BASICS OF MASSIVE SPIN-2 THEORIES

A THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES
OF
MIDDLE EAST TECHNICAL UNIVERSITY

BY

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IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR
THE DEGREE OF MASTER OF SCIENCE
IN
PHYSICS

AUGUST 2015

Approval of the thesis:

BASICS OF MASSIVE SPIN-2 THEORIES

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ABSTRACT

BASICS OF MASSIVE SPIN-2 THEORIES

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August 2015, 78 pages

In this thesis, basics of massive spin-2 theories are studied. The theory of general relativity cannot explain some problems in very small and in very large scales and it needs a modification. The way of modifying general relativity by giving mass to the propagating particle graviton is called massive gravity. The first correct massive gravity theory is the linear theory written by Fierz and Pauli. However, later on it has been found that this theory does not match up with the physical predictions of general relativity in the zero mass limit. This discontinuity in the mass parameter of the theory is called the van Dam-Veltman-Zakharov (vDVZ) discontinuity. When the method of Stueckelberg providing to preserve the gauge symmetry of the massless theory to take a smooth limit is applied to the theory of massive gravity it was seen that the reason of this discontinuity was a scalar field which does not decouple from the source even in the massless limit. Later, it was proposed by Vainshtein that the vDVZ discontinuity may disappear by studying the non-linear theory. However, when the Hamiltonian analysis of the non-linear massive gravity theory was done it was understood that theory consists of an extra unphysical degree of freedom which is called the Boulware-Deser ghost.

Keywords: Massive Gravity, Fierz-Pauli Action, vDVZ Discontinuity, Stueckelberg Formalism, Vainshtein Radius, Boulware-Deser Ghost

ÖZ

KÜTLELİ SPİN-2 TEORİLERİN ESASLARI

Kürekci, Şahin
Yüksek Lisans, Fizik Bölümü
Tez Yöneticisi : Prof. Dr. Bayram Tekin

Ağustos 2015 , 78 sayfa

Bu tezde, kütleli spin-2 teorilerinin esasları çalışıldı. Genel görelilik kuramı çok küçük ve çok büyük ölçeklerde bazı sorunlara çözüm getirememektedir ve değişikliğe ihtiyacı vardır. Yayılan parçacık olan gravitona kütle vererek genel görelilik kuramını değiştirme yöntemine kütleli kütleçekim denir. İlk doğru kütleli kütleçekim teorisi Fierz ve Pauli tarafından yazılan doğrusal teoridir. Ancak daha sonra bu teorinin kütle sıfır limitinde genel göreliliğin fiziksel tahminleriyle örtüşmediği ortaya çıkmıştır. Teorinin kütle terimindeki bu süreksizliğe van Dam-Veltman-Zakharov (vDVZ) süreksizliği denir. Düzgün bir limitin alınması için kütlesiz teorinin ayar simetrisinin sürdürülmesi gerektiğini öngören Stueckelberg yöntemi kütleli kütleçekim teorisine uygulandığında bu süreksizliğin sebebinin kütlesiz limitte dahi kaynaktan ayrılmayan bir skalar parçacık olduğu görülmüştür. Daha sonra Vainshtein tarafından vDVZ süreksizliğinin doğrusal olmayan teori çalışıldığında ortadan kalkabileceği ileri sürülmüştür. Fakat doğrusal olmayan kütleli kütleçekim teorisinin Hamilton analizi yapıldığında teorinin içerisinde Boulware-Deser hayaleti ismi verilen ve fiziksel olmayan fazladan bir serbestlik derecesi barındırdığı anlaşılmıştır.

Anahtar Kelimeler: Kütleli Kütleçekimi, Fierz-Pauli Aksiyonu, vDVZ Süreksizliği, Stueckelberg Yöntemi, Vainshtein Yarıçapı, Boulware-Deser Hayaleti

To the people who have walked alongside with me
✠ *on the paving stones of the life* ✠

ACKNOWLEDGMENTS

One half of me wrote this thesis, the other half always looked for the new stories. I am seeing what I do as telling stories, so I designed this page as a story. A story about passion, wisdom, love and life. A story that consists of no names... A story that includes everyone...

The writer of this thesis (which "surprisingly" will be called "the Writer" from now on) was a young, average height boy who often thought that life was not something worth to live but it was something that has to be lived. One day he realized that the best way to escape from this obligation was writing and so he started to write. He wrote for days, for months and for years. At that time he didn't know the very first thing to publish in his life would be a thesis. "Whatever it is, the first thing should be special," he thought and he decided to write a story about the journey that brought him to the stage. Ladies and gentlemen, we will not go that fast but caution is important: please fasten your seatbelts, the journey starts... right now!

The first part of the story is about passion. In that station, a passionate person, the Supervisor, opened the doors to the Writer's life. The Supervisor was a kind of man that knows everything (not the president of Turkey) and led the Writer on his journey. The Writer learned from him: passion is the fire of the love. The Supervisor had found out that the geodesic for a happy life is the one that occurs when we do what we do with passion. The Writer always watched the Supervisor with envy and appreciation. The Supervisor told him: "Raise your head up, look at the stars. Right there, a hidden treasure lies. Stars tell the story of the past. They play the music of the universe." The Writer learned from him: The passion lies somewhere inside every person and only that person can take it out.

The Writer met the second person on his journey when he was traveling along a sea. So, in this story that second person deserves to be called "the Sea". The Sea was an old aged man who was hosting a baby boy inside. He was watching everything around him with the curiosity of a child and was deciding with the experience of a hundred old man. He was the wisdom. One day, when the Writer was just about to give up, the Sea awakened him. He pointed his finger down and said: "Lower your head down, look at the water. The life there... It is the same as the life here, the one you live. Follow other people like the fishes follow each other. Go after them, turn them around, look at their faces and catch the meaning they carry on. Try to live their lives." The Writer learned from him: The wisdom lies somewhere inside every person, but only other people can take it out.

And the best part of the story comes now: The Love. The only thing that feeds up every other feeling of human being. The theory of everything... The Love was laughing with an unforgettable smile on her face and was looking with an unending shine on her eyes. One day, she said to the Writer: “The only correct path you should follow is the path your heart points." Later that day the Writer was reading a book on his comfortable coach and suddenly he saw the Love inside the book. He closed the book and stepped outside. The Love was winking on the pavements, she was smiling on the surface of the Sun, she was waving her hands behind the showcase glasses of the stores. That day the Love taught him: If you begin to see one thing in everything then you also begin to see everything in that thing. The Writer learned from her: The love lies somewhere inside every person and no one can take it out but only one.

And the Life… Life is all the others. The Writer’s family, his uncle, his friend he walked with, his friend he cried with, his friend he eaten and got drunk with, his friend he laughed with, his friend he lived with, his friend he loved with… The Life has no other name. It is the Life. It is everything behind and everything in front.

As Einstein said once: "You can’t blame gravity for falling in love." This thesis contains Einstein, gravity, falling and love. I hope you enjoy it as much as I did.

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LIST OF ABBREVIATIONS

GR	General Relativity
EH	Einstein-Hilbert
FP	Fierz-Pauli
vDVZ	van Dam-Veltman-Zakharov
ADM	Arnowitt-Deser-Misner
$\eta_{\mu\nu}$	Mostly plus Minkowski metric $(-+++\ldots)$
$g_{\mu\nu}$	Generic metric
$\bar{g}_{\mu\nu}$	Background (or absolute) metric
$h_{\mu\nu}$	Perturbed metric
${}^{(3)}g_{ij}$	Purely spatial three-metric
g	Metric determinant $(g \equiv \det(g_{\mu\nu}))$
∂_μ	Partial derivative
∇_μ	Metric compatible covariant derivative
$F_{\mu\nu}$	Electromagnetic field tensor $(F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu)$
$\partial\partial T$	Double divergence of the source $T^{\mu\nu}$ $(\partial\partial T \equiv \partial_\mu \partial_\nu T^{\mu\nu})$

NOTATION: Latin indices ijk run from 1 to 3 and Greek indices $\lambda\mu\nu$ run from 1 to 4. Unless indicated otherwise we assume all the scalars, vectors, tensors etc. are spacetime dependent, e.g., $\phi \equiv \phi(x)$, $A_\mu \equiv A_\mu(x)$, $h_{\mu\nu} \equiv h_{\mu\nu}(x)$. We use Einstein's summation convention, i.e. $u_\mu v^\mu = \sum_{\mu=0}^{D-1} u_\mu v^\mu$ and Gaussian units in which $\hbar = c = 1$ throughout the text.

CHAPTER 1

INTRODUCTION

1.1 History and Problems of Gravity

In 1687 Isaac Newton wrote the first known theory of gravity in the history of the world [1]. He considered gravity as an attractive force between objects and formulated that idea with the force law

$$F = -G\frac{m_1 m_2}{r^2}, \tag{1.1}$$

where the minus sign indicates the attraction between spherical objects with masses m_1 and m_2 whose centers are separated by a distance r and G is a constant number known as Newton's constant (or universal gravitational constant) first measured a lot later by Henry Cavendish in his famous experiment in 1798 [2]. One of the biggest successes of Newton's theory was the discovery of a new planet, Neptune, which was predicted to be at its location to explain the perturbations in the orbit of the Uranus by using the force law (1.1) [3, 4]. After that discovery all the orbits of the planets in the solar system were tested with Newton's theory and there arose another problem in the Mercury's orbit. The perihelion precession of the planet was slower than the predictions of the theory. It is first thought that there was a new planet, named Vulcan, between the Sun and the Mercury which causes this discrepancy in the orbit. However, that new planet has never been observed. Luckily, in late 1915, Albert Einstein came up with the theory of General Relativity (GR) [5] and in his famous paper he calculated the orbit of the Mercury and explained the perihelion motion without needing a new planet. The field equations proposed by Einstein were

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \kappa T_{\mu\nu}, \tag{1.2}$$

where the left hand side gives the geometry of the space-time and the right hand side covers all the matter and source of energy of the universe. With this equation Einstein simply changed the perception of gravity by redefining it as a manifestation of the curvature of the space-time instead of being a force. Although it is hard to explain Einstein's field equations and his model of gravity in words, John Wheeler does it really good in his famous statement [6]:

"Space-time tells matter how to move; matter tells space-time how to curve."

In the paper which he introduced the theory of general relativity, Einstein also proposed two more tests for scientists to test the theory and later GR also passed these tests. So, people were happy. There was a theory that explains all the gravitational phenomena in the solar system with high precision. That happiness did not last long. Soon it is seen that GR was not giving the correct results out of the solar system. The biggest problem that still remains unsolved in this context is the unexplainable small but non-zero value of the observational cosmological constant (for some nice reviews of the problem see [7, 8, 9]). Existence of a cosmological constant in the universe was also first introduced by Einstein in 1917 [10]. To achieve a static universe as the only solution to GR, Einstein modified his original field equations (1.2) by adding a cosmological constant term Λ,

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}. \tag{1.3}$$

But in 1922, when he was studying Einstein's field equations for a homogeneous and isotropic universe, Friedmann found the interesting fact that such a universe should be either expanding or contracting [11] regardless of all the previous effort put by Einstein to make the universe static. Later, in 1929, Hubble proved the expansion of the universe by observing the relative motion between the galaxies and showing that they are getting away from each other [12]. So, most scientists abandoned the idea of a cosmological constant at that time until up to 1998. At that year, the observed supernova data revealed that the universe is not only expanding but it is also accelerating in its expansion. According to the fundamental physical laws if something is accelerating there should be some mass/energy around it to cause that acceleration and this gave rise to the idea of the dark energy. As of today, around 70% percent of the mass-energy density of the universe is attributed to the dark energy and the

cosmological constant is accepted as the simplest possible form of the dark energy in the current standard model of cosmology.

On the other hand, we can approach the cosmological constant problem theoretically by using an effective local quantum field theory description for the universe. In quantum field theory, because of the pair annihilation and creation processes that holds everytime, the expected energy density of the vacuum is not zero. That means, even if there was nothing in the universe, there would still be an energy density. When we handle the same situation from the point of view of Einstein's equations, we see that even if there was nothing in the universe yet there would be a curvature because of the cosmological constant. Thus, in the most simple and logical way, we can combine these two perspectives and deduce the statement: "Vacuum expectation value of the energy density of the universe is related to the cosmological constant of the universe." But the problem starts right here. Although the above statement looks like very well-founded, at the end of the day it is found to be wrong. The value of the observed cosmological constant from the cosmological principles is about 10^{-122} (in Planck units) and it is much more smaller than the value $(\sim 10^{-2})$ obtained by using the principles of quantum field theories. That inconsistency between the experimental data and the theoretical predictions in the value of the cosmological constant is known as the cosmological constant problem in the literature and today it is accepted as one of the most high-ranked unsolved mysteries of the universe.

The cosmological constant problem is a large scale phenomenon. One can think of it as a low energy problem as well. Such type of problems occurring in large scales show the necessity to modify general relativity in infrared (IR) regimes. But IR modification alone is not a savior of general relativity. When we go to small scales (or high energies) there appear other problems which make it clear that GR also needs an ultraviolet (UV) modification. To understand the events near the center of a black hole or at the early stages of the Big Bang we need to go to high energies at which quantum effects inevitably step into the game and one needs to combine quantum mechanics with general relativity. However, as of today, no consistent quantum gravity theory is found and the type of events mentioned above still remain mysterious.

1.2 Modification of General Relativity

It is now straightforward that general relativity has some problems and it needs to be modified at least in small and in large scales. However, when doing any modification the very first thing to secure is to obtain GR back in the solar system scale. Since GR is such a beautiful theory in the solar system and matching with physical observations in high accuracy, in almost any modification there occur some problems when one tries to re-obtain it in the limits of the new theory. This is both a curse and a beauty.

Before proceeding further let us first introduce some concepts used in quantum field theory to understand what we are trying to do clearly. In quantum field theories we label particles with their masses and related to this with their helicities/spins. If a particle is massive, then we use the particle's *spin* to label it. All the elementary particles should has a spin value s which implies $2s + 1$ degrees of freedom for the particle in question. Basically, spin can be thought of as some kind of an intrinsic angular momentum. On the other hand, for massless particles we use *helicity* instead of spin since it is hard to talk about the angular momentum of a particle when it has zero mass. In this case searching for a Lorentz invariant quantity brings us to introduce the concept of helicity which is defined to be the projection of the particle's spin onto the direction of its momentum. A helicity-h particle has $\pm h$ degrees of freedom.

Using these new tools and by asking questions we can construct new physical theories. For example, asking the question "What is the consistent interacting theory of a helicity-1 particle?" opens the door of massless photon and Maxwell's electromagnetic theory. Similarly, the theory of general relativity is described as the theory of a helicity-2 particle which is called massless graviton (or graviton only). But what about the spin-2? Do we have a theory for it? Or if not, can we find one? When we consider the concepts introduced above, a spin-2 theory corresponds to a massive graviton and that means we need to modify general relativity by finding a way to give mass to the massless graviton. Thus, the simple theoretical field question "What is the theory of a spin-2 particle?" leads us to study massive gravity. So when studying massive gravity what we actually do is to change a helicity-2 particle (massless graviton) to a spin-2 particle (massive graviton). That is, we are modifying gravity by

changing the degrees of freedom of the theory (in four dimensions, from two to five).

So, general relativity is the theory of a propagating massless helicity-2 particle and when modifying the theory we should keep in mind that we do not want to deviate from the beautiful structure of GR much. If we do not respect the predictions of GR it will be like "loosing the bulgur at home when going to Damietta for rice" as in a Turkish idiom. Besides, we expect that a good modified gravity theory could possibly explain all or some of the problems of general relativity which are mentioned in the previous section. The cosmological constant problem, and depending on that the accelerated expansion of the universe, is one of the hardest problems massive gravity challenges to explain. From the point of view of quantum field theory giving mass to graviton means changing the massless wave equation $\Box h_{\mu\nu} = 0$ (where $\Box \equiv \partial^\lambda \partial_\lambda$) to the massive wave equation $(\Box - m^2)h_{\mu\nu} = 0$ [1]. One can immediately note that the term e^{-mr}, called Yukawa suppression factor, is a characteristic of the differential equations of second type. Indeed, when we look at the potentials we see that the massless field conforms a potential proportional to $1/r$ (what we know as the Coulomb potential) while in the massive equation potential comes with a Yukawa term and written proportional to e^{-mr}/r. When $r > \frac{1}{m}$ the Yukawa factor e^{-mr} starts to suppress the potential and it makes a negative effect on the gravitational attraction. That means there is a chance that for a massive interaction particle the gravitational force falls-off in large scales and the universe starts to expand! So, certainly, there is an exciting possibility that massive gravity may solve the cosmological constant problem.

1.3 Outline

After this introduction part, in the following chapter we will start to study massive gravity by presenting the first consistent theory belonging to a free massive graviton written by Fierz and Pauli in 1939 [13]. We will look at the equations of motion and see how Fierz and Pauli became successful in obtaining the five propagating degrees of freedom of a massive spin-2 particle by adding the correct mass term to the

[1] This is a Klein-Gordon equation describing the evolution of a massive integer-spin particle and will be encountered once more in Section 2.1 when we find the equations of motion of the linear massive gravity theory.

linearized Einstein-Hilbert action. Today, their theory is known as *linear massive gravity theory* since they did not pay attention to the non-linearities at that time. Before finishing that section we will also study the Hamiltonian analysis of the linear theory and we will derive the propagators for massive and massless gravitons. In the third chapter, we will start to examine the source added action of linear massive gravity. By choosing a point source we will be able to find the field components belonging to massive and massless gravitons and from there some physical observables like Newtonian potentials and light bending angles will be derived. What we see at the end will be an inevitable mismatch between the observables of massive and massless cases. Namely, the theory obtained after taking the zero mass limit of the massive theory will give different predictions than the theory obtained in the zero mass case. This is first realized in 1970 by van Dam and Veltman and independently by Zakharov and today it is known as the vDVZ discontinuity after their names [14, 15]. In the fourth chapter, we will develop the Stueckelberg formalism [16] to search for the origin of the vDVZ discontinuity. What Stueckelberg does is to introduce new fields to maintain the gauge symmetry of the massless theory in the massive case. We will first start with a toy example by applying the Stueckelberg formalism to the massive photon case and it will be a nice illustration to see how good it works there. Then, after applying the Stueckelberg formalism to massive gravity action it will be seen that, being different from the massive photon case, a scalar particle, which was introduced by Stueckelberg to maintain the gauge symmetry, will be coupled to the source. This scalar field behaves as an attractive force in the theory and affects the Newtonian potential. So that coupling between the source and the Stueckelberg field will be identified as the origin of the vDVZ discontinuity. In the fifth chapter, the Fierz-Pauli action will be examined on curved spaces. Applying Stueckelberg trick will reveal two important and interesting results. The first one will be the existence of partially massless theories which possess one less degree of freedom than the original massive theory for a special choice of the curvature term and the second remarkable result we obtain on curved spaces will be the absence of the vDVZ discontinuity. In the sixth chapter we will start to investigate the non-linearities in gravity theories. When the non-linearities in general relativity are studied we will find a scale called the Scwarzchild radius at which non-linearities starts to become important. After that, non-linearities of massive gravity will be explored and we will find another scale at

which non-linearities enter the game. This new radius is called the Vainshtein radius following the name of Arkady Vainshtein who first introduced it in 1972 [17]. Vainshtein radius will clarify the need of a full non-linear massive gravity theory to solve the problem of the vDVZ discontinuity. Namely, it will be shown that taking the zero mass limit forces us to use the non-linear theory since in this limit the Vainshtein radius goes to infinity and predictions of the linear theory are not reliable. However, when we look at the work of Boulware and Deser presented at the same year [18] we will see the problematic fact that the non-linear massive gravity action studied at that time is unphysical by having an extra degree of freedom in it. This extra ghostlike degree of freedom is called the Boulware-Deser ghost which has not been able to explained for the following forty years after its discovery. Lastly, in the Appendix parts some helpful calculations and tricks will be given to make the reader easily follow the text. In the first Appendix part the terms in the free Fierz-Pauli Lagrangian will be explicitly summed over their indices and from there spatial canonical momenta will be found. Later, in the second Appendix part, we will invert that canonical momenta for the velocities and they together will be used to derive the Hamiltonian density. In the third Appendix part, we will show the derivation of Einstein's field equations and finally in the fourth Appendix part we will make a metric perturbation and derive the curvature terms up to second order in that perturbation.

** The author claims no originality in this work. Most of the material follows the excellent review of Hinterbichler [19] and the references therein. One can also see [20] for another nice review of massive gravity theories.*

CHAPTER 2

THE FREE FIERZ-PAULI ACTION

We begin by writing the unique action describing a massive spin-2 particle (massive graviton) in flat space in which particle is described by a symmetric tensor field $h_{\mu\nu}$,

$$S = \int d^D x \Big\{ -\frac{1}{2}\partial_\lambda h_{\mu\nu}\partial^\lambda h^{\mu\nu} + \partial_\mu h_{\nu\lambda}\partial^\nu h^{\mu\lambda} - \partial_\mu h^{\mu\nu}\partial_\nu h + \frac{1}{2}\partial_\lambda h \partial^\lambda h \\ - \frac{1}{2}m^2\left(h_{\mu\nu}h^{\mu\nu} - h^2\right)\Big\}. \tag{2.1}$$

This is known as the Fierz-Pauli action [13] and will be widely used throughout the thesis. Note that, when $m = 0$ we obtain the linearized Einstein-Hilbert action which describes a massless helicity-2 particle (massless graviton) with the gauge symmetry

$$\delta h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu, \tag{2.2}$$

where ξ_μ is a space-time dependent gauge parameter. This gauge symmetry is a consequence of the general coordinate invariance principle of GR and will be derived in the beginning of Chapter 6. The coefficients of the terms with $m = 0$ are fixed by forcing the action to obey this gauge symmetry. Another important thing to note is the breaking of the gauge symmetry when the mass term is introduced in the theory. Thus, as it is, the action (2.1) is not gauge invariant. In Chapter 4 we will use Stueckelberg's trick to find a way of maintaining gauge symmetry (2.2) even in the massive case. We will also talk about the minus sign between $h_{\mu\nu}h^{\mu\nu}$ and h^2 terms which is known as the Fierz-Pauli tuning and has a crucial role in the theory. We will see how any deviation from this initial form will cause extra unphysical degrees of freedom in the theory.

2.1 Equations of Motion and Degrees of Freedom

One thing we always do when we have an action at hand is to derive equations of motion from it. For the action (2.1), by setting $\frac{\delta S}{\delta h^{\mu\nu}} = 0$, we obtain

$$\Box h_{\mu\nu} - \partial_\lambda \partial_\mu h^\lambda{}_\nu - \partial_\lambda \partial_\nu h^\lambda{}_\mu + \eta_{\mu\nu} \partial_\lambda \partial_\sigma h^{\lambda\sigma} + \partial_\mu \partial_\nu h - \eta_{\mu\nu} \Box h = m^2 (h_{\mu\nu} - \eta_{\mu\nu} h), \tag{2.3}$$

where $\Box \equiv \partial_\mu \partial^\mu$. The left hand side of this equation consists of the linearized form of the Einstein tensor and has zero divergence. Hence, by acting on (2.3) with ∂^μ, we get

$$m^2 \left(\partial^\mu h_{\mu\nu} - \partial_\nu h \right) = 0. \tag{2.4}$$

Assuming $m^2 \neq 0$ and putting this result back into the equations of motion gives $\Box h_{\mu\nu} - \partial_\mu \partial_\nu h = m^2 (h_{\mu\nu} - \eta_{\mu\nu} h)$. By taking the trace of this expression we obtain $h = 0$ which means the field $h_{\mu\nu}$ is traceless. From (2.4) this result implies that $h_{\mu\nu}$ is also a transverse tensor, namely $\partial^\mu h_{\mu\nu} = 0$. Lastly, using transverse and traceless relations of $h_{\mu\nu}$ in the equations of motion, we get the wave evolution equation $(\Box - m^2) h_{\mu\nu} = 0$.

Therefore, we conclude that the equations of motion (2.3) give three equations altogether: a massive wave equation and constraint equations on the field,

$$(\Box - m^2) h_{\mu\nu} = 0, \qquad \partial^\mu h_{\mu\nu} = 0, \qquad h = 0. \tag{2.5}$$

Although these three equations are exactly equivalent to (2.3), it is easier to count the degrees of freedom by using these second set of equations. The first equation is the evolution of a symmetric tensor field $h_{\mu\nu}$ and implies 10 degrees of freedom in $D = 4$. The last equation kills one of these degrees of freedom by forming a single constraint on the system. Moreover, the second equation implies 4 equations which means 4 more constraints. Hence, at the end we are left with $10 - 4 - 1 = 5$ degrees of freedom. As we will verify when doing the Hamiltonian analysis of the theory in the next section, this is the correct number of degrees of freedom for a massive spin-2 particle in four dimensions. So in $D = 4$, massless graviton has 2 degrees of freedom while massive graviton has 5.

Before proceeding to the next section, we lastly discuss the importance of the Fierz-Pauli tuning as we promised. To see how theory deviates from its physical form and

gives unphysical degrees of freedom we modify the Fierz-Pauli coefficient

$$m^2(h_{\mu\nu}h^{\mu\nu} - h^2) \to m^2(h_{\mu\nu}h^{\mu\nu} - \alpha h^2). \tag{2.6}$$

With that replacement the equations of motion become

$$\Box h_{\mu\nu} - \partial_\lambda\partial_\mu h^\lambda{}_\nu - \partial_\lambda\partial_\nu h^\lambda{}_\mu + \eta_{\mu\nu}\partial_\lambda\partial_\sigma h^{\lambda\sigma} + \partial_\mu\partial_\nu h - \eta_{\mu\nu}\Box h = m^2(h_{\mu\nu} - \alpha\eta_{\mu\nu}h), \tag{2.7}$$

and acting on that with ∂^μ gives

$$m^2(\partial^\mu h_{\mu\nu} - \alpha\partial_\nu h) = 0. \tag{2.8}$$

By assuming $m^2 \neq 0$ and plugging this result back into the equations of motions, we get

$$\Box h_{\mu\nu} + (1-2\alpha)\partial_\mu\partial_\nu h + (\alpha-1)\eta_{\mu\nu}\Box h = m^2(h_{\mu\nu} - \alpha\eta_{\mu\nu}h), \tag{2.9}$$

and from the trace of that relation, we obtain a wave equation for the trace of $h_{\mu\nu}$ as

$$\left(\Box - \frac{(1-\alpha D)m^2}{(\alpha-1)(D-2)}\right)h = 0. \tag{2.10}$$

So for this generic case, h becomes a dynamical field. In the previous choice of the Fierz-Pauli tuning what we have found at that point was a traceless field. However, in the present case, unless we work in $D = 2$ or we do not choose $\alpha = 1$, the trace is not zero and hence the theory has an extra degree of freedom which seems to be ghostlike, i.e. unphysical, when the propagator analysis is made. That shows the importance of the Fierz-Pauli tuning for a physically consistent massive gravity theory. For a more throughout way to see the alterations in the theory in the case of a general mass term see [21] in which the significance of the Fierz-Pauli tuning is shown by deriving the propagator of the theory.

2.2 Hamiltonian Analysis

In this section we will use the Hamiltonian analysis to count the degrees of freedom possessed by the action (2.1). At the end, we expect to find 5 degrees of freedom to verify the result of the previous section.

We start by finding the spatial canonical momenta corresponding to the given action[1];

$$\pi_{ij} = \frac{\partial \mathscr{L}}{\partial \dot{h}_{ij}} = \dot{h}_{ij} - \dot{h}_{kk}\delta_{ij} - 2\partial_{(i}h_{j)0} + 2\partial_k h_{0k}\delta_{ij}, \tag{2.11}$$

where $\partial_{(i}h_{j)0} \equiv \frac{1}{2}\left(\partial_i h_{j0} + \partial_j h_{i0}\right)$. Inverting this expression for the velocities $\dot{h}_{ij}$ (see Appendix B), we obtain

$$\dot{h}_{ij} = \pi_{ij} - \frac{1}{D-2}\pi_{kk}\delta_{ij} + 2\partial_{(i}h_{j)0}. \tag{2.12}$$

With some effort (Appendix A might be useful) Hamiltonian density can be derived from these variables,

$$\begin{aligned}\mathcal{H} \;=\; & \frac{1}{2}\pi_{ij}^2 - \frac{1}{2}\frac{1}{D-2}\pi_{ii}^2 + \frac{1}{2}\partial_k h_{ij}\partial_k h_{ij} - \partial_i h_{jk}\partial_j h_{ik} + \partial_i h_{ij}\partial_j h_{kk} \\ & - \frac{1}{2}\partial_i h_{jj}\partial_i h_{kk} + \frac{1}{2}m^2\left(h_{ij}h_{ij} - h_{ii}^2\right),\end{aligned} \tag{2.13}$$

and the Fierz-Pauli action (2.1) can be rewritten in terms of the Hamiltonian variables as

$$S = \int d^D x \Big\{ \pi_{ij}\dot{h}_{ij} - \mathcal{H} + 2h_{0i}(\partial_j \pi_{ij}) + m^2 h_{0i}^2 + h_{00}\left(\vec{\nabla}^2 h_{ii} - \partial_i\partial_j h_{ij} - m^2 h_{ii}\right)\Big\}. \tag{2.14}$$

Note that, for the case $m = 0$, the time-like components of the field, h_{00} and h_{0i}, are interpreted as Lagrange multipliers. They enforce the constraints $\mathcal{C}_1 = \vec{\nabla}^2 h_{ii} - \partial_i\partial_j h_{ij} = 0$ and $\mathcal{C}_2 = \partial_j \pi_{ij} = 0$ respectively and the Poisson bracket of these two constraints gives zero, namely $\{\mathcal{C}_1, \mathcal{C}_2\} = 0$. These type of constraints having zero Poisson bracket with all the other constraints in the theory are known as first class constraints [22, 23]. That means Hamiltonian (2.13) is first class and (2.14) is a first class gauge system. Because of their symmetric nature, both h_{ij} and π_{ij} have 6 independent components in $D = 4$. Thus, together they span a 12 dimensional phase space. Eliminating 4 constraints implies an 8 dimensional constraint surface. Finally by subtracting the dimension of gauge orbits (which is again 4, since four constraints generate four gauge invariances) we are left with a total 4 degrees of freedom. As expected, two of them are the polarization states of the massless graviton and other two belong to their conjugate momenta.

[1] Note that the third term in the canonical momenta cannot be derived straightforwardly. The trick is to apply an integration by parts to one of the h_{0k} terms so that time derivative is removed from that term. The detailed derivations of all the canonical momenta are done in Appendix A.

Now, in the case $m \neq 0$, one of the h_{0i} terms appears quadratically in (2.14). Hence, h_{0i}'s are not Lagrange multipliers any more and are auxiliary variables. Their equations of motion yield

$$h_{0i} = -\frac{1}{m^2}\partial_j \pi_{ij}. \tag{2.15}$$

By putting that result back into the action (2.14), we obtain

$$S = \int d^D x \Big\{ \pi_{ij}\dot{h}_{ij} - \mathcal{H} + h_{00}\left(\vec{\nabla}^2 h_{ii} - \partial_i\partial_j h_{ij} - m^2 h_{ii}\right)\Big\}, \tag{2.16}$$

where Hamiltonian density, in this case, is

$$\begin{aligned}\mathcal{H} \;=\; & \frac{1}{2}\pi_{ij}^2 - \frac{1}{2}\frac{1}{D-2}\pi_{ii}^2 + \frac{1}{2}\partial_k h_{ij}\partial_k h_{ij} - \partial_i h_{jk}\partial_j h_{ik} + \partial_i h_{ij}\partial_j h_{kk} \\ & - \frac{1}{2}\partial_i h_{jj}\partial_i h_{kk} + \frac{1}{2}m^2\left(h_{ij}h_{ij} - h_{ii}^2\right) + \frac{1}{m^2}(\partial_j \pi_{ij})^2. \end{aligned} \tag{2.17}$$

Note that, the term h_{00} is still a Lagrange multiplier in (2.16) and enforces the constraint $\mathcal{C} = \vec{\nabla}^2 h_{ii} - \partial_i\partial_j h_{ij} - m^2 h_{ii} = 0$. But, this time, a secondary constraint $\frac{1}{D-2}m^2\pi_{ii} + \partial_i\partial_j\pi_{ij}$ arises from the Poisson bracket $\{H, \mathcal{C}\}$ which means the Hamiltonian (2.17) is not a first class Hamiltonian now. The resulting set of two constraints is second class and that removes the gauge freedom from the theory. Therefore, the theory now possesses 10 degrees of freedom, obtained by subtracting 2 constraints from 12 dimensional phase space, in $D = 4$. Half of them are the 5 polarizations of the massive graviton and other 5 belong to their conjugate momenta.

2.3 Propagator

In this section we will find the propagators of both massive and massless graviton cases. Propagator tells the probability amplitude for a particle (graviton in our case) to travel from one place to another and it also gives information about the renormalizability properties of the theory. At the end of our examination of the propagators we will find out the first sign of a possible discontinuity between the massive and massless gravity. We will also see that massive gravity propagator is not obeying standard power counting rules and hence we cannot deduce the renormalizability properties for it.

2.3.1 Massive Propagator

To find the propagator of the theory, we first integrate by parts the Fierz-Pauli action and rewrite it as

$$S = \int d^D x \, \frac{1}{2} h_{\mu\nu} \mathcal{O}^{\mu\nu,\alpha\beta} h_{\alpha\beta}, \tag{2.18}$$

where the operator $\mathcal{O}^{\mu\nu,\alpha\beta}$,

$$\mathcal{O}^{\mu\nu}{}_{\alpha\beta} \equiv \left(\eta^{(\mu}{}_{\alpha}\eta^{\nu)}{}_{\beta} - \eta^{\mu\nu}\eta_{\alpha\beta}\right)\left(\Box - m^2\right) - 2\partial^{(\mu}\partial_{(\alpha}\eta^{\nu)}{}_{\beta)} + \partial^{\mu}\partial^{\nu}\eta_{\alpha\beta} + \partial_{\alpha}\partial_{\beta}\eta^{\mu\nu}, \tag{2.19}$$

is a second order differential operator satisfying the symmetries

$$\mathcal{O}^{\mu\nu,\alpha\beta} = \mathcal{O}^{\nu\mu,\alpha\beta} = \mathcal{O}^{\mu\nu,\beta\alpha} = \mathcal{O}^{\alpha\beta,\mu\nu}. \tag{2.20}$$

That is to say, it is symmetric under the exchange of the first and the second pair of indices in between and within themselves. With the help of these symmetries, equations of motion can now be written in terms of this operator in a much simpler form,

$$\frac{\delta S}{\delta h_{\mu\nu}} = \mathcal{O}^{\mu\nu,\alpha\beta} h_{\alpha\beta} = 0. \tag{2.21}$$

To derive the propagator we go to momentum space by making the replacement $\partial \to ip$ in all partial derivatives, so that

$$\mathcal{O}^{\mu\nu}{}_{\alpha\beta} = -\left(\eta^{(\mu}{}_{\alpha}\eta^{\nu)}{}_{\beta} - \eta^{\mu\nu}\eta_{\alpha\beta}\right)\left(p^2 + m^2\right) - 2p^{(\mu}p_{(\alpha}\eta^{\nu)}{}_{\beta)} + p^{\mu}p^{\nu}\eta_{\alpha\beta} + p_{\alpha}p_{\beta}\eta^{\mu\nu}. \tag{2.22}$$

Propagator is the operator $\mathcal{D}_{\alpha\beta,\sigma\lambda}$ which is the inverse of the operator $\mathcal{O}^{\mu\nu,\alpha\beta}$,

$$\mathcal{O}^{\mu\nu,\alpha\beta}\mathcal{D}_{\alpha\beta,\sigma\lambda} = \frac{i}{2}\left(\delta^{\mu}_{\sigma}\delta^{\nu}_{\lambda} + \delta^{\nu}_{\sigma}\delta^{\mu}_{\lambda}\right), \tag{2.23}$$

and also satisfies the symmetries (2.20) of it. Equation (2.23) can be solved for $\mathcal{D}_{\alpha\beta,\sigma\lambda}$ by using the methods mentioned in [24] and the solution is obtained as

$$\mathcal{D}_{\alpha\beta,\sigma\lambda} = -\frac{i}{p^2 + m^2}\left[\frac{1}{2}\left(P_{\alpha\sigma}P_{\beta\lambda} + P_{\alpha\lambda}P_{\beta\sigma}\right) - \frac{1}{D-1}P_{\alpha\beta}P_{\sigma\lambda}\right], \tag{2.24}$$

where $P_{\alpha\beta} \equiv \eta_{\alpha\beta} + \frac{p_{\alpha}p_{\beta}}{m^2}$. Note that for high energies, i.e. large momenta, we have

$$\mathcal{D}_{\alpha\beta,\sigma\lambda} \sim \frac{1}{p^2 + m^2}\frac{p_{\alpha}p_{\beta}p_{\sigma}p_{\lambda}}{m^4} \sim \frac{p^2}{m^4}, \tag{2.25}$$

which implies that the standard power counting arguments [25, 26] are not valid in the present case and hence we cannot deduce the renormalizability properties of the theory.

2.3.2 Massless Propagator

For the massless graviton case, we can rewrite the action (2.1) as

$$S_{m=0} = \int d^D x \, \frac{1}{2} h_{\mu\nu} \mathcal{E}^{\mu\nu,\alpha\beta} h_{\alpha\beta}, \tag{2.26}$$

where the operator $\mathcal{E}^{\mu\nu,\alpha\beta}$ is obtained by taking $m = 0$ in (2.19). That is,

$$\mathcal{E}^{\mu\nu}{}_{\alpha\beta} = \left(\eta^{(\mu}{}_{\alpha}\eta^{\nu)}{}_{\beta} - \eta^{\mu\nu}\eta_{\alpha\beta}\right)\Box - 2\partial^{(\mu}\partial_{(\alpha}\eta^{\nu)}{}_{\beta)} + \partial^{\mu}\partial^{\nu}\eta_{\alpha\beta} + \partial_{\alpha}\partial_{\beta}\eta^{\mu\nu}, \tag{2.27}$$

and it has the same symmetries (2.20) as $\mathcal{O}^{\mu\nu,\alpha\beta}$. However, this time, the operator (2.27) is not invertible and we are not able to find the propagator as straight as we did in the massive case. Hence, we proceed in a different way by investigating some nice properties of the current situation.

First, note that, acting on a symmetric tensor $Z_{\mu\nu}$ with the kinetic operator $\mathcal{E}^{\mu\nu,\alpha\beta}$ gives

$$\mathcal{E}^{\mu\nu,\alpha\beta} Z_{\alpha\beta} = \Box Z^{\mu\nu} - \eta^{\mu\nu}\Box Z - 2\partial^{(\mu}\partial_{\alpha}Z^{\nu)\alpha} + \partial^{\mu}\partial^{\nu}Z + \eta^{\mu\nu}\partial_{\alpha}\partial_{\beta}Z^{\alpha\beta}, \tag{2.28}$$

which is a transverse tensor; $\partial_{\mu}(\mathcal{E}^{\mu\nu,\alpha\beta}Z_{\alpha\beta}) = 0$. Furthermore, we see that the kinetic operator (2.27) annihilates anything which is pure gauge; $\mathcal{E}^{\mu\nu,\alpha\beta}(\partial_{\alpha}\xi_{\beta} + \partial_{\beta}\xi_{\alpha}) = 0$.

Therefore, to find a propagator, we must fix the gauge freedom in the theory. Remember that, the theory has the gauge symmetry (2.2) and if we choose the Lorenz gauge (also called the harmonic gauge),

$$\partial^{\mu}h_{\mu\nu} - \frac{1}{2}\partial_{\nu}h = 0, \tag{2.29}$$

then (2.2) forces the system to satisfy

$$\Box\xi_{\mu} = -\left(\partial^{\nu}h_{\mu\nu} - \frac{1}{2}\partial_{\mu}h\right). \tag{2.30}$$

This fixes the gauge only up to gauge transformations with parameter ξ_{μ} satisfying the harmonicity condition $\Box\xi_{\mu} = 0$. Besides that, with this gauge choice, equations of motion

$$\Box h_{\mu\nu} - \partial_{\lambda}\partial_{\mu}h^{\lambda}{}_{\nu} - \partial_{\lambda}\partial_{\nu}h^{\lambda}{}_{\mu} + \eta_{\mu\nu}\partial_{\lambda}\partial_{\sigma}h^{\lambda\sigma} + \partial_{\mu}\partial_{\nu}h - \eta_{\mu\nu}\Box h = 0, \tag{2.31}$$

obtained from (2.3) by taking $m = 0$, simplify to

$$\Box h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\Box h = 0. \tag{2.32}$$

These are gauge-fixed equations of motion. However, since the gauge condition (2.30) cannot be obtained directly from the current case, we are not able to find these equations of motion from the action without adding a gauge-fixing Lagrangian

$$\mathscr{L}_{GF} = -\left(\partial^\nu h_{\mu\nu} - \frac{1}{2}\partial_\mu h\right)^2, \tag{2.33}$$

to the present Lagrangian $\mathscr{L}_{m=0}$ (the one we used in (2.1) with $m = 0$). By doing that we obtain a new Lagrangian, $\mathscr{L}'_{m=0}$, such that

$$\mathscr{L}'_{m=0} = \frac{1}{2}h_{\mu\nu}\Box h^{\mu\nu} - \frac{1}{4}h\Box h. \tag{2.34}$$

This Lagrangian now gives the equations of motion (2.32). Following the familiar integration by parts procedure we can write this gauge-fixed Lagrangian as

$$\mathscr{L}'_{m=0} = \frac{1}{2}h_{\mu\nu}\tilde{\mathcal{O}}^{\mu\nu,\alpha\beta}h_{\alpha\beta}, \tag{2.35}$$

where the operator is

$$\tilde{\mathcal{O}}^{\mu\nu,\alpha\beta} = \Box\left[\frac{1}{2}\left(\eta^{\mu\alpha}\eta^{\nu\beta} + \eta^{\mu\beta}\eta^{\nu\alpha}\right) - \frac{1}{2}\eta^{\mu\nu}\eta^{\alpha\beta}\right]. \tag{2.36}$$

Going to momentum space by the change $\partial \to ip$ gives

$$\tilde{\mathcal{O}}^{\mu\nu,\alpha\beta} = -p^2\left[\frac{1}{2}\left(\eta^{\mu\alpha}\eta^{\nu\beta} + \eta^{\mu\beta}\eta^{\nu\alpha}\right) - \frac{1}{2}\eta^{\mu\nu}\eta^{\alpha\beta}\right], \tag{2.37}$$

and as in the massive case, the propagator $\tilde{\mathcal{D}}_{\alpha\beta,\sigma\lambda}$ satisfies the relation

$$\tilde{\mathcal{O}}^{\mu\nu,\alpha\beta}\tilde{\mathcal{D}}_{\alpha\beta,\sigma\lambda} = \frac{i}{2}\left(\delta^\mu_\sigma\delta^\nu_\lambda + \delta^\nu_\sigma\delta^\mu_\lambda\right). \tag{2.38}$$

By solving this equation, we obtain the massless graviton propagator

$$\tilde{\mathcal{D}}_{\alpha\beta,\sigma\lambda} = -\frac{i}{p^2}\left[\frac{1}{2}\left(\eta_{\alpha\sigma}\eta_{\beta\lambda} + \eta_{\alpha\lambda}\eta_{\beta\sigma}\right) - \frac{1}{D-2}\eta_{\alpha\beta}\eta_{\sigma\lambda}\right]. \tag{2.39}$$

which at high energies gives consistent results with standard power counting arguments,

$$\tilde{\mathcal{D}}_{\alpha\beta,\sigma\lambda} \sim \frac{1}{p^2}. \tag{2.40}$$

As a final note, notice that the coefficient of the term $\eta_{\alpha\beta}\eta_{\sigma\lambda}$ in massive (2.24) and massless propagators (2.39) are different. For $D = 4$, in the former case we have $1/3$ and in the latter case we have $1/2$ as the coefficient. Hence, even in the $m \to 0$ limit, the coefficients are different. This is the first sign of the vDVZ discontinuity that will be examined in details in the following chapter.

CHAPTER 3

LINEAR RESPONSE TO SOURCES

In this chapter we add a fixed external symmetric source $T^{\mu\nu}$ to the Fierz-Pauli theory and investigate the consequences. The source-added Fierz-Pauli action is

$$S = \int d^D x \Big\{ -\frac{1}{2}\partial_\lambda h_{\mu\nu}\partial^\lambda h^{\mu\nu} + \partial_\mu h_{\nu\lambda}\partial^\nu h^{\mu\lambda} - \partial_\mu h^{\mu\nu}\partial_\nu h + \frac{1}{2}\partial_\lambda h \partial^\lambda h \\ - \frac{1}{2}m^2\left(h_{\mu\nu}h^{\mu\nu} - h^2\right) + \kappa h_{\mu\nu}T^{\mu\nu}\Big\}, \tag{3.1}$$

where the normalizations and the coupling strength to the source, $\kappa = M_P^{-\frac{D-2}{2}}$, are chosen in such a way that compatibility with the GR is maintained.

Equations of motion corresponding to this action is

$$\Box h_{\mu\nu} - \partial_\lambda\partial_\mu h^\lambda{}_\nu - \partial_\lambda\partial_\nu h^\lambda{}_\mu + \eta_{\mu\nu}\partial_\lambda\partial_\sigma h^{\lambda\sigma} + \partial_\mu\partial_\nu h - \eta_{\mu\nu}\Box h \\ - m^2\left(h_{\mu\nu} - \eta_{\mu\nu}h\right) = -\kappa T_{\mu\nu}. \tag{3.2}$$

In the case $m = 0$, acting on the left hand side with ∂^μ gives zero as we mentioned before. Hence, in that case we should have a conserved source; $\partial^\mu T_{\mu\nu} = 0$. But, for $m \neq 0$ we don't have such a condition.

3.1 General Solution to the Sourced Equations

To find the solution to (3.2) we first act on it with ∂^μ and we get

$$\partial^\mu h_{\mu\nu} - \partial_\nu h = \frac{\kappa}{m^2}\partial^\mu T_{\mu\nu}. \tag{3.3}$$

Plugging this result back into the equations of motion gives

$$\Box h_{\mu\nu} - \partial_\mu\partial_\nu h - m^2\left(h_{\mu\nu} - \eta_{\mu\nu}h\right) = -\kappa T_{\mu\nu} + \frac{\kappa}{m^2}\left(\partial^\lambda\partial_\mu T_{\nu\lambda} + \partial^\lambda\partial_\nu T_{\mu\lambda} - \eta_{\mu\nu}\partial\partial T\right), \tag{3.4}$$

where $\partial\partial T \equiv \partial_\mu\partial_\nu T^{\mu\nu}$ is defined as the double divergence of the source. Taking the trace of the above equation by regarding $\eta^{\mu\nu}\eta_{\mu\nu} = D$, we find

$$h = -\frac{\kappa}{m^2(D-1)}T - \frac{\kappa}{m^4}\frac{D-2}{D-1}\partial\partial T. \tag{3.5}$$

Putting this into (3.3) and rearranging gives

$$\partial^\mu h_{\mu\nu} = -\frac{\kappa}{m^2(D-1)}\partial_\nu T + \frac{\kappa}{m^2}\partial^\mu T_{\mu\nu} - \frac{\kappa}{m^4}\frac{D-2}{D-1}\partial_\nu\partial\partial T. \tag{3.6}$$

When (3.5) and (3.6) are both used in the equations of motion (3.2) now, we obtain

$$\begin{aligned}\left(\Box - m^2\right)h_{\mu\nu} = &-\kappa\left[T_{\mu\nu} - \frac{1}{D-1}\left(\eta_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{m^2}\right)T\right] \\ &+ \frac{\kappa}{m^2}\left[\partial^\lambda\partial_\mu T_{\nu\lambda} + \partial^\lambda\partial_\nu T_{\mu\lambda} - \frac{1}{D-1}\left(\eta_{\mu\nu} + (D-2)\frac{\partial_\mu\partial_\nu}{m^2}\right)\partial\partial T\right].\end{aligned} \tag{3.7}$$

Therefore, equation (3.2) implies three equations (3.5), (3.6) and (3.7). However, they are not totally independent. One can work out the trace of (3.7) and obtain the latter two. That means solution to the field $h_{\mu\nu}$ can be obtained by Fourier transforming only the last equation.

At that point, we turn our attention to the case of conserved sources, $\partial_\mu T^{\mu\nu} = 0$. In that case equation (3.7), which is the only one left, becomes

$$\left(\Box - m^2\right)h_{\mu\nu} = -\kappa\left[T_{\mu\nu} - \frac{1}{D-1}\left(\eta_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{m^2}\right)T\right]. \tag{3.8}$$

Now we define the following Fourier transforms which are going to be used in solving the above equation for $h_{\mu\nu}$;

$$h_{\mu\nu}(x) = \int\frac{d^Dp}{(2\pi)^D}e^{ip\cdot x}\tilde{h}_{\mu\nu}(p), \tag{3.9}$$

$$T_{\mu\nu}(x) = \int\frac{d^Dp}{(2\pi)^D}e^{ip\cdot x}\tilde{T}_{\mu\nu}(p), \tag{3.10}$$

$$T(x) = \int\frac{d^Dp}{(2\pi)^D}e^{ip\cdot x}\tilde{T}(p). \tag{3.11}$$

By Fourier transforming (3.8), on the left hand side we obtain

$$\text{(L.H.S.)} = -\int\frac{d^Dp}{(2\pi)^D}\left(p^2 + m^2\right)e^{ip\cdot x}\tilde{h}_{\mu\nu}(p), \tag{3.12}$$

and right hand side transforms as

$$\text{(R.H.S.)} = -\kappa \int \frac{d^D p}{(2\pi)^D} \left[\tilde{T}_{\mu\nu}(p) - \frac{1}{D-1} \left(\eta_{\mu\nu} + \frac{p_\mu p_\nu}{m^2} \right) \tilde{T}(p) \right] e^{ip\cdot x}. \tag{3.13}$$

(3.12) and (3.13) together give

$$\tilde{h}_{\mu\nu}(p) = \frac{\kappa}{p^2 + m^2} \left[\tilde{T}_{\mu\nu}(p) - \frac{1}{D-1} \left(\eta_{\mu\nu} + \frac{p_\mu p_\nu}{m^2} \right) \tilde{T}(p) \right] e^{ip\cdot x}, \tag{3.14}$$

and finally by using (3.9), we get

$$h_{\mu\nu}(x) = \kappa \int \frac{d^D p}{(2\pi)^D} \frac{e^{ip\cdot x}}{p^2 + m^2} \left[\tilde{T}_{\mu\nu}(p) - \frac{1}{D-1} \left(\eta_{\mu\nu} + \frac{p_\mu p_\nu}{m^2} \right) \tilde{T}(p) \right]. \tag{3.15}$$

This is the general solution of $h_{\mu\nu}$ for a conserved source. To find an exact solution one needs to integrate this equation for a given source $T_{\mu\nu}$. That is what we will do in the next section for both massive and massless cases.

3.2 Solution for a Point Source

For the case of a point source of mass M standing at the origin at rest, in $D = 4$, we can express the source tensor as

$$T^{\mu\nu}(x) = M\delta_0^\mu \delta_0^\nu \delta^3(\mathbf{x}), \quad \tilde{T}^{\mu\nu}(p) = 2\pi M \delta_0^\mu \delta_0^\nu \delta(p^0). \tag{3.16}$$

One can also find the trace of the source as $\tilde{T}(p) = -2\pi M \delta(p^0)$. This source is apparently conserved as needed. Hence, we can find the solutions for massive and massless gravitons.

3.2.1 Massive Graviton

The coupling strength to the source, in $D = 4$, is $\kappa = 1/M_P$ and for the point source (3.16) the general solution (3.15) reduces to

$$h_{\mu\nu}(x) = \frac{M}{M_P} \int \frac{dp_0 d^3\mathbf{p}}{(2\pi)^3} \frac{e^{-ip_0 x_0 + i\mathbf{p}\cdot\mathbf{x}}}{-p_0^2 + \mathbf{p}^2 + m^2} \left[\delta_0^\mu \delta_0^\nu + \frac{1}{3} \left(\eta_{\mu\nu} + \frac{p_\mu p_\nu}{m^2} \right) \right] \delta(p_0), \tag{3.17}$$

from which we can easily calculate the field elements and find

$$\begin{aligned} h_{00}(x) &= \frac{2M}{3M_P} \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{e^{i\mathbf{p}\cdot\mathbf{x}}}{\mathbf{p}^2 + m^2}, \\ h_{0i}(x) &= 0, \\ h_{ij}(x) &= \frac{M}{3M_P} \int \frac{d^3\mathbf{p}}{(2\pi^3)} \frac{e^{i\mathbf{p}\cdot\mathbf{x}}}{\mathbf{p}^2 + m^2} \left(\delta_{ij} + \frac{p_i p_j}{m^2} \right). \end{aligned} \tag{3.18}$$

Solving the above integrals gives

$$\begin{aligned} h_{00}(x) &= \frac{2M}{3M_P}\frac{e^{-mr}}{4\pi r}, \\ h_{0i}(x) &= 0, \\ h_{ij}(x) &= \frac{M}{3M_P}\frac{e^{-mr}}{4\pi r}\left[\frac{1+mr+m^2r^2}{m^2r^2}\delta_{ij} - \frac{3+3mr+m^2r^2}{m^2r^4}x_ix_j\right]. \end{aligned} \tag{3.19}$$

Notice that, this solution has the Yukawa supression factor e^{-mr} in its components which are the characteristics of a massive field.

We can express components of the field in spherical coordinates which will be useful later for comparison. We have

$$h_{\mu\nu}dx^\mu dx^\nu = -B(r)dt^2 + C(r)dr^2 + A(r)r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{3.20}$$

from which, in the present case, we read

$$\begin{aligned} B(r) &= -\frac{2M}{3M_P}\frac{e^{-mr}}{4\pi r}, \\ C(r) &= -\frac{2M}{3M_P}\frac{e^{-mr}}{4\pi r}\frac{1+mr}{m^2r^2}, \\ A(r) &= \frac{M}{3M_P}\frac{e^{-mr}}{4\pi r}\frac{1+mr+m^2r^2}{m^2r^2}. \end{aligned} \tag{3.21}$$

In $mr \ll 1$ limit these coefficients reduce to

$$\begin{aligned} B(r) &= -\frac{2M}{3M_P}\frac{1}{4\pi r}, \\ C(r) &= -\frac{2M}{3M_P}\frac{1}{4\pi m^2r^3}, \\ A(r) &= \frac{M}{3M_P}\frac{1}{4\pi m^2r^3}. \end{aligned} \tag{3.22}$$

Keep in mind that we will return to this page for comparison in Chapter 6.

3.2.2 Massless Graviton

For the massless graviton case we turn back to (3.2) and take $m = 0$ so that equations of motion reduce to

$$\Box h_{\mu\nu} - \partial_\lambda\partial_\mu h^\lambda{}_\nu - \partial_\lambda\partial_\nu h^\lambda{}_\mu + \eta_{\mu\nu}\partial_\lambda\partial_\sigma h^{\lambda\sigma} + \partial_\mu\partial_\nu h - \eta_{\mu\nu}\Box h = -\kappa T_{\mu\nu}. \tag{3.23}$$

If we choose the Lorenz gauge,

$$\partial^\mu h_{\mu\nu} - \frac{1}{2}\partial_\nu h = 0, \tag{3.24}$$

then these equations of motion give

$$\Box h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\Box h = -\kappa T_{\mu\nu}. \tag{3.25}$$

By taking the trace of the last equation and substituting it back, we obtain

$$\Box h_{\mu\nu} = -\kappa\left[T_{\mu\nu} - \frac{1}{D-2}\eta_{\mu\nu}T\right]. \tag{3.26}$$

This equation can be solved for the field $h_{\mu\nu}$ by Fourier transforming both sides. Left hand side of the equation after Fourier transform is

$$\text{(L.H.S)} = -\int \frac{d^Dp}{(2\pi)^D} p^2 e^{ip\cdot x}\tilde{h}_{\mu\nu}(p), \tag{3.27}$$

and right hand side transforms as

$$\text{(R.H.S)} = -\kappa\int \frac{d^Dp}{(2\pi)^D} e^{ip\cdot x}\left[\tilde{T}_{\mu\nu}(p) - \frac{1}{D-2}\eta_{\mu\nu}\tilde{T}(p)\right]. \tag{3.28}$$

From (3.27) and (3.28), we get

$$\tilde{h}_{\mu\nu}(p) = \frac{\kappa}{p^2}\left[\tilde{T}_{\mu\nu}(p) - \frac{1}{D-2}\eta_{\mu\nu}\tilde{T}(p)\right]. \tag{3.29}$$

Plugging this into (3.9) gives the general solution for a massless graviton

$$h_{\mu\nu}(x) = \kappa\int \frac{d^Dp}{(2\pi)^D}\frac{e^{ip\cdot x}}{p^2}\left[\tilde{T}_{\mu\nu}(p) - \frac{1}{D-2}\eta_{\mu\nu}\tilde{T}(p)\right]. \tag{3.30}$$

When we again specialize to $D = 4$ and consider the source to be the point particle of mass M at the origin (3.16) we find the components of the field $h_{\mu\nu}$ as

$$\begin{aligned} h_{00}(x) &= \frac{M}{2M_P}\int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{e^{i\mathbf{p}\cdot\mathbf{x}}}{\mathbf{p}^2} = \frac{M}{2M_P}\frac{1}{4\pi r}, \\ h_{0i}(x) &= 0, \\ h_{ij}(x) &= \frac{M}{2M_P}\int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{e^{i\mathbf{p}\cdot\mathbf{x}}}{\mathbf{p}^2}\delta_{ij} = \frac{M}{2M_P}\frac{1}{4\pi r}\delta_{ij}. \end{aligned} \tag{3.31}$$

In spherical coordinates, we read

$$\begin{aligned} B(r) &= -\frac{M}{2M_P}\frac{1}{4\pi r}, \\ C(r) &= \frac{M}{2M_P}\frac{1}{4\pi r}, \\ A(r) &= \frac{M}{2M_P}\frac{1}{4\pi r}. \end{aligned} \tag{3.32}$$

Now we have the fundamental tools to compare the massive graviton case to the massless graviton case which at the end will lead to the vDVZ discontinuity.

3.3 The vDVZ Discontinuity

In this section we will check the validity of our massive theory in $m \to 0$ limit. We first start with a test particle moving in the field $h_{\mu\nu}$ and assume that this particle will behave as if it is under the effect of the metric deviation in GR, so that $\delta g_{\mu\nu} = \frac{2}{M_P} h_{\mu\nu}$. From Chapter 7 of Carroll's book [27] we know that if the metric perturbation $\delta g_{\mu\nu}$ takes the form

$$\delta g_{00} = \frac{2}{M_P} h_{00} = -2\Phi, \quad \delta g_{0i} = \frac{2}{M_P} h_{0i} = 0, \quad \delta g_{ij} = \frac{2}{M_P} h_{ij} = -2\psi\delta_{ij}, \tag{3.33}$$

then the function $\phi(r)$ is interpreted as the Newtonian potential experienced by the test particle. Moreover, if for some constant γ, called the parameterized post-Newtonian (PPN) parameter, we have the equality $\psi(r) = \gamma\phi(r)$ and also if the Newtonian potential $\phi(r) = -\frac{k}{r}$ for some constant k; then we can write down the bending angle of light around a heavy source as

$$\alpha = \frac{2k(1+\gamma)}{b}, \tag{3.34}$$

where b is the impact parameter.

After introducing these new concepts, we first examine the massless graviton case. From the field components (3.31) and their relations to potentials (3.33), we get

$$\phi(r) = -\frac{GM}{r}, \quad \psi(r) = -\frac{GM}{r}, \tag{3.35}$$

where we have used $G = \frac{1}{8\pi M_P^2}$. The PPN parameter in this case is $\gamma = 1$. Therefore, light bending angle at impact parameter b for the massless graviton is

$$\alpha = \frac{4GM}{b}. \tag{3.36}$$

For the massive case, metric components (3.19) are not quite in the right form to read the potential and the bending angle. However, remember that, in the beginning of this section we assumed field $h_{\mu\nu}$ couples to the test particle as GR does. Hence, although massive gravity action (2.1) is not gauge invariant, making a gauge transformation on $h_{\mu\nu}$ in this case will not change the theory and will have no effect on the test particle. When we go back to (3.18) we see that the $\frac{p_i p_j}{m^2}$ term in the last field component h_{ij}

has no effect on any physical observable. Thus, it is a pure gauge and can safely be excluded [1]. Therefore, the field components in (3.18) are gauge equivalent to

$$\begin{aligned} h_{00}(x) &= \frac{2M}{3M_P}\int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{e^{i\mathbf{p}\cdot\mathbf{x}}}{\mathbf{p}^2+m^2}, \\ h_{0i}(x) &= 0, \\ h_{ij}(x) &= \frac{M}{3M_P}\int \frac{d^3\mathbf{p}}{(2\pi^3)}\frac{e^{i\mathbf{p}\cdot\mathbf{x}}}{\mathbf{p}^2+m^2}\delta_{ij}. \end{aligned} \tag{3.37}$$

After solving these integrals, we get

$$\begin{aligned} h_{00}(x) &= \frac{2M}{3M_P}\frac{e^{-mr}}{4\pi r}, \\ h_{0i}(x) &= 0, \\ h_{ij}(x) &= \frac{2M}{3M_P}\frac{e^{-mr}}{4\pi r}\delta_{ij}. \end{aligned} \tag{3.38}$$

With the help of (3.33), in small mass limit ($e^{-mr} \approx 1$), we obtain

$$\phi(r) = -\frac{4}{3}\frac{GM}{r}, \quad \psi(r) = -\frac{2}{3}\frac{GM}{r}\delta_{ij}. \tag{3.39}$$

PPN parameter is $\gamma = 1/2$ in this case and hence bending angle for the light is

$$\alpha = \frac{4GM}{b}. \tag{3.40}$$

Therefore, we immediately conclude that light bending angle is same in both massive and massless graviton cases but Newtonian potentials are different. In massive graviton case we have a greater value of Newtonian potential in magnitude. One can think of rescaling $G \to \frac{3}{4}G$ to provide the equality of Newtonian potentials in both cases but this would cause a difference in the bending angles this time. That means, massive gravity theory does not reduce to GR in the zero mass limit! But since mass is just a parameter in the theory, physically there shouldn't be such kind of a disagreement in $m \to 0$ limit. That discontinuity in the mass parameter of the theory is known as the vDVZ discontinuity after the names of van Dam, Veltman and Zakharov [14, 15].

[1] We can also think of the same concept from the point of view of conserved quantities. We have assumed the conservation of the stress-energy tensor $T_{\mu\nu}$ from the beginning and by Noether's theorem that should correspond to a symmetry. When the relation between the momenta and partial derivatives are considered it becomes clear that forcing $p_\mu p_\nu = 0$ guarantees the conservation $\partial_\mu T^{\mu\nu} = 0$. Moreover one can find the correct gauge paramater in this case by using $h_{ij}^{(\text{massive})} = h_{ij}^{(\text{massless})} + \partial_i\xi_j + \partial_j\xi_i$ where $h_{ij}^{(\text{massive})}$ and $h_{ij}^{(\text{massless})}$ are the field components (3.19) and (3.31) respectively.

CHAPTER 4

THE STUECKELBERG TRICK

In the last chapter we have seen that massless limit of the linear massive gravity is not equal to the linear massless gravity and this discontinuity in $m \to 0$ limit of the theory is called the vDVZ discontinuity. In this chapter we will search for the origin of this discontinuity and we will see that $m \to 0$ limit is not a smooth limit in the theory. Taking this limit causes us to lose one degree of freedom. To handle this problem we will introduce a new scalar field into the theory such that the new action will have a gauge symmetry. Taking the zero mass limit after that operation will give us the correct number of degrees of freedom. This procedure of introducing extra fields, creating gauge symmetries and taking the limit afterwards to preserve the degrees of freedom is known as the Stueckelberg trick [16]. To introduce the idea smoothly we will first consider a simpler case; the case of massive Maxwell electromagnetism.

4.1 Massive Photon Case

We consider the theory of a massive photon A_μ coupled to a source J_μ. The action of this theory is written as

$$S = \int d^D x \left\{ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} m^2 A_\mu A^\mu + A_\mu J^\mu \right\}, \tag{4.1}$$

where $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field tensor. When $m = 0$ we have the usual Maxwell electromagnetism which possesses 2 degrees of freedom (two polarization states of the massless photon) in four dimensions and has the gauge symmetry under the transformation $A_\mu \to A_\mu + \partial_\mu \Lambda$. However, when the mass term is introduced, we lose this gauge symmetry. In more terminological words, mass term

breaks the would-be gauge symmetry. Also, for this massive theory, in $D = 4$, we have 3 degrees of freedom (representing a massive spin-1 particle). Thus, when we take the $m \to 0$ limit we lose one degree of freedom. So this limit is not smooth and to take a smooth limit we should do something else. That is where Stueckelberg fields play their role.

To take a proper limit, we introduce a new scalar field ϕ and we write a new action including this new field. While introducing the new field we try to recover the would-be gauge invariance in the zero mass case. This suggests us to redefine A_μ such that it includes the field ϕ now. Hence, we make the following replacement

$$A_\mu \to A_\mu + \partial_\mu \phi. \tag{4.2}$$

This replacement is just like the gauge symmetry of the massless case as we desired. Hence $F_{\mu\nu}F^{\mu\nu}$ term in the action will not change under this transformation but other two terms are going to change and at the end we will have

$$S = \int d^D x \left\{ -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} - \frac{1}{2} m^2 \left(A_\mu + \partial_\mu \phi\right)^2 + A_\mu J^\mu - \phi \partial_\mu J^\mu \right\}, \tag{4.3}$$

where we have integrated the $(\partial_\mu \phi) J^\mu$ term by parts to obtain the last term. Notice that, this new action now has gauge symmetry

$$A_\mu \to A_\mu + \partial_\mu \Lambda, \quad \phi \to \phi - \Lambda. \tag{4.4}$$

We can fix the gauge to $\phi = 0$ (known as the unitary gauge) at any time and recover the original massive action (4.1). That means theories (4.3) and (4.1) are equivalent to each other. The only difference is that new theory uses an additional field ϕ and an associated gauge symmetry (4.4).

All this procedure shows us the interesting fact that gauge symmetries are just redundancies of physical theories. Any gauge theory, after eliminating the redundant degrees of freedom, loses its gauge symmetry. Conversely, by introducing redundant variables (like the ϕ above) any theory can be put into the form of a gauge theory.

Returning to our problem, the next thing to do before taking the $m \to 0$ limit is to normalize the kinetic ϕ term in (4.3). For that purpose, we rescale the scalar field,

$\phi \to \frac{1}{m}\phi$, and obtain

$$S = \int d^D x \Big\{ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} m^2 A_\mu A^\mu - m A_\mu \partial^\mu \phi \\ - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + A_\mu J^\mu - \frac{1}{m} \phi \partial_\mu J^\mu \Big\}, \tag{4.5}$$

with gauge symmetry

$$A_\mu \to A_\mu + \partial_\mu \Lambda, \quad \phi \to \phi - m\Lambda. \tag{4.6}$$

If we try to take $m \to 0$ limit now, we see that the last term in (4.5) is problematic and limit does not exist. However, if we assume a conserved source, i.e. $\partial_\mu J^\mu = 0$, then this term will disappear and we will be able to take the limit safely. Under this assumption $m \to 0$ limit gives

$$S = \int d^D x \left\{ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + A_\mu J^\mu \right\}, \tag{4.7}$$

with gauge symmetry

$$A_\mu \to A_\mu + \partial_\mu \Lambda. \tag{4.8}$$

In this theory, for $D = 4$, we have a massless vector field A_μ which possesses two degrees of freedom and a massless scalar ϕ which possesses one degree of freedom, adding to a total of three degrees of freedom as in the massive case. Thus, number of degrees of freedom are preserved in this procedure.

4.2 Massive Graviton Case and Origin of the vDVZ Discontinuity

In this section we will apply the technique of Stueckelberg to the sourced massive gravity action (3.1). For simplicity we first rewrite the action by collecting all the massless terms (except the source term) into a separate Lagrangian,

$$S = \int d^D x \left\{ \mathscr{L}_{m=0} - \frac{1}{2} m^2 (h_{\mu\nu} h^{\mu\nu} - h^2) + \kappa h_{\mu\nu} T^{\mu\nu} \right\}. \tag{4.9}$$

For the massless graviton case we have the gauge symmetry

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu, \tag{4.10}$$

which is broken by the mass term. Now we introduce a new field A_μ, the Stueckelberg field, by following the same pattern of that gauge symmetry,

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu A_\nu + \partial_\nu A_\mu. \tag{4.11}$$

Under this transformation $\mathscr{L}_{m=0}$ term remains invariant since this transformation satisfies the gauge symmetry of this massless Lagrangian. However, other two terms in (4.9) changes and we get

$$S = \int d^D x \Big\{ \mathscr{L}_{m=0} - \frac{1}{2} m^2 (h_{\mu\nu} h^{\mu\nu} - h^2) - \frac{1}{2} m^2 F_{\mu\nu} F^{\mu\nu} \\ - 2m^2 \left(h_{\mu\nu} \partial^\mu A^\nu - h \partial_\mu A^\mu \right) + \kappa h_{\mu\nu} T^{\mu\nu} - 2\kappa A_\mu \partial_\nu T^{\mu\nu} \Big\}, \tag{4.12}$$

where we have used the pre-defined electromagnetic field tensor $F_{\mu\nu}$ and we also applied integration by parts to obtain the last term. This action has gauge symmetry

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu, \quad A_\mu \to A_\mu - \xi_\mu. \tag{4.13}$$

We can fix the gauge to $A_\mu = 0$ and recover the original action at any time. Thus, the theories (4.9) and (4.12) are equivalent. However, if we try to take $m \to 0$ limit at that point, we see that it is still not smooth and one degree of freedom is lost. Hence, we need to apply the Stueckelberg trick once more. For that purpose, we introduce a new field ϕ, again a Stueckelberg field, by following the pattern of the gauge symmetry of Maxwell electromagnetism,

$$A_\mu \to A_\mu + \partial_\mu \phi. \tag{4.14}$$

With this transformation, action (4.12) becomes

$$S = \int d^D x \Big\{ \mathscr{L}_{m=0} - \frac{1}{2} m^2 (h_{\mu\nu} h^{\mu\nu} - h^2) - \frac{1}{2} m^2 F_{\mu\nu} F^{\mu\nu} \\ - 2m^2 \left(h_{\mu\nu} \partial^\mu A^\nu - h \partial_\mu A^\mu \right) - 2m^2 \left(h_{\mu\nu} \partial^\mu \partial^\nu \phi - h \partial^2 \phi \right) \\ + \kappa h_{\mu\nu} T^{\mu\nu} - 2\kappa A_\mu \partial_\nu T^{\mu\nu} + 2\kappa \phi \partial\partial T \Big\}, \tag{4.15}$$

where the last term is obtained by integration by parts and $\partial\partial T = \partial_\mu \partial_\nu T^{\mu\nu}$ is the double divergence of the source as defined before. This action has two gauge symmetries

$$\begin{cases} h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu & , \quad A_\mu \to A_\mu - \xi_\mu, \\ A_\mu \to A_\mu + \partial_\mu \Lambda & , \quad \phi \to \phi - \Lambda. \end{cases} \tag{4.16}$$

We can fix the gauge to $\phi = 0$ and this recovers the action (4.12). This means (4.15) is equivalent to (4.12) which was already equivalent to (4.9). Thus, with some

additional fields and gauge symmetries new action does the same job with the initial massive gravity action.

We now make the rescalings, $A_\mu \to \frac{1}{m}A_\mu$ and $\phi \to \frac{1}{m^2}\phi$, so that action (4.15) becomes

$$S = \int d^Dx \Big\{ \mathscr{L}_{m=0} - \frac{1}{2}m^2(h_{\mu\nu}h^{\mu\nu} - h^2) - \frac{1}{2}F_{\mu\nu}F^{\mu\nu} \\ - 2m\left(h_{\mu\nu}\partial^\mu A^\nu - h\partial_\mu A^\mu\right) - 2\left(h_{\mu\nu}\partial^\mu\partial^\nu\phi - h\partial^2\phi\right) \\ + \kappa h_{\mu\nu}T^{\mu\nu} - \frac{2}{m}\kappa A_\mu\partial_\nu T^{\mu\nu} + \frac{2}{m^2}\kappa\phi\partial\partial T \Big\}, \tag{4.17}$$

with gauge symmetries

$$\begin{cases} h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu \quad , \quad A_\mu \to A_\mu - m\xi_\mu, \\ A_\mu \to A_\mu + \partial_\mu\Lambda' \quad , \quad \phi \to \phi - m\Lambda', \end{cases} \tag{4.18}$$

where $\Lambda' \equiv m\Lambda$. At that point, assuming the source is conserved, i.e. $\partial_\mu T^{\mu\nu} = 0$, $m \to 0$ limit can be safely taken and it gives

$$S = \int d^Dx \Big\{ \mathscr{L}_{m=0} - \frac{1}{2}F_{\mu\nu}F^{\mu\nu} - 2\left(h_{\mu\nu}\partial^\mu\partial^\nu\phi - h\partial^2\phi\right) + \kappa h_{\mu\nu}T^{\mu\nu} \Big\}. \tag{4.19}$$

To count the degrees of freedom we will decompose the scalar and tensor by redefining the field $h_{\mu\nu}$ as

$$h_{\mu\nu} = h'_{\mu\nu} + \pi\eta_{\mu\nu}, \tag{4.20}$$

where π is just any other scalar. This is a conformal transformation on the field $h_{\mu\nu}$ and under this change the massless Lagrangian $\mathscr{L}_{m=0}$ becomes

$$\mathscr{L}_{m=0}(h) = \mathscr{L}_{m=0}(h') + (D-2)\left[\partial_\mu\pi\partial^\mu h' - \partial_\mu\pi\partial_\nu h'^{\mu\nu} + \frac{1}{2}(D-1)\partial_\mu\pi\partial^\mu\pi\right], \tag{4.21}$$

and action (4.19), in total, takes the form

$$S = \int d^Dx \Big\{ \mathscr{L}_{m=0}(h') + (D-2)\left[\partial_\mu\pi\partial^\mu h' - \partial_\mu\pi\partial_\nu h'^{\mu\nu} + \frac{1}{2}(D-1)\partial_\mu\pi\partial^\mu\pi\right] \\ - \frac{1}{2}F_{\mu\nu}F^{\mu\nu} - 2\left(h'_{\mu\nu}\partial^\mu\partial^\nu\phi - h'\partial^2\phi\right) + 2\pi(D-1)\partial^2\phi \\ + \kappa h'_{\mu\nu}T^{\mu\nu} + \kappa\pi T \Big\}. \tag{4.22}$$

At that point, all coupled tensor-scalar $h\phi$ terms can be cancelled by taking $\pi = \frac{2}{D-2}\phi$. After applying integration by parts several times, we finally obtain

$$S = \int d^D x \Big\{ \mathscr{L}_{m=0}(h') - \frac{1}{4} F'_{\mu\nu} F'^{\mu\nu} - \frac{1}{2} \partial_\mu \phi' \partial^\mu \phi' + \kappa h'_{\mu\nu} T^{\mu\nu} + \frac{\kappa \phi' T}{\sqrt{(D-2)(D-1)}} \Big\}, \quad (4.23)$$

with gauge symmetries

$$\begin{cases} h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu, \\ A_\mu \to A_\mu + \partial_\mu \Lambda', \end{cases} \quad (4.24)$$

where we have also rescaled $A_\mu \to A'_\mu = \frac{1}{\sqrt{2}} A_\mu$ $\big($which resulted in $F_{\mu\nu} \to F'_{\mu\nu}\big)$ and $\phi \to \phi' = \frac{1}{2}\sqrt{\frac{D-2}{D-1}}\phi$ to satisfy the correct coefficients in front of the kinetic terms of the vector and the scalar fields. It is easy to count the degrees of freedom now. For $D = 4$ we have one massless graviton $h'_{\mu\nu}$ which possesses two degrees of freedom, one massless vector A'_μ which again possesses two degrees of freedom and finally one massless scalar ϕ' which possesses a single degree of freedom, in total making a five degrees of freedom as we expected.

Lastly, we point out to the origin of vDVZ discontinuity by looking at the last term in the action (4.23). In that term, we see that coupling of the scalar to the trace of the stress-energy tensor survives even in the $m \to 0$ limit. This coupling causes an extra degree of freedom which does not affect the light bending angle (because we set $T = 0$ there) but affects the Newtonian potential. The new potential obtained by the contribution of this coupling term accounts for the discontinuity between the massless gravity and massless limit of the massive gravity. That is, the coupling $\phi' T$ is the origin of the vDVZ discontinuity.

CHAPTER 5

MASSIVE GRAVITONS ON CURVED SPACES

What happens when we put the Fierz-Pauli action onto a curved space? In this section we will search for an answer to that question by changing the flat background metric to a more generic one. At the end we will come up with two important results; one which will make us to discover the existence of partially massless theories and the other will prove the absence of the vDVZ discontinuity in curved spaces.

5.1 Partially Massless Theories

To obtain a curved space we change the flat background metric $\eta_{\mu\nu}$ to a more generic metric $g_{\mu\nu}$ and partial derivatives to covariant derivatives. We will modify the Fierz-Pauli action (2.1) in such a way that massless part of the new action will be linearized Einstein-Hilbert action with the cosmological constant Λ. Thus, we start with the action $S = \int d^D x \frac{1}{2\kappa^2}\sqrt{-g}(R - 2\Lambda)$ and then expand it around a solution $\bar{g}_{\mu\nu}$ via the metric perturbation $g_{\mu\nu} \to \bar{g}_{\mu\nu} + 2\kappa h_{\mu\nu}$ up to second order. That solution satisfies the Einstein space conditions

$$\Lambda = \left(\frac{D-2}{2D}\right)\bar{R}, \qquad \bar{R}_{\mu\nu} = \frac{\bar{R}}{D} g_{\mu\nu}, \tag{5.1}$$

which are easily found by tracing the equations of motion, $\bar{R}_{\mu\nu} - \frac{1}{2}\bar{R}\bar{g}_{\mu\nu} + \Lambda\bar{g}_{\mu\nu} = 0$. Since we have a curved space now, we append the Fierz-Pauli term onto it and obtain

the final form of the action as

$$S=\int d^Dx\sqrt{-\bar{g}}\Big[-\frac{1}{2}\bar{\nabla}_\lambda h_{\mu\nu}\bar{\nabla}^\lambda h^{\mu\nu}+\bar{\nabla}_\mu h_{\nu\lambda}\bar{\nabla}^\nu h^{\mu\lambda}-\bar{\nabla}_\mu h^{\mu\nu}\bar{\nabla}_\nu h+\frac{1}{2}\bar{\nabla}_\lambda h\bar{\nabla}^\lambda h$$
$$+\frac{\bar{R}}{D}\left(h^{\mu\nu}h_{\mu\nu}-\frac{1}{2}h^2\right)-\frac{1}{2}m^2\left(h_{\mu\nu}h^{\mu\nu}-h^2\right)+\kappa h_{\mu\nu}T^{\mu\nu}\Big], \tag{5.2}$$

where metric determinant $\sqrt{-\bar{g}}$, curvature term $\bar{R}$ and all covariant derivatives belong to the background metric $\bar{g}_{\mu\nu}$. All contractions over indices are also done by using the background metric.

For the case $m=0$ and $D=4$, action (5.2) has 2 degrees of freedom corresponding to two polarizations of the massless graviton. In this case it also has the gauge symmetry

$$h_{\mu\nu}\to h_{\mu\nu}+\bar{\nabla}_\mu\xi_\nu+\bar{\nabla}_\nu\xi_\mu. \tag{5.3}$$

Our aim is to maintain this gauge symmetry in the massive case by using Stueckelberg's technique studied in Chapter 4. For that purpose we introduce a Stueckelberg field A_μ patterned after the gauge symmetry (5.3),

$$h_{\mu\nu}\to h_{\mu\nu}+\bar{\nabla}_\mu A_\nu+\bar{\nabla}_\nu A_\mu, \tag{5.4}$$

which makes $\mathscr{L}_{m=0}$ term unchanged [1]. Under the assumption of a covariantly conserved source , $\bar{\nabla}_\mu T^{\mu\nu}=0$, the last term of (5.2) also remains invariant. The only term left to deal with is the mass term and after some work on it, this action reduces to

$$S=\int d^Dx\Big\{\mathscr{L}_{m=0}+\sqrt{-\bar{g}}\Big[-\frac{1}{2}m^2(h_{\mu\nu}h^{\mu\nu}-h^2)-\frac{1}{2}m^2F_{\mu\nu}F^{\mu\nu}+\frac{2}{D}m^2\bar{R}A^\mu A_\mu$$
$$-2m^2(h_{\mu\nu}\bar{\nabla}^\mu A^\nu-h\bar{\nabla}_\mu A^\mu)+\kappa h_{\mu\nu}T^{\mu\nu}\Big]\Big\}, \tag{5.5}$$

where we applied integration by parts several times and used the relation $\bar{\nabla}_\mu A_\nu\bar{\nabla}^\nu A^\mu=(\bar{\nabla}_\mu A^\mu)^2-\bar{R}_{\mu\nu}A^\mu A^\nu$. This new action has gauge symmetry

$$h_{\mu\nu}\to h_{\mu\nu}+\bar{\nabla}_\mu\xi_\nu+\bar{\nabla}_\nu\xi_\mu,\quad A_\mu\to A_\mu-\xi_\mu. \tag{5.6}$$

[1] For its future role, we do not put the term with curvature constant into this massless Lagrangian. The source term is also excluded as we did before. Thus, we take

$$\mathscr{L}_{m=0}=\sqrt{-\bar{g}}\Big[-\frac{1}{2}\bar{\nabla}_\lambda h_{\mu\nu}\bar{\nabla}^\lambda h^{\mu\nu}+\bar{\nabla}_\mu h_{\nu\lambda}\bar{\nabla}^\nu h^{\mu\lambda}-\bar{\nabla}_\mu h^{\mu\nu}\bar{\nabla}_\nu h+\frac{1}{2}\bar{\nabla}_\lambda h\bar{\nabla}^\lambda h\Big].$$

Note that, fixing the gauge to $A_\mu = 0$ recovers the original action (5.2).

Now, we introduce another Stueckelberg field, a scalar ϕ, following the gauge symmetry of electromagnetism,

$$A_\mu \to A_\mu + \bar{\nabla}_\mu \phi. \tag{5.7}$$

With this transformation third and fourth terms inside the square brackets in (5.5) changes and action becomes

$$\begin{aligned} S = \int d^D x \Big\{ \mathscr{L}_{m=0} + \sqrt{-\bar{g}} \Big[& -\frac{1}{2} m^2 (h_{\mu\nu} h^{\mu\nu} - h^2) - \frac{1}{2} m^2 F_{\mu\nu} F^{\mu\nu} + \frac{2}{D} m^2 \bar{R} A^\mu A_\mu \\ & - 2m^2 (h_{\mu\nu} \bar{\nabla}^\mu A^\nu - h \bar{\nabla}_\mu A^\mu) + \frac{4m^2 \bar{R}}{D} A^\mu \bar{\nabla}_\mu \phi + \frac{2m^2 \bar{R}}{D} (\partial\phi)^2 \\ & - 2m^2 (h_{\mu\nu} \bar{\nabla}^\mu \bar{\nabla}^\nu \phi - h \bar{\Box} \phi) + \kappa h_{\mu\nu} T^{\mu\nu} \Big] \Big\}, \end{aligned} \tag{5.8}$$

with gauge symmetries

$$\begin{cases} h_{\mu\nu} \to h_{\mu\nu} + \bar{\nabla}_\mu \xi_\nu + \bar{\nabla}_\nu \xi_\mu & , \quad A_\mu \to A_\mu - \xi_\mu, \\ A_\mu \to A_\mu + \bar{\nabla}_\mu \Lambda & , \quad \phi \to \phi - \Lambda. \end{cases} \tag{5.9}$$

Under the conformal transformation

$$h_{\mu\nu} = h'_{\mu\nu} + \pi \eta_{\mu\nu}, \tag{5.10}$$

massless Lagrangian changes into

$$\begin{aligned} \mathscr{L}_{m=0}(h) = \mathscr{L}_{m=0}(h') + \sqrt{-\bar{g}} \Big[(D-2) \Big(& \bar{\nabla}_\mu \pi \bar{\nabla}^\mu h' - \bar{\nabla}_\mu \pi \bar{\nabla}_\nu h'^{\mu\nu} \\ & + \frac{1}{2} (D-1) \bar{\nabla}_\mu \pi \bar{\nabla}^\mu \pi \Big) - \frac{D-2}{D} \bar{R} \left(h' \pi + \frac{D}{2} \pi^2 \right) \Big], \end{aligned} \tag{5.11}$$

and action (5.8), in total, takes the form

$$\begin{aligned} S = \int d^D x \Big\{ \mathscr{L}_{m=0}(h') + \sqrt{-\bar{g}} \Big[& (D-2) \Big(\bar{\nabla}_\mu \pi \bar{\nabla}^\mu h' - \bar{\nabla}_\mu \pi \bar{\nabla}_\nu h'^{\mu\nu} \\ & + \frac{1}{2} (D-1) \bar{\nabla}_\mu \pi \bar{\nabla}^\mu \pi \Big) - \frac{D-2}{D} \bar{R} \left(h' \pi + \frac{D}{2} \pi^2 \right) - \frac{1}{2} m^2 (h'_{\mu\nu} h'^{\mu\nu} - h'^2) \\ & + \frac{1}{2} m^2 (D-1)(2h'\pi + D\pi^2) - \frac{1}{2} m^2 F_{\mu\nu} F^{\mu\nu} + \frac{2}{D} m^2 \bar{R} A^\mu A_\mu \\ & - 2m^2 (h'_{\mu\nu} \bar{\nabla}^\mu A^\nu - h' \bar{\nabla}_\mu A^\mu) + 2m^2 (D-1) \pi \bar{\nabla}_\mu A^\mu + \frac{4m^2 \bar{R}}{D} A^\mu \bar{\nabla}_\mu \phi \\ & + \frac{2m^2 \bar{R}}{D} (\partial\phi)^2 - 2m^2 (h'_{\mu\nu} \bar{\nabla}^\mu \bar{\nabla}^\nu \phi - h' \bar{\Box} \phi) + 2m^2 (D-1) \pi \bar{\Box} \phi \\ & + \kappa h'_{\mu\nu} T^{\mu\nu} + \kappa \pi T \Big] \Big\}. \end{aligned} \tag{5.12}$$

After redefining $\pi = \frac{2}{D-2}m^2\phi$ and applying several integration by parts, we finally obtain

$$
\begin{aligned}
S = \int d^Dx \Big\{ \mathscr{L}_{m=0}(h') + \sqrt{-\bar{g}} \Big[& -\frac{1}{2}m^2(h'_{\mu\nu}h'^{\mu\nu} - h'^2) - \frac{1}{2}m^2 F_{\mu\nu}F^{\mu\nu} \\
& + \frac{2}{D}m^2\bar{R}A^\mu A_\mu - 2m^2(h'_{\mu\nu}\bar{\nabla}^\mu A^\nu - h'\bar{\nabla}_\mu A^\mu) \\
& + 2m^2\left(\frac{D-1}{D-2}m^2 - \frac{\bar{R}}{D}\right)(2\phi\bar{\nabla}_\mu A^\mu + h'\phi) \\
& - 2m^2\left(\frac{D-1}{D-2}m^2 - \frac{\bar{R}}{D}\right)\left((\partial\phi)^2 - m^2\frac{2D}{D-2}\phi^2\right) \\
& + \kappa h'_{\mu\nu}T^{\mu\nu} + \frac{2}{D-2}m^2\kappa\phi T \Big] \Big\},
\end{aligned}
\tag{5.13}
$$

with gauge symmetries

$$
\begin{cases}
h'_{\mu\nu} \to h'_{\mu\nu} + \bar{\nabla}_\mu\xi_\nu + \bar{\nabla}_\nu\xi_\mu + \frac{2}{D-2}\Lambda g_{\mu\nu} & , \quad A_\mu \to A_\mu - \xi_\mu, \\
A_\mu \to A_\mu + \bar{\nabla}_\mu\Lambda & , \quad \phi \to \phi - \Lambda.
\end{cases}
\tag{5.14}
$$

Although the procedure we applied so far was familiar, action (5.13) has some special feature hidden behind the curvature term $\bar{R}$. Namely, for the special value of the curvature term,

$$
\bar{R} = \frac{D(D-1)}{D-2}m^2, \tag{5.15}
$$

that action becomes independent of the Stueckelberg field ϕ. Tracing back through all the field replacements, normalizations, conformal transformations etc., for the initial action (5.2) we arrive a gauge symmetry

$$
\delta h_{\mu\nu} = \bar{\nabla}_\mu\bar{\nabla}_\nu\lambda + \frac{1}{D-2}m^2\lambda g_{\mu\nu}, \tag{5.16}
$$

where λ is a scalar gauge parameter. That gauge symmetry removes the longitudinal degree of freedom from the theory. Hence, theory (5.2) has one fewer degree of freedom than usual because of the gauge symmetry (5.16); in $D = 4$ it has four degrees of freedom instead of five. That theory having one less degree of freedom for some special value of the curvature term is called a partially massless theory [28, 29, 30].

5.2 Absence of the vDVZ Discontinuity

After getting so far, what is left to take $m \to 0$ limit of (5.13) to see the fate of the vDVZ discontinuity. We first make the rescaling, $A_\mu \to \frac{1}{m}A_\mu$, by which action changes to

$$\begin{aligned} S = \int d^D x \Big\{ \mathscr{L}_{m=0}(h') + \sqrt{-\bar{g}} \Big[& -\frac{1}{2}m^2(h'_{\mu\nu}h'^{\mu\nu} - h'^2) - \frac{1}{2}F_{\mu\nu}F^{\mu\nu} \\ & + \frac{2}{D}\bar{R}A^\mu A_\mu - 2m(h'_{\mu\nu}\bar{\nabla}^\mu A^\nu - h'\bar{\nabla}_\mu A^\mu) \\ & + 2m\left(\frac{D-1}{D-2}m^2 - \frac{\bar{R}}{D}\right)(2\phi\bar{\nabla}_\mu A^\mu + mh'\phi) \\ & - 2m^2\left(\frac{D-1}{D-2}m^2 - \frac{\bar{R}}{D}\right)\left((\partial\phi)^2 - m^2\frac{2D}{D-2}\phi^2\right) \\ & + \kappa h'_{\mu\nu}T^{\mu\nu} + \frac{2}{D-2}m^2\kappa\phi T \Big] \Big\}. \end{aligned} \tag{5.17}$$

with gauge symmetries

$$\begin{cases} h'_{\mu\nu} \to h'_{\mu\nu} + \bar{\nabla}_\mu\xi_\nu + \bar{\nabla}_\nu\xi_\mu + \frac{2}{D-2}\Lambda g_{\mu\nu} & , \quad A_\mu \to A_\mu - m\xi_\mu, \\ A_\mu \to A_\mu + m\bar{\nabla}_\mu\Lambda & , \quad \phi \to \phi - \Lambda. \end{cases} \tag{5.18}$$

Upon taking $m \to 0$ limit and rescaling the vector field again by letting $A_\mu \to A'_\mu = \frac{1}{\sqrt{2}}A_\mu$ to achieve the correct coefficients for the kinetic terms, we obtain

$$S = \int d^D x \Big\{ \mathscr{L}_{m=0}(h') + \sqrt{-\bar{g}}\Big[-\frac{1}{4}F'_{\mu\nu}F'^{\mu\nu} + \frac{\bar{R}}{D}A'^\mu A'_\mu + \kappa h'_{\mu\nu}T^{\mu\nu} \Big] \Big\}, \tag{5.19}$$

with gauge symmetry

$$h'_{\mu\nu} \to h'_{\mu\nu} + \bar{\nabla}_\mu\xi_\nu + \bar{\nabla}_\nu\xi_\mu + \frac{2}{D-2}\Lambda g_{\mu\nu}. \tag{5.20}$$

Note that, in taking $m \to 0$ limit, no degrees of freedom are lost. We started with a 5 degrees of freedom and in (5.19) we have one massive vector field A'_μ possessing 3 degrees of freedom plus one massless tensor field $h'_{\mu\nu}$ possessing 2 degrees of freedom adding up to a total 5 degrees of freedom again. Hence, without any need to introduce an additional scalar Stueckelberg field, we were able to take $m \to 0$ limit, i.e. this limit was smooth.

The important thing to observe in (5.19) is the decoupling of the massive vector A'_μ from the source tensor. That coupling between the Stueckelberg field and the trace

of the stress-energy tensor was the main reason of the vDVZ discontinuity as we mentioned before and hence, if there is no such coupling then there is no vDVZ discontinuity. Therefore, on curved spaces vDVZ discontinuity is automatically removed from the Fierz-Pauli theory. Some useful references on that topic are [28, 31, 32, 33].

CHAPTER 6

NON-LINEAR INTERACTIONS

In this chapter we will start to examine the non-linearities for the massive gravity theory. In the first section we will make a review of general relativity and we will discover the role of non-linearities in there. In the second section we will apply the same procedure to massive gravity. In the third section we will examine the spherical solutions and we will see the dependence of non-linearities to a radius called Vainshtein radius. Finally, in the last section we will study the Hamiltonian of non-linear massive gravity through the ADM formalism and we will discover the Boulware-Deser ghost.

6.1 Non-linearities in General Relativity

General relativity is the theory of a dynamical metric $g_{\mu\nu}$ with the action

$$S = \frac{1}{2\kappa^2} \int d^D x \sqrt{-g} R. \tag{6.1}$$

This action is invariant under the gauge symmetry

$$g_{\mu\nu}(x) \to \frac{\partial f^\alpha}{\partial x^\mu} \frac{\partial f^\beta}{\partial x^\nu} g_{\alpha\beta}(f(x)). \tag{6.2}$$

For infinitesimal changes $f^\mu(x) = x^\mu + \xi^\mu(x)$, by using the Taylor expansion we can write the variation in the metric as

$$\delta g_{\mu\nu} = g_{\mu\lambda}\partial_\nu \xi^\lambda + g_{\lambda\nu}\partial_\mu \xi^\lambda + \xi^\lambda \partial_\lambda g_{\mu\nu}, \tag{6.3}$$

where the right hand side is identified as the Lie derivative of the metric tensor $g_{\mu\nu}$ along the gauge parameter ξ, i.e. $\delta g_{\mu\nu} = \mathcal{L}_\xi g_{\mu\nu}$. We can replace all partial derivatives to covariant derivatives in the case of torsion-free connections. Since we have

metric compatibility ($\nabla_\sigma g_{\mu\nu} = 0$) the last term in the above expression will vanish. Regarding this and lowering the indices with the metric we obtain

$$\delta g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu, \tag{6.4}$$

which is the ultimate form of the gauge symmetry for the general relativity.

Taking the variation of (6.1) with respect to inverse metric gives Einstein's field equations

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 0, \tag{6.5}$$

whose detailed derivation is given in Appendix C. Most symmetric solution to these field equations is obtained when we have flat space, i.e. $g_{\mu\nu} = \eta_{\mu\nu}$. In this case we get $R_{\mu\nu} = R = 0$ and (6.5) is automatically satisfied.

We know that general relativity is the theory of an interacting massless helicity-2 particle. However, we cannot read it directly from the action (6.1). In order to see this fact clearly, we need to expand the action around the flat space solution,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{6.6}$$

to second order in metric perturbation $h_{\mu\nu}$. By doing that (See Appendix D for the detailed expansions of curvature terms in the case of a more generic perturbation metric) , we obtain

$$S^{(2)} = \frac{1}{4\kappa^2} \int d^D x \Big\{ -\frac{1}{2} \partial_\lambda h_{\mu\nu} \partial^\lambda h^{\mu\nu} + \partial_\mu h_{\nu\lambda} \partial^\nu h^{\mu\lambda} - \partial_\mu h^{\mu\nu} \partial_\nu h + \frac{1}{2} \partial_\lambda h \partial^\lambda h \Big\}. \tag{6.7}$$

Canonical normalization of the field, $h_{\mu\nu} = 2\kappa \hat{h}_{\mu\nu}$, reduces this linearized action of GR to the one we have used for a spin-2 particle (2.1) in zero mass case.

What would happen if we had continued to expand the action around flat space to higher non-linear orders in $\hat{h}_{\mu\nu}$? The obvious thing is that we would have many interaction terms including the higher powers of $\hat{h}$. But it is crucial to note that none of these terms would have more than two derivatives. That is because what we do, at all, is to expand the combination $\sqrt{-g}R$ in which the first one contains no derivatives and the second one consists of only two derivatives. Hence, at the end of that higher order expansion, schematically we would have

$$S = \int d^D x \left\{ \partial^2 \hat{h}^2 + \kappa \partial^2 \hat{h}^3 + \ldots + \kappa^n \partial^2 \hat{h}^{n+2} + \ldots \right\}, \tag{6.8}$$

where we assume all the coefficients are precisely fixed so that general coordinate covariance (or diffeomorphism invariance) of GR is maintained. Note that the action (6.8) is actually expanded in powers of $\kappa\hat{h}$ and when $\kappa\hat{h} \ll 1$ we end up with the linearized expansion of the action, i.e. the form (6.7).

Under the perturbation (6.6), we can find the new transformation rule for the metric by using (6.2). All the gauge symmetry must be put into $h_{\mu\nu}$ in this case and so the transformation rule is

$$h_{\mu\nu}(x) \to \frac{\partial f^\alpha}{\partial x^\mu}\frac{\partial f^\beta}{\partial x^\nu}\eta_{\alpha\beta} + \frac{\partial f^\alpha}{\partial x^\mu}\frac{\partial f^\beta}{\partial x^\nu}h_{\alpha\beta}(f(x)) - \eta_{\mu\nu}. \tag{6.9}$$

After following the similar procedure of infinitesimal changes and applying Taylor expansion, we obtain

$$\delta h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu + \mathcal{L}_\xi h_{\mu\nu}. \tag{6.10}$$

Note that, this transformation rule is different than the linear form (2.2) by a Lie derivative and this implies that the gauge symmetry of the Einstein-Hilbert action is modified at higher order contributions of h.

6.1.1 Curved Spaces

In writing (6.8) we said that all the contributions should include two derivatives. However, there could be one more interesting case (which we skipped intentionally to examine it here) that includes no derivatives at all. These terms having zero derivatives sum up to a constant which we call the cosmological constant. GR with a cosmological constant Λ has the action

$$S = \frac{1}{2\kappa^2}\int d^D x\,\sqrt{-g}(R - 2\Lambda), \tag{6.11}$$

whose equations of motion are

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 0. \tag{6.12}$$

These equations of motion imply that the background metric $\bar{g}_{\mu\nu}$ satisfies the Einstein space solutions (5.1), that is

$$\Lambda = \left(\frac{D-2}{2D}\right)\bar{R}, \qquad \bar{R}_{\mu\nu} = \frac{\bar{R}}{D}g_{\mu\nu}. \tag{6.13}$$

By expanding the metric around the background

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}, \tag{6.14}$$

up to second order, we obtain the linearized action on curved spaces

$$S^{(2)} = \frac{1}{4\kappa^2}\int d^D x\sqrt{-\bar{g}}\Big[-\frac{1}{2}\bar{\nabla}_\lambda h_{\mu\nu}\bar{\nabla}^\lambda h^{\mu\nu} + \bar{\nabla}_\mu h_{\nu\lambda}\bar{\nabla}^\nu h^{\mu\lambda} - \bar{\nabla}_\mu h^{\mu\nu}\bar{\nabla}_\nu h + \frac{1}{2}\bar{\nabla}_\lambda h\bar{\nabla}^\lambda h + \frac{\bar{R}}{D}\left(h^{\mu\nu}h_{\mu\nu} - \frac{1}{2}h^2\right)\Big] + (\text{total } d), \tag{6.15}$$

where all the indices are raised and lowered by the background metric $\bar{g}_{\mu\nu}$. Note that, canonical normalization $h_{\mu\nu} = 2\kappa\hat{h}_{\mu\nu}$ reduces this action to (5.2) with having no source and zero mass.

As before, all gauge transformations should be put into $h_{\mu\nu}$ when we expand the metric as in (6.14). The transformation rule in this case is

$$h_{\mu\nu}(x) \to \frac{\partial f^\alpha}{\partial x^\mu}\frac{\partial f^\beta}{\partial x^\nu}\bar{g}_{\alpha\beta}(f(x)) + \frac{\partial f^\alpha}{\partial x^\mu}\frac{\partial f^\beta}{\partial x^\nu}h_{\alpha\beta}(f(x)) - \bar{g}_{\mu\nu}, \tag{6.16}$$

and for infinitesimal transformations $f^\mu = x^\mu + \xi^\mu$, we have

$$\delta h_{\mu\nu} = \bar{\nabla}_\mu\xi_\nu + \bar{\nabla}_\nu\xi_\mu + \mathcal{L}_\xi h_{\mu\nu}. \tag{6.17}$$

In the linear order this reduces to (5.3), the gauge symmetry of the massless curved space.

6.1.2 Spherical Solutions and the Schwarzschild Radius

In this section we will look for the spherically symmetric static solutions to Einstein's field equations (6.5) in $D = 4$. The most general spherically symmetric, static metric is written in the form

$$g_{\mu\nu}dx^\mu dx^\nu = -B(r)dt^2 + C(r)dr^2 + A(r)r^2 d\Omega^2. \tag{6.18}$$

The radial coordinate r can be reparametrized such that $A(r) = C(r)$ without loosing the generality. In this case metric reduces to

$$g_{\mu\nu}dx^\mu dx^\nu = -B(r)dt^2 + C(r)(dr^2 + r^2 d\Omega^2), \tag{6.19}$$

which in matrix form is

$$\begin{pmatrix} -B(r) & 0 & 0 & 0 \\ 0 & C(r) & 0 & 0 \\ 0 & 0 & r^2C(r) & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta C(r) \end{pmatrix}. \tag{6.20}$$

Given this metric one can work out the equations of motion [1] and at the end finds

$$3r(C')^2 - 4C(2C' + rC'') = 0, \tag{6.21}$$

$$4B'C^2 + 2(2B + rB')C'C + Br(C')^2 = 0, \tag{6.22}$$

from the tt and rr equations respectively[2]. To solve these equations we first note that, the flat space

$$\eta_{\mu\nu}dx^\mu dx^\nu = -dt^2 + dr^2 + r^2 d\Omega^2, \tag{6.23}$$

is a known solution of Einstein's field equations and in this case the coefficients are

$$B_0(r) = 1, \quad C_0(r) = 1. \tag{6.24}$$

Now, by using our knowledge of solving differential equations we can deduce that exact solutions to $B(r)$ and $C(r)$ can be obtained by linear expansions around these flat space solutions. To count the order of non-linearity we use a parameter ϵ and introduce the expansions

$$\begin{aligned} B(r) &= B_0(r) + \epsilon B_1(r) + \epsilon^2 B_2(r) + \dots, \\ C(r) &= C_0(r) + \epsilon C_1(r) + \epsilon^2 C_2(r) + \dots. \end{aligned} \tag{6.25}$$

When we put these solutions into (6.21) and (6.22) and collect the like powers of ϵ's, at order zero we obtain the trivial result $0 = 0$ as expected. Upon continuing expansion, at order ϵ we get

$$C_1'' + \frac{2}{r}C_1' = 0, \quad B_1' + C_1' = 0. \tag{6.26}$$

Solutions to these equations are

$$B_1(r) = \frac{c_1}{r} + c_3, \quad C_1(r) = -\frac{c_1}{r} + c_2, \tag{6.27}$$

[1] In most of the cases it is hard to solve Einstein's field equations by hand. There are some useful computer softwares that can handle this job easily. We have used a Mathematica code mostly aparted from [34] to solve the field equations used in this thesis.

[2] The $\theta\theta$ and $\phi\phi$ equations are the same by spherical symmetry and that single equation obtained from the angular variables is redundant in our case.

which give three arbitrary constants c_1, c_2 and c_3. But we can fix two of them by demanding that our solution goes to zero as $r \to \infty$. By this assumption, in some sense, we want our curved space to be flat when looked from very far away. That makes $c_2 = c_3 = 0$. After that we behave wisely and choose $c_1 = -2GM$ so that the solution obtained before in (3.32) can be safely recovered at the final stage. With that choice we get

$$B_1 = -\frac{2GM}{r}, \quad C_1 = \frac{2GM}{r}. \tag{6.28}$$

At order ϵ^2, we again have a set of differential equations

$$\frac{3G^2M^2}{r^4} - \frac{2C_2'}{r} - C_2'' = 0, \quad \frac{7G^2M^2}{r^3} + B_2' + C_2' = 0, \tag{6.29}$$

whose solutions are

$$B_2 = \frac{2G^2M^2}{r^2} + \frac{c_1}{r} + c_3, \quad C_2 = \frac{3G^2M^2}{2r^2} - \frac{c_1}{r} + c_2. \tag{6.30}$$

Once more, we demand solutions to vanish at $r \to \infty$ which makes $c_2 = c_3 = 0$. However, this time we also set $c_1 = 0$ so that the term proportional to $1/r^2$ does not compete with the c_1/r term when $r \to \infty$. Therefore we have

$$B_2 = \frac{2G^2M^2}{r^2}, \quad C_2 = \frac{3G^2M^2}{2r^2}. \tag{6.31}$$

Upon expanding all the terms in a similar fashion, we finally obtain

$$B(r) = 1 - \frac{2GM}{r}\left(1 - \frac{GM}{r} + \dots\right), \tag{6.32}$$

$$C(r) = 1 + \frac{2GM}{r}\left(1 + \frac{3GM}{4r} + \dots\right). \tag{6.33}$$

Note that all the non-linear terms are proportional to the factor $\frac{2GM}{r}$ where the numerator can be identified as the Schwarzschild radius,

$$r_S = 2GM, \tag{6.34}$$

and represents the radius at which non-linearities of GR become important. Non-linearities of the theory should be taken into consideration for the cases $r < r_S$. In all other regions starting from r_S going up to infinity linearized theory is valid. The Schwarzschild radius r_S can be written proportional to M/M_P^2 and for the choice $M = M_{\text{Sun}}$ we have $r_S \sim 1$ km. That is, linearized Einstein gravity is good for almost everywhere in the solar system.

6.2 Non-linearities in Massive Gravity

In this section, we will first write a non-linear action describing a massive graviton and then look for the spherical solutions to the theory. Our main purpose is to obtain a full theory of massive gravity. Such a full theory should be non-linear but also should give us the Fierz-Pauli theory (2.1) when linearly expanded around some background metric.

In general relativity, because of the gauge invariance, Einstein gravity is the unique theory at the lowest order in the curvature. But in the non-linear extension of massive gravity there is no obvious symmetry to preserve, so any interaction terms are allowed and the theory is not unique. At the end what we need is a non-linear theory that reproduces the predictions of GR in the solar system scale but also includes the Fierz-Pauli mass term in it. That is to say, we want an action that consists of both $\sqrt{-g}R$ and $m^2(h_{\mu\nu}h^{\mu\nu} - h^2)$ at the same time. But, this now seems a little bit easy since the GR Lagrangian $\sqrt{-g}R$ is already non-linear. Hence, the first extension comes to mind in this context is to add Fierz-Pauli mass term to the full non-linear GR action,

$$S = \frac{1}{2\kappa^2}\int d^Dx\left[(\sqrt{-g}R) - \sqrt{-\bar{g}}\frac{1}{4}m^2\bar{g}^{\mu\alpha}\bar{g}^{\nu\beta}\left(h_{\mu\nu}h_{\alpha\beta} - h_{\mu\alpha}h_{\nu\beta}\right)\right]. \tag{6.35}$$

The important thing about the action (6.35) is that it has two metrics in it. Being different from GR, a fixed metric $\bar{g}_{\mu\nu}$ (called the absolute metric), on which the linear massive graviton propagates, is now explicitly seen in the action. Presence of this metric breaks the diffeomorphism symmetry of general relativity but it is also needed to be able to introduce a Fierz-Pauli type mass term. The indices on $h_{\mu\nu}$ are contracted with the absolute metric and hence it is required to use a non-dynamical metric like that to allow traces and contractions in the theory. Now, by varying this action with respect to $g_{\mu\nu}$, we obtain the equations of motion

$$\sqrt{-g}\left(R^{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right) + \sqrt{-\bar{g}}\frac{1}{2}m^2\left(\bar{g}^{\mu\alpha}\bar{g}^{\nu\beta}h_{\alpha\beta} - \bar{g}^{\alpha\beta}h_{\alpha\beta}\bar{g}^{\mu\nu}\right) = 0. \tag{6.36}$$

Here, indices on $R_{\mu\nu}$ are contracted with the full metric $g_{\mu\nu}$ whereas those on $h_{\mu\nu}$ with the absolute metric. Note that, if the absolute metric satisfies Einstein's field equations (6.5), then $g_{\mu\nu} = \bar{g}_{\mu\nu}$ (which makes $h_{\mu\nu} = 0$) is a solution. In most of the circumstances, this will be the case. But, sometimes, the absolute metric that explicitly breaks the diffeomorphism symmetry of non-linear GR may not be equal to

the background metric which is the solution of full non-linear equations, about which we expand the action. Namely, for certain occasions, we may have two different background structures in the massive non-linear theory. This would be the case, for example, if we were expanding the action around a black hole solution. In such a theory, two different background structures, the background metric of the black hole and the absolute metric, would appear together.

6.2.1 Spherical Solutions and the Vainshtein Radius

In section 6.1.2 we found the static spherical solutions for Einstein's general relativity. Now, we attempt the same thing for the massive gravity theory (6.35). We first choose the absolute metric $\bar{g}_{\mu\nu}$ to be the flat Minkowski,

$$\bar{g}_{\mu\nu}dx^\mu dx^\nu = -dt^2 + dr^2 + r^2 d\Omega^2, \tag{6.37}$$

and then for the dynamical metric $g_{\mu\nu}$ we consider the same spherically symmetric static ansatz we used before

$$g_{\mu\nu}dx^\mu dx^\nu = -B(r)dt^2 + C(r)dr^2 + A(r)r^2 d\Omega^2. \tag{6.38}$$

When these two ansatzs are plugged into (6.36), one obtains

$$\begin{aligned}
&4BC^2m^2r^2A^3 + \left(2B(C-3)C^2m^2r^2 - 4\sqrt{A^2BC}(C - rC')\right)A^2 \\
&+ 2\sqrt{A^2BC}\left(2C^2 - 2r(3A' + rA'')C + r^2A'C'\right)A + C\sqrt{A^2BC}r^2(A')^2 = 0,
\end{aligned} \tag{6.39}$$

$$\begin{aligned}
&\frac{4(B + rB')A^2 + (2r^2A'B' - 4B(C - rA'))A + Br^2(A')^2}{A^2BC^2r^2} \\
&- \frac{2(2A + B - 3)m^2}{\sqrt{A^2BC}} = 0,
\end{aligned} \tag{6.40}$$

$$\begin{aligned}
&-2B^2C^2m^2rA^4 - 2B^2C^2(B + C - 3)m^2rA^3 - B^2C\sqrt{A^2BC}r(A')^2 \\
&- \sqrt{A^2BC}\left(2C'B^2 + (rB'C' - 2C(B' + rB''))B + Cr(B')^2\right)A^2 \\
&+ B\sqrt{A^2BC}\left(CrA'B' + B(4CA' - rC'A' + 2CrA'')\right)A = 0,
\end{aligned} \tag{6.41}$$

from tt, rr and $\theta\theta$ equations respectively. Note that, in this massive case there is no redundancy in the equations. That is because diffeomorphism invariance of GR is broken in massive gravity and we cannot apply any coordinate gauge transformations.

When we apply the same perturbation method used in the case of general relativity and expand these equations around the flat space solution

$$B_0(r) = 1, \quad C_0(r) = 1, \quad A_0(r) = 1, \tag{6.42}$$

by using the expansion

$$\begin{aligned} B(r) &= B_0(r) + \epsilon B_1(r) + \epsilon^2 B_2(r) + \dots, \\ C(r) &= C_0(r) + \epsilon C_1(r) + \epsilon^2 C_2(r) + \dots, \\ A(r) &= A_0(r) + \epsilon A_1(r) + \epsilon^2 A_2(r) + \dots, \end{aligned} \tag{6.43}$$

at order zero we again obtain the trivial result $0 = 0$. Upon proceeding to collect like terms in higher orders, at order ϵ we get

$$2(m^2r^2 - 1)A_1 + (m^2r^2 + 2)C_1 + 2r(-3A_1' + C_1' - rA_1'') = 0, \tag{6.44}$$

$$-\frac{1}{2}B_1m^2 + \left(\frac{1}{r^2} - m^2\right)A_1 + \frac{r(A_1' + B_1') - C_1}{r^2} = 0, \tag{6.45}$$

$$rA_1m^2 + rB_1m^2 + rC_1m^2 - 2A_1' - B_1' + C_1' - rA_1'' - rB_1'' = 0. \tag{6.46}$$

To solve these equations, we first solve them simultaneously for A_1, A_1' and A_1'' in terms of B_1, C_1 and their derivatives. We should end up with something like

$$\begin{aligned} A_1 &= f\left(B_1, B_1', B_1'', C_1, C_1'\right), \\ A_1' &= g\left(B_1, B_1', B_1'', C_1, C_1'\right), \\ A_1'' &= h\left(B_1, B_1', B_1'', C_1, C_1'\right), \end{aligned} \tag{6.47}$$

which implies

$$\begin{aligned} f'\left(B_1, B_1', B_1'', C_1, C_1'\right) &= g\left(B_1, B_1', B_1'', C_1, C_1'\right), \\ g'\left(B_1, B_1', B_1'', C_1, C_1'\right) &= h\left(B_1, B_1', B_1'', C_1, C_1'\right). \end{aligned} \tag{6.48}$$

These two equations can be solved for C_1 and C_1' in terms of B_1 and its derivatives,

$$C_1 = u\left(B_1, B_1', B_1''\right), \quad C_1' = v\left(B_1, B_1', B_1''\right), \tag{6.49}$$

and finally $u' = v$ implies a second order differential equation for B_1,

$$-3rB_1m^2 + 6B_1' + 3rB_1'' = 0, \tag{6.50}$$

whose solution is

$$B_1(r) = c_1 \frac{e^{-mr}}{r} + c_2 \frac{e^{mr}}{2mr}. \tag{6.51}$$

By demanding $B_1(r) \to 0$ when $r \to \infty$ we are forced to choose $c_2 = 0$ and as usual we fix c_1 in such a way that compatibility with Chapter 3 is maintained, namely the solutions agree with 3.21. Now, by proceeding backwards we can similarly find C_1 and A_1 and what we obtain at the end is

$$\begin{aligned} B_1 &= -\frac{8GM}{3}\frac{e^{-mr}}{r}, \\ C_1 &= -\frac{8GM}{3}\frac{e^{-mr}}{r}\frac{1+mr}{m^2r^2}, \\ A_1 &= \frac{4GM}{3}\frac{e^{-mr}}{r}\frac{1+mr+m^2r^2}{m^2r^2}. \end{aligned} \tag{6.52}$$

Following the similar procedure, at order ϵ^2 we obtain B_2, C_2, A_2 and for the final solution, when $mr \ll 1$, we get

$$B(r) = 1 - \frac{8GM}{3r}\left(1 - \frac{1}{6}\frac{GM}{m^4r^5} + \dots\right), \tag{6.53}$$

$$C(r) = 1 - \frac{8GM}{3m^2r^3}\left(1 - 14\frac{GM}{m^4r^5} + \dots\right), \tag{6.54}$$

$$A(r) = 1 + \frac{GM}{3\pi m^2r^3}\left(1 - 4\frac{GM}{m^4r^5} + \dots\right). \tag{6.55}$$

Note that in this case non-linearities depend on the parameter r_V/r where

$$r_V = \left(\frac{GM}{m^4}\right)^{1/5}, \tag{6.56}$$

is known as the Vainshtein radius [17]. In the regions where $r > r_V$, predictions of the linear theory are reliable but when $r < r_V$ non-linear effects start to become important and we need to abandon the linear theory. However, this now uncovers an interesting feature: when the graviton mass m becomes smaller, the Vainshtein radius gets larger. If we try to take $m \to 0$ limit, we see that r_V goes to infinity. Thus, the linear theory loses its credibility in the zero mass limit and if we want to take a smooth limit, we need to use the non-linear theory. So, there is now hope to think that non-linear massive gravity theory may solve the smooth $m \to 0$ limit problem and hence the VDVZ discontinuity might be cured in the non-linear regime. To have some numbers in our mind, if we take $M = M_{\text{Sun}}$ and m to be a very small number at the order of the Hubble constant, $m \sim 10^{-33}$ eV, then we find $r_V \sim 10^{18}$ km which is

about the size of the Milky Way! That means, if we want to understand what is going on inside our galaxy, we need to use the non-linear theory. The linear theory is only reliable after walking away a distance of 10^{18} km from the source.

6.3 Hamiltonian Analysis

In this section, we study the Hamiltonian of the non-linear massive gravity action (6.35) in $D = 4$ by using the Arnowitt-Deser-Misner (ADM) formalism [35, 36]. We first choose the background metric to be the flat Minkowski, i.e. $\bar{g}_{\mu\nu} = \eta_{\mu\nu}$, so that the action reduces to

$$S = \frac{1}{2\kappa^2} \int d^4x \left[(\sqrt{-g}R) - \frac{1}{4} m^2 \eta^{\mu\alpha} \eta^{\nu\beta} \left(h_{\mu\nu} h_{\alpha\beta} - h_{\mu\alpha} h_{\nu\beta} \right) \right]. \tag{6.57}$$

What we have investigated in Chapter 2 was the linearized version of this action, namely we expanded the term $\sqrt{-g}R$ up to second order in the metric perturbation and analyzed the resulting action. We saw that in $D = 4$ that theory carries five degrees of freedom. In the Hamiltonian analysis we found out that the time-like component h_{00} appeared as a Lagrange multiplier in the action and so number of degrees of freedom is reduced from six to the correct value five. However, for the non-linear action (6.57), we will see that there will be no constraints in the theory and hence we will end up with an extra ghostlike degree of freedom.

6.3.1 ADM Variables

In the context of the ADM formalism, we start by foliating the four dimensional space-time into a series of spacelike hypersurfaces Σ_t. The subscript indicates Σ_t is a hypersurface of a constant parameter t which corresponds to the time in our theory. So roughly what we do is to slice four dimensional space-time into three dimensional hypersurfaces flowing in the time coordinate. Next, on each of these hypersurfaces we define unit normal vectors n_i which are necessarily timelike, i.e. $n_i n^i = -1$ [3]. With the help of these normal vectors we can define a purely spatial three-metric ${}^{(3)}g_{ij}$

[3] Note that we use the mostly plus metric convention $(-+++\ldots)$ in this thesis and hence a timelike object should have a minus norm. In some other resources metric convention may be taken to be mostly minus and so the norm of a timelike unit normal vector n_i might be shown with the product $n_i n^i = 1$.

on each hypersurface by

$$^{(3)}g_{ij} = g_{ij} + n_i n_j, \tag{6.58}$$

where g_{ij} is the spatial components of the generic metric $g_{\mu\nu}$. One can immediately see how the definitions we made so far is consistent with each other by checking that $^{(3)}g_{ij}n^i = 0$. Hence n_i being timelike forces $^{(3)}g_{ij}$ to be spacelike as we desired.

A really adorable question might be asked at that point: How can we write the proper length ds between two arbitrary points on two different hypersurfaces by using that newly defined purely spatial three-metrics $^{(3)}g_{ij}$ of the hypersurfaces?

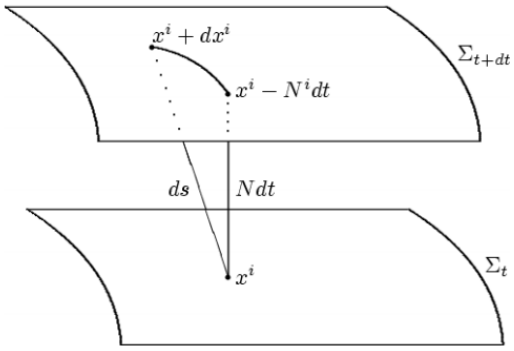

Figure 6.1: Proper length ds is calculated from the Lorentzian version of Pythagorean theorem in terms of the variables $^{(3)}g_{ij}$, N and N^i. Here $^{(3)}g_{ij}$ is the purely spatial three-metric defined on the hypersurfaces Σ_t of constant t, N is the lapse function measuring the rate of flow of proper time with respect to t and N^i is the shift vector responsible from the distance shifted away from the point x^i (The figure is taken from [37]).

To answer this question we first choose a point represented by x^i on a hypersurface Σ_t and another point $x^i + dx^i$ living on the hypersurface Σ_{t+dt} (see Figure 6.1). Since the point $x^i + dx^i$ is chosen arbitrarily, in general case we cannot reach it by starting from the point x^i of the hypersurface Σ_t and moving along the unit normal vector n_i of the same hypersurface. The distance we take by following this path is the proper

time from Σ_t to Σ_{t+dt} and it is shown by Ndt where N is called the lapse function. The lapse function measures the proper time elapsed when a coordinate distance dt is walked. Or in some other words, it represents the rate of flow of proper time with respect to t.

We said we reach a point different than $x^i + dx^i$ when we start from x^i and move along the unit normal vector n_i. Thus, the point we reach is shifted from the point x^i and that enables us to define a new object called the shift vector N^i measuring the tangential movement in the hypersurface Σ_{t+dt} with respect to the hypersurface Σ_t.

By using the Lorentzian version of the Pythagorean theorem we can now write the proper length ds in terms of these new variables known as ADM variables,

$$\begin{aligned} ds^2 &= -\,(\text{proper time})^2 + (\text{coordinate distance})^2 \\ &= -\,(Ndt)^2 + {}^{(3)}g_{ij}(dx^i + N^i dt)(dx^j + N^j dt). \end{aligned} \tag{6.59}$$

Note that if we have defined the proper length in terms of the symmetric generic metric $g_{\mu\nu}$ only, we would end up with ten quantities (remember that we are studying in four dimensions). In the present case, we have the symmetric spatial three-metric ${}^{(3)}g_{ij}$ which has six entries, the shift vector N^i which has three entries and lastly the lapse function N which has only one entry summing to a total of ten entries again. After checking the consistency of our theory once again in that way, we proceed by writing the components of the four dimensional generic space-time metric in terms of the ADM variables by using (6.59),

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0j} \\ g_{i0} & g_{ij} \end{pmatrix} = \begin{pmatrix} -N^2 + {}^{(3)}g^{ij}N_iN_j & N_j \\ N_i & {}^{(3)}g_{ij} \end{pmatrix}. \tag{6.60}$$

The inverse metric is also easily calculated from that to be

$$g^{\mu\nu} = \frac{1}{N^2}\begin{pmatrix} -1 & N^j \\ N^i & {}^{(3)}g^{ij}N^2 - N^iN^j \end{pmatrix}. \tag{6.61}$$

6.3.2 Boulware-Deser Ghost

Since we have the metric and its inverse now, the next step should be rewriting the action (6.57) in terms of the metric components. In Einstein-Hilbert part of the action

we have the metric determinant $\sqrt{-g}$ which easily can be rewritten in terms of the ADM variables as

$$\sqrt{-g} = N\sqrt{^{(3)}g}. \tag{6.62}$$

The second term in Einstein-Hilbert part of the action is the four dimensional Ricci scalar R which consists of the second derivatives of the metric tensor $g_{\mu\nu}$. Upon remembering how we defined the lapse function N, one realizes that when written in terms of the ADM variables R should include some terms proportional to the covariant derivative of the unit normal vectors n_i. This quantity expressing the rate of change of the unit normal vectors to hypersurfaces is known as the extrinsic curvature and defined to be

$$K_{ij} = {^{(3)}g_i}^k \nabla_k n_j. \tag{6.63}$$

Note that, this tensor K_{ij} is symmetric $(K_{ij} = K_{ji})$ and also it is purely spatial $(K_{ij}n^j = 0)$. The second part of the extrinsic curvature, $\nabla_k n_j$, carries the information about the curvature due to the embedding of the surface and the first part, ${^{(3)}g_i}^k$, projects that information onto the three dimensional hypersurfaces. Hence the tensor K_{ij} is roughly representing the acceleration of the surface it is defined on. In terms of the ADM variables we can rewrite it as

$$K_{ij} = \frac{1}{2N}\left({^{(3)}\dot{g}_{ij}} - {^{(3)}\nabla_i N_j} - {^{(3)}\nabla_j N_i}\right), \tag{6.64}$$

where ${^{(3)}\nabla_i}$ is the spatial covariant derivative being the three dimensional counterpart of the space-time covariant derivative ∇. With some effort now (see [38, 39]), we can also write the Ricci scalar as

$$R = {^{(3)}R} + K_{ij}K^{ij} - K^2, \tag{6.65}$$

where ${^{(3)}R}$ is the Ricci scalar of the spatial three-metric ${^{(3)}g_{ij}}$. Finally, combining (6.62) and (6.65) gives the Einstein-Hilbert action written in terms of the ADM variables,

$$S_{EH} = \frac{1}{2\kappa^2}\int d^4x\sqrt{^{(3)}g}N\left[{^{(3)}R} + K_{ij}K^{ij} - K^2\right]. \tag{6.66}$$

The canonical momenta belonging to the action is

$${^{(3)}\pi^{ij}} = \frac{\partial L}{\partial\, {^{(3)}\dot{g}_{ij}}} = \frac{1}{2\kappa^2}\sqrt{^{(3)}g}\left(K^{ij} - {^{(3)}g^{ij}}K\right). \tag{6.67}$$

By inverting this expression for ${}^{(3)}\dot{g}_{ij}$ and plugging the result into (6.64), we obtain the extrinsic curvature in terms of the Hamiltonian variables,

$$K_{ij} = \frac{2\kappa^2}{\sqrt{{}^{(3)}g}} \left({}^{(3)}\pi_{ij} - \frac{1}{2} {}^{(3)}\pi {}^{(3)}g_{ij} \right), \tag{6.68}$$

and then Hamiltonian itself is found to be

$$H = \left(\int_{\Sigma_t} d^3x \, {}^{(3)}\pi^{ij} \, {}^{(3)}\dot{g}_{ij} \right) - L = \int_{\Sigma_t} d^3x \left(N\mathcal{C} + N_i \mathcal{C}^i \right), \tag{6.69}$$

where the quantities $\mathcal{C}$ and $\mathcal{C}_i$ are

$$\mathcal{C} = \sqrt{{}^{(3)}g} \left[{}^{(3)}R + K_{ij}K^{ij} - K^2 \right], \quad \mathcal{C}_i = \sqrt{{}^{(3)}g} \, {}^{(3)}\nabla_j \left(K^{ij} - {}^{(3)}g^{ij} K \right). \tag{6.70}$$

Therefore, without adding the Fierz-Pauli mass term we see that both lapse and shift appear as Lagrange multipliers in the theory and they enforce the constraints $\mathcal{C} = 0$ and $\mathcal{C}^i = 0$. When checked it is seen that $\{C, C^i\} = 0$ and hence these are first class constraints generating 4 diffeomorphism symmetries. Thus, the 12 dimensional phase space degrees of freedom are reduced by 4 constraints and further reduced by 4 gauge symmetries leaving us a total 4 degrees of freedom at the end. Half of them belong to the conjugate momenta and other half are counted as real space degrees of freedom. Hence, for this massless non-linear case we have two degrees of freedom as in the massless linear case. This says nothing but that: The linearized Einstein gravity has the same number of degrees of freedom as the full non-linear Einstein gravity. Of course, this is what we expect from a consistent physical theory. However, we will see in a minute, when the mass term is added to the non-linear theory things are going to change dramatically.

Up to that point we have only dealed with the first part of the action (6.57), namely with the Einstein-Hilbert part. Now we will consider the second part, the Fierz-Pauli mass term, and we will follow the same steps. Keeping in mind that $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$, in ADM variables the mass term is rewritten as

$$\begin{aligned} \eta^{\mu\alpha}\eta^{\nu\beta} \left(h_{\mu\nu}h_{\alpha\beta} - h_{\mu\alpha}h_{\nu\beta} \right) = \eta^{ik}\eta^{jl} \left(h_{ij}h_{kl} - h_{ik}h_{jl} \right) + 2\eta^{ij}h_{ij} \\ - 2N^2\eta^{ij}h_{ij} + 2N_i \left({}^{(3)}g^{ij} - \eta^{ij} \right) N_j, \end{aligned} \tag{6.71}$$

and combining this with the Einstein-Hilbert part of the action gives

$$S = \frac{1}{2\kappa^2}\int d^4x \Big\{ {}^{(3)}\pi^{ij}\,{}^{(3)}\dot{g}_{ij} - N\mathcal{C} + N_i\mathcal{C}^i$$
$$- \frac{1}{4}m^2\left[\eta^{ik}\eta^{jl}\left(h_{ij}h_{kl} - h_{ik}h_{jl}\right) + 2\eta^{ij}h_{ij} - 2N^2\eta^{ij}h_{ij} + 2N_i\left({}^{(3)}g^{ij} - \eta^{ij}\right)N_j\right]\Big\}. \tag{6.72}$$

Here, it is immediately seen that the lapse function N and the shift vector N^i are no longer Lagrange multipliers since they appear quadratically. There are no time derivatives acting on them, so they are not propagating. Thus, they are auxiliary variables and can be solved from their own equations of motion. What we find after this operation is

$$N = \frac{\mathcal{C}}{m^2\eta^{ij}h_{ij}}, \qquad N_i = \frac{\mathcal{C}^j}{m^2\left({}^{(3)}g^{ij} - \eta^{ij}\right)}, \tag{6.73}$$

and by using these back in (6.72) we obtain an action with no constraints or gauge symmetries at all,

$$S = \frac{1}{2\kappa^2}\int d^4x\,\left({}^{(3)}\pi^{ij}\,{}^{(3)}\dot{g}_{ij} - H\right). \tag{6.74}$$

The Hamiltonian in this case is

$$H = \frac{1}{2\kappa^2}\int d^3x \Big\{ \frac{1}{2m^2}\frac{\mathcal{C}^2}{\eta^{ij}h_{ij}} + \frac{1}{2m^2}\frac{\mathcal{C}^i\mathcal{C}^j}{\left({}^{(3)}g^{ij} - \eta^{ij}\right)}$$
$$+ \frac{1}{4}m^2\left[\eta^{ik}\eta^{jl}\left(h_{ij}h_{kl} - h_{ik}h_{jl}\right) + 2\eta^{ij}h_{ij}\right]\Big\}, \tag{6.75}$$

which is non-vanishing, unlike in GR. Since there are no constraints and gauge freedoms in this theory all 12 phase space degrees of freedom are active and that corresponds to 6 real space degrees of freedom. But, what happened just now? That was something we did not expect. From the beginning of this thesis what we always say is "a massive spin-2 particle should has 5 degrees of freedom." Here, Hamiltonian analysis of the non-linear massive gravity theory gives us one extra degree of freedom. Ok, we can save it for the dowry of our daughter if we want but since we are some serious guys studying physics we first want to know what it actually is. When the propagator analysis of the theory is done this additional degree of freedom is found to be a ghost (a particle having negative kinetic energy which, of course, is unphysical) called the Boulware-Deser ghost [18].

CHAPTER 7

CONCLUSION

In this thesis, basics of massive spin-2 theories are examined in details. The necessities to modify Einstein's theory of general relativity are introduced in the first chapter. The reasons for doing massive gravity are also given in the same chapter.

In the second chapter, we first introduced the unique linear massive gravity action known as the Fierz-Pauli action and talked about the properties it has. We found the equations of motion belong to that action and counted the degrees of freedom of the theory which at the end found to be the correct number, five, for a massive spin-2 particle. We also checked our results by doing the Hamiltonian analysis of the theory and by counting the degrees of freedom once more from there. When the propagator analysis of the theory is done for both massless and massive cases, we have seen the first sign of a possible discontinuity in $m \to 0$ limit of the Fierz-Pauli theory.

In the third chapter, we investigated the response of massive graviton to a source term coupled to it. We first found the field solution under the addition of a general source term and then we looked at the same solution for a specifically chosen point source. When we wrote the field components in spherical coordinates and compared the observables such as the Newtonian potentials and the light bending angles in both massive and massless cases, we have found a discontinuity in $m \to 0$ limit. That discontinuity stating that the theory we obtain from massive one by taking zero mass limit is not equivalent to the theory we obtain in the zero mass case is called the vDVZ discontinuity.

In the fourth chapter, we used Stueckelberg formalism to search for the possible rea-

son causing the vDVZ discontinuity. We first studied the massive photon case and review the technique by introducing a new field and its associated gauge symmetry. When $m \to 0$ limit is taken after these new definitions, it is seen that the degrees of freedom are preserved in the limit. In the following section of the same chapter, application of Stueckelberg's technique to the massive graviton case revealed the origin of the vDVZ discontinuity. It is shown that even in the massless limit a scalar particle is coupled to the source and it behaves as an additional attractive force which at the end causes the mismatch between the Newtonian potentials of massive and massless gravity.

In the fifth chapter, we put the massive gravitons onto curved spaces by changing the flat metric to a more generic one and partial derivatives to covariant derivatives. After applying the Stueckelberg trick to the linearized Einstein-Hilbert action with a cosmological constant, we have seen that the coupling between the Stueckelberg field and the source disappeared and hence the vDVZ discontinuity is removed from the theory. Another interesting feature we faced on curved spaces was the existence of partially massless theories in which the number of degrees of freedom of the massive theory is decreased by one when a special value of the curvature term is used in the action.

In the last chapter of this thesis, non-linear interactions are examined. We first studied the non-linearities in the theory of general relativity and showed that we should worry about them for the scales smaller than the Scwarzschild radius r_S that is found to be about 1 km for the solar system. Later, we moved to the non-linear massive gravity action by choosing the non-linearities to be those of GR and leaving the mass term unchanged. The spherical solutions in this case suggested another scale called the Vainshtein radius at which non-linearities of massive gravity starts to become important. We have seen that whenever $m \to 0$ limit is attempted to taken, Vainshtein radius goes to infinity and forces us to use the non-linear theory. Therefore, it is claimed by Vainshtein that a complete non-linear massive gravity theory may cure the vDVZ discontinuity. However, when Boulware and Deser studied the canonical Hamiltonian formalism for the non-linear massive gravity theory the existence of an extra unphysical degree of freedom, the Boulware-Deser ghost, is found.

Boulware-Deser ghost remained as a puzzle for about forty years until de Rham, Gabadadze and Tolley (dRGT) came up with a new theory in 2010 [40]. In their theory, they constructed a new potential instead of the linear Fierz-Pauli mass term and in that potential the coefficients are tuned order by order to avoid the Boulware-Deser ghost. However, although the ghost degree of freedom is lost, the dRGT theory has some other problems (such as the superluminal propagation around nontrivial backgrounds) that are need to be investigated in details. Thus, after all the studies taken place in about eighty years, today we still do not have a complete massive gravity theory at hand. As we said in the introduction part, that is both a curse and a beauty of the theory of general relativity. As it happens all the time in life, if you change something beautiful there is always the possibility that you can never go back. That is what we have shown in this thesis. General relativity is such a perfect theory that even giving a small mass to the graviton messes up everything. But, after all, it is the beauty of physics that makes us keep working as it is the beauty of life that makes us keep living.

REFERENCES

[1] I. Newton, "The Principia," translated by I. Bernard Cohen and Anne Whitman, University of California Press, Berkeley, CA, (1999).

[2] H. Cavendish, “Experiments to determine the density of the earth,” Philos. Trans. R. Soc., **88**, 469, (1778).

[3] M. Grosser, “The Discovery of Neptun,” Harvard Uni. Press, Cambridge, (1962).

[4] A. S. Goldhaber and M. M. Nieto, "Photon and Graviton Mass Limits," Rev. Mod. Phys., **82**, 939, (2010).

[5] A. Einstein, "The Foundation of the General Theory of Relativity," Annalen Phys., **49**, 769, (1916).

[6] J. A. Wheeler, K. Ford, "Geons, Black Holes, and Quantum Foam: A Life in Physics," W. W. Norton & Company, (2000).

[7] L. Abbott, "The Mystery of the Cosmological Constant," Sci.Am., **258**, 106, (1988).

[8] S. Weinberg, "The Cosmological Constant Problem," Rev. Mod. Phys., **61**, 1, (1989).

[9] M. S. Berman, "Introduction to General Relativity and The Cosmological Constant Problem," Nova Science Publishers. Inc, (2007).

[10] A. Einstein, "Cosmological Considerations in the General Theory of Relativity," Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.), **1917**, 142, (1917).

[11] Friedmann, A., "On the curvature of space," Z. Phys., **10**, 377, (1922).

[12] E. Hubble, "A relation between distance and radial velocity among extra-galactic nebulae," Proc. Nat. Acad. Sci. **15**, 168, (1929).

[13] M. Fierz and W. Pauli, "On relativistic wave equations for particles of arbitrary spin in an electromagnetic field," Proc. Roy. Soc. Lond., **A173**, 211, (1939).

[14] H. van Dam, M. Veltman, "Massive and mass-less Yang-Mills and gravitational fields," Nucl. Phys., **B22**, 397, (1970).

[15] V. I. Zakharov, "Linearized gravitation theory and the graviton mass," JETP Letters (Sov. Phys.), **12**, 312, (1970).

[16] E.C.G. Stueckelberg, "Theory of the radiation of photons of small arbitrary mass," Helv.Phys.Acta, **30**, 209, (1957).

[17] A. I. Vainshtein, "To the problem of nonvanishing gravitation mass," Phys. Lett., **B39**, 393, (1972).

[18] D. G. Boulware, S. Deser, "Can gravitation have a finite range?," Phys.Rev., **D6**, 3368, (1972).

[19] K. Hinterbichler, "Theoretical Aspects of Massive Gravity," arXiv: 1105.3735[hep-th], (2011).

[20] C. de Rham, "Massive Gravity," Living Rev. Rel., **17**, 7, (2014).

[21] M. Crisostomi, "Massive Gravity," Unpublished doctoral thesis, University of L'Aquila, Paris, France, (2014).

[22] P. A. M. Dirac, "Generalized Hamiltonian Dynamics," Proc. R. Soc. Lond. A, **246**, 326, (1958).

[23] A. Das, "Lectures on Quantum Field Theory," World Scientific Publishing, (2008).

[24] A. Zee, "Quantum Field Theory in a Nutshell," Princeton University Press, (2010).

[25] S. Weinberg, "High-Energy Behavior in Quantum Field Theory," Phys. Rev., **118**, 838, (1960).

[26] S. Weinberg, "The Quantum theory of fields. Vol. 1: Foundations," Cambridge, UK: Univ. Press, (1995).

[27] S. Carroll, "Spacetime and Geometry," Addison-Wesley, (2004).

[28] A. Higuchi, "Forbidden mass range for spin-2 field theory in de sitter space time," Nucl. Phys., **B282**, 397, (1987).

[29] S. Deser and A. Waldron, "Partial masslessness of higher spins in (A)dS," Nucl. Phys., **B607**, 577, (2001).

[30] S. Deser and A. Waldron, "Null propagation of partially massless higher spins in (A)dS and cosmological constant speculations," Phys. Lett., **B513**, 137, (2001).

[31] M. Porrati, "No van Dam-Veltman-Zakharov discontinuity in AdS space," Phys. Lett., **B498**, 92, (2001).

[32] A. Karch, E. Katz, and L. Randall, "Absence of a VVDZ discontinuity in AdS(AdS)," JHEP, **12**, 016, (2001).

[33] I. I. Kogan, S. Mouslopoulos, and A. Papazoglou, "The massless limit for massive graviton in dS(4) and AdS(4): How to circumvent the van Dam-Veltman-Zakharov discontinuity," Phys. Lett., **B503**, 173, (2001).

[34] G. Baumann, "Mathematica for Theoretical Physics: Electrodynamics, Quantum Mechanics, General Relativity, and Fractals," Springer, (2005).

[35] R. L. Arnowitt, S. Deser, and C. W. Misner, “Canonical variables for general relativity,” Phys. Rev., **117**, 1595, (1960).

[36] R. L. Arnowitt, S. Deser, and C. W. Misner, “The dynamics of general relativity,” arXiv:gr-qc/0405109.

[37] C. S. Rampf, "(Non-)Relativistic Lagrangian Perturbation Theory," Unpublished doctoral thesis, RWTH Aachen University, Germany, (2013).

[38] E. Poisson, "A Relativist’s Toolkit: The Mathematics of Black-Hole Mechanics," Cambridge University Press, (2004).

[39] E. Gourgoulhon, “3+1 formalism and bases of numerical relativity,” gr-qc/0703035 [GR-QC], (2007).

[40] C. de Rham et al., "Resummation of Massive Gravity," arXiv:1011.1232v2 [hep-th], (2010).

[41] İ. Güllü, "Conserved Charges in Gravity Theories," Unpublished masters thesis, METU, (2005).

[42] İ. Güllü, T. Ç. Şişman, B. Tekin, "Unitarity analysis of general Born-Infeld gravity theories," Phys. Rev. D, **82**, 124023, (2010).

APPENDIX A

SPATIAL CANONICAL MOMENTA

We start by dividing the Fierz-Pauli Lagrangian (2.1) into parts. Let

$$\begin{aligned}
\mathscr{L}_1 &= -\frac{1}{2}\partial_\lambda h_{\mu\nu}\partial^\lambda h^{\mu\nu}, \\
\mathscr{L}_2 &= +\partial_\mu h_{\nu\lambda}\partial^\nu h^{\mu\lambda}, \\
\mathscr{L}_3 &= -\partial_\mu h^{\mu\nu}\partial_\nu h, \\
\mathscr{L}_4 &= +\frac{1}{2}\partial_\lambda h\partial^\lambda h, \\
\mathscr{L}_5 &= -\frac{1}{2}m^2(h_{\mu\nu}h^{\mu\nu} - h^2).
\end{aligned} \tag{A.1}$$

Now, by proceeding term by term we will sum over all the indices indicating each index we are working on above the equality sign. The $\downarrow$ indicates that all the indices before that equation are now being lowered.

A.1 First Lagrangian

$$\begin{aligned}
\mathscr{L}_1 &\overset{\lambda}{=} -\frac{1}{2}\partial_0 h_{\mu\nu}\partial^0 h^{\mu\nu} - \frac{1}{2}\partial_k h_{\mu\nu}\partial^k h^{\mu\nu} \\
&\overset{\mu}{=} -\frac{1}{2}\partial_0 h_{0\nu}\partial^0 h^{0\nu} - \frac{1}{2}\partial_0 h_{k\nu}\partial^0 h^{k\nu} - \frac{1}{2}\partial_k h_{0\nu}\partial^k h^{0\nu} - \frac{1}{2}\partial_k h_{j\nu}\partial^k h^{j\nu} \\
&\overset{\nu}{=} -\frac{1}{2}\partial_0 h_{00}\partial^0 h^{00} - \frac{1}{2}\partial_0 h_{0k}\partial^0 h^{0k} - \frac{1}{2}\partial_0 h_{k0}\partial^0 h^{k0} - \frac{1}{2}\partial_0 h_{kj}\partial^0 h^{kj} \\
&\quad -\frac{1}{2}\partial_k h_{00}\partial^k h^{00} - \frac{1}{2}\partial_k h_{0j}\partial^k h^{0j} - \frac{1}{2}\partial_k h_{j0}\partial^k h^{j0} - \frac{1}{2}\partial_k h_{ji}\partial^k h^{ji} \\
&\overset{\downarrow}{=} +\frac{1}{2}\partial_0 h_{00}\partial_0 h_{00} - \frac{1}{2}\partial_0 h_{0k}\partial_0 h_{0k} - \frac{1}{2}\partial_0 h_{k0}\partial_0 h_{k0} + \frac{1}{2}\partial_0 h_{kj}\partial_0 h_{kj} \\
&\quad -\frac{1}{2}\partial_k h_{00}\partial_k h_{00} + \frac{1}{2}\partial_k h_{0j}\partial_k h_{0j} + \frac{1}{2}\partial_k h_{j0}\partial_k h_{j0} - \frac{1}{2}\partial_k h_{ji}\partial_k h_{ji}
\end{aligned}$$

$$
\begin{aligned}
&= +\frac{1}{2}\partial_0 h_{00}\partial_0 h_{00} - \partial_0 h_{0k}\partial_0 h_{0k} \underbrace{+\frac{1}{2}\partial_0 h_{kj}\partial_0 h_{kj}}_{\bigstar} \\
&\quad -\frac{1}{2}\partial_k h_{00}\partial_k h_{00} + \partial_k h_{0j}\partial_k h_{0j} - \frac{1}{2}\partial_k h_{ji}\partial_k h_{ji},
\end{aligned} \tag{A.2}
$$

with corresponding canonical momentum

$$
\pi^{(1)}_{ij} = \frac{\partial \mathscr{L}_1}{\partial \dot{h}_{ij}} = \frac{\partial \bigstar}{\partial \dot{h}_{ij}} = \dot{h}_{ij}. \tag{A.3}
$$

A.2 Second Lagrangian

$$
\begin{aligned}
\mathscr{L}_2 &\overset{\mu}{=} +\partial_0 h_{\nu\lambda}\partial^\nu h^{0\lambda} + \partial_k h_{\nu\lambda}\partial^\nu h^{k\lambda} \\
&\overset{\nu}{=} +\partial_0 h_{0\lambda}\partial^0 h^{0\lambda} + \partial_0 h_{k\lambda}\partial^k h^{0\lambda} + \partial_k h_{0\lambda}\partial^0 h^{k\lambda} + \partial_k h_{j\lambda}\partial^j h^{k\lambda} \\
&\overset{\lambda}{=} +\partial_0 h_{00}\partial^0 h^{00} + \partial_0 h_{0k}\partial^0 h^{0k} + \partial_0 h_{k0}\partial^k h^{00} + \partial_0 h_{kj}\partial^k h^{0j} \\
&\quad +\partial_k h_{00}\partial^0 h^{k0} + \partial_k h_{0j}\partial^0 h^{kj} + \partial_k h_{j0}\partial^j h^{k0} + \partial_k h_{ji}\partial^j h^{ki} \\
&\overset{\downarrow}{=} -\partial_0 h_{00}\partial_0 h_{00} + \partial_0 h_{0k}\partial_0 h_{0k} + \partial_0 h_{k0}\partial_k h_{00} - \partial_0 h_{kj}\partial_k h_{0j} \\
&\quad +\partial_k h_{00}\partial_0 h_{k0} - \partial_k h_{0j}\partial_0 h_{kj} - \partial_k h_{j0}\partial_j h_{k0} + \partial_k h_{ji}\partial_j h_{ki} \\
&= -\partial_0 h_{00}\partial_0 h_{00} + \partial_0 h_{0k}\partial_0 h_{0k} + 2\partial_0 h_{k0}\partial_k h_{00} \underbrace{-2\partial_0 h_{kj}\partial_k h_{0j}}_{\bigstar} \\
&\quad -\partial_k h_{j0}\partial_j h_{k0} + \partial_k h_{ji}\partial_j h_{ki},
\end{aligned} \tag{A.4}
$$

with corresponding canonical momentum

$$
\begin{aligned}
\pi^{(2)}_{ij} &= \frac{\partial \mathscr{L}_2}{\partial \dot{h}_{ij}} = \frac{\partial \bigstar}{\partial \dot{h}_{ij}} = -2\partial_k h_{0j}\frac{1}{2}(\delta_{ki}\delta_{jj} + \delta_{kj}\delta_{ji}) \\
&= -\partial_i h_{0j} - \partial_j h_{0i} \\
&= -2\partial_{(i} h_{j)0}.
\end{aligned} \tag{A.5}
$$

Before proceeding further to the next term let us first work out the trace h of the field $h_{\mu\nu}$. Starting from the definition of the trace, we have

$$
\begin{aligned}
h &= \eta^{\mu\nu} h_{\mu\nu} \\
&= \eta^{00} h_{00} + \eta^{ij} h_{ij} \\
&= -h_{00} + \eta^{ij} h_{ij}.
\end{aligned} \tag{A.6}
$$

By defining $h_{kk} \equiv \eta^{ij} h_{ij}$, we obtain a second form

$$h = -h_{00} + h_{kk}. \tag{A.7}$$

Throughout the coming calculations (A.6) and (A.7) will be used miscellaneously.

A.3 Third Lagrangian

$$\begin{aligned}
\mathscr{L}_3 &\overset{\mu}{=} -\partial_0 h^{0\nu}\partial_\nu h - \partial_k h^{k\nu}\partial_\nu h \\
&\overset{\nu}{=} -\partial_0 h^{00}\partial_0 h - \partial_0 h^{0k}\partial_k h - \partial_k h^{k0}\partial_0 h - \partial_k h^{kj}\partial_j h \\
&\overset{A.6}{=} +\partial_0 h^{00}\partial_0 h_{00} - \partial_0 h^{00}\partial_0 \eta^{ij} h_{ij} + \partial_0 h^{0k}\partial_k h_{00} - \partial_0 h^{0k}\partial_k \eta^{ij} h_{ij} \\
&\quad + \partial_k h^{k0}\partial_0 h_{00} - \partial_k h^{k0}\partial_0 \eta^{ij} h_{ij} + \partial_k h^{kj}\partial_j h_{00} - \partial_k h^{kj}\partial_j \eta^{mn} h_{mn} \\
&\overset{\downarrow}{=} +\partial_0 h_{00}\partial_0 h_{00} \underbrace{-\partial_0 h_{00}\partial_0 \delta_{ij} h_{ij}}_{\mathbf{A}} - \partial_0 h_{0k}\partial_k h_{00} \underbrace{+\partial_0 h_{0k}\partial_k \delta_{ij} h_{ij}}_{\mathbf{B}} \\
&\quad - \partial_k h_{k0}\partial_0 h_{00} \underbrace{+\partial_k h_{k0}\partial_0 \delta_{ij} h_{ij}}_{\mathbf{C}} + \partial_k h_{kj}\partial_j h_{00} - \partial_k h_{kj}\partial_j h_{ii}.
\end{aligned} \tag{A.8}$$

Notice that we have also numbered the last term of the first line which, at first, seems irrelevant on finding the spatial momentum $\pi_{ij}^{(3)}$. However by applying integration by parts twice, we obtain

$$\begin{aligned}
\int d^D x\ \partial_0 h_{0k}\partial_k \delta_{ij} h_{ij} &= -\int d^D x\ h_{0k}\partial_0\partial_k \delta_{ij} h_{ij} + (\text{boundary terms}) \\
&= +\int d^D x\ \partial_k h_{0k}\partial_0 \delta_{ij} h_{ij} + (\text{boundary terms}),
\end{aligned} \tag{A.9}$$

where boundary terms vanish in both cases by Stoke's theorem. Hence we have

$$\mathbf{B} \equiv \partial_k h_{0k}\partial_0 \delta_{ij} h_{ij}, \tag{A.10}$$

so that the corresponding spatial canonical momentum of $\mathscr{L}_3$ is

$$\begin{aligned}
\pi_{ij}^{(3)} &= \frac{\partial \mathscr{L}_3}{\partial \dot{h}_{ij}} = \frac{\partial}{\partial \dot{h}_{ij}}(\mathbf{A} + \mathbf{B} + \mathbf{C}) \\
&= -\partial_0 h_{00}\delta_{ij} + 2\partial_k h_{0k}\delta_{ij}.
\end{aligned} \tag{A.11}$$

A.4 Fourth Lagrangian

$$
\begin{aligned}
\mathscr{L}_4 &\overset{\lambda}{=} +\frac{1}{2}\partial_0 h\partial^0 h + \frac{1}{2}\partial_k h\partial^k h \\
&\overset{A.6}{=} +\frac{1}{2}\partial_0(-h_{00}+\eta^{ij}h_{ij})\partial^0(-h_{00}+\eta^{mn}h_{mn}) \\
&\quad +\frac{1}{2}\partial_k(-h_{00}+\eta^{ij}h_{ij})\partial^k(-h_{00}+\eta^{mn}h_{mn}) \\
&= +\frac{1}{2}\partial_0 h_{00}\partial^0 h_{00} - \frac{1}{2}\partial_0 h_{00}\partial^0\eta^{mn}h_{mn} - \frac{1}{2}\partial_0\eta^{ij}h_{ij}\partial^0 h_{00} \\
&\quad +\frac{1}{2}\partial_0\eta^{ij}h_{ij}\partial^0\eta^{mn}h_{mn} + \frac{1}{2}\partial_k h_{00}\partial^k h_{00} - \frac{1}{2}\partial_k h_{00}\partial^k\eta^{mn}h_{mn} \\
&\quad -\frac{1}{2}\partial_k\eta^{ij}h_{ij}\partial^k h_{00} + \frac{1}{2}\partial_k\eta^{ij}h_{ij}\partial^k\eta^{mn}h_{mn} \\
&\overset{\downarrow}{=} -\frac{1}{2}\partial_0 h_{00}\partial_0 h_{00} + \frac{1}{2}\partial_0 h_{00}\partial_0\delta_{mn}h_{mn} + \frac{1}{2}\partial_0\delta_{ij}h_{ij}\partial_0 h_{00} \\
&\quad -\frac{1}{2}\partial_0\delta_{ij}h_{ij}\partial_0\delta_{mn}h_{mn} + \frac{1}{2}\partial_k h_{00}\partial_k h_{00} - \frac{1}{2}\partial_k h_{00}\partial_k\delta_{mn}h_{mn} \\
&\quad -\frac{1}{2}\partial_k\delta_{ij}h_{ij}\partial_k h_{00} + \frac{1}{2}\partial_k\delta_{ij}h_{ij}\partial_k\delta_{mn}h_{mn} \\
&= -\frac{1}{2}\partial_0 h_{00}\partial_0 h_{00} \underbrace{+\partial_0 h_{00}\partial_0\delta_{ij}h_{ij}}_{\mathbf{A}} \underbrace{-\frac{1}{2}\partial_0\delta_{ij}h_{ij}\partial_0\delta_{mn}h_{mn}}_{\mathbf{B}} \\
&\quad +\frac{1}{2}\partial_k h_{00}\partial_k h_{00} - \partial_k h_{00}\partial_k\delta_{ij}h_{ij} + \frac{1}{2}\partial_k\delta_{ij}h_{ij}\partial_k\delta_{mn}h_{mn}
\end{aligned}
\tag{A.12}
$$

Corresponding spatial canonical momentum is

$$
\begin{aligned}
\pi^{(4)}_{ij} &= \frac{\partial\mathscr{L}_4}{\partial\dot{h}_{ij}} = \frac{\partial}{\partial\dot{h}_{ij}}(\mathbf{A}+\mathbf{B}) \\
&= \partial_0 h_{00}\delta_{ij} - \dot{h}_{kk}\delta_{ij}
\end{aligned}
\tag{A.13}
$$

A.5 Fifth Lagrangian

$$
\begin{aligned}
\mathscr{L}_5 &\overset{\mu}{=} -\frac{1}{2}m^2\left(h_{0\nu}h^{0\nu} + h_{k\nu}h^{k\nu} - h^2\right) \\
&\overset{\nu}{=} -\frac{1}{2}m^2\left(h_{00}h^{00} + h_{0k}h^{0k} + h_{k0}h^{k0} + h_{kj}h^{kj} - h^2\right) \\
&\overset{A.7}{=} -\frac{1}{2}m^2\left(\cancel{h_{00}h^{00}} + h_{0k}h^{0k} + h_{k0}h^{k0} + h_{kj}h^{kj} \cancel{- h_{00}^2} + 2h_{00}h_{kk} - h_{kk}^2\right) \\
&\overset{\downarrow}{=} -\frac{1}{2}m^2\left(-h_{0k}h_{0k} - h_{k0}h_{k0} + h_{kj}h_{kj} + 2h_{00}h_{kk} - h_{kk}^2\right) \\
&= +m^2 h_{0k}^2 - m^2 h_{00}h_{kk} - \frac{1}{2}m^2\left(h_{kj}h_{kj} - h_{kk}^2\right).
\end{aligned}
\tag{A.14}
$$

This term, obviously, has no contribution to canonical momenta since it does not contain any time derivatives at all. However when finding Hamiltonian (2.13) and constructing (2.14) one should use that explicit form.

APPENDIX B

INVERSION OF CANONICAL MOMENTA FOR VELOCITIES

We start by expanding the $\partial_{(i}h_{j)0}$ term of (2.11) on its indices so that we have

$$\pi_{ij} = \dot{h}_{ij} - \dot{h}_{kk}\delta_{ij} - \partial_i h_{j0} - \partial_j h_{i0} + 2\partial_k h_{0k}\delta_{ij}. \tag{B.1}$$

Taking the trace of the above equation gives

$$\begin{aligned}\eta^{ij}\pi_{ij} = \pi_{kk} &= \dot{h}_{kk} - (D-1)\dot{h}_{kk} - \partial^j h_{j0} - \partial^i h_{i0} + 2(D-1)\partial_k h_{0k} \\ &= -(D-2)\dot{h}_{kk} - 2\partial_i h_{i0} + 2(D-1)\partial_k h_{0k} \\ &= -(D-2)\dot{h}_{kk} + 2(D-2)\partial_k h_{0k},\end{aligned} \tag{B.2}$$

which implies

$$\dot{h}_{kk} = -\frac{1}{D-2}\pi_{kk} + 2\partial_k h_{0k}. \tag{B.3}$$

By plugging this result back into (B.1), we obtain

$$\begin{aligned}\pi_{ij} &= \dot{h}_{ij} + \frac{1}{D-2}\pi_{kk}\delta_{ij} - 2\partial_k h_{0k}\delta_{ij} - 2\partial_{(i}h_{j)0} + 2\partial_k h_{0k}\delta_{ij} \\ &= \dot{h}_{ij} + \frac{1}{D-2}\pi_{kk}\delta_{ij} - 2\partial_{(i}h_{j)0}.\end{aligned} \tag{B.4}$$

Thus, velocities $\dot{h}_{ij}$ can be expressed in terms of the Hamiltonian variables as

$$\dot{h}_{ij} = \pi_{ij} - \frac{1}{D-2}\pi_{kk}\delta_{ij} + 2\partial_{(i}h_{j)0}. \tag{B.5}$$

APPENDIX C

EINSTEIN'S FIELD EQUATIONS

Einstein-Hilbert action is given in (6.1) as

$$S = \frac{1}{2\kappa^2} \int d^D x \sqrt{-g} R. \tag{C.1}$$

Variation of this action with respect to the inverse metric $g^{\mu\nu}$ implies [1]

$$\delta S = \frac{1}{2\kappa^2} \int d^D x \left\{ R\delta(\sqrt{-g}) + \sqrt{-g}\delta R \right\}. \tag{C.2}$$

The variation of metric determinant which appears in the first term is

$$\delta(\sqrt{-g}) = -\frac{1}{2}\sqrt{-g} g_{\mu\nu} \delta g^{\mu\nu}. \tag{C.3}$$

Second term can be further expanded by writing $R = g^{\mu\nu} R_{\mu\nu}$ so that

$$\delta R = \delta(g^{\mu\nu} R_{\mu\nu}) = R_{\mu\nu} \delta g^{\mu\nu} + g^{\mu\nu} \delta R_{\mu\nu}. \tag{C.4}$$

By adding these up and regrouping the terms, we obtain

$$\delta S = \frac{1}{2\kappa^2} \int d^D x \sqrt{-g} \left[-\frac{1}{2} R g_{\mu\nu} + R_{\mu\nu} \right] \delta g^{\mu\nu} + \frac{1}{2\kappa^2} \int d^D x \sqrt{-g} g^{\mu\nu} \delta R_{\mu\nu}. \tag{C.5}$$

For the variation of Ricci tensor in the second integral we have

$$\begin{aligned} \sqrt{-g} g^{\mu\nu} \delta R_{\mu\nu} &= \sqrt{-g} g^{\mu\nu} \left(\nabla_\rho \delta \Gamma^\rho{}_{\nu\mu} - \nabla_\nu \delta \Gamma^\rho{}_{\rho\mu} \right) \\ &= \sqrt{-g} \nabla_\rho \left(g^{\mu\nu} \delta \Gamma^\rho{}_{\nu\mu} - g^{\mu\rho} \delta \Gamma^\nu{}_{\nu\mu} \right), \end{aligned} \tag{C.6}$$

where we have used the metric compatibility on the last line. This term is now a total derivative and by Stoke's theorem

$$\int d^D x \sqrt{-g} g^{\mu\nu} \delta R_{\mu\nu} = 0. \tag{C.7}$$

[1] For detailed techniques to take variations (including actions with higher curvature terms) in general relativity see Appendix part of [41].

Looking back at (C.5), we see that $\delta S = 0$ is satisfied if

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0, \tag{C.8}$$

which is a set of ten equations known as Einstein's field equations.

APPENDIX D

CURVATURE TERMS UP TO SECOND ORDER IN METRIC PERTURBATION

Our aim in this chapter is to derive the general form of the curvature terms up to second order when a perturbation is applied to the metric (Appendix part of [42] might be useful). After this is done, (6.7) and (6.15) can easily be obtained.

We start by expanding the generic metric $g_{\mu\nu}$ around a background metric $\bar{g}_{\mu\nu}$ (not necessarily the flat metric)

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \tau h_{\mu\nu}, \tag{D.1}$$

where τ is a dimensionless parameter put by hand to keep track of the order of the perturbation. Since we aim to keep the contributions only up to second order in $h_{\mu\nu}$ we will neglect the terms that include τ^3 and higher powers of τ. Under this assumption the inverse metric is expanded as

$$g^{\mu\nu} = \bar{g}^{\mu\nu} - \tau h^{\mu\nu} + \tau^2 h^{\mu\rho} h^{\nu}{}_{\rho} + \mathcal{O}(\tau^3). \tag{D.2}$$

Now, we can expand all the curvature elements by using (D.1) and (D.2).

D.1 Christoffel Symbols

The very first thing that can be obtained from the metric and its derivatives are Christoffel symbols (or connection coefficients). By definition

$$\Gamma^{\rho}{}_{\mu\nu} = \frac{1}{2} g^{\rho\lambda} \left(\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu} \right), \tag{D.3}$$

and it can be expanded as

$$
\begin{aligned}
\Gamma^{\rho}{}_{\mu\nu} &= \frac{1}{2}\left(\bar{g}^{\rho\lambda} - \tau h^{\rho\lambda} + \tau^2 h^{\rho\alpha} h^{\lambda}_{\alpha}\right)\Big[\partial_\mu\left(\bar{g}_{\lambda\nu} + \tau h_{\lambda\nu}\right) \\
&\quad + \partial_\nu\left(\bar{g}_{\mu\lambda} + \tau h_{\mu\lambda}\right) - \partial_\lambda\left(\bar{g}_{\mu\nu} + \tau h_{\mu\nu}\right)\Big] \\
&= \frac{1}{2}\bar{g}^{\rho\lambda}\left(\partial_\mu \bar{g}_{\lambda\nu} + \partial_\nu \bar{g}_{\mu\lambda} - \partial_\lambda \bar{g}_{\mu\nu}\right) + \tau\frac{1}{2}\bar{g}^{\rho\lambda}\left(\partial_\mu h_{\lambda\nu} + \partial_\nu h_{\mu\lambda} - \partial_\lambda h_{\mu\nu}\right) \\
&\quad - \tau\frac{1}{2}h^{\rho\lambda}\left(\partial_\mu \bar{g}_{\lambda\nu} + \partial_\nu \bar{g}_{\mu\lambda} - \partial_\lambda \bar{g}_{\mu\nu}\right) - \tau^2\frac{1}{2}h^{\rho\lambda}\left(\partial_\mu h_{\lambda\nu} + \partial_\nu h_{\mu\lambda} - \partial_\lambda h_{\mu\nu}\right) \\
&\quad + \tau^2\frac{1}{2}h^{\rho\alpha}h^{\lambda}{}_{\alpha}\left(\partial_\mu \bar{g}_{\lambda\nu} + \partial_\nu \bar{g}_{\mu\lambda} - \partial_\lambda \bar{g}_{\mu\nu}\right). \qquad \text{(D.4)}
\end{aligned}
$$

We define the first term as Christoffel symbol of the background metric

$$
\bar{\Gamma}^{\rho}{}_{\mu\nu} = \frac{1}{2}\bar{g}^{\rho\lambda}\left(\partial_\mu \bar{g}_{\lambda\nu} + \partial_\nu \bar{g}_{\mu\lambda} - \partial_\lambda \bar{g}_{\mu\nu}\right), \qquad \text{(D.5)}
$$

and we also make the replacements

$$
\begin{aligned}
\text{THIRD TERM:} \qquad & h^{\rho\lambda} = \bar{g}^{\rho\alpha}\bar{g}^{\lambda\beta}h_{\alpha\beta}, \\
\text{FOURTH TERM:} \qquad & h^{\rho\lambda} = \bar{g}^{\beta\lambda}h^{\rho}{}_{\beta}, \\
\text{FIFTH TERM:} \qquad & h^{\rho\alpha}h^{\lambda}{}_{\alpha} = \bar{g}^{\alpha\beta}h^{\rho}{}_{\beta}h^{\lambda}{}_{\alpha} = \bar{g}^{\alpha\beta}h^{\rho}{}_{\beta}\bar{g}^{\lambda\sigma}h_{\alpha\sigma}, \qquad \text{(D.6)}
\end{aligned}
$$

so that upon collecting like orders in τ and τ^2, we get

$$
\begin{aligned}
\Gamma^{\rho}{}_{\mu\nu} = \bar{\Gamma}^{\rho}{}_{\mu\nu} &+ \tau\Bigg\{\frac{1}{2}\bar{g}^{\rho\lambda}\left(\partial_\mu h_{\lambda\nu} + \partial_\nu h_{\mu\lambda} - \partial_\lambda h_{\mu\nu}\right) \\
&\quad - \bar{g}^{\rho\alpha}h_{\alpha\beta}\left[\frac{1}{2}\bar{g}^{\lambda\beta}\left(\partial_\mu \bar{g}_{\lambda\nu} + \partial_\nu \bar{g}_{\mu\lambda} - \partial_\lambda \bar{g}_{\mu\nu}\right)\right]\Bigg\} \\
&- \tau^2\Bigg\{\frac{1}{2}\bar{g}^{\beta\lambda}h^{\rho}{}_{\beta}\left(\partial_\mu h_{\lambda\nu} + \partial_\nu h_{\mu\lambda} - \partial_\lambda h_{\mu\nu}\right) \\
&\quad - \bar{g}^{\alpha\beta}h^{\rho}{}_{\beta}h_{\alpha\sigma}\left[\frac{1}{2}\bar{g}^{\lambda\sigma}\left(\partial_\mu \bar{g}_{\lambda\nu} + \partial_\nu \bar{g}_{\mu\lambda} - \partial_\lambda \bar{g}_{\mu\nu}\right)\right]\Bigg\}. \qquad \text{(D.7)}
\end{aligned}
$$

The terms in the square brackets are again Christoffel symbols of the background metric. Therefore,

$$
\begin{aligned}
\Gamma^{\rho}{}_{\mu\nu} = \bar{\Gamma}^{\rho}{}_{\mu\nu} &+ \tau\left\{\frac{1}{2}\bar{g}^{\rho\lambda}\left(\partial_\mu h_{\lambda\nu} + \partial_\nu h_{\mu\lambda} - \partial_\lambda h_{\mu\nu}\right) - \bar{g}^{\rho\alpha}h_{\alpha\beta}\bar{\Gamma}^{\beta}{}_{\mu\nu}\right\} \\
&- \tau^2\left\{\frac{1}{2}\bar{g}^{\beta\lambda}h^{\rho}{}_{\beta}\left(\partial_\mu h_{\lambda\nu} + \partial_\nu h_{\mu\lambda} - \partial_\lambda h_{\mu\nu}\right) - \bar{g}^{\alpha\beta}h^{\rho}{}_{\beta}h_{\alpha\sigma}\bar{\Gamma}^{\sigma}{}_{\mu\nu}\right\}. \qquad \text{(D.8)}
\end{aligned}
$$

After a change in indices

$$
\begin{aligned}
\tau \text{ ORDER:} \qquad & \bar{g}^{\rho\alpha}h_{\alpha\beta}\bar{\Gamma}^{\beta}{}_{\mu\nu} \xrightarrow{\alpha\to\lambda} \bar{g}^{\rho\lambda}h_{\lambda\beta}\bar{\Gamma}^{\beta}{}_{\mu\nu}, \\
\tau^2 \text{ ORDER:} \qquad & \bar{g}^{\alpha\beta}h^{\rho}{}_{\beta}h_{\alpha\sigma}\bar{\Gamma}^{\sigma}{}_{\mu\nu} \xrightarrow{\alpha\to\lambda} \bar{g}^{\lambda\beta}h^{\rho}{}_{\beta}h_{\lambda\sigma}\bar{\Gamma}^{\sigma}{}_{\mu\nu}, \qquad \text{(D.9)}
\end{aligned}
$$

we obtain

$$\begin{aligned}
\Gamma^{\rho}{}_{\mu\nu} &= \bar{\Gamma}^{\rho}{}_{\mu\nu} + \tau\left\{\frac{1}{2}\bar{g}^{\rho\lambda}\left(\partial_\mu h_{\lambda\nu} + \partial_\nu h_{\mu\lambda} - \partial_\lambda h_{\mu\nu}\right) - \bar{g}^{\rho\lambda}h_{\lambda\beta}\bar{\Gamma}^{\beta}{}_{\mu\nu}\right\} \\
&\quad - \tau^2\left\{\frac{1}{2}\bar{g}^{\beta\lambda}h^{\rho}{}_{\beta}\left(\partial_\mu h_{\lambda\nu} + \partial_\nu h_{\mu\lambda} - \partial_\lambda h_{\mu\nu}\right) - \bar{g}^{\lambda\beta}h^{\rho}{}_{\beta}h_{\lambda\sigma}\bar{\Gamma}^{\sigma}{}_{\mu\nu}\right\} \\
&= \bar{\Gamma}^{\rho}{}_{\mu\nu} + \tau\frac{1}{2}\bar{g}^{\rho\lambda}\left\{\partial_\mu h_{\lambda\nu} + \partial_\nu h_{\mu\lambda} - \partial_\lambda h_{\mu\nu} - 2h_{\lambda\beta}\bar{\Gamma}^{\beta}{}_{\mu\nu}\right\} \\
&\quad - \tau^2\frac{1}{2}\bar{g}^{\beta\lambda}h^{\rho}{}_{\beta}\left\{\partial_\mu h_{\lambda\nu} + \partial_\nu h_{\mu\lambda} - \partial_\lambda h_{\mu\nu} - 2h_{\lambda\sigma}\bar{\Gamma}^{\sigma}{}_{\mu\nu}\right\}.
\end{aligned} \tag{D.10}$$

Now applying the trick

$$\begin{aligned}
\tau \text{ ORDER:} &\quad \text{add and substract} \quad h_{\mu\beta}\bar{\Gamma}^{\beta}{}_{\nu\lambda} + h_{\nu\beta}\bar{\Gamma}^{\beta}{}_{\mu\lambda}, \\
\tau^2 \text{ ORDER:} &\quad \text{add and substract} \quad h_{\mu\sigma}\bar{\Gamma}^{\sigma}{}_{\nu\lambda} + h_{\nu\sigma}\bar{\Gamma}^{\sigma}{}_{\mu\lambda},
\end{aligned} \tag{D.11}$$

and then regrouping the terms gives

$$\begin{aligned}
\Gamma^{\rho}{}_{\mu\nu} &= \bar{\Gamma}^{\rho}{}_{\mu\nu} + \tau\frac{1}{2}\bar{g}^{\rho\lambda}\Big\{\left(\partial_\mu h_{\lambda\nu} - h_{\nu\beta}\bar{\Gamma}^{\beta}{}_{\mu\lambda} - h_{\lambda\beta}\bar{\Gamma}^{\beta}{}_{\mu\nu}\right) \\
&\quad + \left(\partial_\nu h_{\mu\lambda} - h_{\mu\beta}\bar{\Gamma}^{\beta}{}_{\nu\lambda} - h_{\lambda\beta}\bar{\Gamma}^{\beta}{}_{\mu\nu}\right) - \left(\partial_\lambda h_{\mu\nu} - h_{\nu\beta}\bar{\Gamma}^{\beta}{}_{\mu\lambda} - h_{\mu\beta}\bar{\Gamma}^{\beta}{}_{\nu\lambda}\right)\Big\} \\
&\quad - \tau^2\frac{1}{2}\bar{g}^{\beta\lambda}h^{\rho}{}_{\beta}\Big\{\left(\partial_\mu h_{\lambda\nu} - h_{\nu\sigma}\bar{\Gamma}^{\sigma}{}_{\mu\lambda} - h_{\lambda\sigma}\bar{\Gamma}^{\sigma}{}_{\mu\nu}\right) \\
&\quad + \left(\partial_\nu h_{\mu\lambda} - h_{\mu\sigma}\bar{\Gamma}^{\sigma}{}_{\nu\lambda} - h_{\lambda\sigma}\bar{\Gamma}^{\sigma}{}_{\mu\nu}\right) - \left(\partial_\lambda h_{\mu\nu} - h_{\nu\sigma}\bar{\Gamma}^{\sigma}{}_{\mu\lambda} - h_{\mu\sigma}\bar{\Gamma}^{\sigma}{}_{\nu\lambda}\right)\Big\}.
\end{aligned} \tag{D.12}$$

We recognize the terms in the parentheses as the covariant derivative of a rank-2 tensor,

$$\bar{\nabla}_\lambda h_{\mu\nu} = \partial_\lambda h_{\mu\nu} - h_{\nu\sigma}\bar{\Gamma}^{\sigma}{}_{\mu\lambda} - h_{\mu\sigma}\bar{\Gamma}^{\sigma}{}_{\nu\lambda}. \tag{D.13}$$

Hence (D.12) takes the form

$$\begin{aligned}
\Gamma^{\rho}{}_{\mu\nu} &= \bar{\Gamma}^{\rho}{}_{\mu\nu} + \tau\frac{1}{2}\bar{g}^{\rho\lambda}\left(\bar{\nabla}_\mu h_{\lambda\nu} + \bar{\nabla}_\nu h_{\mu\lambda} - \bar{\nabla}_\lambda h_{\mu\nu}\right) \\
&\quad - \tau^2 h^{\rho}{}_{\beta}\frac{1}{2}\bar{g}^{\beta\lambda}\left(\bar{\nabla}_\mu h_{\lambda\nu} + \bar{\nabla}_\nu h_{\mu\lambda} - \bar{\nabla}_\lambda h_{\mu\nu}\right).
\end{aligned} \tag{D.14}$$

Now, we define the linearized Christoffel symbols,

$$\left(\Gamma^{\rho}{}_{\mu\nu}\right)_L = \frac{1}{2}\bar{g}^{\rho\lambda}\left(\bar{\nabla}_\mu h_{\lambda\nu} + \bar{\nabla}_\nu h_{\mu\lambda} - \bar{\nabla}_\lambda h_{\mu\nu}\right), \tag{D.15}$$

and obtain the final form of the Christoffel symbols expanded up to second order in metric perturbation as

$$\Gamma^{\rho}{}_{\mu\nu} = \bar{\Gamma}^{\rho}{}_{\mu\nu} + \tau\left(\Gamma^{\rho}{}_{\mu\nu}\right)_L - \tau^2 h^{\rho}{}_{\beta}\left(\Gamma^{\beta}{}_{\mu\nu}\right)_L. \tag{D.16}$$

D.2 Riemann Tensor

Riemann tensor is defined as

$$R^{\mu}{}_{\nu\rho\sigma} = \partial_\rho \Gamma^{\mu}{}_{\sigma\nu} + \Gamma^{\mu}{}_{\rho\lambda}\Gamma^{\lambda}{}_{\sigma\nu} - \partial_\sigma \Gamma^{\mu}{}_{\rho\nu} - \Gamma^{\mu}{}_{\sigma\lambda}\Gamma^{\lambda}{}_{\rho\nu}. \tag{D.17}$$

We start by defining

$$\begin{aligned} \delta\Gamma^{\rho}{}_{\mu\nu} &\equiv \Gamma^{\rho}{}_{\mu\nu} - \bar{\Gamma}^{\rho}{}_{\mu\nu} \\ &= \tau \left(\Gamma^{\rho}{}_{\mu\nu}\right)_L - \tau^2 h^{\rho}{}_{\beta} \left(\Gamma^{\beta}{}_{\mu\nu}\right)_L , \end{aligned} \tag{D.18}$$

where we have used (D.16) to obtain the second line. By rewriting Riemann tensor using this new element, we get

$$\begin{aligned} R^{\mu}{}_{\nu\rho\sigma} &= \partial_\rho \left(\delta\Gamma^{\mu}{}_{\sigma\nu} + \bar{\Gamma}^{\mu}{}_{\sigma\nu}\right) + \left(\delta\Gamma^{\mu}{}_{\rho\lambda} + \bar{\Gamma}^{\mu}{}_{\rho\lambda}\right)\left(\delta\Gamma^{\lambda}{}_{\sigma\nu} + \bar{\Gamma}^{\lambda}{}_{\sigma\nu}\right) \\ &\quad - \partial_\sigma \left(\delta\Gamma^{\mu}{}_{\rho\nu} + \bar{\Gamma}^{\mu}{}_{\rho\nu}\right) - \left(\delta\Gamma^{\mu}{}_{\sigma\lambda} + \bar{\Gamma}^{\mu}{}_{\sigma\lambda}\right)\left(\delta\Gamma^{\lambda}{}_{\rho\nu} + \bar{\Gamma}^{\lambda}{}_{\rho\nu}\right) \\ &= \left(\partial_\rho \bar{\Gamma}^{\mu}{}_{\sigma\nu} + \bar{\Gamma}^{\mu}{}_{\rho\lambda}\bar{\Gamma}^{\lambda}{}_{\sigma\nu} - \partial_\sigma \bar{\Gamma}^{\mu}{}_{\rho\nu} - \bar{\Gamma}^{\mu}{}_{\sigma\lambda}\bar{\Gamma}^{\lambda}{}_{\rho\nu}\right) \\ &\quad \underbrace{+\partial_\rho \delta\Gamma^{\mu}{}_{\sigma\nu}}_{\mathbf{A}} + \delta\Gamma^{\mu}{}_{\rho\lambda}\delta\Gamma^{\lambda}{}_{\sigma\nu} \underbrace{+\delta\Gamma^{\mu}{}_{\rho\lambda}\bar{\Gamma}^{\lambda}{}_{\sigma\nu}}_{\mathbf{B}} \underbrace{+\bar{\Gamma}^{\mu}{}_{\rho\lambda}\delta\Gamma^{\lambda}{}_{\sigma\nu}}_{\mathbf{A}} \\ &\quad \underbrace{-\partial_\sigma \delta\Gamma^{\mu}{}_{\rho\nu}}_{\mathbf{B}} - \delta\Gamma^{\mu}{}_{\sigma\lambda}\delta\Gamma^{\lambda}{}_{\rho\nu} \underbrace{-\delta\Gamma^{\mu}{}_{\sigma\lambda}\bar{\Gamma}^{\lambda}{}_{\rho\nu}}_{\mathbf{A}} \underbrace{-\bar{\Gamma}^{\mu}{}_{\sigma\lambda}\delta\Gamma^{\lambda}{}_{\rho\nu}}_{\mathbf{B}} . \end{aligned} \tag{D.19}$$

Defining the first term in the parenthesis as the Riemann tensor in the background metric,

$$\bar{R}^{\mu}{}_{\nu\rho\sigma} \equiv \partial_\rho \bar{\Gamma}^{\mu}{}_{\sigma\nu} + \bar{\Gamma}^{\mu}{}_{\rho\lambda}\bar{\Gamma}^{\lambda}{}_{\sigma\nu} - \partial_\sigma \bar{\Gamma}^{\mu}{}_{\rho\nu} - \bar{\Gamma}^{\mu}{}_{\sigma\lambda}\bar{\Gamma}^{\lambda}{}_{\rho\nu}, \tag{D.20}$$

and grouping the terms labelled with the same letter **A** or **B** gives

$$\begin{aligned} R^{\mu}{}_{\nu\rho\sigma} = \bar{R}^{\mu}{}_{\nu\rho\sigma} &+ \left(\partial_\rho \delta\Gamma^{\mu}{}_{\sigma\nu} + \bar{\Gamma}^{\mu}{}_{\rho\lambda}\delta\Gamma^{\lambda}{}_{\sigma\nu} - \bar{\Gamma}^{\lambda}{}_{\rho\nu}\delta\Gamma^{\mu}{}_{\sigma\lambda}\right) \\ &- \left(\partial_\sigma \delta\Gamma^{\mu}{}_{\rho\nu} + \bar{\Gamma}^{\mu}{}_{\sigma\lambda}\delta\Gamma^{\lambda}{}_{\rho\nu} - \bar{\Gamma}^{\lambda}{}_{\sigma\nu}\delta\Gamma^{\mu}{}_{\rho\lambda}\right) \\ &+ \delta\Gamma^{\mu}{}_{\rho\lambda}\delta\Gamma^{\lambda}{}_{\sigma\nu} - \delta\Gamma^{\mu}{}_{\sigma\lambda}\delta\Gamma^{\lambda}{}_{\rho\nu}. \end{aligned} \tag{D.21}$$

Now we add and subtract $\bar{\Gamma}^{\lambda}{}_{\rho\sigma}\delta\Gamma^{\mu}{}_{\lambda\nu}$ from the above equation. Precisely we put the minus of that term inside both parentheses. Then,

$$\begin{aligned} R^{\mu}{}_{\nu\rho\sigma} = \bar{R}^{\mu}{}_{\nu\rho\sigma} &+ \left(\partial_\rho \delta\Gamma^{\mu}{}_{\sigma\nu} + \bar{\Gamma}^{\mu}{}_{\rho\lambda}\delta\Gamma^{\lambda}{}_{\sigma\nu} - \bar{\Gamma}^{\lambda}{}_{\rho\nu}\delta\Gamma^{\mu}{}_{\sigma\lambda} - \bar{\Gamma}^{\lambda}{}_{\rho\sigma}\delta\Gamma^{\mu}{}_{\lambda\nu}\right) \\ &- \left(\partial_\sigma \delta\Gamma^{\mu}{}_{\rho\nu} + \bar{\Gamma}^{\mu}{}_{\sigma\lambda}\delta\Gamma^{\lambda}{}_{\rho\nu} - \bar{\Gamma}^{\lambda}{}_{\sigma\nu}\delta\Gamma^{\mu}{}_{\rho\lambda} - \bar{\Gamma}^{\lambda}{}_{\rho\sigma}\delta\Gamma^{\mu}{}_{\lambda\nu}\right) \\ &+ \delta\Gamma^{\mu}{}_{\rho\lambda}\delta\Gamma^{\lambda}{}_{\sigma\nu} - \delta\Gamma^{\mu}{}_{\sigma\lambda}\delta\Gamma^{\lambda}{}_{\rho\nu}. \end{aligned} \tag{D.22}$$

The terms in the parentheses are now easily recognized as the covariant derivative of a rank-3 tensor,

$$\bar{\nabla}_\rho \delta\Gamma^\mu{}_{\sigma\nu} = \partial_\rho \delta\Gamma^\mu{}_{\sigma\nu} + \bar{\Gamma}^\mu{}_{\rho\lambda}\delta\Gamma^\lambda{}_{\sigma\nu} - \bar{\Gamma}^\lambda{}_{\rho\nu}\delta\Gamma^\mu{}_{\sigma\lambda} - \bar{\Gamma}^\lambda{}_{\rho\sigma}\delta\Gamma^\mu{}_{\lambda\nu}, \tag{D.23}$$

so that (D.22) reduces to

$$R^\mu{}_{\nu\rho\sigma} = \bar{R}^\mu{}_{\nu\rho\sigma} + \bar{\nabla}_\rho \delta\Gamma^\mu{}_{\sigma\nu} - \bar{\nabla}_\sigma \delta\Gamma^\mu{}_{\rho\nu} + \delta\Gamma^\mu{}_{\rho\lambda}\delta\Gamma^\lambda{}_{\sigma\nu} - \delta\Gamma^\mu{}_{\sigma\lambda}\delta\Gamma^\lambda{}_{\rho\nu}. \tag{D.24}$$

At that point we can use (D.18) again to expand the $\delta\Gamma$ terms, so that

$$\begin{aligned} R^\mu{}_{\nu\rho\sigma} = \bar{R}^\mu{}_{\nu\rho\sigma} &+ \bar{\nabla}_\rho \Big[\tau \left(\Gamma^\mu{}_{\sigma\nu}\right)_L - \tau^2 h^\mu{}_\beta \left(\Gamma^\beta{}_{\sigma\nu}\right)_L\Big] \\ &- \bar{\nabla}_\sigma \Big[\tau \left(\Gamma^\mu{}_{\rho\nu}\right)_L - \tau^2 h^\mu{}_\beta \left(\Gamma^\beta{}_{\rho\nu}\right)_L\Big] \\ &+ \Big[\tau \left(\Gamma^\mu{}_{\rho\lambda}\right)_L - \tau^2 h^\mu{}_\beta \left(\Gamma^\beta{}_{\rho\lambda}\right)_L\Big]\Big[\tau \left(\Gamma^\lambda{}_{\sigma\nu}\right)_L - \tau^2 h^\lambda{}_\beta \left(\Gamma^\beta{}_{\sigma\nu}\right)_L\Big] \\ &- \Big[\tau \left(\Gamma^\mu{}_{\sigma\lambda}\right)_L - \tau^2 h^\mu{}_\beta \left(\Gamma^\beta{}_{\sigma\lambda}\right)_L\Big]\Big[\tau \left(\Gamma^\lambda{}_{\rho\nu}\right)_L - \tau^2 h^\lambda{}_\beta \left(\Gamma^\beta{}_{\rho\nu}\right)_L\Big]. \end{aligned} \tag{D.25}$$

Collecting the terms in the orders of τ gives

$$\begin{aligned} R^\mu{}_{\nu\rho\sigma} = \bar{R}^\mu{}_{\nu\rho\sigma} &+ \tau\Big\{\bar{\nabla}_\rho \left(\Gamma^\mu{}_{\sigma\nu}\right)_L - \bar{\nabla}_\sigma \left(\Gamma^\mu{}_{\rho\nu}\right)_L\Big\} \\ &- \tau^2\Big\{\bar{\nabla}_\rho h^\mu{}_\beta \left(\Gamma^\beta{}_{\sigma\nu}\right)_L - \bar{\nabla}_\sigma h^\mu{}_\beta \left(\Gamma^\beta{}_{\rho\nu}\right)_L \\ &- \left(\Gamma^\mu{}_{\rho\lambda}\right)_L \left(\Gamma^\lambda{}_{\sigma\nu}\right)_L + \left(\Gamma^\mu{}_{\sigma\lambda}\right)_L \left(\Gamma^\lambda{}_{\rho\nu}\right)_L\Big\} + \mathcal{O}(\tau^3). \end{aligned} \tag{D.26}$$

We ignore the higher order τ terms and also change the summation index from λ to β for the last two terms at order τ^2,

$$\begin{aligned} R^\mu{}_{\nu\rho\sigma} = \bar{R}^\mu{}_{\nu\rho\sigma} &+ \tau\Big\{\bar{\nabla}_\rho \left(\Gamma^\mu{}_{\sigma\nu}\right)_L - \bar{\nabla}_\sigma \left(\Gamma^\mu{}_{\rho\nu}\right)_L\Big\} \\ &- \tau^2\Big\{\bar{\nabla}_\rho h^\mu{}_\beta \left(\Gamma^\beta{}_{\sigma\nu}\right)_L - \bar{\nabla}_\sigma h^\mu{}_\beta \left(\Gamma^\beta{}_{\rho\nu}\right)_L \\ &- \left(\Gamma^\mu{}_{\rho\beta}\right)_L \left(\Gamma^\beta{}_{\sigma\nu}\right)_L + \left(\Gamma^\mu{}_{\sigma\beta}\right)_L \left(\Gamma^\beta{}_{\rho\nu}\right)_L\Big\}. \end{aligned} \tag{D.27}$$

By using the distributive property of the covariant derivative for the first two terms in the τ^2 order, we obtain

$$\begin{aligned} R^\mu{}_{\nu\rho\sigma} = \bar{R}^\mu{}_{\nu\rho\sigma} &+ \tau\Big\{\bar{\nabla}_\rho \left(\Gamma^\mu{}_{\sigma\nu}\right)_L - \bar{\nabla}_\sigma \left(\Gamma^\mu{}_{\rho\nu}\right)_L\Big\} \\ &- \tau^2\Big\{h^\mu{}_\beta \left[\bar{\nabla}_\rho \left(\Gamma^\beta{}_{\sigma\nu}\right)_L - \bar{\nabla}_\sigma \left(\Gamma^\beta{}_{\rho\nu}\right)_L\right] \\ &+ \left(\Gamma^\beta{}_{\sigma\nu}\right)_L \bar{\nabla}_\rho h^\mu{}_\beta - \left(\Gamma^\beta{}_{\rho\nu}\right)_L \bar{\nabla}_\sigma h^\mu{}_\beta \\ &- \left(\Gamma^\mu{}_{\rho\beta}\right)_L \left(\Gamma^\beta{}_{\sigma\nu}\right)_L + \left(\Gamma^\mu{}_{\sigma\beta}\right)_L \left(\Gamma^\beta{}_{\rho\nu}\right)_L\Big\}. \end{aligned} \tag{D.28}$$

The term that appears in both τ and τ^2 orders is defined as the linearized Riemann tensor,

$$\left(R^{\mu}{}_{\nu\rho\sigma}\right)_L = \bar{\nabla}_\rho \left(\Gamma^{\mu}{}_{\sigma\nu}\right)_L - \bar{\nabla}_\sigma \left(\Gamma^{\mu}{}_{\rho\nu}\right)_L , \tag{D.29}$$

and we further reduce (D.28) to

$$\begin{aligned} R^{\mu}{}_{\nu\rho\sigma} &= \bar{R}^{\mu}{}_{\nu\rho\sigma} + \tau \left(R^{\mu}{}_{\nu\rho\sigma}\right)_L - \tau^2 \Big\{ h^{\mu}{}_{\beta} \left(R^{\beta}{}_{\nu\rho\sigma}\right)_L + \left(\Gamma^{\beta}{}_{\sigma\nu}\right)_L \bar{\nabla}_\rho h^{\mu}{}_{\beta} \\ &\quad - \left(\Gamma^{\beta}{}_{\rho\nu}\right)_L \bar{\nabla}_\sigma h^{\mu}{}_{\beta} - \left(\Gamma^{\mu}{}_{\rho\beta}\right)_L \left(\Gamma^{\beta}{}_{\sigma\nu}\right)_L + \left(\Gamma^{\mu}{}_{\sigma\beta}\right)_L \left(\Gamma^{\beta}{}_{\rho\nu}\right)_L \Big\} \\ &= \bar{R}^{\mu}{}_{\nu\rho\sigma} + \tau \left(R^{\mu}{}_{\nu\rho\sigma}\right)_L - \tau^2 \Big\{ h^{\mu}{}_{\beta} \left(R^{\beta}{}_{\nu\rho\sigma}\right)_L \\ &\quad + \left(\Gamma^{\beta}{}_{\sigma\nu}\right)_L \left[\bar{\nabla}_\rho h^{\mu}{}_{\beta} - \left(\Gamma^{\mu}{}_{\rho\beta}\right)_L\right] - \left(\Gamma^{\beta}{}_{\rho\nu}\right)_L \left[\bar{\nabla}_\sigma h^{\mu}{}_{\beta} - \left(\Gamma^{\mu}{}_{\sigma\beta}\right)_L\right] \Big\}. \end{aligned} \tag{D.30}$$

Let us deal with the term $\bigstar \equiv \left(\Gamma^{\beta}{}_{\sigma\nu}\right)_L \left[\bar{\nabla}_\rho h^{\mu}{}_{\beta} - \left(\Gamma^{\mu}{}_{\rho\beta}\right)_L\right]$ now;

$$\begin{aligned} \bigstar &= \left(\Gamma^{\beta}{}_{\sigma\nu}\right)_L \left[\bar{\nabla}_\rho \bar{g}^{\mu\alpha} h_{\alpha\beta} - \frac{1}{2}\bar{g}^{\mu\alpha} \left(\bar{\nabla}_\rho h_{\alpha\beta} + \bar{\nabla}_\beta h_{\rho\alpha} - \bar{\nabla}_\alpha h_{\rho\beta}\right)\right] \\ &= \left(\Gamma^{\beta}{}_{\sigma\nu}\right)_L \left[\frac{1}{2}\bar{g}^{\mu\alpha} \left(\bar{\nabla}_\rho h_{\alpha\beta} + \bar{\nabla}_\alpha h_{\rho\beta} - \bar{\nabla}_\beta h_{\rho\alpha}\right)\right] \\ &= \left(\Gamma^{\beta}{}_{\sigma\nu}\right)_L \left[\frac{1}{2}\bar{g}^{\mu\alpha} \delta^{\lambda}_{\beta} \left(\bar{\nabla}_\rho h_{\alpha\lambda} + \bar{\nabla}_\alpha h_{\rho\lambda} - \bar{\nabla}_\lambda h_{\rho\alpha}\right)\right] \\ &= \left(\Gamma^{\beta}{}_{\sigma\nu}\right)_L \left[\frac{1}{2}\bar{g}^{\mu\alpha} \bar{g}_{\beta\gamma} \bar{g}^{\gamma\lambda} \left(\bar{\nabla}_\rho h_{\alpha\lambda} + \bar{\nabla}_\alpha h_{\rho\lambda} - \bar{\nabla}_\lambda h_{\rho\alpha}\right)\right] \\ &= \bar{g}^{\mu\alpha} \bar{g}_{\beta\gamma} \left(\Gamma^{\beta}{}_{\sigma\nu}\right)_L \left[\frac{1}{2}\bar{g}^{\gamma\lambda} \left(\bar{\nabla}_\rho h_{\alpha\lambda} + \bar{\nabla}_\alpha h_{\rho\lambda} - \bar{\nabla}_\lambda h_{\rho\alpha}\right)\right] \\ &= \bar{g}^{\mu\alpha} \bar{g}_{\beta\gamma} \left(\Gamma^{\beta}{}_{\sigma\nu}\right)_L \left(\Gamma^{\gamma}{}_{\rho\alpha}\right)_L , \end{aligned} \tag{D.31}$$

where we have used the definition of linearized Christoffel symbols (D.15) two times, on the first and on the last lines. Similarly,

$$\left(\Gamma^{\beta}{}_{\rho\nu}\right)_L \left[\bar{\nabla}_\sigma h^{\mu}{}_{\beta} - \left(\Gamma^{\mu}{}_{\sigma\beta}\right)_L\right] = \bar{g}^{\mu\alpha} \bar{g}_{\beta\gamma} \left(\Gamma^{\beta}{}_{\rho\nu}\right)_L \left(\Gamma^{\gamma}{}_{\sigma\alpha}\right)_L . \tag{D.32}$$

Therefore we can rewrite the final form of the Riemann tensor perturbed up to second order in metric as

$$\begin{aligned} R^{\mu}{}_{\nu\rho\sigma} &= \bar{R}^{\mu}{}_{\nu\rho\sigma} + \tau \left(R^{\mu}{}_{\nu\rho\sigma}\right)_L \\ &\quad - \tau^2 \Big\{ h^{\mu}{}_{\beta} \left(R^{\beta}{}_{\nu\rho\sigma}\right)_L + \bar{g}^{\mu\alpha} \bar{g}_{\beta\gamma} \left[\left(\Gamma^{\beta}{}_{\sigma\nu}\right)_L \left(\Gamma^{\gamma}{}_{\rho\alpha}\right)_L - \left(\Gamma^{\beta}{}_{\rho\nu}\right)_L \left(\Gamma^{\gamma}{}_{\sigma\alpha}\right)_L\right] \Big\}. \end{aligned} \tag{D.33}$$

D.3 Ricci Tensor

Ricci tensor is defined by the contraction over Riemann tensor, $R_{\nu\sigma} = R^{\mu}{}_{\nu\mu\sigma}$. Hence by using (D.33), we immediately get

$$\begin{aligned} R_{\nu\sigma} = \bar{R}_{\nu\sigma} + \tau \left(R_{\nu\sigma}\right)_L \\ - \tau^2 \Big\{ h^{\mu}{}_{\beta} \left(R^{\beta}{}_{\nu\mu\sigma}\right)_L + \bar{g}^{\mu\alpha} \bar{g}_{\beta\gamma} \left[\left(\Gamma^{\beta}{}_{\sigma\nu}\right)_L \left(\Gamma^{\gamma}{}_{\mu\alpha}\right)_L - \left(\Gamma^{\beta}{}_{\mu\nu}\right)_L \left(\Gamma^{\gamma}{}_{\sigma\alpha}\right)_L \right] \Big\}. \end{aligned} \tag{D.34}$$

From the definition of linearized Riemann tensor (D.29), we have

$$\begin{aligned} \left(R^{\mu}{}_{\nu\rho\sigma}\right)_L &= \bar{\nabla}_{\rho} \frac{1}{2} \bar{g}^{\mu\lambda} \left(\bar{\nabla}_{\sigma} h_{\lambda\nu} + \bar{\nabla}_{\nu} h_{\sigma\lambda} - \bar{\nabla}_{\lambda} h_{\sigma\nu} \right) \\ &\quad - \bar{\nabla}_{\sigma} \frac{1}{2} \bar{g}^{\mu\lambda} \left(\bar{\nabla}_{\rho} h_{\lambda\nu} + \bar{\nabla}_{\nu} h_{\rho\lambda} - \bar{\nabla}_{\lambda} h_{\rho\nu} \right) \\ &= \frac{1}{2} \big(\bar{\nabla}_{\rho} \bar{\nabla}_{\sigma} h^{\mu}{}_{\nu} + \bar{\nabla}_{\rho} \bar{\nabla}_{\nu} h^{\mu}{}_{\sigma} - \bar{\nabla}_{\rho} \bar{\nabla}^{\mu} h_{\sigma\nu} \\ &\quad - \bar{\nabla}_{\sigma} \bar{\nabla}_{\rho} h^{\mu}{}_{\nu} - \bar{\nabla}_{\sigma} \bar{\nabla}_{\nu} h^{\mu}{}_{\rho} + \bar{\nabla}_{\sigma} \bar{\nabla}^{\mu} h_{\rho\nu} \big), \end{aligned} \tag{D.35}$$

and R_L term is easily obtained from that as

$$\begin{aligned} \left(R_{\nu\sigma}\right)_L &= \left(R^{\mu}{}_{\nu\mu\sigma}\right)_L \\ &= \frac{1}{2} \Big(\bar{\nabla}_{\mu} \bar{\nabla}_{\sigma} h^{\mu}{}_{\nu} + \bar{\nabla}_{\mu} \bar{\nabla}_{\nu} h^{\mu}{}_{\sigma} - \bar{\nabla}_{\mu} \bar{\nabla}^{\mu} h_{\sigma\nu} \\ &\quad - \cancel{\bar{\nabla}_{\sigma} \bar{\nabla}_{\mu} h^{\mu}{}_{\nu}} - \bar{\nabla}_{\sigma} \bar{\nabla}_{\nu} h^{\mu}{}_{\mu} + \cancel{\bar{\nabla}_{\sigma} \bar{\nabla}^{\mu} h_{\mu\nu}} \Big) \\ &= \frac{1}{2} \left(\bar{\nabla}_{\mu} \bar{\nabla}_{\sigma} h^{\mu}{}_{\nu} + \bar{\nabla}_{\mu} \bar{\nabla}_{\nu} h^{\mu}{}_{\sigma} - \bar{\Box} h_{\sigma\nu} - \bar{\nabla}_{\sigma} \bar{\nabla}_{\nu} h \right). \end{aligned} \tag{D.36}$$

D.4 Ricci Scalar

Ricci scalar is defined by $R = g^{\nu\sigma} R_{\nu\sigma}$. By using (D.2) and (D.34), we obtain

$$\begin{aligned} R = &\left(\bar{g}^{\nu\sigma} - \tau h^{\nu\sigma} + \tau^2 h^{\nu\lambda} h^{\sigma}{}_{\lambda} \right) \Big[\bar{R}_{\nu\sigma} + \tau \left(R_{\nu\sigma}\right)_L \\ &- \tau^2 \Big\{ h^{\mu}{}_{\beta} \left(R^{\beta}{}_{\nu\mu\sigma}\right)_L + \bar{g}^{\mu\alpha} \bar{g}_{\beta\gamma} \left[\left(\Gamma^{\beta}{}_{\sigma\nu}\right)_L \left(\Gamma^{\gamma}{}_{\mu\alpha}\right)_L - \left(\Gamma^{\beta}{}_{\mu\nu}\right)_L \left(\Gamma^{\gamma}{}_{\sigma\alpha}\right)_L \right] \Big\} \Big] \\ = &\, \bar{g}^{\nu\sigma} \bar{R}_{\nu\sigma} + \tau \Big[\bar{g}^{\nu\sigma} \left(R_{\nu\sigma}\right)_L - h^{\nu\sigma} \bar{R}_{\nu\sigma} \Big] \\ &- \tau^2 \Big[\bar{g}^{\nu\sigma} \Big\{ h^{\mu}{}_{\beta} \left(R^{\beta}{}_{\nu\mu\sigma}\right)_L + \bar{g}^{\mu\alpha} \bar{g}_{\beta\gamma} \left[\left(\Gamma^{\beta}{}_{\sigma\nu}\right)_L \left(\Gamma^{\gamma}{}_{\mu\alpha}\right)_L - \left(\Gamma^{\beta}{}_{\mu\nu}\right)_L \left(\Gamma^{\gamma}{}_{\sigma\alpha}\right)_L \right] \Big\} \\ &+ h^{\nu\sigma} \left(R_{\nu\sigma}\right)_L - h^{\nu\lambda} h^{\sigma}{}_{\lambda} \bar{R}_{\nu\sigma} \Big] + \mathcal{O}(\tau^3). \end{aligned} \tag{D.37}$$

Making the definitions,

$$\bar{R} = \bar{g}^{\nu\sigma} \bar{R}_{\nu\sigma}, \tag{D.38}$$

$$R_L = \bar{g}^{\nu\sigma} \left(R_{\nu\sigma}\right)_L - h^{\nu\sigma} \bar{R}_{\nu\sigma}, \tag{D.39}$$

and neglecting the higher order contributions gives the final form

$$\begin{aligned} R = \bar{R} + \tau R_L - \tau^2 \Big\{ & \bar{g}^{\nu\sigma} h^{\mu}{}_{\beta} \left(R^{\beta}{}_{\nu\mu\sigma}\right)_L + h^{\nu\sigma} \left(R_{\nu\sigma}\right)_L - h^{\nu\lambda} h^{\sigma}{}_{\lambda} \bar{R}_{\nu\sigma} \\ & + \bar{g}^{\nu\sigma} \bar{g}^{\mu\alpha} \bar{g}_{\beta\gamma} \left[\left(\Gamma^{\beta}{}_{\sigma\nu}\right)_L \left(\Gamma^{\gamma}{}_{\mu\alpha}\right)_L - \left(\Gamma^{\beta}{}_{\mu\nu}\right)_L \left(\Gamma^{\gamma}{}_{\sigma\alpha}\right)_L \right] \Big\}. \end{aligned} \tag{D.40}$$