DEVELOPING MATHEMATICAL PRACTICES IN A SOCIAL CONTEXT: A HYPOTHETICAL LEARNING TRAJECTORY TO SUPPORT PRESERVICE MIDDLE SCHOOL MATHEMATICS TEACHERS' LEARNING OF TRIANGLES

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ABSTRACT

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The purpose of the current study was to document preservice middle school mathematics teachers' (PMSMT) classroom mathematical practices emerged through six-week instructional sequence about triangles. In this respect, the research question of "*What are the classroom mathematical practices that are developed within design research environment using problem-based learning for teaching triangles to preservice middle school mathematics teachers?"* guided the present study. In order to answer this research question and document the mathematical practices, a hypothetical learning trajectory and instructional sequence lasting six weeks related to triangles were formed. The hypothetical learning trajectory for the instructional sequence was performed for PMSMT to document their classroom mathematical practices about triangles.

The classroom mathematical practices were analyzed benefiting from collective learning activity of whole class discussions including individual learning and social aspects of the environment by using emergent perspective. Focusing on taken-as-shared knowledge identified by Toulmin’s argumentation model, the classroom mathematical practices were extracted. The classroom mathematical practices encouraging PMSMT’s learning of triangles in the present study were: PMSMT’s reasoning on (a) the formation of a triangle, (b) the elements of triangles and their properties, and (c) congruence and similarity. Based on these mathematical practice, PMSMT improved their understanding of the concept triangles benefiting from other geometry concepts such as transformation geometry, geometric constructions and argumentations. In this respect, they examined the properties and elements of triangles and related properties by developing their conceptual understanding.

Keywords: Design-based research, Classroom mathematical practice, Triangles, Preservice middle school mathematics teachers.

ÖZ

SOSYAL BİR ORTAMDA MATEMATİKSEL UYGULAMALARIN GELİŞTİRİLMESİ: ORTAOKUL MATEMATİK ÖĞRETMENİ ADAYLARININ ÜÇGENLERİ ÖĞRENMELERİNİ SAĞLAYAN BİR VARSAYIMA DAYALI ÖĞRENME ROTASI

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Bu çalışmanın amacı, geometrik kavramlardan biri olan üçgenler konusuyla ilgili tasarlanmış olan altı haftalık öğretim sürecinde oluşan matematiksel uygulamaları belirlemektir. Bu açıdan, çalışmayı "Ortaokul matematik öğretmeni adaylarının üçgenleri öğrenmeleriyle ilgili problem tabanlı öğrenme stratejisine göre hazırlanmış tasarım tabanlı araştırma ortamında geliştirdikleri sınıf içi matematiksel uygulamaları nelerdir?" araştırma problemi yönlendirmektedir. Bu bağlamda matematiksel uygulamaları belirlemek için üçgenler konusuyla ilgili varsayıma dayalı öğrenme rotası oluşturulmuştur. Altı haftalık bir öğretim dizisi sürecinde kullanılmak ve bu süreci yürütmek amacıyla varsayıma dayalı öğrenme rotası oluşturulmuştur. Varsayıma dayalı öğrenme rotası tasarlanmıştır. Tasarlanan varsayıma dayalı öğrenme rotası tamamlanıp gerekli düzenlemeler yapıldıktan sonra 23 ortaokul matematik öğretmeni adayından oluşan bir gruba katılımcıların matematiksel uygulamalarını belirlemek amacıyla uygulanmıştır.

Bireysel öğrenmelerin ve sosyal öğrenme ortamlarının içerildiği toplu öğrenme ortamında gerçekleşen toplu sınıf tartışmaları incelenerek sınıf içi matematiksel uygulamalar belirlenmiştir. Sınıf içi matematiksel uygulamalar Toulmin'ın bilimsel tartışma modeli kullanılarak paylaşılarak-alınan bilgilere odaklanılması sonucunda belirlenmiştir. Bu çalışmada belirlenen ortaokul matematik öğretmeni adaylarının üçgenleri öğrenmelerini destekleyen sınıf içi matematiksel uygulamalar şunlardır; üçgenlerin oluşumunun, üçgenlerin elemanlarının ve bunların özelliklerinin ve eşlik ve benzerliğin düşünülmesidir. Bu matematiksel uygulamalara göre, katılımcıların üçgenler konusuyla ilgili öğrenme ve anlamalarını dönüşüm geometrisi gibi diğer geometik kavramlardan, geometrik şekillerin inşasından ve argümantasyonlardan faydalanarak geliştirdikleri belirtilebilir. Bu açıdan, onların üçgenlerin elemanlarını ve özelliklerini inceleyerek kavramsal anlamalarını geliştirdikleri belirtilebilir.

Anahtar Kelimeler: Tasarım tabanlı araştırma, Sınıf içi matematiksel uygulamalar, Üçgenler, Ortaokul matematik öğretmeni adayları.

To My Parents,

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LIST OF ABBREVIATIONS

DBR	Design-based Research
HLT	Hypothetical Learning Trajectory
MKT	Mathematical Knowledge for Teaching
NCTM	National Council of Teachers of Mathematics
PMSMT	Preservice Middle School Mathematics Teachers

CHAPTER 1

1. INTRODUCTION

If geometry is not taught and learned effectively, the students and teachers tend to memorize the concepts in geometry rather than to understand them. Furthermore, teachers prefer teaching geometry topics by emphasizing rote memorization rather than to developing conceptual understanding and gradually moving students toward developing formal deductive reasoning (Fuys, Geddes, & Tischler, 1988).

> The mere memorization of a demonstration in geometry has about the same education value as the memorization of a page from the city directory. And yet it must be admitted that a very large number of our pupils do study mathematics in just this way. There can be no doubt that the fault lies with the teaching. (Young, 1925, pp. 4-5).

The students taking education in this way develop procedural understanding and poor performance in geometry. This situation can result from the teachers having little geometry knowledge since the teachers especially middle grade teachers typically have very little experience and knowledge of geometry (Clements, 2003; Fuys, Geddes, & Tischler, 1988; Hershkowitz, Bruckheimer, & Vinner, 1987; Stipek, 1998). Also, it can be stated that the desired learning environments can be provided only by knowledgeable teachers (Putnam, Heaton, Prawat & Remillard, 1992; Van der Sandt & Nieuwoudt, 2003). Also, the roles and importance of the knowledgeable teachers can be described effectively by having content knowledge of mathematics for teaching needed for the teachers to perform their professions of teaching mathematics effectively. In that respect, the knowledge needed for the design of this kind of environments can be explained in the literature in different ways by different researchers such as Shulman (1986), Ma (1999) and Ball, Hill and Bass (2005). First, Shulman suggested the content knowledge is needed for teaching. Also, this knowledge forms the foundation of the mathematical knowledge for teaching proposed by Ball, Sleep, Boerst and Bass (2009). In addition, Ma (1999) proposed the profound understanding of fundamental mathematics as a mathematical understanding of a teacher which is “deep, broad, and thorough” (p. 120). Lastly, Ball, Hill and Bass (2005) suggested the substantive knowledge of mathematics and knowledge of mathematics as the foundations of mathematical knowledge for teaching. The mathematical knowledge for teaching is the knowledge necessitated to perform the responsibility and work of teaching mathematics (Hill, Ball & Schilling, 2008). Substantive knowledge of mathematics is comprised of knowledge of mathematical principles and their meanings and applications, procedural and conceptual knowledge, and connections between them. Also, knowledge of mathematics is connected with knowing mathematics and doing mathematics, applying mathematical procedures and possessing mathematical knowledge (Ball, Hill & Bass, 2005).

In this respect, it is necessary that the mathematics teachers have good knowledge and understanding of geometry for the existence of effective geometry instructions in the classrooms. The teachers are expected to teach geometry in the secondary school classes when they have little knowledge of geometry so they encounter difficulties in geometry lessons (Jones, 2000). This problem can be solved if middle school teachers become well prepared to teach geometry in preservice years (NCTM, 2006). Also, the reason of the situation that the teachers are not equipped with necessary geometry knowledge is related to teacher education. In this respect, it is essential to educate mathematics teachers providing them opportunities in which they obtain rich and deep geometry knowledge. Moreover, subject matter knowledge representing necessary understanding of mathematical concepts have connection with teachers’ teaching performances in their classes by relating to other types of knowledge of Mathematical Knowledge for Teaching (MKT) of Ball, Thames and Phelps (2008). The teachers having deep subject matter knowledge can analyze their students' thinking and organize instructional sequence by making appropriate instructional decisions in their classroom (Hill & Ball, 2004). The situations in which the mathematics teachers having deep and rich subject matter knowledge perform teaching mathematics in their classrooms can be provided in their teacher education programs. In other words, they are supported by rich and deep subject matter knowledge through their preservice stage in order to be effective mathematics teachers in the future (Chapman, 2007).

Design based research is a useful way to provide the opportunities facilitating the development of preservice mathematics teachers' mathematical knowledge for teaching including subject matter knowledge. In teacher education programs, the instructors can help preservice mathematics teachers by designing hypothetical learning trajectories and conducting them in instructional sequences effectively. In this respect, it is crucial to identify the geometrical concepts, the tasks and tools to teach the concepts. In this respect, it is necessary to provide learners experiences and tasks to learn geometry by improving their geometric thinking and broadening their views of geometry content (Han, 2007; Henningsen & Stein, 1997). The geometrical tasks can be designed based on the properties of van Hiele geometric thinking levels since the geometric thinking levels of the PMSMT can be determined and their potential about geometric reasoning can be mad. Also, necessary predictions about thinking and actions of the PMSMT can be produced. Inservice and preservice elementary school mathematics teachers were expected to at least attain the first three van Hiele geometric thinking levels (Aydin & Halat, 2009; Pandiscio & Knight, 2010). Based on this knowledge, the hypothetical learning trajectory was prepared based on the properties of these levels and problem situations in the activity sheets were formed.

In the study, the geometric constructions by compass and straight edge were used since they are good at helping teachers attain good understanding of geometry about the rules and theorems. They provide preservice teachers opportunities to investigate the reasons of theorems, rules and topics in geometry since they need not only comprehend that something is so; s/he must further understand (Shulman, 1986). The geometric constructions provide a non-typical way for the solution of geometry problems with two instruments: the straight edge and compass. They are beneficial to investigate the work of Greek mathematicians such as Euclid and Pythagoras taking important role in mathematics curricula of all grade levels (Sanders, 1998). The geometric constructions are effective since they do not only provide the opportunities about constructing geometric shapes but also the skills of using the tools of the compass and the ruler (Cherowitzo, 2006). In this respect, it can be said that the geometric constructions improve physical and cognitive mathematical skills. In the process of constructing geometric shapes by compass and straight edge, the students examine how to construct the geometric shapes analyzing and understanding their properties (Erduran & Yeşildere, 2010; Napitupulu, 2001; Hoffer, 1981). Hence, it was necessary to incorporate geometric constructions to the present study. These constructions taking place in the hypothetical learning trajectory of the present study also supported the understanding of geometry by using conceptual understanding, problem solving, applications and communication of ideas.

The geometrical tasks taking place in the hypothetical learning trajectories and helping teachers investigating reasons of the topics of geometry may be more beneficial when they are used with the teachers' mathematical discourses such as argumentations illustrated in collective learning environment. Argumentation can increase the communication which is essential in attaining good understanding of geometry since the research show that teachers have deficiency in their understanding of geometry as well as in their skills to communicate geometry (Hershkowitz, 1989; Owens & Outhred, 2006; Sundberg & Goodman, 2005). Argumentation can provide these benefits since it takes role in interactive dialogue of two or more people reasoning together. It is also important to make scientific claims since the people obtained the idea after evaluating alternatives and weighing evidences (Voss & Van Dyke, 2001). Also, argumentations encourage conceptual understanding, problem solving, criticizing and justifying the ideas (Abi-El-Mona & Abd-El-Khalick, 2011; Duschl & Osborne, 2002; Jim´enez-Aleixandre et al., 2000; Jonassen & Kim, 2010; Osborne, Erduran, & Simon, 2004; Zembal-Saul, 2005). In this respect, it is beneficial to use argumentation in geometry, especially in mathematical tasks such as construction activities since the teachers having good understanding of geometry tend to have qualified scientific thinking, articulation of their ideas, and development of clearly structured arguments. Furthermore, argumentation promotes conceptual understanding and learning of the content effectively and deeply (Driver, Newton & Osborne, 2000) with the skills of communication and critical reasoning as two significant features of argumentation. Moreover, the construction steps facilitate problem solving, geometrical justifications and proofs. These steps necessitate the justifications and forming proofs for the process of construction of the geometric shapes and convincing others about the truth of them by examining the shape in a challenge situation (Erduran & Yeşildere, 2010; Smart, 1998).

The instructional sequences designed for particular geometrical concepts help the students learn and make reasoning about the geometrical concept effectively. When these instructions are supported by mathematical discourses, they can improve learning and understanding of the concepts by analyzing, discussing and convincing others about their ideas. In this respect, argumentations can enhance their learning (Lampert, 1990). Also, by discussing and transferring the obtained knowledge in different context, mathematical practices can be used to represent their learning since classroom mathematical practices represent taken-as-shared ways of reasoning and arguing mathematically (Cobb, Gravemeijer, Yackel, McClain & Whitenack, 1997). In this respect, classroom mathematical practices formed by using geometric constructions about a particular geometric concept of triangles were determined in order to determine preservice mathematics teachers' learning and understanding in the present study. Triangles as a geometrical concept was selected to help preservice middle school teachers attain deep and rich subject matter knowledge about it. Triangle is an important geometric concept since it is commonly used geometric shape for producing real life buildings and constructing and examining the properties of other geometric shapes (Fey, 1982). However, triangles have importance in teaching geometry, and learners from all grade levels have difficulty in learning triangles (Damarin, 1981; Vinner & Hershkowitz, 1980). Therefore, it is necessary to examine and develop the preservice middle school mathematics teachers' understanding and reasoning of the geometry concept of triangles. In other words, in spite of the value of geometry in biological and physical world (Fey, 1982), some learners do not achieve complete understanding of the concept of triangle (Vinner & Hershkowitz, 1980). Moreover, it is fundamental concept as a prerequisite for other geometrical shapes, higher levels of geometry concepts and other mathematical learning areas such as algebra (Athanasopoulou, 2008; Kellogg, 2010). Therefore, it is important for teachers to acquire necessary knowledge about triangles in teacher training programs. In this respect, it is crucial to provide mathematics teachers opportunities to learn and understand triangles in their preservice years.

Although there have been many research into preservice middle school mathematics teachers' development and understanding of geometry concepts, there have been necessities to investigate preservice teachers' development and understanding of the specific geometry concepts, especially in an attempt to increase their understanding of triangle concept. In light of the explanations above, it seemed worthwhile to explore their argumentations in geometrical tasks that might contribute to the development of their geometry knowledge, understanding of triangles and classroom mathematical practices. In this respect, the current study examined the communication formed in the classroom including the process of argumentation and collective learning environment related to designed mathematical tasks and tools and imagery. This was provided by performing instructional sequence as well as examining teaching and learning as it occurred in the classroom. In a collective learning environment, preservice middle school mathematics teachers' understanding and reasoning of triangles were investigated. Moreover, their classroom mathematical discourses were analyzed in order to illustrate their geometrical understanding and reasoning of triangles through identifying the classroom mathematical practices. PMSMT's understanding and learning of triangles were examined through mathematical practices since they provided information about individual and social processes, since mathematical practices are formed in a social learning environment including individual and social aspects of learning; neither occurring without the other and nor dominating to each other (Cobb et al., 2011). In this way, PMSMT's learning was investigated in a social environment designed by problem-based learning including argumentations and geometric constructions by considering the effects of individuals' learning on and their contributions to the collective learning environment.

With this design-based research, a lesson sequence was performed based on a designed hypothetical learning trajectory on triangles. Moreover, supposing this designed instructional sequence helped the participants develop their geometric thinking and knowledge about triangles, pretest and posttest were conducted before and after a six-week instructional sequence. Therefore, the effect of hypothetical learning trajectory and instructional sequence including the emergence of classroom mathematical practices about triangles was examined. In this respect, the answer of the research question for "*What are the classroom mathematical practices emerging in design-based research environment designed by problem-based learning for teaching triangles to* *preservice middle school mathematics teachers?*" was examined through the present study.

1.1 Significance of the Study

It is important for the teachers to have in-depth knowledge about the concepts of geometry. Also, it is claimed that both pre and inservice teachers have inadequate geometry knowledge (Clements, 2003; Stipek, 1998). Therefore, it is vital for preservice mathematics teachers to attain necessary geometry knowledge to teach geometry in their classrooms in the future. When the system of teacher education programs is thought, the preservice years of teachers are important since they are the places for preparing future teachers and dominating them with necessary knowledge needed for their professions. These programs provide opportunities for future teachers to learn mathematical knowledge needed for their profession. Many researchers suggest different ideas to improve this situation. Some of them insist on increasing the number of the courses emphasizing more mathematical content to use their teaching in the future (Goldhaber & Brewer, 2000). On the other hand, Suzuka, Sleep, Ball, Bass, Lewis and Thames (2009) explain that "teaching is mathematically demanding work. The requisite knowledge and skills are not necessarily picked up on the job nor are they typically learned in college courses or used in other professions" (p. 7). Therefore, it can be useful to improve the quality of teacher preparation programs in mathematics by providing ideas that they construct through effective learning tasks to develop their mathematical knowledge for teaching and especially subject matter knowledge (Ball, Hill & Bass, 2005).

The preservice mathematics teachers need to be educated to obtain necessary knowledge by understanding their reasons and constructing them rather than memorizing them (Chapman, 2007). This can be achieved by providing them opportunities in instructional sequences. When these sequences include mathematical discourse, the learning can become more meaningful. Research show that classroom discourse practices especially including argumentations have effect on learning and conceptual understanding (Jonassen & Kim, 2010; McNeal & Simon, 2000; Yackel & Cobb, 1996) since classroom discourses include mathematical talks occurring in the classrooms and the interactions among students, teachers about the subject matter. Also, the student learning is affected by the classroom discourse, and negotiating the norms based on argumentation (Cobb, 2002). Argumentation can be defined as a social phenomenon observed while cooperating individuals are trying to adjust their intentions and interpretations in presenting their actions verbally (Krummheuer, 1995). Mathematical argumentation improves the abilities of the articulation of mathematical thinking, justification and explanation of the reasoning (Yackel & Cobb, 1996), communication and critical reasoning (Krummheuer, 1995).

In order to help the preservice middle school mathematics teachers attain these skills, it is aimed to design lessons and produce instructional sequences through ongoing analysis of classroom activity and using the results of these analysis for instructional planning and decision making. Design experiment research "can help create and extend knowledge about developing, enacting, and sustaining innovative learning environments" (Cobb, et al., 2003, p. 5), since it is constructed based on student cognition and instructional materials. Design experiments are pragmatic and theoretical with the function of the design and the resulting collective learning environment (Cobb, et al., 2003). It is beneficial since it connects theory and practice so that it can solve the problems that the theories have lack of practical implications (Roth, 2011). In the process of design-based research, the design produced based on the theories is dynamic and it tends to change with respect to practical issues on the contexts. In other words, the instructions are based on theoretical and practical considerations by being organized in a developmental process (Cobb et al., 2003). In this respect, it can be said that this dynamic lesson design process can provide beneficial learning environment for preservice mathematics teachers to attain necessary knowledge and skills (Simon, 2000). Also, they can attain necessary understanding and skills about how and when to use the mathematical content knowledge when they are taught extracting their ideas by using their learning and teacher preparation programs. Moreover, they can learn this in a flexible way because of the dynamic nature of the design experiment (Cobb et al., 2003; Wheeldon, 2008). Therefore, the design experiment for this study was produced considering mathematical discourse.

The hypothetical learning trajectory was designed with respect to the concept of triangles. The concept of triangles was chosen because it is one of the fundamental concept for geometry (Fey, 1982). All of the students are expected to attain deep knowledge about it since other concepts in the geometry are learned by using it. Moreover, the concept of triangles is a difficult concept and the students have difficulties about it (Fey, 1982). Also, because of the nature of the mathematics education, the other concepts of geometry are constructed by using the concepts of triangles. Therefore, the instructional sequence was designed and performed about this critical concept to provide preservice middle school mathematics teachers attain necessary subject matter knowledge about it.

Although the instructions are made with an accurate understanding of the goals of the lessons, it is possible to face with the problems such as determining the specific practices used to implement the lessons (Hufferd-Ackles, Fuson & Sherin, 2004). In this respect, this study aimed to provide information to the literature on the problems by determining and analyzing the planning and classroom practices designed with the concept of triangles in a collective learning environment. Moreover, this study provided information about the solution of the problem related to this geometrical concept for preservice middle school mathematics teachers to attain necessary geometrical knowledge through the mathematical practices by argumentations.

1.2 Definition of Important Terms

Mathematical practices are taken-as-shared ways of reasoning and arguing mathematically in in a social learning environment. Taken-as-shared way illustrates an environment and process including the discussions about mathematical problems and ideas by mathematical symbolization and notations (Cobb, Gravemeijer, Yackel, McClain & Whitenack, 1997).

Argumentation is a kind of mathematical discourse referring to the ways of mathematical justifications formed and interpreted by the students and used in the communications (Lampert, 1990).

Geometrical Constructions are systematic steps used to produce geometric entities by producing intended geometric shapes following particular basic and complex steps of sequence by compass and straight edge (Demiray & Çapa-Aydın, 2015; Djoric & Janicic, 2004).

Design-based research refers to "a series of approaches, with the intent of producing new theories, artifacts, and practices that account for and potentially impact learning and teaching in naturalistic settings" (Barab & Squire, 2004, p. 2).

Hypothetical Learning Trajectory represents the ways of reasoning in learning context and includes teachers' predictions about the progress in teaching sequence (Smith et al., 2006).

CHAPTER 2

2. THE LITERATURE REVIEW

In this chapter, the mathematical knowledge for teaching is explained in order to understand the mathematical practices on the geometrical concept of triangles as the main purpose of the study. Then, the philosophies; constructivism and social constructivism that the lessons and the study have been designed based on triangles are discussed. Also, the strategy of problem-based learning and van Hiele geometric thinking theory are explained since they have formed the basis of instructional sequence and Hypothetical Learning Trajectory (HLT) of the current study. HLT as a beneficial concept used to create instructional sequence is described. Then, geometrical constructions and proof as the tools used in the instructional sequence are explained. At the end, triangles as the subject of the lessons are stated. In this way, theoretical framework of the study is formed.

2.1 Mathematical Knowledge for Teaching

In order to educate teachers effectively and to provide their improvement, many research have been conducted. Some of them have focused on types of knowledge needed for teachers to teach mathematics effectively (Ball, Hill, & Bass, 2005; Ma, 1999; NCTM, 2000). Through these research, Ball, Thames, and Phelps (2008) identified the concept of Mathematical Knowledge for Teaching (MKT) focusing on teaching rather than teachers. MKT can be defined as “the mathematical knowledge needed to perform the recurrent tasks of teaching mathematics to students" (p. 399). Through this definition, MKT explains the nature of mathematics needed for teachers' professions. In other words, it includes different types of knowledge essential for the teachers while performing their responsibilities in their professions. MKT focuses on the mathematical knowledge and the usage of it in teaching (Ball, Bass & Hill, 2004; Ball, Hill & Bass, 2005).

In the literature, there have been different categorization systems for the types of knowledge needed for teachers to teach students. Initially, Shulman (1986) separated these knowledge domains into different groups. Then, different categorization systems and modifications have been made on his knowledge system. Hill, Ball & Schilling (2008) have changed and made modifications on the Shulman's (1986) categorization system. In their categorization. MKT is composed of four categories under two main titles; pedagogical content knowledge and subject matter knowledge.

Pedagogical content knowledge focuses on pedagogy and content based on teaching and learning (Ball, Hill & Bass, 2005) with two subtitles; knowledge of content and students and knowledge of content and teaching. Knowledge of content and students examines subject matter knowledge based on knowledge about teaching (Ball, Hill & Bass, 2005; Hill, Schilling & Ball, 2004). In this category, knowledge about mathematical concepts needed while teaching on the perspectives of the students is focused on. In this respect, this type of knowledge focuses on tasks or representations formed to understand and model mathematical concepts, designing instructional sequence to teach the concept and making guidance for the students to help them improve their discussions and understanding (Ball, Hill & Bass, 2005). Moreover, knowledge of content and students examines the subject matter knowledge of mathematics through knowledge about the learners' thinking (Ball, Bass & Hill, 2004; Ball, Hill & Bass, 2005; Hill, Schilling & Ball, 2004). It provides teachers opportunities to think about mathematical concepts from the learners' views and performing teaching in this way. It includes thinking of students about specific mathematical concepts, engaging with mathematical tasks, and being motivated and challenging about mathematical concepts (Ball, Hill & Bass, 2005; Hill, Ball & Schilling, 2008).

The other title of MKT is subject matter knowledge as the main focus point of the present study. Hill, Ball and Schilling (2008) have connected Shulman's pedagogical content knowledge (PCK) and content knowledge (CK) domains under the title of subject matter knowledge. It involves mathematical knowledge that the teachers are expected to have in order to perform their responsibilities of teaching (Ball, Hill & Bass, 2005). This title is separated into two categories which are specialized content knowledge and common content knowledge. The category of specialized content knowledge focuses on mathematical knowledge which is unique to the mathematics teachers since it is knowing mathematics for mathematics curriculum in the schools (Hill, Ball & Schilling, 2008; Ball, Hill & Bass, 2005; Hill & Ball, 2004; Hill et al., 2004). In other words, it refers to the knowledge of mathematical concepts in a way that it is placed in mathematics curriculum and in the lessons. It can be exemplified by understanding, reasoning, illustrating and making connections between mathematical topics and expressing, discussing and using them through mathematical ideas (Hill & Ball, 2004) without the concerns of teaching them. This type of MKT is beneficial in a way that teachers are using their knowledge in the process of guiding their students to understand the mathematical concepts and make connections between them and performing their common teaching responsibility while students are constructing their knowledge (Ball & Bass, 2000, 2003; Ball, Bass & Hill, 2004; Ball, Hill & Bass, 2005; Hill & Ball, 2004). The other title of MKT is common content knowledge. It represents the knowledge of mathematical concepts being expected for the students to be known and taking place in mathematics curriculum in the schools and general knowledge of mathematics (Ball, Hill & Bass, 2005; Hill et al., 2004). It can be exemplified by procedural knowledge of mathematical operations, defining mathematical concepts and applying mathematical facts to the problems.

Subject matter knowledge has effect on teachers' teaching practices with respect to the findings of many research since deep and rich knowledge and understanding of the mathematical concepts develop teaching. Moreover, subject matter knowledge has close relationship with other types of knowledge since teachers attaining deep subject matter knowledge can successfully understand students' thinking to design and organize the instructional sequence by making appropriate instructional decisions in their classroom (Hill & Ball, 2004). Hence, the teachers are expected to have deep and rich knowledge about the mathematical concepts that they teach in their classrooms. This situation can be provided for them in their teacher education programs since preservice teachers attain and develop these types of knowledge through their preservice stages in order to be effective mathematics teachers in the future (Chapman, 2007). Through this stage, they improve themselves by being encouraged by the courses about their professions. In this respect, it can be claimed that preservice teachers need opportunities to have experiences in the courses that they take in teacher education programs with the aim of improving their MKT (Philipp, 2007). However, it should be considered that participation of these courses does not guarantee that preservice teachers become effective and equipped with these types of knowledge. Therefore, it is needed to form environments encouraging that preservice teachers attain mathematically rich experiences, and improve other types of knowledge related to subject matter knowledge that they possess (Turner et al., 2012). Through this kind of environments, beneficial opportunities are provided for preservice teachers to improve themselves and become effective as future teachers. In other words, preservice stages are critical for them since they are encouraged to develop their knowledge and skills with various opportunities to reach the resources and to practice their knowledge (Bryan, 2003). These opportunities should be selected and designed carefully based on the views of preservice mathematics teachers. Beneficial tasks help preservice teachers examine, practice and analyze their conceptual learning and understanding improving their mathematical content knowledge (Ball & Forzani, 2009).

The critical importance of subject matter knowledge is emphasized in a way that mathematics teachers must have detailed knowledge of common content knowledge and specialized content knowledge in order to perform their professions effectively (Hill & Ball, 2004). Ma (1999) also insists on the importance of subject matter knowledge by deep understanding of fundamental mathematics needed for making connection between mathematical topics and mathematical ideas in order to perform teaching in the classrooms effectively. This knowledge includes connection between mathematical concepts, structure of it and discussing it in their mathematics classrooms. Teachers may underestimate the difficulty of teaching mathematics to the students by overestimating their subject matter knowledge and considering only their own conceptual and operational knowledge about subjects in the curriculum. Teachers may claim that the elementary mathematics curriculum is not difficult (Sowder, et al., 1998). This ignorance affects the behaviors of mathematics teachers in classrooms and they think teaching mathematics in the classrooms as teaching procedures to follow. However, it is needed to attain the view point of deep conceptual part of this curriculum since they face with actual teaching and realize the difficulty of mathematical concepts. This realization increases the skills of teachers' understanding of mathematical concepts (Sowder, et al., 1998). This realization can be effectively provided by teacher education programs and opportunities in these programs presented to preservice teachers. Moreover, although there have been many research emphasizing the importance of teacher subject matter knowledge in teaching, there exists critical need to study preservice teachers' subject matter knowledge in preparing teachers. This need can be caused by the case that teachers begin teaching in their novice years without having sufficient content knowledge of mathematics (Ball, 1988). Therefore, it is critically needed to obtain detailed information about what preservice teachers have before becoming a real teacher and how they learn teaching mathematics from the view of mathematical content (National Mathematics Advisory Panel, 2008). Therefore, the preservice stages should be designed to help preservice teachers attain necessary subject matter knowledge effectively.

With all of the explained views, HLT are important since they provide opportunities for the instructors to design mathematically rich environments to help preservice mathematics teachers attain necessary knowledge and skills for their professions. In this respect, HLT on the mathematical concept of triangles was designed to help PMSMT attain various and rich experiences on this concept to develop their skills of teaching and other types of knowledge that they needed to teach in the present study. In other words, HLT used in the present study was organized based on the aim of improving the subject matter knowledge of the preservice middle school mathematics teachers in the geometrical concept of triangles. Moreover, in this section, the current study paid attention on preservice teachers' subject matter knowledge paid attention on in teacher education programs among other types of knowledge in MKT. In this respect, the necessities of the opportunities providing deep knowledge and understanding about the mathematical concepts were explained. These necessities were provided in different ways in teacher education programs as explained in many research in the literature. In the present study, the design of the path that preservice middle school mathematics teachers' (PMSMT) understanding the concept of triangles were examined by mathematical practices. Mathematical practices provided a beneficial way to investigate the development of their knowledge and understanding about mathematical concepts related to their subject matter knowledge. Also, the related theories and philosophies used to design hypothetical learning trajectory of the current study were examined in the following sections.

2.2 Mathematical Practices

In the literature, there have been valuable research investigating learning and teaching focusing on learning of communities from sociological points of views by forming classroom practices (Ball & Bass, 2000; Cobb & Bauersfeld, 1995). The researchers focus on social context of learning based on the fact that mathematical learning takes place in the social context of the classroom (Cobb & Bauersfeld, 1995; Cobb, Stephan, McClain, & Gravemeijer, 2001). The social contexts in which learning takes place have been examined based on different explanations and definitions of mathematical practices. Various studies defining and examining the mathematical practices in different ways exist in the literature (Cobb et al., 2011; Font, Godino & Gallardo, 2013; Moschkovich, 2002). In their studies, they focus on the different meanings of mathematical practices. For example, based on the definition of mathematical practices made by Moschkovish (2002), the term is explained by being separated into two groups; every day and academic mathematical practices. Everyday mathematical practices is stated by the students' daily life experiences related to mathematics such as shopping, classifying and ordering. Academic practices are the activities in which the students perform their responsibilities such as forming and testing conjectures, form mathematical arguments and discuss about them in a way that mathematicians do. In another study, Font, Godino & Gallardo (2013) define the mathematical practices in a different way by stating that the mathematical practices consist of operative and discursive practices. Operative practices include the activities of reading, forming and operating the mathematical ideas and skills. On the other hand, discursive practice represents the reflection of the students produced through the former practice. Godino and Batanero (1994) defines the mathematical practice by stating;

We consider mathematical practice [sic] any action or manifestation (linguistic or otherwise) carried out by somebody to solve mathematical problems, to communicate the solution to other people, so as to validate and generalize that solution to other contexts and problems (Godino, Batanero, & Font, 2007, p. 129).

In this definition, the researchers emphasize the role of mathematical activities by the term of mathematical practices.

The definition and meaning of mathematical practices guiding the present study has been produced by Cobb et al. (2011). They have produced this definition based on the view of learning through individual and social processes, neither occurring without the other and nor dominating to each other. The definition and the formation of mathematical practice are made by them based on the emergent approach considering the individual and the community as reference points in learning process. Classroom mathematical practices are defined as “it is feasible to view a conjectured learning trajectory as consisting of an envisioned sequence of classroom mathematical practices together with conjectures about the means of supporting their evolution from prior practices” (Cobb et al., 2011, p. 125). The mathematical practices mean taken-as-shared ways of reasoning and arguing mathematically in the emergent perspective. Taken-as-shared way represents an environment in which the discussions about mathematical problems and ideas by mathematical symbolization and notations take place in the process of emergence of mathematical practices (Cobb et al., 1997). Based on this definition, the mathematical practices emerge in a classroom representing a social environment. The social environment formed in the classroom encourages the students to participate in the classroom mathematical tasks. Also, mathematical tasks make contributions to their mathematical knowledge, participation and skills while they reorganize their individual mathematical activities (Cobb & Yackel, 1996). It is important to examine the ways in which the students participate in the collective learning environment and make contributions to this environment and the process of development of classroom mathematical practices. However, the individual students’ interpretations, activities and understanding should not be ignored since they have effect on the process of emergence of mathematical practices. Although the definition of mathematical practices mainly focuses on collective learning, individual student learning has critical importance based on mathematics in which they engage in. This importance results from their contributions to the development of taken-as-shared mathematical ideas and mathematical practices having connection with the process of individual students' learning and understanding. In this respect, the examination and development of classroom mathematical practices focus on both individual and collective learning of the students. Moreover, this view is emphasized by Cobb and Yackel (1996) by stating that "students actively contribute to the evolution of classroom mathematical practices as they reorganize their individual mathematical activities, and conversely, that these reorganizations are enabled and constrained by the students' participation in the mathematical practices" (p. 180). In other words, the students reorganize their mathematical reasoning representing their individual learning in the process of participating in collective learning. Also, the evolution of mathematical practices occurs by their rearrangements of their individual activities with respect to local social situations where they take place and to whose emergence they make contribution (Cobb et al., 1997). In this respect, the learning takes place in a way that learners improve their individual learning by participating in the social context of learning including the views of all of the learners. The process of development of mathematical practices including mutual practices and students' own reasoning represents the reflexive connection of social and individual perspectives about mathematical tasks in the instruction through their mathematical development (Cobb & Bowers, 1999).

The process of emergence of mathematical practices in a taken-as-shared way is beneficial to develop mathematical practices and also mathematical understanding and individuals' reasoning. These practices represent the ways of understanding, reasoning, explaining and convincing others by justifications in a way that mathematical classroom community make them taken-as-shared for the particular mathematical content by specific mathematical tasks or ideas (Cobb et al., 2011; Stephan, Bowers & Cobb, 2003). They are observed in a social environment in which the instructor and the students take place in challenging situations since “classroom mathematical practices are … localized to the classroom and are established jointly by the students and the teacher through discussion; they emerge from the classroom rather than come in from the outside” (Stephan & Cobb, 2003, pp. 41-42). Mathematical practices identified by taken-as-shared methods of reasoning are ways or representations of the students’ knowledge based on their reflections. The emergence of reflections takes place in classroom discussions rather than their individual knowledge, thinking and strategies specific to a mathematical concept or a problem situation (Stephan, Bowers, Cobb & Gravemeijer, 2003). In other words, mathematical practices focus on social learning including the reflections of learners’ individual learning. In this respect, the information about classroom discourse, the ways of using tools, notations is attained through collective and social part of mathematics learning in the development of classroom mathematical practices (McClain & Cobb, 2001; Stephan & Rasmussen, 2002).

While students participate in the discussions in which the mathematical practices are formed, they develop their mathematical reasoning and understanding at the same time (Cobb et al., 1997) in an environment. This environment encourages the establishment of social and sociomathematical norms providing the development of mathematical practices (Stephan & Akyuz, 2012). These norms are necessary because of their support to the social environment in which the mathematical practices emerge by taken-as-shared strategy. These norms facilitating the emergence of mathematical practices provide information about the participation structure of the clas-srooms about the communication among learners and the instructor and the learners (Cobb et al., 1997). Despite of these norms’ common positive effect of encouraging mathematical practices, they have different properties. The classroom mathematical practices are more content-specific when compared with social and sociomathematical norms since these norms are not specific to mathematical ideas based on this definition (Stephan et al., 2003). In other words, social and sociomathematical norms are important since the mathematical practices emerge in a way specific to classroom community, mathematical content, the path followed in the instructional sequence and the problems that the students engage in. In this respect, the teacher and the students develop the classroom mathematical practices collectively by localized and formed in the classrooms. However, outside mathematical practices have effect on this developmental process based on its connection with local mathematical practices. The students make contributions to the local mathematical practices by the ways of their participation and experiences in outside mathematical practices (Whitenack, Knipping, & Novinger, 2001).

All of these explanations emphasize learning taking place in the classroom environments by the combination of two extreme positions and approaches of learning including individual and social aspects of learning operating equally. This view can be stated based on the views of social constructivism as emergent perspective. Based on this perspective, learning occurs by relating the social and individual processes with strong relationship in which the existence of one of them necessitates the other one by not separating. In the emergent perspective, individuals' mathematical development and understanding are examined through their participation in the social and cultural practices taking place in the classroom community (Cobb, 2000; Yackel & Cobb, 1996). This perspective illustrates a process including two connecting parts; the individual students take roles in developing communal practices and the community produces mathematical practices in taken-as-shared ways through reasoning, symbolizing and producing mathematical arguments (Cobb et al., 2001; Stephan, 2003). Therefore, the properties and perspectives of social constructivism are examined and taken into consideration in the present study. Moreover, classroom mathematical practices are generally examined through design-based research studies connecting instructional design and teaching in the literature (Cobb et al., 2011). There have been many research designed and conducted in this way. In the literature, there exist research studies examining mathematical practices at different grade levels and for different mathematical topics (Stephan & Akyuz, 2012; Stephan et al., 2003 Stephan & Rasmussen, 2002).

Stephan and Akyuz (2012) examined the classroom mathematical practices emerged within design-based research using Realistic Mathematics Education theory. With the teaching experiment taking place in a 7th grade classroom, the students' classroom mathematical practices designed by the concept of integer addition and subtraction were identified by testing and revising a hypothetical learning trajectory as a potential instructional theory for this concept with the tools such as financial contexts and vertical number lines. The learning of the students participating in the instructional sequence with 19 class periods by hypothetical learning trajectory was examined through the content and structure of the students' arguments about the concept of integer addition and subtraction. In the analysis process, Krummheuer's (1995) adaptation of Toulmin's argumentation model was used. Then, the argumentation logs determined in this way were used to identify the collective classroom mathematical practices by a three-phase approach with two criteria constructed and described in Rasmussen and Stephan (2008) and in Stephan and Rasmussen (2002). These mathematical practices representing students' construction of conceptual understandings of integers and their operations were determined through taken-as-shared mathematical ideas. Based on the findings of the study, there have been five mathematical practices with a limited number of mathematical ideas in each mathematical practice. In order to determine the effect of designed hypothetical learning trajectory on students' achievement, pretest and posttest designed by the researchers were conducted to the students. The quantitative findings obtained through these tests showed that students improved their understanding on the concept of integer addition and subtraction through this instructional sequence illustrating more achievement on the subtraction operation than the other operation.

In the study of Akyuz (2014), the researcher examined the classroom mathematical practices developed within design-based research using Realistic Mathematics Education theory to ten preservice mathematics teachers including eight junior and two senior grade students in a university. With the teaching experiment taking place in an elective course in mathematics teacher education program, the participants' classroom mathematical practices designed by the geometric concept of circle were identified by testing and revising a hypothetical learning trajectory with the tool of GeoGebra in dynamic geometry environment. The learning of them in the instructional sequence including 5 weeks and 4 hours in each week by hypothetical learning trajectory was conducted in inquiry-based and technology-supported teaching environment. The content and the structure of the students' arguments about the concept of circles were examined by the Krummheuer's (1995) adaptation of Toulmin's argumentation model. Emergent perspective and the scheme designed and described in Rasmussen and Stephan (2008) and in Stephan and Rasmussen (2002) were used to determine taken-as-shared mathematical ideas representing mathematical practices. Through the analysis, three sequentially emergent mathematical practices ordered based on complexity from lower level to higher one have been formed in the study. Another research designed and analyzed by the same methodology was conducted to preservice mathematics teachers in order to establish their mathematical practices in a dynamic geometry environment by Bowers and Nickerson (2001). In this study, prospective secondary mathematics teachers participated in teaching episodes designed by Geometer's Sketchpad. Their social norms, sociomathematical norms and mathematical practices in the undergraduate course in which the design experiment study was performed. In this environment, their learning was examined through their contributions to the whole class discussions. In a cyclic process including designing, testing, modifying and retesting the learning trajectory, their individual and collective learning were examined by establishing these norms and mathematical practices. Also, four mathematical practices were determined in the study by using Cobb et al.'s (1997) framework.

Stephan and Rasmussen (2002) examined the classroom mathematical practices developed within 15-week classroom teaching experiment using Realistic Mathematics Education theory to university students enrolled in a beginning course about differential equations for engineers. The learning of the students participating in the instructional sequence by a learning trajectory was examined through the content and structure of the students' arguments about the concept of differential equations by the Toulmin's argumentation model. The mathematical practices provided a systematic system for the learning of classroom community. Emergent perspective and the scheme including three phases designed were used to determine taken-as-shared mathematical ideas as mathematical practices. Through the analysis, there have been six mathematical practices with a limited number of mathematical ideas in each mathematical practice. Moreover, the researchers state that there are two important cases for the emergence of mathematical practices, time and structure. Based on these cases, the mathematical practices can emerge in a non-sequential manner.

In the study of Roy (2008), a design-based research was made to establish preservice teachers' classroom mathematical practices in whole number concepts and operations. In the study, a revised learning trajectory and instructional tasks designed based on Realistic Mathematics Education theory used in a previous classroom teaching experiment in the literature (Andreasan, 2006) was conducted in the study. Mathematical practices were determined by the same analysis techniques used in previously explained ones, Toulmin's argumentation model and Rasmussen and Stephan's three-phase methodology (2008). Four classroom mathematical practices were documented. Moreover, Content Knowledge for Teaching Mathematics (CKT-M) database described by Hill, Schilling, and Ball (2004) was conducted to the participants as pretest and posttest for a ten-day instructional sequence. The quantitative findings obtained through these tests showed that students improved their understanding of the whole numbers concept. Also, Wheeldon (2008) made a research for prospective elementary teachers, two classroom mathematical practices have been established by a hypothetical learning trajectory designed by Realistic Mathematics Education theory on the concept of fractions by the same analysis techniques based on the same methodology, Toulmin's argumentation model and Rasmussen and Stephan's three-phase methodology (2008). Furthermore, Andreasan (2006) conducted a hypothetical learning trajectory designed by Realistic Mathematics Education theory on the concept of whole numbers and three classroom mathematical practices were established by the same methodology and analysis techniques.

Argumentation takes place as the flow of the ideas by expressing, challenging and validating them. In this process, it is important for the students to understand others' ideas clearly, examine them based on different perspectives and communicate about them. In this process, when the students use tools, they can effectively represent their ideas.

The research of Johnson (2013) differentiates from the other research explained above. The other research focus on understanding and learning through their discussions but Johnson (2013) examined them through notations and symbols. The researcher examined student learning through mathematical practices as local changes and made implications. They investigated teaching experiment conducted by Realistic Mathematics Education for abstract algebra started with the context of symmetries of an equilateral triangle. The students also examined the notations and symbols. The researcher made analysis by Toulmin's argumentation model and Rasmussen and Stephan's three-phase methodology (2008). The researcher determined two local changes which were emerging symbols and notations, and the way in which they were used.

While all of the explained research focus on the improvements of students learning and understanding, Martin and McCrone (2003) paid attention on the development of a skill. They investigated the classroom mathematical practices about proof-construction ability of the teachers and their pedagogical choices contributing to these practices. They focused on developing the students' proof writing skills rather than their learning of a particular mathematical concept. Two different high school geometry classes instructed by these teachers were observed during four-month period with proof-based geometry lessons. The researchers determined three taken-as-shared classroom mathematical practices. They were *the importance of details in proof writing, the understanding that only certain methods are valid for establishing the congruence of overlapping triangles and marking diagrams as an essential part of the proof-writing process*. This research is also beneficial since it provided information about the effect of classroom micro culture and teachers' pedagogical choices on the learners' proof constructions and the emergence of mathematical practices.

Font and Planas (2008) focus on mathematical practices in a different perspective; that is, they emphasize different meaning of mathematical practices explained by Godino, Batanero and Font (2007) from the definition by Cobb et al. (2011) used in the present study. It represents the efforts to solve mathematical problems by discussing to validate and generalize the solution. They examined the mathematical practices, socio-mathematical norms and semiotic conflicts based on the onto-semiotic approach. Mathematical practices were established focusing on the cognitive conflicts through discussing and forming the solution to a problem. Learning happened through changes in participants' positioning. In this respect, semiotic conflicts were explored while the learners were solving and removing these conflicts. In the study, the data were collected through teaching episodes designed based on problem solving. Learning occurred understanding the experiences and interpretations of others for socio-mathematical norms and mathematical practices.

As it has been observed, mathematical practices have been examined for different mathematical topics (Stephan & Akyuz, 2012; Stephan et al., 2003 Stephan & Rasmussen, 2002). On the other hand, based on the learners' failure in geometry and the difficulties of the students in learning and understanding of geometry, it has become crucial to identify which practices learners form through learning geometry. Also, it is important to think that mathematical practices are connected to social and socio-mathematical norms as other dimensions of interpretative framework since they emerge in a social learning environment. Studying mathematical practices for learning geometrical concept of triangles for preservice middle school mathematics teachers was aimed by the problem-based learning strategy, the problem solving method and van Hiele geometric thinking theory through geometric constructions and argumentations in the present study. In this respect, it was also necessary for the study to design hypothetical learning trajectory to organize the tolls and activities and make predictions about instructional sequence in order to meet these aims.

2.3 Social Constructivism and Mathematical Practices

Social constructivism has importance on the current study for the emergence of classroom mathematical practices since they are developed through emergent perspective called as social constructivism. Social constructivism as a kind of constructivism specifies the context in a social way, culture and learning in a collaborative way (O'Donnell & King, 1998). It proposes that learning should be thought with social interactions because of its socio-cultural aspects. It is created with respect to Vygotsky's ideas (Palmer, 2005) considering the impacts of other people, language used between the learner and the other people, objects benefited from in this social interaction, society and culture taking place in the process of forming knowledge actively (Jones & Brader-Araje, 2002). Also, this philosophy is about the effects of communication, language and culture on the process of learning (Fosnot, 1996; Jonassen et al., 1995). According to Vygotsky, the potential of an individual about learning can be increased by making interactions with the people having knowledge about the related issue. In other words, through the process of interaction taking place between learners having some amount of knowledge, they can improve their knowledge and understanding. The amount of knowledge obtained by communicating is more than the one obtained by spending effort and studying alone (Liang & Gabel, 2005). The social interactions performed with the aim of obtaining knowledge can be encouraged with the help of language and artifacts. These artifacts refer to the tools with the aim of shaping and transferring mental processes. This approach provides individuals opportunities to improve knowledge and skills such as problem solving, synthesis, critical and creative thinking and deep understanding (Terhart, 2003).

In the environments designed with respect to the theory of social constructivism and interpretative framework, the responsibility of the teachers is to organize environments with the aim of acquiring and helping the learners. Therefore, they can obtain and improve skills such as analysis, synthesis, critical and creative thinking and deep understanding (Trigwell, Prosser & Waterhouse, 1999). Moreover, it can be claimed that the social constructivist approach is important because of its positive effect on the learners. In this respect, it is necessary to design experiment based on social constructivist approach since it provides effective learning environments (Woo & Reeves, 2007). Design experiments were used in the current study since it included arranging significant and different kinds of learning and then working on them in a systematic way related to the context encouraging them. Also, it was preferred since it provided deep and effective understanding for learning ecology. In this way, a beneficial lesson could be prepared and tested to use in the education of mathematics teachers by design experiment. There have been research exemplifying these studies and situations in the literature. In a research of design experiment, sessions for teaching with teacher, experimenter and student in order to form small-scale version for ecology of learning were suggested (Steffe & Thompson, 2000).

In the design experiment study of Steffe and Thompson (2000), they collected the data about the learning and reasoning of the students by the first hand. In this way, powerful understanding about the students' constructions of the concepts was obtained. According to the findings of their study, mathematics should be taught by using the mathematical realities belonged to the students rather than mathematical realities of the teachers. They confirmed the fact that the mathematics must be taught considering the students' properties such as prior knowledge, history and achievement. In a different design experiment study, it was conducted to identify classroom practices, a teacher had the role of being member of the research team (Cobb, 2000). Simon (2000) proposed design experiments in which the researchers made organization and work about education to the preservice teachers. The researcher conducted a different application of design experiment as Teacher Development Experience. In this methodology, the researcher served as the instructor of the classroom.

In an example of the design experiment study of Lehrer and Schauble (2000), the researchers helped inservice teachers improve about their professions. They used the design experiment in their study. They investigated the learning of the students and the teachers with 45 teachers and the students in their classes including mathematics and science instructions. The levels of their classes included the range of grades 1-5. This study was conducted with the aim of obtaining knowledge to develop the process of thinking and learning of the students and the daily practices of the teachers through the processes of forming, testing, evaluating and modifying the models. They investigated the learning and thinking processes of the students and the teachers based on longitudinal changes, while they progressed through this range of grade levels. They explained different kinds of professional development and teacher practices that could enhance student learning and thinking. Social constructivist theory has been also effective about educating preservice teachers as the teacher candidates as it has been observed in previous research (Akar, 2003; Holt-Reynolds, 2000; Jadallah, 1996; Kroll & Laboskey, 1996). In other words, in the present study, the design experiment was conducted based on the social constructivism since the environments designed in this way could contribute to the construction of understanding from many perspectives. The learning trajectories were organized based on the roles and responsibilities of the teachers and the learners explained in this philosophy.

The environments including the communication and interaction between the students and the teachers are important in mathematics education (Kovalanien & Kumpulanien, 2007). If these communication processes take places in meaningful manner, they become beneficial to mathematics education. The environments including such communications can also be created with the help of design experiments and hypothetical learning trajectories explaining the teachers' predictions based on the students' learning and geometric reasoning. Also, instructional sequence providing opportunities for mathematical argumentation can be effectively organized with design experiment by a research team. Therefore, mathematical discourse and argumentation as kinds of communication happening in the classrooms are explained in the following section.

The interpretative framework as the emergent perspective or social constructivism was used in the present study. In other words, learning taking place in the designed instructional sequence was examined by using the interpretative framework explaining learning based on psychological (or individual) perspective and social (or group) perspective (Cobb, 2000; Cobb & Yackel, 1996). In the emergent perspective, learners evaluate and determine their mathematical understanding while they make contributions to the mathematical practices for the groups (Cobb & Yackel, 1996; Yackel & Cobb, 1996). In that respect, sense making processes of the individuals and the groups are taken into consideration simultaneously and equally to identify the classroom dynamic (Cobb, 2000; Yackel, 2002). While analyzing classroom environment with the emergent perspective proposed by Cobb and Yackel (1996), the interpretive framework is used to examine learning in this environment in a different way as illustrated in Table 1.

The social and psychological perspectives are separated into three factors. The social part of the emergent perspective includes social norms, sociomathematical norms, and classroom mathematical practices with closed relationship with each other. In DBR, the theories are produced based on particular learning processes related to the design by being named as local instruction theories. Hence, the designed learning environments and collective learning environment based on the actual HLT implemented with supports of learning and the behaviors of the instructor and PMSMT participating in the study. In this way, DBR is examined in light of the designed setting and the participants rather than all environments with curricular goals. Mathematical practices as the focus point of the current study can be effectively examined in a social environment in DBR so the mathematical practices can be clearly examined through social and sociomathematical norms (Roy, 2008). The social perspective main focus of the data collection and analysis of the current study can be illustrated in the Table 1.

Table 1 Interpretative Framework

Social Perspective	Psychological Perspective
Social norms	Beliefs about one’s role, others’ roles, and general nature of mathematical activity in school
Sociomathematical norms	Mathematical beliefs and values
Classroom practices	Mathematical conceptions and activity

The first domain of interpretative framework, social norms, is examined benefiting from regularities taking place in the activities in the classroom and identified jointly by instructor and the learners as the members of the classroom community. They extract the structure of participation taking place in instructional sequence in the classroom (Stephan & Cobb, 2003). In that process, the identity and role of each individual are determined from the interpretations happening in the social interactions (Yackel, 2002). The second domain, sociomathematical norms, include “a different mathematical solution, a sophisticated mathematical solution, an efficient mathematical solution, and an acceptable mathematical explanation” (Cobb & Yackel, 1996, p. 178). The last domain, classroom mathematical practices, is the domain in which the participants produce mathematical explanations while engaging in pedagogical content tools.

Classroom mathematical practices emerge in a social environment based on the connection of social and socio-mathematical norms. In this respect, it is important to design hypothetical learning trajectory to encourage their learning through communicating in the classroom and establishing these norms. Therefore, the hypothetical learning trajectory was designed by geometric constructions referring to problem situations and tools and argumentations in the current study.

2.4 Hypothetical Learning Trajectories

Learning trajectories can be stated in a way that they are “successively more sophisticated ways of reasoning within a content domain that follow one another as students learn” (Smith et al., 2006, p. 1). It can also be added that “a hypothesized description of successively more sophisticated ways student thinking about an important domain of knowledge or practice develops as children learn about and investigate that domain over an appropriate span of time" (Corcoran, Mosher, & Rogat, 2009, p. 37). By HLT, teachers can make predictions on student learning and then testing them in practice. In this respect, it becomes possible to talk about the hypothetical nature of the learning trajectories as a bridge linking the theory of constructivism to practice (Duncan, 2009; Simon, 1995). In other words, in the process of the teaching period, the teachers have the opportunity to test the designed hypothetical learning trajectories (HLT) and make modifications based on the experiences obtained in this process. Also, it is possible to explain the HLT as a construct for teaching since "actual learning trajectory is not knowable in advance" (Simon, 1995, p. 135). HLT as a way of connecting constructivist theory to practice can be defined as ". . . the teacher's prediction as to the path by which learning might proceed. It is hypothetical because the actual learning trajectory is not knowable in advance and it characterizes an expected tendency" (Simon, 1995, p. 135). Hence, the nature of HLT related to not being resistant to change increases its benefits for teaching and learning by making necessary revisions on it. The construct of a HLT can be accepted as a cognitive tool improving mental processes and mathematical learning actions constructed with respect to the philosophy of constructivism (Clements & Sarama, 2004).

In the present study, HLT based on the constructivist philosophy was used since it provided the teachers a framework for supporting an understanding of students' thinking and learning of specific mathematical concepts. HLT includes the teachers' predictions about the progress in teaching sequence. In other words, HLT explains the usage of the teachers' predictions made with respect to the teachers' knowledge and assessments about student knowledge and their history about how the learning may happen or how the learning process may happen. Learning trajectories make the link between teachers' knowledge and their students' actions around three elements such as learning goals, learning activities and hypothetical learning process (Simon, 1995). In other words, in HLT, there exist learning goals of teaching process, learning activities and the ideas about how the process will go on in the classroom designed based on the predictions of the teachers. The teachers make the predictions by examining the student's learning and reasoning carefully considering their actions in the classroom, the results of assessments about them and their history. In this way, HLT helps the teachers understand their students' learning and thinking processes. Learning trajectories are identified as a useful attempt for assessment (Battista, 2004) and teacher education (Wilson, Mojica, & Confrey, 2013). Moreover, when the effects of the argumentations are explored, they provide information about the classroom environment designed with respect to social constructivist approach as the emergent perspective needed for the development of classroom mathematical practices to investigate the classroom environment effectively. Mathematical argumentations help the researchers analyze how the students share their ideas in a systematic and clear way support and refute the others' ideas in a scientific way using their ideas in a collective learning environment. This form of discussion may encourage the students participate in the classroom discourse effectively. Also, the students need to understand the concept carefully while producing mathematical argumentations.

In light of the explanations, it was considered that it could be important to design the lessons for the education of preservice mathematics teachers by using mathematical argumentations in the present study. These lessons designed with the help of HLT were expected to be beneficial through the process of testing the classroom experiences in a hypothetical manner (Andreasan, 2006; Wheeldon, 2008). Then, necessary modifications could be made based on these experiences. Also, mathematical argumentations were expected to be beneficial for the education of preservice mathematics teachers since they could effectively think about the theorems or properties of the concepts of triangle and how they were produced and which kind of properties were connected in a collective learning environment. In other words, mathematical argumentations could direct the students to make reasoning about the concept of the triangles. They could examine how these properties were formed by examining the reasons of them and the ways of this formation. In this respect, preservice mathematics teachers could effectively understand triangles in an environment designed with respect to the social constructivist theory and the lessons designed as HLT and tested regularly through the classroom experiences. HLT was designed by geometric constructions applied by following the way suggested by Simon (1995). Also, by examining the characteristics of the mathematical argumentations formed by the participants through these lessons, beneficial information could be obtained to make implications for the process of their learning and reasoning (Smith, 2010). In this respect, beneficial lessons about triangles for training preservice middle school mathematics teachers could be provided to the literature. This could be an alternative and designed through the process of assessing regularly based on the real experiences in the classrooms in order to educate preservice middle school mathematics teachers effectively.

In the present study, the instructional sequence in which the classroom mathematical practices emerged was performed by the hypothetical learning trajectory. The hypothetical learning trajectory was designed based on constructivism, social constructivism and problem-based learning. The social learning environment helping the establishment of social norms, socio-mathematical norms and mathematical practices was encouraged by argumentations in which stating, analyzing, discussing and convincing the geometrical ideas were made. Moreover, the tasks including the tools of compass and straight edge in the hypothetical learning trajectory were designed considering the properties of van Hiele geometry thinking levels and geometric constructions. In other words, van Hiele geometric thinking levels were considered to determine the preservice middle school mathematics teachers' geometric reasoning to organize their activities referring to problem situations, predicting their possible answers and reasoning in the study. Their van Hiele geometric thinking levels were also provided to help them use the tools of straight edge and compass to examine the triangles, their properties and prove them.

2.5 Argumentation in Mathematics Education

In design-based research, learning taking place in a social setting is an important measure to be evaluated and social interaction is important in providing mathematical learning (Cobb, 2000). In general perspective, by participating in mathematical discourse which is convenient for learner-centered classrooms, the students make reasoning aloud and explanations about what they think and how they think about them (Hufferd-Ackles, Fuson, & Sherin, 2004; Yackel & Cobb, 1996). In the environments including mathematical discourse, the individuals can achieve learning and understanding by thinking and interacting with other people. They can provide this achievement by modifying their thinking schemes when the confusions in their thinking in the process of the mathematical discourse are observed (Steffe & Tzur, 1994). These environments illustrate "communication as a process of mutual adaptation wherein individuals negotiate meanings by continually modifying their interpretations" (Cobb & Bauersfeld, 1995, p. 8). Also, it is clear that there exist positive impacts of communication through interactions of teacher-student and students on learning (Lampert & Cobb, 2003). Mathematical discourse is beneficial for the teachers since they can form an environment including multiple ways of constructing mathematics and solving the mathematical problems for the students (Fullerton, 1995). It is beneficial since it provides opportunities for the students to challenge, make clear, judge and justify their ideas related to mathematics (Andrews, 1997; Owen, 1995).

Argumentation as a kind of mathematical discourse explains how students form mathematical justifications interpreted by them and use them in the communications. It can be claimed that producing the mathematical arguments refers to the understanding of conceptual mathematics (Lampert, 1990). The students use the rules and theorems in a way that they are memorized without questioning and knowing when, how and why to use them. This problem can be removed by producing mathematical argumentations since the learners can acquire the mathematical knowledge and skills necessary for this knowledge about these theorems and rules by questioning and understanding effectively. Also, they can improve reasoning skills necessary for mathematical learning and understanding. Effective learning can be provided by deep engagement of the ideas by problem solving and critical thinking skill of argumentation so that conceptual change occurs through argumentation practices considering their qualities (Abi-El-Mona & Abd-El-Khalick, 2011; Jonassen & Kim, 2010).

The term of the argumentation can be described as a "social phenomenon, when cooperating individuals tried to adjust their intentions and interpretations by verbally presenting the rationale of their actions" (Krummheuer, 1995, p. 229). Argumentation can also be specified as a process with try-outs of an individual with the aim of persuading others about a claim. The learners can form a common shared understanding related to the concepts by discussing and forming mathematical argumentations. While producing argumentations and shared understandings through discussions, there exist justifications, active negotiation of mathematical claims and modifications of the concepts, statements and ideas used in mathematical discussions (Forman et al., 1998). Through mathematical arguments, the importance of previous knowledge cannot be ignored. In the process of producing arguments, the previous knowledge is used actively with the aim of reaching a shared understanding by discussing and producing statements about it (Cross, Taasoobshirazi, Hendricks, & Hickey, 2008). In this respect, argumentations are useful to examine the students' understanding by determining mathematical practices defined by Cobb et al. (2011). By the taken-as-shared way of understanding, the students use the claim produced in previous argumentations in different parts of latter arguments. In other words, they use knowledge or conclusion representing the claim of the previous argumentation as data, warrant, backing or rebuttal of another argument produced in latter teaching episodes. In this respect, there have been studies to examine the mathematical practices of the students from different grade levels (Akyuz, 2014; Stephan & Akyuz, 2012; Stephan & Rasmussen, 2002; Roy, 2008; Wheeldon, 2008). For example, in the studies of Akyuz (2014), Roy (2008) and Bowers and Nickerson (2001), the mathematical practices of preservice mathematics teachers emerged in collective learning environment were examined. In order to analyze their learning in social environments, Toulmin's model of argumentation was used. Based on mathematical argumentations, their learning process was illustrated in taken-as-shared way.

The argumentations are useful for the learners to share and validate their ideas about a particular concept. In this process, the elements of Toulmin's argumentation model represent the ways that the students express, challenge and validate their ideas. They can be either supported by the more than one warrant and backings or refuted by the rebuttals. Through validating their true claims, the students can provide backings and warrants referring to qualifications for the claims. Based on this view, Inglis, Mejia-Ramos and Simpson (2007) conducted a research in order to examine the importance of qualifications. They investigated modelling mathematical argumentation and the role of qualification in the argumentations. Before conducting the study to the participants, they considered that the more mathematically the students think and discuss, the more qualifications they use. In this respect, in order to emphasize that the full argumentation including warrant, backing and rebuttals is needed to be used since they think about the concept in detail and different perspectives. They examined the highly talented postgraduate mathematics students' arguments by Toulmin's model of argumentation. It was found that the restricted form of the model limited the learners' thinking with absolute conclusions and these talented mathematicians needed to use full version of the model to represent their ideas and reasoning effectively. This finding can result from the case that the students validate and express their claims through mathematically rich and connected expressions. By the study, their assumption was confirmed and illustrated in this way. Moreover, when the students were higher grade levels such as preservice teachers, they could use mathematical proofs, theorems and properties in the argumentations.

Argumentations and proofs are related to each other since both of them are composed of justifications and expressions made to convince others about the truth of a statement (Chazan, 1993; Pedemonte, 2007). In the study of Pedemonte (2007), the connection of argumentation with proof was examined through the teaching experiment conducted to 12th and 13th grade students in France and Italy. The researcher analyzed the data in order to test the hypothesis about proof as a particular case of argumentation by Toulmin's model of argumentation. Also, the structures of an argumentation about a conjecture (abduction, induction, etc.) and its proof were examined. Based on the findings, it was observed that they had structural differences in spite of their connection in different perspectives. This structural difference was important while producing argumentations and proofs for the mathematical ideas and conjectures. However, the argumentations and proofs have connected skills encouraging each other.

In argumentations, the tools that the students use to learn the concepts while they are engaging in their mathematical tasks facilitate their understanding and learning. By using the tools, students can form argumentations effectively by understanding others represented using tools. Also, dynamic geometry software as tool enhances the students' understanding and learning of mathematical concepts (Athanasopoulou, 2008). In this respect, there have been research to examine the effect of the environments enhanced by technology on students' argumentations (Hollebrands, Conner & Smith, 2010; Lavy, 2006; Maher et al., 2006). This effect has been examined for the students from different grade levels. Hollebrands, Conner and Smith (2010) investigated the explanations of college students created in a dynamic geometry environment based on the structure of the arguments. Their arguments were separated into three groups with respect to the properties of the warrants used in the classrooms considering the usage of technology. These groups were explained by explicit warrants without technology, an explicit warrant with technology and warrant with merely a diagram on the screen. In the study, it was found that their argumentations and especially the properties of the warrants changed based on the ways of using technology in the classroom. The students produced well-qualified warrants including necessary properties and theorems of the mathematical concepts in technological environment when compared with non-technological environments. Also, the effects of technological tools in argumentations for the students in lower grade levels was examined in the studies of Lavy (2006) and Maher et al. (2006). The structure and content of the arguments formed by the students in a technology enhanced classroom were analyzed by Toulmin's model of argumentation. The students produced arguments in an environment in which the technology was used as mediator between students and means to collect data. These studies found that technology had positive effects on understanding content and explicitness of the warrants, structure of the arguments and challenging the claims and warrants. Therefore, it can be stated that tools such as dynamic geometry software has the facilitating role in forming argumentations. When the tools are used effectively in the classrooms, implicit and clear warrants understood by the students can be produced. Also, when the structure of the arguments are compared, the ones formed by using technological tools are better than the others.

Teachers have important responsibility in the process including effective mathematical argumentations. They should direct the students to participate in the discussions, to form particular claims, explain their ideas and produce mathematical arguments including necessary terms and concepts. There have been different research examining the effect of teachers' roles and behaviors on the mathematical argumentations. Veerman, Andriessen and Kanselaar (2002) examined collaborative argumentations of undergraduate students formed in the eight-week course in Educational Technology and Computer-based learning (CBL). They focused on the connection between questioning and argumentation in different mathematical tasks. The role of the tasks and the context were identified in the study. It was found that question asking was related to argumentation based on the tasks, instruction, medium, role of the participants and how to be represented in the learning situation. Based on this study, it was found that it was important to select and design appropriate instructional tasks and ask question accurately in order to form argumentations. In this respect, the learning environment should be designed including the tasks related to asking questions so that argumentations can be formed effectively and learning can be encouraged. In this respect, the teachers have important role of providing asking questions in the environment.

Yackel (2002) investigated the teacher roles in classrooms of the grade levels from elementary school to college level including argumentations. It has been found that the teachers should provide students' mathematical activity designed including argumentations, the negotiation about the classroom norms in order to enhance the argumentation, opportunities for the students to make interactions with other students with the aim of forming arguments, and argumentative supports such as data, claim, warrants, and backing. In other words, the teachers should support a good start point for mathematical argumentations to the concepts and tools when they are newly produced. Moreover, the teachers should provide instructional sequence including beneficial argumentations by acquiring deep understanding of mathematical concepts. The results of this study provide essential knowledge for the current study about the role of the teacher and how to provide an environment including mathematical argumentations. Based on the roles of teacher explained in this study, the instructor benefited from this knowledge while guiding and forming the discussions in the study.

In the study of Yackel and Cobb (1996), the roles of the teachers have been investigated in the classrooms with argumentations. They explained that the teachers were important since they were the formers of the mathematical community including argumentations and they have crucial roles in this process. With respect to the findings of the study, they proposed that social and sociomathematical norms were important in forming argumentations since they could affect student learning of mathematics. Furthermore, the more the students partcipate in the process of negotiation of sociomathematical norms, the more autonomous they became about learning. In this respect, it could be thought that it was important to produce social and sociomathematical norms in the classrooms to form the lessons including argumentations. Therefore, necessary precautions were taken to provide opportunities for the students to participate in the activities of forming them in the current study.

To conclude, there have been research about argumentations in the literature. Through these research, it can be stated that argumentations are effective to improve learners' academic achievement, understanding, conceptual learning, and reasoning in a way that they share and challenge their ideas by problem solving and making communication with their class-mates. It has still been necessitated to examine their argumentations through geometric reasoning and conceptual knowledge about particular geometry concepts such as triangles. Therefore, the present study was designed to improve preservice middle school mathematics teachers' learning and understanding of triangles through argumentations. Also, geometric constructions were used since the tools were necessary to encourage argumentations in the classrooms.

2.6 Problem-based Learning in Mathematics Education

John Dewey described problem-based learning (PBL) as a teaching strategy in order to solve medical schools' problems initially. Then, it has attained importance and functioned in all grade levels from primary to college levels. Because it activates students to learn by using their prior knowledge and interests and makes connections with the real world (Goodnough, 2006). In this respect, PBL is identified as "focused, experiential learning organized around the investigation, explanation, and resolution of meaningful problems" (Hmelo-Silver, 2004, p. 236). PBL can be explained with the help of constructivist approach because of their common principles. Both of them are designed based on student-focused instructional approach. It should be provided that the students understand and accept the purpose of learning activities so that they can perform their main responsibility of learning effectively. This can be encouraged by designing tasks that can be manipulated by the learners so that they can construct their own learning and understanding by manipulating and analyzing complex parts of the tasks and topics in order to produce critical and creative means (van Tassel-Baska, 1998). In this process by encouraging students to participate actively in their own learning and learning of the others, tasks and environments challenging the students' thinking skills should be designed based on the students.

The teachers attain the responsibility of supporting, encouraging and facilitating the views, discussions, learning of the students and the environment. Moreover, in PBL environment, the role of the teacher can be explained as a facilitator or coach encouraging the learners to ask reflective questions (Wang et al., 1998; Greenwald, 2000; Kolodner et al., 2003) that "force them to justify their approach and explain their conclusions" (Kolodner et al., 2003, p. 505) in a way that they continually test and revise their hypotheses and ideas (Kolodner et al., 2003). In this process, the learners form the solution reasoning and redesigning their thinking with the help of teacher. The teacher asks the questions and provides the guidance for the students to challenge their thinking and organize their own learning (Greeanwald, 2000) by understanding and attaining the knowledge in detail and necessarily (Uyeda et al., 2002). In doing so, the instructor scaffolds students' learning and provides clues with the help of encouraging learning and thinking as the facilitator (Hmelo-Silver, 2004). In this case, the teacher can provide hints to help learners solve the problem and learn it by facilitating the process. The roles of the instructor in PBL are summarized by Torp and Linda (2002) in a way that student learning can be provided encouraging their motivation on problem solving by teachers' actions about modeling and coaching strong cognitive and metacognitive behaviors (Araz, 2007). In addition, PBL focuses on the types of cooperative and independent learning. By cooperative learning, the structure of the students' works in small groups encouraging active learning, participation, interaction and discussion is emphasized (Rivarola & Garcia, 2000; Silberman, 1996). In this respect, it was considered that PBL was appropriate for the current study by its explained properties. Also, it could effectively produce an environment including the discussion and mathematical argumentations.

The learners attain the responsibility of examination of necessary and meaningful questions cooperatively by practicing the skills of decision making and problem solving (Frank & Barzilai, 2004; Kolodner et al., 2003). PBL describes a kind of learning process in which the learners have the responsibility of learning through reasoning, decision making and problem solving. Based on these properties of PBL, another definition of it is "a cognitive apprenticeship approach that focuses on learning from problem-solving experience and promotes learning of content and practices at the same time" (Kolodner et al., 2003, p. 497) and inverting "the order of learning procedures to make it reflect much more realistically the learning and problem solving that occurs in professional practice" (Gallagher, Stepien, Sher, and Workman, 1995, p. 137). In PBL, the instruction is performed by taking the problem at the center of the learning. In this process, the instruction is the strategy of giving students the problem and then providing the learning through problem solving (Burgess, 2004). However, all of the questions representing related mathematical concepts do not always refer to a problem. The problems providing learning in this way are ill-structured problems which are unclear and open-ended problems used to learn new concept with prior knowledge through finding the solution (Greenwald, 2000). In this environment, the process of finding solution can happen with the help of others' views and experiences and instructor guidance (Greenwald, 2000). This way was the critical one for the present study to produce whole class discussion in which the learners formed mathematical argumentations. In addition to improving problem solving skills, PBL also provides opportunities to make connection between various topics and prior knowledge. In this respect, the instructional sequence and hypothetical learning trajectory of the present study was designed in light of these properties of PBL. For example, previous knowledge of preservice middle school mathematics teachers about transformation geometry was used to examine the congruence and similarity of triangles to learn more complex properties of triangles in the present study. Furthermore, Frank and Barzilai (2004) explain that there are four benefits of PBL in instructional sequence; understanding and attaining the deep knowledge of content and process, encouraging the independent learning and taking responsibility and providing student learning by active engagement. Also, it can be stated that problem solving skills are improved through problem-based learning strategy.

The common definitions of problem solving are "a situation where something is to be found or shown and the way to find or show it is not immediately obvious" (Grouws, 1996, p.72), "to have a problem means: to search consciously for some action appropriate to attain a clearly conceived, but not immediately attainable aim" (Polya, 1962, p.117) and "the situation is unfamiliar in some sense to the individual and a clear path from the problem conditions to the solution is not apparent" (Grouws, 1996, p.72) benefiting from prior knowledge (Frensch & Funke, 1995) where a problem is defined as "a situation for which one does not have a ready solution" (Henderson & Pingry, 1953, p.248). Through these definitions, there are assumptions to be provided for a situation to become a problem. In this respect, it can be stated that it is needed to determine whether a situation is problem since it changes based on individuals and their experiences (Henderson & Pingry, 1953; Lester, 1980). Therefore, a situation is a problem in case of holding some criteria. These criteria can be explained in a way that an individual must realize the situation and be willing to remove it, and then he cannot directly move on the solution process but he insists on to reach the solution (Lester, 1980).

Problems are beneficial to design “an environment for students to reflect their conceptions about the nature of mathematics and develop a relational understanding of mathematics” (Skemp, 1978, p.9) with the learning opportunities. When the learners face with the problem, they have cognitive conflict since the situation does not fit their existing knowledge. Then, they start working on it. Through this studying process, they try to make some modifications on their existing knowledge by learning additional ones since “they confirm or redefine their conceptual knowledge, relearn mathematics content and become more open to alternative ways of learning mathematics” (Steele & Widman, 1997, p.190) since problem solving is not remembering the memorized facts or using and following well-learned operations or procedures (Lester, 1994). In other words, through problem solving, learners attain the skills of organizing their mathematical ideas, participating in the discussions, defending their ideas and convincing others on their ideas. Hence, the learners realize the dynamic nature and structure of mathematics and attain deep insight of mathematics (Manuel, 1998; NCTM, 2000). In this respect, the necessity of problem solving in mathematics curriculum is understood. Hence, problem solving is strongly proposed to be placed in school mathematics, used for teachers and practices as much as possible for students (NCTM, 2000).

The problems are at the core of learning and doing mathematics since problem solving provides teachers a good strategy and tool to teach mathematical concepts (Manuel, 1998; Schroeder & Lester, 1989). Problem-based learning including problem solving is used in classrooms as a teaching strategy in a way that “the teaching of a mathematical topic begins with a problem situation that embodies key aspects of the topic, and mathematical techniques are developed as reasonable responses to reasonable problems" (Schroeder & Lester, 1989, p.33). Moreover, problem-based learning including problem solving provides opportunities to present, share and discuss their mathematical ideas and to find the solution through discussing and evaluating (Manuel, 1998). These opportunities facilitates the formation of mathematical argumentations and also emergence of mathematical practices. Therefore, the HLT and instructional sequence were designed based on problem-based learning strategy. Through engaging in problems, learners can learn and improve their understanding of mathematical concepts (NCTM, 2000). Based on this nature of problem-based learning, it was essential to make teaching via problems in the instructional sequence, it was possible to identify the classroom mathematical practices by the argumentation forming the mathematical practices of the present study.

Problem-based learning has effect on achievement of the learners in all grade levels with respect to the findings of many research. In the lessons designed based on this strategy, the students are provided by the opportunities and important practice forming useful learning environments for them. They can attain problem-solving skills by engaging in the different contexts having connection with real-life (Apaçık, 2009; Dochy, Segers, Bossche & Gijbels, 2003; Efendioğlu, 2015; Cantürk-Günhan & Başer, 2009). The problem-based learning occurs in the learning environment by taking the problem at the heart of the lessons. By using problems, the students face with the challenge situations about particular concepts. Through understanding the problem and designing the plan for the solution, they examined their previous knowledge in detail. Then, when they do not form a solution plan for the problem, they try to make connections between the related concepts they learned previously. Hence, they understand and learn the particular concept explained in problem situations deeply by making connections with other mathematical concepts. Also, they illustrate success and achievement while solving the problems since they understand the concept effectively in problem-based learning environment (Polya, 1962; Posamantier, 1998). These processes occur in the classrooms in all grade levels from primary to college level. Moreover, the research in the literature illustrate that problem-based learning improved the learners' achievement of the students in elementary and high school grade levels (Apaçık, 2009; Boren, 2012; Cantürk-Günhan & Başer, 2009) and preservice teachers (Banes, 2013; Efendioğlu, 2015; Hodges, 2010).

Through engaging in problem situations in problem-based learning, the students perform the activities and tasks cognitively. The students acquire new knowledge benefiting from the prior knowledge. They use, elaborate and restructure the previous knowledge (Schmidt, 1993). When the students face with the problems for the first time, they benefit from their prior knowledge to understand, analyze and develop a plan for solution. In this way, their prior knowledge is strengthened. Then, they start to examine necessary knowledge to solve the problem situation. Through problem solving, they acquire new knowledge. In the process of acquiring the new knowledge structured by previous one, they understand the mathematical concept deeply by sense-making and their interests (Kahan & Wyberg, 2003). In the process of the engagement in problem situations, understanding and comprehension of mathematical concepts are provided (Apaçık, 2009; Banes, 2013; Boren, 2012; Cantürk-Günhan & Başer, 2009; McCarthy, 2001). Hence, through engaging in the problems, they improve their understanding and learning the concept.

Through improving the conceptual understanding by problem-based learning, the impact of this strategy on the learners' psychological aspects such as attitudes toward mathematics and mathematical concepts, self-efficacy, motivation and decreasing mathematics anxiety (Banes, 2013; Boren, 2012, Cantürk-Günhan & Başer, 2009). Problem-based learning provides an environment to the students by taking the problems as the focus point of the lessons. When the criteria of individuals' awareness of the situation and willingness to remove it for a situation for becoming a problem are considered, it can be stated that problems and also problem-based learning have psychological effects (Lester, 1980). In this respect, it can be stated that a situation refers to a problem because the students are willing to handle it so that their motivation can be supported. This case encourages their motivation to remove the challenge situation and provide solutions to the problems (Rotgans & Schmidt, 2012). Also, the students understand and learn the mathematical concepts through problem-based learning. Therefore, understanding and learning are related to removing the challenge situation by providing the solution to the problem. In this respect, it can be stated that being motivated to solve the problem supports being motivated to learn the concept. Another psychological aspect is self-efficacy. Self-efficacy beliefs are affected by problem-based learning (Boren, 2012; Pajares & Graham, 1999). Because of the definition of self-efficacy as the individuals' judgment about their own capabilities to perform the required tasks effectively, it affects the time and effort needed for them to perform the tasks successfully by sustaining to complete it (Bandura, 1997). The previous research also indicate that self-efficacy is connected with mathematics problem solving and their performance in solving the problems (Hoffman, 2010). Furthermore, problem-based learning decreases the learners' mathematics anxiety (Banes, 2013). Through solving the problems, they actively engage in them and then they decrease their anxiety. The previous research indicate that when the students and preservice teachers' mathematics anxiety was compared before and after problem-based learning, it has been found that this strategy impacts their anxiety by decreasing their anxiety levels. Also, problem-based learning affects the attitudes of the students toward mathematics positively as it has been observed in research (Banes, 2013; Cantürk-Günhan & Başer, 2009). Attitude is formed through students' past experiences as it happens in mathematics anxiety (Allport, 1935). By problem-based learning, useful opportunities in which the students perform the tasks effectively and represent success about mathematical concepts can be provided to the students. Hence, problem-based learning environment affects the attitudes toward mathematics positively (Banes, 2013; Cantürk-Günhan & Başer, 2009).

To conclude, there have been research about problem-based learning in the literature. Through these research, it can be stated that problem-based learning is effective to improve learners' academic achievement, understanding, conceptual learning, motivation, self-efficacy and skills such as critical thinking, sharing and challenging their ideas, problem solving and making communication with their class-mates. However, it has been necessitated to examine problem-based learning in social environments by improving their argumentation skills through geometric constructions. In is respect, the social learning environment and hypothetical learning trajectory including argumentations and geometric constructions about triangles was designed by problem-based learning in the present study.

2.7 Van Hiele Geometric Thinking Levels

The van Hiele theory examines the learners' levels of reasoning about geometric shapes in a way that learners move through various geometric thinking levels ranging between recognizing geometric shapes and constructing formal proof (van Hiele, 1986; van Hiele, 1999; Clements, 2004). By this theory, understanding and learning of the students in geometry can be examined with hierarchical levels. Instructional implications can also be made based on their levels. Hence, this theory is beneficial for teachers. In other words, through these levels and their properties, the theory proposes opportunities to capture deep inside to learners' difficulties in geometry and to develop geometry instructions (van Hiele, 1986; Fuys, Geddes & Tischler, 1988; Pegg, 1995). The van Hiele theory focuses on five sequential and hierarchical levels of geometric thinking (Hoffer, 1981; Usiskin, 1982; Senk, 1989). These levels can be explained as follows:

Level 0 is named as visualization or recognition level. In this level, students focus on clear physical attributes of geometric shapes so they can determine them in this way without reasoning. They think the shape as a whole and criticize on its visual form with standard orientation since they are not able to analyze the shape using its elements and properties (Battista, 2007; Crowley, 1987). In this respect, the learners at this level are expected to determine, name, compare and contrast the shapes based on their appearances ignoring the properties of them (Fuys, Geddes & Tischler, 1988; Mayberry, 1983). In other words, they "use of imprecise qualities to compare drawings and to identify, characterize, and sort shapes" (Burger & Shaughnessy, 1986, p.43).

Level 1 is named as analysis or descriptive level. In this level, the students consider about geometric shapes with their specific properties. They can recognize and name collections of properties for geometric shapes ignoring relationships between these properties and other geometric shapes. Moreover, while describing and defining them, they are not able to make decisions on the appropriateness and sufficiency of these properties (Mason, 1998). They are not able to classify the shapes based on the relationship with other shapes and common properties with other figures but they can form particular definitions of geometric shapes and use them in clear and definite cases (Battista, 2007).

Level 2 is named as informal deduction, order or theoretical level. In this level, the students can determine the "interrelationships of properties both within figures and among figures" (p. 3), comprehend formal definitions of and informal arguments about geometric shapes with the lack of understanding of deduction process, axioms (Crowley, 1987) and make inferences in simple form (Pegg, 1995). The students can form definitions of geometric shapes and interpret class inclusions of them by forming diagrams or charts representing the relationship between them based on their properties learned by ordering or comparing these properties (Mayberry, 1983; van Hiele, 1999). The students can make decisions about the appropriateness and sufficiency of the sets of properties to describe and define geometric shapes (Fuys, Geddes & Tischler, 1998).

Level 3 is named as deduction. In this level, the students are expected to develop proofs about geometric shapes by reasoning logically and formally and comprehend formal geometric statements and arguments such as axioms, definitions and theorems (Clements & Battista, 1992). They can also form proofs, the role of formal mathematical statements and arguments and criticize the appropriateness and sufficiency of conditions referring to the geometric shapes and their properties by reasoning (Pegg, 1995). Students can comprehend formal proofs and then form their own proofs following different ways and using different types of reasoning and strategies (Crowley, 1987). In this respect, many high school students are expected to reach this geometric thinking level (Shaughnessy & Burger, 1985).

Level 4 as the last van Hiele geometric thinking level and the matured level of geometric thought is named as rigor. In this level, the students are expected to make reasoning formally and logically about mathematical systems and statements focusing on abstract deductions with the necessity of rigor (Usiskin, 1982). They can also examine different deductive and axiomatic systems by comparing them, understanding relationships between them and constructing proofs. In this respect, they can comprehend, make reasoning and compare in the geometries except for plane geometry (Mayberry, 1983; Feza & Webb, 2005).

All of the levels of van Hiele geometric thinking follow a linear way in a fixed hierarchy. They reach a level by completing the necessities of the previous level(s) (Mason, 1998). Another property of them is adjacency. The necessary behaviors of a level are observed in the behaviors of the students on the latter level (Fuys, Geddes & Tischler, 1988). The students at a particular level are expected to represent the behaviors of previous levels. These levels are oriented in a process that these levels are separated with each other in a qualitatively distinct level of thinking (Clements, 2004). Hence, the behaviors of the students for each level can be observed and understood. Based on the property of discontinuity referring to the lack of coherence among the levels, the students can not represent the behaviors of any level unless if they become matured in the previous levels (Pegg, 1995). The students at a particular level attain and represent the properties of previous levels at the matured level. Based on the property of retention, students can represent different van Hiele geometric thinking levels on different geometric thinking levels (Pegg, 1995). The students at a particular van Hiele geometric thinking level can represent different levels. Moreover, with the property of ascendancy, the progression of the students through these levels can be provided by instructional experiences rather than age or biological maturation (Clements, 2004); that is, “the transition from one level to the following is not a natural process; it takes place under influence of a teaching – learning program” (van Hiele, 1986, p.50). The students can represent lower van Hiele geometric thinking levels when compared with younger students.

In order to examine the students from different grade levels, there have been research conducted to examine their reasoning by van Hiele geometric thinking levels. In order to train preservice mathematics teachers about geometry knowledge and geometry teaching, the researchers focus on van Hiele theory. Their geometry thinking was measured by van Hiele geometry thinking levels (Aydın & Halat, 2009; Halat, 2008). For example, Halat (2008) found that most of the preservice elementary teachers at or above level of analysis and half of the secondary school teachers were at or above level of ordering. Then, by comparing the properties of these levels and the required skills and knowledge about geometry, he stated that these finding showed that the participants at these levels had adequate content knowledge in geometry (NCTM, 2000; Mayberry, 1983). In this respect, the preservice elementary teachers should attain the properties of initial two levels while preservice secondary school teachers should do initial three levels. Hence, van Hiele geometric thinking levels can be used as predictor to make estimations about preservice mathematics teachers' academic achievement in college level courses. Based on this view, Watson (2012) conducted a research to analyze the relationship between van Hiele geometric thinking levels and their success in a college level course. Then, the researcher found that the van Hiele geometric thinking level was an important factor to predict their success in this kind of class. In this respect, it can be stated that van Hiele geometry thinking levels can be used to predict their existing geometric reasoning so that their improvement in geometry courses can be provided in this way.

The instructions and activities can be designed based on the predictor role of van Hiele geometric thinking levels since the students at the same level can tend to have similar amount of geometry knowledge and behave similarly. In other words, van Hiele geometric thinking levels are predictors for the achievement of the students because the students at the same levels tend to reason about the geometric concepts similarly (Hill, 2013; Wang, 2011. In this respect, this case provides that Wang (2011) thought that the students at the same van Hiele geometric thinking levels represent similar actions engaging in the activities. The prospective elementary school teachers' geometric thinking through geometric discourses about the classification of quadrilaterals based on van Hiele theory was examined. The study focused on the similarities and differences between the discourses of the participants at the same van Hiele geometric thinking level and the changes on the discourses based on the improvement on their geometric thinking levels. The teaching episodes were designed in order to help them reach the required van Hiele geometric thinking level; that is, informal deduction level. Based on the results of the study, the students reached this level and represent similar actions. The participants at the ordering level named the geometrical shapes correctly and determined their properties. Although they made correct logical expressions using their main elements of angles and sides, they could not use all of the auxiliary elements such as diagonals. Also, they could not effectively construct proofs by deductive reasoning and abstract thinking. Viglieti (2011) also added that the learners at this same level provided incomplete and incorrect answers for the geometric shapes except for triangle, isosceles triangle and quadrilaterals. In this respect, the properties of van Hiele geometric thinking levels are useful to design instructions since useful activities and pedagogical supports can be provided by determining the students' van Hiele geometric thinking levels and separating them into groups. By determining the students at the same van Hiele levels, the instruction can be designed effectively. Also, useful predictions about the instructional sequence for the hypothetical learning trajectory can be made in geometry lessons.

When a problem is defined as a challenge situation, beneficial problem situations can be selected and designed using van Hiele geometric thinking levels. Moreover, the students can learn their geometric reasoning while providing the solutions to the problems. Based on this view, van Putten (2008) examined the van Hiele geometric thinking levels of preservice mathematics educators through using their content knowledge of plane geometry in geometric problem solving situations and their progression of geometric thinking levels. Their progress was investigated through mathematics teacher education system. In this study, the relationship between problem solving and geometric thinking was explored. They provided geometry problems to solve and the students engaged in these problems and they attained and improved geometry knowledge and understanding. It was found that by geometric problem solving and attending the course, they improved their geometric thinking. In this respect, geometric thinking and problem solving is connected. Hence, problem situations can be formed by van Hiele geometric thinking levels and geometric thinking can be improved by problem solving activities.

Van Hiele geometry thinking levels are also used in order to examine their knowledge and skills about geometrical proof having critical importance in geometry education. There have been research to explore their connection in the literature (Aydın & Halat, 2009; Dimakos & Nikoloudakis, 2008; Wang & Kinzel, 2014). Aydın and Halat (2009) and Dimakos and Nikoloudakis (2008) explored the relationship between the proof and geometric thinking. They conducted the research to test this relationship by designing tasks about geometrical proofs and then their geometric thinking improvement through these tasks was explored. It was observed that the students in the course including proof activities represented higher geometric reasoning stages and proof activities improved geometric reasoning. In this respect, while improving the students' proof constructing skills and knowledge in geometry, it is necessary to improve their geometric thinking.

These levels have relationship with pedagogy proposing suggestions for the instruction in geometry. With this aim, van Hiele theory provides five-phase instruction representing the students' progress (van Hiele, 1986; Mason, 1998). The first phase is information in which the students attain deep knowledge about the geometrical concept using their previous knowledge and own language (Pegg, 1995). Then, in guided orientation phase, the students are expected to participate in carefully structured geometrical tasks generally permitting only one solution (Mason, 1998; van Hiele, 1986). After completing these tasks, explication phase begins and the students are aware of what they have learned and begin to use appropriate mathematical terms and symbols to communicate. Afterwards, in the free orientation phase, students engage in the geometrical tasks representing problem situations with more than one path solution (van Hiele, 1986). In the last phase, integration, the students attain necessary deep knowledge about the concepts in geometry and construct an overview about them. These five phases are related to the descriptions of van Hiele geometric thinking levels. Teachers can help student pass from one level to the next level successfully following these phases (Pegg, 1995).

There have been research in the literature in order to illustrate the effect of these phases on the students' geometric thinking and geometry achievement (Abdullah & Zakaria, 2012/2013; Meng & Idris, 2012). Meng and Idris (2012) designed learning environment supported by Geometer's Sketchpad and implemented the activities about solid geometry based on van Hiele phases. By exploring the effect of van Hiele phase-based learning on the students' geometric thinking and achievement about this concept, they realized that this instructional sequence improved the participants' geometric thinking and achievement in solid geometry. Abdullah and Zakaria (2013) also studied about the impact of van Hiele phase-based learning on students' geometric thinking. In this quasi-experimental study, 94 secondary school students were divided into two groups and taught during six weeks. Based on the van Hiele' Geometry Test result, the students taught by van Hiele phase-based learning showed better level of geometric thinking in a way that all of the students showed complete acquisition of initial two levels and most of the students represented the high acquisition of the third level of van Hiele geometric thinking. Moreover, Abdullah and Zakaria (2012) investigated the views of experts and preservice teachers about the activities designed based on van Hiele phases in order to obtain detail information about the effects of these phases. In their study, they designed learning environment by Geometer's Sketchpad as a tool about the geometrical concept of quadrilaterals. Then, these activities were examined by 10 experts and 24 final year preservice teachers. They explained that these activities were beneficial about pedagogical usability criteria and should be used in order to teach and learn geometry. These studies illustrate that van Hiele phases are useful to be used in geometry instructions in order to help students examine geometrical concepts effectively and deeply based on their properties and connections between them.

Through these previous research in the literature, it can be stated that the phases and descriptions of van Hiele geometric thinking levels support an environment designed by problem-based learning. Moreover, it encourages formation of a classroom including collective argumentation. In this respect, it was necessary to examine and use them while designing the hypothetical learning trajectory and applying it in an instructional sequence. In the current study, the van Hiele geometric thinking levels and phases were benefited from in order to design an effective hypothetical learning trajectory and to form instructional sequence in the classrooms about the concept of triangles for preservice middle school mathematics teachers with the aim of determining their mathematical practices. These levels was also used to design and organize the activities of geometric constructions by compass and straight edge to help them use these tools effectively. Furthermore, by examining van Hiele geometry thinking levels, it was aimed to understand the effects of geometric constructions effectively in the study.

2.8 Geometric Constructions in Mathematics Education

Geometric constructions are important in mathematics education to teach Euclidean geometry focusing on constructing geometric shapes by compass and straight edge (Stillwell, 2000; Janicic, 2010). Euclid examined the geometric shapes, their properties and theorems through construction in his book of “Elements” so that construction has taken place in geometry and mathematics education (Karakuş, 2014). Geometric constructions are systematic steps in order to form geometric entities in the way of producing intended geometric shapes following particular basic and complex steps of sequence by compass and straight edge (Demiray & Çapa-Aydın, 2015; Djoric & Janicic, 2004). They also have pedagogical importance in geometry learning and teaching. They are used to explore the work of Greek mathematicians such as Euclid and Pythagoras taking important role in mathematics curricula of all grade levels (Sanders, 1998). They are strongly related to proof, geometric understanding, geometrical knowledge, problem solving, psycho-motor skills, in-depth thinking and relational understanding (Ameis, 2005; Cheung, 2011; Güven, 2006; Karakuş, 2014; Khoh, 1997; Kuzle, 2013; Napitupulu, 2001; Posamentier, 2000; Tapan & Arslan, 2009).

Through geometric constructions, learners engage in the tasks using compass and straight edge. By following the sequence of the steps, they improve their psycho-motor skills. However, by following these steps, they encourage geometry achievement and conceptual knowledge when they are used effectively as instructional tools in a planned way. In this respect, the construction activities do not only develop psychomotor skills by using compass and ruler and also improve cognitive skills, geometric understanding and knowledge about forming geometric shapes (Cherowitzo, 2006). Therefore, the construction activities improve physical and cognitive mathematical skills of the learners. In the process of construction activities, the learners do not only examine how to construct the shapes examining and they also understand its properties (Erduran & Yeşildere, 2010; Napitpulu, 2001; Hoffer, 1981). They form the geometric shapes by discovering their critical attributes and properties based on the relationship between them through constructing them by compass and straight edge. In this way, they improve their conceptual knowledge and relational understanding of the geometric shapes and they think about the shapes in detail (Cheung, 2011; Hoffer, 1981; Napitupulu, 2001). In this process, they examined the geometric shapes and their properties benefiting from other shapes and their properties (Khoh, 1997). For example, they can construct quadrilaterals by triangles, angle and perpendicular bisectors by rhombus and isosceles triangles. Hence, relational understanding can be provided by in-depth thinking and understanding of the shapes by using the tools, compass and straight edge. By improving relational understanding between geometric shapes, geometric constructions develop van Hiele geometric understanding (De Villiers, 2003; Napitupulu, 2001).

Based on the views about the effects of geometric constructions on relational understanding, there have been research in the literature (Erduran & Yeşildere, 2010; Karakuş, 2014; Khoh, 1997; Kuzle, 2013). In this respect, Khoh (1997) conducted a research about geometric constructions by compass and straight edge in order to explore the students' relational understanding and higher order thinking skills. The students engaged in three-stage construction and problem solving activities. They were designed about isosceles triangles, rhombuses, kites, angle and perpendicular bisectors. They constructed and justified these geometric shapes benefiting from other geometric shapes and their properties such as constructing angle bisector by isosceles triangle or parallelogram. By constructing the angle bisector of an angle, some of them used the knowledge that angle bisector of the angle having angle measure from other interior angles of an isosceles triangle was coincident with altitude, perpendicular bisector and median of the edge opposite of this angle. Also, some participants used the knowledge that the diagonal of a parallelogram separates the angles into two parts having equal angle measures. By doing so, they made connection between these knowledge so that they improved their relational understanding and their reasoning. It was found that these activities improved making relationship between the geometrical concepts and thinking skills. Also, Erduran and Yeşildere (2010), Karakuş (2014) and Kuzle (2013) also supported the view that the relational understanding can be encouraged by geometric constructions with compass and straight edge by their research. They state that the constructions support relational understanding by constructing geometric shapes by using other geometric shapes and their properties. Therefore, geometric constructions should be used in geometry classrooms so that the students use other geometric shapes and their properties by examining and learning a particular a geometric shape. In this way, geometrical constructions improve their achievement in geometrical concepts. By relational understanding, they can obtain subject matter knowledge and understand the concepts by increasing their achievement in geometry (Güven, 2006; Napitupulu, 2001; Tapan & Arslan, 2009).

Geometric constructions are not appropriate to use for the learners at van Hiele level-0 since they focus on the appearance of the shapes rather than their properties. However, the learners at other van Hiele levels can engage in construction activities. Also, in transition from Level-I to Level-II, they are beneficial and effective since they can analyze the shapes and their properties based on the relationship between them in detail (De Villiers, 2003;

Napitupulu, 2001; Posamentier, 2000). Therefore, geometric constructions can improve learners' van Hiele geometric thinking levels, critical thinking and mathematical thinking (Cheung, 2011; Güven, 2006; Kuzle, 2013; Napitupulu, 2001). Based on this view, there have been research to examine the effects of geometric constructions on geometric thinking. In the study of Napitupulu (2001), the effect of geometric constructions by compass and straight edge on the preservice mathematics teachers' van Hiele geometric thinking levels and learning geometry was examined. The participants were selected from the students of an undergraduate course. They engaged in basic and complex construction activities about the geometric shapes such as quadrilaterals and triangles. The researcher found that construction activities improved the preservice mathematics teachers' van Hiele geometric thinking levels providing their geometry learning. It was also emphasized that construction activities were beneficial in transition from van Hiele geometric thinking level of *Analytic* to the level of *Abstract*. In another study (Güven, 2006), the effect of geometric constructions on van Hiele geometric thinking levels was examined for the seventh and eighth grade students. In this quasi experimental research design, the results showed that the participants improved their activities of construction and drawing improved geometry achievement and van Hiele geometric thinking.

Smart (1998) explains geometric constructions as a strategy of solving geometry problems based on particular set of rules including basic and complex steps. It is added that they include actions of providing theoretically correct and satisfactory solutions to the problems rather than drawing figures supporting particular conditions. The geometric constructions are accepted as providing solution to a problem rather than drawing shapes based on applying fixed particular rules or steps (Erduran & Yeşildere, 2010). Posamentier (2000) states the connection of geometric constructions with problem solving in a way that they are "reinforcement of many different geometric concepts and relationships and for the development of problem-solving skills" (p.1). In geometric constructions, the learners have difficulty in deciding how to construct a geometric shape when they face with it at the beginning. Initially, they have challenge how to construct the shape by compass and straight edge and how and which steps to follow (Erduran & Yeşildere, 2010). Another challenge situation in constructions occurs in justifying that the solution is accurate and satisfactory. Erduran and Yeşildere (2010) conducted a research in order to examine the relationship between problem solving and geometric constructions. They examined the process of mathematics teachers' constructing geometric shapes by compass and straight edge by collecting data about their learning and ideas about constructing geometric structures. They found that learning occurred effectively when the participants did not follow construction steps in rote manner. By doing so, they engaged in geometric constructions as problem situations, they improved their problem solving skills. Also, it can be stated that a situation becomes a problem if the students are willing to handle and remove it. These cases improve the students' motivation. In other words, problem situations encourage their motivation to remove the challenge situation and provide solution to the problems (Rotgans & Schmidt, 2012). When the view that geometric constructions refer to problems, it can be stated that geometric constructions can improve the learners' motivation toward geometry and geometrical concepts (Erduran & Yeşildere, 2010).

The learners form hypothesis about the possibility and the way of constructing geometric shapes, organize their ideas and solutions, test their hypothesis, evaluate and analyze their solutions, geometric shapes and their reasoning in geometric construction (Cherowitzo, 2006; Karakuş, 2014; Lim-Teo, 1997). Hence, it can be stated that these actions can encourage the scientific skills and facilitate the social environment including argumentations. In this respect, the construction activities could be beneficial to help the learners form the argumentations by which the classroom mathematical practices emerged. Also, the constructions provide opportunities in order to examine postulates, rules, theorems and properties about geometric shapes since it proposes a different way with the straight edge and the compass to justify them. They necessitate to justify the solution of the geometric construction problem since providing accurate constructions means discovering the appropriate knowledge and justifying it in a mathematically correct way (Güven, 2006; Çiftçi & Tatar, 2014). When the justification of the shapes constructed by compass and straight edge is considered, this process can include proof construction and proof writing. Through constructing the geometric shape, the solution of the construction problem necessitate to show and prove that formed figure is the required shape in the problem (Chan, 2006; Napitupulu, 2001). Therefore, geometric constructions improve the learners' constructing and writing proofs (Napitupulu, 2001; Tapan & Arslan, 2009). In order to examine the relationship between geometric constructions and proof, Tapan and Arslan (2009) investigated preservice teachers' drawings and justifications about geometric constructions by asking the participants to solve geometrical constructions by compass and straight edge and prove their constructions. They found that although they had difficulty in transferring their geometrical knowledge to constructing and justifying them previously, they formed and developed arguments by geometrical reasoning and subject matter knowledge while engaging in the geometric constructions. Also, the participants formed different types of justifications through the process of constructing geometric shapes. Moreover, in the study of Cheung (2011), the effect of geometric constructions by compass and straight edge on justification and constructing proofs was confirmed. It was found that construction activities improved geometric knowledge, critical thinking and skills of justification and proving. It was added that construction activities enhance learning environment in which the students developed their communication skills. Therefore, geometric constructions are useful tools to be used in geometry lessons to help the students learn how to justify and prove the expressions by communicating their ideas and strategies.

When the literature is examined, it is observed that there exist limited research about geometric constructions (Demiray & Çapa-Aydın, 2015; Erduran & Yeşildere, 2010). Through these research, it can be stated that geometric constructions are effective to improve learners' geometry achievement, understanding, conceptual learning, motivation, geometric thinking, proving and skills such as critical thinking, sharing and challenging their ideas and problem solving. However, it has been necessitated to examine geometric constructions in social environments with their argumentation skills about particular geometry concepts. In this respect, HLT in the present study was designed using geometrical constructions to be used in an instructional sequence including argumentations. Because of supportive effect of argumentations on learning and reasoning, the level of van Hiele geometric thinking levels of preservice mathematics teachers and geometric constructions took important place in the present study.

2.9 Proof and Geometric Constructions

Geometric constructions explore the properties of the shapes by representing them through drawings and then proving geometrical explanations (Chan, 2006; Napitupulu, 2001). Furthermore, geometric constructions provide opportunities about proofs by explaining that the students learn how to determine whether a statement is conclusion or premise by their causal relationship using the logical structure of if-part and then-part in the expressions (De Villiers, 2003). When van Hiele geometric thinking levels of the participants of the study were considered in the present study, they could be expected to provide proofs for the geometric constructions. In this process, proof is beneficial for constructions since it does not only indicate accuracy or inaccuracy of a statement but also illustrate why it is correct (Hanna, 2000). Even proving is commonly an effort for putting across the correctness or incorrectness of a claim or a result with enough evidences (Garnier & Taylor, 1997). The meaning of proving is removing or creating doubts related to the accuracy of a statement. Proving includes two sub-processes: first process focuses on understanding the truth in order to remove own doubts, the second one concerns about convincing the others to remove their doubts about the statement (Harel & Sowder, 1998).

In literature about proof, several definitions of it have been produced. Traditionally, proof has been accepted as verification of correctness of the statement and used mainly to remove personal and social doubts (Hanna, 2000). Hanna (2000) stated that proof is an argument that may assume several different forms as long as it is justified. According to Bell (1978), proof provides a way of individuals' reaching conclusion by justifying, convicting and communicating rather than producing formal arguments. Selden and Selden (2003) refer to proof as "texts that establish the truth of theorems" whereas Stylianides and Stylianides (2009) define it as an argument for the truth of a statement that is "general, valid and accessible to the members of the community". Proof process consists of three different but interrelated stages: investigation of the subject which will be proved, organization of proof and explanation of it to the others. The individuals analyze the statement or problem, investigate its correctness and design the proof by benefiting from the theorems proved previously. This process ends with forming proof or showing incorrectness of the statement (Lee, 2002).

There are two components of proof. One of them is reasoning as a concept facilitating the generalization along with the processes of explanation, exploring and organization (Mingus & Grassl, 1999). Proof process includes both inductive and deductive reasoning. In deductive reasoning, implication process initiates generally and ends specifically by providing necessary evidence about the accurate of the final statements. In inductive reasoning, inference process proceeds from particular to general and probable without providing necessary evidence for conclusion. Another component of proof is communication. Proof is a way used for sharing the results and arguments, constitution of the mathematical concepts, learning and presenting the generalizations. However, students have difficulties in explaining their results and expressing various ways followed in solution process. Yet, validating the answers by evidences and proving have crucial part in development and alteration of mathematical reasoning (Flores, 2002). Because, when students try to clarify and defend their results, it provides using the mathematical expressions more meaningfully (Forman, McCormick & Donato, 1998). Thurston (1994) believes that proof leads to mathematical understanding and helps learners think more clearly about mathematics. Therefore, proof has important place in mathematics (Hanna, 2000).

In general, there are several functions of proof. The list of functions of proof includes verification of the statement, reasoning about its correctness, documenting the results systematically deductively by axioms and theorems, inventing new conclusions, interacting by mathematical ideas, formation of a theory empirically, investigation the roles of definitions or the conclusions of an assumption and interacting with previous mathematical knowledge based on a new framework (de Villiers, 1990; Hanna, 2000). In addition to these functions, the most important contribution of proof is supporting mathematical understanding (Hanna, 2000). According to Bernard (1989), learning proof is necessary for holding a good awareness and appreciation of mathematics and gaining interest in mathematics. The learners defend that the characteristics such as mathematical curiosity, precision of thought and mathematical proof are needed to be promoted in order to serve the aims of a good education and training, to present a correct picture of mathematics and to provide interest in the subject.

By considering the importance of proofs and proving, there have been research with the aim of identifying the students' proving skills in different grade levels (Güler, Özdemir & Dikici, 2012; McCrone & Martin, 2009; Özer & Arıkan, 2002; Selden & Selden, 2003). Selden and Selden (2003) conducted a research about the concept of validity related to proof to university students.

A range of mathematical expressions referring to correct and incorrect proofs was designed and the students were asked to determine whether these expressions were valid or invalid. They found that the students had difficulties in distinguishing correct and incorrect justifications and their abilities of defining the meaning of proof were poor. In another study about the proof levels of the students, Güler, Özdemir and Dikici (2012) and Özer and Arıkan (2002) found that the levels of the students were not at required level and they could not prove by using material and examining the skills and levels of proving of the students. It was also stated that the ability of using the methods and techniques of proving were not at the required level. Also, the findings of the study of McCrone and Martin (2009) illustrated that students were not aware of what kind of formal proof was required and they might think that they proved a judgement based on only one example. Stylianides, Stylianides and Philippou (2007) stated that preservice teachers had difficulty in proving because teacher training programs did not focus on the concepts related to proof. Also, Jones (2000) explained that they did not have rich proof schemes. The research show that the students' skills and knowledge are poor and not at the required level. However, their levels can be improved. In this respect, the preservice teachers should be trained to improve preservice mathematics teachers' skills and knowledge of proof. Therefore, the present study was conducted in order to improve preservice middle school mathematics teachers' skills and knowledge of proofs necessitated to justify geometric constructions.

The skills and levels about proofs can be developed by providing instructions including the tasks of proofs. The research have showed that effective instructions impact the proof skills. There have been research to examine the impact of instruction including proof activities on learning and understanding of the students about proofs (Martin et al., 2005; Flores, 2006). Martin et al. (2005) investigated the relationship between instruction of proving and learning of it. Researchers focused on interaction between the actions of both teacher and student, classroom discourse and their effects on students' learning of proof through four-week instruction. It was observed that the students began to construct formal proof participating in the teaching episodes supported by communication and discussion. Also, the use of analytic proof schemes showed that the students had good qualities in proving after instruction on proving supported by interaction and communication among the students and teacher. In the study of Flores (2006), the researcher wanted the students to explain why the concepts which they learned were correct. The students were asked to indicate what they learned like rule, assumption, and procedure and to provide justifications about why they worked in engaging in the tasks. It was found that most of the students' shared limited ideas, knowledge and proofs regarding how they thought and what they learned. In addition, it was stated that the students tended to prove the mathematical statements in a way that their teachers did or they saw in the textbooks. Therefore, the instructions should be designed based on proofs and different proving strategies for the students. Hence, the students can prove mathematical expressions effectively by learning proving effectively.

In recent years, there have been tendency to examine the subjects such as proving, reasoning and argumentation in educational research (Heinze & Reis, 2003). This tendency results from different views based on the case that the effective communications occurring in the classrooms among the students and teachers in the classroom (Martin et al., 2005). One of the views is that proof is seen as the one of the most essential topics taken at the hearth of mathematics and mathematics education (Knuth, 2002; Lee, 2002). Many researchers and curriculum developers defend that proof must become an integral part of the students' mathematical experiences from all grades (NCTM, 2000). Another view is provided based on their understanding since proof increases mathematical understanding, mathematical discover and connections among mathematical ideas (Stylianides, 2007). It provides conceptual understanding by exploring the reasons (Carpenter, Franke & Levi, 2003). Moreover, it is stated that proof and argumentations are related since both of them encourage the skills of critical reasoning and evaluating. For example, Stylianides and Stylianides (2009) investigated preservice teachers' skills of proving and evaluating their own proof. Researchers prepared and implemented proving-evaluating activities. The findings showed that most of the students constructed correct proof, some of them formed invalid proofs and some of them represented empirical judgements. In this respect, preservice teachers should be provided to evaluate their own proof so that they can improve their skills of evaluating.

According to Hart (1994), there should be research focusing on cognitive processes. For example, thinking processes of learners such as preservice mathematics teachers should be made to explore their proving processes with the mistakes by reasoning in these processes (Weber, 2001). Besides, most of the students do not know how to prove and they must use initial step for proving, the conceptual knowledge and definitions in proving process (Weber, 2001; Sarı, Altun, & Aşkar, 2007). There are many research aiming to illustrate the opinions related to proof and the processes of proving of the students, preservice teachers and teachers in literature (Jones, 2000; Weber, 2001; Knuth, 2002; Stylianides, Stylianides & Philippou, 2007).

The previous research illustrate that the students in all grade levels have difficulties in understanding, caring and constructing proof (Moore, 1994; Jones, 2000). These difficulties are not appeared only in proving but also in remembering what proof is (Chazan, 1993; Moore, 1994). In general, students have difficulties in comprehending, appreciating, constructing, following the steps of reasoning and formulating their proofs. In addition, most of the students do not know how to prove, start and use the knowledge and definitions of the necessary concepts (Weber, 2001). Weber (2001) evaluates the students' difficulties related to proving in terms of two perspectives. The former is that example(s) for accuracy are enough for proving and students have incorrect considerations about how proof must be. The latter is that students inadequately understand and apply a concept or theorem. Baker (1996) determines that students have difficulties in proof techniques in terms of both conceptual and operational aspects. They give more importance operational aspect of proof rather than conceptual aspect. It is believed that the deficiencies of students in mathematical knowledge cause these difficulties (Tatar & Dikici, 2008). Through these research, it can be stated that proofs are important in mathematics education especially training preservice mathematics teachers. The research show that despite of the importance of proof, preservice mathematics teachers do not have necessary knowledge and skills about proofs. Therefore, they need to educate them to develop their proving skills. In this way, their geometric reasoning and geometry knowledge can be encouraged. Geometric constructions are useful to encourage their improvement in proving and proofs are necessary to justify the truth of the geometric shapes constructed by compass and straight edge. Hence, it has been necessitated to examine geometric constructions in social environments with their argumentation skills about particular geometry concepts in a way that their skill of proving was developed. In this respect, HLT in the present study was designed using geometrical constructions to be used in an instructional sequence including argumentations. Because of the nature of geometric constructions and the level of van Hiele geometric thinking levels of preservice mathematics teachers, proofs were formed through geometric constructions and argumentations about triangles in the present study.

2.10 Triangles and Mathematics Education

Triangles are one of the basic and common geometric shapes developing the geometric world and being used in the design of buildings and bridges in the real life. Moreover, it is important since it can be used to construct other geometric shapes and make calculations on them such as calculating area of a parallelogram and rectangle by a triangle (Fey, 1982). In this respect, triangles are important to understand and learn other geometric shapes and their properties. However, triangles have importance in teaching geometry, and learners from all grade levels have difficulty in learning triangles (Damarin, 1981; Vinner & Hershkowitz, 1980). In other words, although it has importance in geometry for learners, triangles are hard to learn for them. One of the reasons of this situation is the fact that triangles are taught as content-specific facts rather than triangles as a concept (Gillingham & Price, 1987). Moreover, the definition of triangles and applying these definitions on new examples should be emphasized (Vinner & Hershkowitz, 1980). Also, triangles can be learned effectively through the instruction dominated by numerous examples and non-examples of triangles (Wilson, 1982). In addition to the usage of examples and non-examples of triangles in instruction, hierarchical relationships should be used in instructional designs for the geometric objects such as triangles (Novak & Tyler, 1977).

Through the literature and historical development of geometric shapes, triangles have critical importance. Through this developmental process, the necessary properties of triangles needed to learn geometry are determined. Euclid's book of "Elements" (approximately 300 BC) was used as an effective beginning for learning geometry for more than twenty centuries (Morrow, 1970). In this book, he defined geometric objects such as a point, a line, a straight line, a surface, a plane surface, a plane angle and types of angles, a circle, a semicircle, rectilinear figures, trilateral figures, quadrilateral figures, and then the concept of parallel lines. By these definitions, he emphasized the classification of triangles. In this classification, the main elements of triangles were analyzed, the analysis was performed by them (Morrow, 1970). Then, Proclus examined the types of triangles through their definitions and the relationship between them. For example, it was stated that "From these classifications you can understand that the species of triangle are seven in all, neither more nor less. The equilateral triangle is one only and is acute-angled; but each of the other two has three kinds. The isosceles is either right angled, obtuse-angled, or acute-angled; and the scalene likewise has the same three forms" (Morrow, 1970, pp.132-133). The classification necessitates generalization, abstraction and making connections about types of triangles by attaining knowledge of triangles. Nowadays, these explanations are still used in the classrooms while teaching triangles. In this respect, the classification of triangles was placed in the hypothetical learning trajectory of the present study through the PMSMT's learning of triangles because of its importance on the historical development of triangles.

After the definitions and postulates, Euclid examined criteria of congruence of triangles in his book of the "Elements". Also, he insisted on the definitions of similar and congruent triangles. Euclid proposes similarity of two triangles in the Book VI of the "Elements". This process was started benefiting from the definition of two similar rectilinear objects such as "similar rectilinear figures are such as have their angles severally equal and the sides about the equal angles proportional" (Heath, 1956, vol. 2, p. 188). Moreover, the criteria of similar triangles were examined by Euclid. 4^{th}, 5^{th} and 6^{th} propositions taking place in Book VI of Euclid examined the proofs of the criteria of similarity of triangles. Proposition 4 is "in equiangular triangles the sides about the equal angles are proportional, and those are corresponding sides which subtend the equal angles (Heath, 1956, vol. 2, p. 200). Proposition 5 is "if two triangles have their sides proportional, the triangles will be equiangular and will have those angles equal which the corresponding sides subtend" (Heath, 1956, vol. 2, p. 202). Lastly, proposition 6 is "if two triangles have one angle equal to one angle and the sides about the equal angles proportional, the triangles will be equiangular and will have those angles equal which the corresponding sides subtend" (Heath, 1956, vol. 2, p. 204). Then, this process continued with the similarity of triangles having the roots from the study of Thales (624-547 B.C.) with the evidence of Heath (1921/1981, vol. 1) stating the Hieronymus's, a pupil of Aristotle, about it with the quotation of Diogenes Laertius, "Hieronymus says that he even succeeded in measuring the pyramids by observation of the length of their shadow at the moment when our shadows are equal to our own height" (p. 129). The way representing the root of Thales' method is still used in much literature for the concept of triangles. Moreover, there are real life problems taking place in the textbooks and representing it. Through this developmental process of triangles in history, the similarity and congruence of triangles have been important to be emphasized in the study. Moreover, the criteria of similarity on triangles were explained by comparing the main elements of triangles through examples and generalization.

When the literature is examined, it is observed that there are beneficial studies examining learning of the students and teachers and teaching of teachers about the concept of triangles. While working with kindergarten children including 65 children aged 5-6 years-old, Tsamir, Tirosh and Levenson (2008) examined intuitive non-examples and the features making these non-examples intuitive for triangles. By using non-examples of triangles, teachers help students reason by attaining knowledge about their thinking of triangles benefiting from the previous research of Clements et al. (1999). In this process, the researchers stated that the students were encouraged to form concept images by concept definitions so that definitions were important to criticize the visual of triangles and their attributes to understand triangles. Also, the process of determining whether the geometric objects were triangles based on their visual form was the beginning level for geometric thinking and analytical judgment by attributes of geometric objects and their critical attributes. Moreover, Tsamir, Tirosh, Levenson, Barkai and Tabach (2014) examined the concept images and definitions of triangles, circles and cylinders. The participants were composed of early-years teachers. They found that the triangle definitions made by them were examined by considering necessary attributes of the triangle. Also, they added that formation of correct definition of triangle did not mean that they determined the triangle shapes correctly so the definitions and concept images should be examined. Also, Ward (2004) investigated the concept images and mathematical definitions of preservice teachers that would teach mathematics in K-8 about polygons. In the study, the participants were asked to identify the triangles and different types of triangles such as right triangles among a collection of shapes of triangles.

In the literature, there have been research to examine the effects of argumentations in an environment including mathematical argumentations on the students' understanding of the concepts (Abi-El-Mona & Abd-El-Khalick, 2011; Alatorre, Flores & Mendilo, 2012; Smith, 2010). In the study of Smith (2010), the researcher compared the middle school students' mathematical arguments in technological and non-technological environments about the concept of triangles. With the teaching experiment taking place in two 8th grade classrooms; one designed with Geometer's Sketchpad and the other one designed by non-technological tools, the content and the structure of the students' arguments about the concept of triangles were examined by the Toulmin's argumentation model. In this teaching experiment, hypothetical learning trajectory was formed including the learning goals related to classification of different types of triangles, examining common basic theorems about triangles and right triangles and application of their knowledge and understanding of triangles in problems representing real-world contexts. It was stated that the instructional sequence including the tasks about types and classification and sorting of triangles, triangle inequality and triangle interior angle relationships provided middle school students' learning environments in which they formed mathematical arguments. It was also observed that the students in technological environment produced more arguments than their counterparts. Moreover, Alatorre, Flores and Mendilo (2012) investigated the primary teachers' geometric reasoning through argumentations about the concept of triangle inequality. The data were collected from the primary teachers participated in workshop lasting two hours focusing on their common content knowledge and special content knowledge. They found that although the participants had difficulty while producing argumentations satisfactorily initially, they produced different types of argumentations such as authority-based, symbolic, factual, empirical or incomplete analytical. In this respect, the learning and understanding of triangles should be encouraged by argumentations.

The usage of tools helps the learners to learn the mathematical concepts effectively and facilitates producing arguments (Smith, 2010). Hence, there have been research conducted to investigate the effects of manipulatives on the students' understanding of triangles. Athanasopoulou (2008) conducted teaching-experiment research in order to examine preservice and inservice mathematics teachers' skills of understanding, knowledge and proving the properties of triangles. The study was designed about triangles and quadrilaterals. They participated in an inquiry-based geometry course including the tools supported by Geometer's Sketchpad during 30 classroom teaching episodes. The activities were composed of mostly definition and classification of triangles, congruence and similarity of triangles and proving activities about triangles. It was found that their geometric knowledge and geometric thinking about triangles were improved through teaching episodes. Also, their skills about writing geometric arguments and forming clear proofs were developed. Furthermore, Kellogg (2010) organized a design experiment in order to examine preservice elementary teachers' pedagogical content knowledge about the concept of area and perimeter in web-based learning environment. They investigated the area and perimeter of triangles and other geometric shapes such as quadrilaterals benefiting from triangles. Through these teaching episodes, they had procedural knowledge by formulas, used representations ineffectively and did not know the possible misconceptions about the concept at the beginning of the study. They improved their learning and knowledge about triangles by participating in instructional sequence. In addition, Doğan and İçel (2011) investigated the effect of dynamic geometry software of GeoGabra on eighth grade students' learning about triangles. An experimental design was used. It was found that construction activities encouraged by GeoGabra improved their motivation positively.

While working with 25 students with ages ranging from 12 to 15 year-old, Kordaki and Balomenou (2006) examined the strategies used by them in the activities about conservation of area in triangles and discrimination of area and perimeter of equivalent triangles by *Cabri-Geometry (educational software)*. The participants took place in three-week learning experiment where they engaged in learning activities such as construction of equivalent triangles in many different ways and examining their area and perimeter with possible tools. Through these activities, the Cabri facilitated the process by opportunities such as means of construction, connecting and controlling them. They stated that the students attained a broader view rather than the view that they obtained through typical paper and pencil environment. In other words, in order to help students attain deep knowledge about a geometrical concept, various learning activities, tools and problems with solutions in as many ways as possible should be designed and supported except for the typical activities and environment in which they engaged in everyday classrooms. Through this explanation, it was necessary for the learners to design non-typical learning environments. Moreover, it can be stated that problems are beneficial to enhance these environments since they are challenging situations not fitting the situations that they have experienced. The findings of the research show that using manipulatives and technological tools develop the students' understanding and learning of triangles. In this respect, while teaching triangle to the students from different grade levels, it is important to design learning environments by the manipulatives.

In the study of Gutierrez and Jaime (1999), they stated that student learning was related to the way in which their teachers understood mathematics and they taught how to transfer their knowledge to the students. They formed their hypothesis based on this explanation and investigated it through the concept of altitude of a triangle. This concept was difficult to understand by both of teacher and their students so it was necessary to make study about this geometry concept. The effects of concept definitions, concept images, difficulties and errors about the altitude of a triangle on their performance were examined by 190 preservice primary teachers. Through this process, they were determined and explained under the categories of formal definitions and classroom activities about this concept. Based on the findings of the study, the teachers had difficulty and errors about the concept of triangles and grasping this concept was difficult for the preservice teachers. Therefore, it is necessary to help preservice teachers attain deep knowledge about the altitude of a triangle and other auxiliary elements of triangles since some of the errors determined in the study are related to other elements of triangles. Also, it is necessary to examine preservice teachers understanding about auxiliary elements of triangles in addition to the altitude. The altitude of triangles of triangles was also examined by Alatorre and Saiz (2010). They conducted research to inservice and preservice teachers in order to investigate their mathematical content knowledge about triangles by considering the effects of their gender and experiences. The tasks focused on triangle inequality, the altitude and area of triangles. They found that male teachers, secondary school teachers, inservice teachers and highly experienced teachers improved their mathematical content knowledge of triangles better than their counterparts of the participants. They stated that they had difficulty about the triangles and also the application of Pythagorean Theorem.

Kemankaşlı (2010) conducted a quasi-experimental study about the design of a geometry learning environment on triangles for 10^{th} grade students. The effects of this organized environment on academic achievement, cognitive characteristics and their skills were investigated through cooperative learning environment by constructivist learning approach. There were 60 tenth grade students separated into experimental and control groups and participated in the study lasting eight weeks. The activities were prepared about the formation of triangles, the relationship between angles and the edges of a triangle, medians, angle bisectors, similarity and congruence of triangles. It was stated that these activities when applied in the classrooms by the constructivist learning approach, the students participating in this classroom become more successful than the students in the control group taking education by traditional learning method. In this respect, the concept of triangles can be taught effectively in an environment where the group work and social environment are encouraged and constructivist learning approach is benefited from.

While learning mathematical concepts, it is important to understand them by making connections between other concepts and their properties. Therefore, the research show that it is necessary to determine the mathematical concepts related to triangles and to teach the triangles using them (Kellogg, 2010; Paquette, 1971). In the study of Paquette (1971), the tasks for the congruence of triangles were designed through transformation geometry. The study included the activities about forming abbreviations for the congruence of triangles and their image triangles formed through types of transformation geometry by explaining two variables; a transformational variable and a positional variable. There were sheets including the triangles and their images formed through one of the rigid motions in different positions then the participants were asked to examine the congruence of triangles based on their main elements producing abbreviations. In this respect, using representations were beneficial in teaching congruence of triangles. Moreover, rigid motions of transformation geometry provided a non-typical strategy to examine and understand the concept of congruence of triangles. This strategy facilitated learning of congruence and similarity of triangles. In addition, Gerretson (1998) examined the similarity of triangles in dynamic geometry learning environment by transformation geometry. The researchers conducted the study to the preservice elementary teachers. They found that transformation geometry was crucial to teach similarity of triangles.

Through the literature related to historical development of triangles and previous studies about teaching and learning triangles, beneficial information has been obtained for the current study. The hypothetical learning trajectory was designed and instructional sequence was performed based on the information obtained through the examination process of the literature. The learning environment was designed based on the implications produced through this information in order to examine the preservice middle school mathematics teachers' understanding and learning of the geometric concept of triangles with the classroom mathematical practices.

2.11 Summary

The main goal of mathematics education is to emphasize the importance of *doing mathematics*. It can be provided by the knowledgable teachers that can perform mathematical instructions effectively. In this respect, it is important to educate mathematics teachers in their teacher training programs to obtain mathematical knowledge for teaching. The preservice teachers can improve their subject matter knowledge as the dimension of mathematical knowledge for teaching by understanding, reasoning and connecting mathematical topics. Also, they can improve their subject matter knowledge by expressing, discussing and using them through mathematical ideas. In other words, the preservice mathematics teachers can understand the mathematical concepts through sharing, analyzing and discussing their ideas about the concept and then transferring and applying them in different contexts. By doing so, they learn the concepts in a taken-as-shared way by forming mathematical practices. When the emergence of mathematical practices and the way of supporting learning of the concepts are considered, the argumentations are useful to provide this learning process. Emergence of mathematical practices necessitates the social learning environment including social and socio-mathematical norms as the dimension of the social aspect of interpretative framework related to social constructivism. Moreover, it is necessary to design an environment to help the preservice mathematics teachers learn in this way effectively. In order to design this environment, design-based research is used to organize instructional sequence. Hence, a hypothetical learning trajectory including the estimations of pathways of the actions in the classrooms can be formed in order to perform instructional sequence. Also, this hypothetical learning trajectory is organized by determining three elements; learning goals, learning activities and hypothetical learning process. Hypothetical learning progress can be organized based on problem-based learning as a strategy encouraging scientific skills such problem solving, forming, analyzing, testing and discussing their mathematical ideas and reasoning related to argumentations. Geometric constructions by compass and straight edge for learning activities as the second element of hypothetical learning trajectory can provide learning and understanding of the concepts. These tools can encourage the preservice middle school mathematics teachers' reasoning and constructing proof about the concepts while examining their constructions. Lastly, triangles are critical geometrical concepts in order to provide their learning and understanding. In this respect, learning objectives are determined about triangles based on historical development and connection with other mathematical concepts. Moreover, van Hiele geometric thinking levels can provide information to the instructors about how to order and relate learning goals and design learning activities including the tools based on their geometric reasoning.

To sum up, this study maintains mathematical practices are important in improving preservice middle school mathematics teachers' learning and understanding of triangles through geometric constructions, argumentations and justifications in problem-based learning environment. There is a general consensus about the positive effects of geometric constructions and argumentations in mathematics learning environment, however, there have been necessitated the research to examine their reasoning and understanding about triangles through teaching episodes supporting proofs, argumentations and constructions. The present study aims to provide contribution to the field in this way.

CHAPTER 3

3. METHODOLOGY

In this study, design based research methodology was used since it allowed to examine preservice middle school mathematics teachers' understanding of triangles focusing on classroom mathematical practices through collective learning environment in designed environment that supported problem-based learning for the mathematical content of triangles. In this chapter, firstly, properties and the rationale of the use of design based research and case study are discussed. Secondly, the participants in the current study are introduced. Thirdly, how to intervene in the research is explained. Also, HLT and instructional sequence designed in the study are described. Fourthly, data collection and data analysis processes are stated. Finally, how the trustworthiness is provided in the current study is discussed.

3.1 Design-based Research

Design based research (DBR) or design experiments include arranging significant and different kinds of learning. It provides a way to work on learning in a systematic way related to the context encouraging them. They are used with the aim of the development of the theories about the processes of domain-specific learning. Theory constructed in this way illustrates the successive patterns in learners' reasoning with the help of the means encouraging these patterns. Also, design experiments provide deep and effective understanding for learning environment. In other words, design experiments illustrate an interacting system including multiple elements in a complex way and the way in which these elements interact together to enhance learning (Cobb, Confrey, diSessa, Lehrer & Schauble, 2003).

Design experiments can be accepted as pragmatic and theoretic with respect to the design and practice in learning environment. That is to say, the design experiments are conducted with respect to theoretical considerations and the validity studies about these constructs are tested in a pragmatics way. It can be said that this explanation is valid for all design experiments (Cobb et al., 2003). These different studies can be exemplified with studies such as design experiments suggesting sessions for teaching with teacher, experimenter and student in order to form small-scale version for learning environments (Steffe & Thompson, 2000); classroom practices with a teacher as the member of the research team (Cobb, 2000); experiments in which the researchers make organization and work about education to the preservice teachers (Simon, 2000); the researchers help inservice teachers improve about their professions (Lehrer & Schauble, 2000). In this respect, it can be explained that it becomes possible to link the practice with the theory. In other words, the gap between the theory and practice can be removed. The claims that the theories do not have practical benefits and the practices of the teachers have missing aspect of theoretical sides necessary for teaching practices. This can be achieved in a way that the teachers and the theorists study collaboratively in all parts including the design and the experiment in the research (Design-Based Research Collective, 2003).

DBR as “a series of approaches, with the intent of producing new theories, artifacts, and practices that account for and potentially impact learning and teaching in naturalistic settings” (Barab & Squire, 2004, p. 2) have important characteristics (Anderson & Shattuck, 2012). They can be explained by taking place in real and actual educational contexts, designing and testing how to intervene with the learning of the classroom in an iterative refinement, using different methods by the cooperation of researchers and practitioners, relating the theory with the practice and producing domain-specific theory based on the results of the study (Cobb et al., 2003; Wheeldon, 2008). In this respect, through iterative process, collective learning environment and development of social and individual mathematical thinking, revisions and improvements on instructional designs are provided by DBR (Cobb et al., 2001).

DBR have critical importance by taking place in the classrooms (Gravemeijer, 2004). DBR are beneficial since it provides firsthand information about how the individuals learn and reason in a learning context in a way that "students' mathematics is indicated by what they say and do as they engage in mathematical activity, and a basic goal of the researchers in a DBR is to construct models of students' mathematics" (Steffe & Thompson, 2000, p. 269). In this respect, Steffe and Thompson (2000) explain that there are five elements necessary to design and conduct DBR: (a) teaching episodes, (b) a teacher or instructor, (c) one or more students/learners, (d) a witness observer, and (e) recording method of happenings in the teaching episodes (Wheeldon, 2008). Therefore, DBR have a cyclical nature including repetition of the process composed of development of instructional sequence including instructional activities, testing it in classroom instruction, documenting and analyzing the learning and social process and making revisions on the instructional sequence and redesigning it (Gravemeijer, Bowers & Stephan, 2003). At the end of this iterative process, an instructional theory is developed with the aim of improving the greatest progress of all of the learners in a social environment. In this developmental process, an observer as the witness for the instructional sequence and teaching episode takes place in all steps and whole process such as teaching episodes and meetings by being a member of community of learners formed in this process in order to understand and analyze the learning of the students and reasoning of them effectively. They also have the role of interpreter of the classroom environment and learners' reasoning, planner of the instructional sequence and analyzer of the classroom activity and modifications of learning goals and activities (Gravemeijer, 2004). Hence, this iterative process is related to testing, revising and redesigning the instructional sequence and activities on a weekly basis with the help of retrospective analysis in order for redesigning the instructional sequence (Gravemeijer, Bowers & Stephan, 2003; Steffe & Thompson, 2000).

The main purpose of design experiments is to develop theories. These theories explain learning process and the means facilitating the learning (Graveimejer & Cobb, 2006). This aim can be achieved as Graveimejer and Cobb (2006) told by developing theories of instruction which are local and theoretical frameworks illustrating more comprising issues. They also suggested three phases to make design experiment research; preparing for the experiment, conducting the design experiment and then retrospective analyses.

DBR is composed of progressive phases. In the phase of preparing for the experiment, the important result is to formulate local instruction theory. This theory is open to change, revision, elaboration in the process of experimenting in the classroom (Graveimejer & Cobb, 2006). In this respect, it is important to identify theoretical intent (Cobb et al., 2003). Then, the research team makes the learning goals specific or the instructional endpoints and instructional starting points. The first step is to identify learning goals. Graveimejer & Cobb, (2006) suggested that these goals can be collected through history, tradition and assessment. Also, it is necessary to identify the core ideas in the domain. For example, in the present study, the lessons were related to the constructions of triangles. When the history of the participants were examined about the concept of triangles, the participants were preservice middle school mathematics teachers so they were expected to have deep understanding of the triangles concept since it is one of the main concepts in geometry. It was also accepted that they attained knowledge about basic theorems and properties of triangles in elementary and high schools. With this aim, the previous lectures and courses were also examined. Previous lectures and courses that the participants had taken were important in the present study since they could provide information about mathematical knowledge of them and mathematical argumentations explaining claims, data and warrants in these lessons could be predicted. They learned how to construct geometrical figures and how to make argumentations about them. The preservice middle school mathematics teachers could be ready to learn the concept of the constructions of the triangles. The questions about constructing triangles necessitated understanding of triangles and basic construction activities of triangles and the properties of the triangles. Then, the data were collected through the classroom sessions, interviews and the research team meetings. Also, the research team held the meetings to discuss the results of the test and the students' knowledge and reasoning related to the previous instructions to obtain necessary information for design experiment.

In the literature, the concept has been taught through the different kinds of medium including some theorems in some research. The literature shows that the concept of triangles, the properties and the theorems about it have been taught with the help of proving strategy such as deductive or inductive reasoning or proof by contradiction in the geometry lessons of PMT (Durmus, Toluk and Olkun, 2002). Also, they have been examined with the help of technological tools such as Geometry Sketchpad, GeoGebra by proving strategies in the literature (Ceylan, 2012). With this motivation, it has become possible for most beneficial and effective goals by examining the goals related to the domain carefully and already defined in the curriculum in a disciplinary way (Graveimejer & Cobb, 2006). So, useful goals which were heavily related to the construction of triangles were able to be specified in the present study. This phase also includes determining the starting points for instructions (Graveimejer & Cobb, 2006). For this purpose, the previous instructions related to constructions of angles and lines were examined. The current skills, the knowledge of them and the results of the test conducted to the PMSMT before the instructional sequence was started. After the endpoints and starting points for instruction were determined, their task was to formulate the design experiment. The design experiment was formulated with the help of hypothetical learning trajectory (HLT). An HLT was created about the concept of construction of triangles. HLT comprised 6 weeks with three hours in each week.

In the phase of conducting design experiment, after the first phase was completed, conducting the design experiment began (Graveimejer & Cobb, 2006). That is, the learning process based on the concept of construction of triangles started. The responsibility of the research team was to follow and analyze the learning process and to make inferences for the design experiment and the HLT. In this respect, it could be claimed that design experiments had cyclical nature as a characteristic (Cobb et al., 2003). In the study of the design experiment, there existed two research goals as investigation of learning of the students and their cognition and developing the instructional theory and the HLT beneficial for communication of learners and their conceptual understanding. The data collected for the first research goal was analyzed for the second research goal. The design was tested, changed and reorganized based on the data. Moreover, while the learning process was in progress, the understanding of the research about the phenomenon becomes deeper and more meaningful. It was important to interpret the learning process and the learners' reasoning and learning in this process and the means organized to encourage learning. The issue of what was happening in the classroom was critic to interpret and explain explicitly and clearly (Graveimejer & Cobb, 2006). In other words, it was important to make connection between theory and practice by interpreting the happenings in the classroom to make inferences for HLT and the theory. In these respect, the actions, learning and reasoning process of the PMSMT were examined by considering their mathematical argumentations based on the concept of triangles to make inferences for the designed HLT. Graveimejer and Cobb (2006) suggested using interpretative framework to help researchers interpret complex and huge amount of data in the process of retrospective data analysis and teaching episodes. In the process of teaching episodes, formal and informal meetings were made. In these meetings, the problems in the instructional sequence and their solutions were discussed.

The last phase is conducting retrospective data analysis. It includes the actions of analyzing the data in a comprehensive and systematic way and recording the reasons for particular inferences (Graveimejer & Cobb, 2006). In this way, the resulting claims become trustworthy (Cobb et al., 2003).

DBR used in the present study was conducted as the experiences of PMSMT's in a classroom focusing on the improvement of their geometrical understanding and learning by testing and revising an instructional design on the concept of triangles. With the help of the nature of DBR about the development of domain specific theories, it aimed to systematically examine the learning and means of support within a designed environment for specific learning about a particular domain (Cobb et al, 2003) in mainly pilot study. Also, the research team aimed to test a theory in order to design a beneficial lesson plan related to the concept of triangles. According to the results of retrospective analysis made through data obtained from the pilot study, actual HLT was produced. Then, this HLT was used in order to extract classroom mathematical practices in the main study.

3.2 Local Instruction Theory

In DBR, the learners are provided by a designed group of connected supports in a designed environment, also named as design contexts, identified as "interacting systems rather than as either a collection of activities of a list of separate factors that influence learning" (Cobb et al, 2003, p. 9) by producing instructional theories. In this respect, it has an exploratory nature by producing, testing and revising the instructional theory since it focuses on why designs work and which inferences can be made for other environments. Then, this theory can be explained as local instruction theory by designing supports and environment and examining various types of learning in a specific content area. In this respect, it can be stated that learning is performed by obtaining knowledge through social practices (Cobb et al, 2003).

DBR mainly aims to develop a local instruction theory (Gravemeijer & Cobb, 2006). In the process of the applying DBR methodology, a conjectured local instructional theory is designed through empirical evidence such as literature review and proposed learning theories considering specific mathematical domain at the beginning. In the progression of DBR, a conjectured local instruction theory is analyzed in ongoing process and modified based on information obtained from implementation of instructional interventions (Gravemeijer & van Eerde, 2009). Also, revisions can be made on the instructional sequence and the subsequent instructional experiment (Markworth, 2010). For example, in the present study, in the process of the course of a six-week instructional cycle, there existed mini cycles occurring almost six times in each week in the sequence of instruction as a DBR as illustrated in Figure 1 adapted from Gravemeijer and Cobb (2006).

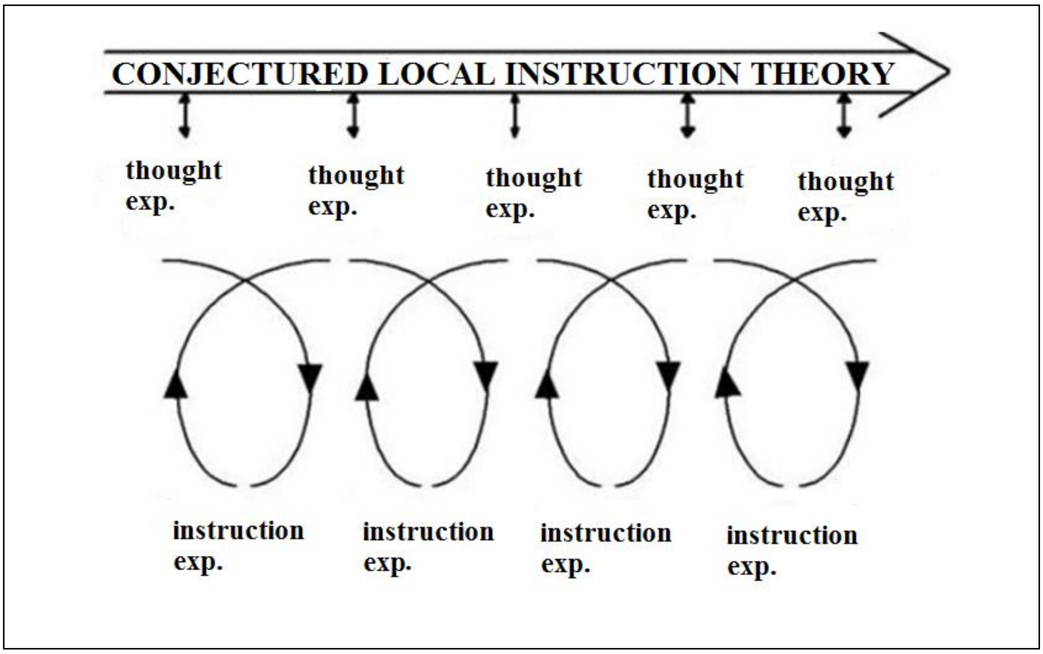


Figure 1 Reflexive Relation between Theory and Experiments

All of the micro cycles represent the long term macro cycle. In DBR, it is important to examine what is happening paying attention on what has taken place in the past and what is going to take place in the future (Fuentes, 2012). Therefore, the HLT implemented in a macro cycle and the instructional sequence was examined and revisions were made on the HLT based on this macro cycle. Then, revised HLT was conducted to another group referring to another macro cycle. For example, completed six-week instructional sequence explained in the above example comprised a macro cycle as illustrated in Figure 2 adapted from Gravemeijer and Cobb (2006). The second macro cycle consisted of the implementation of the revised instructional sequences based on the revisions to the conjectured local instruction theory. Therefore, in the present study, there existed two macro cycles which were the pilot study and implementation of the revised HLT.

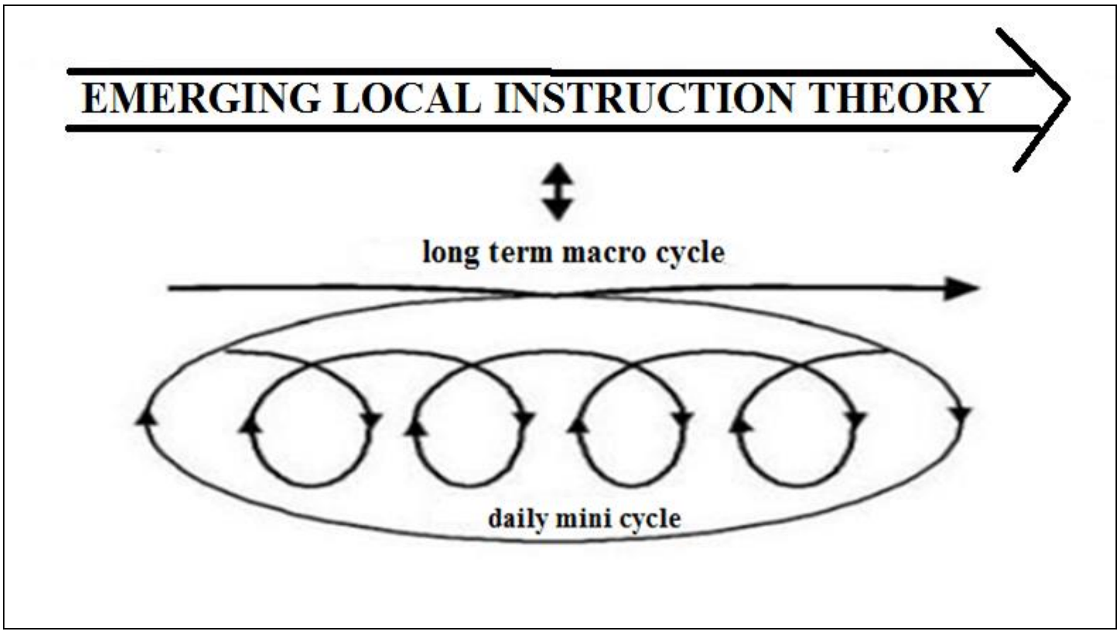


Figure 2 The Micro and Macro Cycles

When the cyclical iterative phases of conducting DBR are considered, they are anticipation, enactment and evaluation whose iterations form the macro cycle related to conjectured local instruction theory (Gravemeijer & van Eerde, 2009; Shavelson, Phillips, Towne, & Feuer, 2003; Simon, 1995). Firstly, in anticipation phase, the sequence is designed. Secondly, the enactment phase happens. The planned and designed instructional sequence takes place through weekly mini cycles, i.e., the phase of DBR. The last phase is evaluation. Retrospective analysis is made and necessary revisions are made on HLT and instructional sequence. The findings obtained through these three phases propose implications for the conjectured local instruction theory.

3.2.1 Anticipation of conjectured local instruction theory

The anticipation phase includes the planning process for the hypothetical learning trajectory (HLT), designing learning activities and the development of conjectured local instruction theory. A HLT was designed in order to solve some of the perceived problems in traditional ways of teaching triangles concept. The theoretical issues and the model of geometrical reasoning and mathematical argumentation constituted the basis of the teaching approach organized and tested in the current study. The main focus of the HLT was the problem solving activities related to triangles designed by considering the properties of van Hiele geometric thinking levels. Moreover, van Hiele geometric thinking theory used to explain the instructor's expectations about the pathway of the instruction and the learners' actions in the classroom since the HLT included the expectations of the instructor about the learners' behaviors related to learning activities and their understanding of the concept. This HLT included imagery/tools such as drawing and constructing triangles for the activities such as equilateral and isosceles triangles, examining the possibility of formation of triangles by some elements, constructing and examining the auxiliary elements of triangles and congruence and similarity of them.

The goal of the HLT used to determine the classroom mathematical practices designed for the concept of triangles was to affect preservice middle school mathematics teachers' (PMSMT) subject matter content knowledge related to triangle concept of geometry. While forming the HLT, means of support was important taking place in this process. In order to design HLT, it was important to determine means of support.

Cobb (2003) separates means of support in DBR into four interrelated groups: the instructional tasks, the tools students use, the nature of the classroom discourse and the classroom activity structure. Instructional tasks refer the mathematical activities on which the learners make reasoning and develop their understanding while dealing with. These activities become more useful when they are designed based on the situations which are problematic to improve their understanding and conception. These situations can be exemplified as "(a) resolving obstacles or contradictions that arise when they attempt to make sense of a situation in terms of their current concepts and procedures, (b) accounting for a surprising outcome, (c) verbalizing their mathematical thinking, (d) explaining or justifying a solution, (e) resolving conflicting points of view, or (f) developing a framework that accommodates alternative solution methods and formulating an explanation to clarify another child's solution attempt" (Wood, Cobb, & Yackel, 1995, p. 413). Therefore, the researcher considered the ways in which the tasks developed conceptual understanding relating problematic situations and facilitated whole class discussions leading learning and understanding while planning the instructional tasks taking place in the HLT on the concept of triangles. The tools as the second means of support (Cobb, 2003) are useful in the process that the learners reorganize their understanding and reasoning while dealing with the problematic situations in the activities. Moreover, Stephan (2003) exemplifies the tools as physical materials, tables, pictures and standard or nonstandard symbols. Gravemeijer (2004) adds other examples for the tools invented by the learners through solving problems in the activities. They are accepted as the basis on imagery with the explanation of Thompson (1996) about grounding mathematical reasoning in imagery. In this respect, similar imagery can be formed through the experiences of individuals participating in the same instruction. Therefore, it is important to determine tools and imagery for the HLT of the study in order to design an environment in which the learners participating in the instructional sequence attain common experiences. In the present study, the tools and imagery for the conception of triangles of PMSMT were examined through literature review and previous research. The other means of support identified by Cobb (2003) is the classroom discourse taking basis from norms which are social and sociomathematical norms referring the participation structure in the classroom (Cobb et al., 2001; Cobb & Yackel, 1996; Stephan & Cobb, 2003). These norms are vital to provide PMSMT opportunities to express their understanding and reasoning on triangles leading the emergence of classroom mathematical practices. The last mean of support is the activity structure of the classroom composed of small group works and whole class discussions taking place in the DBR. The activity structure has strong impact on classroom discourse and emergence of mathematical practices including various interpretations, expressions and solutions of the students becoming taken-as-shared. All explained means of support were examined by planning and designing the HLT on the concept of triangles for the present study. Therefore, this planning process was separated into the titles of the tasks, tools and imagery and possible discourse topics by making closed relationship between them to form classroom activity structure (Cobb, 2003) benefiting from problem solving method and van Hiele geometric thinking levels in the environment of problem-based learning.

In light of the means of support identified by Cobb (2003) and with the aim of accomplishing this goal, the HLT was designed with three phases including the means of support. Also, literature review based on the concept of triangles, van Hiele geometric thinking levels adapted for triangles, the objectives about them in the middle school mathematics curriculum and many textbooks was used in the present study. Through these sources, the important points, necessary knowledge and skills on the concept of triangles were determined. Then, the problems and the activities were formed benefiting from problem solving strategy and van Hiele geometric thinking levels. Hypothetical Learning Trajectory (HLT) specified as theoretical construct explaining the instructors' predictions about the progress in instructional sequence (Simon, 1995). Also, it can be designed effectively in three categories as a learning goal, learning activities, and a hypothetical learning process. In this respect, the HLT was designed by considering them in three ways so learning goal, learning activities, and a hypothetical learning process were explained by the practices. These practices included the period of six weeks and three hours in each week. The cycle of instructional sequence was organized based on the HLT, PMSMT's interactions including mathematical argumentations, and the instructor's knowledge in the current study. This organization process was made and revised regularly by the research team. The ways in which the PMSMT engaged in mental activities supporting them to form mathematical argumentations and the activities reflected the learning goals. In the process of instructional sequence, necessary modifications were made on the HLT by considering experiences. In this process, the instructor was responsible for providing opportunities for the PMSMT to transfer their geometric reasoning about the concept of triangles to more knowledgeable one by the HLT. In order to bring the HLT back to life, it was important to design instructional sequences. There were three goals of the present study forming three phases. The tasks focused on three types of activities in the instructional unit of the first phase. They were definition and classification of types of triangles, studying on exemplars, variants, and palpable (clear) and difficult distractors for triangles and examination of the possibility of construction and drawing of triangles based on some known elements. Phase 2 emphasized the critical properties related to auxiliary elements of triangles such as median, altitude, angle and perpendicular bisectors. Phase 3 as the last phase of the HLT focused on formation of similarity and congruence of triangles and important properties about congruent/similar triangles.

The first phase of the HLT was devoted to basic ideas necessary to construct and develop the expected deep conceptual understanding on triangles. Therefore, fundamental concepts having importance for understanding and development of triangles were placed and emphasized in the HLT. By the end of this phase, the goal was that the PMSMT would not only form triangles, but also evaluate different contexts about the formation of triangles. With this aim, the first phase of the HLT was composed of three activity sheets focusing on different but related learning objectives as in Tale 2. The first activity sheet was designed for the objective about the formation of triangles focusing on the definitions of types of triangles and the classification of them. This knowledge was focused on since the fifth grade students learn different types of triangles based on the mathematics curriculum. Also, seventh grade students are taught the definition and formation of triangles. Moreover, while designing this activity sheet, the properties of the second and the third van Hiele geometric thinking levels, analysis and informal deduction in order, were considered. In other words, through this activity, the PMSMT were expected to attain necessary skills on the concept of triangles based on these levels. This activity sheet was prepared based on the history of the concept of triangles so the definitions produced by Euclid and the study of Proclus in the Commentary on the First Book of Euclid's Elements analyzing the definition of types of triangles for the basis of classification of them based "partly on their sides and partly on their angles" (Morrow, 1970, p. 130). With the activity of classification, the importance of the main elements of triangles was emphasized. Moreover, as explained by Proclus, this activity enhanced understanding the relationship between these types by stating "From these classifications you can understand that the species of triangle are seven in all, neither more nor less. The equilateral triangle is one only and is acute-angled; but each of the other two has three kinds. The isosceles is either right-angled, obtuse-angled, or acute-angled; and the scalene likewise has the same three forms" (Morrow, 1970, pp.132-133). With the help of this activity, the basic knowledge on the formation of triangles was examined to help PMSMT attain deep knowledge about it.

The second activity sheet was designed based on the properties of the first and second geometric thinking levels, visualization and analysis in order. The objective of the second activity sheet was to determine the shapes of triangles and reasoning on this identification. It was designed based on the explanation of Clements and Sarama (2009) about thinking and learning about specific shapes adapted for PMSMT education. They claim that the learners show tendency to seeing and discussing typical forms and appearance of the geometric shapes which are exemplars and ignoring other forms of these shapes which are variants. Also, they emphasize the importance of discussing about non-examples separated into two groups; "*palpable distractors* if they have little or no overall resemblance to the exemplars and *difficult distractors* (for the children, we call them "foolers") if they are highly visually similar to exemplars but lack at least one defining attribute" (Clements & Sarma, 2009, p. 127). In this activity, twelve shapes were formed on the sheet as the representations of exemplars, variants, palpable and distractors for triangles. Then, the participants were asked to determine whether the shapes were triangles or not and to explain the reasons of this identification.

The last activity sheet was designed for the objective about evaluation of the formation of triangles using some of elements or attributes of triangles. Moreover, while designing this activity sheet, the properties of the third and the fourth van Hiele geometric thinking levels, informal deduction and deduction in order, were considered. In this activity, the PMSMT were not expected to represent the properties of the level of deduction completely. That is, through this activity, the PMSMT were expected to attain necessary skills on the concept of triangles based on these levels. In this activity sheet, there were problems examining the possibility of formation of triangles based on some known elements which were main and auxiliary elements. This activity sheet was designed since eigth grade students examine the possibility of formation of triangles with the objective of drawing the triangle by knowing enough number of elements. For these problems, various groups accepting some of these elements as known were formed and then PMSMT were asked to investigate the possibility of formation of triangles having these known elements represented in these groups. The first problem was about making generalization about the formation of triangles based on knowing some elements keeping the number of these elements at the minimum level. These questions take place in Şahin's book (2013). Moreover, when literature review was made about triangles, it was observed that construction steps could be beneficial tools for the formation of triangles and reasoning on them.

Table 2 Phase 1 of the Hypothetical Learning Trajectory

Learning Goals	Evaluating the formation of triangles
Concepts	Definition of types of triangles Examples and Non-examples of triangles Main and Auxiliary elements Forming and drawing triangles based on known elements
Supporting Tasks	Classification of triangles Basic drawings of triangles
Tools and Imagery	Diagrams Compass and straight edge
Possible Discourse	Definition of triangles based on main elements Process of different construction and drawings of specified triangles

The second phase of the HLT was devoted to basic ideas necessary to develop deep conceptual understanding on the auxiliary elements of triangles. Auxiliary elements of triangles were placed in the study since eigth grade students learn the construction and properties of these elements in the mathematics curriculum. Therefore, fundamental concepts having importance for understanding and development of them were placed and emphasized in the HLT. By the end of this phase, the goal was that the PMSMT would not only construct these auxiliary elements, but also attain deep knowledge about properties of them and formation of critical points formed by them such as centroid as the concurrence point of the medians, orthocenter as the concurrence point of the altitudes. In order to examine these elements clearly and effectively, the second phase of the HLT was composed of four activity sheets focusing on each of these elements as in Table 3. They were perpendicular bisectors, angle bisectors, altitudes and medians in respectively. While designing these activity sheets, the properties of the third and the fourth van Hiele geometric thinking levels, informal deduction and deduction in order, were considered. Informal deduction of van Hiele geometric thinking level was expected to be observed while the PMSMT were proving that their construction steps were the construction of expected geometric shapes in the problem situation. Also, deduction of van Hiele geometric thinking level was expected to be observed while PMSMT were proving the concurrence of auxiliary elements of triangles and naming these critical points benefiting from proofs and justifications based on theorems that they knew and reasoning "formally by logically interpreting geometric statements such as axioms, definitions, and theorems" (Clements & Battista, 1992, p.428). The similar problems which were constructions of these auxiliary elements, concurrence of them, naming of them and the places of these concurrent points on different types of triangles through construction and mathematical justification processes were formed on the activity sheets for each auxiliary element.

The tools and imagery used for these activities were construction steps including compass and straight edge. Construction activities has been described as the techniques used to solve problems based on previously determined rules and conditions where the problems are performing drawings by only compass and straight edge. These activities instead of drawings supporting necessary conditions are beneficial with its historical grounds by Euclid (Axler & Ribet, 2005; Smart, 1998). In construction, the learners analyze the properties and elements of geometric shapes to attain deep knowledge about them (Cherowitzo, 2006).

Nowadays, the usage of compass and straight edge has been given importance in the mathematics curriculums. This construction process it provides opportunities to make connection between other geometric shapes such as circles, arcs, lines and polygons. In this respect, this process proposes challenge situation for the learners since they do not realize the construction of the shapes by this tool (Erduran & Yeşildere, 2010) but "doing compass and straightedge construction early in the course helps students to understand properties of figures" (Hoffer, 1981, p. 12). The construction activities help the learners think about the properties of geometric shapes by making relationship between them, reasoning on them to develop their geometric thinking and deep conception about geometric shapes (Napitupulu, 2001; Hoffer, 1981). Also, the process of construction does not mean proving since they refer to the applications supporting expected conditions while proving refers to the representation process of what extend the geometric shapes support expected conditions (Hartshorne, 2000).

PMSMT should be equipped about construction activities. Moreover, the nature of construction activities makes them appropriate for the present study since the learners cannot realize how to begin and construct at first glimpse so they are in challenge situation and a problem situation forms. These problem situations were useful to be used in the present study since the instructional sequence and the HLT of the present study was designed based on problem-based learning approach. Therefore, the activities by compass and straight edge were emphasized in this phase. PMSMT were asked to construct these auxiliary elements, justify the concurrence of them on triangles, name these concurrent points formed by them and identify the places of these critical points for different types of triangles by reasoning. They were also asked to reason about these situations and provide mathematical justifications and representations for them.

Table 3 Phase 2 of the Hypothetical Learning Trajectory

Learning Goals	Reasoning on auxiliary elements of triangles and concurrence of them
Concepts	Medians Angle bisector Altitude Perpendicular bisector
Supporting Tasks	Definitions Constructions Concurrence on a triangle Name and critical importance of concurrent points Changing/unchanging places of these points for different types of triangles
Tools and Imagery	Drawings Compass and straight edge
Possible Discourse	Various ways of construction of these elements Various ways of justifying the concurrence of them Various reasons of changing/unchanging places of these points for different types of triangles

The last phase of the HLT was devoted to basic ideas necessary to develop deep conceptual understanding of congruence and similarity of triangles having critical importance on the concept of triangles as in Table 4. The congruence and similarity of triangles are taught in sixth, seventh and eighth grade classroom as in mathematics curriculum. This concept has historical importance beginning from the book of Euclid and Thales (624-547 B.C.) developed the concept of similarity of triangles. Through this historical development and by the connection of this concept with real life problems such as forming and representing the models in proportion before their real construction process, this concept attains critical importance in curriculum and mathematics teaching and learning from middle school to college level. Therefore, these fundamental concepts having importance for understanding and development of them were placed and emphasized in the HLT. By the end of this phase, the goal was that the PMSMT would not only form these congruent and similar triangles, but also attain deep knowledge about properties of them, criteria of congruence and similarity and application of them in different problem situations.

While designing these activity sheets, the properties of the third and the fourth van Hiele geometric thinking levels, informal deduction and deduction in order, were considered. Informal deduction of van Hiele geometric thinking level was expected to be observed while the PMSMT were proving that the triangles obtained through construction were the image triangles formed by transformation geometry. They made connection between triangles and image triangles concerning their shapes, elements and properties. Also, deduction of van Hiele geometric thinking level was expected to be observed while PMSMT were proving and justifying whether the triangles and their image triangles were congruent/similar and criticizing the congruence and similarity of triangles explained in different contexts with some specific known elements based on theorems that they knew and reasoning (Clements & Battista, 1992). In order to examine them clearly and effectively, this phase of the HLT was designed at two levels; congruence and similarity through transformation geometry and similarity and congruence on triangles. The first level was designed including four activity sheets about translation, rotation, reflection and dilation transformations respectively. In the textbooks, the concept of congruence to reproduce exactly the same object and similarity to form the object in proportion with the prior geometric object is emphasized. When the process of image formation through transformations was examined, the same objects were formed through translation, rotation and reflection referring to congruence and the proportional object is formed through dilation referring to similarity. This explanation about motional aspect of congruence/similarity by construction was also placed in some textbooks (Alexander & Koeberlein, 2011).

French (2004) explains the concept of construction and congruence of triangles with rigid motions. These activity sheets were formed in this way. French (2004) also states the importance of enlargement and similarity so an activity sheet was designed about dilation in the similar way. Therefore, the similarity and congruence of triangles through transformation geometry by construction was used in the present study. The similar problems about formation of the image triangles through transformation by construction on geometric view and drawing benefiting from the coordinate system on algebraic view were formed on the activity sheets for each type of transformation. Moreover, the participants could use proofs in order to show congruence and similarity of triangles and making inferences for the similarity/congruence critaria. The tools and imagery used for these activities were construction steps including compass and straight edge and the coordinate system.

The other level of this phase was designed including two activity sheets about congruence and similarity respectively. In almost all textbooks, the criteria of congruence and similarity have been studied. Therefore, the emergence and formation of these criteria was examined in these activity sheets. Moreover, the relationship between triangles and image triangles based on congruence and similarity was formed at the beginning of these activity sheets. Then, the emergence of congruence/similarity criteria and necessary mathematical justifications or proving was examined. At the end of the second activity sheet, there was a list of explanations representing particular triangles with their some known elements. Then, PMSMT were asked to examine whether they were congruent/similar triangles formed by transformation geometry in specific contexts. Also, they were expected to make mathematical justifications or proving about these contexts. These contexts were formed by the researcher inspiring from the explanations and general statements about triangles placed in various geometry textbooks.

Table 4 Phase 3 of the Hypothetical Learning Trajectory

Learning Goals	Reasoning on congruence and similarity of triangle
Concepts	Translation, Rotation, Reflection, Dilation Congruence & Similarity
Supporting Tasks	Formation of the image triangles with geometric and algebraic views comparing triangles and their images
Tools and Imagery	Dot paper Compass and straight edge
Possible Discourse	Various ways of construction of image triangles Similarities and difference of triangles and their images including main or auxiliary element, concurrent points, orientation, position Criteria of congruence/similarity of triangles by mathematical justifications Determining congruence/similarity of specified triangles

3.2.2 Enactment of conjectured local instruction theory

The phase of the enactment of conjectured local instruction theory of DBR was taken place by the application of the HLT and the instructional sequence. In other words, it included the first macro cycle with weekly mini cycles lasting six-week-period and three hours in each week. While determining the means of support, the researcher examined the literature based on the concept of triangles. They were formed in light of the necessities of van Hiele geometric thinking levels and problem-based learning strategy. Once they were formed, they were implemented to five PMSMT not participating in the instructional sequence in neither pilot study nor main study. The activities were examined by these PMSMT, the researcher and the academician as the non-participant observer of the instructional sequence. After determining the HLT with this team, the HLT was conducted to the pilot study group. In the process of the present study, enactment of conjectured local instruction theory, pilot study was conducted. The pilot study group included 23 PMSMT. The designed HLT was conducted to them in order to attain a source of input for the revisions of the HLT and then to apply the revised form of it in the main study. In other words, the first macro cycle of the sequence was implemented to obtain data for retrospective analysis and designing actual HLT for the instructional sequence of the main study that the classroom mathematical practices about triangles were examined.

The research team including the researcher, non-participant observer and three PMSMT participating in the instructional sequence for the pilot study took responsibility in the first macro cycle to collect data to form actual HLT through retrospective analysis based on the explanations of Cobb (2000) and Simon (2000) about the necessity of the research team. In the instructional sequence, the PMSMT initially worked in small groups in which the participants studied with their peers. While they were working on the activity sheets with their peers in small group works, the instructor visited the small groups to attain knowledge about their knowledge and actions, to determine different mathematical ideas about tasks and to extract whole class discussion topics. After the completion of small group works, the whole class discussion was started. Different interpretations, representations and mathematical ideas were discussed about the tasks with their reasons. The small group works and whole class discussions were mainly taken place in all stages and for all mathematical tasks on the HLT in this way.

The first week of the instructional sequence began with phase one of the HLT with the learning goal of evaluating formation of triangles. This was also the initial phase where PMSMT became acquainted with the triangles. The goal of this lesson was for PMSMT to define a triangle and its types and to evaluate formation of triangles in various contexts in problem situations. In the instructional sequence, the first week was important because of examining and attaining deep knowledge about triangles and began to establish the social and sociomathematical norms of the classroom. The instructional sequence started with the activity having the title of *Classification of Triangles* as the first activity sheet of the week, a task designed to classify the types of triangles based on the relationship between them by defining them. In this activity, the participants had difficulty in defining and classifying triangles by relating them based on their critical attributes. Then, they were challenged to explain the reasons and main ideas about the placement of them. At this problem, they made explanations about main elements of triangles and their roles by relating and classifying triangles. They had difficulty since they knew the names of main elements but did not know how to apply them on relating and classifying them. At the end, they were asked to find another way to classify triangles.

By this task, it was intended to lead into a discussion of different classifications based on main elements of triangles. Moreover, PMSMT produced different representations for the types of triangles and the classification of them. They discussed the definitions of triangles and realized that the classification of triangles could be made with respect to the main elements of triangles; *angle* and *edge*. In this way, they realized the role of triangles' main elements on definitions and classifications of them. Moreover, they used different representations and diagrams. The purpose of the next task for the class was to determine whether the figures on the activity sheet were triangles or not and to explain the reasons of this identification. Therefore, they were asked to determine whether the given shapes were triangles or not and to explain the reasons of this identification. They were wanted to form these explanations mathematically. In both small groups and whole class discussions, it was realized that this task did not produce challenge situations for the participants. They successfully formed the explanations for these shapes with a minimum level of effort. In these initial two activity sheets, the PMSMT focused on the formation of triangles based on their main elements and relationships between them with the help of classification of them. The purpose of the last task for the class was to evaluate the possibility of formation of triangles based on some known elements which were main and auxiliary elements. For these problems, various groups accepting some of these elements as known were formed and then PMSMT were asked to investigate the possibility of formation of triangles having these known elements represented in these groups. In the small group works, they tended to solve these problems based on related theorems and the properties of triangles. In the whole class discussion, they learned the solution of these problems by drawings and constructions of these problems and the importance and necessity of this tools/imagery. In other words, they realized that their solutions were not completed since they thought different types of triangles fitting the explained situation and examining the number of types of triangles for that situation by drawing and construction strategy. This last activity sheet provided opportunities for PMSMT to examine the formation of triangles based on auxiliary elements of triangles so that they realized the roles of them in the formation process. These activities completed phase one of the hypothetical learning trajectory whose learning goal was evaluating formation of triangles. This phase was concluded as the tasks which were aimed directly at providing learning and understanding about formation of triangles based on their main elements by defining, classifying and examining these processes.

The second and third weeks of the instructional sequence continued into phase two of the hypothetical learning trajectory with the learning goal of reasoning on auxiliary elements of triangles and their concurrence. The second week included the tasks about the perpendicular and angle bisectors and the third week focused on angle bisectors and medians. These tasks included similar problems which were construction of one of these elements among three elements for each triangle, the construction of three of them for a triangle and representing the concurrence of them, naming these concurrence points and identifying whether the position of these concurrence points changed or not and the reason of this identification. These tasks were supported by the imagery/tool of construction steps. The process of usage of construction activities in the instructional sequence was performed by the steps explained by Smart (1998). The first step is analysis. The learners perform the construction of the shape assuming that the explained conditions occur and making connections between necessary unknown conditions in the problem and explained the conditions in the problem. The second step is construction. The learners form the shape using compass and drawing straight edge through construction. The third step is proving. It represents the process in which they prove that the constructed shape is the shape wanted to be formed in the problem. The last step is discussion. The possible alternative solutions and situations for the construction and proving are discussed. These steps facilitated the understanding of PMSMT about how they could use the compass and straight edge with their prior geometry knowledge to construct the auxiliary elements of triangles. In this respect, the actions and the ways that they were constructed by geometrical thinking were discussed and learned effectively. In order to help PMSMT attain familiarity with these steps, learn effectively and establish one of the sociomathematical norms of the classroom, a problem about copying a particular triangle through construction was formed in the first activity sheet of the second phase of the HLT. These activities completed phase two of the hypothetical learning trajectory and two weeks of the instructional sequence whose learning goal was examining the auxiliary elements of triangles. This phase was concluded as the tasks which were aimed directly to provide learning and understanding of these elements, concurrence of them, examine their names, critical importance and places on different types of triangles.

The fourth and fifth weeks of the instructional sequence continued into phase three of the hypothetical learning trajectory with the learning goal of reasoning on the congruence and similarity of triangles. These weeks included different kinds of tasks and the learning goal was examined at two levels, one for each week. The fourth week included the tasks about the congruence and similarity of triangles through transformation geometry. These tasks included similar problems which were definition of the types of transformation geometry, construction of the images of triangles by compass and straight edge, formation of images of triangles on the coordinate system, the similarities and differences between triangles and their images obtained by one transformation or composition of two same transformations. There were four activity sheets including these problems adapted for translation, rotation, reflection and dilation respectively in each of these activity sheets. These tasks were supported by the imagery/tool of construction steps by compass and straight edge and dot paper to illustrate the coordinate system. The process of usage of construction activities in the instructional sequence was performed by the steps explained by Smart (1998) and also these steps took place in the sociomathematical norm through previous weeks of the instructional sequence. They attained knowledge about similarity and congruence of triangles and implications for the criteria for congruence and similarity in the whole class discussion. For example, by these imagery/tools, PMSMT realized that the lengths of the edges of triangles were preserved through translation, rotation and reflection. In translation by construction, they focused on vectors and their properties so that they realized that the lengths of the edges were kept same because the distance between two parallel lines was always same. The activities on coordinate system strengthened this realization since they used the formula of finding the length of the line segments whose start and end point places on the coordinate system were known. They found the lengths of the edges of triangles and their image triangles so that they clearly illustrated congruence of triangles formed through translation. Then, they made implications for the congruence criteria of S.S.S. The similar discussions were made about other transformations, rotation and reflection by relating congruence of triangles. The last activity sheet was designed including two problems; one for enlargement of a triangle and the other for reduction of a triangle, to form image triangles through construction. This activity sheet did not include the problem about formation of image of triangle on coordinate system. Through the discussion of these problems, they realized the similarity of triangles and some of the criteria for similarity. These activities completed the first level of phase three of the hypothetical learning trajectory and one week of the last phase of the instructional sequence whose learning goal was about examining the congruence and similarity of triangles. This week was concluded as the task about changing and unchanging properties and elements of triangles through transformation and explanations of reasoning about them and at the end, attaining the deep knowledge about formation of congruent and similar triangles, their properties and motional aspect of congruence and similarity.

The other level representing the fifth week of the instructional sequence related to last phase of the HLT included two activity sheets; one for congruence of triangles and the other one for similarity of triangles. In the first activity sheet including tasks for congruence of triangles, the PMSMT initially discussed the differences and similarities between triangles and the image triangles formed through rigid motions or composition of finite number of these rigid motions. In another task, PMSMT were asked which rigid motions were performed on the triangle to obtain the other. By reasoning and discussing on this activity, they realized the criteria of A.S.A. and then they continued to discuss importance and roles of other criteria of congruence of triangles and to make reasoning on them. The other activity sheet was designed for similarity of triangles. They talked about the differences and similarities of triangles and their images obtained through the dilation and continued to discuss the criteria of similarity of triangles by reasoning. This week was concluded as the task about identifying whether the triangles explained in problem situation and having particular properties were congruent/similar or not and explaining the reasons of this identification. All of the activities completed phase three of the hypothetical learning trajectory and two weeks of the instructional sequence whose learning goal was to understand the congruence and similarity of triangles. This phase was concluded as the tasks which were aimed directly at providing learning and understanding of congruence and similarity of triangles. After all of the phases of the HLT were completed, PMSMT engaged in an activity sheet including problems about triangles related to objectives of all phases in the sixth week of the instructional sequence.

3.2.3 Evaluation of conjectured local instruction theory

The instructional sequence referring to the enactment of the designed HLT and conduction of the pilot study was followed weekly by a research team including five participants which were the researcher, an academician having the Phd. degree in mathematics education and three PMSMT participating in the pilot study. The data were collected weekly through the process of instruction for the pilot study based on the concept of triangles by video recordings of whole class discussions and classroom sessions, audio recordings of small group works and research team discussions, field notes taken by the instructor and artifact collection including worksheets. Through and at the end of instructional sequence, rich and detailed data were collected in order to obtain information about the HLT and making necessary revisions on the HLT and the conjectured local instruction theory. This instructional sequence representing the pilot study formed the first macro cycle of the study. In other words, throughout the implementation of this first cycle of the instructional sequence referring to the pilot study and after the sequence was completed, revisions were made on the sequence for implementation of the main study (Cobb, 2000; Simon, 2000). The revised HLT was then implemented again as the main study representing the second macro cycle by the instructor. The pilot study was illustrated and summarized in Figure 3 where mini cycles representing DBR referred to the weeks of instructional sequence. For the main study, the same figure can be formed by being titled as Macro Cycle 2. Before starting the main study which was the second macro cycle, retrospective analysis was performed and necessary revisions were made on the HLT so that actual HLT for the main study was formed.

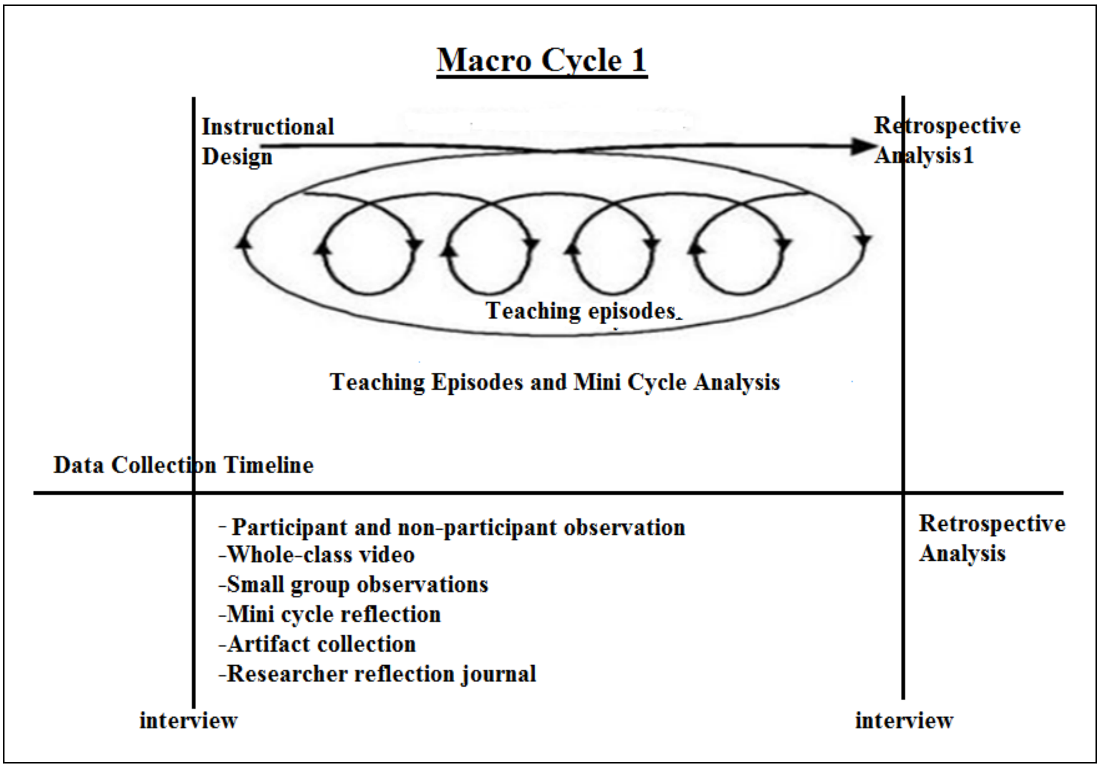


Figure 3 Illustration of the analysis of data collection process of the pilot study

After the completion of the instructional sequence, retrospective analysis was conducted. The retrospective analysis was used for both analyzing the data collected through various sources such as video and audio recordings for the process of pilot study in DBR and forming new data synthesized for the next part of DBR as main study. Whole data set gathered through the first macro cycle was analyzed collectively to attain information about "patterns in the data, framing assumed patterns as conjectures about the data, testing those conjectures on the complete data set, and using the findings as data for a subsequent round of analysis" (Gravemeijer & van Eerde, 2009, p. 517). The data collected through the retrospective analysis based on the first macro cycle were used for the next macro cycle of the study that the mathematical practices emerging in the social learning environment were investigated for the research question of the present study. In light of the findings of the retrospective analysis, revisions were made on the HLT and the actual HLT using for the main study was formed. In initial HLT, three learning goals, or big ideas, were at the center of it. The research team decided not to change these phases and kept them in the actual HLT. They were: (a) evaluating the formation of triangles, (b) reasoning on auxiliary elements of triangles and concurrence of them, and (c) reasoning on congruence and similarity of triangles. The lessons designed based on these learning goals was supported by various tasks and concepts about triangles and they were systematically prepared for the phases of the HLT. The revisions were made on initial HLT based on these phases separately.

In the first phase of the HLT, the research team decided to make two revisions on the tasks. First revision was removing the second activity sheet since this activity sheet did not provide challenge situations for the PMSMT although the problem-based learning strategy was used in the present study. In other words, the PMSMT determined whether or not twelve shapes were triangles by explaining the reasons without having any difficulty. They made identification by illustrating minimum level of effort when compared with other problems in the instructional sequence. Another revision was made on the last activity sheet of the phase. The research team decided to change the place of the first problem. This problem “Which and at least how many elements do we need to know in order to show that it is possible to form a triangle? Explain the groups including some of these elements.” was placed at the end of this activity sheet since the solution of this problem was providing a general statement about the formation of triangles knowing some main and auxiliary elements by summarizing and considering other problems on the same activity sheet. Moreover, suggestion about changing the explanation of this problem was made and performed.

In the second phase of the HLT, the research team decided to make two revisions on the tasks. The research team decided to change the explanation of the problem about showing the concurrence of three auxiliary elements on a triangle. This problem was written by showing the concurrence of them through construction. The research team decided that the usage of construction limited the thoughts of the PMSMT about solution of this problem since they could have provided different solutions for this problem. Therefore, the statement of the construction on the problem was removed. The second revision was made about the order of these activity sheets. The research team decided to engage in the activity sheet about medians initially since the PMSMT had more knowledge about it than others. The researcher and the non-participant observer made the same decision to form sociomathematical norms related to the construction steps of Smart (1998) based on the same reason. Moreover, the activity sheet about perpendicular bisectors was placed as the last one in this phase since PMSMT had least knowledge about it when compared with the others. The researcher and the non-participant observer (witness of the study) made the same decision to provide PMSMT opportunities about realizing that the perpendicular bisector was different from other auxiliary elements since it was not a cevian while the others were the examples of cevian.

In the last phase of the HLT, the research team decided to make two revisions on the tasks. The research team decided to remove the problems about the formation of image triangles through composition of transformations. They decided that these problems could be discussed briefly while discussing the formation of the image triangle. For example, one of the PMSMT in the research team stated,

We found that congruent triangles are formed through translation and the result does not change if we conduct two translations consecutively. We again obtain congruent triangles. Therefore, it is unnecessary to deal with this situation as another problem since we always form congruent triangles no matter how many times we apply translation.

This explanation was valid for all rigid motions. Therefore, the research team decided to remove this problem on the activity sheets for all rigid motions but suggested to discuss it in the problem about formation of image triangle. Another revision was suggested for the last problem on the activity sheet about similarity of triangles. This problem was about identifying whether the triangles explained in problem situation and having particular properties were congruent/similar or not and explaining the reasons of this identification. The research team decided to increase the number of these problem situations and forming different and harder examples. Therefore, the number of these examples was increased by adding more difficult statements related to all learning goals of the HLT.

According to these revisions based on the findings obtained through retrospective analysis, the actual HLT was formed in order to conduct in the main study. The actual HLT was represented in the following table. This table illustrated the concepts, supporting tasks, tools and imagery and possible discourses for each phase of the HLT. This actual HLT was used in the main study to identify the classroom mathematical practices emerging in instructional sequence representing the second macro cycle.

3.3 Case Study

Case study is probably most widely used approach in education research aiming to investigate a specific phenomenon in a bounded system (Creswell, 2009; Merriam, 2009). It can be described as examination of one or more instances of a phenomenon in its real life context externalizing the participants' perspectives in a detailed way carefully. A good case study makes the phenomenon alive and real and provides understandable meanings for the reader (Gall, Gall & Borg, 2007). The case study approach is used when the research focus is finding and stating the holistic and meaningful characteristics of real-life phenomena (Yin, 2003). In the particularistic or intrinsic case studies as a kind of case study research, the case is selected with respect to the researchers' interest and willingness to understand the phenomena (Merriam, 2009; Stake, 1995) with the aim of in-depth investigation of the case.

In this respect, the present study was a particularistic case study because it was aimed to examine the PMSMT's mathematical argumentations to understand their learning understanding and reasoning about the geometrical concept of triangles.

3.4 Participants

The participants in the present study were enrolled in the program of elementary mathematics education at a university in the northern part of Turkey. Fourty-siz Preservice middle school mathematics teachers (PMSMT) participated in the study. The classrooms included preservice middle school mathematics teachers who were junior and registered in the program of elementary mathematics education. The junior PMSMT were selected since participants were expected to have necessary knowledge about the concept of triangles and main theorems related to it. Also, they could make connections between triangles and other concepts in geometry of Geometry and Analytic Geometry courses in previous semesters. Moreover, because of the prior knowledge of PMSMT for the activities and problems in the instructional sequence, they were expected to have knowledge about the course of Analytic Geometry. Therefore, they were selected since they had enrolled in these courses. Twenty-six of fourty-six junior PMSMT were female and twenty were male students.

Fourty-six PMSMT were separated into two groups and half of them took place in the pilot study and other half of them participated in the main study. In other words, there existed two groups of PMSMT; one for the pilot study as the first macro cycle and the other one for the main study as the second macro cycle, and two research teams wereproduced by selecting three PMSMT from these classrooms in the present study. In pilot study, twelve of twenty-three junior PMSMT were female and eleven were male students. Also, in the main study, fourteen of twenty-three junior PMSMT were female and 9 were male students.

There were two research teams; one for the pilot study and the other one for the main study. These research teams included two academicians; one was the doctoral student in the department of mathematics education as the instructor of the lessons and the researcher of the present study, and other academician was assistant professor in the program of mathematics education as the non-particicpant observer (witness observer), and three PMSMT participating in the lessons in the classroom. Three of the PMSMT in the research teams were randomly selected from the classrooms. The research teams came together after the teaching episodes for each week was completed. The researcher instructed the designed lessons by providing opportunities for PMSMT to form mathematical argumentations in the instructional sequence. The study was conducted in six-week instructional sequence and three hours in each week for both of the classes. The classes of weekly cycles composed of 160-minute sessions once per week during the 2015 summer. The PMSMT learned in a social environment designed by problem-based learning strategy in which they engaged in geometrical problems with their peers in small groups and then participating in whole-class discussions.

3.5 Data Collection

The data were collected through the process of instruction based on the concept of triangles by video recordings of classroom sessions, audio recordings of small group works and research team discussions, field notes taken by the instructor and learners' works such as worksheets. The research team constituted learning community. This community shared their ideas and experiences in the instructional sequence about what was happening in the instructional sequences, what were the problems and potential misconceptions in them, what could be done to solve and remove them and what the implications were for the design research. These discussions were beneficial on documenting changes on instructional sequence with their rationales. The instructor was the researcher and member of the research team of the study. All of the participants in the study were referred to by pseudonyms.

Data collection period started approximately three months before the instructional sequence about triangles in the main study. The activities were conducted to five PMSMT, and then pilot study took place. Also, one week before the application of the main study, the first meeting of the research team happened. Firstly, the designed activities and supports were conducted to five PMSMT who were attending in neither pilot study nor main study group. They discussed about the activity sheets and provided feedback for them. Then, necessary revisions were made in light of their suggestions and discussions. Secondly, when all of the activities and HLT were designed and applied to five group members, the pilot study was conducted.

A research team was formed in the pilot study group. Instructional sequence was conducted similarly to the main study group. Based on the discussions in the classroom and the research team, retrospective analysis was made. At the end, retrospective analysis was performed and the actual HLT was produced. Afterwards, the actual HLT was implemented in the main study group. One week before performing the instructional sequence in the main study group, the data collection process was started for this group and the pretests were conducted to them. Also, after the tests were finished, the research team came together and discussed about the tests and the part of actual HLT (Phase 1 of HLT) including the activities and supports of Week 1. In the study, in order to examine and identify PMSMT's mathematical practices in the instructional sequence designed for the concept of triangles, the data were collected through classroom sessions, formal and informal meetings and interviews.

Because of the dynamic nature of DBR and the closed connection of mathematical practices with social and sociomathematical norms, the researcher obtained information and detailed understanding about the phenomenon both while the research was continued and when it was finished. In this respect, it was vital to collect and document various and detailed data about the entire process of the research by examining each step and action of the research (Cobb et al., 2003). Therefore, several data sources were employed in the current study. The sources for the research question of the study used in pilot study and main group study could be illustrated in Table 5 for the relationship between data and the research question for pilot study and main study and the points of usage of these data. For the weekly mini cycle analysis, the teaching episode conducted was discussed and inferences were made for following teaching episodes in pilot study and main study. For the macro cycle analysis, whole instructional process was analyzed. In pilot study, retrospective analysis was completed. In the main study, the mathematical practices emerging in the instructional sequence were extracted.

Table 5 Data sources and the places of usage of them in the analysis

	Weekly Mini Cycle Analysis	Macro Cycle Analysis
Pre-instruction Interview	✓	✓
Post-instruction Interview	✓	✓
Classroom Observation	✓	✓
Whole-class Discussion	✓	✓
Peer Discussion	✓	✓
Weekly Mini Cycle Reflection	✓	✓
Artifact Collection	✓	✓
Researcher Reflection Journal	✓	✓
Pre- and post-tests	✓	✓

In the classroom session, the data were collected through six weeks through participant and non-participant classroom observations, whole-class discussions by recording video cameras, peer discussions by audio recordings, field notes and artifact collection. Participant and non-participant classroom observation was made by the other academician, member of the research team and the members of the research team. She acted as complete observer by not participating in the instruction process. She observed the classroom and took notes about what was happening in the classroom, the roles of the instructor and the behaviors of the participants, environment, supports, and tasks and these notes were examined and discussed twice, once by the research team and also by the researcher and witness. Also, other members of the research team were the participant observers of the instructional sequence. They participated in the instructional sequence by engaging in mathematical tasks and observed the classroom. The video recordings of whole class discussions were the most important part of the data collection process. Each teaching episode was recorded with the help of two video cameras in order to capture the instruction and the behaviors of the instructor, and the activities of the participants and the instructor, collective learning environment and whole class discussions. With this aim, two video cameras were used by placing one of them in front of the classroom and the other at the back of the classroom. In addition to whole class video recordings, audio recordings were used in order to collect data about peer discussions about how to solve the problems on the activity sheets while the participants were engaging in activities with their peers.

These data were beneficial in order to understand clearly the social and sociomathematical norms emerging mathematical practices and to examine their individual learning. In the process of peer group discussions, the instructor interacted with these small groups so these data provided information about how the instructor interacted with these small groups and the connection of these interactions with the whole class discussions through social norms. All audio and video recordings were transcribed. Furthermore, artifact collection was performed and the activity sheets on which the participants worked with their peers and whole group discussions were collected at the end of each teaching episode. These activity sheets were examined in order to clearly represent how the participants interacted with their peers and how they solved the problems discussing and how these processes were transferred to and attracted on the whole class discussion. In addition to artifact collection at the end of the teaching episodes, researcher reflection journals were formed once a week, during each mini cycle, by field notes taken by the researcher and the other academician, witness of the research, through classroom sessions. These journals as reflective tools provided the researcher opportunities to record feelings, thoughts and impressions stepping back from the experienced teaching episode (Holly, 2002) and plans and thoughts about following teaching episodes. These journals were used in data analysis process and formal meetings.

In the main study group, these journals were beneficial in making necessary changes and inferences for instructional sequence. These journals and other data sources explained were also used in the pilot study in order to make changes in instructional sequence and to apply in retrospective analysis. Moreover, the data sources obtained through classroom sessions were watched and examined by the researcher in each week to extract the issues to focus on, to discuss with the members of the research team in formal meetings and to make inferences for following teaching episode. For example, in the whole class discussion of Week 1, it was determined that the participants had difficulty on main and auxiliary elements of triangles and then, the activities of Week 2 and Week 3 about auxiliary elements were conducted in the light of this identification.

In formal and informal meetings, necessary information and inferences were made in order to clearly examine, understand and develop instructional sequence. Through formal meetings, the instructor, who was the researcher of the study, the non-participant observer of the instructional sequence and three PMSMT participating in the instructional sequence met every week to discuss the week's experiences and make inferences, plans and revisions for the following weeks. Through this process, a small learning community was formed where the members of the research team shared their ideas and suggestions for the HLT and the instructional sequence. Moreover, the progression taking place in the instructional sequence for each teaching episode and all prior weeks were evaluated and suggestions were formed in order to remove challenges and difficulties and to provide improvement. In this respect, these meetings were useful in order to make weekly mini cycle reflections. They provided opportunities in identifying evolving conjectures and making reflections on them benefiting from other data sources collected through classroom sessions with the aim of examining these conjectures (Cobb et al., 2003).

The formal meetings of the research team were recorded by video cameras. Moreover, informal meetings were held when critical problems were observed in the instruction and needed to be solved immediately. Also, when the suggestions were needed for the immediate moment, the meetings were held informally. Two academicians discussed about these situations and questions about instructional sequence when they occurred. In this respect, the research team met formally once a week in the process following the instruction and then informal meetings were held throughout the week if necessary. The formal meetings were recorded by video camera and the informal meetings were audio taped. These meetings were held in both pilot study and main study. The instructor as the researcher of the study watched the recordings to determine the issues which were discussed, important issues which were not discussed in necessary time span or completely and issues to discuss in following meetings. Moreover, the necessary immediate issues were determined in order to use in following teaching episodes. In the pilot study, these meetings were also used in retrospective analysis.

The interviews were another data collection tool used in the present study. The participants who were members of the research team at the same time were met individually in order to make pre and post-instruction interviews. These interviews were conducted as semi-structured interviews. They were asked questions about their tasks of the particular week. These interviews were about their experiences and reflections on the teaching episodes. Pre-instruction interviews were made in order to attain information about their initial understanding and prior knowledge about triangles and the concept of triangles of the week. The post-instruction interviews were made based on their tasks of the week. They were asked about their experiences, feelings and thoughts about the problems on the activity sheets so that their improvement through instructional sequence was examined.

For the pilot study, these interviews provided detailed information to be used in retrospective analysis. In the main study, they were beneficial to extract mathematical practices and social and sociomathematical norms in which mathematical practices emerged. They provided opportunities to be flexible in following comments by the participants (Ginsburg, 1981) and to obtain various and rich information to identify and document nature of their thinking and understanding in mathematics (Clement, 2000).

3.6 Data Analysis

The classroom mathematical practices were extracted by analyzing collective activity representing how mathematical ideas became established in a classroom through interactions and accepting the classroom as a whole in the main study since the community as a whole was paid attention on. Therefore, taken-as-shared knowledge and practices were the focus point of the study examining collective activity and implications for individual participants learning taking place in the whole class discussion. Despite accepting individual PMSMT's learning as the providers for the development of taken-as-shared mathematical ideas and classroom mathematical practices emerged in the designed instructional sequence, the focus point was on whole class discussions because of the nature of classroom mathematical practices extracted by taken-as-shared view. Two methods were used with the aim of the analysis of the qualitative data gathered through the DBR. Because of the nature of the design experiment providing the development of the theories (Cobb et al., 2003), data were examined by constant comparative data analysis method of grounded theory. The data were collected through observations, field notes and documents. Glaser and Strauss (1967) explained that the constant comparative method was an inductive procedure since it included the actions of generating and linking categories by making comparisons between different incidents, incidents and categories and different categories with the aim of grounding categories in the data. Also, while the data analysis process was in progress, comparisons were made between different indicators, different codes and different categories constantly. In the broad perspective, the comparison between emerging scheme and raw data with the aim of grounding categories in the information collected in the current study was made (Creswell, 2009). Moreover, the data collection and analysis processes provide opportunities to discover the patterns in the data effectively since conducting the designed lessons took six weeks and eighteen hours in total. Also, the constant comparative method can be used effectively by comparing the data itself collected in the same day and the data gathered across different days. The meanings of the obtained categories and themes are interpreted by reflecting personally on the impact of the findings and on the literature. Moreover, the second method used to analyze the data gathered through whole-class discussion and classroom mathematical practices becoming taken-as-shared on the concept of triangles designed for PMSMT was the methodology of Rasmussen and Stephan (2008) and based on Toulmin's argumentation model (1969).

Rasmussen and Stephan (2008) formed a methodology in order to examine taken-as-shared collective learning by documenting classroom collective learning activities leading classroom mathematical practices and whole class discussions (Stephan & Cobb, 2003). This methodology includes three phases performed with Toulmin's (1969) model of argumentation. This methodology is beneficial to analyze classroom discourse and to document reasoning of the participants during instructional sequence. Moreover, classroom mathematical practices are determined by identifying what has become taken-as-shared knowledge. These phases are composed of different actions, objectives and ideas. These differences produce different products based on the ideas, solutions, strategies and procedures. The formation of mathematical practices by these phases is illustrated in Table 6 formed by Rasmussen and Stephan (2008, pp. 83-84).

Table 6 Phases in documenting collective

Phases of Research	Activity	Product
Phase One	•Transcribe every whole class discussion • Notate claims made by students or instructor • Identify data and conclusions, as well as warrants and/or backings if present • Compare argumentation schemes and come to agreement	Argumentation Log
Phase Two	• Use Argumentation Log as data • Identify taken-as-shared mathematical ideas	Mathematical Ideas Chart
Phase Three	• Use Mathematical Ideas Charts to identify common mathematical activities associated with taken-as-shared mathematical ideas	Classroom Mathematical Practices

The first phase begins by transcribing videotape recordings of each teaching episode so that transcripts for all whole-class discussions from the class periods are formed. Then, these transcripts are examined to note claims produced by either learners or the instructor, i.e., each time that a claim is formed is identified. This is followed by analyzing transcripts to determine data, warrants, backings and rebuttals produced for each claim so that

Toulmin's model of argumentation is used to produce an argumentation scheme for each claim. This process ends with the product of an argumentation log.

The process of extraction and identification of the elements of Toulmin's model from the transcripts obtained was carried out by more than one researcher independently by making meetings. The other researcher taking place in this identification process was the witness and non-participant observer of the teaching episodes. Both of them produced their argumentation logs representing claims, data, warrants and backings for each claim independently. In other words, after identifying claims, related data, warrants, backings and rebuttals as the elements of Toulmin's model were determined specifically in the contexts in which they emerged.

Afterwards, both of the researchers as producers of argumentation logs came together to discuss about their analysis and argumentation logs. They also accepted or refuted each other's opinions on the elements identified by Toulmin's model until they reached an agreement about them. If they did not come to an agreement, they discussed about it until they come to an agreement. The processes including the discussions about coming to an agreement on the argumentation logs and the elements on it strengthened the analysis. Toulmin's (1969) model of argumentation was used to illustrate the structure of arguments. This model is composed of four parts: the claim, data, warrant and backing. The first part, the claim, is the opinions proposed as true by the learners. They are also conclusions of the discussions and the easiest parts of this model since they may be an answer of a problem or a mathematical statement to be questioned. The second part, the data are the expressions encouraging claims. They provide evidences for the claims in a way that the learners participating in the argumentations show the truth of the claims. Moreover, they can be mathematical procedures or methods, mathematical relationships, facts, theorem or definitions leading to the claims. The third part, the warrant, makes the connection between the data and the claim. They provide this connection benefiting from implications of the data. It explains how the data encourages the claim by justifying the reasons that the data lead to the claim. The last part of the model is backing. A backing expresses the reasons of acceptance of an argument by increasing the validity of the claim.

The second phase of analysis focuses on identification of taken-as-shared mathematical ideas by using produced argumentation logs. Hence, these argumentation logs are examined in order to extract evidences for the ideas becoming taken-as-shared focusing on data across all class sessions and teaching episodes. This phase of analysis is about extraction of mathematical ideas taking place in the argumentation logs and classroom's normative ways of reasoning. For this aim, Rasmussen and Stephan (2008) developed two criteria for how mathematical ideas become taken-as-shared. The first criterion explains that the backings and/or warrants of the argumentation disappear in the whole class discussion. In other words, the participants no longer challenge the argumentation since all of them understand the mathematical idea represented in the core of the argument. The other criterion states that the mathematical idea formed and becoming self-evident in an argument is used in future arguments for justifications with the functions of the data, warrant, or backing (Rasmussen & Stephan, 2008). Moreover, Rasmussen and Stephan (2008) propose the researchers a mathematical ideas chart for each class session in order to identify classroom mathematical practices effectively. This mathematical ideas chart includes three columns: "(a) a column for the ideas that now function as if shared, (b) a column of the mathematical ideas that were discussed and that we want to keep an eye on to see if they function subsequently as if they were shared, (c) a third column of additional comments" (p. 200). There is an example represented on Table 7 produced for the current study on Week 1 for the second activity sheet. In other words, this chart helps the researcher to identify mathematical ideas discussed, needed to be investigated in the following process, becoming taken-as-shared and additional interpretations with theoretical or practical links. This chart is produced by making comparisons between lessons so that emerging of taken-as-shared ideas can be made in a progressive process and transfer of the mathematical ideas from "keep an eye on" to "taken-as-shared" with additional comments.

Table 7 Mathematical Ideas Chart for the Second Activity Sheet on Week 1

Ideas that function as-if-shared	Ideas to keep-an-eye-on	Additional comments
Identification of main and auxiliary elements of triangles The possibility of formation of triangles by knowing some of elements of triangles	Construction of triangles	By some of known elements (main or auxiliary elements), how to construct these triangles was examined. The number of types of these triangles was investigated by knowing some elements.

Third phase of the methodology is the stage of the analysis process that the classroom mathematical practices are identified based on determining taken-as-shared mathematical ideas with Toulmin's model of argumentation (Cobb & Yackel, 1996; Rasmussen & Stephan, 2008; Yackel & Cobb, 1996). After the process of identification of taken-as-shared mathematical ideas, they are examined and organized based on the contexts and mathematical ideas in which they become established and turn into taken-as-shared and then they are organized under a common title representing common mathematical activities in which the participants deal with. The general mathematical activities produced in this way are named as classroom mathematical practices (Rasmussen & Stephan, 2008). This organization process for producing classroom mathematical practices based on general mathematical activities has close relationship with Cobb's (2003) criteria that "the analysis should permit documentation of the developing mathematical reasoning of individual students as they participate in communal classroom processes" (p. 11). In this respect, three classroom mathematical practices were produced in the current study; PMSMT's reasoning on (a) the formation of a triangle, (b) the auxiliary elements of triangles and their properties, and (c) congruence and similarity.

3.7 Interpretative Framework

The social part of the emergent perspective includes three domains as social norms, sociomathematical norms, and classroom mathematical practices with closed relationship with each other. The first domain, social norms, focuses on the structure of participation taking place in instructional sequence in the classroom (Stephan & Cobb, 2003). In DBR, two types of participation took place. The first type was small group works where the PMSMT studied on the activity sheets with their peers by discussing and sharing their ideas and reasoning, usage of construction steps by recording their solutions and expressions on the activity sheets to present in whole class discussions. Other type of presentation was observed as whole class discussion. The participants explained their solutions and representations with the reasons. Then, the others investigated further clarification and explanation for them, alternative solutions and mathematical expressions. The interpretations of social interactions included the examination of the participants' (a) providing explanations and justifications for the problems, (b) understanding others' explanations, (c) approving or disapproving the solutions of others, and (d) asking questions when a conflict happens in the process (Cobb & Yackel, 1996). These social norms were also observed in the current study. When these social norms were considered based on Toulmin's argumentation model, they were produced providing data and warrant for the claims in the process that the participants dealt with classroom discussions about the concept of triangles. Therefore, these social norms encouraged the formation of the parts of Toulmin's model so that social norms were vital to provide PMSMT's understanding and learning of triangles and identification of classroom mathematical practices.

The second domain, sociomathematical norm, was extracted from whole class discussions engaging in mathematical activities and about the concept of triangles for the present study. The process of formation of sociomathematical norms was critical in DBR since they provided the formation of mathematical practices by focusing on mathematical solutions. The participants shared their solution of the problems and reasoning with the others by justifying and explaining procedures benefiting from the words, drawings, constructions, models, symbols and representations. They helped the participants in expressing their reasoning while participating in the whole class discussion. Some of the sociomathematical norms emerging in the study were formation of specific triangles through construction, examination of the elements of triangles through construction and examination of critical points for different types of triangles.

Mathematical practices as the last domain were examined by the Toulmin's model in the current study by paying attention on mathematical explanations formed through the tasks. These practices produced a nice way to illustrate the collective mathematical learning but it was not possible to claim that the participants learned the related concept effectively in the classroom setting (Cobb, 1998). Cobb et al. (2001) explains mathematical practices as "taken-as-shared ways of reasoning, arguing, and symbolizing established while discussing particular mathematical ideas" (p. 126) by not requiring further justification and explanation. The mathematical practices were produced by the participants' engagement of the classroom activities. In order to answer the research question of the study, an analysis of classroom mathematical practices were made to illustrate how the designed activities in the HLT provided PMSMT's learning of the concept of triangles. They were extracted by the way of taken-as-shared based on Toulmin's argumentation model.

3.8 Trustworthiness

Through the six-week instructional sequence, the learning and understanding of PMSMT about triangles were examined by collecting data from different sources. With the aim of providing trustworthiness, the data were collected through multiple methods such as observations, interviews, field notes, documents and meetings of the research team in the present study. Making triangulations between the data obtained from these multiple methods provided trustworthiness by decreasing the limitations of the present study (Mathison, 1988). Trustworthiness by triangulation with different data sources could be made by obtaining information investigating evidences from these sources. Using the information in this way supported building coherent justifications for the themes (Creswell, 2009). Also, member checking strategy was used for trustworthiness. It could be explained as taking the data, descriptions, themes or interpretations made based on the data back to the people who participated in the study during the analysis and specifying whether they thought that they were true (Creswell, 2009; Guba & Lincoln, 1981). By using the member checking strategy, the follow-up interviews as a suggested way by Creswell (2009) were made with the PMSMT participating in the current study and wanted to make comments about the results. The findings and interpretations based on them were discussed with the participants of the present study. In addition, rich and thick descriptions were used to communicate the findings. These descriptions gave the readers necessary information about setting and opportunities to share their experiences so that the results became realistic and richer.

3.9 Limitations of the Study

The present study have some limitations especially because of its design of qualitative research and methodology of design based research. The first limitation is that the findings of the study can be less generalized to the population because of its research design. On the other hand, the generalizability can be provided by studying with different groups of PMSMT in other macro cycles benefiting from its cyclical nature. Secondly, another limitation is that the findings of the study has focused on the collective learning of the participants. Hence, this study is limited to social aspects of the emergent perspective without paying attention on the participants' individual learning. The third limitation is about the instructor. The mathematical practices emerged in the study in an environment in which the PMSMT thought about the geometrical ideas in the classroom under the guidance of the instructor. In other words, the instructor guided the whole class discussion and the mathematical ideas emerged in this way. Therefore, the instructor had effect on the process of the emergence of them. The last limitation is about the willingness of the participants since the mathematical practices emerged in whole class discussion and their motivation to participate in this discussion limited the formation of mathematical practices.

3.10 Summary

In this design-based research as a qualitative research, the HLT to be used in six-week instructional sequence was formed and tested based on the findings of the pilot study. Then, conjectured local instruction theory was formed and tested in the study. The process of designing, testing and evaluating this theory by the HLT was explained through the pilot study. Then, revised HLT was conducted to the main study group. The data were collected in different ways such as observations, interviews and tests. In this process, the classroom mathematical practices were identified by analyzing the data through the Toulmin' model of argumentation and Rasmussen and Stephan's (2008) three-phase method. Moreover, the trustworthiness was provided in different ways.

CHAPTER 4

4. RESULTS

The answer is provided to the research question of "*what are the classroom mathematical practices emerging in design-based research environment designed by problem-based learning for teaching triangles to preservice middle school mathematics teachers?*" in this chapter of the study. Qualitative and quantitative findings of the study is explained. Qualitative findings represent classroom mathematical practices formed by the Toulmin's model of argumentation. Quantitative findings illustrate the pre and post-test scores obtained by van Hiele geometry test and the triangles tests produced by the researcher to analyze the preservice middle school mathematics teachers' thinking and learning about triangles.

Classroom mathematical practices are the mathematical ideas that have become taken-as-shared through the process in which the learners do not need mathematical justification in order to show its truth (Cobb & Yackel, 1996). Moreover, the way of taken-as-shared happens by using a conclusion produced in an argument to produce a different conclusion for another argument (Cobb, & Yackel, 1996; Rasmussen & Stephan, 2008; Yackel, 2001). In this respect, the present study with the aim of identifying the mathematical practices in a classroom teaching experiment designed about the concept of triangles, the following classroom mathematical practices of preservice middle school mathematics teachers (PMSMT) emerged: PMSMT's reasoning on (a) the formation of a triangle, (b) the elements of triangles and their properties, and (c) congruence and similarity with limited number of mathematical ideas in these practices as in Table 6.

Table 8 Three Classroom Mathematical Practices

Classroom Mathematical Practices
Mathematical practice 1: Reasoning on the formation of a triangle
Reasoning on the definition of triangles and classification of them
Reasoning on the construction of triangles
Mathematical practice 2: Reasoning on the elements of triangles and their properties
Reasoning on construction of auxiliary elements of triangles
Reasoning on the concurrence of auxiliary elements of triangles
Reasoning on the names of concurrent points of auxiliary elements of triangles and their places
Mathematical practice 3: Reasoning on congruence and similarity
Reasoning on the formation of congruent or similar triangles through transformation geometry
A.S.S. is not a congruence/similarity criterion

4.1 Mathematical practice 1: Reasoning on the formation of a triangle

The first mathematical practice emerging through the instructional sequence conducted in light of the designed HLT on triangles was reasoning on the formation of a triangle. The mathematical ideas included in this mathematical practice were related to definition of a triangle and types of triangles and construction of triangles based on their some known elements. These practices mainly emerged from the activities in which the participants engaged on Week 1. In that week, the participants examined the classification of triangles based on their definitions and the possibility of constructing them by knowing the measures of some of their main or auxiliary elements. For these activities, they initially worked with their peers and then participated in the whole class discussion. While engaging in these problems, they used the strategy of construction and related mathematical theorems, and the definition of triangles.

4.1.1 Mathematical idea 1: Reasoning on the definition of triangles and classification of them

The first mathematical idea included in the first mathematical practice was observed on the first week of the instructional sequence while the participants were engaging in the activities about defining and classifying the triangles. In this activity on Week 1, there were problems about placing different types of triangles on a diagram by relating them for the classification of them as illustrated in the Figure 4.

CLASSIFICATION OF TRIANGLES

Place the following words on the diagram by making their definitions.

Triangle, isosceles triangle, equilateral triangle, scalene triangle, right triangle, acute-angled triangle, obtuse-angled triangle.

Figure 4 The first problem about placing the types of triangles in the diagram by using the definition of them in the first activity sheet on the first week

Then, they were asked to determine the places of the types of triangles on the diagram by defining and relating them. While they were working on the problems in the activity sheet with their peers in small group works, the instructor visited the small groups to determine different mathematical ideas and use them in whole class discussion. Through observing the studies of the participants with their peers in the small groups, the instructor realized that they were not able to define triangles accurately and necessarily. Also, they were not able to successfully benefit from the main elements of triangles in formation and classification of triangles. Therefore, the instructor initiated the discussion by asking a question in order to help the participants understand their errors and accumulate them. So, a discussion was initiated by the instructor in order to reach the definition of a triangle based on its main elements as follows:

Instructor: How can you form a triangle? Or, which elements are used to construct and define a triangle?

Selim: When we think about triangles, we can say that there are two kinds of main elements which are corners and edges. Therefore, we can define and classify triangles based on these elements.

In the explanation of Selim, it was observed that he stated accurate main elements of triangles to construct and define them but it was not made in a sufficient way since all of the polygons included these elements. He did not tell the formation of triangles by these elements. Hence, his explanation was a general statement valid for all polygons. The instructor asked them whether these elements were sufficient to construct a triangle in order to make the participants realize this case but they were not able to answer it. At that point, the instructor made the suggestion of thinking about the definition of a triangle to guide the whole class discussion about understanding the definition and formation of a triangle by these elements. Then, the participants began to make the definition of a triangle.

Ayşe: Triangles are geometrical figures formed by three points where two line segments intersect at a point.

Mehmet: This explanation does not refer to the definition of a triangle. We have three points and line segments intersecting on these points in the case of two line segments for each point. According to your definition, this figure is also a triangle but it is not actually.

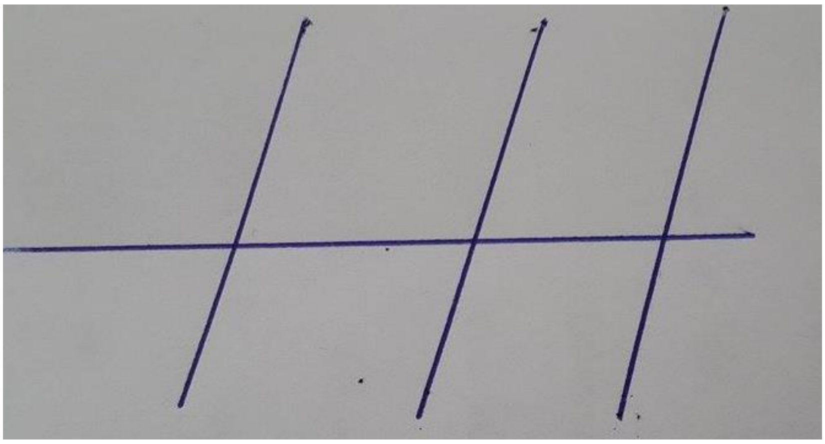


Figure 5 The counter example by Mehmet for the definition of Ayşe.

In the explanation of Ayşe, she told the main elements in a way different from Selim's explanation. She stated the corners as the intersection points of the line segments so her explanation was necessary but not sufficient. She might benefit from the idea of the corner as the intersection point of two line segments or two edges on a polygon but she ignored the non-linearity of three points and the necessity of closeness of a triangle. In order to help the participants realize these missing points, the instructor wanted them to criticize the appropriateness of her explanation. At that point, Mehmet provided an appropriate example for her explanation but not a triangle. Then, the instructor asked the definition of a triangle to continue the discussion.

Instructor: Ok. What is the definition of a triangle?

Halit: Triangles are geometrical figures formed by intersecting three non-parallel line segments in the plane.

Selim: In the figure, there are intersecting three non-parallel lines but the figure formed in this way is not a triangle. According to the definition of Halil, it must be a triangle and we can talk about coincident line segments as it is seen in the figure.

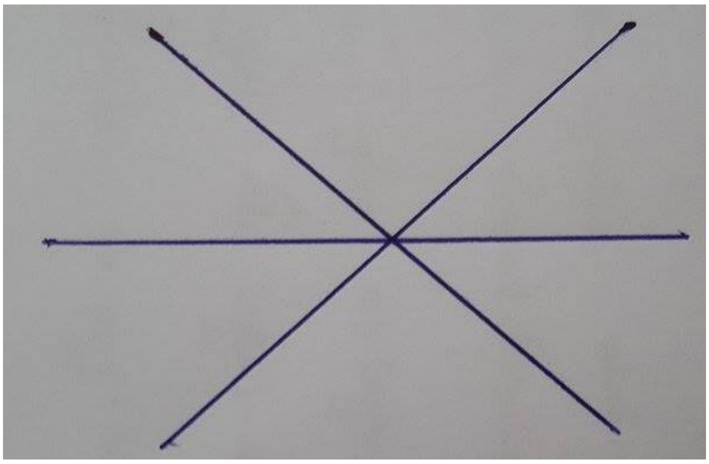


Figure 6 The counter example by Selim for the definition made by Halit.

By asking the definition of a triangle again, the instructor expected that they realized the non-linearity of three points and the necessity of closeness of a triangle but they focused on a different point which was the intersecting three non-parallel line segments. His explanation was not sufficient and Selim showed its insufficiency by providing an example appropriate for Halit's definition but not a triangle. Also, Halit provided a nice point for the definition of a triangle by stating the necessity of placing the line segments and the intersection points as the vertices on the same plane. Then, the instructor confirmed the appropriateness of Selim's explanation and encouraged the participants to make the right definition.

Özlem: In that respect, we say that the geometrical figures formed by combining three non-linear line segments are triangles.

Merve: The edges which are the main element of the triangles refer the line segments. These line segments are linear.

Instructor: Let's summarize the right points that we discussed. What are they?

Mine: … three non-linear points on the plane, three line segments combining these points…the edges and the corners… the angles of the triangles are formed at these corners.

Instructor: Well. Can you define the triangle using them?

Mine: … triangles are closed convex geometrical figures formed in a way that three non-linear points are combined by three line segments on the same plane.

In the light of the explanations and definitions made by the participants, the necessary points of three non-linear points, placing on the same plane and closeness were realized and then they formed the appropriate and sufficient definition of a triangle. To sum up, by stating "triangles are closed convex geometrical figures formed in a way that three non-linear points are combined by three line segments on the same plane", the discussion finished and they understood the definition of a triangle based on the main elements of it.

In this debate, Selim first attempted to explain how to define a triangle. In other words, he made a claim that it was important to think about the triangles' edges and corners which were main elements in order to define the triangles. However, Ayşe used data benefiting from the appearance of a triangle by explaining three points and line segments. However, she provided warrant for the same claim in a wrong way where two line segments intersected at a point. Then, Mehmet stated rebuttal for the argument by providing counterexample for her warrant to refute her explanation. Afterwards, another data for the definition of triangles and warrants for this emerging argument were constructed by Halit and Özlem based on the claim in order to emphasize three non-parallel line segments. Their explanations were followed by rebuttals provided by Selim and Merve, respectively. They refuted the explanations referring the definition of a triangle incorrectly by providing counterexample. At the end of the debate, Mine explained data and warrant by defining triangles truly emphasizing the main elements of triangles under the guidance of the instructor. Through the discussion process, the participants formed the mathematical idea about the definition of a triangle based on its main elements under the guidance of the instructor. At the end of her explanation, nobody in the classroom challenged this argument. According to Toulmin's model of argumentation, the structure of the discussion about the definition and classification of triangles considering the main elements can be summarized as shown in Figure 7.

DATA

Mine: …triangles are closed convex geometrical figures formed in a way that three non-linear points are combined by three line segments on the same plane

CLAIM

Selim: When we think about triangles, we can say that there are two kinds of main elements which are corners and edges…

WARRANT

Mine: We can define and classify the triangles based on their three line segments and three corners. Two line segments intersect at each point. Moreover, the angles of the triangles are formed at these corners.

Figure 7 Toulmin's model of argumentation for reasoning on the definition of a triangle based on main elements

In the first week on the advancing hours and the third week of the instructional sequence, it was illustrated that the mathematical arguments produced by the participants, and knowledge and skills about reasoning on the definition and classification of triangles attained during this debate in the first week became taken-as-shared. They used this one as data and warrant in their arguments on Week 1 and Week 3 without necessitating backings, and by confirming that it became taken-as-shared. At Activity Sheet 1 on Week 1, also in the same problem as illustrated in Figure 4, the participants were asked to define different types of triangles and place them on the diagram by connecting them. In this process, they used the knowledge about the definition of a triangle formed in this mathematical idea in order to define other types of triangles and determine their places on the diagram. In other words, while producing the definitions for other types of triangles, they used the definition of a triangle and particular properties of the main elements of triangles; edges and vertices. For example, an equilateral triangle is a triangle whose measure of all angles and the length of the edges are equal. Also, they benefited from the knowledge about the definition of a triangle in the definition of a right triangle as illustrated in the following part of this mathematical idea.

At Activity Sheet 2 on Week 1 as in Figure 11 represented in the following mathematical idea in this mathematical practice, there were problems examining the possibility of the formation of a triangle such as "When the values of h_a and b and m(BAC) = 90^0 in the triangle of ABC were known, is it possible to draw/construct this triangle? How?". The participants benefited from the knowledge related to the definition of a triangle about combining three non-linear points representing the vertices of the triangle with line segments to examine the possibility of the formation of a triangle as it was examined in Mathematical Idea 2 in the Mathematical Practice 1. In addition, the mathematical idea about the definition of a triangle based on its main elements was used in order to construct a triangle which was the image of a triangle formed through transformation geometry as it was illustrated and discussed in Mathematical Practice 3. Through the formation of the image of a triangle by transformation geometry, the participants focused on the formation of a triangle through its definition and main elements in a way that it was discussed previously. For example, through translation, the participants focused on the identification of the vertices of the image triangle formed by moving the vertices of the former triangle by a particular vector. Then, by combining these moved points by line segments, the image triangle was formed as it happened in the definition of a triangle. To conclude based on these discussions taking place at different time points in the instructional sequence, the mathematical idea about reasoning on the definition and classification of triangles based on their main elements became taken-as-shared.

The participants criticized the regions on a plane formed by a triangle in the same problem represented in Figure 4 while discussing the formation and definition of triangles with the property of the placement of their main elements on the same plane. Through the argumentation about the definition of a triangle, they also made the definition of a triangle based on the regions formed on a plane. In this way, the participants produced another discussion period about this mathematical idea based on the regions on the plane formed by a triangle and its main elements as follows:

Büşra: Triangles are geometrical figures including three line segments intersecting three non-linear points. When this formation process is thought, triangles are geometrical figures separating the plane into three regions.

Ayşe: Two regions are formed not three ones.

In this definition, Büşra formed a different definition of a triangle based on the regions formed on a plane. Then, Ayşe did not accept the truth of this definition although it was right. The instructor realized that the participants had problem in understanding the regions formed by a triangle on a plane. Hence, the instructor guided the discussion in order to them focus on the process of formation of these regions as follows:

Instructor: How are these regions formed? Think about the elements of triangles and the formation of these elements.

Selim: Three line segments formed by infinitely many points with respect to the definition of line segment intersect each other at three corner points of a triangle...

Instructor: How many regions are formed by these critical elements? What are they?

Ayşe: The points on the edges and inside the edges of a triangle form the interior region and the remained points on the plane form the exterior region. The edges are the borders of the interior region so that two regions are separated apart.

Instructor: Is it correct? What do you think?

Nuray: There are two regions but they are interior one and the other one including the points of exterior and edges.

Özge: There are two regions. For example, think about the circles. Circle is the set of specific points and its interior part is empty. When we think about the interior points, we begin to talk about the sphere. We can compute the area of a triangle or we do not have different names for triangles with/without interior points. Therefore, we accept interior points with the points forming the edges as interior region and examine based on two regions…

At this episode of the argument, the instructor tried to have the participants realized that the lines segments took up place on a plane so that the edges of a triangles could take place on the plane. Therefore, she guided the discussion about the definition of line segments. After the instructor provided the participants focus on the definition of a line segment as the set of points equidistant to two particular points, they talked about this process but they were not able to reach a consensus about the regions were formed on a plane by a triangle. Then, Özge provided an explanation by claiming that two regions were formed by making connection with circle and sphere and the possibility of computing the area of it. She stated that the interior regions was connected with the line segments because it was possible to compute the area of a triangle since they considered that if the area of it were not be able to be computed, it would attain a different name as it happened in the circle and sphere. In other words, she claimed that the interior region was composed of the points on the interior region limited by the edges of the triangle and the line segments forming the triangle since it was possible to compute the area of triangle. The idea that if it were not possible to compute the area of a triangle, it would have attained a different name as it happened for circle and sphere was produced. In order to help the participants realize the point that they confused with the area of a triangle in this way, the instructor focused on the perimeter and area of triangles by using representations.

Instructor: Ok. Let's model this. Think about a real life example by the area and perimeter of a triangle. Think about the difference between a triangle plate and a triangle frame.

Yücel: Both of them represent a triangle but the interior part of the frame is empty while the other's is not.

Instructor: It is a good point. So…

Mehmet: Let's form a triangle, by combining three pens for the edges. This is a triangle and its interior part is empty. We cannot say that it is not a triangle for this reason.

Instructor: Yes. In this example, although we have a triangle, we are not able to engage in the area of a triangle since it does not exist. So, what can you say about the regions formed on a plane?

Yücel: Therefore, the interior region is not connected with the region of the line segments. Also, we have stated that triangles are formed by line segments. These segments occupy places in the plane. Moreover, line segments are formed by points and the points occupy places in the plane. Therefore, the edges can form a region alone.

Instructor: Well, it is a good point.

Büşra: In this respect, we can state that triangles separate the plane into three regions which are the set of interior points and the set of exterior points and the set of the points on the edges.

In the discussion, they criticized the idea of the possibility of computing the area of a triangle. They realized that although a triangle's interior region was empty and there was a case without the possibility of computing its area, it was still a triangle. By benefiting from the cases related to the possibility of computing the area of triangles and connecting the formation process of main elements of triangles, they reached the accurate and sufficient explanation related to the regions formed by a triangle on a plane.

In this debate, Büşra first extended the discussion about the definition and classification of triangles based on their main elements by adding a different definition including the regions formed by a triangle on a plane. In other words, she extended the claim, data and warrant by adding the idea that these main elements separated the plane into three regions in the discussion challenged by the other participants. However, Ayşe and Özge provided rebuttals for this debate by stating that there were two regions and the edges of the triangle was belonged to the interior region. It was observed that some of the participants had confusion about the regions formed by a triangle on a plane so the instructor guided the discussion in order to help them reason about three regions accurately. Then, Mehmet, Selim and Yücel showed that the rebuttals provided by Ayşe and Özge did not represent the expected true explanation for the claim. Selim provided the definition of a line segment. Also, Mehmet and Yücel stated that both of the cases of a triangular frame with/without its interior region represented the geometrical shape of triangle. Through this part of the discussion process, they examined two of three parts of the region formed on a plane by a triangle which were interior region and the region formed by the points of the edges of triangle. At the end, Büşra added true claim, data and warrant by separating these regions appropriately under the guidance of the instructor and the ideas explained by the others. By doing so, through the discussion process, the participants formed the accurate and necessary mathematical idea under the guidance of the instructor. The Toulmin's model of argumentation for some parts of this debate is shown in Figure 8.

DATA

Büşra: Triangles are geometrical figures including three line segments intersecting three non-linear points.

CLAIM

Büşra: Triangles are geometrical figures separating the plane into three regions.

WARRANT

Yücel: We have stated that triangles are formed by line segments. These segments occupy places in the plane…

Figure 8 Toulmin's model of argumentation for reasoning the definition and classification of a triangle adding the regions on the plane formed by triangles.

The evidence that the idea about three regions formed by a triangle on a plane used as if shared was observed. This evidence came from analyzing the argumentation structures constructed on the second and third weeks of the instructional sequence. They used this one as data and warrant in their arguments on Week 2 and Week 3 without necessitating backings, providing that it became taken-as-shared. The activity sheets designed for the auxiliary elements of triangles in these weeks included the problems about the concurrence points of auxiliary elements of triangles as critical center points at different places for different types of triangles. For example, in one of the activity sheets, this problem was stated "Does/Do the place of the intersection point(s) of the all perpendicular bisectors of a triangle change based on the types of triangles? Why? If it/they change(s), predict the place(s) of it/them for different types of triangles by showing its truth." Then, they explained that this concurrence point was circumcenter and its place changed based on the types of triangles at the end of the discussion. The place of it was told by stating that it was in the interior region for acute triangles, in the region of the points forming the hypotenuse of the triangle for right triangles and in exterior region for obtuse triangles. Hence, the mathematical idea about the regions formed on a plane was used as data and warrant in the argument about discussing this problem. Hence, the notion of the regions formed by triangles on a plane was used in order to predict the places of these center points for different types of triangles as it was explained in the third mathematical idea for the second mathematical practice. In other words, the idea about the regions formed by triangles on the plane re-emerged on the second and third weeks of the instructional sequence. The participants used this notion in the argumentation about the concurrence points of auxiliary elements of triangles as critical center points at different places on triangles.

In Activity Sheet 1 on Week 1, at the same problem represented in the Figure 4, while the participants were engaging in this problem with their peers, the instructor realized that they had difficulty in defining a right triangle and they were not able to produce an accurate and sufficient definition for a right triangle. Hence, the participants were asked to define the types of triangles by the instructor in order to produce the accurate definition for a right triangle. The instructor initiated the discussion by asking a question in order to discuss this issue:

Instructor: How can you define a right triangle?

Mine: We know that a triangle is composed of three non-linear points. For three non-linear points, two equidistant points to a specific point refer to right triangles when they are combined with line segments.

When the explanation of Mine was examined, it was observed that she benefited from the mathematical idea about the definition of a triangle based on its main elements. When her definition of a right triangle was examined, her definition was unnecessary and insufficient since the idea about two points equidistant to another point was not necessary for right triangles. Also, she did not emphasize the perpendicularity. Hence, the instructor asked the following question to everybody in the class with the aim of helping them realize the unnecessary parts in the definition of Mine.

Instructor: Is equal distance necessary in defining and forming right triangles?

Mine: While finding the shortest distance between a point and a line, we behave based on this idea since this distance is perpendicular to the line.

At that point, it was observed that although Mine realized the unnecessary parts of her definition, she insisted on the necessity of them and continued making unnecessary explanations for defining a right triangle. Therefore, the instructor thought that more guidance and clues needed to remove them. In order to help Mine reason her explanation's unnecessary parts, the instructor asked the following question:

Instructor: What is the relationship between equal distance and perpendicularity?

Mine: Between two parallel lines, equal distances form perpendicular lines.

Nuray: As you said, this perpendicular distance is between two lines but you talked about the distance between two points in your definition. We cannot determine the perpendicular distance between two points.

Instructor: Right. It is a good point.

Halit: Let's think about a circle or an arc. For example, we have three points; two of them are equidistant to the remaining one so we talk about them in this definition as in the figure.

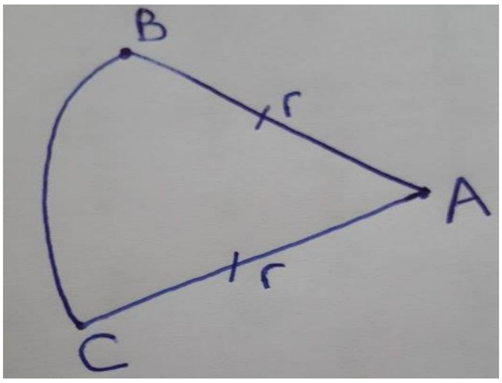


Figure 9 Two equidistant points to the one and an arc by Halit.

Merve: When we combine the points of B and C with a line segment, we have an isosceles triangle based on this definition.

Instructor: It is a good explanation. By the same distance, we can only reach the idea of equal length.

Mine: Yes, you are right. We need to emphasize the perpendicularity.

In this episode of the discussion, Nuray, Halit and Merve made the explanations in order to refute the idea stated by Mine. They told their ideas with the aim of helping Mine reason accurately. While Nuray was explaining the impossibility of forming a perpendicular line segment between two points, Halit emphasized that two line segments equal in length did not intersect perpendicularly in Figure 9. Also, by extending his explanation, Merve claimed that an isosceles triangle could be produced in the case of having two line segments equal in length. At that point, the instructor guided them to think about the definition of a triangle that they produced previously and the main properties of a right triangle and then composing the necessary definition in this way.

Halit: We can define right triangles based on angles and edges which are main elements of triangles ... Moreover, we must emphasize three non-linear points necessary for the formation of a triangle. Right triangles are triangles whose two of the edges intersect perpendicularly at a corner.

Yücel: We can say that right triangles are triangles whose angle measure of one of interior angles is 90^0.

Through the whole class discussion guided by the instructor, Mine realized that the property of having equal length of edges was not necessary and related with perpendicularity needed for right triangles. To conclude, the participants produced the necessary and sufficient definition for a right triangle based on the evidences of the definitions of Halit and Yücel at the end of the discussion.

In this debate, Mine first attempted to explain how to define a right triangle based on the definition of triangles. She used the definition of triangle as data for this argumentation. It was observed that a prior argument about the definition of triangles served as data for the argument about the definition of right triangles. In other words, she produced a claim that it was important to think about the definition of triangles while defining the types of triangles. However, she used warrant for the claim in an incorrect way. While making the connection between data and claim, she could not reason effectively for the warrant. She considered about his data by emphasizing the perpendicular distance between points but she dismissed the idea that perpendicular distance could not be formed between two points. Then, Halit, Merve and Nuray provided rebuttals for her explanation. Nuray refuted her explanation by emphasizing this dismissed point. Halit made explanation based on the equal distances between points by the radius of a circle. Then, Merve ended the refutation period with the formation of an isosceles triangle. At the end of the debate, Halit and Yücel provided correct data and warrant by defining right triangles accurately. They benefited from angles and edges which were main elements of triangles as in the definition of triangles. In this way, the evidence that the notion of the definition of a triangle was provided by being used as data in the discussion. Therefore, this discussion period included the second instance that the notion of the definition of triangles was used. Through the discussion process, the participants formed the mathematical idea under the guidance of the instructor. Hence, the mathematical idea about the definition of right triangles as represented in the core of the argument in the Figure 9 was produced appropriately through whole class discussion. At the end of the discussion, all of the participants agreed with the claim and reasoning about it using the data by Mine and warrant by Halit so that the claim was taken for granted being unchallenged no longer. The Toulmin's model of argumentation for some parts of this debate is shown in Figure 10.

DATA

Mine: We know that a triangle is composed of three non-linear points…

We need to emphasize the perpendicularity.

CLAIM

Halit: Right triangles are triangles whose two of the edges intersects perpendicularly at a corner.

WARRANT

Halit: We can define right triangles based on angles and edges which are main elements of triangles as we have done to define triangles. Moreover, we must emphasize three non-linear points necessary for the formation of a triangle.

Figure 10 Toulmin's model of argumentation for reasoning on the definition of a right triangle.

In the first week on the advancing hours, third, fifth and sixth weeks of the instructional sequence, it was illustrated that the mathematical arguments produced and the knowledge and skills about reasoning with the definition of right triangles attained by the participants during this debate in the first week became taken-as-shared. They used this one as data and warrant in their arguments in these weeks without necessitating backings, confirming that it became taken-as-shared. Firstly, at Activity Sheet 2 on Week 1 illustrated in Figure 11, the participants used this knowledge in order to examine the possibility of the formation of triangles when the measures of some of their elements were known. Moreover, while the participants were engaging in this kind of activities, they used some theorems related to right triangles such as Euclidean or Pythagorean theorems. In right triangles, they investigated the possibility of determination of known elements by using these theorems. In this activity, they benefited from the definition of right triangles as it was illustrated from the whole class discussion placed in the second mathematical idea for the same mathematical practice.

Secondly, on Week 3, one of the activity sheets was about the altitude of triangles. In this activity, they engaged in the problem about how to construct the altitude of a triangle and produced the claim about the definition and construction of the altitudes. In this argumentation, they used the definition of right triangle as data for that claim and also it was benefited from in the process of stating warrant. While constructing the altitudes of a triangle, they used the notion of the definition of a right triangle since an altitude separated a triangle into two right triangles. When the altitude of the edge of BC was drawn at the point of H on ABC triangle, it could be claimed that AHC was a right triangle based on the definition of right triangle. The notion of the definition of right triangles was used as data and warrant in the argumentation about the reasoning on the altitude as an auxiliary element of triangles. This discussion taking place in the third week by providing evidence of its functioning as if shared as it was observed in the argumentation in the related first mathematical idea in the second mathematical practice.

Thirdly, in Activity Sheet 2 on Week 5 as in Figure 16, the last problem was composed of the statements about triangles and the participants were asked to determine their appropriateness and to explain their reasons. In this problem, there was three-column table. The first column included the statements related to triangles such as “Right triangles are sometimes similar”. The second column was the place that the participant stated as true or right for the statement in the first column and the last column was the one where they wrote the reason of the truth or error of the statement. Therefore, on the similarity and congruence content, this idea was used in order to determine whether the triangles were similar explained in the statement of “Two right triangles are always similar when the measure of one of their interior acute angles are same”. The teacher initiated the discussion by reading this problem on Activity Sheet. With respect to interior angle measures of right triangles, the claim of these right triangles were similar was explained with the definition of right triangles. In other words, they explained appropriately the reason of their similarity with the definition of a right triangle since one of the interior angles' measures were 90^0 by the definition of a right triangle and the measures of one of the interior angles were also same as explained in the statement. Therefore, the idea that the remaining acute angle measures were same based on the fact that the sum of interior angle measures of a triangle was always 180^0 was reasoned accurately and necessarily. In this discussion, they provided it as data that all of opposing angles' measures were same for these right triangles. Then, Merve used this data in the warrant by the similarity criterion of A.A. Through this discussion, they showed the truth of this statement by the A.A. similarity criterion under the instructor's guidance so that they successfully showed the truth of the statement using the mathematical idea related to the definition of a right triangle. This case provided evidence for the notion of the definition of right triangles in a way that it became taken-as-shared. Moreover, the statements of “Isosceles right triangles are always similar.”, “When the length of the hypotenuse and the altitude of the hypotenuse are same for two triangles, they are always similar.”, “When the length of the radius of incircle and the altitude of the hypotenuse are same for two triangles, they are sometimes similar.” and “Some right triangles are similar.” were discussed in the same way based on the reasons of “Right triangles are triangles whose two of the edges intersects perpendicularly at a corner” and “right triangles are triangles whose angle measure of one of interior angles is 90^0” and data for the argumentation. Lastly, in the last week, the participants were given the problems that could be solved benefiting from the properties about right triangles and theorems related to them. Some of these problems were also solved benefiting from the definition of right triangles as explained in detail under the title of following mathematical idea. Based on this definition, they determined whether they were right triangles and then they used related theorems about them. The discussions occurring in this period provided evidence for the notion taking the function as if shared by using as data and warrants. In this respect, further evidences were provided for the process of becoming taken-as-shared for the notion of definition of right triangles. To conclude based on these discussions, the mathematical idea about reasoning on the definition of right triangles became taken-as-shared.

4.1.2 Mathematical idea 2: Reasoning on the construction of triangles

The second mathematical idea was observed on the first week of the instructional sequence while the participants were engaging in the activities related to basic constructions of triangles. In this activity, different groups including the known values of some main and auxiliary elements of a triangle were provided as illustrated in Figure 11.

When we know the measures of h_a and V_a and m(BAC) = 90^0 in the triangle of ABC, is it possible to draw/construct this triangle? How?   When we know the measures of h_a and a and m(BAC) = 90^0 in the triangle of ABC, is it possible to draw/construct this triangle? How?

Figure 11 The problems asking the possibility of the construction of particular triangles by knowing the values of the explained main and auxiliary elements of triangles on Activity Sheet 2 on Week 1.

Then, they were asked to investigate if it was possible to form or construct the triangle by using their known values of some of its elements provided in the problem. In these problems, they investigated whether they were able to form a triangle by reasoning differently. Also, they examined the types of triangles that could be formed with known/given measures of elements. For this activity, they worked with their peers and participated the whole class discussion. While engaging in these problems, they used the strategy of computation of the measures of specific unknown elements based on the known/given ones with related theorems such as Pythagorean, definition of these elements and right triangles and construction activities. During the whole class discussion, they debated how they used these mathematical ideas in the solution of the problem. The instructor initiated the discussion by reading one of the problems on Activity Sheet of the Week 1:

Instructor: When we know the measures of h_a and b and m(BAC) = 90^0 in the triangle of ABC, is it possible to draw/construct this triangle? How?

Selim: Yes, we can construct this triangle. We have the measures of two main elements and one auxiliary one. Also, we can construct a triangle when we know its two of the lengths of the edges and the angle measure of one of its angles. Moreover, when we know the lengths of two edges of

> the triangle, we can make predictions about the length of the third edge and the possibility of the formation of the triangle benefiting from triangle inequality. They are enough to construct it…

In this explanation, Selim made good reasoning but it had inaccurate parts. His reasoning was different from the other participants' reasoning since he focused on the construction steps even if the way that he followed was not appropriate while the others engaged in the main theorems about triangles. He assumed that a right triangle of AHC illustrated in Figure 12.a was formed by known elements explained in the problem. Then, he claimed that it was possible to compute the value of α which was the measure of the angle of HAC by using the known values of the explained elements. His reasoning about computing the value of the angle measure of α was not possible but he was not aware of it. Then, he continued by constructing a right angle which was the angle measure of BAC. He constructed this angle in a way that one of the rays of this angle was the edge of AC. Hence, the other ray formed the edge of AB of the triangle of ABC so that the triangle asked in the problem was formed as in Figure 12.b. Selim ended his explanation by constructing the triangle finding the places of all vertices of the triangles as main elements of the triangle. Although Selim benefited from the construction steps of a particular angle, he could not use these steps appropriately. The critical point for his explanation was that he assumed that it was possible to compute and construct the angle of α. In order to help the participants realize this inappropriate part, the instructor guided discussion to focus on this point. After they realized this unrelated part, they started to discuss the problem again.

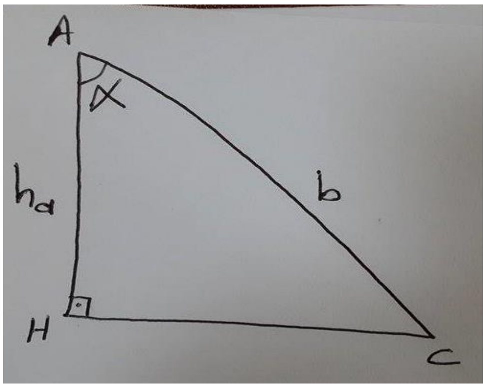
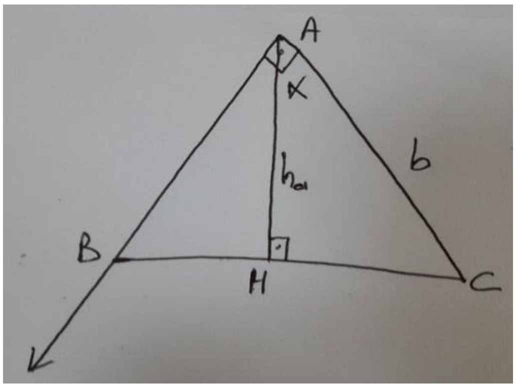
 a. The triangle of AHC	 b. The triangle of ABC

Figure 12 The triangles formed through the explanations of Selim

Mehmet: …By the definition of an altitude, h_a starts on the vertex of A and ends on the edge of BC by intersecting it perpendicularly. Here, we do not have the edge of BC so we cannot construct the specific h_a. Moreover, if we are able to construct a triangle with two edges and an angle, this angle places between these two edges.

Yücel: We can solve this problem in a different way. We know the values of h_a and b and m(BAC) = 90^0 for this problem. This is a right triangle as we know from the definition of right triangles. It makes possible to use Pythagorean and Euclidean theorems. By using related theorems with known elements, we can find the necessary unknown elements to draw the triangle…

In his explanation, Yücel drew a triangle of ABC similar to the triangle drawn by Selim as in Figure 12.b. For this triangle, he made computations for the length of the edge of HC by Pythagorean Theorem, $|HC|^2 + h_a^2 = b^2$, the length of the edge of BH by Euclidean theorem, $h_a^2 = |BH|.|HC|$ (by knowing the lengths of h_a and the edge of HC) and $|BC| = |BH| + |HC|$. Then, he determined the length of the edge of BC by Pythagorean Theorem, $|BC|^2 = |AB|^2 + |AC|^2$.

He assumed that the triangle was formed by these known elements. Then, he thought that he used these values to find the unknown measures of necessary elements of triangles such as the edges. He claimed that the edges were the main elements so by knowing the measures of all edges, the formation of a triangle could be determined. In his explanation, he made good reasoning about the problem and then he gave the correct answer that the triangle could be constructed by these known elements. All of the participants except for Selim in the classroom solved this problem by reasoning similarly in the same way. This explanation was appropriate since it provided the correct answer that the triangle could be constructed by these known elements. However, it had missing part since the answer was related to just the possibility of the formation of this triangle but it did not examine the types of triangles formed by these known measures of particular elements. Therefore, the instructor guided them to examine the types of triangles formed by the measures of these known specific elements. The instructor emphasized the correctness of the possibility of the formation of the triangle and then asked different solutions for this problem to provide them examine alternative triangles formed by these known elements. Hence, the instructor continued the discussion as follows:

Instructor: Ok. Let's solve the problem by using another solution strategy? How can we solve?

Nuray: We can solve it by using construction.

Instructor: How can you do this? …Focus on the construction of the known elements.

Nuray: ... Firstly, we construct an angle with the measure of 90^0 as the measure of the angle of BAC… Secondly, we construct an arc belongs to the circle with the center of A and the radius in the length of b. This arc intersects the rays and then we name one of these intersection points as the vertex of C. Lastly, we repeat the steps that we made to find the vertex B for h_a to find the point of H…

The instructor guided them to examine the formation of the triangle through the construction of given elements in the problem. In the explanation of Nuray, she found a good point for the problem by reasoning through the construction. She produced the construction based on the definition of a triangle by emphasizing that it was formed through three non-linear points. It was a nice starting point since she explained the necessity of determining the vertices of the triangle. She constructed a right angle referring the right angle of the triangle by using compass and straight edge. She drew a line segment and two circles having the equal length of radius but different from each other based on their center points; end and starting points of the line segment. Then, by combining the intersection points of these circles with a line segment, the perpendicular bisector of the line segment was constructed. This perpendicular bisector represented a right angle. Afterwards, the place of the vertex of C determined using the length of b as illustrated in Figure 13.a. Until this point, she produced the known elements by construction accurately. However, she was not able to maintain her success about constructing the triangle. At the second part of her explanation, she repeated this process on the line segment of AC and drew the perpendicular bisector of this line segment as illustrated in Figure 13.b. Then, she determined the place of the point of H using the length of h_a. At the end, she formed the triangle by connecting the points of H and C and extending this line segment to intersect other ray of the right angle constructed initially. She constructed a triangle different from the triangle asked in the problem because she followed inaccurate steps for constructing the altitude. In order to help her realize the missing and inappropriate parts of her explanation and construction, the instructor asked questions to help her and the others in the class articulate her ideas and steps. The instructor asked "Why did you draw the perpendicular bisector of the edge of AC?" and "How does this perpendicular bisector help to construct the altitude?". By these questions, the participants reasoned on the construction steps and then they realized the unrelated parts of the solution. They claimed that the second perpendicular bisector was unnecessary and the altitude was not constructed appropriately. They produced different explanations refuting the ideas and steps explained by Nuray in the second part of her explanation. Then, the discussion was continued by the last idea refuting Nuray's explained by Kader as follows:

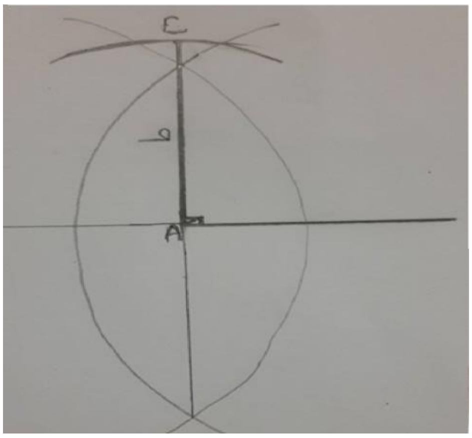
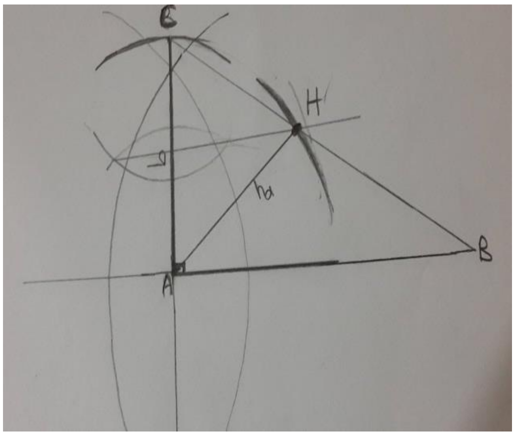
	
a. Construction of right angle and the edge of AC	b. Construction of the triangle of ABC

Figure 13 The triangles formed based on the explanations of Nuray.

Kader: It is clear that these two arcs formed by using the lengths of h_a and b do not intersect since when we think about the triangle of HAC, the edge of AC in the length of b is the hypotenuse and h_a is belonged to the right edge of this triangle. Hence, we form bigger circle by using the length of b than the one by the length of h_a.

Instructor: Can you say that the strategy of construction cannot be used for the solution of this problem?

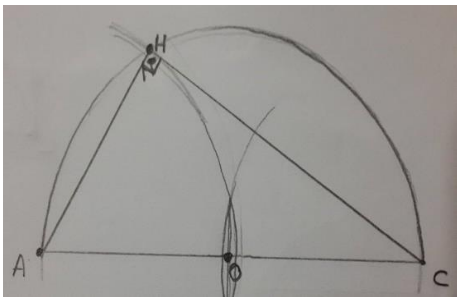
Ali: No, we cannot say. We followed wrong construction steps.

Instructor: How can you articulate these steps?

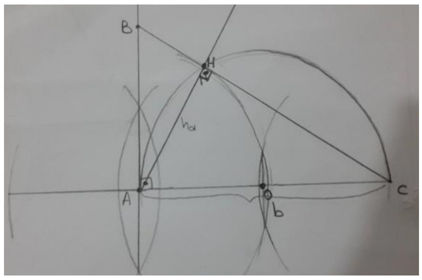
After this question, the participants focused on identification of true parts of the explanation of Nuray and they decided to determine the place of the points of A, C and H. Then, with the help of the clues about the construction process, they constructed triangle appropriately as follows:

Ali: … the edge of AC in the length of b becomes the hypotenuse of the triangle of AHC. Firstly, we find the midpoint of the edge of AC and construct a semi-circle with the center of this midpoint and the radius in the length of the half of the length of this edge. Secondly, we construct an arc belonged to the circle with the center of A and the radius in the length of h_a…

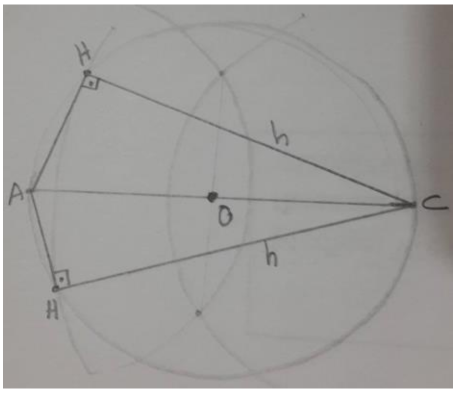
Ali initially constructed the right triangle of AHC as in Figure 14.a with the hypotenuse having the length of b benefiting from the property that the inscribed angle opposing of the diameter had the angle measure of 90^0. Then, he constructed a right angle whose one of the rays was the edge of AC as in Figure 14.b. Afterwards, he extended the line segment passing through the points of H and C by providing that it intersected the other ray of the right angle on the vertex of A. This intersection point was stated as the vertex of B so that the triangle of ABC was constructed. This explanation was a clear way of constructing the triangle with the measures of known particular elements stated in the problem. By following these steps, other participants and the instructor asked questions about them and mathematical explanations and justifications for them. Through explaining them in a way that the steps were challenged, they constructed the triangle. After completing the construction steps, another solution and answer was provided about the possibility of the formation of the triangle explained in the problem. Then, the instructor guided the discussion to examine the types of triangles formed by these known elements. The reasoning was made for these questions as follows:



a. Construction of the triangle of AHC formed by Ali



b. Construction of the triangle of ABC formed by Ali



c. Construction of a whole circle and two right triangles by the length of h formed by Büşra

Figure 14 The triangles formed based on the explanations of Ali and Büşra.

Büşra: In the first step, if we construct a whole circle instead of a semi-circle, we have two triangles at both sides of the diameter of the circle in the second step so that we form two triangles of AHC. Then, we can have more than one type of triangle of ABC by following construction steps (Figure 14.c).

Ali: You are right but the number of the types of triangles does not change since these triangles are congruent…

Büşra explained that two types of triangles could be formed when the construction was made by using a whole circle and then repeating the steps for other part of the semi-circle illustrated in Figure 14.a. She focused on the number of triangles considering their places. However, the idea that changing the position and the orientation of triangles did not provide different types of triangles was dismissed by Büşra. However, her explanation was refuted by Ali since he stated that two triangles formed based on her explanation were congruent. He benefited from the idea that when the triangle was constructed on the other part of the whole circle, the other part was the image of the previous triangle obtained through reflection with the symmetry line referred by the diameter of this circle. Then, the instructor guided the participants to focus on the types of triangles formed by the known elements stated in the problem as follows:

Instructor: Ok. This way did not form a different type of triangle. What does it mean? You cannot construct another type of triangle with these known elements, can you?

…

Selim: Let's think about the semi-circle formed in the first step. It is possible to draw more than one triangle in this problem. The place of the point of H changes with respect to the length of the altitude of h_a by the steps of construction. There are many possible places for the point of H on the circle since it is determined by the intersection point of the arc with the circle. This point can be any of the points forming the circle.

Instructor: Well. It is a good point. How do these possible places help you construct different types of triangles?

At this episode of the argumentation, Selim made good and necessary explanation for the problem. He made explanation based on the idea that the length of h_a was assumed as known but not in a way that it's length was explained by comparing the lengths of other elements in the problem. By constructing the triangle of AHC as in Figure 14.a, a semi-circle was drawn with the radius in the length of h_a so the place of H could be any point forming this semi-circle. This mathematical idea was important since the participants might state that various triangles were constructed based on all of the points forming this semi-circle. Therefore, the instructor asked the question by emphasizing the types of triangles constructed in this problem. In the previous example, Büşra explained that congruent triangles could be formed through reflection by preserving all of the properties although its position changed. Then, the instructor reminded that the congruent triangles were not different types of triangles by referring Büşra's this explanation. At the end of the discussion, Selim provided accurate and necessary explanation related to the types of triangles that could be constructed with known elements. He stated that two types of triangles which were scalene and isosceles right triangles could be constructed considering the necessity of $b > h_a$. While the isosceles right triangle was formed in the case of $b = 2h_a$, the scalene right triangle was formed in the case of $b \neq 2h_a$. At the end of the discussion, the participants successfully stated that it was possible to construct the triangle by knowing the values of h_a, b and $m(BAC) = 90^0$ and the possible types of triangles that might be constructed by the known elements in the problem. Moreover, the participants used the mathematical knowledge that a line segment had three positions for a circle; not intersecting, intersecting as a tangent line and intersecting at two points although they did not realize. This was an appropriate and necessary part of the solution. In this problem, the positions of the intersection and tangent points on the circle were considered and the case of not intersecting was ignored since the triangle was formed. Hence, scalene right triangle was formed in the case intersecting at two points and isosceles right triangle was done in the case of intersecting at a tangent point. Then, at the end of the discussion, the instructor emphasized this mathematical knowledge about the formation of a triangle by knowing some of its elements and their reasoning about it using construction effectively.

In this debate, Selim first attempted to explain how to construct a specific triangle by the measures of known particular elements but he used wrong data and also warrants for his claim. In other words, he made a claim that it was possible to draw a triangle with these given elements. He focused on finding the measures of the angles between the edges of the triangle and the altitude. Mehmet provided rebuttal for his conversation by talking about the points that Selim did not reason correctly. He stated that it was not possible to compute the measures of these angles. Then, Yücel provided another data that related theorems could be used for this right triangle and warrant about finding the values of the unknown elements by using known ones for the claim as a solution strategy for the problem. He showed the possibility of finding the measures of necessary unknown elements to form the triangle by using the related theorems such as Pythagorean Theorem. He confirmed the claim and Ali provided backing for his debate by explaining the construction strategy under the guidance of the instructor. The instructor guided them about the construction since they showed the possibility of the formation of the triangle by measures of known particular elements. However, the answer had missing part about identification of the types of triangles formed in this process. So the instructor helped them determine the types of triangles formed by the elements stated in the problem. However, they were unsure about the types of triangles. Ali made a claim that there existed more than one triangle formed with these known elements and then provided data that the point of H was placed with respect to the length of the altitude and warrant by the steps of construction with the help of the instructor. By making these explanations about the possibility of the construction of triangles and types of triangles, the argument was finished because of providing no backing or rebuttal and agreeing with the discussed notion. The Toulmin's model of argumentation for some parts of this debate is shown in Figure 15.

DATA

Yücel: We know the values of h_a and b and m(BAC) = 90^0 for this problem. This right triangle with two right angles on it makes possible to use Pythagorean and Euclidean theorems.

CLAIM

Selim: We can construct a triangle with these known elements.

WARRANT

Yücel: By using related theorems with known elements, we can find the necessary unknown elements to draw the triangle. For this triangle, we can compute the length of the edge of HC by Pythagorean Theorem…

BACKING

Ali: By constructing known elements with reasoning related properties of geometrical objects such as angle on the circle and intersection of a line or arc of the circle, we can find critical intersection points representing the corners and critical points on the triangle…

Figure 15 Toulmin’s model of argumentation for reasoning on the construction of a triangle when the measures of some of its elements are known.

In the first week, on the activity sheet as in Figure 11, triangles knowing different groups of main and auxiliary elements and their measures were explained and the participants were asked if these triangles were able to be constructed. On the fifth week of the instructional sequence, it was illustrated that the mathematical arguments formed by the participants and knowledge and skills obtained by them about reasoning with the formation of a triangle by using some of its elements during this debate in the first week became taken-as-shared. The second instance that the notion about the reasoning on the possibility of the construction of triangles based on some known elements was observed in the fifth week. In terms of Toulmin's model, we saw that this notion as a prior argument taking place on the first week served as the data in the arguments on Week 5 without necessitating backings by confirming that it became taken-as-shared. This problem was about the statements about triangles and the participants were asked to determine their truth and the reasons. In this problem, there was three-column table. The first column included the statements related to triangle such as "Right triangles are sometimes similar". The second column was the place that the participant stated as true or right for the statement in the first column and the last column was the one where they wrote the reason of the truth or error of the statement. Therefore, on the similarity and congruence content, this idea was used in order to determine whether the triangles were similar explained in this statement as illustrated in Figure 16. The teacher initiated the discussion by reading this problem on the Activity Sheet.

4. Determine whether the explanations in table below are true or false by making mathematical explanations.

	True(T)	False(F)	Explanation
Two triangles are similar only if one of them is the image of the other triangle formed by dilation.			
Some triangles having the interior angle measures of (30^0-60^0-90^0) are similar.			
All triangles having the interior angle measures of (45^0-45^0-90^0) are similar.			
When triangles with the edges in the length of (3k-4k-5k, k€R) are sometimes similar.			
When triangles with the edges in the length of (7k-24k-25k, k€R) are always similar.			

Figure 16 The figure of the last on Activity Sheet 2 on Week 5.

On the similarity and congruence content, the idea discussed above related to the possibility of formation of triangles and types of them by knowing the measures of its some elements was used in order to determine whether they were congruent/similar. The instructor initiated the discussion by reading the problems on this activity sheet as follows:

Instructor: When the lengths of the hypotenuses and the altitudes of the hypotenuses are equal for two right triangles, are they always congruent?

İlkay: We can think about this question in a way that we engaged in the activities based on determining the possibility of forming triangles when we know the values of some of its elements.

Instructor: That is a different point. How can we benefit from the idea that we learned in this activity.

Halit: By constructing a triangle with these known elements, we can determine the types of triangles that we can form. Then, it becomes possible to identify whether they are congruent based on the types of triangles.

Through the discussion, the participants produced a connection between the construction of triangles and determined the congruence of them considering some known elements successively. At this episode of the discussion, the instructor reminded the previous mathematical idea that they discussed by stating "…we examined the possibility of the formation of the triangle by some known measures of the particular elements and types of triangles" and then asked how to use this idea to determine whether the triangles were congruent by knowing the measures of some known elements. Through the discussion under the guidance of the instructor's questions, the participants reached the consensus that the congruent triangles had same properties but their positions and orientations changed. Moreover, they produced the idea that by determining the types of triangles that could be constructed by the given elements, it became possible to determine whether they were congruent. Then, the instructor asked the participants to show the solution by construction. In this process, the instructor realized that the participants had difficulty in the construction process so she asked the question and directed the discussion as follows:

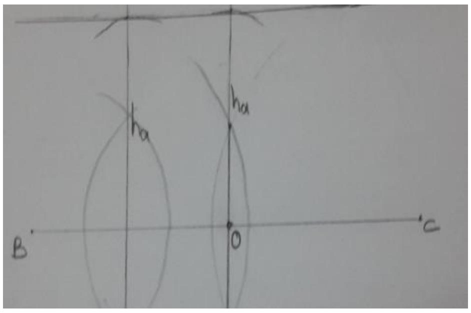
Instructor: In the activity that we remember, we constructed the triangles based on the measures of some known particular elements. So, which elements do you know in this problem?

İlkay: We know that measure of the one of the interior angles is 90^0, the lengths of the hypotenuse and its altitude.

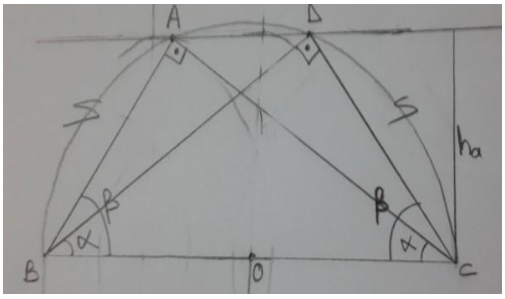
Instructor: Well. Let's construct the triangle.

Halit claimed that constructing two perpendicular line segments far away in the distance of the length of the altitude was necessary to construct a parallel line to the hypotenuse. In other words, he explained that a line parallel to the hypotenuse and far away in the distance of the length of the altitude was constructed as in Figure 17.a. The end points of these perpendicular lines were combined by a line so that a parallel line far away in the distance of the altitude to the hypotenuse was constructed. Then, they constructed a semi-circle having the radius in half of the length of the hypotenuse. Then, they explained with the help of the instructor that when the constructed parallel line intersected the circle at a point, an isosceles triangle was formed. With the same values of the explained elements, only one isosceles right triangle could be constructed in different orientations and positions so they were always congruent. The case of isosceles right triangles was explained successfully by Halit. Also, he stated that when the constructed parallel line intersected the circle at two points, two scalene right triangles were formed as in Figure 17.b. Then, the instructor asked how these two scalene right triangles were congruent and Halit answered as follows:

Halit: … since when we determine the measures of the interior angles of these triangles are equal since the measures of the arcs opposite of the angles which are α and β are equal. These angles having equal measures are opposite of the arcs having the equal measure on the circle. Hence, they are congruent by A. A…



a. Construction of the line in the distance of h with the edge of BC



b.Construction of the triangles of ABC and DBC

Figure 17 The triangles formed through the explanations of Halit.

In this explanation, Halit successfully explained that the triangles explained in the problem were congruent by using the mathematical idea produced through the discussion related to the possibility of formation of triangles with some known elements.

This argumentation provided evidence that the mathematical idea about reasoning on the formation of triangles based on the measures of some known elements functioned as if shared in a discussion under the guidance of the instructor. This evidence came from the explanation made by İlkay as seen in the argumentation core. In this conversation, İlkay used the claim produced in the discussion of Week 1 as the data that the number and types of triangles formed with some known elements could be benefited from determining the congruence without necessitating backings or warrants. This connection was made with the help of the instructor and understood by the participants by challenging the construction steps and discussion process. In this respect, it was provided that reasoning with the formation of a triangle by using some of its elements became taken-as-shared. The claim produced on Week 1 was used as the data in the argumentation made in Week 5.

On the same activity sheet, there was a problem similar to the previous one in Figure 16. In this problem, the measures of different elements were explained as known ones and it was asked whether it was possible to construct the triangle. In this activity, the problem was "When the measures of h_a and V_a and m(BAC) = 90^0 in the triangle of ABC were known, is it possible to draw/construct this triangle? How?" The discussion about this one flowed through the same way as it happened for the previous problem. The participants made an appropriate claim which was "It is possible to form the triangle". They provided an appropriate data about the measures of these known elements and their places on the triangle and their connections with the triangle and other elements of the triangle in a similar way happened for the previous discussion. Then, they provided warrant by using some theorems such as Euclidean and Pythagorean theorems in order to determine the possibility of the formation of this triangle. In this episode of the argumentation, although they provided accurate and necessary claim, data and warrant, there was a missing part in the discussion since they ignored the types of triangles that could be formed by measures of these known elements. In order to help the participants realize this insufficiency, the instructor asked questions how to construct this triangle by compass and straight edge. They constructed a triangle similar to the triangle constructed in Figure 14. In this representation belonged to the previous problem, a right triangle whose hypotenuse's length was known was constructed. The hypotenuse and right angle were known parts of the triangle asked in this problem. The instructor asked how they knew the length of the hypotenuse and they answered that the length of the hypotenuse was equal to the double of the length of the median of the hypotenuse in a right triangle necessarily and accurately. Then, they were constructed the altitude of the triangle by following the similar steps represented in Figure 17. Through this process, they explained the steps of construction of this triangle under the guidance of the instructor. As this discussion process was examined, the argumentation modeled in Figure 15 was used as data in this construction process while producing this claim. Moreover, in the construction process, they realized that there were two types of triangles as it happened in the previous problem with the help of the instructor's questions. Furthermore, they provided two different construction processes in addition to the previous construction process.

In these construction processes, the instructor helped them to complete these construction steps and made explanations by emphasizing necessary and important parts of these processes. Hence, it could be stated that the participants provided three backings as different construction processes for the claim. In the advancing hours on the first week and on the fifth week of the instructional sequence, it was illustrated that the mathematical arguments formed by the participants and knowledge and skills obtained by them about reasoning with the formation of a triangle by using some of its elements during this debate in the first week similar to the previous one became taken-as-shared. They used this one as data and warrant in their arguments on Week 1 and 5 without necessitating backings, confirming that it became taken-as-shared. In advancing hours on the first week, the activity about the formation of other triangles was followed. The next problem on the same activity sheet was "When the values of h_a and a and m(BAC) = 90^0 in the triangle of ABC were known, is it possible to draw/construct this triangle? How?". The participants made the accurate claim that "it is possible to construct this triangle". Then, they used the mathematical idea as the data and the warrant for this question based on the knowledge that the length of the hypotenuse was equal to the double of the length of the median of the hypotenuse in a right triangle accurately. By this knowledge, the instructor provided that the participants realized that the problem was changed in a way for the problem of "When the values of h_a and V_a and m(BAC) = 90^0 in the triangle of ABC were known, is it possible to draw/construct this triangle?". The discussion process happened sufficiently in a way similar to the previous one explained above. Therefore, this mathematical idea became taken-as-shared necessarily and accurately by providing data which was the conclusion of the previous discussion.

On the similarity and congruence content, this idea was used appropriately in order to determine whether they were congruent/similar in the second activity sheet illustrated in Figure 16 on the fifth week. The problem causing the debate in the fifth week was "When the values of the lengths of the altitude and the median of the hypotenuse are equal for two right triangles, are they always congruent?". They discussed in order to determine whether this statement was mathematically true in an environment directed by the instructor's questions. In the discussion process, they determined the connection between the mathematical ideas about the possibility of formation of triangles and determining whether they were congruent/similar with some known elements. They reasoned in a way that the congruent triangles had similar properties but their positions and orientations changed and by determining the types of triangles that could be constructed by the known elements, it became possible to determine whether they were congruent necessarily and appropriately. In this process, with the help of the previous discussions taking place for similar problems explained above, the discussion process was guided with less help provided by the instructor. With the help of the previous similar discussions and the construction steps performed for the previous problems, they made the connection more easily than the ones happening for the previous similar problems. In this conversation, they used the mathematical idea produced in the discussion of Week 1 as the data that the possibility of the formation of the triangle and the types of the triangles formed with the measures of some known particular elements could be benefited from determining the congruence of triangles having the equal measure of the same elements of the opposing edges and vertices of the triangles. In this respect, it was provided that reasoning with the formation of a triangle by using some of its elements became taken-as-shared by challenging the ideas of the others and questions of the instructor to guide the discussion. The claim that it was possible to construct a triangle by knowing the lengths of h_a and V_a and the angle measure of BAC equal to 90^0 produced in the problem of "When we know the measures of h_a and V_a and $m(BAC) = 90^0$ in the triangle of ABC, is it possible to draw/construct this triangle? How?" in Week 1 was used as data in the argumentation made in Week 5 as it happened in the mathematical idea exemplified in the previous claim necessarily and sufficiently. In this problem, they remembered that there were two types of triangles which were right and scalene triangles that could be formed by these known elements. In both cases, the triangles formed by these known elements were congruent triangles. For example, in the case of right triangles, by using these known elements, the length of the other right edge could be determined. Then, by the congruence criterion of S.A.S., it was identified that these right triangles were congruent. Also, similar reasoning and explanations could be made for the other case about scalene triangles.

4.2 Mathematical practice 2: Reasoning on the elements of triangles and their properties

The second mathematical practice was reasoning on the elements of triangles and their properties. The mathematical ideas included in this mathematical practice were about the formation of these auxiliary elements, concurrence of them and the importance of these points. They were mainly emerged from the activities that the participants engaged on Week 2 and 3. In this week, they examined auxiliary elements which were medians, angle and perpendicular bisectors and the altitudes. They engaged in the formation and the definition of them, concurrence of them and changing or unchanging positions of these points based on the types of triangles and finding the name of them and their critical importance. For these activities, they worked with their peers and participated in the whole class discussion. While engaging in these problems, they used the strategy of construction and related mathematical theorems, definitions of these elements and right triangles.

4.2.1 Mathematical idea 1: Reasoning on construction of auxiliary elements of triangles

The first mathematical idea which was reasoning on construction of auxiliary elements of triangles emerged on Week 2 and Week 3. They investigated the construction of the angle bisector and the altitude of a triangle. For this activity, they studied by using compass and ruler with their peers and participated in the whole class discussion by explaining their construction strategies and steps and mathematical expressions for them. These processes were followed in the construction of auxiliary elements of medians, altitudes, angle bisectors and perpendicular bisectors of triangles and they became taken-as-shared by being used in similar whole class discussions in the similar ways. The following discussion processes illustrated the argumentations about two of these auxiliary elements. During the whole class discussion, they debated how they used formation of altitude and angle bisector and showing that the formed line segment was the altitude or angle bisector. The discussion as the first instance of the argument explaining the construction of altitude of triangle was initiated by the definition of them and their reflection on construction steps on Week 3:

Instructor: Can you construct the altitude of the edge of BC on the triangle? How?

Selim: ... An altitude is a line segment passing through a vertex of a triangle, and intersecting the opposite side perpendicularly. In this respect, we need to construct a perpendicular line segment from the vertex of A to the edge of BC. Firstly, place the compass on the vertex of A and set the compass width as exceeding the distance between the vertex and the edge…

In this construction process, Selim constructed the triangle drawing an arc intersecting the edge belonged to a circle with the center on the vertex A as in Figure 18. Then, these intersection points were named as the points of D and E. By following the construction steps of perpendicular bisector, the perpendicular bisector of the line segment starting and ending with the points of D and E was constructed. The midpoint of this line segment was determined. Then, the altitude of the triangle was constructed by combining the vertex of A with this midpoint as in Figure 18. By following these construction steps, Selim appropriately and necessarily constructed the altitude of this triangle for the edge of BC as in Figure 18. At the end of the discussion, the instructor and the other participants challenged the truth of the result of this construction process. Then, Selim provided a necessary and appropriate mathematical justification for this process. He claimed that when the arc of DE intersecting the edge of BC and belonged to the circle with the center point of A was drawn, the line segment of DE became the chord of this circle. Then, the line segment of AH intersected this chord perpendicularly based on the mathematical idea that the perpendicular bisector of a chord passed through the center of a circle represented by the vertex of A as the center of the circle in Figure 18. He successfully and mathematically justified the process of the construction of the altitude. Afterwards, the instructor asked another strategy and way to construct any altitude of a triangle. Then, Merve explained another construction process and different mathematical justification for construction of an altitude.

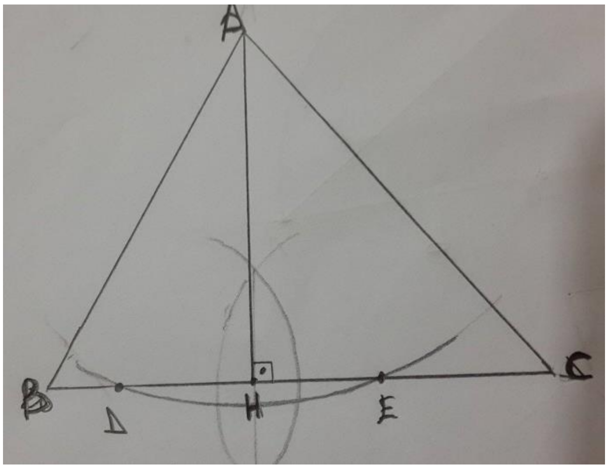


Figure 18 The construction of the altitude of a triangle with respect to the explanation of Selim.

Merve: … initially, we find the midpoint of the edge of AC and we draw a circle with the center as this midpoint and having the diameter in the length of half of the length of this edge. The intersection point of the circle with the edge of BC is the point of H. When we combine the vertex of A with the point of H by a line…

In this construction process, in order to construct the altitude of the edge of AB, the midpoint of the edge of BC was determined by construction. Then, a semi-circle was drawn with the center of this midpoint and the diameter in the length of half of the length of the edge of BC. Then, the intersection point of this semi-circle on the edge of AB was determined and this intersection point was combined with a line segment as in Figure 19. Merve appropriately reasoned the construction process and formed the altitude. Afterwards, the instructor asked the mathematical justification for this construction process. She explained that it was the altitude because of the definition of a right triangle and the knowledge that inscribed angle opposing the radius of the circle had the angle measure of 90^0. She made reasoning necessarily and successfully. Moreover, the instructor asked "what happens when you combine the vertex of B with the intersection point of semi-circle on the edge of AC on the triangle?". They answered that the altitude of the edge of AC was also constructed. By this answer, the strategy in which two altitudes of the triangle was constructed at the same time was emphasized.

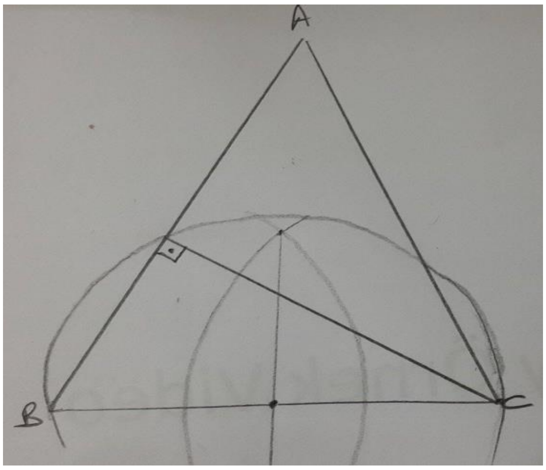


Figure 19 The construction of the altitude of a triangle with respect to the explanation of Merve.

In this debate, Selim first made a claim that it was possible to construct an altitude of a triangle. However, Betül and the instructor challenged the truth of the claim explaining the strategy of construction. Then, Selim provided data and warrant for his explanation about the construction process for the altitude of a triangle. Afterwards, Merve provided another data for the definition of triangles and backing representing another construction strategy of an altitude of a triangle. Merve explained a different way for the construction of an altitude of a triangle benefiting from the definition of the right triangles.

Through the construction processes, the participants and the instructor wanted them to make mathematical justifications for the processes and the truth of the result. According to the Toulmin's model of argumentation, the structure of the argument about the construction of an altitude of a triangle can be summarized as in Figure 20.

DATA

Selim: An altitude is a line segment passing through a vertex of a triangle...

CLAIM

Selim: We can construct the altitude of a triangle.

WARRANT

Selim: … we need to construct a perpendicular line segment from the vertex of A to the edge of BC. Firstly, place the compass on the vertex of A and set the compass width as exceeding…

BACKING

Merve: …when we assume that the altitude intersects the edge at the point of H, the triangles of AHC and AHB are the right triangles based on the definition of right triangles. In this construction process, initially, we find the midpoint of the edge of AC…

Figure 20 Toulmin's model of argumentation for reasoning the construction of the altitude of a triangle.

On the second week, there was a problem similar to the previous one. In this problem, the participants engaged in how to construct an angle bisector of a triangle. The discussion about this one flowed through the same way as it happened for the discussion of previous mathematical idea about the altitude.

The discussion including the first instance about the construction of angle bisector of a triangle was initiated by the definition of them and their reflection on construction steps on Week 3:

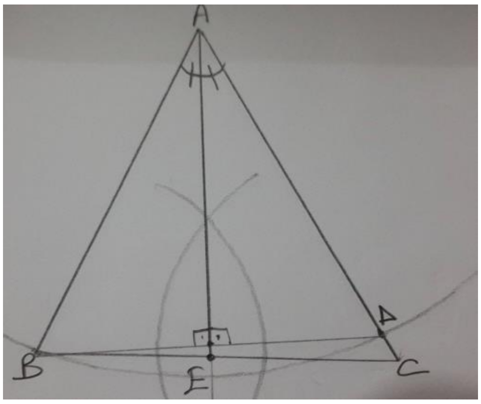
Instructor: Can you construct the angle bisector of the angle of BAC on the triangle? How?

Büşra: It is possible to construct the angle bisector of a triangle.

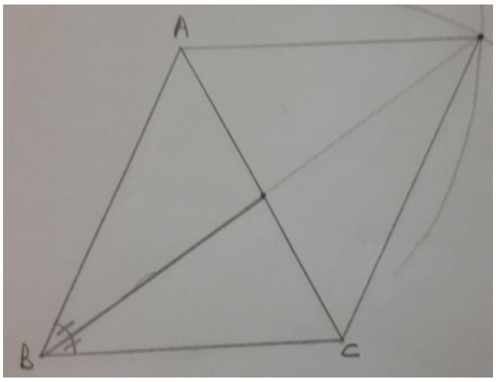
This was the conclusion of the argumentation produced by Büşra. All of the participants used the definition of angle bisector of a triangle in order to construct under the guidance of the instructor.

Büşra: Initially, I form an isosceles triangle. By drawing an arc passing through the vertex of B, the intersection point of this arc on the other edge is identified. When this intersection point is combined with the vertex of B with a line segment, we form the isosceles triangle of ABD… (Figure 21.a)

Then, two different ways of steps representing the construction steps of angle bisector was produced through the discussion. One of them was provided as warrant and the other way was stated as backing. In the way told as warrant, they constructed the angle bisector by forming an isosceles triangle since angle bisector of it was the median of the opposing edge as in Figure 21.a. In other words, the warrant was produced by Büşra benefiting from the property of angle bisector of an isosceles triangle since the angle bisector of an isosceles triangle was coincident with the median of the edge not having the same length with the other edges. Hence, she formed an isosceles triangle and then constructed the angle bisector of it benefiting from the construction process of median. An arc was drawn with the center of the vertex of A and the intersection point of this arc on the edge of AC was combined by a line segment with the vertex of B in order to form the isosceles triangle of ABD. Then, the median of the edge of BD was constructed by the construction steps of the perpendicular bisector of this edge as it was used in previous discussions. This median was at the same time the angle bisector of the angle on the vertex of A because of the nature of the isosceles triangles. In this construction process, the angle bisector was constructed accurately by making good reasoning and providing necessary mathematical justification about the property of isosceles triangles. In this construction process, the instructor asked the question about the process and its mathematical justification to help them reason on the process effectively and appropriately.



a. Based on median of isosceles triangle



b. Based on the diagonal of a parallelogram

Figure 21 Construction of angle bisector of a triangle in two different ways

Also, in the construction way represented as the backing in Figure 21.b, Mehmet formed a parallelogram with its diagonals as the angle bisectors of the interior angles of it. In these construction processes, the possibility of construction of an angle bisector of a triangle was discussed.

Selim: I form the angle bisector of this angle by forming a parallelogram. I adjust compass width with the length of the edge of BC and I draw an arc without changing this width by placing compass on the point as the vertex of A. Then, I adjust compass width with the length of the edge of

AB and I draw an arc placing the compass on the vertex of C. I identify the intersection point of these arcs and I combine this intersection point with the vertices of the triangle by using line segments…

In this process, two arcs with the center of A and the radius in the length of the edge of BC, and with the center of C and the radius in the length of the edge of AB were constructed as in Figure 21.b. This intersection point was combined with all vertices of the triangle so that a parallelogram was constructed. Then, the line segment combining this intersection point with the vertex of B formed the diagonal of the parallelogram and also the angle bisector of the triangle of ABC. All of the participants knew that the diagonal of the parallelogram bisected the angle on the vertices of it. Based on this knowledge, the participants made a good reasoning for the construction process and the claim of the discussion by providing accurate mathematical justification for this construction process. In this construction and justification process, the instructor asked the questions to emphasize and make them reason the process accurately and successfully.

On the advancing hours on the second week, the third week and the fourth week of the instructional sequence, it was illustrated that the mathematical arguments formed by the participants, knowledge and skills obtained by them about reasoning with the construction of the altitude and the angle bisector of triangles during this debate in the second and third weeks became taken-as-shared. In other words, the construction of the altitude and angle bisector of triangles became taken-as-shared in similar ways through similar discussions. They used this one as data and warrant in their arguments on Week 2 and 3 without necessitating backings, confirming that it became taken-as-shared. Firstly, on advancing hours on the second and third weeks, the participants used this knowledge in order to determine whether all of altitudes/angle bisectors were concurrent at a point when all of them were constructed. The construction processes of the elements explained and represented above for an altitude and an angle bisector were repeated for all of the altitudes or angle bisectors of a triangle to examine the concurrence of all of these auxiliary elements. Moreover, the names of these concurrent points were determined benefiting from this mathematical idea. Furthermore, they used this idea while examining whether these points changed based on the types of triangles. In this way, this mathematical idea became taken-as-shared by being used as warrant. The discussion process in which this mathematical idea became taken-as-shared was guided by the instructor so that the participants made good and appropriate reasoning by producing an accurate claim. Secondly, on Week 4, the participants examined the image of the triangles after applying the transformation geometry and the relationship between triangles and their images. They investigated the concurrence of these auxiliary elements and the name of these points for their images. While the distances between these points and the edges of the triangles did not change for congruent triangles (for triangles formed through rigid motions), they changed with respect to the scale factor for similar triangles (for triangles formed through dilation). They discussed these knowledge benefiting from this mathematical idea. The last problem on the last activity sheet on the fourth week was about changing and unchanging elements and the properties of triangles after applying transformation geometry. They were also asked the reasons of the cases of changing and unchanging properties. For example, Ali claimed and provided data that "the length of the altitude of the edge of BC does not change after applying translation because they are congruent triangles". He also stated warrant that "when we construct the altitude of the image triangle and we put two triangles on end providing the vertices are on mutual vertices as we have done by constructing the altitude, they remain same and on end". Hence, this mathematical idea being used as data and warrant by reasoning appropriately and effectively became taken-as-shared by being used as warrant. To conclude based on these discussions, the mathematical idea about construction of auxiliary elements of triangles became taken-as-shared in two ways. All of these arguments were the other instances that the notion of construction of auxiliary elements were observed serving as data and warrant to conclude new claims and functioned as if shared.

4.2.2 Mathematical idea 2: Reasoning on the concurrence of auxiliary elements of triangles

The second mathematical idea which was the reasoning on the concurrence of auxiliary elements of triangles emerged on Week 2 and 3. They investigated these points for median, angle bisector, altitude and perpendicular bisector respectively. For these activities, they thought about the relationship between these elements, related theorems and properties about triangles with their peers and participated in the whole class discussion by explaining their thoughts and mathematical expressions for them. During the whole class discussion, they debated how they illustrated that these elements concurred at a point on triangles and explained the reason of this case. The discussion was initiated by asking at how many points these elements concurred for a triangle on Week 2 and 3. These four elements were investigated separately at the second mathematical idea. Through the extension of the first mathematical idea about constructing all of these auxiliary elements of a triangle, the second mathematical idea of the second mathematical practice was emerged. Firstly, the concurrence of the medians was examined on Week 2. Initially, the participants were asked to construct all of the medians of a triangle in order to determine at how many point(s) these medians concurred. Through the process of peer discussions, the instructor realized that some of them constructed three medians of the triangle in a way that they concurred at a point and the others did it in a way that they intersected at more than one point. The instructor asked the question of "How many points do the medians of a triangle intersect each other at?". The participants answered accurately by explaining that they were concurrent. Then, the instructor showed some of the participants' constructions that all of the medians intersected at more than one point. They repeated their answer by being sure and explaining that it was possible to make errors in construction and drawing so they did not concur at a point on the figures they constructed. Afterwards, the instructor wanted them to justify their answer mathematically since although they gave correct answer, they could not provide necessary mathematical expression and justification. Hence, they formed their explanations about their reasoning in the discussion taking place as follows:

Merve: When we form two of the medians, they concur at a point since non-parallel two lines intersect at a point. Then, we see that we can apply Ceva Theorem. Then, we assume that the third median passes through this point. When we apply this theorem, we can confirm our idea as it is seen.

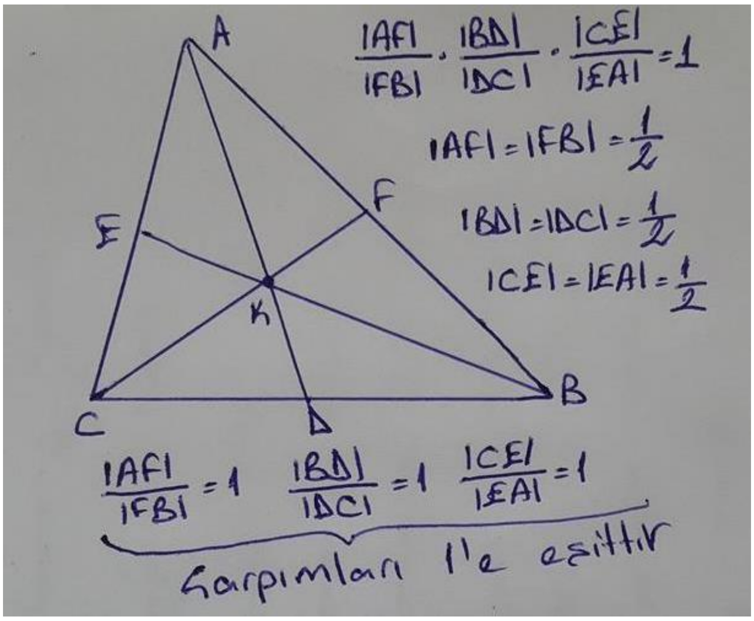


Figure 22 Ceva theorem for medians

At this explanation, Merve justified the concurrence of medians of a triangle by Ceva Theorem appropriately. Initially, she assumed that the medians were concurrent. Then, she applied the Ceva Theorem by using the ratios between the lengths of the parts of the edges formed through medians. When the result was equal to 1, it could be stated that the theorem met the formed ratios and the medians became concurrent. By this way, she showed that the medians were concurrent in a correct way. In this process, although they used this theorem, they were not aware of the knowledge that the median was a type of cevian. Hence, they used the correct answer and showed the concurrence of them by applying this theorem on the problem and reasoning unnecessarily. After Merve completed her explanation, Ali immediately explained that the concurrence of them could be showed by Menelaous Theorem. The instructor did not focus on the missing part of their knowledge about the median as cevian since the angle bisectors and altitudes as the topic of the following activity sheets in the instructional sequence were different types of cevian and the perpendicular bisector was not cevian. The instructor considered that the participants could understand what the cevian was when they examined examples and non-examples of cevian together so that they could define a cevian as any line segment drawn in a triangle whose end points were placed on a vertex of the triangle and on the opposite side of this vertex. Hence, the instructor postponed to discuss about the cevian until they examined examples and non-examples of it. In this respect, the instructor continued to talk about the medians which was the topic of the activity sheet they engaged in. Then, she guided the discussion by asking another strategy or solution that the medians were concurrent on a triangle.

Sevim: Let's form the line segment of KL. Then, KL // BC and |KL| / |BC| = ½ since |AK| / |AB| = |AL| / |AC| = ½. Find the places of the points of D and E as the midpoints of the lines of BL and KC. Hence, we find that DK//AC and EL//AB. Also, |DK| / |AL| = |EL| / |AK| = ½ and |KM| / |AC| = |LM| / |AB| = ½ and KM // AC and LM // AB since |DK| = |DM|

and |EM| = |EL|. Therefore, the triangles of BKM and ABC are congruent (the criterion of A. A.). Then, the point of M is the midpoint of the edge of BC based on the scale factor of ½. Also, when we combine the points of A and M, we form the median of the edge of BC which is the line segment of AM. This line segment passes through the point of intersection of other two medians. In other words, all medians are concurrent at a point.

In the explanation of Sevim, she assumed that the midpoints of the edges of the triangle were accepted as determined and the medians were formed. The points of K, L and M were combined by the line segments as in Figure 23. Based on the similarity of triangles, the triangles of AKL, BKM and LMC were similar to the triangle of ABC with the scale factor of ½. Then, the relationships of 2|KF| = 2|FL| = |BM| = |MC|, 2|KD| = 2|DM| = |AL| = |LC| and 2|EL| = 2|EM| = |AK| = |KB| were determined. Therefore, the scale factor and similarity of them appropriately and necessarily showed that all of the medians were concurrent. By making good reasoning in an appropriate way, the concurrence of them were justified sufficiently. After reaching a consensus about its truth, the instructor asked another solution to show the concurrence of them.

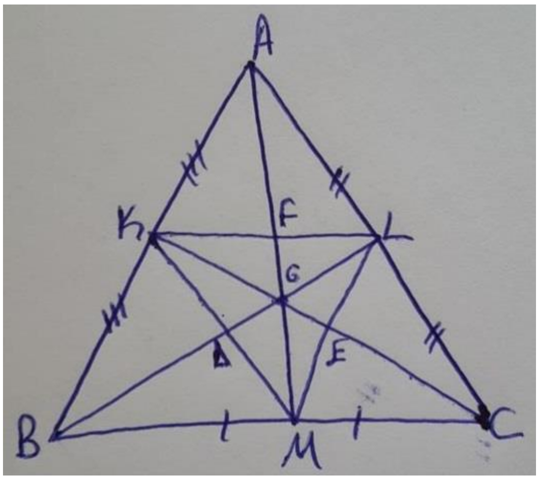


Figure 23 The figure of the concurrence of the medians on a triangle.

In this debate, Kader first explained the claim about the concurrence of the medians of a triangle by providing the data of the construction of a median. She added the warrant that when all of the medians were constructed, they all intersected at a point through construction. Moreover, Merve provided the backing about the concurrence of the medians based on the theorems of Ceva and Menelaous since when all of the medians were explained, it was observed that they concurred at a point. Then, Sevim provided backing for the claim by the guidance of the instructor. Sevim stated her explanation based on the content of similarity by the medians with their scale factors. The structure of the argument taking place in this discussion can be illustrated as shown in Figure 24.

DATA

Kader: …we examined the construction of a median as we did previously.

CLAIM

Kader: We need to know at least two of the medians since they concur at a point.

WARRANT

Kübra: … repeat the steps of the construction for all medians of a triangle…

BACKING

Merve: … they concur at a point since non-parallel two lines intersect at a point. Then, we see that we can apply the theorem of Ceva…

BACKING

Sevim: …KL // BC and |KL| / |BC| = ½ since |AK| / |AB| = |AL| / |AC| =

Figure 24 Toulmin's model of argumentation for reasoning on the concurrence of the medians of a triangle.

On the same week on the advancing hours of the instructional sequence, it was illustrated that the mathematical arguments produced by the participants and knowledge and skills attained about the concurrence of the medians at a point on triangles during this debate in the second week became taken-as-shared. They used this one as data and warrant in their arguments on Week 2 without necessitating backings, confirming that it became taken-as-shared. The participants used this knowledge in the debates made in order to determine the name of this concurrence point as centroid and whether this point changed based on the types of triangles. The data was produced based on this mathematical idea since the centroid was formed basically by the concurrent point of the medians. Then, this mathematical idea was also used as the warrant of the discussion. The processes of showing separation of the medians into ratio 2;1 and the regions with equal areas were provided benefiting from the process of the concurrence of the medians. Moreover, the change of the place of the concurrent point of the medians was examined based on the mathematical idea about the concurrence of the medians of a triangle in the similar way.

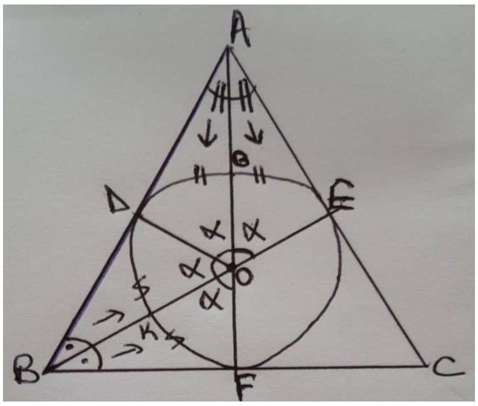
In the second week, the second activity sheet was about angle bisectors. Initially, the participants constructed all of the angle bisectors of a triangle in order to determine at how many point these medians concurred. While the participants were talking about the problem with their peers in the small groups, the instructor realized that there were different construction examples representing that the angle bisectors of the triangle concurred at a point and at more than one point. Then, in order to examine the participants' thoughts about that examples, the instructor asked the question of "How many points do the angle bisectors of a triangle intersect each other at?". Through answering this question, it was identified that the participants were aware of the fact that they were concurrent appropriately. Then, the instructor showed some of the participants' constructions that all of the angle bisectors intersected at more than one point and wanted them to provide explanations for these constructions. They insisted on their answer by being sure and explaining that they did not concur at a point because of drawing errors through construction. In this activity sheet, the participants discussed how to show that angle bisectors concurred at a point. Afterwards, the instructor wanted them to justify their answer mathematically and they explained their reasoning in the discussion taking place as follows:

Fulya: We need to know at least two of the angle bisectors of a triangle because of the concurrence of them at a point. …we constructed the angle bisector of a triangle. When we repeat the steps of the construction for all angle bisectors of a triangle, we see that they concur at a point…

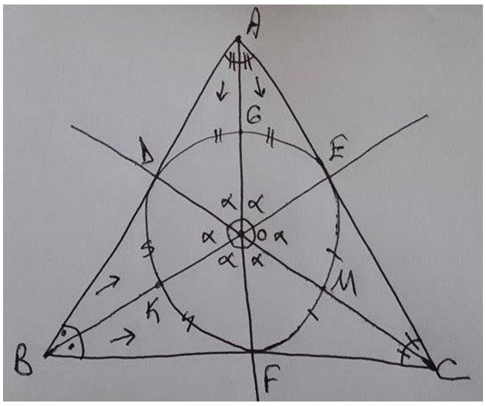
Ali: When we see the figure of a triangle with all angle bisectors on it, we realize that we can apply the theorems of Ceva and Menelaous. We can show the concurrence of angle bisectors of a triangle…

Fulya claimed the concurrence of angle bisectors of a triangle benefiting from the construction steps as it was discussed in the previous mathematical idea about the formation of angle bisector. She stated that when all of the angle bisectors were constructed, they concurred at a point by emphasizing the existence of drawing and construction errors. Then, Ali provided explanations for the concurrence of angle bisectors benefiting from the theorems of Ceva and Menelaous. In other words, they stated that when the theorems of Ceva and Menelaous applied, the necessary results were obtained by applying them and the concurrence of them was justified in this way. As it happened for applying this theorem for the concurrence of the medians, although they used these theorems, they were not aware of the fact that the angle bisector was cevian. He assumed that they were concurrent then by showing the applicability of the theorems for the concurrent angle bisectors, the concurrence of them was showed and justified mathematically in a correct way. Then, the instructor asked the others in the classroom whether the theorems could be used to show the concurrence of them and they agreed with Ali's explanation. Then, the instructor asked another solution or strategy about showing and justifying the concurrence of them.

İlkay: Let the lines of AF and BE are the angle bisectors of the angles of A and B. The angle of A is opposite of the arc of DGE with two equal parts of this arc. Then, it becomes that the measures of angles of DOG and GOE are equal to α…



a. Intersection of two angle bisectors



b. Concurrence of all angle bisectors

Figure 25 The figure of the concurrence of angle bisectors on a triangle benefiting from arcs based on angles.

In this solution, İlkay assumed that two angle bisectors of the angles of A and B intersected each other at the point of O and the incircle of it was constructed by combining the incenter with the tangent points by line segments of OD, OE and OF as in Figure 25.a. In this circle, the arcs of DG and GE had the equal measures because they were opposite of two equal parts of the angle of A separated into two equal parts by its angle bisector as in the same figure. The angle measure of KOF was equal to the angle measure of GOE (equal to α) since they were alternate interior angles. Then, the angle measure of DOK became equal to this angle measure since they were opposite of the arcs of DK and KF separated into two equal parts by the angle bisector of the angle of B as in Figure 25.a. Then, she added that when the line segment of OD was extended to intersect the vertex of C, the angle bisector of the angle of C was constructed as in Figure 25.b. Because of the same reasoning made for the other angle bisectors, the line segment of OM separated the arc of EMF into two equal parts having the same angle measure of α. Therefore, two parts of the angle of C were equal in angle measure since they were opposing the arcs having same measure. In his explanation, he made reasoning successfully but he ignored some important points so his explanations could not appropriately justify the concurrence of angle bisectors. He thought that the angle bisectors passed through the points of the incenter. He knew that the angle bisectors were concurrent and this point was named as incenter but it was observed that he memorized this knowledge. In order to help the participants realize this unrelated part of the explanation, the instructor asked questions to challenge the validity of this justification.

Özge: The explanations of İlkay are valid for equilateral triangles. Here, all of the angles at the center of the circle are equal to α having the angle measure of 60^0 since full angle at the incenter was separated into six equal parts. Then, the measures of the angles at the vertices are equal to 60^0 and the triangle becomes an equilateral triangle.

Instructor: Assume that the triangle is a scalene triangle. What can we say about the position of these angle bisectors?

Özge: Also, the angle bisectors do not pass through the tangent points of incircle.

Instructor: That is a good point. So, how can you show that the angle bisectors concur at a point? Focus on a scalene triangle.

Esra: Assume that we have the angle bisectors of the angles of A and B. We form the perpendicular lines from the intersection point of them to the edges of the triangle… When we draw the perpendicular lines from the

> intersection point of angle bisectors to the edges, |BG| = |BK| and |GO| = |OK| for the angle bisector of the angle of B and |AG| = |AH| and |GO| = |OH| for the angle bisector of the angle of A. Then, |OK| = |OH|. If we draw the line from the point of H to the point of K, we get isosceles triangle of OHK…(in Figure 26)

In her explanation, Esra benefited from the knowledge about angle bisector theorem stating that when a point was placed on an angle bisector, then it was far away in equal distance from the rays forming the angle. She formed perpendicular lines to the edges of the triangle as in Figure 26 so that she determined the line segments in equal length. This point that she reached encouraged important and necessary reasoning for the justification. In order to show that the third angle bisector belonged to the angle of C passed through the intersection point of other two angle bisectors, she showed |CK| = |CH| and |OK| = |OH|. Through the process, she drew the line segments to combine the point of K with H, and C with O so that a deltoid was formed with its diagonals. Based on the property that one of the diagonal of the deltoid separated it into two isosceles triangles (isosceles triangle of OHK and CHK) and the other diagonal divided it into two congruent triangles (congruent triangles of OHC and OHK). Therefore, these diagonals were also angle bisectors of the interior angles. The diagonal passing through the points of O and C was the angle bisector of the angle on the vertex of C so that the concurrence of angle bisectors of the triangle of ABC was showed accurately and necessarily. At the end of her explanation, the instructor summarized the reasoning process and emphasized the important parts. Then, the discussion continued with the instructor's question as follows:

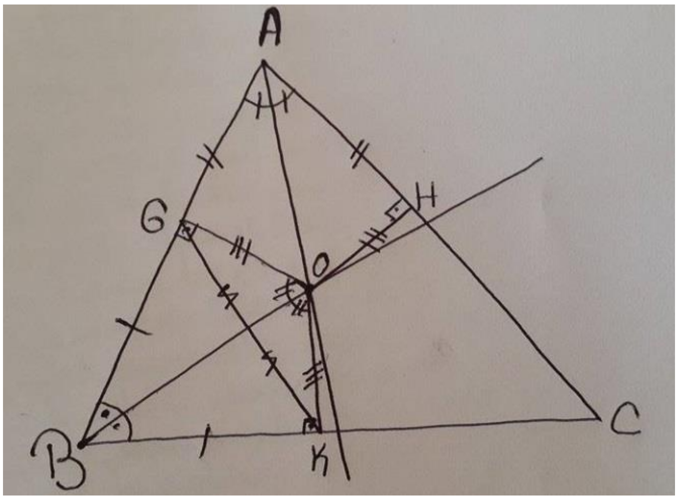


Figure 26 The figure of the concurrence of the angle bisectors on a triangle based on the theorem of angle bisector.

Instructor: Is there anybody else who adds something or explains different solution.

Efsa: ... In the explanation of Esra, we extend it by drawing the line segments of DE, DF and EF and we obtain three deltoids composed of isosceles triangles. However, we continue our explanation based on the angles at all points by drawing arcs as it is illustrated in the figure on the board. Then, we draw the line segment from the vertex of C to the edge of AB, this line segment bisects the arc of EF as the similar cases happened for other angle bisector lines. This line segment becomes the angle bisector of the angle on the vertex of C.

Efsa continued the explanations of Esra to state another justification for the concurrence of angle bisectors. She stated that the perpendicular line segments passing through the concurrence point of them to intersect the edges were the tangent points of the incircle. Also, she formed three deltoids of ADOE, BDOF and ECFO. She determined the line segment in equal length and angles with equal angle measure. The angles of ODE and DOE had the angle measure of Θ, the angles of ODF and DFO with the measure of β and the angles OEF and OFE with the angle measures of α. Then, she found the angle measures of angles of DOE as 2(α+β), DOF as 2(α+Θ) and EOF as 2(Θ+β). She stated that the arcs opposite of these angles had the same measures as in Figure 27. Then, the measures of the angles on the vertices of the triangle were determined benefiting from these measures of the arcs, it was showed that the line segments passing through the incircle and the vertices became the angle bisectors of this triangle. Hence, Efsa showed the concurrence of angle bisectors based on Esra’s idea. This was a good and different justification. Moreover, Efsa used the idea of İlkay by accumulating it and using correctly. At the end of her explanation, the instructor finished discussion by summarizing the reasoning process and emphasized the important parts and the discussion about the concurrence of angle bisectors.

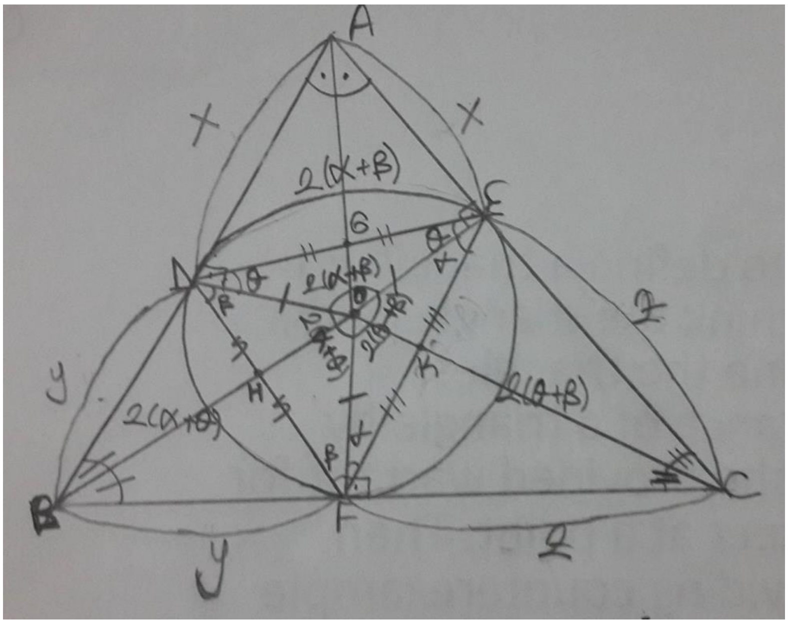


Figure 27 The figure of the concurrence of the angle bisectors on a triangle based on the angle measures.

In this debate, Fulya first explained the claim about the concurrence of angle bisectors of a triangle. In other words, she made a claim that it was important to think that triangles' angle bisectors concurred at a point on a triangle by providing the data by the construction of an angle bisector as it was discussed in the first mathematical idea in the same mathematical practice, reasoning on the elements of triangles and their properties. She added the warrant that when all of the angle bisectors were constructed, it was observed that they all intersected at a point through construction. Moreover, Ali provided the backing about the concurrence of angle bisectors based on the theorems of Ceva and Menelaous since when all of them were formed and it was observed that they concurred at a point. Then, İlker provided a backing but his explanation had missing parts since he talked about the concurrence of angle bisectors of equilateral triangles. Özge confirmed him by stating this truth. Afterwards, Efsa and Esra provided backings for the claim. They explained their backings by forming deltoid composed of two isosceles triangles. In addition, Esra stated her explanation based on the theorem of angle bisectors. Efsa explained the concurrence of angle bisectors using the content of angles on the vertices of deltoids based on arcs.

On the same week on the advancing hours of the instructional sequence and a problem on Week 6, it was illustrated that the mathematical arguments formed by the participants and knowledge and skills obtained by them about the concurrence of the angle bisectors at a point of triangles during this debate in the second week became taken-as-shared. They used this one as data and warrant in their arguments on Week 2 without necessitating backings, confirming that it became taken-as-shared. The participants used this knowledge in the debates made in order to determine the name of this point as incenter and whether this point changed based on the types of triangles discussed and represented in the third mathematical practice about reasoning on the names of concurrent points of auxiliary elements of triangles and their places. This way was similar to the way happened for the medians. Also, it became taken-as-shared in the discussion process of the concurrence of perpendicular bisectors. They stated that there were three deltoids including isosceles triangles as it happened previously. When the horizontal diagonals of them were formed since vertical diagonals were angle bisectors of the triangle, a triangle was formed and the parts of angle bisector referred to line segments became the perpendicular bisectors of this formed triangle. Therefore, the concurrence point of angle bisectors of a triangle became the concurrence point of perpendicular bisectors of another interior triangle. Moreover, this knowledge became taken-as-shared by using in the solution of a problem on Week 6. The teacher initiated the discussion by reading the problem on Activity Sheet of the Week 6 as in Figure 28:

PROBLEMS

In the triangle of ABC, the angle bisectors of the angles on the vertices of A and B intersect the edges of BC and CA at the points of D and E. If |AE| + |BD| = |AB|, find the angle measure of the angle on the vertex of C.

Figure 28 The problem on the activity sheet of the last of instructional sequence.

Instructor: In the triangle of ABC, the angle bisectors of the angles on the vertices of A and B intersect the edges of BC and CA at the points of D and E. If |AE| + |BD| = |AB|, find the angle measure of the angle on the vertex of C.

Ahmet: We know that we need at least two angle bisectors in order to determine the point of concurrence of angle bisectors since all of them are concurrent. Let these angle bisectors in the problem intersect at the point of I. This is an equilateral triangle since we know |AE| + |BD| = |AB|.

Instructor: How do you find this solution?

After this question, Ahmet stated that the point of I was the concurrence point of angle bisectors. He made reasoning in order to explain necessary part of the solution benefiting from the theorem of the point on angle bisector theorem stating that when a point was placed on an angle bisector, then it was far away in equal distance from the rays forming the angle. He explained that when the perpendicular line segments passing through the intersection point of angle bisectors and intersecting the edges were constructed, the intersection points on the edges were equidistant to the vertices of the triangle for each angle bisector as in Figure 29. After stating the result, the participants used İlkay's idea about tangent points of incircle on the triangle were also the intersection points of the angle bisectors on the edges of the triangle for equilateral triangles with the help of the instructor's questions.

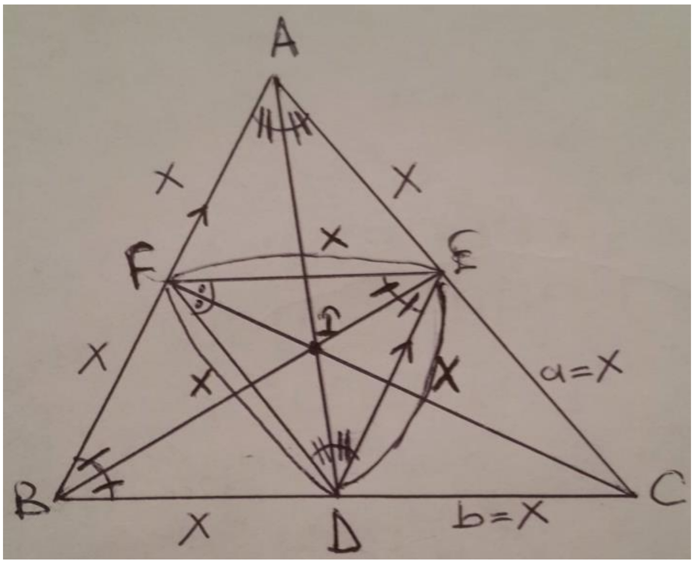


Figure 29 The figure of the concurrence of the angle bisectors on an equilateral triangle

In this conversation, Ahmet used the claim produced in the discussion of Week 2 as the data that angle bisectors of a triangle were concurrent and two of them were enough to determine this point. Moreover, he used the warrant produced in the same discussion on the same week as the warrant for that claim. He also benefited from the expression of İlkay made in the second week using equilateral triangle. Although it was not appropriate in the previous discussion, it was used by making accurate reasoning in this episode of the argumentation. In this respect, it was provided that reasoning with the concurrence of angle bisectors of a triangle became taken-as-shared. The claim produced on Week 2 was used as data and warrant in the argumentation made in Week 6. Therefore, there were two instances that the mathematical idea, reasoning on the concurrence of auxiliary elements about angle bisectors was observed and became taken-as-shared through the process of argumentation.

In the third week, the first activity sheet was about perpendicular bisectors. In this activity sheet, the participants discussed how to show the concurrence of perpendicular bisectors. Although all of the participants explained that they concurred at a point, there were constructions representing that they concurred at more than one point. Then, the reason of the case of concurrence at more than one point was explained by possible mistakes occurred in drawing and construction process. In this activity sheet, the participants discussed how to show and reason that perpendicular bisectors concurred at a point. Afterwards, the instructor wanted them to justify their answer mathematically and they explained their reasoning in the discussion taking place as follows:

Ahmet: Perpendicular bisectors of a triangle intersect at just a point since they are concurrent. When we construct all of the perpendicular bisectors of a triangle as we did previously, we see that they concur at a point.

After Ahmet's explanation, the instructor asked how they concurred at a point and they suggested applying Ceva Theorem. They tried to apply it and then they observed that it was not valid for perpendicular bisectors. Then, the discussion about the definition of cevian was made as follows. The mathematical idea which was reasoning on cevian emerged on Week 3. They investigated what the cevian was through the construction and concurrence of medians, angle bisectors, perpendicular bisectors and altitudes respectively. The discussion about cevian was emerged in the discussion about the concurrence of perpendicular bisectors. The episode of the discussion about the concurrence of perpendicular bisectors about the cevian was examined at this part of the study. The discussion about cevian was initiated and guided by asking how perpendicular bisectors concurred at a point on the plane on Week 3. This mathematical idea was observed while discussing the concurrence of perpendicular bisectors through the investigations of auxiliary elements on Week 3:

Merve: While examination of this, we applied the theorems of Ceva and Menalous but they were not valid.

Instructor: Why did not it happen? What is the different point for perpendicular bisectors?

Merve: While trying to apply the Ceva Theorem, the line segment representing medians and angle bisectors for the theorem begin from the vertices of the triangle and end on the opposing edge of it. However, it is not appropriate for the perpendicular bisector except for the equilateral triangles. Therefore, we cannot apply this theorem for perpendicular bisectors.

Instructor: So. What does this difference mean? This difference can make the other auxiliary elements cevian while the perpendicular bisector is not, cannot it?

Halit: In this respect, this difference tells what the cevian is so a cevian is the line segments drawn from the vertices to the edges of a triangle. Hence, the perpendicular bisector is not cevian based on this definition of perpendicular bisectors. In other words, the perpendicular bisectors are the line segment bisecting the edges perpendicularly.

While the participants were discussing the construction and the concurrence of perpendicular bisectors, they realized that these elements were different from the others accurately benefiting from the non-applicability of Ceva Theorem. In this debate, Halit made a claim about the definition of a cevian. Then, Merve provided the data and warrant based on the applicability of Ceva Theorem. The structure of the argument including some parts of this debate can be illustrated as shown in Figure 30.

DATA

Halit:…the perpendicular bisectors are the lines bisecting the edges perpendicularly.

CLAIM

Halit: … this theorem is about the cevians and cevian can be defined as the lines drawn from the vertices to the edges of a triangle.

WARRANT

Merve: While examination of this, we applied the theorems of Ceva and Menalous but they were not valid.

Figure 30 Toulmin's model of argumentation for reasoning on cevian.

On the same week on the advancing hours of the instructional sequence and the problems on Week 6, it was illustrated that the mathematical discussions produced by the participants and knowledge and skills attained by them about cevian during this debate in the second week became taken-as-shared. The participants made the claim that the altitudes concurred at a point. Then, they provided the data that the altitudes were cevian based on the definition of it accurately. Moreover, they explained the warrant about the applicability of the theorems of Ceva and Menalous necessarily. They used this one as data and warrant in their arguments on Week 3. Moreover, this knowledge was used as data and warrant about the discussions of the solutions of the problems on Week 6 accurately since there were problems could be solved through the line segments as cevian and Ceva Theorem. Therefore, it became taken-as-shared in two ways.

After the discussion about the explanation of cevian, they continued to discuss about the concurrence of perpendicular bisectors as follows:

Merve: Let three perpendicular bisectors intersect at a point. We know that a + b + c = 2u. When we find the sum of the length of the edges on the figure, we obtain this formula. In this respect, we confirm this formula. Then, we show that perpendicular bisectors are concurrent.

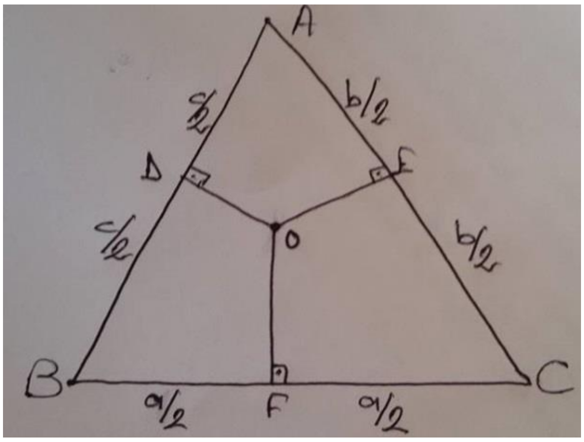


Figure 31 The figure of the concurrence of the perpendicular bisectors of a triangle

Esra: Here, it does not show that they are concurrent. Any value that we use the length of the edges of a triangle, we obtain this formula since we make operations on a triangle.

Instructor: Well. It is the correct point. So, how do they concur at a point?

Yücel: Let two of the perpendicular bisectors for the edges of AB and BC are formed and they intersect on the point of K. For the triangle of AKB, |AK| = |KB|, since the altitude of an isosceles triangle separates the edge into two equal parts. Based on the same reason, for the triangle of KBC, |KB| = |KC|. Then, |AK| = |KB| = |KC|. Therefore, the triangle of AKC is an isosceles triangle. When we form the altitude of the edge of AC for this triangle, it bisects this edge perpendicularly. We form the perpendicular bisector of the edge of AC passing through the point of K which is the intersection point of other two one. Hence, we show that perpendicular bisectors are concurrent.

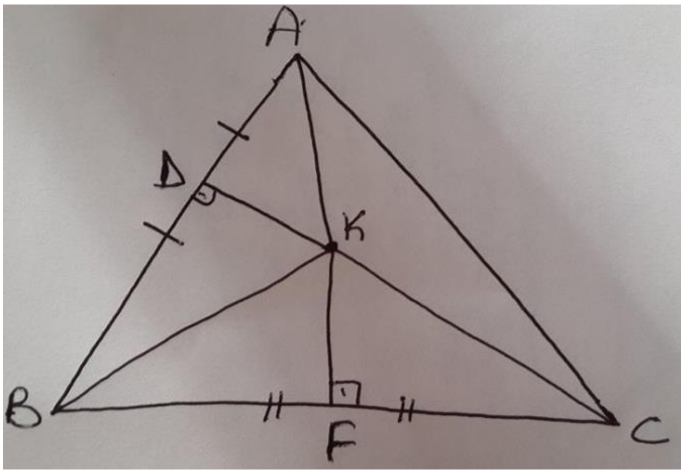


Figure 32 The figure of the concurrence of the perpendicular bisectors of a triangle by isosceles triangles.

In his explanation which was valid for isosceles triangles, Yücel made the correct and necessary explanation to justify the concurrence of perpendicular bisectors. Also, his explanation was understood and accepted by the others since nobody challenged its truth or asked any question about the process.

Hence, the instructor continued discussion by wanting them to explain another solution and justification for the concurrence of them.

Efsa: We stated that we obtain three deltoids including isosceles triangles while showing the concurrence of angle bisectors. When the horizontal diagonals of them are formed since vertical diagonals are angle bisectors of the triangle, a triangle is formed and the parts of angle bisector lines become the perpendicular bisectors of this formed triangle…

In her explanation, she initially assumed that all of the perpendicular bisectors were formed and they concurred at the point of O as in Figure 33. Then, the midpoints of the edges were combined by the line segments of GH, GK and HK. In this way, three deltoids of AGOH, BGOK and CKOH were formed by their diagonals. She added that the point of O became the concurrent point of angle bisectors since the line segments of AO, BO and CO became the angle bisectors of the angles on the vertices of the triangle as in Figure 33. While saying this, she benefited from the property that one of the diagonal of the deltoid separated it into two isosceles triangles (isosceles triangle of OHK and CHK) and the other diagonal divided it into two congruent triangles (congruent triangles of OHC and OHK). Then, by using the mathematical idea about the concurrence of angle bisectors discussed in the previous week, she stated that the line segments of AO, BO and CO concurred at the point of O. Therefore, these deltoids' edges of GO, OK and OH (which were also perpendicular bisectors of the triangle of ABC) intersected at the point of O as the vertex of these deltoids. Therefore, this point became the concurrence point of perpendicular bisectors. The points that the some of the edges of the deltoids were the perpendicular bisectors and the concurrence of perpendicular bisectors were critical to understand the justification process so the instructor asked questions and made explanations in order to emphasize them and help the participants realize and understand them. After the process was completed, the instructor summarized the solution process and the discussion ended.

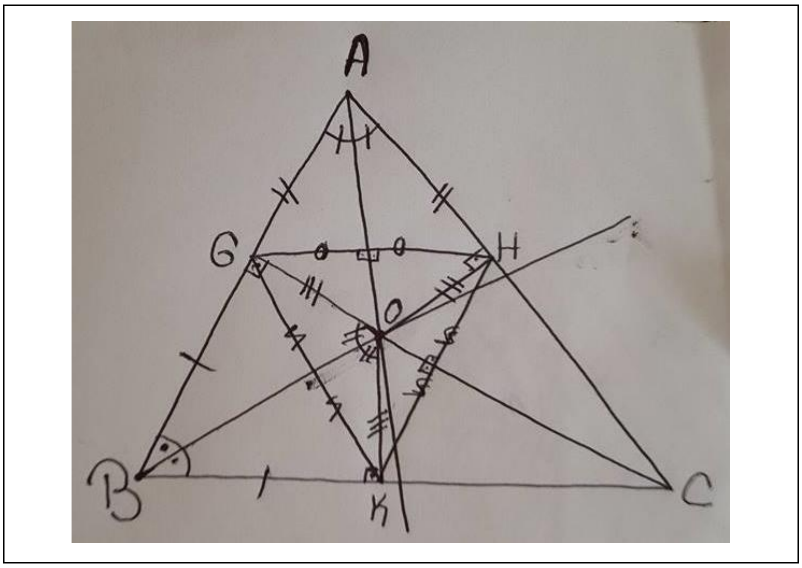


Figure 33 The figure of the concurrence of the perpendicular bisectors of a triangle by the concurrence of angle bisectors

In this debate, Ahmet initially explained the claim about the concurrence of perpendicular bisectors of a triangle. In other words, he made a claim that it was important to think that the triangles' perpendicular bisectors concurred at a point on triangles by providing the data about the construction of a perpendicular bisector. She added the warrant that when all of the perpendicular bisectors were constructed, they all intersected at a point through construction. Moreover, Merve provided a wrong rebuttal about the concurrence of them based on the formula of circumference and area of a triangle. Then, Esra stated that this explanation was not valid. Afterwards, Yücel provided backing for the concurrence of perpendicular bisectors on a triangle. He showed it benefiting from the knowledge of $V_a = h_a$ for isosceles triangles. Three isosceles triangles were formed and then perpendicular bisectors of the main triangle became the altitudes and medians of these isosceles triangles. In happening so, the mathematical idea about the concurrence of perpendicular bisectors expressed in the core of the argument was understood by the participants.

On the same week on the advancing hours of the instructional sequence, it was illustrated that the mathematical arguments produced by the participants, knowledge and skills about the concurrence of the perpendicular bisectors at a point of triangles attained during this debate in the third week became taken-as-shared. They used this one as data and warrant in their arguments on Week 3 without necessitating backings, confirming that it became taken-as-shared. The way of taken-as-shared happened in emergence of third mathematical idea in this mathematical practice. The participants used the knowledge about the concurrence of perpendicular bisectors on a triangle in order to determine the name of this concurrence point as circumcenter and whether this point changed based on the types of triangles. This way was similar to the way happened for the medians and the angle bisectors. Moreover, this knowledge was used in order to show the concurrence of the altitudes of a triangle and became taken-as-shared. In this way, the altitudes of a triangle were transformed into perpendicular bisectors of another triangle. While examining the concurrence of the altitudes, a triangle was formed by making the former triangle as the orthic triangle of the latter one in a way that the altitudes of the former triangle became the perpendicular bisectors of the latter triangle as in Figure 34. By the way, the orthic triangle is the triangle formed combining the feet of any triangle by line segments. Therefore, it could be stated that the altitudes were concurrent since the perpendicular bisectors were concurrent on a triangle. In this respect, this mathematical idea served as data and warrant for other parts of the same activity sheet and other activity sheets in the same week.

In the third week, the last activity sheet was about the altitudes. In this activity sheet, the participants discussed how to show that altitudes concurred at a point. The discussion was started benefiting from the construction with drawing errors and representing that the altitudes did not concur at a point as it happened about the similar problems for other auxiliary elements in previous activity sheets. The discussion about the concurrence of the altitudes of a triangle was stated on Week 3 as follows:

Nuray: ... when we repeat the steps of the construction for all altitudes for the edges of a triangle, we show that they concur at a point …

Instructor: How can you show that they are concurrent mathematically?

Ahmet: We can apply the theorems of Ceva and Menelaous since the altitudes of a triangle are cevian… We can show the concurrence of the altitudes of a triangle based on them as we did for medians and angle bisectors.

At that point, they used the mathematical idea about what the cevian was taking place in the previous activity sheet on the perpendicular bisectors. This mathematical idea was used as data in this explanation.

Instructor: Well. An altitude is a cevian. Then, how can you show their concurrence differently?

Özge: We know the equation of $a.h_a = b.h_b = c.h_c$ from the area formula. When we know the values of the length of the edges and one of the altitudes, we can determine the lengths of other two altitudes. For example, let the length of the altitude of the edge of BC is known…

Özge explained that by using the area formula, she could find the lengths of all edges and the altitudes. Then, by knowing these measures, she could construct the triangle and its altitudes by using compass and straight edge. She made the correct explanation but this was not the necessary and sufficient one for justifying the concurrence of the altitudes of a triangle. Then, by asking questions, the instructor got the participants realized this unrelated part of the explanation and continued the discussion by focusing on the problem.

Esra: But by knowing the measure of one of the altitudes, we cannot determine the place of concurrence point of them. For the intersection point, we have at least two line segments so we need to know the measures of at least two altitudes of a triangle...

Instructor: It is a good point. Let's turn back our problem. How can you show that the remaining altitude passes through this intersection point that Esra said?

…

Buse: We can show the concurrence of the altitudes by using Carnot theorem. When we form the altitudes of the triangle, they intersect the edges at the points of A^{I}, B^{I} and C^{I}.

$|CC^{I}|^2 + |AC^{I}|^2 = |AC|^2$

$-|CC^{I}|^2 - |BC^{I}|^2 = -|BC|^2$

$|AA^{I}|^2 + |BA^{I}|^2 = |AB|^2$

$-|AA^{I}|^2 - |CA^{I}|^2 = -|AC|^2$

$|BB^{I}|^2 + |CB^{I}|^2 = |BC|^2$

$-|BB^{I}|^2 - |AB^{I}|^2 = -|AB|^2$

When we add all of these equations, we get $|AC^{I}|^2 - |BC^{I}|^2 + |BA^{I}|^2 - |CA^{I}|^2 + |CB^{I}|^2 - |AB^{I}|^2 = 0$. Then, we end the showing the concurrence of the altitudes of a triangle.

Instructor: Well. It is a good point. Is there another different explanation?

Halit: We can form a different triangle having the perpendicular bisectors which are the altitudes of the triangle of ABC as it is in the figure since the edges of this triangle are parallel to the edges of the main triangle so that the altitudes are perpendicular to the edges of the formed triangle. We know that all perpendicular bisectors concur at a point. In this respect, when we show that the perpendicular bisectors of the formed triangle are concurrent, we show that altitudes of the main triangle are concurrent since the line segments representing the perpendicular bisectors are also the altitudes so they are concurrent.

Buse and Halit provided accurate and necessary explanations in order to justify the concurrence of the altitudes of a triangle by reasoning successfully. Buse applied the Carnot Theorem appropriately using the lengths of the edges and necessary line segments formed by the altitudes on the triangle as it was stated. This theorem was a theorem formed by applying the Pythagoras Theorem. Buse used Carnot Theorem for the concurrence of the altitudes of triangles benefiting from the right triangles formed by the altitudes of the triangle. Then, Halit explained different justification by reasoning accurately. Halit formed a bigger triangle by making this former triangle as the orthic triangle of the latter one. In the latter triangles, the line segments referring to the altitudes of the former triangle became the perpendicular bisectors of the bigger and latter triangle as in Figure 34. Moreover, he used the mathematical idea about the concurrence of perpendicular bisectors was used as data in this part of the argumentation. The line segments of AD, BE and FC referring to the perpendicular bisectors of bigger triangle and also the altitudes of the smaller triangles concurred at a point because of the concurrence of perpendicular bisectors. In other words, the line segments referring to different auxiliary elements for different triangles concurred at a point common for these triangles as in Figure 34. Based on concurrence of perpendicular bisectors, Halit formed orthic triangle whose perpendicular bisectors of AD, FC and BE were at the same time the altitudes of the smaller triangle; i.e., the triangle of ABC. Based on the notion of concurrence of perpendicular bisectors, it could be stated that the line segments of AD, FC and BE concurred at a point. Hence, it was showed and justified that the altitudes of the triangle of ABC which were AD, FC and BE concurred at a point accurately. At the end of the discussion, the instructor emphasized the important points of different justifications produced by the participants.

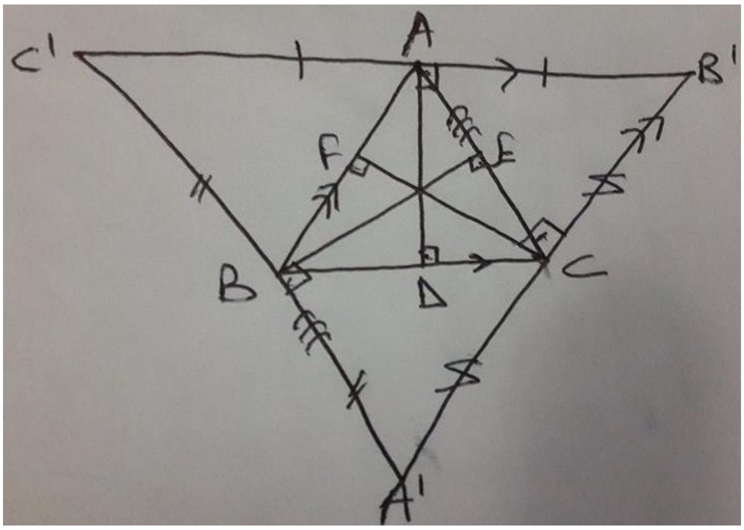


Figure 34 The figure of the concurrence of the altitudes of a triangle by perpendicular bisectors

In this debate, Nuray initially explained the claim about the concurrence of the altitudes of a triangle. In other words, she made a claim that it was important to think that triangles' altitudes concurred at a point on a triangle by providing the data about the construction of an altitude. She added the warrant that when all of the altitudes were constructed, they all intersected at a point through construction. Moreover, Ahmet provided a backing about the concurrence of them based on the theorems of Ceva and Menalous since the altitudes of a triangle were cevian. Then, Özge supported the backing by the area formula with the altitude and Buse supported another explanation done by the Carnot Theorem. Lastly, Halit made an explanation as a backing benefiting from the concurrence of the perpendicular bisectors through the orthic triangle. At this point, this argument was understood by them since none of the participants in the classroom challenged the elements of this argument.

On the same week on the advancing hours of the instructional sequence, it was illustrated that the mathematical arguments produced by the participants, knowledge and skills about the concurrence of altitudes at a point on triangles attained during this debate in the second week became taken-as-shared. They used this one as data and warrant in their arguments on Week 3 without necessitating backings, confirming that it became taken-as-shared. The participants used this knowledge in the debates made in order to determine the name of this point as the orthocenter and whether this point changed based on the types of triangles. This way was similar to the way happened for the medians, angle bisectors and perpendicular bisectors. The discussion about this case happened on Week 3 as follows:

Instructor: Ok. What can you say about the place of orthocenter based on the types of triangles such as obtuse and right triangles?

Nuray: All of the altitudes of a triangle are concurrent at a point named as orthocenter. When we think about the process of the concurrence of the altitudes and the definition of a right triangle and the altitude, each altitude forms a right triangle in the main triangle. Therefore, the place of the orthocenter changes since the place of each of the altitudes of these triangles changes. For a right triangle, the place of orthocenter is the vertex including right angle on the region of the set of points forming the triangle since the altitudes of the perpendicular edges on a right triangle themselves. Also, the altitude of the hypotenuse passes through this vertex because of the definition of the altitude and the concurrence of them.

Nuray made accurate and necessary explanation about the place of the orthocenter on right triangles by reasoning correctly and effectively. Because nobody challenged her explanation and reasoning and it was correct, the instructor confirmed its truth and continued the discussion by asking about obtuse triangles.

Instructor: Right. What about on obtuse triangles?

Buse: When we form all of the altitudes of an obtuse triangle, the orthocenter is on the figure and it takes place on the region of the set of exterior points on the plane.

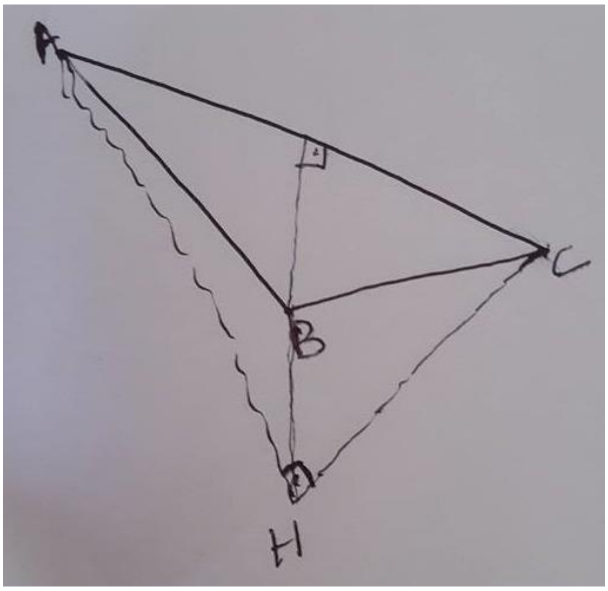


Figure 35 The point of concurrence of the altitudes as orthocenter of an obtuse triangle

Instructor: What do you think about Buse's explanation and drawing?

Selim: In this explanation, we do not form the altitudes. We made a right triangle but not correct. The place of the orthocenter of an obtuse triangle is on the region including the set of exterior points. Based on the definition of the altitude, it begins from the vertex and ends on the opposite edge by intersecting it perpendicularly and forming a right triangle. Therefore, the altitudes of two edges opposite of the acute angles take places outside the triangle since they need to form right angles. When we think that the rays of the obtuse angle are these edges.

Therefore, the altitudes are on the outside region in order to form right angles by $90^0 + \alpha = \beta$ as it is on the figure.

Buse made an explanation about the altitudes of obtuse triangle but it was not the expected one. Selim accumulated her explanation by providing the accurate and necessary one. He formed the altitudes and its truth benefiting from the property of angle measures of interior and exterior angles of a triangle as in Figure 36. Moreover, his drawing was the necessary one representing the formation of the orthocenter for an obtuse triangle.

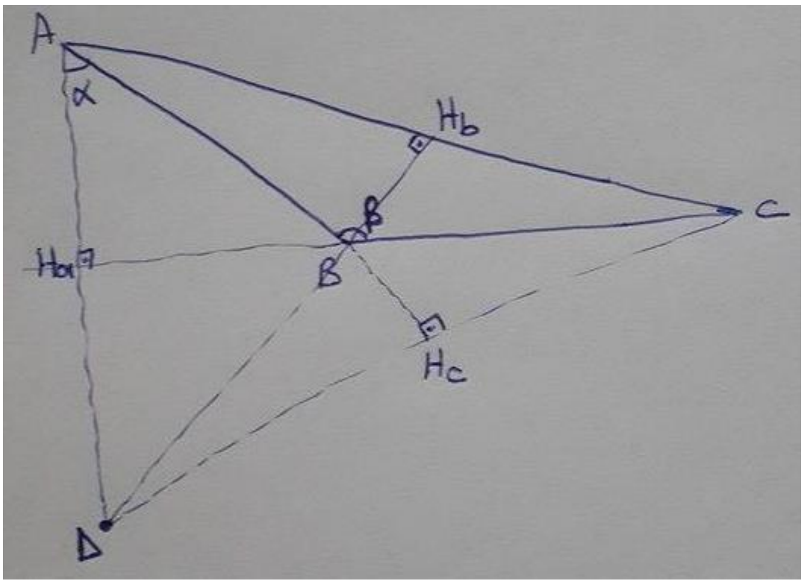


Figure 36 The point of concurrence of the altitudes as orthocenter of an obtuse triangle.

In this debate, as it was observed, the participants used the knowledge of the point of the concurrent point of the altitudes as orthocenter as data and warrant in these debates made in order to determine whether the place of this concurrent point representing the orthocenter changed for obtuse and right triangles as it was observed in this discussion. In other words, by thinking about the concurrence point of the altitudes as orthocenter and this point's formation process, they talked about the place of orthocenter for these kinds of triangles. Nuray provided claim, data and warrant for a right triangle benefiting from these knowledge as data that there was a point that altitudes concurred at a point and warrant about the formation and concurrence of altitudes of triangles. Also, the participants benefited from the definition and construction of this element. Moreover, Buse provided the claim for obtuse triangle and then provided the data with the concurrence of them but he used wrong warrant. In other words, she explained the concurrence of the altitudes and its place for obtuse triangles correctly but she could not provide expected representation for it correctly. She could not form the altitudes of this obtuse triangle for the edges forming the obtuse interior angle of the triangle correctly. Then, Selim provided rebuttal and true data and warrant for this discussion. He benefited from drawing of the altitudes and the knowledge as data and warrant. He used the property that the sum of the measures of two interior angles was equal to the measure of the exterior angle belonged to the remaining interior angle. This discussion period was the second instance that the mathematical idea about the concurrence of the altitudes of triangles was observed since it was used in order to determine whether the place of this concurrent point changed or not for different types of triangles.

4.2.3 Mathematical idea 3: Reasoning on the names of concurrent points of auxiliary elements of triangles and their places

The third mathematical idea which was the reasoning on the names of concurrent points of auxiliary elements of triangles and their places on different types of triangles emerged on Week 2 and 3. The participants investigated these points as centroid for medians, incenter for angle bisectors, circumcenter for perpendicular bisectors and orthocenter for altitudes respectively. Moreover, they continued the discussion whether the place of these points changed based on the types of triangles. For these activities, they thought about the relationship between these elements, related theorems and properties about triangles with their peers and participated in the whole class discussion by explaining their thoughts and mathematical expressions for them. During the whole class discussion, they debated how these concurrent points became critical points and changing/unchanging critical places. The discussion was initiated by asking how these concurrent elements attained critical importance on the plane on Week 2. These four elements are investigated separately and grouped as the third mathematical idea.

In the second week, the first activity sheet was about the medians. In this activity sheet, the participants discussed how to name the concurrent point of the medians and to attain critical importance in geometry. It was important since a point was formed through the concurrence of the medians and this point with the name of centroid had some properties such as separation of the medians through the ratio of 2;1 based on it and the separation of triangle into equal areas by the medians and centroid. The discussion about the centroid as concurrence point of medians and its importance was examined through the explanations of the participants for the instructor's question of "What is/are the name(s) of the intersection points of the medians on a triangle?" on Week 2 as follows:

Büşra: When we think about the concurrence of the medians, we can state that all of the medians of a triangle are concurrent on a point. This point is the centroid of the triangle because this point is the center of gravity on the triangle.

Instructor: How can you identify this point as the centroid?

Merve: When we think about the process of the concurrence of the medians, we see that these line segments separate the edges of the triangle into equal parts and then we observe that the medians of a triangle divide one another in the ratio 2; 1 ... For example, on the triangle, the triangles of KFG and GMC; AKF and ABM are similar triangles. |FG|

= |GM| = ½ and |AF| = |FM| = ½ based on the scale factors of these similar triangles. Therefore, the relationship between these lengths can be described as |AG| = 2|GM| = 4|FG|. When we repeat this process for the other medians, we find the same ratio.

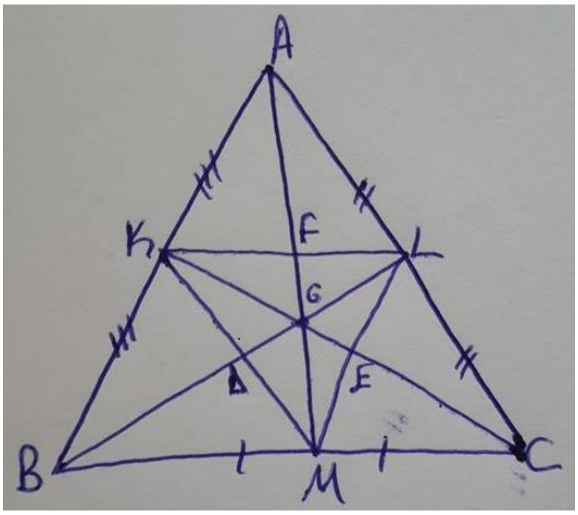


Figure 37 The separation of the medians through the ratio of 2;1 based on the concurrence of the medians

Instructor: Is this ratio enough to name the concurrent point of the medians as the centroid?

Halit: It is not enough since we need to show that a triangle is dissected by its medians into six smaller triangles having equal area. The triangles of BXP and CXP; BPZ and APZ; CPY and APY have equal area. Then, the areas of the triangles of ABX and ACX; ACZ and BCZ are equal in measure so that we find y = z and z = x and then x = y = z. Therefore, we show that the medians separate the triangle into regions having the equal area.

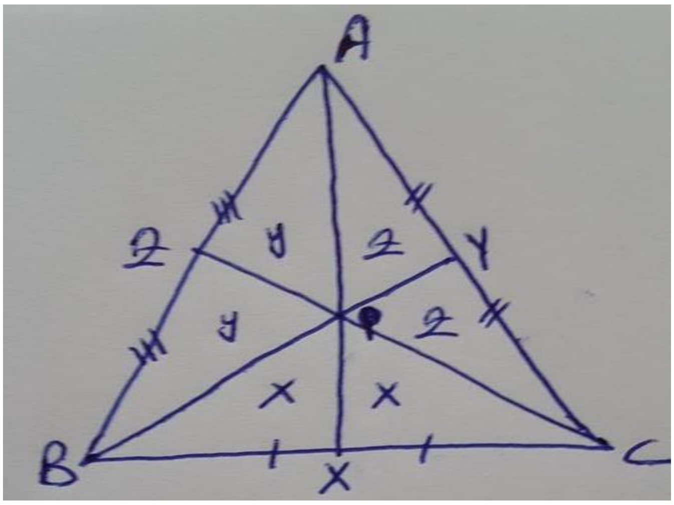


Figure 38 Dissection of a triangle into regions having equal areas with medians

In this discussion, Büşra initially made a claim about the name of the concurrence point of the medians as centroid based on its nature of the gravity center of the triangle. Then, Merve provided data by emphasizing the concurrence of the medians of the triangle benefiting from the mathematical idea about the concurrence of them so that the evidence that the mathematical idea about the concurrence of auxiliary elements functioned as if shared was obtained. In her explanation, Merve found the ratio between the lengths of the edges of the triangle in Figure 37 benefiting from similar triangles. She made good reasoning and her explanation might be useful for determining the concurrence point of the medians as centroid. However, she realized that her explanation was not sufficient for this identification Then, Halit provided a different explanation and justification for this identification since he stated that Merve’s explanation was not sufficient to identify the concurrence point as centroid since she found the ratios between the line segments formed through the intersection points of the line segments formed through the medians. Although her explanation represented the ratios between the line segments formed by the medians, her explanation had missing parts for justifying that the concurrent point of the medians was centroid. Halil made accurate and necessary explanation at this episode of the argumentation to complete the warrant of the argumentation. In other words, another part of the warrant for that claim was provided benefiting from the same mathematical idea by Halil. The name of the concurrence point of the medians as centroid was identified based on the discussion about the separation of the medians into ratio of 2;1 with respect to this concurrence point into regions and the triangle into regions having equal area. Based on the area formula which was $a.h_a$, the triangles having the equal area were determined benefiting from the idea that the lengths of the edges and the altitude of these edges having the common vertex were same. They were examined based on the process and the mathematical idea of the concurrence of the medians. Then, by showing them, the concurrence point was determined as the centroid of the triangle accurately and necessarily so that it was named as the centroid. The structure of the argument including some parts of this debate can be illustrated as shown in Figure 39.

DATA

Büşra: When we think about the concurrence of the medians…

CLAIM

Büşra: This point is the centroid of the triangle because this...

WARRANT

Merve: When we think about the process of the concurrence of the medians, we see that these lines separate the edges of the triangle …

Halit: It is not enough since we need the show that a triangle is dissected by its medians into six smaller triangles of equal area…

Figure 39 Toulmin’s model of argumentation for reasoning on the formation of the centroid.

On the same week on the advancing hours of the instructional sequence, it was illustrated that the mathematical discussions produced by the participants, knowledge and skills about the point of concurrence of the medians attained was the centroid in the second week became taken-as-shared. They used this one as data in their arguments on Week 2 without necessitating backings, confirming that it became taken-as-shared. They made the claim explaining the centroid of the triangles took place in the region of interior points formed by the triangle on a plane for all types of triangles. They produced this claim for the problem of “Estimate the place(s) of the intersection point(s) of the medians on right, obtuse and acute triangles. Do(es) the place(s) of this/these intersection point(s) change for these triangles? Why?”. The instructor directed the participants to answer this problem using the mathematical ideas about the concurrence of the medians and this point as centroid. Then, they provided data that the centroid was the centroid of the triangle by dissecting the triangle into six regions with equal area in a similar way made in Figure 38. Therefore, they explained the warrant “when the medians are formed for all these types of triangles, it is seen that they exist on the same region including interior points as one of three regions formed by a triangle on a plane” (the mathematical idea about regions formed on a plane by a triangle as discussed in the first mathematical practice) and “six regions with equal area always exist in the same region so the centroid always take place in the same region for all types of triangles” as in Figure 38. By making this explanation, they stated that the centroid always existed in the interior region of all triangles by separating the medians into ratio 2;1 accurately and appropriately. By the way, it could be stated that the mathematical idea about the regions formed by the triangle by separating the plane into three parts was also became taken-as-shared. This was the other instance that the notion of the regions formed by triangles on a plane were observed in an argumentation functioned as if shared. The concurrence of the medians and the name of this concurrence point as the centroid used as data and it became taken-as-shared. Moreover, the mathematical idea about the centroid as the concurrence point of the medians became taken-as-shared by being used in the discussion taking place on Week 6 about the question related to two rotating and coinciding equilateral triangles on the point of the centroid and examining the area of the overlapping region. In other words, the concurrence of the medians and the name of this concurrent point as the centroid used as data and warrant and it became taken-as-shared as it happened in the following discussion:

Instructor: Two congruent equilateral triangles (n units) overlap as shown in the figure. Vertex of C of one triangle is at the centroid of the other triangle. If the triangle with the vertex of C is allowed to rotate about the centroid, C, of the other triangle, what is the largest possible value of the overlapping area?

Two congruent equilateral triangles (n units) overlap as shown in the figure. Vertex of C of one triangle is at the centroid of the other triangle. If the triangle with vertex C is allowed to rotate about the centroid, C, of the other triangle, what is the largest possible value of the overlapping area?

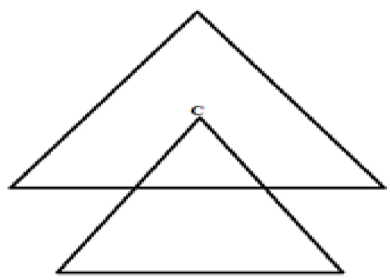


Figure 40 Figure of the problem about two rotating and overlapping equilateral triangles.

The instructor read the problem on the activity sheet represented in Figure 40. Then, the instructor asked some participants to solve the problem. Through the process of peer discussion, the instructor saw that there were two different answers for this problem so she selected two participants to represent typical examples of these different answers.

Merve: The area of the overlapping region is $2{,}6a^2$ as a maximum area for the overlapping region and the area of whole triangle is $15{,}6a^2$. In the problem, it is explained that the centroid of triangle is coincident with the vertex of the other triangle. The centroid is the concurrent point of the medians of a triangle by separating the triangle into six regions having equal areas and the edges with the ratio 2;1…

In her explanation, Merve placed the front triangle in a way that the edge of the front triangle was coincident with the median of the back triangle as in Figure 41 benefiting from the ratio of 2:1 formed on the parts of medians. She thought that the largest overlapping area that could be determined by forming a right triangle whose median was coincident with the median of the back triangle.

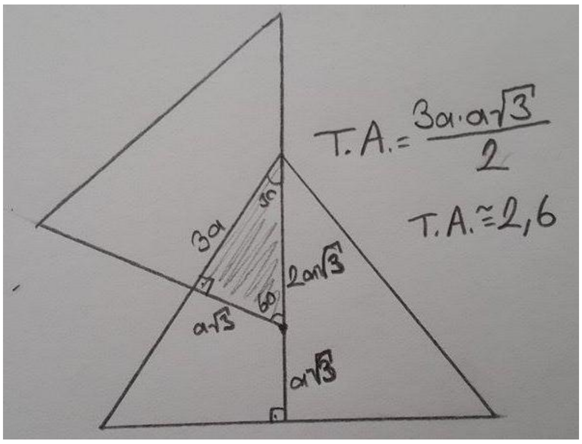


Figure 41 Figure of the two overlapping equilateral triangles by Merve.

Instructor: Ok. İlkay, Could you please explain your solution?

İlkay: I found bigger value than this one. The maximum area that we can compute is $4a^2$ in the case of the area of whole triangle is $15{,}6a^2$. When the place of the front triangle is determined as the medians of these triangles are coincident because they are connected each other on the centroid of the back triangle, the centroid of the front triangle becomes coincident with the vertex of the back triangle as it is in the figure by Merve's explanation… Then, we obtain two isosceles obtuse triangles with the angle measure of 120^0…

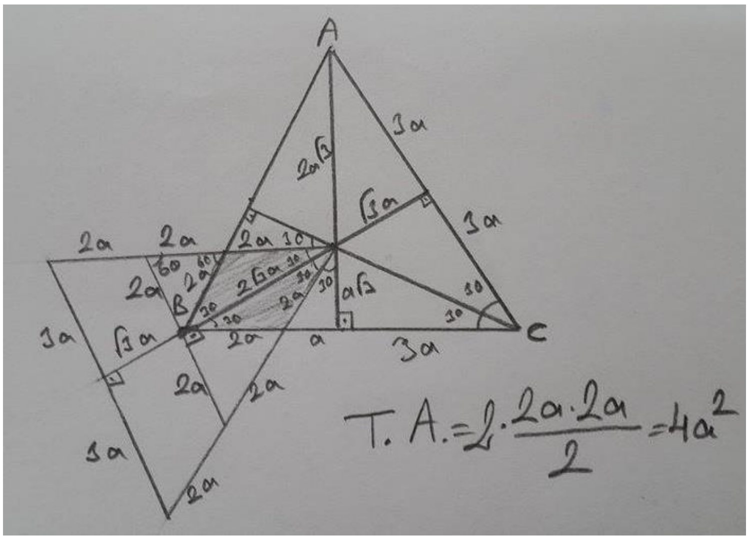


Figure 42 Figure of the two overlapping equilateral triangles by İlkay.

In this discussion, the mathematical idea about reasoning on naming the concurrence point of auxiliary elements for medians became taken-as-shared. Merve made claim about the calculation of the largest area for overlapping region as $2{,}6a^2$. For this claim, she provided the data "In the problem, it is explained that the centroid of triangle is coincident with the vertex of the other triangle. The centroid was the concurrent point of the medians of a triangle by separating the triangle into six regions having equal areas" benefiting from the mathematical idea of the centroid as the concurrence point of the medians and this idea functioned as if shared. Then, she provided warrant by explaining the position of the front triangle and the process of determination of it through the medians. She provided her answer by positioning the triangle with vertex C in a way that one of its edges passed through the vertex of the other one in Figure 41. She made reasoning accurately and positioned the triangle appropriately benefiting from the medians and the position of the centroid on the medians based on the idea that the medians of a triangle divided the other in the ratio of 2; 1. However, although the reasoning way was correct, this was not the largest possible value of area for overlapping region since there was another position to produce the largest overlapping area. Then, İlkay provided correct answer by explaining largest value for the area of overlapping region. He used the same data provided by Merve. Then, he stated the warrant based on the place of the front triangle in a way that the medians were coincident by obtaining two obtuse isosceles triangles as in Figure 42. He placed the front triangle by benefiting from the ratio of 2:1 formed by the parts of the medians of a triangle. Different from Merve's representation, he placed the front triangle whose median was coincident with the vertex of the back triangle. The largest area was determined in a way that the median of one of the triangles was coincident with the edges or median of the other triangle. In this respect, İlkay reasoned that when the medians of both triangles were coincident, the largest lengths of the parts of the edges of the triangle were formed. He assumed that n = 6a representing the length of the edges of these triangles. Then, he found that the overlapping parts of the edges had the lengths of 2a. Hence, he obtained a larger value for overlapping region than the value found by Merve. Merve found smaller value since the overlapping triangle's edges had smaller lengths. As it was observed in the discussion, the mathematical idea was used as data and warrant for the discussion of the problem in order to examine the maximum value of the overlapping region of two coincident equilateral triangles. With this motivation, the mathematical idea about the centroid as the concurrence point of the medians became taken-as-shared since this mathematical idea previously emerged took place by functioning in other parts of new argument analyzed by Toulmin's model of argumentation.

In the second week, the next activity sheet was about angle bisectors. In this activity sheet, the participants discussed how to name the concurrence point of angle bisectors and to attain critical importance in geometry. The discussion about the incenter as concurrence point of angle bisectors and its importance was examined and stated on Week 2 as follows:

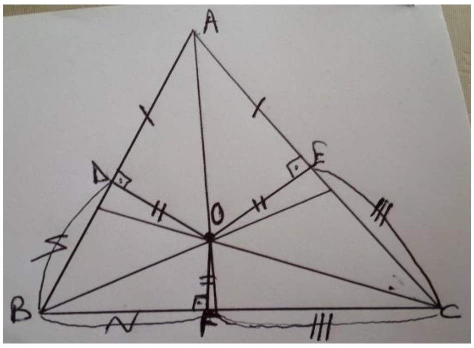
Instructor: What can we say about the concurrent point of angle bisectors and the process of concurrence of them.

…

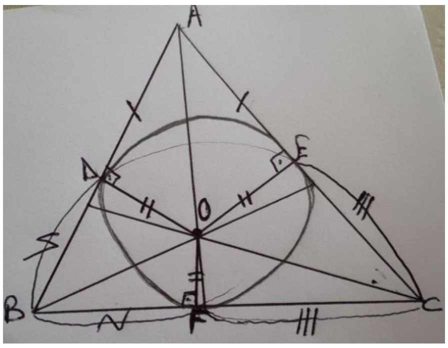
Merve: The concurrence point of angle bisectors has critical importance since any two of them are enough to determine this point. When we examine the process of showing the concurrence of them, we can name this point as incenter.

Instructor: How do you show that this point is incenter?

Halit: … we have four points that three of them are equidistant to the specific one. The circle is the set of points equidistant to a point which is the center of it. When we combine three points which are equidistant to the concurrent point of angle bisectors on the triangle by arcs, we obtain incircle of this triangle whose center is the concurrent point of angle bisectors… Also, this concurrent point becomes the incenter…



a. Tangent points and the point equidistant to these points



b. Incenter with its radius on incircle

Figure 43 The formation of incenter by angle bisectors

In this debate, Merve first made a claim that the name of the concurrence point of angle bisectors was incenter. Then, Halit provided data and warrant for that claim. The data was about the concurrence of the angle bisectors and the definition of circle. The warrant was about the formation of a circle benefiting from the points through the concurrence of them. Halit remembered the process of the concurrence of angle bisectors and its justification as in Figure 43.a. In this figure, there was the concurrence point and three points which were D, E and F equidistant to this concurrence point. Then, he made the connection between this mathematical idea and the formation and definition of a circle. In this figure, when these three points were combined by arcs, the combination of these arcs formed a circle with the center point as the concurrence point of the angle bisectors as in Figure 43.b. The distances of these three points to the center represented the radius of this circle. Also, this circle became the incircle of the triangle since these three points were the tangent points on the triangle. In this way, the concurrent point became the incenter. Through this process, Halit made necessary and accurate explanation for the problem by reasoning correctly. The structure of the argument can be illustrated by including some parts of this debate as shown in Figure 44.

DATA	CLAIM
Halit: When we think about the process of showing the concurrence of them…	Merve: …we can name this point as incenter.

WARRANT
Halit: …Therefore, when we combine three points which are equidistant to the concurrent point of angle bisectors on the triangle by arcs…

Figure 44 Toulmin’s model of argumentation for reasoning on the formation of the incenter.

On the same week on the advancing hours of the instructional sequence, it was illustrated that the mathematical discussions produced by the participants, knowledge and skills attained by them about the point of concurrence of angle bisectors was the incenter in the second week became taken-as-shared. They used this one as data and warrant in their arguments on Week 2 without necessitating backings, confirming that it became taken-as-shared. The concurrence of angle bisectors and the name of this concurrent point as the incenter used as data and warrant in other mathematical arguments and it became taken-as-shared as it happened in the following discussion:

Instructor: Ok. Does the place of the incenter change based on the types of triangles? How?

Selim: The place of it does not change. … the place of incenter for all types of triangles is always in the set of interior points which is one of the regions formed by triangles on the plane. The concurrent point of angle bisectors is the incenter of the triangle, therefore incircle is always formed in the triangle and also the incenter is made. Moreover, three tangent points of the incircle on the triangle is on the region of the set of points forming the triangle.

In this debate, as it was observed, the participants used the knowledge of the concurrence point of angle bisectors as the incenter as data and warrant in the debates made in order to determine whether the place of the point representing the incenter changed based on the types of triangles. This reasoning was necessary and sufficient for the problem. Therefore, it became taken-as-shared as functioning in other parts of the argumentation model of the discussion. Moreover, the knowledge about the formation of the incenter point was used as data and warrant in another two discussions made in the content of similarity/congruence on Week 5. In other words, there were other instances that this mathematical notion functioned as if shared in a way that they produced evidences for becoming taken-as-shared. In the first debate, the topic that the radius of incircle of congruent triangles had always same length was discussed. Esra made the claim that they were same accurately. Then, she provided the data that the point of the concurrence of angle bisectors was incenter and congruent triangles' edges had the equal length and the measures of angles of them were equal. She explained the warrant that the congruent triangles' angle bisectors separated the angles into two angles having equal angle measures and the distance between the incenter and tangent points were equal. By doing so, this mathematical idea became taken-as-shared in an argumentation reasoned correctly under the guidance of the instructor. In another discussion, this idea became taken-as-shared again. The teacher initiated the discussion by reading the problem on Activity Sheet 2 represented in Figure 16 about the content of congruence/similarity on Week 5:

Instructor: When the lengths of the radius of incircle and the altitude of the hypotenuse are equal for two right triangles, they are sometimes congruent, are not they?

Mehmet: They were always congruent. By construction, I can show that they are congruent.

Instructor: How can you do that?

Mehmet: By identifying the possibility and types of triangles constructed by known elements, I can do it. Initially, the incenter is the point of concurrence of angle bisectors. Therefore, we know the distances between the incenter and the tangent points.

Instructor: Well. It is a good point. So.

Mehmet: Also, we know the angle measure of one of the angles as 90^0 and the length of the altitude of the hypotenuse. ... We construct a right angle, a circle having the radius in the length of the altitude and incircle. At the end, we draw a line tangent to the incircle since the hypotenuse is tangent to incircle. The intersection points of this line on the lines of AX and AY are vertices of the triangle and the hypotenuse and the triangle were formed…

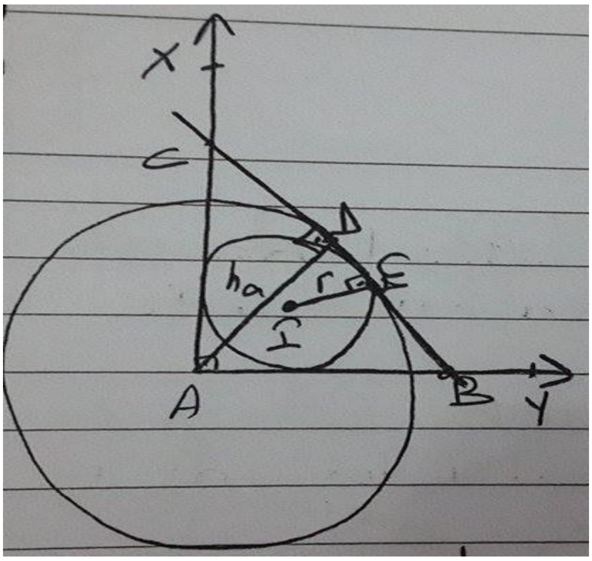


Figure 45 Construction of the problem by Mehmet.

In his explanation, he provided accurate and necessary explanation for the problem by reasoning successfully. He constructed the right angle and the circle with the radius in the length of the altitude as in Figure 45. Then, the incircle was constructed by being tangent to the rays of the right angle. Then, by constructing the tangent line for the incircle to intersect the rays of right angle, the triangle was constructed. Through his explanation, he did not state the place of the altitude on the hypotenuse. Then, the instructor asked question about its place to fulfill this gap. Then, nobody answered so the instructor answered by stating that there were two cases that the tangent line of the incircle intersected the circle with the radius equal to the length of the altitude; it intersected at two points or one point which was also the tangent point of the incircle. Therefore, two cases for the place of the altitude were formed based on the idea that when the position of a line to a circle was examined there were three cases; not intersecting, intersecting at two points and tangent. Because the triangle was formed, the case of not intersecting was eliminated. In order to help the participants understand the idea and use it to determine their congruence, she asked "What do these cases mean?" and they answered that there were two types of triangles constructed by these elements. In the case of intersecting at two points, one was the tangent point of incircle and the other one was the intersection point of the circle constructed for the altitude. In this case, a scalene triangle was formed. The other case was the hypotenuse was tangent to the circle formed for the altitude. This tangent point was also the tangent point of the incircle with the hypotenuse so that an isosceles triangle was formed. Then, the instructor asked to reach the complete answer about congruency "By these knowledge, what can you say about the congruence of triangles?". Then, they answered that the scalene triangles were congruent since by following construction steps, all of the scalene and isosceles right triangles formed by these known elements had the main and auxiliary elements having the same properties and measures so they were always congruent. Through the discussion, sufficient and accurate solution and justification was provided for the problem. After the discussion was completed, Halit provided another formation steps for this triangle as follows:

Halit: … We construct the right angle and its angle bisector. Then, we draw a circle having the center point of the intersection point of rays and with the radius of the altitude. Then, we can determine the place of incenter by the point which is $\sqrt{2}\,r$ far away from the point of A and construct incircle in the figure. The point of E is the tangent point of incircle to the hypotenuse and the point of D is the intersection point of the hypotenuse and right angle's angle bisector. When we draw a line passing through these points, the intersection points of this line on the rays are the vertices of the triangle…

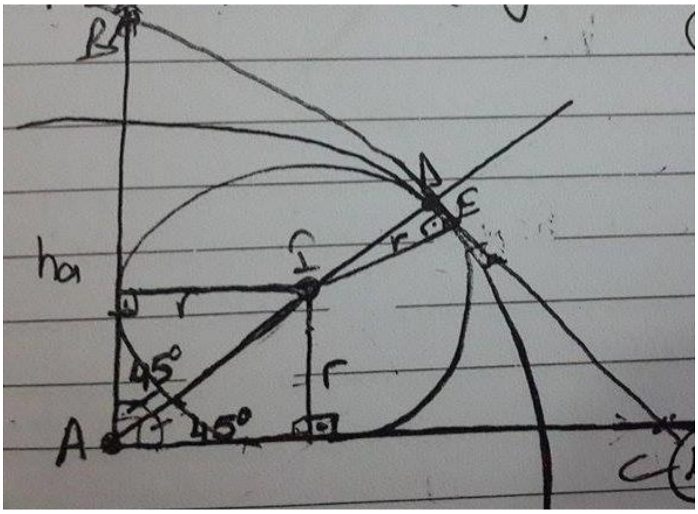


Figure 46 Figure for construction of the problem by Halit.

The construction steps of Halit were similar to the steps followed by Mehmet. However, the differences came from the construction of incircle. While Mehmet was constructing incircle, he reasoned on the construction steps of a tangent circle to the particular lines representing the rays of the right angle. In other words, Mehmet constructed a right angle and he formed a circle on the center as the point of the vertex of this right angle in the length of the altitude. Then, he constructed an incircle tangent to two rays forming the right angle on the vertex of A as in Figure 45. On the other hand, Halit determined the place of the incenter based on the angle bisector theorem stating that when a point was placed on an angle bisector, then it was far away in equal distance from the rays forming the angle. He formed the angle bisector of the right angle and determined the place of the incenter by Pythagorean Theorem as in Figure 46 since the incenter was far away in the distance of $\sqrt{2}r$ to the point of A. Afterwards, there were two cases in order to form the triangle having the values explained in the problem; isosceles and scalene triangle. In the case of isosceles triangle, the point of D representing the altitude foot on the hypotenuse and the point of E representing the perpendicular bisector starting on the incenter and ending on the hypotenuse were coincident. Then, the hypotenuse was constructed passing through this coincident point and intersecting the rays with the equal distance to the vertex of A. All of the isosceles triangles formed by these known measures of the elements were congruent since the lengths of the opposing edges were equal by the criteria of S.S.S. On the other hand, in the case of scalene triangles, the points of D and E were not coincident. The scalene triangle was formed passing through these points intersecting the rays of the right angle. All of the scalene triangles formed by these known measures of the elements were congruent since the lengths of the opposing edges were equal by the criteria of S.S.S. Then, the congruence of the triangles was showed and justified by using the idea formed through the discussion taking place in Mehmet's explanation process guided by the instructor.

In this discussion, Mehmet used the claim produced in the discussion of Week 2. The data that the point of concurrence of angle bisectors as incenter was used in order to determine the congruence and warrant was provided benefiting from the steps of construction. Moreover, Halit provided backing explaining the steps of construction in a different way. In this respect, it was provided that reasoning with the names of concurrence points of auxiliary elements of triangles and their places for angle bisectors in order to determine the place of incenter became taken-as-shared. The claim produced on Week 2 was used as data in the argumentation made on Week 5.

Lastly, the critical importance of concurrence point of perpendicular bisectors was examined on Week 3. The discussion about reasoning on the names of concurrence points of auxiliary elements of triangles and their places for perpendicular bisectors was made in similar way made for the incenter as the concurrence point of the angle bisectors. In this debate, İlkay first made a claim that the name of the concurrence point of perpendicular bisectors was circumcenter. Then, the instructor challenged him to explain how this concurrence point became circumcenter. He provided data that three perpendicular bisectors of a triangle concurred at a point appropriately. He also added the warrant that there were three isosceles triangles of AOB, BOC and AOC where two of their edges' lengths were equal as in Figure 47. Also, the vertices of the triangle were equidistant from the concurrence point of perpendicular bisectors so that a circle could be formed based on the definition of a circle by combining these vertices with the arcs based on the center of this concurrence point. Then, the perpendicular bisectors represented the case that the perpendicular line segments passing through the center bisected the chords in a circle as in Figure 47. As it was observed, the concurrence of perpendicular bisectors and the process of showing their concurrence were used while forming the warrant accurately and necessarily.

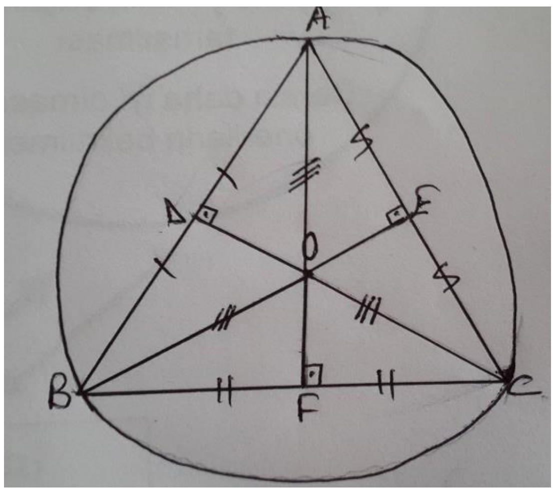


Figure 47 The circumcenter as the concurrent point of perpendicular bisectors.

On the same week on the advancing hours of the instructional sequence, it was illustrated that the mathematical discussions produced by the participants and knowledge and skills attained by them about the point of concurrence of the perpendicular bisectors was the circumcenter in the third week became taken-as-shared. They used this one as data and warrant in their arguments on Week 3 without necessitating backings, confirming that it became taken-as-shared. The concurrence of perpendicular bisectors and the name of this concurrence point as the circumcenter used as data and warrant and it became taken-as-shared as it happened in the following discussion:

Instructor: Ok. Does the place of the circumcenter change based on the types of triangles? How?

Mehmet: The place of it changes based on the types of triangles. … The concurrence point of perpendicular bisectors is the circumcenter of the triangle; therefore circumcircle is always formed on the outside region...

Mehmet made an explanation about the place of the concurrence point of perpendicular bisectors but his explanation had unnecessary and incorrect parts. He made reasoning about the placement of a right triangle in a circle incorrectly. He ignored the fact that the hypotenuse of the right triangle was coincident with the diameter of the circle. At this point, the instructor wanted Mehmet to represent his explanation. He drew Figure 48. In order to help him realize the incorrect part of his explanation, the instructor wanted him to estimate the measures of the arcs formed by the vertices of the triangle on circumcircle. By estimating these measures, he correctly stated that the hypotenuse must be on the diameter of circumcircle. Then, the necessary and appropriate explanation about the place of the circumcenter for a right triangle was made by Nuray as follows:

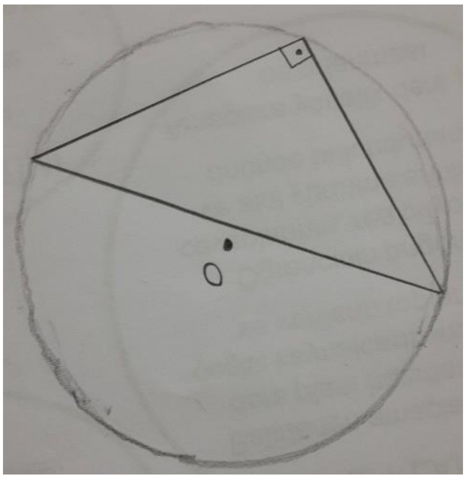
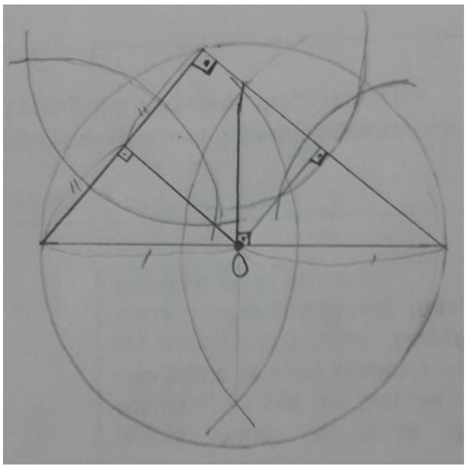


Figure 48 The circumcircle of a right triangle by Mehmet

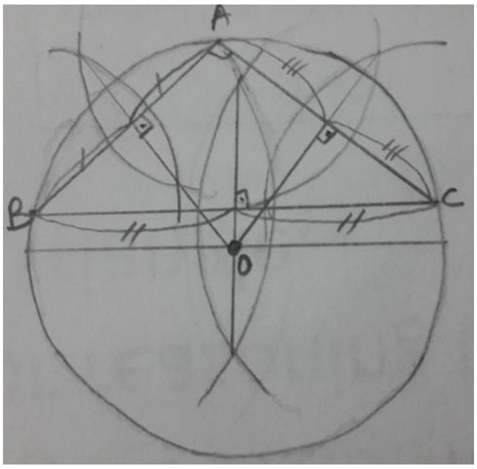
Nuray: The place of the circumcenter of a right triangle is the midpoint of the hypotenuse. The diameter of circumcircle is the hypotenuse and the inscribed angle of a circle opposite of the diameter has the measure of 90^0. When we follow the concurrence of perpendicular bisectors and the formation of the circumcenter, this case becomes valid for right triangle (in Figure 49.a).

Instructor: What can you say about an obtuse triangle?

Halit: We know that the inscribed angle of a circle opposite of the diameter has the measure of 90^0 so the arc opposite of the obtuse angle of the triangle must exceed the diameter of the circumcenter. Also, the concurrence point of perpendicular bisectors is the center of circumcenter. Therefore, this center point takes place on the region including the set of exterior points near the largest edge (in Figure 49.b).



a. Construction of circumcircle for right triangle by Nuray



b. Construction of circumcircle for obtuse triangle by Halit

Figure 49 The circumcircle of right and obtuse triangles

In this episode of the discussion, Halit provided necessary and accurate explanation and justification about the place of circumcenter for an obtuse triangle. In this debate, the participants used the knowledge about the concurrence point of perpendicular bisectors as the circumcenter as data and warrant in the debates made in order to determine whether the place of the point representing the circumcenter changed for obtuse and right triangles as it was observed in this discussion. In other words, by thinking the concurrence point of perpendicular bisectors as circumcenter and the process of formation of this center point, they discussed the place of circumcenter for these kinds of triangles. Hence, reasoning on the names of concurrent points of auxiliary elements of triangles and their places for perpendicular bisectors in order to determine the place of circumcenter became taken-as-shared. Moreover, the knowledge about the formation of the point of circumcenter was used in another discussion made in the content of congruence/similarity on Week 5. In this debate, the topic was that the radius of circumcircles of congruent triangles had sometimes equal length and the distances of circumcenter to the edges were sometimes equal. Efsa made the claim that they were always equal by stating that the explanation was wrong. Then, she provided the data that the point of the concurrence of perpendicular bisectors was circumcenter and congruent triangles' edges had the equal length and the measures of angles of them were equal under the guidance of the instructor. She explained the warrant that the congruent triangles' perpendicular bisectors had equal length. In order to show the truth of their claim, the instructor wanted them to draw and they formed a triangle as in Figure 47. Then, they constructed another triangle having equal length of radius of circumcirle with the previous one and equal length of distances for the opposing edges of the triangle with the previous one. When they found the length and angle measure necessary for these triangles by using these measures, they became congruent triangles since all of these measures were equal for these triangles. In this way, they showed the congruence of these triangles accurately and sufficiently. In doing so, this knowledge which was the reasoning with the names of concurrence points of auxiliary elements of triangles and their places for perpendicular bisectors became taken-as-shared. Moreover, similar discussions made for angle and perpendicular bisectors were observed for the altitudes. Then, the mathematical idea about the reasoning on the concurrence point of the altitudes as the orthocenter became taken-as-shared in similar way as it happened for angle and perpendicular bisectors.

4.3 Mathematical practice 3: Reasoning on congruence and similarity

The last mathematical practice was reasoning on congruence and similarity of triangles. The mathematical ideas included in this mathematical practice were about the formation of congruent and similar triangles through transformation geometry and Angle-Side-Side (A.S.S.) was not a criterion for congruence or similarity. They had been mainly emerged from the activities that the participants engaged in the fourth and fifth weeks. On the fourth week, they examined the construction of images of triangles through the types of transformation geometry and the relationship between triangles and image triangles by using the compass and ruler, and coordinate system. On the fifth week, they engaged in the activities about congruence and similarity of triangles and the criteria for them. While they were talking about criteria, they proposed a congruence/similarity criterion of A.S.S. and discussed its incorrectness. For these activities, they worked with their peers and participated in the whole class discussion.

4.3.1 Mathematical idea 1: Reasoning on the formation of congruent or similar triangles through transformation geometry

The last mathematical practice was observed on the fourth week of the instructional sequence while the participants were engaging in the activities about forming congruent and similar triangles through transformation geometry. In this activity, the participants were asked to find the image of the triangles by following the steps of construction through transformation geometry. After forming triangles which were the images, they determined the congruent or similar triangles. While they were engaging in this activity, they benefited from the definition of types of transformation geometry and a triangle. For this activity, they worked with their peers and participated in the whole class discussion. The instructor initiated the discussion by asking a question in order to discuss the activities on the first activity sheet of the Week 4:

Instructor: What is the relationship between the triangle and its image formed through the translation?

Selim: We attain a triangle through translation. The triangle and its image triangle are congruent triangles.

Instructor: How do they become congruent triangles?

Selim: Translation is moving a shape. A triangle is moved through specific way, direction and distance through translation so the image triangle and the triangle are congruent.

Instructor: How can you form the image of a triangle through translation?

Nuray: A triangle is composed of three non-linear points. While forming the image of triangle, we find the places of three non-linear points as the vertices by preserving the distances and the directions between them. In other words, we move the vertices through the same vector, find the vertices of the image triangle and combine these vertices by the line segments. At the end, we form the image triangle composed of three non-linear points moved by the same vector. In this respect, we have congruent triangles since they have the equal length of edges.

Nuray explained the process of construction of the image of the triangle through translation. In her explanation, Nuray benefited from the knowledge that the distance between parallel lines were preserved and the vectors represented the line segments having magnitude and direction. In this respect, the edges of the triangle were moved by using parallel lines preserving the angles between the edges and the lengths of them. Hence, the lengths of the edges and the measures of the interior angles of the image triangle were equal to the previous triangle.

Instructor: Ok. Is knowing that the triangles have equal length of edges enough to say that they are congruent?

Büşra: Let's think about the triangles of ABC and DEF. We know the lengths of the edges and then we try to compute the measures of the angles of the triangle. In this respect, the known values are written on the cosine formula in order to show the equality of the angle measures opposite of the edges having equal lengths. When the process is repeated for all opposing angles of triangles, the equalities of the measures of corresponding angles are shown. Therefore, it can be claimed that when the lengths of the corresponding edges of the triangles are equal, these triangles are congruent.

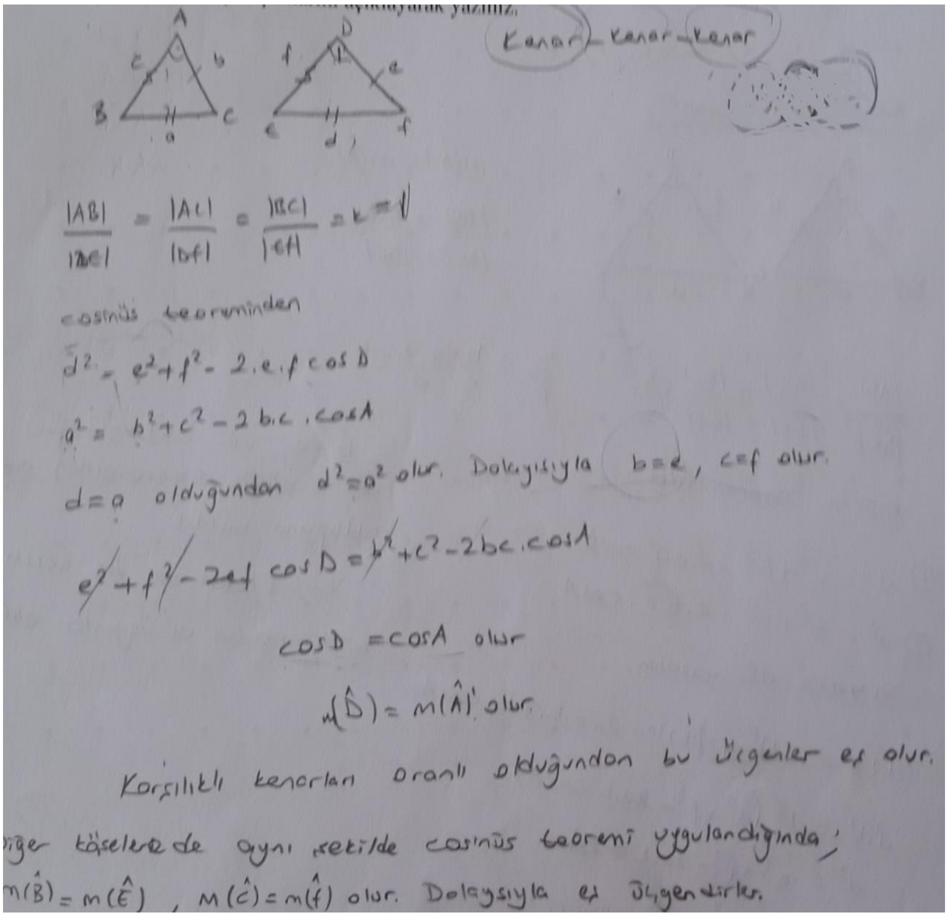


Figure 50 Figure of the congruence and similarity citeria of S.S.S.

In this debate, Selim initially made a claim that the triangle and its image triangle were congruent triangles correctly. Then, he provided data by the definition of translation considering the formation of image triangle preserving the properties of the triangle. He also provided warrant by the process of translation applied on triangle necessarily and appropriately. He constructed the triangle by following geometric construction steps using compass and straight edge. Nuray and Büşra provided sufficient and appropriate backings for this mathematical idea under the guidance of the instructor and the help of their ideas and discussion process. With the help of the instructor's question and guidance, Nuray provided backing benefiting from the definition of translation from a different perspective. She examined the formation of image triangle by using a vector from algebraic view. She insisted on moving the dots forming the triangle by the same vector necessarily. In her explanation, she benefited from the definition of triangles appropriately since by forming the image of the triangle, the vertices were critical since a triangle could be formed accurately by identifying the places of them and combining them with line segments. This process was produced based on the critical attributes of triangles necessitated to form and define them. By the way, this was the other instance that the notion of definition of triangles was observed in a way that it functioned as if shared. In other words, this was another case in which the fisrt mathematical idea in the first mathematical practice became taken-as-shared. Moreover, she benefited from the knowledge that the distance between parallel lines were preserved and the vectors represented the line segments having the same magnitude and direction. Based on the knowledge that the vectors were parallel and any opposing points on parallel lines are equidistant. Then, the instructor directed the discussion to talk about the congruence criteria necessarily since they needed the criteria in order to represent and justify the congruence of the triangles. Afterwards, Büşra made backing by explaining that two triangles were congruent with equal lengths of corresponding edges by the cosine formula since the measures of the corresponding angles were equal by reasoning necessarily and sufficiently. The Toulmin's model of argumentation for some parts of this debate can be represented as shown in Figure 51.

DATA

Selim: Translation is moving a shape…

CLAIM

Selim: We attain a triangle through translation. The triangle and the image triangle are congruent triangles.

WARRANT

Selim: … A triangle is moved through specific way, direction and distance through translation so the image triangle and the triangle are

BACKING

Nuray: A triangle is composed of three non-linear points. While forming the image of triangle, we find the places of three non-linear points as the vertexes by preserving the distances and the directions between

BACKING

Büşra: Let’s think about the triangles of ABC and DEF. We know the lengths of the edges and then we try to compute the angle measures of the angles of the triangle…

Figure 51 Toulmin’s model of argumentation for reasoning on congruent triangles by translation.

On the same week on the advancing hours of the instructional sequence, it was illustrated that the mathematical discussions produced by the participants, knowledge and skills got by them about the formation of congruent triangles during this debate in the fourth week became taken-as-shared. They used this one as data and warrant correctly in their arguments on Week 4 without necessitating backings, confirming that it became taken-as-shared. The participants used this knowledge in the debates in order to decide whether the triangles and their image triangles formed through reflection were congruent. The process followed through the translation was similar to the steps made for reflection appropriately. The similar claim, data, warrant and backings were provided. The claim that the triangles and their image triangles formed through reflection were congruent triangles as a true mathematical explanation. Then, the data was explained by the definition of it. At the end, they provided warrant and backing benefiting from the process of the formation and construction of the images through reflection. Moreover, the mathematical idea that triangles whose lengths of corresponding edges were equal, were congruent triangles was used as data and warrant accurately. Hence, the mathematical idea about the congruence by the equal lengths of corresponding edges became taken-as-shared. The process of formation of congruent and similar triangles through transformation geometry happened effectively under the guidance of the instructor. Moreover, on Week 5, the activities were about congruence and similarity of triangles. The participants discussed the criteria of congruence and similarity. Then, by explaining the criteria of S.S.S., they used this mathematical idea as data and warrant. In other words, they claimed that S.S.S. was a congruence and similarity criterion. They also produced the data about the definition of congruence and similarity correctly. Then, they used the warrant that two triangles were congruent since when the corresponding edges had equal lengths, the measures of the corresponding angles were equal benefiting from this mathematical idea reasoning accurately. In this way, these two triangles fit the definition of congruent triangles. Hence, it became taken-as-shared. Moreover, the activity sheet on Week 6 was composed of the problems on the content of congruent and similar triangles. Therefore, this mathematical idea was used as data and warrant in the solution of these problems. Therefore, this mathematical idea became taken-as-shared again.

The instructor initiated the discussion by asking a question in order to discuss the activities on the second activity sheet of the Week 4 about the rotation as a type of transformation geometry. In this activity sheet, they were asked to construct the image of the triangle through rotating by 45^0 based on the reference point of O using compass and straight edge as in Figure 52 representing the activity sheet on rotation. For this problem, after the participants completed the construction process, the instructor asked them a question to emphasize the congruence of triangles and the discussion was flowed as follows:

ROTATION

Define the rotation.

Construct the image of the triangle ABC by rotating with the angle measure of 45^0 and the reference point and justify this construction mathematically.

Figure 52 The figure of the activity sheet about rotation

Instructor: What is the relationship between the triangle and its image formed through the rotation?

Merve: The geometrical object obtained through rotation is a triangle as the image of a triangle. These triangles are congruent triangles.

Instructor: How do you show that they are congruent triangles?

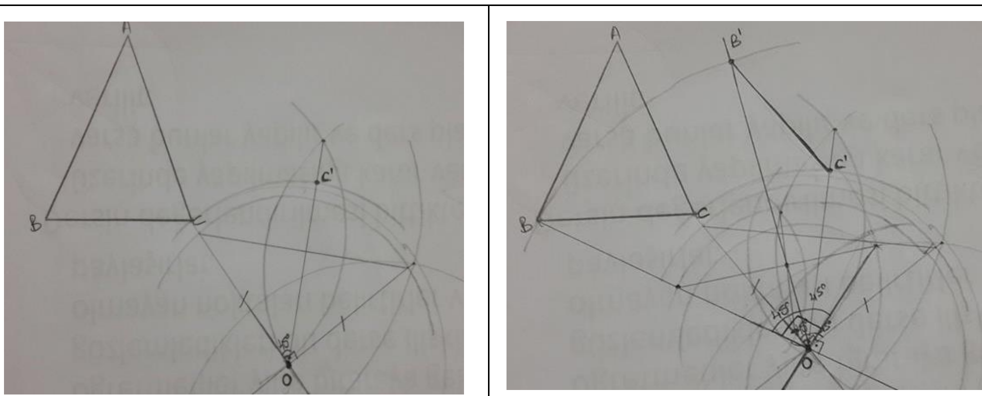
Merve: Rotation is also moving a shape. Every point composing the triangle rotates about a reference point by a given particular angle so that the image triangle and the triangle are congruent. In other words, a triangle is moved on a circular way with respect to the angle so they are congruent.

Instructor: Selim. How did you construct the image triangle?

Selim: We know that a triangle is composed of three non-linear points and also it includes three interior angles. For example, when the angle measure of rotation is 45^0, we form an angle whose rays pass through the vertex of C and the image of this vertex is formed. Then, we repeat this step for the vertex of B and we form the image of the edge of CB. Afterwards, by copying the angle of ABC and determining the place of the image of the vertex of C through construction, we draw the image of the triangle by rotation.

Through the discussion process of formation of congruent triangles by rotation, all of the participants constructed the image triangle by determining the vertices of the triangle after rotating by 45^0 on the reference point and the image triangle was formed by combining these points with line segments. Merve provided claim that congruent triangles were formed through rotation. She also provided data and warrant by using the definition of rotation. She stated that the image triangle was formed moving it through circular way based on the angle measure of 45^0. Selim provided backing for the argumentation benefiting from the construction steps. He determined the places of the vertices by rotating them with this angle measure following construction steps. Then, by combining these vertices using line segments, the triangle congruent to the previous triangle was formed as in Figure 53. In order to determine the image of the vertex, a line was constructed beginning on vertex of C and exceeding the rotation reference point. A line perpendicular to this line passing through the point of reference point was constructed. Then, an isosceles right triangle was constructed as in Figure 53.a. The median of the hypotenuse was constructed and this median was extended. The compass was placed on the reference point and an arc was constructed passing through this vertex and intersecting the median. Hence, the image of the vertex of C was determined. When the similar construction steps were repeated for the vertex of B, the image of this vertex was identified as in Figure 53.b. Then, by combining these image vertices with a line segment, the image of the edge of BC was formed.

Then, the angle of ABC was copied by construction using the edge of BC as one of the rays forming this angle. By using the width of the compass, the length of the edge of AB and the place of the vertex of A were determined. At the end, by combining the images of the vertices of A and C with a line segment, the image triangle was formed.



a. Rotating the vertex of C	b. Rotating the vertex of B and forming the edge of BC

Figure 53 Constructing the image of the edge of BC through rotation

As happened in the translation, the congruence of the triangle and its image triangle could be justified by the congruence criterion of S.S.S. Because this criterion was discussed in translation, different construction strategies that could represent different congruence criteria were examined. With this aim, the instructor wanted Selim to explain his construction process. In this process, he determined the place of the vertex of C through construction as in Figure 53.a. Then, by repeating the similar construction steps, he identified the place of the vertex of B as in Figure 3.1.4.b. By combining these points with a line segment, he formed the edge of BC by reasoning appropriately and sufficiently. Afterwards, by construction, he copied the angle of A having the rays represented by the edges of AB and AC. Through this construction process, the image triangle was constructed accurately and the way to show their congruence could be happened by the congruence criterion of S.A.S. In order to represent and justify this congruence criterion, the instructor directed the discussion to focus on this idea.

Instructor: By following these steps, how can you claim that these triangles are congruent?

Nuray: Let's think about two triangles of ABC and DEF as the image of the prior triangle. When we think about the formation of its image through the steps explained by Selim, we know the lengths of two of the edges and the measure of the angle between these edges since we copy it. If we write these known values on the cosine formula in order to find the length of the remaining edge, we find that the values of these edges for both triangles are same. Therefore, it can be claimed that when the lengths of the corresponding two of the edges of the triangles and the measure of the corresponding angles are equal, these triangles are congruent.

In this debate, Merve initially provided an accurate claim stating that the triangle and its image triangle formed through rotation were congruent triangles accurately. Then, she provided data by the definition of rotation. The warrant was also explained by Selim stating the process of rotation applied on the triangle. One of the edges of triangle was moved through rotation, i.e., moving on a circular way based on a reference point and by particular angle measure. Then, the angle was copied through construction since one of the rays of the angle was drawn and the place of the remaining vertex was determined. Through this process, Selim provided a different construction process and reasoning process for justifying their congruence by a different congruence criterion successfully and necessarily. By completing the formation of the image triangle, the instructor directed the discussion about congruence of the triangle and its image triangle. Nuray explained that the lengths of two of the edges and the measure of the opposing angles between these edges were equal. Then, by using the cosine formula, she showed that the lengths of the opposing remaining edges were equal so the triangles were congruent. Afterwards, under guidance of the instructor, Nuray provided an accurate backing by stating that knowing that the lengths of two edges and the angle measure between these edges were equal was enough to justify the congruence of these triangles. Benefiting from the cosine formula, she showed the equivalence of the lengths of the remaining edges of the triangles sufficiently. In this way, the image was formed by drawing a congruent triangle. The Toulmin's model of argumentation was used in order to represent the structure of the argument including some parts of this debate as shown in Figure 54.

DATA

Merve: Rotation is also moving a shape. Every point composing the triangle rotates about…

CLAIM

Merve: The geometricl object obtained through rotation is a triangle as the image of a triangle. These triangles are congruent triangles.

WARRANT

Selim: We know that a triangle is composed of three non-linear points and also it includes three interior angles. For example, when the angle measure of rotation is 45^0, we form an angle…

BACKING

Nuray: Let's think about the triangles of ABC and DEF as the image of the prior triangle. When we think about the formation of its image through the steps explained by Selçuk, …

Figure 54 Toulmin's model of argumentation for reasoning on congruent triangles by rotation.

On Week 4, it was illustrated that the mathematical arguments produced by the participants, knowledge and skills attained by them about the formation of congruent triangles through rotation during this debate became taken-as-shared. They used this one as data and warrant in their arguments on Week 4 without necessitating backings, confirming that it became taken-as-shared. The participants used the knowledge about the congruence of two triangles when one of them was the image of the other formed by rotation following the steps represented in Figure 53 in the discussions about the congruence criterion of S.A.S. They talked about congruence of these two triangles in order to decide whether knowing that the lengths of two corresponding edges of the triangle and the measures of angle between these edges were equal was enough to claim that they were congruent triangles, i.e., congruence criterion of S.A.S. In the activity on Week 4, there was a problem asking the congruence and similarity criteria. It was claimed that S.A.S. was congruence/similarity criterion and the data was provided by explaining that there were two triangles having the property of the lengths of the corresponding two edges and the measures of the corresponding angles were same. Then, by explaining the criterion of S.A.S., they used this mathematical idea as warrant in a way that it was explained for the formation of congruent triangles through rotation. In other words, they claimed that S.A.S. was a congruence/similarity criterion and produced the data about the definition of congruence/similarity. Then, they used the warrant that two triangles were congruent/similar since when the corresponding known values were equal in measure, the measures of the corresponding elements were equal benefiting from this mathematical idea. In this way, these two triangles fit the definition of congruent triangles. Hence, it became taken-as-shared. Moreover, the activity sheet on Week 5 was composed of the problems on the content of congruent and similar triangles. The criteria of congruence formed through this discussion period was considered in order to solve these problems. Therefore, this mathematical idea was used as data and warrant in the solution of these problems. Therefore, this mathematical idea became taken-as-shared again.

The last activity on Week 4 was about the dilation on triangles. The teacher initiated the discussion about how to produce a bigger or a smaller form of a triangle based on a scale factor through dilation with respect to a reference point. In this discussion, the participants claimed that the triangle and its image triangle formed through dilation were similar triangles. Then, they provided necessary data sufficiently benefiting from the definition of dilation which was resizing the triangle or enlarging or reducing the triangle based on a scale factor focusing on a particular center point. They also used the definition of a triangle as data accurately since they stated that identifying the places of three nonlinear points representing the vertices of the triangle were used to form a triangle by combining them with line segments. As warrant, they explained the process of formation of the image triangle through dilation appropriately. For example, in the process of reducing a triangle with a scale factor of 1/2, three lines were drawn beginning from the reference point and passing through the vertices of the triangle. Through this process, the instructor asked the questions to help them construct its image and understand the process accurately related to the possible position of the triangle and place of it. With the help of this guidance, they stated that because of reducing, the image triangle was formed on the place between the triangle and the reference point. The places of the image of the vertices were the midpoints of these lines passing through the vertices and the reference point. In order to produce this mathematical idea, the instructor helped the participants remember Thales theorem. When the vertices of the triangle were combined with the reference point using line segments, the applicability of this theorem was appeared. As in this theorem, when the triangle of OBC was considered, the image of the edge of BC could be constructed through dilation with the scale factor of ½ as in Figure 55. For the line segment of OB, the compass was placed on the point of O, its width was set to exceed the approximate midpoint of it and an arc was constructed. Then, by preserving the width of the compass, the compass was placed on the point of B and an arc was constructed. The intersection point of these arcs was combined by a line segment passing through the line segment of

OB. The intersection of this constructed line segment on the edge of OB was the midpoint of the line segment of OB. After, repeating the same construction steps for the other vertices of the triangle of ABC and the edges of OA and OC, the image points of A and C were identified. Then, by compounding these midpoints by the line segments, the image triangle similar to the triangle was formed as in Figure 55. As backing, they stated that the corresponding line segments representing the edges of the triangles were parallel necessarily and accurately. Therefore, they found that the measures of the corresponding angles were equal. Also, they stated that the measures of the opposing angles were equal by Thales Theorem.

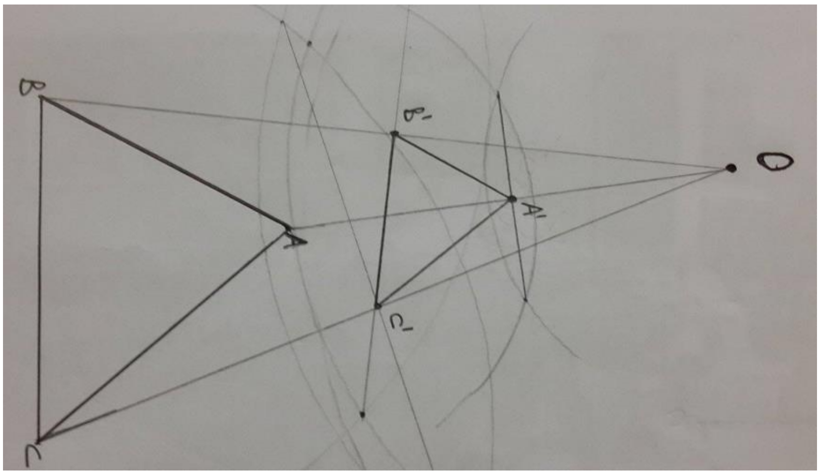


Figure 55 Construction of the image triangle through dilation by reducing.

This mathematical idea became taken-as-shared in the discussion made on Week 4 by reasoning appropriately under the guidance of the instructor. In the problem on the activity sheet on Week 4 about similarity criteria, they claimed that A.A. was a similarity criterion. They used the necessary and accurate information that the corresponding edges of the triangle were parallel as data for the argumentation. Then, they provided the appropriate warrant benefiting from the mathematical idea about formation of similar triangles through dilation. They stated that when the corresponding edges of triangles enlarged and reduced had the scale factor and always had equal angle measures, they were similar triangles. Because of parallel edges, the angle measures were preserved and the lengths of the edges were changed with the same ratio by the Thales theorem. In this respect, it was illustrated that the mathematical arguments formed by the participants and knowledge and skills obtained by them about the formation of similar triangles through rotation during this debate in the third week became taken-as-shared. Moreover, the activity sheet on Week 5 was composed of the problems on the content of congruent and similar triangles. Therefore, this mathematical idea was used as data and warrant in the solution of these problems. Therefore, this mathematical idea became taken-as-shared again. In this way, different instances took place in the instructional process in a way that this mathematical idea became taken-as-shared.

4.3.2 Mathematical idea 2: A.S.S. is not a criterion

The last mathematical idea in the last mathematical practice was observed on the fifth week of the instructional sequence while the participants were engaging in the activities about congruent and similar triangles and congruence and similarity criteria. In this activity, the participants were asked to define the similar and congruent triangles and criteria and the properties about them. While they were engaging in this activity, they made discussions about congruence and similarity criteria. For this activity, they worked with their peers and participated in the whole class discussion. The instructor and the participants made the discussion by asking a question of “A.S.S. is a similarity criterion” in order to discuss the criteria and their reasons on the activity sheets of the Week 5:

Merve: A.S.S. is not a similarity criterion.

Ayşe: Why not! In the rotation, we know when the lengths of two edges and the measure of the angle between them are equal... A.S.S. can be a criterion.

Instructor: What do you think about this explanation?

Kader: When we think about the right triangles and the angle is not the angle measure in the criterion, this is a criterion. Also, this is valid for isosceles triangles.

Ayşe made reasoning based on the lengths of two edges and the angle measure of one of the interior angles were equal but she dismissed the point that the order of these elements having equal measures was important and necessary for using as criterion of A.S.S. The instructor asked the question to help the participants determine this missing point. Then, Kader explained the cases that A.S.S. was used to identify congruence or similarity. This explanation was good but the similarity of right or isosceles triangles since by knowing two sides' lengths and the angle measure of one of interior angles, the other elements of these triangles could be determined so that their congruence/similarity could be determined by other related criteria appropriately.

Buse: This is special situations so we cannot generalize this for all triangles.

Instructor: Well. Think about scalene triangles.

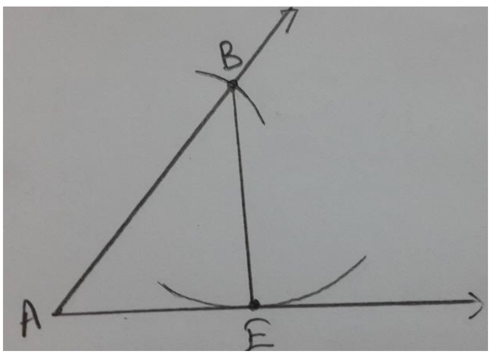
İlkay: We have information about two edges and the angle which is not between these edges. This case is very different from the criterion of S.A.S. When we determine corresponding edges and angles, there is uncertainty about corresponding ones. We cannot be certain about which edge is near to which triangle.

Instructor: That is a good point.

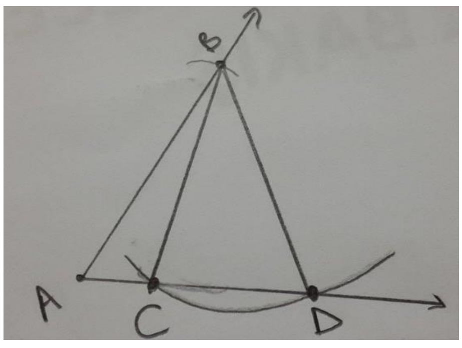
Ayşe: However, when we put the known values for the lengths of two edges and the angle measure on the cosine formula, we can provide equivalence of the edges and the angle measures.

Selim: We cannot use this formula effectively since we do not know which edge is opposite of which angle and we have two unknown angle measures... Firstly, we construct the angle. Then, we place one of the edges on one of the rays. Afterwards, we form a circle having the radius in the length of the other edge. This circle can intersect the other ray on two points or one point or no point. In this process, the edges are placed on the rays randomly and there are alternative situations for the vertices of the triangle. Hence, we do not have a particular triangle and we do not claim that A.S.S. is not similarity criterion.

At this episode of the discussion, Buse made an explanation about the non-applicability of A.S.S. for all situations. Then, by agreeing with the explanation of Buse, İlkay stated that this case could not be accepted as similar to the criterion of S.A.S. In his explanation, Selim focused on the construction process of types of triangles by knowing the lengths of two edges and angle measure of one of interior angles which was not between these edges. In this process, he constructed the angle of A. Then, he constructed an arc with the center of A and the radius having the equal length of the edge of AB. Afterwards, in order to construct the other edge, he constructed an arc with center of the point of B and the radius having the equal length of this edge. This reasoning way provided successful explanation for the problem. He focused on two cases which were the tangent and intersecting the arc at two points when the arc represented one of the known edges as in Figure 56. Based on this knowledge, he examined these cases. In the case of tangent, it was possible that the triangle was formed and the criterion of A.S.S. could be used as in Figure 56.a. In this case, a right triangle was formed. In a right triangle, A.S.S. could be used since the measures of all interior angles could be computed. An angle measure was 90^0 and the measure of one of the acute angle was known so that the remaining angle measure could be computed. Moreover, when the lengths of two edges were known, it was possible to compute the length of the remaining edge by Pythagorean Theorem. Hence, it could be stated that A.S.S. was valid for right triangles since the unknown measures of some elements in these triangles could be computed by some properties and theorems about these triangles. This was not sufficient to extend the applicability of A.S.S. for all triangles. On the other hand, in the case of intersecting at two points, two triangles of ABD and ABC were formed by having the equal measures for the elements of this criterion and the A.S.S. could be used as in Figure 56.b. It could be stated that although it was appeared that A.S.S. could be applied for right triangles, the actual reason was the explained properties and theorems of right triangles. In this case, obtuse and acute triangles were formed. We could obtain different triangles by having this property so we could not use A.S.S. for acute and obtuse triangles. Therefore, the idea that A.S.S. was not congruence/similarity criterion was justified necessarily and accurately since there were different triangles having the equal measures stated in this criterion.



a. Triangle with tangent point



Triangle with intersection points

Figure 56 The formation of a triangle by knowing the lengths of two edges and one angle measure.

In this debate, Merve initially made a claim that A.S.S. was not similarity criterion. Then, Ayşe refuted the claim by explaining the opposite idea of this explanation. Kader also exemplified her refutation with the help of the isosceles and right triangles and Ayşe tried to use the cosine formula for the truth of her explanation. Afterwards, İlkay stated that there were uncertain elements and their orders in this criterion as warrant. Selim encouraged him by telling formation of a triangle based on the elements on this criterion. Because, the uncertainty about the placement of the elements and their intersection of the edges and formation of the vertices so that he provided backing. The Toulmin's model of argumentation for some parts of this debate can be represented as shown in Figure 57.

DATA

İlkay: We have information about two edges and the angle which is not between these edges. This case is very different from the criterion of S.A.S.

CLAIM

Merve: A.S.S. is not a similarity criterion.

WARRANT

İlkay: …When we determine corresponding edges and angles, there is uncertainty about corresponding ones. We cannot be certain about which edge is near to which triangle.

BACKING

Selim: We cannot use this formula effectively since we do not know which edge is opposite of which angle and we have two unknown angle measures…

Figure 57 Toulmin's model of argumentation for reasoning on A.S.S.

On Week 5, it was illustrated that the mathematical arguments formed by the participants and knowledge and skills obtained by them about the similarity criterion of A.S.S. during this debate became taken-as-shared. They used this one as data in their arguments on Week 5 without necessitating backings, confirming that it became taken-as-shared. There was a problem "when the lengths of one of the right edges and the hypotenuse of a right triangle are equal, they are always similar triangles". They made the claim that these right triangles were always similar. While discussing this problem, they benefited from the mathematical idea that A.S.S. was not a similarity criterion as data and the discussion process about this mathematical idea was established for the first time. Also, the data were explained in a way that the lengths of two edges which were one of right edges and the hypotenuse, and right angle were known. Therefore, the lengths of two edges and the measure of an angle which was not between these edges were known was explained. However, it was possible to compute the measure of remaining angle by the property that the sum of the interior angle measures of a triangle was equal to 180^0 and the length of remaining edge by Pythagorean Theorem. As a warrant, they stated that this problem fit the idea of A.S.S. but it was not similarity criterion. Therefore, the statement in the problem could not be justified by it. By computing the values of the unknown elements of the triangle, their similarity could be explained by other similarity criterion. For example, Halit explained that A.A.A. criterion could be used by showing that the measures of the remaining interior angles were equal. Also, Selim added that S.S.S. criterion could be used by showing that the lengths of the remaining edges were equal by using Pythagorean Theorem. Therefore, the claim was verified successfully and appropriately. In this respect, the mathematical idea about A.S.S. became taken-as-shared by using it as data in the discussion under the guidance of the instructor.

4.4 Summary of the Findings

With the design experiment research, a beneficial lesson sequence by the hypothetical learning trajectory designed about triangles based on problem-based learning for preservice middle school mathematics teachers was performed. Through the analysis of PMSMT's classroom mathematical discourses taking place in this instructional sequence in order to illustrate their geometrical understanding and reasoning of triangles, the classroom mathematical practices were identified by taken-as-shared ways of reasoning and arguing mathematically. By using Toulmin's (1969) model of argumentation and two-criterion methodology of Rasmussen and Stephan (2008), classroom mathematical practices were emerged by examining classroom collective learning activities leading whole class discussions. Through the whole class discussions, three mathematical practices were identified in the instructional sequence about triangles. The first mathematical practice was reasoning about the formation of a triangle. There existed two mathematical ideas contributed to this mathematical practice; (a) reasoning on the definition of triangles and classification of them, and (b) reasoning on construction of them. The second mathematical practice was established as the reasoning about the elements of triangles and their properties. Mathematical ideas contributed to this practice were: (a) reasoning on construction of auxiliary elements, (b) reasoning on concurrence of them and (c) reasoning on the names of these concurrence points and their places. The last mathematical practice established in the sequence was reasoning about congruence and similarity. The mathematical ideas included in this practice were (a) reasoning on the formation of congruent or similar triangles through transformation geometry, and (b) A.S.S. is not a congruence/similarity criterion. Moreover, the instructional sequence including construction activities with compass and straight edge improved the PMSMT's van Hiele geometric thinking levels and conceptual knowledge of triangles.

CHAPTER 5

5. DISCUSSION AND CONCLUSION

The overall purpose of the current study was to document the learning of triangle formed in a classroom community and shared development of triangle concepts in an elementary education mathematics classroom. In other words, the goal was to determine the mathematical practices emerging in the collective discourses and documenting the situations in which they developed and became taken-as-shared. By doing so, the study has represented a window into the classroom designed based on the van Hiele geometric thinking and problem solving skills in problem-based learning in order to enhance learning and understanding of the content of triangles. In the study, with the aim of examination of the emergence of mathematical practices, a classroom teaching experiment was conducted letting the participants of the classroom community to study mathematics learning in a classroom setting designed by problem-based learning (Cobb, 2000).

5.1 Discussion of Hypothetical Learning Trajectory

The preservice middle school mathematics teachers' (PMSMT) understanding and development of subject matter knowledge about triangles through argumentations were examined in the problem-based learning environment including geometric constructions in the present study. The argumentations improved their conceptual knowledge of triangles. For example, while defining triangles, they produced triangle definitions without all necessary critical attributes and properties. However, through argumentation, they challenged the definitions formed by them, determined the missing and unrelated parts and then they produced correct definition including critical attributes and properties necessarily and sufficiently. When the PMSMT's learning of triangles through mathematical practices was considered, it was observed that discussion period including argumentations improved their geometric thinking and knowledge of triangles in the study. The previous research validated this finding as in the study of Olkun and Toluk (2004) who found that in-class discussions improved the learners' geometric thinking. Also, research in the literature illustrated that discussions including argumentations taking place in problem solving activities facilitated and improved problem solving abilities, scientific thinking by criticizing and justifying claims, knowledge production and conceptual understanding (Abi-El-Mona & Abd-El-Khalick, 2011; Duschl & Osborne, 2002; Jim´enez-Aleixandre et al., 2000; Jonassen & Kim, 2010; Osborne, Erduran, & Simon, 2004; Zembaul-Saul, 2005). In this respect, argumentations facilitates doing mathematics and discussing claims in a social environment in which the learners communicate and make reasoning to form the discourse, imagery/tools, and classroom culture (Abi-El-Mona & Abd-El-Khalick, 2011). In this respect, the argumentations in problem solving activities also encouraged the role of instructor, instructional sequence and HLT.

In order to provide a social learning environment for PMSMT to develop their subject matter knowledge of triangles though argumentations, problem-based learning was used to design this environment and instructional sequence. In this respect, geometric constructions were used in the study since they represented useful problem situations in the study. The geometric constructions are solutions of a problem because the learners do not decide easily how to start constructing the shapes at first glance and then they have challenge to complete the constructions (Erduran & Yeşildere, 2010). In this respect, geometric constructions represented problems for PMSMT in the problem and they had challenge to solve these problems. These activities applied in problem-based learning environment improved the participants' geometric thinking and knowledge. The previous research in the literature has confirmed that by stating that problem-based learning increases geometric thinking and knowledge (Cantürk-Günhan & Başer, 2009; Dochy et al., 2003; Hodges, 2010). Also, these problems were followed by argumentations in order to help the PMSMT understand the triangles effectively in the study. Through following four construction steps of Smart (1998), PMSMT analyzed, constructed, proved and discussed their thoughts and knowledge about the problem. By following these steps, geometric constructions facilitated thinking skills of analyzing, evaluation, forming hypothesis, organizing, testing the hypothesis and proving the solutions benefiting from the previous knowledge (Lim-Teo, 1997). These scientific skills were encouraged by argumentations and justifications in whole class discussions to learn triangles effectively since geometric constructions were beneficial in argumentations and proofs related to scientific thinking skills. Hence, geometry concepts could be learned through geometric constructions with argumentation and justification (Wiley & Voss, 1999) so that the skills of critiquing the ideas and claims, evaluating evidences and justifications, explaining and evaluating counter positions/examples could be improved (Asterhan & Schwarz, 2007; Sadler & Fowler, 2006; von Aufschnaiter, Erduran, Osborne, & Simon, 2008). Moreover, Erduran and Yeşildere (2010) stated that the learners can follow the construction steps in a rote manner. In order to prevent this case in the study, argumentations and proofs were used since they reasoned and discussed each steps through argumentations in the study. Hence, the tools of constructions were placed in the instructional sequence in the planned way under the guidance of the instructor appropriately. The tools of compass and straight edge were used to teach triangles by supporting the claim about the construction of triangles and justifying its truth to encourage learning effectively. For example, while defining the triangles, they examined the critical attributes and the relationship between them by compass and straight edge. They made explanations about definition of triangles, they tested their explanations through constructions and argumentations and then produced correct definition by making revisions, refuting and convincing their ideas. In this process, geometric constructions facilitated argumentations, learning and justification of the PMSMT. In this respect, it can be stated that geometric constructions should be used, discussed and argued in order to learn geometric concepts consciously by improving scientific thinking skills (Spear-Swerling, 2006).

The connections of geometric constructions and argumentations with proving geometrical explanations were observed in the study. Through following construction steps and discussing their solutions in the classroom, they provided explanations in order to verify their reasoning and solutions. They made these explanations in order to remove others' doubts about the process and examine its correctness. In this respect, it can be stated that this skill is related to proving based on its definition (Hanna, 1989). In this study, the PMSMT were expected to attain the properties of initial three van Hiele geometric thinking levels and obtain previous knowledge about geometry. In this respect, while validating their ideas, they could be expected to provide proofs for the geometric constructions and then they were able to provide geometrical proofs. For example, after showing the concurrence of auxiliary elements of triangles by geometric construction, they proved their concurrence using related theorems and properties and how these concurrence points were named. In this process, they used geometric constructions to represent the concurrence of the auxiliary elements and prove their concurrency. By doing so, they understood this concept by reasoning, proving and constructing. In this process, proof is beneficial for constructions since it does not only indicate accuracy or inaccuracy of a statement but also illustrate why it is correct (Hanna, 2000). Moreover, conceptual understanding and geometric reasoning of the PMSMT could be improved by geometrical proofs and constructions (Cheung, 2011; Napitupulu, 200; Tapan & Arslan, 2009). Geometrical constructions are beneficial to teach geometrical concepts such as triangles and facilitate constructing and writing proofs.

The development of PMSMT's understanding and reasoning of triangles was explored in the present study. It was observed that the participants had missing knowledge about triangles although they were expected to know in order to become a mathematics teacher. For example, the participants did not have sufficient knowledge about definitions of triangles, justifying the concurrence of auxiliary elements, proving congruence and similarity of triangles and their criteria. Also, their learning about these missing knowledge was provided through geometric constructions and argumentations. The period of acquiring the subject matter knowledge about triangles was represented by the classroom mathematical practices. Their development of subject matter knowledge about triangles was encouraged by geometric constructions with compass and straight edge. There have been research in the literature explaining that geometric constructions improve the learners' conceptual knowledge and understanding (Cheung, 2011; Çiftçi and Tatar, 2014; Doğan & İçel, 2011; Karakuş, 2014; Napitupulu, 2001). In the process of learning triangles, they examined triangles based on their elements, properties and theorems about them in problem-based learning environment in the study. For example, they examined the definition of triangles. In this process, they used geometric constructions and other geometric shapes such as circles and quadrilaterals in order to determine the critical properties of triangles and the connection between them. Through following construction steps, the necessity and importance of the properties were understood. The critical properties of triangles for definitions of triangles were examined as identified in previous research in the literature (Tsamir, Tirosh & Levenson, 2008; Tsamir et al., 2014). By participating in whole class discussions and argumentations, they acquired necessary knowledge and modified their errors about triangles. Then, in different contexts in other activities in the following phases of HLT, they used what they previously learned in later parts of the instructional sequence.

Therefore, geometric constructions are useful to learn triangles by examining the elements, properties and theorems about them and proving the theorems related to them. By following construction steps, the learners can analyze them and make connection between them by reasoning. Hence, it can be concluded that learning triangles can be performed effectively by geometric constructions.

In order to examine the process of the understanding of PMSMT on triangles, HLT was designed. The resulting HLT included instructional tasks, tools, and possible discourse as the support of the classroom formed by the research team. In order to form an effective mathematical discourse including the argumentation, the instructor focused on and initiated this process by misconceptions, errors and different strategies in instructional sequence. When the instructor realized the emergence of them, she asked questions and provided necessary clues to help the participants become aware of them and make accumulations on their ideas and expressions. By doing so, they formed new mathematical knowledge correctly by accumulating their previous knowledge with the help of the others' ideas expressed in the discussion under the guidance of the instructor. Through instruction, the instructor focused on the knowledge of the students. Based on the knowledge, errors and misconceptions of them about the concept, the discussion flowed. For example, the mathematical practice about the definition of triangles was produced in this way. While the PMSMT were classifying the triangles, the instructor realized that they had difficulty in defining the triangles. Then, the instructor initiate the discussion about definition of triangles and forming them by geometric constructions. In this respect, it is important for the instructors to determine the errors of the students to form an effective social learning environment (Gökkurt, Şahin, Soylu & Doğan, 2013; Gökkurt, 2014). By benefiting from their errors, their errors can be removed by reasoning and accurate knowledge can be formed. Moreover, explaining and representing different solution strategies for the problems can improve their understanding and learning. Also, they can examine the concepts from different points of views.

By determining the knowledge and errors of the PMSMT, it was important for the instructor to have through understanding about "their students' current mathematical conceptual possibilities and constraint and the relevant underlying mathematical concepts" (Yackel, 2002, p. 439). Therefore, the instructor did not only identify the misconceptions, errors and different strategies but also examined how and why they were formed in the present study. In this respect, she carefully focused on them by using her conceptual knowledge and the knowledge about the participants on triangles. In this way, she directed the argumentations while reaching accurate conclusion and improving sufficient understanding through the emergence of mathematical practices. It was observed that identification of knowledge of content and students was important to provide their learning and geometric understanding. One of the dimensions of MKT, knowledge of content and students focuses on the subject matter knowledge of mathematics through knowledge about the learners' thinking (Ball, Bass & Hill, 2004; Ball, Hill & Bass, 2005; Hill, Schilling & Ball, 2004). This dimension is critical for teachers to be possessed and teach effectively.

Through the instructional sequence directed based on the hypothetical learning trajectory, the PMSMT improved their knowledge and understanding about triangles through geometric constructions. When the participation structure and flow of the discussions in the classrooms were considered, their knowledge and errors about the concepts were determined and they obtained correct knowledge through instructional sequence. This process was also encouraged by geometric constructions with compass and straight edge. Through constructions, they also improved their knowledge and understanding benefiting from other geometrical concepts. In this respect, it could be stated that they improved their relational understanding. For example, in the construction steps, a geometric shape was constructed by using simple structures such as constructing the perpendicular bisector of a line, parallel line to a line and benefiting from other geometric shapes and their properties such as arcs, circles, quadrilaterals. The emergence of the mathematical practices illustrated this process in detail. In this respect, the quantitative findings showed that PMSMT improved their subject matter knowledge about triangles through instructional sequence providing useful opportunities to teach triangles in the future (Ball & Forzani, 2009; Chapman, 2007; Turner et al., 2012). Also, it can be stated that geometric constructions improved PMSMT's understanding of triangles as it was suggested by previous research (Cherowitzo, 2006; Cheung, 2011; Çiftçi and Tatar, 2014; Erduran & Yeşildere, 2010; Karakuş, 2014; Khoh, 1997; Napitpulu, 2001; Hoffer, 1981).

Based on the previous research, inservice and preservice elementary school mathematics teachers are expected to at least attain the first three van Hiele geometric thinking levels (Aydin & Halat, 2009; Pandiscio & Knight, 2010). Based on this view, the hypothetical learning trajectory and mathematical tasks were designed in the present study. It was aimed to help PMSMT attain the properties of initial three van Hiele geometric thinking levels completely and begin to acquire the properties of remaining levels. It was observed that their geometric thinking was improved by geometric constructions with compass and straight edge since they engaged in the tasks successfully and solved the problems appropriately. The related research in the literature have validated that the geometric constructions improve the learners' van Hiele geometric thinking levels (De Villiers, 2003; Güven, 2006; Napitupulu, 2001). In this respect, it could be stated that the PMSMT acquired the properties of initial three levels of van Hiele geometric thinking through engaging in these tasks. By examining the constructions of the geometric shapes and reasoning about proving that the constructed shapes was the shape asked to be solved in the problem, they improved their geometric thinking levels. That is, they reasoned about the geometric shapes in the problems and their properties effectively to construct and prove their truth so they improved their geometric thinking levels. By constructing the geometric shapes by compass and ruler, the learners progressed step by step. In each step, they made geometric reasoning by using simple and complex geometrical structures such as finding the midpoint of a line segment, constructing perpendicular lines (Djoric & Janicic, 2004). In this respect, the geometric constructions are useful to be used in teaching geometry concepts effectively so that their geometric thinking levels can be improved.

According to the findings of the study and the flow of the ideas in the classroom, the PMSMT improved their conceptual understanding and knowledge about triangles. The tasks and instructional sequence encouraged their subject matter knowledge about the particular concept of triangles. Also, argumentations and geometric constructions facilitated this process and understanding of them. A solution can be provided for the problem that the preservice mathematics have little knowledge and experience of geometry about the particular concept of triangles by the present study.

5.2 Discussion of Mathematical Practices

With the aim of describing PMSMT's understanding and development about the concept of triangles, both qualitative and quantitative analysis were conducted in the study. The qualitative part of it was closely linked with constant-comparative method (Glaser & Strauss, 1967), Toulmin's (1969) method of argumentation and three-phase and two-criterion methodology of Rasmussen and Stephan (2008) in order to document taken-as-shared classroom mathematical practices emerged through collective discourses in six-week instruction sequence by HLT on the concept of triangles. In other words, Toulmin's (1969) model of argumentation was crucial as a methodological tool to identify the instances that the mathematical practices emerged becoming self-evident and then re-emerged in way of functioning if-shared based on the methodology of Rasmussen and Stephan (2008). All of them can be provided by the analysis of the six-week instructional sequence by the technique of

Glaser and Strauss (1967). The methodology and analysis techniques were useful to examine the preservice mathematics teachers' understanding, reasoning and knowledge as suggested in some previous research (Akyuz, 2014; Roy, 2008; Stephan & Rasmussen, 2002; Wheeldon, 2008). Through the qualitative analysis, classroom mathematical practices becoming taken-as-shared were PMSMT's reasoning on: (a) the formation of a triangle, (b) the auxiliary elements of triangles and their properties, and (c) congruence and similarity.

The first mathematical practice, PMSMT's reasoning on the formation of a triangle addresses the underlying concepts about the critical elements of triangles as main elements and the process in which these elements produce a triangle. The mathematical ideas related to this practice are reasoning on: (a) the definition of triangles and classification of them, and (b) construction of them. The learning goal for Stage One of the HLT was classifying triangles based on the definitions and main elements of them and the examining the possibility of construction of triangles about the situations including the groups of some of the elements of triangles. This stage was intended to provide background to the participants about triangles through a general perspective. The first mathematical idea about the definition and the classification of triangles were examined through the definition of triangles and right triangles by main elements of them and the regions formed by them in a plane. While this idea emerged on the first week of the instructional sequence, it became taken-as-shared by being used in the activities on all weeks including the first week since this idea was the main knowledge about this concept. By the emergence of mathematical idea, the process of PMSMT's understanding about the definitions of triangles was observed. In this phase, PMSMT examined the critical attributes of triangles and the relationship between them. Through construction, this examination was performed effectively by compass and straight edge. For example, they determined the main elements of edges and angles by providing other critical attributes such as closeness. They identified the places of three non-linear points in a plane and combined them by line segments through geometric construction. Moreover, this definition formation process became more effective when they were encouraged by argumentations. By discussing their ideas and construction strategies, they talked about the related and unrelated properties of triangles in order to form the correct definition of triangles. An important knowledge and skill of formation of definitions was attained by geometric constructions and argumentations focusing on the attributes of triangles as suggested in previous research in the literature (Leiken & Zazkis, 2010; de Villiers, Govender, & Patterson, 2009; Tsamir, Tirosh, Levenson, Barkai & Tabach, 2014). In this respect, geometric constructions are useful to examine the necessity and sufficiency of the critical properties of triangles. They facilitate formation of definitions by understanding through mathematical argumentations. In other words, mathematical argumentations improved their knowledge about the definitions of triangles. The other mathematical idea in this practice is about the construction of triangles by some of their elements. For this idea, the construction steps and processes as the main tools and also the main theorems such as Pythagorean were important. While this idea emerged on the first week, it did not become taken-as-shared until the advancing hours of the first week and the fifth week of the instructional sequence. These mathematical ideas were used by the participants in order to support their reasoning and explanations in different situations and activities such as congruence and similarity, auxiliary elements of triangles by using constructions, geometrical justifications and proofs. Moreover, the possibility of formation of triangles by knowing some of their elements was examined by geometric constructions. For example, they examined the possibility of construction of particular triangles by making connections between main and auxiliary elements of triangles by geometric constructions. Also, in the process of construction and justifications, they benefited from other theorems and rules about triangles and other geometric shapes. In other words, they focused on the relationship between the elements of triangles and other geometric shapes so that the relational understanding was provided by geometric constructions. This case can be confirmed by the previous research (Erduran & Yeşildere, 2010; Karakuş, 2014; Khoh, 1997; Kuzle, 2013). Therefore, the critical attributes of triangles and the relationship between the critical attributes, main and auxiliary elements and other geometric shapes of triangles are useful to obtain basic knowledge about triangles such definition of triangles. In this respect, these knowledge are necessary to learn other knowledge about triangles. Hence, they can be learned through geometric constructions and argumentations. In this way, relational understanding about triangles can be supported.

The second mathematical practice, PMSMT's reasoning on the elements of triangles is important for the formation of auxiliary elements and their critical importance of them for triangles. The mathematical ideas related to this practice are reasoning on: (a) construction of auxiliary elements, (b) concurrence of them and (c) the names of these concurrent points and their places. The learning goal for Stage Two of the HLT was examining the auxiliary elements of triangles. This stage provided background to the participants about how to form them and what happened in case of the formation of all elements. Through the second phase of HLT, they obtained beneficial knowledge about these mathematical ideas. These knowledge are necessary for other geometric concepts needed for preservice mathematics teachers. For example, they were fundamental for understanding main theorems related to the concurrence points of auxiliary elements and other important geometry concepts such Euler line and nine-point circle. Euler line is a line on the triangle including the orthocenter, the circumcenter, the centroid and the center of the nine-point circle of a triangle. These concepts are important to prove important theorems in geometry and in the courses of advanced geometry and to obtain deep understanding about it. In this way, geometric constructions were useful to explore the auxiliary elements and their properties and prove related theorems benefiting from argumentations. They provided opportunities to examine the geometric knowledge and prove them as it has been suggested in previous research (Chan, 2006; Napitupulu, 2001; Tapan & Arslan, 2009). Also, through PMSMT's understanding about these three mathematical ideas in the second mathematical practice, they benefited from the geometric constructions by compass and straight edge, justifications and geometrical proofs as stated in previous research so that they could form the expected subject matter knowledge about auxiliary elements of triangles effectively (Axler & Ribet, 2005; Cherowitzo, 2006; Clements & Battista, 1992; Doğan & İçel, 2011; Erduran & Yeşildere, 2010; Martin & McCrone, 2003; Smart, 1998). In this respect, necessary knowledge about the auxiliary elements of triangles can be learned by geometric constructions and argumentations.

It was observed that PMSMT had difficulty in constructing the altitudes of different types of triangles and determining and justifying the concurrence of the altitudes on the triangles as orthocenter in the study. In the study, in order to provide them to construct the altitudes, the perpendicularity of right triangles and the knowledge that the altitude separated a triangle into two right triangles were used. When they made connection with right triangles, they could easily constructed and defined the altitude of triangles. Moreover, while determining the altitudes and the places concurrence points of the altitudes as orthocenter for right and obtuse triangles, they had difficulty. Benefiting from definition of right triangles and altitudes, PMSMT was provided to determine the altitudes and orthocenter of a right triangles. Through argumentations and justifications, they found and understood their places. Moreover, through geometric constructions benefiting from right triangles, constructing perpendicular bisectors and definition of altitudes, they were supported to construct the altitudes for obtuse triangles. They constructed and justified their places by argumentations. In this respect, although altitudes are crucial to learn because of its critical connection between other concepts such as trigonometry and other theorems such as Pythagorean Theorem, it is difficult to understand and learn (Alatorre & Saiz, 2010; Gutierrez & Jaime, 1999; Kellogg, 2010). In this respect, altitudes can be taught through geometric constructions by examining the places of altitudes and orthocenter points for different types of triangles by producing argumentations and proofs. Through geometric constructions, they can analyze the properties and definitions of altitudes and orthocenter and understand them in a way that they argue, analyze, represent and refute/justifying their explanations to reach the correct and necessary explanation. In this respect, it can be stated that the difficulty of the students from different grade levels about the altitudes of triangles can be removed by geometric constructions and argumentations.

In the second and third weeks of the instructional sequence, it was observed that geometric constructions were useful to convince others by justifying their explanations and producing the proofs. The PMSMT examined the concurrence of medians and naming them as centroid, angle bisectors naming as incenter, perpendicular bisectors as circumcenter and altitudes as orthocenter and the places of the concurrence points of them for different types of triangles. In order to show that all of these auxiliary elements concurred at a point, they constructed all of these elements and then proved their concurrence. When they were asked whether all of these elements concurred at a point or not, they were answered correctly but they were not able to explain why they were concurrent. However, in the process of representing and examining their concurrence by geometric constructions, they understood how they were concurrent. Also, they provided proofs about concurrence of auxiliary elements of triangles. Then, they were asked whether the places of these concurrence points change for different types of triangles or not and why they changed or did not change, they sometimes provided incorrect answers except for medians. In order to determine whether these concurrence points changed for different types of triangles, they constructed these elements for obtuse, right and acute angled triangles. By doing so, they determined their places correctly and proved their explanations and strategies. In this respect, geometric constructions are useful to justify the truth of geometrical explanations and providing proof for them. In this respect, proof and geometric constructions are related and improved similar skills needed for learning geometry (Chan, 2006; De Villiers, 2003; Napitupulu, 2001).

In the second phase of the HLT in the study, it was observed that the usage of examples and non-examples together was useful to teach the concept. The participants learned the cevian by examining examples and non-examples in the problem about proving the concurrence of auxiliary elements of triangles. Except for the perpendicular bisector, concurrence of other auxiliary elements could be proved through the theorems of Ceva and Menelaous. This exception produced an environment including the discussion of this exception leading the understanding of cevian. The participants were engaged in examples and non-examples of cevian in the instructional sequence. By determining the related and unrelated properties of the cevian, they formed the accurate and sufficient definition of the cevian. The examples representing the attributes used for defining the concept were medians, angle bisectors and altitudes while the non-example referring to attributes unrelated to the definition of the concept was perpendicular bisector. Özyürek (1984) and Senemoğlu (1997) suggest that the examples and non-examples should be used together while teaching the concept. Learning in this way encourages that the students identify the particular attributes and critical properties of the concept and then form a generalization about examples. This process is important and useful since the students can understand the particular properties of the concept represented by the examples and determine the difference between examples and non-examples based on critical properties. By doing so, the students can form correct definition of the concept which is accurate and sufficient.

The last mathematical practice, PMSMT's reasoning on congruence and similarity of triangles as one of the concept about triangles has critical importance. The mathematical ideas related to this practice are: (a) reasoning on the formation of congruent or similar triangles through transformation geometry, and (b) A.S.S. is not a congruence/similarity criterion. The learning goal for Stage Three of the HLT was examining the triangles through transformation geometry. This stage was intended to provide background to the participants about how to form the image triangle through the types of transformation geometry and the relationship between the triangles and their images as in a way suggested by French (2004). Through the examination of them, the beginning for the understanding the concept of congruence and similarity of triangles was aimed and performed. While these ideas emerged on the fourth week of the instructional sequence, they became taken-as-shared by being used in the activities on following weeks. Moreover, the first mathematical practice about the definition and formation of triangles was used by the participants to support their reasoning on the similarity and congruence of triangles and the congruence/similarity criteria for them. The other mathematical idea in this practice was proposed and discussed by the participants under the guidance of the instructor using the first mathematical idea and the related congruence/similarity criteria. The participants attained knowledge about congruence and similarity of triangles by geometric constructions and transformation geometry. By geometric constructions, PMSMT determined the places of the vertices of the image triangles and these vertices were combined by line segment. Then, they compared the triangles and image triangles based on main and auxiliary elements of triangles. Also, benefiting from the definitions of transformation geometry and geometric constructions, they proved their congruence/similarity. For example, in translation, they moved the vertices of the triangle by the same vector by geometric constructions. Then, they claimed that by combining these vertices by line segments, a congruent triangle was formed since the distance between parallel lines were preserved and the vectors represented the line segments having the same magnitude and direction. Moreover, they justified this conclusion by geometric constructions. Through conducting transformation geometry by geometric constructions, they formed congruent and similar triangles. In addition to geometric constructions, transformation geometry facilitated understanding and learning congruency and similarity of triangles

(French, 2004; Gerretson, 1998; Paquette, 1971; Park City Math Institute [PCMI], 2010). They also proved this process with the help of geometric constructions. Through the emergence of this mathematical practice about PMSMT's understanding of similarity and congruence of triangles, they benefited from geometric constructions, transformation geometry by motion-based reasoning, justifications and proofs. The research in the literature emphasizes the effects of transformation geometry and geometric constructions about congruence and similarity of triangles (Finzer & Bennet, 1995; PCMI, 2010; Yanık, 2013). The students can learn congruent triangles based on the knowledge that rigid motions preserve all of their properties except for their orientation. Also, dilation is useful to form similar triangles and understand them since their image triangles are formed proportionally.

The current study aims the examination of mathematical practices in collective learning environment designed by problem-based learning for a classroom community with the help of HLT formed about triangles and documents these mathematical practices. It can also be stated that the PMSMT could think and reason effectively through the instructional sequence and the tasks in the HLT on the concept of triangles. The findings of qualitative and quantitative data analysis processes showed that the reasoning of the participants on geometry about the concept of triangles could be improved by geometric constructions. Three mathematical practices emerged in the present study through collective learning environment can be beneficial for other researchers studying about teaching triangles or related topics in a similar setting.

5.3 Implications and Recommendations

The present study was designed in order to make contribution to the research base about preservice middle school mathematics teachers' (PMSMT) development of profound understanding of triangles. In the study, the learning of PMSMT encouraged by the HLT was documented in the study. This process can be used in teacher education programs to teach triangles. Their thinking and errors identified in the study can be considered while making instructions about triangles for preservice mathematics teachers. Also, the difficulty and the ways of removing this difficulty documented in the study can be considered in the instructions about triangles.

Their understanding was examined in a collective learning environment designed by a hypothetical learning trajectory (HLT) and then applied in a six-week instructional sequence about the concept of triangles. With the help of design-based research methodology used in this classroom teaching experiment, the main purpose was to identify PMSMT's understanding of triangles encouraged by geometric constructions from a social perspective. This identification process was performed by determining classroom mathematical practices. The effects of particular instructional activities, actual HLT obtained through making necessary revisions by the pilot study and the instructor's guidance were explained in the process of emergence of mathematical practices. By generalizing the HLT, the understanding of PMSMT in different environments and cultures can be improved on the concept of triangles. By benefiting from them and making necessary revisions on the HLT, instructors and researchers interested in PMSMT's understanding of triangles can design their classroom environments and improve their participants' understanding of it.

Geometric constructions are useful to improve students' geometrical thinking and form geometrical justifications and proofs (Chan, 2006; Cheung, 2011; De Villiers, 2003; Napitupulu, 2001). Learning in this way can be encouraged effectively by argumentations. By following construction steps, the students can examine the geometrical objects, their properties and their connections with other geometrical shapes providing relational understanding (Erduran & Yeşildere, 2010; Karakuş, 2014; Khoh, 1997; Kuzle, 2013). In the present study, the students understood and learned the triangles by making connections with other geometrical concepts such as quadrilaterals and circles through geometrical constructions, proofs and argumentations. In this respect, the process of learning other geometric shapes such as quadrilaterals and circles can be provided by geometric constructions, argumentations and proofs. When the positive effects of geometric constructions on understanding and learning of PMSMT are considered, these tools are useful to be used in geometry lessons in mathematics teacher education programs. Hence, the geometry courses can include the opportunities for preservice mathematics teachers to explore the geometric concepts by geometric constructions.

When the argumentations formed in the whole class discussion and the tasks encouraging the emergence of the mathematical ideas are examined, the instructor have important role. The instructor visited the small groups, determined the mathematical misconceptions, errors and different solution strategies for the problem and then guided the discussion using them in the study. In this respect, the knowledge of the instructor and the participants are important in understanding and learning of the concepts (Gökkurt, Şahin, Soylu & Doğan, 2013; Gökkurt, 2014; Yackel, 2002). Moreover, errors and difficulties of PMSMT about triangles such as concurrence of auxiliary elements, definitions of triangles and altitudes of triangles determined in the study can be used in different research about triangles. Moreover, these errors and misconceptions about triangles should be considered while teaching triangles by removing them. Also, the PMSMT had more difficulty in altitudes when compared with other auxiliary elements (Alatorre & Saiz, 2010; Gutierrez & Jaime, 1999). Moreover, altitudes can be examined in detail by considering the errors and misconceptions about it identified in the study. Also, whole class discussion was initiated and flowed under the guidance of the instructor based on the ideas of PMSMT in the classroom. Therefore, the study can be repeated by a different instructor and different group of PMSMT. By making comparisons between the mathematical practices formed by different group of classroom community and the instructor, generalized and detailed knowledge can be obtained about learning and understanding of PMSMT about triangles in an environment designed by problem-based learning.

Analysis of the present study was performed by using mathematical practice dimension of social perspective of the interpretative framework. The other dimensions of this perspective which are social and sociomathematical norms should also be examined in a similar study (Andreasan, 2006; Roy, 2008; Wheeldon, 2008). In this way, the nature and structure of the collective learning environment providing the emergence of mathematical practices can be explained and analyzed to have a complete picture of the PMSMT's learning and understanding of triangles in the instructional sequence. The research in the literature examining social dimension of the framework established the social and sociomathematical norms generally in an environment designed by Realistic Mathematics Education (RME). By the present study, different norms can be established for a problem-based learning environment different from the most of design-based research in the literature. Moreover, interpretative framework also has psychological perspective and there have been research to examine this dimension of the framework (Stephan, Bowers, Cobb, & Gravemeijer, 2003). This perspective should be investigated in a study similar to the present one. Generally, this dimension was explored in a RME learning environment. Differently, this dimension can be explored in a problem-based learning environment as it has been used in the present study. Moreover, also by connecting this perspective with the social one in a different study, individual PMSMT's understanding and learning of triangles can be examined encouraged by instructional sequence and the HLT. By conducting this kind of study including all perspectives and dimensions of interpretative framework, the suggestion of "the results from the analyses should feedback to improve the instructional designs" (p. 11) made by Cobb (2003) can be performed. Moreover, making comparisons between the findings of the research obtained from different learning environments designed by different strategies, detail information and the effects of the strategies can be acquired.

In the study, PMSMT engaged in the activities about geometric constructions by compass and straight edge. They examined the triangles through constructing them as explained in the problem situation. While constructing the shapes, they made drawing errors. Although they followed true construction steps, they made drawing errors such as not fixing the compass truly and strictly on a point. Hence, some of the participants could not form clear drawings for the shapes. For example, some of them could not show the auxiliary elements concurred at a point on their drawings because of this kind of drawing errors. Furthermore, the participants spent much time and effort of the participants in complex geometric constructions. For example, while examining the possibility of formation of triangles by knowing the measures of some of their main and auxiliary elements, they made constructions including many steps and formed complex drawings. The technological tools can be more useful while constructing complex drawings in short time period by spending less effort. Moreover, drawing errors made by following true construction steps by compass and straight edge can be removed by technological tools so that completely true drawing can be produced effectively. Çiftçi and Tatar (2014) state that there is not statistically significant difference in geometry achievement of the groups taught by compass and straight edge and technological tools. However, dynamic geometry environment facilitates complex constructions in short time period while the compass and straight edge could provide learning permanently. In this respect, basic geometric constructions can be used to teach the geometrical concepts such as line, angle, different kinds of triangles by compass and straight edge but the concepts necessitating complex constructions such as triangles with their auxiliary elements can be taught in technological learning environments. The future research about understanding and learning geometrical concepts through constructions can be designed in this way.

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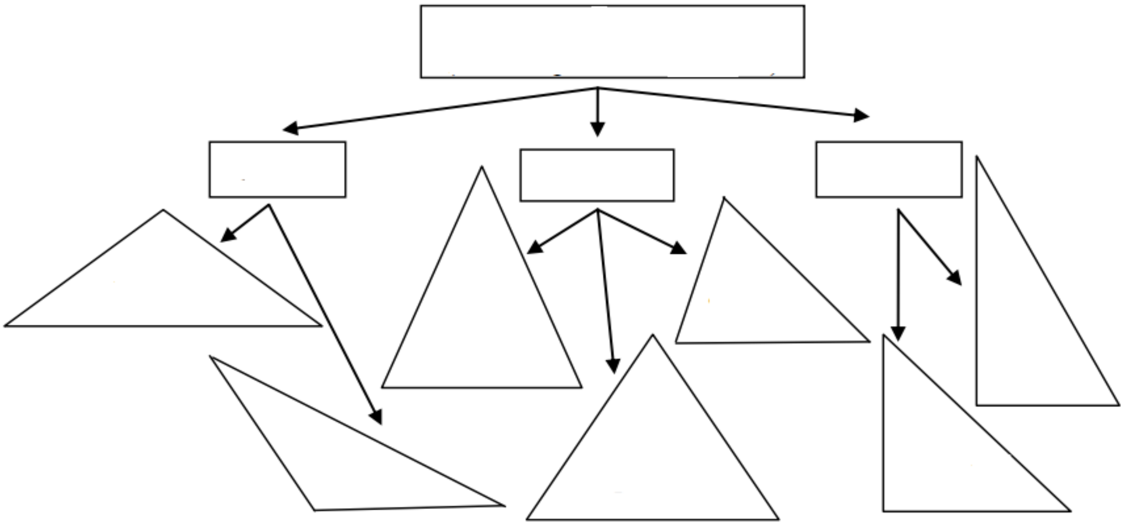
APPENDICES

Appendix A: Activities

1. ÜÇGENLERİN SINIFLANDIRILMASI

1. Aşağıdaki kavramları tanımlayarak bu kavramlardan uygun olanları diyagramdaki uygun yerlere yerleştiriniz.
   Üçgen, ikizkenar üçgen, eşkenar üçgen, çeşitkenar üçgen, dik üçgen, dar açılı üçgen, geniş açılı üçgen.



2. Soruda üçgen çeşitlerini kullanarak yaptığınız sınıflandırmayı açıklayınız. Bu sınıflandırmayı yaparken üçgenlerin hangi özellik ve elemanlarını göz önünde bulundurduğunuzu ve bunların sınıflandırmada nasıl kullanıldığını belirtiniz.

3. Üçgen çeşitlerini sınıflayabileceğiniz farklı bir yol varsa çizerek gösteriniz. Bu sınıflamada göz önünde bulundurduğunuz üçgenlerin hangi özellik veya elemanlardır? Bunlar nasıl kullanılmıştır? Açıklayınız.

2. ÜÇGENLERİN OLUŞTURULMASI

1. Üçgenlerin temel ve yan elemanları nelerdir? Açıklayınız.

2. Bir ABC üçgeninin m(BAC) = 90^0, h_a ve b elemanları bilindiğine göre bu üçgenin çiziminin mümkün olup olmadığını belirtiniz. Bu durumu açıklayınız (şahin, 2013).

3. ABC dik üçgeninde m(BAC)= 90^0, h_a ve V_a elemanları bilindiğine göre bu üçgenin çiziminin mümkün olup olmadığını belirtiniz. Bu durumu açıklayınız (Şahin, 2013).

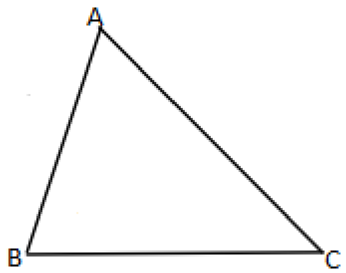
4. Bir ABC üçgeninin m(BAC) = 90^0, h_a ve a elemanları bilindiğine göre bu üçgenin çiziminin mümkün olup olmadığını belirtiniz. Bu durumu açıklayınız (Şahin, 2013).

5. Üçgenlerin sahip olduğu elemanlardan en az hangileri bilindiğinde bir üçgenin çiziminin mümkün olduğu veya belirli bir üçgenin oluşturulabildiği söylenebilir? Bu elemanlarla oluşturulabilecek grupların üçgen çizimini nasıl mümkün kıldığını açıklayınız.

3. ÜÇGENLERDE KENARORTAY

1. Kenarortay nedir? Tanımlayınız.

2. Aşağıda verilen ABC üçgeninin BC kenarına ait olan kenarortayı pergel ve ölçüsüz cetvel kullanarak çiziniz ve doğruluğunu gösteriniz.



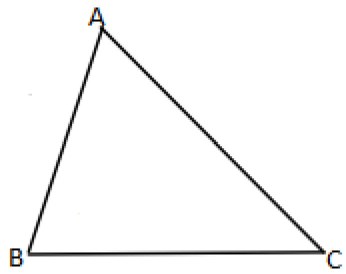
3. Bir ABC üçgeninin bütün kenarlarına ait kenarortayların kaç noktada kesiştiğini gösteriniz. Bu kesişim noktasının/noktalarının genel adı nedir?

4. Dik, dar veya geniş açılı üçgenlerin kenarortaylarının kesim noktasının/noktalarının yerini tahmin ediniz. Bu noktanın/noktaların yerinin bu üçgen çeşitleri için değişip değişmediğini gösteriniz.

4. ÜÇGENLERDE AÇIORTAY

1. Açıortay nedir? Tanımlayınız.

2. Aşağıda verilen ABC üçgeninin ABC açısının açıortayını pergel ve ölçüsüz cetvel kullanarak çiziniz ve doğruluğunu gösteriniz.

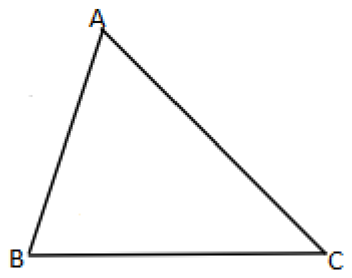


3. Bir ABC üçgeninin bütün açılarına ait açıortayların kaç noktada kesiştiğini gösteriniz. Bu kesişim noktasının/noktalarının genel adı nedir?

4. Dik, dar veya geniş açılı üçgenlerde açıların açıortaylarının kesim noktasının/noktalarının yerini tahmin ediniz. Bu noktanın/noktaların yerinin bu üçgen çeşitleri için değişip değişmediğini gösteriniz.

5. ÜÇGENLERDE ORTA DİKME

1. Pergel ve üzerinde ölçüm işaretleri olmayan cetvel kullanarak aşağıda verilen ABC üçgenini A noktası A^I noktasına gelecek şekilde kopyalayarak çiziniz. Kopyalanan üçgeni A'B'C' üçgeni olarak ifade ettiğimizde, bu üçgenin ABC üçgeninin kopyası olduğunu gösteriniz.



· A^I

2. Orta dikme nedir? Tanımlayınız. 1. Soruda verilen ABC üçgeninin BC kenarının orta dikmesini pergel ve ölçüsüz cetvel kullanarak çiziniz ve doğruluğunu gösteriniz.

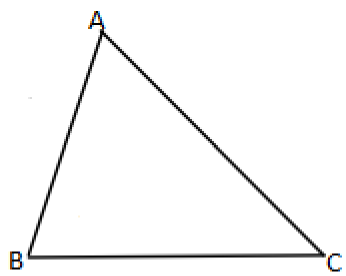
3. Bir ABC üçgeninin bütün kenarlara ait orta dikmelerinin kaç noktada kesiştiğini gösteriniz. Bu kesişim noktasının/noktalarının genel adı nedir?

4. Dik üçgende, dar açılı üçgende ve geniş açılı üçgende kenarlara ait orta dikmelerin kesişim noktasının/noktalarının yerini tahmin ediniz. Bu noktanın/noktaların yerlerinin bu üçgen çeşitleri için değişip değişmediğini gösteriniz.

6. ÜÇGENLERDE YÜKSEKLİK

1. Yükseklik nedir? Tanımlayınız.

2. Aşağıda verilen ABC üçgeninde A noktasından BC kenarına indirilen yüksekliği pergel ve ölçüsüz cetvel kullanarak çiziniz ve doğruluğunu gösteriniz.



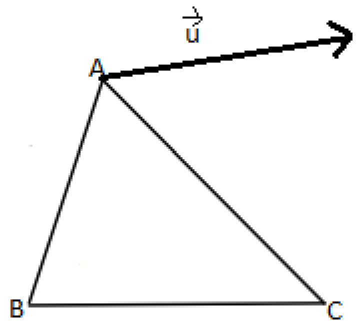
3. Bir ABC üçgeninin bütün kenarlarına ait yüksekliklerin kaç noktada kesiştiğini gösteriniz. Bu kesişim noktasının/noktalarının genel adı nedir?

4. Dik, dar veya geniş açılı üçgenlerde yüksekliklerin kesim noktasının/noktalarının yerini tahmin ediniz. Bu noktanın / noktaların yerlerinin bu üçgen çeşitleri için değişip değişmediğini gösteriniz.

7. ÖTELEME

1. Öteleme nedir? Tanımlayınız.

2. Aşağıda verilen ABC üçgeninin verilen u vektörünü kullanarak öteledikten sonra oluşan görüntüsünü pergel ve ölçüsüz cetvel kullanarak çiziniz ve doğruluğunu gösteriniz.

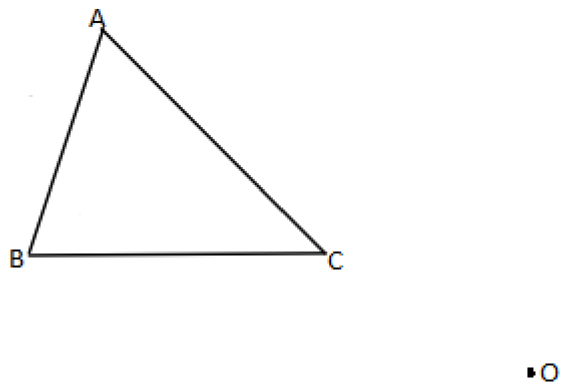


3. Analitik düzlemde köşeleri A(0, 3), B(2, 5) ve C(2, -3) olan ABC üçgeninin 2 cm sola ve 1 cm yukarıya ötelendikten sonraki görüntüsünü çiziniz.

8. DÖNDÜRME

1. Döndürme nedir? Tanımlayınız.

2. Aşağıda verilen ABC üçgeninin belirtilen O noktası etrafında 45^0 döndürülmesiyle elde edilen görüntüsünü pergel ve ölçüsüz cetvel kullanarak çiziniz ve doğruluğunu gösteriniz.

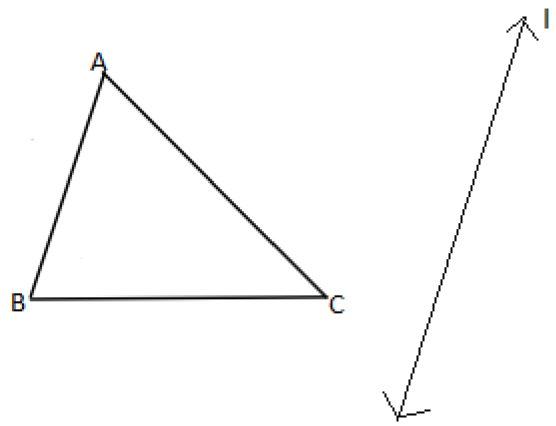


3. Analitik düzlemde köşeleri A(0, 3), B(2, 5) ve C(2, -3) olan ABC üçgeninin orjin etrafında 60^0 döndürülmesiyle elde edilen görüntüsünü çiziniz.

9. YANSITMA

1. Yansıtma nedir? Tanımlayınız.

2. Aşağıda verilen ABC üçgeninin *l* doğrusunda yansımasının yapıldıktan sonra elde edilen görüntüsünü pergel ve ölçüsüz cetvel kullanarak çiziniz ve doğruluğunu gösteriniz.

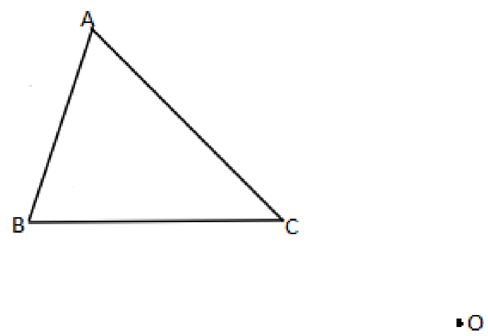


3. Analitik düzlemde köşeleri A(0, 3), B(2, 5) ve C(4, 2) olan ABC üçgeninin x + y = 0 denklemli doğrusu kullanılarak yansıması yapıldıktan sonra elde edilen görüntüsünü çiziniz.

10. BÜYÜTME/KÜÇÜLTME

1. Büyütme/küçültme nedir? Tanımlayınız.

2. Aşağıda verilen ABC üçgeninin O noktası kullanılarak 2 oranında büyütülmüş modelini pergel ve ölçüsüz cetvel kullanarak çiziniz ve doğruluğunu gösteriniz.



3. 2. Soruda verilen ABC üçgenini ve O noktasını kullanarak, bu şeklin O noktasına göre ½ oranında küçültülmüş modelini pergel ve ölçüsüz cetvel kullanarak çiziniz ve doğruluğunu gösteriniz.

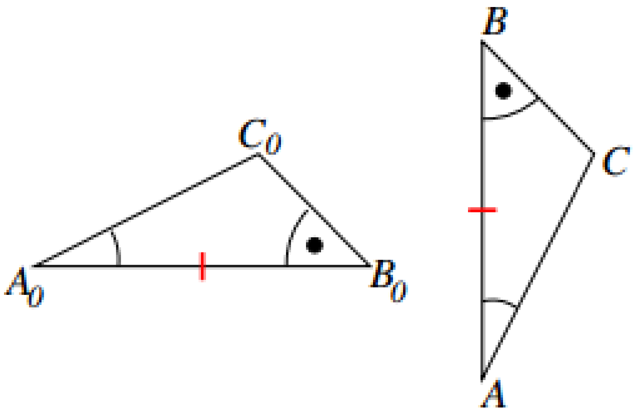
4. Aşağıda verilen tabloyu, bir ABC üçgeninin ve onun tabloda belirtilen dönüşüm çeşidini kullanarak elde edilen görüntüsü arasındaki ilişkiyi, değişen ve değişmeyen özellik ve elemanlarını göz önünde bulundurarak nedenleriyle açıklayınız.

Dönüşüm Çeşidi	Değişmeyen özellik	Değişen özellik	Geometrisel bağlam
Öteleme			
Döndürme			
Yansıtma			
Büyütme/Küçültme			

11. ÜÇGENLERDE EŞLİK

1. Bir üçgene art arda öteleme, döndürme ve yansıtma yaptığımızda veya bunlardan birinin yapılma sayısı sınırlı sayıda arttırıldığında üçgen ile bu üçgenin görüntüsü arasında nasıl bir ilişki vardır?

2. Aşağıdaki şekilde ABC üçgenine birtakım dönüşüm işlemleri uygulanarak $A_0B_0C_0$ üçgeni elde edilmiştir. Hangi dönüşüm işlemlerinin yapıldığını açıklayarak çizimlerle gösteriniz.



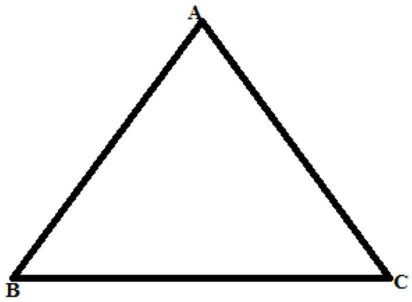
3. 3. soruda verilen ABC ve $A_0B_0C_0$ üçgenlerinin eş olduğunu gösterirken, üçgenin hangi elemanları kullanılmıştır? Bu elemanların eş olması üçgenlerin eş olduğu sonucuna varmak için yeterli midir? Eğer yeterliyse, bu eşlik çeşidi nasıl adlandırılabilir?

4. Bildiğiniz eşlik şartlarını açıklayarak yazınız.

12. ÜÇGENLERDE BENZERLİK

1. Bir üçgen ve onun büyütme/küçültme sonucu oluşan modelleri arasında nasıl bir ilişki vardır? Açıklayınız.

2. Aşağıdaki şekilde belirtilen ABC üçgenini kullanarak, bu üçgenle yeni oluşturulan üçgenin benzerlik oranını 3 olacak şekilde oluşturulan $A_0B_0C_0$ üçgenini pergel ve ölçüsüz cetvel kullanarak çiziniz ve doğruluğunu gösteriniz.



3. Herhangi iki üçgenin benzer olduğunu nasıl gösterirsiniz? Açıklayınız.

4. Aşağıda yer alan ifadelerin doğru veya yanlış olduklarını belirterek nedenlerini açıklayınız.

	Doğru (D)	Yanlış (Y)	Açıklama
İki üçgen ancak ve ancak biri diğerinin büyütme/küçültme sonucu oluşturulan modeli olduğunda benzerdir.			
İç açılarının ölçüleri (30^0-60^0-90^0) olan üçgenler bazen benzerdir.			
İç açılarının ölçüleri (45^0-45^0-90^0) olan üçgenler her zaman benzerdir.			
Kenar uzunlukları (7k-24k-25k, k€R) olacak şekilde oluşturulan üçgenler her zaman benzerdir.			
AKK bir eşlik/benzerlik şartıdır.			
AA benzerlik şartında üçüncü açının kullanılması gereksizdir.			
4 cm ve 6 cm kenar uzunluklarına sahip ve bu kenarlar arasında kalan açının 45^0 olduğu üçgenle; 2cm ve 3 cm kenar uzunluklarına sahip ve bu kenarlar arasında kalan açının 45^0 olduğu üçgen benzerdir.			
Eş üçgenler aynı zamanda benzer			

üçgenlerdir.			
Eşkenar üçgenler her zaman benzerdir.			
İkizkenar üçgenler her zaman benzerdir.			
Dik üçgenler bazen benzerdir.			
İkizkenar dik üçgenler her zaman benzerdir.			
Benzerlik oranı 4 olan benzer iki üçgenin, yükseklikleri oranı da 4’tür.			
Bir üçgenin ağırlık merkezi sabit kalacak şekilde büyütüldüğünde/küçültüldüğünd e diklik merkezi, iç teğet ve dış teğet çemberlerinin merkezleri asla değişmez.			
Eş üçgenlerin iç teğet çemberlerinin yarıçaplarının uzunluklarının ölçüsü her zaman aynıdır.			
Eş üçgenlerin dış teğet çemberlerinin yarıçaplarının uzunluklarının ölçüsü ve bu merkez noktalarının kenarlara olan uzaklıkları bazen aynıdır.			
Bir üçgen onun yansıma işlemi sonucu oluşan görüntüsü bazen			

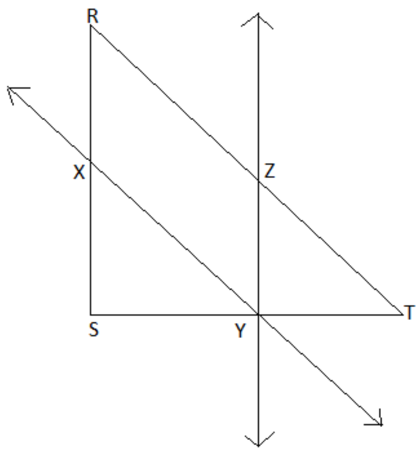
benzerdir.			
İki dik üçgenin hipotenüs ve hipotenüse indirilen yüksekliklerinin uzunlukları aynı olduğunda, bu üçgenler her zaman benzerdir.			
İki dik üçgenin, iç teğet çemberlerinin yarıçap uzunlukları ve hipotenüse indirilen yüksekliklerin uzunluklarının ölçüleri aynı olduğunda, bu üçgenler bazen eştir.			
İki ikiz kenar üçgenin eşit uzunluktaki kenarları arasındaki açıların ölçüleri eş olduğunda, bu üçgenler her zaman benzerdir.			
İki dik üçgenin dar açılarından birin açısının ölçüsü eş olduğunda, bu üçgenler her zaman benzerdir.			
Benzer üçgenlerin, açıortaylarının ve kenarortaylarının uzunluklarının oranı her zaman benzerlik oranına eşittir.			
İki dik üçgenin dik kenarlarından birinin ve hipotenüslerinin			

uzunlukları eşit olduğunda, her zaman eş üçgenler olurlar.			
İki ikizkenar üçgenin, eş açılarından birinin olduğu köşede kesişen iki kenarının uzunlukları eşit olduğunda, bu üçgenler benzerdir.			

13. PROBLEMLER

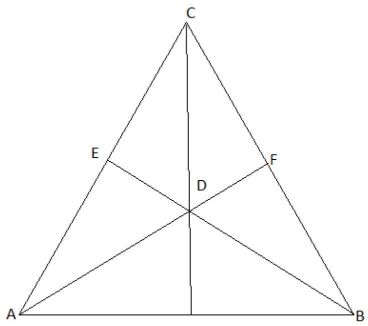
1. Bir ABC üçgeninde, AD uzunluğu A açısının açıortayı ve D noktası BC kenarı üzerinde olsun. D noktasından AB kenarına paralel olan ve AC kenarını E noktasında kesen DE doğru parçası oluşturulsun. E noktasından da BC kenarına paralel olan ve AB kenarını F noktasında kesen EF doğru parçası oluşturulsun. Bu durumda, AE ve BF kenarlarının eş olduğunu gösteriniz (Şahin, 2013).

2. RST üçgeninde, XY doğrusu ile RT doğru parçası ve YZ doğrusuyla RS doğru parçası paralel olduğuna göre; (RX/XS) = (ZT/RZ) olduğunu gösteriniz.



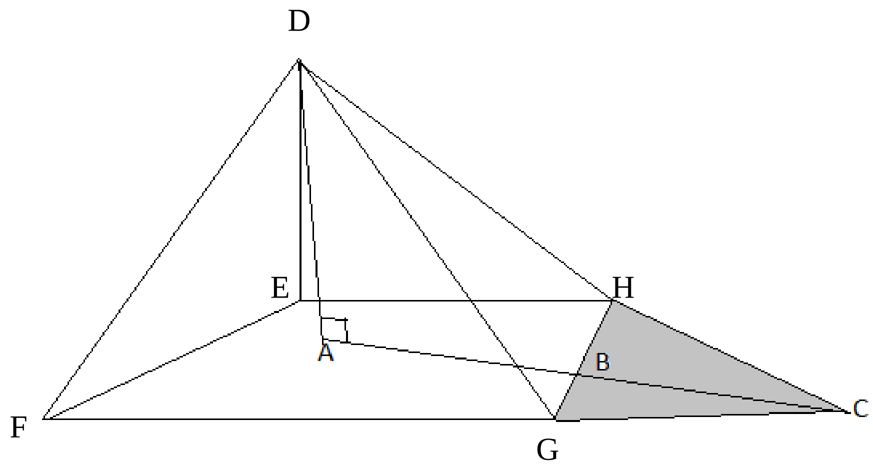
3. Aşağıda verilen ABC üçgeninde, doğru parçaları arasındaki oranlar çarpımının 1 e eşit olduğunu gösteriniz.

$$\frac{AG}{GB}.\frac{BF}{FC}.\frac{CE}{EA}=\frac{AE}{EC}.\frac{CF}{FB}.\frac{BG}{GA}=\frac{CF}{FB}.\frac{BG}{GA}.\frac{AE}{EC}=1$$

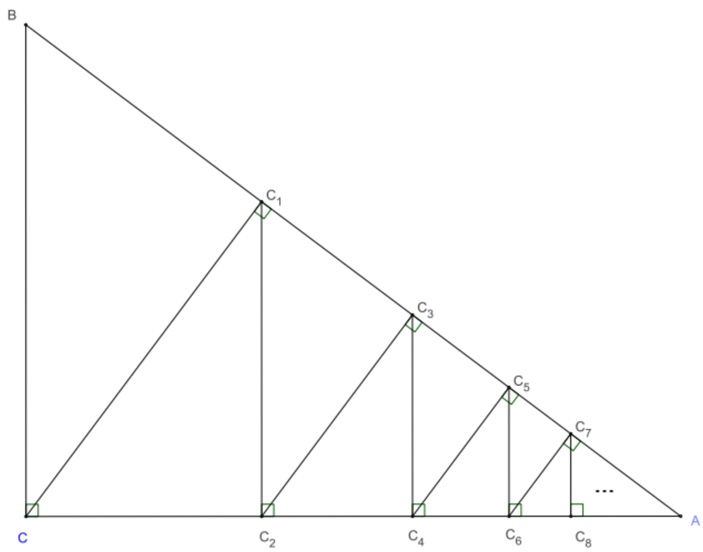


4. 3. soruda verilen eşitliğin, herhangi bir üçgenin açı ortaylarının, kenar ortaylarının ve yüksekliklerinin kesişim noktaları için de doğru olup olmadığını gösteriniz.

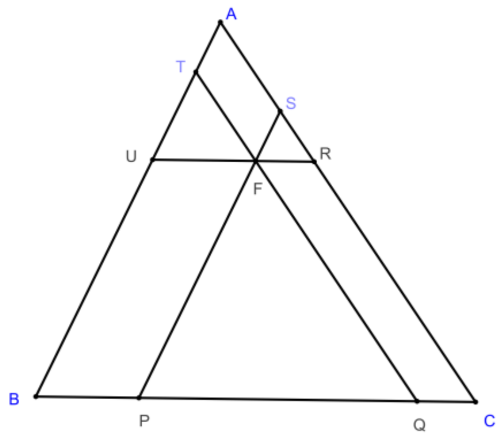
5. Aşağıdaki şekilde verilen DEFGH kare piramidin tabanın bir kenarı 23 m dir ve güneşli bir günün belirli bir saatindeki gölgesi de gösterildiği gibidir. Gölgenin tepe noktasının piramide olan uzaklığı yani BC doğru parçasının uzunluğu 10,5 m'dir. Ayrıca, 6cm boyunda olan bir kibrit de yere dikey olacak şekilde yerleştirildiğinde 9cm'lik gölge oluşturduğu gözlemlenmiştir. Bu bilgilere göre, piramidin tepe noktasından tabanın merkez noktasına indirilen dik yüksekliğin uzunluğunu bulunuz.



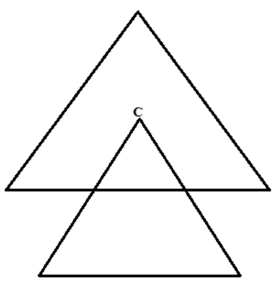
6. Aşağıdaki ABC dik üçgeninde, IBCI = 3cm, ICAI = 4 cm ve IABI = 5 cm'dir. $|CC_1| + |C_1C_2| + |C_2C_3| + \ldots = ?$



7. Aşağıdaki ABC üçgeninde, SP, TQ UR doğru parçaları F noktasından geçmektedir. AB // SP, AC // TQ ve BC // UR olduğuna göre, $\frac{PQ}{BC} + \frac{RS}{CA} + \frac{TU}{AB} = 1$ ifadesinin doğru olduğunu gösteriniz.



8. Aşağıdaki şekilde gösterildiği gibi iki eşkenar üçgen bulunmaktadır. Bu iki üçgen birbiri üzerine çakışık durumdadır v kenar uzunlukları n birimdir. C noktası üçgenlerden birinin tepe noktasıyken diğerinin ağırlık merkezidir. Tepe noktası C olan eşkenar üçgen diğerinin üzerinde döndürülebilmektedir. Bu durumda üst üste gelen taralı alanın alabileceği en büyük değer nedir?



Appendix B: Turkish Summary

SOSYAL BİR ORTAMDA MATEMATİKSEL UYGULAMALARIN GELİŞTİRİLMESİ: ORTAOKUL MATEMATİK ÖĞRETMENİ ADAYLARININ ÜÇGENLERİ ÖĞRENMELERİNİ SAĞLAYAN BİR ÖĞRETİM DİZİSİ

Giriş

Geometriyi öğrenmek çok önemli olmasına rağmen öğrencilerin geometri bilgi ve başarılarının istenilen seviyede olmadığı görülmektedir. Bunun nedenlerinin başında öğrencilere geometrinin kavramsal öğretilmesinden ziyade ezberleyerek ve işlemsel bilgi odaklı öğretilmesidir (Fuys, Geddes, & Tischler, 1988; NCTM, 2000; Young, 1925). Öğretmenlerin özellikle de ortaokul matematik öğretmenlerinin geometriyi bu şekilde öğretmelerinin nedeni geometrisel kavramlarla ilgili bilgi ve tecrübelerinin yetersiz olmasıdır (Clements, 2003; Fuys, Geddes, & Tischler, 1988; Hershkowitz, Bruckheimer, & Vinner, 1987; Stipek, 1998). Ayrıca, etkili matematik öğretiminin gerçekleştirilebilmesi için gerekli öğrenme ortamları da bilgili öğretmenler tarafından hazırlanabilir (Putnam, Heaton, Prawat & Remillard, 1992; Van der Sandt & Nieuwoudt, 2003). Bu açıdan, bilgili öğretmenlerin sahip olması gereken bilgiler literatürde araştırmacılar tarafından farklı şekilde açıklanmıştır (Ball, Sleep, Boerst, & Bass, 2009; Ma, 1999; Shulman, 1986). Bunlardan en yaygın kullanılanlarından biri de öğretim için matematiksel bilgi adı altında önemli matematiksel bilgi ve temel matematik öğretimi bilgilerinin açıklandığı yöntemdir (Hill, Ball & Schilling, 2008). Bu sınıflama yönteminde araştırmacılar, matematiği bilmenin ve öğretmenin ancak gerekli matematiksel bilgiye sahip olarak gerçekleştirilebileceğini belirtmişlerdir. Bu nedenle, etkili geometri öğretiminin geometri bilgisi iyi olan öğretmenler tarafından yapılabileceği söylenebilir çünkü öğretmenlerin konu alan bilgileri onların sınıflarında yaptıkları seçimleri ve sergiledikleri performansları etkilemektedir (Ball, Thames & Phelps, 2008). Bu yüzden, sınıflarda etkili geometri öğretiminin gerçekleşebilmesi için, öğretmen adaylarına eğitim gördükleri süreçte gerekli geometrik bilgiyi edinmelerini sağlayacak olanak sunulmalıdır (NCTM, 2006; Chapman, 2007).

Tasarım-tabanlı araştırma öğretmen adaylarının matematik öğretimi için gerekli bilgi çeşitlerinden biri olan konu alan bilgilerini geliştirmek için çeşitli olanakların sunulduğu faydalı bir yöntem olabilir. Bu yolla, öğretmen yetiştirme programlarında, eğitmenler öğretmen adayları için öğretim süreçleri planlayıp sınıflarında uygulayarak onların gelişimlerini sağlayabilirler. Bu açıdan, öğretim sürecinde kullanılacak geometrik kavramların, etkinliklerin ve materyallerin öğretmen adaylarının geometrik anlama ve düşünmelerini geliştirecek şekilde seçilip uygulanması gerekmektedir (Han, 2007; Henningsen & Stein, 1997). Geometrik şekillerin pergel ve çizgeç kullanılarak inşa edilmesi öğretmen adaylarının gerekli geometrik bilgiyi kazanmaları ve geometrik düşünmelerinin gelişimini sağlayabilir. Öğretmen adayları pergel ve çizgeç kullanarak geometrik teoremleri, kuralları ve konuları inceleyebilir ve anlayabilirler (Erduran & Yeşildere, 2010; Napitupulu, 2001; Hoffer, 1981). Ayrıca, şekillerin geometrik inşası pergel ve çizgeç kullanarak geometrik problemlerin çözümü için standart olmayan bir çözüm yolu sunmaktadır. Öğretmen adayları bu materyallerle geometrik konuların öğrenilmesine ek olarak gerekli fiziksel becerilerin kazanılmasını da sağlamaktadır (Cherowitzo, 2006). Bu açıdan, bu geometrik şekillerin inşa edildiği etkinliklerine çalışmada yer verilerek ortaokul matematik öğretmeni adaylarının kavramsal öğrenme, problem çözme, uygulama ve iletişim kurma gibi becerilerini geliştirerek gerekli konu alan bilgilerini edinmelerini sağlamak amaçlanmıştır.

Geometrik etkinlikler ve materyaller öğretim sürecinde matematiksel söylemler ve tartışmalarla birlikte kullanıldığında daha etkili olabilirler. Argümantasyon geometrik bilginin etkili bir şekilde kazanılması için gerekli olan iletişimi sağlayacak etkili bir yöntem sunabilir çünkü öğretmenlerin bilgilerinin öneminin yanında geometrik bilgileri aktarabildikleri bir iletişim süreci oluşturmada da yeterli değillerdir ve bunu geliştirmelidirler (Hershkowitz, 1989; Owens & Outhred, 2006; Sundberg & Goodman, 2005). Argümantasyon öğretmenlerin ve öğretmen adaylarının bu konuda gelişimlerini sağlayabilir. Ayrıca, bilimsel bilginin üretilebilmesi için de gereklidir çünkü insanlar alternatifleri ve kanıtları değerlendirerek fikirler üretirler (Voss & Van Dyke, 2001). Ayrıca, argümantasyon kavramsal anlama, problem çözme, eleştirel bakış açısı, doğrulama ve kanıtlama gibi becerilerin kazanılmasında da etkilidir (Abi-El-Mona & Abd-El-Khalick, 2011; Duschl & Osborne, 2002; Jim´enez-Aleixandre ve ark., 2000; Jonassen & Kim, 2010; Osborne, Erduran, & Simon, 2004; Zembal-Saul, 2005). Bu yüzden, argümantasyon özellikle de geometrik şekillerin inşasında kullanılarak öğretmen adaylarının geometri bilgilerini ve bilimsel düşünme becerilerini gerekli ön bilgilerini düzenleyerek yeni bilgileri yapılandırarak geliştirebilir. Böylelikle, argümantasyon iletişim kurma ve kritik düşünme becerileriyle kavramsal anlamayı ve derinlemesine kavramsal öğrenmeyi sağlayabilir (Driver, Newton & Osborne, 2000). Ayrıca, argümantasyon ve geometrik şekillerin inşası öğrencilerin problem çözme ve geometrik ispat yazma becerilerini geliştirmektedir. Geometrik şekillerin inşası sürecinde öğretmen adayları şekil oluşturma süreçlerini anlatmak ve diğerlerine bu sürecin doğruluğunu göstermek için diğer geometrik şekillerle ilgili özelliklerden ve geometrik ispatlardan faydalanmaktadır (Erduran & Yeşildere, 2010; Smart, 1998).

Belirli geometrik kavramlara odaklanılarak hazırlanan öğretim süreci, öğretmen adaylarının bu kavramları anlamaları ve öğrenmelerini sağlayacak şekilde tasarlanmalıdır. Ayrıca bu süreç matematiksel söylemler ve tartışmalarla desteklendiğinde, öğretmen adaylarının birbirlerinin fikirlerini analiz ederek, tartışarak ve birbirlerini kanıtlarla ikna ederek öğrenmelerini ve anlamalarını geliştirir. Böylelikle, argümantasyon, onların öğrenmelerini sağlamaktadır (Lampert, 1990). Ek olarak, öğrenenler tartışarak ve bilgilerini başka durumlara transfer ederek ürettikleri bilgilerini belirten matematiksel uygulamalarla da öğrenmeleri incelenebilir. Matematiksel uygulamalar matematiksel tartışma ve düşünmelerle elde edilen paylaşılan bilgileri belirtmektedir (Cobb, Gravemeijer, Yackel, McClain & Whitenack, 1997). Bu açıdan, bu çalışmada ortaokul matematik öğretmeni adaylarının üçgenlerle ilgili geometrik inşalar kullanarak oluşturdukları sınıf içi matematiksel uygulamaları sayesinde bu konuyla ilgili öğrenme ve anlamalarını incelemek amaçlanmıştır. Bu çalışmada, ortaokul matematik öğretmeni adaylarının üçgenler konusuyla ilgili konu alan bilgilerini geliştirmek amaçlanmıştır. Üçgenler günlük yaşamda kullanılan en yaygın şekillerden biridir. Ayrıca, diğer disiplinler, diğer matematiksel kavramlar ve geometrik şekillerle de yakından ilgilidir (Athanasopoulou, 2008; Kellogg, 2010). Üçgenler bu önemine rağmen, öğrenenler üçgenler konusuyla ilgili yeterli bilgiye sahip olma konusunda zorluk yaşamaktadırlar (Vinner & Hershkowitz, 1980). Ayrıca, üçgenler çeşitli yaş gruplarında yer alan birçok öğrenci açısından da önemlidir (Damarin, 1981; Vinner & Hershkowitz, 1980). Bu yüzden, ortaokul matematik öğretmeni adaylarının üçgenler konusuyla ilgili anlama ve öğrenmelerini sağlamak gerekmektedir.

Tasarım tabanlı araştırmayla, bu çalışmada etkili bir öğretim süreci oluşturmak için varsayıma dayalı öğrenme rotası tasarlanmış. Hazırlanan bu varsayıma dayalı öğrenme rotası altı haftalık bir öğretim sürecinde uygulanmıştır. Pilot çalışmada bu rota üzerinde gerekli bulunan düzeltmeler yapılmıştır. Bu açıdan, bu öğrenme rotasının etkisini değerlendirmek ve sınıf içi matematiksel uygulamaları belirlemek amacıyla bu çalışma tasarlanmıştır. Diğer bir ifadeyle çalışmada "Ortaokul matematik öğretmeni adaylarının üçgenleri öğrenmeleriyle ilgili problem tabanlı öğrenme stratejisine göre hazırlanmış tasarım tabanlı araştırma ortamında geliştirdikleri sınıf içi matematiksel uygulamaları nelerdir?" araştırma probleminin cevabı araştırılmıştır.

Kaynak Bildirişleri

Çalışmanın odak noktası olan matematiksel uygulamalar bireysel ve sosyal öğrenme ortamlarının etkisini birlikte düşünerek oluşturulmuştur. Matematiksel uygulamalar oluşturularak sağlanan öğrenmelerde bireysel ve sosyal öğrenme birbirini destekleyecek ve biri diğerine hakim olamayacak şekilde gerçekleşmektedir (Cobb ve ark., 2011). Bu açıdan, matematiksel uygulamalar sosyal öğrenme ortamını yansıtan sınıf ortamlarında gerçekleşmektedir. Bu sosyal öğrenme ortamında öğrenciler sınıf içi matematiksel etkinlere etkin bir şekilde katılırlar. Ayrıca, öğrencilerin etkin katılım sürecinde matematiksel bilgilerini ve becerilerini yeniden düzenleyerek öğrenirler (Cobb & Yackel, 1996). Diğer bir ifadeyle öğrenciler tartışma sürecine aktif katılım göstererek bireysel bilgilerini düzenleyerek konuları öğrenmektedir. Bu açıdan, öğrenmeler ve sosyal etkileşim yakından ilgilidir. Böyle sosyal bir sınıf ortamı içerisinde ortaya çıkan sınıf içi matematiksel uygulamalar ancak bu ortamda ortaya çıkan sosyal ve sosyomatematiksel normlar yardımıyla oluşabilir. Sosyal normlar sınıf içerisinde oluşan genel davranışları belirtmektedir (Cobb, Yackel, & Wood, 1992). Bunlar sınıf içerisinde öğrencilerden sergilemeleri beklenen davranışlar, cevap ve çözümlerin açıklaması, diğerleriyle paylaşılması ve tartışılması olarak örneklendirilebilir (Yackel & Cobb, 1996). Bu açıdan bakıldığında, her sınıfın kendine özgü normları vardır ve matematiksel uygulamaların ortaya çıkması için katılımcıların fikirlerini nedenleriyle birlikte paylaştığı ve birbirlerinin fikirlerini değerlendirdiği sınıf içi normlar oluşturulmalıdır. Bu normlardan da matematik sınıflarına özgü olan sosyomatematiksel normlar oluşmalıdır. Bu normlar sınıf içerisinde oluşan matematiksel tartışmaları belirtmektedir (Yackel & Cobb, 1996). Sosyomatematiksel normlar farklı çözüm yolları, detaylı ve önemli çözümler olarak örneklendirilebilir (Yackel, 2002). Bu yüzden bu çalışmada toplu öğrenme ortamı tasarlamak, bu ortama öğrencilerin nasıl katıldığı ve katkıda bulunduğunu ve sınıf içi matematiksel uygulamaların oluşum sürecini araştırma amaçlanmıştır.

Matematiksel uygulamaların etkili bir şekilde ortaya çıkması ve bahsedilen normların oluşması için sınıf içerisinde etkili tartışmaların oluşması sağlanmalıdır. Bu tartışmalar, öğrencileri hedeflenen bilgi ve beceriyi kazandıracak şekilde oluşturulmalıdır. Argümantasyonlar kullanılarak öğretmen adayları üçgenleri öğrenebilir, birbirlerinin üçgen inşa süreçlerini irdeleyerek denetleyebilir ve geometrik düşünme ve ispat becerilerini geliştirebilir. Bir çeşit matematiksel söylem olarak ifade edilebilen argümantasyon öğrencilerin fikirlerini nasıl doğruladıklarını ve bu matematiksel doğrulamaları iletişimlerinde nasıl kullandıklarını göstermektedir. Bu açıdan, matematiksel argümanlar üretmek aynı zamanda matematiksel kavramları anlamakla da ilişkilidir (Lampert, 1990). Öğrenciler genellikle teorem ve kuralları nerede, nasıl ve niye oluştuklarını bilmeden ve sorgulamadan ezberleyerek öğrenme eğilimindedirler. Bu problem, ancak matematiksel argümanlar üreterek giderilebilir çünkü öğrenciler argüman üretirken bunları sorgulamaya başlar ve sonunda da soru işaretlerini gidererek konuyu anlamayı amaçlar. Etkili bir yönlendirme yapıldığında anlamlı ve doğru öğrenme gerçekleşebilir. Ayrıca, sorgulama becerisiyle matematiksel anlama ve öğrenme sağlanabilir. Bu açıdan, argümantasyon yardımıyla öğrenciler problem ve fikirleri derinlemesine inceleyerek ve anlayarak etkili öğrenmeyi gerçekleştirebilirler (Abi-El-Mona & Abd-El-Khalick, 2011; Jonassen & Kim, 2010). Argümantasyonun bu süreci düşünüldüğünde öğrenciler fikirlerini rahatlıkla ifade ederek, savunarak, gerekçe ve deliller sunarak tartışmaya katılırlar. Ayrıca, diğerleri de bunlar üzerinde destekleyici ya da çürütücü kanıtlar ve düşünceler sunarak fikirlerin doğruluğunu ve geçerliğini sınarlar. Böylelikle, doğru ve anlamlı bilgi oluşturulmaya çalışılır. Bu süreç, öğretmen adaylarıyla gerçekleştirildiğinde ve katılımcıların düşünme düzeyleri göz önünde bulundurulduğunda matematiksel veya geometrik ispatlara ihtiyaç duyulabilir. Bu açıdan, argümantasyon ispatla yakından ilgilidir (Chazan, 1993; Pedemonte, 2007).

Çalışmada ortaokul matematik öğretmeni adaylarının üçgenlerle ilgili anlama ve öğrenmelerini matematiksel uygulamalar kullanarak incelemek için varsayıma dayalı öğrenme rotası oluşturulmuştur. Eğitmen öğrenme süreciyle ilgili tahminlerini van Hiele geometri düzeylerinin, geometrik şekillerin inşasının ve argümantasyonun özelliklerini düşünerek oluşturmuştur. Yapılan çalışmalar, ortaokul matematik öğretmen ve adaylarının van Hiele geometrik düşünme düzeylerinden ilk üçünün özelliklerini kazanmış olmalarının gerektiğini önermektedir (Aydın & Halat, 2009; Halat, 2008). Buna göre ortaokul matematik öğretmeni adaylarının geometrik düşünme becerileri açısından geometrik şekiller ve özellikleri arasında bağlantı kurabilmeleri, formal ve formal olmayan tanımları, argümanları ve açıklamaları anlayabilmeleri beklenmektedir. Ayrıca, geometrik şekillerle ilgili özelliklerin doğruluğu ve yeterliliği konusunda karar verebilmeleri ve açıklamalar üretebilmeleri beklenmektedir (Crowley, 1987; Fuys, Geddes & Tischler, 1998; van Hiele, 1999; Pegg, 1995).

Öğretim süreci ve öğretim sürecinde kullanılacak materyal ve etkinlikler, problem tabanlı öğrenme stratejisi kullanılarak tasarlanmıştır. Bu süreçte öğretmen adaylarına çeşitli problem durumları sunulmuş ve bunlara çözüm oluşturarak öğrenmeleri sağlanmıştır. Probleme dayalı öğrenme argümantasyon oluşturulması açısından da önemlidir çünkü probleme dayalı öğrenme düşünme, karar verme, sorgulama ve problem çözme gibi argümantasyon sürecinde de gerekli becerileri geliştirmektedir (Frank & Barzilai, 2004; Kolodner ve ark., 2003). Probleme dayalı öğrenme, öğrencilerin kavramla ilgili derinlemesine bilgi edinmelerini, bireysel öğrenmelerinin desteklenmesi, sorumluluk alınması ve aktif öğrenmeyle öğrenmelerin gerçekleştirilmesi sağlanmaktadır (Frank & Barzilai, 2004). Ek olarak probleme dayalı öğrenme öğrencilere matematiğin doğasıyla ilgili kavramsal düşünmelerini yansıttıkları ve bağlantısal öğrenmelerini geliştirdikleri olanaklar sunmaktadır (Skemp, 1978). Problem çözerek, öğrencilerin matematiksel fikirlerini organize ettikleri, tartışmalara katıldıkları, fikirlerini savundukları ve diğerlerini bu fikirler konusunda ikna etmeye çalıştıkları olanaklarla sunulmaktadır (Manuel, 1998; NCTM, 2000). Bu açıdan, probleme dayalı öğrenme argümantasyonların kullanılmasını desteklemektedir.

Varsayıma dayalı öğrenme rotasının oluşturulmasında pergel ve çizgeç kullanılarak yapılan geometrik şekillerin inşasından faydalanılmıştır. Öğretmen adaylarının üçgenler konusundaki öğrenme ve anlamalarını sağlamak amacıyla geometrik şekillerin inşası kullanılarak etkinlikler oluşturulmuştur. Pergel ve çizgeç kullanılarak geometrik şekillerin inşa edilmesi Öklid geometrisinin öğrenilmesi açısından önemlidir (Stillwell, 2000; Janicic, 2010). Öklid, "Elements" adlı kitabında geometrik şekilleri, özelliklerini ve teoremleri geometrik şekilleri inşa ederek incelemiş ve böylece geometrik şekillerin inşası geometri ve matematik eğitimde yer edinmiştir (Karakuş, 2014). Geometrik şekillerin inşasında oluşturulmak istenilen geometrik şeklin belirli temel ve karmaşık adımları takip ederek pergel ve çizgeç kullanılarak çizilmesi olarak belirtilmektedir (Demiray & Çapa-Aydın, 2015; Djoric & Janicic, 2004). Öğrenciler geometrik şekilleri inşa ederken ve çizilen şeklin belirtildiği şekilde oluşturulduğunu gösterirken ispattan yararlanmışlardır. Pergel ve çizgeç kullanılarak yapılan geometrik şekillerin inşası geometrik anlama, geometrik düşünme, problem çözme, psiko-motor, derinlemesine düşünme ve bağlantısal düşünme gibi becerileri geliştirmektedir (Ameis, 2005; Cheung, 2011; Güven, 2006; Karakuş, 2014; Khoh, 1997; Kuzle, 2013; Napitupulu, 2001; Posamentier, 2000; Tapan & Arslan, 2009).

Geometrik şekillerin inşasında, öğrenciler pergel ve çizgeç kullanarak öğrenme sürecine katılırlar. Geometrik şekillerin inşası sürecinde takip edilen adımlar planlı ve farkında olunarak etkili bir şekilde yerine getirilerek öğrencilerin geometri başarıları ve kavramsal öğrenmeleri sağlanmaktadır (Cherowitzo, 2006). Geometrik şekillerin inşasında öğrenciler sadece şeklin oluşturulmasını değil aynı zamanda onun özelliklerini ve diğer geometrik şekil ve özellikleriyle bağlantısını da incelemektedir (Erduran & Yeşildere, 2010; Napitpulu, 2001; Hoffer, 1981). Pergel ve çizgeç kullanılarak, geometrik şekillerin kritik özellikleri ve bunlar arasındaki bağlantı şekil inşa edilerek incelenip öğrenilebilir. Bu açıdan, öğrenciler geometrik şekillerin kavramsal ve bağlantısal öğrenmelerini geliştirebilir ve onlar hakkında ayrıntılı ve etkili şekilde düşünerek geometrik düşünme düzeylerini ilerletebilirler (Cheung, 2011; Hoffer, 1981; Napitupulu, 2001). Böylelikle yapılan çalışmalar pergel ve çizgeç kullanılarak oluşturulan geometrik şekillerin inşası öğrencilerin van Hiele geometrik düşünme düzeylerini geliştirdiği görülmüştür (De Villiers, 2003; Napitupulu, 2001).

Pergel ve çizgeç kullanılarak yapılan geometrik şekillerin inşası etkinliklerinde, öğrencilerin şekillerin özelliklerinin incelenmesi ve geometrik açıklamaları doğrulaması ve ispatlaması gerekmektedir (Chan, 2006; Napitupulu, 2001). Diğer bir ifadeyle, geometrik şekillerin inşası bir ifadenin sonucu veya doğruluğu ile ilgili sebep sonuç ilişkisiyle kanıtlar sunularak belirtildiği ve ispatlama becerisinin geliştirildiği olanaklar sunmaktadır (de Villiers, 2003) çünkü bu süreçte olduğu gibi ispatlarda ifadelerin doğruluğunun yanında niye doğru olduğunu belirtmek de faydalıdır (Hanna, 2000). Ayrıca, geometrik şekil inşa edildikten sonra oluşturulan şeklin belirtildiği şekilde çizildiğini doğrulamak amacıyla ortaokul matematik öğretmeni adaylarının ispatlardan faydalandığı belirtilebilir çünkü onların van Hiele geometrik düşünme düzeylerinde ilk üçünün özelliklerini elde etmeleri gerekmektedir. Bu açıdan, bu doğrulama sürecinde ispat oluşturmaları beklenebilir veya bu süreçte geliştirilebilir.

Bu çalışmada ortaokul matematik öğretmeni adaylarının üçgenler konusuyla ilgili anlama ve öğrenmeleri incelenmiştir. Çalışmada üçgenler konusu temel alınmıştır çünkü günlük yaşamda kullanılan birçok yapının oluşturulmasında ve tasarlanmasında faydalanılan geometrik şekillerin başında gelir. Ayrıca, diğer geometrik şekillerin inşasında, onların belirli özelliklerinin incelenmesinde ve alan gibi bazı hesaplamaların yapılmasında kullanılmaktadır

(Fey, 1982). Örneğin, paralelkenarın ve dikdörtgenlerin alanlarının hesaplanmasında üçgenlerden faydalanılabilir. Fakat üçgenlerin bu önemine rağmen çeşitli yaş seviyesinde olan öğrenciler üçgenler konusunda zorluk yaşamaktadır (Damarin, 1981; Vinner & Hershkowitz, 1980).

Yöntem

Nitel araştırma desenlerinden biri olan durum çalışmasına göre tasarlanan bu çalışma ortaokul matematik öğretmeni adaylarının altı haftalık süreçten oluşan sınıf içi matematiksel uygulamalarının belirlendiği tasarı-tabanlı araştırma modeli kullanılarak yürütülmüştür. Tasarı tabanlı araştırma modeli alana özgü öğrenme süreciyle ilgili teorilerin gelişiminde kullanılmaktadır. Bu yolla oluşan öğrenme teorileri öğrencilerin öğrenmelerinde gerçekleşen ve birbirini takip eden örüntüleri resmetmektedir. Ayrıca, bu modelle etkili ve derinlemesine anlamanın gerçekleştiği öğrenme ortamı sağlanmaktadır. Diğer bir ifadeyle, tasarı tabanlı araştırma modeli birbiriyle ilişkili karmaşık elemanların ve onların birlikte nasıl öğrenmeyi sağladıklarının gösterildiği bir süreçtir (Cobb, Confrey, diSessa, Lehrer & Schauble, 2003). Bu açıdan bu model teori ve uygulamayı birbirine bağlayan etkili bir yol sunmaktadır. Tasarı tabanlı araştırma modelinin öğrenme ortamındaki tasarı ve uygulamaya göre eğitici ve teorik yapısının olduğu kabul edilebilir. Diğer bir ifadeyle, bu modelde teorik düşünceler uygulanır ve geçerliliği eğitici bir yolla test edilir (Cobb ve ark., 2003). Birbirini tekrarlayan bir süreçte, sosyal öğrenme ortamında, sosyal ve bireysel matematiksel düşünme, öğretimsel tasarının geliştirilmesi ve düzenlenmesi tasarı tabanlı araştırma modeliyle sağlanabilir (Cobb ve ark., 2001).

Tasarı tabanlı araştırma modeli sınıfta uygulama olanakları sunarak eğitimde de önem kazanmıştır (Gravemeijer, 2004). Sınıfta uygulanan bu model öğretim modeli, öğretmen/eğitmen, bir veya daha fazla öğrenci, şahit gözlemci ve neler olduğunun kaydedilmesi olmak üzere beş bileşenden oluşmaktadır (Wheeldon, 2008). Bu yüzden, tasarı tabanlı araştırma modeli öğretim etkinliklerini içeren öğretim dizisinin tasarlanması, sınıf içi ortamda test edilmesi, kaydedilmesi, analiz edilmesi ve gerekli düzenlemelerin yapılması gibi süreç ve adımların yer aldığı döngüsel bir süreçten oluşmaktadır (Gravemeijer, Bowers & Stephan, 2003). Bu birbirini tekrarlayan süreçte, sosyal bir ortamda en iyi gelişimin amaçlandığı bir öğretim teorisi geliştirilir. Bu gelişimsel süreçte, şahit gözlemci varsayıma dayalı öğrenme rotasının planlanması, uygulanması, analiz edilmesi ve yorumlanması gibi süreçlerde yer almıştır (Gravemeijer, 2004).

Tasarı tabanlı araştırma modelinin temel amacı teori geliştirmektir. Bu teoriler öğrenme sürecini ve öğrenmeyi sağlayan araçları açıklamaktadır (Gravemeijer & Cobb, 2006). Bu teorilerin geliştirilmesi süreci üç aşamada gerçekleştirilebilir; deneyin tasarlanması, uygulanması ve geçmişe yönelik analiz. Deneyin tasarlanması sürecinde, önemli sonuç değişime ve düzenlemeye açık sınırlı öğretim teorisinin oluşturulmasıdır (Gravemeijer & Cobb, 2006). Bu açıdan, teorik niyetin belirlenmesi önemlidir (Cobb ve ark., 2003). Sonrasında, araştırma takımı öğrenme amaçlarını, öğretimin için başlama ve bitiş noktalarını belirler. Bu süreçte ilk adım öğrenme amacını belirlemektir ve bu amaçlar tarih, gelenek ve değerlendirmeyle sağlanabilir. Örneğin, öğrencilerin geçmişi, sahip olduğu bilgiler, konunun tarihsel gelişimi düşünülebilir. Bu çalışmada, ortaokul matematik öğretmeni adaylarının üçgenler konusuyla ilgili temel bilgilere sahip olması beklenmektedir. Literatürde, üçgenler konusunun çeşitli araçlar kullanılarak öğretildiği görülmektedir. Bu çalışmanın amacı, geometrik kavramlardan biri olan üçgenler konusuyla ilgili tasarlanmış olan bu altı haftalık öğretim sürecinde oluşan matematiksel uygulamaları belirlemektir. Bu açıdan, çalışmayı "Ortaokul matematik öğretmeni adaylarının üçgenleri öğrenmeleriyle ilgili problem tabanlı öğrenme stratejisine göre hazırlanmış tasarım tabanlı araştırma ortamında geliştirdikleri sınıf içi matematiksel uygulamaları nelerdir?"

araştırma problemi yönlendirmektedir. Bu bağlamda matematiksel uygulamaları belirlemek için geometrik kavramlardan biri olan üçgenler konusuyla ilgili varsayıma dayalı öğrenme rotası oluşturulmuştur. Her bir haftada üç saatlik uygulamaların olduğu altı haftalık bir öğretim dizisi oluşturulup pilot çalışma süresince uygulanarak elde edilen deneyimler neticesinde öğrenme rotası yeniden düzenlenmiştir. Öğretim dizisi sürecinde uygulanan varsayıma dayalı öğrenme rotası üç aşamadan oluşmaktadır; üçgenlerin oluşturulmasının sorgulanması, yardımcı elemanlar ve bunların bir noktada çakışmasının düşünülmesi ve eşlik ve benzerliğin düşünülmesidir. Bu aşamalar üçgenlerin tarihsel gelişiminden, temel ve yardımcı elemanlar ve ilgili özelliklerden ve dönüşüm geometrisinden faydalanarak üçgenlerin eşlik ve benzerliğinden oluşturulmuştur. Öğretim dizisinde kullanılan etkinliklerin büyük bir çoğunluğu pergel ve çizgeç kullanılarak yapılan üçgenlerin inşası etkinliklerinden oluşmaktadır. Bütün aşamalar bu materyaller kullanılarak ilgili özellikler ve teoremlerin de yardımıyla incelenmiştir. Böylelikle ortaokul matematik öğretmeni adaylarının üçgenler konusuyla ilgili matematiksel uygulamaları belirlenmiş ve bu konuyla ilgili kavramsal öğrenmeleri ve anlamaları da geliştirilmiş ve araştırılmıştır.

Tasarı tabanlı araştırma modelinin ikinci aşamasında oluşturulan sınırlı öğrenme teorisi ve tasarlanan öğretim dizisinin uygulanma süreci gerçekleşir (Gravemeijer & Cobb, 2006). Pergel ve çizgeç kullanılarak yapılan üçgenlerin inşası yardımıyla öğrenme süreci gerçekleştirilir. Üçgenlerin inşası etkinlikleri Smart (1998) tarafından tavsiye edilen dört aşama kullanılarak uygulanmıştır. Birinci aşama analizdir. Problem içerisinde belirtilen geometrik şekille ilgili bilinen, bilinmeyen ve gerekli durumlar belirlenir. İkincisi inşa aşamasıdır. Problem durumunda belirtilen geometrik şekil pergel ve çizgeç kullanılarak oluşturulur. Üçüncüsü ispat aşamasıdır. Öğrenciler oluşturdukları şeklin problem durumunda belirtilen şekil olduğunu ispatlarlar. Sonuncusu tartışma aşamasıdır. Olası çözümler, durumlar, inşa adımları ve ispat süreçleri araştırılır ve tartışılır. Bu süreç araştırma takımı tarafından takip edilmiş, incelenmiş ve araştırma için çıkarımlar yapılmıştır.

Son aşamada geçmişe yönelik analiz yapılır. Bu aşamada iki temel amaç vardır; öğrencilerin öğrenmelerinin araştırılması ve sınırlı öğrenme teorisinin ve varsayıma dayalı öğrenme rotasının test edilerek geliştirilmesidir. Tasarlanan öğrenme ortamı toplanan veriye göre test edilir, değiştirilir ve yeniden düzenlenir. Pilot çalışma tamamlandıktan sonra geçmişe yönelik analiz tekniği yardımıyla gerekli düzenlemeler yapılarak ana uygulamada kullanılacak olan varsayıma dayalı öğrenme rotası oluşturulmuş ve sonrasında 23 ortaokul matematik öğretmeni adayından oluşan bir gruba katılımcıların matematiksel uygulamalarını belirlemek amacıyla uygulanmıştır.

Katılımcılar

Araştırmaya toplamda ilköğretim matematik öğretmenliği programına kayıtlı kırk altı üçüncü sınıf öğrenciden oluşmaktadır. Bu öğrenciler pilot ve ana çalışma gruplarını oluşturmak üzere iki gruba ayrılmıştır. Pilot ve ana çalışma gruplarında yer alan yirmi üç öğrenciden üç tanesi araştırma takımını oluşturmak amacıyla rasgele seçilmiştir. Ayrıca, araştırmacı (aynı zamanda sınıfın eğitmeni) ve şahit gözlemci de araştırma takımlarında bulunmuştur.

Veri Toplama

Araştırma verileri pilot çalışma ve ana uygulama olmak üzere iki makro döngüde uygulanan öğretim dizisinin uygulanması sürecinde araştırma grubu ve toplu sınıf tartışmalarının video kayıtları, katılımcı ve katılımcı olmayan gözlemci kayıtları, küçük grup çalışmalarının ses kayıtları, araştırmacı notları, yazılı dokümanlar ve ön ve son görüşme kayıtları gibi birçok kaynaktan faydalanarak toplanmıştır.

Veri Analizi

Sınıf içi matematiksel uygulamaların belirlenmesi için veriler gömülü teorinin analiz tekniği olan sürekli karşılaştırmalı analiz tekniği kullanılarak incelenmiştir. Ayrıca, matematiksel fikirleri belirlemek için Toulmin'ın argümantasyon modeli de kullanılmıştır. Sınıf içi matematiksel uygulamalara dönüşen matematiksel fikirleri belirlemek amacıyla Rasmussen ve Stephan (2008) tarafından geliştirilen üç aşamalı iki kriterli analiz modeli kullanılmıştır. Bu modele göre ilk tartışma sürecinde üretilen sonuç cümlesi ilerleyen süreçte Toulmin'in modelinin diğer kısımlarda yer alarak matematiksel uygulama haline gelmektedir.

Araştırma verileri kullanılarak yapılan analizlerin geçerliği ve güvenirliğini sağlamak amacıyla çeşitli yöntemler kullanılmıştır. Veri çeşitlemesi kullanılarak ortaokul matematik öğretmeni adaylarının üçgenler konusuyla ilgili öğrenmeleri incelenmiştir. Gözlem, mülakat, doküman ve buluşmalarla çeşitli yöntemlerle veriler toplanmıştır. Ayrıca, üye kontrolü kullanılarak verilerin analizi neticesinde yapılan yorumlar tartışılmış ve sorgulanmıştır. Ayrıca, analiz sonuçları ayrıntılı ve zengin açıklamalar kullanılarak bulgular tartışılmış ve sunulmuştur.

Sonuç ve Tartışma

Bireysel öğrenmelerin ve sosyal öğrenme ortamlarının içerildiği toplu öğrenme ortamında gerçekleşen toplu sınıf tartışmaları incelenerek sınıf içi matematiksel uygulamalar belirlenmiştir. Diğer bir ifadeyle, amaç toplu tartışma ortamlarındaki matematiksel uygulamaların belirlenmesi ve nasıl geliştirilip paylaşılarak-alınan haline geldiğinin belirtilmesidir. Bu yolla, çalışma ortaokul matematik öğretmeni adaylarının van Hiele geometrik düşünme ve problem tabanlı öğrenme stratejine göre hazırlanan derslerde üçgenler konusundaki öğrenmelerinin nasıl gerçekleştiğinin incelenmesidir.

Bu çalışmada ortaokul matematik öğretmeni adaylarının probleme dayalı öğrenme strateji kullanılarak ve geometrik inşa etkinlikleriyle desteklenerek hazırlanan öğrenme ortamlarında üçgenler konusuyla ilgili konu alan bilgilerini nasıl geliştirdikleri incelenmiştir. Toplu sınıf tartışması sürecinde gerçekleşen argümantasyonlar katılımcıların üçgenlerle ilgili kavramsal bilgilerini geliştirmiştir. Örneğin, katılımcılar başlangıçta üçgeni tanımlarken üçgenin gerekli kritik yön ve özelliklerini tam ve doğru bir şekilde içerildiği tanımlar oluşturamamışlardır. Fakat argümantasyon sürecinde, katılımcılar birbirlerinin tanımlarını inceleyerek eksik ve ilgisiz kısımlarını belirlemişlerdir ve sonrasında kritik özelliklerin doğru ve beklenen şekilde ilişkilendirilerek kullanıldığı doğru ve tam üçgen tanımına ulaşmışlardır. Ortaokul matematik öğretmeni adaylarının üçgenlerle ilgili oluşturdukları matematiksel uygulamalar incelendiğinde, argümantasyonlardan oluşan bu tartışma sürecinin onların geometrik düşünme düzeylerini ve üçgenlerle ilgili bilgilerini geliştirdiği görülmüştür. Önceki çalışmalarda elde edilen bulgular bu sonucu desteklemektedir çünkü Olkun ve Toluk (2004) sınıf içi tartışmaların öğrencilerin geometrik düşünmelerini geliştirdiklerini belirtmiştir. Ayrıca, literatürde yer alan geçmiş çalışmalar, argümantasyon içeren sınıf içi tartışmalar kritik düşünerek ve iddiaları doğrulayarak oluşturulan bilimsel düşünme, problem çözme, bilgi üretme ve kavramsal anlama gibi becerileri geliştirdiğini göstermişlerdir (Abi-El-Mona & Abd-El-Khalick, 2011; Duschl & Osborne, 2002; Jim´enez-Aleixandre ve ark., 2000; Jonassen & Kim, 2010; Osborne, Erduran, & Simon, 2004; Zembaul-Saul, 2005). Bu açıdan, argümantasyon öğrencilerin söylemleri, materyalleri ve sınıf ortamını oluşturmak için iletişim kurdukları ve nedensel düşündükleri sosyal bir öğrenme ortamı içerisinde öğrencilerin matematik yapmalarını ve iddialarını tartışmalarını sağlamaktadır (Abi-El-Mona & Abd-El-Khalick, 2011). Ayrıca, problem çözme etkinliklerindeki argümantasyonlar eğitmenin rollerini, öğretim dizisini ve varsayıma dayalı öğrenme rotasının geliştirilmesini desteklemektedir.

Ortaokul matematik öğretmeni adaylarının üçgenler konusuyla ilgili konu alan bilgilerini geliştirmek amacıyla hazırlanan argümantasyon içeren sosyal öğrenme ortamları probleme-dayalı öğrenme stratejisi kullanılarak tasarlanmıştır. Bu açıdan, pergel ve çizgeç kullanılarak yapılan geometrik şekillerin inşası etkinlikleri kullanılmıştır çünkü bu etkinlikler ortaokul matematik öğretmeni adayları için faydalı problem durumları oluşturmaktadır. Öğrencilerin bu etkinliklerle ilk karşılaştıklarında şekli nasıl oluşturacaklarına karar verememeleri ve zorlanmaları onlar için problem durumu teşkil etmektedir (Erduran & Yeşildere, 2010). Probleme dayalı öğrenme stratejisiyle kullanılan bu etkinliklerin katılımcıların geometrik düşünmelerini ve bilgilerini geliştirdiği görülmüştür. Literatürde yer alan önceki çalışmalar da probleme dayalı öğrenme stratejisinin öğrencilerin geometrik düşünme ve bilgilerini geliştirdiği düşüncesini desteklemektedir (Dochy ve ark., 2003; Cantürk-Günhan & Başer, 2009; Hodges, 2010). Ayrıca, geometrik şekillerin inşası etkinlikleriyle oluşturulan problem durumları argümantasyonlarla desteklenerek ortaokul matematik öğretmeni adaylarının üçgenleri etkili bir şekilde anlamalarını sağlamıştır. Bu etkinlikler Smart'ın (1998) dört adımlı çözüm aşamaları kullanılarak gerçekleştirilmiştir. Bu aşamalarda, katılımcılar problemi analiz etmiş, şekli inşa etmiş, doğruluğunu ispat edip tartışmışlardır. Ayrıca, bu aşamalar takip edildiğinde, geometrik inşa etkinliklerinin analiz etme, değerlendirme, hipotez kurma, organize etme, hipotezi test etme ve sonuçları ispatlama gibi düşünme becerilerini geliştirdiği belirtilmiştir (Lim-Teo, 1997). Bu bilimsel düşünme becerileri tartışma sürecinde argümantasyonlar ve ispatlarla desteklendiğinde geometrik inşa etkinliklerinin argümantasyon ve ispat becerilerini geliştirdiği ve üçgenlerin öğrenilmesini sağladığı görülmüştür. Bu yüzden, geometrik kavramlar argümantasyon ve ispatlarla desteklenen geometrik inşa etkinlikleri kullanılarak öğretilebilir (Wiley & Voss, 1999). Böylece, fikir ve iddiaların kritik edilmesi, kanıt ve doğrulamaların değerlendirilmesi ve örnek olmayan durumların incelenmesi gibi beceriler geliştirilebilir (Dochy ve ark., 2003; Cantürk-Günhan & Başer, 2009; Hodges, 2010). Ayrıca, Erduran ve Yeşildere (2010) öğrencilerin geometrik inşa adımlarını bazen ezbere ve farkında olmayarak yaptıklarını belirtmiştir. Bu durumu önlemek için, argümantasyonlar ve ispatlar kullanılmıştır çünkü her adımı tartışarak sorgulamışlardır. Bu yüzden, pergel ve çizgeç kullanılarak yapılan geometrik inşa etkinlikleri planlı bir şekilde tasarlanıp eğitmen kontrolünde yapılmış ve tartışılmıştır. Pergel ve çizgeç araçları üçgenleri öğrenmek için faydalı bir şekilde kullanılmıştır. Örneğin, üçgenleri tanımlarken, üçgenlerin kritik yönü, özellikleri ve onların arasındaki ilişki pergel ve çizgeç kullanılarak incelenmiştir. Üçgenin tanımı olabilecek açıklamalar yapmışlar ve bu açıklamaların doğruluğunu, üçgen tanımı olup olmadığını geometrik inşa ve argümantasyonlarla incelemişlerdir. Sonrasında, katılımcılar gerekli düzenlemeler yaparak ve birbirlerini ikna ederek doğru üçgen tanımını oluşturmuşlardır. Bu süreçte, geometrik inşa etkinliklerinin argümantasyonları, öğrenmeyi ve ispatları desteklediği görülmüştür. Bu sebeple, Geometrik inşa etkinliklerinin geometrik kavramları öğretirken bilimsel düşünme becerilerini de geliştirdiği söylenebilir (Spear- Swerling, 2006).

Pilot uygulama sonrasında gerekli düzenlemeler yapılarak oluşturulan altı haftalık bir öğretim sürecini gösteren varsayıma dayalı öğrenme rotası ana gruba uygulanmıştır. Bu gruba uygulamadaki amaç sınıf içi matematiksel uygulamaları belirlemektir. Bu belirleme sürecinde çeşitli kaynaklardan elde edilen verilerden faydalanılmıştır. Böylelikle süreçte oluşan matematiksel uygulamalar daha anlaşılır hale gelmiş ve altı haftalık öğretim süreci daha derinlemesine olanağı sağlamıştır. Ayrıca, katılımcıların öğrenme süreci ve anlamaları daha iyi bir şekilde araştırılabilmiştir. Sınıf içi matematiksel uygulamalar Toulmin’ın bilimsel tartışma modeli kullanılarak paylaşılarak-alınan bilgilere odaklanılması sonucunda belirlenmiştir. Altı haftalık öğretim dizisinde oluşan argümantasyonlar Rasmussen ve Stephan (2008) tarafından geliştirilen yöntem ve Glaser ve Strauss (1967) tarafından önerilen sürekli karşılaştırmalı analiz tekniği kullanılarak incelenmiştir. Bu çalışmada belirlenen ortaokul matematik öğretmeni adaylarının üçgenleri öğrenmelerini destekleyen sınıf içi matematiksel uygulamalar şunlardır; üçgenlerin oluşumunun, üçgenlerin elemanlarının ve bunların özelliklerinin ve eşlik ve benzerliğin düşünülmesidir.

Tablo 1. Sınıf İçi Matematiksel Uygulamalar

Matematiksel Uygulamalar
Matematiksel uygulama 1: Üçgenlerin oluşumunun sorgulanması • Üçgenlerin tanımlarının ve sınıflandırılmalarının sorgulanması • Üçgenlerin inşasının sorgulanması
Matematiksel uygulama 2: Üçgenlerin elemanlarının ve özelliklerinin sorgulanması • Üçgenlerin yardımcı elemanlarının inşa edilmesinin sorgulanması • Üçgenlerin yardımcı elemanlarının bir noktada kesişmesinin sorgulanması • Yardımcı elemanların noktadaşlığının ve bu noktaların yerlerinin sorgulanması
Matematiksel uygulama 3: Eşlik ve benzerliğin sorgulanması • Dönüşüm geometrisiyle eş ve benzer üçgen oluşumlarının sorgulanması • Açı-Kenar-Kenar eşlik/benzerlik kriteri değildir

İlk matematiksel uygulama, ortaokul matematik öğretmeni adaylarının üçgenlerin oluşumunu sorgulamasıdır. Bu süreçte üçgenleri tanımlamak için gerekli olan temel elemanlar ve kritik özellikler belirlenip üçgenlerin oluşumları incelenmiştir. Bu matematiksel uygulama ile ilgili iki matematiksel fikir oluşmuştur; üçgenlerin tanımlamalarının ve sınıflamalarının sorgulanması ve üçgenlerin oluşturulması. Varsayıma dayalı öğrenme rotasının ilk aşamasının amacı üçgenleri temel elemanlarını ve tanımlarını kullanarak sınıflamak ve temel ve yan elemanlarından bazılarının değerleri bilinen üçgenlerin oluşumunun incelenmesidir. Bu aşamada katılımcılara üçgenler ve üçgenlerin oluşumuyla ilgili genel bir bakış açısı ve bilgi kazandırmaktır. Araştırmada belirlenen ilk matematiksel uygulamadaki matematiksel fikirlerden birincisi üçgenlerin tanımlarının ve sınıflandırmalarının sorgulanması, üçgenlerin tanımı, dik üçgenlerin tanımı ve üçgenlerin düzlemde ayırdığı bölgelere ilişkin bilgiler tartışılarak incelenmiştir. Bu matematiksel fikir ilerleyen haftalarda yer alan etkinlik ve argümantasyonlarda kullanılarak paylaşılarak-alınan bilgi haline gelir. Üçgenlerin tanımlarının ve sınıflandırmalarının sorgulanması ile ilgili matematiksel fikir katılımcıların doğru üçgen tanımı oluşturmaları sürecinde oluşmuştur. Burada, katılımcılar üçgenlerin kritik özellikleri ve bunlar arasındaki ilişkiye odaklanarak üçgen tanımıyla ilgili yapılan açıklamaları tartışmışlardır. Bu tartışma ve inceleme süreci geometrik şekillerin inşasıyla desteklenerek katılımcıların birbirlerinin açıklamalarını ve kritik özellikler arasındaki ilişkiyi incelemeleri daha etkili hale getirilmeye çalışılmıştır. Örneğin, katılımcılar pergel ve çizgeçle geometrik şekillerin inşası etkinlikleriyle üçgenlerin temel elemanları olan köşe ve kenarları incelemiş bunların paralel olmama ve kapalılık özelliklerini belirlemişlerdir. Daha sonra, aynı düzlemde doğrusal olmayan üç noktanın doğru parçaları kullanılarak oluşturulmasını geometrik şekillerin inşasıyla incelemişlerdir. Ayrıca, bu inceleme sürecinin argümantasyonlarla desteklendiğinde daha etkili olduğu görülmüştür. Katılımcılar birbirlerinin fikirlerini ve çözüm stratejilerini tartışarak üçgenlerin tanımlaması için gerekli ve yeterli özellikleri belirlemiş ve doğru ve tam üçgen tanımını oluşturmuşlardır. Önceki çalışmalarda da yer alan doğru ve tam üçgen tanımıyla ilgili matematiksel bilgi bu çalışmada da geometrik şekillerin inşası ve argümantasyonlar yardımıyla kazanılmıştır (Leiken & Zazkis, 2010; de Villiers, Govender, & Patterson, 2009; Tsamir, Tirosh, Levenson, Barkai & Tabach, 2014). Bu açıdan, geometrik şekillerin inşasının üçgenlerin tanımıyla ilgili gerekli ve yeterli kritik özelliklerin incelenmesinde faydalı olduğu görülmüştür. Bu matematiksel uygulamada yer alan ikinci matematiksel fikir üçgenlerin temel ve yardımcı elemanlarının bazılarının bilinmesi ile oluşturulmasının sorgulanmasıdır. Burada, katılımcılar üçgenlerin temel ve yan elemanlarından bazılarının değerlerini bilerek pergel ve çizgeç kullanarak geometrik inşa yardımıyla bu üçgenlerin çiziminin mümkün olup olmadığını araştırmışlardır. Ayrıca, bu süreçte üçgenlerle ilgili bazı teorem ve kurallardan faydalanmışlardır. Diğer bir ifadeyle, üçgenlerin elemanlarından ve diğer geometrik şekiller ve bunlar arasındaki ilişkilerden bu üçgenlerin inşası sürecinde faydalanılmıştır. Bu süreç ve bulgu literatürde yer alan önceki çalışmaların sonuçlarıyla da paralellik göstermektedir (Erduran & Yeşildere, 2010; Karakuş, 2014; Khoh, 1997; Kuzle, 2013). Bu yüzden, üçgenlerin kritik özellikleri, bunlar arasındaki ilişki, temel ve yan elemanlar ve üçgen oluşumu sürecinde bunlar arasındaki ilişki üçgenlerle ilgili temel bilgi edinmede faydalıdır. Böylelikle, üçgenlerle ilgili bağlantısal öğrenme sağlanmış olur.

Araştırmada belirlenen ikinci matematiksel uygulama yardımcı elemanların ve öneminin sorgulanmasıdır. Bu matematiksel uygulama ile ilgili matematiksel fikirler şunlardır; yardımcı elemanların inşası, bir noktada kesişmesi ve bu kesişim noktalarının ismi ve yeri. Varsayıma dayalı öğrenme rotasının ikinci aşamasının amacı üçgenin yardımcı elemanlarının incelenmesidir. Bu aşama sürecinde, bu matematiksel fikirler ile ilgili gerekli bilgi ve beceri pergel ve çizgeç kullanılarak yapılan geometrik şekillerin inşasıyla incelenmiş ve kazanılmıştır. Varsayıma dayalı öğrenme rotasının ikinci aşamasının uygulandığı öğretim dizisinin ikinci ve üçüncü haftalarında gerçekleşen öğretim etkinliklerinde katılımcılar üçgenlerin yardımcı elemanlarını incelemişlerdir. Öncelikle bu elemanların pergel ve çizgeç kullanılarak nasıl inşa edildiği araştırılmıştır. Bu inşa sürecinde bu yardımcı elemanların oluşumu diğer geometrik şekillerle olan ilişkisi göz önünde bulundurularak öğrenilmiştir. Daha sonra, bu elemanların bir noktada kesişmesi geometrik şekillerin inşasıyla araştırılmıştır. Bu süreçte ortaokul matematik öğretmeni adayları üçgenlerin yardımcı elemanları, özellikleri ve ilgili teoremleri ispatlarıyla gerekli bilgi ve beceriler geometrik şekillerin inşası ve argümantasyonlar kullanılarak geliştiği görülmüştür. Burada, katılımcılar kenarortayların kesişim noktasının ağırlık merkezini, açıortayların kesişim noktasının içteğet çemberin merkezini, orta dikmelerin kesişim noktasının çevrel çemberin merkezi ve yüksekliklerin kesişim noktasının diklik merkezi olduğunu geometrik şekillerin inşasıyla incelemiş ve ispatlamışlardır. Ayrıca, bu kesişim noktalarının yerinin üçgen çeşitlerine göre değişip değişmediğini geometrik şekillerin inşasıyla araştırmış ve ispatlamıştır. Burada, üçgenin düzlemi üç bölgeye ayırdığı düşünülerek bu noktaların üçgen çeşitlerine göre hangi bölgelerde yer aldığı araştırılmış ve ispatlanmıştır. Örneğin, diklik merkezinin geniş açılı üçgenlerde dış bölgede, dik üçgenlerde dik açının olduğu köşede ve dar açılı üçgenlerde üçgenin iç bölgesinde yer aldığını geometrik şekillerin inşasıyla gösterip matematiksel açıklamalarla ispatlamışlardır. Çalışmada elde edilen bu bulgu önceki çalışmalardaki geometrik bilgi ve ispat becerilerinin gelişimiyle ilgili sonuçlarla desteklenmektedir (Chan, 2006; Napitupulu, 2001; Tapan & Arslan, 2009). Bu matematiksel uygulamanın oluşumu sürecinde ortaokul matematik öğretmeni adayları üçgenin yan elemanları geometrik şekillerin inşası ve ispatlar yardımıyla araştırılmıştır. Katılımcıların yan elemanlarla ilgili geometrik bilgilerinin ve düşüncelerinin geliştiği görülmüştür. Bu bulgu önceki çalışmaların sonuçlarıyla da desteklenmektedir (Axler & Ribet, 2005; Cherowitzo, 2006; Clements & Battista, 1992; Doğan & İçel, 2011; Erduran & Yeşildere, 2010; Martin & McCrone, 2003; Smart, 1998).

Çalışmada elde edilen son matematiksel uygulama üçgenlerin eşliğinin ve benzerliğinin sorgulanmasıdır. Bu uygulama ile ilgili matematiksel fikirler şunlardır; dönüşüm geometrisiyle eş ve benzer üçgen oluşumunun sorgulanması ve Açı-Kenar-Kenar eşlik/benzerlik kriteri değildir. Varsayıma dayalı öğrenme rotasının son aşaması ile ilgili öğretim dizisinin uygulanması sürecinde, ortaokul matematik öğretmeni adayları dönüşüm geometrisiyle eş ve benzer üçgenlerin nasıl oluştuğu geometrik şekillerin inşası etkinlikleriyle nasıl oluştuğu araştırılmış ve gerekli ispatlar yapılmıştır. Ayrıca, üçgen ve bu üçgenlerin dönüşüm geometrisiyle oluşturulan görüntüleri arasındaki ilişki tartışılmıştır. Katılımcılar, pergel ve çizgeçle geometrik inşası etkinlikleri, argümantasyon, dönüşüm geometri ve üçgen tanımı ve kritik özelliklerini kullanarak eş ve benzer üçgenler oluşturmuştur. Üçgen ve görüntü üçgen arasındaki ilişki tartışılarak üçgenlerin eşliği, benzerliği ve bunlarla ilgili kriterler öğrenilmiştir. Örneğin, ötelemeyle üçgenler belirli bir vektör yardımıyla taşınmıştır. Görüntü üçgenin köşeleri diğer üçgenin köşelerinden vektörler inşa edilerek belirlenmiştir. Daha sonra, bu köşeler doğru parçaları yardımıyla birleştirilerek görüntü üçgen oluşturulmuştur. Katılımcılar, vektörlerin boy, yön ve doğrultularının aynı olduğunu belirterek görüntü üçgen oluştururken başlangıçtaki üçgenin konumu dışındaki bütün özelliklerinin korunduğunu belirtmişlerdir. Geometrik şekillerin inşası desteklenerek üçgenlerin eşliği ve benzerliği konularının öğrenildiği görülmüştür. Eşlik ve benzerliğin dönüşüm geometrisiyle öğretilmesinin faydasıyla ilgili benzer bulgular önceki çalışmalarda da görülmektedir (French, 2004; Gerretson, 1998; Paquette, 1971; Park City Math Institute [PCMI], 2010).

Geometrik şekillerin inşası ve probleme dayalı öğrenme stratejisi kullanılarak oluşturulan varsayıma dayalı öğrenme rotası ve uygulanan altı haftalık öğretim dizisinin ortaokul matematik öğretmeni adaylarının üçgenlerle ilgili konu alan bilgilerinin ve geometrik düşünme düzeyleri üzerindeki etkisini araştırılmıştır. Bu amaçla, ortaokul matematik öğretmeni adaylarının üçgenlerle ilgili etkinliklere katılımları, paylaştıkları fikirler, fikirler ve çözümlerindeki değişimler incelenmiştir. Çalışmadaki etkinlikler van Hiele geometri düşünme düzeyleri düşünülerek tasarlanmıştır. Yapılan çalışmalar ortaokul matematik öğretmenlerinin ilk üç seviyenin özelliklerini kazanmaları gerektiğini göstermektedir. Bu açıdan, katılımcılar bu özellikler düşünülerek hazırlanan etkinliklere katılmışlar ve etkinliklerle kazandırılması amaçlanan bilgiyi argümantasyonlar yardımıyla öğrenmişlerdir. Bu açıdan geometrik şekillerin inşası etkinliklerinin ve argümantasyonların öğretmen adaylarının geometrisel düşünme düzeylerini geliştirdiği belirtilebilir. Ayrıca, katılımcılarda gerçekleşen bilgi değişimleri ve bu edinilen bilgileri farklı problem durumlarına uyarlama ve kullanmaları incelendiğinde altı haftalık uygulamanın katılımcıların üçgenlerle ilgili bilgilerini geliştirdiği ifade edilebilir. Böylelikle, geometrik şekillerin inşası etkinliklerinin ve argümantasyonların öğrencilerin geometrik anlama, geometrik düşünme ve anlamalarını geliştirdiği belirtilebilir.

Van Hiele geometri testinde elde edilen sonuçlara göre ortaokul matematik öğretmeni adayların geometrik düşünme düzeylerinden ilk üçünün özelliklerini kazandığı ve öğretim dizisi sürecinde bu geometrik düşünme düzeylerini geliştirdikleri görülmüştür. Önceki çalışmalarda da ortaokul matematik öğretmeni adaylarının geometrik düşünme düzeylerinin ilk üçünü kazanmasının beklendiği belirtilmiştir (Aydin & Halat, 2009; Hoffer, 1988; Pandiscio & Knight, 2010; Spear, 1993). Bu açıdan, katılımcıların geometrik düşünme düzeylerinin beklenen seviyede olduğu ve geometrik inşa etkinlikleri ve probleme dayalı öğrenme stratejisiyle de bu seviyelerini geliştirdikleri görülmektedir (De Villiers, 2003; Güven, 2006; Napitupulu, 2001). Ayrıca, bunlarla tasarlanan sosyal öğrenme ortamının katılımcıların üçgenlerle ilgili konu alan bilgilerini geliştirdiği de belirtilebilir.

Appendix C: Vita

PERSONAL INFORMATION

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EDUCATION

Degree	Institution	Year of Graduation
Phd	METU, Elementary Education	2016
MS	YYU, Mathematics Education	2012
BS	METU, Mathematics Education	2009

WORK EXPERIENCE

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2014-Present	Bartin University, Faculty of Education	Research Assistant
2010-2014	YYU, Faculty of Education	Research Assistant

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REFEREED PUBLICATIONS

Temiz, T. & Topcu, M. S. (2013). Preservice teachers' teacher efficacy beliefs and constructivist-based teaching practice. *European Journal of Psychology of Education*, *28*, 1435-1452.

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Guner, P., & **Uygun, T.** (2015). *Examining creative drama class with community of practice.* International Congress of Social Sciences Education. Cappadocia, Turkey, 08 – 10 May.

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Appendix D: Tez Fotokopisi İzin Formu

ENSTİTÜ

Fen Bilimleri Enstitüsü

Sosyal Bilimler Enstitüsü

Uygulamalı Matematik Enstitüsü

Enformatik Enstitüsü

Deniz Bilimleri Enstitüsü

YAZARIN

Soyadı : UYGUN

Adı : TUĞBA

Bölümü : İLKÖĞRETİM

TEZİN ADI: DEVELOPING MATHEMATICAL PRACTICES IN A SOCIAL CONTEXT: A HYPOTHETICAL LEARNING TRAJECTORY TO SUPPORT PRESERVICE MIDDLE SCHOOL MATHEMATICS TEACHERS' LEARNING OF TRIANGLES

TEZİN TÜRÜ : Yüksek Lisans Doktora

1. Tezimin tamamından kaynak gösterilmek şartıyla fotokopi alınabilir.
2. Tezimin içindekiler sayfası, özet, indeks sayfalarından ve/veya bir bölümünden kaynak gösterilmek şartıyla fotokopi alınabilir.
3. Tezimden bir bir (1) yıl süreyle fotokopi alınamaz.

TEZİN KÜTÜPHANEYE TESLİM TARİHİ: