DISCRETE SYMMETRIES IN QUANTUM THEORY

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ABSTRACT

DISCRETE SYMMETRIES IN QUANTUM THEORY

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In this thesis, one of the most central problems of modern physics, namely the discrete symmetries, is discussed from various perspectives ranging from classical mechanics to relativistic quantum theory. The discrete symmetries, namely charge conjugation (C), parity (P), time reversal (T), which are connected by the so-called CPT Theorem are studied in detail. The anti-particles with a view to matter-anti-matter symmetry is also addressed and the anti-unitarity nature of the time reversal, as well as the CPT, is worked out in detail. Another issue, which have been devoted special attention to, is the CP violation in the context of neutral Kaon mixing and oscillations. Although there have been recent discoveries of CP violation in the framework of other neutral systems, like B and D mesons, this historical problem is taken up because of its simplicity and beauty.

Keywords: Discrete symmetries in Quantum systems, CPT Theorem, matter-anti-matter symmetry, Neutral K meson systems

ÖZ

KUANTUM TEORİSİNDE KESİKLİ SİMETRİLER

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Bu tezde, modern fiziğin en önemli problemlerinden biri olan kesikli simetriler klasik mekanikten göreli kuantum teorisine uzanan geniş bir perspektiften tartışılmıştır. Kesikli simetrileri, yük eşleniği (C), parite (P), ve zaman tersinmesi (T), birbirine bağlayan CPT teoremi ayrıntılı bir biçimde çalışılmıştır. Madde-antimadde simetrisi bağlamında antiparçacıklar kavramının incelenmesi odaklanılan bir başka konuyu oluşturmaktadır. Bu bağlamda, zaman tersinmesinin anti-üniter doğası ve CPT teoremine de özellikle dikkat çekilmiştir. Çalışmada yoğunlaşılan bir başka konu da, CP kırılmasının yüksüz Kaon karışımları ve salınımları çerçevesinde incelenmesidir. CP kırılmasının B ve D mesonları gibi diğer nötr sistemler üzerindeki etkilerini içeren yeni sonuçlar olmasına rağmen, bu tarihsel problem basitliği ve estetik albenisi nedeniyle özellikle ele alınmıştır.

Anahtar Kelimeler: Kuantum sistemlerinde kesikli simetriler, CPT Teorisi, madde-antimadde simetrisi, nötr K meson sistemleri

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CHAPTER 1

INTRODUCTION

The aim in this thesis is to study the discrete symmetries in Quantum theory. The two of them are inversions (reversals), concerning the space and time operations, and have been around since the formulation of Classical Mechanics. The third one is of totally different nature; charge conjugation has emerged with the advent of relativistic quantum mechanics and the antiparticles. The classical version of space and time reversal symmetries gained new meanings in the context of quantum theory, and eventually the combination of the three has risen to a very central position in the context of Quantum Field Theory, via the Charge-Parity-Time reversal (CPT) theorem [1].

Particularly the time reversal operation was shown to involve rather unusual properties in the context of quantum theory, which carries itself to the CPT transformation. This is not too surprising, since the time, unlike the space coordinates, could not be associated with well-defined quantum operators. Therefore at the stage of unification of quantum theory with the theory of special relativity, which brings the space and the time coordinates to the same status, one faces a serious difficulty with this asymmetry in the quantum descriptions of these entities [2].

Because of these peculiarities of time in the context of quantum theory, it may be a useful exercise to review the historical evolution of the concept of time briefly. It is probably the oldest but least understood concept of science and philosophy.

Indeed, the concept of time has been around since the antiquity. In the scientific revolution period with the advancement of physics and philosophy, formally more devoloped concepts of time have emerged [3]. There are presently different concepts of time, applied to different sorts of physical situations, though they are all called by the same name: Time in physics from Newton-Leibnitz to Einstein, biological time, geologic time, and so on [3].

In ancient Greece, Aristotle defined the time as the potentiality for the motion of matter. For him the reason for a thing to move was that it absorbed its motion from an earlier motion, and the earlier motion was preceded by a still earlier motion, and so on. By its very definition, this time had no beginning and no end. This concept was criticized by medieval philosophers based on the theological belief of a created universe. According to this belief, the moment of creation was the beginning of every thing including the time. Modern cosmological discoveries of the 20th century resolved all these issues yielding a complete understanding of the universe, including its history. Ironically these discoveries put forward solidly that the universe had a beginning, nowadays known as Big Bang. However the time scale is large that it is beyond comparison to those of the theological predictions [3].

Newton defined time as an external and absolute parametric representation of the spatial trajectory of a (point-like) material object. This concept of absolute time changed radically with the advent of Einstein's theory of relativity. The primary change in relation with time was that in the relativity theory the interaction between material entities propagates at a finite speed as compared to instantaneous action-at-a-distance in Newton's framework. In relativity theory, the space and time were put on the same footing. They are not absolute any more, as the values measured depend on the observer [3].

There was another new concept of time emerged in the context of the second law of thermodynamics in relation with the entropy of a system. Entropy is a measure of disorder, and for closed systems increases with time. The time defined by the direction of increasing entropy is radically different than that of Newton's. That is, the entropic time has an arrow, and is irreversible. Newton's equations of motion instead, because of their very mathematical nature, are insensitive to the reverting the direction of time [3].

Symmetry principles occupy special position in the formulation of physical laws. For instance, the special theory of relativity was discovered by the observation that the space-time symmetries possessed by two different set of classical laws, namely that of Newton's and Maxwell's, did not match. The resolution was essentially to change the Newton's law instead of giving up the principal of relativity so that it possessed the same symmetry as the Maxwell's equations (along with many other fundamental changes like elevating the time to the same status of space coordinates, etc.) [3, 4].

There was another very profound development in this respect at the beginning of the 20th century, relating the symmetries and the conservation laws by Noether [5], in relation with continuous, analytic changes of the space and time coordinates. Conservation laws follow by requiring that the laws of nature must be covariant under these changes, or in the contemporary language, should leave the action invariant. For instance, the energy is conserved if there is invariance with respect to the continuous changes of time. And, invariance of action under (continuous) space translations lead to conservation of momentum. Furthermore, these conserved (constant) charges generate these symmetry transformations. It should be underlined that this theorem does not apply to any discrete transformations, and thus the invariance under these symmetries does not lead to conserved charges as in the continuous cases.

With the advent of quantum theory in the first quarter of the twentieth century, the discrete spatial symmetries like space inversion (P) and time reversal (T) gained a new significance. For instance, parity was introduced in quantum physics in 1927, and the time reversal in 1932, both by Wigner [2]. To this group of discrete space-time symmetries, a new quantum discrete symmetry of different nature, namely the particle-antiparticle symmetry or charge conjugation (C), was added by Dirac in 1931 [6].

Then the relativistic quantum field theory was formulated for the description of elementary-particle interactions. At the time the laws governing these interactions were thought to posses these discrete C, P and T symmetries, in addition to the Poincare symmetry. That these discrete symmetries were connected by the so-called CPT theorem was demonstrated [7] in 1952, showing that the combination of C, P, and T is a general symmetry of physical laws.

Not long afterward, it was discovered that the weak interactions violated parity[8, 9]. The discovery of parity violation has opened a new avenue towards understanding the true nature of the weak interactions. Namely, the weak interactions did not posses P and C symmetries; these were thought to be maximally broken, in such a way that CP and T were still intact, up to 1964

In 1964 decays of neutral K mesons into two charged pi (π) mesons at relatively long lifetimes indicated that CP symmetry was also broken in the weak interactions [10].

CP violation[11], like the P violation is one of the most important discoveries in particle physics, which had significant consequences in particle physics, as well as cosmology [11].

Experimental works focusing on all K^0 decays in 1970's have revealed that, although the T symmetry is largely violated in neutral K-oscillations, the CPT symmetry is still valid, however. As this system is extremely simple, and offers a beautiful setting on the quantum mechanical interference effects, a separate section is devoted to this issue in this thesis.

This thesis is organized as follows: Parity is studied in Chapter 2, time reversal in Chapter 3, charge conjugation in Chapter 4. Then, the CPT theorem which connects the discrete symmetries is covered in Chapter 5. The consequences of the CPT theorem, namely the equality of masses and lifetimes of particles and anti-particles, are also discussed in this chapter. Next, in Chapter 6, the problem of strangeness oscillations in the neutral Kaon system in the CP violating regime is worked out. Finally, some of the technical details are given in the Appendices.

CHAPTER 2

PARITY

The first discrete symmetry that will be considered is space inversion, which is also known as parity. The parity operation, as applied to transformation on the coordinate system, changes a right handed (RH) system into a left handed (LH) one[12]. In other words, space reflection transformation denoted by $\mathcal{P}$ changes the position vectors $\mathbf{x}$ as follows[1]:

$$\mathbf{x} \xrightarrow{\mathcal{P}} \mathbf{x}' = -\mathbf{x}. \tag{2.1}$$

For systems described by the Hamiltonian H,

$$H(\mathbf{p}, \mathbf{x}) = \frac{\mathbf{p}^2}{2m} + V(|\mathbf{x}|), \tag{2.2}$$

which corresponds to a special case, as it excludes the velocity dependent interactions, the equation of motion under the parity transforms as:

[1] One notes that any vector which transforms in this way is said to be odd under parity.

$$m\ddot{\mathbf{x}} = -\frac{dV(|-\mathbf{x}|)}{d\mathbf{x}}. \tag{2.3}$$

It is obvious that to ensure the parity invariance, the condition $V(\mathbf{x}) = V(-\mathbf{x})$ is to be satisfied (central force problems fall into this class).

Since time is unchanged by the parity transformation, the behavior of the momentum under parity follows from eq. (2.1),

$$\mathbf{p} \xrightarrow{\mathcal{P}} -\mathbf{p}, \tag{2.4}$$

which is also of odd parity. On the other hand, the angular momentum is of even parity:

$$\mathbf{L} = m\mathbf{r} \times \mathbf{p} \xrightarrow{\mathcal{P}} \mathbf{L}. \tag{2.5}$$

One notes that when electromagnetic interactions are included, the force entering in the Newton's equation becomes the Lorentz Force involving both electric and magnetic fields. Therefore, the question of parity invariance at the level of equations of motion necessitates to question the transformation of these fields as well. For this reason, one has to address the parity invariance issue at the level of Maxwell Equations which are the governing equations of these fields. Applying the parity operation on the position coordinates only in the Maxwell

Equations, one gets;

$$\nabla \cdot \mathbf{E} = 4\pi\rho \xrightarrow{\mathcal{P}} -\nabla \cdot \mathbf{E}' = 4\pi\rho', \tag{2.6}$$

$$\nabla \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} = 4\pi\mathbf{j} \xrightarrow{\mathcal{P}} -\nabla \times \mathbf{B}' - \frac{\partial \mathbf{E}'}{\partial t} = 4\pi\mathbf{j}', \tag{2.7}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \xrightarrow{\mathcal{P}} -\nabla\times\mathbf{E}' = -\frac{\partial \mathbf{B}'}{\partial t}, \tag{2.8}$$

where the primed entities denote the parity transforms of these. One notes that eqs. (2.6),(2.7) and (2.8) are invariant if and only if,

$$\mathbf{E}' = -\mathbf{E},\ \mathbf{B}' = \mathbf{B},\ \rho' = \rho,\ \mathbf{j}' = -\mathbf{j}. \tag{2.9}$$

Thus, the Lorentz Force $\mathbf{F_L}$ is transformed under parity as follows:

$$\mathbf{F_L} = q[\mathbf{E} + \mathbf{v} \times \mathbf{B}] \xrightarrow{\mathcal{P}} \mathbf{F_L}' = q[(-\mathbf{E}) + (-\mathbf{v}) \times \mathbf{B}] = -\mathbf{F_L}. \tag{2.10}$$

Clearly, this insures the invariance of the equations of motion of a charged particle moving in an external electromagnetic field[2].

After these simple essentially geometrical classical considerations, we now turn our attention to the more complex case of quantum mechanics involving opera-

[2] If one focuses only on the particle itself, by treating the electromagnetic field exclusively as external, then this formulation is not parity invariant.

tors and state vectors etc. in an infinite dimensional Hilbert Space.

2.1 Parity in Non-Relativistic Quantum Mechanics

The parity operation can be defined by its action on the eigenket of the position operator $|\mathbf{x}\rangle$ as[3]:

$$\mathcal{P}|\mathbf{x}\rangle = |-\mathbf{x}\rangle. \tag{2.11}$$

Here, $|-\mathbf{x}\rangle$ represents the eigenket of the reverted position operator. The application of the parity operator $\mathcal{P}$ twice yields:

$$\mathcal{P}^2|\mathbf{x}\rangle = \mathcal{P}|-\mathbf{x}\rangle = |\mathbf{x}\rangle. \tag{2.12}$$

Since $|\mathbf{x}\rangle$ is an arbitrary eigenket, one obtains:

$$\mathcal{P}^2 = I. \tag{2.13}$$

It follows from eq. (2.13) that $\mathcal{P}$ is Hermitian as well as unitary, $\mathcal{P}^\dagger = \mathcal{P} = \mathcal{P}^{-1}$. Denoting the eigenvalue of $\mathcal{P}$ as $\lambda_{\mathcal{P}}$ and its eigenket as $|\varphi\rangle$, the eigenvalue equation can be defined as:

[3] There are other equivalent approaches which actually start defining the parity operation in terms of the transformation of the observables like $\mathbf{X}$ and $\mathbf{P}$, and then moving in the opposite direction, to construct its action on the eigenkets

$$\mathcal{P}|\varphi\rangle = \lambda_{\mathcal{P}}|\varphi\rangle. \tag{2.14}$$

Acting $\mathcal{P}$ on $|\varphi\rangle$ twice gives:

$$\mathcal{P}^2|\varphi\rangle = \mathcal{P}\lambda_{\mathcal{P}}|\varphi\rangle = \lambda_{\mathcal{P}}\mathcal{P}|\varphi\rangle = (\lambda_{\mathcal{P}})^2\,|\varphi\rangle. \tag{2.15}$$

Using eq. (2.13), one observes that the eigenvalues of $\mathcal{P}$ are:

$$(\lambda_{\mathcal{P}})^2 = +1 \Rightarrow \lambda_{\mathcal{P}} = \pm 1. \tag{2.16}$$

That one obtains real eigenvalues is very natural, since $\mathcal{P}$ is Hermitian. Next, to see how the position operator $\mathbf{X}$ transforms under $\mathcal{P}$, namely to determine what $\mathcal{P}\mathbf{X}\mathcal{P}$ equals to, one needs to act on $|\mathbf{x}\rangle$ by $\mathcal{P}\mathbf{X}\mathcal{P}$**.** Starting with eq. (2.11), and considering this action, one gets,

$$\mathcal{P}\mathbf{X}\mathcal{P}|\mathbf{x}\rangle = \mathcal{P}\mathbf{X}|-\mathbf{x}\rangle = \mathcal{P}\,(-\mathbf{x})\,|-\mathbf{x}\rangle = (-\mathbf{x})\,\mathcal{P}|-\mathbf{x}\rangle, \tag{2.17}$$

$$= (-\mathbf{x})\,|\mathbf{x}\rangle, \tag{2.18}$$

since, $|\mathbf{x}\rangle$ is an arbitrary eigenket, from which it follows that:

$$\mathcal{P}\mathbf{X}\mathcal{P} = -\mathbf{X}. \tag{2.19}$$

Next, we turn our attention to the behavior of momentum operator $\mathbf{P}$ under parity. To analyse the parity operation on the eigenket of the momentum operator, one has to express $|\mathbf{p}\rangle$ in terms of $|\mathbf{x}\rangle$[4]:

$$|\mathbf{p}\rangle = (2\pi)^{-3/2} \int_{-\infty}^{\infty} d^3x \, e^{i\mathbf{p}\cdot\mathbf{x}} |\mathbf{x}\rangle. \tag{2.20}$$

In constructing this expression, one takes into account of the fact that$\{|\mathbf{x}\rangle\}$ and $\{|\mathbf{p}\rangle\}$ are complete orthonormal eigenkets of the Hermitian position and momentum operators: That is, $|\mathbf{p}\rangle$ can be written as $|\mathbf{p}\rangle = \left[\int d^3x |\mathbf{x}\rangle\langle\mathbf{x}|\right] |\mathbf{p}\rangle$. Also, using $\mathbf{P}|\mathbf{x}\rangle = i\frac{\partial}{\partial \mathbf{x}}|\mathbf{x}\rangle$, $\langle\mathbf{x}|\mathbf{p}\rangle$ can be determined as:

$$\begin{aligned} \langle\mathbf{p}|\mathbf{P}|\mathbf{x}\rangle &= i\hbar\frac{\partial}{\partial \mathbf{x}}\langle\mathbf{p}|\mathbf{x}\rangle, \\ \mathbf{p}\langle\mathbf{p}|\mathbf{x}\rangle &= i\hbar\frac{\partial}{\partial \mathbf{x}}\langle\mathbf{p}|\mathbf{x}\rangle. \end{aligned} \tag{2.21}$$

The solution of eq. (2.21) yields:

$$\langle\mathbf{p}|\mathbf{x}\rangle^* = \langle\mathbf{x}|\mathbf{p}\rangle = Ne^{i\mathbf{p}\cdot\mathbf{x}}. \tag{2.22}$$

Here, the normalization constant $N = (2\pi)^{-3/2}$ is obtained by using the dirac delta functions recipe.

Applying the parity operator on eq. (2.20), one gets

[4] In this thesis, $\hbar = 1 = c$ natural unit system is used.

$$\mathcal{P}|\mathbf{p}\rangle = (2\pi)^{-3/2} \int_{-\infty}^{\infty} d^3x\, e^{i\mathbf{p}\cdot\mathbf{x}} \mathcal{P}|\mathbf{x}\rangle, \tag{2.23}$$

$$= (2\pi)^{-3/2} \int_{-\infty}^{\infty} d^3x\, e^{i\mathbf{p}\cdot\mathbf{x}} |-\mathbf{x}\rangle. \tag{2.24}$$

Replacing $\mathbf{x}$ with $-\mathbf{x}$ inside the integral, the action of the parity operator on the eigenket of the momentum operator can be found as:

$$\begin{aligned}
\mathcal{P}|\mathbf{p}\rangle &= (2\pi)^{-3/2} \int_{\infty}^{-\infty} d^3\,(-x)\, e^{i\mathbf{p}\cdot(-\mathbf{x})}|\mathbf{x}\rangle, \\
&= (2\pi)^{-3/2} \int_{-\infty}^{\infty} d^3x e^{-i\mathbf{p}\cdot\mathbf{x}}|\mathbf{x}\rangle, \\
&= |-\mathbf{p}\rangle.
\end{aligned} \tag{2.25}$$

One can also compute $\mathcal{P}\mathbf{P}\mathcal{P}|\mathbf{p}\rangle$ by using eq. (2.25),

$$\mathcal{P}\mathbf{P}\mathcal{P}|\mathbf{p}\rangle = \mathcal{P}\mathbf{P}|-\mathbf{p}\rangle = \mathcal{P}\,(-\mathbf{p})\,|-\mathbf{p}\rangle = (-\mathbf{p})\,\mathcal{P}|-\mathbf{p}\rangle, \tag{2.26}$$

$$= (-\mathbf{p})\,|\mathbf{p}\rangle, \tag{2.27}$$

from which, for arbitrary $|\mathbf{p}\rangle$, it follows that:

$$\hat{\mathcal{P}}\mathbf{P}\hat{\mathcal{P}} = -\mathbf{P}. \tag{2.28}$$

The transformation law for the angular momentum operator $(L_i = \epsilon_{ijk}X_jP_k)$ under parity, can also be worked out as:

$$\mathcal{P}L_i\mathcal{P} = \mathcal{P}\left[\epsilon_{ijk}X_jP_k\right]\mathcal{P}. \tag{2.29}$$

Inserting identities $I = \mathcal{P}^2$ to the right hand side (RHS) of the eq. (2.29),

$$\mathcal{P}L_i\mathcal{P} = \epsilon_{ijk}\left(\mathcal{P}X_j\mathcal{P}\right)\left(\mathcal{P}P_k\mathcal{P}\right), \tag{2.30}$$

and using eqs. (2.19) and (2.28), one finally gets:

$$\mathcal{P}L_i\mathcal{P} = L_i. \tag{2.31}$$

Therefore, angular momentum is invariant under parity. This property distinguishes it from other vectors like $\mathbf{x}$. Indeed, vectors are the entities which are defined through their behavior under space rotations:

$$V_i \to V_i' = R_{ij}V_j, \tag{2.32}$$

with

$$R^T R = I. \tag{2.33}$$

Ordinary vectors like position, momentum etc. change sign under parity. However, there are other types of vectors which transform under space rotations like $\mathbf{x}$, but do not change sign under parity. These vectors are called axial vectors or pseudo vectors. Thus, angular momentum is an axial vector.

As the orbital angular momentum is the generator of the space rotations it follows from eq. (2.31) that space-reflection and the space rotations commute. It is natural to postulate that same relation holds for the general rotation, $\mathbb{R}(R)$ [12]:

$$\mathcal{P}\mathbb{R}(R) = \mathbb{R}(R)\mathcal{P}. \tag{2.34}$$

Denoting the generator of general rotation by $\mathbf{J}$, it follows from eq. (2.34) that

$$[\mathcal{P}, \mathbf{J}] = 0, \tag{2.35}$$

or equivalently,

$$\mathcal{P}\mathbf{J}\mathcal{P} = \mathbf{J}. \tag{2.36}$$

For the special case in which $\mathbf{J} = \mathbf{L} + \mathbf{S}$, since the parity operator commutes with both $\mathbf{J}$ and $\mathbf{L}$, than it should also commute with $\mathbf{S}$.

Finally, wave functions under parity transformations will be considered: If $\psi(\mathbf{x})$ is a wave-function of a spinless particle whose state ket is given by $|\psi\rangle$, it can be represented as:

$$\psi(\mathbf{x}) = \langle \mathbf{x}|\psi\rangle. \tag{2.37}$$

On the other hand, representing the wave function of a space inverted state by the state ket $\mathcal{P}|\psi\rangle$ and using the Hermiticity of the parity, and the eq. (2.11), one gets:

$$\langle \mathbf{x}|\mathcal{P}|\psi\rangle = \langle -\mathbf{x}|\psi\rangle = \psi(-\mathbf{x}). \tag{2.38}$$

In the case where $|\psi\rangle$ is an eigenket of parity, since the eigenvalues of parity are ± 1, one gets

$$\mathcal{P}|\psi\rangle = \pm|\psi\rangle. \tag{2.39}$$

Then the corresponding wave function will be,

$$\langle \mathbf{x}|\mathcal{P}|\psi\rangle = \pm\langle \mathbf{x}|\psi\rangle, \tag{2.40}$$

or equivalently

$$\psi(-\mathbf{x}) = \pm\psi(\mathbf{x}). \tag{2.41}$$

However, not all wavefunctions are in such a relationship with parity. For example, a plane wave which is an eigenstate of momentum operator is not expected to be a parity eigenstate, since momentum operator does not commute with the parity operator.

An eigenket of orbital angular momentum is expected to be a parity eigenket since $\mathbf{L}$ and $\mathcal{P}$ commute. To see how a common eigenket of $\mathbf{L}^2$ and $\mathbf{L}_z$ behaves under parity, one can examine the properties of its wavefuntion $\langle\hat{\mathbf{x}}|lm\rangle$ which is defined as,

$$\langle\hat{\mathbf{x}}|lm\rangle = Y_l^m(\theta,\phi), \tag{2.42}$$

under space inversion. Here $Y_l^m(\theta,\phi)$ represents the spherical harmonics, and $\hat{\mathbf{x}}$ is the unit vector. Starting from the explicit expression of $Y_l^m(\theta,\phi)$,

$$Y_l^m(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}e^{im\phi}(sin\theta)^{-m}\frac{d^m}{d(cos\theta)^m}(sin\theta)^{-2l}, \tag{2.43}$$

and using the transformation $\hat{\mathbf{x}} \to -\hat{\mathbf{x}}$, in spherical coordinates:

$$\theta \to \pi - \theta \text{ and } \phi \to \phi + \pi, \tag{2.44}$$

one gets,

$$
\begin{aligned}
Y_l^m(\pi-\theta,\pi+\phi) &= \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}e^{im(\phi+\pi)}(sin\theta)^{-m}(-1)^{l-m}\frac{d^{l-m}}{d(cos\theta)^{l-m}}(sin\theta)^{-2l}, \\
&= (-1)^l\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}e^{im\phi}(sin\theta)^{-m}\frac{d^m}{d(cos\theta)^m}(sin\theta)^{-2l}, \\
&= (-1)^l Y_l^m(\theta,\phi).
\end{aligned}
\tag{2.45}
$$

Therefore, it is concluded that for an integer m:

$$
\mathcal{P}|lm\rangle = (-1)^l|lm\rangle. \tag{2.46}
$$

This result cannot be generalized to general angular momenta, however. That is, the parity of an angular momentum state, in general, is not determined by its total angular momentum.

2.2 Parity in Relativistic Quantum Mechanics

The Dirac equation for an electron in an external electromagnetic field is expressed in the following form [1, 3]:

$$
(\not{p} - e\not{A} - m)\psi(\mathbf{x},t) = 0. \tag{2.47}
$$

Here $\not{p} = \gamma^\mu i\partial_\mu$ with the standart representation of the Dirac matrices[5]:

[5] For the massive fermions (which are Dirac fermions) this is the standart representation for the Dirac matrices. For Majorana fermions (neutral fermions) and Weyl fermions (massless fermions) there are different representations.

$$\gamma^i = \begin{bmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{bmatrix} \text{ and } \gamma^0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{2.48}$$

One notes that in eq. (2.47) $\not{A}$ is defined as $\not{A} = \gamma_\mu A^\mu$, where $A^\mu = \{\Phi, \mathbf{A}\}$ represents the four-vector-potential whose scalar and vector parts are $\Phi(x)$ and $\mathbf{A}(\mathbf{x}, t)$, respectively (Appendix A).

Under parity operator $\mathcal{P}$, position 4-vector are transformed as:

$$x^\mu \xrightarrow{\mathcal{P}} x^{\mu\prime} = (\Lambda_\mathcal{P})^\mu_\nu \, x^\nu, \tag{2.49}$$

where $(\Lambda_\mathcal{P})^\mu_\nu$ is the space reflection matrix in the 4-dimensional space-time, which is given by:

$$(\Lambda_\mathcal{P})^\mu_\nu = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}. \tag{2.50}$$

The corresponding transformation in the Dirac Spinor space is defined as:

$$\psi(\mathbf{x}, t) \xrightarrow{\mathcal{S}_\mathcal{P}} \psi'(\mathbf{x}', t) = \mathcal{S}_\mathcal{P}\psi(\mathbf{x}, t). \tag{2.51}$$

First focusing on the free particle equation, one can see that to ensure its covariance, the parity operator $\mathcal{S}_\mathcal{P}$ in the spinor space should satisfy the following equation [1]:

$$\mathcal{S}_{\mathcal{P}} \gamma^{\mu} \mathcal{S}_{\mathcal{P}}{}^{-1} = (\Lambda_{\mathcal{P}})^{\mu}_{\nu} \, \gamma^{\nu}. \tag{2.52}$$

The solution to eq. (2.52) can easily be obtained as:

$$\mathcal{S}_{\mathcal{P}} = e^{i\varphi} \gamma^{0}. \tag{2.53}$$

Here, φ is a phase factor with the values of $0, {}^{\pi}/_{2}, \pi, {}^{3\pi}/_{2}$. This is obtained by requiring that four inversions is necessary to bring the spinor back to its original value, in the same spirit as one needs a 4π spatial rotation for the same purpose. Thus, $\mathcal{S}_{\mathcal{P}}$ should satisfy the equation,

$$\mathcal{S}_{\mathcal{P}}{}^{4} = I, \tag{2.54}$$

so that the phase $e^{i\varphi}$ can take only four values of ± 1 and $\pm i$.

Next turning our attention on the interaction case, one has:

$$\mathcal{S}_{\mathcal{P}} (\not{p} - e \not{A} - m) \mathcal{S}_{\mathcal{P}}{}^{-1} \mathcal{S}_{\mathcal{P}} \psi(\mathbf{x}, t) = 0. \tag{2.55}$$

Using eq. (2.52), eq. (2.55) takes the form,

$$\left[(\Lambda_{\mathcal{P}})^{\mu}_{\nu} \, \gamma^{\nu} \, (p_{\mu} - e A_{\mu}) - m \right] \mathcal{S}_{\mathcal{P}} \psi(\mathbf{x}, t) = 0. \tag{2.56}$$

Covariance in the interaction case means this should be equivalent to

$$(\tilde{\not{p}} - e\tilde{\not{A}} - m)\psi'(-\mathbf{x}, t) = 0, \tag{2.57}$$

where $\tilde{p}^\mu = (p^0, -\mathbf{p})$. Then, one sees that the Dirac Equation, with electromagnetic interaction is covariant, if and only if A^μ transforms as

$$A^\mu \to \tilde{A}^\mu = (A_0, -\mathbf{A}), \tag{2.58}$$

which is in agreement with the results that we have obtained in the classical case[6].

Therefore under parity transformation, scalar potential and zeroth component of the momentum keep its sign, whereas vector potential does not.

[6] This is natural, since the A^μ entering in the Dirac Equation here is treated as an external field, i.e. classical.

CHAPTER 3

TIME REVERSAL

What is adapted in this thesis concerning the meaning of time reversal is the reversal of motion. In other words, if one records some physical event and watches this film backward, if it can not be understood whether the physical event is taking place forward or backward, then the time reversal is said to be a symmetry [3].

Denoting the time reversal operation by $\mathcal{T}$, one observes that

$$\frac{d^n\mathbf{x}}{dt^n} \xrightarrow{\mathcal{T}} \frac{d^n\mathbf{x}}{d(-t)^n} = (-1)^n \frac{d^n\mathbf{x}}{dt^n}. \tag{3.1}$$

Thus, velocity picks a minus sign under time reversal, whereas position and acceleration do not. As a result, every element in mathematical equations describing the physical event that has a linear velocity part is odd under time reversal.

This, for instance, brings in a subtlety at the classical level in the presence of electromagnetic interactions, since the Lorentz force involves a linear term in velocity. To see how this issue is resolved, one needs to study the time-reversal problem for the Maxwell's equations. Indeed, applying time reversal operation, on the time variable only, in the Maxwell's Equations, one gets[12];

$$\nabla \cdot \mathbf{E} = 4\pi\rho \overset{\mathcal{T}}{\rightarrow} \nabla\cdot\mathbf{E}' = 4\pi\rho', \tag{3.2}$$

$$\nabla \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} = 4\pi\mathbf{j} \overset{\mathcal{T}}{\rightarrow} \nabla\times\mathbf{B}' - \frac{\partial \mathbf{E}'}{\partial(-t)} = 4\pi\mathbf{j}', \tag{3.3}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \overset{\mathcal{T}}{\rightarrow} \nabla \times \mathbf{E}' = -\frac{\partial \mathbf{B}'}{\partial(-t)}, \tag{3.4}$$

where the primed entities denote the transformed values of the corresponding entities, under time reversal transformation. It can be noted that eqs. (3.2)-(3.4) are invariant if and only if,

$$\mathbf{E}' = \mathbf{E},\ \mathbf{B}' = -\mathbf{B},\ \rho' = \rho,\ \mathbf{j}' = -\mathbf{j},\ \text{and}\ \mathbf{v}' = -\mathbf{v}. \tag{3.5}$$

These total set of transformations given in eq. (3.5) leave the Lorentz Force $\mathbf{F_L}$ invariant under time reversal:

$$\mathbf{F_L} = q[\mathbf{E} + \mathbf{v} \times \mathbf{B}] \overset{\mathcal{T}}{\rightarrow} \mathbf{F_L}' = q[\mathbf{E} + (-\mathbf{v}) \times (-\mathbf{B})] = \mathbf{F_L}. \tag{3.6}$$

On the other hand, the action of time reversal on angular-momentum operator $\mathbf{L} = \mathbf{X} \times \mathbf{P}$ gives,

$$\mathbf{L} = \mathbf{X} \times \mathbf{P} \xrightarrow{\mathcal{T}} \mathbf{X} \times (-\mathbf{P}) = -\mathbf{L}. \tag{3.7}$$

Generalizing this result to total angular momentum $\mathbf{J}$, one gets

$$\mathbf{J} \xrightarrow{\mathcal{T}} -\mathbf{J}. \tag{3.8}$$

3.1 Time Reversal in Non Relativistic Quantum Mechanics

Starting with the schrödinger Equation which describes velocity independent interactions,

$$i\frac{\partial\Psi(\mathbf{x},t)}{\partial t} = \left[-\frac{1}{2m}\nabla^2 + V(\mathbf{x})\right]\Psi(\mathbf{x},t), \tag{3.9}$$

one notes that, if $\Psi(\mathbf{x},t)$ is a solution, $\Psi(\mathbf{x},-t)$ is not, due to the first order nature of time-derivative. In fact, replacing "t" by "$-t$" in eq. (3.9) gives

$$-i\frac{\partial\Psi(\mathbf{x},-t)}{\partial t} = \left[-\frac{1}{2m}\nabla^2 + V(\mathbf{x})\right]\Psi(\mathbf{x},-t). \tag{3.10}$$

However, if one takes complex conjugation of the new inverted equation, one gets $\Psi^*(\mathbf{x},-t)$ is the solution of the conjugated equation.

$$i\frac{\partial\Psi^*(\mathbf{x},-t)}{\partial t} = \left[-\frac{1}{2m}\nabla^2 + V(\mathbf{x})\right]\Psi^*(\mathbf{x},-t). \tag{3.11}$$

which is the same as the original equation. Thus, if $\Psi(\mathbf{x}, t)$ is the solution to schrödinger equation so is$\Psi^*(\mathbf{x}, -t)$. Given the fact that the physically meaningful quantities are the expectation values of the observables, which involve both $\Psi(\mathbf{x}, t)$ and $\Psi^*(\mathbf{x}, -t)$, then one should think of the eq. (3.9) together with its conjugate to have a complete physical description.

To support the argument presented above one can consider the stationary-state solutions with positive energy:

$$\Psi(\mathbf{x}, t) = \psi(\mathbf{x}) e^{-iE_n t}, \; \Psi^*(\mathbf{x}, -t) = \psi^*(\mathbf{x}) e^{-iE_n t}. \tag{3.12}$$

This together with the above observation shows that time reversal operation involves complex conjugation.

However in the presence of electromagnetic fields one needs to be more careful. The new schrödinger equation for this case is given as [13],

$$i\frac{\partial \Psi(\mathbf{x}, t)}{\partial t} = \frac{1}{2m}\left[(-i\nabla - q\mathbf{A})^2 + eA^0\right]\Psi(\mathbf{x}, t), \tag{3.13}$$

The above recipe, namely, if $\Psi(\mathbf{x}, t)$ is a solution so is $\Psi^*(\mathbf{x}, -t)$, does not seem to work, without an additional condition imposed on the potentials. Namely, to ensure the recipe, one has to further transform the vector potential $\mathbf{A}$ without touching scalar potential A^0, according to $\mathbf{A} \xrightarrow{\mathcal{T}} -\mathbf{A}$ and $A^0 \xrightarrow{\mathcal{T}} A^0$, as in the classical case. That is if there is only electric fields in the medium the time reversal invariance is intact. However, in the presence of magnetic field in order not to violate T-reversal one has to make further arrangements.

Because of the entry of the complex conjugation operation in reverting the time in the schrödinger equation, the first issue to settle is how to define the time reversal operator, which acts on the states of the Quantum Hilbert Space.

In classical mechanics, the states are defined by specifying the set of positions and momenta at each instant of time. If the time is reversed, the elements of the state set transform as $\mathbf{x} \xrightarrow{\mathcal{T}} \mathbf{x}$, $\mathbf{p} \xrightarrow{\mathcal{T}} -\mathbf{p}$. Based on this classical observation, one can postulate that the quantum mechanical time reversal operator obeys the following operator transformation laws for the position and momentum operators:

$$\mathcal{T}\mathbf{X}\mathcal{T}^{-1} = \mathbf{X}, \qquad \mathcal{T}\mathbf{P}\mathcal{T}^{-1} = -\mathbf{P}. \tag{3.14}$$

The postulated transformation law $\mathcal{T}\mathbf{P}\mathcal{T}^{-1} = -\mathbf{P}$ deserves extra attention. Because, a "straightforward" application of this law, together with the $\mathcal{T}\mathbf{X}\mathcal{T}^{-1} = \mathbf{X}$, to the basic commutator of the quantum mechanics leads to an inconsistency. Indeed, if one considers

$$[\mathbf{X}_i, \mathbf{P}_j] = i\delta_{ij}, \tag{3.15}$$

and applies the time reversal transformation on it, one gets:

$$\mathcal{T}\,[\mathbf{X}_i, \mathbf{P}_j]\,\mathcal{T}^{-1} = \mathcal{T}i\delta_{ij}\mathcal{T}^{-1}. \tag{3.16}$$

Inserting identities $\mathcal{T}\mathcal{T}^{-1} = I$, and using eq. (3.14), the LHS of eq. (3.16) yields:

$$
\begin{aligned}
(LHS)_{\hat{\mathcal{T}}} &= \left[\mathcal{T}\mathbf{X}_i\mathcal{T}^{-1}, \mathcal{T}\mathbf{P}_j\mathcal{T}^{-1}\right], \\
&= [\mathbf{X}_i, -\mathbf{P}_j] = -[\mathbf{X}_i, \mathbf{P}_j] = -LHS.
\end{aligned}
$$

Computing the RHS is more subtle however, as it involves "i" in the light of the above discussion concerning complex conjugation. Therefore, one needs to decide on the mathematical nature of $\mathcal{T}$ first. Namely, if $\mathcal{T}$ is chosen to be a unitary operator, $\mathcal{T}^\dagger = \mathcal{T}^{-1}$, then one gets for the RHS of the eq. (3.16) as,

$$
(RHS)_{\hat{\mathcal{T}}} = \mathcal{T}i\delta_{ij}\mathcal{T}^{-1} = i\delta_{ij} = RHS, \tag{3.17}
$$

which obviously is a contradiction, given the fact that LHS has changed sign.

The solution proposed by Wigner [14] to remedy this contradiction is to take $\mathcal{T}$ as an anti-unitary operator, so that $\mathcal{T}\, i\, \mathcal{T}^{-1} = -i$, and thus, the RHS also behaves in line with the LHS, by changing sign, and the contradiction is removed[1].

In computing the LHS of eq. (3.15), the postulated transformation law eq. (3.14) for $\mathbf{P}$, which overlooks this anti-unitarity issue, is used: One would have faced with the same kind of contradiction, if the explicit form of the $\mathbf{P}$ operator in $\mathbf{x}$-representation, namely $\mathbf{P} = -i\frac{\partial}{\partial \mathbf{x}}$, is used. Indeed, focusing on,

[1] The general properties of the anti-linear unitary operators are discussed in detail in the Appendix B.

$$\mathcal{T}\mathbf{P}\mathcal{T}^{-1} = \mathcal{T}\left(-i\frac{\partial}{\partial \mathbf{x}}\right)\mathcal{T}^{-1}, \tag{3.18}$$

one notes that if $\hat{\mathcal{T}}$ is unitary, since it does not affect the space variable, then RHS of eq. (3.18) becomes

$$\mathcal{T}\left(-i\frac{\partial}{\partial \mathbf{x}}\right)\mathcal{T}^{-1} = -i\frac{\partial}{\partial \mathbf{x}} = \mathbf{P}, \tag{3.19}$$

which contradicts the postulated transformation law given by eq. (3.14). To overcome this difficulty, $\hat{\mathcal{T}}$ is to be taken as anti-unitary (Appendix B), then eq. (3.18) becomes,

$$\begin{aligned}
\mathcal{T}\mathbf{P}\mathcal{T}^{-1} &= \mathcal{T}\left(-i\frac{\partial}{\partial \mathbf{x}}\right)\mathcal{T}^{-1}, \\
&= \mathcal{T}\left(-i\right)\mathcal{T}^{-1}\frac{\partial}{\partial \mathbf{x}}, \\
&= i\frac{\partial}{\partial \mathbf{x}} = -\mathbf{P},
\end{aligned}$$

which is in full agreement with eq. (3.14).

As the above discussion shows, how to represent $\mathcal{T}$ explicitly is a very subtle issue. One can convince oneself that this representation depends on the physical system under consideration.

To illustrate this, for instance, one can consider the eigenket systems of position and momentum operators. It can be demonstrated that for these specific examples, $\mathcal{T}$ is the simple complex conjugation operation.

Indeed, considering the eigenvalue equation of the position operator, acted upon by $\mathcal{T}$, one gets:

$$\mathcal{T}X_i|x_i\rangle = \mathcal{T}\left(x_i|x_i\rangle\right) = x_i\left(\mathcal{T}|x_i\rangle\right) \tag{3.20}$$

Inserting the identity $\mathcal{T}^{-1}\mathcal{T} = I$ in the LHS, and also using eq. (3.14), one gets for the LHS;

$$\mathcal{T}X_i\mathcal{T}^{-1}\mathcal{T}|x_i\rangle = X_i\left(\mathcal{T}|x_i\rangle\right), \tag{3.21}$$

which together with eq. (3.20) means,

$$\mathcal{T}|x_i\rangle = |x_i\rangle. \tag{3.22}$$

Similarly, starting with $P_i|x_j\rangle$, using eq. (3.14), and inserting $\mathcal{T}^{-1}\mathcal{T} = I$, it follows that:

$$\mathcal{T}\left(P_i|x_j\rangle\right) = \mathcal{T}P_i\mathcal{T}^{-1}\mathcal{T}|x_j\rangle = -P_i\left(\mathcal{T}|x_j\rangle\right). \tag{3.23}$$

Further, using eq. (3.22), one gets:

$$\mathcal{T}\left(P_i|x_j\rangle\right) = -P_i|x_j\rangle. \tag{3.24}$$

Recalling $P_i|x_j\rangle = i\frac{\partial}{\partial x_i}|x_j\rangle$, it is seen that $\mathcal{T}$ is simply the complex conjugation operator $\mathcal{T} = K$, because $KP_iK^{-1} = -P_i$. Thus, for the spinless systems, one can simply take $\mathcal{T}$ as K.

To complete the discussion, the time evolution equation, namely the schrödinger equation will be considered:

$$i\frac{d}{dt}|\Psi(t)\rangle = H|\Psi(t)\rangle. \tag{3.25}$$

In this discussion, it will be assumed that H is invariant under time reversal, that is, it commutes with $\mathcal{T}$: $[\mathcal{T}, H] = 0$.

It will be demonstrated that, if $|\Psi(t)\rangle$ is a solution to eq. (3.25), so is $|\Psi'(t)\rangle \equiv \mathcal{T}|\Psi(-t)\rangle$. To see this, starting from,

$$i\frac{d}{dt}|\Psi'(t)\rangle = i\frac{d}{dt}\left(\mathcal{T}|\Psi(-t)\rangle\right), \tag{3.26}$$

replacing $\frac{d}{dt}$ by $-\frac{d}{d\tau}$, and remembering that $\mathcal{T}$ is assumed not to depend on t explicitely, one gets

$$
\begin{aligned}
i\frac{d}{dt}|\Psi'(t)\rangle &= -i\frac{d}{d(-\tau)}\left(\mathcal{T}|\Psi(\tau)\rangle\right), \\
&= i\mathcal{T}\frac{d}{d\tau}|\Psi(\tau)\rangle.
\end{aligned}
$$

At this point, using the above-proven fact that $\mathcal{T}$ is anti-unitary meaning $-i\,\mathcal{T} = \mathcal{T}\,i$ (see, Appendix B),

$$
i\frac{d}{dt}|\Psi'(t)\rangle = \mathcal{T}\left(i\frac{d}{d\tau}|\Psi(\tau)\rangle\right), \tag{3.27}
$$

is obtained. Using the schrödinger equation, for the generic variable τ in the RHS of eq. (3.27), one gets:

$$
i\frac{d}{dt}|\Psi'(t)\rangle = \mathcal{T}\left(H|\Psi(t)\rangle\right). \tag{3.28}
$$

Taking into account of the starting assumption, that H commutes with $\mathcal{T}$, it can be shown that,

$$
i\frac{d}{dt}|\Psi'(t)\rangle = H\left(\mathcal{T}|\Psi(t)\rangle\right) = H|\Psi'(t)\rangle, \tag{3.29}
$$

which completes the proof.

It would be useful to add further observations on the rather non-trivial nature of the time reversal transformation in the context of quantum mechanics by introducing a new perspective in the classical field theory framework. First let us note that schrödinger equations can be obtained as Euler-Lagrange equations starting from a classical field lagrangian (which is discussed in detail in the Appendix C).

$$\mathcal{L}(x) = \frac{i}{2}\left[\psi^*(x)\,\frac{\partial\psi(x)}{\partial t} - \frac{\partial\psi^*(x)}{\partial t}\,\psi(x)\right] - \frac{1}{2m}\nabla\psi^*(x)\cdot\nabla\psi(x). \tag{3.30}$$

Transforming $t \to -t'$, $\mathcal{L}$ changes as

$$\begin{aligned}\mathcal{L}(\mathbf{x},-\,t') &= \frac{i}{2}\left[\psi^*(\mathbf{x},-t')\frac{\partial\psi(\mathbf{x},-t')}{\partial\,(-t')} - \frac{\partial\psi^*(\mathbf{x},-t')}{\partial\,(-t')}\psi(\mathbf{x},-t')\right]\\ &- \frac{1}{2m}\nabla\psi^*(\mathbf{x},-t')\cdot\nabla\psi(\mathbf{x},-t'),\end{aligned} \tag{3.31}$$

$$\begin{aligned}\mathcal{L}(\mathbf{x},-t') &= \frac{i}{2}\left[\frac{\partial\psi^*(\mathbf{x},-t')}{\partial t'}\psi(\mathbf{x},-t') - \psi^*(\mathbf{x},-t')\frac{\partial\psi(\mathbf{x},-t')}{\partial t'}\right]\\ &- \frac{1}{2m}\nabla\psi^*(\mathbf{x},-t')\cdot\nabla\psi(\mathbf{x},-t').\end{aligned} \tag{3.32}$$

One observes that eq. (3.32) is the same as (C.1), for the identifications $\psi^*(\mathbf{x},t) \longleftrightarrow \psi(\mathbf{x},-t)$ and $\psi^*(\mathbf{x},-t) \longleftrightarrow \psi(\mathbf{x},t)$. Namely,

$$\psi(\mathbf{x},t) \xrightarrow{\mathcal{T}} \psi_{\tau}(\mathbf{x},t) \equiv \psi(\mathbf{x},-t) = \psi^*(\mathbf{x},t). \tag{3.33}$$

and

$$\psi^*(\mathbf{x},t) \xrightarrow{\mathcal{T}} \psi^*_{\tau}(\mathbf{x},t) \equiv \psi^*(\mathbf{x},-t) = \psi(\mathbf{x},t). \tag{3.34}$$

Thus the action S (also $\mathcal{L}$) stays invariant under time reversal,

$$S = \int_a^b \mathcal{L}dt \xrightarrow{\mathcal{T}} \int_b^a \mathcal{L}(-dt) = \int_a^b \mathcal{L}dt = S. \tag{3.35}$$

3.1.1 Time Reversal and Angular Momentum

Next, the attention can be turned on the behaviour of the angular momentum operator under time reversal: From eq. (3.14), it follows automatically that, the orbital angular-momentum operator $\mathbf{L} = \mathbf{X} \times \mathbf{P}$, transforms as:

$$\mathcal{T}\mathbf{L}\mathcal{T}^{-1} = -\mathbf{L}. \tag{3.36}$$

This transformation law can be generalized to all kinds of angular-momentum-like operators (interpreted as the generators of the generalized rotations) such that:

$$\mathcal{T}\mathbf{J}\mathcal{T}^{-1} = -\mathbf{J}. \tag{3.37}$$

3.1.1.1 Time Reversal for Spinless System

One can now formulate how the common eigenkets of $\{\mathbf{L}^2, \mathbf{L}_z\}$ transform under $\mathcal{T}$. First noting that, since $[\mathcal{T}, \mathbf{L}^2] = 0$, from eq. (3.36), then, using this fact for

$\mathbf{L}^2$, and eq. (3.36) for $\mathbf{L}_z$, one gets:

$$\mathbf{L}^2 \left(\mathcal{T}|l,m\rangle\right) = l(l+1) \left(\mathcal{T}|l,m\rangle\right), \tag{3.38}$$

$$\mathbf{L}_z \left(\mathcal{T}|l,m\rangle\right) = -m \left(\mathcal{T}|l,m\rangle\right), \tag{3.39}$$

which means

$$\mathcal{T}|l,m\rangle = (constant)|l,-m\rangle. \tag{3.40}$$

As it is observed above that time reversal operation involves complex conjugation, then one needs first to consider the transformation properties of the wave function $\langle\hat{\mathbf{x}}|l,m\rangle = Y_l^m(\theta,\phi)$, defined above in equation eq. (2.42) under complex conjugation:

$$\begin{aligned} \langle\hat{\mathbf{x}}|l,m\rangle &= Y_l^m(\theta,\phi) \xrightarrow{K} Y_l^{m*}(\theta,\phi) = (-1)^m Y_l^{-m}(\theta,\phi), \\ &= (-1)^m \langle\hat{\mathbf{x}}|l,-m\rangle. \end{aligned} \tag{3.41}$$

Finally, one gets;

$$\mathcal{T}|l,m\rangle = (-1)^m|l,-m\rangle. \tag{3.42}$$

Applying time reversal operator once more, eq. (3.42) yields:

$$\mathcal{T}^2|l,m\rangle = (-1)^{2m}|l,m\rangle. \tag{3.43}$$

This gives

$$\mathcal{T}^2 = I, \tag{3.44}$$

because for the case in hand, m's are integer.

3.1.1.2 Time Reversal for half-integer Spin Systems

Spin being an angular momentum (angular momentum in the rest frame of the particle) satisfies the usual angular momentum commutation relations. Therefore, the consistency with the commutation relations requires that it should transform under time reversal as,

$$\mathcal{T}\mathbf{S}\mathcal{T}^{-1} = -\mathbf{S}, \tag{3.45}$$

where $\mathbf{S}$ is spin operator defined by $\mathbf{S} = \frac{1}{2}\sigma$ for spin $1/2$ systems.

Since $\mathcal{T}$ is anti-unitary, namely $\mathcal{T} = \mathcal{T}_0 K$ where $\mathcal{T}_0$ is unitary part, one can rewrite eq. (3.45) as:

$$(\mathcal{T}_0 K)\,\mathbf{S}\,(\mathcal{T}_0 K)^{-1} = -\mathbf{S}. \tag{3.46}$$

Therefore,

$$\frac{1}{2}\mathcal{T}_0 \left(K\sigma K^{-1}\right)\mathcal{T}_0^{-1} = -\frac{1}{2}\sigma, \tag{3.47}$$

so,

$$\mathcal{T}_0\, \sigma_i^*\, \mathcal{T}_0^{-1} = -\sigma_i. \tag{3.48}$$

From eq. (3.48), one can deduce that $\mathcal{T}_0$ should commute with σ_2 and anti-commutes with σ_1 and σ_3. The only operator which satisfies these properties in the 2-dimensional Pauli Spin Space is σ_2: $\mathcal{T}_0 = c\sigma_2$ where c is a complex number [13]. Next, to determine c we will make use of the unitarity property of the $\mathcal{T}_0$:

$$\mathcal{T}_0^\dagger \mathcal{T}_0 = |c|^2 \sigma_2^\dagger \sigma_2 = |c|^2 = I, \tag{3.49}$$

one sees that c is a pure phase. With the definition $c = -i\eta$, $\mathcal{T}_0$ can be expressed as;

$$\mathcal{T}_0 = -i\eta\sigma_2, \tag{3.50}$$

which can also be rewritten as:

$$\mathcal{T}_0 = \eta e^{-iS_y\pi}. \tag{3.51}$$

Using the definition $\mathcal{T} = \mathcal{T}_0 K$, one finally gets:

$$\mathcal{T} = \mathcal{T}_0 K = \eta e^{-iS_y \pi} K. \tag{3.52}$$

To evaluate the action of the $\mathcal{T}^2$ on the spin-1/2 system, one can for instance consider the basis of the $\{\mathbf{S}^2, \mathbf{S}_z\}$ system which is defined as (suppressing the $s = 1/2$ quantum number in the definition of the eigenstates),

$$\mathbf{S}^2 |m\rangle = \frac{3}{4} |m\rangle, \tag{3.53}$$

$$\mathbf{S}_z |m\rangle = m|m\rangle, \tag{3.54}$$

with $m = \pm\frac{1}{2}$. So $\mathcal{T}$ acts on $|m\rangle$ as:

$$\mathcal{T}|m\rangle = -i\eta\sigma_2 K|m\rangle = -i\eta\sigma_2|m\rangle. \tag{3.55}$$

Using eq. (3.45), one gets:

$$S_z \left(\mathcal{T}|m\rangle\right) = -\frac{m}{2} \left(\mathcal{T}|m\rangle\right). \tag{3.56}$$

It follows from eq. (3.56) that $\mathcal{T}|m\rangle$ is an eigenstate with eigenvalue $-m$:

$$\mathcal{T}|m\rangle = c|-m\rangle \tag{3.57}$$

Here c is a pure phase because of normalization. Next, using the explicit form of $\mathcal{T}_0$ in eq. (3.50) (basis vectors are real under K) on the individual $|m\rangle$, one gets:

$$\mathcal{T}|+\rangle = -i\eta\sigma_2|+\rangle = \eta|-\rangle, \tag{3.58}$$

$$\mathcal{T}|-\rangle = -i\eta\sigma_2|-\rangle = -\eta|+\rangle. \tag{3.59}$$

One notes that, eqs. (3.58) and (3.59) enable to determine the pure phase c in eq. (3.57), as $c = (-1)^m$, provided that the formerly undetermined pure phase η is choosen as $\eta = i$. Namely,

$$\mathcal{T}|m\rangle = (-1)^m|-m\rangle, \tag{3.60}$$

which means that $\mathcal{T}_0$ becomes $\mathcal{T}_0 = \sigma_2$.

Next, acting by $\mathcal{T} = -i\eta\sigma_2 K$ on $|m\rangle$ once more, one gets:

$$
\begin{aligned}
\mathcal{T}^2|m\rangle &= \mathcal{T}\left(-i\eta\sigma_2|m\rangle\right), \\
&= \left(-i\eta\sigma_2\right)\left(i\eta^*\sigma_2^*\right)|m\rangle, \\
&= |\eta|^2\sigma_2\sigma_2^*|m\rangle, \\
&= -\sigma_2^2|m\rangle, \\
&= -|m\rangle. && (3.61)
\end{aligned}
$$

Thus, one obtains on the spin-1/2 systems,

$$\mathcal{T}^2 = -I \tag{3.62}$$

independent of the phase η (which means it is totally arbitrary).

3.1.1.3 Generalizations to arbitrary half-integer spin systems

To determine the action of $\mathcal{T}$ on arbitrary half-integral spin system one starts with eq. (3.37). The first step is to see how $\mathcal{T}$ acts on the basis formed by the common eigenstates of $\{\mathbf{J}^2, J_z\}$ which is defined as:

$$\mathbf{J}^2|j,m\rangle = j(j+1)|j,m\rangle, \tag{3.63}$$

$$J_z|j,m\rangle = m|j,m\rangle. \tag{3.64}$$

Following the steps adapted in the spin-1/2 system, one can define $\mathcal{T}$ for this system as:

$$\mathcal{T} = \eta e^{-iJ_y\pi}. \tag{3.65}$$

As $\mathbf{J}^2$ commutes with any components J_i, one sees that $\mathcal{T}$ commutes with $\mathbf{J}^2$. Thus, on acting $|j,m\rangle$, $\mathcal{T}$ does not change the value of j.

Next, to determine the action of $\mathcal{T}$ on $|j,m\rangle$, one has to first determine the transformation property of J_z under $\mathcal{T}$. Carrying out some algebra one first gets:

$$e^{i\pi J_y} J_z e^{-i\pi J_y} = -J_z. \tag{3.66}$$

Using eq. (3.66) one gets:

$$\begin{aligned} J_z\left(e^{-i\pi J_y}|j,m\rangle\right) &= e^{-i\pi J_y}\left(-J_z|j,m\rangle\right), \\ &= (-m)\, e^{-i\pi J_y}|j,m\rangle. \end{aligned} \tag{3.67}$$

Thus, one sees that, $e^{-i\pi J_y}|j,m\rangle$ should be an eigenstate of $|j,-m\rangle$

$$e^{-i\pi J_y}|j,m\rangle = \xi_1|j,-m\rangle. \tag{3.68}$$

Here, ξ_1 is a pure phase because of the normalization.

Using this expression, one can attempt to determine the action of $\mathcal{T}^2$ on $|j,m\rangle$:

$$\mathcal{T}|j,m\rangle = \eta e^{-iJ_y\pi} K|j,m\rangle. \tag{3.69}$$

Acting once more, one gets:

$$\begin{aligned} \mathcal{T}^2|j,m\rangle &= \eta e^{-iJ_y\pi} K \left(\eta e^{-iJ_y\pi} K|j,m\rangle\right), \\ &= |\eta|^2 e^{-i2\pi J_y}|j,m\rangle, \\ &= e^{-i2\pi J_y}|j,m\rangle. \end{aligned} \tag{3.70}$$

To determine the action of $e^{-i2\pi J_y}$ on $|j,m\rangle$, one has to repeat the above procedure used in eq. (3.66). With some algebra one gets:

$$e^{i2\pi J_y} J_z e^{-i2\pi J_y} = +J_z. \tag{3.71}$$

Using eq. (3.71),

$$\begin{aligned} J_z\left(e^{-i2\pi J_y}|j,m\rangle\right) &= e^{-i2\pi J_y} J_z|j,m\rangle, \\ &= m\left(e^{-i2\pi J_y}|j,m\rangle\right). \end{aligned} \tag{3.72}$$

One sees from eq. (3.72) that the state $e^{-i2\pi J_y}|j,m\rangle$ is an eigenstate $|j,m\rangle$;

$$e^{-i2\pi J_y}|j,m\rangle = \xi_2|j,m\rangle, \tag{3.73}$$

where ξ_2 is also a pure phase because of the normalization.

One may be inclined to choose this phase ξ_2 as $(-1)^{2j}$ based on the experience attained on spin-1/2 system, by noting that $(-1)^{2s} = -1$, for $s = 1/2$.

It was noted [12] that for a general half-integer spin-j, the state $|j,j\rangle$ can be interpreted as being constructed from $2j$ spin-1/2 particles with each individual spins in the up direction, as far as the transformation properties under rotations are conserved. With the purpose of determining the pure phase ξ_2 explicitly, one can adapt this point of view, and take the time reversal operator for this general half-integer spin-j system as a tensor product (involving $2j$-terms) of the individual time reversal operators for each spin-1/2 component:

$$\mathcal{T}_j = \mathcal{T}_1 \otimes \mathcal{T}_2 \otimes ... \otimes \mathcal{T}_{2j}, \tag{3.74}$$

on

$$|j,m_j\rangle = |m\rangle_1 \otimes |m\rangle_2 \otimes ... \otimes |m\rangle_{2j}, \tag{3.75}$$

each $\mathcal{T}_i$ acting on the corresponding partner $|m\rangle_i$ in the tensor product composite state, producing a pure phase factor -1. Thus, the cumulative effect on the composite state is $(-1)^{2j}$:

$$\mathcal{T}_j^2|j,m\rangle = (-1)^{2j}|j,m\rangle. \tag{3.76}$$

Next, to determine the phase $\xi_1$which appeared in eq. (3.68) one needs to repeat the same analysis. That is we have to consider an arbitrary half integer spin as a composite of $2j$ spin-1/2 as in eqs. (3.74) and (3.75). Then using eqs. (3.60), (3.74),(3.75), for each individual spin system, one gets:

$$\mathcal{T}|j,m\rangle = \left[\prod_i (-1)^{m_i}\right]|j,-m\rangle. \tag{3.77}$$

Then using $m = \sum_i^{2j} m_i$ one finally gets:

$$\mathcal{T}|j,m\rangle = (-1)^m|j,-m\rangle. \tag{3.78}$$

3.1.2 Kramers Degeneracy

An interesting consequences of eq. (3.62) is the so called Kramers Theorem which was first discovered in the context of schrödinger equation, Wigner reached this degeneracy as a result of time-reversal symmetry [12]. To demonstrate this theorem, one considers a Hamiltonian which commutes with $\mathcal{T}$. Then acting on the energy eigenvalue equation by $\mathcal{T}$ one gets:

$$H\left(\mathcal{T}|\psi_n\rangle\right) = \mathcal{T}H|\psi_n\rangle = \mathcal{T}E_n|\psi_n\rangle = E_n\left(\mathcal{T}|\psi_n\rangle\right). \tag{3.79}$$

Here, $|\psi_n\rangle$ is the energy eigenket with the energy value E_n. One sees that, the states $\mathcal{T}|\psi_n\rangle$ and $|\psi_n\rangle$ have the same energy. If they are the same states, they should differ only by a phase factor $e^{i\alpha}$:

$$\mathcal{T}|\psi_n\rangle = e^{i\alpha}|\psi_n\rangle. \tag{3.80}$$

Acting time reversal operator once more gives,

$$\mathcal{T}|\psi_n\rangle = \mathcal{T}e^{i\alpha}|\psi_n\rangle = e^{-i\alpha}\mathcal{T}|\psi_n\rangle = e^{-i\alpha}e^{i\alpha}|\psi_n\rangle = +|\psi_n\rangle, \tag{3.81}$$

which is consistent only for bosonic systems. However, for the half-integer spin systems, in the light of eq. (3.62), $\mathcal{T}|\psi_n\rangle$ and $|\psi_n\rangle$ should represents different systems with the same energy. Namely, these states are degenerate.

This is true, not only for a single electron, but also for a system of an odd number of electrons, in the light of construction (of the general half-integral spin-j systems in terms of the spin-1/2 ones) and discussion presented in section 3.1.3.

Furthermore, this is also true for a system with an odd total number of fermions, e.g., electrons, protons, and neutrons. In this respect, this theorem has interesting implications for the atomic systems. For instance, a system constructed with odd number of electrons under external electric field, each energy level has to be at least twofold degenerate independent of the nature of the electric field.

3.2 Time Reversal in Relativistic Quantum Mechanics

As in the non-relativistic case, one can define time reversal operator as $\mathcal{S}_{\mathcal{T}} = U_T K$ where U_T is some unitary matrix, such that the free Dirac equation is covariant [1, 13].

Under time reversal operator $\mathcal{T}$, position 4-vector are transformed as:

$$x^{\mu} \xrightarrow{\mathcal{T}} x^{\mu'} = (\Lambda_{\mathcal{T}})^{\mu}_{\nu} x^{\nu}, \tag{3.82}$$

with the time-reversal matrix $(\Lambda_{\mathcal{T}})^{\mu}_{\nu}$ in 4-dimensional space-time:

$$(\Lambda_{\mathcal{T}})^{\mu}_{\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{3.83}$$

The corresponding transformation in the Dirac Spinor space is defined as:

$$\psi(\mathbf{x}, t) \xrightarrow{\mathcal{S}_{\mathcal{T}}} \psi'(\mathbf{x}, t') = \mathcal{S}_{\mathcal{T}} \psi(\mathbf{x}, t) \tag{3.84}$$

where $\mathcal{S}_{\mathcal{T}}$ is the time reversal operator in the spinor space.

In this case the covariance of the free Dirac equation is not a simple one, as it should be between the ψ and ψ^* equations. One starts with the free Dirac equation:

$$(i\gamma^{\mu}\partial_{\mu} - m)\,\psi(\mathbf{x},t) = 0. \tag{3.85}$$

where $\partial^{'}_{\mu} = (-\partial_0, \nabla)$. Next, one needs to complex conjugate eq. (3.85) first, and act on it by the unitary part of time reversal operator U_T in the spinor space:

$$U_T\,(-i\gamma^{\mu *}\partial_{\mu} - m)\,\psi^{*}(\mathbf{x},t) = 0. \tag{3.86}$$

After inserting identities $U_T^{-1}U_T = I$, one gets:

$$\left(-iU_T\gamma^{\mu *}U_T^{-1}\partial_{\mu} - m\right)U_T\psi^{*}(\mathbf{x},t) = 0. \tag{3.87}$$

Then requiring covariance, eq. (3.87) should be matched against:

$$\left(i\gamma^{\mu}\partial^{'}_{\mu} - m\right)U_T\psi^{*}(\mathbf{x},t) = 0. \tag{3.88}$$

For this, ∂_{μ} in eq. (3.87), should be converted to $\partial^{'}_{\mu}$ in eq. (3.88), which yields;

$$\left(-iU_T\gamma^{\mu *}U_T^{-1}\left(\Lambda_{\mathcal{T}}\right)^{\nu}_{\mu}\partial^{'}_{\nu} - m\right)U_T\psi^{*}(\mathbf{x},t) = 0, \tag{3.89}$$

where $\partial^{'}_{\nu} = (-\partial_0, \nabla)$. Comparing eqs. (3.88) and (3.89), one gets:

$$U_T\gamma^{\mu *}U_T^{-1} = -\left(\Lambda_{\mathcal{T}}^{-1}\right)^{\nu}_{\mu}\gamma^{\nu}. \tag{3.90}$$

Noting that γ^2 is complex, and the others ($\gamma^{0,1,3}$) are real, from eq. (3.90) one obtains:

$$\left[U_T, \gamma^{0,2}\right] = 0 \quad \text{and} \quad \left\{U_T, \gamma^{1,3}\right\} = 0. \tag{3.91}$$

The solution to (3.91) is given as

$$U_T = c\gamma^1\gamma^3. \tag{3.92}$$

That is, the Dirac equation stays form-invariant, if and only if $\mathcal{S}_T\psi(\mathbf{x},t) = U_T\psi^*(\mathbf{x},t)$ with U_T chosen as $U_T = i\gamma^1\gamma^3$. Therefore $\psi'(\mathbf{x},-t)$ can be defined as,

$$\psi'(\mathbf{x},-t) \equiv \mathcal{S}_T\psi(\mathbf{x},t) = U_T\psi^*(\mathbf{x},t). \tag{3.93}$$

Moreover, to see the effect of time reversal on electromagnetic field, one can consider the interacting Dirac equation with minimal coupling and arrives at:

$$\left[\left(U_T\gamma^{\mu *}U_T^{-1}\right)\left(p_\mu^* - eA_\mu\right) - m\right]\left(U_T\psi^*(\mathbf{x},t)\right) = 0. \tag{3.94}$$

The covariance of the Dirac equation requires,

$$(\tilde{\not{p}} - e\tilde{\not{A}} - m)\psi'(\mathbf{x},-t) = 0, \tag{3.95}$$

where $\tilde{p}^\mu = (p_0, -\mathbf{p})$ and $\tilde{A}^\mu = (A_0, -\mathbf{A})$ as defined in Chapter 2.

Thus, under time reversal transformation, the scalar potential keeps its sign whereas the vector potential is reverted.

CHAPTER 4

CHARGE CONJUGATION

After some unsuccessful efforts on what to do with the negative energy solutions of the Dirac equation, it was soon realized that those solutions can be interpreted as the positive energy antiparticles of those described by the positive energy solutions, first by Dirac and later by Stueckelberg and Feynman independently in a different context [13]. In Dirac's version, which is known as the "hole theory", he filles up the negative energy states with electrons, in line with the Pauli exclusion principle [1]. The vacuum state becomes a state with all negative-energy states are occupied whereas all positive-energy states are empty.

If, one of the electrons in negative energy sea absorbs a photon, it gets excited to positive energy state. Then, the result is a positive-energy electron and a hole in negative-energy sea. The absence of an electron with negative energy interpreted as a particle of positive energy and opposite charge.

In Stuckelberg and Feynman [15, 16] version, they associated the negative-energy solutions of the Dirac equation with the positive energy physical particles moving backward in time. In the following discussion this subject will be taken up in the context of discrete symmetry, charge conjugation. With the judicious choice of identical mass, and opposite charge for the positron, it becomes a straightforward analysis.

This will be done by trying to construct the charge conjugated positive energy Dirac equation for the positron from the negative energy one for the electron. In this discussion it will be assumed that the particle and the anti-particle have the identical masses (an issue which will be discussed in detail in the consequences of CPT theorem) and opposite charges.

First one starts with the Dirac equation for the electron in an external electromagnetic field expressed as:

$$(\not{p} - e\not{A} - m)\psi(\mathbf{x}, t) = 0. \tag{4.1}$$

The next step is to change the sign in front of the electric charge e in eq. (4.1). That is one proposes that the wave function of positron $\psi_c(\mathbf{x}, t)$ should satisfy the following equation:

$$(\not{p} + e\not{A} - m)\psi_c(\mathbf{x}, t) = 0, \tag{4.2}$$

Here the masses of the electron and positron are taken to be equal [1].

It is clear that this sign change of electric charge can be achieved by taking complex conjugation of eq. (4.1), since $\not{p}$ and $\not{A}$ differ by the imaginary factor inside $\not{p}$.

[1] To be proven later in the context of CPT theorem

As A_μ is real (treated classically), one gets:

$$(i\gamma^\mu \partial_\mu)^* = -i\gamma^{\mu^*} \partial_\mu, \tag{4.3}$$

$$(\gamma^\mu A_\mu)^* = \gamma^{\mu^*} A_\mu. \tag{4.4}$$

Eq.(4.1), after complex conjugation becomes,

$$(-i\gamma^{\mu^*} \partial_\mu - e\gamma^{\mu^*} A_\mu - m)\psi^*(\mathbf{x}, t) = 0. \tag{4.5}$$

Therefore, introducing a unitary operator U_c, and acting on eq. (4.5), leads to:

$$U_c \left[-\gamma^{\mu^*} (i\partial_\mu + eA_\mu) - m \right] U_c^{-1} U_c \psi^*(\mathbf{x}, t) = 0. \tag{4.6}$$

By requiring that,

$$U_c \gamma^{\mu^*} U_c^{-1} = -\gamma^\mu, \tag{4.7}$$

one obtains

$$\left[\gamma^\mu (i\partial_\mu + eA_\mu) - m \right] U_c \psi^*(\mathbf{x}, t) = 0, \tag{4.8}$$

or equivalently,

$$(\not{p} + e\not{A} - m)\psi_c(\mathbf{x}, t) = 0, \tag{4.9}$$

where the definition $\psi_c(\mathbf{x}, t) \equiv \mathcal{S}_\mathcal{C}\psi(\mathbf{x}, t) = U_c\psi^*(\mathbf{x}, t)$ is introduced to describe the Dirac spinor corresponding to the positron.

Recalling the explicit expression for the γ^μ matrices, one observes that (4.7) reduces to (as γ^2 is the only complex one),

$$\left[U_c, \gamma^2\right] = 0, \tag{4.10}$$

$$\left\{U_c, \gamma^{0,1,3}\right\} = 0. \tag{4.11}$$

The solution to eqs. (4.11) and (4.10) are obtained as $U_c = i\gamma^2$. Thus the charge conjugation operation becomes $\mathcal{S}_\mathcal{C} = U_c K = i\gamma^2 K$.

To illustrate all these on a simple example, one can consider the spin-up negative-energy "rest solution", $\psi_R^3(t)$, of the free Dirac equation:

$$\psi_R^3(t) = N \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} e^{+imt}. \tag{4.12}$$

Acting on $\psi_R^3(t)$ by the $\mathcal{S}_\mathcal{C}$ operator, one gets;

$$
\begin{aligned}
\mathcal{S}_{\mathcal{C}}\psi_R^3(t) &= \mathcal{S}_{\mathcal{C}} N \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} e^{+imt}, \\
&= -N \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} e^{-imt},
\end{aligned}
$$

which corresponds to the rest-solution of spin-down positive energy spinor. Now, associating the negative energy solution with anti-particles, it is seen, in this simple case of rest solution, that the absence of a spin-up negative energy electron at rest corresponds to the existence of a spin-down positive energy positron at rest, and vice versa (provided that the masses are assumed to be equal).

CHAPTER 5

THE CPT SYMMETRY

5.1 CPT Theorem and its Consequences

The combined action of all the discrete symmetries on the wave-function of electron, by using eqs. (2.53), (4.9) and (3.93) yields[1];

$$\psi_{CPT}(\mathbf{x},t) \equiv \mathcal{S}_{\mathcal{C}}\mathcal{S}_{\mathcal{P}}\mathcal{S}_{\mathcal{T}}\psi(\mathbf{x},t) = i\gamma^5\psi(\mathbf{x},t) = \psi_c(-\mathbf{x},-t), \tag{5.1}$$

where $\gamma^5 = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$ is in the Dirac representation and $\psi_{CPT}(\mathbf{x},t)$ represents a positron moving backward in space-time. In other words, an electron wave function multiplied by $i\gamma^5$ and moving backward in space-time represents a positron moving forward in space-time.

The CPT theorem is one of the most important characteristics of Relativistic Quantum Field Theory. It is built on three assumptions: Locality, Lorentz Invariance, and the Hermiticity of the Hamiltonian. If CPT is violated, the implications would be very serious. Because, this would mean that at least one of the above three assumptions would not be valid [17].

The most important physical implications of the CPT symmetry is that the masses and the lifetimes are equal for particles and anti-particles. These issues are discussed in the following sections.

5.1.1 Equality of Masses of Particles and Anti-particles

A particle state may be represented as the common eigenstate of a Hamiltonian H (which includes all the relevant interactions), $\mathbf{J}^2$ and $\mathbf{J}_z$. The mass of the particle is defined as the expectation value of the Hamiltonian H in its rest frame, this may be denoted as $|\mathbb{P}; m_j\rangle$ where $\mathbb{P}$ represents the particle and m_j is the 3^{rd} component of spin (because it is in the rest frame). That is, the mass of the particle $M_{\mathbb{P}}$ is given by:

$$M_{\mathbb{P}} = \langle \mathbb{P}; m_j | H | \mathbb{P}; m_j \rangle. \tag{5.2}$$

If H is Hermitian (which is necessary for the validity of the CPT theorem), and does not depend on m_j explicitly, then $M_{\mathbb{P}}$ is real:

$$M_{\mathbb{P}} = \langle \mathbb{P}; m_j | H | \mathbb{P}; m_j \rangle^* = \langle \mathbb{P}; m_j | H | \mathbb{P}; m_j \rangle. \tag{5.3}$$

Before attempting to the define the action of the combined $\Theta = \mathcal{CPT}$ on $|\mathbb{P}; m_j\rangle$, one has to first review the action of individual operators P, $\mathcal{T}$ and $\mathcal{C}$ on $|\mathbb{P}; m_j\rangle$. Paying attention that $|\mathbb{P}; m_j\rangle$ is an eigenstate of $\{\mathbf{J}^2, \mathbf{J}_z\}$ (or $\{\mathbf{S}^2, \mathbf{S}_z\}$ in the rest frame), and using eq. (3.78), one first gets:

$$\mathcal{PT}|\mathbb{P}; m_j\rangle = (-1)^{j+m}|\mathbb{P}; -m_j\rangle. \tag{5.4}$$

Denoting the antiparticle of $\mathbb{P}$ by $\bar{\mathbb{P}}$, one can define the charge conjugation operation as:

$$\mathcal{C}|\mathbb{P}; m_j\rangle = |\bar{\mathbb{P}}; m_j\rangle. \tag{5.5}$$

Therefore, the combined $\mathcal{P}$, $\mathcal{T}$ and $\mathcal{C}$ operations can be represented as;

$$\Theta|\mathbb{P}; m_j\rangle = (-1)^{j+m}|\bar{\mathbb{P}}; -m_j\rangle, \tag{5.6}$$

where $\Theta = \mathcal{CPT}$. Inserting the identities, $I = \Theta^{-1}\Theta$ to both sides of H, in equation (5.2), one obtains:

$$M_{\mathbb{P}} = \langle\mathbb{P}; m_j|\Theta^{-1}\Theta H\Theta^{-1}\Theta|\mathbb{P}; m_j\rangle. \tag{5.7}$$

Using the CPT invariance of the Hamiltonian, $\Theta H\Theta^{-1} = H$, eq. (5.7) yields:

$$M_{\mathbb{P}} = \left(\langle\mathbb{P}; m_j|\Theta^{-1}\right) H \left(\Theta|\mathbb{P}; m_j\rangle\right). \tag{5.8}$$

Next, taking into account of eq. (5.6), eq. (5.8) results in the following form:

$$\begin{aligned} M_{\mathbb{P}} &= (-1)^{2(j+m)}\langle \mathbb{P}; -m_j|H|\mathbb{P}; -m_j\rangle, \\ &= \langle \bar{\mathbb{P}}; -m_j|H|\bar{\mathbb{P}}; -m_j\rangle. \end{aligned} \tag{5.9}$$

One notes that the right hand side of (5.9) is nothing but the expectation value of the Hamiltonian $\bar{H} \equiv H_{CPT} = H$ in the rest frame of the antiparticle, namely the mass of the anti-particle. Thus, we finally get the result what we have aimed to prove:

$$M_{\mathbb{P}} = M_{\bar{\mathbb{P}}}. \tag{5.10}$$

5.1.2 Equality of Lifetimes (and the masses) of Particles and Anti-particles

The purpose here is to show that the lifetimes of the decays $\mathbb{P} \to f$ and $\bar{\mathbb{P}} \to \bar{f}$ are equal. These decays are governed by the weak part of the Hamiltonian $H = H_{st} + H_{wk}$. Both parts are invariant under combined $\Theta = \mathcal{CPT}$, and the strong part of the Hamiltonian H_{st} is also invariant under individual $\mathcal{P}$, $\mathcal{T}$ and $\mathcal{C}$ symmetries. One also notes that $\mathbb{P}$ and $\bar{\mathbb{P}}$ decay into different products because they are oppositely charged.

a) For proving the equality of lifetimes, eq. (B.14) of the Appendix B will be used. With the choices $\mathcal{A} = \Theta = \mathcal{CPT}$, $\mathcal{O} = H$, $|\alpha\rangle = |\mathbb{P}; m_j\rangle$, $|\beta\rangle = |f\rangle$ (where f denotes the final decay products of $\mathbb{P}$), eq. (B.14) becomes:

$$\left(\widetilde{\langle f|}\right) H_{wk} \left(\widetilde{|\mathbb{P}; m_j\rangle}\right) = \langle \mathbb{P}; m_j | H_{wk} | f \rangle. \tag{5.11}$$

This can be reduced to

$$\begin{aligned} \langle \bar{f} | H_{wk} | \bar{\mathbb{P}}; -m_j \rangle &= \langle \mathbb{P}; m_j | H_{wk} | f \rangle, \\ &= \langle f | H_{wk} | \mathbb{P}; m_j \rangle^*, \end{aligned} \tag{5.12}$$

which means that the decay rates are equal: $\Gamma_{\mathbb{P}\to f} = \Gamma_{\bar{\mathbb{P}}\to \bar{f}}$. As the lifetime τ is defined as $\tau = {}^1\!/\Gamma$, it follows that lifetimes are equal.

b) For the choice $|\alpha\rangle = |\beta\rangle = |\mathbb{P}; m_j\rangle$, eqs. (5.6) and (B.14) yield:

$$\langle \bar{\mathbb{P}}; -m_j | H | \bar{\mathbb{P}}; -m_j \rangle = \langle \mathbb{P}; m_j | H | \mathbb{P}; m_j \rangle, \tag{5.13}$$

which is nothing but the equality of masses of the particles and the anti-particles.

CHAPTER 6

NEUTRAL KAON MIXING & OSCILLATIONS

Neutral K-meson has two states which are called as K^0 and $\bar{K}^0$ in the context of strong interactions. Strangeness is conserved in strong interactions but, not in weak interactions.

Gell-Mann and Pais [17] analyzed the K^0-$\bar{K}^0$ system as a quantum mechanical 2-state system, with the basis states chosen as $|K^0\rangle$ and $|\bar{K}^0\rangle$ which are eigenstates of the strangeness operator $\mathcal{S}$. Since the weak interactions violate strangeness, they mix these states together.

K^0 and $\bar{K}^0$ with the quark structures $s\bar{d}$ and $\bar{s}d$ with strangeness values $+1$ and -1, respectively, and transform to each other via second-order weak interaction which in the modern notation (Standart Model) represented by the box diagrams given in Figure 6.1.

Any state $|\psi\rangle$ of the neutral K^0 system can be described by giving the amplitudes in either basis state [19].

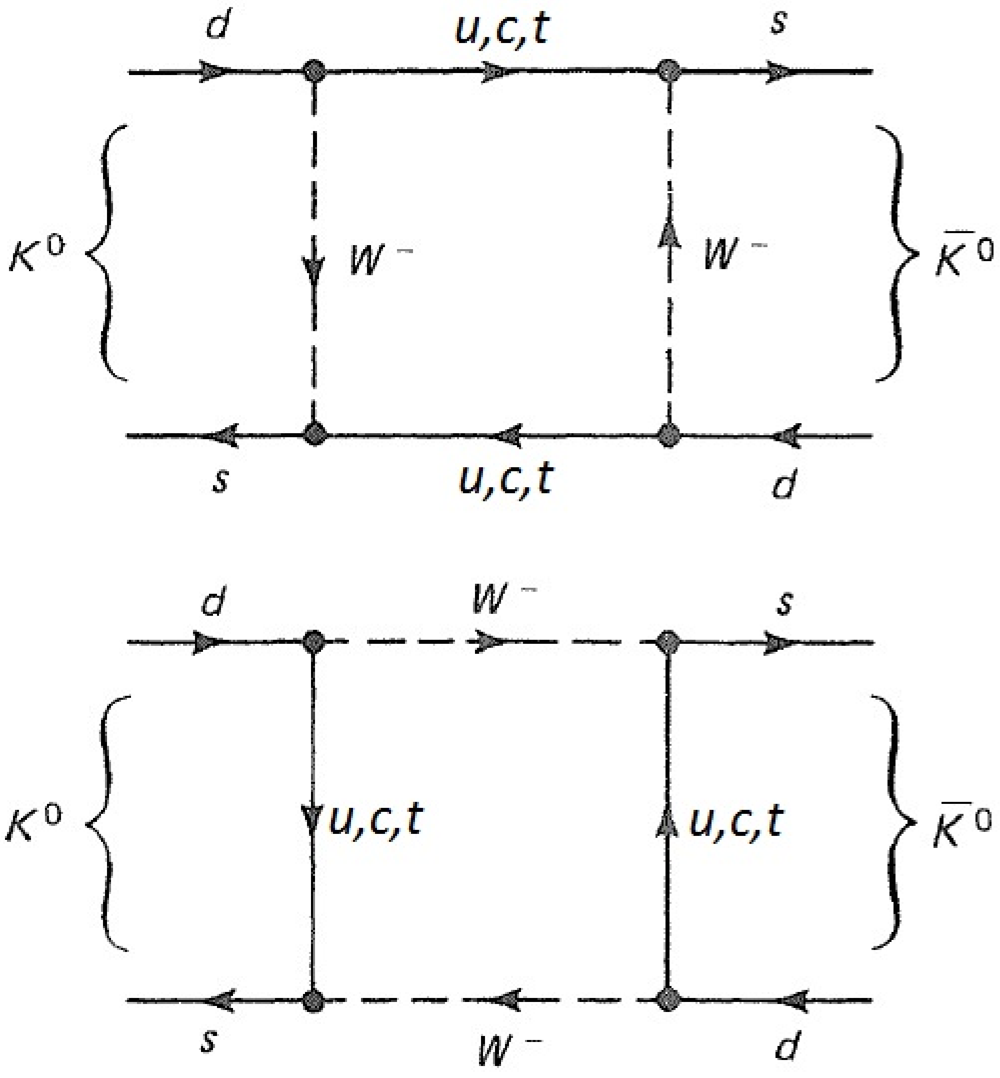

Figure 6.1: Feynman diagrams contributing to $K^0 \leftrightarrow \bar{K}^0$

$$|\psi(t)\rangle = c_1|K^0\rangle + c_2|\bar{K}^0\rangle \doteq c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \tag{6.1}$$

where $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ represent $|K^0\rangle$ and $|\bar{K}^0\rangle$, respectively.

The next step is to construct the Hamiltonian H including the weak interaction. Since the Weak interactions is CP-conserving, one can write the most general form of Hamiltonian in the 2-dimensional state space as,

$$H = H_0 + H_1, \tag{6.2}$$

where H_0 is the diagonal strong part, reflecting the equality (degeneracy) of the K^0 and $\bar{K}^0$ masses, and H_1 is the CP-conserving weak part.

Before focusing on the solution of the eigenvalue equation of H eq. (6.2), one needs to understand how the decay process is described in quantum mechanics.

The time evolution of the state ψ of a stable particle with mass m in its rest frame is governed by the schrödinger equation

$$i\frac{d\psi(t)}{dt} = H_R\psi(t) = m\psi(t), \tag{6.3}$$

with the solution

$$\psi(t) = Ne^{-imt}. \tag{6.4}$$

The simple eq. (6.3) is however not sufficient for describing the decay processes. As decaying of a particle means to disappear (or to get lost), one needs an additional non-Hermitian piece denoted by $\frac{i\Gamma}{2}$ in the Hamiltonian to describe the decay process where Γ is the total decay width. Thus, the modified schrödinger Equation in the rest frame, becomes[1];

$$i\frac{d\psi}{dt} = H_{eff}\psi = \left(m - \frac{i\Gamma}{2}\right)\psi, \tag{6.5}$$

with the solution

$$\psi(t) = e^{-imt-\frac{1}{2}\Gamma t}. \tag{6.6}$$

Working out the continuity equation, one gets;

$$\frac{d|\psi(t)|^2}{dt} = -\Gamma|\psi(t)|^2 \neq 0, \tag{6.7}$$

implying an average life-time for the decay, $\tau = {}^1\!/_\Gamma$.

Next, the representations of strangeness $\mathcal{S}$, parity $\mathcal{P}$ and $\mathcal{CP}$ operators in the 2-dimensional neutral K-state space need to be constructed. As, the K^0-$\bar{K}^0$ states were chosen as the initial basis, in order to represent these discrete operators in this basis, one needs to work out their actions on these states first:

i) The K^0, $\bar{K}^0$ particles carry ± 1 values of strangeness, namely:

[1] Although H_{weak} is Hermitian, the H_{eff} is taken to be non-Hermitian to describe the decay, following Breit-Wigner[19].

$$\mathcal{S}|K^0\rangle = |K^0\rangle, \tag{6.8}$$

$$\mathcal{S}|\bar{K}^0\rangle = -|\bar{K}^0\rangle. \tag{6.9}$$

Thus, in the $|K^0\rangle$ and $|\bar{K}^0\rangle$ basis, $\mathcal{S}$ can be represented as:

$$\mathcal{S} = |K^0\rangle\langle K^0| - |\bar{K}^0\rangle\langle\bar{K}^0| \doteq \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \sigma_3. \tag{6.10}$$

ii) K-mesons are pseudoscalars, thus the parity operator act on them as,

$$\mathcal{P}|K^0\rangle = -|K^0\rangle, \tag{6.11}$$

$$\mathcal{P}|\bar{K}^0\rangle = -|\bar{K}^0\rangle. \tag{6.12}$$

Thus in the K-state space, $\mathcal{P}$ is represented as the negative identity matrix:

$$\mathcal{P} = -I = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{6.13}$$

This is an expected result, as the strong interactions is invariant under $\mathcal{P}$.

iii) K^0 and $\bar{K}^0$ are anti-particles of each other: Thus the charge conjugation operator should convert them into each other:

$$\mathcal{C}|K^0\rangle = |\bar{K}^0\rangle, \tag{6.14}$$

$$\mathcal{C}|\bar{K}^0\rangle = |K^0\rangle. \tag{6.15}$$

iv) So that, the combined $\mathcal{CP}$ operator should act on $|K^0\rangle$ and $|\bar{K}^0\rangle$ as:

$$\mathcal{CP}|K^0\rangle = -|\bar{K}^0\rangle, \tag{6.16}$$

$$\mathcal{CP}|\bar{K}^0\rangle = -|K^0\rangle. \tag{6.17}$$

Therefore, in the K-state space it can be represented as:

$$\mathcal{CP} = -\left(|K^0\rangle\langle\bar{K}^0| + |\bar{K}^0\rangle\langle K^0|\right) \doteq \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = -\sigma_1. \tag{6.18}$$

In the light of these, the total (strong+weak) CP-conserving Hamiltonian can be constructed by adding to H_0 the only CP-conserving term in the 2-state space, namely σ_1:

$$H = m_0 I + \delta\sigma_1. \tag{6.19}$$

The eigenstates of this total CP-conserving Hamiltonian will be the eigenstates of the σ_1-operator, this is because as the strong part is proportional to the identity matrix (guaranteeing the degeneracy of the masses of strangeness eigenstates of K^0 and $\bar{K}^0$)[2].

These new eigenstates of the full Hamiltonian, the CP eigenstates (the eigenstates of σ_1), can be expressed in the σ_1-basis as

$$|+\rangle_x = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } |-\rangle_x = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}. \tag{6.20}$$

Calling $|+\rangle_x$ and $|-\rangle_x$ as $|K_1^0\rangle$ and $|K_2^0\rangle$, respectively, one can write;

$$|K_1^0\rangle = \frac{1}{\sqrt{2}} \left(|K^0\rangle + |\bar{K}^0\rangle \right), \tag{6.21}$$

$$|K_2^0\rangle = \frac{1}{\sqrt{2}} \left(|K^0\rangle - |\bar{K}^0\rangle \right), \tag{6.22}$$

with the corresponding eigenvalues:

$$\lambda_1^0 = m_0 + \delta, \tag{6.23}$$

$$\lambda_2^0 = m_0 - \delta. \tag{6.24}$$

[2] An important note is in order here: The H_0 is just the identity operator which commutes with any operator in 2-state space. Thus one can choose its eigenstates freely, physics leads the way in this choice, and they are taken as the eigenstates of the strangeness operator $\mathcal{S} = \sigma_3$.

To summarize, the starting point was the strangeness eigenstates, $|K^0\rangle$ and $|\bar{K}^0\rangle$, with equal masses. This is natural due to the CPT theorem, as K^0 and $\bar{K}^0$ are the anti-particles of each other. In the presence of weak interactions, the eigenstates $|K_1^0\rangle$ and $|K_2^0\rangle$ emerge with different masses. Note that for the full Hamiltonian $H = H_0 + H_1$, strangeness is no more conserved:

$$[H, \mathcal{S}] = [m_0 I + \delta\sigma_1, \sigma_3] = -i\delta\sigma_2. \tag{6.25}$$

It is further to be re-iterated that; $H_{weak} = \delta\sigma_1$ term mixes strange particle with opposite strangeness; and δ is real (due to CP-symmetry, imaginary part of δ is forbidden by CP)

The CP properties of these new eigenstates $|K_1^0\rangle$ and $|K_2^0\rangle$ are given by:

$$\mathcal{CP}|K_1^0\rangle = (+1)|K_1^0\rangle, \tag{6.26}$$

$$\mathcal{CP}|K_2^0\rangle = (-1)|K_2^0\rangle. \tag{6.27}$$

Here, $|K_1^0\rangle$ and $|K_2^0\rangle$ are the actual decaying states of the neutral K-system. In fact, two different decay modes were observed: A short-lived K_S meson, which decays into 2π, and a long-lived K_L meson which decays into 3π (among other things). The K_S meson was identified as being the $CP = +1$ state (K_1^0). The K_L was identified with $CP = -1$ state (K_2^0), due 3π final state has $CP = -1$.

This fit very well into the simple framework described above, until the discovery in 1964 [10], that the long-lived K_L, decayed also into 2π's. In 1964, Fitch and Cronin showed that there was 45 number of 2π events out of 22,700 events (nearly 1 in 500) at the end of the beam. Therefore, some of the long lived meson should also decayed into 2π state which was the evidence for $\mathcal{CP}$-violation.

This should be due to the fact that the decaying states, K_S and K_L have an admixture of the "opposite type of CP's".

This means that one has to modify the Hamiltonian given in eq. (6.19) by adding a CP-violating term. CP-violating term, in principle could be a superposition of σ_2 and σ_3, as both of these do not commute with σ_1, and thus with the old CP-conserving Hamiltonian given in eq. (6.2).

However CPT theorem forbids a σ_3-type breaking. Indeed, as will be demonstrated later, CPT requires that in H_{eff} the diagonal terms should be same, implying equality of the masses of particles and anti-particles[3].

Thus, only σ_2 type breaking is left to be considered for the purpose of breaking CP-invariance. Here, the requirement of Hermiticity has to be relaxed to allow decay. Therefore, one can take the general, non-Hermitian, CP-violating Hamiltonian as;

$$H = \begin{bmatrix} A & B/z \\ Bz & A \end{bmatrix}, \tag{6.28}$$

with complex A, B, and z. Notice that in eq. (6.28) one recovers the CP-conserving Hamiltonian, for $z = 1$. Thus any deviation of z from 1, will be a

[3] Furthermore, one can show that this type of breaking does not induce an admixture of wrong CP-components to lowest order, to the CP-eigenstates.

measure of CP-violation.

The eigenvalues $\lambda_{1,2}$ of eq. (6.28) can be determined as:

$$\lambda_{1,2} = A \pm B. \tag{6.29}$$

As H is not Hermitian, $\lambda_{1,2}$ are not to be real anymore. Similarly the corresponding eigenstates would not be necessarily orthogonal.

The normalized eigenstates, corresponding to $\lambda_{1,2}$ can be determined and expressed in the K^0-$\bar{K}^0$ basis as

$$|K_1\rangle = N \begin{bmatrix} 1 \\ z \end{bmatrix}, \text{ and } |K_2\rangle = N \begin{bmatrix} 1 \\ -z \end{bmatrix}, \tag{6.30}$$

where $N = 1/\sqrt{1+|z|^2}$ is the normalization constant. This requires some clarification: As one gets the CP-conserving Hamiltonian in the limit $z \to 1$, one should recover the corresponding CP-eigenstate K_1^0 , K_2^0 also in this limit. That this is indeed the case can be seen from eq. (6.30). As the expressions for K_1^0 , K_2^0 are given in the K^0 , $\bar{K}^0$ basis in eqs. (6.21) and (6.22), so the new expressions are also to be in the same (original) K^0 , $\bar{K}^0$ basis.

The new eigenstates are not indeed orthogonal:

$$\langle K_1|K_2\rangle = \frac{1-|z|^2}{1+|z|^2}. \tag{6.31}$$

Next, the CP-violation in K decays to pions, and the oscillation problem will be discussed:

To set the stage, one needs to invert the expressions (6.30), and express K^0, $\bar{K}^0$ in terms of K_1 and K_2:

$$|K^0\rangle = \frac{1}{2N}\left(|K_1\rangle + |K_2\rangle\right), \tag{6.32}$$

$$|\bar{K}^0\rangle = \frac{1}{2zN}\left(|K_1\rangle - |K_2\rangle\right). \tag{6.33}$$

To discuss the oscillation, one can start with an initial state $|\psi(t=0)\rangle = |K^0\rangle$ and follow, how it evolves in time:

$$|\psi(t)\rangle = U(t)|\psi(t=0)\rangle. \tag{6.34}$$

Here $U(t)$ is the time evolution operator for the Hamiltonian given in eq. (6.28):

$$U(t) = e^{-iHt}. \tag{6.35}$$

To compute the action of $U(t)$ on $|K^0(t=0)\rangle$, one has to use eqs. (6.32) and (6.33), expressing K^0 in terms of the eigenstates of Hamiltonian:

$$|K^0(t)\rangle = U(t)|K^0(t=0)\rangle = \frac{1}{2N}e^{-iHt}\left(|K_1\rangle + |K_2\rangle\right), \tag{6.36}$$

$$= \frac{1}{2N}\left(e^{-it\lambda_1}|K_1\rangle + e^{-it\lambda_2}|K_2\rangle\right). \tag{6.37}$$

Converting $|K_1\rangle$ and $|K_2\rangle$ back to $|K^0\rangle$, $|\bar{K}^0\rangle$ again one gets the time evolved $|K^0(t)\rangle$ in terms of $|K^0\rangle$ and $|\bar{K}^0\rangle$:

$$|K^0(t)\rangle = \frac{1}{2N}\left(e^{-it\lambda_1}N\left(|K^0\rangle + z|\bar{K}^0\rangle\right) + e^{-it\lambda_2}N\left(|K^0\rangle - z|\bar{K}^0\rangle\right)\right), \tag{6.38}$$

$$|K^0(t)\rangle = \frac{1}{2}\left[\left(e^{-it\lambda_1} + e^{-it\lambda_2}\right)|K^0\rangle + z\left(e^{-it\lambda_1} - e^{-it\lambda_2}\right)|\bar{K}^0\rangle\right]. \tag{6.39}$$

Thus, finally one gets the probability amplitude of the surviving K^0-component or $\bar{K}^0$-component after a time t,

$$\langle K^0|K^0(t)\rangle = \frac{1}{2}\left(e^{-it\lambda_1} + e^{-it\lambda_2}\right), \tag{6.40}$$

$$\langle \bar{K}^0|K^0(t)\rangle = \frac{z}{2}\left(e^{-it\lambda_1} - e^{-it\lambda_2}\right). \tag{6.41}$$

The same analysis can be repeated for an initial $\bar{K}^0$ beam:

$$\langle \bar{K}^0|\bar{K}^0(t)\rangle = \frac{1}{2}\left(e^{-it\lambda_1} + e^{-it\lambda_2}\right), \tag{6.42}$$

$$\langle K^0|\bar{K}^0(t)\rangle = \frac{1}{2z}\left(e^{-it\lambda_1} - e^{-it\lambda_2}\right). \tag{6.43}$$

It should be noted that,

$$\langle K^0|K^0(t)\rangle = \langle \bar{K}^0|\bar{K}^0(t)\rangle, \tag{6.44}$$

which is not unexpected. This is a consequence of the CPT theorem, as the Hamiltonian is chosen to be CPT invariant.

As one possible test of CP-violation, one may calculate the following asymmetry, using the eqs. (6.41) and (6.43):

$$\mathcal{A}_{CP} = \frac{|\langle \bar{K}^0|K^0(t)\rangle|^2 - |\langle K^0|\bar{K}^0(t)\rangle|^2}{|\langle \bar{K}^0|K^0(t)\rangle|^2 + |\langle K^0|\bar{K}^0(t)\rangle|^2} \approx 2Re(\epsilon) + O(\epsilon^3). \tag{6.45}$$

Here z is parametrized as $z = 1 + \epsilon$, where ϵ is the small CP-violating complex parameter.

Setting $z = 1$, or equivalently $\epsilon = 0$ one gets, $\mathcal{A}_{CP} = 0$, as one should. Thus, by measuring this asymmetry, one can determine the real part of the CP-violating complex phase.

Next, we turn our attention to the π decays. First, one notes that the mass differences, namely the available decay energies in the $CP = +1$ channel and $CP = -1$ channels are different:

$$m_K - 2m_\pi \approx 220MeV \tag{6.46}$$

$$m_K - 3m_\pi \approx 80MeV \tag{6.47}$$

That is, there is more phase space available for the 2π decay. Thus using a rough uncertainty argument, one expects that 2π decay should go faster than the 3π. The actual explanation however requires a detailed phase space analysis.

Indeed, K_1^0's mostly decay in a few centimeters, on the other hand K_2^0's can live through many meters. In fact, the lifetimes of K_1^0 and K_2^0 are found as:

$$\tau_1 = 0.895 \times 10^{-10}\, seconds, \tag{6.48}$$

$$\tau_2 = 5.11 \times 10^{-10}\, seconds. \tag{6.49}$$

That is why they are called short-lived, and long-lived, respectively. As they are not anti-particle of each other, they have different masses however small they may be. Indeed the tiny mass difference Δm is given as:

$$\Delta m = m_2 - m_1 = 3.48 \times 10^{-6}\, eV. \tag{6.50}$$

To find the probabilities of decaying in the 2π and 3π final states for K_S and

K_L in the CP-violating case, one first needs to express these states in terms of the CP-eigenstates:

$$|K_S\rangle = N\left(|K^0\rangle + z|\bar{K}^0\rangle\right) = \frac{N}{\sqrt{2}}\left((1+z)\,|K_1^0\rangle + (1-z)\,|K_2^0\rangle\right), \qquad (6.51)$$

$$|K_L\rangle = N\left(|K^0\rangle - z|\bar{K}^0\rangle\right) = \frac{N}{\sqrt{2}}\left((1-z)\,|K_1^0\rangle + (1+z)\,|K_2^0\rangle\right). \qquad (6.52)$$

These two sets of eigenstates $|K_{1,2}\rangle$ and $|K_{1,2}^0\rangle$ are indeed the same in the $z \to 1$ limit (in the CP-symmetric limit).

Thus, to see the CP-violating effects, one should observe the decay products towards the end of K-beam[4]:

$$\langle 2\pi|K_S\rangle = \frac{1+z}{\sqrt{2\left(1+|z|^2\right)}}\langle 2\pi|K_1^0\rangle, \qquad (6.53)$$

$$\langle 3\pi|K_L\rangle = \frac{1-z}{\sqrt{2\left(1+|z|^2\right)}}\langle 3\pi|K_2^0\rangle. \qquad (6.54)$$

Thus, choosing $\langle 2\pi|K_1^0\rangle$, $\langle 3\pi|K_2^0\rangle$ real for simplicity; one obtains

$$\frac{|\langle 3\pi|K_S\rangle|^2}{|\langle 2\pi|K_S\rangle|^2} = \frac{|1-z|^2}{|1+z|^2}\frac{|\langle 2\pi|K_1^0\rangle|^2}{|\langle 3\pi|K_2^0\rangle|^2} \approx \frac{1}{4}|\epsilon|^2\frac{|\langle 2\pi|K_1^0\rangle|^2}{|\langle 3\pi|K_2^0\rangle|^2}. \qquad (6.55)$$

Similarly, repeating the same analysis for K_L, one gets

[4] If CP was conserved, all the $K_S = K_1^0$ would have decayed out by the time they reach to the end of tube, which was about 17m long in the Fitch-Cronnin[10] experiment.

$$\langle 2\pi | K_L \rangle = \frac{1-z}{\sqrt{2\,(1+|z|^2)}} \langle 2\pi | K_S^0 \rangle, \tag{6.56}$$

$$\langle 3\pi | K_L \rangle = \frac{1+z}{\sqrt{2\,(1+|z|^2)}} \langle 3\pi | K_L^0 \rangle, \tag{6.57}$$

and, from the ratio of these, one obtains,

$$\frac{|\langle 2\pi | K_L \rangle|^2}{|\langle 3\pi | K_L \rangle|^2} = \frac{|1-z|^2}{|1+z|^2} \frac{|\langle 2\pi | K_1^0 \rangle|^2}{|\langle 3\pi | K_2^0 \rangle|^2} \approx \frac{1}{4} |\epsilon|^2 \frac{|\langle 2\pi | K_1^0 \rangle|^2}{|\langle 3\pi | K_2^0 \rangle|^2}, \tag{6.58}$$

that the equality of these two CP-violating measures is very satisfying.

One can repeat the decay analysis for an initial K^0-state, and to see how the 2π and 3π decay modes follow in an oscillating sequence. To obtain this, starting by expressing the $|K^0(t)\rangle$ as given in eq. (6.39), in terms of the CP-eigenstate $|K_1^0\rangle$ and $|K_2^0\rangle$, one gets:

$$\begin{aligned} |K^0(t)\rangle &= \frac{1}{2\sqrt{2}} \left[\left(e^{-it\lambda_1} + e^{-it\lambda_2} \right) \left(|K_1^0\rangle + |K_2^0\rangle \right) \right] \\ &+ \frac{z}{2\sqrt{2}} \left[\left(e^{-it\lambda_1} - e^{-it\lambda_2} \right) \left(|K_1^0\rangle - |K_2^0\rangle \right) \right], \end{aligned} \tag{6.59}$$

$$\begin{aligned} K^0(t)\rangle &= \frac{1}{2\sqrt{2}} \left[(1+z)\, e^{-it\lambda_1} + (1-z)\, e^{-it\lambda_2} \right] |K_1^0\rangle \\ &+ \left[(1-z)\, e^{-it\lambda_1} + (1+z)\, e^{-it\lambda_2} \right] |K_1^0\rangle. \end{aligned} \tag{6.60}$$

Thus, the decay probability amplitudes are:

$$\langle 2\pi|K^0(t)\rangle = \frac{1}{2\sqrt{2}}\left[(1+z)\,e^{-it\lambda_1} + (1-z)\,e^{-it\lambda_2}\right]\langle 2\pi|K_1^0\rangle, \tag{6.61}$$

$$\langle 3\pi|K^0(t)\rangle = \frac{1}{2\sqrt{2}}\left[(1-z)\,e^{-it\lambda_1} + (1+z)\,e^{-it\lambda_2}\right]\langle 3\pi|K_2^0\rangle. \tag{6.62}$$

In the CP-symmetric limit, eqs. (6.61) and (6.62) reduce to the following form:

$$\langle 2\pi|K^0(t)\rangle = \frac{e^{-it\lambda_1}}{\sqrt{2}}\langle 2\pi|K_1^0\rangle, \tag{6.63}$$

$$\langle 3\pi|K^0(t)\rangle = \frac{e^{-it\lambda_2}}{\sqrt{2}}\langle 3\pi|K_2^0\rangle. \tag{6.64}$$

Again, choosing as before the $\langle 2\pi|K_1^0\rangle$, $\langle 3\pi|K_2^0\rangle$ real, one can get for the probability ratio of the $CP = +1$ and $CP = -1$ decay channels, by defining $\lambda_{1,2}^0 = m_{1,2}^0 - \frac{i}{2}\Gamma_{1,2}^0$

$$\frac{|\langle 2\pi|K^0(t)\rangle|^2}{|\langle 3\pi|K^0(t)\rangle|^2} = e^{-\left(\Gamma_1^0-\Gamma_2^0\right)t}\frac{|\langle 2\pi|K_1^0\rangle|^2}{|\langle 3\pi|K_2^0\rangle|^2}. \tag{6.65}$$

The main purpose of this section is just to illustrate and discuss the beautiful phenomenon of quantum mechanical interference. Therefore we do not enter into a detailed discussion of the phenomenology of the CP-violation in neutral K-meson system, which is quiet involved, because it involves six parameters.

CHAPTER 7

CONCLUSION

In physics, symmetry plays a key role in gaining an understanding of the physical laws governing the structure and the behavior of matter. It provides a short-cut for getting at some of Nature's innermost secrets. For instance, the laws of conservation of energy and momentum emerge because the laws of physics are the same at any time, translational symmetry in time, and any location, translational symmetry in space. In addition to these continuous symmetries, there are also discrete symmetries which play a particularly important rule in the context of quantum theory, namely space and time reversals, and the charge conjugation.

Time probably is the oldest but also the least understood concepts in physics. It involves rather special philosophical and mathematical peculiarities, like the anti-unitarity of the corresponding operators in the quantum space. Thus, special attention is devoted to address those peculiarities in detail in this thesis.

The combination of these three discrete symmetries has risen to a very central position in the context of Relativistic Quantum Field Theory, namely the CPT theorem. In the present level of our understanding, there are solid reasons for expecting that nature has CPT symmetry; that is laws of physics stay the same after reversing charge, space, and time all together. Recently there is also intense research going on in relation with the correlations between possible violations of CPT and the Lorentz symmetries. This problem is interesting, because one of the inputs which goes into the proof of CPT theorem is the Lorentz invariance. Because of its primary implication concerning the equality of masses of particles and antiparticles, the violation of the CPT symmetry would have significant physical consequences.

Therefore, if CPT is a symmetry of nature, CP symmetry implies time-reversal invariance. Conversely, violation of one implies the violation of the other. Given the fact that we live in a matter dominated universe, the explanation of the lack of matter-antimatter symmetry, or equivalently the CP violation is one of the challenging problems awaiting for a solution.

Neutral K meson system was the first physical system in which CP violation was discovered. Eventually this was extended to other neutral meson systems like the B and D mesons. Presently the only viable theoretical framework which introduces some sort of CP violating mechanism in the standard model of micro world is the CKM mechanism. However this also did not prove to be the ultimate explanation, because this violation is too small to explain the cosmological CP violation, namely the matter-antimatter asymmetry of the universe.

Because of its extreme beauty and simplicity, the neutral K meson system both in the CP conserving and the violating cases, is taken up in this thesis in the framework of quantum mechanical 2-state problem.

APPENDIX A

RELATIVISTIC NOTATION

In the relativistic case, space and time are on the same footing and are described by a contravariant four-vector x^μ, which is defined as:

$$x^\mu = \left(x^0, x^1, x^2, x^3\right), \tag{A.1}$$

where $x^0 = t$ is the time, and x^is are the 3 dimensional space components, respectively. Here, i runs from 1 to 3, and μ from 0 to 3. Since the space-time is flat in Special Relativity, the metric $g^{\mu\nu}$ is chosen as:

$$g^{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}. \tag{A.2}$$

With the help of this metric, one can pass from the contravariant entities to the covariant ones: Namely $g^{\mu\nu}$ and $g_{\mu\nu}$ play the role of raising and lowering operators, respectively. Therefore, one can construct the covariant position and momentum vectors (as well as the similar entities, like gradients etc.) as:

$$x_\mu = g_{\mu\nu} x^\nu = \left(x^0, -x^1, -x^2, -x^3\right), \tag{A.3}$$

$$\partial^\mu = \frac{\partial}{\partial x_\mu} = \left(\frac{\partial}{\partial x_0}, \frac{\partial}{\partial x_i}\right) = \left(\frac{\partial}{\partial t}, -\nabla\right), \tag{A.4}$$

$$p^\mu = i\partial^\mu = i\left(\frac{\partial}{\partial t}, -\nabla\right). \tag{A.5}$$

One notes that the contravariant four-momentum vector and the can be easily described in terms of the energy and momentum as:

$$p^\mu = (E, \, \mathbf{p}), \tag{A.6}$$

whereas the four-potential of the electromagnetic field is given by,

$$A^\mu = \left(A^0, \, \mathbf{A}\right). \tag{A.7}$$

In eq. (A.7), $A^0(x)$ and $\mathbf{A}(x,t)$ are scalar and vector potentials respectively.

APPENDIX B

ANTI-UNITARY OPERATORS

If there exists an operator $\mathcal{A}$ affecting transformation between two arbitrary vectors of a Hilbert Space $|\alpha\rangle$ and $|\beta\rangle$:

$$|\alpha\rangle \to |\widetilde{\alpha}\rangle = \mathcal{A}|\alpha\rangle \text{ and } |\beta\rangle \to |\widetilde{\beta}\rangle = \mathcal{A}|\beta\rangle, \tag{B.1}$$

with the property

$$\mathcal{A}(c_1|\alpha\rangle + c_2|\beta\rangle) = c_1^*\mathcal{A}|\alpha\rangle + c_2^*\mathcal{A}|\beta\rangle, \tag{B.2}$$

then, $\mathcal{A}$ is said to be *anti-linear [14]*. Furthermore, if it satisfies the following additional property

$$\langle\widetilde{\beta}|\widetilde{\alpha}\rangle = \langle\beta|\alpha\rangle^* = \langle\alpha|\beta\rangle, \tag{B.3}$$

then, $\mathcal{A}$ is said to be *anti-unitary*. Any anti-unitary operator $\mathcal{A}$ can be represented as

$$\mathcal{A} = UK, \tag{B.4}$$

where U is a unitary operator, and K is the complex conjugation operator in some given basis. K satisfies the following,

$$Kc|\alpha\rangle = c^* K|\alpha\rangle, \tag{B.5}$$

and if eq. (B.5) multiplied by K once more, one gets

$$K^2 c|\alpha\rangle = Kc^* K|\alpha\rangle = cK^2|\alpha\rangle, \tag{B.6}$$

which means

$$K^2 = I. \tag{B.7}$$

That is,

$$K^{-1} = K. \tag{B.8}$$

One can show that anti-unitary $\mathcal{A}$ satisfies the *antilinearity* condition:

$$\begin{aligned}\mathcal{A}(c_1|\alpha\rangle + c_2|\beta\rangle) &= UK(c_1|\alpha\rangle + c_2|\beta\rangle), \\ &= c_1^* UK|\alpha\rangle + c_2^* UK|\beta\rangle, \\ &= c_1^* \mathcal{A}|\alpha\rangle + c_2^* \mathcal{A}|\beta\rangle. \end{aligned} \tag{B.9}$$

In the litreature there is a controversy concerning the definition of the adjoint for the anti-linear operators. Some authors take positive standing, some not. Authors like Weinberg[1] [18], and Messiah [20] define and proceed with adjoint conjugate. Sakurai is one of those who is against the introduction of the adjoint. Therefore, in this thesis, in order to avoid taking part in the polemics, Sakurai's approach will be adopted and followed.

In the following some interesting properties of the anti-unitary operators will be presented, for completeness.

Before reviewing the properties of the anti-unitary operators, one needs to first note that the complex conjugation operation does not affect the base kets $|\kappa\rangle$, as they are composed of real parameters:

$$K|\kappa\rangle = (|\kappa\rangle)^* = |\kappa\rangle. \tag{B.10}$$

In order to avoid the introduction of the adjoint one has to first investigate how the bra's are effected by the action of anti-unitary operators $\mathcal{A}$. This will be done by first considering the action on the kets and then taking the dual

[1] Weinberg defines the adjoint of an anti-unitary operator $\mathcal{A}$ as $\langle\alpha|\mathcal{A}^\dagger\beta\rangle \equiv \langle\mathcal{A}\alpha|\beta\rangle^* = \langle\beta|\mathcal{A}\alpha\rangle$.

correspondence of the kets, as summarized below:

$$
\begin{aligned}
|\alpha\rangle \overset{\mathcal{A}}{\to} |\widetilde{\alpha}\rangle &= \mathcal{A}\sum_{\alpha'} |\alpha'\rangle\langle\alpha'|\alpha\rangle, \\
&= \sum_{\alpha'} \left(\langle\alpha'|\alpha\rangle\right)^* \mathcal{A}|\alpha'\rangle, \\
&= \sum_{\alpha'} \left(\langle\alpha'|\alpha\rangle\right)^* UK|\alpha'\rangle, \\
&= \sum_{\alpha'} \left(\langle\alpha'|\alpha\rangle\right)^* U|\alpha'\rangle, \qquad \text{(B.11)}
\end{aligned}
$$

where $\{|\alpha'\rangle\}$ and $\{|\alpha''\rangle\}$ are arbitrary base kets. Repeating the same for the ket $|\beta\rangle$, and taking the dual correspondence of the transformed $|\beta\rangle$ one gets:

$$
\langle\widetilde{\beta}| = \sum_{\alpha''} \left(\langle\alpha''|\beta\rangle\right) \langle\alpha''|U^\dagger. \qquad \text{(B.12)}
$$

Then by using eqs. (B.11) and (B.12), one can form the inner product $\langle\widetilde{\beta}|\widetilde{\alpha}\rangle$:

$$
\begin{aligned}
\langle\widetilde{\beta}|\widetilde{\alpha}\rangle &= \sum_{\alpha''}\sum_{\alpha'} \langle\alpha''|\beta\rangle\langle\alpha''|U^\dagger U|\alpha'\rangle\langle\alpha|\alpha'\rangle, \qquad \text{(B.13)} \\
&= \sum_{\alpha'} \langle\alpha|\alpha'\rangle\langle\alpha'|\beta\rangle = \langle\alpha|\beta\rangle, \\
&= \langle\beta|\alpha\rangle^*.
\end{aligned}
$$

Next, an important property will be demonstrated:

$$
\langle\widetilde{\beta}|\mathcal{A}\mathcal{O}\mathcal{A}^{-1}|\widetilde{\alpha}\rangle = \langle\alpha|O^\dagger|\beta\rangle. \qquad \text{(B.14)}
$$

where $\mathcal{O}$ is an arbitrary linear operator .

To demonstrate this identity, first B.11 and B.12 will be used, to express $|\widetilde{\alpha}\rangle$ and $\langle\widetilde{\beta}|$, and the reality of the basis:

$$
\begin{aligned}
\langle\widetilde{\beta}|\mathcal{A}\mathcal{O}\mathcal{A}^{-1}|\widetilde{\alpha}\rangle &= \left(\sum_{\alpha''}\left(\langle\alpha''|\beta\rangle\right)\langle\alpha''|U^{\dagger}\right)\left(\mathcal{A}\mathcal{O}\mathcal{A}^{-1}\right)\left(\sum_{\alpha'}\left(\langle\alpha'|\alpha\rangle\right)^{*}U|\alpha'\rangle\right) \\
&= \sum_{\alpha''}\sum_{\alpha'}\langle\alpha''|\beta\rangle\langle\alpha''|U^{\dagger}\left(\mathcal{A}\mathcal{O}\mathcal{A}^{-1}\right)U|\alpha'\rangle\langle\alpha|\alpha'\rangle, \\
&= \sum_{\alpha''}\sum_{\alpha'}\langle\alpha''|\beta\rangle\langle\alpha''|U^{\dagger}\left(UK\mathcal{O}K^{-1}U^{-1}\right)U|\alpha'\rangle\langle\alpha|\alpha'\rangle, \\
&= \sum_{\alpha''}\sum_{\alpha'}\langle\alpha''|\beta\rangle\langle\alpha''|\left(K\mathcal{O}K^{-1}\right)|\alpha'\rangle\langle\alpha|\alpha'\rangle, \\
&= \sum_{\alpha''}\sum_{\alpha'}\langle\alpha''|\beta\rangle\langle\alpha''|\left(\mathcal{O}\right)^{*}|\alpha'\rangle\langle\alpha|\alpha'\rangle, \\
&= \sum_{\alpha''}\sum_{\alpha'}\langle\alpha''|\beta\rangle\left(\langle\alpha''|O|\alpha'\rangle\right)^{*}\langle\alpha|\alpha'\rangle, \\
&= \sum_{\alpha''}\sum_{\alpha'}\langle\alpha|\alpha'\rangle\left(\langle\alpha'|O^{\dagger}|\alpha''\rangle\right)\langle\alpha''|\beta\rangle.
\end{aligned}
$$

Using the completeness of the bases $\{|\alpha'\rangle\}$ and $\{|\alpha''\rangle\}$, one finally obtains eq. (B.14).

This identity can also be proven in a more compact manner. Defining:

$$|\chi\rangle \equiv \mathcal{O}|\alpha\rangle, \tag{B.15}$$

and taking the dual correspondence one gets:

$$|\chi\rangle \overset{DC}{\rightarrow} \langle\chi| = \langle\alpha|\mathcal{O}^{\dagger}, \tag{B.16}$$

Then, $\langle\alpha|\mathcal{O}^\dagger|\beta\rangle$ becomes:

$$\langle\alpha|\mathcal{O}^\dagger|\beta\rangle = \langle\chi|\beta\rangle. \tag{B.17}$$

By using eq. (B.13), one gets;

$$\langle\alpha|\mathcal{O}^\dagger|\beta\rangle = \langle\widetilde{\beta}|\widetilde{\chi}\rangle. \tag{B.18}$$

Using eq. (B.15), $|\widetilde{\chi}\rangle$ becomes,

$$|\widetilde{\chi}\rangle = \mathcal{A}|\chi\rangle = \mathcal{A}\mathcal{O}|\alpha\rangle. \tag{B.19}$$

Thus, eq. (B.18) becomes,

$$\begin{aligned}\langle\alpha|\mathcal{O}^\dagger|\beta\rangle &= \langle\widetilde{\beta}|\widetilde{\chi}\rangle,\\ &= \langle\widetilde{\beta}|\mathcal{A}\mathcal{O}|\alpha\rangle.\end{aligned} \tag{B.20}$$

Inserting identity $I = \mathcal{A}^{-1}\mathcal{A}$, in the RHS of eq. (B.20),

$$\begin{aligned}\langle\alpha|\mathcal{O}^\dagger|\beta\rangle &= \langle\widetilde{\beta}|\mathcal{A}\mathcal{O}\mathcal{A}^{-1}\mathcal{A}|\alpha\rangle,\\ &= \langle\widetilde{\beta}|\mathcal{A}\mathcal{O}\mathcal{A}^{-1}|\widetilde{\alpha}\rangle,\end{aligned} \tag{B.21}$$

which is the identity intended to be proven.

In the following further properties will be briefly summarized:

i) The product of two anti-unitary operators are unitary: Labelling them as $\mathcal{A}_1 \equiv U_1 K$ and $\mathcal{A}_2 = U_2 K$, one can construct the product:

$$\mathcal{A}_1 \mathcal{A}_2 = U_1 K U_2 K = U_1 U_2^* K^2 = U_1 U_2^*. \tag{B.22}$$

To demonstrate that the RHS of eq. (B.22) is indeed unitary one can proceed as:

$$(U_1 U_2^*) (U_1 U_2^*)^\dagger = U_1 U_2^* U_2^{*\dagger} U_1^\dagger = U_1 \left(U_2 U_2^\dagger\right)^* U_1^\dagger = U_1 U_1^\dagger = I. \tag{B.23}$$

ii) The product of anti-unitary operator $\mathcal{A}$ and linear operator L gives anti-linear operator:

$$\begin{aligned} \mathcal{A} L \left(c_1 |\alpha\rangle + c_2 |\beta\rangle\right) &= \mathcal{A} \left(c_1 L |\alpha\rangle + c_2 L |\beta\rangle\right) \\ &= c_1^* \mathcal{A} L |\alpha\rangle + c_2^* \mathcal{A} L |\beta\rangle \end{aligned}$$

iii) The product of a number of anti-unitary operators $\mathcal{A}$ and linear operators L gives:

$$\mathcal{A}_1...\mathcal{A}_m L_1...L_n = \left\{ \begin{array}{c} Linear\, for\, m\, is\, even \\ \\ anti - Linear\, for\, m\, is\, odd \end{array} \right\}. \tag{B.24}$$

iv) Moreover, from eq. (B.3), it can be seen that anti-unitary operators preserve the norm:

$$|\langle\beta|\alpha\rangle^*| = |\langle\alpha|\beta\rangle|. \tag{B.25}$$

APPENDIX C

SCHRÖDINGER FIELD THEORY

The schrödinger formulation of quantum mechanics can be reinterpreted as a classical field theory, treating the quantum wave function as a classical field. The Lagrangian density for this classical field theory is given by[1]:

$$\begin{aligned} \mathcal{L}(x) &= \frac{i}{2}\left[\psi^*(x)\,\frac{\partial\psi(x)}{\partial t} - \frac{\partial\psi^*(x)}{\partial t}\,\psi(x)\right] \\ &- \frac{1}{2m}\nabla\psi^*(x)\cdot\nabla\psi(x). \end{aligned} \tag{C.1}$$

An immediate observation one should make is that this lagrangian density is not relativistic. Because it involves first derivative in time and second derivatives in space. However, it obeys Galilean invariance. To find the Euler-Lagrange equations, one can start from the variation of the action,

$$\delta S = \delta \int_{t_1}^{t_2} dt \int_V d^3x\mathcal{L}, \tag{C.2}$$

[1] The classical field is denoted by ψ, as in the case of quantum mechanical wavefunction.

which yields;

$$\delta S = \int_{t_1}^{t_2} dt \int_V d^3x \left(\frac{\partial \mathcal{L}}{\partial \psi} \delta\psi + \frac{\partial \mathcal{L}}{\partial \dot{\psi}} \delta\dot{\psi} \right), \tag{C.3}$$

$$\delta S = \int_{t_1}^{t_2} dt \int_V d^3x \left[\left(\frac{\partial \mathcal{L}}{\partial \psi} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\psi}} \right) + \left(\frac{\partial \mathcal{L}}{\partial \psi^*} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\psi}^*} \right) \right] \delta\phi, \tag{C.4}$$

where $\delta\psi(\mathbf{x}, t_1) = \delta\psi(\mathbf{x}, t_2) = 0$ is used in the last step. Since;

$$\delta S = 0, \tag{C.5}$$

from the least action principle, Euler-Lagrange equation for classical field can be derived as,

$$\frac{\delta \mathcal{L}}{\delta \psi} - \frac{d}{dt}\frac{\delta \mathcal{L}}{\delta \dot{\psi}} = 0, \tag{C.6}$$

and,

$$\frac{\delta \mathcal{L}}{\delta \psi^*} - \frac{d}{dt}\frac{\delta \mathcal{L}}{\delta \dot{\psi}^*} = 0. \tag{C.7}$$

From (C.6), one gets the equation of motion in the following form:

$$-\frac{i}{2}\left(\frac{\partial}{\partial t}\psi^* \right) + \frac{1}{2m}\nabla^2\psi^* = \frac{i}{2}\left(\frac{\partial}{\partial t}\psi^* \right), \tag{C.8}$$

$$i\frac{\partial}{\partial t}\psi^* = [\frac{\nabla^2}{2m}]\psi^*, \tag{C.9}$$

and similarly (C.7) yields;

$$\frac{i}{2}\left(\frac{\partial}{\partial t}\psi\right) + \frac{1}{2m}\nabla^2\psi = -\frac{i}{2}\left(\frac{\partial}{\partial t}\psi\right), \tag{C.10}$$

$$i\frac{\partial}{\partial t}\psi = [-\frac{\nabla^2}{2m}]\psi. \tag{C.11}$$

An interesting feature in this new classical field theoretical framework is that one can understand the conservation of probability as a conserved charge related to the phase invariance of the schrödinger lagrangian in the context of Noether Theorem.

Namely, it can be seen that Lagrangian density given in eq. (C.1) is invariant under the space independent transformations $\psi \to \psi' = e^{-i\alpha}\psi \approx (I - i\delta\alpha)\psi$ and also its conjugate $\psi^* \to \psi'^* = e^{i\alpha}\psi^* = (I + i\delta\alpha)\psi^*$ for arbitrary parameter α. This symmetry leads to a conserved current via Noether Theorem, and is given by:

$$\begin{aligned}
\delta j_\mu &= \frac{\partial \mathcal{L}}{\partial\left(\partial^\mu \psi\right)}\delta\psi + \delta\psi^* \frac{\partial \mathcal{L}}{\partial\left(\partial^\mu \psi^*\right)}, \\
\delta\alpha j_\mu &= \frac{\partial \mathcal{L}}{\partial\left(\partial^\mu \psi\right)}\left(-i\delta\alpha\right)\psi + i\delta\alpha\,\psi \frac{\partial \mathcal{L}}{\partial\left(\partial^\mu \psi^*\right)}.
\end{aligned}$$

Therefore, using

$$j_\mu = -i\left[\frac{\partial\mathcal{L}}{\partial\left(\partial^\mu\psi\right)}\psi - \psi^*\frac{\partial\mathcal{L}}{\partial\left(\partial^\mu\psi^*\right)}\right], \tag{C.12}$$

,

one can construct the constant charge as:

$$\begin{aligned} Q &= \int d^3x j_0 = \int d^3x\,(-i)\left[\frac{\partial\mathcal{L}}{\partial\left(\partial^0\psi\right)}\psi - \psi^*\frac{\partial\mathcal{L}}{\partial\left(\partial^0\psi^*\right)}\right], \\ &= \int d^3x\frac{1}{2}\left[\psi^*\psi + \psi^*\psi\right] = \int d^3x|\psi|^2. \end{aligned}$$

It is seen that the conservation of total probability is obtained from the Noether theorem, as the conserved charge associated with phase symmetry of the schrödinger field theory.

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