USE OF LEARNING TRAJECTORIES BASED INSTRUCTION TO RESTRUCTURE MATHEMATICAL CONTENT AND STUDENT KNOWLEDGE OF PRE-SERVICE ELEMENTARY TEACHERS

A THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF SOCIAL SCIENCES
OF
MIDDLE EAST TECHNICAL UNIVERSITY

BY

ZUHAL YILMAZ

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY
IN
THE DEPARTMENT OF ELEMENTARY EDUCATION

JUNE 2015

Approval of the Graduate School of Social Sciences

Prof. Dr. Meliha ALTUNIŞIK
Director

I certify that this thesis satisfies all the requirements as a thesis for the degree of Doctor of Philosophy.

Prof. Dr. Ceren ÖZTEKİN
Head of Department

This is to certify that we have read this thesis and that in our opinion it is fully adequate, in scope and quality, as a thesis for the degree of Doctor of Philosophy.

Assoc. Prof. Dr. Çiğdem HASER
Supervisor

Examining Committee Members

Prof. Dr. Erdinç ÇAKIROĞLU	(METU, ELE)	
Assoc. Prof. Dr. Çiğdem HASER	(METU, ELE)	
Prof. Dr. Emine ERKTİN	(BOUN, ELE)	
Assist. Prof. Dr. Didem AKYÜZ	(METU, ELE)	
Assist. Prof. Dr. Engin ADER	(BOUN, ELE)	

I hereby declare that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.

Name, Last name: Zuhal YILMAZ

Signature :

ABSTRACT

USE OF LEARNING TRAJECTORIES BASED INSTRUCTION TO RESTRUCTURE PRE-SERVICE ELEMENTARY TEACHERS' MATHEMATICAL CONTENT AND STUDENT KNOWLEDGE

Yılmaz, Zuhal

Ph.D., Department of Elementary Education

Supervisor: Assoc. Prof. Dr. Çiğdem HASER

June 2015, 322 pages

The purpose of this study was to explore elementary pre-service teachers' (PTs) mathematical content knowledge (MCK) and student knowledge (SK) restructuring practices for equipartitioning related mathematical ideas. Nine senior PTs from a private university located southeastern part of the Turkey were voluntarily participated to Learning Trajectories Based Instruction (LTBI) teaching experiment. Classroom video data, written works of PTs, pre-post tests and field notes were the main sources of the data collected in the study.

The analysis of pre-test data showed that majority of PTs had a limited MCK and SK of equipartitioning related ideas. They exhibited serious mathematical misconceptions and errors. They rarely utilized multiple representations and strategies in their solutions. In addition, PTs exhibited a limited ability to anticipate students' mathematics. The analysis of the post-test data showed that LTBI helped

PTs to enhance their prior MCK and SK. They remediated their misconceptions and errors and utilized multiple mathematical strategies and representations, They started to anticipate a variety of students' mathematical strategies and misconceptions along with rich and accurate.mathematical explanations.

The findings of this study suggested an emergent framework of knowledge restructuring practices of PTs. They employed seven practices for their MCK and four practices for SK. PTs exhibited remediating and shifting, expanding and challenging practices for restructuring Common Content Knowledge, internalizing and sizing up practices for Specialized Content Knowledge, connecting and generalizing practices for Horizon Content Knowledge. They exhibited distinguishing and recognizing, anticipating, ordering and empathizing practices for restructuring their SK.

Keywords: Pre-Service Elementary Teachers, Learning Trajectories Based Instruction, Mathematical Content Knowledge, Student Knowledge, Equipartitioning

ÖZ

ÖĞRENME ROTALARI TEMELLİ ÖĞRETİMİN SINIF ÖĞRETMEN ADAYLARININ MATEMATİKSEL ALAN VE ÖĞRENCİ BİLGİLERİNİ YENİDEN YAPILANDIRILMASINDA KULLANIMI

Yılmaz, Zuhal

Doktora, İlköğretim Bölümü

Tez Yöneticisi: Doç. Dr. Çiğdem HASER

Haziran 2015, 322 sayfa

Bu çalışmanın amacı sınıf öğretmen adaylarının eşpaylaşım kavramı ile alakalı matematiksel alan bilgilerini ve öğrenci bilgilerini nasıl yeniden yapılandırdıklarını araştırmaktır. Türkiye'nin güneydoğusundaki özel bir üniversitede okuyan dokuz öğretmen adayı, öğrenme rotaları temelli öğretim deneyine gönüllü olarak katılmışlardır. Sınıf etkinliklerinin video çekimleri, adayların yazılı çalışmaları, ön ve son test ve alan notları bu çalışmanın veri toplama araçlarıdır.

Ön testin analizleri öğretmen adaylarının büyük bir çoğunluğunun eş paylaşım ile alakalı matematiksel düşüncelere dair kısıtlı bir alan bilgisine ve öğrenci bilgisine sahip olduğunu ortaya koymuştur. Adaylar ön testte ciddi kavram yanılgıları ve matematiksel hatalar göstermişlerdir. Adaylar nadiren birden fazla çözüm yolu ve gösterim kullanmışlardır. Adaylar öğrencinin matematiksel öğrenmesini anlamada kısıtlı bir beceri ortaya koymuşlardır. Son testin analizi öğrenme rotaları temelli öğretimin adayların matematiksel alan bilgilerini ve öğrenci bilgilerini iyileştirdiğini ortaya koymuştur. Adaylar sahip oldukları kavram yanılgı ve hatalarını düzeltmişlerdir ve çözümlerinde farklı matematiksel yol ve gösterim kullanmışlardır. Ek olarak, adaylar öğrencilerin matematiksel stratejilerini ve kavram yanılgılarını tahmin etmiş ve bunları doğru ve zengin bir matematiksel dille açıklamışlardır.

Bu çalışmanın sonucunda öğretmen adaylarının öğrenme rotaları temelli öğretimde bilgilerini yeniden yapılandırma eylemleri çerçevesi önerilmiştir. Adaylar matematiksel alan bilgilerini yedi ve öğrenci bilgilerini dört yeniden yapılandırma eylem çeşidi ile göstermişlerdir. Adaylar genel alan bilgilerini düzeltme ve değiştirme, genişletme ve meydan okuma, özel alan bilgilerini içselleştirme ve boyutlarını ortaya çıkarma, ufuk alan bilgilerini ilişkilendirme ve genelleme eylemlerini göstererek yeniden yapılandırmışlardır. Öğrenci bilgilerini yapılandırırken ise ayırt etme ve tanıma, öngörme, sıralama ve empati kurma eylemlerini göstermişlerdir.

Anahtar Kelimeler: Sınıf Öğretmen Adayları, Öğrenme Rotaları Temelli Öğretim, Matematiksel Alan Bilgisi, Öğrenci Bilgisi, Es Paylaşım

To my family
For their support, care and encouragement
To my fiancé, Hasan
For his love, patience, and understanding
To the people who make my life meaningful

ACKNOWLEDGEMENTS

I would like to thank my advisor and committee chairperson, Dr. Cigdem Haser for her insightful comments, support and patience throughout my doctoral studies. Dr. Haser has been an awesome mentor and has encouraged me to work hard on this dissertation.

I would also like to thanks my other committee members, Prof Dr. Emine Erktin, Prof. Dr. Erdinç Çakıroğlu, Assist. Prof. Dr. Didem Akyüz and Assist. Prof. Dr. Engin Ader for their time, support, and feedbacks to make this study better.

In addition, I would like to thank my friends Rabia, and Becky, Nadia and Ayanna for their support, expertise, friendship and feedbacks especially in my data analysis. I also would like to thank my colleagues and friends Isil, Lara, Cyndi, Marielle, Zeynep, Denise, and to many that I cannot list. Thank you for your friendship and encouragement.

Last but not least, I would like to thank the pre-service elementary teachers who agreed to participate in this study. Without you, this study could not be done.

Finally, I would like to thank my family dad, mom, my sister and brother, Muhittin Birinci and Prof. Dr. Esra Macaroglu Akgul, you have been always a support for me in this process.

TABLE OF CONTENTS

LIST OF TABLES

TABLES

LIST OF FIGURES

FIGURES

LIST OF ABBREVIATIONS

CCK: Common Content Knowledge

HCK: Horizon Content Knowledge

LTs: Learning Trajectories

LTBI: Learning Trajectories Based Instruction

MCK: Mathematical Content Knowledge

NCTM: National Council of Teachers of Mathematics

MEB: Ministry of Education

PTs: Pre-Service Teachers

SCK: Specialized Content Knowledge

SK: Student Knowledge

CHAPTER I

INTRODUCTION

Researchers have deeply examined how students understand, reason, and learn, and how they develop sophisticated thinking within the past two decades (Clements, Sarama & Julie, 2009; Fennema & Franke, 1992; Ma, 1999). Although many researchers studied how children learn over time, possible learning obstacles that they may encounter during learning and skills they need to acquire to learn a concept are only one side of the their learning. On the other side, teachers' knowledge and abilities to analyze students' mathematical reasoning and learning are important issues to consider because teachers have great influence on what and how students learn over time (Darling-Hammond & Ball, 1998; Ma, 1999). Their knowledge of mathematics has a critical influence on their practices in classroom (Clements, Sarama & Julie, 2009; Fennema & Franke, 1992; Ma, 1999). Although studies indicated the importance of teacher role in mathematics teaching process, several studies (Baki, 2013; Phillip, 2008; Spitzer, Phelps, Beyers, Johnson & Sieminski, 2011; Thipkong & Davis, 1991) have revealed that both pre-service and in-service teachers lack complete and comprehensive content knowledge required for the mathematics they teach. Therefore, in order to construct a better teaching and learning environment, teachers should be trained to develop an understanding for how children learn mathematics and to possess deep content knowledge to teach mathematics.

Knowing deeply the mathematics itself and understanding how students learn over time are the two critical issues that have impact on revisions and refinements in mathematics education (Darling-Hammond & Ball, 1998; Ma, 1999). Initially researchers suggested that Hypothetical Learning Trajectories (HLT) (Simon, 1995) have potential to contribute our understanding of how students' mathematical understanding evolves over time. They stated that HLT can also assist teachers in teaching and learning mathematics in their classrooms (Clements & Sarama, 2004; Duncan & Hmelo-Silver, 2009).

In his seminal work, Simon (1995) defined HLT as “the learning goal, the learning activities, and the thinking and learning in which the students might engage” (p. 133). He perceived these trajectories as hypothetical because individual students’ progression could not be predicted in advance (Sztajn, Wilson, Confrey & Edington, 2012). Although Simon (1995) named these trajectories as hypothetical, mathematics educators constructed learning trajectories (LT) that were derived from empirical data in recent approaches. There are various working definitions of these LTs. For instance, Corcoran, Mosher and Rogat (2009) stated LTs “...are hypothesized descriptions of the successively more sophisticated ways student thinking about an important domain of knowledge or practice develops as children learn about and investigate that domain over an appropriate span of time” (p.37). In addition, Clements and Sarama (2004) indicated LTs consisted of three parts that are a mathematical goal, children’s developmental route to achieve that goal and a set of instructional activities, or tasks for each level of the LT to support children higher level of mathematical thinking.

Learning trajectory definitions have generally used comprehensive research synthesis as a base, then they all identified a particular domain and examined how students’ mathematical thinking and learning proceeded over time from least to more complex nets of constructs (Yilmaz, 2011). Although these commonalities exist among definitions, Confrey, Maloney, Nyguyen, Mojica and Myers’s (2009) LT definition have been employed in this study because their definition had distinct LT features:

> A researcher-conjectured, empirically supported description of the ordered network of constructs a student encounters through instruction (i.e., activities, tasks, tools, forms of interaction, and methods of evaluation), in order to move from informal ideas, through successive refinements of representation, articulation and reflection, towards increasingly complex concepts over time (p.2).

The distinct features in Confrey et al.’s (2009) definition are embedded in the inclusion of the following phrases:

> *Researcher-conjectured* refers to the fact that LTs are models created by researchers who work on students' likely paths.
> *Empirically supported* refers to a three-step process: reviewing the literature, asking outside experts to review the syntheses, and conducting further studies.
> *Through instruction* is the recognition that students will only progress if provided appropriate opportunities, technology, and tools to learn the material and that the sequence of those activities must be designed intentionally to support the trajectory.
> *Through successive refinements* indicates the needs for students' active involvement in the learning process and engagement in cycles of problem-solving behavior (Confrey et al., 2009, p. 2–3).

Integrating LTs into education has great potential for contributing current stage of knowledge on how students learn (Clements, Sarama, & Julie, 2009; Confrey et al., 2009; Duncan & Hmelo-Silver, 2009). Yet the field has recently started explore to what extent LTs could be integrated into mathematical teaching practices, how LTs could be used in teacher education and used as a tool for planning phase of the instruction, and assessing students' learning (Clements, Sarama & Julie, 2009; Daro, Mosher & Corcoran, 2011; Sztajn, Wilson, Confrey & Edington, 2012). Although there are recent studies (e.g. Niess & Gillow-Wiles, 2014; Sztajn, Wilson, Edgington & Myers, 2014; Wilson, Sztajn, Edgington & Confrey, 2013) conducted on utilization of LTs in in-service teacher training to develop in-service teachers' mathematical content knowledge and pedagogical content knowledge, there are only a few studies on how pre-service teachers (PTs) utilize LTs for the similar aim (Butterfield, Forrester, McCallum & Chinnappan, 2013; Wilson, Mojica & Confrey, 2013a). Butterfield and his colleagues' (2013) study was in the form of proposing utilization of LT about area and perimeter concepts in teacher education. Wilson and his collegues (2013) study was a study conducted with PTs through utilizing LT.

Working with PTs is an important task since PTs need necessary skills to support students' mathematical learning and understanding before they actively work in the field. Preparing PTs for creating meaningful mathematical practices that will engage their students in doing and learning mathematics is one of the main aims of teacher education programs (Stein & Smith, 2011). To realize this aim, teacher education programs should be designed according to students' learning models, effective instructional practices, and well-integrated current learning approaches (Elmore, 2002). Pedagogy of these programs should be grounded in assessment of students' understanding and reasoning (Elmore, 2002). Courses in these programs should support PTs to acquire certain skills such as determining students' initial knowledge level, understanding how students' thinking evolves over time, launching mathematical tasks that elicit important mathematical strategies (Philipp, 2008; Stein & Smith, 1998; Smith & Stein, 2011) and ideas, eliciting possible learning obstacles and misconception of students (Confrey, 2006), and actively refining their mathematical content and pedagogical content knowledge (Graeber, Tirosh, & Glover, 1989).

There may be a certain level of agreement among researchers about how teacher-education programs should be designed (Elmore, 2002), yet it is still unclear which design is the most effective one or how effective existent programs are. This ambiguity aroused since several problems related to teacher education and quality of PTs were documented in the existing literature. Some of the problems can be stated briefly as PTs do not know the mathematics they are supposed to teach (Ball, 1990; Phillip, 2008), the teacher education courses are insufficient to tie the theory and practice (Ubuz, 2009), PTs do not have much opportunity to work with actual students (Hacıömeroğlu & Taşkın, 2010; Jansen & Spitzer, 2009), building the connection between the mathematics courses provided at the university and the mathematics that PTs would teach is hard (Eraslan, 2009). Thus, mathematics educators try to bring new approaches to redesign mathematics teacher education (Elmore, 2002). These new approaches should have the potential to support PTs for acquiring these skills and prepare PTs for creating a learning environment in which all students engage in cognitively demanding mathematics.

Simon (1995) and Clement and Sarama (2013) suggested that research-based learning trajectories are tools that educators can use to improve mathematics learning and teaching. In addition, several researchers (Butterfield et al., 2013; Clements & Sarama, 2013; Sztajn et al., 2012; Wilson et al., 2013a) indicated that a learning trajectory may serve as a tool for realizing above mentioned reference design in which PTs have the opportunity to experience practices that emphasize students' mathematical thinking and aims for a high level success for all students.

Recently, Sztajn et al. (2012) coined the concept of learning trajectories based instruction (LTBI) in which LTs are used as an instructional tool. This emergent theory emphasized knowing students' developmental progression in mathematics learning (Sztajn et al, 2012). Also, this theory is a comprehensive explanatory theory of teaching since it included the accumulated knowledge deduced from various teaching frameworks and from the LTs research. These two strengths of the LTBI theory have the potential to address the existent problems and issues in teacher education. Because, in a LTBI environment, PTs have access to a rich body of knowledge related to subject they would teach and also deep knowledge about how the students learn that subject. Thus, they can build more effective mathematics learning environments with the help of the experiences in the LTBI. Yet the question is, whether in-service teachers and teacher candidates in teacher education programs are aware of the progressions of students' mathematical learning along with influence of their own mathematical content knowledge.

In this instance, examination of documented literature indicated that integrating LTs in teacher education courses is one of the areas that needs further investigations. There is only a few number of studies that have been conducted on how to use LTs in teacher education (such as Andreasen, 2006; Mojica, 2010; Wilson et al., 2013a). Andreasen (2006) utilized a hypothetical learning trajectory on place value and operations concept with 16 elementary pre-service teachers. He examined pre-service teachers' social interaction within the classroom while engaging the presented mathematical tasks. Mojica (2010) conducted a study to examine how to train PTs so that they would use a LT to teach mathematics. She worked with a specific PTs group in which PTs had an intense mathematics courses at the university and had intense experience on working with students prior to study. She found that usage of LT supported these PTs' understanding of students' learning. She also found that PTs' subject matter knowledge could be improved through usage of LT.

Although the studies mentioned above utilized LT to assess teachers' and PTs' progression in their mathematical knowledge, earlier studies (Andreasan, 2006; Mojica, 2010) did not situate their studies in a theory specifically linked to LTs. Also, although these studies (Andreasen, 2006; Mojica, 2010; Wilson et al., 2013a) reported that PTs enhanced their mathematical content knowledge (MCK) as a result of utilization of a LT, a detailed examination of how this change occured in teachers' and PTs' knowledge levels still remained as a question. As Ball et al. (2012) suggested the dimensions of MCK should be delineated further. Since LTBI combined the comprehensive knowledge deduced from various teaching approaches and LTs research, examination of the actions and practices related to enhancement in MCK and also student knowledge (SK), as it is closely related with LTs research, under a comprehensive explanatory teaching theory may give us a holistic view and a rich detailed knowledge about the learner actions in their mathematical knowledge and student knowledge construction processes. This present study has potential to inform us about the actions of learners in LTBI teaching experience.

Clements and Sarama (2013) suggested that there is no one stable LT for every learner and every culture. Therefore, research findings about how to use LTs in teacher education might have limited implications for learners in other cultures. As a result, researchers should translate LTs and the embedded instructional knowledge within LTs for specific cultural, school, and individual contexts. This action underlies "…re-think[ing] mathematics education, … [and] re-considering the cultural and sociopolitical contexts children experience unique to our educational system" (Wager & Carpenter, 2012, p.123).

1.1 Aim of the Study and Research Questions

Based on the previously discussed needs, this study will investigate knowledge restructuring practices of pre-service elementary teachers (PTs) in a LTBI teaching experiment conducted in a mathematics education course of a teacher education program in Turkey. Restructuring practices refer to PTs' repeatedly encountered actions in which they exhibited a change, a revision and a progression in their mathematical content knowledge (MCK) and student knowledge (SK) during the LTBI.

In order to realize this aim, this study first examined the PTs' current mathematical content knowledge on a particular concept: equipartitioning also called fair sharing. Equipartitioning can be defined as:

> Cognitive behaviors that have the goal of producing equal-sized groups (form collections) or equal-sized parts (from continuous wholes), or equal-sized combinations of wholes and parts, such as is typically encountered by children initially in constructing "fair shares" for each of a set of individuals. (Confrey, et al., 2009, p. 2).

Then, how PTs restructured their mathematical content knowledge and student knowledge in relation to an equipartitioning-learning trajectory (ELT) was investigated. ELT was selected as a tool in this study since ELT have established a sound ground for rational number reasoning (RNR) which is one of the most challenging mathematics topics to understand (Confrey, Maloney, Nguyen, Wilson, & Mojica, 2008).

This study did not only focus on PTs' mathematical strategies, it also provided how PTs interacted with the LT and their peers throughout the research and documented their progression in MCK. LT interpretation of MCK is evolved around Ball, Thames and Phelps (2008)'s definitions. Ball et al. (2008) indicated there is a further need for eliciting the meanings of Common Content Knowledge (CCK), Specialized Content Knowledge (SCK) and Horizon Content Knowledge (HCK). These are the types of mathematical content knowledge and from a LT point of view, (i) CCK which refers to knowing mathematical ideas and performing mathematical strategies that are the embedded in each level of the LT (Sztajn et al., 2012) (ii) SCK that refers to unpacking the mathematical ideas, strategies, misconceptions and representations that are addressed in each level of the trajectory (Sztajn et al., 2012) and (iii) HCK that means building connections beyond and within the mathematical ideas in the LT and deducing mathematical generalizations for the ideas embedded in the LT (Adapted from Sztajn et al., 2012). This research also examined the processes of how PTs started to restructure their student knowledge. From an LTBI stand point, student knowledge refers to understanding how students think mathematically and also realizing the difference between the adults and students' mathematical thinking (Sztajn et al., 2012). Within the frame of this research aim, this study sought answers for the following research questions:

1) What are differences between pre-service elementary teachers' (PTs) knowledge level before and after the LTBI teaching experiment?
    - What is PTs' initial knowledge about the equipartitioning/fair sharing concepts, which they are supposed to teach?
    - Do PTs hold any misconceptions, difficulties, errors and knowledge gaps related to concept of fair sharing? If yes, what are those?
    - What is PTs' knowledge about the equipartitioning/fair sharing concept, which they are supposed to teach, after the LTBI teaching experiment?
2) What are pre-service teachers' restructuring practices for mathematical content knowledge in a Learning Trajectories Based Instruction (LTBI)?
    - In what ways does LTBI support PTs to detect their own mathematical misconceptions, errors and knowledge gaps and remediate them?
    - In what ways does LTBI support PTs to make sense of mathematical ideas and knowledge of equipartitioning?
    - To what ways PTs connect the mathematical ideas embedded in the ELT to further mathematics topics?
3) What are PTs' restructuring practices for student knowledge in a LTBI?
    - In what ways does LTBI support PTs' ability to understand students' mathematical thinking and learning?

These questions were investigated by a LTBI teaching experiment in a mathematics education course for nine pre-service elementary education teachers. An equipartitioning learning trajectory were utilized in the experiment and instructional tasks and items related to fair sharing ideas were created and revised prior to this study in a three weeks pilot study where 10 elementary education PTs participated in three hours teaching sessions per week. In the actual teaching experiment, nine PTs participated the six weeks of sessions each lasted approximately three hours. The data were collected through videotaped recordings of teaching sessions, field notes, observational notes, pre and post tests and PTs' written works.

1.2 Significance of the Study

Findings of the studies on teacher education (Ambrose, 2004; McDonough, Clarke, & Clarke, 2002; Tirosh, 2000) conveyed a common message that PTs' experiences with children played an important role in changes in PTs' initial understanding about how students might think mathematically, what kinds of mathematical strategies they might employ, possible mathematical misconceptions students might have, and how PTs should design their instructional activities. However, many PTs in Turkey do not have much opportunity to work with students in practicum and method courses in the teacher education programs due to time constraints and an extensive curriculum to cover both in teacher education programs and in schools (Görgen, Çokçalışkan & Korkut, 2012; Manouchehri, 1997; Uçar Toluk & Demirsoy, 2010).

The studies on existing practicum courses in Turkey indicated that the practices in these courses did not meet the aims of the courses such as providing PTs with teaching and learning experience before they actively worked as teachers in the field (Mete, 2013). Moreover, PTs have perceived themselves as a guest in the practicum schools. They thought that the practicum teaching was a formality (Eraslan, 2009). Mentor teachers usually told PTs not to come to schools or did not allow them to work with students in the classroom (Eraslan, 2009). All these findings raised an important concern that although there existed practicum courses in the teacher education programs, PTs still lacked actual experiences of how students learned when they graduated. They also did not have chance to test whether their own mathematical knowledge was sufficient to meet student needs in the classroom.

Methods courses might seem to contribute to PTs' knowledge, however, their effectiveness are limited and they are not specifically designed to decrease the gap between practice-theory (Elmore, 2002; Paker, 2008; Philipp, 2008; Zembat, 2007). In addition, although teacher candidates took some mathematics courses, little correlation was found between the number of the higher mathematics courses teachers have taken and the level of their students' mathematical learning (Akbayir & Tas, 2009; Baştürk, 2009; Swars, Hart, Smith, Smith, & Tolar, 2007). The mathematics and methods courses in the teacher education programs seem to have a limited role in providing the necessary mathematical content knowledge for PTs (Baştürk, 2009; Clements & Sarama, 2013; Görgen et al., 2012; Paker, 2008; Zembat, 2007). PTs engage in these method courses with little or no experience of working with students on mathematical ideas and concepts (Ball, Thames & Phelps, 2008; Clements & Sarama, 2013; Mojica, 2010). They utilize their own reasoning lenses and beliefs while trying to make sense of students' mathematical understanding and teaching mathematics to address their understanding. As a result, PTs encounter difficulties to distinguish their own mathematical thinking from students' thinking (Ball & Forzani, 2009; Jacop, Lamb & Philipp, 2010). In addition, teacher education courses focusing on mathematics teaching might not provide PTs with reflection on the completeness and accurateness of their own mathematical understanding (Jansen & Spitzer, 2009). This lack of reflection might direct PTs to conduct mathematics instruction without essential knowledge of students' understanding of mathematics and the mathematical understanding required for teaching (Philipp, 2008; Spitzer et al., 2011). Therefore, there is a need for a tool that has potential to support PTs' in depth understanding of students' mathematical learning and mathematics that they are supposed to teach in these mathematics education courses.

LTs are constructed based on empirical evidences from students' actual work and have provided detailed descriptions of students' mathematical strategies and misconceptions. As a result, embedding LTBI in methods courses may provide a comprehensive approach to how to teach a particular mathematics concept through integrating students' knowledge of that concept, such as rational number reasoning, across and among grade levels (Confrey et al., 2008). PTs can acquire the ability to use LTs in methods courses to decide their instruction based on evidence of students' improvement (Corcoran et al., 2009) before they actually start their in-service teaching. Confrey and Maloney (2011) listed conjectured values of usage of LT for teachers as follows: "1) Know what to expect about students' preparation, 2) More readily manage the range of preparation of students in your class, 3) Know what teachers in the next grade expect of your students, 4) Identify clusters of related concepts at grade level, 5) Have the clarity about the student thinking and discourse to focus on conceptual development, and 6) Engage in rich uses of classroom assessment" (p.31).

The mentioned benefits indicated that LTBI could be used as a reference tool for teacher training programs to help PTs in several ways. First, PTs can complete and test their knowledge of the mathematics they are supposed to teach. Second, they can develop an understanding on how students learn and understand mathematics. Third, they will have opportunities to learn to design instructional activities that count students' knowledge, misconceptions, and learning obstacles. However, these potential practical benefits of LT should be examined through empirical research to strengthen its influence.

With the scope of this study, utilization of ELT has the potential to address students' mathematical strategies and identify their misconceptions in outcome descriptions of each level (Confrey et al., 2008). PTs can diagnose gaps in students' understanding through checking LTs' description of students' cognitive strategies and misconceptions related to certain mathematics topics. This will support PTs when they develop or improve an idea about students' strategies, conceptions and possible learning obstacles and misconceptions before designing and practicing instructional tasks. As a result, "a general theoretical framework related to cognitive processes and sources of misconceptions could support teachers in their attempts to foresee, interpret, explain, and make sense of students' ways of thinking" (Tirosh, 2000, p. 23) in a methods course. Thus, LTBI has the potential to improve PTs' both subject matter and student knowledge (Butterfield et al., 2013).

1.3 Definitions of Important Terms

This section includes definitions of the key concepts utilized in this study.

Hypothetical Learning Trajectory (HLT): "The teacher's prediction as to the path by which learning might proceed. [HLT includes] the learning goal, the learning activities, and the thinking and learning in which the students might engage" (Simon, 1995, p. 133).

Learning Trajectory: "A researcher-conjectured, empirically supported description of the ordered network of constructs a student encounters through instruction (i.e., activities, tasks, tools, forms of interaction, and methods of evaluation), in order to move from informal ideas, through successive refinements of representation, articulation and reflection, towards increasingly complex concepts over time" (Confrey et al., 2009, p.2).

Learning Trajectories Based Instruction (LTBI): An explanatory framework for teaching or an emergent theory of teaching that utilize the research on LTs to combine and revise several frameworks on teaching deduced from the existing research (Sztajn et al., 2012).

Equipartitioning: "Cognitive behaviors that have the goal of producing equal-sized groups (form collections) or equal-sized parts (from continuous wholes), or equal-sized combinations of wholes and parts, such as is typically encountered by children initially in constructing "fair shares" for each of a set of individuals" (Confrey et al., 2009, p.2).

Common Content Knowledge (CCK): Knowing and performing the mathematical ideas embedded in each level of the LT (Sztajn et al., 2012).

Specialized Content Knowledge (SCK): Utilizing personal perspective to unpack each levels of the LT. This unpacking process includes sizing up the mathematical errors, misconceptions and testing effectiveness of multiple the mathematical ideas, strategies and representations (Sztajn et al., 2012).

Horizon Content Knowledge (HCK): Connecting various mathematical ideas across LT and beyond LT with further mathematical topics. Also, HCK refers to reach a generalizable mathematical conclusions and utilization of symbols to represent these generalizations (Adapted from Sztajn et al., 2012).

Mathematical Content Knowledge (MCK): The knowledge type that contains CCK, SCK and HCK.

Student Knowledge (SK): Understanding students' mathematical thinking and learning and empathizing with them on how they can exhibit certain mathematical misconceptions and errors (Adapted from Sztajn et al., 2012).

1.4 Organization of the Dissertation

This dissertation was organized into seven chapters. First chapter introduced the problems that suggested the need for this study, then the significance of the study and the aim and research questions of the study. The second chapter started with the discussion on the theoretical orientation of the study along with introduction of emergent theoretical model of the LTBI that guided this study. This followed by review of the mathematical concepts and ideas related to equipartitioning and learning trajectories. Then, the review on problems in teacher education was examined both in global and local context and the emergent literature on the benefits of LT utilization in teacher education. The third chapter introduced the methodological approach of the study through describing the context, participants, data sources, the data analysis method, limitations and assumptions of the study. The fourth chapter documented detailed findings related to first research question. The fifth chapter documented detailed findings of the restructuring practices of the PTs that cover research questions two and three. The sixth chapter discussed the findings of the study and reported the conclusions that are deduced from the findings of the study to answer the research questions. The last chapter included the closing thoughts that reported the limitations, implications and possible future research suggestions.

CHAPTER II

LITERATURE REVIEW

2.1 Theoretical Background

This study aims to understand pre-service elementary teachers' (PTs) restructuring process of their mathematical content knowledge and student knowledge in a learning trajectories based instruction (LTBI) teaching experiment. This chapter will introduce a review of relevant literature to situate the study. The chapter starts with introducing theoretical perspective in which the research situated. Then, a review on the existing literature on equipartitioning concept, which is an essential terminology for understanding Equipartitioning LT, will be presented. Also, a review on pre-service teachers and their mathematics knowledge will be presented.

2.1.1 Constructivism

According to Cobb, Yakel and Wood (1992) many researchers in the mathematics education field perceived learning as a process of constructing internal mental representations. In order to understand learning and develop learning theories, they utilized different underlying assumptions. Over the past two decades, the researchers, educational reformers and teachers grounded mathematics learning in a constructivist view.

Constructivism emphasizes the idea of learner constructing their own knowledge through engaging mathematical practices mostly through social interaction. The learner holds an active role in the learning process and makes sense of the knowledge through utilizing his or her own experiences, existing beliefs, and knowledge (Cole, 1992). Thus, the core element of the constructivism can be stated as: Learners do not store the presented information in separate pieces; instead they develop arguments to understand the information, and relate the information with each other to construct and internalize new knowledge (Perkins, 1991).

At this point, understanding the various types of constructivism explaining knowledge construction and internalization process is essential. Karagigorgi and Symeou (2005) suggested there are two types of constructivism that are loosely attached: first one is radical constructivism and the second one is social constructivism. Radical constructivists assert that the reality and knowledge construction are unique to individual and are more isolated from the social context. On the other hand, social constructivists state that knowledge construction is not merely an individual process; also this construction grows out of a social context (Tobin & Tippings, 1993). This study adapted social constructivism's assumptions while creating a learning environment as an integral part of the Learning Trajectories Based Instruction (LTBI).

2.1.2 Social Constructivism

Social constructivism built upon the works of two pioneer researchers: Piaget and Vygotsky (Palinscar, 1998). Bryant (2003) stated that Piaget's theory forms a base for development of constructivism. The underlying reason for that assertion is that children inflict own concepts to understand the world (Byrnes, 1996). Similar to Piaget's theory, Vygotsky's theory forms the other pillar of the social constructivist theory. In this section, these two pioneer researchers' contribution to social constructivism will be introduced.

Vygotsky (1978) emphasized the role of social context in the learning process and discussed the facilitator role of the social communication in learning (Scrimsher & Tudge, 2003). Thus, according to Vygotsky (1978) social constructivism underpins the interaction between the learner and the social environment in the process of knowledge acquisitions. It examines how this process ends in restricting and refining both skills and knowledge (Cobb & Bowers, 1999).

In addition, Piaget acknowledged the role of social context in learning. He stated "…individual would not come to organize his operations in a coherent whole if he did not engage in thought exchanges and cooperation with others…" (Piaget, 1947, p. 174). In their accounts, both Piaget (1970) and Vygotsky (1978) identified a clear role for social exchange in intellectual development and cognitive change (Smith, 1997). Yet, there existed a difference between Vygostky (1978) and Piaget's (1965) perception of social interaction. Vygosky's approach mainly oriented towards social interaction between learner and more capable peer. Different from Vygostky's approach, Piaget (1965) valued social relationships between equal peers. This discrepancy added a great value into this study since both orientations were merged in the study. Thus, interaction between both teacher-learner and learner-learner enhanced the knowledge [re]construction and learning.

According to Vygotsky (1978) "learning awakens a variety of internal development processes that are able to operate only when the …[learner] is interacting with people in the environment and with his peers" (p.90). On the other hand, the social context is not merely enough to construct mathematical knowledge. The learner's own ability, prior experience and knowledge also play an important role in learning. To this account, Vygotsky (1978) coined the construct of zone of proximal development (ZPD) and defined as:

> There is a gap between any student's…actual developmental level as determined by independent problem-solving and the level of potential development as determined through problem-solving under adult guidance or in collaboration with more capable peers (p.86).

Based on the definition, one can deduce that ZPD has two developmental levels. First level describes what an individual learner can do or perform independently. The second level describes what this learner can do with support. There is a zone between these two levels. According to Vygotsky (1978) "the distance between the actual developmental level as determined by independent problem solving and the level of potential development through problem solving under adult guidance or in collaboration with more capable peers" (p.85) is the ZPD. As Steffe (1991) stated ZPD of a specific mathematical concept could be determined in a constructivist-learning environment as a result of interaction. Thus, interaction is a key construct to support learner's capacity to reconstruct mathematical concepts through modifications. According to Vygotksy (1978), the more capable peer or teacher plays an important role in the modification process through exchanging ideas with the learner. As a result, the learner can close the gap between the two developmental levels.

The assistance or intervention plays a crucial role for the learner in the process of moving into next level in ZPD (Pritchard & Woollard, 2010). In this study, LTBI was designed to build a bridge across this zone. A common practice in the intervention studies is called instructional scaffolding. Instructional scaffolding is also an inherent idea of Vygotsky's ZPD. Because, ZPD is the determination of the difference what a learner can do by self and with support.

Zhoe and Orey (1999) stated, "scaffolding is a metaphor to characterize a special type of instructional process which works in a task-sharing situation between teacher and the learner" (p.6). These ideas can be further delineated into two key elements: The first one is to set the task elements beyond the learners' capabilities and let learner to work on the task without help. The second one is to support the learner to attend and skilled at the features of the task. (Puntambekar & Hübscher, 2005; Zhoe & Orey, 1999). As a result, learner can finally grasp the idea of the presented task.

Applebee and Langer (1983) identified critical features of scaffolding in instruction. Scaffolding is achieved through giving appropriate time and levels of sophistication within each task with appropriate support for the learner to meet their needs (Applebee & Langer, 1983). According to social constructivist perspective, this scaffolding can be achieved through designing appropriate task as a first step. Then, these tasks are implemented within instruction. Finally, learner is given enough space to engage with the task independently and then receives support from either their equal peers or capable peers.

The processes of scaffolding should be examined by observing how learners are engaged in shared activities. As learners become more proficient, teacher deduces the guidance and learners start to perform independently (Brown & Campione, 1984). After learners complete working individually, they exchange, present their ideas both mathematically and verbally, and challenge each other's ideas. Teacher encourages and guides them while discussing each other's ideas and support their cooperation on solving complex situations (Resnick, Salmon, Zeitz, Wathen & Holowchak, 1993).

In relation to Vygostky's perspective of social interaction in ZPD, according to Piaget (1970), creating cognitive conflicts [in this zone, during intervention and interaction] as a result of social interaction is a key component for learning. This cognitive conflict creates disequilibrium between learners' existing understanding and their experiences with the newly encountered knowledge and situation (Palinscar, 1998). This disequilibrium also leads learner to question his or her existing beliefs, understanding, and knowledge. Yet, creation of these conflicts might not be sufficient to restructure their mathematical understanding and knowledge. Forman and Kraker (1985) suggested that verbal interaction is a key component while reconstructing knowledge and understanding. In this disequilibrium state, learner exchanges ideas with more equal friends and tries to restructure his or her own understanding, knowledge, or beliefs to achieve the equilibrium again. Thus, different from Vygostky (1978), Piaget (1965) suggested, in the process of achieving equilibrium, the interaction takes place either between two equal friends or there exists a respective relationship between more capable peers.

Both Vygotsky (1978) and Piaget (1965)'s perspectives addressed the role of social process in learning as the main integral component of socially constructed learning environment. Based on previously discussed perspectives, in this study, social process in learning can be framed as follows:

Social interaction in the learning process should enable learners to work collaboratively. Collaborative learning entails enabling learners to develop, compare, and discuss a variety of perspectives and conjectures on the issue (Bednar, Cunnigham, Duffy & Perry, 1992). In this learning environment, the ultimate aim is to test the viability of the developed arguments and work toward reaching a shared meaning (Cobb, 1994). To test the arguments, learners should be able to communicate their ideas and solutions to given instructional activities. Also, they should attentively listen to their peer's way of thinking and solutions. Then, they discuss how they interpret the tasks and solve them (Cobb, 1994). At the end, this learning environment provides learners with the opportunity to construct new knowledge that interrelates both own and peers' interpretations, conjectures, and solutions. This process finally leads to a shared understanding about the task and a system of knowledge largely consistent with one another.

To sum up, based on both researchers' view on social constructivism, this study adapted a theoretical position that benefits both orientations. This orientation is closely related to Davydov's (1988) perspective on mathematics learning achieved in a constructive social process. According to him, in a socially constructed instruction following characteristics should be achieved: (i) construction of mathematical knowledge; (ii) social communications, debates and exchanging ideas; (iii) problem solving as a part of learning activities; and (iv) both verbal and symbolic representation of mathematics (Davydov, 1988).

2.2 Learning Trajectories

Learning trajectories ideas are mainly rooted in Simon's (1995) Hypothetical Learning Trajectories (HLTs) work. Simon's (1995) HLT refers to "the teacher's prediction as to the path by which learning might proceed" (p.135). He perceived these trajectories as hypothetical since individual students' progression could not be predicted in advance (Sztajn et al., 2012). Although Simon named these trajectories as hypothetical, recently mathematics education researchers constructed learning trajectories (LT) that are rooted in empirical data. Confrey et al., (2009) referred LTs as "A researcher-conjectured, empirically supported description of the ordered network of constructs a student encounters through instruction" (p.347). Corcoran, Mosher and Rogat (2009) stated that students' progression of cognition and learning is demonstrated in LT and a LT is also rooted in actual research conducted on how students learn and reason mathematically. Clements and Sarama (2004) stated LTs "have three parts: a mathematical goal, a developmental path along which children develop to reach that goal, and a set of instructional activities, or tasks, matched to each of the levels of thinking in that path that help children develop ever higher levels of thinking" (p.1).

Although there are myriad of definitions exists for LTs, common features of LTs can be deduced from the literature. LTs are based on a specific mathematics domain (Clements & Sarama, 2004; Daro et al., 2011), are developed out of empirical data on students' thinking and learning progression (Clements, Sarama & Julie, 2009; Confrey et al., 2009; Corcoran et al., 2009), emphasized the importance of using tasks to create interaction between students and mathematical concepts (Battista, 2004; Clement & Sarama, 2004; Wilson, Sztajn & Edgington 2013b), and LTs require ongoing revisions and refinements which are called validation (Confrey & Maloney, 2011; Duncan & Hmelo-Silver, 2009). In addition, all LTs, in connection with the previously carried out research on learning, examine how students' mathematical understanding and thinking evolve overtime. Also, LTs examine where mathematical learning is started and where the students are in terms of mathematical understanding (Confrey & Maloney, 2011).

According to Daro et al. (2011) there are currently 18 different LTs on different mathematical topics. Despite the common features among these LTs, existing LTs are still varied in mathematical content coverage, the way they diagnose the misconceptions, targeted grade levels and the detailed description of proficiency levels (Daro et al. 2011). For instance, Confrey and her research team (Confrey, Maloney, Nguyen, Mojica & Myers, 2009) developed a LT for equipartitioning that underlies rational number reasoning. They developed a LT consisting of 16 proficiency levels of mathematical thinking through grades K-8. Nguyen (2010) constructed a LT on length and area. More comprehensively, Clements, Sarama and Julie (2009) constructed 10 LTs about various mathematical content topics, through kindergarten to 8th grades, such as numbers and operations, and geometry. Existence of multiple learning trajectories raised important concerns: (i) how a teacher would negotiate these LTs in their instruction, and (ii) how LTs would be an integral part of teacher education programs. These concerns indicated a tremendous need for further empirical examination of existing LTs. These attempts should aim to provide both pre-service teachers and in-service teachers with a mean to use multiple LTs for instructional and educational purposes (Daro et al., 2011;

Sztajn et al., 2012). This need leads emergence of a new theory called Learning Trajectories Based Instruction (LTBI). This theory will be introduced in the following section.

2.2.1 Learning Trajectories Based Instruction

Taking students' mathematical thinking and learning as a base for instruction is not a new field for mathematics educators. Earlier work showed great examples of this approach such as cognitively guided instruction (Carpenter, Fennema & Franke, 1996).

All these studies were a precursor for utilization of LTs in teacher education or teacher professional developments. There existed some studies conducted with teachers as a part of professional development. A few studies (Butterfield, Forrester, McCallum & Chinnapan, 2013; Mojica, 2010) were conducted with pre-service teachers, but these studies did not cover a whole LT. Also at the time of these studies, a specific theoretical approach had not evolved specifically linked to LTs. As a result, in depth examination of how mathematical knowledge evolved were not examined under a theoretical framework linked to learning trajectories. Yet, these studies contributed to shaping the theoretical framework LTBI.

Sztajn et al. (2012) defined LTBI as "using research on LTs to refine and unify various frameworks from research on teaching" (p.152). They perceived LTBI as a theory of teaching that is a possible explanatory framework for instruction. LTBI instruction places students' learning as a central construct in the instruction. In this process, teachers' knowledge of LTs shapes the instructional decisions in a great extent (Sztajn et al., 2012).

Next section will deeply examine sub-constructs of LTBI. These sub-constructs are various knowledge types that are the main interest of this study.

2.2.2 Mathematical Content and Student Knowledge: Conceptualization around LTBI

Shulman (1986), in his seminal work, identified two main dimensions of knowledge: Subject Matter Knowledge and Pedagogical Content Knowledge. Built upon Shulman's work Ball et al. (2008) categorized subject matter knowledge as "Common Content Knowledge (CCK), or mathematical knowledge that is needed in contexts other than teaching; Specialized Content Knowledge (SCK), or the ways of knowing mathematics that are particularly useful in understanding students' mathematics; and, Horizon Content Knowledge (HCK), or knowledge of more advanced topics supported by the current mathematical idea of study" (p.106). Mathematical Content Knowledge (MCK) is a kind of subject matter knowledge that includes all these knowledge components.

Researchers in the field of mathematics education found that teachers' MCK is a crucial index for teaching, and teacher should acquire a certain level of MCK to be able to teach mathematics at a curricular level (Petrou & Goulding, 2011). Yet, unfortunately, studies on PTs' MCK clearly documented that majority of elementary PTs lacked of conceptual MCK (Behr, Khoury, Harel, Post, & Lesh, 1997; Tall 1991) and at the same time they lacked an in depth understanding of mathematics that they are required to teach (Ball, Hill & Bass, 2005). Also, the research clearly documented that teachers could successfully perform calculations to solve mathematical problems, yet they could not explain the procedures and mathematical meaning of the concepts they performed (Ball, Lubienski & Mewborn, 2001).

These are all important concerns to be addressed since teachers' MCK level has a great impact on how they shape their classroom practices (Clements et al., 2009; Ma, 1999) and on students' meaningful understanding of mathematics overtime (Darling-Hammond & Ball, 1998; Ma, 1999). All these mathematical understanding also becomes an essential predictor of students' success in mathematics (Hill, Rowan, & Ball, 2005).

Knowing merely what subject content looks like and how to design classroom practices around the subject content required different skills. Yet, there is no specific framework in the field of teacher education to support teachers to decide what to teach, how to teach, how to design instruction task about it, how to represent it, and how to address and remediate misconceptions about it. Ball and colleagues (Ball, Hill & Bass, 2005) stated that teaching PTs more content knowledge does not supply an answer for these questions. They indicated that along with the mathematical content knowledge, teaching for understanding is a must. As a result, to supply effective answers for these questions in teacher education programs, a substantial body evidence should be utilized on how students learn mathematics and how teachers enhance their both content and pedagogical knowledge of mathematics. In the LT construction process, researchers considered all these substantial body of evidence from existing research on learning and also conducted empirical research with students, and with both pre-service and in- service teachers.

From a LTBI standpoint, Wilson, Mojica and Confrey (2013a) conceptualized students' mathematical thinking similar to content knowledge. As a result, LT can be perceived as a referenced tool that combines both MCK and student knowledge (SK) (Sztajn et al., 2012; Wilson et al., 2013a). Figure 1 represents the theoretical framework of the present study that represents the relation between LTBI and MCK's sub-knowledge components and SK framed by social constructivism.

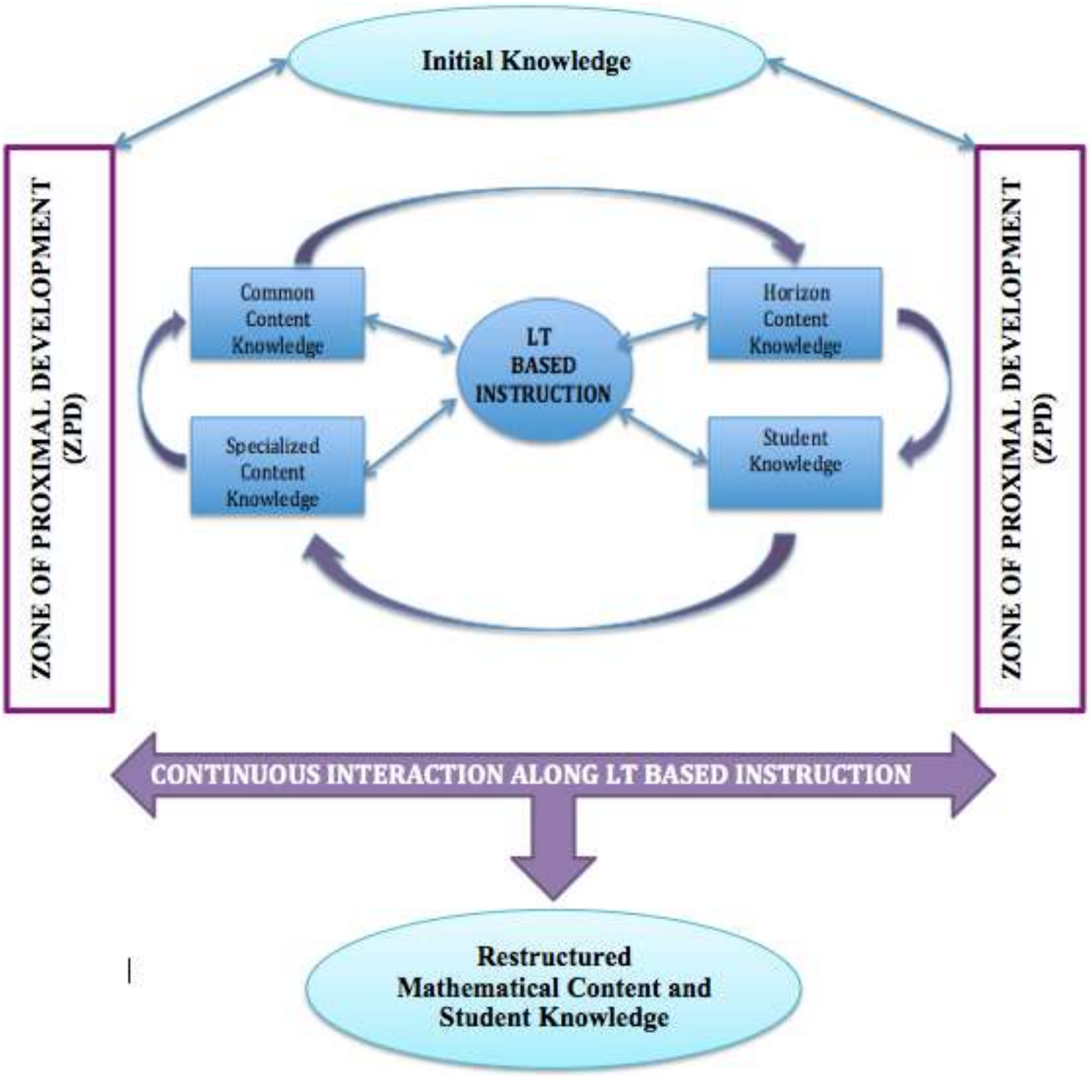


*Figure 1***.** A model of relation between LTBI and MCK and SK (Adapted from Sztajn et al., 2012 & Wilson et al., 2013a)

Figure 1 depicts how MCK is conceptualized by locating LTBI at the center of the knowledge types. The blue arrows indicated the interaction between the constructs and LTBI. The purple colored arrows indicated that all types of knowledge have an influence on each other, and knowledge refinement and reconstruction is an iterative cycle.

In Figure 1, the relation between *common content knowledge* and LTBI refers to both PTs' and in-service teachers' understanding of mathematical concepts and procedures they teach (Clements et al., 2009; Wilson et al., 2013a). These concepts and procedures are addressed in each level of the trajectory and support individuals when they perform related tasks in each respected levels (Sztajn et al., 2012). The comprehensive final goal can be reached through mastering these concepts and procedures through navigating each level. Then, one can reach the ultimate mathematical goal at the highest level of the LT.

The relation between *specialized content knowledge* and LTBI refers to one's ability to devise and test multiple mathematical strategies, explanations and representations. This process begins with the learner's initial state of mathematical content knowledge (Sztajn et al., 2012; Wilson et al., 2013a). In order to acquire this sub-knowledge of MCK, one should unpack each level of LT (Sztajn et al., 2012). Unpacking each level of the trajectory means articulation of mathematics behind each level and sizing up possible mathematical errors and misconceptions (Sztajn et al., 2012; Wilson, Sztajn, Edgington & Confrey, 2013, 2014).

As a result of this unpacking process, PTs first can make sense of multiple mathematical representations and explanations embedded in the trajectory (Sztajn et al., 2012). Second, PTs should make sense of how mathematical learning functions as a process in which their students will construct their own mental mathematical representations and strategies to embody the mathematical aspects of the external representation (Cobb, Yackel &Wood, 1992). Last, PTs should learn how to represent their MCK into understandable formats by unpacking levels of LTs (Fennema & Franke, 2007). From a holistic view of engaging with unpacking process of LT's levels, PTs also have the chance to identify their MCK gaps (i.e. misconceptions, mathematical errors) and close them (Sztajn et al., 2012; Wilson et al., 2014).

Horizon content knowledge is perceived as the most complicated mathematical knowledge that is targeted at the highest level of LTs (Sztajn et al., 2012). This kind of knowledge requires mathematical abstraction, connections and generalization that are comprised by whole LT and later beyond LT. For instance, the highest level of Confrey and her colleagues constructed LT for fair sharing underlies the following generalization:

> Generalize that *a* objects shared among *b* persons results in *a*/*b* objects per person, applying strategies based on both the distributive property and ratio reasoning, and asserting their equivalence (Confrey et al., 2009, p.10).

This type of MCK enables both PTs and in-service teachers to distinguish imitation of mathematical ideas from actual abstraction of that mathematical idea, which is one the most important aims of mathematics teaching and learning (Confrey, 2006; Maher & Martino, 1997). To achieve this, prior levels of LTs laid out routes for learning a particular mathematical knowledge, and these routes started from least complex ideas and reached the more complex mathematical ideas. In this process, description of each level and the mathematical strategies assist conceptual understanding of the most sophisticated targeted mathematics that PTs required to know (Maher, 1996). At the final stage, learner can connect mathematical ideas within LT with further mathematical topics (Sztajn et al., 2012).

Student knowledge refers to knowing how students' progress through levels of LT (Stein & Smith, 2011; Sztajn et al., 2012). Then, it refers to knowing students' cognition in each level (Carpenter, Fennema, Peterson & Carley, 1988) and knowing how students deal with the LT based tasks (Franklin, Yilmaz & Confrey, 2010; Wilson et al., 2013b). This type of knowledge guides teachers on how they utilize their MCK while planning their instruction. Planning an instruction includes designing learning activities (Clements et al., 2009; Sztajn et al., 2012, Wilson et al., 2013a, 2013b), predicting students' mathematical strategies (Stein & Smith, 2011), sequencing those strategies (Stein & Smith, 2011), and assessing students' learning (Confrey, 2012; Webb, 2007).

It is conjectured that during LTBI, PTs' current level of knowledge will expand. This knowledge enhancement occurred as a result of both social interaction and the engagement in activities in LTBI. This continuous interaction and engagement is represented in Figure 1. This interaction happens in the *zone of proximal development***.** LT is used as a tool that helps PTs to navigate from their existing knowledge to a higher and complex knowledge. *Continuous reconstruction* in teaching sessions (Dewey, 1902, p.11) yields a restructured mathematical content and student knowledge**.**

Although LTs provide PTs with knowledge of various possible mathematical strategies, learning obstacles, misconceptions of students and the learning routes that student most likely to follow, there may be remaining contingency knowledge. Since teaching and learning is a complex process, one or one theory could not encounter all the possible scenarios in educational settings. Yet, LTBI has the potential to combine the puzzles from various educational researches on teaching and learning, and also perceive mathematics in relation to learner and perceive learner in relation to mathematics based on empirical evidences (Sztajn et al., 2012). All these knowledge types will be deduced into sub-categories based on the empirical data from this study. The descriptions will be provided in the methodology section as a coding schema of the study.

2.3 Concepts, Ideas and Issues

Literature review in this part is composed into three sections. The first section underlies the mathematical ideas and concepts that formed a base for this study. Also, it discusses the mathematical concepts that are related with equipartitioning. Second, the specific learning trajectory called equipartitioning-learning trajectory (ELT) is briefly introduced. Finally, review of literature on pre-service teachers and mathematics is discussed briefly. This review included pre-service teacher and mathematics research in both global and local context. Then at last, potential benefits of using learning trajectories in teacher education will be discussed tied with the discussed issues in the literature review.

2.3.1 Equipartitioning Literature

In the existing literature, rational number reasoning has been interpreted in myriad of ways. Although different constructs were raised in those studies, one repeated theme is called partitioning. Partitioning has been defined throughout literature in different ways. Kieren and Nelson (1981) defined partitioning as dividing a whole into parts. McGee, Kervin and Chinnappan (2006) defined partitioning different from the other perspectives. They added two important criteria into partitioning action as: (i) exhaustion of the whole object(s) and (ii) creation of disjoint pieces that exhaust the whole object(s). In line with McGee et al. (2006) perspective on partitioning, English and Halford (1995) stated the two criteria of partitioning as; the pieces should not overlap and one should exhaust the whole object.

Several studies (Charles & Nason, 2000; Lamon, 1996; Pothier & Sawada, 1983) were conducted to examine students' partitioning strategies on continuous wholes. DELTA (Diagnostic E-Learning Trajectories Approach) research team made a comprehensive literature synthesis (can be found at the url: gismo.fi.ncsu.edu) on rational number reasoning and classified children's partitioning strategies under four -later collapsed into three- cases which are discussed in more detail in the following section. Different from previously mentioned definitions of partitioning Confrey et al., (2008, 2009) referred partitioning as creating only fair shares. They disagreed with the definitions of partitioning which included breaking into uneven groups (Steffe, 2004). In order to clarify the ambiguity of partitioning definition, they introduced a new concept called equipartitioning. DELTA team defines equipartitioning as:

> Cognitive behaviors that have the goal of producing equal-sized groups (form collections) or equal-sized parts (from continuous wholes), or equal-sized combinations of wholes and parts, such as is typically encountered by children initially in constructing "fair shares" for each of a set of individuals" (Confrey, Maloney, 2010, p.3).

This is a comprehensive definition of partitioning action since it covers equipartitioning of both collections and wholes with any size and shape. The next section will introduce four equipartitioning cases (A, B, and C and D) through discussing students' reasoning on the tasks related with particular case.

2.3.1.1 Case A

Many research (Confrey et al., 2009, Davis & Hunting, 1991; Hunting & Sharpley, 1991; Pepper & Hunting, 1998) examined children's ability to fairly share collections and their strategies as they equipartitioned collections such as *counting.* Case A included fair sharing collection tasks (Confrey et al., 2008). In Case A, children are presented with a certain amount of objects to be shared fairly among a certain number of people.

Researchers stated that children are usually successful at sharing collections among two or more people. They employed different strategies as they fairly share collection. Pepper (1991) classified children's sharing strategies under three categories: (i) *systematic strategies produced even shares*; (ii) *unsystematic strategies produced even shares*; and (iii) *unsystematic strategies produced uneven shares*.

Pepper (1991) conducted two interviews with 75 children of ages four and five. In the first interview, he examined children's counting ability and in the second interview he examined children's partitioning strategies. In the second interview, he asked them to share 12 biscuits between two dolls. Based on his data, he categorized children's actions into three. According to him, a good sharer shared collection systematically and produced fair share at most four moves. An intermediate sharer shared collection somehow systematically and produced fair shares using four to seven moves. Poor sharers shared collection unsystematically and could not produce a fair share. He found that 80% of children used *dealing strategy* which is "a cyclic distribution of discrete objects (regarded as identical) with the same number distributed to each place on each round of the cycle until there are none left" (Davis & Pitkethly, 1990, p.145).

DELTA team also created assessment items to examine children's equipartitioning strategies (Yilmaz, 2011). They also found that children used both systematic and unsystematic dealing strategies (Confrey, et al., 2008). In systematic dealing, children tended to use initially *1-1 correspondence*. For instance, they asked students to share 15 candies among three people. Figure 2 demonstrates one child's 1-1 correspondence strategy.

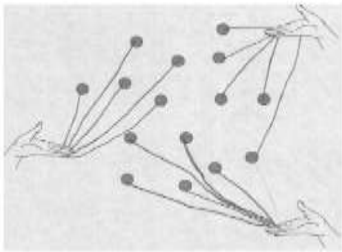


Figure 2. Systematic dealing: 1-1 correspondence (Retrieved from Yilmaz, 2011, p.18)

DELTA team also found that as children gained proficiency, they started to use many to one strategy that is called *composite units*. As needed, they switched between many to one and 1-1 correspondence strategy. For instance, a student systematically shared 18 objects among three people by initially giving five objects to each person and then giving one object to each person.

In the existing literature, researchers also closely examined children's justification strategies for their shares. Hunting and Sharpley (1991) reported that *counting* of items in each group, *visual and height comparisons* of items per pile were used to justify fair shares. In systematic dealing, some children were aware of the fact that systematic dealing produced fair shares yet the others not. The ones who were not aware of systematic dealing produced fair shares used formerly mentioned strategies to justify fair shares.

Using counting as a strategy to justify fair shares generated a question: Is there any recognizable cognitive relation between children's ability to count and fairly share? This question was intensively researched. Pepper (1991) asked 75 children of ages four and five to fair share 12 biscuits among three people. He found that eighty percent of children used systematic dealing regardless of their counting competency. Some of those children used visual or height comparisons to justify generated fair shares even if they used systematic dealing. Seventy six percent of the children who were identified as poor counters also used systematic dealing to generate fair shares and they succeeded. As a result, he concluded that children's counting competency was not directly related with their fair sharing competency.

Then, in 1992, Davis and Pepper deeply examined the relationship between counting ability (i.e., good, medium, poor) and fair sharing ability of children. Built upon Pepper's (1991) study task (12 Biscuits among two people), Davis and Pepper reported that children participated in this study could mentally split six discrete objects in the ratio of 2:1 regardless of their counting ability.

Pepper and Hunting (1998) also indicated that sharing discrete items was not directly related to the counting ability level of children. In their study, they interviewed with 25 preschool children. Children were asked to solve three presented tasks: 1) Sharing 12 crackers among two dolls, 2) one more dolls joins the group, sharing 21 cookies among three dolls, and 3) sharing 15 coins among three dolls. They found that children's systematic dealing strategy did not involve counting skills. Moreover, they found that there was variability among good sharers' counting competence that reinforced Pepper's (1991) and Davis and Pepper's (1992) study findings.

All these studies indicated that counting and sharing have distinct cognitive roots (Confrey et al., 2009). Children's use of systematic dealing was not directly related with children's counting ability; rather, it was related with forming equal groups as a result of systematic dealing actions.

2.3.1.2 Case B

Case B tasks involved equipartitioning a single whole that yields *unit fractions* (Confrey et al., 2008). In early research, Piaget, Inhelder, and Szeminska (1960) investigated children's strategies as they partitioned single wholes. They observed a progression in children's ability to share single wholes as follows: First they performed *general fragmentation (chopping)*, and then they progressed to make equal parts through *dichotomous, trichotomous, or* both *methods of division*. At the final stage, they could equipartition a whole into five and six parts (Piaget et al., 1960).

Later on, Pothier and Sawada (1983, 1989) investigated children's partitioning strategies on wholes. They generated a theory about how children's understanding of equipartitioning a single whole progressed from the least complex to more complex skills. They generated five proficiency levels of equipartitioning a single whole. The first level was called *sharing*. In this level, children first learned halving the whole and then they learned to create 4-splits. In this level, although children learned halving and constructing fourths, they sometimes created unequal shares through breaking the rectangular and circular whole. The second level was called *algorithmic halving* in which children mastered repeated halving which supported children's ability to construct halves "to the *n*th power shares by doubling the number of partitions" (Yilmaz, 2011, p.22) on circles and rectangles. The third level was called *evenness* in which children could distinguish whether result of sharing produced a fair share or not. They could check for equality of the parts. Also in this level, child could equipartition whole(s) through algorithmic halving strategies for even numbers of people. Pothier and Sawada (1983) demonstrated this algorithmic halving strategy to create six equal parts as in Figure 3.

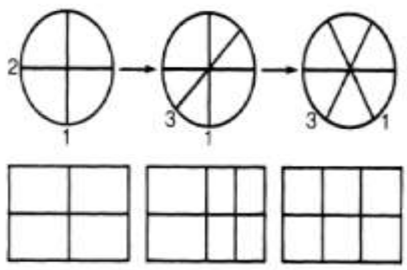


Figure 3. Repeated halving strategy: Construct sixths

The next level was called *oddness* in which children became aware of the fact that algorithmic halving failed to produce odd number of fair shares such as thirds and fifths. The highest level was called *composition* in which children used multiplication facts to create larger number of fair shares. Confrey and Maloney (2010) disagreed with Pothier and Sawada's (1983) last level. They disputed that knowing multiplicative factors or facts did not come before composition. Indeed, they suggested that children could learn multiplicative factor derived from the act of composition of splits. For instance, children could create a 12-split on a rectangle through composition of splits. They could first create fair 4-splits on a rectangle through vertical cut then, they could create 3-splits on the same rectangle through horizontal cut. As a result, they fairly shared a rectangle into 12 splits (3 x 4).

In a follow up study, Empson and Turner (2006) examined the relation between students' understanding of algorithmic halving and multiplicative reasoning. Algorithmic halving includes repeatedly splitting a continuous whole into two fairly. They worked with 30 students from grades 1st, 3rd and 5th grades. These students were engaged with paper folding tasks. They grouped students' thinking of folding into three categories. The first one was *non-recursive thinking about folding* (Empson & Turner, 2006) in which student connected the number folding and the parts yielded as a result of folding in a non-recursive ways. The second one was *emergent recursion* which "involved the insight that folding any number of parts in half, at any point in a sequence of folds, doubled the number of parts" (Empson & Turner, 2006, p.51). The last one was called *recursion*. Only a few students made sense of this strategy. This strategy indicated that there existed a recursive relationship between the folding sequence and the resultant number of share (Empson & Turner, 2006, p.51).

One major dispute on Empson and Turner (2006) work came from Confrey et al. (2008). They argued that importance of creating equal sized parts through folding was essential for creating fair shares. As a result, DELTA team proposed three criteria for equipartitioning as follows: (i) create the correct number of parts; (ii) create equal-sized parts; and (iii) exhaust the whole.

Researchers also conducted studies on how children verified whether they created fair shares or not on the continuous whole. Pothier and Sawada (1989) conducted over 200 clinical interviews with students from grades K-6. Students were asked to fairly share geometric shapes or giant cookies. They grouped students' verifications under following categories: (i) visual estimation in which students visually approximated whether created parts were fair or not, (ii) techniques resulted in fair shares in which students believed that the employed technique automatically created fair shares, (iii) compensatory description of parts in which students tried to compensate unequal parts, (iv) use measurement to justify the fair share in which students compared the widths and lengths of the shares, (v) check for congruency in which students checked whether the created parts were the same size and shape, and (vi) use geometry of the parts in which students did not necessarily focus on the shape of the parts; instead, focused on geometric aspects of wholes and parts to justify fair shares, such as area congruence. In their study, they suggested that children should perform different ways to create same number of splits on the same whole.

As suggested, Franklin et al. (2010) examined students' strategies to justify fair shares formed on the same whole with different strategies. They identified three strategies as follows: Qualitative compensation, decomposition and composition and transitivity, later named as property of equality of equipartitioning (Confrey, Maloney & Corley, 2014). For instance, in Figure 4, a rectangle is shared fairly among four parts through employing diagonal cuts. The part A and part B are fair but not congruent.

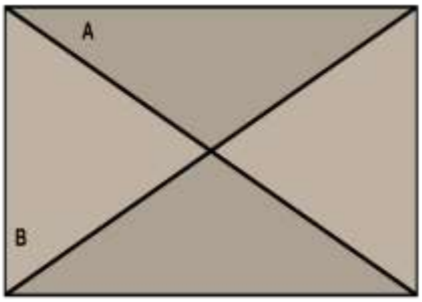


Figure 4. Diagonal cuts form fair but non-congruent shares in terms of shape

A student who used qualitative compensation strategy to verify the equivalence of part A and part B stated that triangle A was tall and skinner and triangle B was short and fat so they were equal to each other. A student who employed composition and decomposition strategy to verify the equivalence of part A and part B, cut the triangle A and B into half (*decomposition*), then reassembled (*composition*) them to show the congruence. A student who employed transitivity strategy to verify the equivalence of part A and part B, knew that creating fair shares on a rectangle formed the parts which were congruent in terms of area.

The above literature indicated that students used different strategies to equipartition the whole and they employed a variety of justification ways to show they constructed fair shares.

2.3.1.3 Cases C and D

Case C and D tasks dealt with sharing multiple wholes among multiple people. Case C tasks produced a *proper fraction* outcome while Case D tasks yielded an *improper fraction* (Confrey et al., 2008).

A limited number of studies (Charles & Nason, 2000; Lamon, 1996) examined students' strategies on equipartitioning multiple wholes. The reason might be that Case A and Case B tasks were mainly related with the instruction in early grades (K-2) and they were investigated in depth, yet Case C and D tasks were the later tasks and they were examined in limited number of studies with higher grades (4-6) (Mojica, 2010).

Lamon (1996) investigated partitioning strategies of 346 students from grades 4 to 8. Students were asked to fairly share multiple meals (i.e. cookies, pizzas) among multiple people in 11 tasks. She identified three partitioning strategies from students' work: (i) *preserved-pieces* which meant student first dealt with wholes and then fairly shared the remaining parts, (ii) *mark all* which meant students first partitioned each whole in a certain number and then dealt with the parts from each whole, and (iii) *distribution* which meant students first partitioned each whole into an appropriate number of shares and then dealt with all.

Charles and Nason (2000), based on their literature review on equipartitioning and conducting interviews with 12 students from 3rd grade, developed a taxonomy for children's partitioning strategies. They classified 12 partitioning strategies derived from their research and literature review into three categories: Partitive quotient construct strategies, multiplicative strategies, and iterative sharing strategies. All partitive quotient construct strategies shared a common feature as they used the relationship between numbers of sharers and generated fractional name for each share. In multiplicative strategies, a multiplicative algorithm was used to generate required number of fair shares. Iterative sharing strategies included four different types "(i) halving the object then halving again and again, (ii) half the objects between half the people, (iii) repeated sizing strategy, and (iv) repeated halving and repeated sizing strategy" (Charles & Nason, 2000, p. 203).

2.3.2 Equipartitioning Learning Trajectory

Confrey et al. (2008) have constructed a learning trajectory for equipartitioning (ELT) as a two-dimensional matrix display LT. The construction of the ELT started with comprehensive literature synthesis on rational number reasoning (RNR) (Confrey & Maloney, 2010). Then, based on theoretical perspectives of the LT construction combined with the knowledge deduced from the RNR assessment pilot items were constructed. These items assessed students' progression on equipartitioning related ideas. Initially these items were piloted in interview settings and the students' responses were gathered and ongoing revisions were made to finalize the items (Confrey & Maloney, 2010). An initial LT was constructed within this process. After this construction, paper-pencil assessment items were utilized in the field-testing at the four North Carolina school districts (Confrey & Maloney, 2010). The DELTA research team categorized the responses of the students in the field test and the mathematical strategies were classified in the rubrics. Also, students' mathematical errors and misconceptions were coded under specific codes. Through utilizing these rubrics, each student's work were scored and evaluated. Ongoing revisions and refinements were made to finalize the ELT during these times. At the end, the research team constructed and revised the two-dimensional matrix display for the ELT (Confrey & Maloney, 2010). "The vertical dimension demonstrates the progression of the LT's proficiency levels with the sophistication increasing from bottom to top, and the horizontal dimension represents the task classes" (Yilmaz, 2011, p.33) in the matrix. Duncan and Hmelo-Silver (2009) stated that LP development and validation processes were interconnected, and these processes took place in iterative rounds of testing for empirical evidence and theoretical modification. DELTA team iteratively revised ELT that forms as a sound ground for rational number reasoning. Figure 5 demonstrated ELT that was developed by DELTA team and used as a reference tool in this study.

The ELT included outcome description for each levels and included more levels related to each selected levels presented in Figure 5. These levels described level of the progression of knowledge: using multiple methods, justification, naming, reassembly and properties (Confrey et al., 2008). This progression is not necessarily linear. According to Confrey, Maloney and Corley (2014) the LTs do not follow a stage approach in which a prior level must be mastered to move into next one. Instead, the levels in the LT ordered carefully through counting students' prior knowledge and existing research. This order is a probabilistic claim in which students might show various mathematical justifications and methods (Confrey, Maloney & Corley, 2014). For instance, if a child fairly shares a rectangular whole into four, he should justify the fair shares. Second, he should show different ways of creating four fair shares such as through two diagonal cuts and three vertical cuts. Then, he should name each share as $1/4^{th}$ or one out of four. He can demonstrate the understanding that the whole is four times larger than a part, which is reassembling. At the final stage, he can understand that although both methods (diagonal cuts and vertical cuts) created fair shares with different shapes, he should show both shares (one triangular to one rectangular part) are congruent in terms of area or using composition and decomposition.

Equipartitioning Learning Trajectory Matrix (grades K-8) Task Parameters→ / Proficiency Levels		A Collections	B 2-split (Rect/Circle)	C 2^n split (Rect)	D 2^n split (Circle)	E Even split (Rect)	F Odd split (Rect)	G Even split (Circle)	H Odd split (Circle)	I Arbitrary integer split	J $p = n + 1$; $p = n-1$	K p is odd, and $n = 2^i$	L $p >> n, p$ close to n	M all p, all n (integers)
16	*Generalize*: a among $b = a/b$													
15	*Distributive property*, multiple wholes													
14	*Direct-, Inverse- and Co-variation*													
13	*Compositions* of splits, multiple wholes													
12	Equipartition *multiple wholes*													
11	Assert *Continuity principle*													
10	*Transitivity* arguments													
9	Redistribution of shares (*quantitative*)													
8	Factor-based changes (*quantitative*)													
7	*Compositions* of splits; *factor-pairs*													
6	*Qualitative compensation*													
5	*Re-assemble*: *n* times as much													
4	*Name* a share w.r.t. the referent unit													
3	*Justify* the results of equipartitioning													
2	*Equipartition single wholes*													
1	*Equipartition Collections*													

Figure 5. The equipartitioning learning trajectory. Adapted from Confrey (2012)

On the other hand, this probabilistic claim also indicated a student's mathematical belief could be correct and functional in an early stage. However, this claim might become incorrect at higher levels. As a result, the students should revise their thoughts. Confrey et al. (2014) provided an example for this situation. A student who equipartitioned a single whole may arrive at a conclusion that the parts must be same size and shape to be a fair share. However, when this student worked on two identical rectangles, first one partitioned into half diagonally and the second one partitioned into half horizontally, this student might think the halves from each rectangle are not same size. This example showed that although this student could justify fair shares (level 3) when working on single whole level, same student could not justify the equality of shares when working on distinct identical multiple wholes. Confrey et al. (2014) concluded that when the student realized the equivalence of the halves in this example, this change in students' mathematical conclusion is not based on logical order but on experience.

2.4 Mathematics and Pre-Service Teachers

The practices of learning and teaching are intertwined, which means that teachers and students learn from each other. According to Jacop et al., (2010) and Stein and Smith (2011) knowledge of students' mathematical understanding helps in shaping teachers' instructional practices. As a result, the ability to learn about students' mathematics has become an important issue in teacher education programs (Jacobs et al., 2010; Philipp, 2008). Thus, the main goal of teacher education programs in relation to mathematics education is to prepare future teachers with sound knowledge of both pedagogical and mathematical content knowledge. Consequently, PTs should possess the skills necessary to create meaningful mathematical practices when they begin their in-service practices (Smith & Stein, 2011).

Mathematics education research has shown that meaningful mathematical practices at schools should include important components that should be integrated into teacher education programs. The first practice involves modeling students' learning. This practice is based on the students' initial level of mathematical understanding. Also, this practice calls for teachers to determine their students' initial level of understanding (van de Walle, 2007). The second practice utilizes high-level tasks that engage students in mathematics activities (Smith & Stein, 2011). The third practice involves the assessment of students' progression (Elmore, 2002). Although training teachers for the components is one of the main goals of teacher education programs, the question remains in many countries of whether teacher education programs prepare teacher candidates completely for their future teaching practices (Dick, 2013; Philipp, 2008).

Several studies (Jacobs, Lamb & Philipp, 2010; Mewborn, 2000; Philipp, 2008; Wilson et al., 2013a) have focused on how examination of students' mathematical thinking and understanding might support teacher education. Their findings seem to address two important results. The first is related to PTs' mathematical difficulties, knowledge gaps and misconceptions and the second is concerned about how different approaches can be used to fix these problems. In addition, the results of these studies led to similar conclusions. The studies generally indicated that teachers' quality is a widespread main problem and teachers' development should be supported (Rowe, 2004). The same conclusion holds for the mathematics education field. In addition, unique problems were found in elementary education.

Existing research conducted with pre-service elementary education teachers showed that PTs have difficulty in distinguishing between their way of mathematical thinking and children's mathematical thinking. They perceived that students' correct mathematical answers indicated that the child had a conceptual understanding of mathematics (Crespo, 2000; Jansen & Spitzer, 2009; Morris, 2006). During teacher education programs, PTs should acquire ability to move beyond the simple evaluation of right or wrong answers (Jansen & Spitzer, 2009). To obtain this ability, PTs should be exposed to students' mathematical thinking during their undergraduate studies and should be trained on how to meaningfully analyze students' mathematical works. However, the studies showed that PTs had limited experiences on actually working with students (Mewborn, 2000; Philipp, 2008).

Findings of several studies (Mewborn, 2000; Philipp, Ambrose, Lamb, Sowder, Thanheiser et al., 2007) have indicated that exposure to children's mathematical thinking supports the development of PTs' mathematical knowledge in relation to teaching. These studies suggested that future teachers could develop knowledge about how students learn mathematics by working with children in a mathematical field experience course. Although PTs participate in field experience courses, they show weakness in understanding students' mathematics and analyzing students' mathematical thinking (Jacobs et al., 2010). This weakness demonstrate the need for PTs to practice analyzing and understanding students' mathematical thinking by working with actual students' work before taking a field experience course (Philipp, 2008).

Some of the studies have focused on how the analysis of students' work helps PTs understand their future students' mathematical thinking (Bartell, Webel, Bowen & Dyson, 2012; Crespo, 2000; Jacobs & Philipp, 2004; Jansen & Spitzer, 2009). These studies indicated that examining students' work and understanding their mathematical thinking could contribute to PTs' development. However, this development could not occur on its own. In order to have this benefit, PTs need opportunities to experience students' mathematical thinking and support to learn ways to understand their mathematical thinking. In addition, PTs should acquire the necessary mathematical content knowledge to capture the mathematics behind students' work.

Bartell et al. (2012) conducted a short-term intervention study on PTs' ability to recognize students' conceptual understandings of mathematics. They stated that the majority of PTs initially accepted procedural calculations that yielded mathematically correct responses as evidence of conceptual understanding. After the intervention, PTs exhibited significant improvement in their analysis of children's mathematics. They were able to pay more attention to evidences of conceptual understanding compared to their initial perception of accepting procedures as evidences. In some areas such as multiplication of fractions, PTs were less successful while attending children's conceptual understanding because PTs did not possess sufficient content knowledge of multiplication of fractions (Bartell et al., 2012). Thus, they concluded that mathematical content knowledge plays an important role in understanding students' mathematical responses yet it is not only factor that influence the PTs ability to understand students' mathematical responses deeply.

Similarly, Ball (1990) conducted a study with 252 secondary mathematics and elementary education PTs. The result of this study indicated that the majority of PTs lacked conceptual understanding of the mathematics they would eventually teach. According to Ball's (1990) findings, the majority of PTs provided procedural answers without demonstrating their understanding of the mathematics involved and considered rules as explanations. Based on the findings of this study, Ball (1990) challenged three common assumptions held by PTs that were related to teaching school mathematics: "mathematics content is easy, traditional K-12 education includes most of what teachers need to know about mathematics, and … mathematics majors possess the necessary subject matter knowledge" (p.449).

Findings from several studies (Bartell et al., 2012; Lowery, 2002; Philipp, 2008; Spitzer, Phelps, Beyers, Johnson & Sieminski, 2011) could be utilized as evidence to argue against PTs' assumptions. For instance, pre-service teachers and students hold the same serious learning misconceptions about topics such as the multiplication of rational numbers, decimals, and place value (Graeber, Tirosh & Glover, 1989; Thipkong & Davis, 1991). Likewise, the majority of PTs also lack mathematical content knowledge:

> Most PTs do not know what mathematics they need to know to teach effectively, and many are not open to approaching the content anew in a deeper and more conceptual way that they experienced in elementary school know something, then children would not be expected to know it, and if I do know something, I certainly don't need to learn again" (Philipp, 2008, p.8).

Philipp (2008) indicated that if PTs knew how to solve a mathematical problem, they were not interested in the mathematics behind it because they assumed that mathematics was a set of rules and the explanations based on these rules. If they did not know the content, they believed that children did not need to learn about the content. Ball (1990) addressed a possible solution for these situations:

> Attending seriously to the subject matter preparation of elementary and secondary math teachers implies the need to know much more than we currently do about how teachers can be helped to transform and increase their understanding of mathematics, working with what they bring and helping them move toward the kinds of mathematical understanding needed in order to teach mathematics well (p.465).

Several studies (Ball, Levis & Thames, 2008; Ball, Sleep, Boerst & Bass, 2009; Philipp, 2008; Sztajn et al., 2012, 2014) later have been conducted on how to draw boundaries for this sort of mathematical understanding. These studies discussed the current quality of teacher education programs, the information teacher training programs should provide to PTs, whether mathematics and mathematics teaching methods courses were sufficient for PTs to acquire a conceptual mathematical understanding to use in their instruction. The results of these studies indicated that mathematics teaching methods courses were not sufficient for PTs to acquire the necessary mathematical understanding and knowledge (Philipp, 2008). Because, it is hard to fully examine the conceptual meaning behind mathematics during a short mathematics teaching methodology course (Manouchehri, 1997). In addition, PTs could not connect what they learned in basic mathematics courses to what they would teach at the elementary school level. Thus, some other steps should be incorporated in teacher education programs.

Ball (2000) and Philipp (2008) suggested that mathematics education content courses should be taught at universities covering elementary-level mathematics. Philipp (2008) indicated that these courses should combine aspects of both mathematics courses and mathematics teaching methodology courses. Also, Ball (2000) described how one could design these courses:

> To improve our sense of what content knowledge matters in teaching, we would need to identify core activities of teaching, such as figuring out what selecting, and modifying textbooks; and deciding among alternative courses of action, and analyze the subject matter knowledge and insight entailed in these activities (p.244).

Spitzer et al. (2011) conducted a two-week intervention study with elementary PTs as a brief example of this course design. Their study showed that even relatively short interventions could produce improvement in PTs' ability to attend students' mathematical thinking. Similarly, Lowery (2002) conducted a study and indicated that the intervention helped PTs to organize their own mathematical content knowledge (Lowery, 2002). However, no specific type of intervention was found to be the most effective. Instead, a variety of opportunities should be embedded in those interventions (Boyd et al., 2009). The focus should be on exposing PTs to the experiences evolved around and centered in big ideas (Clements & Sarama, 2013; Confrey, 2006; Confrey et al., 2012) in mathematics and students' mathematical thinking and understanding (Ball & Forzani, 2009).

Although several studies documented the problems in PTs' mathematical content knowledge and knowledge of students' mathematics in relation to teacher education programs, other problems unique to elementary mathematics education programs have also been documented in the literature. Wolf (2003) stated that the college advisor has a great responsibility while guiding elementary PTs in their field experiences. Yet, majority of advisors are not trained in mathematics education field. This may cause insufficiency while guiding PTs for how to understand students' mathematical thinking, what kind of mathematical knowledge is needed to capture important mathematical ideas and how to learn from students' mathematics (NCATE, 2010). Therefore, especially elementary PTs need mathematics education specific support during their education at universities.

The significant issues deduced from the review of the literature on PTs and mathematics could be summarized as:

1. Most of the PTs did not know the mathematics they would be teaching in the future.
2. Most of the PTs were unaware of how teach mathematics effectively and to help students develop meaningful mathematical ideas.
3. Most of the PTs could not establish a connection between the mathematics courses and the mathematics they are supposed to teach at school.
4. Most of the PTs perceived procedural mathematical solutions as evidence of the conceptual understanding of students' mathematics.
5. Most of the PTs assumed that if they did not know a mathematics topic, the students also did not need to know the same topic.
6. Elementary mathematics was assumed as simple.
7. Methods of mathematics teaching courses did not equip PTs sufficiently with conceptual mathematical understanding required for teaching due to time limitation.
8. Most of the PTs had limited experience with actual students during their teacher education studies. Thus, they had a hard time to recognize the difference between their mathematics and students' mathematics.

9. New mathematics education content courses should be designed around the big ideas of mathematics to address the missing components of PTs' mathematical content knowledge and the student knowledge.

Thus, this study seeks to provide an exemplar course design in which a learning trajectory was utilized as a tool to situate students' mathematical thinking in the center of learning a long with utilization of appropriate instructional tasks. Then, the outcomes of the study were documented in terms of how elementary PTs restructured their mathematical content knowledge and student knowledge, as they were engaged in a Learning Trajectory Based Instruction.

2.4.1 The Context of Turkey

The Ministry of National Education (MEB) in Turkey has initiated major changes within school system and the curriculum to increase the quality of education in Turkey (MEB, 2006; 2013). Within this major move, a constructivist view of teaching and learning was adopted (MEB, 2013). As the constructivist view suggests, the new curriculum placed students at the center of the learning activities. The revisions also altered the traditional roles of teachers. The traditional teacher role was to provide the necessary knowledge to the students directly. The new curriculum describes a teacher's role as creating a learning environment rich in context. Considering mathematics, the teacher's role is to guide the students to undertake mathematics and to participate actively in meaningful activities. The role of the teacher has moved from being a direct knowledge supplier to guiding the students to construct their own knowledge.

However, merely changing the system and the curriculum does not alter the current problems within learning environments. Because regardless of curriculum quality; teachers are implementers of the curriculum and the system (Arslan & Özpınar, 2008). Thus, the quality of education is linked closely to teachers' quality and effectiveness (Baki & Gökçek, 2007; Seferoğlu, 2004). This indicates that teachers play an important role in teaching and learning mathematics. As a result, one could say that even when the curriculum and system are well-designed, if the teachers do not possess the necessary skills and attributes to implement the curriculum, the desired learning outcomes will not be achieved (Demirel & Kaya, 2006). Therefore, the importance of teacher education and teacher quality is evident.

Several factors shape teacher candidates' teaching quality (Baki & Gökçek, 2007). The first one is their prior experiences before entering the university teacher training programs. The second one is the experiences that they acquire within teacher training programs. Teacher candidates have several experiences prior to entering teacher education programs as they have observed different teachers and experienced a variety of teaching and learning environments. All these prior experiences and the experiences in the teacher training programs are reliable sources in determining effective teachers (Baki & Gökçek, 2007). Highly qualified teachers should acquire the content knowledge of their subject and the knowledge of how to teach and of how students learn (Işık, Çiltaş & Baş, 2010). Yet, existing literature indicated that both teachers and teacher candidates lack this essential knowledge, especially in the field of mathematics (Baki; 2013; Uçar, 2010; Zembat, 2007).

The elementary education program holds an important position among rest of the teacher education program because elementary school teachers have an important effect on young students' cognitive and emotional development (Eraslan, 2009). Within these early years, if a student establishes a good academic background, this student is more likely to be successful in advanced learning (Aydın, Şahin & Topal, 2008). This assertion is also valid for mathematics learning, as learning the essential mathematical ideas and skills in elementary school supports later, more in-depth mathematical learning and understanding (Eraslan, 2008).

In Turkey, several researchers (Akbayır & Taş, 2008; Baki & Gökçek, 2007; Baştürk, 2007; Eraslan; 2009; Hacıömeroğlu & Taşkın, 2010; Ubuz, 2009) conducted studies to detect deficiencies in teacher training programs and of how these deficiencies influenced teacher candidates' quality. Findings from these studies documented crucial issues to be considered related to mathematics education. Based on the review of the Turkish studies, the general problems and deficiencies, similar to the findings of international studies discussed earlier, could be listed as follows:

1. There were few opportunities for teacher candidates to experience mathematics teaching in real classroom settings (Baştürk, 2007; Hacıömeroğlu & Taşkın, 2010; Ubuz, 2009).
2. Teacher candidates had difficulty in understanding how students learn mathematics (Baştürk, 2007).
3. Elementary education teacher candidates did not perceive themselves personally proficient to teach mathematics (Hacıömeroğlu & Taşkın, 2010).
4. Teacher candidates could not establish the connection between the mathematics that they were supposed to teach and the basic mathematics courses they took at the university (Eraslan, 2009).
5. Mathematics education method courses should tie the theory and practice (Ubuz, 2009; Zembat, 2007).
6. Teacher candidates found it difficult to verbalize mathematical thoughts upon graduation to meet the thought complexity of the student (Ubuz, 2009).

Many researchers (Baki, 2013; Çıkla & Duatepe, 2002; Gökkurt, Şahin, Soylu, & Soylu, 2013; Haser & Ubuz, 2002; Işık, 2011; Işıksal, 2006; Toluk-Uçar, 2010, 2011; Zembat, 2007) also examined both pre-service elementary mathematics education teachers' and elementary education teachers' mathematical content knowledge of various mathematics topic, specifically fractions, ratio, multiplication and division (Haser & Ubuz, 2002; Zembat, 2007). The common major findings of these studies could be summarized as follows. Due to the lack of mathematical content knowledge, teacher candidates failed to provide in-depth mathematical explanations for their solutions or to why the rules were working (Çıkla & Duatepe, 2002; Zembat, 2007; Baki, 2013). Instead, they provided procedural explanations. Although they solved the given problems procedurally and produced a correct answer, they failed to explain why they employed a particular mathematical strategy (Baki, 2013; Işıksal, 2006; Toluk- Uçar, 2010; Zembat, 2007). Due to lack of knowledge on how students would learn mathematics and employ mathematical thinking, teacher candidates experienced difficulty in identifying students' mathematical errors and producing strategies to eliminate those errors. Because teacher candidates did not have sufficient common content knowledge, they did not the use correct mathematical language while explaining their mathematical thoughts

(Toluk-Uçar, 2011). Moreover, some of the candidates utilized a language irrelevant to mathematical language while explaining both their and students' mathematical solutions (Baki, 2013; Gökkurt et al., 2013). All these results conveyed that the majority of teacher candidates graduated without having a strong mathematical understanding regarding the mathematics that they would be teaching.

Although these studies determined the problems, they did not practically remediate the problems. First, they all indicated some possible reasons that might underlie these problems. Then, they provided further suggestions for handling and solving the determined deficiencies and problems. Throughout the review of the studies (Baki; 2013, Çıkla & Duatepe, 2002; Işık; 2011; Işıksal; 2006; Toluk-Uçar; 2010; Zembat; 2007), one possible reason that caused the problems seemed to be that the basic mathematics courses in the universities did not equip teacher candidates with the necessary mathematical knowledge of the topics that they were supposed to teach. This also led to questioning of the quality of both mathematics and mathematics education courses provided in the universities (Zembat, 2007). Toluk-Uçar (2010) also brought the issue of reducing the hours of teaching mathematics method courses in the elementary education program. She suggested that current time allocation for covering the content in elementary school mathematics was not enough in the teaching mathematics courses in the elementary education programs. As a result, in-depth examination of mathematical ideas could not be achieved in these courses.

Thus, based on the above-mentioned problems and issues, researchers stated (Baki; 2013, Çıkla & Duatepe, 2002; Işık; 2011; Işıksal; 2006; Toluk-Uçar; 2010; Zembat; 2007) that these courses should be redesigned to emphasize student's mathematics through considering the evidences gathered from the studies. To achieve this, the time allocation for method courses should be increased (Toluk-Uçar, 2010) or new courses should be offered in elementary teacher education programs (Toluk- Uçar, 2010). These courses should establish a sound base for teacher candidates to understand how to teach the mathematics**.** Also, these courses should act as a tool to improve PTs' conceptual knowledge in elementary mathematics.

As a result, the balance between theory and practice and the compelling method course contents play an important role to educate highly qualified teacher candidates. Yet, the studies in Turkey showed that the courses towards linking practice and theory remains insufficient in the sense of the time allocation and the content of the method courses and teacher practicum (Yar, 2013; Yesilyurt & Karakus, 2011). This situation causes major gaps in both mathematical content knowledge and student knowledge of teacher candidates. Thus, the present study sought to provide a way to design a course that aimed to close that gap through utilization of LTBI with elementary PTs.

2.4.2 Use of Learning Trajectories in Teacher Education: Potentials

As previously discussed, empirical evidences of students' mathematical learning and thinking, students' prior experiences and students' diagnosed mathematical misconceptions, difficulties and errors are the three components that are used to construct LTs. Also, LTs show how students navigate through the least complex to more complex mathematical ideas by engaging instructional tasks. This nature of the LTs have encouraged the recent studies (Butterfield et al., 2013; Clements & Sarama, 2013; Duncan & Hmelo-Silver, 2009; Wilson et al., 2013a) to conjecture that coordinating teacher education programs that reflect an emphasis on usage of LTs has great potential to develop PTs' conceptual mathematical understanding. This utilization has potential to support PTs to gain in-depth knowledge about students' mathematical thinking progression over time before they start to teach in schools.

Although a number of LTs were recently constructed in the field of mathematics education, practical utilization of these trajectories in teaching is in the early stages of investigation (Butterfield et al., 2013). There is a limited number of research on examining the use of LTs in enhancing teachers' knowledge for mathematics teaching and improving their instructional practices (Sztajn et al., 2012). As a result, a need for further studies to examine the use of LTs in teacher training and teaching practices has emerged.

A few emergent studies documented the potential benefits of usage of LTs in both teacher education and in-service teacher professional development. The use of the LTs in teacher training enhanced PTs' own mathematical content knowledge (Sztajn et al., 2012; Wilson et al., 2013a). The sequential task-based structure of the LT on a specific mathematics content topic has the potential to serve as an instructional and assessment guide for a novice teacher who has no prior teaching experience of teaching that topic (Clements et al., 2009; Wilson et al., 2014). The rich information about the complexity level of students' mathematical thinking, behaviors and understanding covered in the LTs have a potential for teachers to maintain the cognitive demand of the presented mathematics tasks in their instruction (Stein, Grover, & Henningsen, 1996) and sequence their instruction around students' mathematics.

Designing the instruction around the tasks that are deduced from individual student's way of learning mathematics is a key aspect of LT integration (Clements et al., 2009). Sullivan, Mousley and Zevenbergen (2004) found that task differentiation empowered students with diverse abilities to succeed in mathematics. They indicated "carefully sequenced activities" and "prompts" helped students to be proficient in the expected learning trajectory.

Wilson and his colleagues utilized a theoretical framework called LTBI with in-service teachers as a part of a professional development program (Wilson et al., 2014). They reported three teachers' cases on how working though a LT improved their mathematical content knowledge and pedagogical content knowledge. They examined teachers' discussions during professional development program and concluded that one teacher contributed to the group discussion based on subject matter knowledge and extended existing concepts into further mathematics, one teacher clarified the discussion within the group and the other teacher related the subject with pedagogical aspects necessary to implement in the classroom. All these findings indicated that teachers utilized LT in various extents that supported their teaching in the classroom.

A recent design of using area and perimeter-learning trajectory as theoretical lens to examine PTs' both content and pedagogical knowledge is proposed by Butterfield, Forrester, McCallum and Chinnappan (2013). They suggested that LTs could be used to improve PTs' mathematical knowledge for teaching for the field of area and perimeter. In their proposal, they also referred the emerging theory of LTBI as a model of teaching. They supported Sztajn et al.'s (2012) opinion about LTs' potential to build connections within complex and multifaceted mathematical knowledge for teaching. They utilized Ball, Thames and Phelps's (2008) framework of mathematical content knowledge for teaching as a tool to analyze how PTs would progress in the study. Butterfield, Forrester, McCallum and Chinnappan plan to work with PTs in a Graduate Diploma of Education Primary program in three phases. The first phase aims to identify PTs' current knowledge level of measurement. In second phase the PTs will work with actual primary students, and in the third phase the data from the primary school students will be gathered and examined. Then, they listed expected outcomes of the proposed research. They stated the area and perimeter-learning trajectory would yield an exquisite data on both PTs' progression on how students learn and how their own understanding of the concepts could be developed. They suggested the result of the study might inform teacher education course designs.

Although several researchers stated the potential benefits of LTs, Empson (2011) considered potential pitfalls of LTs. She asserted that although LTs focused on conceptual development in a particular area of mathematics, they might be insufficient in addressing other features of a curriculum. Because, learning is a complex and multidimensional process. Thus, it is so hard to embed all the characteristics of learning in one trajectory. In addition, she suggested that the trajectory might be subject to change in different context, with different learners and within different countries' educational systems. As Clements and Sarama (2013) suggested, there is no one single stable learning trajectory. Similarly, Confrey (2006) stated that a learner faces different learning barriers and obstacles in the conceptual corridor of the trajectory, thus each individual has unique LTs. However, this is the case where LTs has potential to capture the landmarks and possible learning obstacles in this trajectory in advance.

Empson's (2011) analysis on the LTs merged into a similar need of testing the benefits of LT usage in teaching practices as a tool. She indicated researchers had a critical mission of generating resources that could be used in the mathematics education field to optimize mathematics learning of students. To achieve that mission, first, researchers should find ways to equip teachers with the conceptual knowledge of mathematics and to incorporate students' mathematics learning in their teaching practices (Empson, 2011). Several researchers (Butterfield et al., 2013; Confrey et al., 2012; Daro et al., 2011; Simon & Tzur, 2004; Steffe, 2004; Sztajn et al., 2012, 2014) supported the claim that LTs could be utilized as a teaching framework and integrating LTs into teaching practices has the potential to realize these missions. These researchers also acknowledged the mutual role of teachers and learners in the learning process.

LT utilization in a teacher education program as a reference tool has the potential to provide PTs with opportunities to learn how to (i) count students' knowledge states that are situated in learning theories, misconceptions and learning obstacles (Confrey, 2006), (ii) understand students' prior mathematical experiences, (iii) launch mathematical tasks that elicit important mathematical strategies (Stein & Smith, 1998; Stein & Smith, 2011) and (iv) iteratively revise their own subject and pedagogical content knowledge (Graeber, Tirosh, & Glover, 1989), and (v) enhance developing conceptual understanding of mathematics (Simon & Tzur, 2004). As a result, integrating LT in a teacher education program have potential to inform PTs in a systematic way about all these previously discussed pedagogical and academic teaching skills rather than leaving them to learn those skills through trial and error during their initial years in the profession.

2.5 Summary of Literature Review

Understanding students' mathematical learning has been intensely researched especially in the learning trajectories research area. Development of students' mathematical thinking and learning is documented in several learning trajectories. These trajectories embedded a rich body of knowledge gathered both from existing literature and empirical evidences of students' work. This feature of the LTs makes them a powerful potential tool for teaching practices that entail both conceptual mathematical content knowledge and knowledge of students. Although these trajectories are essential tools to capture students' mathematics, their practical usage in teaching is in the early stages of the research. Also, the need for a framework that can be utilized to train teachers have led the development of an emergent theory called, Learning Trajectories Based Instruction (LTBI). This emergent theory of instruction has the potential to address the problematic issues in teacher education and teacher professional development.

Major problems related to teacher quality and their education are (1) lacking sufficient mathematical content knowledge, (2) perceiving procedural calculations as evidence of conceptual mathematical understanding, (3) underestimating both students' mathematics and complexity of elementary school mathematics, (4) having a limited amount of experience with actual students, (5) lacking opportunity to encounter with students' mathematics in their university courses, (6) possessing same mathematical misconceptions and errors as students, (7) the disconnection between the mathematics courses at the university and the mathematics that the teacher candidates will teach, and (8) lacking sufficient mathematics education course hours to possess the PTs with conceptual understanding of mathematics. Thus, the discussed issues and problems indicate a need for conducting an in depth study that aims to address these problems. Designing mathematics teaching courses for teacher candidates is one possible way to handle these problems. Thus, in this dissertation study, LTBI teaching experiment was designed around the big mathematical idea of equipartitioning that laid a foundation for important mathematical topics such as rational numbers, fractions, multiplication, division and ratio. In the experiment, the ultimate aim was to capture PTs' restructuring practices of their MCK and SK and to document the PTs' progression in their knowledge thought out the experiment. In addition, this teaching experiment design has the potential to show an example of redesigned method courses around big ideas of mathematics such as measurement.

CHAPTER III

METHODOLOGY

This study aimed to address how usage of Learning Trajectories Based Instruction (LTBI) in a 6-week teaching experiment helped elementary pre-service teachers (PTs) to restructure their mathematical content knowledge and student knowledge. In order to examine individual PT's restructuring process, this study employed a constructivist teaching experiment method (Steffe & Thompson, 2000). Table 1 shows data sources utilized for answering each research questions. In the following sections, these data sources and how they informed the data analysis will be discussed in detail.

Table 1

Related data sources informed each research question

Main Research Questions	Data Sources
1) What are differences between pre-service elementary teachers' (PTs) knowledge level before and after the LTBI teaching experiment?	Pre-Post tests Video Recordings
2) What are pre-service teachers' restructuring practices for mathematical content knowledge in a Learning Trajectories Based Instruction (LTBI)?	Video Recordings Observation Notes Field Notes PTs' written works
3) What are PTs' restructuring practices for student knowledge in a LTBI?	Video Recordings Observation Notes Field Notes PTs' written works

3.1 Study Design

3.1.1 Teaching Experiment Methodology

The teaching experiment methodology was not a widely accepted method in mathematics education research until 1970s (Steffe & Thompson, 2000). Several driving reasons contributed to the acceptance of this methodology. One main reason was the need for new models that would examine the progress of students as a result of socially constructed mathematics learning (Confrey, 1986; Sinclair, 1987). Then, post-modern period in mathematics education research accelerated the acceptance of teaching experiment methodology. In this period, over the past three decades, research on understanding students' mathematical learning and mathematical knowledge construction overtime have increased rapidly (Steffe & Thompson, 2000). In these studies, the main focus was on considering students' live mathematical experiences within a classroom setting instead of merely addressing effects of different variables on students' learning in a quantitative research setting (Steffe & Thompson, 2000).

Teaching experiment method evolved over time because researchers utilized and contributed to it. In addition, further questions that a classical experimental design could not completely answer appeared. According to Steffe and Thompson (2000) one of the main questions was about how students created meanings. Another question was interested in "how students learn specific mathematical concepts rather than become interested in these issues in a pure form" (Steffe & Thompson, 2000, p.272). These questions were needed to be addressed in the field of mathematics education research (Confrey, 2006; Kilpatrick, 1987). In addition, Steffe and Thompson (2000) indicated that this methodology was not a standardized method; instead it was a tool for researchers to organize their activities. Therefore, this methodology has been subject to ongoing revisions.

Several reseachers (Confrey, 2006; Steffe, 1991; Steffe & Thompson, 2000) indicated that teaching experiment methodology evolved from Piagetian clinical interview method. However, teaching experiment suggests more than Piagetian clinical interview (Engelhardt, Corpuz, Ozimek & Rebello, 2004; Steffe, 1991). Because the aim of clinical interview is to understand the current state of students' knowledge structure and thinking without aiming to alter them (Clements, 2000; Engelhardt et al., 2004). Yet, in the teaching experiment, one aims to understand how teaching influence the students' existing knowledge structure and reasoning (Steffe & Thompson, 2000).

3.1.2 Purposes of Teaching Experiment

The main purpose of teaching experiments for researchers is "to experience, firsthand, students' mathematical learning and reasoning" (Steffe & Thompson, 2000, p.267). Another aim of teaching experiments is to guide instructional decisions and they also produce a mechanism that will help the enhancement of the learning (Cobb, Confrey, diSessa, Lehrer & Schauble, 2003). The constraints the researchers faced during the teaching help to enhance the learning. Steffe and Thompson (2000) stated that the constraints refers to two meanings: (i) the effect of researchers' own language usage and guidance on students' learning and, (ii) student's own misconceptions rooted in their existing mathematical knowledge. To experience and determine these constraints, one can conduct a teaching experiment. Finally, the experiment creates a *living model* of students' mathematical activities (Steffe & Thompson, 2000). To understand the living model of students' mathematical activities, this research method involves "engineering particular forms of learning and systematically studying those forms of learning with the context defined by the means of supporting them" (Cobb et al., 2003, p.9).

In the teaching experiment settings, the researcher tries to understand two main issues: students' mathematics and mathematics of students (Steffe & Thompson, 2000). First one refers to students' own mathematical images and realities independent of us (i.e. an external researcher). The second one refers to our interpretation of students' mathematics. Steffe and Thompson (2000) suggested that the researcher would like to understand students' mathematics with the support of mathematics of students. As a result, examining deeply what students produce mathematically and attempt to understand the underlying reasoning and thinking is a major goal of a teaching experiment. This attempt is named as conceptual analysis of students' mathematical thoughts and reasoning by von Glasersfeld (1995). This process of analysis is also called “mathematizing” (Treffers, 1987, p.51). For the researcher, determining a learner’s mathematical knowledge and examining how the learner constructs this knowledge could be called as mathematizing (Steffe, 1991). In the mathematizing process, the researcher does not only aim to understand learner’s initial mathematical knowledge but also tries to understand how intervention or instructional activities help learners’ to reconstruct their mathematical knowledge and reasoning. The mathematizing process is also one major reason for conducting a teaching experiment.

3.1.3 Teaching Experiment Method Structure

Steffe (1983) stated that a sequence of teaching episodes forms a teaching experiment. One or more students, a teaching agent and a recording method of teaching episodes are the main components of a teaching experiment. Before starting a teaching, the researcher should identify a learning objective for students and the theory in mind, existing research on the mathematics topics, and the students’ readiness are three important constructs to consider while creating these objectives (Steffe & Thompson, 2000).

In a constructive teaching experiment, the “researcher acts as a teacher” (Steffe, 1991, p.177). Teacher-researcher assigns mathematical attributes to students rather than his/her mathematical realities (Steffe, 1983). Assigning mathematical attributes to students means determining “mathematical concepts and operations […] that [students] have constructed” (Steffe & Thompson, 2000, p.267). Through this theoretical lens, learning objectives place students’ readiness, prior knowledge and further knowledge construction in the center of the learning activities.

The main orientation of the teaching episodes is to understand how learner [re]constructs knowledge and generate ways to foster this process (Steffe, 1991). To achieve this goal, the teacher-researcher owns two main roles. The first role is to ask critical questions and create situations in which learner can actively learn. The interactive mathematical discourse is a main characteristics of the situations that teacher-researcher aims to create. The second role is to analyze how learning takes place in teaching episodes (Steffe, 1991). In the analysis, learner’s interactions, language, and actions should be considered. The results of the continuous analysis should be utilized for revising and refining future teaching episodes.

In the first stage of the teaching experiment in this present study, the teacher-researcher “formulates an image of the students' mental operations and an itinerary of what they might learn and how they might learn it” (Steffe & Thompson, 2000, p.280). This learning route is determined through utilizing a body of knowledge gathered from existing research and the learning trajectory. Learning objectives and related instructional activities are constructed according to this possible learning routes. The teacher-researcher knows initial learning objectives accompanied with the knowledge of possible situations how the intended learning objectives would be achieved in the teaching episodes. (Steff & Thompson, 2000). Although, the teacher-researcher has a sense of likely pathways of students’ learning, these routes are subject to revisions and refinements during teaching (Confrey, 2006; Steffe & Thompson, 2000).

To realize learning objectives in the second stage, the teacher-researcher constructs a series of interventions such as instructional activities (Confrey & Lachance, 2000) that are implemented in the classroom (Cobb, 2000). Based on classroom interactions during implementation, researchers capture and conceptualize how learning processes become more effective and productive for students (Cobb, 2000; Steffe & Thompson, 2000). These features of teaching experiment studies support testing innovative instructional approaches in the classroom such as using LTBI in mathematics method courses in teacher education programs (Cobb et al., 2003).

3.2 Study Procedure: LTBI Teaching Experiment

3.2.1 The Participants

Nine senior female elementary pre-service teachers studying at an elementary education program at a private university in the southeastern region of Turkey participated in this study. Each PT had completed a basic mathematics course and two mathematics education method courses. In the basic mathematics course they covered the mathematics topics starting from the rational numbers up to limits. In the mathematics education courses, they learned about various teaching and learning approaches related mathematics they would teach in the elementary schools and also examined the elementary mathematics curriculum of Turkey. They will teach all subject areas including mathematics at elementary school level from grades 1 to 4 upon graduation. In addition to mathematics related courses, each PT had two teaching practicum, one of which was in an urban public school, the other one in a rural public school, both with mathematics as an integrated part of it. In this practicum courses, the PTs indicated that they generally just observed the mentor teacher's classroom.

Purposeful sampling method was used to select the participants. Also, I was the instructor of these PTs before the teaching experiment and I had considerable information about each PTs' academic background and personal characteristics that I gained through my interaction with them in and out of the class. This acquaintance helped me to reach each PT easily. All the participants contributed to the study voluntarily outside of their course work at the university. I was not a part of their university at the time of the study. There were two reasons for the employed sampling procedure. First, selecting PTs from elementary education program was in line with the aim of this study since the scope of the mathematics covered in Equipartitioning Learning Trajectory (K-4) is covered at elementary school level in Turkey. This sampling technique enabled me to examine my research questions in the most efficient way. Second, participating PTs did not have any pre-instruction on equipartitioning and they were accessible at the time of this study in order to obtain more in-depth data. Necessary permissions and informed consent from each PT was obtained (See Appendix C). In the consent forms, procedure and emerging nature of the study was explained. Then, the participants were informed about how the findings of the study could be utilized.

Academic background of the PTs was categorized under four categories: GPA, scholarship status, the same and last mathematics education course grade and teaching experience (in the form of private tutoring). Some of the PTs received scholarship from the university based on their scores on the university entrance examination. This scholarship status was maintained through out their formal span of education regardless of the GPA in their programs. GPA of each student was reported in grade-bands since PTs did not give their permissions to announce their exact GPA. Each grade-bands consisted of 0.25 intervals. Table 2 showed the academic background of each PT.

Table 2

Each PT's academic background

PTs	GPA	Scholarship Status	Last Math. Ed. Grade	Teaching Experience
1	3.00-3.25	Full	BB (80-85)	No
2	2.25- 2.50	None	CB (75-80)	No
3	2.25- 2.50	None	CB	No
4	2.25- 2.50	None	CB	No
5	3.5-3.75	Partial	BA (85-90)	No
6	2.5-2.75	None	BA	No
7	2.75-3.00	Full	BB	No
8	2.75-3.00	Partial	BA	Yes
9	3.75-4.00	Partial	BA	No

Table 2 shows that two PTs received full scholarship, three PTs received partial scholarship (50%) and four of them did not receive any scholarship from the university based on their university entrance scores. PT2, PT3 and PT4 had lower academic background in terms of their GPAs and scholarship status. Only one PT had private tutoring experience. This showed that majority of PTs did not have opportunity to work with actual students. Their later mathematics education course grades were relatively close to each other. Three PTs' GPAs was above 3.00 out of 4.00, two PTs' GPAs was between 2.75-3.00, one PT's GPA was between 2.5-2.75 and three PTs' GPAs was between 2.25-2.50. All these distributions showed that the experiment classroom consisted of a variety of PTs who had different levels of course success.

After collecting information related to each PT's academic background, as their instructor at the university, I also paid attention to their actions in my prior courses. PT1, PT2, PT5 and PT7 were very expressive students based on my observation during my instruction at the university. PT1, PT5 and PT7 could verbalize their mathematical thoughts yet they had difficulty with expressing these verbal thoughts mathematically. PT2 had difficulty with mathematics-related courses and she had difficulty in utilizing both symbolic and verbal language of mathematics. PT7 had difficulty with utilizing correct mathematical terms while expressing her mathematical thoughts.

PT3 and PT4 were quiet students. They felt comfortable with expressing their mathematical thoughts in written language. Yet, they had some difficulty with using symbolic language of mathematics and using correct mathematical terminology.

PT9 was very expressive student. Also, this PT could use mathematical language clearly to express her mathematical thoughts. Although PT9 performed well at mathematics courses, she stated that she "would not pursue a career with mathematics education." She was a double major student. She would purse another career pathway in another field.

PT8 and PT6 both were very expressive students based on my observation during my instruction at the university. PT8 was particularly successful at mathematics related courses; she could use symbolic language of mathematics effectively. PT8 had a great sense of anticipating students' possible strategies. She stated that her tutoring with 2nd grade student helped her to understand how students might learn.

3.2.2 Context of the Teaching Sessions

The major goal of the teaching sessions was to ensure that PTs would have a strong mathematical background in teaching equipartitioning related concepts that established a base for rational numbers, fractions, multiplication and ratio for grades 1 to 4. The second goal was to develop each PT's ability to encounter students' mathematical thinking and learning, and eventually support them in their reflective teaching. The teaching experiment lasted for 6 weeks. The procedure for each week utilized instructional tasks that will be explained in the following sections in more detail.

I was the teacher-researcher of the experiment. This study was conducted in two phases. The first phase was the pilot study conducted with 10 elementary pre-service teachers who did not participate in the main study and it lasted for 3 weeks. Initial instructional activities and pre and post tests were designed and implemented in this phase. The results of the pilot study were utilized to shape the design of the final teaching sessions. Necessary revisions and refinements were created on the instructional tasks and test items, which will be explained later in this chapter. The actual teaching experiment lasted for 6 weeks. In each week, PTs and I gathered approximately for 2 - 3 hours and certain equipartitioning learning trajectory (ELT) levels were covered.

Learning objectives for each teaching sessions were determined based on the proficiency levels of ELT. Each objective followed the characteristics of ELT; that is, they were constructed from the least complex to the more complex mathematical strategies and thinking that students might have as documented in ELT. ELT has been studied extensively by the DELTA (Diagnostic E-Learning Trajectories Approach) team, which I worked as a member in the past, for several years. This team has been continuously revising and refining ELT. The specified levels of the LT retrieved from the work of DELTA team (as cited in Pellegrino, 2009, p.16) were used in this study as in Table 3:

Table 3

Utilized levels of equipartitioning learning trajectory in the study (Retrieved from Pellegrino, 2009, p.16).

Levels	Description
1	Equipartition collections by dealing single units or composite units
2	Equipartition a single whole (circles and rectangles) Criteria: correct number of parts, equal- sized parts, exhaust the whole
3.	Justify fair shares - by counting, stacking, arrays, or patterns
4	Name the fair shares in relation to referent units a. Of collections: sharing 12 among 2: *half* or *six* b. Of single wholes: sharing a whole among n: *1/n* or *1/n* of
5	Re-assemble equal groups or parts to produce the collection or the single whole as "*n* times as many" or "*n* times as much" as a single group or part
6	[Predict (qualitatively) the] effect of changes in number of people sharing on size of shares (qualitative compensation)
7	Predict, [demonstrate, and justify] outcomes of compositions of splits (splits of a split of a whole) [or on collections or a single whole]. *a. Two or more splits, and identification of factor- based pairs*
8	Demonstrate and justify the effect of factor- based changes in number of persons sharing on the *size of shares,* and vice versa*, for collections or single whole (quantitative compensation 1)*
9	"Demonstrate and justify how extra shares can be redistributed for fewer people (additive changes) sharing collections [equipartitioning over breaking to quantify compensation]" (Yilmaz, 2012, p.5).
10	Demonstrate equivalence of non-congruent parts across or within methods of non-prime equipartitioning a. *Decomposition/composition*: *b. Transitivity*: if $X = Y$, $x = 1/2X$, and $y = 1/2Y$, then $x = y$
11	Assert that a whole can be *equipartitioned* for all natural numbers greater than 1 (*continuity principle*)
12	Equipartition multiple wholes among multiple persons and name the resulting share in relation to referent units
13	Predict the outcome of a *composition of splits* on *multiple wholes.*
14	Make factor or split-based changes in a number of objects, number of people sharing, the size of fair shares, or any combination thereof and predict the effects on the other variables (direct, inverse, and covariation to quantify compensation)

The levels represented in Table 3 included the detailed descriptions of the levels showed in Figure 5. Two of levels of ELT represented in Figure 5 were not included since those levels covered mathematical ideas beyond the scope of grade 1-4 mathematics in mathematics textbooks and standards (MEB, 2006; MEB, 2013). In addition, Yilmaz (2011) stated that those levels were not administered to grades K-3 in actual field-testing of ELT assessment tasks.

3.2.3 General Characteristics of Teaching Sessions

The PTs could understand the English language in written materials so each tasks and items were constructed in English. However, the PTs felt more comfortable with Turkish, when they expressed their thoughts aloud in the teaching sessions. As a result, the medium for the instruction was in Turkish. Before the teaching sessions, PTs were briefly informed about what a learning trajectory was. I explained them we would cover the ELT together in this study. After these explanations, the PTs received the instructional tasks. The PTs worked either individually or in pairs on the given tasks. Instructional tasks were usually composed of three main parts. The first part of the tasks allowed PTs to work on given problems alone and utilize their own mathematical knowledge and strategies. In the second part, PTs were asked to predict students' mathematical strategies and possible misconceptions. The last part asked what kind of revisions or utilization techniques made this task accessible to younger children and how PTs would implement this task. For some tasks, the last part asked PTs to determine which further mathematical ideas these tasks were connected and contributed to.

After completing the second and third phases, PTs were asked to discuss and exchange their ideas with their peers. The whole classroom discussion focused on conceptual ideas rather than merely providing procedural calculations (Lambert & Cobb, 2003). Teacher-researcher utilized ELT to guide discussion and supported the PTs in gaining the essential knowledge related to equipartitioning-related concepts and ideas. The general norms for this whole classroom discussion were adapted from Cobb, Stephen, McClain and Gravemeijer (2001) as follows:

1. PTs have to explain their reasoning related to solution way(s),
2. PTs have to listen attentively their peers' explanations for solution way(s),
3. If there is an instance that a PT did not understand any mathematical issues emphasized in the class, this PT has to ask further questions to clarify his or her understanding,
4. PTs have to evaluate whether presented solutions are valid or not, and exchange ideas and support their explanations with evidences, and
5. A shared meaning of the activity should be deduced before moving into the next activity.

In the beginning of each teaching session, a brief summary discussion related to the previous weeks' content was conducted. This discussion helped in designing new interventions based on detecting any knowledge gaps, misconceptions, or unclear understandings about past weeks' content.

3.3 The Teaching Experiments

3.3.1 Pilot Study

ELT was utilized as an intervention tool in a 3-weeks teaching experiment to understand and expand the 10 sophomore elementary education PTs' understanding and conceptions related to equipartitioning. The PTs were the students in the same private university and informed consent from the university were received. They would teach elementary school mathematics from the grades 1-4 upon graduation. None of the PTs had experience in private tutoring. They received one pure mathematics course. At the time of the study, they only received an educational method course that included mathematics as a part of it. The pilot and actual group were similar in terms of teaching experience and both groups had a limited background in mathematics. Two groups would teach same grade levels.

The assessment items and tasks that were developed for the actual experiment was piloted. Each week, class was held for approximately 2.5-3 hours. In the first week, PTs received pilot items and task related to equipartitioned a single whole. In the second and third weeks, LT based instructional tasks were utilized as intervention tools. Also, PTs received pilot items related to fair sharing multiple wholes, reallocation and covariation. In the pilot study, items and the tasks related to the levels 2-4, 9, 12 and 14 were covered. Each week's session was voice recorded with the exception of the first session, accompanied with field notes and PTs' written work. The data were analyzed analytically. The analytical model will be explained in the data analysis section.

The findings of the pilot study were utilized to revise the tasks and pre-post tests' items that were prepared for the actual teaching experiment. The implications of the pilot study will be reported in the actual experiment preparation section. Moreover, the analysis of the data gathered from the pilot study was implemented in producing the final coding frame of the study. Detailed explanation of this will be provided in the data analysis section.

3.3.2 Main Teaching Experiment Initial Phase: Preparation

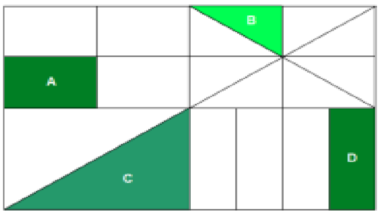
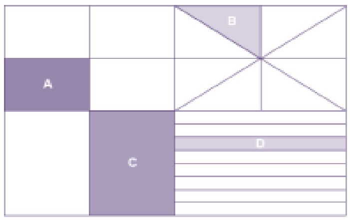
3.3.2.1 Construction of Pre- and Post-Tests

A test adapted from Wilson (2009) and Mojica (2010) was utilized as the pre-test to understand each PT's initial knowledge level related to the ELT levels covered in the study. A post-test was devised as a parallel form of the pre-test. The post-test was utilized to assess the progress of each PT. These tests were referred as parallel since the tests included adapted versions of six items and retrieved two items from the parallel forms utilized in Wilson (2009) and Mojica (2010) studies. Only the numbers and the minor changes in the wordings of the rest of the items were changed between the tests while maintaining the cognitive demand and mathematical skills required by the items. One mathematics educator, one educational measurement and statistics doctoral student and I worked collaboratively to establish parallel cognitive load for each item.

There were 17 open-ended items in both tests (See chapter IV for the items). Six items were adapted and two items were retrived from previously used tests (Mojica, 2010; Wilson, 2009). Four items were adapted from existing research (Confrey, Nguyen, Lee, Corley & Maloney, 2012; Empson & Turner, 2006; Lamon, 1996; Yilmaz, 2011). Five items were developed for the study and piloted. Ten elementary school PTs who did not participate the actual teaching experiment received these pilot items and the final version of the items were constructed based on their feedbacks and performances. For both tests, item 10 were anchor items and the rest were parallel items. Anchor items address the items, which were utilized in both pre, and post tests without any adaptation of the item. These items were selected as anchor items since in the pre-test PTs performed poorly on these items. Table 4 shows an example parallel item (item 17) from pre and post tests.

Table 4

Pre/post test: A parallel item example

Pre-Test Item	Parallel Post-Test Item
Ali's mum fairly shared a cake among four of his son's friends. Ahmet received piece A, Kaan received piece B, Gulsen received piece C and Mehtap receives piece D. Figure 1 shows how Ali's mum shared the cake.  Answer the following questions: a. Did Ali's mum fairly share the cake? How do you know? b. Did each person get the same amount of cake? Justify your answer. c. Describe the relation among four friends' shares mathematically.	Ali's mum fairly shared a cake among four of his son's friends. Ahmet received piece A, Kaan received piece B, Gulsen received piece C and Mehtap received piece D. Figure 1 shows how Ali's mum shared the cake.  Answer the following questions: a. Did Ali's mum fairly share the cake? How do you know? b. Did each person get the same amount of cake? Justify your answer. c. Describe the relation among four friends share mathematically

In the pilot study, the PTs received five items before the instructional activities. First, three items related to collections and a single whole case were administered in the 1st week. Then, two reallocation and one covariation items were administered in the 2nd week. Sharing multiple wholes items were administered in the 3rd week. In all weeks, PTs had 30 to 40 minutes to work on the items. Based on the analysis of pilot data, items were revised. Table 5 shows two examples of the pilot and revised items.

Table 5

Example Item Revisions

<table>
<tr><th>Pilot Item</th><th>Revised Item</th></tr>
<tr><td>Item 9. Didem invited her friends to her birthday party and 4 friends showed up. What happened to each person’s fair share of candies when…</td><td>Item 9. Didem invited her friends to her birthday party and 4 friends showed up. She fairly shared the cake among 4 friends. What happened to each friend’s fair share of birthday cake when…
Please justify your answers</td></tr>
<tr><td>a. More friends showed up for the party?
b. Fewer friends showed up for the party?
c.Half of the friends showed up for the party?
d.Double the number of the friends show up for the party?</td><td>a.More friends showed up for the party?
b.Fewer friend showed up for the party?
c.Half of the friends showed up for the party?
d.Double the number of the friends show up for the party?</td></tr>
<tr><td>Item 16. Mustafa knows that 12 carrots will feed 4 rabbits if they are shared fairly. Predict the number of carrots needed for each number of rabbits listed in the table below, so that each rabbit will get the same share of carrots (Adapted from Yilmaz, 2011).<table><tr><th>Number of rabbits</th><th>Number of carrots</th></tr><tr><td>2</td><td></td></tr><tr><td>4</td><td>12</td></tr><tr><td>8</td><td></td></tr></table></td><td>Item 16. Mustafa knows that 6 carrots will feed 4 rabbits if they are shared fairly. Predict the number of carrots needed for each number of rabbits listed in the table below, so that each rabbit will get the same share of carrots (Adapted from Yilmaz, 2011).<table><tr><th>Number of rabbits</th><th>Number of carrots</th></tr><tr><td>2</td><td></td></tr><tr><td>4</td><td>6</td></tr><tr><td>8</td><td></td></tr></table>In how many ways can you figure out your answers? Explain your reasoning mathematically.</td></tr>
</table>

There were four main reasons for item revisions. First, the items in the pilot study did not elicit sufficient justification of what PTs were actually thinking and employing. For instance, Item 9 did not ask for any justification in the pilot study. PTs only provided the short answers for the item such as increase and decrease. Yet, the reason for selecting this item was to understand whether PTs could explain the result of factor based change on each person fair share. Thus, in the revised item justification was requested. Second, the items did not elicit multiple mathematical strategies. For item 16 in the pilot study, PTs only provided one strategy to find the solution. Yet, this item aimed to elicit different mathematical strategies. As a result, a sentence explicitly asking for various solution ways was added. Third, the numbers utilized in the item were friendly numbers that made item very easy. For instance, the first set of numbers in item 16 yields an integer number as a result of the fair sharing action. Yet, in the revised item the fair sharing action produced an improper fraction. Lastly, the language of the item was mathematically unclear. For instance, item 9 required fair sharing a rectangular cake among 4 people. Yet in the pilot item, this number was not clear. Some PTs understood that the cake would be fairly shared among 5 people, Didem and her four friends.

After all revisions, at the final stage one mathematic education researcher with a doctoral degree and researcher-teacher examined the items in terms of how they met the description of targeted levels of ELT. The final versions of the items were constructed. Table 6 indicates the targeted level(s) and brief targeted-content description of the items in both tests.

Table 6

ELT Levels and Pre-Post Tests' Items Alignment

Items		Levels
1	Fair sharing collections and naming fair shares	1 and 4
2	Quotient construct – Generalization of n amount of object can be fairly shared among p amount of people	11 and16
3	Reassembly- Reversibility of discrete equipartitioning	4 and 5
4	Times as many- Comparing size of the whole to size of the one share	4 and 5
5	Reallocation and justification of fair shares (case of discrete collections)	3 and 9
6	Reallocation and justification of fair shares (case of discrete collections)	3 and 9
7	Reallocation and justification of fair shares (case of discrete collections)	3 and 9
8	Sharing multiple wholes among multiple people	12
9	Compensation/ Factor based change	6, 7, 8 and 14
10	Sharing single circle through utilizing multiple strategies and indicating possible misconceptions.	2, 3, 7 and 10
11	Sharing single rectangle through utilizing multiple strategies and indicating possible misconceptions - Ordering strategies in terms of difficulty.	2, 3,7 and 10
12	Ordering tasks related to fair sharing whole(s) among multiple people-Levels 1-3 and Level 12	1-3 and 12
13	Predicting number of fair parts as a result of folding through repeated halving. (Connecting equipartitioning idea to rational number reasoning content area) Repeated halving	2, 4,7 and 13
14	Predicting the multiplicative relation between number of folds and number of fair parts created. Composition of splits	7 and 13
15	Sharing multiple wholes and compensation and detecting students' mathematical strategies and misconception	10,12 and 13
16	Covariational reasoning and utilization of multiple mathematical strategies	2,5,10,12 and 14
17	Area congruence and transitivity argument	2-5 and 10

3.3.2.2 Construction of LT-Based Tasks

Although research on LT-based tasks utilization in mathematics teaching is an emerging and new field of research, recent studies have indicated that teachers' MCK could be enhanced through the usage of LT-based tasks (Wilson et al., 2013b). Confrey and Lachance (2000) suggested that these tasks should have two aspects: mathematical and pedagogical. The mathematical aspect deals with what should be taught and the pedagogical aspect deals with how it should be taught meaningfully. As a result, the process of task development based on LTs considers students' present conceptual understanding of mathematics; effective instructional practices, and clearly articulated current mathematics teaching and learning approaches (Confrey, 2006; Duncan & Hmelo-Silver, 2009; Elmore, 2002; Wilson et al., 2013b).

A learning trajectory based task should provide the opportunity for students to articulate and examine their mathematical ideas through ongoing revision and refinements of earlier mathematical ideas (Clements et al., 2009; Confrey, 2012). In addition, this task should be an open-ended one in which students have the opportunity to devise multiple solution strategies and mathematical representations, and reflect on those (Stein & Smith, 2011). An instructional task in a specific level of LT should provide a base for moving to the next level of LT (Franklin et al., 2010). It should also be related to other prior mathematical experiences and the knowledge of students, and can be used for filling gaps in students' prerequisite knowledge if needed (Battista, 2004).

For a LT-based task, one needs to identify the following design principles: 1) recognizing underlying mathematical ideas in each level of LT, 2) capturing the relationship between the mathematical goal of the task and students' current proficiency level (Stazjn et al., 2013), 3) identifying students' possible mathematical strategies and misconceptions in conceptual corridor of LT (Confrey, 2006), 4) ordering these mathematical strategies according to complexity levels presented in LT, and 5) making use of all these information embedded in LT to finalize the task (Stazjn et al., 2013).

In this study, LT-based task design was utilized as a reference (Confrey et al., 2009; Sztajn et al., 2013; Wilson et al., 2013) for the tasks used. Instructional activities used in the study were designed according to learning objectives and targeted proficiency levels within ELT. Each task was open-ended and included two main phases. In the first phase, PTs were expected to solve the given task, produce different solution strategies, and provide justifications for their answers. Initially asking PTs' own solution strategy of the given task aimed to guide PTs to check their own MCK and detect any mathematical misconception they might hold. In the second phase, PTs were expected to predict K-4 students' strategies, misconceptions and justifications. This part of the task aimed to capture the relationship between the mathematical goal of the task and PTs' ability to predict students' proficiency levels, mathematical strategies, and misconceptions in relation to ELT. This step helped me to realize design principles 1 and 2. In addition to these two main parts, a third part addressed whether PTs could connect the mathematical ideas embedded within each task to other mathematical ideas and topics. This part was included in the task on the activity sheet. If it was not written on the activity sheet, this part was addressed in verbal discussions.

Upon completion of each task, PTs were asked how they would utilize this knowledge in their teaching. This discussion as a verbal part of the task design aimed to realize design principles 3 and 4. Furthermore, the missing misconceptions, mathematical strategies, and learning difficulties were discussed upon the completion of each task. This step satisfied the last design principle of LT-based task by utilizing all the embedded information within LT.

A total of ten LT-based tasks were utilized in this experiment (See Appendix B for task examples). Three tasks were directly adapted from existing research (Empson & Turner, 2006; Mojica, 2010; Stein and Smith, 2011). The rest of the tasks was created based on the literature and design principles of LT-based tasks. One researcher with a doctoral degree and one PhD candidate in the field of educational measurement and statistics and background in mathematics education helped me to examine each task. They analyzed each task separately while keeping in mind the learning objectives for the particular level that the tasks aimed to realize. They provided their feedbacks. Then, pilot study implications were used to modify and create the tasks. Table 7 provides an example task revision.

Table 7

Utilized Pilot Task and Their Revised Version Example

<table>
<tr><th>Pilot Task</th><th>Revised Task</th></tr>
<tr><td>Please fair share a rectangular birthday cake into four. Name each child's share.

How would elementary school children fairly share a rectangle into four? Please indicate any misconceptions.

Try the same task for 3-splits.

Please fair share a circular whole into six. Name each share.

How would elementary school children fairly share a circular whole into four? Please indicate any misconceptions.</td><td>1. You and your group are given a set of color pencils and a rectangular paper that represents a garden. You will plant different fruits in this garden. The rules are:
• Each fruit should have the same amount of space.
• Color each space for each different fruit with a different color.
• Try as many as possible ways and make sure each fruit has the same space.

Answer the following questions:

1. If you plant for n different fruits, how you would fairly share the rectangular garden in different ways? Try for n: 4, 6, 10, 12.

a. How do you make sure each fruit has the same amount of space?
b. Name the number of parts that each of you paint.
c. Compare the size of the whole shape to one fruit's share.
d. Compare the size of one fruit's share to the whole shape.
e. What mathematical ideas does this task serve as a base?
f. Can you fairly share a single whole for any amount of people? If yes, why? If no, why not?

Work on the same activity for circles (n=4, 6, 10, 12). Be sure to address students' misconceptions or learning difficulties while working on the task.</td></tr>
</table>

Table 7 (cont' d.)

	At the end of the activity answer the following questions: a. How is this task different from or similar to the task of fair sharing discrete collections? b. What kinds of misconceptions may you encounter while implementing this task in an elementary school classroom? c. How is fairly sharing a circle different from or similar to fairly sharing a rectangle? **Summary** a. How many cuts were needed to create 4, 6, 8, and 10 fair shares if only horizontal or vertical cuts were used? What about creating n fair shares? How should a circle be marked so that it can be fairly shared easily?

This task was modified since, the first pilot task limited PTs' ability to create different number of splits on a given rectangle or circle. Creating different numbers of shares also yielded different sharing strategies and helped PTs to examine comprehensively the differences between odd and even splits. For instance, creating 12 splits on a given rectangle could be achieved through 11 *parallel cuts*, 3 *parallel cuts* and 4 *vertical cuts* (3x4) or different *composition of splits* (6x2). These different strategies all yield congruent fair shares. In addition, the first task did not allow PTs to generate a generalization. It also did not emphasize the connection of the task with other mathematics topic. All these constructs need to be embedded in a LT-based design since without having a prior set of questions in mind, the verbal discussion in the classroom in the pilot study left some missing pieces.

3.3.2.3 Video Case Selection

At the end of the 2nd, 3rd and 5th weeks, a video of K-2 students' fair sharing activities were displayed. Videos were gathered from my own master thesis (Yilmaz, 2011). Only the covariation video in the 5th week was captured as a part of an interview with a 2nd grade student in the United States. Informed consent was obtained from the school, the parents and the students for the videos to be used for research purposes as a part of my master thesis study. The videos were selected based on three criteria: utilization of multiple strategies, displaying a common misconception, and explicitly indicating progression in students' mathematical thinking. Each video was played after the instructional activities were completed. I showed the video and paused in the necessary parts and guided the discussion about students' mathematical thinking. Also, in some instances, I asked the PTs what they could ask the student if they were the ones who conducted the interview.

Utilizing video cases within the teaching sessions had three main reasons. Although PTs had a theoretical understanding of fair sharing related concepts and also had an understanding about how students learn fair sharing, they might not actively apply this knowledge in real teaching practices (Kersting, Givvin, Sotelo & Stigler, 2010). As a result, the video case activities aimed to provide PTs with the opportunity to observe and evaluate embedded information within ELT in an actual real student work practically. The second reason was that these analysis not only revealed PTs' MCK but also provided evidences of how they related their MCK with the students' actual engagement and their ability to bring that [fair sharing] knowledge [embedded in ELT] to bear on a mathematical tasks segment depicted in the video clip (Kersting et al., 2010). The last reason was that the video case analysis allowed researcher to assess PTs' MCK more easily than traditional assessment (Kersting, Givvin, Thompson, Santagata & Stigler, 2012). This assessment allowed me to ask questions promptly and elicit more information about how PTs internalized the presented video clip as they restructured their MCK and SK. Video show case analysis brought the opportunity of observing not just PTs' mathematical knowledge but also how PTs mobilize their mathematical knowledge to analyze the actual teaching and learning cases. As Ball et al. (2000) suggested the concrete learning and teaching situations used as a measure for understanding the progress in PTs' MCK and SK. Since, during the analysis the PTS required to apply their theoretical knowledge about students learning to capture evidences of students' mathematical thinking and learning in the video segments.

3.3.3 Main Teaching Experiment Second Phase: Implementation

3.3.3.1 Weeks 1 and 6

Pre-test was implemented in the first week of the experiment. Each PT received the tests individually. Post-test was implemented in the 6th week in the same way. Each test consisted of 17 open-ended items. Explanations for each item were provided at the beginning of both tests for any unclear parts related to the language of item or what was exactly being asked. During the administration of the test, if PTs had any additional questions about the structure or the language of the tests, they are allowed to ask to me individually. PTs were asked to clearly write their mathematical thoughts along with the justifications on the given tests. Although there was not any time restriction for both tests, the pre- and post- tests each lasted for two and a half hours.

3.3.3.2 Week 2

The equipartitioning collections were the main topic of interest. Collection cases in levels 1, 3, 4 and 5 of ELT were covered. In these levels, various equipartitioning collection strategies, justification ways, naming practices for fair shares and reassembly of equipartitioning collection cases were covered. Also, the differences between complexity of the fair sharing collection strategies were discussed. Learning objectives related to each level were determined as in Table 8:

Table 8

Week 2 Learning Objectives: At the end of the Week PTs will be able to…

1	Conceptualize equipartitioning criteria (exhaust the whole, create equal sized and correct number of shares).
2	Utilize ELT to anticipate students' mathematical strategies, misconceptions, difficulties and justifications related to fair sharing discrete collection.
3	Produce mathematical strategies to facilitate their students' ability to name fair shares and reassembly.
4	Construct connections between fair sharing collections and further mathematical topics and ideas.
5	Produce factors of given amount of object (n) and finally formulate a generalization related to fair sharing n objects among p people yields n/p amount per person.
6	Practically examine students' actual work related to mathematical ideas embedded in levels 1, 3, 4 and 5.

In the second week, three tasks were utilized and implementation of these tasks lasted for two and half hours. The first two tasks included the previously mentioned three main parts of LT-based tasks (See section 3.3.2.2). Since the last task was designed for deducing a generalization, it did not follow the same patterns as prior tasks. The task briefly asked to deduce a mathematical generalization for n number of objects fairly shared among p amount of people. Appendix B shows reflection questions utilized as a guide for the classroom discussion. The question in the first task was:

> *"Eight children were coloring a picture using a box of 32 crayons. They shared all the crayons in the box fairly. How many crayons did each child get?"*

The second task aimed to realize learning objectives 2-4. The task question was:

> *"Eight friends are playing a Lego game. They plan to build a city. They altogether had one pack of Lego. They shared the packet and each got 9 Legos."*

1. *How much of the whole collection of Legos did each friend get? Name each friend's share.*
2. *Compare the total number of Legos in the packet to the number of Legos each friend has.*

3. *Compare the number of Legos each friend has to the total number Legos in the packet.*

The last task aimed to detect whether teachers could arrive at a general statement that n objects can be fairly shared among p people and each share will be n/p. The implementation of these three tasks within the teaching experiment lasted for 2 hours. Table 9 shows the related learning objectives of each task and a video utilized in this week. The objectives were also achieved through additional verbal discussion built upon the task or video analysis.

Table 9

Alignment of each Activity with Learning Objectives: Week 2

Activities	Learning Objectives					
	1	2	3	4	5	6
Task 1	X	X		X		
Task 2		X	X	X		
Task 3				X	X	
Video Analysis 1	X	X		X		X

After the implementation of these tasks, a video case analysis was done in the last 15-20 minutes. In this video, a 1st grade student was asked to fairly share 24 candies among four friends. The student used one to one correspondence to fairly share 24 candies and he counted each pile of candies to justify each friend's fair share. The analysis of this video was made through verbal discussion by paying attention to the student's mathematical strategies, what he might know or not know, and justification methods.

3.3.3.3 Weeks 3 and 4

In the third and fourth weeks of the teaching experiment, equipartitioning a single whole was the main topic of interest. Single whole cases in levels 2-7, and 10 of ELT were covered. Fair sharing strategies, mathematical misconceptions, area equivalence argument, part-whole relations, naming practices and folding were the main mathematical ideas covered in these weeks. Learning objectives were determined as in Table 10.

Table 10

Weeks 3 and 4 Learning Objectives: At the end of the Weeks PTs will be able to

1	Utilize ELT to anticipate students' mathematical strategies, misconceptions, difficulties and justifications related to fair sharing a single whole.
2	Produce mathematical strategies to facilitate their students' mathematical understanding and relations (transitivity argument also called property of equality of equipartitioning).
3	Construct connections between fair sharing a single whole and further mathematical topics and ideas (multiplication, area, ratio, improper fractions, commutative property).
4	Generate a generalization that a single whole could be fairly shared among any amount of people.
5	Practically examine students' actual work related to mathematical ideas in respected levels of ELT.
6	Generalize the relation between the number of folds and the number of fair shares created (for generalization utilize composition of splits and repeated halving).

In the third week, two tasks were utilized and implementation of these tasks lasted for 2 hours. In the fourth week the last two tasks were utilized. Appendix B shows reflection questions utilized as a guide for the classroom discussion. Table 11 shows the alignment of each objective with the presented tasks.

Table 11

Alignment of each Activity with the Learning Objectives: Weeks 3 and 4

Activities	Learning Objectives					
	1	2	3	4	5	6
Task 1	X	X	X	X		
Task 2	X					
Task 3	X	X	X			
Task 4	X	X	X		X	X
Video Analysis 2	X	X			X	

In the third week, PTs dealt with fair sharing a rectangular whole case in the first task. Then in the second task, they worked on fair sharing a circular whole case (See Table 7 above for both tasks). For both tasks, PTs were asked to create several fair sharing strategies (such as *repeated halving, composition of splits, utilization of diagonal cuts, parallel cuts or radial cut*) and comparing the size of the fair shares with different shapes. After the implementation, a video of a 2nd grade student was presented. In this video, the student was asked to fair share a rectangular cake into four fair parts. He fairly shared this cake into four through the utilization of diagonal cuts. PTs observed the student's fair sharing actions and took notes on student's verbal descriptions. Then, a whole class discussion was conducted on how students think about and engage with the task.

In week 4, PTs started to work on the difficulty of ordering tasks that involved cases of comparing a circular whole versus a rectangular whole. In addition, the tasks addressed the difference between creating odd versus even splits on a single whole. At last, the PTs compared utilization of *composition of splits* on a circle case and a rectangle case of different tasks with different difficulty levels. PTs ordered fair sharing single whole cases from the least complex to more complex ones, based on their experiences in the third week and the first task in the fourth week. They worked in groups of two and then came up with conjectured order. These orders were written on the board and each group challenged other groups' ideas. Then, differences and similarities of the conjectures were discussed. At the end, the shared meaning on difficulty levels of sharing was deduced based on the information suggested by ELT and PTs' experiences in the instructional activities.

The last task was a folding activity (See Appendix B) adapted from Empson and Turner (2006). This activity involved utilization of *repeated halving and composition of splits*. The mathematical ideas covered by the folding activities were predicting the result of the folding actions, understanding the role of folding to understand radial cuts, relating folding activities with *missing factor* questions in mathematics, identifying factors of a number such as 12, and understanding composition of split as a precursor to understanding area. In addition, this activity directly utilized students' work of Empson and Turner's (2006) folding activities. PTs tried to understand each student's work and decide what these students might know or not know mathematically.

At the end of both weeks, PTs summarized what they learned and compared the similarities and differences between fair sharing a single whole and discrete collections.

3.3.3.4 Week 5

Reallocation, equipartitioning multiple wholes and covariation were the main topics of interest. Mathematical ideas related to levels 9, 10 and 12 were covered. In the prior weeks, *composition of splits* and the *argument of area* *congruence* (equivalence of the fair parts) were deeply examined. Therefore, fair sharing of multiple wholes was discussed at a greater pace. Table 12 presents the learning objectives of this week.

Table 12

Week 5 Learning Objectives: At the end of the Week PTs will be able to

1	Utilize ELT to anticipate students' mathematical strategies (i.e. benchmarking, split all), misconceptions (n+1 cuts to create n cuts, cut results in uneven shares, parallel cut on circles), difficulties (recognizing area congruence of different shaped fair shares), and justifications related to fair sharing multiple wholes.
2	Differentiate fair sharing discrete collections and reallocation strategies.
3	Produce multiple strategies and utilize knowledge related to covariational reasoning.
4	Construct connections between engaged levels and further mathematical topics and ideas (ratio, multiplication, fraction types, factors of a number).
5	Practically examine students' actual work related to mathematical ideas embedded in levels 9,10, and 12.

To realize each learning objective presented in Table 11, three tasks and two video analyses were utilized in a teaching session that lasted 3 hours. The first task was related to reallocation level. The second task was related to fair sharing multiple whole and the last task was a covariation task. First video was related to reallocation level and secon video was related to covariation level. Table 13 shows the alignment of each learning objective with the related tasks.

Table 13

Alignment of each Activity with Learning Objectives: Week 5

Activities	Learning Objectives				
	1	2	3	4	5
Task 1		X		X	
Task 2	X		X	X	
Task 3			X	X	
Video Analysis 3 & 4	X	X		X	X

The necessary background for reallocation was established by discussing in prior weeks both qualitative and quantitative compensations in relation to single wholes and collection cases. As a result, the implementation of first task on reallocation was relatively easier than the other tasks. PTs came up with solutions to what happened to each person’s share when factor based changes occurred in the number of people.

In the second task, the mathematical strategies of benchmarking and split all for fair sharing multiple wholes were discussed. The first part of the task had a context of a birthday party and four children trying to fairly share 6 small birthday cakes. The second part of the task had the same context with different numbers 7 cakes among 4 people. These tasks were adapted from Wilson et al. (2013). Each task had previously mentioned three components of LT based tasks. Upon the completion of the task, different mathematical strategies were discussed. Then, the equivalence of fractions, improper fractions, proper fractions and mixed numbers was discussed.

The third task demonstrated the covariation concept. Before the task implementation, I asked PTs the meaning of ratio, covariation, unit ratio and fractional unit. Since, knowing the meaning of these mathematical concepts is important to capture the idea of covariation and relate the covariation with further mathematics topics. In addition, up to this week, the PTs had discussions on the meaning of fraction and also ratio terms were utilized. Then, I gave the PTs the third task (see appendix B, adapted from Stein and Smith, 2011). When the PTs finished the task, several mathematical strategies (such as *unit ratio, additive strategy, scaling up, scaling factor* and others) of solving a covariation task were discussed. Then, I asked the PTs to came up with the definition of unit ratio, ratio unit, unit fraction, covariation and ratio concepts based on what they had experienced in the week 5 teaching session. The PTs worked in two groups to define these concepts. At the end, the whole class with the help of the teacher-researcher saturated their definitions for these concepts.

At the last part of this week's teaching session, two video analyses were done in approximately 35 minutes. In the first video, a 2nd grade student was asked to solve a reallocation departure task. In this task, initially 40 crayons were fairly shared among five children. Then, one child left the group. The task asked how the remaining friends could fairly share the crayons after one child left the group. The student first represented the original share of each child in an array format by using manipulatives (8 crayons per child). Then, she took the last column and distributed 8 crayons among the rest of the four children by utilizing composite unit as 2 crayons per child.

In the second video, one 3rd grader was constructing relation between fair sharing discrete collections to covariational reasoning. This video comprehensively covered fair sharing collections, naming each fair share, justification of fair share and covariation. This feature of the video helped both the PTs and me to revisit the big ideas that were discussed in the prior teaching sessions. In the video, initially a 2nd grade student was asked to fairly share 6 candies among 2 people and asked to find one person share. I showed the children a ratio table shown in Figure 6.

# candies	# people

Figure 6. Ratio table

This table helped children to realize there were two quantities involved in the questions, and that they both vary together as the size of the one person's share remained the same. In the case of the number of people being doubled or halved, what would happen to each person's share was discussed and represented on the table. The relation between the numbers on the table was examined. Eventually the student stated an informal definition of ratio concept.

Upon the completion of the video case analyses, a whole class discussion took place. The PTs shared their analysis of what the student did in the videos and they exchanged ideas. In addition, PTs discussed the differences between the reallocation and covariation concepts based on the video analysis and their experience on the tasks.

3.4 Data Collection Tools

The main data sources in this study were classroom observations and field notes, video recordings of each teaching experiment session, written works of PTs and pre and post-tests.

3.4.1 Pre/Post-Tests

A pre-test was utilized to understand each PT's current level of understanding related to the ELT levels covered in the study. The post-test was utilized to assess the progress of each PT after the experiment. PTs' written responses were utilized to examine changes in their mathematical misconceptions, knowledge gaps, thinking strategies and the representations they employed. As a result, the PTs' written work on both tests provided additional evidence to document how each PT restructured and enhanced their MCK and SK after the intervention. The implementation of pre- and post-tests took a total of 5 Hours. The construction of the tests was explained in detail in the previous sections of this chapter.

3.4.2 Classroom Observations: Video Data

In this study, a video recording of each teaching session was obtained for observation purposes. In the third and fifth week, due to technical problems with card memory, the last half hour of the teaching session were not recorded, yet observation and field notes were taken. Observation notes were taken during the experiment, yet the parts that I did not have time to take notes simultaneously were recorded as field notes upon the completion of the teaching session. The reason for selecting video recording was that it would be difficult to collect and remember a great amount of detailed information related to each teaching session without capturing each moment with a video recording (Powell, Francisco, & Maher, 2003). Video recordings have a great potential to provide rich audio and visual data about the participants' strategies and actions during teaching sessions (Bottorff, 1994). In addition, it helped the researcher to reexamine PTs' actions during teaching sessions and ensure the triangulation of the data. This reexamination also illuminated the possible discrepancies between what PTs performed in post-test and what they actually performed in the teaching sessions.

There were two video cameras in the classroom. One camera captured the classroom view and the other captured the board view. The microphones of the videos were on. As a teacher-researcher, I also utilized my phone to capture critical discussions within groups or the interesting work of an individual PT while circulating the classroom. In addition, due the resolution quality of the video, one PT took the photographs of the PTs' written work on the board when instructed.

3.4.3 PTs' Written Work and Field Notes

The PTs written works on the tasks utilized during the teaching sessions were gathered in each week. The written works were utilized as supportive evidence for the video analysis. These written works also helped teacher-researcher to examine each PT's mathematical thinking and mathematical strategies on each item and the tasks being engaged. Although teacher-researcher tried to give the chance to every PT to express their mathematical thoughts during the classroom discussion, sometimes this was not possible. In such instances, the PTs' written work was an effective tool to capture all the PTs' mathematical thinking and understanding.

After each teaching session, teacher-researcher wrote field notes. The main focus of these notes were:

1. Describing the critical events that took place within a particular group of PTs or each individual PT's work that the video recording might not have captured (Polman, 2006a). For instance, the instance of a PT tried to teach another PT a particular mathematical strategy was recorded in the notes.
2. Describing the particular action of a PT while working a specific task that the video recording might not have captured closely. For instance, paying close attention to how manipulative materials or different mathematical representations were used by PTs during the course of tasks (Polman, 2006b).
3. Describing the context and the PTs' actions in the instance of technical difficulties in which video recording was not functioning properly.

These field notes were utilized as a tool that enriched the content of the data analysis. Notes also provided the missing details from the video data.

3.5 Data Analysis

3.5.1 Analysis of Pre-Post Tests

In order to analyze data from pre- and post-tests, rubrics were created (See Appendix A). Each rubric aimed to categorized PTs' responses into performance levels. These levels were determined based on three criteria. First, the utilization of existing literature and Confrey's et al. (2008; 2009; 2012) and DELTA research team's studies on the related items. For instance, to create the rubric for the folding tasks, Empson and Turner's (2006) study was utilized. Second, the previously tested items' rubrics were adapted or retrieved from Wilson (2009) and Mojica (2010). They were adapted because scorers could not categorize some PTs' responses under any specified levels. Third, the rubrics were constructed by ongoing revisions and refinements on the newly created items through categorization of each PT's responses with the support of the PhD candidate student and the mathematics education researcher with doctoral degree.

Revisions on the rubrics were made based on the shared scoring of items by one PhD candidate in the field of educational measurement and statistics with a background in mathematics education. She and I scored a small sample of the data (we randomly selected four PTs' pre and post tests) and then we compared our scoring. When there was a disagreement in the scoring, the rubrics were revised and refined. In addition, in the instance of failure to categorize PTs' response categories the rubrics were revised and refined. This process of revisions and refinements on rubrics continued until each scorer reached a consensus of the PTs' scores on the selected items. Table 14 presents a finalized example rubric for item 17 (see Table 4 for the item).

Table 14

Rubric for Item 17: Area Congruence , Naming and Justification of Fair Shares

Correct responses	Complete Explanations	Incomplete Explanations
Parts a and b: no For part c: B<A=D<C or C>A=D>B	Utilize decomposition or composition of shapes or area congruence as a personal strategy to reach the declaration of equivalent fractions. Then order the fractions. & Name parts correctly and order the fractions correctly.	Verbally compare the size of each share (B is the smallest one because it is skinnier or C is the largest since it is wider and taller). & Name the parts erroneously.
Score	**Description**	
3	Correct response with a complete explanation	
2	Correct response for part a and b but failed to declare the relation between two parts. (ie A&C, A&B)	
1	Correct response with an incomplete explanation	
0	Incorrect response with incomplete, unreasonable, or no explanation Correct response with unreasonable explanation No response	

In Table 14, each score on the rubric indicates a performance level. The top score indicates the PTs provided a correct answer along with the required justifications and mathematical strategies. The lowest score indicates the PTs failed to provide the correct response along with the correct justifications. The score levels on the rubrics for each test are varied between 2-4 points. However, majority of the items scored on the levels between a maximum score of 3 and a minimum score of 0. The reason for that variation is the performances levels of a PT could show on different items varied.

In the scoring process, inter and intra rater reliability were found by employing Miles and Huberman's (1994) approach. In this approach, first the number of the agreements among the scorers was determined. Then, this number divided by the total number of agreements and disagreements. In this approach, 90% and higher results are perceived as high reliability. Then, the result is converted into the percent. When two separate researchers scored the pre-post tests, in both tests there were two disagreements in the scoring of PTs' answers. As a result the inter-rater reliability was calculated as 2/17=0.117, 0.12x100=12, and 100-12=88. Thus the inter-rater reliability was 88%. Also as a researcher, I scored each test twice two months apart. The intra-rater reliability was found as 94%.

Throughout the process, I presented two items, upon one of which the PhD candidate and I failed to agree on the scoring, to the experts in the field of mathematics education in a national research meeting. The scoring for these two items was determined after the experts and I reached a consensus.

The written responses of each PT on both tests were utilized to provide evidences for their mathematical content knowledge and student knowledge level and how it is progressed or restructured. Although each PT's performance was scored through utilization of rubric, the final score were not communicated with PTs. The reason for scoring was to determine individual PTs' performances on both tests. Since the rubrics were not utilized as a mean of determining a grade at the end of the teaching experiments, the data yielded from the rubrics were utilized only as descriptive and qualitative evidences of each PT's performance and progression after the teaching experiment.

3.5.2 Analysis of Teaching Sessions' Video Data

3.5.2.1 Model for Data Analysis

In this study, Powell et al.'s (2003) analytical model for analyzing video data was used. This model included seven steps as follows: "1) viewing attentively the video data, 2) describing the video data, 3) identifying critical events, 4) transcribing [necessary sections and constructing video clips], 5) coding, 6) constructing storyline, and 7) composing narrative" (p.413).

According to Powell et al. (2003) in the first step, the entire video should be watched carefully several times to understand the overall flow. In this phase, I kept in mind the purpose of the research and watched the videos accordingly. This phase helped me to identify the points for starting the analysis. In the second step, I tried to organize the rich information in the video data. Organizing this massive amount of information in a meaningful way was a considerable challenge. To overcome this challenge in this study. I "… noted [data] in an ethnographic-like fashion particular time-coded transitions of situations, activities, or meanings" (Powell et al. 2003, p. 416). Verbal and written descriptions of related activities were composed to map out the video data.

In the third step, significant moments and critical events related to the research questions were determined. I referred to Maher's (2002) critical events description. According to Maher, critical events would show evidence of significant change and contrasting act of participants from their initial understanding. In this study, I looked for a significant change in PTs' understanding of how students reasoned mathematically and PTs' understanding of equipartitioning. In addition, I detected the changes in PTs' mathematical strategies and their varieties.

In the fourth step, the time intervals for these critical events were determined. Then, video clips of these critical events were composed. After that, transcription of these sections was done. I had been trained and gained the experience on how to transcribe and accumulate transcription during my work in the DELTA research team at North Carolina State University. These experiences helped me in the transcribing the critical events. In each transcription, a verbal description of the context was provided. Context included gestures, silence, and the objects in the environment, time, and materials.

In the fifth step, each transcription was examined deeply and the video was re-examined if necessary. First, significant themes and statements were selected. Then, video-data transcription was examined iteratively to form categories that captured the commonality or pattern of significant themes, statements, and actions. This iterative examination continued until saturation of the coding schema was achieved. In the coding procedure, shared examination of the data between the mathematics education researcher, elementary school teacher and I ensured the reliability of the formed codes. The elementary school teacher had four years of teaching experience. She shared her opinions on the examination of the data as a consultant in the process of the data analysis. In addition, the mathematics education researcher with a doctoral degree and I used this coding schema to test whether it measured the intended outcome and gave consistent qualitative results. In the following sections the coding schema formulation will be explained in detail and the final codes will be provided.

In the last two steps, the determined coding schema was utilized to interpret the data. While constructing the story line, I tried to " come up with insightful and coherent organizations of the critical events, often involving complex flowcharting" (Powell et al., 2003, p.430). Repeated and shared viewing of the video data was utilized to refine and revise interpretation of the particular critical event coded in the data until a clear and coherent interpretation of the meaning was established. In composing the narrative stage, the evidences from the video data were utilized to report the findings of the study. Actual quotes from the participants and photos of PTs' works were used for reporting purposes as a form of empirical evidence for each finding. In addition, narrative communicated the overall meaning of the coded data.

3.5.2.2 Qualitative Analysis: Pilot Study

The findings of the pilot study revealed three main restructuring evidences of PTs: *Changes in content, misconceptions and learning difficulties,* and *understanding students' thinking models*. Below, how these categories appeared in the pilot study was given with evidences of classroom instances recorded in the pilot study.

The initial evidence was that PTs changed their prior knowledge related to fair sharing. For instance, PTs did not know the difference between fair sharing collections and a single whole. They assumed both cases of fair sharing demonstrated same mathematical ideas. After the experiment, PTs realized both cases of fair sharing actually related but had some differences. If a PT stated that "I did not know ...", these sorts of statements were gathered and classified under the first category: *Changes in content knowledge*. For instance, in the second week of the pilot, PTs were engaged with reallocation items and tasks. One PT stated that "I did not think marking the common amount in each friend's share and dealing the rest. I would combine all collections and share them all." This PT was engaging with reallocation uneven shares task and she exhibited an indication of learning a new strategy called reallocation.

The second evidence of PTs was categorized under *misconceptions and learning difficulties* code. Although PTs knew the word 'misconceptions', they could not come up with any student misconceptions prior to the experiment. Through examining students' work and working on the tasks, all PTs in the pilot study indicated some possible misconceptions and provided solution ways to remediate these misconceptions during the intervention. For instance, Figure 7 shows one PT's work that emphasized one of the coded misconceptions of students in ELT called additive misconception (Confrey et al., 2010).

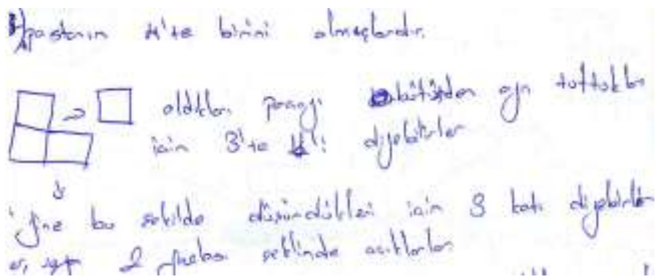


Figure 7. Additive misconception: Times as many

In Figure 7, PT stated "when sharing [the rectangle] for four, since students perceive the part [one person receive] separate from the whole, student might say: part is one third of the whole." Similarly, the PT indicated, students could state that "the whole is three times larger than a part or two more than the part." To over come this misconception, this PT suggested a solution strategy as:

> PT: As we learned in class, I would emphasize multiplicative relation between the whole and a person's share. And let the students iterate a person's share 4 times on the given whole. Students can understand that the whole is four times larger.

The rest of the misconceptions anticipated during intervention were; utilizing n+1 cuts to create n parts. PTs examined some students' works. In these works, one of the students exhibited this misconception and all PTs could detect n+1 misconceptions. None of the PTs indicated utilizing parallel cut strategy on circles could be a misconception prior to experiment. When PTs examined a student's work on this case, they suggested that in the prior task (fair sharing a rectangle), students could use parallel cuts, so they used the same strategy on circles. These evidences from PTs' works in the pilot study indicated that utilization LTs actually helped PTs to learn about these misconceptions.

In addition to detecting possible students' misconceptions, some PTs also presented the same misconceptions and learning difficulties as students did. For instance, one PT stated diagonal cuts did not create fair shares on rectangles in the third week of the pilot. This PT was having difficulty to understand the area congruence between two different shaped triangles formed as a result of diagonally equipartitioned a rectangular whole into four. Another example for this case was that two PTs did not know how to split a circular cake into six and asked for help from another PT. The main reason for this learning difficulty was that this PT did not know how to employ a radial cut. The interaction took place as follows:

> PT1: Can you show me how to create three splits on circles? (PTs having hard time to locate the radial cuts to create 3-splits on the given circle)
> PT2: Start here (pointed to the center of the circle). Then you share the circle into three, and then share each one-third into two.
> T: How do you determine this? (Pointing out the center of the circle)

PT2: Measure diameter and take the midpoint of it.
T: What about younger elementary school students who cannot measure?
PT2: I don't know.

This conversation indicated that PT2 had a hard time to think in terms of students' thinking level. Also, PT1 did not know how to create radial cut to split a circle into three. After this conversation, a folding task was utilized to show how to determine and utilize the center of a circle to create odd number of splits on a circle. Through this activity, both PTs learned how to find the center of a circle and then how to create radial cuts. This action also categorized under the changes in content knowledge. PT1 learned a new mathematical strategy called *radial cut utilization*. PT2 expanded her existing knowledge on fair sharing circles through learning how to determine center of a circle without measuring.

PTs generally thought that students would engage the tasks the way PTs did. Yet, the discussions and the mathematical strategies of students ordered in LT showed that students actually engaged in these tasks in a different way. As indicated above, PT2 had a hard time finding another way for showing the center of the circle that a younger student could understand or employ. Prior to pilot intervention, majority of PTs (n=8) were having hard time to distinguish students' mathematical thinking from their mathematical thinking. These sorts of instances coded under a third category called *understanding students' thinking model*. PTs usually utilized the statements such as "They used division, if they did not use division how can they solve this? I think they cannot." Yet, with the help of engagement with tasks and instructors' guidance, the PTs finally came up with a sense of how students thought mathematically. Such instances were coded as *understanding students' thinking models*.

Another clear example for this category was detected when PTs were asked "how could young students name each person's share, when a rectangle is split into four?" Eight out of 10 PTs thought students would say each [friend] got $\frac{1}{4}$ of the cake. Those PTs also predicted students' naming strategy. They stated, " students would tell, each got a quarter." Although this answer was not an incorrect answer, it was less likely to be an answer of young elementary school children. After the intervention, PTs showed evidence of an understanding variation in students' responses. For instance, one PT stated, “I think younger students cannot solve this problem directly with mathematical symbols. They can use a concrete object or drawing to show each friend’s share and name it as a part or piece.” Figure 8 represents this PT’s expectations of students’ drawing of fair sharing a rectangular cake for four friends.

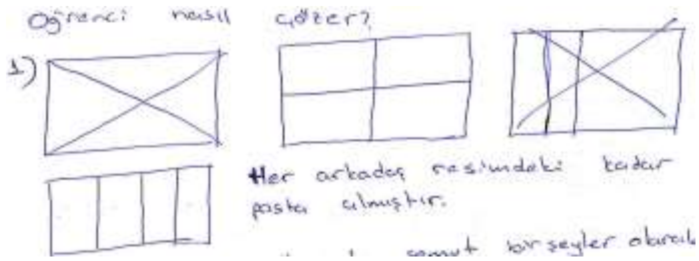


Figure 8. Fair sharing for four people and each person’s share

In this figure, the PT drew three different correct representations and also included an incorrect representation of fair sharing. And she wrote on her paper “each friend got the amount of cake as shown in the picture.” This example indicated that PTs became aware of the fact that younger students may not think in the same way and as complex as they solved the problems.

As a result, findings of the pilot study helped me to generate three initial codings as (i) changes in content knowledge, (ii) misconceptions and learning difficulties, and (iii) understanding students’ thinking.

These preliminary findings of the pilot study led to further examination of how PTs would become aware of students’ mathematical learning and thinking, and also how they restructure their mathematical content knowledge for what they were supposed to teach in a mathematics education course that is integrated with use of LT. Each PT showed evidences of utilizing LT to change their initial mathematical content knowledge. Data from PTs eventually led the codes. For instance, one set of PT’s behavior was coded as anticipating. This code was deduced since individual

PTs indicated that they were not aware of how students engaged with the fair sharing tasks before. Table 15 shows the initial action codes under three main categories.

Table 15

Initial Categories and Examples from Individual PTs

General Category	Sub-Category Name	Description	Example
Changes in Content Specific Knowledge	Revising	PTs are aware of their incomplete initial knowledge on equipartitioning and they revise them.	"I thought that children can do …but actually it is not like that."
	Expanding	PTs add newly presented information on equipartitioning into their existing knowledge.	"Wow I did not think before that children can do …"
Misconceptions and Learning Difficulties	Supporting	PTs seek for support while dealing with presented tasks.	"Can you show me how to …?"
	Identification and remediation	PTs use ELT to identify and remedy students' misconceptions.	"One out of four means you have one whole and four parts in it."
Understanding students' thinking	Ordering (Adapted from Smith & Stein, 2011)	PTs can order students' possible strategies from the less to the more complex ones using ELT.	"I think this is easier than the other…why students use this strategy." "I can teach first repeated halving, then teach odd-splits to him."
	Anticipating (Smith & Stein, 2011)	PTs anticipate students' possible mathematical strategies and misconceptions through usage of ELT.	"They can use parallel cuts on circles."

In this step of analysis, although an initial coding schema was formulated based on individual experiences of PTs, mainly PTs' verbal or written explanations were recorded individually. Then, the narrative for each PT's experiences of pilot teaching experiment was composed.

3.5.2.3 Qualitative Analysis: Main Study

During the time of the pilot study, although some research existed on how LTs might effect PTs' and teachers' knowledge, the theoretical framework for examining this effect had not been determined. Yet, upon the completion of the pilot, an emergent theory started to be constructed as LTBI. The findings of the pilot study indicated that the PTs had changed their content knowledge and their knowledge about how students learned mathematics. These findings were parallel with the approach of LTBI. As a result, the coding schema of the pilot study guided me to select LTBI theoretical framework for the main study. Then, through analytical examination of the main study data along with revisiting the existing literature on LTBI, the initial coding was refined and elaborated.

Examination of the data across participants delineated categorical data that were utilized to construct categorical combinations that helped me to develop initial themes: *Changes in mathematical content knowledge* and *changes in student knowledge*. Then, examination of the critical events of each participant provided action categories that represented the process. The final product for this research consisted of a cross-participant analysis constructed from the knowledge reconstruction practices of each participant when LTBI was used in the teaching sessions.

In actual experiment analysis, I also worked on generating "abstractions across participants" (p.195). Although some details might exhibit differences for each participant, I would seek for "a general explanation that fits each of the individual [participants]" (Yin, 1994, p.112). To achieve this abstraction and generality, the findings of each participant were examined holistically. Through deducing a pattern within data, a general coding schema was generated. This pattern helped me to see "processes and outcomes that occur across many cases, to understand how they are qualified by local conditions, and thus develop more sophisticated descriptions and more powerful explanations" (Miles & Huberman, 1994, p.172).

The changes in PTs' content specific knowledge were later coded as *Common Content Knowledge.* Misconceptions and learning difficulties were handled under Mathematical Content Knowledge and coded as *Specialized Content Knowledge*. When the PTs connected the equipartitioning related ideas with each other and with further mathematics, this was coded as *Horizon Content Knowledge.* Understanding students' thinking model was later coded as *Student Knowledge*. Yet, there were not specific categories that addressed each PT's reconstruction practices under each category. Thus, each PT's experiences in LTBI was documented through systematic interpretive data analysis (Schwandt, 1998). Their responses in the classroom and the ways they developed an understanding of fair sharing were recorded across points in time. These experiences helped me to create categories that conceptualized the utility of LTs. Table 16A and 16B showed final categories of knowledge restructuring practices of PTs

.

Table 16A

Final Categorization of PTs' Mathematical Content Knowledge Restructuring Practices during LTBI

General Theme	Main Category	Restructuring Practices	Description of PTs Knowledge Restructuring Actions
RESTRUCTURING MATHEMATICAL CONTENT KNOWLEDGE	Horizon Content Knowledge	*Connecting (Adapted from Wilson et al., 2013)*	PTs connect the embedded mathematical ideas within LT and with further mathematical ideas (including curricular knowledge).
		Generalizing	PTs realize equipartitioning relates with several subject areas that he or she was not aware of before. PTs extend mathematical concepts, ideas.
	Specialized Content Knowledge	*Internalizing*	PTs: "Make sense of multiple mathematical explanations, [strategies] and representations for ideas within the trajectory."(Wilson et al., 2013)
		Sizing Up	PTs explain possible reasons behind the mathematical errors and misconceptions.
	Common Content Knowledge	*Remediating Shifting*	PTs hold some mathematical misconceptions and learning difficulties. PTs utilize LT to remediate existing misconceptions and support learning process. PTs are aware of their incomplete or different initial knowledge on equipartitioning and change them.
		Expanding	PTs learn a new mathematical content (such as mathematical strategy, misconception, and concept). PTs add newly presented information on equipartitioning into their existing knowledge. PTs use mathematical concepts correctly.
		Challenging	PTs reject the presented information embedded in LT and develop a counter argument. PTs develop out of sequence strategy. PTs argued against their peer's mathematical claims.

Table 16B

Final Categorization of PTs' Student Knowledge Restructuring Practices during LTBI

General Theme	Main Category	Restructuring Practices	Description of PTs Knowledge Restructuring Actions
RESTRUCTURING STUDENT KNOWLEDGE	Student Knowledge	*Distinguishing and Recognizing (Adapted from Mojica, 2010)*	PTs separate their own mathematical thinking and strategies from elementary grades students' mathematical thinking and strategies by using ELT. PTs captured students' thinking in the analysis of the actual students' work.
		Ordering (Adapted from Smith & Stein, 2011)	PTs can order students' possible strategies from the least to the most complex ones using ELT.
		Anticipating (Adapted from Smith & Stein, 2011)	PTs anticipate students' possible mathematical strategies and misconceptions through usage of ELT.
		Emphatizing	PTs realize that they can also exhibit misconception, errors and difficulty related to equipartitioning ideas as students did. PTs realized the importance of the elementary school mathematics and acknowledge the complexity of that mathematics

3.6 Reliability and Validity Issues

According to Lincoln and Guba (1985) "…there can be no validity without reliability, a demonstration of the former [validity] is sufficient to establish the latter [reliability]" (p.316). In addition, in order to ensure the reliability of the data collected from the observation, intra-observer reliability and inter-observer reliability should be established. According to Creswell (2007), the intra-observer reliability referred to the extent the observer was consistent with his or her coding schema. In order to ensure this reliability, the video data were coded twice in this study by the researcher herself. Creswell (2007) stated that inter-observer reliability referred to the extent independent observers would come to an agreement about the coded data. As a result, one more mathematics educator with doctoral degree coded the video data in this study. Video excerpt of critical events were determined in the teaching sessions. Within each excerpt various restructuring practices of the PTs was captured and coded. The intra-rater reliability was found as 95%. The inter-rater reliability was 90%. The pre-post tests were scored also twice by the teacher-researcher. The intra-rater reliability was found as 94%. Also, randomly selected four PTs' pre and post-tests were scored separately by a PhD candidate in educational measurement and I. The inter-reliability was found as 88%. All reliabilities were calculated by Miles and Huberman's (1994) approach.

In this study, in order to ensure the validity, triangulation was used. According to Mathison (1988) triangulation was a typical method to improve validity and reliability of the study. Patton (1990) stated that triangulation could be achieved in a study through combining multiple data collection methods. In this study, video data of teaching sessions, classroom observations and field notes, and PTs' written works were collected to answer the research questions. In addition, according to Creswell and Miller (2000), triangulation also involved "a validity procedure where researchers search for convergence among multiple and different sources of information to form themes or categories in a study" (p.126). In this study the qualitative data gathered from different data sources (PTs written works, pre-post test data, observations, field notes etc.) were examined two times and coding schema of the study was composed. Codes' descriptions were provided in detail and empirical evidences from the gathered data were provided for each code. This detailed description of codes also ensured that other researchers could take these codes and test whether they could be used as a framework for other studies.

Mathison (1988) stated that triangulation could be achieved through multiple viewing of the data. In this study, each video data for each teaching session were examined through shared multiple viewing with other colleagues including, the doctoral candidate in educational measurement field with a background in mathematics education, one elementary teacher, one researcher with a doctoral degree in mathematics education, and several mathematics education researchers. In this shared discussion, the coding schema of the study was discussed by mathematics education researchers and saturated. These gatherings contributed to the saturation of the codes and creating a common meaning for the observed events or behaviors in the video data (Creswell, 2007).

At last, Merriam (2002) indicated that one could increase the validity of the findings through long-term observations in a qualitative study. Because one could verify or refute the findings of the study via cross checking findings against data. In this study, an emergent pattern related to PTs' knowledge restructuring process was deduced within 3 weeks pilot study. Then, in the main study, 6 weeks of observation took place. Within this time frame, the mathematics educator and I continuously discussed the data and picked the negative cases to test the findings of the study. This time frame (9 weeks total) helped me to collect data that yielded a more accurate and detailed picture of the phenomenon under examination.

3.7 Researcher's Role

The aim of the teaching experiments is to observe the learners in their settings and report this to the audience. The challenge is in such kind of qualitative study "is to combine participation and observation so as to become capable of understanding the program as an insider while describing the program for outsiders" (Patton, 1990, p.207). Thus, creating a balanced relation between the participant and the teacher-researcher is a key for accurate and unbiased reporting.

In this teaching experiment, I was the teacher-researcher. The PTs knew the researcher-teacher from their previous university course. This acquaintance between the PTs and I helped the PTs to share their ideas freely without having a fear of being judged. This also saved time for me from the preparation time in which the PTs got used to the atmosphere of the teaching experiment. As a result, this time was used for instructional purposes.

During the implementation, since I knew each PT personally, this helped me to select which PT to present his or her work first in the class. In the instance of a PT remaining silent, I could ask this PT to share her thoughts, solution ways freely and the PT answered positively to this request. This ensured an ongoing interaction even in the silent moments during the experiment. In addition, this helped me to stabilize the amount of my inputs during the teaching. Thus, I could set myself aside, let PTs to discuss and then I observe the PTs.

Being a teacher-researcher of the course, I could use my knowledge of the learning trajectory to plan the tasks, orchestrating the discussion in the implementation in LTBI. With this knowledge, I assisted the PTs to think on the tasks in different ways, to reflect on the more complex mathematical ideas.

In the data analysis part, knowing the academic background of the PTs and their mathematical success level helped me to interpret the significance of the improvement in each PT’s mathematical content knowledge and student knowledge.

One of the disadvantages of knowing each PT was that the PTs were very comfortable in the class and this rarely caused some distractions in the classroom. In such instances, I addressed the situation through warning the PTs with a warm tone to focus on the presented tasks and reminding them there would be whole class discussion after they completed individual works. These moments lasted for very short times, a maximum of one or two minutes.

Second disadvantage of being the teacher-researcher was that the PTs might perceive me as an authority in the classroom in terms of mathematical knowledge. However, in the beginning of the experiment, I made it clear that the PTs’ contribution regardless of their correctness or incorrectness was valuable for this course and through active engagement in the course we should all progress together.

I also informed the PTs that I would also learn from their experiences and mathematical thinking in the experiment.

3.8 Assumptions and Limitations

In the study, the PTs participated to the study outside of the their course time at the university. The course content was not graded and did not affect their GPAs. Each PT willingly allocated their personal time to enhance their mathematical understanding. Also, at the beginning of the experiment, all the PTs were informed that speaking their thoughts aloud was very critical. Then, as a researcher-teacher, I emphasized that I was interested in their mathematical thinking and this interest was independent of evaluation of their answers as correct or incorrect. As a result, I assumed that the PTs participated the study without having an intention to impress the researcher-teacher. They all gave their sincere responses in the teaching sessions.

The first limitation of the study was time. Since the time was limited, only three weeks of pilot study were conducted. As a result, only newly developed items could be piloted. Although the adapted and retrieved items were utilized in prior research studies, these items could not be piloted with the PTs in Turkey for this study. The second limitation was, since the course was based on voluntary participation and the PTs allocated a limited time (maximum of 3 hours per week) for the study, each week's content was very intense. In addition, the formative assessment of the PTs' mathematical understanding was realized through analysis of the students' works and videos not through real-world in class application in an elementary school. The familiarity with the researcher-teacher did not seem to be a limitation from the PTs' point of view. Because, the PTs felt free to express their thoughts in the teaching sessions and were easily adapted to the course flow.

Another limitation related to time was that the pre-post tests were conducted within a long duration. This long time duration might have led changes in PTs' performances in the items when they became exhausted. However, the examination of the PTs reactions during the tests did not convey a message of exhaustion. Also, voluntary participation was another motivation factor that kept PTs actively working on the presented items.

Another limitation of the study was that I was the researcher-teacher of the course. My theoretical orientation might have the potential to affect the results of the study. To minimize this effect triangulation was employed. The data from multiple sources (i.e., pre-post tests, written works, observational and field notes, video data) were compared and contrasted to test the validity of the results. In addition, a mathematics educator, one doctoral student in education measurement field and an elementary school teacher analyzed the data and shared their own interpretations of the data.

CHAPTER IV

PRE/POST-TEST FINDINGS

This section will compare the results of pre- and post-tests through providing descriptive analysis and qualitative evidences from PTs' written works. Then, each PT's progress will be examined on each item. In order to document the progress of each PT, both descriptive and qualitative analysis results will be disseminated. The PT's scores on each item denote a certain performance level represented in rubrics (See Appendix A). All these findings will be reported to provide answers for the first research question and its related sub-questions as follows:

1) What are differences between pre-service elementary teachers' (PTs) knowledge level before and after the LTBI teaching experiment?
    - What is PTs' initial knowledge about the equipartitioning/fair sharing concepts, which they are supposed to teach?
    - Do PTs hold any misconceptions, difficulties, errors and knowledge gaps related to concept of fair sharing? If yes, what are those?
    - What is PTs' knowledge about the equipartitioning/fair sharing concept, which they are supposed to teach, after the LTBI teaching experiment?

4.1 Individual Analysis of Each Item and the PT

Item 1

The first item was scored on the base of 4 points and asked to fairly share discrete collections and then provide different mathematical names for each share. In the pre-test the question asked to *fairly share 18 candies among 3 friends through utilizing the line on the given picture of three friends and 18 candies* (Adapted from

Mojica, 2010). A parallel item was asked in the post-test that included 24 candies among 3 friends case.

In both tests, all PTs fairly shared the collection. Yet, PTs performed poorly on naming the fair shares. Figure 9 shows the comparisons of each PT's score on this item in both tests.

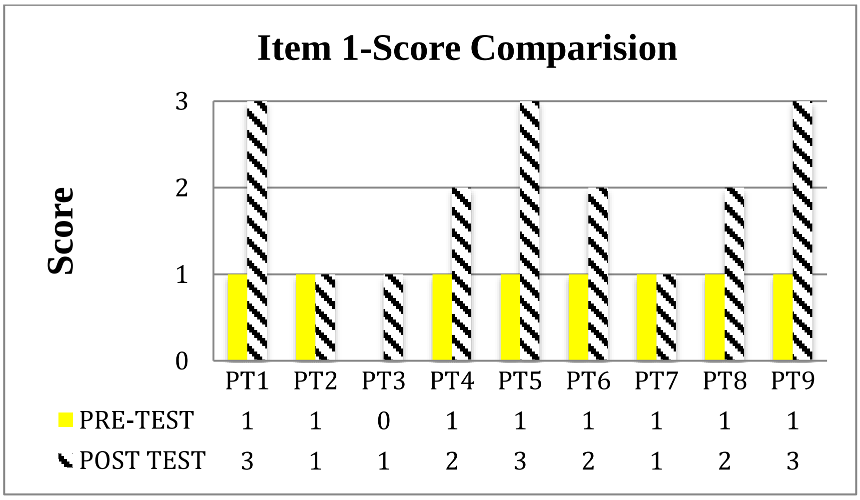


Figure 9. Each PT’s score on pre-post tests: Item 1

Figure 9 shows that the overall performance of PTs increased in the item 1 in the post-test. The mean score of item 1 in pre-test was found as 0.88 and as 2.00 in the post-test. Figure 9 also shows that all PTs could fairly share the given collection, yet they failed to provide different mathematical names for each person share in the pre-test. All the PTs, except PT3, named each share only as a count, six candies, in the pre-test and did not provide any other name. PT3 could not name each person’s share even if she fairly distributed candies among three friends. This finding indicated that many PTs were not knowledgeable about the various mathematical naming for fair shares such as ratio (6 candies per person), fractions ($\frac{1}{3}$ of whole candies, $\frac{6}{18}$) and operators (Confrey et al., 2010) before the experiment. Yet, in the post-test, all PTs except PT2 and PT7 could utilize at least two different mathematical naming for fair shares.

In the pre-test, PT1, PT3, PT5, PT7, and PT9 shared the collection by ones to each friend. This strategy is called *1-1 correspondence*. The rest of the PTs utilized *composite unit strategy* and they gave three candies at once to each friend.

In the post-test, PT1, PT5 and PT9 showed a considerable progress. These PTs provided three different mathematical names for each fair share including ratio, fractions, and count. PT4, PT6 and PT8 provided two different mathematical names of fair shares including either ratio or fractions and count. PT3 could name each person's share as a count in the post-test. These findings indicated that majority of PTs (n = 7) seemed to expanded their mathematical content knowledge related to mathematical naming of a fair share. They internalized the embedded knowledge within ELT.

No change was recorded in PT2 and PT7 in terms of knowledge about naming a fair share. Contrary to their performances in the post-test, these PTs actively participated the discussions and their contributions to the discussion provided evidences related to naming a fair share through using ratio or fraction in the teaching sessions. This contradiction might show that PT2 and PT7 did not completely retain the knowledge upon teaching sessions. PT4, PT6 and PT8 failed to receive full credits in the post-test since full credits required providing at least three mathematical names. My personal communication with the PTs showed that these PTs did not think it was necessary to use general mathematical name for each person's share such as a part, some portion of the whole collection. Yet, younger students used these names frequently (Confrey et al., 2008, 2010). PTs thought that students would not use these general naming. This showed that PTs could not completely anticipate students' mathematical thinking.

Item 2

The second item was an anchor item. This item assessed whether PTs could find a general name for a share resulting from equipartitioning. The question was "*For given any amount of objects (n) and any amount of people (p). How would you name each person share and why*?"

This item was scored on the base of 3. The overall mean for the item in the pre-test was 2.77 and in the post-test was 2.55. There was a slight decline in the overall mean score in the post-test that I did not anticipate. Seven PTs could provide a correct answer and complete justifications for the generalization in both tests. Although PT3 and PT7 mentioned a specific case and some sorts of generality in the pre-test, both PTs only utilized a specific case in the post-test without mentioning a generality. For instance, in the pre-test, PT3 utilized an example, 6 candies and 6:3=2, and said each individual would get n/p candies. On the other hand, in the post-test, she only gave the example of 10 candies among 5 people. Then she did not relate this example with n (number of objects) and p (number of people). One possible reason for the thought shift in PT3 and PT7 might be failed to connect concrete examples with the abstract generalization. These PTs might have thought that young children could not achieve this kind of generalization. Thus, they perceived their answers as sufficient. These findings indicated that many PTs (n = 7) produced a generalizable mathematical idea as a result of the LTBI teaching experiment.

Item 3

The third item was identified as one of the easiest items on the tests. The item required finding the total number of pencils through utilizing different strategies. The question on the pre-test was asked to *find the total number of pencils if a box of pencils were shared among three friends and each got 13 pencils*. A parallel item was asked in the post-test in which each of *seven friends got six pencils*.

This item focused on the reversibility of discrete collections. PTs were expected to employ additive and multiplicative strategies. This item was scored on the base of 4. The overall mean for the pre-test was 3.22 and for the post-test was 3.88. In the pre-test, PT1, PT2 and PT7 utilized additive strategy such as, 13+13+13=39. Rest of the PTs utilized both additive and multiplicative strategies. In the post test, only PT7 did not receive full credits, since she only showed multiplicative strategy to find total number of crayons as 6 (pencils) x 7 (students)= 42 pencils. The rest of the PTs used both multiplicative and additive strategy to find the number of pencils in the original collection. These findings indicated that although PTs knew multiplication and that it could be used for finding the total number of pencils in the collection, they did not exhibit the ability to reverse the operation of equipartitioning in the pre-test. This ability indicated the recognition of combining equal groups to re-create the entire collection. This recognition later entails "n times as many relation" between the size of whole collection and equal groups. These findings indicated PTs started to make sense of multiple mathematical strategies and used them upon intervention.

Item 4

Item 4 assessed the PTs' ability to anticipate and explain students' mathematical strategies and misconceptions related to reassembly. Reassembly underlies the part-whole relation (n times as many or n times as much), one of the hardest concepts for the young children to understand (Confrey et al., 2009; Thompson & Saldanha, 2003). Pre-test item was as follows:

> *Three friends had total 13 same size Legos. They combined all the Legos to build a Lego tower. Then, they compared the size of one Lego piece with the size of the Lego tower. Three friends provided different answers:*

a) Fatma suggested the height of Lego tower was 12 times as long as one Lego piece.
b) Ayse suggested the height of Lego tower 2 times as long as one Lego Piece.
c) Berrin suggested the height of Lego tower 13 times as long as one Lego Piece.

Then, PTs were asked first to determine which statement (s) was correct or incorrect. They were asked to explain why the statements were correct or incorrect. At the end, they were also required to explain the each friend's understanding of *reassembly*. In the post-test a parallel item that included three friends and each has 7 seven Legos. Each friend also suggested parallel mathematical statements.

This item was scored on the base of 4. Figure 10 compared PTs' scores on pre-and post-tests.

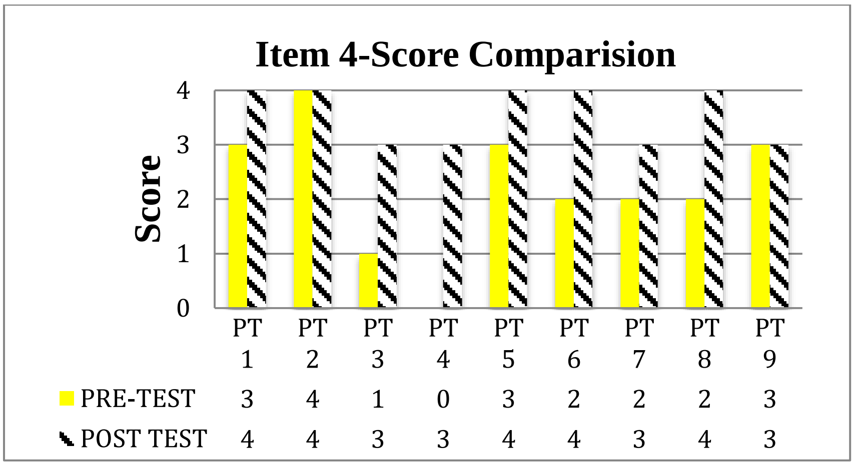


Figure10. Each PT's score on pre-post tests: Item 4

The overall mean for the pre-test was 2.22 and post-test was 3.55. Figure 10 shows that except PT4, all of the PTs could select which of the students' response descriptions was correct in the pre-test. However, PT3 received only 1 point since she did not provide a complete sensible mathematical explanation for the correct response. As Wilson (2009) stated, even some teachers may hold the same mathematical misconceptions as students do. This was demonstrated in the pre-test by PT4's additive misconception. Figure 11 shows PT4's additive misconception.

Figure 11. PT4's additive misconception: Part-whole relation

In figure 11, PT4 described the relation between the size of the Lego tower and one Lego piece as 12 times more and decided that Fatma's answer was correct.

In the pre-test, only PT2 explained the correct strategy, detected both students' additive misconception and described why the students perceived the relation between size of the tower and a single Lego piece as two times as long as. Researchers (Confrey et al., 2009; Pothier & Sawada, 1983) suggested that young students might perceive each share as half regardless of their size. For some children breaking into half could create any number of parts such as half in five pieces.

In the post-test, the PTs were asked to compare the size of a Lego tower composed of 7 Legos to 1 Lego piece. In the post-test, PT1, PT2, PT5, PT6, PT7 and PT8 provided correct answers and explained the underlying mathematical reasons behind each friend's answers completely. PT4 corrected her misconception; she also reached a level that she could identify students' misconceptions in their responses. PT4 stated that Fatma exhibited a misconception since she put each Lego on top of each other and did not count the Lego in the bottom.

In the pre-test, although only PT1 and PT5 identified additive misconception in Fatma's response, in the post-test, 5 of the PTs could completely articulate students' mathematical thinking and misconceptions. Four PTs (PT3, PT4, PT7 and PT9) still could not provide an explanation for Ayse's mathematical thinking. However, when these 4 PTs' work were deeply examined and their progress were documented through comparing their performances on pre- and post-tests, it appeared that PT3 and PT4 made considerable progress. Both PT3 and PT4, unlike their performance in the pre-test, could explained both Fatma's and Berrin's mathematical reasoning behind their answers in the post-test. Only PT9 did not show any progress, yet she could still provide a complete explanation for Fatma's and Berrin's responses. PT9 still could not explain how Ayse stated the relation as 2 times as long. These findings indicated that PTs could anticipate the reasons behind both students' misconceptions and correct mathematical strategies. Also, they could explain the reasons behind these misconceptions after teaching sessions. This indicated that the PTs started to enhance their knowledge about students' mathematics. Correcting their own mathematical misconception that an elementary student could also possess seemed to be an indicator of restructuring their CCK.

Items 5 and 7

Both item 5 and item 7 assessed the reallocation of uneven shares (Yilmaz, 2011). Yilmaz (2011) defined this type of reallocation tasks as "Given a set of objects *unfairly* shared among a number of people but that could have been fairly shared, participants adjust the shares to obtain fair shares" (p. 6). The difference between item 5 and item 7 was that item 5 included a picture of the initial shares while item 7 did not. Yilmaz (2011) found that when younger students were presented with the items with pictures, they utilized reallocation strategy. But when the item did not include a pictorial representation, they added up all the collection and fairly shared among the number of the existing people. This feature of the item assessed whether PTs who took the test could make sense of the role of different representations.

The pre-test item 5 included the picture of cookies unevenly shared among four people and the PTs were asked to fairly share it and justify their fair shares. The picture in the item 5 stem is shown in the Figure 12.

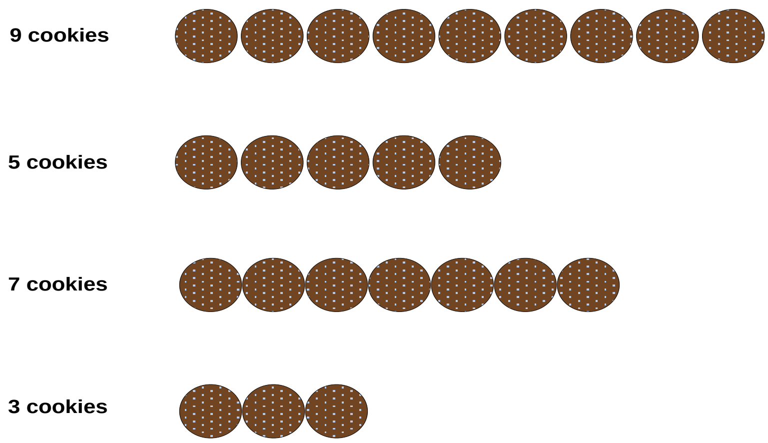


Figure 12. The picture of uneven shares in the pre-test item 5's stem

The post-test included a parallel item with a picture of the uneven shares as shown in the Figure 13 below. Item 7 included a verbal description of the each friend's uneven shares in the pre-test as *Pelin had three chips, Meryem had 6 chips, Tamer had 4 chips and Sinan had 7 chips*. In the post-test the verbal description was; *Pelin had 4 marbles, Meryem had 7 marbles, Tamer had 5 marbles and Sinan had 8 marbles*. For both items, the PTs were asked to fairly share the collection and justify fair shares.

In the pre-test, the overall mean for these two items did not differ greatly. The mean for item 5 was 1.88 out of 3 and the mean for item 7 was 1.55 out of 3 in the pre-test. In the post-test, the mean for item 5 increased up to 2.77 and item 7 increased up to 2.55. When each PT's responses to both items were examined, the same trend as Yilmaz (2011) observed with young children could be observed. PTs also employed reallocation strategy more frequently (n = 7) when the item stem included the picture of original share, similar to the young children. Figure 13 shows PT4's reallocation strategy on item 5.

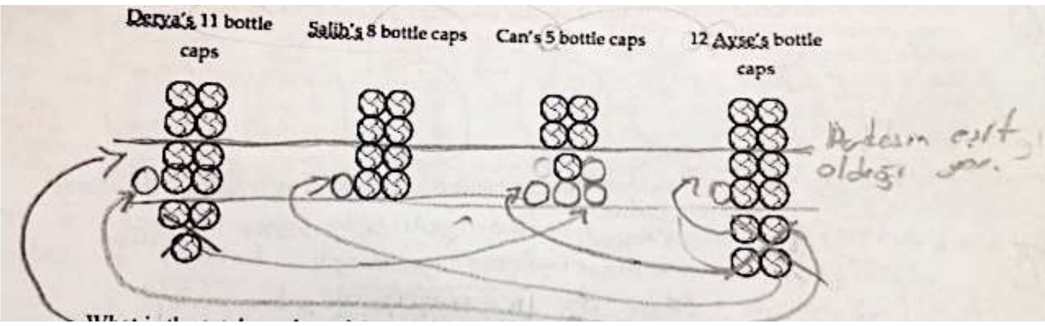


Figure 13. Reallocation of uneven shares: Detecting common shares in each share strategy

In Figure 13, PT4 drew a line that showed each friend's had equal common amount of bottle caps (n = 4). Then, she drew a second line in which she completed the missing bottle caps (in Can's and Salih's shares) by taking one bottle cap that exceeded this line (Derya's and Ayse's shares). Then, she represented redistribution of remaining parts by utilizing arrows.

In the post-test, except PT2 and PT7, all seven PTs could utilize reallocation strategy with correct justifications. For item 7, five PTs utilized reallocation strategy with correct justifications. In addition to these findings, PT3 exhibited great progress in both items upon intervention. Although she received 0 credit for item 5 in the pre-test, she received full credit (3) in the post-test. Similarly she increased her score from 1 to 3 for item 7. None of the PTs exhibited a decline after the intervention for both items. All these findings and the increase in the mean indicated that PTs expanded their mathematical knowledge and learn a new strategy called reallocation. Utilization of a different strategy also seemed to be an indication of expansion in their CCK.

Item 6

Item 6 assessed the reallocation departure case (Yilmaz, 2011). Yilmaz defined this type of reallocation tasks as "Given a set of objects already fairly shared among a number of people, participants adjust the shares based on the *departure* of one or more people" (p.6). Yilmaz (2011) stated that younger children redistribute the share of the person(s) who left the group in this type of tasks. The items on the pre-test and post-test are briefly described below:

> ***Pre-test:*** *Ali has a sleepover party with his five friends. His mother gave him six bags of strawberry candies. Each bag contained 5 candies. Mustafa is allergic to strawberry and could not eat his candies.*
> ***Post-test:*** *Ali has a sleepover party with his six friends and Mustafa was allergic to the candy. They had seven bags of candies each contained six candies.*
> . ***Common item stem for each item:*** *Show how the boys can share the candies and describe each boy's share after Mustafa leave his candies. Explain your answer.*

This item was scored on the base of 3. Figure 14 shows each PT's performance on item 6 in both tests.

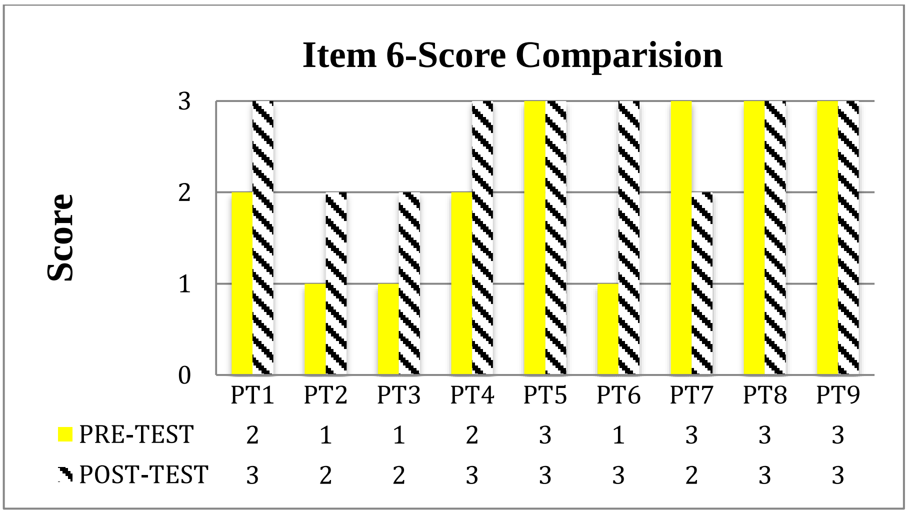


Figure 14. Each PT's score on pre-post tests: Item 6

In the pre-test, only PT5, PT7, PT8 and PT9 utilized both reallocation and collection strategy to solve the task. The other five PTs utilized only collection strategy. Yet, only two out of five PTs (PT1 and PT4) could completely justify their collection strategy. For instance, PT1 drew a picture of utilization of 1-1 correspondence strategy to show how she distributes the whole collection. On the other hand, PT2 and PT3 only wrote 30÷5=6 without explaining the relationship between this division operation and the action of equipartitioning collections.

After the intervention, six PTs utilized both collection and reallocation strategies and the rest utilized collection strategy with complete justifications. One of the justifications was commutative property of multiplication. In her study, Yilmaz (2011) stated that reallocation arrival and departure tasks serve as an important base for understanding commutative property of multiplication. Similarly, in the post-test, three PTs indicated that 6x7 (first number indicates number of objects in each person share and second number indicates number person) and 7x6 should be equal. PT3 utilized both commutative strategy and array representation to justify the answer. The PTs drew two arrays with the dimensions 6x7 and 7x6 and wrote area on top two arrays. These finding indicated that PTs actually started to utilize variety of mathematical strategies embedded in LT. Also, they enhanced their ability to justify their responses through utilizing multiple mathematical explanations and representations.

Item 8

Item 8 (Adapted from Lamon, 1996 retrieved from Mojica, 2010) required fair sharing multiple wholes among multiple people in two different contexts with the same number of wholes and people. PTs were asked whether these two tasks were mathematically equivalent or not (see Figure 16 below for the tasks). PTs could justify the mathematical equivalence of two tasks by utilizing a mathematical model, an area model, by indicating each friend's share in relation to a whole, and by stating context was an extraneous variable. This item was scored on the base of 3. The overall mean of this item in the pre-test was 1.66 and overall mean in the post-test 2.55. Figure 15 shows each PT's comparative scores on this item in both tests.

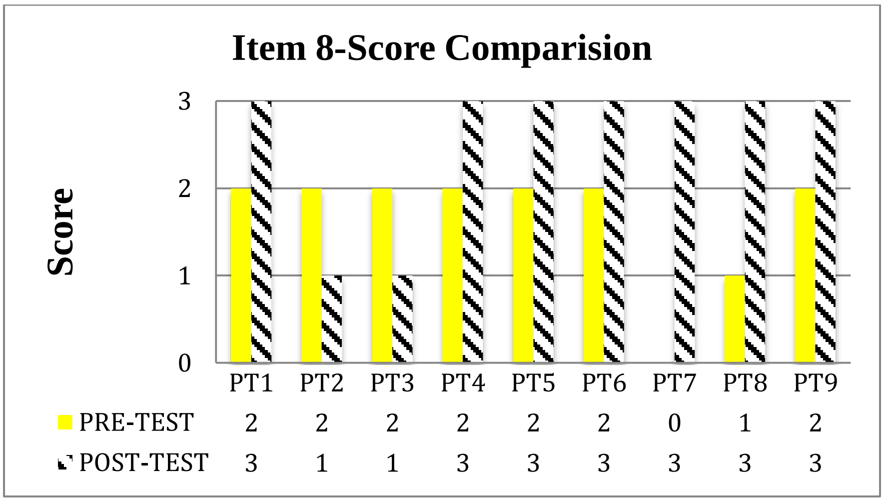


Figure 15. Each PT's scores on Pre-Post tests: Item 8

Although there was an increase in the mean of this item in the post-test, PT2 and PT3's scores declined as shown in the Figure 15. On the other hand, PT7 and PT8's scores increased dramatically. Although none of the PTs provided a complete explanation for why two tasks were mathematically equivalent in the pre-test, after the LTBI, six PTs could provide a complete explanation in the post-test.

In the pre-test, except PT7 and PT8, the rest of the PTs received 2 points since they all indicated the tasks were mathematically equivalent. Yet they failed to provide a complete justification for that claim. Their justifications lacked the mathematical name of each friend's share and lacked the mathematical model or area model for that share. For instance, PT3 only wrote a 3÷5 operation under each task as a justification without explaining the meaning of the division operation.

PT7 did not received any credit for this item since she made a mathematical error while determining the referent whole. Figure 16 represents PT7's written work on the pre-test.

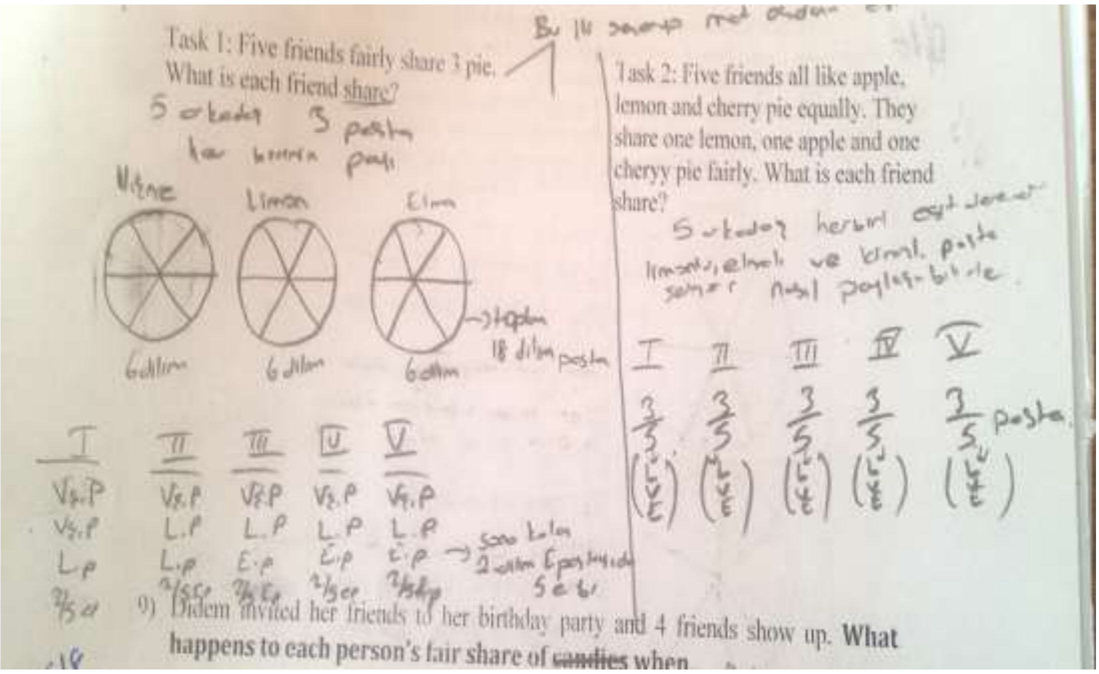


Figure 16. Mathematical error: Determine the referent whole

Figure 16 shows that PT7 fairly shared three pies among six people instead of five. This action of the PT resulted in 18 equal pieces. Then, she distributed each $\frac{1}{6}$ piece to each friend. At the end, she stated, "She is left with two pieces." Actually, she should have 3 pieces left. Then, she tried to fairly share the two pieces and gave each person $\frac{2}{5}$ of an apple pie. This action of the PT was not mathematically correct since she was not fairly sharing the original whole pie; instead she was sharing $\frac{1}{6}$ of an apple pie among 5 friends in each distribution cycle. The resulting share should be $\frac{3}{6}$÷5=$\frac{1}{10}$. As a result, she should have concluded each friend got: $\frac{1}{6}+\frac{1}{6}+\frac{1}{6}+\frac{1}{10}=\frac{3}{5}$. Yet, she concluded each person got $\frac{1}{6}+\frac{1}{6}+\frac{1}{6}+\frac{2}{5}$. In the second case, she gave each person directly $\frac{3}{5}$ of a pie. Based on her mathematical errors, she concluded these two situations were not mathematically equivalent.

According to Confrey et al. (2009; 2010), students could utilize two strategies while fair sharing multiple wholes. The first strategy is called *benchmarking* and the second strategy is called *split all*. PT7 tried to utilize split all strategy in both parts. In the first part, she distributed pieces ($\frac{1}{6}$ of a pie) to each person, and then she tried to fairly share the remaining pieces. Unfortunately, she could not determine her referent whole correctly. She made mathematical error while redistributing the remaining pieces. Her solution to the second task was to utilize the split all strategy. She split each pie into five and then deal each $\frac{1}{5}$ th among five people. In the post-test, PT7 used both strategies: benchmarking for the first task and split all for the second one. Then she concluded that both tasks were mathematically equivalent but their problem context was different. This showed that PT7 corrected her mathematical misunderstanding about referent whole and utilized multiple mathematical strategies as students did.

In the post-test, PT1, and PT4, PT5, PT6, PT7, PT8 and PT9 stated that both tasks were mathematically equivalent and justified that claim. All the mentioned PTs except PT9 modeled both tasks mathematically as either 5 ÷ 4 or $\frac{1}{4}+\ldots+\frac{1}{4}$, or 5x($\frac{1}{4}$). For instance, PT5 utilized this justification. Figure 17 showed PT5's work on the item.

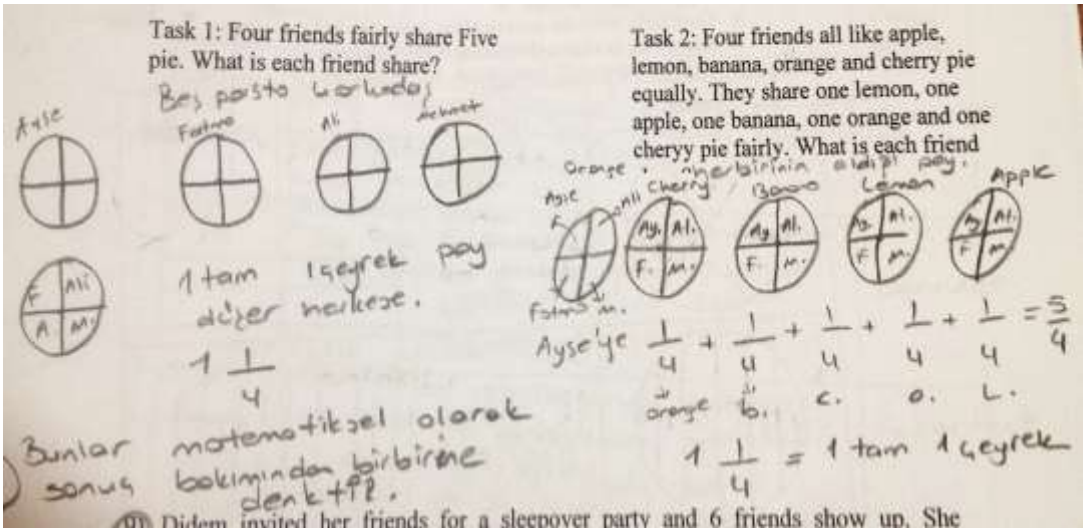


Figure 17. Mathematical modeling to show mathematical equivalence of tasks

As seen in Figure 17, PT5 modeled each person's share and wrote each person "*received one whole and a quarter, 1* $\frac{1}{4}$" under the first task. Then she modeled each person's share for the second task, mathematically explaining the model as $\frac{1}{4}+\ldots+\frac{1}{4}=\frac{5}{4}$ and stating "*1* $\frac{1}{4}$ *= 1 whole and a quarter*". Then she concluded, "*these are mathematically equivalent yet the context is not equivalent.*" Only PT9 explained that pie type was an extraneous variable. She stated "*Task 1 & 2 are mathematically equivalent, because type of pie doesn't matter*."

Figure 15 shows a performance decline in PT2 and PT3's efforts in post-test after LTBI. PT2 modeled each friend's share for each task correctly, yet she did not state whether the two tasks were mathematically equivalent or not. In the pre-test, PT2 used a similar strategy with a conclusion sentence. PT3 exhibited an interesting mathematical error while fair sharing multiple wholes among multiple people. She divided the number of people with the number of the whole. Thus, she concluded that the tasks were not mathematically equivalent.

All the findings of item 8 indicated that in the pre-test, six PTs found the amount of pie each friend got for each task, yet they failed to provide a sensible explanation for why these two tasks were mathematically equivalent. This showed that simply carrying out the calculations to find a friend's share did not address a coherent understanding of the mathematical explanations behind the calculations. Most of the PTs did not have this understanding prior to the LTBI experiment. After experiment, in the post-test, the evidence indicated that seven PTs have acquired this understanding.

Item 9

Item 9 assessed the effects of factor-based change that happened in the number of the people who shares the collection or the whole, on the size of the new shares of each person. The item in both tests was briefly described as:

> ***Pre-test:*** *Didem invited her friends to her birthday party and 4 friends showed up. What happens to each person's fair share of a cake when… (a) More friends shows up for the party (b) Fewer friends shows up for the party. (c) Half of the friends show up for the party. (d)Number of the friends doubles show up for the party. Please justify you answers.*
>
> ***Post-test:*** *Didem invited her friends to a sleepover party and 6 friends showed up. She prepared a bag of chips for each friend and each contained same amount of chips. What happens to each person's fair share when…… (a) More friends shows up for the party (b) Fewer friends shows up for the party. (c) One thirds of the friends show up for the party (d)Number of the friends triples show up for the party. Please justify you answers.*

The first two parts of the item assessed *qualitative compensation*, in which the PTs could verbally describe the changes in the size of the share as a result of factor-based change. The last two parts of the item assessed quantitative compensation in which the PTs were required to show the result of factor base change mathematically.

This item was scored on the base of 4. The overall mean of this item the in pre-test was 1.88, and the overall mean in the post-test was 3.11. Figure 18 showed each PT's score on both tests.

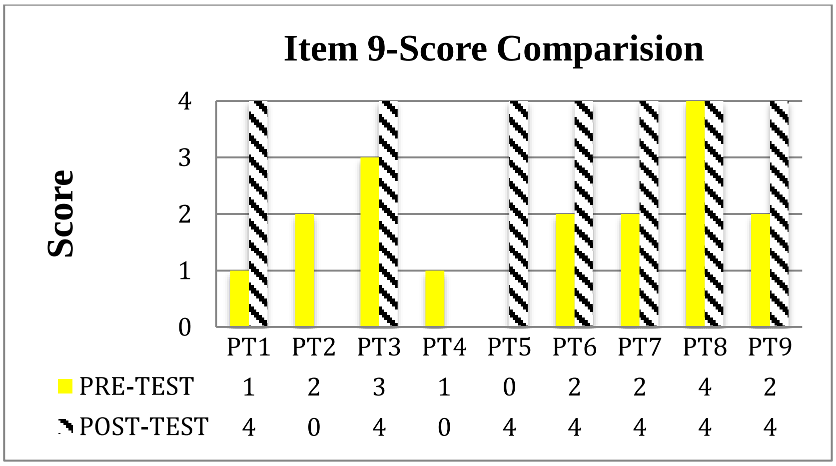


Figure 18. Each PT's score on pre-post tests: Item 9

Figure 18 shows that in the pre-test, eight PTs could not provide a complete explanation for why each person’s share size had changed based on a factor-based change in the number of sharers. Four PTs received 2 points since they produced correct answers along with limited mathematical explanations for their answers or they could provide complete explanation for some cases. For instance, PT2 utilized verbal and pictorial representations to show the factor-based change on each friends share when the number of friends halved in the pre-test. Figure 19 shows her work.

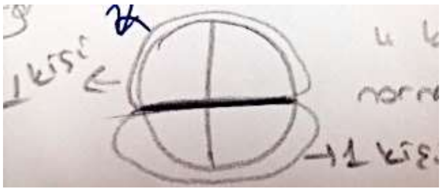


Figure 19. Pictorial representation to show the effects of factor-based change

In Figure 19, PT2 initially fairly shared a cake for 4 people (the original number of people) then she marked one person’s share as half of the circle, drew an arrow, and wrote "one person." She explained, "*Each person gets two times larger than the size of each person's share normally*."

PT5 received 0 point in the pre-test since she did not even conclude whether a person share size increased or decreased correctly when the number of people increased or decreased. Only PT8 provided a complete explanation. PT8 wrote, "*There is an inverse relation between the amount of cake in [each friend's share] and the number of the friends*." This showed that none of the PTs (expect PT8) had a robust mathematical understanding that led to a generalizable idea such as inverse relation.

In the post-test, seven PTs (PT1, PT3, PT5, PT6, PT7, PT8 and PT9) provided a correct response along with complete explanations. Table 17 shows the distribution of the mathematical explanations of each PT.

Table 17

Distribution of PT's Sensible Explanation Type(s) in the Post-test

Explanation	PT1	PT2	PT3	PT4	PT5	PT6	PT7	PT8	PT9
Inverse Relation with Mathematical Explanation(s)	x		x			x		x	x
Area Model with Mathematical Explanation(s)					x		x	x	

Table 17 shows that five PTs perceived the inverse relation between the number of the sharers and the share size. PT9 showed this relation through both verbal and mathematical representations. She wrote "*n/p*" on the top of item 9 and said, "*When the denominator (p) increases, the amount of each person's share decreases; when p decreases, the amount of each person's share increases.*" Also, PT9 represented this inverse relation mathematically. Figure 20 showed her mathematical response for the case of one-third of the friends showing up for the party.

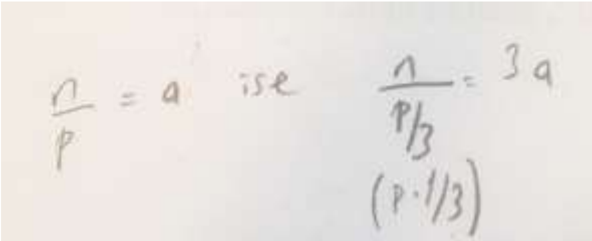


Figure 20. Mathematical representation for the inverse relation between the number of people and each person share size

In Figure 20, PT9 showed mathematically how each person's share changed from "a" to "3a". Then she concluded, "*The amount of each person's share increases 3 times*."

Unexpectedly, PT2 and PT4 exhibited misconceptions and produced incorrect responses in the post-test unlike their performances in the pre-test. They could not perceive the inverse relation between the number of sharers and the size of each fair share. For instance, when they were asked, "What happens to each person's share when number of friends showing up for the party triples?", PT2's response was, "3 x (chips/6)" where 6 represented the initial number of the people expected to come to the party. In the post-test, PT2 tried to provide a symbolic representation to generate the answer. Piaget (1960) suggested that providing a general abstract response would be more difficult than providing a concrete response. Thus, in the post-test, trying to mathematically represent the inverse relation between number of the people and the share size might be harder than providing a pictorial representation as she did in the pre-test for PT2.

PT4 exhibited a misconception that showed she failed to grasp multiplicative roots of fair sharing in the post-test. She responded that each friend's share "*decreases/increases as the number of the friends decreases/increases*." To find the change when one-third of the friends came to the party, she first found the number of friends that came to party as $6\times\frac{1}{3}=2$. Then, she stated each friend's share was "*decreased by 2 friends*" and failed to provide a sound reasoning for her response. It seemed that PT3 did not understand the problem situation and was confused about the relationship between the decrease rate between the number of people and its relation to the share size.

The findings of item 9 indicated that majority of the PTs (n = 7) failed to provide a sensible mathematical explanation for effects of factor-based change in the number of sharers on each share size prior to LTBI. After the teaching experiment, they could explain the effects of the factor-based change on the size of the fair shares through utilizing multiple representations including symbolic and pictorial.

Item 10

Item 10 was an anchor item that required anticipation of students' various fair sharing strategies and misconception of fair sharing a circular whole (Retrieved from Mojica, 2010, p.251). The PTs were asked to *draw and describe the variety of strategies that they anticipated an elementary school children might use to fairly share a circular whole into six. Then, the item asked for classification of the strategies in which (I) meant unsophisticated, (II) meant intermediate and (III) meant sophisticated.*

I made a general explanation of the meanings for these levels of sophistication for PTs. I told them the unsophisticated strategies included the incorrect strategies, the intermediate and sophisticated strategies included the correct ones with different mathematical complexity levels. (See rubrics in the appendix A for the detailed description of these levels of sophistication). This item was scored on the base of 3. The overall mean of this item in the pre-test was 0.33 and the overall mean in the post-test was 2.44. Figure 21 shows each PT's score comparison.

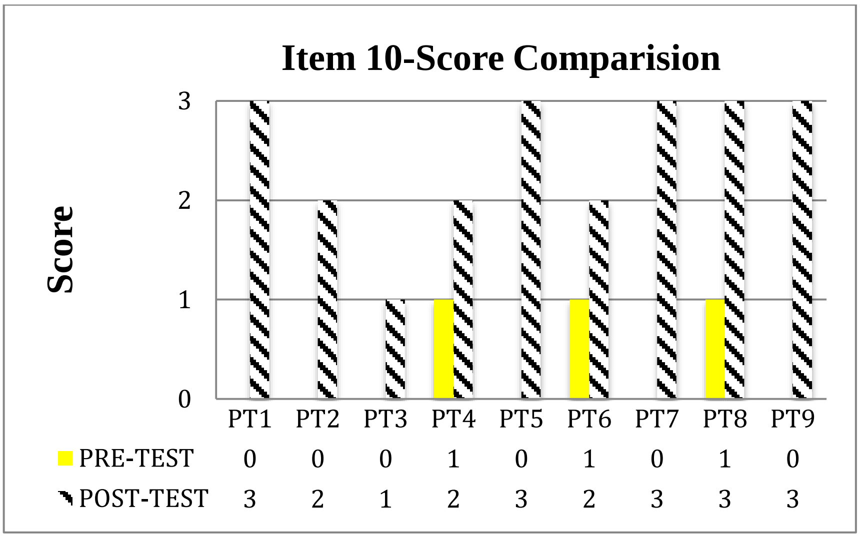


Figure 21. Each PT's score on pre-post tests: Item 10

Figure 21 shows that in the pre-test, the many PTs (n=6) could not anticipate the variety in students' mathematical strategies for fair sharing multiple wholes. Also, none of the PTs anticipated misconceptions. Only PT4, PT6 and PT8 showed one way of correct fair sharing strategy along with one incorrect strategy. Yet, these PTs failed to categorize the complexity of these mathematical strategies. The only correct mathematical strategy with a correct description recorded in the pre-test was "a composition of cuts to create all congruent pieces." The PTs first split a circle into half or one-third then split the half into three or the third into half by utilizing radial cuts. In the pre-test, none of the PTs could provide correct and complete mathematical explanations for other strategies. Also, some of their explanations exhibited mathematical misconceptions and showed that PTs lacked common content knowledge. PT2 was an example of this lack of knowledge. Figure 22 shows PT2's work on the pre-test.

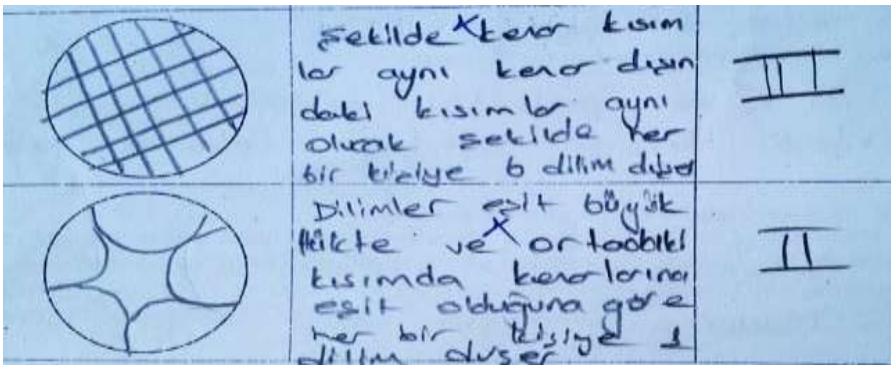


Figure 22. PT2' incorrect ways for fair sharing a circular whole

Figure 22 shows that PT2 considered both representations as fair shares. She classified the first fair sharing strategy as the most sophisticated (III) and the other one as intermediate (II). She explained the first strategy as "*In the figure, the ones [parts] on the side are same and the parts outside the edge [the 16 parts inside circle, the 4x4 square] are the same. Each person receives 6 pieces.*" She explained the second strategy as "*Pieces are equal sized: the middle part is equal to the parts on the sides. Each person receives 1 piece.*" The PT2's mathematical explanations included several problems. Mathematically, her rationale could be inscribing a square in the circle, and stating that the "parts on the sides" -meaning the parts between the circle and the inscribed square- were the same. However, if she shared for 4 or 8, the argument would make more sense. It seems she wanted to produce a proof of her response but she was unable to elaborate one. This might indicate that she did not understand what the parts in her representation constituted mathematically.

For the second strategy, she found parts that could potentially be equal (the arcs), but could not prove that the inside piece was the same size. She called the boundary of the circle and arcs as sides. For both strategies, PT2 had met two criteria of equipartitioning: exhaust the whole and create the correct number of parts. She failed to meet the creating same size parts criteria. PT2 corrected her misconception in the post-test. She classified those sorts of strategies as incorrect in the post-test.

PT3's response was another example of incomplete mathematical descriptions. She tried to utilize successive radial cuts to create six fair shares on the given circle. Then she explained one of her strategy, "*This looks more equal*." Figure 23 shows another strategy of PT3.

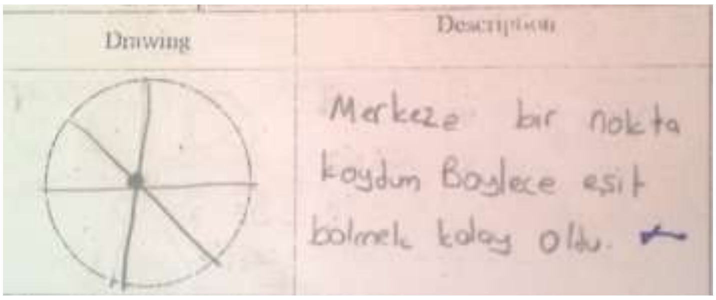


Figure 23. Incomplete mathematical description

Figure 23 shows that in the pre-test PT3 could employ two criteria of equipartitioning: exhaust the whole and create correct number of share. Yet, she failed to identify the equivalence of each share. This figure also shows that PT3 was aware of the importance of "center" concepts while utilizing radial cuts. She wrote "*I put a point on the center, thus fair sharing becomes easy*."

In the post-test, she could identify all equipartitioning criteria including; creating equal parts. She utilized the center of the circle correctly to employ radial cuts to create six fair shares. In addition, PT3 could provide a complete mathematical explanation as "*first I cut the circle into half through the center, then I split the halves into three*." Although this PT showed small progress in terms of scores, the post-test result documented that a basic misunderstanding related to fair sharing and justification of fair shares were remediated through LTBI. A variety of responses derived from all PTs' efforts in post-test are shown in Table 18.

Table 18

PTs' Anticipation of Students' Mathematical Strategies to Fairly Share a Circular Whole in Pre-post Tests.

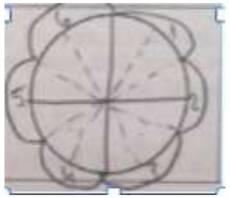
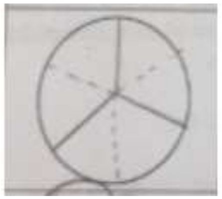
Correct Strategies	Example Works from The Post Test	PTs' Explanations	f – Pre-Test	f - Post-Test
Equivalence			2	4
Composition of Splits	 	First split into three equal parts/ into half starting from center, and then split each third/half into half/third.	2	8
Landmark		First, split into four, then, split each 1/4th into half. Gave one 1/8th to each friend. Two 1/8ths remains. Split those into three and deal by 1's.	-	1

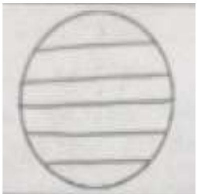
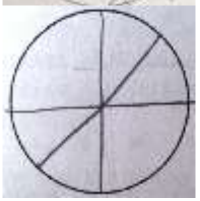
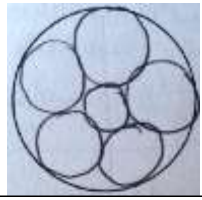
Table 18 (cont.'d) Incorrect Strategies	Example Works from The Post Test	PTs' Explanations	f – Pre-Test	f – Post-Test
Parallel Cut		Utilize parallel cuts to create fair parts as in fair sharing rectangular whole(s).	-	8
Chopping/ Unreasonable Cuts		This strategy was not recorded in the post-test	-	-
Failed to Create Fair Shares as a result of Radial Cut(s)		First split into two, then split again into two [repeated halving strategy]; to get 6 pieces split two 1/4ths into half. Yet the pieces are not equal	2	4
Failed to Exhaust the Whole		I used a glass to create small cakes. However, I did not exhaust the original cake.	-	1

Table 18 shows that in the post-test, PTs began to anticipate a variety of strategies and misconceptions that were not recorded prior to the LTBI teaching experiment in the pre-test. PT1, PT5, PT7, PT8, and PT9 were able to anticipate possible strategies as represented in Table 18. These five PTs showed at least two correct and two incorrect strategies along with correct mathematical descriptions of the strategies showed in Table 18. Within these PTs, PT1, PT5, and PT7 could not fairly share a circular cake into six along with correct mathematical description in the pre-test. This finding showed that after LTBI, these PTs made considerable progress and they learned new fair sharing strategies that a student might use along with correct mathematical justifications.

When the strategies deduced from the PTs' responses were examined in both tests, LTBI helped PTs to internalize a variety of fair sharing strategies along with correct mathematical descriptions and classifications. The incorrect mathematical descriptions did not arise again in the post-test. Although PTs had several misconceptions as discussed above in the pre-test, they did not exhibit any misconception in the post-test. They were also able to anticipate students' mathematical strategies and presentations. Then, they could order these strategies from the least to the most sophisticated ones.

Item 11

Item 11 was retrieved from Mojica (2010, p.252) and focused on anticipating students' several fair sharing strategies and misconceptions of a rectangular whole. The PTs were asked to *draw and describe the variety of strategies that they anticipated an elementary school children might use to fairly share a rectangular whole into four in the pre-test and into eight in the post-test. Then, the items asked for classification of these strategies as correct and incorrect*.

This item was scored on the base of 4. The overall mean of this item in pre-test was 2.22 and overall mean in the post-test was 3.77. Charles and Nason (2000), and Confrey et al. (2009; 2010) stated that the tasks were ordered according to difficulty levels in ELT. They indicated that based on students' works, fair sharing rectangular whole(s) was easier than fair sharing circular whole(s), since the utilization of radial cuts on circles make fair sharing a circle more difficult. The same pattern of difficulty level was observed in the PTs' performances on item 10 and item 11. PTs performed better on fair sharing the rectangular whole task than the fair sharing the circular whole in the pre-test. Figure 24 shows each PTs score comparison on this item on both tests.

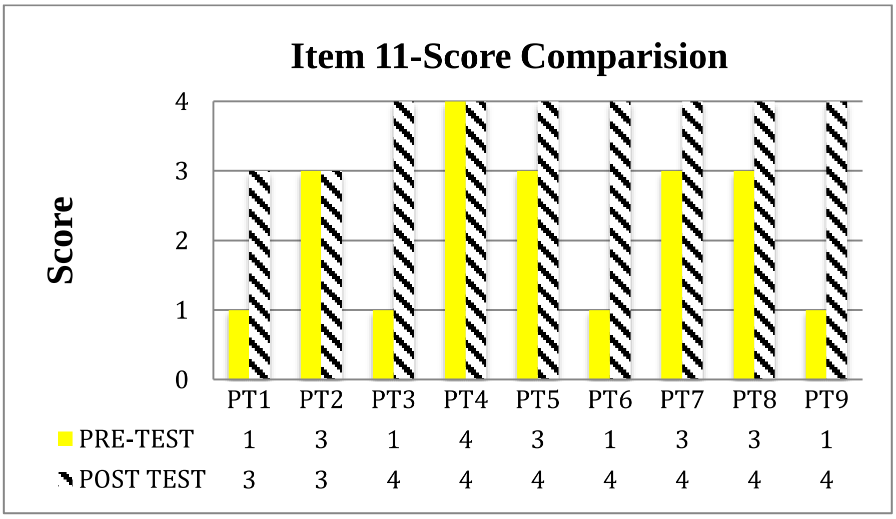


Figure 24. Each PT's scores on pre-post tests: Item 11

Figure 24 shows that PTs showed a considerable progress in LTBI. All PTs both expanded their mathematical content knowledge and revised their existing misconceptions or awareness of their misconceptions. PT1, PT3, PT6 and PT9 showed a substantial progress after the teaching sessions, since they all could show at least 3 correct strategies and 1 misconception in the post-test. On the other hand, on the pre-test, these four PTs utilized 3 successive parallel or vertical cuts to split a rectangle into four fair parts. Also, they all used two diagonal cuts to create four parts. Yet, they all stated utilization of diagonal cuts did not create a fair share and stated that this strategy was incorrect. In addition, PT2 exhibited this misconception too. This was one of the major misconceptions observed in the PTs' answer(s) on the pre-test. Figure 25 shows PT9's response, which represented this misconception.

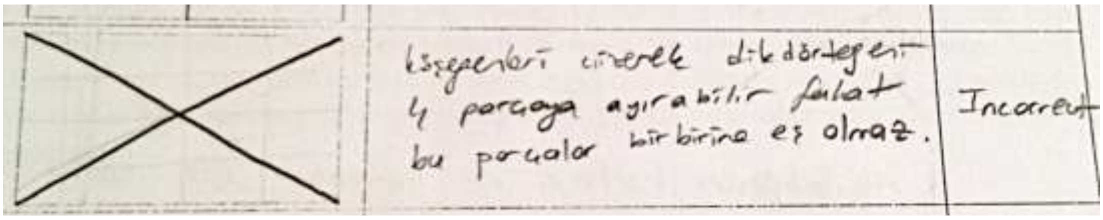


Figure 25. Utilization of diagonal cuts: PT9's misconception

In Figure 25, PT9 identified the utilization of diagonal cuts as an incorrect strategy to create four fair shares on a rectangle. She explained, "*Although 4 parts are created, they are not equal.*" This showed that these PTs were not aware that each part was congruent in terms of area. In the post-test, none of the PTs showed this misconception. Instead, they used diagonal cuts to create four equal parts and justified their reasoning. This documented that LTBI helped PTs to remediate their misconceptions and they connected fair sharing actions with another mathematical topic, such as area.

In the pre-test, except PT4, all the PTs had difficulty with generating a sensible mathematical description of their fair sharing actions. For instance, although PT8 utilized parallel cuts, composition of splits (2x2) and diagonal cuts, her description for composition of split strategy (2x2) was "*I folded [rectangle] in four.*" She tried to explain that she folded the paper in half twice horizontally so that four equal parts were created. However, she expressed the number of folds employed incorrectly. She confused the concepts of split into four and fold into four.

Another illustrative example of the utilization of incorrect/insufficient mathematical language and explanations in the pre-test work was from PT9. In the pre-test, PT9 utilized two diagonal cuts, giving each person two parts. She did not provide a sensible mathematical description of fair sharing action.

In the post-test, all the PTs anticipated both incorrect and correct strategies and classified them correctly. In addition, the PTs such as PT8 and PT9 revised their insufficient/incorrect mathematical language and explanations. In the post-test, unlike her performance in the pre-test, PT8 explained her composition of split strategy correctly as "*first, split the [rectangle] into half, then split each half into* *four. [To split each half into four] first split each part into half, then half again*." This explanation indicated that after LTBI session, PT8 learned a new strategy called repeated halving and made sense of mathematical strategies embedded in ELT. PT9 utilized diagonal cuts as well as parallel and vertical cuts to split a rectangle into four equal parts. Figure 26 shows PT9's work and mathematical description.

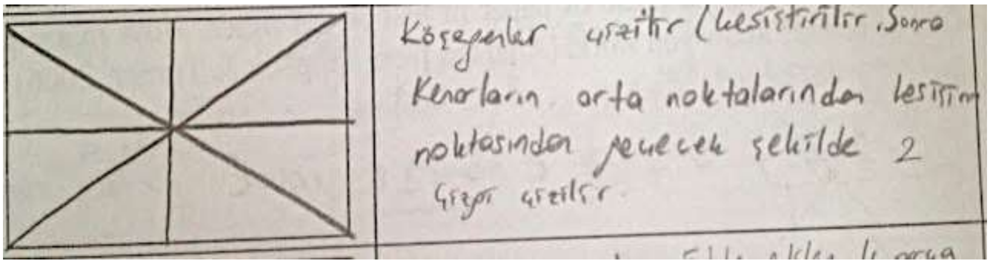


Figure 26. Equivalence of non-congruent parts strategy and its justification

In the figure above, PT9 utilized the equivalence of non-congruent part strategy and classified this strategy as correct. The triangular parts are congruent in terms of area. She explained the fair sharing actions as "*diagonals were marked and intersected. Then, a line that passes through the intersection point from the mid point of the sides was marked.*" This PT could explain correctly what she did in each step mathematically.

Figure 24 shows that after LTBI teaching experiment were completed, seven out of nine PTs (PTs 3-9) included all possible fair sharing strategies and predicted the incorrect strategies in the post-test. A variety of responses derived from all PTs' efforts in the post-test are showed in Table 19. Also, the frequency (f) of these strategies in both tests are presented.

Table 19

PTs' Anticipation of Students' Mathematical Strategies to Fairly Share a Rectangular Whole in Pre-post Tests.

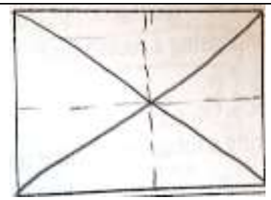
Correct Strategies	Example Works from The Post Test	PTs' Explanations	f – Pre-Test	f - Post-test
Equivalence of non-congruent parts		First split the rectangle into four diagonally. Then split each part into half.	2	7
Equivalence (Equivalent Fractions)		Create 16 fair shares and distribute 2 parts to each receiver. 1/16+1/16=2/16 and 2/16=1/8	-	1
Composition of Splits		First split into two then split each half into two. Then split each quarter into two.	6	8
Landmark	-	This strategy was not recorded in the post-test	-	-

Table 19 (cont.'d)

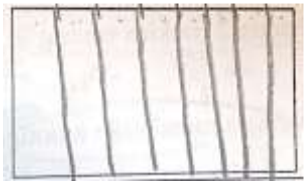
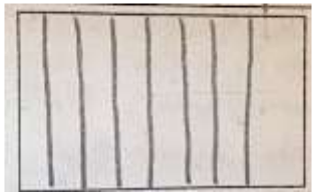
Repeated Halving Repeated Folding into Half		First (1) cut into half. Then, the resulted parts split into 8 equal parts through the repeated halving method	4	7
Utilization of Successive Parallel or Vertical Cuts		Each of them[parts] are equal. I divided the long side into 8 equal parts. Then I drew horizontal lines.	4	4
Incorrect Strategies				
n+1 Cuts		Utilized 8 cuts to create 8 equal parts but 9 parts were yielded.	1	7
Chopping/ Unreasonable Cuts		Randomly cut the rectangle without paying attention to the size and number of the parts	-	1

Table 19 (cont.'d)

Failed to Create Fair Shares (correct number of parts and exhaust the whole)		Split into 8 parts. But failed to create equal parts. Last part is larger than the rest.	5	7
Failed to Exhaust the Whole	-	This strategy was not recorded in the post-test	-	-

All the findings above indicated the PTs could anticipate a variety of students' strategies in post-test more than in the pre-test (see the frequencies in Table 19 above). Also, after intervention, the PTs could produce mathematically acceptable descriptions and justification for the anticipated strategies. In addition, within these descriptions, the PTs started to utilize their knowledge from other mathematics topics such as measurement and geometry. These evidences of PTs' progress could be considered as indicators of PTs' restructuring of their prior MCK and SK.

Item 12

Item 12 was adapted from Mojica (2010) and assessed the PTs' ability to order and justify which equipartitioning task would be difficult for K-2 students. For instance, in the pre-test one comparison was *fair sharing a rectangular cake among four friends versus three friends*. Knowing the difficulty level of each task helps teachers to design their instructional sequences (Smith & Stein, 2012). This item was scored on the base of 4. The overall mean of this item in the pre-test was 1.88 and overall mean in the post-test was 3.00. Figure 27 shows each PT's score comparison.

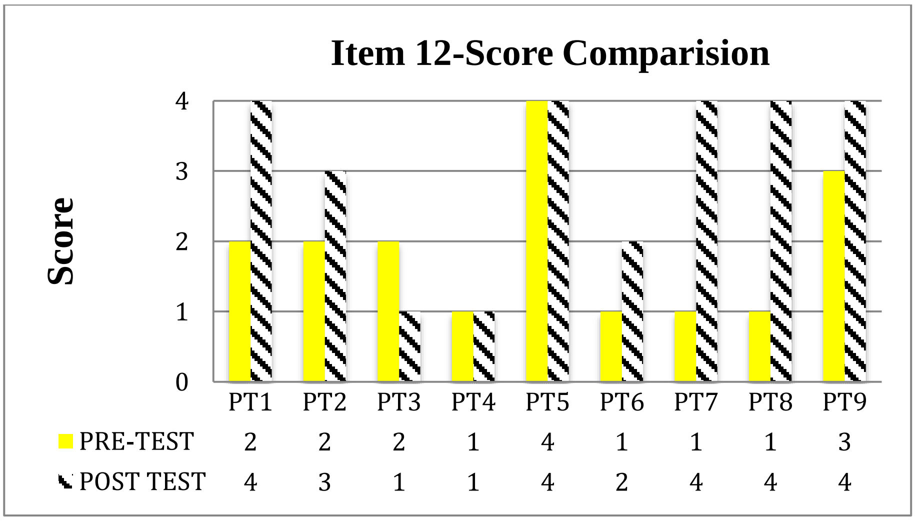


Figure 27. Each PT's scores on pre-post tests: Item 12

Figure 27 shows that, all the PTs performed better in the post-test upon intervention, except PT3 and PT4. Among all the PTs, PT7 and PT8 showed a particularly great progress in the post-test. For both tests, there were five characteristics of fair sharing tasks that determined the difficulty of the task: (i) utilization of radial cuts is harder than the utilization of parallel cuts (Confrey et al., 2008); (ii) creating an odd number of splits is harder than the creating even number of splits (Charles & Nason, 2000); (iii) the tasks that require a composition of splits are harder than the ones that do not (Confrey et al., 2008); (iv) dichotomous fair sharing is easier (Piaget et al. 1960); and (v) fair sharing circular whole(s) is harder than fair sharing rectangular whole(s) (Ball, 1993; Charles & Nason, 2000).

In the pre-test, the majority of the PTs (n = 7) utilized two criteria to order the tasks and justify their decisions. PT1, PT2, and PT3 anticipated that dichotomous fair sharing was easier for young students. Thus, they marked the tasks that involved repeated halving as easy. For the item part that asked to compare "*sharing 7 cookies among 2 children versus sharing 2 cookies among 7 children cases*", P1, PT2, PT3, PT4, PT6, PT7 and PT8 thought if the number of the objects were greater than the number of the receivers, these sorts of tasks would be easy for students.

Only PT9 and PT5 utilized other characteristics to decide the difficulty level along with correct justifications. For one instance, PT9 decided fair sharing a rectangular cake among four friends was more difficult than fair sharing a rectangular cake into three. This judgment was not correct according to Ball (1993) and Charles and Nason (2000). Yet, she justified her reasoning as "*Since there is more than one way for splitting into four but there is only one way for splitting into three*." This may indicate that this particular PT paid attention to producing multiple ways of splitting to decide the difficulty level of the item. According to her, if a certain number of shares could be created through multiple ways, this sharing action was the most difficult one. Parallel to the pre-test response but with a minor difference, PT9 challenged the required response in the post-test. This time she suggested that *sharing a rectangular cake among eight friends was as difficult as sharing a rectangular cake among three friends*. In this case, in item 11, PT9 had a valid argument in terms of comparing difficulty levels according to odd versus even number of splits on a rectangular whole. At the same, she still held her prior belief that multiple ways for splitting was also another criteria for deciding difficulty level of the task in addition to odd versus even splits criteria.

Although in the pre-test seven PTs utilized two task characteristics to determine their answers, the PTs realized new characteristics of equipartitioning tasks that should be considered while ordering the difficulty in the post-test. In the pre-test, except PT5 and PT9, none of the PTs were aware of the last characteristics. When the same number of splits was created on a circle and a rectangle (i.e. 3 splits on a rectangle versus a circle), they could not differentiate the difficulty level of these tasks. Yet in the post-test, except PT3, all the PTs decided that splitting circle was more difficult than splitting a rectangle. All the PTs had parallel statements, such as: "*for splitting a circle, one should use radial cuts. To do that one should determine the center of the circle first.*" In addition, except PT3, the rest of the PTs realized that splits that required compositions were more difficult than those that did not require composition. As a result, in the post-test, except PT3, all the PTs were aware of all the characteristics and ordered the tasks correctly, along with mathematically sound explanations.

PT3 and PT4 did not show any increase in their scores in the post-test. Yet, PT4 only marked the easier tasks in the post-test along with the correct justifications. Since the rubric of the item required to mark harder tasks and noted that 1 point would be subtracted for each wrongly marked response, PT4 failed to receive full credit. From a conceptual viewpoint, PT4 also utilized various fair sharing task characteristics to make her decisions. Only PT3 did not restructure and enhance her prior mathematical content knowledge and student knowledge based on the correct mathematical knowledge. She drew each fair sharing task and stated under this drawing which was easier. Then, she decided which task was easier for students. Although, in the pre-test she performed better, there was a decline in the post-test.

Overall the findings of this item indicated that eight of the nine PTs could anticipate the task difficulty for students. In addition, they provided answers for why a particular task was harder than another. They changed their initial thoughts and reasons for which task was more difficult. This showed a shift in their mathematical thinking, that the PTs' became capable of detecting their incomplete or different prior knowledge on equipartitioning and revised them after LTBI.

Item 13

Item 13 was adapted from Empson and Turner (2006) and it was an item about repeated halving through the utilization of folding. This folding task demonstrated the emergent multiplicative thinking in partitioning (Empson & Turner, 2006). The pre-test item asked "*Ceren folded a rectangular piece of paper in half five times. How many equal parts did she create? Show your work*." The post test item asked to *find number of equal parts if the paper was folded into half seven times.*

This item was scored on the base of 2. The overall mean of this item in the pre-test was 1.33 and overall mean in the post-test was 2.00. Kieran (1995) stated that even young children "are aware of the multiplicative nature of the patterns in such folding" (p.51). In the pre-test, PT2, PT4 and PT7 could not perceive this multiplicative pattern or misinterpreted this multiplicative pattern. For instance, PT4 replied, "*When I fold once, [the whole] is divided into 2 parts. When I fold twice, 4 parts. So, the relation between the number of folds and the number of the parts created is two times. Thus, when I fold [the rectangular whole] 5 times [into halves] 2x5=10 fair parts*." This PT did not correctly interpret the multiplicative relation. She could not see that the each fold created 2^n parts in which n represented the number of folds. Instead, her interpretation of multiplicative relation was that 2xn parts created. In the post-test, she drew the picture of first three folds in the instance of folding a rectangular paper into half seven times. Then, she wrote 8x2=16, 16x2=32, 32x2=64, 64x2=128. This answer indicated that this PT revised her formerly misinterpreted knowledge of multiplicative relation in folding.

PT7's work in the pre-test is documented below in Figure 28. She failed to link different representations, as she could not link pictorial representation with mathematical representation.

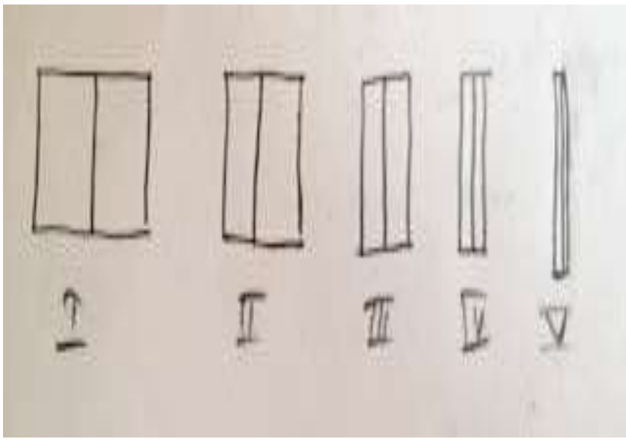


Figure 28. Repeated folding into half pictorial representation: No explicit link to symbolic representation

Figure 28 shows that PT7 tried to model each folding through drawing. She failed to represent explicitly the number of the equal parts created in the rectangular whole as a result of each fold into half. Yet, she showed that the size of resulting fair parts was reducing. She did not state any information related to resultant part such as the name. One could assert that she received one half of the whole, then she halved again this half part. She repeated this 5 times. Unfortunately, there was not a solid evidence for this claim in PT7's work.

Different from the pre-test, in the post-test PT7 first drew each folding action up to three folds in half and she could utilize the representations correctly. In her drawings, she worked on the same whole and drew the resultant fair parts. Then, she mathematically showed how many fair parts were created, utilizing both count of 128 parts and naming each part as fraction $\frac{1}{128}$. Figure 29 shows the PT's work.

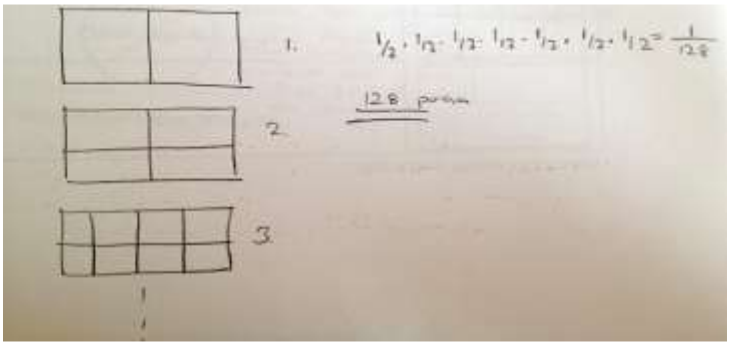


Figure 29. Folding into half: Pictorial and symbolic representation

When we compared the progression of PT4 and PT7, one can say that PT7 showed more advanced progressions, since she perceived the relation between number of folds and fair parts apart from thinking about the resultant share of each halving. She perceived the relation *functional multiplication* which was “how a given outcome could be produced using a sequence of more than two folds” (Empson & Turner, 2006, p.51). Whereas, PT4 saw the relation as emergent recursion in which she first drew three and saw the result of each action. Then, she took the resultant number of equal parts and recursively doubled it. These strategies were more apparent in the teaching session. This evidence showed that after the LTBI, the PTs started to exhibit different representations and strategies embedded within LT. Thus, one could say that upon intervention, PTs could perceive the multiplicative relation between number of folds and number of fair shares created in the instance of repeated halving through folding. Also, the PTs connected the embedded repeated halving strategy within LT and with further mathematical ideas multiplication and early functions.

Item 14

Different from item 13, item 14 (Adapted from Empson & Turner, 2006) involved the utilization of a series of folds to create a targeted number of fair parts, instead of predicting the number of equal parts resultant from folding actions. The pre-test item was:

*Engin learns origami in his school. In order to create a ship, he needs to fold a rectangular paper to create 24 equal parts. How many ways can he fold the paper? Explain your answer. (*The post-test item included *36 equal parts*)

Also, I reminded the PTs to explain their answers in many ways as they can for the item.

This item was more complex than the previous item since it required reaching target numbers that had prime factorization and any combination of the prime factors, such as 2 and 3 forms 6 (Empson & Turner, 2006). This item addressed the composition of splits idea of equipartitioning a single whole. This item was scored on the base of 3. The overall mean in pre-test was 0.77 and in the post-test was 1.66. Figure 30 shows the score comparison of each PT on both tests.

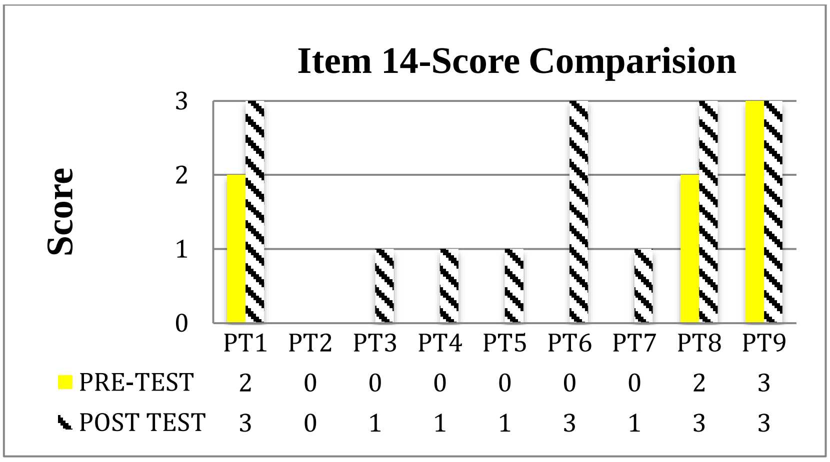


Figure 30. Each PT's scores on pre-post tests: Item 14

Figure 30 shows that this item was difficult for many PTs (n = 6) in the pre-test. They could not come up with several ways of folding to create 24 equal-sized parts on a rectangular paper. Five PTs could not produce any answer. PT2 and PT4 thought there existed an additive relation between number of folds and the number of equal parts created. Both PTs thought when they folded a rectangular paper into half it would create 2 equal-sized parts. Then, if you would fold again into half, it would create 4-equal sized parts. PT3 wrote, "*If I fold once, it would split into 2; if I fold twice, it would split into 4, if I fold three times, it would split into 6… As a result, I should fold 12 times to create 24 parts. [They wrote] 12x2=24.*" PT6 wrote that she needed to fold the rectangular paper five times. Yet, she did not specify the number of folds for each time. PT7 wrote "*I would reach the result [24-equal sized parts] in five steps, in the first step I would fold into half, in the second step I would fold into 4, in the third step I would fold into eight, in the fourth step I would fold into 16 and in the fifth step I would fold into 24*." This explanation of the PT had a mathematical language error; she would say, "split into" instead of "fold into". Since, the folding is a recursive action, if this student fold into 24 it directly created 24 parts. She probably meant that she would split each part created as a result of folding into a certain number. If she would fold into 2 then fold into 4, then this action would create 2x4=8 equal sized parts. PT3 said if she would fold the paper into half six times, she would create 24 parts. Because, in each time the number of parts would increase by 4. And on the sixth fold she would have 24 parts. Thus, she exhibited an additive misconception.

PT1 and PT8 showed three different ways of folding to create 24 equal-sized parts. Both PTs came up with a combination of folds such as 2-3-4 (fold into half, then fold into three and fold into four). Yet, these PTs did not come up with a mathematical deduction based on this combination of the folds and 24. Only PT9 came up with a mathematical conclusion and said, "*I could use the combination of factors of 24.*" Then she wrote down various combinations of the factors such as 2x2, 2x3, 3x8, 4x6 and 12x2 (2x3x4).

In the post-test, PT6 showed a great progress and reached a mathematical conclusion that any combinations of the prime factors of 36 could be used. In addition, PT1 and PT8 also reached this conclusion. As discussed above, although PT1 and PT8 came up with several ways of folding, they did not come up with this mathematical conclusion. PT3 and PT4 corrected their initial additive misconception; they perceived the multiplicative relation between number of folds and the number of the parts created. For instance, PT3 wrote the result of each fold as follows:

1 ➔2 [Fold into] Half
2 ➔4 [Fold into] Third
3 ➔12 [Fold into] Third
4 ➔36

This PT utilized a folding action four times. She represented 3 folding actions (fold into half, third, and third). Yet, she started with 2 equal parts. She did not explicitly write folding into half took place twice. Mathematically, PT3 employed 2x2x3x3=36.

PT5 represented the result of each fold through utilizing area model. She drew a picture in which she first folded the paper into half horizontally, and then folded it in half vertically ($\frac{1}{2}$x$\frac{1}{2}$=$\frac{1}{4}$). Then she folded it into three horizontally ($\frac{1}{4}$x$\frac{1}{3}$=$\frac{1}{12}$) and this created 12 parts. Then, she folded again into three vertically ($\frac{1}{12}$x$\frac{1}{3}$=$\frac{1}{36}$). This created 36 equal sized parts. In the pre-test, PT7 could not specify the result of each fold, but in the post-test she used an area model along with verbal mathematical description of the folding actions to show how to create 36 equal-sized parts as a result of folding. She stated, "*First I would fold into four [PT7 drew horizontal cuts] ($\frac{1}{4}$th, quarter). Then I would fold into nine [PT7 drew vertical cuts] ($\frac{1}{9}$ of each quarter).*" PT2 left the answer for this item blank in the post-test.

Overall findings of this item indicated that four PTs could come up with a general mathematical conclusion—the combination of the prime factors of the given number of parts— to state how many ways of folding could be employed and they corrected their misconceptions. PT7 and PT5 newly learned how to utilized area model to represent the result of each folding. Only PT2 did not exhibit any difference in her performance in the post-test. However, her orientation towards these types of tasks in the teaching session will be discussed in the next part of the findings chapter.

Item 15

Item 15 was related to sharing multiple wholes and compensation and adapted from Mojica (2010). The items in the pre and post-tests were as follows:

Pre-test: *18 friends went to a restaurant and ordered 12 pizzas. The friends were sitting into two tables. How can 12 pizzas be fairly shared between the tables?*
Hasan suggested that 9 friends can sit on each table and receive 6 pizzas.
Ahmet suggested that10 friends sit at one table and receive 7 pizzas and 8 friends sit at other table and receive 5 pizzas.
Post-test: *9 friends went to a restaurant and ordered 15 pizzas. The friends were sitting into two tables. How can 12 pizzas be fairly shared between the tables?*
Hasan suggested that 3 friends sit one table and receive 10 pizzas and 6 friends sit on the other and receive 4 pizzas.
Ahmet suggested that 4 friends sit at one table and receive 6 pizzas and 5 friends sit at other table and receive 9 pizzas.
Pre-post tests common questions:

- *Decide which of the suggested strategies is correct and why?*
- *What is unclear about mathematical understanding of the friend who produced incorrect strategy?*

This item was scored on the base of 3. Figure 31 shows each PT's score.

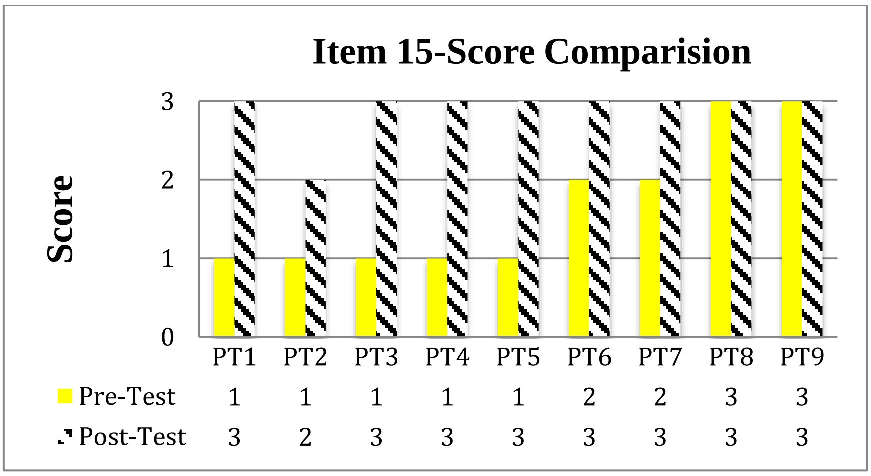


Figure 31. Each PT's scores on pre-post tests: Item 15

Figure 31 shows that the overall mean of this item in pre-test was 1.66 and the overall mean in the post-test was 2.88. In the pre-test, all of the PTs could determine which friend suggested the correct strategy. Yet, the justification ways for the correct strategy varied. PT1 could explain why Ahmet's strategy was incorrect, yet they failed to provide a complete explanation for why Hasan's strategy was correct. PT1 stated that in Hasan's strategy each friend received fair shares but she did not mathematically justify why each friend received fair shares. Similarly, PT2, PT3, PT4 and PT5 only determined the correct strategy and found friends' shares indicating they all received $\frac{6}{9}$ of a pizza. Yet, these PTs failed to explain why Ahmet's strategy was incorrect and what was unclear about Ahmet's understanding of fair shares.

In the pre-test, PT7 could determine correct strategy and yet failed to give a sensible and complete mathematical justification ($\frac{6}{9}$ pizzas per friend, ratios are equal). This PT stated that each friend's share was not equal in Ahmet's strategy, but did not provide a complete mathematical explanation for this claim. Conversely, PT6 stated, "*the shares are not equal in [Ahmet's strategy], Ahmet thought more people should receive more pizza. Also he might think 10-7=8-5 and the difference is 3, the same*." On the other hand, PT6 decided that Hasan's strategy was correct since *"the same number of people received the same number of pizza.*" Since PT6 did not mathematically show how this situation led to fair shares, she did not receive full credits. Only PT8 and PT9 provided correct answers along with correct justifications.

Figure 31 shows that in the post-test, majority of PTs (n = 8) could produce correct answers along with correct justifications. Eight PTs could identify each friend's share and decide whether the shares were equal or not. Also, they concluded Ahmet employed an additive thinking instead of multiplicative thinking while comparing each friend's share on each table.

The findings related to this item indicated that, although in the pre-test the PTs could determine correct and incorrect strategies, they failed to understand and explain students' mathematical thinking. Ball and Thames (2008) suggested, "recognizing a wrong answer is common content knowledge (CCK), while sizing up the nature of the error [is] specialized content knowledge (SCK)" (p.401). As a result, in the post-test eight of the PTs restructured their SCK.

Item 16

Item 16 was a covariation item. The pre and post-test items were:

Pre-test: *Mustafa knows that 6 carrots will feed 4 rabbits if they are shared fairly. Predict the number of carrots needed for each number of rabbits listed in the table below, so that each rabbit will get the same share of carrots (Adapted from Yilmaz, 2011).*

Number of rabbits	Number of carrots
2	
4	6
8	

Post-test: *Mustafa knows 5 carrots will feed 12 rabbits if they are shared fairly. Predict the number of carrots needed for each number of rabbits listed in the table below, so that each rabbit get the same share of carrots (Adapted from Yilmaz, 2011).*

Number of rabbits	Number of carrots
4	
12	5
36	

PTs were expected to utilize several mathematical strategies and explain their reasoning mathematically. Stein and Smith (2012) determined the five mathematical ways to solve this sort of tasks as:

1. Unit rate: Find the number of carrots eaten by a rabbit and multiply by the number of rabbits to find the required number of carrots.
2. Scale factor: Perceive the vertical multiplicative relations: if the number of the rabbits doubles, so does the number of carrots.
3. Scaling up: Add 3 carrots for every two rabbits until reaching the required number of rabbits.
4. Additive: Add 1.5 carrots 8 times to find the number of carrots.
5. Others: Drawing pictures to show each rabbit's share and repeatedly copy this drawing to represents required number of carrots for required number of rabbits.

This item was scored on the base of 4. The overall mean of this item in the pre-test was 1.88 and in the post-test was 3.00. Figure 32 shows each PT's score comparison on this item in both tests.

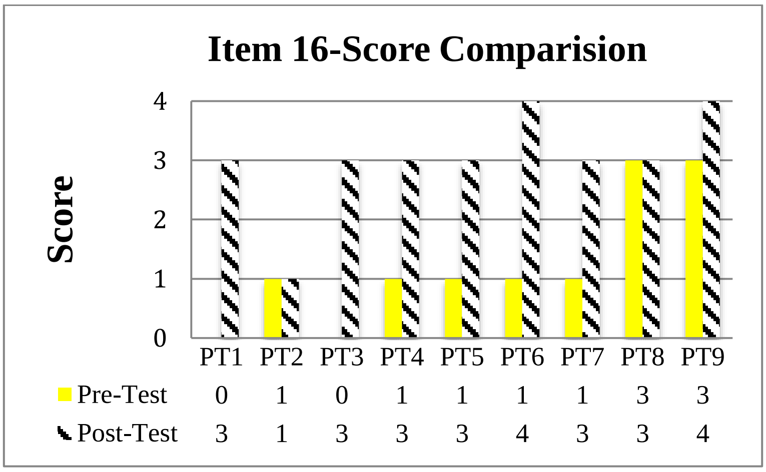


Figure 32. Each PT's scores on pre-post tests: Item 16

The score's on the Figure 32 indicated that the majority of PTs (n = 7) failed to produce various mathematical solutions. PT1 could not produce a correct answer for the item. PT3 also produced an incorrect answer. Figure 33 shows PT3's work in the pre-test.

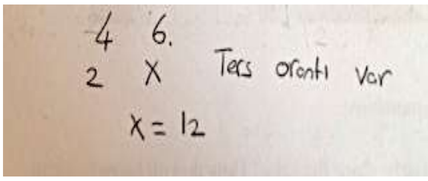


Figure 33. Mathematical error of PT3: Assuming existence of inverse relation between two quantities in covariation task

In the figure above, PT3 set a proportion, yet she failed to identify the direct proportion and instead she stated, "*there is an inverse proportion*." Prior to the experiment, this PT knew that she would set a proportion, yet she did not acquire a conceptual understanding of the proportion concept. PT2, PT4, PT5, PT6, and PT7 only showed a single mathematical solution. PT8 and PT9 provided three different strategies, yet these strategies did not include unit rate, scaling factor or scaling up strategies at once.

In the post-test, PT1 and PT3 showed a considerable progress. Unlike their performance in the pre-test, they could come up with three different strategies to solve the task. PT3 perceived the direct relation between the number of carrots and the number of rabbits and seemed to remediate her initial incorrect mathematical understanding (inverse relation). Also, PT4, PT5, PT6, and PT7 produced at least three different mathematical strategies to solve the task in the post-test unlike their pre-test performance where they could produce only one correct strategy. Table 20 shows the PTs' mathematical strategies that were correctly performed to solve the item in both tests.

Table 20

Each PT's Mathematical Strategies to Solve Covariation Item in Pre-Post Tests

	Pre-test					Post-Test				
	Unit Rate	Scaling Factor	Sc. Up	Additive	Proportion	Unit Rate	Scaling Factor	Sc. Up	Additive	Proportion
PT1						x	x			x
PT2					x					x
PT3						x			x	x
PT4		x				x	x			x
PT5	x					x	x		x	
PT6	x					x	x	x		x
PT7		x				x	x			x
PT8	x	x			x	x	x			x
PT9	x	x				x	x	x		x

Table 20 shows that after the teaching sessions the most of the PTs utilized both unit rate and scaling factor strategies. This shows that all the PTs could recognize equivalence either within a ratio as a:b (Noelting, 1980) or between ratios as $a2$:$a1$ (Noelting, 1980). Then, PTs could preserve this "ratio while covarying the number of the collections or whole to be shared [with] the number of receivers" (Yilmaz, 2011, p.92).

Seven PTs utilized the scaling factor strategy in which they recognized that the number of rabbits triples as the number of carrots does. They recognized between ratios in which $a1$ (12) represented the initial number of rabbits and $a2$ (36) represented the new number of the rabbits a2:a1 (36:12) as 3. Then they preserved this ratio and found the new number of carrots (15) required for the new number rabbits (36). Picture of PT1's work is a clear example of this understanding.

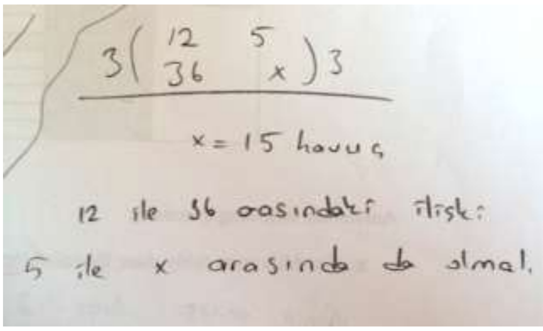


Figure 34. Scaling factor strategy: Preserving between ratio in covariation item

In the figure above, PT1 showed the relation between 12 and 36 with lines and wrote 3. In her explanation, the relationship between 12 and 36 should exist between 5 and x. Then, she found x=15.

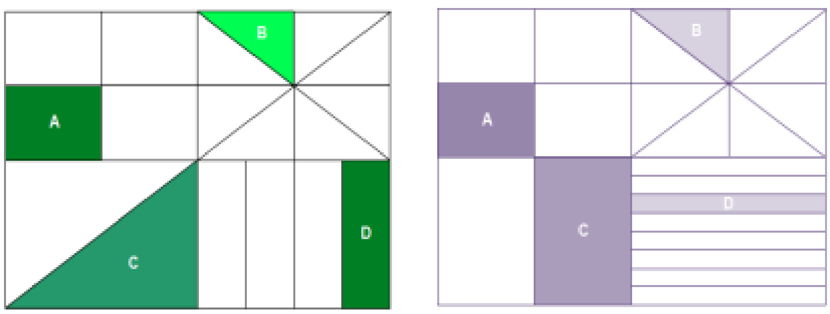
Table 20 shows that eight PTs utilized unit rate strategy in the post-test. These PTs first found each rabbit's share as $\frac{5}{12}$ and that was called unit ratio. Then they preserved this ratio while covarying the quantities. PT9's work was one of the clear examples for this strategy. PT9 wrote, "*First, the amount of carrot required to* *feed one rabbit should be found. Then the amount [of carrot] should be found for the required number of rabbits. If* $\frac{5}{12}$ *is the required amount for one rabbit, then for 4 rabbits:* $4\times\frac{5}{12}=\frac{5}{3}$. *[For 36 rabbits]*, $36\times\frac{5}{12}=15$."

Findings from item 16 indicated that PTs became aware of several mathematical strategies to solve the covariation task. This means they enhanced their common content knowledge, since this knowledge type required PTs to know the mathematics itself to solve a task (Wilson et al., 2013). Also, some PTs were aware of their prior misunderstandings or misconceptions that seemed remediated. This meant that these PTs elaborated errors and developed strategies for how to fix it. In addition, eight PTs linked the covariation concept with direct proportion strategy. This evidence also showed that PTs had perceived the connections between various mathematical topics.

Item 17

Item 17 was the last item in both tests. This item assessed whether PTs could demonstrate the equivalence of non-congruent parts created by non-prime splits on a given rectangle. The items in the pre and post test were:

> *Ali's mum shared a cake among four of his son's friend. Ahmet receives part A, Kaan receives part B, Gulsen receives part C and Mehtap receives part D. The Figures below is shown Ali's mum sharing respectively in pre and post-tests.*



In both test, first the PTs required to decide and explain whether the rectangular cake was equipartitioned. Then, they were asked to compare each share of A, B, C and D. Finally, PTs asked to indicate the mathematical relation among the each share.

When a single whole was equipartitioned into a number of parts, these parts were all equivalent in size, yet these parts did not have to be "*congruent*" in terms of shape. There were three mathematical strategies to indicate the equivalence of non-congruent parts. These were qualitative compensation, composition and decomposition, and indicating area congruence. This item was scored on the base of 3. The overall mean of this item in the pre-test was 1.55 and the overall mean in the post-test was 2.66. Figure 35 shows score comparison of each PT on both tests.

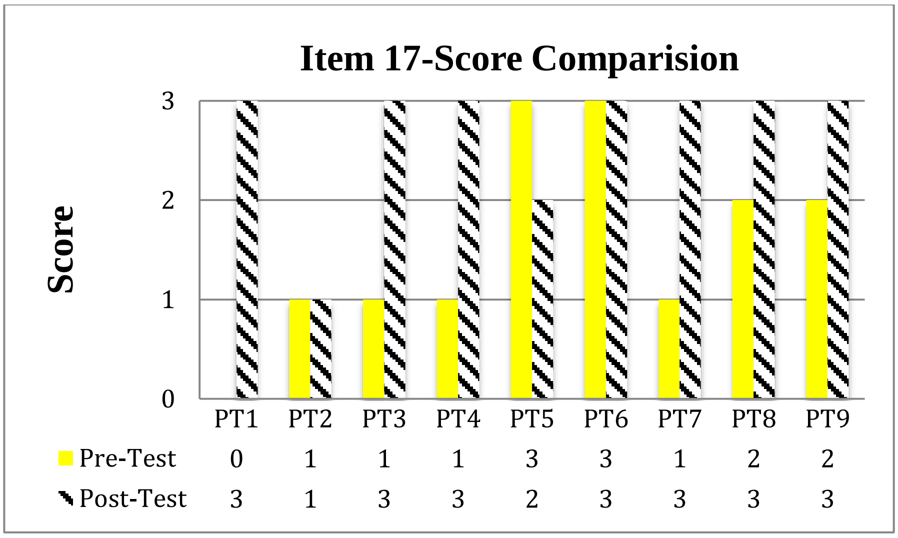


Figure 35. Each PT's score on pre-post tests: Item 17

In the pre-test only PT5 and PT6 could indicate the mathematical relations among parts A, B, C and D. For instance, PT6 utilized the area congruence strategy to indicate the equivalence of the fractions. Figure 36 shows this PT's work.

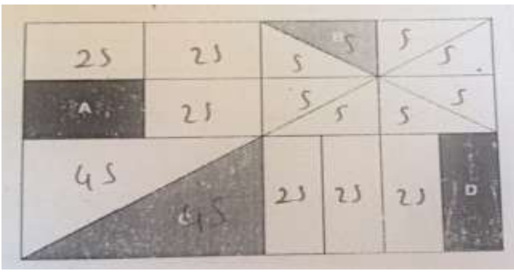


Figure 36. Symbolic notation for each share area to evaluate equivalence of the shares

In the figure above, PT6 utilized "S" as an area unit for $\frac{1}{32}$ of the rectangle. Then she determined the size of each part by utilizing unit "S". Then she concluded that, A and D received the same amount, B received less than these and C received more than these. Then she mathematically indicated this relation as "8B=4A=4D=2C". PT5 found the fractional name for each part. For parts A and D, she wrote " $\frac{1}{4} \div 4 = \frac{1}{4} \times \frac{1}{4}$ and this equals to $\frac{1}{16}$. For part B, she wrote $\frac{1}{8} \div 4 = \frac{1}{8} \times \frac{1}{4}$ and this equals to $\frac{1}{32}$. For part C, she wrote $\frac{1}{4} \div 2 = \frac{1}{4} \times \frac{1}{2}$ and this equals to $\frac{1}{8}$. Then, she concluded, "the shares A and D are equal and part B is 2 times larger than part C. Also, Part B is half of parts A and D and one-fourth of part C." PT8 and PT9 also indicated the mathematical relations between the sizes of the parts. Yet, they failed to provide a complete mathematical explanation. For instance, Figure 37 shows PT8's work.

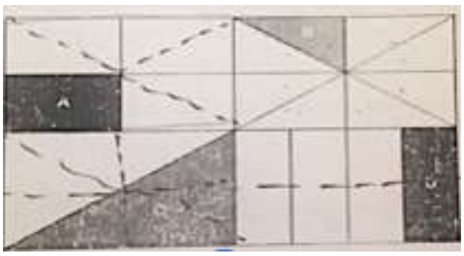


Figure 37. Composition-decomposition strategy to evaluate equivalence of shares A, B, C & D

Figure 37 shows PT8 utilized decomposing for each part. Then she indicated friend A received 2 pieces, B received 1 piece, C received 4 pieces and D received 2 pieces. Then this PT concluded C>D=A>B. Although the PT's response was correct, she did not mathematically indicate the size of "one piece".

The rest of the PTs (n = 5), except PT1, could provide correct answer. Yet, they failed to provide a sensible mathematical explanation for their claim or they named the parts erroneously. For instance, PT4 stated "C received 2 times as much as A, A received 2 times as much as B, A and D received equal parts". Yet, this PT4 named each part erroneously as A=$\frac{1}{4}$, B= $\frac{1}{8}$, C=$\frac{1}{2}$ and D=$\frac{1}{4}$. This response indicated PT failed to identify the referent whole. PT2, PT3 and PT7 also made the same error.

In the post-test, majority of the PTs (n = 7) received the full score on the item. Except PT2 and PT5, all the PTs provided mathematical justifications for why the parts are congruent in terms of area. PTs1-4 and PT7 could name each share and fixed their mathematical error from the pre-test. Only PT2 failed to identify the referent whole as she did in the pre-test. The rest could identify the referent whole correctly and utilize fractional names for each part to decide the mathematical relations among the parts. In the pre-test, PT1 could not generate any response. Yet in the post-test, this PT also utilized the fractional name of each part, then order the fractions. Based on her ordering, the PT determined the mathematical relations among the parts correctly. PT9, PT6, and PT4 indicated area equivalence and also utilized composition of splits.

4.2 Analysis of Overall Performance of PTs

A comparison of the PTs' mathematical performances on both tests showed a noticeable increase in performance. The total scoring for both tests was 56. The pre-test scores' mean was 30.11 and the post-test scores' mean was 47.44. The difference between these two means was 17.33 points. Figure 38 shows comparison of each PT's total scores on both tests.

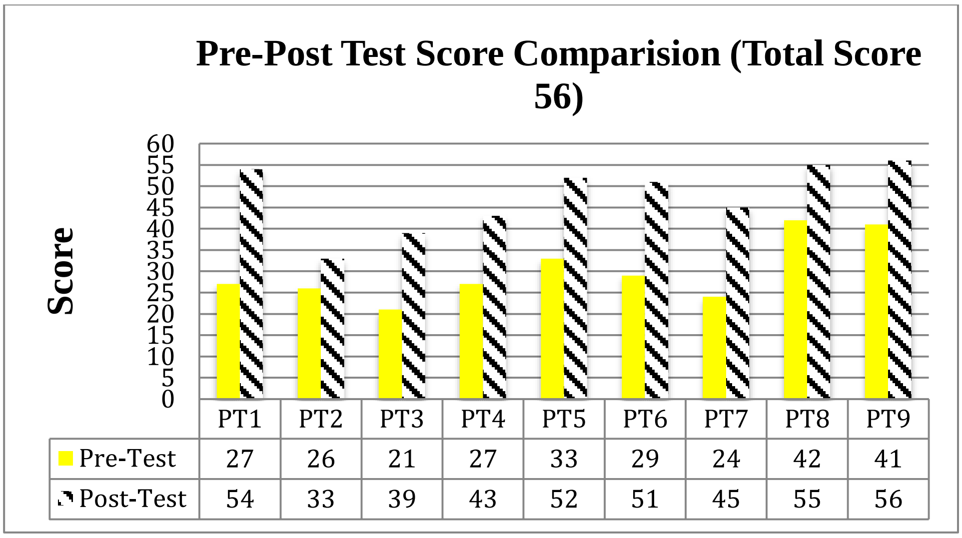


Figure 38. Overall performances of each PT on both tests

Figure 38 shows that all the PTs' performances were increased by LTBI teaching sessions. Some PTs showed more progress than other PTs. PT1 exhibited the best progress. PT2 showed the least progress. As discussed, PT2 had difficulty with mathematics related courses in utilizing both the symbolic and verbal language of mathematics prior to study. This issue was addressed briefly as a result of this study. In the teaching sessions she could communicate her mathematical ideas verbally correctly. Yet, she still had some problems with symbolic use of mathematical language.

4.3 Summary of PTs' Knowledge Levels Before-After LTBI

The findings deduced from the PTs' performance on the pre-test showed that PTs did not initially have an in-depth knowledge about equipartitioning. They did not put intense thought on what equipartitioning was and which mathematical ideas and concepts were related with the equipartitioning. They generally focused on a single idea of equipartitioning as creating same groups or parts and employed single mathematical strategy to generate answers. Also, they exhibited a limited ability to highlight the connections between the equipartitioning and further mathematical topics.

After the LTBI, the PTs generated responses that could be counted as an evidence for that they could focus on various aspects of equipartitioning in the post-test. They learned different cases of equipartitioning, utilized different mathematical naming practices, new mathematical strategies, and representations. Also, their responses to the items showed that they utilized different mathematics topics such as proportion, ratio, fractions, exponential numbers and area on equipartitioning items. This would be considered as an indication for expanding their knowledge about equipartitioning through creating a web of connections between equipartitioning and related mathematical topics.

The pre-test findings also showed that the PTs acquired serious mathematical misconceptions and mathematical errors related to equipartitioning. These were (1) failing to perceive multiplicative relation between the size of the share and the size of the whole (2) failing to identify the equivalence of the shares that are congruent in terms of area yet not in shape (3) creating n cuts to create n fair parts (4) utilizing combination of parallel and vertical cuts on circles to create fair parts similar to what they employed on the rectangles (5) failing to identify multiplicative relation between folds and the resultant number of fair shares (6) employing additive misconceptions in detecting the patterns and (7) failing to identify whether direct or inverse relation existed between variables in the factor based change item and covariation item. All these misconceptions and errors showed that PTs had mostly incorrect and incomplete MCK prior to the LTBI experiment.

After the LTBI experiment, PTs did not exhibit these misconceptions and errors in the post-test. Also, they could employ correct strategies along with correct explanations for the items that they exhibited a misconception or error in the pre-test. This showed that PTs enhanced their incomplete and incorrect MCK prior to experiment as a result of LTBI.

The pre-test findings also indicated that the PTs exhibited serious mathematical difficulties especially when they worked with the items out of PTs' initial conception of equipartitioning. These items were fair sharing circular whole in a variety of ways, engaging with covariation item in variety of ways, and the items related the folding with equipartitioning ideas. The first difficulty was that PTs could not produce various mathematical strategies to solve a item. The second difficulty was related to employing fully correct mathematical language to justify their answers. The third difficulty was employing new strategies on the items that they did not face frequently the mathematical context before such as folding and creating odd splits on circles. At last, the PTs had difficulty with visualize the mathematical context in the item and then expressing their mathematical thoughts by utilizing various representations.

After the LTBI experiment, majority of PTs overcame their mathematical difficulties. They utilized various mathematical representations in their solutions. They utilized various mathematical strategies as required. Also, their mathematical language became more precise, rich and accurate when they justified their responses. They started to use the terminologies related to equipartitioning that they utilized in the LTBI accurately. Some of these terminologies were ratio, multiplicative relation, inverse relation, parallel cut, radial cut, split, equivalent, and congruent.

They exhibited a limited proficiency in predicting students' possible strategies. The general tendency among PTs was predicting only possible correct responses that were also affected by their mathematical orientation towards the presented item. Also, when they presented with cases of students' responses on a particular item, majority of them could detect whether the students' response was correct or not. However, only a few PTs could partially explain why the responses are incorrect or correct mathematically.

After the LTBI, the findings related to post-test showed that the PTs improved their ability to predict students' various mathematical strategies including both correct and incorrect ones. Also, majority of them could provide mathematically more precise and accurate explanations about the cases of students' incorrect responses. They could identify students' possible mathematical thinking, misconceptions or errors behind the presented responses.

All these findings indicated that LTBI seemed to remediate PTs' misconceptions and contributed the restructuring of CCK. Moreover, through learning new mathematical ideas and concepts related to equipartitioning seemed to enhance their CCK. Then, LTBI helped to enhance their student knowledge as the PTs started to anticipate a variety of students' mathematical strategies and misconceptions and could explain the underlying reasons behind students' mathematical solutions. Also, the PTs enhanced their SCK as their descriptions of fair sharing actions in the post-test became mathematically more accurate. In addition, they acknowledged that different mathematical representations carried out different mathematical meanings and these representations could be used to teach different mathematical ideas and concepts. Lastly, the post-test findings indicated that PTs enhanced their HCK since they related equipartitioning related ideas with further mathematics topics such as area, equivalent fractions, ratio, direct proportion, exponential numbers, multiplication and division. In addition, the PTs could produce generalizable mathematical ideas.

CHAPTER V

TEACHING SESSIONS FINDINGS

This part will introduce how each PT restructured her mathematical content knowledge (MCK) and student knowledge (SK) in each week of the LTBI teaching experiment. The findings related to research questions two and three will be reported. The questions are:

Research Question 2: What are pre-service teachers' restructuring practices for mathematical content knowledge in a Learning Trajectories Based Instruction (LTBI)?

- In what ways does LTBI support PTs to detect their own mathematical misconceptions, errors and knowledge gaps and remediate them?
- In what ways does LTBI support PTs to make sense of mathematical ideas and knowledge of equipartitioning?
- To what ways PTs connect the mathematical ideas embedded in the ELT to further mathematics topics?

Research Question 3: What are PTs' restructuring practices for student knowledge in a LTBI?

- In what ways does LTBI support PTs' ability to understand students' mathematical thinking and learning?

In this chapter, the restructuring process of MCK will be examined under three components of MCK and related practices. These components are Horizon Content Knowledge (HCK), Specialized Content Knowledge (SCK) and Common Content Knowledge (CCK). Restructuring process of student knowledge (SK) will be examined under four practices called distinguishing, anticipating-recognizing, ordering and empathizing. The findings were reported in a chronological order of each week's content. The content of the each week were determined according to the suggested route in the ELT. In each week, the findings related to knowledge restructuring practices of the PTs were reported through considering the logical flow within the instruction.

The following sections of this chapter presented how PTs restructured SK and MCK in each week. The findings will be reported starting from the second week of the study, since pre-test was administered to see the PTs current knowledge level of equipartitioning in the first week. The findings related to pre and post tests were reported in the first part of the findings Chapter IV.

5.1 Week 2

5.1.1 Restructuring Student Knowledge

The first task asked the PTs to fairly share 32 crayons among eight children. Their written responses showed that they solved this task through division. When the PTs were asked to predict several strategies of elementary school students to solve the task, they came up with different mathematical strategies and representations of equipartitioning 32 crayons among eight children. To achieve this anticipation the PTs utilized the given manipulatives. This showed that the PTs could distinguish their own mathematical thinking from the students since they produced three different representations and three different mathematical strategies for equipartitioning the 32 crayons when they had opportunity to manipulate concrete materials to reflect on possible students' strategies. Figure 39 shows the PT1, PT7 and PT8's representation for fairly sharing 32 crayons among eight children as an anticipation of students' strategies.

Figure 39. Dealing by ones and forming groups of fair shares strategy

Figure 39 shows the initial process and the final process of the sharing. PT1 started the process. PT1 gave one item at a time to each friend. PT7 and PT8 helped PT1 and verbally described their dealing by ones strategy as "*here one for you and so on*." While they were working on completing each cycle, PT8 dealt an object without paying attention to the order. PT7 warned PT8 and said "*don't change the order; this may confuse the children's mind*." At the end of the fourth cycle, the PTs exhausted the 32 objects. PT8 wrote under each group "4" then she wrote "4 + 4+… 4=32, [n=8]". PT1 and PT7 counted one group share as four then they counted by four to make sure the exhausted the whole collection as "4, 8, 12, 16, 20, 24, 28, 32". They utilized repeated addition and counting by groups of four as ways for checking whether they exhausted the whole collection. Also, this was an indication that the PTs employed additive reasoning in their reassembly actions.

PT8 suggested a different fair sharing strategy. She stated, "*we can deal by twos, we can give two [objects] to each [child]*." Then, she gave two objects to each child systematically, completed the first cycle, and stated "*they could deal the rest of the objects by twos, too*." After that, PT7 stated, "*They could give three objects then give one object to each friend*." These actions of the PTs shows that they anticipated that students could start with composite unit strategy and switch back to 1-1 correspondence strategy if needed to fairly share a collection.

PT3 and PT5 in one group and PT6 and PT9 in another group also utilized the dealing by ones strategy. PT3 and PT5 utilized the same grouping representation and 1-1 correspondence strategy as shown in Figure 39. PT6 and PT9 also systematically gave one object per child at a time. However, they created a different representation of the shares. Figure 40 shows PT6 and PT9's representation.

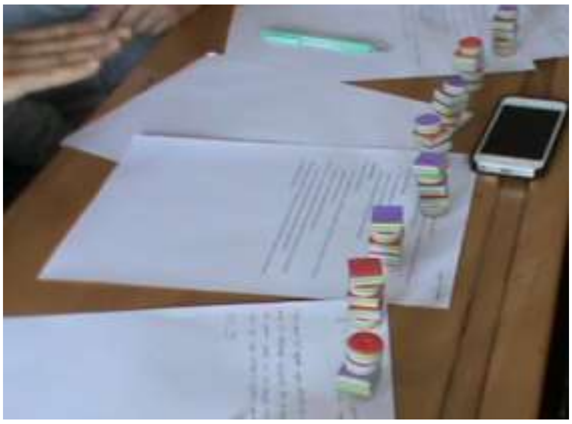


Figure 40. Stacking representation of each fair share

PT2 and PT4 also utilized systematic dealing by ones strategy. They created an array representation. Figure 41 shows their representation.

Figure 41. Array representation of fair shares

When all the PTs finished their works on the first two parts of the first task, a class discussion took place on the characteristics of different representations and students' predicted fair sharing strategies. The written responses of the PTs only included one predicted strategy as "systematic dealing by ones" for fair sharing 32 crayons among eight friends. As discussed above only PT8 utilized "composite unit" strategy. In the whole class discussion, the PTs selected the similar strategies and described their similarities. They all indicated either a student could deal one object or groups of objects at a time. PT8 also suggested that the students could deal in an order. At the end of this interaction, I introduced the formal descriptions of the strategies utilized and described by the PTs. The PTs were introduced other possible students' mathematical strategies for equipartitioning collections as (i) unsystematic dealing by ones and creating even shares, (ii) systematic dealing by ones and creating even shares, and (iii) utilizing composite unit and creating even shares.

PTs focused on the strategies that were systematic and yielded only even shares until this stage of the teaching session. However, the students could also employ the strategies that were unsystematic and might or might not create fair shares. None of the PTs predicted these strategies. This was a general tendency of the PTs observed in the first week of the LTBI teaching experiment. The PTs did not anticipate that a student might generate an incorrect solution and these solutions could be an effective tool for further mathematical discussion. To guide the PTs to become aware of possible incorrect strategies, I started a conversation as follows:

> T: Ok, all of you provided the ways that yielded fair shares. Do you think that this will be the general response trend of your students in an elementary school classroom?
> PTs: No
> T: Why?
> PTs: [Thinking, no response]

PTs could not completely anticipate the way an elementary school student might be engaged with fair sharing collection tasks. I prompted the PTs to think on different strategies by suggesting scenarios of students' solution ways:

T: Ok, if a student gives me one [object], gives you [pointing a PT] two, and gives you four. Is this a systematic way of dealing?
PT9: Yes, It is systematic.
PT1, PT4 and PT7: It is not systematic.
T: For those who said systematic, why? And those for who said unsystematic, why?
PT2: Actually, it is systematic since student deals the objects to all receivers.
PT4: It is not systematic since student did not give equal amount to each.
PT8: For instance, one can deal the objects increasing by ones. Gives one [object] to first person, two objects to second person and three objects to third person. Then reverse the order. One can create their own systematic way.
T: What if a student deals randomly?
PT3, PT5, PT7 and PT9: It is not systematic.
T: Why?
PT3: They did not deal the objects in a predictable pattern.
PT6: Yes, in a same way.
T: Ok, is there any way a student starts fair sharing unsystematically and ends up with fair shares?
All the PTs: Yes. They can count and make them fair at the end.

The discussion above showed that the PTs elaborated on the meaning of the systematics versus unsystematic dealing. In the prior classification, the characteristics of both dealing strategies were assumed as explicit. However, PT2 and PT4 did not conceptualize differences between systematic and unsystematic strategies correctly. The discussion on the difference between systematic and unsystematic dealing on the presented scenarios helped PTs to understand the distinction between these strategies. This situation also was an evidence for restructuring SCK since the PTs made sense of several mathematical strategies for the fair sharing collection ideas embedded in the ELT. Also, PTs started to anticipate students' mathematical thinking correctly as a result of identifying students' mathematical strategies in the presented cases. In addition, they learned a new strategy of students to create fair shares called unsystematic dealing.

I carried the discussion into a further context of focusing on incorrect strategies by asking whether the students always employed the strategies that created fair shares. PTs agreed on a common conclusion that they did not. PT2 said, "*The student can confuse the order when she gives objects to people.*" PT8 agreed with her and stated, "*At the end, the children should show that they fairly shared especially in unsystematic dealing*." These responses of the PTs leaded a new area of discussion on how elementary school children can make sure that they created fair shares. Before discussing the justifications ways, I asked the PTs to order these mathematical strategies from the least to the most complex:

> T: Ok, what might be the most complex strategy?
> PT9: Systematic one.
> T: Which of the strategies were systematic?
> PT9: 1-1 correspondence.
> PT3: Dealing by twos.
> T: Ok. Which one is the most complex?
> The PTs all together: Dealing by twos.
> T: Why?
> PT9: They learn counting by ones at first (other PTs approved through nodding heads).
> PT6: Counting by ones is the easy.
> PT7: You have to form group in the other strategy [referring dealing by twos].
> T: What did we call this strategy?
> PT3, PT8 and PT9: Composite unit.

The discussion above showed that PT6 and PT9 utilized their HCK to support their claim about the order. They related the complexity of each strategy with the students' counting skills. At the end, the PTs ordered the strategies from the least to most complex as addressed in the ELT. All the PTs came up with the following order of the possible strategies for the least and the most complex: the least complex strategies as unsystematic dealing and creating unfair shares, unsystematic dealing and creating fair shares, and systematically dealing by 1's (1-1 correspondence), and creating fair shares; and the most complex strategies as composite unit strategy and creating fair shares.

After discussing various dealing strategies, PT3 commented, "*different from students, we always use short cut operations as divisions. I would not show the other ways in my teaching until now. Since, anyone specifically asked for it before.*" The rest of the PTs also showed their agreement with their friends either with similar comments or mimics. I completed her comment and said "*to develop an understanding of the ultimate complex mathematical ideas, we should consider these foundation strategies.*" PT3's comment indicated that she realized students' way of mathematical learning which were different than her perspectives. Her experience in LTBI helped her to realize that distinction and the connection between her mathematics and students' mathematics. This is an indication of distinguishing practice for restructuring SK.

5.1.2 Restructuring Mathematical Content Knowledge

In this week, none of the PTs challenged the presented information within the LTBI teaching experiment. Also, none of the PTs exhibited a misconception related to equipartition discrete collections. The PTs expanded their prior knowledge through learning several strategies of students for distributing and justification ways for the fair shares.

In this week, students' possible incorrect strategies or responses generated a new discussion on how PTs could size up students' errors while dealing with fair sharing collection tasks. For this purpose, the task asked the ways elementary school student might justify their solutions. To search answer for this question, I turned back to prior representations of the PTs (see figures 40 and 41 above). These representations were utilized as a tool to elaborate on how PTs could guide their students to justify their fair shares.

I initially selected PT6 and PT9's representation of stacking objects. The conversation took place as follows:

> T: Why did you put the objects on top of each other?
> PT9: We showed they all were equal height.
> T: What is the advantage of showing "they all were equal heights"?
> PT9: They all received equal amounts. Same height convinced visually that all received the same amount. The same height showed that this is a fair share.
> PT6: They are both physically and numerically same.

This conversation showed that this representation could be useful to visually justify the fair shares. All the PTs indicated that this could be a way for students to justify their fair shares. As a result, making sense of a particular representation helped the PTs to learn a justification way called: visual height comparison. This evidence was also coded under restructuring SCK since the PTs realized that different representations conveyed different mathematical meanings. This discussion on visual height comparison strategy started a new discussion in which the PTs elaborated on the meaning of the collection concept:

> PT3: Ok, I said the same thing. For instance, you asked for fair sharing three objects and you had two different erasers and one pencil. How did the child compare?
> PT4: Yes, the height comparison did not work for this.
> T: Ok, if the height comparison did not work for those situations, what did we call something as "collection"?
> PT3 and other PTs: The objects should all have same properties.

The discussion above showed that, although PTs fairly shared the given collections in the task one, they could not put specific thoughts on "What is a collection?" until they realized height comparison can be a way of justification of the fair shares. This discussion on the meanings of the collection seemed to result in confusion in PT's ideas of "collection" concept where they clarified the concept by determining the characteristics of "a collection". They concluded that all the objects in a collection should be identical. PT3's leading question helped the other PTs to internalize the collection concept embedded in ELT.

After the discussion, I directed the attention back to the representation types:

> T: Are there any differences or similarities between array representation and stacking?
> PT2: Yes, there are. In our representation [showed the array representation, see Figure 41 above] children thought in a simple way and could see easily [showed the objects within each group one by one].
> PT4: In here [showed the array], student saw the number of objects explicitly [showed each friend's share]. Yet, in the stacking representation, student could not see that number explicitly and might perceive all objects in a stack as a one piece.
> PT9: Yes, the number of objects in each group is more explicit in the array representation. This is one by one [showed the array].

The discussion above indicated that the PTs were aware that each representation communicated different mathematical meanings. The first representation (stacking) communicated height comparison (measurement) as a way of justification, and the second representation communicated count (number) as a way of justification. At the end, all the PTs indicated that utilization of these justification strategies had the potential to address students' incorrect strategies.

At the beginning of the week, all the PTs asserted that they knew the fair sharing concepts and could fairly share the discrete collections. Then, I asked them to define fair sharing or equipartitioning concepts and the PTs constructed some informal definitions. They stated that fair sharing means, "*getting same amount of something.*" This definition showed that the PTs could not fully internalize the required criteria of equipartitioning (fair sharing) although they could fairly share. After the PTs engaged with the first task (fair sharing 32 crayons among 8 children) they made an in-depth examination of the concept equipartitioning. A part of the discussion took place is as follows:

> T: Up to now, we examined fair sharing and its strategies. To be an equipartition, what characteristics a sharing should have?
> PTs: [waited]
> T: In order to say an action is an equipartition, what are necessary conditions?
> PT8: At the end, each group should have the same number of elements in it. Each set should have the same number of elements.
> T: Ok. Anything else?
> PTs: Thinking [no response for a while approximately 1.5 minute]
> T: Ok. Let's think on this situation; I have eight objects and want to fairly share these objects between two people. I give two objects for each and leave the rest.
> PT9: Oo, all needs to be consumed.
> PT2: We should finish them all [at the same time with PT9's comment].
> Other PTs: [Approved]

I wrote the two criteria on the board: Exhaust the whole and each groups has equal amounts. Then, the discussion continued as follows:

> T: Now, what do you think about these two criteria, are they enough to say that something is equipartitioned?
> PT8: Yes.
> PT6 and PT5: No.
> PT9: No. I would check whether I deal to each group. We should create the right number of the group.
> PT5: [I] check the number of the group [At the same time with PT9].

I wrote the third criteria on the board as creating correct number of groups.

The discussion above showed that only one criteria of equipartitioning came into the PTs minds immediately. Then, my inputs helped them to realize two other criteria of the equipartitioning. In this restructuring process, PT2, PT5, PT6 and PT9 more explicitly expressed their mathematical thoughts. They reflected on their experience in the teaching session and evaluated the case of “I have 8 objects and fairly share these objects between 2 people. I give 2 objects for each and leave the rest” that I presented in the light of this experience.

In the first and the second tasks, the PTs were also asked to examine the relations between the part and the whole. All the PTs connected the initial fair sharing situation with the division concept and the reverse action called *reassembly* with multiplication concept:

> T: Based on your experiences in first two tasks, which mathematical topics equipartitioning collections lay a foundation for?
> PT9: I think division and multiplication [PT1, PT6 and PT8 responded at the same time].
> T: Why do you think so?
> PT6: Yes I agree, we divide whole collection to find each friend’s share.
> PT9: In task 1, we had a whole collection of 32 pencils, and we shared this among eight friends. To find each friend’s share, I can use division. 32 divided by eight equals to four. So each friend has four pencils.
> PT8: Division gives us the result of fair sharing [action].
> PT1: Yes, it gives number of the objects in each group.
> PT3: We used grouping in fair sharing.
> T: Yes, we called this division “partitive division”. In this division, one knows how many groups to be created and find the number of object in each group.

The conversation above indicated that PTs connected the equipartitioning collections with partitive division and they were aware of this division would answer how many in one group (one person’s share). The conversation continued as follows:

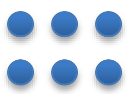
> T: What about the second task?
> PT3: Reverse of the first.
> T: What do you mean by saying “reverse”?
> PT3 [along with similar comments from whole class]: Reverse of the division, [that is] multiplication.
> PT8: Multiplication is the inverse operation of the division.

T: Ok. How do you all relate second task with multiplication?
PT6: To find number or the objects in the original collection, I multiplied four pencils and 8 friends.
T: Any other comments?
PT9: In the first task, division gives me the number of object in each group. In the second task, I know the number of pencil each friend had, and know the number of friends. Thus, I multiplied these numbers to find the total.

The conversation above showed that all the PTs agreed that multiplication was the reverse operation of division as reassembly was the reverse action of equipartitioning. All the PTs utilized a verbal mathematical language in some extent. Among these PTs, PT8 and PT9 utilized more precise and correct mathematical language to indicate that reassembly was the reverse action of fair sharing and indicate how these equipartitioning ideas were connected to multiplication and division. This an indication of the PTs connection equipartitioning related ideas with further mathematical topics as a part of enhancing their HCK. Utilization of more accurate mathematical terminology is also an indication for restructuring their existed CCK.

Another restructuring of HCK practice was detected when the whole class discussed array representation of the fair shares. PT1 and PT6 stated, "*Array representation of the fair shares of each friend in each task will set a foundation for area concept.*" Then a conversation took place between PTs as follows:

T: [I drew an array representation for 2x3 on the board. Figure below shows the representation]



Which [topic] can this representation lay a foundation for?
PT6: Area.
PT7: How? I did not understand. [PT7 and PT1 discussed together]
PT1: Think, as it is a multiplication. You multiply the number of the unit(s) on the sides.
PT8: Also for multiplication.
T: How?
PT6: Teacher, if we look at the thing [referred the array representation] on the board both horizontally and vertically, we have two multiplications, 2x3 and 3x2, both give same result. [PT8 nodded her head to show her agreement]
PT5: We can explain commutative property of multiplication with this.

PT4: Also the concrete representation of multiplication exists.

The discussions above show that same representation led PT4, PT5, PT6, PT7, and PT8 to come with various connections between array representation of the shares and further mathematics topics. PT8 initiated an idea that this representation served a base for multiplication concept. Then, PT6 explained how multiplication operation was embedded in the representation. PT5 came up with a more specific statement after PT6 indicated that both 2x3 and 3x2 could be deduced from this representation and both operations yielded the same results. This statement of PT6 underlined the 2-dimensional nature of multiplication. PT5 captured this nature and stated that these sorts of representations could be used for teaching commutative property of multiplication. Then, PT4 concluded that this representation could be used as a way to represent multiplication concept concretely. As a result, this representation and thought exchange on the representations helped the PTs to associate rectangular array display with the idea of division and multiplication.

Although PTs did not partition a space and construct an array that was composed of rectangular parts as units in the discussion above, PT1's explanation showed the emergence of this idea. Because, she stated two concepts "*units* [on the] *sides*". This usage indicated that PT1 visualized this array as a rectangular space partitioned into six and perceived each part as a rectangular unit that covered the whole space. This discussion continued in the next week.

The last task was asked finding various "n"s for 36 jellybeans fairly shared among "n" people. In this task, all the PTs indicated different "n"s have to divide 36 without a remainder. PT3 and PT4 explained this deduction in a more precise way. They stated that this task would help students to understand finding factors of a number. I revised the PTs' conclusion and corrected as positive factors. PT6 and PT9 added that this task also served for understanding prime factorization. After that, a generalization question was posed: "Can you fairly share discrete collections for any amount of people? If yes, why? If no, why?" PT1, PT2, PT3, PT5, PT6, PT8, and PT9 wrote "no" as an answer. They all merged a common conclusion that discrete collection could be fairly shared among "n" many people, if n divided the number of objects in the collection evenly. PT9 explained this generalization in a more formal way. She wrote "*If n (amount of object) and "p" (number of people), n* *could be any positive integer greater than 0 and* $n \div p = k$ *and* $k \times p = n$.” This mathematical explanation of PT9 showed both equipartitioning ($n \div p = k$) and reversibility of partitioning ($k \times p = n$). The findings reported above showed that all of the PTs could associate the fair sharing collection tasks with several further mathematical ideas such as, prime factorization idea, multiplication and division idea and fraction.

The second task asked the PTs to compare the size of the each person share to the whole collection and compare the size of the whole collection to one person share in the instance of each eight friends owned nine Legos and wanted to build a city plan through utilizing all Legos. Many PTs (n = 6) named each person share as $\frac{1}{8}$ of the whole collection. PT1 and PT8 named each person share by first finding the total number of Legos as 8x9=72, then naming each person’s share as $\frac{9}{72}$. PT2 named each person’s share as one out of eight. All the PTs indicated the relation between the whole collection size and the each share size as 8 times. This showed that the PTs perceived the multiplicative relation between the size of the whole compared to the size of the part.

5.2 Week 3

In this week, equipartitioning single rectangular and circular whole was the content of the experiment. Various partitioning and justification strategies, naming practices, and transitivity argument (property of equality of equipartitioning) were covered. First, the findings related to restructuring MCK, then restructuring SK were reported. This order of reporting was based on the logical and chronological flow of the LTBI that restructuring MCK was followed by the tasks in the experiment focusing on SK.

5.2.1 Restructuring Mathematical Content Knowledge

The first task required PTs to utilize several fair sharing strategies to create different number of splits on a rectangular and circular whole. At first, the PTs created four fair shares on a given rectangle. They utilized different fair sharing strategies such as utilizing successive parallel or vertical cuts, utilizing vertical and parallel cut together to form composition of splits, and utilizing diagonal cuts. Although the PTs utilized these strategies, the way they employed the cuts differed in terms of mathematical complexity. PT2, PT4 and PT6 utilized successive parallel or vertical cuts however, they utilized a visual approximation to determine the size of the shares as they located the cuts.

Three PTs utilized folding to locate the cuts. They first folded the rectangles to create four fair shares. Two PTs used repeated halving strategy to create four fair shares instead of using successive parallel cuts. One PT utilized measurement strategy to mark the long side of the rectangle. After the PTs completed their individual works, I selected several strategies of the PTs and asked them to group the similar ones. Based on PTs comments, I grouped the rectangles on the board as shown in Figure 42.

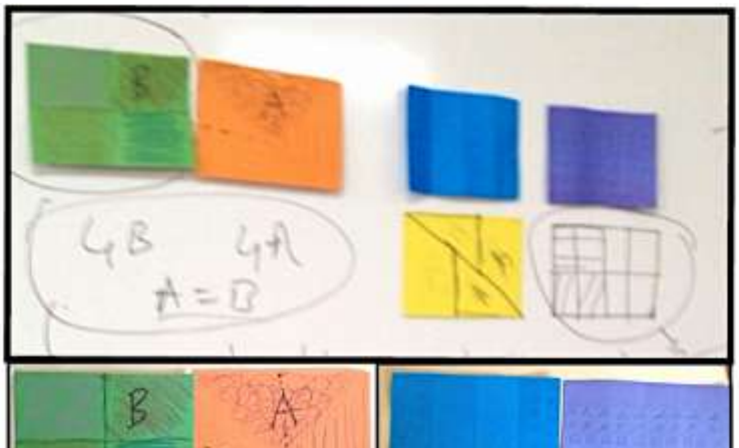


Figure 42. The PTs' various splitting strategies

The PTs discussed whether all the rectangles on the board were equipartitioned into four. PT1, PT2, PT3, PT4 and PT7 believed the orange rectangle was not split into four evenly. The discussion took place as:

T: What do you think about this group? [Pointed the green rectangle and orange rectangle]
PT7: I think, the green one is more reasonable.
T: Why?
PT7: Because the parts are equal. Yet, the part on the sides and the parts on the bottom of the other one [orange rectangle], are not equal. Because, when you fold the orange rectangle, the parts do not overlap completely. However, when you fold the green one, the parts overlap completely.
PT4: I agree, since this is a rectangle, the angles are not equal [pointed the angles formed in the intersection of diagonal and the angle at the vertex of the triangles formed].
PT5 and PT8: No, the rectangle is fairly shared. The parts are equal [sized].
PT1: Do you think this is fairly shared? [Pointed the orange rectangle partitioned into four with two diagonal cuts]
PT8: Yes, I am sure. When you multiply side and height of the triangles, their areas are equal.

The PTs exhibited a mathematical misconception that the diagonal cuts would not create fair shares. PT7 tried to justify this claim with the folding explanation mentioned above. Also, PT4 only focused on the angle at the vertex of the triangle to decide whether the triangles were congruent. This showed that both PTs tried to justify their mathematical thinking with incorrect mathematical explanations that also pointed the lack of conceptual understanding of area congruence of the shares. PT8's explanation was also supported by PT5 and PT9. PT9 marked one part from both rectangles with the letters A and B as shown in Figure 43.

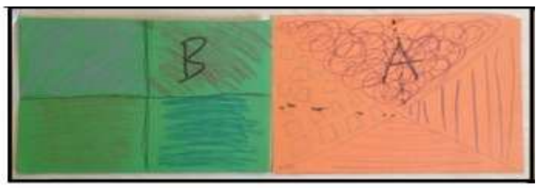


Figure 43. Labeled the parts from both rectangles

PT9 drew dot line that split the part A into half. Then, eight PTs except PT2 indicated that they could cut the part [A] into two equal triangles and rotate one of them and form a rectangle that was exactly the same size as part [B]. This was a composition and decomposition strategy to indicate the congruence of the shares. Also, PT5, PT8 and PT9 indicated the areas of them [part A and B] were equal. In addition to these two strategies to show the equivalence of the shares, PT6 and PT9 indicated that since both rectangular wholes were the same size and they were both equipartitioned into four parts, the parts were equal. PT6 indicated this relation mathematically as "*when you symbolized one part with A and the other one with B. The first rectangle is composed of 4As and the second rectangle is composed of 4Bs. Thus, 4A=4B that is A=B.*" Then, PT5 stated "*This is a very highly complicated [mathematical] thinking level.*" These interactions among the PTs showed that majority of PTs (n = 8) produced several mathematical strategies and explained them correctly within the classroom. This is an indicator for that the PTs internalized transitivity argument embedded in the ELT. This discussion served for making sense of the underlying mathematical explanations in the representations and different fair sharing strategies.

The PTs utilized a correct mathematical language since they emphasized the correct referent whole. They did not merely say both share was $\frac{1}{4}$. They indicated that the same size wholes equipartitioned into four. This showed that the PTs had a good understanding of the quantity meaning of the fraction. At the end, all of the PTs explained the area congruence of the shares through utilizing the same strategies. This showed that the transfer of the learned mathematical ideas was successfully achieved. As a result, the five PTs remediated their incorrect mathematical understanding and also shifted their way of mathematical thinking. These PTs reacted as "*Yes, areas are equal, how did I make this mistake?*"

The PTs utilized an incomplete mathematical language when they expressed their mathematical thinking. The PTs (n = 3) utilized mathematically incomplete description to refer the equality of the share size. They stated that the parts were equal. However, they did not supply an answer for which quality of the parts were equal. I asked PTs in the follow-up discussion about what they referred to when they said "equal". Four PTs stated "area" and the rest said, "size". This showed that although the PTs employed a correct mathematical thinking, they still needed support to utilize correct mathematical language to communicate their mathematical thought aloud and I tried to provide this support to help them communicate their thoughts in the discussions.

Another discussion on the purple and blue rectangle pair followed the examination of the first pair of rectangles in class:

> T: Can you explain how you shared the rectangle? [Referring the purple one]
> PT6: I split the rectangle into four horizontally.
> T: Can you explain the process of the splitting? Which cut did you draw first?
> PT4: [intervene] Visual approximation. [PT2 also nodded her head]
> PT6: First, I drew the cut on the top. Then, I drew the cut underneath of that cut based on my visual approximation of the share size.
> T: Is there anyone who performed this differently?
> PT5 and PT7: With folding.
> PT9: I drew the first cut from the middle [of the rectangle].
> PT1: First, I split the rectangle into half, then I split the halves into two again.
> PT5: The shares are not fair enough with visual approximation.

The discussion above indicated the three PTs utilized a visual approximation to adjust the location of the cuts for maintaining the size of the shares. The other PTs claimed that this strategy was ineffective to create fair shares. Because, the visual approximation did not ensure the equality of the share size. This examination of the mathematical strategies and their representations helped the PTs to understand the pitfalls of their employed strategy of visual approximation. PT1, PT5, PT7 and PT9 did not merely point out the pitfalls of the visual approximation; they also suggested alternative strategies of repeated halving and folding into half twice. They claimed these strategies would produce more precise fair shares. They employed the strategies correctly and could explain why they were appropriate to use in this particular problem.

The three PTs who utilized visual approximation became aware of the pitfalls of their strategy and learned new strategies to ensure the equality of the shares. Only PT3 suggested measurement could be used to adjust the location of the shares. She stated, "*We can measure the side length of the rectangle and divide it into four and marked each part.*" This comment of PT3 illuminated the connection between the measurement concept and equipartitioning. After this PT3 input, the rest of the PTs realized this connection.

One main issue aroused when I drew a rectangle that was split into eight and two parts were allocated to each fruit type:

> T: Let’s compare this [pointed the rectangle split into four by 2x2] representation of the fair sharing and this one [pointed the rectangle split into four by 2x4].
> PT9: Fractions.
> T: Which contents of the fractions?
> PT5, PT6 and PT9: Reducing and expanding fractions.
> T: What do we call them?
> PT1 and PT4: Equivalent fractions.

The discussion pointed that my input helped the PTs to see the connection between composition of splits and the equivalent fractions. To clarify whether the PTs actually built a connection among these mathematics topics, I asked the PTs to name each share. PTs (n = 8) named each share as $\frac{1}{4}$ and $\frac{2}{8}$. Then, they indicated these fractions represented the same amount. As a result, these fractions were equivalent. PT3 also stated that this equipartitioning tasks could be utilized to represent the idea of equivalent fractions concretely.

When PTs were asked to equipartition a rectangular whole into eight and later in fifteen, they made use of the same strategies. They utilized combinations of vertical, horizontal and diagonal cuts to create eight fair shares. To create eight fair parts, many PTs (n = 5) used repeated halving. Three PTs marked the long side of the rectangle seven times to create eight fair shares. Only PT2 used visual approximation in the process of splitting. These findings demonstrated that eight PTs improved their inefficient strategy of fair sharing task they exhibited before.

At the end of splitting into four and eight tasks, the PTs compared the mathematics behind each task. PT8 stated since the whole remained the same, the amount of the share reduced. PT7 added, “*When you split into four, you created four parts. When you split into eight, since the number of parts increased, the share size reduced*” These comments indicated that these PTs started to develop an understanding of factor based change. This showed that the PTs started to build connection across the mathematical ideas embedded in the ELT.

Another connecting practice was observed when PT1, PT5, PT6, PT7 and PT9 employed composition of splits to create eight fair shares. They indicated that they used factors of eight while employing composition of splits strategy. Then, I asked "*Think of a case that you want to create 12 equal parts, how would you do this sharing?*" Same PTs stated, "*It could be 6x2, 12x1 and 4x3*." After these comments, all the PTs indicated factors of the number were another mathematics topics related to these tasks. I corrected the PTs' mathematical wording by saying "the positive factors".

When the PTs were asked to create 15 equal parts on the rectangle. They had difficulty with creating equal sized shares. For instance, PT2 and PT3 stated that it was difficult for them. Then, the PTs developed arguments about why equipartitioning into 15 parts was harder than creating the prior splits of four and eight. PT6 said, "*This means as the magnitude of the numbers increases and the size of the shares decreases, the task becomes harder.*" PT9 argued against this response and said, "*No, it is because of odd numbers. We could not employ repeated halving*." PT4 later added:

> When we fold this [rectangular] paper, in each time [the number of parts] increases by two. Thus, we create equal shares however, to create 15 equal shares, this is not the case. We should fold every time to create each part. When we fold into half [she showed this action on the rectangle], the parts overlapped. Thus, we are sure they are equal [sized].

After the discussion, the PTs realized why they experienced difficulty with creating fifteen fair shares. They eventually reached a common conclusion as PT7 stated, "*There is a serious difference between creating 2-splits and 3-splits in terms of [mathematical] difficulty. Thus, we should start from even splits then move into odd splits*."

In addition to the findings on fair sharing strategies, the PTs had discussion on naming practices. All PTs could correctly name each share. However, they exhibited different perceptions about the meaning of the fractional naming such as $\frac{1}{15}$. The interaction went as follows:

PT6: The number at the top [numerator] represents the single whole.
T: Does anyone argue differently?
PT1: For one part, we named as $\frac{1}{15}$. Thus, this [number 1 in the numerator] represents one of the parts.
T: So, this one in the numerator represents the whole or not?
PT1: 15 is denominator and 1 is the numerator, so 1 represents the share.
PT6: We split the whole into 15, we did not split the 15 into one.
PT8: But, we received one out of 15 [parts].
PT5: Yes, one out of fifteen.
PT8: What if it is two-fifteenth?
PT6: But, in here my statement is also correct since we split a whole into 15.
PT7: In the elementary school teaching, teachers also teach this to the students as comparing one part to the total number of parts.
PT6: We multiply a number with $\frac{1}{15}$ means dividing the whole into 15. However, if we take one of the parts, we received $\frac{1}{15}$ [of the whole].
PT1: Let say $\frac{3}{15}$, so in this instance do we split 3 wholes into 15?

The discussion between the PTs (n = 5) showed that the PTs interpreted meaning of the fraction differently. PT1, PT5, PT7 and PT8 knew the part-whole meaning of the fraction (Lamon, 1999). PT6 knew two meanings; the division and part-whole. PT1's last question also pointed the division meaning of the fraction if multiple wholes were fairly shared among 15 receivers. PT7's comment also indicated how teachers, that they observed, also focused on merely one meaning of the fraction. After this discussion, various meanings of the fraction including part-whole, division, ratio and operator were briefly introduced and discussed. However, I allocated less time on the ratio and division meaning since these would be the focus of the following weeks. At the end, the PTs agreed that fraction had multiple meanings in various settings.

After working on the equipartitioning rectangular single whole tasks, the PTs were asked to work on the circular singular whole and create the same number of splits respectively 4, 6, 12 and 15 splits. All of the PTs could split the circular paper into four fairly. However, PT3, PT4 and PT8 had difficulty with creating odd number of splits on the given circular paper. They had difficulty to locate the cuts and adjust their degrees to create equal sized parts on the circle. For instance, PT3 tried to use repeated halving strategy on the circle, as she utilized in the fair sharing a rectangle task. Then, she realized, this strategy did not work in the circle when creating 15 splits. Next, she erased her cuts and created three equal parts and tried to fairly share each one third. As she split each third, she counted how many parts she created. This showed that PT3 did not think of the positive factors of 15 as she created split of splits. Based on this, a discussion took place on split of splits strategy:

> T: Why do you split the circle differently?
> PT6: Their factors, teacher. For instance, to create six parts we utilized the factors of six; two and three. We can do the same for 10 as two and five.
> T: Alright, can we reverse this splitting action?
> PT4 and PT6: Yes.
> T: How?
> PT3: We can first split into five and split each part into two again.
> T: What does this show to us mathematically?
> PT5 and PT8: 2x5 and 5x2 give us the same number.
> PT1, PT6 and PT8: It is commutative property of multiplication.

The discussion above helped PT3 to restructure her thoughts through shifting her way of operating split of splits strategy. She was able to decide how many parts should be created in each part after the discussion. At the end, she first split the circle into three and split each third into five evenly without counting the number of parts formed. In addition, the PTs realized that the orders of the factors while creating the cuts would not affect the outcome. Three PTs connected this idea with commutative property of multiplication and area. PT6, PT8 and PT9 realized the multiplication of the factors gave the number of the parts in the whole and these parts were the unit to find the area.

Other connecting practice was captured when the PTs engaged with partitioning a circle into 12 evenly.

> T: What are the similarities and differences between fair sharing a circle into six and 12?
> PT1: A person who knows to split in six can also split into 12.
> PT5: I agree. Because, a whole equipartitioned into 12 could be fairly shared among six people.
> T: Good, can we connect PT1’s and PT5’s statements with further mathematical topic(s)?
> PT5: Teacher, when we fairly shared a rectangle, we did not come up with this strategy but you drew and asked us. Similarly, if a student

equipartitioned [a circle] into 12, [the student] could give 2 parts at a time to each person.
PT9: It is like composite unit [strategy].
PT8: Teacher, we could teach equivalent fractions.
PT6: Yes yes, $\frac{1}{6}$ and $\frac{2}{12}$ are equivalent fractions.
Rest of the PTs: [agreed with their friend's statement]

The discussion above helped PTs to reach a common conclusion that equivalent fractions ideas inherently embedded in these tasks. PT5 started with a verbal description of the idea of equivalent fractions. Her comment triggered other thoughts, PT8 and PT6 named the thought initiated by PT5's mathematically. At the end, the collaborative effort of these PTs helped other PTs to connect the fair sharing single whole tasks with equivalent fractions. They supported their claims with their prior experience on fair sharing a rectangle in the experiment.

After finishing employing different number of cuts on the circular whole, I asked the PTs to discuss the difficulties that they experienced as they tried to create the fair shares. PT7 and PT8 indicated that without measuring the angles, it was difficult to locate the cuts onto exact position in the instance of creating odd splits on a circle. Other PTs also shared their experiences of creating into odd versus even number of splits as follows:

PT2 and PT3: 15 [split] is so difficult, teacher.
PT7: I agree.
PT1: Actually it is not so difficult. If you can split into three, you can also split into 15 evenly.
T: Ok, good point. What about the rest who found creating 15 splits harder?
PT4: When we created even splits, we directly cut [the circle] into half. Then, we split the parts [each half] again. But, in here [15-splits] we did not do this.
PT3: I halved the circle, then I tried to create 15 parts starting from there. Yet, I could not. Then I erased my initial cut and split the circle into three.
Rest of the PTs: Yes, we agree.
PT9: Teacher, initially it was really difficult to split the circle into five evenly. For instance, I first created three equal parts. Then, split each part into five again.
T: How many of you experienced the same difficulty?
All PTs: [They all raised hand or verbally agreed.]

The discussion above showed that majority of the PTs (n = 8) had difficulty with creating odd number of splits. They realized creating three equal parts on a circle was easier than creating five equal parts. The PTs also realized creating two and four splits were the easiest ones. These findings showed two important results. First, the PTs experienced the mathematical task difficulty exactly in the same order that was suggested in ELT. In addition, the PTs identified the possible reasons behind their own mathematical difficulty.

5.2.2 Restructuring Student Knowledge

The PTs were asked to anticipate various equipartitioning strategies of the students and they were able to produce both correct and incorrect strategies. Only PT9 predicted chopping strategy of the students in which students randomly created cuts without paying attention to the three criteria of the equipartitioning. Rest of the PTs indicated their agreement with her. However, none of them came up with this thought until PT9 suggested it. PT3, PT5 and PT8 anticipated a student misconception that was also coded in the ELT. They anticipated employing n cuts to create n fair parts. For instance, PT5 anticipated this misconception and showed this through drawing it. Then she explained her drawing as "*I drew 15 cuts to create 15 splits [on the rectangle]*." After that, I guided a discussion on generalization for finding the number of parallel or vertical cuts to create fair parts on a rectangular whole:

> T: To create four fair parts, how many vertical or parallel cut should be used?
> PTs: Three.
> T: For five?
> PTs: Four.
> T: What about n parts?
> PTs: n-1,

This conversation showed that the PTs were aware of the correct mathematical way to create the required number of fair parts. Producing a generalizable response for creating n parts one should employ n cuts was an example of practice for HCK that was aroused after examination of possible students' misconceptions.

The PTs also anticipated various incorrect mathematical strategies of the students when they worked on the circle. Figure 44 shows the selected of work of PTs.

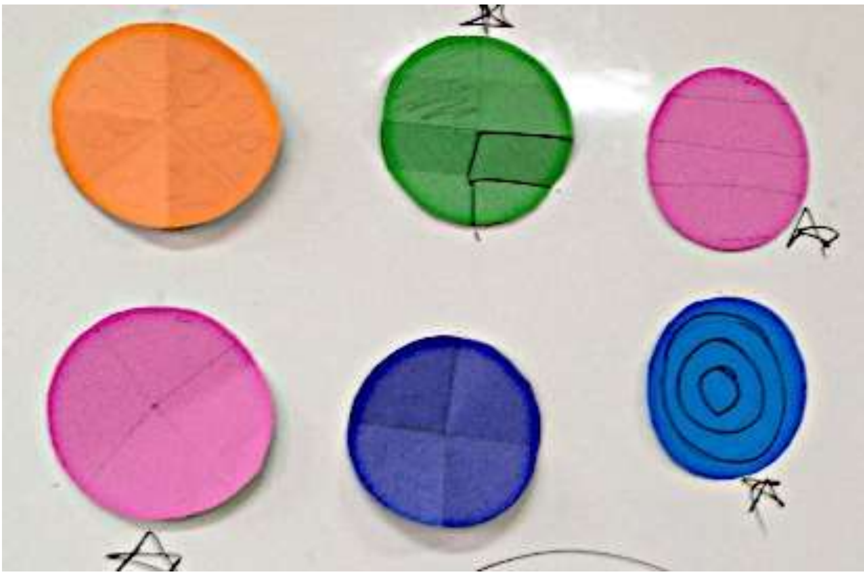


Figure 44. Various splitting strategies on the circle

I asked PTs to decide whether these were all fairly shared or not. All the PTs indicated the ones marked with star were not fairly shared. The discussion on why they were not fairly shared took place as the following:

> T: Why do you think these are not fairly shared?
> PT4 and PT5: They do not have middle points.
> PT1: Mine [the green one in the middle] has that.
> T: Ok,... what is the mathematical misconception or error under these?
> PT1: The children utilized repeated having strategy in the way that they employed on the rectangle.
> T: What about the pink one in the first row?
> PT3: The student employed parallel cut as s/he did in the rectangles.

Although PT3 did not anticipate the parallel cut misconception when she worked on the given task alone, she recognized the misconception and explained why a student might employ this strategy.

I asked the PTs the possible ways to eliminate this parallel cut misconception. PT1, PT4, PT5, PT7 and PT8 indicated that the students could cut the parts and put on top of each other. Thus, they could see that the parts would not completely overlap. PT8 also indicated that the importance of the radial cut should be emphasized for creating fair parts on the circle. She suggested that a teacher should stand and open both arms. Then, the teacher could turn around and form a circle.

At the end of the discussion on the strategies to fairly share both rectangles and circles, the PTs reached a common conclusion about the mathematical complexity of the strategies and they exhibited ordering practice. Although they did not specify such an order at the beginning of the experiment, at the end they stated that fair sharing rectangles was easier than fair sharing circles. Also, they ordered the tasks from easy to difficult as creating two-splits, 2^n splits, odd number of splits, and at last composition of splits based on their experiences in the LTBI. This order was also suggested in the ELT. This pointed that the PTs also experienced the same mathematical difficulty as students did when they created into odd number of splits on single whole. This experience seemed to helped them to empathize the mathematical difficulty that students might encounter as they learned equipartitioned single whole.

After my guiding question, another student misconception was anticipated related to reassembly practices:

T: How many times is the whole larger from the part? How can a student respond to this question?
PT4: 14.
PT1, PT2, PT7, PT8 and PT9: 15 times.
T: Why do you think “14 times”?
PT5: The student did not count one part. Counted the rest of the parts.
PT9: But, this is an incorrect response.
T: Ok. What can you do if a student cannot see the whole is 15 times larger than size of a part?
PT7: The student can combine the parts to create the whole and count how many parts are needed to form the whole again. [She drew 15 parts

separately and an empty whole on a paper.] The student could cut the parts and placed on the whole like a puzzle.

The discussion above showed that PT4 captured possible additive misconception of the students. However, the majority of the PTs (n=5) focused on the correct response. PT9 clearly indicated that 14 was an incorrect response and it was unnecessary to focus on that response. This showed that the PTs mainly focused on correct responses rather than thinking the possibility of incorrect responses that would be produced by the students. However, at the end, all the PTs learned this misconception and also suggested some strategies to remediate it. Furthermore, they started to realize that the anticipation of students' mathematical thinking did not merely entail correct responses.

PT7 suggested a way to remediate this misconception that included iteration, an important measurement idea. PT7 suggested making repeated the measurement of the parts to go back to the original whole. I explained this inherent idea of measurement in PT7's suggestion. My direct input and the interaction among PTs finally led them to restructure the way they anticipated students' mathematics.

At the end of this week, an analysis of 2nd grade student video, in which the student fairly shared a rectangular cake among four friends through utilization of diagonal cuts, was performed in the class. All of the PTs stated that the student utilized the correct way. I paused the video and asked:

T: What do you ask this student next?"
PT3: Yes, I learned the diagonal cut would produce four fair shares. However, I will ask to the students after he employed the cut "How do you know that the rectangle fair shared?"
PT1: Yes, we should check whether he really understands it.
T: Ok, after you asked for justifications, what are the possible responses of the student?
PT2: He cuts the parts in half and puts one on top of the other.
T: What did we call this strategy?
PT7: Composition-decomposition.
T: Any other?
PT6: Teacher, since he is a 2nd grade student, I think he will not use any other way.
PT7: Yes, I don't think he will compare the length of the sides or the thickness of the parts.

The student in the video started to explain his strategy as "*When people cut this way they think the parts will be different*." Then, I paused the video and asked PTs what they understood from this statement:

> PT9: He may want to say the parts are not fairly shared.
> PT1: Yes, but we don't know for sure.
> T: Why do you think so?
> PT1: Teacher, we don't know what different means to him.
> Rest of the PTs: This is very important.

PT1 stated, "*Although we assume the student try to mean something, we cannot be sure until we ask the student*." Rest of the PTs (n = 8) agreed that understanding students' mathematical thinking could be achieved first observing the students' actions and then asking them directly what they meant or what they did.

Different from the PTs anticipation, the student cut each $\frac{1}{4}$ out and combined the ones congruent to each other in terms of shape and indicated both forms a parallelogram and they were fair. Many PTs stated that they would never think in this way. This showed that although the PTs could anticipate the students' strategy, there is always a possibility for encountering a different strategy when they worked with students. Thus, differentiating their own way of mathematical thinking from the students was a key practice. In this video analysis activity, all of the PTs realized that the student could think differently than what they learned or anticipated so far about students' mathematics. This realization was evidence for that all of the PTs could recognize the student's justification strategy even if they did not anticipate it in advance. The discussion on this issue continued when I paused the video and asked:

> T: Ok, he used a different composition and decomposition strategy from what you anticipated. What do you want to ask to the student?
> PT8: I will show the two different shaped parts and ask if those are fair?

The PTs (n = 5) wanted to ask similar questions. For instance, PT4 stated "*What about these parts?*" and PT1 "*Is there any other way to show these parts are equal?*" These showed that majority of the PTs started to acquire questioning skills to elicit the student's mathematical thinking.

5.3 Week 4

In this week, two main courses of activities took place. The first activity included a task that asked PTs to order the given tasks in terms of difficulty along with the mathematical justification. The second activity included the folding tasks. The findings related to first course of activities will be reported together under restructuring practices for SK and MCK since the task asked PTs' to anticipate the difficulty of the given tasks for elementary school children. Even if the task primarily assessed the PTs' knowledge of students, the PTs utilized their MCK and restructured it in the progress of this task engagement.

The findings related to second course of activity will be reported in the regular order that was also followed in week 3 and week 5.

5.3.1 Restructuring Student Knowledge and Mathematical Content Knowledge

The PTs (n = 5), including PT1, PT3, PT5, PT7 and PT8, could compare correctly the difficulty level of the tasks that included creating the same number of splits separately on a circular whole and rectangular whole. In addition, these PTs provided correct mathematical justifications for their claims. For instance, the PTs were asked to compare equipartitioning a rectangular and a circular cake into eight parts. These PTs indicated that fair sharing a rectangular cake into eight was easier than fair sharing a circular cake into eight. They all indicated radial cut utilization made fair sharing a circular harder for the students. Also, they stated a student could use repeated halving to create eight equal parts on rectangle. Only PT5 stated students could use folding easily on a rectangle.

Although PT6 indicated splitting a circular cake into eight evenly is harder than the rectangular case, she could not provide a reasonable and correct mathematical justification for her claim and only stated the reason as "*one could draw more cuts while sharing the circle*." PT2 correctly ordered the tasks yet she did not provide any justification.

PT4 and PT9 asserted both tasks were equivalent in terms of difficulty. Both PTs indicated since eight was an even number, creating eight splits on the circle and rectangle could be achieved through repeated halving. PT9 wrote this assertion as:

> One could form eight splits on a rectangle through composition of splits four and two. To form eight splits on a circle, one could split the circle into four by drawing diagonals, then one could split each part into half.

I opened these PTs' responses to the discussion. Other PTs indicated that for some students this might be the case. However, for the students who did not employ radial cut before, fair sharing the circle was still harder than fair sharing the rectangle. These findings related to first question of the first task revealed that majority of PTs could anticipate students' mathematical thinking complexity while engaging the tasks. They used this anticipation to justify their task difficulty order. Also, PTs (n = 5) could order the tasks in terms of difficulty considering an elementary school student mathematical thinking, not their own mathematical thinking. Their conjectured order was consistent with the order suggested by the ELT. This was an evidence of these PTs could distinguish their own mathematical thinking from the students' thinking. However, the two PTs challenged the suggested task difficulty order in the task. PT4 and PT9's challenge was approved at a certain degree among the rest of the PTs.

Many PTs correctly compared the difficulty level of the tasks that included creating odd versus even number of splits separately on a circular whole or rectangular whole. The first problem asked PTs to order fairly sharing a rectangular cake among four versus five people cases. Six of the PTs including, PT1, PT4, PT6, PT7, PT8 and PT9 ordered the task correctly along with a complete mathematical justification. They all indicated that creating odd number of splits on a rectangular whole was harder than creating 2^n splits. Because, to create four splits, a student can use repeated halving. PT3 and PT5 understood wording of the problems incorrectly since the problems were in English. They confused the meaning of the word rectangular with circular and compared the difficulty level correctly based this understanding. For instance, PT3 indicated that creating odd number of splits on circle was harder than creating even number of splits and that to create four parts on a circle, student could use folding through diagonals. Although PT2 correctly ordered the task difficulty, she did not provide any justification. After classroom discussion, PT2 indicated she also thought in this way. These showed that the PTs could order the tasks that they anticipated for a student from the least difficult to most difficult. Also, they could anticipate what a student could employ to solve each fair sharing case and use this anticipation in their justification.

The second problem asked the PTs to compare difficulty level of the tasks that included fair sharing a circular cake among six people versus five people. Eight PTs indicated creating six splits is the easy one. The common conclusion deduced from these eight PTs responses was, although creating six splits on the circular cake required composition of splits of two and three, fairly sharing a circle into half was an easy task and fairly sharing each half into third was also easier than directly creating five splits on the circle. This finding showed a contradictory conclusion with the knowledge embedded in the ELT. Since, the ELT suggested the order of the tasks from the least to the most difficult ones should be half, 2^n-splits, odd splits then composition of splits. In here, all of the PTs challenged the existing knowledge of ELT with a reasonable mathematical explanation.

Only PT2 came up with an incorrect response. She stated "*because the number six is greater than number five, creating six fair parts is harder than the other one.*" I opened this explanation for discussion. The rest of the PTs explained their above-mentioned reasoning for how they decided to the order. Also, some of them (n = 3) gave contradictory examples such as creating seven versus eight splits. PT3 indicated creating eight splits on the circle or rectangle could be achieved by repeated halving, on the other hand creating seven splits evenly was very difficult. Although seven was smaller than eight, creating seven fair shares was more difficult than creating eight fair shares. After these kinds of examples, when I examined her returned written work, I saw that she made a note under her work stating "*Look at whether the number is even or odd. Also, look at the numbers when you multiply them that gives you the number you shared for*." This written note of the PT2 showed, she revised her incorrect response and shifted her mathematical thinking.

The last question was related to comparing difficulty level of composition of splits to creating odd number of splits on either a rectangular or a circular whole.

The question asked for fairly sharing a rectangular cake among five versus 15 people. Except PT3, all PTs indicated that creating 15 fair shares was more difficult than the other case. Common points deduced from their justifications indicated that one should know creating five and three splits to create 15 equal parts. PT3 gave an interesting response: "*We could split 3x5. This is easier for splitting into 15. Since, children played games such as XOX and SOS and they employed this splits frequently. [This is not] difficult for them]*." Although, this PT grounded her claim to children's real life experience in which the children employed composition of splits strategy to create 15 parts, she was not aware of the fact that to create 15 equal parts, children had to know how to create 5 equal parts. The composition splits of three and five formed 15 equal parts and the task asked to compare five splits versus 15 splits on the rectangle. These findings indicated majority of PTs (n = 8) could order the tasks correctly and they produced complete mathematical explanations for conjectured orders. This showed that PTs could predict the possible learning paths of elementary school children in the equipartitioning single whole topic. Based on this, they decided in which order instructional tasks should be presented to the students.

5.3.2 Restructuring Mathematical Content Knowledge: Folding Activities

The PTs were asked to fold a rectangular paper into half four times and find the number of the fair shares created as a result of folding. More than half the PTs (n = 5) had difficulty with processing the task. They could not generate a procedure that would directly produce the solution. Thus, they asked whether they could use the rectangular paper to show. This introduction to the basic folding task indicated that many PTs (n=5) displayed a mathematical difficulty in building the mathematical relation between the number of folds and the number of fair shares created. As a result, they could not solve the problem mathematically in the first place. Therefore, I asked them not to try folding the given rectangular paper first, but to try to imagine the folding action abstractly and tie to mathematics behind each folding action with equipartitioning of the single whole.

PTs started to work on the task individually after my suggestion. Then, each worked to form his or her own strategy. Within these strategies, PT8 and PT9 utilized exponential numbers and supported their claim with pictorial representation. PT7 utilized fractions along with the pictorial representation. PT1, PT5 and PT6 merely used drawings to show the result of each folding. PT2 and PT3 exhibited mathematical misconceptions and PT4 could not come up with a strategy. PT2 exhibited an additive misconception. She drew a picture for each fold. Figure 45 shows her drawings.

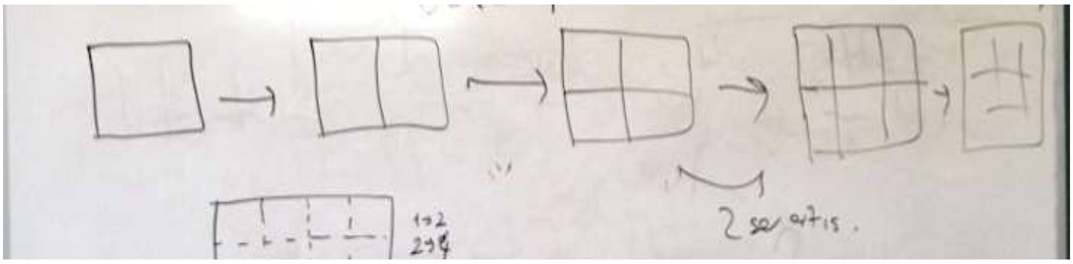


Figure 45. Additive misconception exhibited in folding task

She explained her strategy as she drew it on the board:

> PT2: First, I folded the rectangle into half (the second drawing from the left in the Figure 45). Then, I folded this half-folded rectangle into half again. [She drew a rectangle partitioned into fourths]. In the next stage, six parts were created. Then, eight parts were created.
> T: Can you explain how you created six [parts]?
> PT2: I thought that it [number of the parts] increases by two. Then, two- four then six and then eight.

After this conversation, some of the PTs (n = 4) argued against her explanation and solution strategy. For instance, PT7 said, "*I perceived it not additively, instead it increases multiplicatively. So, two times each time*." Then, I asked if anyone used the drawing strategy as PT2 employed and find a different answer. PT5, PT6 and PT9 responded. PT5 first showed result of each folding by drawing and also she folded concrete rectangle into half three times. After both representation, PT2 was convinced and she said, "*Yes it did create 16 equal parts*." At that point, PT3 said, "*I thought, every time I folded, 2 equal parts were created. When I folded once, it created 2 equal parts. Then, I asked if I fold four times, how many equal parts I will create. So, I created eight equal parts.*" Figure 46 shows her strategy.

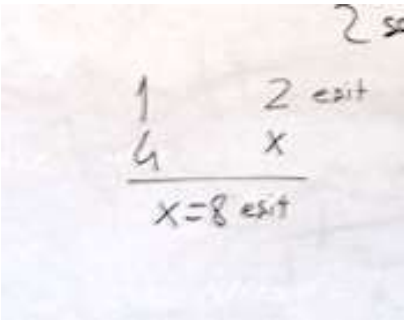


Figure 46. Setting direct proportion to find the number of parts created as a result of folding

Based on the discussion on the strategies of others, PT3 started to realize that there was something wrong in her mathematical strategy, yet she could not explain why and asked, "*What is wrong with my strategy?*" Then, I drew a table on the board that included the suggested the equal number of parts created as a result of each folding actions. The numbers in the parentheses represented the correct responses that the other PTs provided. I encourage PTs to get help from this representation to explain the problem in PT3's strategy.

Number of Fold	Equal Parts Created
1	2
2	4
3	6 (8)
4	8 (16)

The PTs stated there was multiplicative relation between the number of the parts created and each repeated folding action as multiplication by two. Thus, in each step one should multiply the number of parts by two to find the new resultant number of parts. PT8 and PT9 also stated the relation could be represented by exponentials. They said the relation was not between the order number of the folds and the number of parts created. PT9 said "*1x2=2, 2x2=4 but, 3x2=6 does not produce the correct answer. Only the number of parts increases multiplicatively by two as a result of each folding*." After PT9's explanation PT3 indicated that she understood why her strategy did not work. She stated, "*I set a proportion between the number of fold in order and the number of parts, instead I should have focused on the results of each folding into half.*"

After these discussions, I asked for other strategies. PT8 and PT9 stated that they used exponential numbers. PT8 explained her strategy. Figure 47 shows PT8's work.

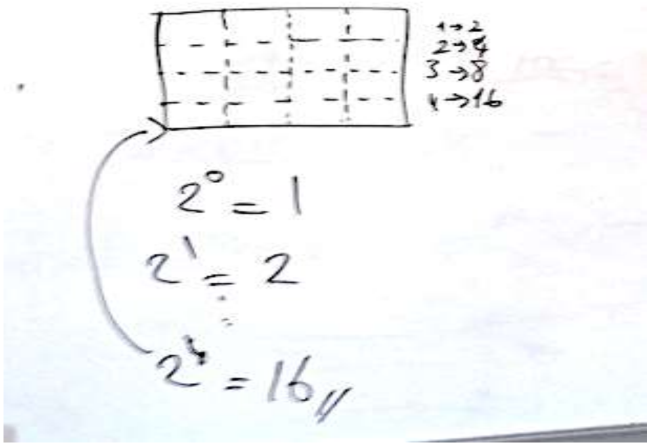


Figure 47. Relating results of the folding action with exponentials

In the figure above, the PT explained the strategy as: "*For the first step, I took* 2^0 *since there is no folding [she drew a rectangle on the board]. Then, I folded once into half, this is* 2^1*. So, it created 2 equal parts. [Then she drew the results of each* *fold on the rectangle]. Thus, through 2^4, it created 16 equal parts.*" This PT found the resultant number of share by utilizing another mathematics topic exponential numbers. Other PTs also agreed with their friend. PT2, PT3 and PT4 stated that they had never thought in that way before. At this point, PT7 raised her hand and shared her thoughts:

> PT7: I used a similar strategy to PT8's strategy. Mine is the reverse one. [She explained her strategy with mathematical symbols as: $\frac{1}{2}$x$\frac{1}{2}$x$\frac{1}{2}$x$\frac{1}{2}$=$\frac{1}{16}$.]
> T: How many equal parts did you create?
> All PTs: 16 equal parts.
> T: How do you know?
> PT7: $\frac{1}{16}$ represents a person's share. Thus, 16 equal parts were created.
> PT8: So, to represent this [strategy] with exponential numbers, we should say 2^{-1} and then so forth.

This conversation indicated that PT7 also tied her prior knowledge of reassembly to find the total number of parts created. She found one person's share and indicated if one person's share was $\frac{1}{16}$, the whole should be 16 times larger. Thus, she stated 16 parts were created. PT8 also revealed the relation between two strategies. She used exponential numbers to explain both strategies. Rest of the PTs (n = 7) also agreed with this explanation. For instance, PT5 said "*We could find directly either the number of parts created or we could name each person's share, then find number of the parts in the whole*." These findings pointed out connecting various strategies and making sense of multiple mathematical explanations for the same task. PTs used multiple mathematical explanations and representations in the process of the discussion.

At the end of the task, the PTs were asked to deduce a general mathematical conclusion about how many parts would be created as a result of repeated folding into half. All PTs, except PT4, concluded that 2^n parts would be created. PT1, PT6, PT8 and PT9 also concluded that this repeated folding into half was similar to repeated halving strategy, as they learned in the last week. These instances showed that PTs reached a general solution that could be called generalization practice and they connected the mathematical ideas embedded within LT.

Next folding task aimed to test whether PTs could transfer their knowledge to different settings and capture the connections across tasks. In the task, they were asked to fold a rectangular paper into half then third two times. In this task, many PTs (n = 5) showed a consistent pattern with their solution ways as in prior task. Differently, PT4 could generate a correct pictorial representation for this task and was able to explain her strategy. PT2 and PT3 did not exhibit the same misconceptions again. They could produce correct strategy along with correct mathematical explanations. PT2, PT4 and PT6 drew pictures, and PT7 utilized multiplication strategy in two ways: 2.3.3=18 equal parts and $\frac{1}{2}\text{x}\frac{1}{3}\text{x}\frac{1}{3}=\frac{1}{18}$ then she concluded $\frac{1}{18}$ was the size of one share. PT1, PT3 and PT5 utilized multiplication along with pictorial representation. PT8 and PT9 used exponential numbers strategy as $2^1\text{x}3^1\text{x}3^1=2^1\text{x}3^2=18$.

In this task, discussion on different pictorial representations also yielded conceptual understanding of transitivity concepts embedded in the LT. Figure 48 shows PT4 and PT6's drawings.

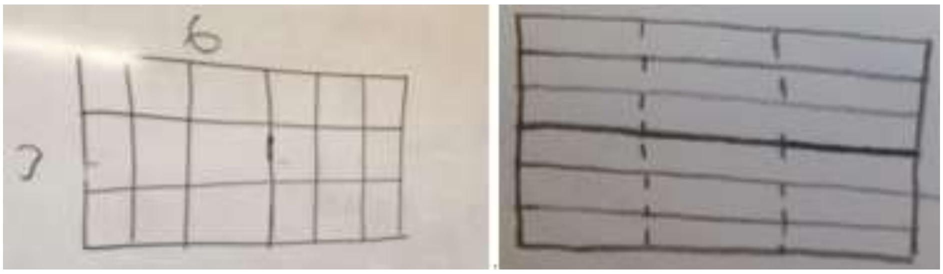


Figure 48. PT4 and PT6's drawings: The transitivity argument

The drawing on the left, first, PT4 folded the rectangle into half vertically. Then, she folded into three horizontally and then she folded into three again vertically. On the other hand in the drawing on the right, PT6 first folded into half horizontally, then folded into three vertically and then folded again into three horizontally. Both PTs created 18 equal parts, yet the parts were not congruent in terms of shape. Then a discussion started as follows:

T: Is there anything we can talk about mathematically when we look at both drawings?
PT6: This part and that part, their shapes are not same.
T: What do you think all, these parts are equal or not?
PTs: They are equal.
T: How do we show they are equal?
PT5, PT6 and PT7: We can put one part on top of the other one. Then, we could cut the longer one into half [decomposing], then we put them on top of each other [composing]. Thus, one covers the other completely.

This discussion indicated that the PTs deeply examined the mathematical ideas behind each representation and tied with previously learned equipartitioning idea called transitivity. None of the PTs could conclude that each part was $\frac{1}{18}$ of the same sized rectangle. Thus, they were equal sized parts. I reminded this strategy. PTs indicated they would not think of that.

Only PT9 stated their areas were equal. She wrote 3.6=6.3 in both representations. I probed this idea further:

T: What is the result of 18 represented as a result of both calculations?
The majority of the PTs (n = 6): The area of the rectangular whole.
T: Ok, what is the unit for the area?
The PTs: [Thinking]
PT1: The parts
PT8 and PT9: The number of parts in each whole.
T: What are the shapes of the parts?
PTs: They are different rectangles.
T: Different in terms of?
PTs: Shape.
T: How is the number of rectangles equal in both wholes even if they are different shape rectangles?
Majority of the PTs (PT1, PT4, PT5, PT6, PT7, PT8 and PT9): We showed previously, their sizes are equal.

The discussion above showed that the PTs realized different shaped rectangular units could be utilized to find the area of the rectangular whole. Also, the PTs indicated area equivalence of different rectangular units and as a result the area of the both whole was found as 18.

The third task focused on utilizing different number of folds to create the same number of equal parts. In this task, two scenarios were presented to the PTs. In the first scenario, Ayşe folded a paper into four then into three. In the second task, Fatma folded same sized paper into six and then into an unknown number. The question asked to find that unknown number. All PTs could find this number as two. To reach that answer, first all of the PTs found the number of equal parts created by Ayşe. Then, they found which folding action was necessary for Fatma to create the same number of equal parts. The performance of each PT showed that they all correctly utilized mathematical strategies to solve the problem. Although all PTs found the correct solution, there existed some problems. For instance, PT3 had a problem while utilizing drawing strategy to find 12 equal parts. Although she found the correct answer, the way that she showed her work on the drawing revealed a mathematical error in employing parallel cuts. PT1 and PT5 captured the error and explained why it was incorrect:

> PT3: [To create four fair shares, she drew four horizontal cut and erased the extra piece]
> All PTs: This would not create four fair shares.
> PT1: In primary schools, teachers erase the extra piece.
> All PTs: [Nodded their heads.]
> PT5: They said students to split into four. Then, they drew five parts and erased the extra part.
> PT1: I would do that too, if I were a student. Because my teacher is doing it.

This showed that the PTs started to capture mathematical errors in other friends' strategy and correct it. They could even build connections between the errors they observed in the LTBI sessions and their experiences in the school. Then, they acknowledged the importance of having a correct mathematical knowledge while teaching mathematics to students. After this discussion, I checked whether PTs addressed the factors that led this mathematical error. The conversation carried out as:

> T: Which criterion of equipartitioning was not achieved in PT3's representation?
> PT1, PT4 and PT5: She did not exhaust the whole and create the correct number of fair shares.

This showed that the PTs were able to build connections across levels of LT and also they applied basic mathematical criteria of equipartitioning across all levels.

I asked the underlying mathematical ideas or topics in this type of folding tasks. Two PTs stated as finding the missing factors. They said, "*Four times three and six times two is equal to each other*." PT6, PT8 and PT9 shared thought was "*Through this, we are finding the area of the rectangle*." PT3 said, "*This could be a concrete representation of multiplication.*" PT5 also approved PT3's statement. PT8 tried to build a connection as the following:

> PT8: I think this also sets a base for the least common multiple (LCM).
> T: How?
> PT8: The LCM of four and three is 12. I would find the unknown number and six and their LCM should be 12. So this number is two.
> T: Does this hold for every case?
> PT8: [Thinking] It holds for some but not all.
> T: Can you give a counter example?
> PT8: [She thinks for a while] Actually, it did not even work in here.

These findings indicated that majority of PTs (n = 7) also connected various mathematical ideas such as exponentials, missing factors, positive factors of a positive number, multiplication, equipartitioning and fractions to the folding activity. They did not perceive folding activity merely as folding a paper and creating equal parts. In addition, PT8 came to a level that she started to test her own claim.

Last activity focused on finding different combination of folds to create the same number of equal parts. All of the PTs found various strategies and represented the result of each fold mathematically. This showed that PT1, PT2, PT3, PT4 and PT5 progressed into pictorial representation to more complex mathematical representation when their initial solution strategies were compared to current strategy. This indicated that these PTs enhanced their existing knowledge through employing more complex mathematical strategies to solve the task. In addition to these findings, this task generated an important interaction among PTs:

PT9: This task could form a base to find [positive] factors of a positive integer. 36 equal parts could be created through various combinations of these factors as a fold.
PT6: This is another way for what we did last week that was fairly sharing a single whole. We called that to get 12 we could fairly share [the whole] by 12x1, 4x3 and 6x2.
Other PTs: [agreed with PT5 and PT9] Yes.
T: What did we call this splitting strategy, the one PT6 described?
PTs: [All were thinking, none of them remembered the name.]
T: Composition of splits.
PTs: Yes.

Also, PT9's commented on how to connect the task with other mathematics topic called finding positive factors of a positive number helped the rest of the PTs to see this connection. In the pre-test item 14, many PTs could not utilize this mathematical reasoning to find the various combinations.

As a result, through analyzing different representations including pictorial, table and mathematical representations, all of the PTs (n = 9) made sense of how these representations were connected to each other and understood the mathematics behind them. Also, through discussing their peers' strategies and representations, PT2 and PT3 *remediated* their misconceptions. In addition, PT4 could solve the next similar folding task and she developed ideas for how folding actions created fair shares. Moreover, the PTs who started with concrete representations to solve the given task moved into abstract mathematical representations.

PT8 and PT9 thought in a more abstract way than their peers. They could generalize the mathematical ideas and apply in other settings. Moreover, they also argued against their own generalization, which they admitted as a new experience for them. Unlike their prior performances in the pretest, all of the PTs could explain their mathematical thought at the end of the activities. This showed that they made sense of the possible explanations behind procedural calculations and be able to communicate these explanations. This showed that the PTs started to internalize the mathematical ideas and strategies embedded in the ELT.

5.3.3 Restructuring Student Knowledge: Folding Tasks

In the first task, thinking about the result of folding action abstractly was difficult for many PTs (n = 5). They urged a need for using a concrete material to employ the folds. Also, PT4 indicated "*Even we could not imagine it easily, how can a student [imagine]?*" and PT1 stated, "*I could not solve this; I am not expecting a student can solve this.*" These two comments of the PTs potentially addressed two findings: First, elementary mathematics is not always elementary (Phillips, 2008) and second, PTs had a perception about mathematics that they could not solve a problem, a student also could not (Phillips, 2008).

PTs started to realize students actually could solve these tasks too. When they analyzed students' actual work adapted from Empson and Turner (2006) the same PTs saw that students worked on the folding tasks by drawing. Thus, they also started to anticipate students' mathematical strategies while engaging the tasks.

As discussed previously, in the first task, two PTs exhibited misconceptions. PT3 utilized ratio reasoning and PT2 used additive reasoning while trying to figure out the number of the resultant equal parts created as a result of the folds employed. The discussion on these misconceptions also merged into a shared conclusion among PTs. They concluded that they might also hold the same misconceptions as students did. This was an indication that the PTs started to understand how a student could exhibit a mathematical misconception.

In the third task, PT6 indicated that folding a rectangular paper into four and then into three, and folding the same sized paper into six and two would yield 12 equal parts. Yet, the shares did not look same. The justification ways to indicate the each share from each rectangle was listed earlier as the composition and decomposition and area congruence of the shares. Many PTs (n = 6) indicated that these strategies could be also employed by students to indicate the equivalence of these shares. In addition, PT3, PT5, PT6 and PT8 indicated that a student might compare the length of the sides. For instance PT5 said, "*Student could say, this part [pointed the 4x3, rectangle] is fatter and [pointed the other part] and this part is skinner but taller.*" This finding indicated that the PTs distinguished their own justification ways from students' justification ways. Because, they discussed an additional justification way called qualitative compensation in the ELT.

In the session, PTs discussed the ways that a student could employ to find the total number of fair shares created as a result of folding(s):

> PT4 and PT5: They could use multiplication. [They gave an example from a task.] They could say three times six.
> PT7: They could count by ones.
> PT8: They can add the parts formed as a result of each folding. For instance, when you folded into half, it created 2 parts and when you folded it into three again, in each half there would be three parts, so [the result is] 3+3.
> T: I would not suggest leading your students to find the number of equal parts created through addition. Why do you think, I would not suggest this?
> PT6: Student could say five. If you folded into two, it created two equal parts. Student might think, if folding into two created two parts, folding into three would create three parts and they could add them [2+3] and said five.
> PT2, PT5 and PT9: Yes, this could happen.

In the discussion above, the PTs started to challenge their peers' suggestions. To argue against a statement, the PTs developed reasonable mathematical explanations. In this case, PT6 realized the potential misleading danger of employing additive strategy while finding the total number of fair shares created. This situation indicated that this particular PT internalized possible mathematical strategies that students could utilize. This internalization entailed perceiving potential benefits and danger in the suggested mathematical strategies, representations and explanations.

One task in the session included the case of Ayşe's solution. She stated that there would be 12 equal parts, if you fold a rectangular paper into half four times. While working on this task, all the PTs recognized the answer of Ayşe was incorrect. However, they (n = 8) had difficulty with explaining the possible error or misconception that might lead Ayşe to generate this incorrect response. In this instance, PT4 could not figure out the possible underlying reason behind the response and wrote, "*I think, the student just threw a random answer.*" PT2 and PT3 only wrote a general statement. For instance, PT3 wrote, "*The student did not understand fractions*." She did not explain how she figured out this based on the evidence deduced from Ayşe's work. PT1 tried to connect Ayşe's answer with the misconception that was stimulated earlier in the teaching sessions. PT1 wrote her interpretation of Ayşe's answer as:

> It is not an additive misconception, as we saw earlier. If it was an additive misconception, Ayşe would say eight equal parts. First fold yielded two equal parts, then the second four. So, plus two six and then eight. I am nervous, I could not figure out why she responded as 12.

PT1's response indicated that she was aware that there was something wrong in the answer. She tried to address this through using her prior experience. However, this seemed a new situation, and her existing knowledge was not sufficient to anticipate and explain Ayse's mathematical thinking. This situation made PT1 uncomfortable. However, she also indicated, "*I learned why it is important to understand and think through students' mathematical thinking.*"

PT7 and PT9 focused on possible drawing errors while representing each fold. PT9 wrote in her written work that Ayşe correctly folded the rectangle into half twice. However, then she might fold each half of the rectangle into three. PT9 drew this as follows:

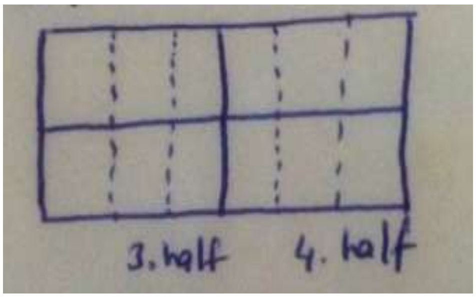


Figure 49. PT9's anticipation of Ayşe's mathematical thinking

PT7 also utilized same argument to explain possible mathematical thought of the student. Both explanations indicated that the PTs were aware that the answer was not correct. However, PTs tried to suit an explanationfor how this answer might be produced without having a solid ground for their claims. PT5, PT6 and PT8 focused on possible misconceptions that Ayşe exhibited when she tried to perceive the number pattern. This patterning activity was a widely encountered strategy of the students (Van de Walle, 2007). PT6's written work on how student might think is shown in Figure 50.

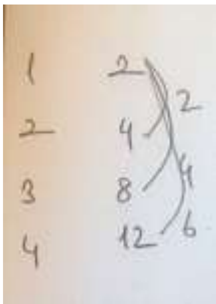


Figure 50. PT6's anticipation: Patterning activity

PT6 explained the work as:

> First, student think 2x2=4, I think the student knew the first fold created two parts, then the second fold created four parts. Then, she realized that the multiplier was increased by two. Thus, in the next fold, she multiplied two by four, 2x4=8. Similarly, 2x6=12. The student correctly solved first two steps [folding into half twice]. Then she did not check the rest.

PT6's explanation indicated that the student merely employed a patterning activity in the third and fourth step without paying attention whether the perceived pattern reflected the correct relation between the folding actions and the resultant number of share. Another anticipation was from PT5, she indicated that student might have employed right mathematical thinking in the first two folds since the pattern was deduced from these steps. PT5 stated, "*The first fold created two equal sized parts, the second fold created four [equal parts]. Then, student might think fourth one would create 6 (4+2). Then, she might fold all the parts into half again* *and produced 12 equal parts*." PT8 also provided similar explanation for what this student might think as she solved the problem. These explanations of the PTs showed that the PTs started to size up students' errors and anticipate their mathematical thinking.

In conclusion, all PTs (n = 9) tried to develop some conjectures on what the student might think when she gave this particular response of 12. However, none of the PTs were sure about their analysis of students' mathematical thinking and response. Some PTs (n = 5) explicitly stated that they were not comfortable with this situation. At that point, I asked, "*Think about a situation, as a teacher, you asked this problem in the examination or used this problem in the classroom activity. And your student responded as 12 parts as Ayşe did. What would you do?*" The discussion was developed as follows:

> PT1: I would ask directly, why she did in this way.
> PT4: I think, we should ask this [justification, why] to the students even if the students produced correct response.
> PT3: Yes, if teacher did not ask, we could not know whether student really understood it.
> Rest of the PTs: [Approved with similar comments.]

This final discussion showed that PTs realized that it was not entirely possible to anticipate students' mathematical thinking. Moreover, the PTs also came to conclusion that they should not only focus on the incorrect answers while examining students' mathematical thinking. Focusing on incorrect answers was a general tendency among majority of the PTs when the teaching experiment started.

In the last activity, the PTs were asked to examine two students' work examples of folding a rectangular paper that yielded 12 equal sized parts. All of the PTs indicated the students' answers were correct. Many PTs (n = 6) could explain students' possible mathematical thinking while employing the particular strategy based on the evidence shown in the students' works. Rest of the PTs (n = 3) started with some assumptions while evaluating the students' responses. This also showed that these PTs had difficulty with distinguishing their own mathematical thinking from students' mathematical thinking. For instance, PT8 stated, "*I think the student did not know that six was a factor of 12. Since, student knew three and four were the* *factors of 12, student utilized 4.3=12 while folding.*" Here, she assumed that student did not know six times two was equal to twelve. However, there was no evidence about this in the student's work.

During the discussion on several occasions, the PTs started their explanations with the assumptions that had no evidence from student's work. In such cases, I asked PTs how they inferred their assumptions. These kind of questions guided PTs to look for evidence from the students' works. For instance, PT1's explanation of student's thinking based on the student's work is shown in the Figure below.

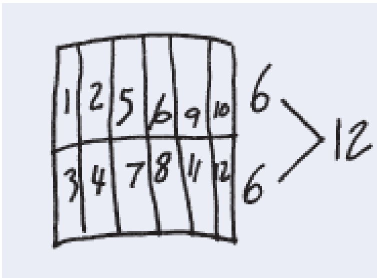


Figure 51. Student's representation of folding resulted in 12 equal parts (Retrieved from Empson & Turner, 2006)

> Since the child goes one, two, three and four and so forth. This means this child used this numeration pattern for three parts; columns [pointed the 1/3 of the rectangle]. Thus, this means the child folded [the rectangle first] into third vertically. Then, the child folded [the rectangle] into half horizontally. Since, the child first numbered the upper parts first [pointed numbers 1 and 2], then the lower parts [pointed numbers 3 and 4]. Since, the child wanted to produce 12 parts; child folded the rectangle into half again.

PT1 grounded each inference about student's mathematical strategy to the observable evidence from the student's work.

5.4 Week 5

In week, three main courses of activities were took place. The first course of activities was related to reallocation level of ELT. The second course of activities was related to fair sharing multiple whole level of the LT. The last course of activities was focused on the covariation level of the LT. The order of the levels in LT formed the logical flow the activities took place in the week5. Thus, the findings first related to reallocation, then sharing multiple wholes and at last covariation will be reported in this week.

5.4.1 Reallocation

In the second and third weeks of the teaching experiment, PTs were engaged with the idea of factor-based change. In week 5, the first task was a reallocation task. Since, PTs examined the relation between factor-based change and the size of the share, conceptualizing the reallocation navigated rather smoothly in week 5.

5.4.1.1 Restructuring Mathematical Content Knowledge

PTs were presented with a reallocation departure task in which 24 cookies were fairly shared among three people, and then one people left the group. Then, the new share of each remaining person was asked. Majority of the PTs (n=8) used fair sharing a collection strategy. They recompiled the cookies and divided 24 cookies into two. Thus, they found each person's share as 12 cookies. Only one PT utilized a different strategy. PT6 redistributed the extra share among the number of the people left. She fairly shared four cookies between two people and concluded each received two more cookies. She stated that they had four cookies earlier, now they had six cookies in total. Other PTs (n = 8) indicated that they did not think of this strategy.

To check whether other PTs conceptually understood the reallocation concept, I posed "If you have an unknown number of marbles fairly shared among six children and then, two children left the group, what will be each child's new share?" Six PTs could produce a general solution for the task. These PTs both represented collection and reallocation strategies mathematically. For instance, PT4 mathematically explained her reallocation strategy as follows:

> PT4: x marbles among six [children], each child got $\frac{x}{6}$ marbles. 6 – 2 = 4 child left. $\frac{\frac{x}{6}+\frac{x}{6}}{4}=\frac{x}{12}$ Child [who left] gave to each child [remains]. $\frac{x}{6}+\frac{x}{12}=\frac{x}{4}$.
> PT3: Directly saying is $\frac{x}{4}$ easier than this.
> T: What do you think about PT3's claim?
> PT1: [PT5, PT7, and PT9 also disagreed with PT3] Yes, when we write the reallocation strategy mathematically, it seems hard. However, in reality it is easy to redistribute left children's shares [extra shares]. It is so easy if you have huge numbers to divide.
> PT8: Yes, this is true. If the question includes three digit numbers initially [number of objects in the initial collection], I think reallocation is easy. At least, students work with small numbers.

This discussion above showed that PTs could discuss on the mathematical representations of the strategies and their effectiveness in the practical application. They understood the connection between the symbolic representation of the strategy and the real actions to employ this strategy.

Another issue was raised related to the effect of the task representation on students' mathematical strategy. PT1, PT5 and PT6 claimed that the representation of the initial share in the problem would more likely to influence students' mathematical strategy. As PT5 explained:

> For the first task, if the representation was composed of rows and columns [the array representation] students more likely combined all objects back together. Then, they would split the collection again into existing number of people.

These discussions helped PTs made sense of how different representations carried various mathematical messages to the students. PT1, PT5, and PT6 were able to explain how different representations might affect the way students engage with the problem. Rest of the PTs also agreed with their peer's explanations.

In the instance of describing the new size of each child's share, PT9 built a connection between the reallocation and factor-based change ideas. She stated:

> As we did earlier [in the pre-test], if more people show up to the party, in the instance of fairly sharing a cake, the amount of cake each person got decreases and vice versa. In reallocation, we worked with collections but number of the sharer changes [a factor-based change occur], so the share changes.

Built upon PT9's comment, PT8 indicated that this was the equivalence of multiplying the factors of the same number. She supported this thought with her experience in the first task. Six cookies times four children was 6x4 and this should be equal to two children times the number of cookies. Then, she wrote 6 x 4 = 2 x □. Since two was half of four, the unknown number [number of cookies per child] should be two times six, which was 12. However, PT8 also questioned into what extent her mathematical inference could be applied to other cases:

> PT9: In this case, two is half of four so six needs to be half of a number that is 12. Students may think this way every time. Similarly, they believe that every fair sharing action results in half.
> T: Think about this case. For instance, 24 objects among six people and two people left. You can make different factor based changes in here and observe how these affect each person's share.
> PT8: OK 6 x 4 = 4 x 6. This is commutative property [of multiplication]. But the case is not half and twice times.

In the interaction above, PT9 pointed the result of factor-based change qualitatively and PT8 addressed the effect quantitatively. These two PTs connected the mathematical ideas within the ELT with each other respectively reallocation and factor-based change. Also, PT8 particularly tested her general conclusion about the effect of factor-based change and the reallocation. As a result of these discussions, rest of the PTs (n=7) also agreed with their friends through verbally stating their approval. PT1 and PT7's written works also indicated the same line of mathematical reasoning.

In the instance of evenly distributing the uneven shares, all the PTs indicated that they would combine all the collection and divide it to the number of the people. Then, I asked if they could think of other ways. PTs produced different redistribution strategies in which some coins were taken from one-person share and dealt among the others until everyone got a fair share. Also, the PTs justified why they selected the particular person's share to redistribute:

> PT1: There were seven people with different numbers of coins in their shares. Then, one person left. I would get rid of the person who had the minimum amount of coins, which was two coins. Then, I started to redistribute [the coins] until all had the same number of coins in their shares.
> PT6: After, you distributed the two coins, you would distribute from the share which included more coins to the ones had fewer coins.
> PT4: I would distribute the two coins to the person who had few coins [she pointed the share who had four coins]. Then, I would mark with a line from the place everyone has the same amount of coins, which is now five coins. Then, I would redistribute coins above this line among the six people.
> PT1: Also, they could use height comparison to justify their fair shares.
> T: Ok, think about the situation in which the task did not include the pictorial representation of each person's share.
> PT8: Students would not reallocate.
> PT1: They would use division. [It is also called] collection strategy.

In this discussion, the PTs attempted to justify their solutions strategies. Four different redistribution strategies were stated by the PTs verbally. This helped other PTs to internalize their peer's mathematical solution. In addition, PTs discussed the pedagogical aspect of the task by focusing on the possible effects of pictorial representation of the initial shares on students' mathematical solution strategy. It also created variation to decide how to redistribute the objects. Then, all of the PTs acknowledged that they would think the effects of the different representations when they would present a task to their students since multiple representations conveyed different messages to the students. Also, PT1 and PT4 indicated the height comparison justification strategy could be helpful in this task to understand whether the shares became equal. Rest of the PTs also indicated with different words that the stacking representation of the shares in the problem might lead students to use height comparison to decide whether the shares became even at the end of the reallocation. At the end, majority of PTs (n = 8) indicated they did not think of the reallocation strategy initially. Thus, they learned a new mathematical strategy to produce correct mathematical solution

5.4.1.2 Restructuring Student Knowledge

For the reallocation tasks, PTs initially could not distinguish their own way of thinking from students' thinking. Because, they all thought students would use division to find the new share size. As they progressed in LTBI teaching experiment, with the help of guidance and social interaction within classroom, PTs started to realize that students might solve the presented tasks with different strategies. The PTs' (n = 9) written works showed that they anticipated that students would draw pictures and then evenly shared the collection. Within these PTs, seven of them drew both reallocation and collection strategy of the students. For instance, Figure 52 shows two PTs' written works of student's possible reallocation strategies.

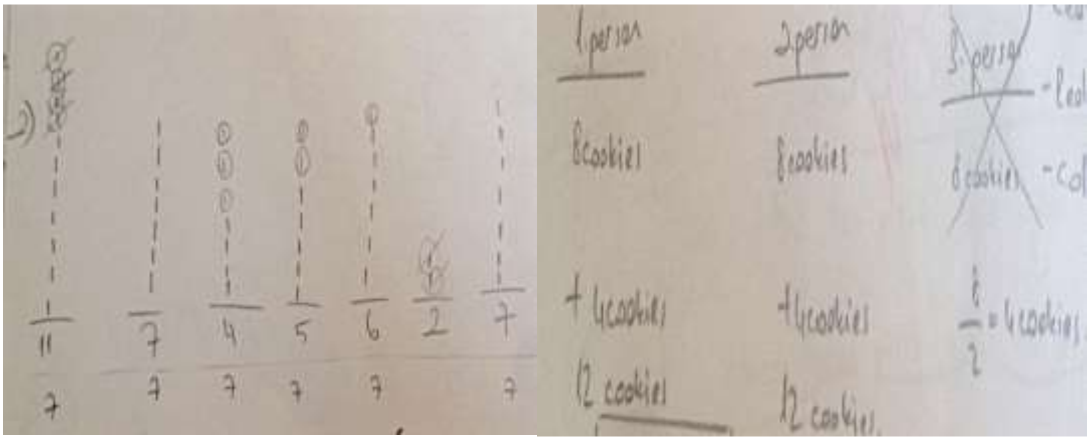


Figure 52. Two reallocation strategy representations

In the Figure 52 PT1 redistributed extra share, then she redistributed from the share that had more coins to the other shares. In the picture on the right in Figure 52, PT6 shared the extra share among the remaining number of people. She divided extra eight coins into two. Then, she gave 4 coins to each remaining person and found each remaining person had 12 coins. PT2 and PT3 showed only collection strategy in their written work as anticipation for students' mathematical strategy. The PTs also indicated they could use division based on the students' grade level. They suggested that in higher grades, students could use division and in early grades, they could use reallocation or collection strategy. These evidences showed that the PTs started to distinguish their mathematical thinking from younger students' mathematical thinking. Also, PTs ordered the students' preferred strategies based on grade levels.

At the end of the week, a video of 2nd grade student was shown in the classroom. Initially, I only showed the introduction part of the video that included the problem stem. The problem stated that five children fairly shared 40 crayons and each got eight crayons. One child left the group and the student was asked to find new share of each child. Then, all PTs asked to predict the student's solution strategy.

> PT3: Which grade is this student in?
> T: 2nd grade.
> PT1, PT3, PT5: Ok, she can use division to solve the task, since they know division at that grade.
> PT2: But, the number may be great for the children to divide.
> PT4: Yes, it may. Also, it [the question's stem] gave the initial situation [size] of the share [eight crayons per children].
> PT8: Yes, this is very important. If this is given, I think student may not use division.
> PT1: Ok then, can we say, presence of the picture of the initial share different than no presence of it?
> PT5: Yes, it was. We discussed before.

The discussion above indicated that the PTs could analyze the factors that might influence the way the students engage with the presented problem. As they analyzed the problem, initially PTs thought the grade level of the student was an indicator of the strategy being used by the student. Later, as a result of the idea exchange among PTs, they considered the possibility of utilization of different strategies as a result of different factors. Two influential factors were the magnitude of the numbers utilized in the problem and the initial share size information provided in the problem stem. Thus, PTs could reason about the factors that might affect students' strategies. Then, based on these factors the PTs shaped their anticipation of students' mathematical strategies.

When I resumed the video, the PTs realized the problem did not include the pictorial representation. As a result, two lines of thought were discussed in the class. The first one was, the student could picture the initial situation or could use manipulative if given. Then, she could reallocate the left friend's share. The second one was, the student could use division or collection strategy. PT5, PT7 and PT9 supported this claim. The rest of the PTs (n = 6) supported the first claim. However, both groups indicated both strategies produced correct answer and could be utilized. This showed that although PTs predicted this particular student's strategy differently by employing different line of reasoning, they also acknowledged the possibility of utilization of both strategies. When the PTs saw that the student used manipulative to model the initial situation, they all agreed the student would reallocate the extra share. They predicted the student's strategy correctly.

I also asked the PTs to pay attention to the student's mathematical language when she explained her solution strategy. All the PTs immediately indicated the student used 1-1 correspondence strategy when she redistributed the extra eight crayons. PT8 paid attention to her naming practice "*She stated that I gave two crayons per each person. She did not turn back and count what she gave. This showed that she could keep track of what she gave to each*." These findings showed that the PTs could understand the students' mathematical thinking and supported their claims with the evidence that they observed from the student's work and actions.

5.4.2 Fair Sharing Multiple Wholes

In the third and fourth weeks of the teaching experiment, PTs were engaged with the concepts of transitivity and sharing single whole. The second task of this week was related to the cases of fair sharing multiple wholes. Because, the PTs encountered the transitivity argument and justification of fair shares argument before, the mathematical strategies and naming practices that a person could employ while working on sharing multiple wholes was the main focus of the tasks in this week.

5.4.2.1 Restructuring Mathematical Content Knowledge

All the PTs successfully produced correct responses along with meaningful explanations and justifications as they engaged with sharing multiple whole tasks. Four PTs did not pay attention to the referent whole while naming each share. This was an important issue to address in the experiment. In the first problem, PTs were asked to fairly share 12 pizzas among four tables. Then, they asked to find how much of the pizza received by each table. PT5, PT6, PT8 and PT9 stated as one fourth of the whole. Rest of the PTs approved their peers' responses by nodding their heads; also their written responses included the similar responses. I asked what they meant by the whole. After this question, PT9 asked number of pizzas in the beginning. PT6 responded as12. Then, she stated, "*Each table received* $\frac{1}{4}$ *of the whole pizza, the whole pizza is 12 pizzas*." This discussion aimed to clarify the ambiguity in PTs' responses. After the discussion, PTs agreed on each table received $\frac{1}{4}$ of the 12 pizzas.

The second problem asked PTs to fairly share 15 pizzas among four people. PTs utilized different strategies and all strategies were discussed together. Five PTs utilized benchmarking strategy. Among these PTs, PT3, PT5 and PT6 utilized composite unit while fairly sharing pizzas. PT3 and PT6 gave three pizzas to each person, and then they split the remaining three pizzas into four evenly and distributed the parts between four people. Differently, PT5 split two of the remaining pizzas into half and the one pizza into four evenly. Then, she distributed a half and a quarter to each person. She concluded each person received "$3 + \frac{1}{4} + \frac{1}{2} = 3\frac{3}{4} = 3.75$". PT2 and PT4 dealt the pizzas by ones, then split the remaining pizzas into four and evenly distributed the one fourth to each person. PT7 and PT9 used division to find each person's share as, 15÷4= 3.75. Also, they indicated $\frac{15}{4} = 3.75$.

PT9 generalized her strategy as the number of pizza was divided by the number of people

Two strategies could be employed for sharing multiple wholes (Confrey et al., 2009; 2010). First one is *benchmarking* employed by five PTs. Second one is *split all,* in which PTs should split each pizza into the number of the people and distribute the shares evenly. None of the PTs used the second strategy in this problem. PT1 used a totally different strategy. Figure 53 showed her strategy:

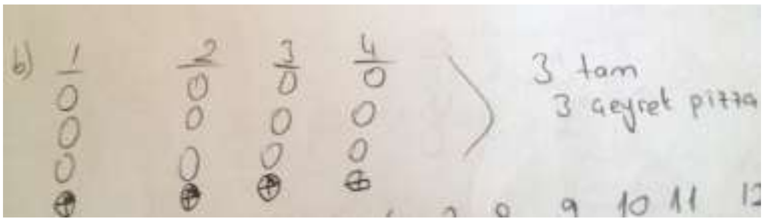


Figure 53. Out of sequence fair sharing multiple wholes strategy

She explained her strategy shown in the Figure 53 as:

> It could be this way: three pizzas assigned to each table [she pointed the last row in the picture above]. The last one is remains empty [the last table did not receive the last pizza]. Then one quarter could be taken and redistributed to the fourth table. So, there are three pizzas and three quarter pizzas on each table. This is $3\frac{3}{4}$.

This PT employed a redistribution strategy in sharing multiple whole tasks. This strategy was not documented in the ELT.

The PTs could represent each share through connecting division, fractions and decimals concepts. PT5's answer "$3 + \frac{1}{4} + \frac{1}{2} = 3\frac{3}{4} = 3.75$" was one of the explicit responses that illuminated the connection among these mathematical concepts. After this task, a discussion took place on the connection between mixed fractions and improper fractions:

T: OK, we see various responses here. Think of these responses. What sorts of mathematical ideas can we deduce from these responses?
PT6: These tasks can also help student to convert mixed fractions to improper fractions. PT5 showed the mixed fraction in her answer, PT8 gave the improper fraction answer. So, if we can discuss these two answers, we could see the connection [between these fraction types].
PT3: The pictures help students to see concrete representation of the mixed fraction.
PT5: We know 0.25 is a quarter. Thus, we have three wholes and three quarters that means 3.75. [This is] decimal representation of mixed fraction.

The discussion above showed that PTs built their mathematical conclusions on their peers' mathematical strategies and responses. They reached a generalizable conclusion: in the instance of number of people was fewer than the number of the objects to be fairly shared, improper fraction or mixed fraction was produced. PT3, PT5, PT8 and PT9 stated this conclusion in their written works. PT3 and PT5 stated this conclusion verbally. PT8 and PT9 represented this conclusion in a mathematically generalized form. PT8 symbolized the number of objects to be fairly shared with the letter "n" and the number of people with the letter "p". Then, she wrote $\frac{n}{p}$ was an improper fraction.

The last task asked the PTs to fairly share four cakes among seven people. All PTs produced correct response. The majority of PTs (n = 7) gave the response directly by dividing the number of cake by number of people, as $\frac{4}{7}$. This is an example for the division meaning of fractions. I reminded this meaning of the fraction to the PTs. In the whole class discussion, all PTs also indicated if the number of objects to be shared was fewer than the number of people; the resultant share was represented by a proper fraction.

After the intervention, majority of PTs (n=7) realized the difference between sharing single whole tasks and multiple wholes tasks that the sharing multiple wholes tasks could produce both proper and improper fractions. PT3 stated that to perform these [sharing multiple wholes] tasks, students should gain a certain level of proficiency in fractions since improper fractions were involved. In addition, only PT8 and PT9 indicated a major difference between equipartitioning single whole and multiple wholes cases. PT8 stated:

PT8: In the single whole our numerator is always one and this is a proper fraction. In the multiple whole [sharing] numerator is not one and the fraction could be proper or improper fraction.
T: What do we call the fractions in which the numerator is one and denominator is a positive integer number?
PTs: [silence]
T: We call that unit fraction.

This interaction showed although PT8's comment illuminated the connection between the fair sharing a single whole tasks and unit fraction, the PTs did not know the name of the mathematical concept they were referring to. Or, even if they knew, they could not connect the ideas.

5.4.2.2 Restructuring Student Knowledge

In the first task, PTs were asked to fairly share 12 pizzas among four tables. To find the number of pizza on each table, all PTs used division as their strategy and students' strategy. Then, I guided PTs to think more deeply on other possible strategies:

T: Think about a student who does not know division.
All PTs: [They could use] one to one correspondence strategy.
T: Anything else?
PT9: They could deal by forming groups.
T: What you mean by saying "forming groups"?
PT9: They could deal by twos or threes.
PT1: Student might know counting by twos. So, they could give two pizzas at a time to one table.
PT5: They could also count by three. Thus, three pizzas per table.
T: What do we call the mathematical concept behind these strategies?
PT2, PT3, PT5, PT8 and PT9: Composite unit.

Initially, all the PTs did not fully distinguish their own way of mathematical thinking from the students' thinking since they assumed all students would use division as they did to solve the task. With my help and guidance, PTs started to anticipate students' strategies and addressed why students might employ different strategies rather than their initial perspectives on students' possible strategies. This finding showed that although PTs knew the strategies, there should be guidance for them to retrieve that knowledge from their internal cognitive schema and use it.

In this week, one possible student misconception was captured in an actual work of a student. In this work, the student thought for fair sharing 7 cakes among 4 people and 5 cakes between 2 people, each person received the same size cake piece. All the PTs indicated that the student's claim was incorrect. Many PTs (n=5) explained why the student produced such conclusion and why the response was incorrect. Then, they suggested possible ways to eliminate this misconception. PT6's response was a clear example for this. Figure 54 shows her work.

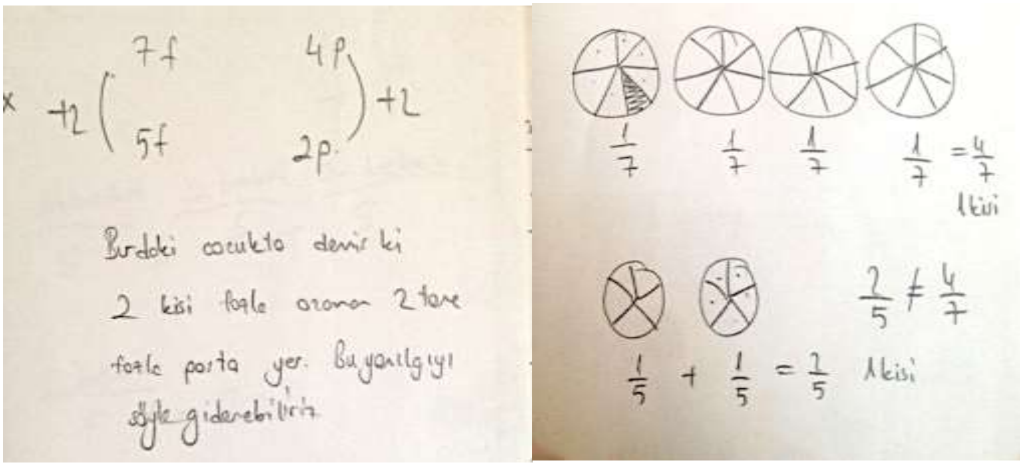


Figure 54. Eliminating additive misconception in fair sharing multiple whole: Showing each person share

In the figure above, PT6 first determined students' additive misconception. Then, she suggested utilization of concrete representation of split all strategy would help the student to understand his/her mathematical error. However, PT6's suggestion for eliminating the error did not illuminate the multiplicative change under factor based change; which addressed that the number of people was halved so the number of cakes.

Four PTs explained why the response was incorrect. However, they could not determine the misconception of the student. Then, they showed the amount of cake that each person got. They showed the unevenness of the shares through comparing the fractions. PT3 indicated that the ratios of the cases were not equivalent. She wrote, "*Seven to four and five to two is not equal. Thus, this is not a fair share.*" PT3's response was an initial reasoning for understanding the concept of ratio with units attached. After, PT3's comment, three PTs perceived the between ratio (Noelting, 1980) which was a ratio with no unit. For instance, PT4 stated, "*Since the number of the people is halved, the number of the cake is needed to be halved.*" These detections of PTs also tied sharing multiple wholes with covariation level of the LT that was also covered in the activities in this week.

5.4.3 Covariation

The covered contents, which were understanding multiplicative relationship between part and the whole(s) (called as naming practices and reassembly) and understanding qualitative and quantitative compensation laid a foundation for understanding covariation concept in this week.

5.4.3.1 Restructuring Mathematical Content Knowledge

In this week, PTs were asked to find the total number of cookies to feed 12 babies if 2 babies could eat 5 cookies. All the PTs could produce correct answer as 30 cookies. Yet, they all set a direct proportion to solve the task. On the other hand, they also indicated other mathematical solution strategies could be employed and these strategies were the strategies a student would more likely employ. Each PT employed variety of strategies is shown in Table 21.

Table 21

Distribution of each PT's Mathematical Strategies on Covariation Task: Teaching Sessions

PTs	Proportion	Unit Ratio	Scale Factor	Scaling Up	Equivalent Fractions	Equivalence Class	Others (Mixed strategies)
PT1	x						x
PT2	x	x		x			
PT3	x	x					
PT4	x			x		x	x
PT5	x	x		x			
PT6	x	x					
PT7	x			x			
PT8	x	x	x	x			
PT9	x	x		x			

In the unit ratio strategy, PTs found the number of cookies required to feed one baby. Then, PTs utilized this unit to find the number of total cookies to feed 12 babies.

PT5 wrote that for one baby 2.5 [cookies] were needed and for 12 babies. Then, she multiplies 2.5 and 12 and found 30 [cookies] were needed. Five out of six PTs who utilized this unit ratio strategy also employed a quantitative reasoning in which they directly utilized mathematical symbols and operations in their solutions. Only PT2 utilized informal reasoning in which she drew picture to find each baby's share. Then, she found the total number of cookies required to feed 12 babies with both utilizing multiplication and addition. This also showed that the PT2 utilized scaling up strategy through preserving each baby's share while employing repeated addition. Figure 55 shows her both strategies.

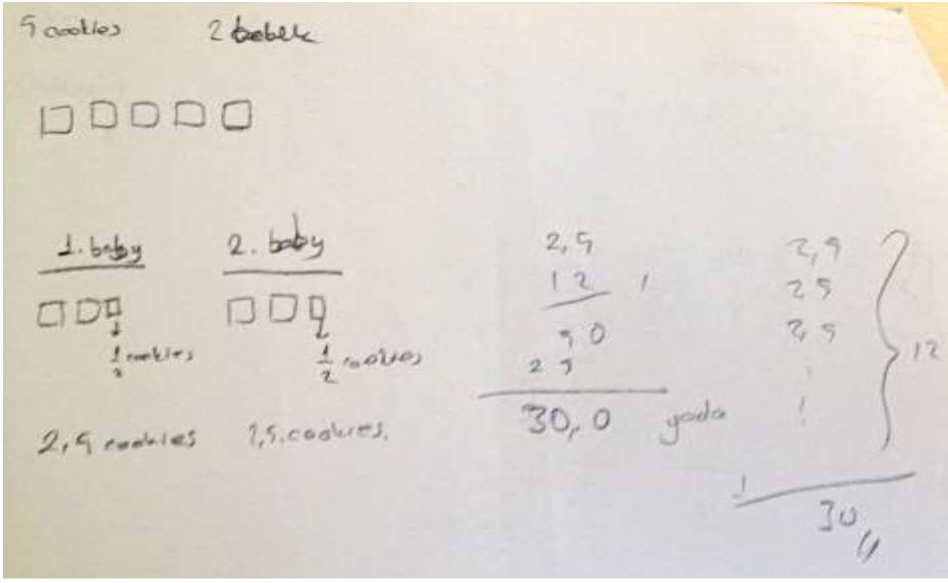


Figure 55. Scaling up and repeated addition strategies of PT2

Four out of six PTs who utilized unit ratio strategy also showed scaling up strategy. Different from PT2’s strategy, rest of the PTs utilized scaling up strategy without finding each baby’s share. Figure 56 shows PT4’s work is an example for this kind of scaling up strategy.

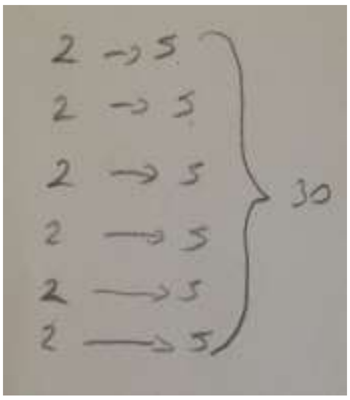


Figure 56. Scaling up strategy: Preserving 2 babies share

PT4 explained her strategy as "*Two babies received five cookies, I repeatedly grouped this two [babies] and five [cookies] until I reached 12 babies, then I counted by fives: 5, 10, 15, 20, 25, 30*." The PTs who did not employ both strategies, agreed with their peers and some of them also explained their strategies. For instance, PT1 found each baby's share as two whole and one half cookies. After that, she added two cookies 12 times and found 24 whole cookies. Then, she added half cookie 12 times and said 12 half cookies made six whole cookies. Thus, she stated that she had 24 + 6= 30 [cookies]. This finding showed that PTs learned new strategies to solve the problem through examining and understanding their peer's strategies and representations.

Only PT4 and PT8 used two strategies different than the rest of the strategies utilized by other PTs (n = 7). These strategies were respectively scaling factor and equivalence class. PT8 recognized the ratio without a unit. The number of babies scaled by the factor of six, so the number of babies should also be scaled by the factor of six. As a result, the proportion between two ratios was set. PT8 explained her strategy as "*Through 2x6=12 [a person] found the number of babies, and again 6x5= 30*." After her explanation, PT3, PT5, PT6, PT7 and PT9 acknowledged that the multiplicative relation between quantities remained same. This finding showed that although the PTs' made sense of the mathematical strategy that the PT8 utilized in the classroom, they did not directly think of that strategy when they first worked on the task alone.

A follow up discussion on the scale factor strategy took place among six PTs:

> PT8: The [multiplicative] relation between the quantities is six times. A student might think this as five times more so [the student] can give 25 cookies.
> PT9: Additive misconception.
> PT3: Also, they can fail to distinguish whether there is a direct relation or inverse relation existed when they set the proportion.
> T: How would you address these misconceptions?
> PT8: We could discuss whether the amount increases together or not.
> PT5: As the number of babies increases, the number of cookies should increase.
> PT7: We should understand the relation between the quantities.
> T: What are those quantities?

PT5: Number of babies.
PT4: Number of cookies.
T: [I constructed a ratio table]. What sorts of relationships can a child capture in this table?
PT7: Both additive and multiplicative.
PT8: Multiplicative.

The discussion above indicated that PT8 and PT9 anticipated possible student misconception. Then, many PTs (n = 6) discussed how they could eliminate the misconception. Also, they realized the key issue for eliminating the misconception was to capture the relation between two quantities. This was an important understanding to develop a complete ratio and proportional reasoning when PTs worked with students. They anticipated student misconception and they sized up the misconception by first dealing with the possible underlying reasons such as failing to perceive multiplicative relation or direct relation. Then, they produced ways to remediate them such as discussing the effects of change in one quantity on the other one.

Only PT4 utilized equivalence class strategy (Cikla & Duatepe, 2002) that rest of the PTs did not think of. After PT4's work, rest of the PTs stated that they had never thought of this way before. In her written work, PT4 found 5 cookies to 2 babies and the equivalence classes of the fraction $\frac{5}{2}$ and wrote: "*10 cookies to 4 babies, 15 cookies to 6 babies, then continued like these and found 30 cookies to 12 babies.*"

PT5 brought the issue of difference between reallocation, factor based change and covariation. PT5 stated in reallocation and factor based change that the whole to be fairly shared remained the same even if the number of people changed. However, in covariation both quantities changed. This comment started a discussion on the formal definition of covariation:

T: So, what is the definition of covariation?
PTs: [thinking]
T: Think about what we did in this week's covariation task.
PTs: [thinking]
T: OK, let's throw some key words that you think that are related to covariation concept.
PT4: Two quantities.

PT6: Unit ratio.
PT5: One person's share.
PT9: Proportion.
PT8: Multiplicative relation.
T: Cool, now try to come up with a definition that includes the essence of the thought PT5 suggested [difference between covariation and reallocation]. What is the thing that you preserve?
PT9: The size of the one-person's share.
PT1: Rate

After, determination of the key words and understanding the share size preserved in the covariation, the PTs worked in two groups. In the first group, PT2, PT4, PT6 and PT9 worked together and came up with the definition as "*The same change occurs on two related quantities in which a person's share remains same*." Then the interaction continued:

T: Nice. What are those quantities?
PTS: The number of objects and the number of people.
PT9: Can we say the ratios are same?
T: What you mean by saying "the same"?
PT6: Not the same, they are equivalent.
PT9: Yes, I mean that but [I chose] wrong word for it.

The discussion above showed that PT9 initially did not differentiate the same ratios and equivalent ratios concepts. As a result of my probe, PT6 helped PT9 to see the error in her mathematical language. Then, PT9 remediated the error and shifted her language from an incorrect term to the correct one. Then I concluded:

T: OK, PT6 and PT9 suggested the equivalence of ratios. So, can anyone tell me, what are those ratios?
PT5: Number of cookies to babies.
Rest of the PTs: Yes.
T: Who can show the equivalence of the ratios?
PT8: 5/2= 30/12, they are also equivalent fractions.

After the discussions mentioned above, I asked other group to come up with a new definition. They defined covariation in fair sharing context as "*Same change occurs in the number of people and objects that results in preserving ratios: objects to person.*"

T: This is a nice definition. We could say when a factor-based change occurs in the number of the sharer, the same factor based change occurs in the number of objects, in which the size of each person's share is preserved or the equivalence of ratios is preserved.

Although they could solve the covariation tasks and intuitively understand the covariation concept, they could not produce a formal definition initially. In the discussion, PTs utilized their own experience to form a formal definition of covariation concept in the context of equipartitioning. Thus, at the end of the interaction, all of the PTs learned the formal definition of covariation.

In this week, none of the PTs exhibited any misconception or mathematical error. They anticipated possible student misconceptions and discussed the possible underlying reasons behind these misconceptions. The interaction occurred as:

T: Can you anticipate any misconception of students?
PT5: Here, it is increased by 10 [2 babies to 12 babies] so this could also be increased by ten [5 cookies to 15 cookies].
T: So, what would be a student response for this case?
PT8: Fifteen cookies.
PT6: Fifteen.
T: OK, how could you remediate this misconception?
PT8: I would ask to the students to find each baby's share. This means, I would turn to equipartitioning again.
PT3 and PT4: I would ask, too.
PT5: [Build upon PT8's response] Child could show that each baby gets 2.5 cookies, and then when [this child] split 15 cookies into 2.5 cookies, the child could see there are six babies not 12 babies.
PT8: Or, child could distribute one cookie per baby, then [the child] could split the remaining 3 cookies into small pieces and distribute those pieces. As a result, the child could see 15 cookies are not enough to feed each baby so that a baby receives 2.5 cookies.
PT9: Or child could add 2.5 cookies until reach 15 cookies and track how many times the child used 2.5 cookies.

In the discussion, PT5 anticipated the possible additive misconception. In this misconception, students added a certain number to one quantity and added the same number to the other quantity to find the unknown quantity. In this task, since the number of babies changed from 2 to 12, the student added 10 cookies to 5 cookies and found 15 cookies. PT6 and PT8 also agreed with PT5's anticipation and they directly stated the response of student who had this kind of additive misconception. The rest of the PTs also indicated with their gestures that they agreed with PT5. PT1, PT4 and PT9 had written this misconception into their notes before this discussion.

PT5, PT8 and PT9 suggested some ways to fix this misconception when I asked for it. They both utilized their knowledge of ELT. PT5 utilized the reassembly reasoning. She found the answer of how many 2.5 cookies existed in 15 cookies. PT8 suggested two ways for remediating the misconception. The first one was finding each person share and check whether each baby received 2.5 cookies if 15 cookies were available. This strategy could be used in fair sharing multiple whole tasks. She elaborated the second one on PT5's suggestion. Because, PT5 did not specify the way to find answer whether 15 cookies could actually feed six babies. PT8 used benchmarking strategy to show this. PT9 employed a measurement perspective. She iterated 2.5 cookies as a unit, then she found 2.5 cookies should be utilized 6 times to reach 15 cookies. As a result of this, a child could see 15 cookies only enough to feed 6 babies not 12 babies.

These findings showed that PT5, PT8 and PT9 utilized their existing ELT knowledge to size up and remediate the misconception. The other PTs (n = 6) understood how these were. Also, PT1, PT2, PT3, PT4 and PT7 stated that they did not think like the way the PT9 suggested to remediate the misconception.

As a closure, I drew a ratio table on the board. The PTs examined this table representation and tried to find out what kind information this table conveyed. Also, they discussed the possible benefits of using this representation. PT4 indicated the table could help students to see the connections between the numbers given in the problem. PT9 also agreed with PT4 and added the table helps us to organize the given data in the task. PT5 indicated students could see horizontal relation (within-state ratio) and PT2 indicated students could see also vertical relation (between ratio). After PT2 and PT5's comment, PT1 asked a question:

> PT1: Teacher, the difference between these quantities is the unit, right?
> T: Can you please construct the ratios? What do you observe about the unit?
> PT1: In the first one, babies to babies, there is no unit. In the second one, babies to cookies, there is a unit.

PT1's written work also named the ratios addressed by PT2 and PT5. Figure 57 shows the PT1's written work:

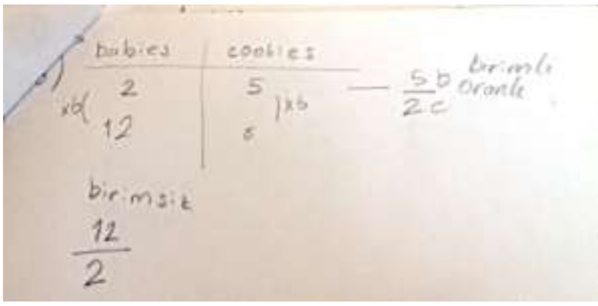


Figure 57. Recognition of ratio types

She wrote the ratio between the number of babies [12:2] and the ratio between the number of cookies and babies [5b:2c, b=babies and c=cookies]. Then, she indicated that the ratio between number babies was a ratio without a unit. The ratio between the number of cookies and babies was a ratio with unit. This horizontal relation was called within-state ratio (Noelting, 1980), *a*:*b*, in which *a* represented the number of the cookies and *b* represented the number of babies. After this interaction between PT1 and me, the rest of the PTs indicated that they did not know these ratio types. For instance, PT6 stated that she did not know the ratio concept completely.

The initial examination of the table representation helped PTs to understand the embedded relations within the table. This was an evidence of internalizing multiple representations for the ideas in the. Also, this examination illuminated the knowledge gap in PTs' ratio understanding. They learned new concepts such as ratio with/without units.

At the end of the week, I asked to PTs which mathematics topics covariation task laid a foundation. PT5 indicated changes in the number of cookies and the number of babies was same. PT7 stated that two ratios were equivalent. When we set a proportion, we utilized the equivalence between these two ratios. Also, PT7 wrote that the equivalence of the ratios formed a proportion. She wrote: " $\frac{5}{2}=\frac{t}{12}$ *and proportion, two ratios is equivalent*."

After PT7's comment, many of PTs (n=5) admitted that they did not know the meaning of the proportion as equivalence of two ratios. PT1, PT2, PT3, PT4 and PT5 admitted that in the direct proportion, they only performed cross multiplication and knew if one quantity increased, the other quantity also increased. This was an evidence of lack of conceptual understanding of mathematics. Then, they realized the conceptual relationship between ratio and proportion. They closed a gap and thought about the proportion in a different way. At the end of the teaching session, all PTs reached a shared conclusion that covariation set a base for ratio and proportion.

5.4.3.2 Restructuring Student Knowledge

In this week, all PTs could distinguish their own mathematical thinking from younger students' thinking. Although PTs set up a direct proportion to solve the problem, they stated that the elementary school students would solve the task differently. Then, they anticipated several mathematical strategies that a student might use to solve the task. These strategies were represented in Table 21 above. Also, majority of PTs (n = 7) utilized drawings that modeled the task when they anticipated students' possible strategies. In addition, among those strategies the PTs found the incorrect strategy. This incorrect strategy inherited an additive misconception. The students could perceive the change in one quantity additively (change in number of babies, 12-2=10) and they could add this change to the other quantity (the number of babies, 5+10=15). Another additive misconception was anticipated as, the student might perceive the factor-based change additively. Factor based change was six times (2x6=12 babies), yet the student might think the change as five times, neglecting the initial relation of 5 cookies to 2 babies. Then, they could reflect this change on the other quantity as 5x5= 25 cookies.

The PTs also concluded the least complex strategy was scaling up since, in this strategy students used additive reasoning instead of perceiving multiplicative relationship between quantities. Many PTs (n = 6) thought that it was hard for students to perceive multiplicative relationship since they worked on addition in early grades in school. Then, a discussion took place on the factors that could help the students to see the multiplicative relation easily:

> T: 5:2 is not an easy relation to capture in the task. What kind of relations may be easy for students?
> PT5: Half.
> PT4: Quarters.
> Two PTs: 2 times.

These PTs' (n = 4) responses indicated that PTs ordered the tasks according to difficulty levels. They concluded in fair sharing tasks that students first learned the concepts of half, two times and quarters. Also, students could learn repeated halving or doubling. Thus, these students could recognize these relations easily. This showed that PTs could order the task difficulty by taking into consideration the students' readiness level including their prior experiences and current mathematical knowledge.

The last activity of this week was watching a 2nd grade student's video. In this video, PTs tried to capture the student's mathematical strategies, difficulties and errors. Also, they tried to illuminate the connections between the student's actions and mathematical knowledge of equipartitioning they have been learned so far.

I informed the PTs that the student knew division and fair sharing. The first task was fairly sharing 24 candies among six children. PTs recognized that the student utilized 1-1 correspondence to deal the candies among six children. The children formed groups of four candies in an array format. The teacher in the video asked to the student, "How did you decide to stop giving the candies?" Then, the student counted the candies by ones up to 24.

> T: Why do you think that she asked this question?
> PT5: She checked whether the student exhausted the whole candies.
> PT1: Yes, one criteria of equipartitioning.
> PT7: She counted by ones to check whether she consumed all 24 candies.

The interaction above shows that the PTs could capture students' mathematics and situate their anticipation of the student's mathematical thinking and reasoning on the evidence gathered from her actions and language. In addition, PTs utilized their mathematical knowledge of equipartitioning to make sense of the student's mathematics.

The student circled six candies in the array format and the teacher asked the reason for that particular action. The student stated "Since 24 divided by four is six, so I circled six candies per child." Then PT1 said that "*Now, the student used division to find each person share, not 1-1 correspondence? She only drew the array structure [3x8] by drawing one candy at a time*."

A discussion took place on how this explanation of the student informed us. Majority of PTs (n = 7) indicated that to understand student's mathematical thinking and solution, we should evaluate the whole process. For instance, PT3 stated, "*Here, if the teacher thought that the student gave correct answer and moved to the next mathematics problem, we would not be able to see that student utilized division to find each person share.*" PT5 added, "*Yes, it is important to ask students how and why you did it even though child gave a correct response*." These PTs' comments showed that they have developed the idea that to understand a student mathematical understanding was not focusing on the correct answer and accepting it an evidence for conceptual understanding. This finding showed that the PTs restructured their knowledge about how they anticipated students' mathematical thinking, strategies and how they made sense of student's mathematical knowledge.

I checked whether the PTs captured the naming practices of the student:

T: How did the child name each share?
PT6: Six candies per child.
PT2: Six candies.
T: What did we call these naming practices?
PT9: The first one is ratio.
PT1: The second one is counting.

The comments of the PTs above showed that they learned the different naming practices embedded in the ELT and utilized this mathematical knowledge of equipartitioning to recognize the student's mathematical naming practices.

The teacher in the video asked students if there was any relation between the numbers 24 and 4. The student could not answer the question. Then, the teacher presented another problem. The problem asked for fairly sharing six candies among three children. A discussion on this teacher input took place as follows:

> T: Why did the teacher change the problem?
> PT3: Teacher, 24 is really greater than four.
> PT2: Yes, six and three is easy since six is two times larger than three.
> PT1: Teacher, we had learned that the young students could understand the relations such as half, double easily. The teacher may think this. The student could not see the six times relation between 24 and 4.
> PT5: [The teacher] wanted to start from the student's [readiness] level.

This discussion showed that the PTs could utilize their knowledge of students and equipartitioning to understand the logic behind the teacher's input. Also, they recognized the importance of counting students' prior knowledge level while presenting the mathematical tasks in an order.

After the discussion, I resumed the video. The student shared the six candies among three children and indicated each child got two candies. The teacher asked for justification. The student stated, "Two plus two plus two is six" and the following discussion took place among PTs:

> PT7: The student used composite unit.
> PT3: The student verified her answer with addition.
> T: Which idea of equipartitioning this verification is related with?
> PT6: Reassembly.
> T: OK, do you think this student exhibits a complete understanding of reassembly?
> PT8: The student did not establish the multiplicative relation.
> PT5: Yes, two times three is equal to six.

The discussion above suggested that the PTs could identify level of mathematical thinking complexity of the student based on the evidence gathered from the student's response. Because, they learned the student's progressions along the LT in the teaching experiment.

5.5 Summary of Teaching Sessions' Findings: Restructuring Practices for Knowledge Types

The findings revealed that the PTs exhibited seven knowledge restructuring practices for Mathematical Content Knowledge and four for Student Knowledge. The findings above also indicated the PTs enhanced their MCK and SK as they progressed in the LTBI teaching experiment as a result of social interactions among peers and the researcher, the LT-based tasks supported this progress.

The findings showed that the PTs exhibited two practices to restructure their HCK. The first practice is called "*connecting*" (Adapted from Wilson et al., 2013). In this practice, PTs were engaged in two activities. First, PTs built connections across the mathematical ideas embedded in the ELT. Also, they discussed the interdependence among the various ideas of the equipartitioning concept. Second, PTs associated the equipartitioning ideas with the further mathematical topics including ratio, rate, proportion, multiplication, division, fractions and area. The second practice is called "*generalizing*" in which the PTs expressed the mathematical ideas in generalizable forms and extended the mathematical concepts and ideas.

The PTs exhibited two restructuring practices of SCK. The first one is called "*internalizing*". In this practice as PTs progressed in the experiment, they made sense of a variety mathematical explanations, strategies and representations for the ideas in the trajectory (Wilson et al., 2013). The PTs discussed various strategies for solving a mathematical task and explaining and revealing the mathematics behind the task. Also, they demonstrated the mathematical concepts and strategies in different ways including drawings, and material usage. This practice provided PTs with more opportunities to understand the mathematics behind each equipartitioning tasks and concepts. In this internalization process, PTs also learned to utilize accurate mathematical language to communicate their mathematical thinking. The second SCK practice is called "*sizing up*" that refers to examining the underlying reasons behind the mathematical errors, difficulties and misconceptions. The PTs examined either their own or students' errors, difficulties and misconceptions. Prior to experiment and in the early phase of the experiment, the PTs exhibited a tendency to focus merely on correct response and ignore the incorrect ones. However, as they progressed, they developed an understanding and knowledge for examining both types of responses. As a result, they exhibited a deeper conceptual understanding about the mathematical aspects of the errors, difficulties and misconceptions.

The PTs exhibited three restructuring practices for CCK. The first practice is called "*remediating and shifting*". In the experiment, the PTs showed that they possessed various mathematical errors, difficulties and misconceptions, with the help of the LT, the PTs remediated these existing errors and misconceptions and overcame their mathematical difficulties. At the end, they changed their way of mathematical understanding and perception of the task being engaged. The second practice is called "*expanding*". The findings revealed that prior to experiment and at the initial phase of the teaching sessions, PTs had knowledge gaps in their skills for solving a mathematical task correctly. After the LTBI, PTs learned new mathematical strategies, concepts, ideas and representations to solve the problem. At the end, PTs seemed to acquire necessary skills to solve a mathematical task along with a conceptual understanding of the mathematics that was being employed in the solution. The third emergent practice is called "*challenging*". These actions of PTs were coded as emergent practice since challenging actions were recorded in limited numbers in the experiment. One of the possible reasons of this rare observation of the action could be that the PTs challenged the presented information within the ELT in this emergent practice and developed a reasonable mathematical counter argument. This is a hard task for a PT to exhibit frequently. This emergent practice appeared in two forms. First, the PTs directly challenged the information provided by the researcher-teacher. Second, the PTs challenged their peers' mathematical claims including solutions, strategies, explanations and representations.

The first restructuring practice for SK is called "*distinguishing and recognizing*" (Adapted from Mojica, 2010). In many instances at the beginning of the teaching sessions, the PTs solved the given task in their own way and occasionally these strategies were the ones that young elementary school students could employ. Realizing this difference between their own mathematical thinking and students' mathematical thinking was recorded as distinguishing (Adapted from

Mojica, 2010). In the LTBI, the PTs actively analyzed the students' work, then they could recognize the students' mathematical thinking and they justified this recognition from the evidence gathered from students' work and behavior. The second practice is called "*anticipating*" (Adapted from Stein & Smith, 2011). In this practice, the PTs anticipated student's possible strategies and misconceptions and explained the possible underlying reasons behind them. Prior to experiment, the PTs exhibited a limited proficiency in anticipating students' mathematics in advance. As they engaged in the LTBI and exchanged their anticipation with their friends, they developed an understanding of students' mathematical thinking and strategies that included both the correct and incorrect ones. The third practice is called "*ordering*" (Adapted from Stein & Smith, 2011) in which the PTs ordered the students' mathematical strategies from the least complex to the most with the help of their CCK, SCK and the LT. Also, the PTs identified the possible factors that affected the equipartitioning task complexity. The curricular knowledge of the PTs was also an effective factor in displaying ordering practice since the PTs ordered the task complexity and the strategy complexity through counting the students' readiness level, in other words, their grades in elementary school. The fourth practice is called "*empathizing*". In the instance of PTs showed a misconception, mathematical error or had a mathematical difficulty while engaging with the task, they indicated that the elementary school mathematics was not so easy, as they assumed prior to experiment. They restated their understanding of how a student could acquire a mathematical misconception, difficulty and error that seemed very easy to them. They acknowledged that as teacher candidates, they could also possess the same misconception, error or difficulty as students did.

CHAPTER VI

CONCLUSION AND DISCUSSION

Schoenfeld (2011) simplified the differences between a theory and a framework by stating that, “A framework tells you what to look at and what its impact might be. A theory tells you how things fit together. It says how and why things work the way they do, and it allows for explanations and even predictions of behavior” (p.4). A theory aims to give a clear picture of the phenemona in a particular domain and it aims to provide explanations for the predicted events (Liehr & Smith, 1999). A theory consists of interelated structures and concepts that can be used to systematically explain the phenomena under examination (Chinn & Kramer, 1999; Liehr & Smith, 1999). On the other side, a framework demonstrated the impact of a theory on practice (Liehr & Smith, 1999) and tested the theory in emprical settings. According to Liehr and Smith (1999) a framework could be used to explain the consistencies or discrepancies in the predicted events through utilizing the findings of the research. Practices are the ways of testing a theory and they are deduced from conducting research (Liehr & Smith, 1999).

Learning Trajectories Based Instruction (LTBI), an emergent teaching theory, combines both the theoretical perspectives deduced from the existing research on particular mathematics content and the empirical evidence related to how students learn mathematics. The present study utilized LTBI and investigated how it could be practically implemented within pre-service teachers’ (PTs) training. The findings of the study revealed the knowledge restructuring practices of the PTs through an examination of their initial knowledge levels and their progression and actions in the teaching experiment. The progressions were addressed through common categories in which PTs exhibited evidence for revisions, refinements, and changes in mathematical content knowledge (MCK) and student knowledge (SK). Thus, documenting the practical utilization and impact of a theory in a particular context, I propose a framework for PTs’ knowledge restructuring practices when they engage in LTBI. Analysis on both pilot and actual experiment data illustrated the similarities and differences between the ways the PTs participated in LTBI and exhibited evidence for restructuring MCK types and SK. As a result, this study examined the importance and practical implications of the LTBI theory of teaching in teacher education.

6.1 Restructuring Practices for Mathematical Content Knowledge: Emergent Framework

Several researchers (Ball, 1990; Fernandez, Llimares, & Valls, 2013; Philipp, 2008; Wilson et al., 2013) suggest that developing MCK is an important index for enhancing teaching practices. However, documentation detailing how PTs develop their MCK is needed in the field of mathematics education (Ball et al., 2008; Butterfield et al., 2013; Sherin, Jacobs, & Philipp; 2011). The results from this study documented the MCK restructuring practices of PTs. The PTs were capable of restructuring their CCK, SCK, and HCK, which are integral parts of MCK. This restructuring also resulted in enhancement in PTs' MCK when compared to the level of knowledge prior to this experiment.

At the end of the study, the PTs were capable of producing multiple solutions, representations and strategies for the presented tasks along with a conceptual understanding of the content employed in the tasks. In addition to these capabilities, the PTs reached a mathematical knowledge level at which they were capable of either arguing against the presented information in the LT or generating mathematical strategies not addressed in the LT.

Moreover, the PTs captured their own mathematical misconceptions and errors and corrected them. They were capable of determining underlying reasons behind the misconceptions, errors, and difficulties. They built connections across the levels of the LT and beyond the LT with further mathematical topics along with sensible mathematical explanations that illuminated these connections.

6.1.1 Restructuring Practices for Common Content Knowledge

6.1.1.1 Remediating and Shifting

Parallel with findings of earlier studies (Ball, 1990; Baki, 2013; Philipp, 2008; Spitzer et al., 2011; Zembat, 2007), prior to the experiment in this study, more than half of the PTs exhibited serious mathematical misconceptions, errors, and difficulties in the pretest. However, as they engaged with the LT-based tasks and interacted with their peers and the researcher-teacher, they gained proficiency and remediated their existing misconceptions and errors. Then, they shifted their prior mathematical orientations.

The pretest results also indicated that although majority of the PTs generally produced a single correct response for the presented items, some of the PTs produced incorrect responses for some equipartitioning items. PTs encountered different mathematical strategies utilized by their peers and they became capable of producing multiple solutions and performed these solutions correctly.

During the experiment, some PTs showed the same mathematical errors and misconceptions that elementary school students showed. The medium of the experiment let them think aloud their mathematical thought processes and exchange and discuss their mathematical thoughts with their peers. The discussion and the understanding of their peers' strategies helped those PTs procedurally solve the given problems correctly. Although this was an indication for remediating their initial incomplete CCK, the PTs who solved the given tasks correctly still lacked the sufficient SCK to differentiate the responses indicating procedural knowledge and those indicating conceptual understanding (Morris, 2006). They usually could not produce a complete mathematical explanation for why their procedure worked on the given problem in the early stages of the experiment and in the pretest which raised the need for restructuring their SCK. The conclusions about the SCK restructuring practices will be addressed in the following section.

6.1.1.2 Expanding

The results of this study have shown that prior to the experiment in the pretest, the majority of the PTs performed a single procedural solution to generate the correct answer. As they engaged with LT- based tasks and interacted with one another, they realized there were more to equipartitioning related ideas. They encountered new mathematical strategies, concepts, and representations in reviewing their peers' solutions and in some instances I brought some cases.

During the experiment and in the pretest, majority of the PTs exhibited a certain degree of understanding about a mathematical concept and idea. They focused on some aspects of the concept and the idea without considering the other aspects. In such instances, I selected several PTs' works focused on one aspect of the concept and used them to initiate a discussion. The social exchange in the class during the discussions supported the PTs in learning multiple aspects of the concepts. As Bransford, Derry, Berliner, and Hammerness (2005) suggested, the initial knowledge levels of people can influence understanding the others knowledge levels. Exchanging those strategies in the classroom discussion helped the PTs learn different justification strategies for fair shares; hence, they expanded their CCK. Wilson et al. (2013) stated that teachers with different mathematical content knowledge influenced the way they engaged and learned mathematics. They stated that the different knowledge levels played a mediator role in learning. Similarly, in this study, the ways the PTs participated in the discussions and the solutions and representations they produced for their mathematical arguments were affected by their initial CCK. Because of these existing differences, the PTs shaped each other's CCK restructuring practice and mathematical learning.

In this study, expanding CCK of the PTs supported them in identifying various students' mathematical thinking and strategies in advance. Also, they used their CCK to deeply examine the mathematical thought processes of their peers and their students in their teaching practices. In this study, building a comprehensive and correct CCK played an important role in restructuring PTs' student knowledge.

6.1.1.3 Challenging

Parallel with the Empson's (2011) argument about the pitfall of LTs, the results of the study showed that the PTs' mathematical actions in the LTBI were not fully coherent with the suggested mathematical ideas and progression in the LT. In addition, as Clements and Sarama (2013) suggested, there is no one single trajectory for every learner. Such a trajectory may be subject to change based on the learner's knowledge level, experience, and the learning setting. The results of this study addressed these concerns through practical usage of the LT in a learning and teaching setting and opening such instances for discussion.

In this study, challenging was recorded as emergent practice of the PTs since the instances for challenging were observed occasionally in the teaching experiment and in the tests. Challenging was observed in two forms. In the first form, the PTs produced mathematical strategies for solving equipartitioning LT-based tasks, which were "out of sequence" (Empson, 2011, p.380). These strategies were referred as "out of sequence" since the LT did not include descriptions of these strategies. In the second form, a few PTs challenged the suggested ideas and progression in the LT and produced alternative reasonable mathematical explanations for the progression. In both instances, the PTs supported their claims with reasonable mathematical arguments and provided counter examples. This practice of the PTs indicated that they reached a level at which they started to develop sound mathematical arguments against the presented knowledge through utilizing their mathematical content knowledge and experiences in the LTBI. Therefore, as PTs progressed in the LTBI, they were able to produce new lines of mathematical reasoning and understanding that were more independent from my guidance and the sequence suggested by the LT.

In the experiment, these instances were utilized as a tool for creating rich learning opportunities and fruitful discussion opportunities for the PTs (Phillipp, 2008). Thus, an important conclusion of this study is that "out of sequence" instances helped the rest of the PTs to learn more about the equipartitioning related ideas in addition to the existing ideas in the LT. Moreover, the challenging practice of some PTs also influenced the expanding practices of the rest of the PTs. Based on this, one could conclude that a particular PT's challenging restructuring practice could trigger another knowledge restructuring practice for another PT. Therefore, these knowledge restructuring practices were interconnected and an enhancement in one of them had the potential to influence other practices.

6.1.2 Restructuring Practices for Specialized Content Knowledge

6.1.2.1 Internalizing

The present study has shown that, although the PTs utilized a mathematical concept and idea while engaging in the LT-based equipartitioning tasks, they did not fully internalize the meaning of that concept or idea. However, the socially constructed learning environment in the LTBI teaching experiment helped PTs to gain insight regarding the meaning of these concepts and ideas. This elaboration led PTs to learn multiple aspects of the mathematical ideas and concepts, including both working and formal definitions. For instance, although the PTs could perform equipartitioning, they could not define what equipartitioning was. As they worked on different equipartitioning tasks and reflected on their mathematical actions in each task, they began to internalize three criteria of equipartitioning and then they were able to clearly define the concept.

The classroom interaction experiences and guidelines for searching multiple solutions were found to be essential elements to understand the mathematics behind the multiple representations, including table, verbal, pictorial and mathematical representations. The PTs realized that different representations conveyed various mathematical meanings, for example they presented the same quantity with two different representations when they fairly shared a single whole. The analysis of the different representations led PTs to discuss different mathematical topics such as the equivalent fractions. The PTs also connected these various representations and went beyond the verbal descriptions of their mathematical solutions. Similar to Simon and Tzur's (2004) findings, the PTs in this study also could integrate their reasons for utilizing a particular representation to show their SCK. To achieve this, the PTs did not provide the initial explanations of their solutions that included only a verbal explanation of the algorithm being employed in the pre-test and in initial phases of the experiment. At the end of the LTBI, they could describe the mathematical concepts and solutions by connecting mathematical symbols and the representations being used. In addition, they could utilize mathematically more precise and correct language.

As a result, as PTs' SCK was restructured and broadened, they started to realize and create links between different ways of thinking, representations, and strategies produced by their peers and themselves. Internalizing the mathematical meanings behind these strategies, explanations, and representations guided the PTs to enrich their SCK of equipartitioning.

6.1.2.2 Sizing Up

One of the important implications of this study was that PTs might not be sufficiently challenged in the teacher education program to examine mathematical solutions deeply. The results of this study support Bartell et al.'s (2012) findings in that the PTs assumed that solving a given mathematical problem procedurally was an indication of an internalized understanding of the mathematics involved in the problem. In addition, when a task was solved incorrectly, they assumed there was nothing to discuss. As Crespo (2000) suggested, the PTs directly assumed the student did not know the required mathematics if they produced an incorrect answer.

Prior to the LTBI, the PTs did not have a clear understanding of sizing up the underlying reasons behind students' mathematical errors, misconceptions, and difficulties. The majority of them indicated that the students did not know the mathematics required to solve the problem or they did not generate a sensible explanation that illuminated the factors that led to the incorrect student response. My encouragement for the PTs to discuss and explore each step of the solution and then to compare them with the correct solutions helped them begin to extract the underlying reasons behind the incorrect responses. As PTs engaged with more incorrect solutions, the PTs' reasoning to solve the mathematical meanings behind the responses became more detailed and more explanatory rather than judgmental.

The results of the study showed that in addition to gaining the knowledge of sizing up the reasons behind incorrect solutions, the PTs also possessed the knowledge of examining the conceptual knowledge behind the mathematically correct solutions. In parallel with the findings of several studies (Bartell et al., 2012; Phillipp, 2008), the PTs did not think about examining the correct responses. They assumed that producing correct response was enough evidence for a comprehensive mathematical knowledge of the topic. This was the case for the PTs in the early phase of the LTBI teaching experiment. Nonetheless, as PTs developed the habit of asking questions about the completeness and meanings of the calculations and the strategies and representations in the teaching experiment, they were capable of eliciting the mathematical thinking processes and interconnectedness of the mathematical ideas even in the correct responses. One PT explained this situation as; they should search for an answer even when they see a correct response. This showed that the PTs restructured their SCK by acquiring the skills for sizing up the reasons behind each mathematical misconception, error, and difficulty that they faced in the students' responses. In this sizing up process, the PTs utilized their CCK of the equipartitioning and related mathematics.

During the experiment and in the pretest, some of the PTs also exhibited serious mathematical misconceptions, errors, and difficulties. In such instances, the PTs who did not exhibit the same misconception or error argued against their friends. In these arguments, the PTs utilized their CCK and HCK to show that their claims were correct. They also utilized counter examples to show that their claims were valid. Forming such arguments and exchanging the mathematical ideas within the classroom helped the PTs to restructure their SCK; as a result, they could size up their peers' incorrect responses. In addition, the implication of sizing up practice rooted in peers' incorrect responses triggered another restructuring practice called remediating and shifting for the ones who exhibited the misconception, error, or difficulty. Also, the PTs practiced testing their arguments and producing counter examples by utilizing their mathematical knowledge while trying to discredit their peer's claims. This phenomenon in the study formed the basis for restructuring their

HCK in the generalization practice. These interactions showed, once again, that these knowledge restructuring practices were interconnected, and advances in one had the potential to start the process in other practices.

6.1.3 Restructuring Practices for Horizon Content Knowledge

6.1.3.1 Connecting

Sztajn et al. (2012) indicated that *connecting* is an integral part of LTBI. According to them, connecting refered to addressing the relations between the students' strategies and mathematical idea development. They also stated that HCK refered to the understanding of the most complicated mathematical idea that was situated at the highest level of the LT. In this study, connecting practice included the PTs' ability and knowledge to build the connections across levels of the LT, and it also entailed the PTs' ability to perceive the connections between the mathematical ideas in the LT and further mathematical topics. This practice was achieved not only through examining the relations between the different mathematical strategies, solutions, and ideas suggested by the PTs, but also, with the help of the further prompting, connecting practices were observed when PTs detected the possible contribution of the presented mathematical idea with further mathematics.

Equipartitioning serves as a basis for further mathematics, including multiplication, division, measurement, ratio and proportion, and fractions (Confrey et al., 2008). At the end of the study, the PTs connected (1) equipartitioning collections with partitive division; (2) the reverse action of equipartitioning with multiplication; (3) equipartitioning single whole with area, unit concept, and multiplicative factors of a positive integer; (4) utilization of a composite unit of the composition of factors to split a whole fairly with equivalent fractions; and (5) comparison of the equipartitioning single whole and multiple wholes with fraction types and meaning of the fractions, including (6) covariation with ratio types and equivalence of two ratios in covariation tasks with proportion.

The important conclusion based on these results could be that perceiving this interconnected web of mathematical knowledge was an evidence for that the PTs were building a conceptual MCK (Ball & Bass, 2009). Therefore, utilizing LTBI in teaching resulted in better conceptual understanding of the mathematics that the PTs were supposed to teach. Another important conclusion of this study was that I and the PTs did not merely focus on covering the equipartitioning concept. Rather, within implementation of the LTBI, the socially constructed learning environment provided the PTs a means to think of the equipartitioning related ideas more deeply and to see the interconnected structure of the mathematics.

6.1.3.2 Generalizing

Prior to the teaching experiment, many of the PTs exhibited a limited ability to determine a ground for their mathematical assertions and deductions. They failed to explain how they reached mathematical conclusions. Initially, in the teaching sessions, when the PTs were asked for generalizations, they performed a "pattern-spotting" activity (Noss, Healy, & Hoyles, 1997). Besides, in the pretest, when they asked for a mathematical generalization, the majority of the PTs could not even perceive the pattern between the variables.

The findings of this study indicated that opportunities to engage in LT-based tasks that asked for mathematical generalizations led the PTs to enhance their knowledge about abstracting and generalizing a detected pattern. LTBI supported the PTs in engaging in the activities and interactions to perceive the links between what they were doing and observing and the mathematical meanings behind their actions. Thus, at the end of the study, the PTs utilized their restructured SK and MCK to determine a sound basis for their claims, and they explained what led them to this conclusion. They finally possessed the knowledge of generalizing to express these links in mathematically abstract and generalizable forms.

The findings of this study showed that LTBI helped the PTs to understand a method sufficient to calculate mathematical objects, conditions, or generalizations (Carraher, Martinez & Schliemann, 2007). Moreover, PTs became capable of examining and discussing underlying structures behind the mathematical generalizations by manipulating the given concrete materials or conditions, arguing against their peers' mathematical explanations, and deductions. All these have indicated that the PTs restructured their HCK.

6.2 Restructuring Practices for Student Knowledge: Emergent Framework

According to Sztajn et al. (2012) and other studies (e.g., Bartell et al., 2012; Jacobs et al., 2010; Philips, 2008), knowledge of students entails the ability to recognize their mathematical thinking. This recognition requires the knowledge of students' mathematics (Jacobs et al., 2010). This mathematics includes the knowledge of students' misconceptions, errors, difficulties, and strategies as they learn mathematics (Jacobs et al., 2010; Stein & Smith, 2011; Wilson et al., 2013). Parallel to previous studies, the present study documented the actual practices of the PTs to reshape their existing knowledge to identify students' mathematics. As Sherin et al. (2011) indicated, there has been a need for this line of research in the mathematics education field. The knowledge restructuring practices deduced from the data of this study exhibited how PTs' knowledge of students changed over the course of the LTBI teaching experiment on equipartitioning related mathematical ideas. The results of the study showed that the LTBI teaching experiment was successful in supporting PTs in restructuring their knowledge about students' mathematics of equipartitioning.

Prior to the teaching experiment, the majority of the PTs exhibited a limited knowledge about how students engaged with mathematical thinking and how they learned mathematics. The majority of the PTs failed to provide a robust understanding of students' mathematics. The results of this study indicated that a LTBI teaching experiment helped the PTs to restructure and enhance their limited student knowledge, and it supported them in understanding and explaining students' mathematical learning and thinking. This conclusion was also supported by the earlier studies of LTs conducted with either teachers or PTs (Mojica, 2010; Wilson, 2009; Wilson et al., 2013). The restructuring practices documented in the study were anticipating, distinguishing and recognizing, ordering and emphaticizing. These practices are important for PTs to decide which part of mathematics learned is important for their future teaching practices through focusing on students' mathematical thinking (Philipp, 2008; Wilson et al., 2013). Also, this helps them to avoid imposing their mathematical thinking onto students (Crespo, 2000). Enhancement in student knowledge motivates the PTs to achieve conceptual mathematics learning rather than mere procedural knowledge accumulation (Philipp, 2008).

6.2.1 Anticipating

Similar to Stein and Smith's (2011) perspective about anticipating practice, in this study, anticipating practice required the PTs to know in advance the various mathematical ways students might exhibit as they worked on the presented mathematical tasks. Knowledge of students' possible mathematical approaches could help PTs set the learning paths in future teaching that considers the students' input. The findings of the study have shown that, initially, the PTs exhibited a tendency of predicting only mathematically correct strategies. Also, as Crespo (2000) suggested, the PTs in this study had a tendency to impose their mathematical approaches in their prediction. This also showed that distinguishing practice was also essential to acquire the knowledge for anticipating student strategies. As PTs encountered more student work and drew upon the experienced ideas of the LT in the study, they exhibited distinguishing practices. As a result, they could think more independently from their own mathematical lenses when they envisioned the students' mathematics in advance.

On the other hand, when PTs were asked to examine students' mathematical thinking on a given actual student work, they focused on merely incorrect answers because they assumed that students producing correct answers had acquired sufficient mathematical knowledge (Philipp, 2008; Spitzer et al., 2011). This implied that the PTs did not envision the possible mathematical approaches of the students in advance, including correct and incorrect ones, in the early phases of the LTBI. The LTBI allowed me to further elaborate on this issue. At the end, the PTs acquired the knowledge of students' mathematical approaches toward a task including strategies, errors, misconceptions, and difficulties. Furthermore, they started to realize that the anticipation of students' mathematical thinking did not merely entail correct responses. They acknowledged the importance of knowing that the students could exhibit mathematical misconceptions and errors. They came to an understanding that the detailed analysis of these incorrect ways in their teaching could yield fruitful mathematical learning.

As the LTBI progressed, the PTs used the prior knowledge of the LT and they interacted more on the variety of the mathematical approaches of the students. They also utilized a correct mathematical terminology to describe the anticipated strategies. At this level, one important implication of this study was recorded. The PTs asked for the grade levels of the students so they could predict the mathematical approaches in advance by considering students' prior knowledge level and future mathematical goals. As the in-service teachers acknowledged this need in Wilson et al.'s (2013) study, the PTs in this study also needed this information. Although the PTs acquired the knowledge of ordering students' strategies described in the LT regarding their sophistication levels, they still needed the grade level information. The result of the study acknowledged the importance of the Empson's (2011) critique of the LT on the ways children undertand and utilize the mathematical strategies influenced by various factors including which classroom they are in and their individual charecteristics. The ELT did not include the grade level or age information for the particular proficiency level. Another important implication of this result is that both PTs and in-service teachers needed to see this information in the LT. This lack of information in the LT was handled through the input by me and the PTs during classroom interaction.

6.2.2 Distinguishing and Recognizing

In this study, the PTs initially solved the given tasks in their own way, and occasionally these strategies were the ones a young elementary school student could employ. Realizing this difference between their own mathematical thinking and students' mathematical thinking was called "distinguishing" (adapted from Mojica, 2010). As Philipp (2008) reported, the PTs assumed that the children would use the same line of mathematical reasoning as they solved the problem. For instance, in many instances, many of the PTs thought students would use division to solve simple equipartitioning collection tasks. However, the students, especially the younger ones, used counting, drawing, and dealing to solve a given task. By the end of the experiment, the PTs were capable of distinguishing between their mathematical strategies and the children's strategies. As a result, they generated a variety of mathematical strategies and representations in addition to their own mathematical solution methods. Some of the PTs noted these strategies and representations, and students would solve this way to show they could distinguish their mathematical thinking from the students' thinking.

Before the experiment, the majority of the PTs were capable of deciding whether a student produced mathematically correct response or not. Also, a few PTs sometimes failed to decide the correctness of the students' response or strategy in the pretest. Because, they also shared the same mathematical misconceptions or errors the students did. In addition, when PTs were asked to notice students' mathematical thinking in students' actual work, the PTs did not recognize the evidences of students' mathematical thinking in the actual work; instead, they only evaluated whether students could generate a correct response. As Spitzer et al. (2011) found, the PTs in this study also initially exhibited a tendency to doubt students' mathematical understanding when their work included an incorrect response.

Initially, the PTs who could detect the correctness of the response or the strategy exhibited a limited ability of attending to the significant details about students' mathematical thinking. Parallel with the Bartell et al.'s (2012) findings, this study has shown that the PTs had difficulty in recognizing students' mathematical thinking. Thus, the design of the study allowed for PTs to attend to the students' possible mathematical strategies by bringing different strategies into discussion from each PT's suggestion and actual student work. Also, similar to Morris' (2006) findings, the PTs often recognized unrelated evidences to explain and argue about students' mathematical thinking. Morris (2006) stated that the PTs utilized the teacher's explanation to back up their claims. In this study, different from the Morris's (2006) findings, the PTs utilized their mathematical content knowledge to back up their claims without rooting their assumptions into actual evidence in the student work. They created assumptions about the students' mathematical thinking based on their mathematical reasoning and knowledge, yet they could not show actual evidence to validate their assumptions from the students' work. This tendency of the PTs was restructured during the experiment and at the end, the PTs could recognize actual evidence of students' mathematical thinking in the presented works and utilized this evidence for explaining student mathematics. In addition, the PTs acknowledged the importance of this recognition for creating fruitful mathematical discussion in their teaching.

As a result, at the end of the LTBI experiment, the PTs' ability to attend to the students' mathematical thinking and to provide evidence for that mathematical thinking from actual student work has been enhanced. This also showed that the PTs started to recognize significant details about students' mathematics in their actual work, regardless of whether students produced correct or incorrect responses. This restructuring practice was supported through social interactions between me and PTs, and active engagement in LT-based tasks.

6.2.3 Ordering

Prior to the experiment, the PTs exhibited a limited proficiency in ordering the task difficulty and mathematical ideas' complexity. As Jacobs et al. (2010) stated, the ability of ordering required knowledge of students' mathematics. As PTs encountered students' mathematical thinking in the LTBI experiment and engaged in the activities that helped them to discuss and interpret students' mathematical strategies during the LTBI, the PTs started to pay attention to various dimensions that made a task easy or difficult for the students. The PTs used their restructured SCK and CCK to understand why the task might be difficult for the students. Because SCK required PTs to identify mathematical sophistication for the task in terms of internalizing hidden mathematical structure (Ball et al., 2008). In addition, in restructuring practice, the PTs also discussed the underlying reasons behind the mathematical difficulties. CCK helped PTs employ various strategies correctly on given tasks. The PTs drew their identification of the task order upon both on their CCK and SCK, and their personal mathematical difficulties faced while engaging LT-based tasks.

As a result, they could order the mathematical strategies, tasks from the least to the most complex in the LTBI as it was suggested by the LT with an exception in the challenging practice. This ordering practice helped PTs to understand in which order they could navigate their future students (Mojica, 2010; Stein & Smith, 2011). One PT explained: "I would start my lesson on fair sharing single whole differently. … However, now I know the mathematical thinking behind each strategy and which strategy is the least complex one." Also, this knowledge helped the PTs to understand where to start and where to finish their teaching (Clements & Sarama, 2013; Mojica, 2010).

In addition to utilizing their experience in the LTBI and their CCK and SCK, the PTs took into consideration of students' grade level in their ordering. They asserted that the students' readiness level played an important role in their proficiency in solving given mathematical tasks. As Confrey (2006) suggested, all students brought their personal experiences and prior knowledge into the classroom and that shaped their learning route within the trajectory. The PTs considered this prior knowledge of students to decide how students navigate the least complex to the most complex mathematical ideas as they progressed in the experiment.

6.2.4 Empathizing

Several studies (Bartell et al., 2012; Crespo, 2000; Jansen & Spitzer, 2009; Philipp, 2008; Spitzer et al., 2011; Wilson et al., 2013) focused on understanding the differences between the students' mathematical thinking and our thinking level. These studies documented the importance of recognizing students' thinking ability. They also addressed that students thought differently from the way teachers and teacher candidates thought. As a result, perceiving mathematics through the lens of students was an important asset for teachers and PTs. This ability and knowledge of understanding student work and examining their mathematical thinking is an essential element for teaching mathematics. This study's findings also supported the results documented from the previous studies. However, this study also revealed an important practice related to enhancing student knowledge. Realizing the difference between the ways of mathematical thinking was important, yet it was not merely enough to possess a comprehensive understanding of student's mathematics. Being capable of empathizing with students' thinking was also important in teaching and learning practices. This practice helped PTs avoid being judgmental and evaluative about students' mathematical work immediately. Instead, they perceived them as an opportunity for rich learning.

In the beginning of the study, the PTs thought that the content related to equipartitioning was very easy. However, as they engaged with the tasks, the PTs actually uncovered a misconception, mathematical error, or had mathematical difficulty. The discussion on the various mathematical strategies and solutions produced by the PTs helped them to remediate their misconceptions and errors. Thus, they realized that elementary school mathematics was not as easy as they assumed (Philipp, 2008). Also, they acknowledged that they also shared some mathematical misconceptions that the elementary school students did. One of the PTs in the study explained, "we understand better how a student can show a misconception that seems so easy to us. Even if as teacher candidates we exhibited the same [misconception]."

CHAPTER VII

IMPLICATIONS

As Lowery (2002) suggested, the PTs should be trained in ways that will be similar to what they will be teaching. Demands of teaching mathematics, such as examining the validity of a mathematical argument, selecting the most effective mathematical representation of the content being taught, mathematical knowledge, and skills required for teaching are rarely addressed in university level mathematics courses (Ball et al., 2008; Toluk- Uçar, 2010; Zembat, 2007). Based on that, the mathematical knowledge learned in university level courses remains insufficient for meeting these sorts of mathematics demands. One important conclusion of this study is that the LTBI teaching experiment at the university level helped the PTs to acquire a deep and comprehensive MCK and the necessary knowledge to teach conceptual mathematics. Also, it helped them to recognize students' mathematical thinking in a particular big mathematical idea called equipartitioning. As Ball et al. (2005) suggested, acquiring MCK is not sufficient to capture the conceptual evidence of learning in student mathematics. The structure of LTBI in this study provided opportunities for PTs to engage with both aspects and to restructure their MCK and SK.

The findings of the study have implications on how to prepare PTs. The courses at the university level typically do not provide PTs with immediate access to actual students (Lowery, 2002; Philipp, 2008; Philipp et al., 2007). Within the design of this LTBI research, the PTs had opportunities to possess a body of rich knowledge of students deduced from the mathematics teaching and learning research and LT based research. Designing a teaching environment that situated around students' mathematics is an effective alternative for the current course design provided at the universities. PTs realized both the similarities and differences between their own way of mathematical thinking and the students' thinking. This understanding could help future teachers overcome their existing anxiety

(Hacıömeroğlu & Taşkın, 2010) related to teaching elementary school mathematics to young children.

Another important implication of the study is that the PTs' judgmental disposition toward students' mathematical thinking was challenged. Prior to the LTBI, they made assumptions about students' mathematical abilities, which Clements and Sarama (2013) referred underestimate students' mathematics and perceiving correct response evidence for conceptual understandings and incorrect responses as evidence for not knowing the mathematics. However, restructuring their SK helped the PTs to deeply understand elementary mathematics and how students learned it. As a result, the PTs became less judgmental about students' mathematical thinking; instead, they tried to understand "what sense the [student] is or is not making" (Philipp, 2008, p. 23). This helped them to avoid their initial judgmental discourse (Ball & Chazan, 1994; Philipp, 2008) and set realistic mathematical learning goals for their students (Clements & Sarama, 2013).

Researchers indicated that the current mathematics education course structures at the university levels might not challenge the PTs mathematically (Ball, 1990, Ball et al., 2008; 2008; Philipp, 2008; Ubuz & Yayan, 2010). The design of this study has an important implication on how PTs could be challenged to generate various mathematical strategies and justifications based on solid mathematical evidence. In addition, the PTs were engaged in a learning environment in which they asked to utilize and restructure their MCK. The structures of the LTBI teaching experiment resulted in a better mathematical understanding and the PTs transformed their prior mathematical knowledge, which will be more useful for their future teaching practices.

Grossman and McDonald (2008) stated that voicing teaching work was the action of composing connected and learnable elements without making them isolated and unrelated, this has been one of the important challenges of teacher education. Ball and Forzani (2009) stated that this challenge was rooted in the difficulty for teachers to situate their teaching practices in relation to various connected mathematics ideas and the students being taught. The findings of this study have an implication on addressing this challenge because the PTs' HCK and SK were interconnected in this study. The PTs did not experience equipartitioning concepts in the teaching session as an isolated mathematical construct. Instead, the LT-based tasks and the social interactions in the classroom helped them to relate various mathematics topics with equipartitioning. Also the knowledge of mathematical sophistication of the content or tasks based on the students' levels counted as a part of ordering practice for SK. Thus, PTs had a chance to face the challenge suggested by Grossman and McDonald (2008) prior to their future teaching practices.

Hammerness, Darling-Hammond, and Bransford (2005) indicated, "The knowledge, skills, and attitudes needed for optimal teaching are not something that can be fully developed in teacher education programs. Instead, teacher education candidates need to be equipped for lifelong learning" (p.358). At the end of the experiment, the PTs realized it was not possible to completely understand students' mathematics in advance, but they acknowledged the importance of interacting with students and asking about their mathematical thinking directly. This realization for the PTs showed that they recognized the importance of learning from their students' mathematical thinking as they experienced a variety of contexts for their future teaching practices. Spitzer et al. (2011) also supported the claim that developing this understanding was essential for preparing PTs for a rewarding teaching career.

The interaction between the practices documented in this study also showed that the socially constructed learning environment was a key practice for creating a fruitful learning environment. In this environment, the PTs learned from each other not merely by perceiving as the instructor or the given LT as an ultimate authority of knowledge. As Vygostky's (1978) ZPD suggested, in this study, the PTs were capable of producing independent mathematical line of reasoning with the help of their MCK, SK and their experiences in the experiment. As Piaget (1965) suggested the interaction between the PTs through bringing their different mathematical ideas, solution ways even exhibiting different mathematical misconceptions and difficulties helped the PTs to reconstruct their MCK. The PTs' feedback to their peers' mathematical works led improvements and revisions in their current knowledge level. This is also an essential part of social contructivist learning environment. Experiencing this socially constructed learning environment in which the PTs discussed both their own and their peers' mathematical arguments with respect would helped them to create this social learning dynamics in their future classroom teaching.

Ball et al. (2008) examined the sub-categories of MCK and they indicated there is a further need for detailing the mathematical practices of the teachers. As Ball et al. (2008) suggested, this study also framed the mathematical practices that a teacher candidate should engage so that they learn how to teach mathematics to their students. Defining these practices as clear as possible has a potential to inform the design of the mathematics education courses including the instructional materials utilized for teaching.

Final implications of this study would be on the dynamics of LTBI teaching experiment based on my personal experiences while conducting this present study. In this LTBI, the PTs occasionally jump around the ideas within the LT. This showed that the levels of the LT were not linear for the PTs. In the instances of the jumps between the levels of ELT, the researcher-teacher should give them flexibility to examine this jump. For instance, the transitivity level is placed at the level 10 of the ELT and the fair sharing single whole is placed at the level of 3 the ELT. The PTs' misconceptions of diagonal cuts would not create fair shares in the case of splitting a rectangle into four led PTs and I to discuss the transitivity argument. Also, this also had an implication on the way that researcher-teacher perceived the design of LTBI. One should not perceive each level as a set of facts followed by each other. Instead, one should focus on the connections across the levels and encourage the PTs to perceive those relations. This perception was also important for the PTs to utilize their prior experiences and knowledge to reconstruct their knowledge that is an essential part of socially constructed learning.

Another important implication on the dynamics is that PTs could produce unexpected mathematical strategies, solutions and explanations in the LTBI. In such instances, one should focus on the mathematics behind the unexpected actions and use this as an opportunity for further mathematical discussions rather than stick on only the suggested mathematical ideas by the LT. A final comment on the dynamics can be, although the LT did not include the knowledge related to a variety components of the learning environment such as initial knowledge level, students' grade level information, motivation level, the researcher should consider these dimensions while designing the LTBI. Also, knowing the national mathematical curriculum scope of the topic being engaged in the experiment would help the PTs to transfer their knowledge in their future teaching practices in their local country.

The instructional tasks, mathematical goal, students' individual progressions commonly included descriptions in all LT definitions, and the social interaction appeared as one of the most influential factor that determined the trajectory of each PT in this present study. This implies that socially constructed learning environment is a key aspect that determines the learning trajectory of the individuals. Thus, utilization of LTs in socially constructed learning environment has potential to alter the trajectory descriptions for individuals. Also, it might add new dimensions to existent description of LTs such as learners' capability to communicate in classroom settings, exchange ideas, verbalize and mathematize their thoughts in the actual socially constructed classroom environment. The impact of socially constructed learning environment on each learner's trajectory would be examined in further studies.

7.1 Future Research Suggestions

This research qualitatively provided analysis of PTs' knowledge restructuring practices as a part of LTBI. Only 9 PTs participated in the actual teaching experiment. This number also limits the generalization of findings. But in qualitative research the major aim is not generalization but deeply examination the activity being studied (Creswell, 2007). The knowledge restructuring practices frame generated in this study could be utilized in different research settings such as with different PTs, in different universities and in different countries. These case-based replications would contribute the validation of the framework generated from the study. Moreover, well-controlled quantitative studies could be conducted to determine the effects of LTBI on PTs' MCK and SK. In addition, these sorts of research results may provide a useful guidance for the researchers, educators and policy makers designing method courses that are based on students' mathematical thinking and learning.

Existing LT based research conducted with both PTs and in-service teachers were targeted teaching mathematics to younger students. One future suggestion can be conducting LT based research that targets to train PTs and teachers that will teach high school students.

In this study, a complete ELT was utilized as an intervention tool to restructure PTs' mathematical understanding of fair sharing. There are currently 18 LTs and new research studies can be conducted through utilization of these LTs in teacher education. Also combinations of these LTs around the big mathematical ideas could be used in a designed teacher education course.

There are several instructional approaches that existed in mathematics education field (Clements & Sarama, 2013), the framework deduced from this study also can be tested in different teaching designs. This testing will add into existing knowledge related to how PTs or teachers restructure their existing knowledge under different learning and teaching environment. This testing will also explain the possible external factors (i.e. utilization of technology) that also have potential to contribute the enhancement in their knowledge levels.

Finally, the study also showed that the PTs' followed learning trajectories within the LTBI. This LT is determined based on the PTs' mutual efforts, interactions and experiences in the LTBI. This route includes how PTs engaged with the LT, their ways of progression, how they perceive the students' learning route across LT, and how they capture the relations among the levels of LT. These experiences of the PTs in the experiment that utilized LT as a reference tool for teaching is potentially emphasized a new construct that could be called Learning to Teach Trajectory (LTT). LTT progressions of PTs and also teachers can be examined under both qualitative and quantitative research designs. The dynamics of how PTs and teachers learn to teach trajectory can be determined.

7.2 Implications on my Future Career

This research gave me the opportunity to document the practices of PTs in a LTBI teaching experiment. The practices deduced from this study would be my guideline especially in the preparation phase of the further studies. While conducting this research, I utilized only a particular LT called ELT. In my future career, I plan to utilize multiple learning trajectories documented in the literature. One of the big challenges in this process would be deciding which part of the LTs I should negotiate and then, designing the instruction around them through utilizing the framework deduced from this study. Another challenge would be aligning the embedded information in multiple LTs with the mathematics curriculum in Turkey. Searching responses to address these challenges would be one of the further research goals for my future research career.

Another important implication deduced from this study was that each PT shaped and influenced their peers' learning trajectories and knowledge restructuring practices during the experiment. From this point of view, I would like to deeply examine which characteristics and abilities of the PTs and learning environment have an influence on individual LT of a PT and also have an influence on the shared LT followed by the PTs during the instruction. To examine this process, I would like to work with diverse group of PTs. In addition, if possible, I would like to conduct a longitudinal study to observe which of the factors, such as prior knowledge, mathematical ability, and ability of verbalizing thoughts, are the prominent ones.

Last important implication would be determining the practices and progress of PTs with diverse backgrounds (such as academic, social) to determine the impact of LTBI on different groups of PTs. This examination would help me to test further whether employing LTBI in different settings with different groups of learners yield fruitful results in terms of learning and teaching mathematics, as it is defined a comprehensive explanatory emergent theory of teaching (Sztajn et al., 2012).

REFERENCES

Akbayır, K. & Taş, Z. (2009). Preservice teacher's view about mathematic education and teacher training. *Journal of Qafqaz University, 26*, 190-197.

Ambrose, R. (2004). Integrating change in prospective elementary school teachers' orientations to mathematics teaching by building on beliefs. *Journal of Mathematics Teacher Education*, *7*(2), 91-119.

Applebee, A. N., and Langer, J. A. (1983). Instructional scaffolding: Reading and writing as natural language activities. *Language Arts*, *60*(2), 168-175.

Arslan, S. & Özpınar, İ. (2008). Öğretmen nitelikleri: İlköğretim programlarının beklentileri ve eğiim fakültelerinin kazandırdıkları. *Necatibey Eğitim Fakültesi Elektronik Fen ve Matematik Eğitimi Dergisi*, 2*(1)*, 38-63.

Aydın, R.; Şahin, H. & Topal, T. (2008). Türkiye'de İlköğretime Sınıf Öğretmeni Yetiştirmede Nitelik. Arayışları. *Türkiye Sosyal Araştırmalar*, *12*(2), 119-142.

Baki, M. (2013). Pre-service classroom teachers' mathematical knowledge and instructional explanations associated with division. *Education and Science, 38*(167), 300-311.

Baki A. & Gökçek, T. (2007). Matematik öğretmeni adaylarının benimsedikleri öğretmen modeline ilişkin bazı ipuçları. *Hacettepe Üniversitesi Eğitim Fakültesi Dergisi*, 20, 16-25.

Ball, D. L. (1990). The mathematical understanding hat prospective teachers bring to teacher education. *The Elementary School Journal*, *90*(4). 449-466.

Ball, D. L. (2000). Bridging practices: Intertwining content and pedagogy in teaching and learning to teach. *Journal of Teacher Education, 51*(3), 241-247

Ball, D. L. & Bass, H. (2009, March). With an eye on the mathematical horizon: Knowing mathematics for teaching to learners' mathematical futures. Paper based on keynote address at the *43rd Jahrestagung für Didaktik der Mathematik* held in Oldenburg, German.

Ball, D.L., & Forzani, F.M. (2009). The work of teaching and the challenge for teacher education. *Journal of Teacher Education*, *60*(5), 497-511.

Ball, D.L., Hill, H.C. & Bass, H. (2005) Knowing mathematics for teaching: Who knows mathematics well enough to teach third grade, and how can we decide? *American Educator*, 2, 14-22.

Ball, D. L., Lewis, J., & Thames, M. H. (2008). Making mathematics work in school. In A. H. Schoenfeld & N. Pateman (Eds.), *A Study of Teaching: Multiple Lenses, Multiple Views* (pp. 13-44). Reston, VA: National Council of Teachers of Mathematics.

Ball, D. L., Lubienski, S. T. & Mewborn, D. S. (2001). Research on teaching mathematics: The unsolved problem of teachers' mathematical knowledge. In V. Richardson (Ed.), *Handbook of research on teaching* (4th ed., pp. 433-456). Washington, DC: American Educational Research Association.

Ball, D. L. , Sleep, L., Boerst, T., & Bass, H. (2009). Combining the development of practice and the practice of development in teacher education. *Elementary School Journal*, *109*, 458-476.

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education*, 59(5), 389–407.

Bartell, G. T., Webel, C., Bowen, B., & Dyson, N. (2012). Prospective teacher learning: Recognizing evidence of conceptual understanding. *Journal of Mathematics Teacher Education*, *16*, 57-79.

Baştürk, S. (2007). *Öğretmen adaylarının öğretmenlik uygulaması dersiyle ilgili deneyimleri*. Makale 16. Ulusal Eğitim Bilimleri Kongresinde bildiri olarak sunulmuştur. Gaziosmanpaşa Eğitim Fakültesi, Tokat.

Baştürk, S. (2009). Mutlak değer kavramı örneğinde öğretmen adaylarının öğrenci hatalarına yaklaşımları. *Necatibey Egitim Fakultesi Dergisi*, *3*(1), 174-194.

Battista, M. T. (2004). Applying cognition-based assessment to elementary school students' development of understanding of area and volume measurement.

Mathematical Thinking and Learning, 6(2), 185-204.

Bednar, A. K., Cunningham, D., Duffy, T. M., & Perry, J. D. (1992). Theory into practice: How do we link? In Duffy, T. M. & Jonassen, D. H. (Eds.), Constructivism and the technology of instruction: a conversation, Hillsdale: Lawrence Erlbaum Associates, 17-34.

Behr, M., Khoury, H., Harel, G., Post, T & Lesh, R., (1997). Conceptual units analysis of preservice elementary school teachers' strategies on a rational-number-as-operator task. *Journal of Mathematics Education*, *28*(1), 48-69.

Bottorff, J. L. (1994). Using videotaped recordings in qualitative research. In: J. M. Morse (Ed.), *Critical issues in qualitative research methods* (pp. 244–261). Thousand Oaks, CA: Sage.

Boyd, D. J., Grossman, P. L., Lankford, H., Loeb, S. & Wyckoff, J. (2009). Teacher preparation and student achievement. *Educational Evaluation and Policy Analysis*, *31*(4), 416-440.

Brown, A. L., & Campione, J. C. (1984). Three faces of transfer: Implications for early competence, individual differences, and instruction . In M. E. Lamb, A. L. Brown, & B . Rogoff (Eds.), *Advances in developmental psychology* (Vol. 3, pp. 143-192). Hillsdale, NJ: Erlbaum.

Bryant, A. (2003). A constructive/ist response to Glaser. About barney g. glaser: constructivist grounded theory? Published in FQS 3(3). Forum Qualitative Sozialforschung / Forum: Qualitative Social Research, 4(1). Retrieved from http://www.qualitative- research.net/index.php/fqs/article/view/757.

Byrnes, J. P. (1996). *Cognitive development and learning in instructional contexts*, Boston, MA: Allyn and Bacon.

Butterfield, B., Forrester, T., McCallum, F. & Chinnappan, M. (2013). Use of learning trajectories to examine pre-service teachers' mathematics knowledge for teaching area and perimeter. In V. Steinle, L. Ball & C. Bardini (Eds.), *Mathematics Education: Yesterday, Today and Tomorrow* (pp. 122-129). Adelaide, Australia: MERGA.

Carpenter, T. P., Fennema, E., & Franke, M. L. (1996). Cognitively Guided Instruction: A knowledge base for reform in primary mathematics instruction. *Elementary School Journal*, *97*, 3-20.

Carpenter, T. P., Fennema, E., Peterson, P. L., & Carey, D. A. (1988). Teachers' pedagogical con- tent knowledge of students' problem solving in elementary arithmetic. *Journal for Research in Mathematics Education,* 19, 385–401.

Charles, K., & Nason, R. (2000). Young children's partitioning strategies. *Educational Studies in Mathematics*, 43(2), 191-221.

Chinn, P. L. & Kramer, M. K. (1999). *Theory and nursing: A systematic approach*. 5th Edition. St Louis, USA: Mosby.

Clements, J. (2000). Analysis of clinical interviews: Foundations and model viability. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 547-589). Mahwah, NJ: Lawrence Erlbaum Associates, Publishers.

Clements, D., & Sarama, J. (2004). Learning trajectories in mathematics education. *Mathematical Thinking and Learning*, *6(2),* 81-89.

Clements, D. H., & Sarama, J. (2013). Rethinking early mathematics: What is research-based curriculum for young children?. In L. D. English and J. T. Mulligan (Eds.), *Reconceptualizing Early Mathematics Learning* (pp. 121-147). The Netherlands: Springer.

Clements, D., Sarama, J., & Julie, A. (2009). Learning and teaching early math. In The Learning Trajectories Approach (pp. 18). New York: Routledge.

Cobb, P. (1994). Constructivism and Learning. In Husen, T. & Postlethwaite, T. N. (Eds.), *International Encyclopedia of Education*, Oxford: Pergamon Press, 1049-1051.

Cobb, P. (2000). Conducting design studies in collaboration with teachers. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 307-333). Mahwah, NJ: Lawrence Erlbaum Associates, Publishers.

Cobb, P., & Bowers, J (1999). Cognitive and situated perspectives in theory and practice. *Educational Researcher, 28* (2), 4–15.

Cobb, P., Confrey, J., diSessa, A., Lehrer, R., & Schauble, L. (2003). Design experiments ineducational research. *Educational Researcher, 32*(1), 9-13.

Cobb, P., Stephan, M., McClain, K., & Gravemeijer, K. (2001). Participating in classroom mathe- matical practices. *The Journal of the Learning Sciences*, 10, 113–163.

Cobb, P., Yackel, E., & Wood, T (1992). A constructivist alternative to the representational view of mind in mathematics education, *Journal for Research in Mathematics Education*, 23(1), 2-33.

Cole, P. (1992). Constructivism revisited: A search for common ground. *Educational Technology, 33* (2), 27-34.

Confrey, J. (2006). *The evolution of design studies as methodology. In R. K. Sawyer (Ed.), The Cambridge Handbook of the Learning Sciences* (pp. 135-152). New York: Cambridge University Press.

Confrey, J. (2012). Better measurement of higher-cognitive processes through learning trajectories and diagnostic assessments in mathematics: the challenge in adolescence. In V. Reyna, M. Dougherty, S. B. Chapman, & J. Confrey (Eds.), The adolescent brain: learning reasoning, and decision making. Washington, DC: American Psychology Association.

Confrey, J., & Lachance, A. (2000). Transformative design studies through conjecture-driven research design. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 231-266). Mahwah, NJ: Lawrence Erlbaum Associates, Publishers.

Confrey, J., & Maloney, A. P. (2011). A next generation of mathematics assessments based on learning trajectories: Diagnostic learning profiles. In J. Confrey, A. P. Maloney & K. H. Nguyen (Eds.), *Learning over time: Learning trajectories in mathematics education:* Information Age.

Confrey, J., Maloney, A. P., & Corley, A. K. (2014). Learning trajectories: a framework for connecting standards with curriculum. *ZDM - The International Journal on Mathematics Education*, *46*(5), 719-733.

Confrey, J., Maloney, A., Nguyen, K., Wilson, P. H., & Mojica, G. (2008, April). *Synthesizing research on rational number reasoning*. Working Session at the Research Pre-session of the National Council of Teachers of Mathematics, Salt Lake City, UT.

Confrey, J., Maloney, A., Nguyen, K., Mojica, G., & Myers, M. (2009) *Equipartitioning/splitting as a foundation of rational number reasoning using learning trajectories*. Paper presented at The 33rd Conference of the International Group for the Psychology of Mathematics Education, Thessaloniki, Greece.

Confrey, J., Nguyen, K. H., Lee, K., Panorkou, N., Corley, A. K., and Maloney, A. P. (2012). Turn-On Common Core Math: Learning Trajectories for the Common Core State Standards for Mathematics. Retrieved from www.turnonccmath.net

Corcoran, T., Mosher, F.A., & Rogat, A. (2009). *Learning progressions in science: An evidencebased approach to reform.* NY: Center on Continuous Instructional Improvement, Teachers College-Columbia University.

Crespo, S. (2000). *Learning mathematics while learning to teach: Mathematical insights prospective teachers experience when working with students*. Paper presented at the Twenty-Second Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, October 7-10, Tucson, AZ.

Creswell, J. W. (2007). *Qualitative inquiry & research design: Choosing among five approaches* (2nd ed.). Thousand Oaks, CA: Sage.

Creswell, J. W., & Miller, D. L. (2000). Determining validity in qualitative inquiry. *Theory into Practice*, 39(3), 124-130.

Çıkla Akkuş, O. & Duatepe, A. (2002). A qualitative study on the proportional reasoning skills of the preservice elementary mathematics teachers. *Hacettepe Üniversitesi Eğitim Fakültesi Dergisi*, *23*, 32-40.

Darling-Hammond, L., & Ball, D. (1998). *Teaching for high standards: What policymakers need to know and be able to do*. Consortium for Policy Research in Education, Philadelphia, PA, National Commission on Teaching &America's Future, New York, NY. Retrieved from http://www.cpre.org/images/stories/cpre_pdfs/jre04.pdf.

Daro, P., Mosher, F., & Corcoran, T. (2011). *Learning trajectories in mathematics: A foundation for standards, curriculum, assessment, and instruction*. CPRE Research Report #RR-68. Philadelphia: Consortium for Policy Research in Education.

Davis, G., & Hunting, R. (1991). Preschoolers' spontaneous partitioning of discrete items. In R. Hunting & G. Davis (Eds.), *Early fraction learning*. New York: Springer-Verlag.

Davis, G., & Pepper, K. (1992). Mathematical problem solving by pre-school children. *Educational Studies in Mathematics*, *23*, 397-415.

Davis, G., & Pitkethly, A. (1990). Cognitive aspects of sharing. *Journal for Research in Mathematics Education*, *21*(2), 145-153.

Davydov, V.V. (1998). The concept of developmental teaching. *Journal of Russian and East European Psychology*, *36*, 11-36.

Demirel, Ö., & Kaya, Z. (2006, Ed.). *Eğitim Bilimine Giriş*. İstanbul: Pegema Yayıncılık.

Dewey, J. (1902), The child and the curriculum, *in The School and Society and The Child and the Curriculum*, (pp. 67-77), Digireads.com Publishing, 2010.

Duncan, R.G., & Hmelo-Silver, C.E. (2009). Learning progressions: Aligning curriculum, instruction, and assessment. *Journal of Research in Science Teaching*, *46*, 606–609.

Elmore, F. (2002). *Bridging the gap between standards and achievement: The imperative for professional development in education*. Albert Shanker Institute. Retrieved July 7, 2013, from http://www.shankerinstitute.org/Downloads/Bridging_Gap.pdf.

Empson, S. B. (2011). On the idea of learning trajectories: Promises and pitfalls. *The Mathematics Enthusiast*, *8*(3), 571 – 596.

Empson, S. B., & Turner, E. (2006). The emergence of multiplicative thinking in children's solutions to paper folding tasks. *Journal of Mathematical Behavior*, 25, 46-56.

Engelhardt, P. V., Corpuz, E. G., Ozimek D. J. & Rebello, N. S. (2004). 2003 Physics Education Conference, *AIP Conference Proceedings*, 720, 157-160.

English, L. D., & Halford, G. S. (1995). *Mathematics education: Models and processes*. Mahwah, NJ: Lawrence Erlbaum.

Eraslan, A. (2009). İlköğretim matematik öğretmen adaylarının ‘öğretmenlik uygulaması’ üzerine görüşleri. *Necatibey Eğitim Fakültesi Elektronik Fen ve Matematik Eğitimi Dergisi, 3*(1), 207-221.

Eraslan, L. (2008). *Yenilenen Öğretmen Yetiştirme Programları Bağlamında Sınıf Öğretmenliği Programının Değerlendirilmesi*. Çanakkale Onsekiz Mart Üniversitesi Eğitim Fakültesi VII. Ulusal Sınıf Öğretmenliği Eğitimi Sempozyumu, Çanakkale.

Fennema, E., & Franke, M. L. (1992). Teachers' knowledge and its impact. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 147-164.) New York: Macmillan.

Fernández, C., Llinares, S., & Valls, J. (2013). Primary teacher’s professional noticing of students’ mathematical thinking. *The Mathematics Enthusiast, 10*(1&2), 441-468.

Franklin, A., Yilmaz, Z., & Confrey, J. (2010, November). *Reconciling student thinking and theory: The delta learning trajectory and the case of transitivity*. Paper presented at the Thirty- Second Annual Conference of the North American Chapter of the International Group for the Psychology of Mathematics Education, Columbus, OH.

Görgen, İ., Çokçalişkan, H., & Korkut, Ü. (2012). Öğretmenlik uygulamasi dersinin öğretmen adaylari, uygulama öğretmenleri ve uygulama öğretim üyeleri açisindan işlevselliği. *Muğla Üniversitesi Sosyal Bilimler Enstitüsü Dergisi*, *28,* 56-72.

Graeber, A. O., Tirosh, D., & Glover, R. (1989). Preservice teachers' misconceptions in solving verbal problems in multiplication and division. *Journal for Research in Mathematics Education*, *20*(1), 95-102.

Grossman, P & McDonald, M. (2008). Back to the future: Directions for research in teaching and teacher education. *American Educational Research Journal*, *45*(1), 184-205.

Hacıömeroğlu, G., & Şahin-Taşkın, Ç. (2010). Sınıf öğretmeni adaylarının matematik öğretimi yeterlik inançları. *Uludağ Üniversitesi Eğitim Fakültesi Dergisi*, *23*(2), 539-555.

Haser, Ç. & Ubuz, B. (2002). Conceptual and procedural performance in fractions. *Education and Science*, *27*(126), 53-61.

Hill, H.C., Rowan, B., & Ball, D. (2005). Effects of teachers' mathematical knowledge for teaching on student achievement. American Educational Research Journal, 42 (2), 371- 406.

Hunting, R. P., & Sharpley, C. F. (1991). Pre-fraction concepts of preschoolers. In R. P. Hunting & G. Davis (Eds.), *Early Fraction Learning* (pp. 9-26). New York: Springer-Verlag.

Işık, C. (2011). Conceptual analysis of multiplication and division problems in fractions posed by preservice elementary mathematics teachers. *Hacettepe University Journal of Education*, *41*, 231-243.

Işık, A., Çiltaş, A., & Baş, F. (2010). Öğretmen yetiştirme ve öğretmenlik mesleği. *Atatürk Üniversitesi Sosyal Bilimler Enstitüsü Dergisi*, 14*(1)*, 53-62.

Işıksal, M. (2006). *A study on pre-service elementary mathematics' subject matter knowledge and pedagogical content knowledge regarding the multiplication and division of fractions*. Unpublished doctoral dissertation, Middle East Technical University.

Jacobs, V. R., & Philipp, R. A. (2004). Mathematical thinking: Helping prospective and practicing teachers focus. *Teaching Children Mathematics*, *11*(4), 194-201.

Jacobs, V. R., Lamb, L.L.C., & Philipp, R.A. (2010). Professional noticing of children's mathematical thinking. *Journal for Research in Mathematics Education*, *41*(2), 169-202.

Jansen, A., & Spitzer, S. M. (2009). Prospective middle school mathematics teachers' reflective thinking skills: Descriptions of their students' thinking and interpretations of their teaching. *Journal of Mathematics Teacher Education*, *12*(2), 133-151.

Karagiorgi, Y., & Symeou, L. (2005). Translating Constructivism into Instructional Design: Potential and Limitations. *Educational Technology & Society*, *8*(1), 17-27.

Kersting, N.B., Givvin, K.B., Sotelo, F.L. & Stigler, J.W. (2010). Teachers' analyses of classroom video predicts student learning of mathematics: Further explorations of a novel measure of teacher knowledge. *Journal of Teacher Education*, *61*(1-2), 172-181.

Kersting, N. B., Givvin, K. B., Thompson, B. J., Santagata, R., & Stigler, J. W. (2012). Measuring usable knowledge: Teachers' analyses of mathematics classroom videos predict teaching quality and student learning. *American Educational Research Journal*, *49*(3), 568-589.

Kieren, T. E., & Nelson, D. (1981). Partitioning and unit recognition in performances on rational number tasks. In T. Post & M. Roberts (Eds.), *The proceedings of the third annual meeting of the International Group for the Psychology of Mathematics Education -North American Chapter*, pp. 91-102).

Kilpatrick, J. (1987). What constructivism might be in mathematics education. In J. C. Bergeron, N. Herscovics, & C. Kieran (Eds.), *Proceedings of the 11th PME International Conference,* 1, pp. 3– 27.

Lampert, M. & Cobb, P. (2003) Communication and learning in the mathematics classroom. In J. Kilpatrick & D. Shifter, Eds. *Research Companion to the NCTM Standards* . Reston, VA: National Council of Teachers of Mathematics, pp. 237-249.

Lamon, S. (1996). The development of unitizing: Its role in children's partitioning strategies. *Journal for Research in Mathematics Education*, 27(2), 170-193.

Liehr, P. & Smith, M. J. (1999). Middle range theory: Spinning research and practice to create knowledge for the new millennium. *Advances in Nursing Science, 21*(4), 81-91.

Lincoln, Y. S., & Guba, E. G. (1985). *Naturalistic inquiry*. Beverly Hills, CA: Sage.

Lowery, N. V. (2002), Construction of teacher knowledge in context: Preparing elementary teachers to teach mathematics and science. *School Science and Mathematics, 102*, 68–83.

Ma, L. (1999). *Knowing and teaching elementary mathematics: Teachers' understanding of fundamental mathematics in China and the United States*. Mahwah, NJ: Erlbaum.

Maher, C. A. (1996). Constructivism and constructivist teaching: Can they co-exist? In *Proceedings of the Eighth International Congress for Mathematics Education*. Spain: Sevilla.

Maher, C. A. (2002). How students structure their own investigations and educate us: what we've learned from a fourteen year study. In: A. D. Cockburn, & E. Nardi (Eds.), *Proceedings of the Twenty-Sixth Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 31–46). Norwich, UK: School of Education and Professional Development, University of East Anglia.

Maher, C. A., & Martino, A. M. (1997). Conditions for conceptual change: From pattern recognition to theory posing. In H. Mansfield (Ed.), *Young children and mathematics: Concepts and their representations*. Durham, NH: Australian Association of Mathematics Teachers.

Manouchehri, A. (1997). School mathematics reform: Implications for mathematics teacher preparation. *Journal of Teacher Education*, *48*, 197-209.

Mathison, S. (1988). Why triangulate? *Educational Researcher, 17*(2), 13-17.

McDonough, A., Clarke, B., & Clarke, D. M. (2002). Understanding, assessing, and developing children's mathematical thinking: The power of a one-on-one interview for preservice teachers in providing insights into appropriate pedagogical practices. *International Journal of Education Research, 37,* 211–226.

McGee, L., Kervin, L., & Chinnappan, M. (2006). Emerging issues in the investigation of the construct of partitive quotient. In P. Grootenboer, R. Zevenbergen & M. Chinnappan (Eds.), *Identities, Cultures and Learning Spaces*. Proceedings of the 29th annual conference of the Mathematics Education Research Group of Australasia (Vol. 2, pp. 360-367). Adelaide: MERGA.

MEB. (2006) *Talim ve Terbiye Kurulu ilköğretim Matematik Dersi 1-5 Sınıflar Öğretim Programı*. Ankara.

MEB. (2013) *Talim ve Terbiye Kurulu Ortaokul Matematik Dersi 5-8 Sınıflar Öğretim Programı*. Ankara.

Merriam, S. B. (2002). Assessing and evaluating qualitative research. In S. B. Merriam (Ed.), *Qualitative research in practice: Examples for discussion and analysis* (pp. 18-33). San Francisco: Jossey-Bass.

Mewborn, D. S. (2000). *An analysis of the research on K-8 teachers' mathematical knowledge*. Paper presented at the Annual Meeting of the American Educational Research Association, New Orleans, LA.

Miles, M. B. & Huberman, M. A. (1994). *Qualitative Analysis: An Extended Source Book* (2nd ed.) Thousand Oaks, CA:Sage.

Mojica, G. (2010). *Preparing pre-service elementary teachers to teach mathematics with learning trajectories.* Unpublished doctoral dissertation. North Carolina State University, Raleigh, NC. Raleigh, NC.

Morris, A. (2006). Assessing pre-service teachers' skills for analyzing teaching. *Journal of Mathematics Teacher Education*, *9*(5), 471-505.

National Governors Association Center for Best Practices Council of Chief State School Officers. (2010). Common Core State Standards Mathematics. Washington, DC: National Governors Association Center for Best Practices, Council of Chief State School Officers.

Niess, L. M. & Gillow-Wiles, H. (2014). Learning trajectory for transforming teachers' knowledge for teaching mathematics and science with digital image and video technologies in an online learning experience. *Journal of Digital Learning in Teacher Education*, *31*(1), 5-17.

Nguyen, K. (2010). *Investigating the role of equipartitioning and creating internal units in the construction of a learning trajectory for length and area* (unpublished doctoral dissertation). North Carolina State University, Raleigh, NC.

Paker, T. (2008). Ögretmenlik uygulamasında ögretmen adaylarının uygulama ögretmeni ve uygulama ögretim elemanının yönlendirmesiyle ilgili karsılastıkları sorunlar. *Pamukkale Üniversitesi, Eğitim Fakültesi Dergisi*, *1*(23), 132-139.

Palinscar, A. P. (1998). Social constructivist perspectives on teaching and learning. *Annual Review Psychology,* 49, 345-75.

Patton, M. (1990). *Qualitative evaluation and research methods* (pp. 169-186). Beverly Hills, CA: Sage.

Pellegrino, J. W. (2009). The challenges of conceptualizing what low achievers know and how to assess their competence. In M. Perie (Ed.), *Considerations for the alternate assessment based on modified achievement standards* (*AA-MAS): Understanding the eligible population and applying that knowledge to their instruction and assessment*. New York, NY: New York Comprehensive Center.

Pepper , K. L. (1991). Preschoolers Knowledge of Counting and Sharing in Discrete Quantity Settings. In R. P. Hunting & G. Davis (Eds.), *Early Fraction Learning* (pp. 103-129). New York: Springer-Verlag.

Pepper, K. L., & Hunting, P. P. (1998). Preschooler's counting and sharing. *Journal for Research in Mathematics Education*, *29*(2), 164-183.

Perkins, D. N. (1991). What constructivism demands of the learner. *Educational Technology, 31*(9), 19-21.

Petrou, M. & Goulding, M. (2011). Conceptualizing teachers' mathematical knowledge in teaching. In T. Rowland & K. Ruthven (Eds.), *Mathematical Knowledge in Teaching*, (pp. 9 – 27). Newyork, NY: Springer.

Philipp, R. A. (2008). Motivating prospective elementary school teachers to learn mathematics by focusing upon children's mathematical thinking. *Issues in Teacher Education*, *17*(2) 7-26.

Philipp, R. A., Ambrose, R., Lamb, L. L., Sowder, J. T., Schappelle, B. T., Sowder, L., Thanheiser, E., & Chauvot, J. (2007). Effects of early field experiences on the mathematical content knowledge and beliefs of prospective elementary school teachers: An experimental study. *Journal for Research in Mathematics Education*, *38*(5), 438 – 476.

Piaget, J. (1947). *La psychologie de VinteUigence.* Paris: Armand Golin.

Piaget, J. (1965). *The child's conception of number*. New York: W. W. Norton and Company. (Original work published in 1941).

Piaget, J. (1970). *Genetic epistemology*. New York: W. W. Norton and Company.

Piaget, J., Inhelder, B., & Szeminska, A. (1960). *The child's conception of geometry*. (E.A. Lunzer, Trans.) New York: Basic Books. (Original work published 1948).

Polman, J. L. (2006a). True stories, storied truth: Stitching narrative and logico-scientific discourse together in an age of "spin". In S. A. Barab, K. E. Hay, & D. T. Hickey, *Proceedings of the International Conference of the Learning Sciences 2006*, pp. 557-563. Mahwah, NJ: Erlbaum.

Polman, J. L. (2006b). Mastery and appropriation as means to understand the interplay of history learning and identity trajectories. *The Journal of the Learning Sciences*, 15 (2), 221-259.

Pothier, Y., & Sawada, D. (1983). Partitioning: The emergence of rational number ideas in young children. *Journal for Research in Mathematics Education*, 14(5), 307-317.

Pothier, Y., & Sawada, D. (1989). Children's interpretation of equality in early fraction activities. *Focus on Learning Problems in Mathematics*, 11(3), 27-38.

Powell, A. B., Francisco, J. M., & Maher, C. A. (2003). An analytical model for studying the development of learners' mathematical ideas and reasoning using videotape analysis. *Journal of Mathematical Behavior, 22*, 405-435.

Pritchard, A., & Woollard, J. (2010). *Psychology for the classroom: Constructivism and social learning*. London and New York: Routledge.

Puntambekar, S., & Hübscher, R. (2005). Tools for scaffolding students in a complex learning environment: What have we gained and what have we missed?. *Educational Psychologist, 40*(1), 1-12.

Resnick, L. B., Salmon, M., Zeitz, C. M., Wathen, S. H., & Holowchak, M. (1993). Rea- soning in conversation. *Cognition and Instruction*, *11*, 347–364.

Rowe, K. (2004). In good hands? The importance of teacher quality [online]. *Educare News: The National Newspaper for All Non-government Schools*, *149*, 4-14.

Schoenfeld, A. (2011). *How we think: A theory of goal-oriented decision making and its educational applications*. New York: Routledge.

Schwandt, T. A. (1998) ‘Constructivist, interpretivist approaches to human inquiry’, in Denzin, N. K. and Lincolm, Y. S. *The landscape of qualitative research: theories and issues*. Sage Publications, pp. 221-259.

Scrimsher, S., & Tudge, J.R.H. (2003). The teaching/learning relationship in the first years of school: Some revolutionary implications of Vygotsky's theory. *Early Education & Development*, *14*, 293-312.

Seferoğlu, S. (2004). Öğretmen yeterlikleri ve mesleki gelişim. *Bilim ve Aklın Aydınlığında Eğitim*, 58, 40-45.

Sherin, M. G., Jacobs, V. R., & Philipp, R. A. (2011). *Mathematics Teacher Noticing: Seeing Through Teachers' Eyes*. New York, NY: Taylor & Francis.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher*, *15*(2), 4-14.

Simon,M . (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education*, *26*, 114-145.

Simon, M. A., & Tzur, R. (2004). Explicating the role of mathematical tasks in conceptual learning: an elaboration of the hypothetical learning trajectory. *Mathematical Thinking and Learning*, *6*(2), 91-104.

Sinclair, H. (1987). Constructivism and the psychology of mathematics. In J. C. Bergeron, N. Herscovics, & C. Kieran (Eds.), *Proceedings of the 11th PME International Conference*, 1, pp. 28– 41.

Spitzer, S. M., Phelps, C. M., Beyers, J. E. R., Johnson, D. Y., & Sieminski, E. S. (2011). Developing pre-service elementary teachers' abilities to identify evidence of student mathematical achievement. *Journal of Mathematics Teacher Education*, *14*(1), 67 – 87.

Steffe, L. P. (1983). Children's algorithms as schemas. *Educational Studies in Mathematics*. 14, 109-25.

Steffe, L. P. (1991). The constructivist teaching experiment: Illustrations and implications. In E. Von Glasersfeld (Ed.), *Radical Constructivism in Mathematics Education* (pp. 177-194). New York: Kluwer Academic Publishers.

Steffe, L. P. (2004). On the construction of learning trajectories of children: The case of commensurate fractions. *Mathematical Thinking and Learning*, *6*(2), 129-162.

Steffe, L. P. & Thompson, P. (2000). Teaching experiment methodology: underlying principles and essential elements. In A. Kelly & R. Lesh (Eds.), *Handbook of Research Design in Mathematics and Science Education* (pp.267 – 306). Mahwah, NJ: Lawrence Erlbaum Associates.

Stein, M. K., Grover, B. W., & Henningsen, M. (1996). Building student capacity for mathematical thinking and reasoning: An analysis of mathematical tasks used in reform classrooms. *American Educational Research Journal*, *33*, 455-488.

Stein, M. K., & Smith, M. S. (1998). Mathematical tasks as a framework for reflection: From research to practice. *Mathematics Teaching in the Middle School*, *3*(1), 268–275.

Stein, M. K., & Smith, M. S. (2011). *Five practices for orchestrating productive mathematics discussions.* Reston, VA: Corwin.

Sullivan, P., Mousley, J., & Zevenbergen, R. (2004). Describing elements of mathematics lessons that accommodate diversity in student background. In M. Johnsen Joines & A. Fuglestad (Eds.), *Proc. 28th Conf. of the Int. Group for the Psychology of Mathematics Education* (pp.257–265). Bergen: PME.

Swars, S., Hart, L.C., Smith, S. Z. Smith, M.E., & Tolar, T. (2007). A longitudinal study of elementary pre-service teachers' mathematics beliefs and content knowledge. *School Science and Mathematics*, 107(8), 325-335.

Sztajn, P., Wilson, P. H., Edgington, C., & Myers, M. (2014). Mathematics professional development as design for boundary encounters. ZDM *Mathematics Education*, *46*, 201-212.

Sztajn, P., Wilson, P. H., Confrey, J., & Edgington, C. (2012). Learning trajectory based instruction: Toward a theory of teaching. *Educational Researcher*, *41*(5), 147–156.

Tall, D. (1991). The psychology of advanced mathematical thinking. In D. Tall, *Advanced Mathematical Thinking* (pp. 3-21). Boston: Kluwer Academic Publishers.

Thipkong, S., & Davis, E. J. (1991). Preservice elementary teachers' misconceptions in interpreting and applying decimals. *School Science and Mathematics, 91*(3), 93-99.

Tirosh, D. (2000). Enhancing prospective teachers' knowledge of children's conceptions: The case of division of fractions. *Journal for Research in Mathematics Education, 31*(1), 5-25.

Tobin, K., & Tippings, D. (1993). Constructivism as a referent for teaching and learning. In Tobin, K. (Ed.), *The practice of constructivism in science education* (pp. 3-21), Hillsdale: Lawrence Erlbaum.

Toluk-Uçar, Z. (2010). Sınıf Öğretmeni adaylarının matematiksel bilgileri ve öğretimsel açıklamaları. *e-Journal of New World Science Academy,* 5(3), 911-920.

Toluk-Uçar, Z. (2011). Öğretmen adaylarının pedagojik alan bilgileri ile matematiksel alan bilgileri arasındaki ilişki. *Turkish Journal of Computer and Mathematics Education, 2*(2), 87-102.

Toluk- Uçar, Z. ve Demirsoy, N. H. (2010). Eski-yeni ikilemi: matematik öğretmenlerinin matematiksel inançları ve uygulamaları. *Hacettepe Üniversitesi Eğitim Fakültesi Dergisi, 39,* 321-332.

Treffers, A. (1987). *Three Dimensions: A Model of Goal and Theory Description in Mathematics Instruction – The Wiskobas Project*. Boston D. Reidel Publishing Company.

Ubuz, B. (2009). Üniversite eğitimi ve öğretmenlik: Matematik öğretmenlerinin ve adaylarının görüşleri.

Ubuz, B. & Yayan, B. (2010). Primary teachers' subject matter knowledge: Decimals. *International Journal of Mathematical Education in Science and Technology, 41*(6), 787-804.

van de Walle, J. (2007). *Elementary and middle school mathematics: Teaching developmentally* (7e). Boston: Pearson Education.

Vygotsky, L. S. (1978). *Mind in society: The development of higher psychological processes*. Cambridge, MA: Harvard University Press.

Wager, A. A., & Carpenter, T. P. (2012). Learning trajectories through a socio-cultural lens. In 1. S. Carlson & 1. R. Levine (Eds.), *Psychological perspectives on contemporary educational issues: Vol.* 3. *Instructional strategies for improving student learning: focus on early mathematics and reading* (pp. 197-204). Charlotte: Information Age.

Webb, L. N. (2007). Assessment of students' knowledge of mathematics: Steps toward a theory. In F. Lester (Ed.), *Handbook for Research on Mathematics Education* (2nd ed), p. 111-155. Charlotte, NC: Information Age Publishing.

Wilson, P. H., Mojica, G., & Confrey, J (2013a). Learning trajectories in teacher education: Supporting teachers' understanding of students' mathematical thinking. *The Journal of Mathematical Behavior*, *32*, 103-121.

Wilson, P. H., Sztajn, P., & Edgington, C. (2013b). Designing Professional learning tasks for mathematics learning trajectories. PNA, 7(4), 135-143.

Wilson, P. H., Sztajn, P., Edgington, C., & Confrey, J. (2013/2014). Teachers' use of their mathematical knowledge for teaching in learning a mathematics learning trajectory. *Journal of Mathematics Teacher Education*. *17*, 149-175.

Wolf, N. (2003). Learning to teach mathematics for understanding in the company of mentors. *Teachers and Teaching: Theory and Practice--The Journal of the International Study Association on Teacher Thinking*, *9*(2), 87–106.

Von Glasersfeld, E. (1995). *Radical constructivism: A way of knowing and learning*. London, UK: Falmer Press.

Yilmaz, Z. (2011). *Toward an understanding of students' strategies on reallocation and covariation items: In relation to an equipartitioning learning trajectory*. Unpublished master's thesis. North Carolina State University, Raleigh, NC.

Yin, R. (1994). *Case Study Research: Design and Methods*. Thousand Oaks, London, New Delhi: Sage.

Zembat, I, O. (2007). Working on the same problem – Concepts; with the usual subjects – Prospective Elementary Teachers. *Elementary Education Online*, *6*(2), 305-312.

Zhao, R., & Orey, M. (1999). *The scaffolding process: Concepts, features, and empirical studies*. Unpublished manuscript. University of Georgia.

APPENDICES

APPENDIX A – PRE/POST TESTS RUBRICS

The same scoring guide were utilized both tests. Since both tests included parallel items and the numbers in the item stem changes the numbers in the correct answers is subject to change. The rubric below showed the correct answers for the pre-test and post-test subsequently that are separated with the sign "/" when needed.

ITEM 1. Fair sharing collections and naming fair shares (Adapted from Mojica, 2010, p. 215)
1a. Correct Answers:

- Draw lines from each candies or group of candies to the each friend
  or
- Indicate explicitly each friends get 6 candies (drawing, mathematical expressions)

2b. A Variety of Naming Practices:

1. Count – 6 candies or numerical expressions (e.g. 18 ÷ 3).
2. Ratio – 6 candies per friend, 12 candies per 2 friends, 18 candies per 3 friends; 6 candies to each friend.
3. Fraction – 6/18, 1/3, other equivalent fractions, or word equivalents (e.g. "one third").
4. Operator – 6/18 of the coins, 1/3 of the coins.
5. General mathematical name – thirds, fraction, quotient, fair share, a part, equal group of candies, equal portions, equipartition, partitions, portions.

Scoring guide

Score	**Description**
4	1a correct AND includes at least four naming practices from the 1b list
3	1a correct AND includes three naming practices from the 1b list
2	1a correct AND includes two naming practices from the 1b list
1	1a correct AND includes one naming from 1b list
0	1a correct or incorrect AND no attempt or superfluous names 1a and 1b incorrect No response

ITEM 2. Quotient construct (Retrieved from Mojica, 2010, p. 216)

Possible Correct Responses:

- "n/q"
- "n ÷ q"
- "n divided by q"
- "1 qth of n"
- "n one-qths"

Scoring Guide

Score	Description
3	Correct response
2	Specific case used as explanation with some mention of generality (e.g., "1/qth" without a reference to the unit or n ÷ q = 1/3 or "n =18, q = 3, 18/3 = 6") Note: If includes a general statement and then makes a specific case as an example, score as 3.
1	Specific case used as explanation with no mention of generality (e.g. "6" or "1/3") Specific case used that is incorrect with some mention of generality (e.g., "6/3" with reference to "n/q")
0	No mention of generality or specific case used Specific case used with incorrect generality Superfluous answer Incorrect Response No response

ITEM 3. Reassembly- Reversibility of discrete equipartitioning.

Correct response: 39 pencils / 42 pencils

Scoring guide

Score	Description
4	Correct response and employ both multiplicative and additive strategies to find the solution
3	Correct response and employ merely multiplicative strategy to find the solution
2	Correct response and employ merely additive strategy to find the solution
1	Correct response but no or unreasonable explanation of solution way
0	Incorrect answer with unreasonable explanation OR no response

ITEM 4. Times as many- Comparing size of the whole to size of the one share

Correct response: Berrin's answer is correct.

Complete explanations:

1. Fatma's understanding: Perceive the relation between one lego piece and the whole tower additively (whole tower 12 lego more than one lego piece/ whole tower 6 lego more than one lego piece) and state this relations as lego tower 12 times as long as one lego piece/ 6 times as long as one lego piece.
2. Ayse's understanding: Name the part whole relation regardless of the size of the share and called two times as long as. Because, young children used to call a fair share half regardless of the size of the share (Yilmaz, 2011). The reversibility of a half is two times longer.
   OR
   Perceive one piece of lego as one share and the rest of the whole tower (12 lego pieces) as another share, in total there are two pieces. State the relation between size of the one lego piece and size of the whole lego tower as lego tower two times as long as one lego tower.
3. Berrin's understanding: Perceive the relation between one lego piece and the whole tower multiplicatively and state the relation correctly as 13 times as long as / 7 times as long as.

Scoring guide

Score	Description
4	Correct response and complete explanation for each friend's understanding of reassembly
3	Correct response, but fail to provide correct explanation for at most one friend's understanding
2	Correct response, but fail to provide correct explanation for at most one friends' understanding and provide incomplete explanations for others Provide one correct explanation and some sorts of explanation for other two friends but the explanations are not quite complete
1	Correct response, but provide incorrect or unreasonable explanation Provide some sorts of explanation for two friends but the explanations are not quite complete
0	Incorrect response and provide no/incorrect/unreasonable explanation No response

ITEM 5. Reallocation and justification of fair shares (case of discrete collections)

Correct response: 6 cookies per friend and justify fair share of each friend correctly/9 bottle caps per friend and justify fair share of each friend correctly.

Correct justifications:

1. Explaining their careful use of one-to-one correspondence in rounds.
2. Counting the number of objects in each group.
3. Stacking the objects and comparing the height of each stack.
4. Constructing arrays from objects and compare them (Confrey et al., 2010; Yilmaz, 2011).
5. Utilizing composite units to create fair shares
6. Utilize inverse operations division and multiplication
7. Asserts equipartitioning creates automatically fair shares (Yilmaz, 2011).

Scoring Guide

Score	Description
3	Reallocate from each friends share and provide both correct response and justification(s)
2	Reallocate from each friends share and provide both correct response but incorrect/unreasonable justification(s) Utilize fair sharing collection strategies and provide both correct response and justification(s)
1	Utilize fair sharing collection strategies and provide correct response but no/ incorrect/unreasonable justification(s)
0	Incorrect response and provide no/incorrect/unreasonable explanation No response

ITEM 6.

Correct Response: 6 candies per Friend

ITEM 7.

Correct Response: 5 chips per Friend / 6 marbles per Friend

Scoring Note: For both Item 6 & 7 Scoring Guide and Justifications same as Item 5

ITEM 8 Sharing Multiple Wholes among Multiple People (Adapted from Mojica, 2010, p.217)

Correct response: Yes – both result in 3/5 / Yes – both result in 5/4

Complete explanations:
The explanations were same for the post-test with different numbers.

- Mathematically modeled the first task as 3 ÷ 5 and the second task as 1/5 +… +1/5 or 3(1/5)

- Utilize an area model to illustrate this argument
- Indicates each task includes the situation of five friends received 1/5 of the total
- Explains that pie type is an extraneous variable

Incomplete explanations:

- Both result in 3/5
- Both involve sharing 3 things among 5 people
- Both involve the same values/units; may use an area model to illustrate this argument
- Both involve same operation 3:5

Scoring Guide:

Score	**Description**
3	Correct response with complete explanation
2	Correct response with one of the incomplete explanations
1	Respond no but state second problem demands $1/5^{th}$ of each pizza Respond no but provides a complete explanation
0	Incorrect response and provide no/incorrect/unreasonable explanation No response

Note: If students showed a complete work but missed stating conclusion sentence about mathematical equivalence of tasks, deduce 1 point.

ITEM 9 Compensation/ Factor Based Change (Adapted from Mojica, 2009, p. 221)

Correct responses:

- **a.** Less than
- **b.** More than
- **c.** Two times larger / Three times larger
- **d.** Two times smaller OR Half of the initial share / Three times smaller OR one third of the initial share

Complete explanations:

A. Utilize qualitative compensation to state the changes (larger or smaller) in size of shares based on changes in the number of persons sharing.
B. Utilize qualitative compensation to state the changes (larger or smaller) in size of shares based on changes in the number of persons sharing
C. One fourth < one half
D. Uses an area model to illustrate this argument with text or symbols
E. One-fourth > One eighth
F. Uses an area model to illustrate this argument with text or symbols

Incomplete explanations:

1. Implied comparisons of fractions that do not use text or symbols (e.g. an area model or fraction notation with out words or “> < =” symbols

Scoring Guide

Score	**Description**
4	Correct answer and complete explanation for a-d
3	Correct answer and complete explanation for any combination of three (A,B &C; A, B & D etc..)
2	Correct answer and complete explanation for either A&B or C&D or A&C or B&D Correct response and provide at least one complete explanation for a-d and provide incomplete explanation for the rest.
1	Correct response and incomplete explanation for any combination of three (A&B&C or A&C&D or B&C&D)
0	Correct response and unreasonable or no explanation for a-d Incorrect response No response

ITEM 10 Knowledge of variety and correct/incorrect strategies and ability to sort the strategies from least complex to more complex ones (Retrieved from Mojica, 2010, p.228)

I. Unsophisticated

- Cannot be done
- Breaking with no attention to creating equal-sized pieces and correct number of equal sized pieces
- Creating correct number of pieces but of unequal size without composition
- Creating the wrong number of equal-sized pieces
- Failure to exhaust the whole
- n or n – 1 parallel cuts
- Sequential radial cuts
- Use of landmark fractions and then dividing the remainder because cannot do the split

II. Intermediate

- Must exhaust whole
- A composition of cuts to create all congruent pieces
- Incorrect compositions

III. Sophisticated

- Must exhaust whole
- A composition of cuts to create incongruent pieces
- Correct or incorrect use of equivalence (e.g. creating 8 congruent pieces and giving 2 per person)

Notes:

1. Strategies that simply change the orientation of the cuts should be counted only once (e.g. 5 horizontal parallel cuts and 5 vertical parallel cuts count as only one strategy)
2. Unless specifically labeled or described as a composition, count a strategy as parallel cuts.
3. Repeated examples with the distinction of measuring should only count once (i.e. if three strategies are repeated but say 'actually measure to find the center' count all three of these only once).
4. Landmark strategy is different from a composition (e.g. a 3-split on a 2-split) because with a composition, still attending to each piece, whereas with landmarks, the actions focus on remaining piece after distribution.

Scoring notes:

1. Subtract one point for incorrect or no labeling of correct/incorrect strategies
2. Subtract one point for 50% of the strategies have complete descriptions. Descriptions may be written or numbers but specific (e.g. 'cut into fourths' is specific, '1/4ths' is not)

Scoring Guide

Score	Description
3	Indicates 2 correct and 2 incorrect strategy Indicates 3 correct and 1 incorrect strategy
2	Indicates 2 correct and 1 incorrect strategy Indicates 1 correct and 2 or 3 incorrect strategy Indicates 3 correct strategies and no incorrect strategy Indicates 3 or 4 incorrect strategies
1	Indicates 1 correct and 1 incorrect strategy with correct labels Indicates 2 correct strategy and no incorrect strategy Indicates 2 incorrect strategy and no correct strategy
0	Indicates 1 correct strategy Indicates 1 incorrect strategy Indicate no strategy

ITEM 11 Knowledge of variety and correct/incorrect strategies (Adapted from Mojica, 2010, p. 229).

Incorrect Strategies

1. Breaking with no attention to creating equal-sized pieces and correct number of equal sized pieces
2. Creating correct number of pieces but of unequal size without composition
3. Creating the wrong number of equal-sized pieces
4. Failure to exhaust the whole
5. n or n – 1 parallel cuts
6. Use of landmark fractions and then dividing the remainder because cannot

do the split

Correct Strategies

Prerequisite: Exhaust whole and create correct number of fair shares

1. A composition of cuts to create all congruent pieces in terms of area
2. Correct or incorrect use of equivalence (e.g. creating 8 congruent pieces and giving 2 per person)

Notes

1. Strategies that simply change the orientation of the cuts should be counted only once (e.g. 4 horizontal parallel cuts and 4 vertical parallel cuts count as only one strategy)
2. Unless specifically labeled or described as a composition, count a strategy as parallel cuts.

Scoring Guide

Score	Description
4	Indicates 3 correct strategies and at least 2 incorrect strategies with correct labels
3	Indicates 3 correct and fewer than 2 incorrect strategy with correct labels Indicates 2 correct and 2 incorrect strategy with correct labels
2	Indicates 2 correct and 1 incorrect strategy with correct labels
1	Indicates 1 correct and 1 incorrect strategy with correct labels Indicates 2 correct strategy with correct label Indicates at least three strategy without labeling incorrect and correct ones
0	Indicates fewer than 3 strategies without labeling incorrect and correct ones Indicates strategies that are all incorrect Indicates no strategy

ITEM 12 Ordering task difficulty (Adapted from Mojica, 2010).

Correct Responses: Pre-Test

a. 7 cookies among 2

Complete Explanations:

1. Indicates sharing among 2 easier than sharing among 7
2. Dealing is easier
3. Odd splits harder than even splits
4. Dividing a larger number into a smaller number is easier

b. a round cake among 3

Correct Responses: Post-Test

a. 2 cookies among 9

Complete Explanations:

1. Indicates sharing among 2 easier than sharing among 9
2. Dealing is easier
3. Odd splits harder than even splits
4. Dividing a larger number into a smaller number is easier

b. a round cake among 6

Complete Explanations:
1. Indicates 3-splits is harder
2. Indicates repeated halving is easier

c. a rectangular cake among 3

Complete Explanations:
1. Indicates 3-splits is harder
2. Indicates repeated halving is easier
3. Indicates odd splits harder than even splits
4. Use symmetry because 4 is an even number

d. a round cake among 3

Complete Explanations:
1. Indicates radial cut is harder
2. Indicates parallel cut is easier

Complete Explanations:
1. Indicates 3-splits is harder
2. Indicates repeated halving is easier

c. a rectangular cake among 3

Complete Explanations:
1. Indicates 3-splits is harder
2. Indicates repeated halving is easier
3. Indicates odd splits harder than even splits
4. Use symmetry because 8 is an even number

d. a round cake among 3

Complete Explanations:
1. Indicates radial cut is harder
2. Indicates parallel cut is easier

e. A rectangular cake among 12

Complete Explanations:
1. Indicates 3-splits is harder while creating 12 equal parts
2. Repeated halving is easier (4-splits)
3. Use symmetry because 4 is an even number
4. 12 fair parts required composition of splits

Scoring Guide

Score	Description
4	At least 4 correct responses and provide complete explanation
3	4 correct responses and 2 or 3 complete explanations 3 correct responses with 3 complete explanations
2	3 or 4 incorrect responses with 3 or 4 complete explanations (it may be possible that the student circled the easier tasks) 3 correct responses and provide less than 3 complete explanations 2 correct responses with 2 complete explanations
1	Less than 4 correct responses with incomplete explanation 1-4correct response(s) with a complete explanation
0	No response Incorrect responses Correct responses with unreasonable, or no explanation

ITEM 13 Repeated halving (Adapted from Empson & Turner, 2006 and Mojica, 2010)

Correct response: 32 equal parts / 128 equal parts

Show your work: Utilizes one the ways presented below:

1. Indicates the multiplicative relation between number of folds and number of fair parts created and denotes this relation with exponential numbers (e.g. 2.2.2.2.2.2.2=128 parts 2^n)
2. Indicates relation between the a fair part compared the size of the whole as a result of folding. (e.g. ½.½.½.½.½.½.½.½=1/128)
3. Finds resulting number of fair parts after each folding until all the folding action completed. (1^{st} fold: 2 parts, 2^{nd} fold: 4 parts, 3^{rd} fold: 8 parts….)
4. Show the resultant number of fair shares as a result of each folding.

Scoring guide:

Score	**Description**
2	Correct response with correct explanation of the work
1	Incorrect response as a result of missing one folding action (ie. 64 fair parts)
0	Incorrect response Unreasonable response No response

ITEM 14 Composition of splits (Adapted from Wilson, 2009)

Correct ways: Includes the possible ways of permutation of the folds to create 24 fair shares in the pre-test and 36 fair shares in the post-test.

Pre-Test

1. Indicates the generalization of how to find required number of fair shares.
2. Half, half, half, third –Category 1
3. Fourth, half, third – Category 2
4. Half, half, sixth or fourth, sixth – Category 3
5. 23 parallel folds– Category 4

Post-Test

1. Indicates the generalization of how to find required number of fair shares.
2. Half, half, third, third –Category 1
3. Fourth, third, third – Category 2
4. Half, third, sixth – Category 3
5. 35 parallel folds– Category 4

Note: The order of the folds does not important and the drawing of the folds for the categories above is accepted as correct response.

Scoring Guide

Score	Description
3	Indicates way from all 4 categories or indicates mathematical generalization and show at least one example that fits into category 1,2 or 3.
2	Indicates way from 3 categories
1	Indicates way from 2 categories OR Indicates 1 way from categories 1, 2 or 3
0	Indicates 1 way from category 4 only OR No response

ITEM 15 Compensation, sharing multiple wholes and justification of fair shares

Correct response: Hasan's strategy

Complete explanations:

1. Explains in Hasan's strategy people at both tables get the same amount of pizza, and in Ahmet's strategy people at both tables do not get the same amount of pizza (i.e. person in Hasan's table gets 2/3rd of a pizza but in Ahmet's 10 people sits in a table and each receives 7/10th of the pizza, and on the other table each received 5/8th pizza)
2. Utilize models or drawings to show the argument stated in 1.

For the post test: State number of people doubles so as number of pizzas doubles

Complete Explanations for Ahmets' Understanding of Fair Sharing

1. Ahmet shows an additive misconception, since 10-7=3 and 8-5=3 the differences between amount of pizza is same so they are fair
2. Ahmet could not use ratios of people per pizza correctly and could not fairly share multiple wholes among multiple people

Incomplete Explanations for part a:

1. Utilizes qualitative compensation and stated the shares determined in Hasan's tables are smaller/larger or are different than Ahmet's tables or vice versa while not explicitly addressing the other.

Incomplete Explanations for part b:

1. Ahmet could not fairly share the multiple wholes, give few pizzas to more people.

Scoring Guide

Score	Description
3	Correct response with complete explanation for part a and b
2	Correct response with complete explanation for part a and provide incomplete explanation for part b or vice versa Incorrect response by computational error with complete explanation
1	Correct response with incomplete explanation for part a and part b Correct response with one complete explanation for either part a or part b Incorrect response with complete explanation for either part a or part b
0	Incorrect response with incomplete, unreasonable, or no explanation Correct response with unreasonable explanation No response

ITEM 16 Covariation and utilization of multiple strategies

Correct Responses

Pre-Test

Number of rabbits	Number of carrots
2	3
4	6
8	12

Post-Test

Number of rabbits	Number of carrots
4	5/3
12	5
36	15

Correct Strategies and explanations (Adapted from Smith & Stein, 2011)

1. Unit rate: find the number of carrots eaten by a rabbit and multiply by the number of rabbits to find the required number of carrots.
2. Scale factor: perceive the vertical multiplicative relations: number of the rabbits triples so the number of carrots
3. Scaling up: e.g. Add 5 carrots for every twelve rabbits until reaching required number of rabbits
4. Additive: e.g. Add $\frac{3}{2}$ carrots 8 times to find the number of carrots
5. Utilize direct proportion or other strategies not listed above

Scoring Guide

Score	Description
4	Utilizes 4 or 5 different strategies with correct answer Utilize 3 different strategies including first three strategy
3	Utilizes 3 different strategies with correct answer but not include all first three strategies Utilizes 2 strategies that certainly includes first two strategies with correct answer
2	Utilizes 2 different strategies with correct answer but not include

	necessarily first two strategy
1	Utilizes 1 strategy with correct answer
0	No response Incorrect answer

ITEM 17- Area congruence, justification fair shares and transitivity argument

Correct response:

Pre Test: Not fairly shared and the relation between the parts: $B<A=D<C$ or $C>A=D>B$

Post Test: Not fairly shared and the relation between the parts: $B=D<A<C$ or $C>A>B=D$

Complete Explanations

1. Utilize decomposition or composition of shapes or area congruence personal strategy to reach his declaration of equivalent fractions. Then order the fractions.

Incomplete Explanations

1. Verbally compares the size of each share (B is the smallest one because it is skinnier or C is the largest since it is wider and taller)
2. Made errors while naming each share

Scoring Guide

Score	**Description**
3	Correct response with complete explanation
2	Correct response for part a and b but fail to declare the relation between two parts. (ie A&C, A&B)
1	Correct response with incomplete explanation
0	Incorrect response with incomplete, unreasonable, or no explanation Correct response with unreasonable explanation No response

APPENDIX B

EXAMPLE INSTRUCTIONAL TASKS AND REFLECTION QUESTIONS

The instructional tasks and reflection questions were subject to revisions throughout the teaching experiment based on the interaction and the needs aroused within the classroom.

Week 2: Reflection questions for equipartitioning collection

1. What do you about the mathematical contribution of activities in this teaching session for learning advance mathematical topics?
2. What have you learned about fair sharing today?
3. What kind of learning difficulty or misconceptions that you may observe as you work with students on this fair sharing collection tasks.
4. How do you plan to address these difficulties and misconceptions?
   a. How does ELT help in this process?
   b. Can we always create a fair share for any given numbers of groups and objects? Why or why not?

Week 3: Task 1 – Equipartitioning single whole

ACTIVITY 1

1. You and your group are given a set of color pencils and a rectangular paper that represents a garden. You will plant different fruits in this garden. The rules are:

- Each fruit should have the same amount of space.
- Color each space for each different fruit with a different color.
- Try as many as possible ways and make sure each fruit has the same space.

Answer the following questions:

2. If you plant for n different fruits, how you would fairly share the rectangular garden in different ways?
   Try for n: 4, 6, 10, 12.

   a. How do you make sure each fruit has the same amount of space?
   b. Name the number of parts that each of you paint.
   c. Compare the size of the whole shape to one fruit’s share.
   d. Compare the size of one fruit’s share to the whole shape.

e. What mathematical ideas does this task serve as a base?
f. Can you fairly share a single whole for any amount of people? If yes, why? If no, why not?

Work on the same activity for circles (n=4, 6, 10, 12). Be sure to address students' misconceptions or learning difficulties while working on the task.

At the end of the activity answer the following questions:

a. How is this task different from or similar to the task of fair sharing discrete collections?
b. What kinds of misconceptions may you encounter while implementing this task in an elementary school classroom?
c. How is fairly sharing a circle different from or similar to fairly sharing a rectangle?

Summary

a. How many cuts were needed to create 4, 6, 8, and 10 fair shares if only horizontal or vertical cuts were used? What about creating n fair shares?
b. How should a circle mark so that it can be easily fairly shared?

Week 4: Folding task (Adapted from empson & turner, 2006)

ACTIVITY 4

1. If you fold a paper into half, in half again twice and finally in half again how many equal parts will you have when you opened the paper?

    - *How would you solve the task?*
    - *How would an elementary school student solve the task?*
    - *If you fold the paper half then, 2 times, 3 times, 4 times…. n times in half how many equal parts will you have when you opened the paper?*

2. If you fold a paper into half, then in thirds and finally in third again how many equal parts will I have when I opened the paper? (Adapted from Turner et al., 2007).

    - *How would you solve the task?*
    - *How would an elementary school student solve the task?*

3. Fatma folded a rectangular paper into four equal parts then 3 equal parts. Ahmet folded his rectangular paper into 6 equal parts. How many equal parts does Ahmet need to fold to make exact same equal parts as Fatma did?
    - *How would you solve the task?*
    - *How would an elementary school student solve the task?*

- *Which mathematical idea can be emphasized through this problem?*

Note: Be sure to include any drawings or other representations that support your explanation.

CASE ANALYSIS

1. When Ayse a first grader were asked to predict number of fair shares created by folding a piece of rectangular paper in half 4 times her reply was “12 parts” (Adapted from Turner et al., 2007).

    - What is Ayse’s understanding of creating fair shares through paper folding?
    - How would you help Ayse to perceive the relation between number of parts created and each fold?

2. Two students’ strategies to fold the same size rectangular paper to create 12 equal parts are shown below.

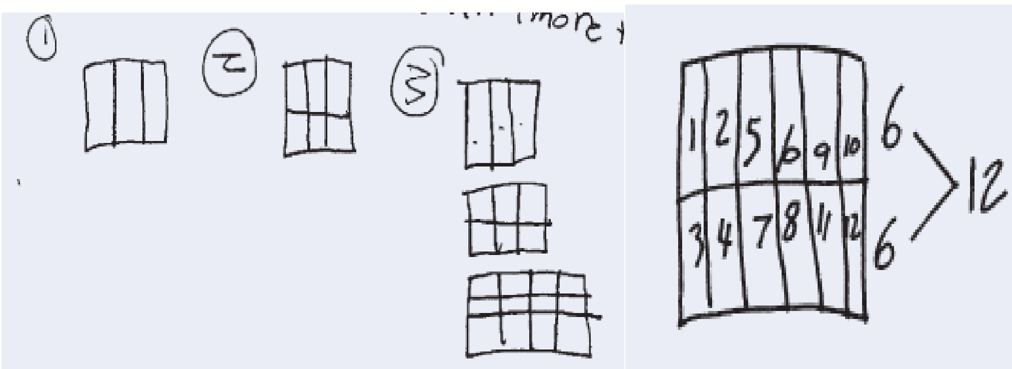


Figure 1. Two strategies to create 12 equal parts (Retrieved from Empson & Turner, 2006, p.)

- *What does each step represent?*
- *What can you tell about each student’s strategy?*
- *What are the possible mathematical ideas can be deduced from these two strategies? (Hint: Think previously discussed lessons).*

REFLECTION ACTIVITY

1. Which mathematical idea/topic folding tasks underlies?
2. What is the connection between using folding to create fair shares and multiplication?
3. What is the connection between using folding to create fair shares and division?

4. What is the connection between using folding to create fair shares and fractions? (Hint: Think about naming the fair shares).

Week 5 – Covariation Activity (Adapted from Smith & Stein, 2011)

Five cookies are enough to feed 2 babies for one day in their breakfast. In order to feed 12 babies how many cookies that you need? (Adapted from Stein and Smith, 2012)

First Part

Please answer followings.

1. For the presented task, provide your own solution **ways.**
2. Explain how you will solve the task.
3. Did each baby receive a fair share? How do you know?

Second Part

Please answer followings.

1. For the presented task, provide the solution ways that you expect to see in an elementary school classroom.
2. In what ways elementary school student might justify their solutions.

Note: Be sure to include any drawings or other representations that support your anticipation of student justification(s).

APPENDIX C

SAMPLE INFORMED CONSENT FORM

Gönüllü Katılım Formu

Ben Zuhal Yılmaz, Orta Doğu Teknik Üniversitesi (ODTÜ) Eğitim Fakültesi İlköğretim Bölümü'nde doktora öğrencisiyim. İlköğretim Bölümü'nde Y.Doç.Dr. Çiğdem Haser tarafından danışmanlığı yürütülen doktora tez çalışmam kapsamında gönüllülük esaslı dizayn edilmiş matematik eğitimi metot dersinde sizlerle çalışmak istemekteyim. Çalışmanın amacı, öğrenme rotalarını dizayn edilmiş metot dersi içinde kullanımının öğretmen adaylarının matematiksel anlamalarını nasıl yapılandırdıklarını incelemektir. Çalışmaya katılım tamimiyle gönüllülük temelinde olmalıdır. Bu çalışma süresince Eğitim Fakültesi Sınıf Öğretmeni adayları ile bire-bir görüşmeler yapıp eş paylaşım öğrenme rotasını kullanarak gönüllük esaslı oluşturulan bir matematik öğretimi metot dersini birlikte işlemeyi planlamaktayım.

Çalışma kapsamında öğrenme rotasının siz öğretmen adaylarının kullanabilecekleri matematiksel stratejileri anlamadaki etkisi, öğrenme rotasında gömülü olan matematiksel bilgiyi sizlerin nasıl kullandığınız video ya kayıt edilecek ve gözlemlenecektir. Sınıf içi gözlemde ve yapılan video kayıtları 6 hafta ve her hafta 2.5 saat sürecektir. Ders içinde kullanılacak olan aktiviteler sizinle paylaşılacak olup, içeriği eş paylaşım konusu hakkında matematik görevleri içermektedir. Ders içi etkinliklerin video çekimine ek olarak sizlere 17 açık uçlu matematik sorusunun yer aldığı ilk ve son test çalışmanın başında ve sonunda uygulanacaktır. Videoya çekilen dersler ve test değerlendirmeleri sadece araştırmanın data analizinde araştırmacılar tarafından kullanılacak olup, elde edilen veriler doktora tezinde ve bilimsel yayınlarda kullanılacaktır.

Bu çalışma bağlı bulunduğunuz programın zorunlu katılım gerektiren bir süreci değildir. Bu çalışmaya katılmanız ilköğretim öğrencilerinin matematiği nasıl öğrendiği hakkında bilgi edinmek ve bu bilgiyi kendi öğretmenlik hayatınızda kullanmanız açısından fayda sağlayacaktır. Eğer bu çalışma için gönüllü olursanız bana sağladığınız analiz edilmemiş bilgi bölümünüz, size ders veren öğretim üyeleri ya da diğer kuruluşlarla paylaşılmayacaktır. Bu görüşme, katılanlara zarar getirebilecek herhangi bir psikolojik ya da fiziksel bir iş içermemektedir. Araştırma sonuçlarının üniversite çapında öğretmenlik eğitiminde metot derslerinin içeriğinin yapılandırılması ve geliştirilmesi amaçlı çalışmalara ve uygulamalara yararlı bir etki yapması beklenmektedir.

Ders içi aktiviteler ve birebir görüşme kişisel rahatsızlık verecek sorular içermemektedir. Ancak, katılım sırasında sorulardan ya da herhangi başka bir nedenden ötürü kendinizi rahatsız hissederseniz cevaplama işini yarıda bırakıp çıkmakta serbestsiniz. Böyle bir durumda araştırmayı yapan kişiye devam etmek istemeyeceğinizi bildirmeniz yeterli olacaktır. Bu çalışmaya katıldığınız için şimdiden teşekkür ederiz. Çalışma hakkında daha fazla bilgi almak için Orta Doğu

Üniversitesi Eğitim Fakültesi İlköğretim Bolumu Doktora Öğrencisi Zuhal Yılmaz (E-posta: yilmaz.zuhal@metu.edu.tr/zyilmazncsu@gmail.com) ve Orta Doğu Teknik Üniversitesi Eğitim Fakültesi İlköğretim Matematik Öğretmenliği Y.Doç. Dr. Çiğdem Haser (Oda No:105; Tel: 210 6415; E-posta: chaser@metu.edu.tr) ile iletişim kurabilirsiniz.

Eğer bu çalışma için gönüllü olmak istiyorsanız lütfen aşağıda verilen yere adınızı, soyadınızı ve tarihi yazıp imzalayınız. Lütfen aşağıdaki iletişim yollarından tercih ettiğiniz birinin bilgisini veriniz. Size o yolu kullanarak görüşme için tercih ettiğiniz zamanı soracağım.

Teşekkürler.

İsim, soy isim: ____________________ İmza : ____________________

Tarih : ____________________

Tercih ettiğiniz iletişim yolu bilgisi:

Tlf: ______________ (Ofis) _________________ (Cep)

Elektronik posta: ___________________________

Eğer bu çalışma kapsamında yapacağımız ders içi etkinliklerinizin ve görüşmelerin ses kaydının alınmasına izin veriyorsanız lütfen aşağıda verilen yere adınızı, soyadınızı ve tarihi yazıp imzalayınız. Görüşme sırasında dilediğiniz zaman kaydın durdurulmasını isteyebilir ya da en başından itibaren kayıt edilmemesini isteyebilirsiniz.

İsim, soy isim: ____________________ İmza : ____________________

Tarih : ____________________

APPENDIX D

APPROVAL OF THE ETHICS COMMITE OF METU RESEARCH CENTER FOR APPLIED ETHICS

UYGULAMALI ETİK ARAŞTIRMA MERKEZİ
APPLIED ETHICS RESEARCH CENTER

DUMLUPINAR BULVARI 06800
ÇANKAYA ANKARA/TURKEY
T: +90 312 210 22 91
F: +90 312 210 79 59
ueam@metu.edu.tr
www.ueam.metu.edu.tr

ORTA DOĞU TEKNİK ÜNİVERSİTESİ
MIDDLE EAST TECHNICAL UNIVERSITY

Sayı: 28620816/

09 Aralık 2013

Gönderilen : Y. Doç. Dr. Çiğdem Haser
İlköğretim Bölümü

Gönderen : Prof. Dr. Canan Sümer
IAK Başkan Vekili

İlgi : Etik Onayı

Danışmanlığını yapmış olduğunuz İlköğretim bölümü doktora öğrencisi Zuhal Yılmaz'ın **"Öğrenme Rotalarının Metot Dersi Kapsamında Sınıf Öğretmeni Adaylarının Matematiksel Bilgisinin Yapılandırılmasında Kullanımı"** isimli araştırması "İnsan Araştırmaları Komitesi" tarafından uygun görülerek gerekli onay verilmiştir.

Bilgilerinize saygılarımla sunarım.

Etik Komite Onayı

Uygundur

09/12/2013

Prof.Dr. Canan Sümer
Uygulamalı Etik Araştırma Merkezi
(UEAM) Başkan Vekili
ODTÜ 06800 ANKARA

APPENDIX E

TURKISH SUMMARY

Öğrencilerin matematiği nasıl öğrendikleri, anladıkları ve matematik hakkında akıl yürütme becerilerini nasıl geliştirdikleri üzerine birçok araştırma (Clements, Sarama ve Julie, 2009; Fennema ve Franke, 1992; Ma, 1999) yapılmıştır. Fakat öğrencilerin matematiği zaman içerisinde nasıl öğrendikleri, öğrenmenin sadece bir boyutunu oluşturmaktadır. Diğer bir önemli boyutunu ise öğretim ortamını oluşturmada en önemli faktörlerden biri olan öğretmenlerdir (Ma, 1999). Bu sebeple, öğretmenin alan bilgisi seviyesi ve öğrencilerin öğrenmesini anlama bilgisi ve becerisi, öğrencilerin neyi nasıl öğrendikleri üzerinde çok önemli bir role sahiptir (Darling-Hammond ve Ball, 1998; Ma, 1999). Araştırmacılar Varsayımsal Öğrenme Rotalarının (Hypothetical Learning Trajectories) öğrencilerin zaman içerisinde nasıl matematiği öğrendiklerini anlamamızda önemli role sahip olduklarını ileri sürmüşlerdir (Clements ve Sarama, 2004; Duncan ve Hmelo-Silver, 2009). Aynı zamanda varsayımsal öğrenme rotası bilgisinin öğretmenlerin matematiği öğretme ve öğrenmelerinde etkisinin olacağını belirtmişlerdir (Clements ve Sarama, 2004; Duncan ve Hmelo-Silver, 2009).

Öğrenme rotaları literatürde farklı şekillerde kavramsallaştırılmışlardır. Bütün farklı kavramsallaştırmalar temelde Simon'ın 1995 yılında varsayımsal öğrenme rotası olarak ifadelendirdiği kavrama dayanmaktadır. Simon (1995) varsayımsal öğrenme rotalarını öğretmenin öğrenmenin izleyeceği yol hakkındaki tahminleri seklinde tanımlamıştır. Simon öğrenme rotasını varsayımsal olarak nitelendirmesinin sebebini su şekilde ifade etmiştir " …. Varsayımsaldır, çünkü öğrenme rotası kesin olarak önceden öngörülemez ve o beklenen eğilimi karakterize etmektedir" (1995, s.135). Simon'ın (1995) ifadelendirdiği bu yaklaşım, öğrenme rotalarının deneysel veriler ile yapılandırılmasının temelini oluşturmuştur. Bu deneysel veriler ile çalışmalar yapan bir çok araştırmacı öğrenme rotası kavramını tanımlamışlardır. Bunlardan bazıları şu şekildedir. Corcoran, Mosher ve Rogat (2009) öğrenme rotalarını belirli bir konu alanında ve uygun bir süre içerisinde öğrencilerin az karmaşıktan çok karmaşık düşünmeye geçiş surecinde izleyecekleri varsayımsal bir rota olarak tanımlamışlardır. Confrey ve arkadaşları (2009) öğrenme rotalarını araştırma sentezi, deneysel veriler ve öğretim faaliyetleri ve araçları boyutunda tanımlamışlardır. Öncelikle belirli bir matematik konu alanında yapılan araştırmaların kapsamlı bir sentezi yapılıp, bu çalışmaların öğrencinin o konu alanını nasıl öğrendiğinin rotası belirlenir. Daha sonra araştırmacı bu bilgiler ışığında etkinlikler tasarlar ve öğrencilerle birebir çalışıp öğrencinin zaman içerisinde az karmaşık düşüncelerden daha karmaşık düşünmeye nasıl geçtiğini deneysel veriler ile saptar (Franklin, Yilmaz ve Confrey, 2010). Bu süreç her zaman devamlı bir değerlendirmeyi, yansımayı ve öğrenme rotasının revizyon edilmesini içeren dinamik bir süreçtir. Bu çalışmada ise Confrey ve arkadaşlarının (2009) tanımlaması benimsenmiştir.

Türkiye'deki ve uluslararası çalışmalar, öğretmen adaylarının öğretecekleri matematiği tam olarak bilmediklerini (Eraslan, 2009; Hacıömeroğlu ve Taşkın, 2010), öğrencilerin sahip oldukları matematiksel kavram yanılgılarına ve zorluklara sahip olduklarını (Butterfield, Forrester, McCallum ve Chinnapan, 2013), kavramsal matematik bilgi eksikliklerinin yanlış öğretme uygulamalarına sebebiyet verdiğini (Phillip, 2008), ilköğretim matematiğini basit bulma eğilimi gösterdiklerini (Phillip, 2008), üniversitelerde verilen matematik dersleri ile öğretecekleri matematik arasındaki bağlantıyı kurmada zorlandıklarını (Eraslan, 2009), öğrencilerin matematiği nasıl öğrendiğini anlayacakları çok az imkana üniversite eğitimleri sırasında sahip olduklarını (Hacıömeroğlu ve Taşkın, 2010; Jansen ve Spitzer, 2009) ve üniversite seviyesinde verilen matematik eğitimi derslerinin gerekli kavramsal alt yapıyı sağlamada yetersiz olduğunu (Ubuz, 2009) ortaya koymuştur. Bu sebeple, öğretmen adayları öğrencilerin matematiği öğrenme bilgileri ve kendi matematiksel alan bilgilerini birlikte geliştirebilecekleri, araştırmanın ve gerçek öğrenci çalışmalarının pratiğinin birleştirildiği ortamlarda eğitilmelidir. Bu eğitim süresinde öğretmen adaylarının davranışları ve gelişmeleri tespit edilmeli ve değerlendirilmelidir.

Elmore'a (2002) göre araştırmacılar arasında bu öğretmen eğitimi ortamının nasıl tasarlanması gerektiği hakkında belirli bir derecede görüş birliği olmasına karşın, hangi programın en etkili şekilde kullanılabileceği hakkında bir birlik bulunmamaktadır. Yeni çalışmalar (Butterfield, Forrester, McCallum ve Chinnappan, 2013; Wilson, Mojica ve Confrey, 2013) öğrenme rotalarının mevcut öğretmen eğitiminde kullanılmalarının öğretmen adaylarının kavramsal matematik anlamalarını geliştirme potansiyeline sahip olduğunu öne sürmektedirler. Buna ek olarak, öğrencinin matematik öğrenmesini temel alarak oluşturulan bu rotaların kullanımının, adayların öğrencinin matematiğini daha iyi öğrenmelerine temel hazırlayacağını da belirtmektedirler. Bu şekilde, öğretmen adayları meslek hayatlarına başlamadan önce öğrencilerin matematiksel kavram yanılgılarını, akıl yürütmelerini ve zorluklarını öğrenme fırsatını elde edebileceklerdir (Mojica, 2010; Sztajn, Confrey, Wilson ve Edgington, 2012).

Öğrenme rotalarının eğitimin içerisine entegre edilmesinin öğrencilerin matematiği nasıl öğrendiklerinin anlaşılmasındaki faydalı ve etkili rol oynadığını bir çok araştırma ortaya koymuştur (Clements, Sarama, ve Julie, 2009; Confrey ve diğerleri, 2009; Duncan ve Hmelo-Silver, 2009). Buna ek olarak, matematik eğitimi alanında son zamanlarda öğretme ve öğrenme ortamının önemli bir bileşeni olan öğretmenlerin öğrenme rotalarını kullanımı üzerine araştırmalar yapılmaya başlanmıştır (Daro, Mosher ve Corcoran, 2011; Niess ve Gillow-Wiles, 2014; Sztajn ve diğerleri, 2012; Wilson, Edgington ve Myers, 2014; Wilson, Sztajn, Edgington ve Confrey, 2013). Çok az sayıda çalışma ise öğretmen eğitiminde kullanımı üzerine yapılmıştır (Butterfield ve diğerleri, 2013; Wilson, Mojica ve Confrey, 2013a). Bu çalışmalardan Mojica (2010) nın çalışması öğrenme rotaları kullanımının, öğrenciler ile birebir çalışmanın yoğun olarak yapıldığı özel bir programa kayıtlı olan öğretmen adaylarının matematik alan bilgisini geliştirmesinde etkili olduğunu ortaya koymuştur. Ancak, bu çalışmadaki gibi olmayıp, genel olarak eğitim fakültelerinde öğrenciler ile birebir çalışma tecrübesini yoğun şekilde yaşamayan öğretmen adayları için böyle bir çalışma yapılmamıştır. Diğer bir araştırma ise Butterfield ve diğerleri (2013) tarafından yapılmış olup bir çalışma önerisi şeklindedir.

Yakın zamanda, öğrenme rotaları araştırmacılarından Sztajn ve diğerleri (2012), öğretmen eğitiminde kaynak olarak kullanılabilecek yeni bir öğretim kuramını ileriye sürmüşlerdir. Sztajn ve diğerleri (2012) bu öğretim kuramını öğrenme rotaları temelli öğretim (learning trajectories based instruction) olarak isimlendirmişlerdir. Bu kuramı "araştırmalarda ortaya konulan öğrenme üzerine olan çeşitli çerçeveleri birleştirmek ve yenilemek için öğrenme rotaları araştırmalarını kullanmaktır" (s.152) seklinde tanımlamışlardır. Dolayısı ile bu çalışma, yukarıda tartışılan öğretmen eğitimindeki sorunlara ve öğretmen adaylarının kalitesine dair durumlara dayanarak, mevcut durumun iyileştirilmesi adına çözüm olabileceği araştırmacılar tarafından önerilen, öğrenme rotaları temelli öğretim kuramını sınıf öğretmen adayları ile gerçekleştirilen bir öğretim deneyinde uygulamıştır. Bu çalışmanın amacı ise öğretmen adaylarının eşpaylaşım ile ilgili mevcut matematik alan bilgilerini ve öğrenci bilgilerini ne şekilde yapılandırdıklarını incelemektir. Eşpaylaşım konusunun seçilme amaçlarından en temeli bu kavramın ilköğretimde öğretilen, öğrencilerin öğrenmekte zorluk yasadığı ve birçok ileri matematik konusuna temel hazırlanmasına katkısı olan bir konu olmasıdır. Eşpaylaşım çarpma, bölme, rasyonel sayılar, kesirler, oran, orantı gibi öğrenciler tarafından anlaşılması zor olan matematik konularına temel hazırlamaktadır (Confrey, Maloney, Nguyen, Wilson, ve Mojica, 2008).

Bu çalışmada sadece öğretmen adaylarının matematik alan bilgilerinin ve öğrenci bilgilerinin ne kadar ilerlediği incelenmemiştir. Bu ilerleme sürecinde, öğretmen adaylarının Ball, Thames ve Phelps' (2008) in ortaya koymuş olduğu matematik alan bilgisinin türleri olan genel alan bilgisi, özel alan bilgisi ve ufuk alan bilgilerini yeniden yapılandırma sürecindeki eylemleri belirlenmiştir. Öğrenci bilgilerinin de gelişmesinde öğretmen adaylarının hangi eylemleri gerçekleştirdikleri araştırılmıştır. Bu çalışmada bu bilgi çeşitleri öğrenme rotaları temelli öğretim kuramının getirdiği yaklaşım ile tanımlanmıştır. *Genel alan bilgisi* öğrenme rotasının her bir düzeyinde belirtilen matematiksel düşünceleri bilme ve uygulayabilme, *özel alan bilgisi* öğrenme rotasının içermiş olduğu matematiksel bilgileri, kavram yanılgılarını, hataları irdeleyebilme ve öğrenme rotasındaki matematiksel bilgileri ve fikirleri birden fazla gösterim çeşidi ve stratejisi ile inceleyebilme ve *ufuk alan bilgisi* ise öğrenme rotasındaki matematiksel fikirleri ve bilgileri birbirleri ile ve ileri matematik konuları ile ilişkilendirme ve buna ek olarak genellenebilir matematiksel sonuçlara ulaşabilme olarak tanımlanmıştır (Sztajn ve diğerleri, 2012). Son olarak öğrenci bilgisi Sztajn ve diğerlerinin (2012) tanımlamalarından uyarlanarak tanımlanmıştır. Bu bilgi çeşidi, öğrencinin matematiksel öğrenmesini ve düşünmesini anlama ve öğrencilerin düşünme yaklaşımları ile empati kurabilmedir.

Yukarıda irdenilen ihtiyaçlar ve çalışmanın gerekliliği kapsamında, bu çalışma üç ana araştırma sorusuna cevap bulmayı amaçlamıştır:

1. Öğretmen adaylarının öğrenme rotaları temelli öğretimden önceki ve sonraki matematik alan bilgileri ve öğrenci bilgileri arasındaki farklar nelerdir?
    - Öğretmen adaylarının öğrenme rotaları temelli öğretimden önce öğretmeleri gereken eşpaylaşım kavramına dair olan bilgi düzeyleri nedir?
    - Öğretmen adayları eşpaylaşım konusuna dair herhangi bir kavram yanılgısı, bilgi eksikliği, zorluk ya da hataya sahipler midir? Sahiplerse, bunlar nelerdir?
    - Öğretmen adaylarının öğrenme rotaları temelli öğretimden sonra öğretmeleri gereken eşpaylaşım kavramına dair olan bilgi düzeyleri nedir?
2. Öğretmen adaylarının öğrenme rotaları temelli öğretim deneyindeki matematik alan bilgilerini yeniden yapılandırma eylemleri nelerdir?
    - Öğrenme rotaları temelli öğretim öğretmen adaylarının matematiksel kavram yanılgılarını, hatalarını ve zorluklarını tespit etmelerinde ve iyileştirmelerinde nasıl bir rol oynamıştır?
    - Öğrenme rotaları temelli öğretim hangi şekillerde öğretmen adaylarının eşpaylaşım ile ilgili matematiksel stratejileri, gösterimleri ve fikirleri anlamalarına destek olmuştur?
    - Öğretmen adayları öğrenme rotasındaki eşpaylaşım ile ilgili matematiksel fikirleri ileri matematik ile nasıl ilişkilendirmişlerdir?
3. Öğretmen adayların öğrenme rotaları temelli öğretim deneyinde öğrenci bilgilerini yeniden yapılandırma eylemleri nelerdir?

- Öğrenme rotaları temelli öğretim hangi şekillerde öğretmen adaylarının öğrenciler ile ilgili matematiksel bilgileri ve fikirleri anlamalarına destek olmuştur?

Yöntem

Öğretim deneylerinin ana amacı öğrencilerin ilk elden matematik öğrenmelerini ve akıl yürütmelerini anlamak (Thompson, 2000) ve öğretim kararlarını buna göre yönlendirmek ve aynı zamanda daha iyi bir öğrenme sağlamaktır (Cobb, Confrey, diSessa, Lehrer ve Schauble, 2003) Öğretim deneyleri öğrencilerin matematiksel etkinliklerinin, davranışlarının modelini ortaya çıkarmada etkili bir yöntemdir (Steffe ve Thompson, 2000). Bu sebeple, bu çalışmada sınıf öğretmeni adaylarının matematik alan bilgilerini ve öğrenci bilgilerini öğrenme rotaları temelli öğretimde hangi yollar ve eylemler ile yeniden yapılandırdıklarını incelemek amacı ile yapılandırmacı öğretim deneyi yöntemi (Steffe ve Thompson, 2000) kullanılmıştır.

Bu çalışmada, öğretim deneyi üç aşamada gerçekleşmiştir. İlk aşamada öğretim etkinlikleri ve değerlendirme soruları yaklaşık 3 haftalık bir süreç içerisinde araştırmacı ve doktora eğitimine sahip bir matematik eğitimci ile birlikte geliştirilmiştir veya mevcut bulunan araştırmalardan (Empson ve Turner, 2006; Mojica, 2010; Wilson, 2009) uyarlanmıştır. İki aşamada geliştirilen bu materyallerden yeni geliştirilenler 3 hafta süren bir pilot çalışmada kullanılmış ve eksiklikleri, çalışan ve çalışmayan yönleri tespit edilmiştir. Son aşamada ise 6 haftalık asıl öğrenme deneyi uygulanmıştır. Bu aşamalar aşağıda detaylandırılmaktadır.

Asıl öğrenme deneyinden önce, yeni geliştirilen değerlendirme sorularının ve kullanılacak etkinliklerin 3 hafta boyunca haftada yaklaşık 3 saat boyunca 10 sınıf öğretmeni adayı ile bir araya gelerek pilot çalışmaları yapılmıştır. Araştırmacı aynı zamanda öğretim deneyinde öğretmen rolündedir. Bu pilot çalışmada öğretmen adaylarının verilen etkinliklerdeki ve sorulardaki stratejileri ve karşılaştıkları zorlukların saptanması için her bir derste ses kaydı alınmıştır. Ses kaydı verilerini desteklemek amacı ile öğretmen adaylarının yazılı çalışmaları ve sınıf içi önemli etkinliklerin fotoğrafları da alınmıştır. Pilot çalışmanın sonunda etkinlikler ve değerlendirme soruları revize edilmiştir. Örneğin, öğretmen adaylarının verdikleri cevapları detaylandırılmalarının beklenildiği değerlendirme sorularına doğrulama soruları eklenmiştir. Pilot çalışmada toplanan verilerin analizleri bilgi yapılandırma eylemleri kategorilerinin ilk taslağını oluşturmada kullanılmıştır.

Bu pilot çalışmanın sonucuna göre belirlenen kategoriler şu şekildedir: (1) *alan bilgisindeki değişiklik*, (2) *kavram yanılgıları ve öğrenme zorlukları* ve (3) *öğrencilerin düşünme yapılarını anlamak*. Bu üç kategorinin her birine bağlı olan alt eylemler de kodlanmıştır. Birinci kategori altında *genişletme ve değiştirme*, ikinci kategori altında destekleme *tanımlama ve düzeltme*, üçüncü kategori altında ise *sıralama ve öngörme* eylemleri (Smith ve Stein, 2011'den uyarlanmıştır) yer almaktadır. Bu kategoriler ve onlara bağlı eylemlerin her biri asıl çalışmada revize edilmiş, genişletilmiş ve yeniden düzenlenmiştir. Bulgular kısmında her bir eylemin açıklamaları ve örneklendirmeleri ayrıntılı olarak verilmiştir.

Pilot çalışmanın ardından Türkiye'de özel bir üniversitede son sınıf olan 9 sınıf öğretmeni adayı ile asıl öğrenme rotaları temelli öğretim deneyi gerçekleştirilmiştir. Araştırmacı aynı zamanda öğretim deneyinde öğretmen rolündedir. Öğretim deneyi başlamadan önce araştırmacı çalışmanın içeriğini ve amacını belirten gönüllü katılım formunu bütün öğretmen adaylarından imzalamaları rica etmiş ve her bir öğretmen adayı gönüllü olarak üniversitedeki ders saatlerinin dışında araştırmacı ile çalışmayı gerçekleştirmek üzere buluşmuşlardır. Öğretmen adaylarının eşpaylaşım öğrenme rotasına dair bir ön tecrübeleri bulunmamak ile birlikte, her bir aday bir temel matematik dersi ve gözleme dayalı okul deneyimi dersi almıştır. Öğretmen adaylarının iki tanesi tam bursa, üç tanesi yarım eğitim bursuna sahiptir. Her bir öğretmen adayının matematik konu alanında başarı seviyeleri birbirinden farklı olup, başarı seviyesi her bir düzeyden dengeli dağılım olacak şekilde bilinçli olarak seçilmişlerdir. Öğretmen adaylarından sadece birisinin özel ders tecrübesi bulunmaktadır. Diğer öğretmen adayları gözlem dersleri dışında öğrenciler ile birebir çalışmamıştır.

Asıl öğretim deneyi altı hafta sürmüş olup, bu altı haftanın ilk ve son haftalarında açık uçlu 17 sorudan oluşan eşpaylaşım öğrenme rotasındaki düzeylerdeki matematiksel fikir ve kavramların içerildiği Mojica (2010) ve Wilson (2009) nın kullanmış olduğu ön-son testin uyarlanarak genişletilmiş hali kullanılmıştır. İlk ve son testin uygulanması arasında geçen süre altı haftadır. Ön-testten sonra bir hafta ara verildikten sonra öğretim etkinliklerinin uygulandığı dört haftalık bir öğretim deneyi başlamıştır. Her hafta araştırmacı-öğretmen ve öğretmen adayları yaklaşık 3 saat süren öğretim deneyini uygulamıştır. Öğretim deneyinin ikinci haftasında belirli bir topluluğu eş paylaştırma, paylaşımın adil gerçekleştirildiğini doğrulama, her bir paylaşımı isimlendirme üzerine odaklanmış toplam 3 adet öğretim etkinliği ve bir öğrenci video analizi kullanılmıştır. Üçüncü haftasında, bir bütünü eş parçalara ayırma, parçaların eşitliğini doğrulama ve erken geçişkenlik argümanın gelişimi, payları isimlendirme, yeniden birleştirme (bütün-parça arasındaki çarpımsal ilişki) ve birleşik bölme üzerine iki etkinlik ve bir öğrenci video analizi kullanılmıştır. Dördüncü hafta katlama, dikdörtgensel ve dairesel bütünleri eş paylaştırma arasındaki karmaşıklık ve zorluk düzeyi farklarının tespiti, Empson ve Turner'den (2006) uyarlanan eşpaylaşım ile katlama arasındaki ilişkiyi bulma üzerine iki etkinlik ve yazılı gerçek öğrenci katlama çalışmalarının incelenmesi yapılmıştır. Beşinci haftada ise birden fazla bütünü eşpaylaştırma, kovaryasyon ve yeniden dağıtma üzerine toplam 3 adet etkinlik ve 2 öğrenci video analizi yapılmıştır.

Öğretim deneyinin uygulanması sırasında gözlem notları, her bir öğretim oturumunun video kayıtları, öğrencilerin yazılı çalışmaları ve resimlenen öğrenci çalışmaları veri toplama gereçleri olarak kullanılmıştır. Buna ek olarak, araştırmacı-öğretmen ders sırasında gözlemlediği fakat çekilen videonun tam olarak içeriğini yansıtmayacağını düşündüğü ya da teknik olarak video çekmede sorun yaşandığı kısımları her bir dersin bitiminde alan notlarına yazmıştır.

Ön- ve son-testin analizleri, Wilson (2009) ve Mojica'nın (2010) ortaya koymuş olduğu rubriklerin uyarlanması ve yeni değerlendirme sorularına araştırmacı ve ölçme değerlendirme alanında doktora yapmakta olan bir öğrenci ile birlikte rubrik geliştirilmesinden sonra, iki araştırmacı tarafından her bir öğretmen adayının testlerde vermiş oldukları cevapların puanlanması yapılmıştır. İki araştırmacının puanlamaları arasındaki güvenirlik %88 olarak bulunmuştur. Puanlamalarının uyuşmadığı değerlendirme sorularında puanlayıcı ve araştırmacı ortak karara varmak üzere tartışmışlardır. Ortak karara varılamayan durumları araştırmacı matematik eğitimi araştırmaları üzerine yapılan bir çalıştaydaki araştırmacılar ile tartışmış ve bir sonuca ulaştırmıştır. Bununla birlikte araştırmacı ön- ve son-testleri ara ile iki kez puanlamış ve puanlamalar arasındaki uyum yüzdesi %94 olarak çıkmıştır.

Video kaydı verileri ve öğretmen adaylarının çalışma kağıtları analizlerde ana veri kaynağı olarak kullanılırken, diğer veri kaynakları bulguların netleştirilmesinde ve desteklenmesinde kullanılmışlardır. Video kaydı verileri, Powell, Francisco, ve Maher'in (2003) analitik modeli kullanılarak analiz edilmiştir. Bu model yedi basamaktan oluşmaktadır; 1) video verisinin dikkatli şekilde izlenmesi 2) video verisinin tanımlanması 3) kritik ve önemli olayların tespiti 4) video verisinin gerekli kısımlarının yazıya dökülmesi ve video kısımlarının kesilmesi 5) kodlama, 6) videodaki içeriğin ana temasının oluşturulması, 7) anlatının oluşturulması" (s.413). Birinci basamakta araştırmacı, video kaydı verisini araştırma sorularını akılda tutarak dikkatli bir şekilde izlemiştir. İkinci basamakta, araştırma sorularına cevap taşıyabilecek nitelikte video kaydı verisinin içeriğinin nelerden oluştuğunu, ve analiz için nelere dikkat edilebileceğini tanımlamıştır. Üçüncü basamakta, öğretmen adaylarının davranışlarındaki kritik olayları belirlemiştir. Bu kritik olaylar öğretmen adaylarının farklı bir strateji kullandıkları, matematiksel alan bilgilerini ve öğrencilerinin hakkındaki bilgilerini geliştirdikleri, değiştirdikleri gibi önemli olaylardan oluşmaktadır. Beşinci basamakta kodlama iki kişi tarafından ayrı ayrı yapılmıştır. Araştırmacı ve doktora derecesine sahip bir matematik eğitimcisi belirlenen kritik olayların ortak noktalarını ifadelendiren kodlamaları pilot verideki kodlamaları göz önüne alarak geliştirmişlerdir. Bununla birlikte, dört yıllık öğretmenlik tecrübesi olan bir sınıf öğretmeni ise veri analizi ve kodlama süresinde fikirlerini kodyalan araştırmacılarla paylaşmıştır. Verilerin kodlanmasının ayrı yapılması güvenirliği sağlamanın bir yöntemi olarak bu çalışmada kullanılmıştır.

Verilerin analizi neticesinde ortaya çıkan kodlar şu şekildedir. Öğretmen adayları ufuk alan bilgilerini iki tür eylemi ile yeniden yapılandırmıştır. Bunlar *ilişkilendirme* (Wilson ve ark.'dan (2013) uyarlanmıştır) ve *genellemedir*. Özel alan bilgileri *içselleştirme* ve *boyutlarını ortaya çıkarma* eylemleri ile yeniden yapılandırmışlardır. Genel alan bilgilerini ise *düzeltme ve değiştirme, genişletme ve meydan okuma (aksini iddia etme)* eylemlerini ortaya koyarak yeniden yapılandırmışlardır. Öğretmen adayları öğrenci bilgilerini ise dört tür eylem ile yeniden yapılandırmışlardır. Bunlar, *ayırt etme* (Mojica'dan (2010) uyarlanmıştır) *ve tanıma, sıralama* (Stein veve Smith'ten (2011) uyarlanmıştır), *öngörme* (Stein ve Smith'ten (2011) uyarlanmıştırve) *ve empati kurmadır.* Bu kodlamalar çalışmanın aynı zamanda bulguları olduğu için bulgular içerisinde ileriki paragraflarda detaylandırılmışlardır.

Creswell'e (2007) göre nitel çalışmalarda güvenirlik aynı kodlayıcının veriyi iki farklı zamanda kodlaması ve birbirinden farklı kodlayıcıların verileri bağımsız şekilde kodlamaları ile sağlanır. Bu çalışmada araştırmacı kendisi veriyi iki kez kodlamış ve iki kodlama arasındaki tutarlılık %95 olarak bulunmuştur. Bir matematik eğitimcisi ve araştırmacı bağımsız olarak aynı video verisinden tespit edilen kritik olayları kodladıklarında 90% olarak bulunmuştur. Bu çalışmada güvenilirliği sağlamak için üçleme (triangulation) metodu kullanılmıştır. Bu yöntem iki şekilde kullanılmıştır. Birincisi birden fazla veri kaynağından öğretmen adaylarının bilgilerini yeniden yapılandırmalarına dair ayrıntılı bilgi toplanmıştır (Patton, 1990). Bu farklı kaynaklardan toplanan veriler elde edilen sonuçların doğruluğunu netleştirmiştir. İkinci olarak, aynı video verileri birden fazla alanında uzman kişi ile birlikte araştırmacının ortak izleme ile analizi süresince bir sonuca ulaştırılmıştır (Mathison, 1988). Buna ek olarak, araştırmacı da video kaydı verisini birden fazla sayıda izlemiş ve analiz etmiştir. Bu ortak ve tek başına izleme sürecinde negatif ve alternatif sonuçlar karşılaştırılmış ve incelenmiş (Merriam, 2002), kodlamalara ayrıntılı alıntılar ya da görsel deliller seçilmiştir. Bütün bu süreç yapılan nitel çalışmanın kalitesini arttırmaya yönelik adımlar olarak uygulanmıştır.

Bu çalışmanın varsayımları ve olası sınırlılıkları ise şu şekilde tespit edilmiştir. Öğretmen adayları üniversitedeki dersleri için ayırdıkları zamanlarının dışında bu çalışmaya gönüllü olarak haftada yaklaşık 3 saat katılmışlardır. Bu çalışmadaki öğretmen adaylarının performansları notlandırılmamış ve performansları üniversitedeki not ortalamalarını etkilememiştir. Öğretmen adayları gönüllü olarak matematik bilgilerini geliştirmek için bu çalışmaya katıldıklarını belirtmişlerdir. Aynı zamanda, araştırmacı öğretmen adayları tarafından önceden tanındığı için her bir öğretmen adayı görüşlerini öğretim deneyi sırasında açıkça belirtmekten çekinmemiştir. Bununla birlikte, araştırmacı öğretmen adaylarını çalışmadan önce bu çalışmada onları yargılamayacağını ve birlikte öğreneceklerini belirtmiştir. Bu sebeplerden yola çıkarak öğretmen adaylarının vermiş oldukları cevaplarda ve öğretim deneyi sırasındaki düşüncelerinde samimi ve doğru oldukları varsayılmıştır.

Bu çalışmanın ilk kısıtlılığı zaman olarak belirlenmiştir. Zaman kısıtlaması sebebi ile pilot çalışma 3 haftada gerçekleştirilmiş ve bu süreç içerisinde sadece yeni değerlendirme soruları test edilmiş, önceki çalışmalarda kullanılan değerlendirme soruları direkt olarak asıl çalışmada kullanılmıştır. İkincisi, öğretmen adaylarının her hafta sadece azami 3 saatlerini gönüllü olarak bu çalışmaya vermiş olmalarıdır. Bunun neticesinde, her haftada işlenen konunun yoğunluğu artmıştır. Buna ek olarak, öğretmen adaylarının öğrendikleri bilgilerini staj okullarında denemeleri için fırsat olmamıştır. Bu durumun giderilmesi için öğretim deneyi içinde gerçek öğrencilerin matematiksel çalışmaları ve videoları öğretmen adayları ile birlikte analiz edilmiştir. Araştırmacının öğretmen adaylarının karşılıklı birbirini tanıması bir sınırlılık olarak ele alınmamıştır. Bu durum öğretmen adaylarının öğretim deneyine alışma sürecini kısaltarak, öğretim deneyindeki zamanın azami derece verimli kullanılmasına olanak sağlamıştır. Araştırmacı aynı zamanda öğretim deneyinin öğretmeni olma durumunun analizler üzerindeki olası yönlendirici etkisini ortadan kaldırmak için üçleme metodunu kullanmıştır.

Bulgular ve Sonuçlar

Bu çalışmanın ön- ve son-test sonuçları öğretmen adaylarının matematiksel alan bilgilerini ve öğrenci bilgilerini iyi bir şekilde ilerlettiklerini göstermektedir. Ön-testin nitel analizi öğretmen adaylarının öğretim deneyine katılmadan önce ilköğretim seviyesinde öğretilecek bir konu olan eşpaylaşım konusu ile ilgili birden fazla kavram yanılgısına sahip olduğunu ortaya koymuştur. Bununla birlikte, öğretmen adaylarının öğrencilerin eşpaylaşım konusu ile ilgili matematiksel akıl yürütmeleri ve düşünceleri hakkında kısıtlı bir bilgiye sahip oldukları açığa çıkmıştır. Öğretmen adaylarının büyük çoğunluğu öğrencilerinin de kendilerinin uygulamış olduğu matematiksel stratejiyi kullanacaklarını düşünmüşlerdir. Ayrıca, öğretmen adayları ön-testteki soruları tek bir çözüm yolu çözmeye çalışmışlardır. Bununla birlikte testlerde yer alan birçok soru çözüm yollarının doğrulanmasını gerektirmektedir. Ön-testte öğretmen adaylarının bir kısmı üretmiş oldukları cevapları sadece kısıtlı bir şekilde açıklayabilmiş ve bu açıklamalarında kısıtlı bir şekilde kavramsal olarak doğru bir dil kullanabilmişlerdir. Son-testte ise, öğretmen adayları üretmiş oldukları cevapları öğretim deneyinde öğrenmiş oldukları matematiksel kavram ve düşünceleri kullanarak açıklayabilmişlerdir. Ek olarak, öğretmen adayları sahip oldukları kavram yanılgılarını düzeltmiş ve çözüm yolları için geçerli matematiksel açıklamalar getirebilmişlerdir. Öğretmen adayları öğrencilerin matematiksel düşünme yollarını düşünmeleri gereken sorularda ise, küçük çocuklar için somut gösterimler içeren matematiksel stratejileri içeren tahminlerde bulunurken, daha ileri yaştaki çocuklar için sembolik gösterimlerin olduğu stratejileri belirleyebilmişlerdir. Bütün bu ön- ve son-test arasındaki öğretmen adaylarının performansları arasındaki fark, öğrenme rotaları temelli öğretim deneyinin, amacını gerçekleştirmede başarılı olduğunu destekler niteliktedir.

Öğretim deneyi sırasında, öğretmen adaylarının matematiksel alan bilgisinin alt türleri olan özel alan bilgisi, genel alan bilgisi ve ufuk alan bilgilerini toplam yedi eylem türü sergileyerek yeniden yapılandırdıkları bulunmuştur. Ufuk alan bilgilerini yeniden yapılandırma eylemleri *genelleme ve ilişkilendirme* olarak bulunmuştur. *Genelleme* eylemini gösteren öğretmen adayları eşpaylaşım ile ilgili bir durumu genel geçer kurallar ile ifade edebilmişlerdir. Örneğin, öğretmen adaylarının büyük bir çoğunluğu katlama etkinliğinde her bir katın sonucu oluşan eş parça sayısını üslü sayılar ile genel olarak gösterememiştir. Öğretim deneyinde ise, tüm öğretmen adayları her bir katlama ile oluşan parça sayısını üslü sayıları kullanarak genelleyebilmişlerdir. Diğer bir örnek ise, öğretmen adaylarının hangi eş paylaşım durumunun basit, bileşik ve tam sayılı kesir oluşturduğunu matematiksel olarak ifadelendirmesi olarak verilebilir. Öğretmen adayları paylaşılan nesne sayısını ya da bütün sayısını “n” ile ve paylaştırılanların sayısını “p” ile temsil etmişler ve eğer n> p ise basit ve n<p ise bileşik kesir oluşur genellemesine

ulaşmışlardır. *İlişkilendirme* eyleminde ise öğretmen adayları iki türde bilgilerini yapılandırmışlardır. İlk başta sadece bölme işlemi ile ilgili olduğunu düşündükleri ve bildikleri eş paylaşımı daha sonra da kesir türleri, kesrin anlamı, çarpma, bölme, oran ve orantı, üslü sayılar, ölçme gibi ileri matematik konuları ile ilişkilendirmişlerdir. Ek olarak, bu ilişkileri nasıl kurduklarını matematiksel delillere ve öğretim deneyindeki tecrübe ve bilgilerine dayandırarak açıklayabilmişlerdir. İkinci olarak öğretim deneyi öncesinde ve başlarında, öğretmen adayları eşpaylaşımın kendi içerisindeki farklı uygulamalarını düşünmemişler ve sadece eş paylaşımı bir bütünü eş parçalara bölme olarak ele almışlardır. Öğretim deneyine katıldıkça öğretmen adayları eş paylaşımın çeşitli uygulamalarını öğrenmiş ve bunları öğrenme rotasındaki bilgilere paralel olarak ilişkilendirmişlerdir.

Bu çalışmada, adayların özel alan bilgilerini yeniden yapılandırma eylemleri *içselleştirme ve boyutlarını ortaya çıkarma* olarak bulunmuştur. Öğretmen adayları öğretim deneyinin başlarında, bir matematiksel kavramı, stratejiyi ya da düşünceyi verilen sorularda ya da etkinliklerde kullandıkları halde tam olarak tüm boyutları ile tanımlamakta güçlük çekmişlerdir. İçselleştime eyleminde öğretmen adayları kullanmış oldukları bu stratejilerin, kavramların, fikirlerin arka planındaki matematiği anlamış ve tecrübelerinden yola çıkarak bu matematiği açıklayabilmişlerdir. Örneğin, öğretmen adayları öğretim deneyinin başında kesir kavramının anlamını sadece bir bütünü eş parçalara bölme olarak bilirken, öğretim deneyinin son haftasında kesirlerin paylaştırma, bölme ve oran anlamını sırası ile bir bütünü eş parçalara ayırma, bir çokluğu dağıtma ve kovaryasyon etkinlikleri aracılığı ile öğrenmişlerdir. *Boyutlarını ortaya çıkarma* eyleminde ise öğretmen adayları eşpaylaşım ile ilgili kavram yanılgılarını, hataları ve zorlukları incelemiş ve bunlara sebebiyet verebilecek faktörleri saptamışlardır. Öğrenme deneyinde tecrübe kazandıkça öğretmen adayları bu faktörleri saptamak için nasıl genel alan bilgilerini kullanacaklarını, nasıl soru sorulacağını sınıf içi etkileşim ve etkinliklerde öğrenmişlerdir. Bununla birlikte örneğin bir kavram yanılgısının temelinde yatan etkenleri açığa çıkardıktan sonra ortadan kaldırmak için hangi matematik bilgisinden ve gösteriminden yararlanacakları bilgisini edinmişlerdir.

Öğretmen adayları genel alan bilgilerini yapılandırırken ise üç ayrı eylem ortaya koymuşlardır. Bunlardan birincisi *düzeltme veve değiştirme*'dir. Bu eylemde ön-testte ya da öğrenme deneyi sırasında üzerinde çalıştığı soruda ya da etkinlikte matematiksel bir hata veya kavram yanılgısı gösteren öğretmen adayları, sınıf içerisinde bütün cevapların sunulup ve tartışılması esnasında yapmış olduğu hatayı ya da yanılgıyı fark etmiş ve düzeltme eylemi göstermiştir. Örneğin, kovaryasyon etkinliğinde kurabiye sayısındaki artışın ve paylaştıran bebek sayısı arasındaki artışın aynı oranda olduğunu fark etmeyen bir öğretmen adayı, bu soruyu çözerken toplamsal bir artış miktarı olduğunu ileri sürmüştür. Bu öğretmenin vermiş olduğu bu yanıtı diğer öğretmen adayları her bir bebeğin aldığı miktarı görseller ile ifade ederek, bu öğretmen adayının akıl yürütmesine ve çözüm yolunun yanlış olduğunu anlamasına ve daha sonrasında doğru çözüm yolunu bulmasına yardımcı olmuştur. İkinci eylem ise, *genişletme*dir. Bu eylemde öğretmen adayları mevcut bilgilerinde olan eşpaylaşım konusunu, önceden bilmedikleri bilgileri edinerek genişletmiştir. Bu eylem iki şekilde gerçekleşmiştir. İlk olarak, öğretmen adaylarının her birinin farklı genel alan bilgisine sahip olmaları ve bunları birbirleri ile paylaşmaları neticesinde, birbirlerinin bilgilerini genişletmişlerdir. Örneğin, bir öğretmen adayının kullanmış olduğu farklı bir matematiksel stratejiyi diğer öğretmen adayı ile paylaşması neticesi öğretmen adayı bu stratejiyi dogru şekilde kullanabilir hale gelip öğretim deneyinin devamında kullanmıştır. İkinci olarak, araştırmacı - öğretmenin öğretmen adaylarının düşünemedikleri durumları ve matematiksel fikirleri öğrenme rotasının yardımıyla soru ve etkinlik halinde onlara deneyin içerisinde gelişecek şekilde sunması ile gerçekleşmiştir. Son eylem çeşidi ise *meydan okuma* ya da *aksini iddia etmedir*. Bu eylem, öğretim deneyinde yer yer karşılaşılan bir eylemdir. Bu eylemde, öğretmen adayı sahip olduğu matematik alan bilgisini ve öğretim deneyindeki öğrenme rotasına dair tecrübelerini kullanarak, öğrenme rotasında öne sürülen matematiksel fikirlere birebir uyuşmayan zorlayıcı argümanlar geliştirmiş ve bunu delillendirmiştir. Örneğin, eşpaylaşım öğrenme rotası eşpaylaştırma konusunda bir bütünü bir sayının pozitif çarpanlarını kullanarak parçalara ayırmanın tek sayıya ayırmaktan daha zor olduğunu ifade etmektedir. Fakat, bir öğretmen adayı bu durumun her sayı için geçerli olamayacağını savunmuş ve bu savını matematiksel örnekler ile desteklemiştir. Bu öğretmen adayı, "*sekize ayırma gibi ikinin kuvvetini kullanarak veya sekiz sayısının çarpanlarını kullanarak yapılabilecek bir eşpaylaşım durumu, bir bütünü mesela üçe eş olarak* *paylaştırmaktan daha kolaydır*" örneğini vermiştir.

Öğretmen adayları öğrenci bilgilerini yeniden yapılandırırken ise dört ayrı eylem çeşidini göstermişlerdir. Birincisi, *ayırt etme* (Mojica'dan (2010) uyarlanmıştır) ve *tanıma* dır. *Ayırt etme* eyleminde öğretmen adayları kendi matematiksel düşünme şekillerinin öğrencinin matematiksel düşünmesinden farklı olduğunu anlamışlardır. Örneğin, öğretmen adaylarının bir çoğu ön-testte bir topluluğu eşpaylaştırmada bölme işlemini kullanmış ve küçük yaştaki öğrencilerin de bölme kullanacaklarını düşünmüşlerdir. Öğrenme deneyi içerisinde ise, öğretmen adayları araştırmacının vermiş olduğu somut materyalleri kullanarak öğrencilerin direkt bölme işlemini kullanmadan önce birebir dağıtma, ya da birden fazla nesneyi aynı anda dağıtma gibi stratejiler kullanabileceklerinin farkına varmışlardır. Bir öğretmen adayı bu durumu şu şekilde ifadelendirmiştir: "*Öğrencilerden farklı olarak, biz her zaman en kısa yol olan bölme işlemini kullandık. Ben diğer yöntemleri öğrencilerime derslerimde göstermezdim. Çünkü, kimse bana daha önce [başka stratejileri] sormadı.*" Tanıma eyleminde ise, öğretmen adayı bu ayrımı öğrencilerin gerçek çalışmalarında saptamış ve kendi düşüncesinden nasıl farklı olduğunu ortaya koymuştur.

İkincisi ve üçüncüsü Stein ve Smith'den (2011) uyarlanan eylemlerdir. Bunlar sırası ile *sıralama ve öngörmedir*. Sıralama eyleminde öğretmen adayları, bir öğrenci için eşpaylaşım konusu ile ilgili matematiksel stratejileri, görevleri ve soruları az karmaşık olandan çok karmaşık olana doğru sıralama bilgisine sahip olmuşlardır. Örneğin, öğretim deneyinin başında öğretmen adayları dikdörtgensel bir bütün ile dairesel bir bütünü eşpaylaştırma arasında zorluk farkının olmadığını düşünmüşlerdir. Öğretim deneyi sırasında ise, dairesel bir bütünü eş paylaştırmayı somut materyaller üzerinde yapmaya çalışmışlar ve özellikle tek sayıda ayırım kullanacakları durumlarda dairesel kesmenin kullanılmasının dairesel bir bütünü eş paylaştırmayı daha zor hale getirdiğini fark etmişlerdir. İkinci eylem olan *öngörmede* ise, öğretmen adayları bir öğrencinin sunulan bir eşpaylaşım görevinde kullanabileceği olası stratejileri, gösterimleri ve bu görevin çözüm sürecinde gösterebileceği olası kavram yanılgılarını, hataları ve zorlukları önceden tahmin edebilmiştir. Örneğin, hiçbir öğretmen adayı ön-testte öğrencilerin bir dairesel bütünü eşpaylaştırmadaki kavram yanılgılarını tahmin edemez iken, son-testte öğretmen adaylarının hepsi en azından bir kavram yanılgısını öngörmüştür. Son saptanan eylem ise *empati kurmadır*. Öğretmen adayları öğrencilerin kavram yanılgılarına sahip olmaları durumu ile kendilerinin de benzer kavram yanılgılarına sahip olma durumları arasında empati kurmuşlardır. Öğretim deneyinin başlarında ve ön-testte, öğretmen adaylarının yanlış cevap veren bir öğrencinin matematik bilgisini değerlendirirken konuyu bilmediği yargısı ile geçiştirdikleri gözlenmiştir. Öğretim deneyi ilerledikçe, öğretmen adayları da öğrencilerin sahip olduğu yanılgıları sergilemişler ve bunları nasıl düzelteceklerini tartışmışlardır. Bu durum, öğretmen adaylarının öğrencilerin hatalarını ve yanılgılarını yargılamamalarını sağlamıştır. Bir öğretmen adayı bu durumu şöyle ifade etmiştir: "*Biz öğretmen olarak bu yanılgılara sahip isek, öğrencilerin sahip olması çok normal. Önemli olan bu yanılgıları gidermeye çalışmaktır*."

Tartışma

Schoenfeld (2011) kuram ve çerçeve arasındaki farkı basit bir dille şu şekilde ifade etmiştir: Çerçeve neye bakmanız gerektiğini ve görmeniz gereken olası etkiyi söylerken; kuram parçaların bir araya nasıl geldiklerini ve uyum sağladıklarını söyler. Kuram bir şeyin niçin ve nasıl çalışacağını söylerken, çerçeve bunun uygulamada nasıl açığa çıktığının göstergesidir. Öğrenme rotaları temelli öğretim son yıllarda ortaya çıkan ve gelişen bir öğretim kuramıdır. Bu kuram mevcut öğretim kuramlarını ve öğrenme rotaları çalışmalarının perspektifi ile birleştirip, matematik öğretiminde kullanımının nasıl olacağını ve potansiyel faydalarını ifadelendirmektedir. Araştırmacıların (Butterfield ve diğerleri, 2013; Sherin, Jacobs, ve Philipp; 2011) belirttiği gibi bilginin sadece ilerlediğinin tespiti değil, bu ilerleme sürecinde onun nasıl yapılandırıldığının tespiti, kullanılan kuramlarının işlerliğini anlamak açışından önemlidir ve bu alanda çalışmalara ihtiyaç duyulmaktadır.

Bu çalışmanın önemli sonuçlarından biri bu ihtiyaca cevap verme ve daha sonra yapılacak farklı öğrenme rotalarının kullanılacağı çalışmalara yön gösterme potansiyelinin olmasıdır. Bu çalışma, öğrenme rotaları temelli öğretim kuramını öğretmen adaylarının katılmış olduğu bir öğretim deneyinde kullanarak, bu kuramın kullanımı sırasında ve neticesinde ortaya çıkan etkileri ve eylemleri sınıflandırmıştır. Bu kuramın pratikte nasıl sonuçlar ortaya çıkardığını ve bilgiyi geliştirmede hangi eylemlerin gerçekleştirilmesine zemin hazırladığını gösteren bir çerçeve ortaya konulmuştur. Bu çerçeve öğretmen adaylarının bilgilerindeki ilerlemenin hangi davranış türlerinin neticesinde açığa çıktığının tespiti açısından önem taşımaktadır.

Bu çalışmanın önemli bir diğer sonucu öğretmen adayları matematiğin birbirinden ayrı yapıya sahip olan konular topluluğu olmadığını öğrenmeleri olmuştur. Başta eşpaylaşım konusunu çok basit olarak nitelendiren öğretmen adayları, ufuk alan bilgilerinin yeniden yapılandırmalarıyla bu konunun birçok ileri matematik konusu ile ilişkili olduğunu öğrenmiş ve bu ilişkileri öğrenme deneyindeki deneyimleri ve alan bilgileri ile ilişkilendirmişlerdir. Bunun neticesinde öğretmen adayları öğretecekleri ilköğretim matematiğinin düşündükleri gibi basit olmadığı kanısına ulaşmışlardır. Bu kanı Ball (1990) ve Phillipp (2008) tarafından belirtilen matematik eğitimine dair öğretmen ve öğretmen adaylarının sahip olduğu ana zorluklardan biridir. Bu öğretim deneyindeki tecrübeler ve etkileşimler neticesinde öğretmen adaylarının sahip oldukları bilgileri yeniden yapılandırmaları ve genişletmeleri bu zorluğun aşılmasında da rol oynamıştır.

Bu çalışma öğrenme rotalarını kullanan önceki çalışmalardan farklı olarak, öğretmen adaylarının öğrencilerden farklı bir matematiksel düşünme sistemine (Mojica, 2010) sahip oldukları bilgisini yapılandırmalarını sağladığı kadar, onlarla birlikte ortak kavram yanılgılarına ya da hatalara sahip olabileceklerini anlamalarını sağlamıştır. Birçok araştırma yetişkinlerin matematiksel düşünmesinin çocuklardan farklı olduğu tespitini yapmış ve bu farkın anlaşılmasının çocuklara matematik öğretiminde ne kadar önemli olduğunu vurgulamıştır. Bu çalışmada ise düşünceler arasındaki bu farkın anlaşılmasının önemi vurgulandığı kadar, öğretmen adaylarının öğrencilerin matematiksel düşünmelerinde ya da matematik sorularını çözme süreçlerinde izledikleri yollarda sahip oldukları yaklaşımlar ile kendi izledikleri süreçlerdeki benzerlikleri göz önüne alarak empati kurmalarının önemli olduğunu açığa çıkarmıştır. Öğretmen adayları öğretim deneyinin başlarında ve ön-testte, yanlış bir öğrenci cevabı gördüklerinde bu öğrencinin soruyu çözecek yeterli matematiği bilmediğini ya da yanlış bildiğini düşünmekteydiler. Fakat, öğrenme deneyinin içinde yaşadıkları tecrübeler neticesinde öğretmen adayları kendilerinin bazı eşpaylaşım durumlarına dair öğrencilerin göstermiş oldukları kavram yanılgılarına ve hatalarına sahip olduklarını gördüler. Bunun neticesinde öğretmen adayları, bunların nasıl düzeltileceği üzerine yoğunlaşmaları ve kendi bilgilerini de kavramsal düzeyde arttırmaları gerektiğini belirtmişlerdir. Öğretmen adaylarının öğretim deneyi sonunda öğrenciler ile bu empatiyi kurmaları, yanlış cevap üreten bir öğrenciyi yargılamaktan ziyade, yanlışa götüren sebepleri irdeleme farkındalığına sahip olduklarını göstermektedir. Bu öğretim deneyinin sonucunda gelişen anlayış aynı zamanda öğretmen adayları için öğrenciler bu hatayı nasıl yaparlar, ilköğretimde öğretilen matematik çok kolaydır (Philipp, 2008) gibi olan ilk varsayımlarını da değiştirmelerini sağlamıştır. Bu anlayışa sahip öğretmen adaylarının söylemleri öğrencilerinin kavram yanılgılarını tespit ettiklerinde, öğrencinin matematik öğrenmesini yargılamak yerine, bu yanılgının sebeplerini aramayı ve bu durumları nasıl üretken sınıf tartışmalarına çevireceğini öğrendikleri yönünde olmuştur.

Öğretim deneyinin sonucunda literatürde tespit edilen öğretmen adaylarının öğretecekleri matematiği tam olarak bilmemeleri (Ball, 1990; Zembat, 2007), öğrenciler ile benzer kavram yanılgılarına sahip olmaları (Butterfield ve diğerleri, 2013) ve öğrencilerin bu matematiği daha nasıl iyi öğrenirler noktasındaki bilgi eksiklikleri (Phillips, 2008) problemleri eşpaylaşım konusu içerisinde ele alınmıştır. Öğretim deneyinin sonunda, öğretmen adayları hem genel alan bilgilerini hem de özel alan bilgilerini genişleterek ve iyileştirerek öğretecekleri konuyu birden fazla boyutu ile öğrenmişlerdir. Aynı zamanda, çeşitli matematiksel gösterimleri, stratejileri ve açıklamaları öğretim deneyinde kullanmışlardır. Bununla birlikte, bu gösterimlerin ve stratejilerin altındaki matematiksel mesajları irdeleyerek öğrenmişlerdir. Öğretmen adaylarının sahip oldukları bilgi seviyelerinin bu noktalarda ilerlemesi, öğretecekleri matematiğin ileride öğrencileri için daha ulaşılabilir ve anlamlı olmasına katkı sağlamaktadır (Philipp, 2008; Sherin ve diğerleri, 2011).

Bu çalışmanın bir diğer önemli sonucu ise, öğretmen adaylarının öğrenme rotasının ileri sürdüğü sıralamayı, yapılandırmış oldukları matematiksel alan bilgilerini ve öğrenme deneyindeki tecrübelerini kullanarak alternatif fikirler üretmeleridir. Bununla birlikte, öğrenme rotasının içinde belirtilmeyen matematiksel stratejileri de kullanmışlardır. Bu durum, her bir kişi için sabit bir öğrenme rotasının olmadığını göstermektedir (Clements ve Sarama, 2013). Bu durum, öğrenenin farklı hazırbulunuşluk düzeyine, bilgi birikimine sahip olması sonucu ortaya çıkmıştır ve aynı zamanda öğretmen adaylarının izledikleri öğrenme rotasının farklılaşmasını sağlamıştır.

Sınırlılık ve Öneriler

Bu çalışmanın birinci sınırlılığı çalışmaya az sayıda öğretmen adayı katılmasıdır. Bu sınırlılık her ne kadar her bir öğretmen adayının mevcut matematiksel alan bilgilerini ve öğrenci bilgilerini yeniden yapılandırma eylemlerinin ortaya çıkarılmasına ve derinlemesine incelenmesine zemin hazırlasa da çalışmanın genellenebilirliği açısından bir sınır teşkil etmektedir. Bu çalışmada sunulan çerçeve farklı büyüklüklerdeki örneklemler ile test edilmeli ve geliştirilmelidir. Bu çeşit çalışmalar iki farklı tasarımla yapılabilir. Birinci olarak nicel deneysel çalışmalarda, öğrenme rotaları temelli öğretim kullanılarak bu bilgilerin yeniden yapılandırılmasında öğretim ortamındaki başka faktörlerin etkileri ve kuramın etkileri araştırılabilir. İkinci olarak, bu çalışmada sadece eşpaylaşım öğrenme rotası kullanılmıştır, başka öğrenme rotalarının kullanıldığı nitel çalışmalar ile bu çerçevenin işlerliği test edilmeli ve öğretim faaliyetleri açığa çıkarılmalıdır.

Clements ve Sarama'nın (2013) ifadelendirdiği gibi her bir bireyin kendine özgü bir öğrenme rotası vardır ve bu rota bireyin içinde bulunduğu şartlara, eğitim durumuna ve tecrübelerine göre şekillenir. Her ne kadar bu çalışmada farklı akademik yeterliliğe sahip öğretmen adayları seçilmiş olsa da, öğretmen adaylarının hepsi özel bir üniversitede eğitim görmektedirler. Dolayısıyla, devlet üniversitelerinde okuyan öğretmen adayları ile bu çalışmaya benzer çalışmalar yürütülebilir ve bu çalışmaların sonuçları birbirleri ile karşılaştırılarak bu çalışmada ortaya konulan çerçevenin geçerliği sınanabilir. Bu şekildeki çalışmalar aynı zamanda çalışmanın bulgularının genellenebilirliğinin test edilmesi adına önem taşımaktadır.

Daro, Mosher veve Corcoran'a (2011) göre matematik eğitimi alanında mevcut 18 adet öğrenme rotası bulunmaktadır. Bu öğrenme rotalarının öğretmen adaylarının alan bilgilerini ve öğrenci bilgilerini geliştirmesindeki rolleri araştırmalı ve birden fazla öğrenme rotasının birlikte kullanılmasıyla yapılacak çalışmaların da etkileri araştırılmalıdır. Birden fazla öğrenme rotasının bir arada kullanılmasıyla yapılacak çalışmalar, matematik eğitiminin nasıl yapılandırılması, hangi sıra ile yapılması gerektiğine ışık tutabilecek niteliktedir (Clements veve Sarama, 2013). Buna ek olarak, bu öğrenme rotalarının içermiş olduğu bilgilerin ve matematiksel sıralamanın, eğitim müfredatlarının içine entegre edilmesi (Confrey, Maloney, ve Corley, 2014) ve bu entegrasyonun araştırmalar ile test edilmesi mevcut müfredat geliştirme çalışmalarına da ışık tutar nitelikte olacaktır.

Yukarıda belirtilen ve dikkatli bir şekilde tasarlanmış hem nitel hem de nicel çalışmaların sonuçları öğretmen eğitimine dair alanyazın taramasında saptanan problemlerin çözümüne yardımcı olma potansiyeline sahiptir. Aynı zamanda, öğretmen adaylarına öğrencilerin matematiği nasıl öğrendiklerini anlamlandırabilecekleri bir matematiksel alt yapı kazandırılması için, öğretmen eğitimindeki matematik eğitimi derslerinin nasıl yapılandırabileceğine dair bilgi verme açısından bu çalışmaların yapılması önemlidir.

APPENDIX F

CURRICULUM VITAE

ZUHAL YILMAZ
June, 1986

Cell Phone #: +90 553 5923379
E-mail zyilmazncsu@gmail.com

ACADEMIC BACKGROUND

Degree	Major	Institution	Graduation Date
MS	Mathematics Education	North Carolina State University	December 2011
BS	Mathematics Education	Bogazici University	June 2009

Foreign Language(s): English

WORK EXPERIENCE

Year	Institution	Position
January 2014 -Current	Yeditepe University	Graduate Research Asistant
January2012 – November 2013	Zirve University	Research Asistant
August 2009 – December 2011	North Carolina State University	Graduate Research Asistant

PUBLICATIONS

Books

International Books

Yilmaz, Z. (2012). *Students' Strategies on Reallocation and Covariation Items: In Relation to an Equipartitioning Learning Trajectory*. Saarbrücken, Germany: Lambert Academic Publishing.

Edited Translation Books

Derek, H. & Cockburn, A. (2014, April). Küçük çocuklar için matematiği anlama (Understanding Mathematics for Young Children). **Yilmaz, Z.** (Ed.). Nobel Akademic Publishing.

Journal Papers

Social Science Indexed Journal Papers

Yilmaz, Z. (2013). Usage of Tinker Plots to Address and Remediate 6th Grade Students' Misconceptions about Mean and Median. *Anthropologist*, *16*(1-2), 21-29.

Ulker, R., **Yilmaz, Z**., Solak, A. & Erguder, L. (2013). Classroom environment: What does students' drawings tell? *Anthropologist*, *16*(1-2), 209-215.

Yilmaz, Z., Kubiatko, M., Topal, H. (2012). Czech Children's Drawing of Nature, *Educational Sciences: Theory & Practice*, *12* (4), p. 3111-3119.

Other Journal Papers

Yilmaz, Z. (2012). Çocukluğumuzun matematik kutusunu beraber açalım (Lets open our childhood mathematics box). *Çoluk Çocuk Anne Baba Eğitim Dergisi (Children, Mother and Father* Education Journal) 94, p. 8-10.

Conferences

International Conference Proceedings Papers

Yilmaz, Z & Topal, O. Z. (2014). Connecting mathematical reasoning and language art skills: The case of common core state standards. *Procedia - Social and Behavioral Sciences*, 116, 3716 – 3721.

Presentations in International Conferences

Yilmaz, Z. & Haser, Ç. (2015, June). Restructuring pre-service teachers' mathematical content knowledge and knowledge of students in learning trajectories based instruction. Presented at *ISER 2015 World Conference on Education*, *Yeditepe University*, Istanbul, Turkey.

Yilmaz, Z. (2011). Usage of Tinkerplots to address and remediate 6th grade students' misconceptions about measures of central tendency: The case of mean and median. Paper presented at the *11th International Educational Technology Conference*, Istanbul, Turkey.

Yilmaz, Z. (2011). Developing Number Sense of Pre-Kindergarten and Kindergarten Students: Presenting Three Instructional Tasks**.** Presented at *North Carolina Council of Teachers of Mathematics Conference*, Greensborough, North Carolina.

Workshops/Working Sessions in International Conferences

Yilmaz. Z. & Olgun, B. (2015, June). Donüsüm Geometrisinin Sanal Dinamik Geometri Ortaminda Kavramsal Olarak Irdelenmesi (Conceptually Studying Transformational Geometry in a Virtual Dynamic Geometry Environment). Workshop presented at *ISER 2015 World Conference on Education*, Yeditepe University, Istanbul, Turkey.

Confrey, J., Maloney A., Nyguyen K., Monrose N., & **Yilmaz, Z.** (2012). Interactive Diagnostic Assessments for Rational Number Reasoning: LPPSync, A working session *presented at National Council of Teachers of Mathematics Research Pre-session*, Philadelphia, US.

Presentations in National Conferences

Yilmaz, Z.; Ader. E., & Olgun, B. (2014, Eylül). *Öğretmen adaylarının istatistiksel ve matematiksel düşünme ile ilgili görüş ve becerilerinin karşılaştırmalı incelenmesi*. Paper presented at 11. National Science and Mathematics Education Congress, Adana, Turkey.

Karatoprak, R.; **Yilmaz, Z.** & Ubuz, B. (2014, Eylül). *Türkiye'deki matematik eğitimi araştırmaları: 2007-2013*. (Mathematics education researches in Turkey: 2007-2013) Paper presented at 11. National Science and Mathematics Education Congress, Adana, Turkey.

International Poster Presentations

Franklin, A., **Yilmaz, Z.**, & Confrey, J. (2010). *Reconciling Student Thinking and Theory: The Delta Learning Trajectory and the Case of Transitivity.* Poster presented at the thirty-second Annual Conference of the North American Chapter of the International Group for the Psychology of Mathematics Education, Columbus, OH.

PROJECT WORKS AND SCIENTIFIC EVENTS

Year	**Project**	**Institution**	**Role**
2014- Current	Matematik Atölyesi	Celal Yardımcı İlköğretim Okulu Yeditepe University	Researcher
2009-2012	Diagnostic E-Learning Trajectories Approach (DELTA)	The Friday Institute & *National Science, Foundation*. North Carolina State University	Researcher
2014 (June)	Proje Yazma Eğitimi	*2237-TUBITAK* Ted University.	Participant
2014 (May)	Matematik Eğitimi Araştırmaları Çalıştayı	*2237-TUBITAK* Karadeniz Technical University	Presenter & Participant
2010 (November)	A National Conference on Diagnostic Assessments	William & Ida Friday Institute for Educational Innovation at North Carolina State University	Reporter Documenting Member of Planning Commitee

GRANTS AND AWARDS

2013 Middle East Technical University, Academic Performance Award
2009- 2012 Graduate Research Asistantship Award– 24.000 $ / Year
2009 Opportunity Grant, Education USA, 3000 $

APPENDIX G

TEZ FOTOKOPİSİ İZİN FORMU

ENSTİTÜ

Fen Bilimleri Enstitüsü	
Sosyal Bilimler Enstitüsü	x
Uygulamalı Matematik Enstitüsü	
Enformatik Enstitüsü	
Deniz Bilimleri Enstitüsü	

YAZARIN

Soyadı : YILMAZ
Adı : ZUHAL
Bölümü : İLKÖĞRETİM

TEZİN ADI (İngilizce): USE OF LEARNING TRAJECTORIES BASED INSTRUCTION TO RESTRUCTURE MATHEMATICAL CONTENT AND STUDENT KNOWLEDGE OF PRE-SERVICE ELEMENTARY TEACHERS

TEZİN TÜRÜ : Yüksek Lisans [ ] Doktora [x]

1.	Tezimin tamamından kaynak gösterilmek şartıyla fotokopi alınabilir.	
2.	Tezimin içindekiler sayfası, özet, indeks sayfalarından ve/veya bir bölümünden kaynak gösterilmek şartıyla fotokopi alınabilir.	
3.	Tezimden bir bir (1) yıl süreyle fotokopi alınamaz.	x

TEZİN KÜTÜPHANEYE TESLİM TARİHİ: