \widetilde{y}(\theta_i+) &= \textstyle\sum_{-\infty}^{k=j} Y(\theta_i,\theta_k) \int_{\zeta_{k-1}}^{\zeta_k} \Psi(\theta_k,s)2(n-1)q(s)ds \\ &+ \int_{\zeta_j}^{\theta_i} \Psi(\theta_i,s)2(1-n)q(s)ds \\ &= \widetilde{y}(\theta_i-) \end{aligned}$$

Next, we show that $\widetilde{y}(t)$ is bounded and separated from zero. One can confirm existence of positive numbers h and H such that $h \leq \|\Psi(t,s)\| \leq H$ for all $t, s \in \big[\theta_i, \theta_{i+1}\big]$, $i \in \mathbb{Z}$. Thus, we have that,

$$\begin{aligned} 0 &< \frac{2\underline{\theta}mhk(n-1)}{1-e^{-\alpha\bar{\theta}}} + 2\underline{\theta}mh(n-1) \leq ||\widetilde{y}(t)|| \\ &\leq \frac{4\bar{\theta}MKH(n-1)}{1-e^{-\alpha\underline{\theta}}} + 4\bar{\theta}MH(n-1) < \infty, \end{aligned}$$

for $t \in \big[\theta_j, \theta_{j+1}\big]$. On the other hand, by (C1) and (C3) one can verify that $\widetilde{y}(t) > 0$, i.e. $\widetilde{y}(t)$ is separated from zero.

Finally, one can see that $y(t) + y(\gamma(t)) = 2x^{1-n}(t)x^{-n}(\gamma(t))$ implies $x(\gamma(t)) = y^{\frac{1}{2n-1}}(\gamma(t))$. Thus, the result follows from (C2) and the relation $x^{n-1}(t) = \dfrac{2y^{\frac{n}{2n-1}}(\gamma(t))}{y(t) + y(\gamma(t))}$.

We omit the proof of the case $\alpha > 0$, since it can be obtained in the similar manner. This finalizes the proof of lemma. □

Corollary 1 *If (C1)-(C3) are satisfied, then (4.1) possesses nonzero bounded on $\mathbb{R}$ solutions $\overline{x}(t)$ which satisfy the following equations*

$$\overline{x}^{n-1}(t) = \begin{cases} \dfrac{2\left(Y(\zeta_i,t_0)x_0^{-2m}+\int_{t_0}^{\zeta_j}\Sigma(\zeta_j,s)2(n-1)q(s)ds\right)^{\frac{n}{2n-1}}}{\int_{-\infty}^{t}\Sigma(t,s)2(n-1)q(s)ds+Y(\zeta_i,t_0)x_0^{-2m}+\int_{t_0}^{\zeta_j}\Sigma(\zeta_j,s)2(n-1)q(s)ds}, & \text{if } \alpha < 0, \\ \dfrac{2\left(Y(\zeta_i,t_0)x_0^{-2m}+\int_{t_0}^{\zeta_j}\Sigma(\zeta_j,s)2(n-1)q(s)ds\right)^{\frac{n}{2n-1}}}{-\int_{t}^{\infty}\Sigma(t,s)2(n-1)q(s)ds+Y(\zeta_i,t_0)x_0^{-2m}+\int_{t_0}^{\zeta_j}\Sigma(\zeta_j,s)2(n-1)q(s)ds}, & \text{if } \alpha > 0, \end{cases}$$

where $t \in [\theta_j, \theta_{j+1}]$.

In what follows, we have different bifurcation scenarios depending on the parity of n. In the next sections we deal with pitchfork and transcritical bifurcations respectively.

4.3 The Pitchfork Bifurcation

Consider (4.1) for $n = 2m + 1$. That is,

$$x'(t) = p(t)x(t) - q(t)x^{2m+1}(t)x^{2m+1}(\gamma(t)). \tag{4.6}$$

Theorem 26 *Suppose that (C1)-(C3) are fulfilled for (4.6). Then, for $\alpha > 0$ the trivial solution is asymptotically stable whereas the nonzero bounded solutions $\overline{x}(t)$ are asymptotically unstable, and for $\alpha < 0$ the trivial solution is asymptotically unstable and the nonzero bounded solutions $\overline{x}(t)$ stable and $\widetilde{x}(t)$ are asymptotically pullback stable.*

Proof. One can verify that the solution of (4.6) satisfy the following equation, [2, 8, 14],

$$x^{2m}(t, t_0, x_0) = \frac{2\left(Y(\zeta_i,t_0)x_0^{-2m}+\int_{t_0}^{\zeta_i}\Sigma(\zeta_i,s)4mq(s)ds\right)^{\frac{2m+1}{4m+1}}}{Y(t,t_0)x_0^{-2m}+\int_{t_0}^{t}\Sigma(t,s)4mq(s)ds+Y(\zeta_i,t_0)x_0^{-2m}+\int_{t_0}^{\zeta_i}\Sigma(\zeta_i,s)4mq(s)ds}. \tag{4.7}$$

In the previous section, we have shown that (4.6) admits the trivial solution and bounded solutions $\widetilde{x}(t)$ and $\overline{x}(t)$. Asymptotic behavior of (4.1) depends on the sign of α. We start with the case $\alpha > 0$. One can confirm the following equation.

$$x^{2m}(t,t_0,x_0) = \frac{2\Big(\widetilde{y}(\zeta_i)+Y(\zeta_i,t_0)\big(x_0^{-2m}-\widetilde{y}(t_0)\big)\Big)^{\frac{2m+1}{4m+1}}}{\widetilde{y}(t)+\widetilde{y}(\zeta_i)+Y(t,t_0)(x_0^{-2m}-\widetilde{y}(t_0))+Y(\zeta_i,t_0)(x_0^{-2m}-\widetilde{y}(t_0))}. \tag{4.8}$$

From (4.8) it follows that $x^{2m}(t,t_0,x_0) \to 0$ as $t \to \infty$. So, $x(t,t_0,x_0) \to 0$ as $t \to \infty$, replying that all solutions are attracted forwards to the point $\{0\}$. On the other hand, $x(t)$ converges to $\overline{x}(t)$ as $t \to -\infty$ whenever $||x_0|| < ||\widetilde{y}(t_0)||^{\frac{1}{2m}}$. Thus, the nonzero bounded solutions $\overline{x}(t)$ are asymptotically unstable.

If $\alpha < 0$, we notice that the expression (4.8) holds. Thus, one can see that $x(t)$ converges to $\widetilde{x}(t)$ as $t_0 \to -\infty$ and to $\overline{x}(t)$ as $t \to \infty$ whenever $||x_0|| < ||\widetilde{y}(t_0)||^{\frac{1}{2m}}$. Thus, $\widetilde{x}(t)$ is asymptotically pullback stable whereas $\widetilde{x}(t)$ forward stable. Moreover, $x^{2m}(t,t_0,x_0) \to 0$ as $t \to -\infty$ whenever $||x_0|| < ||\widetilde{y}(t_0)||^{\frac{1}{2m}}$. Therefore, the origin is asymptotically unstable. The theorem is proved. □

4.4 The Transcritical Bifurcation

In this section we consider (4.1) for $n = 2m$. That is,

$$x'(t) = p(t)x(t) - q(t)x^{2m}(t)x^{2m}(\gamma(t)). \tag{4.9}$$

Theorem 27 *Suppose that (C1)-(C3) are fulfilled for (4.9). Then, for $\alpha > 0$ the trivial solution is asymptotically stable, and for $\alpha < 0$ the trivial solution is asymptotically unstable and the nonzero bounded solution $\overline{x}(t)$ is forward stable and $\widetilde{x}(t)$ pullback stable.*

Proof. One can show that the solution of (4.9) satisfy the following equation, [2, 8, 14],

$$x^{2m-1}(t,t_0,x_0) = \frac{2\left(Y(\zeta_i,t_0)x_0^{-2m+1}+\int_{t_0}^{\zeta_i}\Sigma(\zeta_i,s)2(2m-1)q(s)ds\right)^{\frac{2m}{4m-1}}}{Y(t,t_0)x_0^{-2m+1}+\int_{t_0}^{t}\Sigma(t,s)2(2m-1)q(s)ds+Y(\zeta_i,t_0)x_0^{-2m+1}+\int_{t_0}^{\zeta_i}\Sigma(\zeta_i,s)2(2m-1)q(s)ds}. \tag{4.10}$$

In Section 4.2 we have shown that (4.9) admits the trivial solution and the nonzero bounded solutions $\widetilde{x}(t)$ and $\overline{x}(t)$. As in Section 4.3, it is clear that asymptotic behavior of (4.9) depends on α. Consider the case $\alpha > 0$. From the equation (4.10) it follows that $x(t, t_0, x_0) \to 0$ as $t \to \infty$ as long as $x(\tau, t_0, x_0)$ exists for all $\tau \in [t_0, t]$. If $x_0 > 0$, observe that

$$\begin{aligned} Y(t,t_0)x_0^{-2m+1} &+ \int_{t_0}^{t} \Sigma(t,s)2(2m-1)q(s)ds + Y(\zeta_i,t_0)x_0^{-2m+1} \\ &+ \int_{t_0}^{\zeta_i} \Sigma(\zeta_i,s)2(2m-1)q(s)ds > 0, \end{aligned}$$

for $\tau \in [t_0, t]$. Thus, $x(\tau, t_0, x_0)$ exists for all $\tau \in [t_0, t]$ and does not blow up as $t \to \infty$.

If $x_0 < 0$, to ensure the existence of the solution $x(\tau, t_0, x_0)$ it is sufficient to show that

$$Y(\tau,t_0)x_0^{-2m+1} + \int_{t_0}^{\tau} \Sigma(\tau,s)2(2m-1)q(s)ds < 0,$$

for $\tau \in [t_0, t]$. Since $\int_{t_0}^{\tau} \Sigma(\tau,s)2(2m-1)q(s)ds > 0$, we require

$$|x_0| < \left(\frac{Y(\tau,t_0)}{\int_{t_0}^{\tau} \Sigma(\tau,s)2(2m-1)q(s)ds} \right)^{\frac{1}{2m-1}}.$$

However, one needs to show that right-hand side of the last inequality is bounded from below. One can find that

$$\begin{aligned} &\frac{Y(\tau,t_0)}{\int_{t_0}^{\tau} \Sigma(\tau,s)2(2m-1)q(s)ds} = \\ &\frac{1}{\int_{t_0}^{\zeta_i} \Psi(\tau,s)2(n-1)q(s)ds + \sum_{k=i}^{k=j} Y(t_0,\theta_k) \int_{\zeta_{k-1}}^{\zeta_k} \Psi(\theta_k,s)2(n-1)q(s)ds + Y^{-}(\tau,t_0) \int_{\zeta_j}^{\tau} \Psi(\tau,s)2(1-n)q(s)ds}, \end{aligned} \tag{4.11}$$

for $\theta_j \leq \tau \leq \theta_{j+1}$. It is easy to see that the last expression is bounded from below since $Y^{-1}(\tau, t_0)$ is bounded for a large enough τ.

Finally, we consider the case $\alpha < 0$. To show that the trivial solution is asymptotically unstable notice that

$$x^{2m-1}(t,t_0,x_0) = \frac{2\Big(\widetilde{y}(\zeta_i)+Y(\zeta_i,t_0)\big(x_0^{-2m+1}-\widetilde{y}(t_0)\big)\Big)^{\frac{2m}{4m-1}}}{\widetilde{y}(t)+\widetilde{y}(\zeta_i)+Y(t,t_0)(x_0^{-2m+1}-\widetilde{y}(t_0))+Y(\zeta_i,t_0)(x_0^{-2m+1}-\widetilde{y}(t_0))}. \tag{4.12}$$

From the last expression it follows that $x(t)$ converges to 0 as $t \to -\infty$ for all $0 < x_0 < \widetilde{y}^{\frac{1}{2m-1}}(t_0)$.

It remains to show that $\overline{x}(t)$ is forward and $\widetilde{x}(t)$ pullback stable. If $x_0 > 0$, then it is clear that

$$Y(\tau, t_0)x_0^{-2m+1} + \int_{t_0}^{\tau} \Sigma(\tau, s)2(2m-1)q(s)ds > 0,$$

for $\tau \in [t_0, t]$. Thus, the solution $x(\tau, t_0, x_0)$ exists for all $\tau \in [t_0, t]$ and the (4.12) implies that $\overline{x}(t)$ is forward and $\widetilde{x}(t)$ pullback stable for all $0 < x_0 < \widetilde{y}^{\frac{1}{2m-1}}(t_0)$.

If $x_0 < 0$, then to ensure the existence of the solution $x(\tau, t_0, x_0)$ it is sufficient to show that

$$Y(\tau, t_0)x_0^{-2m+1} + \int_{t_0}^{\tau} \Sigma(\tau, s)2(2m-1)q(s)ds < 0,$$

for $\tau \in [t_0, t]$. Since $\int_{t_0}^{\tau} \Sigma(\tau, s)2(2m-1)q(s)ds > 0$, we require

$$|x_0| < \left(\frac{Y(\tau, t_0)}{\int_{t_0}^{\tau} \Sigma(\tau, s)2(2m-1)q(s)ds} \right)^{\frac{1}{2m-1}}.$$

The right-hand side of the last inequality is bounded from below because (4.11) holds. The theorem is proved.

4.5 Illustrative Examples

In this section, to illustrate theoretical results of Theorem 27 we consider two examples.

Example 1. Let us consider the following system.

$$x'(t) = (1.1 + \sin(5 + t^3/5))x(t) - (4 + 2.5\tanh(t/2))x^4(t)x^4(\gamma(t)), \quad (4.13)$$

where we have taken $p(t) = 1.1 + \sin(5 + t^3/5)$, $q(t) = 4 + 2.5\tanh(t/2)$, $\theta_k = \frac{k-1}{2}, k \in \mathbb{Z}$, $\zeta_k = \frac{k-1}{2}$ and $n = 4$. One can guarantee that $\alpha > 0$ and the conditions of Theorem 27 are satisfied with $m = 1.5$, $M = 6.5$ and $\underline{\theta} = \overline{\theta} = 1/2$. Theorem 27 guarantees that (4.13) has nonzero bounded solutions $\widetilde{x}(t)$ and $\overline{x}(t)$. Figure 4.1

reveals that all solutions starting near the origin diverge from the origin and converge to the nonzero bounded solutions. Therefore, the origin is asymptotically unstable and the bounded solutions are stable as expressed in the numerical simulations.

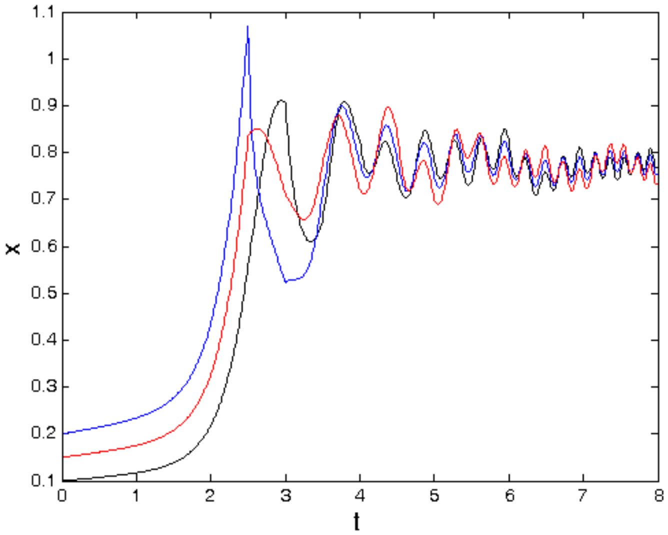

Figure 4.1: Asymptotic behavior of (4.13) for $t \in [0, 8]$. In the figure, the black color corresponds to the solution with initial value $x_0 = 0.1$; the red color corresponds to the solution with initial value $x_0 = 0.15$ and the blue color corresponds to the solution with initial value $x_0 = 0.2$. One can see that all solutions which start in the neighborhood of the origin diverge from the origin and converge to the nontrivial bounded solutions $\overline{x}(t)$.

Example 2. We consider the following system.

$$x'(t) = -(1.1 + \sin(5 + t^3/5))x(t) - (4 + 2.5\tanh(t/2))x^4(t)x^4(\gamma(t)), \quad (4.14)$$

where for this example we have taken $p(t) = -1.1 - \sin(5 + t^3/5)$, $q(t) = 4 + 2.5\tanh(t/2)$, $\theta_k = \frac{k-1}{2}, k \in \mathbb{Z}$, $\zeta_k = \frac{k-1}{2}$ and $n = 4$. One can guarantee that $\alpha < 0$ and the conditions of Theorem 27 are satisfied with $m = 1.5$, $M = 6.5$ and $\underline{\theta} = \overline{\theta} = 1/2$. We present in figure 4.2 the solution of (4.14) with initial values $x_0 = -0.15, 0.1$ and $x_0 = 0.2$. Numerical simulations support our theoretical discussion and reveal that all solutions starting near the origin converge to the origin.

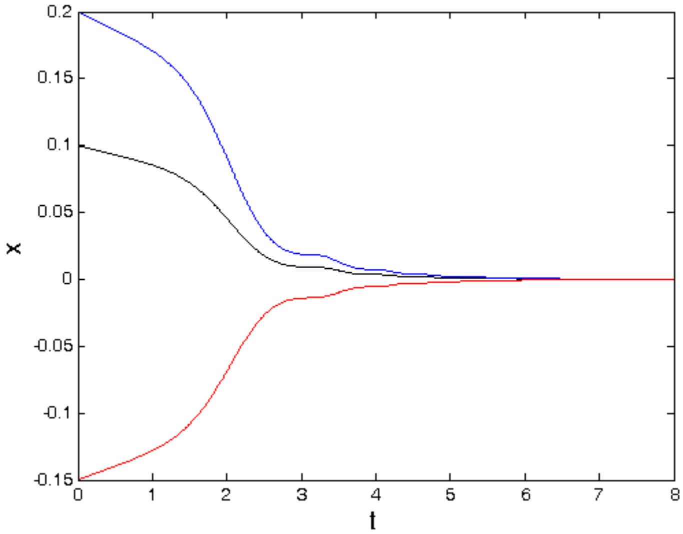

Figure 4.2: Asymptotic behavior of (4.14) for $t \in [0, 8]$. In the figure, the black color corresponds to the solution with initial value $x_0 = 0.1$; the red color corresponds to the solution with initial value $x_0 = -0.15$ and the blue color corresponds to the solution with initial value $x_0 = 0.2$.

4.6 Discussion

In the present chapter, it is the first time the Bernoulli equations with piecewise constant argument of generalized type has been studied. This chapter provides new sufficient conditions guaranteeing the existence and the separation from zero of the nonzero bounded solutions. Moreover, both forward and pullback asymptotic behavior of the trivial and the nonzero bounded solutions and different nonautonomous bifurcation scenarios are obtained.

CHAPTER 5

CONCLUSION

This thesis is devoted to nonautonomous bifurcations in impulsive differential equations as well as differential equations with alternating piecewise constant argument of generalized type. Moreover, we study bifurcation patterns in continuous Bernoulli equations. For the first time in this thesis we show that different concepts of attraction and repulsion such as pullback and forward attraction/repulsion remain as a fruitful idea in impulsive and hybrid systems. We investigate asymptotic behavior of a solution in terms of different convergence analysis and explore different bifurcation scenarios depending on these analysis. In particular, throughout the thesis, we study nonautonomous transcritical and pitchfork bifurcations in continuous, impulsive and hybrid systems. Furthermore, we introduce the Bernoulli equations, well-known in continuous differential equations, in impulsive and hybrid systems. The results mentioned in [24, 25, 26, 27, 28] constitute the main part of this thesis.

Chapter 2 deals with nonautonomous transcritical and pitchfork bifurcations in continuous as well as discontinuous system. We implement the definitions of pullback attractor and forward attractor to study asymptotic behavior of systems. In the beginning of the chapter, we study pitchfork bifurcation scenarios based on pullback convergence which depend on the properties of the system in the past. Sufficient conditions to have transcritical bifurcation are obtained. In the remaining part of the chapter, we generalize the results obtained in the first part as well as attain less restrictive conditions to ensure nonautonomous bifurcation patters. Besides, the Bernoulli equations in impulsive systems were introduced for the first time in the literature. The jump equation of the Bernoulli system is chosen in special manner so that the whole system is converted to a linear non-homogeneous system under the Bernoulli transformation. We carry out pullback as well as forward asymptotic behavior of the original system to analyze. Moreover, conditions to have bounded solutions for the Bernoulli equations are achieved. Illustration of theoretical results attained through numerical simulations.

In Chapter 3, we study nonautonomous transcritical and pitchfork bifurcations in attraction and transition sense. Bifurcation patterns in systems which are not explicitly solvable are under investigation. In other words, we consider impulsive systems for which it is not possible to obtain an explicit solution by any means of transformation. We observe qualitative change in the attractor reppeller pair. Namely, the trivial attractor/repuller become nontrivial one as parameter varies. In addition to these, finite-time analogues of nonautonomous transcritical and pitchfork bifurcations are presented. Moreover, a new results concerning asymptotic behavior of linearized systems depending on entire time are obtained. The theoretical results obtained in this chapter strengthened by means of simulation results.

Chapter 4 concerned with nonautonomous transcritical and pitchfork bifurcations in differential equations with alternating piecewise constant argument. We carry out analysis based on the book of Akhmet [2] where equivalent integral equations we introduced. The Bernoulli equation is presented for the hybrid systems. We construct special type of transformation so that original nonlinear system is converted to a linear non-homogeneous system. We premise that bifurcation scenarios depend on the sign of Lyapunov exponents. Besides of this results, future and past asymptotic properties of bounded solutions are discussed. Appropriate examples with numerical simulations are given to illustrate the theoretical results.

This study suggests many aspects in which bifurcation theory in nonautonomous dynamical systems could be further developed.

- The saddle-node bifurcation remains as an open problem in impulsive and hybrid systems even for one-dimensional systems. One may follow either studies carried out in [63] or in [88] to give definitions of the fold bifurcation in discontinuous systems.

- There are many interesting problems in higher dimensional systems regarding the theory of nonautonomous bifurcation studied in this thesis. Currently, there is no study devoted to this subject. Series of studies carried out in [109, 110, 111] which deal with nonautonomous bifurcation in Banach space might be one of the directions to follow. Attractive topics of the bifurcation theory such as center manifold or normal forms should be treated.

- Structural stability played a big role in the formulation of the bifurcation theory in autonomous systems [35]. However, for nonautonomous dynamical systems it has no proper generalization even in continuous case.

- Issuing from the numerical simulations one can see that results of Chapter 3 can be further developed. In particular, existence and pullback and forward convergence of bounded solutions should be considered.

- Application of theories established in this thesis to mathematical biology such as population dynamics, neural networks, epidemiology models and tumor models are subjects to be addressed. Moreover, other real world problems as in [3, 4, 90] need to be mentioned.

- Nonautonomous bifurcation patterns based on the skew product flows approach is another interesting direction to follow.

- There is no systematic study which addresses nonautonomous Hopf bifurcation theory. Appearance and disappearance of periodic solutions as parameter varies could be interesting issue to study.

- In autonomous dynamical systems, the concept of bifurcation and chaos are related topics [3]. Relatively simple models played an great role in the study of chaotic systems. For instance, there are strong ties between period doubling bifurcation and chaos. It remains as a fruitful subject to investigate relations between nonautonomous bifurcation theory and chaos.

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2. M. U. Akhmet, A. Kashkynbayev, Nonautonomous bifurcation in hybrid systems, submitted.
3. M. U. Akhmet, A. Kashkynbayev, Non-autonomous bifurcation in impulsive sys-

tems, Electronic Journal of Qualitative Theory of Differential Equations, (2013) 74, 1–23.

4. M. U. Akhmet, A. Kashkynbayev, Nonautonomous transcritical and pitchfork bifurcations in impulsive systems, Miskolc Mathematical Notes, 14 (2013) 737–748.

International Conference Publications

1. M. U. Akhmet, A. Kashkynbayev, Finite-time nonautonomous bifurcation in impulsive systems, accepted to Proceeding of 10th QTDE Colloquium, Electronic Journal of Qualitative Theory of Differential Equations.

CONFERENCES ATTENDED

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  Talk: *Non-autonomous transcritical and pitchfork bifurcation in impulsive systems.*
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SEMINARS

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PROJECTS

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