# RECENT DEVELOPMENTS IN PORTFOLIO OPTIMIZATION
# VIA DYNAMIC PROGRAMMING

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# ABSTRACT

RECENT DEVELOPMENTS IN PORTFOLIO OPTIMIZATION
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Optimal control is one of the benchmark methods used to handle portfolio optimization problems. The main goal in optimal control is to obtain a control process that optimizes the objective functional. In this thesis, we investigate optimal control problems for diffusion and jump-diffusion processes. Consequently, we present and prove concepts such as the Dynamic Programming principle, Hamilton-Jacobi-Bellman Equation and Verification Theorem. As an application of our results, we study optimization problems in finance and insurance. In this thesis, we use the Dynamic Programming approach to solve optimal control problems. In the applications, we provide a detailed study of optimal strategies that maximize the expected utility of investors and insurers in finite, random and infinite time horizons. In all applications considered, explicit solutions are obtained for the optimal value function and optimal control processes.



*Keywords* : Hamilton-Jacobi-Bellman Equation, Dynamic Programming Principle, Stochastic Optimal Control, Financial Mathematics, Actuarial Sciences

# ÖZ

DİNAMİK PROGRAMLAMA YOLUYLA PORTFOLYO OPTİMİZASYONUNDAKİ GÜNCEL GELİŞMELER

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Portföy optimizasyonu problemlerinin çözümlerinde, optimal kontrol temel yöntemlerden biridir. Optimal kontrolde esas amaç, hedeflenen fonksiyonu optimize eden kontrol sürecini elde etmektir. Bu tezde, difüzyon ve sıçramak-difüzyon sürecleri için optimal kontrol problemleri incelenmiştir. Bu nedenle, dinamik programlama ilkesi, Hamilton-Jacobi-Bellman Denklemi ve Verification Teoremleri gibi kavramlar sunulmuştur. Sonuçlarımızın uygulaması olarak, dinamik programlama yaklaşımı ile finans ve aktüerya bilimlerinde var olan optimizasyon problemleri incelenmiştir. Uygulamalar kısmında, sonlu, rassal ve sonsuz zaman dilimler altında; yatırımcıların ve sigorta şirketlerinin beklenen fayda fonksiyonunu maksımize eden optimal stratejiler üzerine detaylı bir calışma yapılmıştır. İncelenen tüm uygulamlarda, optimal değer fonksiyonu ve optimal kontrol süreci için analitik sonuçlar elde edilmiştir.

*Anahtar Kelimeler* : Hamilton-Jacobi-Bellman Denklemi, Dinamik Programlama İlkesi, Stokastik Optimal Kontrol, Finansal Matematik, Aktüerya Bilimleri

*To My Family*

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# LIST OF ABBREVIATIONS

| | |
|---|---|
| CRRA | Constant Relative Risk Aversion |
| DP | Dynamic Programming |
| HJB | Hamilton-Jacobi-Bellman |
| i.i.d. | independent and identically distributed |
| ODE | Ordinary Differential Equation |
| PDE | Partial Differential Equation |
| $\mathbb{R}$ | Real Numbers |
| $\mathbb{R}^n$ | n-dimensional Euclidean space |
| $\mathbb{R}^{n\times d}$ | real-valued $n \times d$ matrices |
| SDE | Stochastic Differential Equation |

# CHAPTER 1

# INTRODUCTION

The work of Markowitz (see [22], [23]) led to the emergence of the theory of Portfolio Optimization. Markowitz sought for the highest portfolio return for a given level of risk where the risk is measured by the standard deviation or variance of the portfolio's rate of return. Markowitz's approach to portfolio optimization was termed the Mean-Variance approach. The mean-variance approach has evolved in the literature. Sharpe [31], building on the work of Markowitz, developed the single-index model in which portfolio returns depend only on market index and the covariance between return of assets is ignored. This result yielded the Capital Asset Pricing Model (CAPM).

Optimal control methods is one of the benchmark methods used to handle portfolio optimization problems. Optimal control problems can be viewed as dynamic problems that involve choosing the best path in a system with many feasible paths. In general terms, we describe optimal control problems and worry about technical details in chapters 3 and 4. An optimal control problem consists of a state process $X$, a control process $u$ and a cost functional $J(u)$. The state process which is influenced by the control takes values in $\mathbb{R}^n$ and the control process $u$ takes values in control set $U \subset \mathbb{R}^n$ for $n \geq 1$. The choice of $U$ depends on the problem to be solved. In this thesis, control systems described by Stochastic Differential Equations (SDE) are considered.

The cost functional $J(u)$, which is to be optimized, appropriates the expected cost value to every admissible control [19]. It is of the form

$$J(u) = \mathbb{E}\left[\int_t^T f(s, X_s^{t,x}, u_s)ds + g(T, X_T^{t,x})\right],$$

where $t$ is the initial time, $T$ (finite or infinite) is the terminal time, $x$ is the initial state, $X_T$ is the terminal state and $f$ and $g$ are given functions known as the running cost and terminal cost, respectively. Furthermore, $J(u)$ is called a functional because the control process $u$ is a function of time. The main problem (or the main goal) in optimal control is to obtain a control process $u$ that optimizes the cost functional $J(u)$ over all admissible controls. There are several forms of the optimal control problem described above, and some applications will be explored chapters 3 and 4. Some illustrations of optimal control problems from different applications include:

- Attempt the softlanding of a spacecraft on the lunar space using least amount of fuel,

- Power a railroad car by rocket engines on both sides and attempt to arrive at the origin at zero velocity in the shortest amount of time,
- Optimize reinsurance and dividend policy in the presence of risks so that the insurance company maximizes the expected total dividend payout in a given time interval.

Optimal control problems can be solved via Pontryagin's Maximum Principle or Dynamic Programming (DP) principle [19]. The maximum principle and the DP principle were developed simultaneously but independently. This thesis focuses on the use of DP principle to solve optimization problems. The Markovian property of the problem makes the use of DP principle suitable. Dynamic Programming was initiated by Richard Bellman in the 1950s in [5]. Richard Bellman said, 'in place of determining the optimal sequence of decisions from the fixed state of the system, we wish to determine the optimal decision to be made at any state of the system. Only if we know the latter, do we understand the intrinsic structure of the solution' [5]. The principle claims, in other words, that for any period, if the value of the state variable at the period is a point on an optimal path, then the remaining decisions made after this period must incorporate an optimal policy whose initial condition is the value of the state variable at this period. DP principle reduces the optimal control problem to the problem of solving the Hamilton-Jacobi-Bellman (HJB) Equation. When the HJB equation can be solved, then the optimal value function and optimal control process are found. Moreover, the Verification Theorem that guarantees this. Materials that cover stochastic optimal control in details include Fleming & Soner [12], Kyrlov [2], Fleming & Rishel, [11] and Yong & Zhou[35].

In [24], Merton established the framework for dynamic portfolio choice under uncertainty and applied DP principle to solve the problems. In the paper, Merton sought to maximize the expected utility of running consumption and terminal wealth. In this seminal paper, a closed-form solution to the stochastic control problem faced when the utility function of the investor was assumed to be a power function was derived. Merton's approach has been extended in many studies. For example, portfolio selection with trading constraints, limited borrowing and no bankruptcy was studied by Zariphopoulou [36]. Davis & Norman [10] considered the Merton problem and obtained explicit solutions for the case where there is proportional transaction cost, i.e., the transaction costs are proportional to the amount transacted. Fleming & Hernandez [13] extended Merton's problem to the case where volatility is assumed to be stochastic.

Stochastic control for insurers has been of great interest in recent years. One of the fundamental applications in insurance theory is to use stochastic control theory to minimize the infinite time ruin probability (or maximize the survival probability). The survival probability is the probability that, for a given initial surplus, the surplus will not become negative [21]. Control variables are chosen as investment, reinsurance or dividend payments. Since insurance companies actively participate in investment activities, it is only natural to find trading strategies that will maximize their utilities. In [7], Browne studied the surplus process of the insurance company that is described by a Brownian motion with drift. In this pioneering work, the optimal investment strategy to maximize the expected exponential utility of terminal wealth was found. Moore & Yong [25] incorporated reinsurance policy into Merton's classical optimal investment and consumption framework. The optimal consumption, investment and insurance strategies that maximize the insurer's expected discounted utility of consumption and bequest over a fixed or random horizon were found. Hipp & Plum [15] used the Cramer-Lundberg model to describe the risk process when the surplus of an insurer is invested in risky assets. They were able to find explicit solutions for the case where the claim size follows an exponential distribution. Yang & Zhang [34] were also able to find closed form solutions for the optimal investment strategy of an insurer whose utility function is taken to be exponential. The risk process of the insurer was modelled by a jump-diffusion process. The models can also be generalized to where reinsurance is present. Lin & Yang [20] considered the problem of obtaining the optimal investment and reinsurance strategy that will maximize the exponential utility of terminal wealth of an insurer. The insurer in the problem was allowed to invest in a risk-free asset and a risky asset whose dynamics allows for jumps modelled by a jump-diffusion process. Also, the surplus process of the insurer was modelled by a jump-diffusion process. They were able to obtain closed-form solutions. Cao & Wan [8] obtained the optimal reinsurance and investment policy of an insurer when the surplus process of the insurer follows a Brownian motion with drift. Inspired by the AIG bailout case, Zou & Cadenillas [37] obtained explicit solutions to optimal investment and risk control strategies that maximize the expected utility of terminal wealth of an insurer for various utility functions. Stochastic control in insurance is studied in details in a recent book by Schmidli in [30].

In this thesis, we introduce stochastic optimal control through their applications in finance with and without presence of jumps and particularly deal with cases in which closed-form solutions are obtained. In general, the financial market consists of a risk-free asset and a risky asset. We also review results to exemplify applications of stochastic control in insurance. In the insurance applications that will be considered, we maximize the expected utility of the insurer and their preferences are modelled by exponential utility function because it is the only utility function under which the principle of 'zero utility' gives a fair premium that is dependent on the level of reserve of an insurance company. The thesis is organized as follows. Chapter 2 introduces some mathematical tools that will be needed to model financial assets and solve optimization problems. In Chapter 3, we set up the mathematical framework of stochastic optimal control problems and outline the dynamic programming principle. After, the HJB Equation with its proof is provided. Finally, in Chapter 3, we discuss an application of stochastic optimal control in engineering followed by applications in finance. The financial applications focus on Merton's portoflio consumption problem for two cases. In the first case, the investor exhibits Constant Relative Risk Aversion (CRRA) in a fixed time horizon. In the second case, the investor's preferences is a power utility function in a random time horizon. Lastly, we deal with Merton's portfolio application for the case where the utility function is CRRA. Chapter 4 looks more closely at optimal control problems for jump-diffusion processes and three applications are discussed. The first application is in finance and the final two applications are in insurance. In the first application, the dynamics of the asset price is represented with a jump-diffusion model. The aim in this application is to find a closed-form expression for the optimal investment and consumption strategy that will maximize the utility of the investor over an infinite time horizon. In the second application, the wealth process of the insurer is affected by the presence of a stochastic cash flow which is given by:

$$\text{Surplus} = \text{Initial Capital} + \text{Premium Income} - \text{Cumulated Insurance Claims}.$$

The objective for this application is to find the optimal investment strategy that will maximize the expected exponential utility of terminal wealth of an insurer. In the third application, the insurer is allowed to invest in a financial market and purchase proportional reinsurance to reduce and share risk. The aim in the problem is to obtain the optimal investment and reinsurance strategy which maximizes the expected exponential utility of terminal wealth of an insurer where the insurer's risk is modelled by a jump-diffusion process.

# CHAPTER 2

# PRELIMINARIES

In this chapter, we introduce some mathematical tools needed to model financial markets and solve optimization problems. We shall work in the continuous time setup in this thesis. The time interval $\mathbb{T}$ can be bounded $\mathbb{T} = [0, T], 0 < T < \infty$, or unbounded $\mathbb{T} = [0, \infty]$. In this chapter, we refer to Lamberton & Lapeyre [18], Gan, Ma & Xie [14] and Cont & Tankov [9] for further details.

## 2.1 Diffusion Models

**Definition 2.1.** (***Stochastic Process***) A continuous stochastic process in a space $E$ endowed with a $\sigma$-algebra $\xi$ is a family $(X_t)_{t\geq 0}$ of random variables from a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ into $(E, \xi)$.
The measurable space $(E, \xi)$ is referred to as the state space. For each $\omega \in \Omega$, the mapping $X(\omega) : t \mapsto X_t(\omega)$ is called the path of the process for the event $\omega$. In some cases, we also study vector-valued continuous stochastic processes.

**Definition 2.2.** (***Filtration***) A filtration on $(\Omega, \mathcal{F}, \mathbb{P})$ is an increasing family $(\mathcal{F}_t)_{t\geq 0}$ of $\sigma$-algebras of $\mathcal{F}$ such that $\mathcal{F}_s \subseteq \mathcal{F}_t$ for all $s \leq t$.
$\mathcal{F}_t$ represents the information available up till time $t$. It increases as time elapses.

**Definition 2.3.** (***Natural Filtration***) Let $(X_t)_{t\geq 0}$ be a stochastic process. The natural filtration $\left(\mathcal{F}_t^X\right)_{t\geq 0}$ of $X$ is defined as

$$\mathcal{F}_t^X := \sigma(X_s : s \leq t),\ t \geq 0.$$

Here, $\mathcal{F}_t^X$ can be interpreted as the whole information concerning the process that can be observed from its paths between time 0 and $t$.

**Definition 2.4.** (***Adapted Process***) A process $(X_t)_{t\geq 0}$ is adapted to $(\mathcal{F}_t)_{t\geq 0}$ if for all $t$, $X_t$ is $\mathcal{F}_t$- measurable.
In other words, an adapted process is a process whose value can be determined by the information available at time $t$.

**Definition 2.5.** (***Stopping time***) A random variable $\tau$ taking values in $\mathbb{R}^+ \cup \{\infty\}$ is a stopping time with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$ if for any $t \geq 0$,

$$\{\tau \leq t\} \in \mathcal{F}_t.$$

**Definition 2.6.** (***Standard Brownian Motion***) A continuous stochastic process $(W_t)_{t\geq 0}$ is called a standard Brownian motion if it satisfies:

a. $W_0 = 0, \mathbb{P}$ a.s.

b. For all $0 \leq s \leq t$, the increment $W_t - W_s$ is a random variable normally distributed with expectation 0 and variance $t - s$.

c. For all $0 \leq s \leq t$, the increment $W_t - W_s$ is independent of $\sigma(W_u, u \leq s)$.

*Remark* 2.1. The distribution of $W_t$ is given by $\frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{x^2}{2t}\right) dx$.
Brownian motion is used to model random behaviour that evolves over time. Such random behaviour represents the fluctuations of an asset's price.

**Definition 2.7.** (***Martingale***) Consider the filtered space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. An adapted family $(X_t)_{t\geq 0}$ of integrable random variables, i.e., $\mathbb{E}[|M_t|] < \infty$ for all $t \geq 0$ is a supermartingale if

$$\mathbb{E}[X_t|\mathcal{F}_s] \leq X_s, \text{ a.s., } \forall s \leq t;$$

$(X_t)_{t\geq 0}$ is called a submartingale if

$$\mathbb{E}[X_t|\mathcal{F}_s] \geq X_s, \text{ a.s.} \forall s \leq t.$$

Finally, $X$ is said to be martingale if

$$\mathbb{E}[X_t|\mathcal{F}_s] = X_s, \text{ a.s.} \forall s \leq t.$$

**Definition 2.8.** (***Itô process***) Let $(W_t)_{t\geq 0}$ be an $\mathcal{F}_t$-Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. A stochastic process $(X_t)_{0\leq t\leq T}$ is called an Itô process if it has the form

$$X_t = X_0 + \int_0^t U_s ds + \int_0^t V_s dW_s, \tag{2.1}$$

where $X_0$ is $\mathcal{F}_0$-measurable, $(U_t)_{0\leq t\leq T}$ and $(V_t)_{0\leq t\leq T}$ are $\mathcal{F}_t$-adapted, $\int_0^t |U_s| ds < \infty$ $\mathbb{P}$ a.s., $\int_0^t |V_s|^2 ds < \infty$ $\mathbb{P}$ a.s.

For convenience, Eqn. (2.1) can be written in its differential form as

$$dX_t = U_t dt + V_t dW_t.$$

**Theorem 2.1.** ***(Itô Formula)*** *Let $X_t$ be an Itô process $dX_t = U_t dt + V_t dW_t$. Suppose $f(\cdot) \in C^2(\mathbb{R})$ is a twice continuously differentiable function. Then,*

$$Y_t = f(X_t) = f(X_0) + \int_0^t f'(X_s) U_s ds + \int_0^t f'(X_s) V_s dW_s + \frac{1}{2}\int_0^t f''(X_s) d\langle X, X\rangle_s,$$

*where $\langle X, X\rangle_t = \int_0^t V_s^2 ds$.*
*Furthermore, suppose $f(\cdot,\cdot) \in C^{1,2}$ is a function in $(t, x)$ which is once differentiable with respect to $t$ and twice differentiable with respect to $x$, then the Itô formula gives us*

$$f(t, X_t) = f(0, X_0) + \int_0^t f_s^{'}(s, X_s)ds + \int_0^t f_x^{'}(s, X_s)U_s ds + \int_0^t f_x^{'}(s, X_s)V_s dW_s \\ + \frac{1}{2}\int_0^t f_{xx}^{''}(s, X_s)d\langle X, X\rangle_s .$$

*Proof.* The detailed proof of the theorem can be seen in Karatzas & Shreve [16] □

**Definition 2.9. (*Martingale Representation Theorem*)** Consider the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. Assume that $(\mathcal{F}_t^W)_{t\geq 0}$ is the natural filtration generated by the Brownian motion $W_t$. Let $M_t$ be a square integrable martingale relative to this filtration. Then, there exists an $\mathcal{F}$-adapted process $\phi$ such that

$$M_t = M_0 + \int_0^t \phi_s dW_s.$$

In other words, any martingale adapted with respect to a Brownian motion can be expressed as a stochastic integral with respect to the Brownian motion.

**Definition 2.10. (*Stochastic Differential Equations*)** Consider the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. Suppose we have the equation

$$X_t = x + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s, \tag{2.2}$$

where $x$ is a $\mathcal{F}_0$-measurable random variable, $b : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$, $\sigma : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ and $(W_t)_{t\geq 0}$ is an $\mathcal{F}_t$-Brownian motion. Such equations are called Stochastic Differential Equations (SDEs).

*Remark* 2.2. The stochastic process $(X_t)_{t\geq 0}$ that solves Eqn. (2.2) is $\mathcal{F}_t$-adapted and is called a *diffusion process* and satisfies:

a. For all $t \geq 0$, $\int_0^t b(s, X_s)ds$ and $\int_0^t \sigma(s, X_s)dW_s$ exist.
   That is, $\int_0^t |b(s, X_s)|ds < \infty$ and $\int_0^t |\sigma(s, X_s)|^2 ds < \infty\, \mathbb{P}$ a.s.

b. Eqn. (2.2). That is,

$$\forall t \geq 0\, \mathbb{P}\,\text{a.s.},\ X_t = x + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s.$$

**Theorem 2.2.** ***(Existence and Uniqueness of a solution)*** *Let $b(\cdot)$ and $\sigma(\cdot)$ be continuous functions. Suppose there exists a constant $C > 0$ such that the following conditions are satisfied for all $x$, $y$ and $t$:*

$$|b(t, x) - b(t, y)| + |\sigma(t, x) - \sigma(t, y)| < C|x - y|, \qquad \text{(\textit{Lipschitz condition})}$$

$$|b(t,x)| + |\sigma(t,x)| < C(1+|x|), \qquad \text{(\textit{Linear growth condition})}$$

$$\mathbb{E}\left[|x|^2\right] < \infty.$$

*Then there exists a unique solution to the SDE in Eqn.* (2.2).
*Furthermore, the solution* $(X_s)_{0\le s\le T}$ *satisfies*

$$\mathbb{E}\left[sup_{0\le s\le T}|X_s|^2\right] < \infty.$$

*Proof.* The proof of the theorem can be seen in Lamberton & Lapeyre [18]. □

The SDEs can be extended to the multidimensional case. The existence and uniqueness theorem of the multidimensional case is similar to the theorem above. We refer readers Lamberton & Lapeyre [18] for more details.

**Definition 2.11. (*Markov Property*)** Let $f$ be a bounded Borel function. The $\mathcal{F}_t$-adapted process $(X_t)_{t\ge0}$ is said to satisfy the Markov property if for all $t$ and $0 < s < t$,

$$\mathbb{E}\left[f(X_t)|\mathcal{F}_s\right] = \mathbb{E}\left[f(X_t)|X_s\right].$$

The Markov property states that the future states of a process $(X_t)_{t\ge0}$ that satisfies the property depends only on the present state $x$ and not on the previous history of the process that preceded it.
The diffusion process $(X_t)_{t\ge0}$ which is the solution of the SDE in Eqn. (2.2) satisfies the Markov property. Therefore, we can denote the solution by $(X_s^{t,x} : s \ge t)$. That is, $X_t$ is the solution of the SDE in Eqn. (2.2) starting from $x$ and time $t$. The Markovian property of the diffusion process makes the use of the dynamic programming approach to solve optimal control problems appropriate.
Note: $X_s^{0,x} = X_s^{t,X_t^x}$ $\mathbb{P}$ a.s. $\forall s \ge t$.

**Definition 2.12. (*Infinitesimal generator*)** Let $X = (X_t)_{t\ge0}$ with $X_t \in \mathbb{R}^n$ be a diffusion process. Then the infinitesimal generator $\mathcal{L}$ of $(X_t)_{t\ge0}$ is defined on functions $f : \mathbb{R}^n \to \mathbb{R}$ by

$$\left(\mathcal{L}f\right)(x) = \lim_{t\downarrow0}\frac{\mathbb{E}[f(X_t)] - f(x)}{t},\ x \in \mathbb{R}^n \text{ (if the limit exists).}$$

**Theorem 2.3.** *Let* $(X_t)_{t\ge0}$ *be an n-dimensional Itô diffusion process,* $dX_t = b(X_t)dt + \sigma(X_t)dW_t$ *where* $b$, $\sigma$ *are continuous functions and* $W_t$ *is an n-dimensional Brownian motion. Moreover, suppose* $f \in C_0^2(\mathbb{R}^n)$. *Then* $\left(\mathcal{L}f\right)(x)$ *exists and*

$$\left(\mathcal{L}f\right)(x) = \sum_i b_i(x)\frac{\partial f}{\partial x_i}(x) + \frac{1}{2}\sum_{i,j}(\sigma\sigma^T)_{ij}(x)\frac{\partial^2 f}{\partial x_i \partial x_j}(x).$$

*Proof.* The proof can be found in Øksendal [26]. □

As previously noted, diffusion processes have continuous paths and are useful in modeling price movements. Black Scholes model is one of the well known examples of diffusion models. Diffusion models are very practical in complete markets where claims can be perfectly hedged. In diffusion models, large sudden price movements do not happen.
In real markets, asset prices undergo abrupt jumps in some periods, have strong price movements in a short time and often display discontinuous behaviours. Hence, not all empirical studies are solved by diffusion models. Models with jumps take these into consideration and seem to be more superior than diffusion models. Furthermore, they take into account the risks that cannot be hedged and thus integrate the risk into the exposure of the portfolio. Finally, models with jumps reproduce more realistic properties of presence of jumps in observed prices.

## 2.2 Jump-Diffusion and Lévy Processes

Financial models with jumps can be categorized into two. One category consists of jump-diffusion models and the other consists of models that have infinite number of jumps in each interval (Lévy process). Every jump-diffusion model has two main parts. A diffusion part that has a Brownian component and a jump part which is a compound Poisson process with finite activity. The class of infinite activity models do not always contain a Brownian component as the dynamics of the process essentially move by jumps. These models are said to give more realistic description of observed price movements. Now, we introduce jump-diffusion processes and some of the well-known examples. We finalize with the Itô-Doeblin Formula.

### 2.2.1 Jump-Diffusion Processes

A jump diffusion is a process of the form:

$$X_t = X_0 + \int_0^t b_s ds + \int_0^t \sigma_s dW_s + J_t. \tag{2.3}$$

As previously mentioned, jump diffusion processes have two parts; the diffusion part and the jump part. The jump part is represented by $J_t$. Here, $J = (J_t)$ is an adapted, right continuous pure jump process. $J_t$ is the value of $J$ after the jump and $J_{t-}$ is the value of $J$ immediately before the jump. The jump process $(J_t)$ has finitely many jumps in every time interval and is constant between jumps. The compound Poisson process which will be examined in details below, is a jump process with finite activity, i.e., it has finite number of jumps at any interval so $(J_t)$ covers for compound Poisson process here. Now, we define Poisson and compound Poisson processes.

**Definition 2.13.** **(*Poisson Process*)** Consider the sequence $(\tau_j)_{j\in\mathbb{N}}$ of independent exponential random variables with parameter $\lambda$ and $T_n = \sum_{j=1}^n \tau_j$. The process $(N_t : t \geq 0)$ given by

$$N_t = \sum_{n\in\mathbb{N}} 1_{\{t \geq T_n\}}$$

is called a Poisson process with intensity $\lambda$.

***Properties:***

- The Poisson process $(N_t : t \geq 0)$ counts the number of jumps which occur between time 0 and $t$.
- The jumps occur at $T_j$ with size of 1 only and the interval between jumps are exponentially distributed.
- The Poisson process $(N_t : t \geq 0)$ takes values in $\mathbb{N} \cup \{0\}$ with the relation
$$\mathbb{P}\{N_t = n\} = \frac{(\lambda t)^n e^{-\lambda t}}{n!}, \ (n = 0, 1, 2, ...).$$
- A Poisson process is a Lévy process.
- The characteristic function of a Poisson process is given by
$$\mathbb{E}\left[e^{iuN_t}\right] = \exp\left(\lambda t(e^{iu} - 1)\right).$$

**Definition 2.14. (*Compound Poisson process*)** A compound Poisson process is a stochastic process $(R_t : t \geq 0)$ with intensity $\lambda > 0$ and jump size distribution $G$ defined as

$$R_t = \sum_{i=1}^{N_t} Y_i,$$

where $(N_t : t \geq 0)$ is the Poisson process with intensity $\lambda$ and $(Y_j)_{j\in\mathbb{N}}$ is a sequence of i.i.d. random variables with distribution $G$.

***Properties:***

- In the compound Poisson process, the interval between jumps are exponential but the jump sizes have arbitrary distribution.
- A compound Poisson process is a Lévy process.
- The characteristic function of the compound Poisson process has the form
$$\mathbb{E}\left[e^{iuR_t}\right] = \exp\left(\lambda t \int_{\mathbb{R}} (e^{iu} - 1)G(dr)\right).$$

For a calculation of the characteristic function, see Cont & Tankov [9].

**Theorem 2.4.** ***(Itô-Doeblin Formula)*** *Let $(X_t)$ be a jump process given by Eqn.* (2.3). *Suppose $f(\cdot) \in C^2(\mathbb{R})$ is a twice continuously differentiable function. Then,*

$$f(X_t) = f(X_0) + \int_0^t f^{'}(X_s)b_s ds + \int_0^t f^{'}(X_s)\sigma_s dW_s + \frac{1}{2}\int_0^t f^{''}(X_s)\sigma_s^2 ds + \sum_{0\leq s\leq t} [f(X_s) - f(X_{s-})]. \quad (2.4)$$

*Proof.* The proof can be found in Shreve [32]. □

The theorem follows from the fact that if there is a jump in $X$ from $X_{s-}$ to $X_s$, it generally leads to a jump in $f(X)$ from $f(X_{s-})$ to $f(X_s)$. It is not always possible to write Eqn. (2.4) in differential form but for the scope of this thesis, it can be written as we shall be dealing with cases where the jump part is a compound Poisson process:

$$f(X_t) = f(X_0) + \int_0^t f^{'}(X_s)b_s ds + \int_0^t f^{'}(X_s)\sigma_s dW_s + \frac{1}{2}\int_0^t f^{''}(X_s)\sigma_s^2 ds \\ + \int_0^t \left[f(X_s) - f(X_{s-})\right] dN_s,$$

where $(N_s)$ is a Poisson process as defined in Definition 2.13

### 2.2.2 Lévy Processes

So far, we have given useful details about jump-diffusion processes that are needed in application. Now, important results about Levy processes will be detailed. For further details on properties of Levy processes, see Øksendal [27], Okur [28], Cont & Tankov [9] and Applebaum [1].

**Definition 2.15.** **(*Lévy Process*)** Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space. The càdlàg (i.e. left continuous with right sided limits) stochastic process $(L_t : t \geq 0)$ is called a Lévy process if it has the following properties:

i. *Stationary increments*: for any $s \leq t$, $L_t - L_s$ is equal in distribution to $L_{t-s}$.

ii. *Independent increments*: for any $0 \leq t_1 < t_2 < ... < t_n$, the random variables $L_0$, $L_t - L_0$,...,$L_{t_n} - L_{t_n - 1}$ are independent.

iii. *Stochastic continuity*: $\forall \epsilon > 0, \lim_{h\downarrow 0} \mathbb{P}\left(|L_{t+h} - L_0| \geq \epsilon\right) = 0$.

iv. $L_0 = 0$ almost surely.

The jump of $L_t$ at time $t \geq 0$ is given by

$$\Delta L_t = L_t - L_{t-}.$$

Define

- $\mathbb{B}$ as the family of all Borel subsets $U \subseteq \mathbb{R}$ such that the closure of $U$ does not contain 0.
- For $U \in \mathbb{B}$, the jump measure $M([t_1, t], U)$ is the number of jumps of size $\Delta L_t \in U$ that occur between times $t_1$ and $t$, $(t_1 < t)$.

**Definition 2.16.** ***(Lévy measure)*** The Lévy measure of $(L_t : t \geq 0)$ is defined by

$$v(U) := \mathbb{E}\left[M([0,1], U)\right].$$

That is, the Lévy measure $v(U)$ of the process $(L_t)$ is the expected number, per unit time, of jumps whose sizes are in $U$.
The differential form of the jump process $M([0,1], U)$ is written as $M(dt, dz)$. The compensated jump measure of $(L_t)$ is given by

$$\tilde{M}(dt, dz) = M(dt, dz) - v(dz).$$

**Theorem 2.5.** ***(Lévy-Khintchine Formula)*** *Let $(L_t)$ be a Levy process. Then its characteristic function is given by the Lévy-Khintchine formula:*

$$\mathbb{E}\left[e^{iuL_t}\right] = \exp\left\{t\left(iu\mu - \frac{\sigma^2 u^2}{2} + \int_{\mathbb{R}} \left(e^{iuz} - 1 - iuz1_{|z|<1}\right) v(dz)\right)\right\},$$

*where $\mu \in \mathbb{R}$ and $\sigma^2 \geq 0$ are constants and $v$ is the jump measure on $\mathbb{B}$ satisfying*

$$\int_{\mathbb{R}} \min(1, z^2) v(dz) < \infty.$$

**Theorem 2.6.** ***(Ito-Lévy Decomposition)*** *Let $(L_t)$ be a Lévy process and $v$ its Lévy measure. Then, there exist $a_1$ and $b \in \mathbb{R}$ such that*

$$L_t = a_1 t + bW_t + \int_0^t \int_{|z|<1} z\tilde{M}(ds, dz) + \int_0^t \int_{|z|\geq 1} zM(ds, dz).$$

If the Lévy process is square integrable, i.e., $\mathbb{E}[L_t^2] < \infty$, then $\int_{|z|\geq 1} |z|^2 v(dz) < \infty$. Herewith, the representation becomes

$$\begin{aligned}
L_t &= a_1 t + bW_t + \int_0^t \int_{|z|<1} z\tilde{M}(ds, dz) + \int_0^t \int_{|z|\geq 1} zM(ds, dz) \\
&= a_1 t + bW_t + \int_0^t \int_{|z|<1} z\tilde{M}(ds, dz) + \int_0^t \int_{|z|\geq 1} z(\tilde{M}(ds, dz) + v(dz)dt) \\
&= a_1 t + bW_t + \int_0^t \int_{\mathbb{R}} z\tilde{M}(ds, dz) + t\int_{|z|\geq 1} zv(dz) \\
&= at + bW_t + \int_0^t \int_{\mathbb{R}} z\tilde{M}(ds, dz),
\end{aligned}$$

where

$$a = a_1 t + t\int_{|z|\geq 1} zv(dz).$$

The Decomposition Theorem points out that every Lévy process is a linear combination of Brownian motion with drift and a possibly infinite sum of independent compound Poisson process.

*Remark* 2.3. Lévy process demonstrates strong Markov property, i.e., the process $(L_{t+s} - L_t)_{s\geq 0}$ has the same probability law as process $(L_s)_{0\leq s\leq t}$.

*Remark* 2.4. Consider the SDE of the form

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t + \int_{\mathbb{R}^n} h(t, X_{t-}, z)\tilde{M}(dt, dz); X_0 = x, \tag{2.5}$$

where the deterministic functions $b : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, $\sigma : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$ and $h : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times l}$ satisfy the Lipschitz continuity condition with respect to $l$ and the linear growth condition in $l$, uniformly in $t$. Thereby, guaranteeing that there exists a unique càdlàg adapted solution $(X_t)$ such that $\mathbb{E}[|X_t|^2] < \infty$, $\forall t$ to Eqn. (2.5). Such processes are called *Itô-Lévy processes*.

*Remark* 2.5. Jump diffusion processes can be looked at in another way which is outlined below:
If $b(t, l) = b(l)$, $\sigma(t, l) = \sigma(t)$ and $h(t, x, z) = h(x, z)$, then the corresponding SDE is of the form

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t + \int_{\mathbb{R}^n} h(X_{t-}, z)\tilde{M}(dt, dz). \tag{2.6}$$

Solutions of the SDE (2.6) are called *Lévy-diffusions*. Again, the above model is punctuated by jumps at random intervals. The jumps could represent cases like crashes or large movements in asset price.

**Theorem 2.7.** ***(Infinitesimal generator)*** *Let $(X_t)$ be a Lévy-diffusion process as defined in Eqn.* (2.6) *and suppose $f \in C_0^2(\mathbb{R}^n)$. Then, $(\mathcal{L}f)(x)$ exists and*

$$\begin{aligned}(\mathcal{L}f)(x) = &\sum_{i=1}^{n} b_i(x)\frac{\partial f}{\partial x_i}(x) + \frac{1}{2}\sum_{i,j=1}^{n}(\sigma\sigma^T)_{ij}(x)\frac{\partial^2 f}{\partial x_i \partial x_j}(x) \\ &+ \int_{\mathbb{R}}\sum_{k=1}^{l}\left\{f(x + h^{(k)}(x, z)) - f(x) - \nabla f(x)\cdot h^{(k)}(x, z)\right\} v_k(dz_k).\end{aligned}$$

# CHAPTER 3

# OPTIMAL CONTROL FOR DIFFUSION PROCESSES: APPLICATIONS IN ENGINEERING AND FINANCE

In stochastic control problems in finance, the aim is to find trading strategies that motivate minimal costs and maximal expected utility. Typically, there is a state process whose dynamics is altered with a control process. In this chapter, the state processes are described by Stochastic Differential Equations (SDEs) called diffusion models. Among all possible decisions, we choose the optimal one to achieve the best expected result depending on the objective. The decisions, also called control processes, are made based on the available information. We try to obtain the optimal control process that maximizes the value of the state process. The maximal value is called the optimal value function. The Dynamic Programming (DP) principle for the stochastic control problem leads to the Hamilton-Jacobi-Bellman (HJB) Equation. Since the value function is unknown, we make a guess and then show that it satisfies the equation. Furthermore, the verification theorem is used to show that the guess is indeed the value function. In the applications, we focus on problems where explicit solutions can be found. Stochastic control problems have applications in economics, finance, insurance, engineering, to name a few.

The content of this chapter is as follows. In the first section, the mathematical framework for optimal control problems is set up. In the second part, we state the general control problem, outline the dynamic programming principle and then present the HJB Equation. The proofs of the HJB Equation and the Verification Theorem are also outlined. Finally in the third section, an application of stochastic optimal control in engineering is given. The application, called The Linear Quadratic Regulator, involves finding an optimal process that keeps an initially excited system close to its equilibrium position. Afterwards, we focus on applications in finance. In the first financial application, the optimal investment and consumption problem is studied for an investor who has an initial endowment and is allowed to consume and invest in a financial market with a risk-free asset and a risky asset. In the second application, we consider Merton's portflio allocation problem where the aim is to maximize expected utility of terminal wealth of an investor over finite time horizon.

## 3.1 The Formal Problem

In this section, the general class of optimal control problems is studied. Consider the following controlled system of SDEs:

$$\begin{cases} dX_t = b(t, X_t, u_t)dt + \sigma(t, X_t, u_t)dW_t; \\ X_t = x, \end{cases} \tag{3.1}$$

where $b : \mathbb{R} \times \mathbb{R}^n \times U \longrightarrow \mathbb{R}^n$, $\sigma : \mathbb{R} \times \mathbb{R}^n \times U \longrightarrow \mathbb{R}^{n \times d}$ are two continuous functions, respectively, satisfying the conditions

1. $\|b(t, x, u) - b(t, y, u)\| + \|\sigma(t, x, u) - \sigma(t, y, u)\| \leq C \|x - y\|$ ,
2. $\|b(t, x, u) + b(t, y, u)\| \leq C(1 + \|x\| + \|y\|)$,

where $C < +\infty$ is a constant. We note that $\|\cdot\|$ is a generic notation for Euclidean norms in the corresponding Euclidean spaces.

Here, $X = (X_t)$ with $X_t \in \mathbb{R}^n$ is an $n$-dimensional state process controlled by a process $u = (U_t)$ with $u_t \in U \subseteq \mathbb{R}^k$, and $(W_t)$ is a $d$-dimensional Brownian motion. The process $(u_t)$ is $\mathcal{F}_t^d$ - adapted since the decision at time $t$ depends on past observed values of state process $X$. So we shall be dealing with control process $u$ defined by

$$u_t = u(t, X_t).$$

The function $u_t$ is called a feedback control because the control depends on the state process at time $t$, for all $t \in \mathbb{R}$.

**Definition 3.1.** **(*Admissible Processes*)** A control process $u$ is admissible if $u(t, x) \in U$ for all $t \in \mathbb{R}$ and all $x \in \mathbb{R}^n$ and if for any given initial point $(t, x)$, Eqn. (3.1) has a unique solution. The set of all admissible control processes is denoted by $\mathcal{A}$.

**Objective Function:**
Let $f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^k \longrightarrow \mathbb{R}$ , $g : \mathbb{R}^n \longrightarrow \mathbb{R}$ be given functions. We can then define the value function as the function

$$J : \mathcal{A} \longrightarrow \mathbb{R},$$

defined by

$$J(t, x, u) = \mathbb{E}\left[\int_t^T f(s, X_s^{t,x}, u_s)ds + g(T, X_T^{t,x})\right]$$

for all $(t, x) \in [0, T] \times \mathbb{R}^n$ and $u \in \mathcal{A}$, where $(X_s^{t,x})$ denotes the solution of Eqn. (3.1) starting from $x$ at time $t$: $X_t = x$. We note that $f$ represents the running cost while $g$ represents the terminal cost.
Thus, the objective is to maximize the value function $J$ over feedback control processes $u \in \mathcal{A}$. We introduce the value function:

$$V(t, x) := \sup_{u \in \mathcal{A}} J(t, x, u).$$

If $V(t, x) = J(t, x, \widehat{u})$, we call $(\widehat{u}(t, x))$ an *optimal control process* and $V(t, x)$ is called the *optimal value function* for the problem.

## 3.2 Hamilton-Jacobian-Bellman Equation

### 3.2.1 The Control Problem

The control problem is to maximize the value function

$$J(t,x,u) = \mathbb{E}\left[\int_t^T f(s, X_s^{t,x}, u_s)ds + g(T, X_T^{t,x})\right]$$

given the dynamics

$$\begin{aligned} dX_t &= b(t, X_t, u_t)dt + \sigma(t, X_t, u_t)dW_t; \\ X_t &= x. \end{aligned}$$

We note that the SDE above is as described in Eqn. (3.1).

### 3.2.2 Dynamic Programming Principle

We shall now describe the Dynamic Programming principle which closely follows [6]. The principle is based on the intuition that the control problem can be divided into two parts. The first part involves the notion that the optimal control $\widehat{u}$ on the interval $[t,T]$ can be obtained by looking for the optimal control on $(t+h,T]$ and the second involves maximizing over all control processes on $[t, t+h]$.
Fix $(t,x) \in [0,T] \times \mathbb{R}^n$ and consider $h \in \mathbb{R}$ such that $t+h < T$. On time interval $[t,T]$, we first use the optimal control $\widehat{u}$ and then use the admissible control process $\overline{u}$ defined by

$$\overline{u}(s,y) = \begin{cases} u(s,y), \text{if } (s,y) \in [t, t+h) \times \mathbb{R}^n, \\ \widehat{u}(s,y), \text{if } (s,y) \in (t+h, T] \times \mathbb{R}^n. \end{cases}$$

As for the optimal control $\widehat{u}$, we know that $J(t,x,\widehat{u}) = V(t,x)$.
For the control process $\overline{u}$,

$$\begin{aligned} J\left(t,x,\overline{u}\right) &= \mathbb{E}\left[\int_t^T f(s, X_s^{t,x}, \overline{u}_s)ds + g(T, X_T^{t,x})\right] \\ &= \mathbb{E}\left[\int_t^{t+h} f(s, X_s^{t,x}, u_s)ds\right] + \mathbb{E}\left[\int_{t+h}^T f(s, X_s^{t,x}, \widehat{u}_s)ds + g(T, X_T^{t,x})\right] \\ &= \mathbb{E}\left[\int_t^{t+h} f(s, X_s^{t,x}, u_s)ds\right] + \mathbb{E}\left[V\left(t+h, X_{t+h}^{t,x}\right)\right]. \end{aligned}$$

Since $V(t,x) = J\left(t,x,\widehat{u}\right) \geq J(t,x,\overline{u})$ on $[0,T]$,

$$V(t,x) \geq \mathbb{E}\left[\int_t^{t+h} f(s, X_s^{t,x}, u_s)ds + V\left(t+h, X_{t+h}^{t,x}\right)\right]. \tag{3.2}$$

Equality in Eqn. (3.2) holds if and only if the arbitrarily chosen control process $u$ is an optimal control $\widehat{u}$. This leads us to the following important theorem.

**Theorem 3.1.** ***(Hamilton-Jacobi-Bellman Equation)*** *Assume that there exists an optimal control process $\widehat{u}$ and that the optimal value function $V$ is regular in the sense that $V \in C^{1,2}$, then:*

*i)* *$V$ satisfies the HJB Equation*

$$V_t(t,x) + \textit{sup}_{u\in\mathcal{A}}\left(f(t,x,u) + \left(\mathcal{L}^u V\right)(t,x)\right) = 0,\ \forall (t,x) \in [0,T]\times\mathbb{R}^n, \quad (3.3)$$

*with*

$$V(T,x) = g(x),\ \forall x \in \mathbb{R}^n, \quad (3.4)$$

*where $\left(\mathcal{L}^u V\right)(t,x) = b(x,u)V_x + \frac{1}{2}tr(\sigma(x,u)\sigma^T(x,u))V_{xx}$ is the infinitesimal generator as defined in Definition 2.12.*

*ii)* *For each $(t,x) \in [0,T]\times\mathbb{R}^n$ the supremum in the HJB Equation above is attained by $u = \widehat{u}(t,x)$.*

*Remark* 3.1. We sometimes write the partial differential equation of Eqn. (3.3) in the form

$$V_t(t,x) + H(t,x,V_x(t,x),V_{xx}(t,x)) = 0, \forall (t,x) \in [0,T]\times\mathbb{R}^n.$$

The function $H$ is called the *Hamiltonian* of the associated control problem.
For $(t,x,r,s) \in [t,x) \in [0,T]\times\mathbb{R}^n\times\mathbb{R}^n\times S_n$, where $S_n$ is an $n\times n$ square matrix.

$$H(t,x,r,s) = \sup_{u\in\mathcal{A}}[b(x,u)\cdot r + \frac{1}{2}tr(\sigma(x,u)\sigma^T(x,u)\cdot s + f(t,x,u)].$$

*Remark* 3.2. Eqn. (3.3) is called the *Hamilton-Jacobi-Bellman Equation* or *Dynamic Programming Equation*.

*Proof.* Assume that $V$ is smooth enough. So, apply Itô formula to $V\left(t+h, X^{t,x}_{t+h}\right)$ between $t$ and $t+h$:

$$\begin{aligned} V(t+h, X^{t,x}_{t+h}) = V(t,x) + \int_t^{t+h} V_t(s, X^{t,x}_s)ds + V_x(s, X^{t,x}_s)dX^{t,x}_s \\ + \frac{1}{2}\int_t^{t+h} V_{xx}(s, X^{t,x}_s)d\left\langle X, X\right\rangle_s, \end{aligned}$$

$$\begin{aligned} V(t+h, X^{t,x}_{t+h}) = V(t,x) + \int_t^{t+h} V_t(s, X^{t,x}_s)ds + \int_t^{t+h} V_s(s, X^{t,x}_s)bds \\ + \int_t^{t+h} V_x(s, X^{t,x}_s)\sigma dW_s + \frac{1}{2}\int_t^{t+h} V_{xx}(s, X^{t,x}_s)d\left\langle X, X\right\rangle_s, \end{aligned}$$

$$\begin{aligned} V(t+h, X^{t,x}_{t+h}) = V(t,x) + \int_t^{t+h} V_t(s, X^{t,x}_s) + V_x(s, X^{t,x}_s)b + \frac{1}{2}V_{xx}(s, X^{t,x}_s)ds \\ + \int_t^{t+h} V_x(s, X^{t,x}_s)\sigma dW_s, \end{aligned}$$

$$V(t+h, X_{t+h}^{t,x}) = V(t,x) + \int_t^{t+h} \left[V_t(s, X_s^{t,x}) + (\mathcal{L}^u V)(s, X_s^{t,x})\right] ds + \int_t^{t+h} V_x(s, X_s^{t,x}) \sigma dW_s.$$

Now, we take expectation and substitute into the Eqn. (3.2) to get

$$V(t+h, X_{t+h}^{t,x}) \geq \mathbb{E}\left[\int_t^{t+h} f(s, X_s^{t,x}, u_s) ds + V(t+h, X_{t+h}^{t,x})\right] + \mathbb{E}\left[\int_t^{t+h} \left[V_t(s, X_s^{t,x}) + (\mathcal{L}^u V)(s, X_s^{t,x})\right] ds + \int_t^{t+h} V_x(s, X_s^{t,x}) \sigma dW_s\right].$$

Assuming enough integrability, the expected value of the integral with Brownian motion is zero.
So,

$$0 \geq \mathbb{E}\left[\int_t^{t+h} f(s, X_s^{t,x}, u_s) ds + V_t(s, X_s^{t,x}) ds + (\mathcal{L}^u V)(s, X_s^{t,x}) ds\right].$$

Let us divide the inequality through by $h$ and take the limit as $h$ goes to zero and by the Mean Value theorem, we understand

$$0 \geq V_t(t,x) + (\mathcal{L}^u V)(t,x) + f(t,x,u).$$

Since the control process $u$ is arbitrary, the inequality holds for all $u \in U$. So, we have the following:

$$V_t(t,x) + \sup_{u \in U} \left((\mathcal{L}^u V)(t,x) + f(t,x,u)\right) = 0,$$

since the equality holds if and only if the arbitrarily chosen control process is the optimal control process $\widehat{u}$.
Consequently, $V(T,x) = g(x)$ for all $x \in \mathbb{R}^n$. □

*Remark* 3.3. The theorem states that if an optimal control $\widehat{u}$ exists, then the optimal value function $V$ satisfies the HJB equation in Eqn. (3.3) and ***at point*** $t$**,** $f(t,x,\widehat{u}) + \left(\mathcal{L}^{\widehat{u}} V\right)(t,x)$**,** $(u \in U)$ ***attains its maximum***. Thus, the theorem has the form of a necessary condition. Luckily, the HJB equation is also a sufficient condition for our optimal control problem. The next result, commonly referred to as the verification theorem, states the condition.

**Theorem 3.2.** ***(Verification Theorem)*** *Let $P$ be a function in $C^{1,2}([0,T] \times \mathbb{R}^n)$ that solves the HJB Equation*

$$P_t(t,x) + sup_{u \in \mathcal{A}} \left(f(t,x,u) + (\mathcal{L}^u P)(t,x)\right) = 0, \ \forall (t,x) \in [0,T] \times \mathbb{R}^n \tag{3.5}$$

*with terminal condition*

$$P(T,x) = g(x), \ \forall x \in \mathbb{R}^n.$$

*Suppose there exists an admissible control process $\widehat{u} \in \mathcal{A}$ such that for each $(t,x)$, $u$ attains the supremum in the HJB equation. In other words,*

$$sup_{u \in \mathcal{A}} \left\{f(t,x,u) + (\mathcal{L}^u P)(t,x)\right\} = f(t,x,u) + \left(\mathcal{L}^{\widehat{u}} P\right)(t,x),$$

*such that the SDE which is as defined in Eqn.* (3.1)

$$\begin{aligned} dX_s &= b\left(s, \widehat{u}(s, X_s)\right) ds + \sigma\left(s, \widehat{u}(s, X_s)\right) dW_s; \\ X_t &= x, \end{aligned}$$

*admits a unique solution* $\widehat{X}_s^{t,x}$*. Then,* $P = V$*, the optimal value function and* $\widehat{u}$ *is the optimal feedback control.*

*Proof.* Suppose $P$ and $\widehat{u}$ are as given. Fix $(t, x) \in [0, T] \times \mathbb{R}^n$ and choose an arbitrary control process $u \in \mathcal{A}$.
Apply Itô formula to $P(T, X_T^{t,s})$

$$\begin{aligned} P(T, X_T^{t,s}) = P(t, x) + \int_t^T P_t(s, X_s^{t,x}) ds + \int_t^T P_x(s, X_s^{t,x}) dX_s^{t,x} \\ + \frac{1}{2} \int_t^T P_{xx}(s, X_s^{t,x}) d\left\langle X, X \right\rangle_s, \end{aligned}$$

$$\begin{aligned} P(T, X_T^{t,s}) = P(t, x) + \int_t^T \left(P_t(s, X_s^{t,x}) + \left(\mathcal{L}^u P\right)(s, X_s^{t,x})\right) ds \\ + \int_t^T P_x(s, X_s^{t,x}) \sigma dW_s. \end{aligned} \quad (3.6)$$

We know that $P$ solves the HJB Equation in Eqn. (3.5) . So,

$$P_t(t, x) + f(t, x, u) + \left(\mathcal{L}^u P\right)(t, x) \leq 0, \; \forall u \in \mathcal{A}.$$

Thus,

$$\begin{aligned} P_t(t, X_s^{t,x}) + f(s, X_s^{t,x}) + \left(\mathcal{L}^u P\right)(s, X_s^{t,x}) &\leq 0 \\ P_t(t, X_s^{t,x}) + \left(\mathcal{L}^u P\right)(s, X_s^{t,x}) &\leq -f(s, X_s^{t,x}) \end{aligned}$$

From the boundary condition, we conclude that $P(T, x) = g(x)$.
Hence,

$$P(T, X_T^{t,s}) = g(X_T^{t,s}) \leq P(t, x) - \int_t^T f(s, X_T^{t,s}) ds + \int_t^T P_x(s, X_s^{t,x}) \sigma dW_s.$$

So,

$$P(t, x) \geq \int_t^T f(s, X_T^{t,s}) ds + g(X_T^{t,s}) - \int_t^T P_x(s, X_s^{t,x}) \sigma dW_s.$$

Now we take the expectation of both sides and notice that the expectation of the integral with Brownian motion is zero. This implies that

$$P(t, x) \geq \mathbb{E}\left[\int_t^T f(s, X_T^{t,s}) ds + g\left(X_T^{t,s}\right)\right] = J(t, x, u).$$

Since $u \in \mathcal{A}$ has been arbitrary,

$$P(t, x) \geq \sup_{u \in U} J(t, x, u) = V(t, x).$$

It is left to show that $P(t,x) \leq V(t,x)$.
We have, by assumption, that

$$P_t(t,x) + f(t,x,\widehat{u}) + \left(\mathcal{L}^{\widehat{u}} P\right)(t,x) = 0,$$

Thus,

$$P_t(t,x) + \left(\mathcal{L}^{\widehat{u}} P\right)(t,x) = -f(t,x,\widehat{u}).$$

Applying Itô formula to $P\left(T, \widehat{X}_T^{t,s}\right)$ yields

$$P(T, \widehat{X}_T^{t,s}) = P(t,x) + \int_t^T P_t(s, \widehat{X}_s^{t,x}) ds + \int_t^T P_x(s, \widehat{X}_s^{t,x}) d\widehat{X}_s^{t,x} \\ + \frac{1}{2}\int_t^T P_{xx}(s, \widehat{X}_s^{t,x}) d\left\langle \widehat{X}, \widehat{X} \right\rangle_s,$$

$$P(T, \widehat{X}_T^{t,s}) = P(t,x) + \int_t^T P_t(s, \widehat{X}_s^{t,x}) ds + \left(\mathcal{L}^{\widehat{u}} P\right)(s, \widehat{X}_s^{t,x}) ds \\ + \int_t^T D_x p(s, \widehat{X}_s^{t,x}) \sigma dW_s.$$

So,

$$P(T, \widehat{X}_T^{t,s}) = g(\widehat{X}_T^{t,s} = p(t,x) - f(s, \widehat{X}_s^{t,x}) + \int_t^T P_x(s, \widehat{X}_s^{t,x}) \sigma dW_s.$$

Let us take expectation of both sides and notice that the expected value of the integral with Brownian motion is zero. Hence,

$$P(t,x) = \mathbb{E}\left[\int_t^T f(s, \widehat{X}_T^{t,s}) ds + g(\widehat{X}_T^{t,s})\right] = J(t,x,u) \leq V(t,x).$$

Therefore, $P(t,x) \geq V(t,x) \geq J(t,x,\widehat{u}) = P(t,x)$.
That is, $P = V$ and $\widehat{u}$ is an optimal feedback control process.

□

*Remark* 3.4. We have proved the HJB Equation and the Verification Theorem for a maximization problem. The results are the same for a minimization problem. In that case, the objective function is represented by

$$J(t,x,u) = \mathbb{E}\left[\int_t^T f(s, X_s^{t,x}, u_s) ds + g(T, X_T^{t,x})\right],$$

the value function is given by

$$V(t,x) = \inf_{u \in \mathcal{A}} J(t,x,u)$$

and the associated HJB Equation is

$$V_t(t,x) + \inf_{u \in \mathcal{A}} f(t,x,u) + \left(\mathcal{L}^u V\right)(t,x) = 0, \ \forall (t,x) \in [0,T] \times \mathbb{R}^n, \\ V(T,x) = g(x), \ \forall x \in \mathbb{R}^n.$$

*Remark* 3.5. We can extend the optimal control problem to cases where the state variable is constrained within a domain. Consider the SDE as described in (3.1):

$$\begin{aligned} dX_t &= b(t, X_t, u_t)dt + \sigma(t, X_t, u_t)dW_t; \\ X_t &= x. \end{aligned}$$

We define the stopping time by $\tau = \inf\{t > 0 | (t, X_t) \in \partial D\} \wedge T$, where we have the time interval $[0, T]$, the domain $D \subseteq [0, T] \times \mathbb{R}^n$ and $a \wedge b := \min(a, b)$. Also, the control process is admissible, i.e., $u_t \in \mathcal{A}$. The value function to be optimized is the form

$$J(t, x, u) = \mathbb{E}\left[\int_0^\tau f(s, X_s^{t,x}, u_s)ds + g(\tau, X_\tau^{t,x})\right].$$

The problem is solved using the HJB equation and Verification Theorem.

*Remark* 3.6. **(*Infinite Horizon*)**: The infinite horizon problems follow similar arguments as the finite horizon problems. In such problems, the time horizon is $T = \infty$. To guarantee that the value function is finite, the running cost is exponentially discounted. The value function to be optimized is of the form

$$J(x, u) = \mathbb{E}\left[\int_0^\infty e^{-\beta s} f(X_s^x, u_s)ds\right],$$

$V(x) = J(x, \hat{u})$ represents the optimal value function and $\hat{u}$ the optimal control process. The associated HJB equation is of the form

$$\beta V(x) - \sup_{u \in \mathcal{A}} \left(f(x, u) + (\mathcal{L}^u V)(x)\right) = 0, \ \forall x \in \mathbb{R}^n.$$

The proof is the same as that for the finite horizon case.

## 3.3 Applications

In this section, we first deal with a well known application of optimal control in engineering: The Linear Quadratic Regulator. In the control problem, the objective is to find the optimal control process that will keep an initially excited system close to its equilibrium position. Then, we study Merton's Optimal Consumption and Portfolio problem for two cases. In the first case, the objective is to maximize the expected utility of consumption and terminal wealth of an investor over finite time horizon. In the second case, the objective is to maximize the expected utility from consumption over random time horizon. We end the section with the optimization problem of an investor with initial endowment who is allowed to invest in a financial market that consists of a risk-free asset and a risky asset. The aim is to maximize expected utility of terminal wealth of an investor over finite time horizon.

### 3.3.1 The Linear Quadratic Regulator

First, we study the following optimal control problem in engineering. The system dynamics are linear and the cost is quadratic. The objective is to minimize the quadratic cost:

$$J(t,x,u) = \int_0^{t_1} \left\{ x^T(t)Q(t)x(t) + u^T(t)R(t)u(t) \right\} dt + x^T(t_1)Sx(t_1), \tag{3.7}$$

given the dynamics

$$dX_t = \{A(t)x(t) + B(t)u(t)\}\, dt + P(t)dW_t,$$

where $X_t \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^k$, the control $u$ is unconstrained. Here, $Q$, $R$, $S$, $A$, $B$ and $P$ are known matrices; $S$ and $Q(t)$ satisfy $S = S^T \succeq 0$ and $Q = Q^T \succeq 0$, that is, $S$ and $Q(t)$ are symmetric positive semidefinite; $R(t) = R^T(t) \succ 0$, i.e., symmetric positive definite and, hence, invertible for all $t \in [0, t_1]$.
Our aim in the problem is to keep $x_t$ close to $0$, mainly at the final time $t_1$, while using minimum control $u$.
Before giving the solution of the problem, here is a motivation. Consider a system that is initially excited and is not in equilibrium with initial state $x(t) = x \neq 0$. We regard the initial state state $x \neq 0$ as undesirable so that the objective of the control problem is to find a control $u(t)$ that will return the system back to its equilibrium position $x(t) = 0$ in the shortest possible time. Minimizing the quadratic cost functional in Eqn. (3.7) gives the quantitative measure of the objective. Notice that the quadratic nature of the terms in Eqn. (3.7) guarantees that the quadratic cost functional remains non-negative for all $t$. In Eqn. (3.7), $Q$ and $R$ penalize the state and higher values are more heavily penalized since the objective is to minimize the cost of value while $S$ penalizes the control effort.
We now proceed to the solution of our optimal control problem. Assume there exists an optimal control $\hat{u}$ and an optimal value function $V$. By Theorem 3.1, the corresponding HJB equation is given by

$$\begin{aligned} V_t(t,x) + \inf_{u \in \mathbb{R}^k} \big( x^T Q(t) x + u^T R(t) u + V_x(t,x)(A(t)x + B(t)u) \big) \\ + \frac{1}{2} \sum_{i,j} V_{xx}(t,x)[PP^T]_{i,j} = 0 \end{aligned} \tag{3.8}$$

with boundary condition

$$V(t_1, x) = x^T S x. \tag{3.9}$$

For arbitrary $(t,x)$, we will find the minimizing control $\hat{u}$. Since $R(t) \succ 0$, the infimum of $u^T R u$ is a minimum and we set the gradient of Eqn. (3.8) to be zero. The quantity inside the braces in the HJB equation is maximized by $u$ satisfying

$$2u^T R(t) + V_x(t,x)B(t) = 0,$$

which gives the optimal strategy

$$\hat{u} = -\frac{1}{2} R^{-1}(t) B^T(t) V_x(t,x). \tag{3.10}$$

To apply the verification theorem, we need to know the value function $V$ that solves the HJB equation in Eqn. (3.8). We make a guess about the structure of $V$. It is reasonable that $V$ is a quadratic function because of the boundary condition of Eqn. (3.9). Hence, we make the guess

$$V(t,x) = x^T M(t) x + N(t),$$

where $M(t)$ is a symmetric matrix function of time for all $t$ and $N(t)$ is a scalar function. So,

$$\begin{aligned} V_t(t,x) &= x^T M'(t) + N'(t), \\ V_x(t,x) &= 2x^T M(t) = 2M(t)x, \\ V_{xx}(t,x) &= 2M(t). \end{aligned}$$

If we substitute the above results into the candidate optimal strategy in Eqn. (3.10), we get

$$\begin{aligned} \hat{u} &= -\frac{1}{2} R^{-1}(t) B^T(t) \cdot 2x^T M(t) \\ &= -R^{-1}(t) B^T(t) M(t) x. \end{aligned}$$

Substitute this into the HJB Equation of Eqn. (3.8)

$$\begin{aligned} & x^T M'(t) + N'(t) + x^T Q(t) x + x^T M(t) B(t) R^{-1}(t) R(t) R^{-1}(t) B^T(t) M(t) x \\ & \quad + 2x^T M(t) A(t) x - 2x^T M(t) R^{-1}(t) B^T(t) M(t) x + \sum_{i,j} M(t)_{i,j} [PP^T]_{i,j} = 0. \end{aligned}$$

So,

$$\begin{aligned} & x^T M'(t) + N'(t) + x^T Q(t) x + x^T M(t) B(t) R^{-1}(t) R(t) R^{-1}(t) B^T(t) M(t) x \\ & \qquad + x^T A^T(t) M(t) x + x^T M(t) A(t) x - 2x^T M(t) R^{-1}(t) B^T(t) M(t) x \\ & \qquad\qquad + tr \left[ P^T M(t) P \right] = 0. \end{aligned}$$

This implies that

$$\begin{aligned} & x^T \left\{ M'(t) + Q(t) - M(t) B(t) R^{-1}(t) B^T(t) M(t) + A^T(t) M(t) + M(t) A(t) \right\} \\ & \qquad + N'(t) + tr \left[ P^T M(t) P \right] = 0. \end{aligned}$$

For the equation to hold for all $t$ and $x$, we have the equations

$$M'(t) + Q(t) - M(t) B(t) R^{-1}(t) B^T(t) M(t) + A^T(t) M(t) + M(t) A(t) = 0$$

and

$$tr \left[ P^T M(t) P \right] + N'(t) = 0.$$

Thus, we have the following pairs of systems of Ordinary Differential Equations (ODEs)

$$\begin{cases} M'(t) = M(t) B(t) R^{-1}(t) B^T(t) M(t) - A^T(t) M(t) - M(t) A(t) - Q(t); \\ M(t_1) = S, \end{cases} \tag{3.11}$$

$$\begin{cases} N'(t) = -tr(P^T M(t) P); \\ N(t_1) = 0. \end{cases} \tag{3.12}$$

Eqn (3.11) is known as the Ricatti equation and it can be solved numerically for $M(t)$ and Eqn. (3.12) is integrated to obtain $N(t)$. It can then be concluded by the Verification Theorem that the optimal value function is given by

$$V(t,x) = x^T M(t) x + \int_t^{t_1} tr[P^T M(s) P] ds,$$

and the optimal control is given by

$$\hat{u} = -R^{-1}(t) B^T(t) M(t) x.$$

### 3.3.2 Merton's Optimal Consumption and Portfolio Problem

In this section, we formulate the optimization problem of an investor who has an initial endowment and is allowed to consume and invest in a financial market with a risk-free asset and a risky asset. The problem is considered for two cases. In the first case, the objective is to maximize the expected utility of consumption and terminal wealth of an investor over finite time horizon. In second case, the objective is to maximize the expected utility from consumption over random time horizon.

Consider a market with two assets: a risk-free asset and a risky asset. The risk-free asset follows the price process

$$dS_t^0 = r S_t^0 dt,$$

where $r$ is the risk-free interest rate. The price process of the risky asset is given by the Black Scholes model so that it solves the SDE

$$dS_t = S_t \left[ \mu dt + \sigma dW_t \right],$$

where $\mu$ and $\sigma$ are the rate of return and the volatility of the risky asset respectively and $W_t$ represents the Brownian Motion on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$. We denote the proportion of wealth invested in risky asset at time $t$ by $w_t$, the proportion of wealth invested in the risk-free asset at time $t$ by $(1 - w_t)$ and the consumption rate at time $t$ by $c_t$. Let $X_t$ represent the investor's total wealth at time $t$. Given that at time $t$, the initial wealth of an investor is $x$, we assume that the trading strategy is self-financing, that there is continuous trading and that unlimited short selling is allowed. The wealth process of the investor therefore evolves according to

$$\begin{aligned} dX_t^{w,c} &= \frac{w_t X_t dS_t}{S_t} + \frac{[1 - w_t] X_t dS_t^0}{S_t^0} - c_t dt, \\ dX_t^{w,c} &= w_t X_t [\mu dt + \sigma dW_t] + [1 - w_t] X_t r dt - c_t dt. \end{aligned}$$

Hence, we have the initial value problem

$$\begin{cases} dX_t^{w,c} &= X_t [w_t(\mu - r)] dt + [r X_t - c_t] dt + w_t \sigma X_t dW_t; \\ X_0 &= x. \end{cases} \tag{3.13}$$

The wealth process is influenced by the admissible strategy $u = (w, c) \in \mathcal{A}(x)$ with $\int_0^\infty [|w_t|^2 + c_t]dt < \infty$.
The aim of the investor is to determine his/her optimal investment and consumption up to a final time $T$:

$$J(t, x, u) = \mathbb{E}\left[\int_0^T f(t, c_t)dt + g(X_T)\right], \tag{3.14}$$

subject to Eqn. (3.13), $c_t \geq 0$, $X_t > 0$, $x > 0$, where $f(t, c_t) = e^{-\beta t}U(c_t)$; $U(c)$ is a strictly concave utility function. Let $V(t, x) = \sup_{u\in\mathcal{A}} J(t, x, u)$. That is, $V(t, x)$ is the optimal value function.

**Case 1: Optimal consumption and optimal terminal wealth over finite time horizon**

We consider the problem for the case whereby the investor exhibits constant relative risk aversion. So, we let

$$U(c) = \frac{c^\gamma}{\gamma};\ \gamma \neq 0;\ \gamma < 1.$$

Then, the utility is an increasing concave function of resources used in consumption. Here, $g(x) = \frac{x^\gamma}{\gamma}$. So, the investor wants to maximize

$$\mathbb{E}\left[\int_0^T e^{-\beta t}\frac{c^\gamma}{\gamma}dt + \frac{X_T^\gamma}{\gamma}\right],$$

given the dynamics

$$dX_t^{w,c} = X_t[w_t(\mu - r)]dt + [rX_t - c_t]dt + w_t\sigma X_t dW_t.$$

It follows from Theorem 3.1 that the corresponding HJB Equation is:

$$V_t + \sup_{c\geq 0, w\in R}\left(e^{-\beta t}\frac{c^\gamma}{\gamma} + wx(\mu - r)V_x + (rx - c)V_x + \frac{1}{2}\sigma^2 w^2 x^2 V_{xx}\right) = 0 \tag{3.15}$$

with terminal condition

$$V(T, x) = \frac{x^\gamma}{\gamma}. \tag{3.16}$$

The first-order conditions for a regular interior maximum are:
(1) $e^{-\beta t}c^{\gamma-1} - V_x = 0$
This implies that the optimal consumption is given by

$$\hat{c} = \left[e^{\beta t}V_x\right]^{\frac{1}{\gamma-1}}, \tag{3.17}$$

(2) $x(\mu - r)V_x + \sigma^2 x^2 w V_{xx} = 0$.
This implies that the optimal investment is given by

$$\hat{w} = \frac{-V_x}{xV_{xx}} \cdot \frac{\mu - r}{\sigma^2}. \tag{3.18}$$

To apply the verification theorem, the optimal value function $V$ is needed. We therefore make a guess about the structure of $V$:

$$\begin{cases} V(t,x) = \frac{m(t)x^\gamma}{\gamma}; \\ m(T) = 1. \end{cases}$$

Hence,

$$V_t = \frac{m'(t)x^\gamma}{\gamma},$$
$$V_x = m(t)x^{\gamma-1},$$
$$V_{xx} = (\gamma-1)m(t)x^{\gamma-2}.$$

Now, we insert the above partial derivatives into Eqns. (3.17) and (3.18) to get

$$\hat{c}(t,x) = \left[e^{\beta t}m(t)x^{\gamma-1}\right]^{\frac{1}{\gamma-1}} = \left[e^{\beta t}m(t)\right]^{\frac{1}{\gamma-1}} x, \tag{3.19}$$

$$\hat{w}(t,x) = \frac{-m(t)x^{\gamma-1}}{x(\gamma-1)m(t)x^{\gamma-2}}.\frac{\mu-r}{\sigma^2} = -\frac{\mu-r}{(\gamma-1)\sigma^2}. \tag{3.20}$$

We wish to show that the value function $V(t,x) = \frac{m(t)x^\gamma}{\gamma}$ solves the HJB Equation of Eqn. (3.15). To do this, we substitute its partial derivatives and Eqns. (3.19) and (3.20) into the HJB Equation (3.15):

$$\frac{m'(t)x^\gamma}{\gamma} + \frac{e^{-\beta t}[e^{\beta t}m(t)]^{\frac{\gamma}{\gamma-1}}x^\gamma}{\gamma} - \frac{x^\gamma(\mu-r)^2 m(t)}{(\gamma-1)\sigma^2} + (rx - (e^{\beta t}m(t))^{\frac{1}{\gamma-1}}x)m(t)x^{\gamma-1} + \frac{\sigma^2(\mu-r)^2x^2(\gamma-1)m(t)x^{\gamma-2}}{(\gamma-1)^2\sigma^4} = 0.$$

Thus,

$$\frac{m'(t)x^\gamma}{\gamma} + \frac{e^{\frac{\beta t}{\gamma-1}}m(t)^{\frac{\gamma}{\gamma-1}}x^\gamma}{\gamma} - \frac{x^\gamma(\mu-r)^2m(t)}{(\gamma-1)\sigma^2} + rm(t)x^\gamma - e^{\frac{\beta t}{\gamma-1}}m(t)^{\frac{\gamma}{\gamma-1}}x^\gamma + \frac{x^\gamma(\mu-r)^2m(t)}{2(\gamma-1)\sigma^2} = 0,$$

i.e.,

$$\frac{m'(t)x^\gamma}{\gamma} + \frac{e^{\frac{\beta t}{\gamma-1}}m(t)^{\frac{\gamma}{\gamma-1}x^\gamma}(1-\gamma)}{\gamma} - \frac{x^\gamma(\mu-r)^2m(t)}{(\gamma-1)\sigma^2} + rm(t)x^\gamma + \frac{x^\gamma(\mu-r)^2m(t)}{2(\gamma-1)\sigma^2} = 0.$$

This gives us the equation of the form

$$x^\gamma\left(m'(t) + Am(t) + Bm(t)^{\frac{\gamma}{\gamma-1}}\right) = 0,$$

where $A = -\frac{(\mu-r)^2}{\sigma^2(\gamma-1)}$ and $B = \left(\frac{1-\gamma}{\gamma}\right) e^{\frac{\beta t}{\gamma-1}}$.
For the equation to hold for all $t$ and $x$, $m(t)$ must be a solution of the ODE with terminal condition:

$$\begin{cases} m'(t) + Am(t) + Bm(t)^{\frac{\gamma}{\gamma-1}} = 0; \\ m(T) = 1. \end{cases} \tag{3.21}$$

If we take $n(t) := m(t)^{\frac{1}{1-\gamma}}$, then $m(t) = n(t)^{1-\gamma}$ and $n'(t) = \frac{m'(t)m(t)^{\frac{\gamma}{1-\gamma}}}{1-\gamma}$. We notice that Eqn. (3.21) becomes

$$(1-\gamma)n'(t)m(t)^{\frac{-\gamma}{1-\gamma}} + An(t)^{1-\gamma} + Bn(t)^{-\gamma} = 0,$$

implying that

$$\begin{cases} n'(t) + \frac{An(t)}{1-\gamma} + \frac{B}{1-\gamma} = 0; \\ n(T) = 1. \end{cases} \tag{3.22}$$

Solving the Linear Differential Equation of Eqn. (3.22), we obtain the explicit solution

$$n(t) = e^{\frac{A}{1-\gamma}(T-t)} + \frac{1-\gamma}{\gamma(A-B)}\left[e^{\frac{(A-B)T}{1-\gamma}} - e^{\frac{(A-B)t}{1-\gamma}}\right] e^{\frac{A(T-t)}{(1-\gamma)}}.$$

Hence,

$$m(t) = n(t)^{1-\gamma} = \left[e^{\frac{A}{1-\gamma}(T-t)} + \frac{1-\gamma}{\gamma(A-B)}\left[e^{\frac{(A-B)T}{1-\gamma}} - e^{\frac{(A-B)t}{1-\gamma}}\right] e^{\frac{A(T-t)}{(1-\gamma)}}\right]^{1-\gamma}.$$

**Conclusion:**
We have shown that if we define the value function as $V(t,x) = \frac{m(t)x^\gamma}{\gamma}$, where $m(t)$ is as explicitly obtained as above, and if we define the optimal investment $\hat{w}$ as in Eqn. (3.19) and optimal consumption $\hat{c}$ as in Eqn. (3.20), then $V(t,x)$ satisfies the HJB Equation and $\hat{w}$ and $\hat{c}$ give the maximum value in the equation. Therefore, by the Verification Theorem, the optimal consumption and investment strategy ($\hat{w}$, $\hat{c}$) of the investor is given by

$$\hat{c}(t,x) = \left[e^{\beta t}m(t)\right]^{\frac{1}{\gamma-1}} x,$$

$$\hat{w}(t,x) = -\frac{\mu-r}{(\gamma-1)\sigma^2} = \frac{\mu-r}{(1-\gamma)\sigma^2}.$$

*Remark* 3.7. It is noticeable that for an investor with a Constant Relative Risk Aversion utility, the optimal consumption is proportional to the wealth of the investor at each stage and that the optimal investment is independent of the wealth at each stage.

**Case 2: Optimal consumption over random time horizon**

We consider the problem for the case where $g = 0$ in Eqn. (3.14). In this problem, the investor closes his/her position when the wealth process is zero. That is, the position is closed at a random time horizon $\tau$:

$$\tau := \inf\{t > 0 | X_t = 0\}.$$

The objective function is given by the value function

$$V(x) = \sup_{(w,c)\in\mathcal{A}(x)}\mathbb{E}\left[\int_0^{\tau} e^{-\beta t}U(c_t)\right],$$

where $U(c_t) = c^{\gamma}$ , $\beta > 0$, subject to

$$\begin{cases} dX_t^{w,c} & = X_t[w_t(\mu - r)]dt + [rX_t - c_t]dt + w_t\sigma X_t dW_t; \\ X_0 & = x. \end{cases}$$

The corresponding HJB equation is given by:

$$V_t + \sup_{c\geq 0, w\in\mathbb{R}}\left(e^{-\beta t}c^{\gamma} + wx(\mu - r)V_x + (rx - c)V_x + \frac{1}{2}\sigma^2 w^2 x^2 V_{xx}\right) = 0; \quad (3.23)$$

$$V(T, x) = 0. \quad (3.24)$$

The first-order conditions for a regular interior maximum are:
(1) $\gamma e^{-\beta t}c^{\gamma-1} - V_x = 0$.
This implies that the optimal consumption is given by

$$\hat{c} = \left[\frac{e^{\beta t}V_x}{\gamma}\right]^{\frac{1}{\gamma-1}}, \quad (3.25)$$

(2) $x(\mu - r)V_x + \sigma^2 x^2 w V_{xx} = 0$.
This implies that the optimal investment is given by

$$\hat{w} = \frac{-V_x}{xV_{xx}} \cdot \frac{\mu - r}{\sigma^2}. \quad (3.26)$$

To apply the Verification Theorem, the optimal value function $V$ is needed. We guess that

$$\begin{cases} V(t,x) = e^{-\beta t}m(t)x^{\gamma}; \\ m(T) = 0. \end{cases}$$

Therefore,

$$\begin{aligned} V_t &= e^{-\beta t}m'(t)x^{\gamma} - \beta e^{-\beta t}m(t)x^{\gamma}, \\ V_x &= \gamma e^{-\beta t}m(t)x^{\gamma-1}, \\ V_{xx} &= \gamma(\gamma - 1)e^{-\beta t}m(t)x^{\gamma-2}. \end{aligned}$$

Now, we substitute the above partial derivatives into Eqns. (3.25) and (3.26) to get

$$\hat{c}(t, x) = \left[\frac{e^{\beta t}\gamma e^{-\beta t}m(t)x^{\gamma-1}}{\gamma}\right]^{\frac{1}{\gamma-1}} = m(t)^{\frac{1}{\gamma-1}}x \quad (3.27)$$

$$\hat{w}(t, x) = \frac{-\gamma e^{-\beta t}m(t)x^{\gamma-1}}{x\gamma(\gamma - 1)e^{-\beta t}m(t)x^{\gamma-2}} \cdot \frac{\mu - r}{\sigma^2} = -\frac{\mu - r}{(\gamma - 1)\sigma^2}. \quad (3.28)$$

Now we shall substitute the partial derivatives of Eqns. (3.27) and (3.28) into the HJB Equation in Eqn. (3.23) to show that $V(t,x) = e^{-\beta t} m(t) x^{\gamma}$ solves the HJB Equation:

$$e^{-\beta t} m'(t) x^{\gamma} - \beta e^{-\beta t} m(t) x^{\gamma} + e^{-\beta t} x^{\gamma} m(t)^{-\frac{\gamma}{1-\gamma}} + \frac{x(\mu - r)^2 \gamma e^{-\beta t} m(t) x^{\gamma - 1}}{(1-\gamma)\sigma^2}$$
$$+ r x \gamma e^{-\beta t} m(t) x^{\gamma-1} - x m(t)^{-\frac{1}{\gamma - 1}} \gamma e^{-\beta t} m(t) x^{\gamma - 1}$$
$$+ \frac{x^2 \sigma^2 (\mu - r)^2 \gamma(\gamma - 1) e^{-\beta t} m(t) x^{\gamma - 2}}{2(1-\gamma)^2 \sigma^4} = 0.$$

This implies that

$$e^{-\beta t} x^{\gamma} \left[ m'(t) - m(t) \left[ -\beta + \frac{\gamma(\mu - r)^2}{\sigma^2 (1-\gamma)} + r\gamma \right] + (1-\gamma) m(t)^{\frac{-\gamma}{1-\gamma}} \right] = 0,$$

giving us

$$x^{\gamma} \left[ \dot{m(t)} + A m(t) + B m(t)^{\frac{-\gamma}{1-\gamma}} \right] = 0,$$

where

$$A = -\beta + \frac{\gamma(\mu - r)^2}{\sigma^2(1-\gamma)} + r\gamma \text{ and } B = 1 - \gamma.$$

For the equation to hold for all $t$ and $x$, $m(t)$ must be a solution of the ODE:

$$\begin{cases} m'(t) + A m(t) + B m(t)^{\frac{\gamma}{\gamma - 1}} = 0; \\ m(T) = 0. \end{cases} \tag{3.29}$$

The explicit solution of the Bernoulli Equation in Eqn. (3.29) above is obtained as in Case 1.

**Conclusion**:

We have now proved that if the value function is defined by $V(t,x) = e^{-\beta t} m(t) x^{\gamma}$, where m(t) is the solution of the Bernoulli Equation in Eqn. (3.29) and if we define the optimal investment $\hat{w}$ as in Eqn. (3.27) and optimal consumption $\hat{c}$ as in Eqn. (3.28), then $V(t,x)$ satisfies the HJB Equation and $\hat{w}$ and $\hat{c}$ give the maximum value in the equation. Therefore, by the Verification Theorem, the optimal consumption and investment strategy ($\hat{w}$, $\hat{c}$) of the investor is given by

$$\hat{c}(t,x) = m(t)^{\frac{1}{\gamma - 1}} x,$$

$$\hat{w}(t,x) = -\frac{\mu - r}{(\gamma - 1)\sigma^2}.$$

### 3.3.3 Merton's Portfolio Allocation Problem

In the previous section, we obtained the optimal consumption and investment strategies for an investor with an initial endowment who is allowed to consume and invest in a financial market. Now, we consider a portfolio allocation problem under a similar setup but without consumption. Again, the financial market consists of a risk-free asset and a risky asset. The risk-free asset follows the price process

$$dS_t^0 = rS_t^0 dt.$$

The price process of the risky asset is given by the Black Scholes model so that it solves the SDE

$$dS_t = S_t(\mu dt + \sigma dW_t).$$

The proportion of total wealth invested in the risky asset at any time is denoted by $w_t$. The remaining proportion of wealth $(1 - w_t)$ is invested in the risk-free asset. The wealth process $X(t)$ of the investor corresponding to the strategy $w(t)$, is the solution of the SDE with initial condition

$$\begin{cases} dX_t^w = w_t X(t) \frac{dS(t)}{S(t)} + [1 - w_t] X_t \frac{dS_t^0}{S_t^0}; \\ X_0 = x > 0, \end{cases}$$

i.e,

$$\begin{cases} dX_t^w = X_t[w_t(\mu - r) + r]dt + \sigma w_t X_t dW_t; \\ X_0 = x > 0. \end{cases}$$

The control process is said to be admissible if $\mathbb{E}\left[\int_0^T |w_t^2| dt\right] < \infty$. Here, we recall that $\mathcal{A}(x)$ denotes the set of all admissible portfolios.
The objective of the investor is to choose an allocation of the wealth in such a way that will maximize the expected utility of his/her terminal wealth. That is, the value function is defined by:

$$V(t,x) = \sup_{w \in \mathcal{A}(x)} \mathbb{E}\left[U(X_T)\right].$$

We choose the power utility of CRRA type $U(x) = \frac{x^\gamma}{\gamma}$ for $\gamma < 1$, $\gamma \neq 0$. We note that $\gamma$ is called the relative risk aversion coefficient.
The corresponding HJB Equation is

$$\begin{cases} V_t + \sup_{w \in \mathcal{A}(x)} \left[ (xw(\mu - r) + rx) V_x + \frac{\sigma^2 w^2 x^2 V_{xx}}{2} \right] = 0; \\ V(T,x) = \frac{x^\gamma}{\gamma}. \end{cases} \tag{3.30}$$

Taking the second order derivative with respect to $w$, we get that $\sigma^2 x^2 V_{xx} < 0$, since $V(x)$ is a strictly concave utility function. Hence, from the first-order condition for a regular interior maximum which is:

$$x(\mu - r)V_x + w\sigma^2 x^2 V_{xx} = 0,$$

we get that the $w$ attains a maximum at

$$\hat{w} = \frac{-V_x}{xV_{xx}} \cdot \frac{\mu - r}{\sigma^2}. \tag{3.31}$$

We shall guess that the value function is of the form

$$V(t,x) = p(t)\frac{x^\gamma}{\gamma}.$$

Therefore,

$$\begin{aligned}
V_t &= p'(t)\frac{x^\gamma}{\gamma}, \\
V_x &= p(t)x^{\gamma-1}, \\
V_{xx} &= (\gamma-1)p(t)x^{\gamma-2}.
\end{aligned}$$

Substitute the partial derivatives into Eqn. (3.31) to obtain the candidate optimal investment

$$\hat{w} = \frac{-V_x}{xV_{xx}} \cdot \frac{\mu-r}{\sigma^2} = \frac{-p(t)x^{\gamma-1}}{x(\gamma-1)p(t)x^{\gamma-2}} \cdot \frac{\mu-r}{\sigma^2} = \frac{\mu-r}{(1-\gamma)\sigma^2}.$$

Now, substitute the value function into the HJB Equation (3.30) to show that it solves the following equation:

$$\frac{x^\gamma p'(t)}{\gamma} + \frac{x^\gamma(\mu-r)^2\gamma p(t)}{(1-\gamma)\sigma^2} + rx^\gamma\gamma p(t) + \frac{\sigma^2(\mu-r)^2x^\gamma\gamma(\gamma-1)p(t)}{2(1-\gamma)^2\sigma^4} = 0,$$

thus,

$$p'(t) + r\gamma p(t) + \frac{(\mu-r)^2}{2(1-\gamma)\sigma^2}\gamma p(t) = 0.$$

Hence, $p(t)$ must satisfy the ODE with terminal value:

$$\begin{cases} p'(t) + Ap(t) = 0; \\ p(T) = 1. \end{cases}$$

where $A = \gamma\left[r + \frac{(\mu-r)^2}{2(1-\gamma)\sigma^2}\right]$.
Thus,

$$p(t) = e^{A(T-t)}.$$

**Conclusion:**
We have shown by the Verification Theorem that the value function given by

$$V(t,x) = e^{A(T-t)}\frac{x^\gamma}{\gamma}$$

satisfies the HJB Equation and so it is the optimal value function and the optimal proportion of wealth to invest in risky asset is given by

$$\hat{w} = \frac{\mu-r}{(1-\gamma)\sigma^2}.$$

*Remark* 3.8. The verification theorem is an important step in the use of DP principle to solve optimal control problems. The DP principle with the application of Itô formula is used to derive the HJB Equation for an optimal control problem. Afterwards, the verification theorem is used to guarantee that the value function that satisfies the HJB equation is indeed the optimal value function. It is assumed in the verification theorem that the value function is smooth enough. However, this is not always the case and it is sometimes hard to check for the smoothness of the value function. In such cases that the value function is not smooth enough, the theory of *viscosity solutions* is used. The theory of viscosity solutions can be applied to linear and non-linear PDEs regardless of the order. In [36], Zariphopoulou initiated the study of viscosity solutions in finance. A detailed treatment of viscosity solutions in finance is done by Pham in [29].

# CHAPTER 4

# OPTIMAL CONTROL FOR JUMP-DIFFUSION PROCESSES: APPLICATIONS IN FINANCE AND INSURANCE

Empirical studies in finance and insurance have found evidence of discontinuities, called jumps, in financial and insurance variables. In this chapter, we present three applications of optimal control for jump processes. In the first application, the investor is faced with a portfolio-consumption problem. The investor can consume and invest in a financial market that consists of a risk-free and a risky asset. Risky asset prices are known to make sudden large movements in cases of rare events such as wars and economical crisis. Therefore, the dynamics of the price of the risky asset is represented with a Lévy-diffusion model and a closed-form expression for the optimal investment and consumption strategy that will maximize the utility of the investor over an infinite time horizon is obtained. In the second application, we are concerned with the optimization problem of finding the optimal investment strategy that will maximize the expected utility of terminal wealth of an insurer. The wealth process of the insurer is affected by the risk process (presence of a stochastic cash flow) which is given by:

$$\text{Surplus} = \text{Initial Capital} + \text{Premium Income} - \text{Cumulated Insurance Claims}.$$

The amount of cumulated insurance claims, which is paid by the insurer to the insured, is represented by a compound Poisson process. The premium income is the amount paid by the insured to the insurer for the insurance policy. Hence, the risk process is modelled by a jump-diffusion process. The third and final application is devoted to obtain the optimal investment and reinsurance strategy that maximizes the expected utility of terminal wealth of an insurer. To reduce risk, the insurer is allowed to invest in a financial market and purchase proportional reinsurance. Similar to the second application, a jump-diffusion process is used to model the insurer's risk process. In the last two applications, the utility preferences of the insurers are assumed to be exponential. It is common to use exponential utility in insurance because it is the only utility function under which the principle of 'zero utility' gives a fair premium that is dependent on the level of reserve of an insurance company.

***The Control Problem:***
The general control problem is to maximize the value function

$$J(t,x,u) = \mathbb{E}\left[\int_t^T f(s, X_s^{t,x}, u_s)ds + g(X_T^{t,x})\right]$$

given the dynamics

$$dX_t = b(X_t, u_t)dt + \sigma(X_t, u_t)dW_t + \int_{\mathbb{R}} h(X_{t-}, u_{t-}, z)\tilde{M}(dt, dz);$$
$$X_t = x,$$

where the $b$, $\sigma$, $h$ are deterministic functions satisfying the Lipschitz continuity and linear growth conditions. Hence, guaranteeing that there exists a unique càdlàg adapted solution $(X_t)$ to the equation. More details can be found about existence of uniqueness in Chapter 2.
The objective in optimal control is to obtain a control process $u \in \mathcal{A}$ that maximizes the value function $J$ over all admissible controls. We introduce the value function:

$$V(t, x) := \sup_{u\in\mathcal{A}} J(t, x, u).$$

If $V(t, x) = J(t, x, \widehat{u})$, we call $\widehat{u}(t, x)$ an *optimal control process* and $V(t, x)$ is called the *optimal value function* for the problem.

## 4.1 Merton's Optimal Portfolio and Consumption Problem under Jump-Diffusion Process

In this problem, we study the portfolio-consumption selection of an investor facing risks that are modelled by Brownian motion and Lévy-diffusion processes. The wealth process accounts for events that lead to large price movements and sudden breaks in the prices of the risky asset. The random changes in the risky asset are modelled by Brownian motion and a compound Poisson process which has constant jump sizes at random intervals. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space. Furthermore, $(W_t)$ is a Brownian motion defined on the filtered probability space and is adapted to the filtration $(\mathcal{F}_t : t \geq 0)$. In the financial market, there is one risk-free asset and one risky asset. The risk-free asset follows the price process

$$dS_t^0 = rS_t^0 dt,$$

where $r > 0$ is the risk-free interest rate. The price process of the risky asset satisfies the following SDE with jumps

$$dS_t = S_t \left( \mu dt + \sigma dW_t + \int_{-1}^{\infty} z\tilde{M}(dt, dz) \right),$$

where $\mu > 0$ and $\sigma \in \mathbb{R}$ represent the rate of return and volatility of the risky asset respectively. It is assumed that $\int_{-1}^{\infty} |z| dv(z) < \infty$ and $\mu > r$.
Let $(c_t)$ which is an adapted, càdlàg (right continuous with left limits) process be the consumption rate at time $t \geq 0$, $(w_t)$ be the proportion of the total wealth invested in risky asset at time $t \geq 0$. Hence, $(1 - w_t)$ is the proportion of total wealth invested in risk-free asset at time $t$ for $t \geq 0$. It is assumed that there is no transaction cost. The investor has initial wealth $x > 0$. The wealth process of the investor evolves according to

$$dX_t = \frac{w_t X_{t-}}{S_{t-}} dS_t + \frac{(1 - w_t)X_{t-}}{S_t^0} dS_t^0 - c_t dt,$$

or,

$$dX_t = w_t X_{t-}\left[\mu dt + \sigma dW_t + \int_{-1}^{\infty} z\tilde{M}(dt,dz)\right] + (1-w_t)X_{t-}rdt - c_t dt.$$

So,

$$\begin{cases} dX_t = w_t X_{t-}\left[(\mu - r)dt + \sigma dW_t + \int_{-1}^{\infty} z\tilde{M}(dt,dz)\right] + r\left(X_t - c_t\right)dt; \\ X_0 = x. \end{cases} \tag{4.1}$$

The objective of the investor is to choose the optimal investment proportion and consumption rate in a way that will maximize his utility

$$J(t,x,u) = \mathbb{E}\left[\int_0^{\infty} e^{-\beta t} U(c_t)dt\right],$$

where $\beta > 0$, $U(\cdot)$ is a differentiable, bounded and strictly concave utility function. The investor exhibits constant relative risk aversion. That is, the utility function is given by $\frac{c^\gamma}{\gamma}, 0 < \gamma < 1$. The investment criterion is to choose the admissible control process, i.e., the optimal investment-consumption strategy $u_t = (w_t, c_t) \in \mathcal{A}$ such that

$$V(t,x) = \sup_{(w_t,c_t)\in\mathcal{A}} J(t,x,u).$$

Note that the control $u_t = (w_t, c_t)$ is admissible if $w_t, c_t$ are $\mathcal{F}_t$-adapted and càdlàg and the total wealth $X_t$ is non-negative for all $t \geq 0$. We write $u_t \in \mathcal{A}$, where $\mathcal{A}$ is the set of all admissible strategies.
The following theorem gives the corresponding HJB equation for the value function $V(t,x)$.

**Theorem 4.1.** ***(Hamilton-Jacobi-Bellman Equation)*** *Assume $V(t,x)$ is continuously differentiable in $t \in [0,T]$ and twice continuously differentiable in $x \in \mathbb{R}$. Then $V(t,x)$ satisfies the HJB equation*

$$e^{-\beta t}\frac{c^\gamma}{\gamma} + sup_{u\in\mathcal{A}}\left(\mathcal{L}^u V\right)(t,x) = 0, \tag{4.2}$$

*where the infinitesimal generator of $V$ is given by;*

$$\begin{aligned}\left(\mathcal{L}^u V\right)(t,x) = V_t + \left[(\mu - r)wx + xr - c\right]V_x + \frac{1}{2}\sigma^2 w^2 x^2 V_{xx} \\ + \int_{-1}^{\infty}\left\{V(t,x+zwx) - V(t,x) - V_x zwx\right\}v(dz).\end{aligned}$$

*Proof.* The proof follows from the Theorem 3.1

□

To apply the Verification Theorem that guarantees that the solution to the HJB Equation in Eqn. (4.2), we shall make a guess about the optimal value function $V$. Let us define $V$ as:

$$V(t,x) = Ne^{-\beta t}x^\gamma.$$

This implies

$$\begin{aligned}
V_t &= -\beta N e^{-\beta t} x^{\gamma}, \\
V_x &= \gamma N e^{-\beta t} x^{\gamma-1}, \\
V_x &= \gamma(\gamma-1) N e^{-\beta t} x^{\gamma-2}.
\end{aligned}$$

Considering the partial derivatives, we get that

$$\begin{aligned}
e^{-\beta t}\frac{c^{\gamma}}{\gamma} + \left(\mathcal{L}^u V\right)(t,x) &= e^{-\beta t}\frac{c^{\gamma}}{\gamma} - \beta N e^{-\beta t} x^{\gamma} \\
&+ \left[(\mu-r)wx + xr - c\right]\gamma N e^{-\beta t} x^{\gamma-1} + \frac{1}{2}\sigma^2 w^2 x^2 \gamma(\gamma-1) N e^{-\beta t} x^{\gamma-2} \\
&\quad + \int_{-1}^{\infty} \left(N e^{-\beta t}(x+zwx)^{\gamma} - N e^{-\beta t} x^{\gamma} - \gamma N e^{-\beta t} x^{\gamma-1} zwx\right) v(dz).
\end{aligned}$$

This implies that

$$\begin{aligned}
&e^{-\beta t}\frac{c^{\gamma}}{\gamma} + \left(\mathcal{L}^u V\right)(t,x) = e^{-\beta t}\frac{c^{\gamma}}{\gamma} - c\gamma N e^{-\beta t} x^{\gamma-1} \\
&+ N e^{-\beta t} x^{\gamma}\left[-\beta + \gamma[(\mu-r)w + r] + \frac{1}{2}\sigma^2 w^2 \gamma(\gamma-1) + \int_{-1}^{\infty}\left\{(1+zw)^{\gamma} - 1 - \gamma wz\right\} v(dz)\right].
\end{aligned}$$

Now, we shall differentiate $e^{-\beta t}\frac{c^{\gamma}}{\gamma} + \left(\mathcal{L}^u V\right)(t,x)$ with respect to $c$ and $w$ to find the critical points where the maximum is attained. It can be checked that the expression is concave with respect to $c$ and $w$.

First, we differentiate with respect to $c$ and equate to zero to obtain the maximum:

$$\begin{aligned}
e^{-\beta t} c^{\gamma-1} - N e^{-\beta t} x^{\gamma-1}\gamma &= 0, \\
c^{\gamma-1} - N e^{-\beta t} x^{\gamma-1}\gamma &= 0.
\end{aligned}$$

This implies the optimal consumption is given by

$$\hat{c} = (N\gamma)^{\frac{1}{\gamma-1}} x.$$

Second, we differentiate with respect to $w$ and equate to zero to obtain the maximum:

$$N e^{-\beta t} x^{\gamma}\left[\gamma(\mu-r) - \sigma^2 w\gamma(1-\gamma) + \int_{-1}^{\infty}\left\{\gamma(1+zw)^{\gamma-1}z - \gamma z\right\} v(dz)\right] = 0.$$

This implies that the optimal investment in risky asset $\hat{w}$ satisfies $L(w) = 0$, where $L(w)$ is given by:

$$L(w) = \mu - r - \sigma^2 w(1-\gamma) - \int_{-1}^{\infty}\left\{1 - (1+zw)^{\gamma-1}z\right\} v(dz).$$

We notice that

$$L(0) = \mu - r > 0$$

and

$$L(1) = \mu - r - \sigma^2(1-\gamma) - \int_{-1}^{\infty} \left\{1 - (1+z)^{\gamma-1} z\right\} v(dz).$$

So if $\mu - r < \sigma^2(1-\gamma) + \int_{-1}^{\infty} \{1 - (1+z)^{\gamma-1} z\} v(dz)$, then there exists $w = \hat{w} \in (0,1]$.

Substitute $\hat{c}$ and $\hat{w}$ into the HJB Equation in order to show that the value function solves the HJB Equation

$$\begin{aligned} 0 = e^{-\beta t} \frac{(N\gamma)^{\frac{\gamma}{\gamma-1}} x^{\gamma}}{\gamma} - (N\gamma)^{\frac{1}{\gamma-1}} x \gamma N e^{-\beta t} x^{\gamma-1} + N e^{-\beta t} x^{\gamma} \Big[ -\beta + \gamma[(\mu - r)\hat{w} + r] \\ + \frac{1}{2}\sigma^2 \hat{w}^2 \gamma(\gamma-1) + \int_{-1}^{\infty} \{(1+z\hat{w})^{\gamma} - 1 - \gamma \hat{w} z\} v(dz) \Big], \end{aligned}$$

thus,

$$\begin{aligned} (N\gamma)^{\frac{1}{\gamma-1}} - (N\gamma)^{\frac{1}{\gamma-1}} \gamma - \beta + \gamma[(\mu - r)\hat{w} + r] + \frac{1}{2}\sigma^2 \hat{w}^2 \gamma(\gamma-1) \\ + \int_{-1}^{\infty} \{(1+z\hat{w})^{\gamma} - 1 - \gamma \hat{w} z\} v(dz) = 0, \end{aligned}$$

hence

$$\begin{aligned} (N\gamma)^{\frac{1}{\gamma-1}}(1-\gamma) &= \beta - \gamma[(\mu - r)\hat{w} + r] + \frac{1}{2}\sigma^2 \hat{w}^2 \gamma(\gamma-1) \\ &+ \int_{-1}^{\infty} \{(1+z\hat{w})^{\gamma} - 1 - \gamma \hat{w} z\} v(dz). \end{aligned}$$

So,

$$\begin{aligned} N &= \frac{1}{\gamma}\Bigg[\Bigg(\frac{1}{1-\gamma}\big[\beta - \gamma[(\mu - r)\hat{w} + r] + \frac{1}{2}\sigma^2 \hat{w}^2 \gamma(\gamma-1) \\ &+ \int_{-1}^{\infty} \{(1+z\hat{w})^{\gamma} - 1 - \gamma \hat{w} z\} v(dz)\big]\Bigg)^{\gamma-1}\Bigg] \end{aligned}$$

Below, we summarize our result.

**Theorem 4.2.** *Consider the wealth process in Eqn.* (4.1) *and the utility function defined by* $\frac{c^{\gamma}}{\gamma}, 0 < \gamma < 1$, *with the objective to maximize the utility over infinite time horizon, the optimal value function of the investor is given by*

$$V(t,x) = N e^{-\beta t} x^{\gamma},$$

*where* $N$ *is as given in Eqn.* (4.1). *Furthermore, the optimal consumption is given by*

$$\hat{c} = (N\gamma)^{\frac{1}{\gamma-1}}\, x$$

*and* $\hat{w}$ *satisfying* $L(w) = 0$ *is the optimal investment in the risky asset.*

## 4.2 Optimal Investment for Insurers with Jump-Diffusion Risk Processes

In this application, we focus on the control problem of finding the optimal investment strategy that maximizes the expected utility of terminal wealth of an insurer which was studied by Yang & Zhang in [34]. A closed form expression is obtained for the optimal strategy of an insurer that is allowed to invest when the utility function is exponential.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with filtration $(\mathcal{F}_t : t \geq 0)$. We have two standard Brownian motions $(W_t^1 : t \geq 0)$ and $(W_t^2 : t \geq 0)$ adapted to $(\mathcal{F}_t : t \geq 0)$ on the probability space.
The wealth process of the insurer is affected by the presence of a stochastic cash flow modelled by:

$$\text{Surplus} = \text{Initial Capital} + \text{Premium Income} - \text{Cumulated Insurance Claims}.$$

The stochastic cash flow denoted by $(P_t : t \geq 0)$ is modelled as:

$$dP_t = adt + bdW_t^2 - dR_t, \tag{4.4}$$

where $a$ and $b > 0$ are constants; $R_t = \sum_{i=1}^{N_t} Y_i$ as defined in Definition 2.14 is a compound Poisson process which denotes cumulated insurance claims with intensity $\lambda > 0$ and jump size distribution $G$. Here, $(N_t : t \geq 0)$ is a Poisson process with parameter $\lambda$ which denotes the total number of claims up to time t and $Y_i$ is the size of the *i*th claim and assumed to be independent of the claim number process. We shall assume that $(W_t^1 : t \geq 0)$ and $(W_t^2 : t \geq 0)$ are two correlated Brownian motions with correlation coefficient $\rho$ such that $\rho^2 \neq 1$. In our case, the stochastic flow $(P_t)$ guarantees that the risk to the insurer cannot be totally eliminated. We also assume that continuous trading is allowed, neither transaction cost nor tax is involved, and all assets are infinitely divisible.
The market consists of two assets: a risky asset and a risk-free asset. The risk-free asset follows the price process

$$dS_t^0 = rS_t^0 dt, \tag{4.5}$$

where $r(\cdot)$ is the risk-free interest rate. The price process of the risky asset satisfies the SDE

$$dS_t = \mu S_t dt + \sigma S_t dW_t^1, \tag{4.6}$$

where $\mu(\cdot)$ and $\sigma(\cdot)$ are the rate of return and volatility of the risky asset, respectively. We assume that $\mu > r$.

Let $K_t$ be the amount in the risky asset at time $t$ and $X_t - K_t$ be the amount invested in the risk-free asset at time $t$ so that $X_t$ represents the company's total wealth. The wealth process of the insurer, denoted by $X_t^K$ satisfies

$$dX_t^K = K_t \frac{dS_t}{S_t} + \{X_t - K_t\} \frac{dS_t^0}{S_t^0} + dP_t \tag{4.7}$$

or, if we insert, Eqns. (4.4) , (4.5) and (4.6) into Eqn. (4.7) ,

$$dX_t^K = K_t \left[\mu dt + \sigma dW_t^1\right] + [X_t - K_t] \, rdt + adt + bW_t^2 - dR_t.$$

So,

$$dX_t^K = [(\mu - r)K_t + rX_t + a]\,dt + \sigma K_t dW_t^1 + bW_t^2 - dR_t;\; X_0 = x \in \mathbb{R}, \quad (4.8)$$

where $x > 0$ is the initial wealth. The wealth process is influenced by the investment strategy $K = (K_t : t \geq 0)$ with $\int_0^T |K_t|^2 dt < \infty,\; T < \infty$. The investment strategy $K_t \in \mathcal{A}(x)$ is admissible. Here, $\mathcal{A}$ is the set of all admissible strategies.
The objective of the investor is to maximize the expected utility of terminal wealth

$$V(t, x) := \sup_{K\in\mathcal{A}} \mathbb{E}\left[U(X_T^K) | X_t^K = x\right],$$

where $V(\cdot)$ is the value function, $U(\cdot)$ is a differentiable, bounded and strictly concave utility function and $\left(X_t^K : t \geq 0\right)$ is the wealth process under the investment policy $K$. Since $U(\cdot)$ is concave, there exists a unique optimal strategy $K^*$ such that the value function attains its maximum.
Suppose that the preferences of the insurer are exponential. That is, the utility function is given by

$$U(x) = m - \frac{\alpha}{\eta} e^{-\eta x},\; \alpha > 0,\; \eta > 0.$$

The utility function has constant absolute risk aversion parameter $\eta$, i.e., $\eta = \frac{-U_{xx}(x)}{U_x(x)}$.
The following theorem gives the associated HJB equation for the value function $V(t, x)$.

**Theorem 4.3.** ***(Hamilton-Jacobi-Bellman Equation)*** *Let $\left(X_t^K : t \geq 0\right)$ satisfy the dynamics of the wealth process in Eqn.* (4.8) *and G be the jump-size distribution of the compound Poisson process $R_t$. Assume $V(t, x)$ is continuously differentiable in $t \in [0, T]$ and twice continuously differentiable in $x \in \mathbb{R}$. Then $V(t, x)$ satisfies the HJB equation*

$$sup_{K\in\mathcal{A}} \left(\mathcal{L}^K V(t, x)\right) + \lambda \int_0^\infty \left[V(t, x - y) - V(t, x)\right] G(dy) = 0 \quad (4.9)$$

*with terminal condition*

$$V(T, x) = U(x) = m - \frac{\alpha}{\eta} e^{-\eta x}, \quad (4.10)$$

*where*

$$\left(\mathcal{L}^K V\right)(t, x) = V_t + [(\mu - r)K + rx + a]\,V_x + \frac{1}{2}\left[\sigma^2 K^2 + b^2 + 2\rho b\sigma K\right] V_{xx}.$$

*Proof.* The proof follows from the Theorems 2.4 and 3.1.

□

The Verification Theorem is important to show that under some conditions, a solution to the HJB equation above gives us the optimal investment strategy. Below, we give the theorem and derive the proof.

**Theorem 4.4.** ***(Verification Theorem)*** *Suppose that there exists a smooth function $P(t,x) \in C^{1,2}([0,T],\mathbb{R})$ that satisfies the HJB Equation of Eqn.* (4.9) *with terminal condition of Eqn.* (4.10) *subject to the boundary conditions, then the value function $V(t,x) = P(t,x)$.*
*Furthermore, suppose that there exists a $K^* \in \mathcal{A}$ such that $K^*$ attains a supremum in the HJB equation for all $(t,x) \in [0,T] \times \mathbb{R}$, then $K^*$ defines the optimal investment strategy.*

*Proof.* Let $K \in \mathcal{A}$. By Itô formula, it follows that for $t \in [s,T]$,

$$P(T,X_T^K) = P(t,x)+\int_t^T P_t(s,X_s^K)ds+\int_t^T \left[(\mu-r)K_s + rX_{s-}^K + a\right] P_x(s,X_s^K)ds$$
$$+\int_t^T \sigma K_{s-}P_x(s,X_s^K)dW_s^1+\int_t^T bdW_s^2+\frac{1}{2}\int_t^T \left[\sigma^2 K_{s-}^2 + b^2 + 2\rho b\sigma K_{s-}\right] P_{xx}(s,X_s^K)ds$$
$$+\int_t^T \int_0^\infty \left[P(t,X_{s-}^K - y) - P(t,X_{s-}^K)\right] \tilde{M}(ds,dy),$$

thus,

$$P(T,X_T^K) = P(t,x)+\int_t^T P_t(s,X_s^K)ds+\int_t^T \left[(\mu-r)K_s + rX_{s-}^K + a\right] P_x(s,X_s^K)ds$$
$$+\int_t^T \sigma K_{s-}P_x(s,X_s^K)dW_s^1+\int_t^T bdW_s^2+\frac{1}{2}\int_t^T \left[\sigma^2 K_{s-}^2 + b^2 + 2\rho b\sigma K_{s-}\right] P_{xx}(s,X_s^K)ds$$
$$+\int_t^T \int_0^\infty \left[P(t,X_{s-}^K - y) - P(t,X_{s-}^K)\right] \left(\tilde{M}(ds,dy) - \lambda G(dy)dt\right)$$
$$+\lambda\int_t^T \int_0^\infty \left[P(t,X_{s-}^K - y) - P(t,X_{s-}^K)\right] G(dy)dt.$$

Taking expectation of both sides gives

$$\mathbb{E}\left[P(T,X_T^K)\right] = P(t,x)$$
$$+\mathbb{E}\left[\left(\mathcal{L}^K P\right)(t,X_{s-}^K)ds + \lambda\int_t^T \int_0^\infty \left[P(t,X_{s-}^K - y) - P(t,X_{s-}^K)\right] G(dy)dt\right].$$

This implies that

$$\mathbb{E}\left[P(T,X_T^K)\right] \le P(t,x)+$$
$$\mathbb{E}\left[\int_t^T \sup_{K\in\mathcal{A}}\left(\mathcal{L}^K P\right)(t,X_{s-}^K)ds + \int_t^T \lambda\int_0^\infty \left[P(t,X_{s-}^K - y) - P(t,X_{s-}^K)\right] G(dy)dt\right].$$

Therefore,

$$\mathbb{E}\left[P(T,X_T^K)\right] \le P(t,x);$$

so,

$$P(t,x) \ge \sup_{K\in\mathcal{A}}\mathbb{E}[P(T,X_T^K)] = \sup_{K\in\mathcal{A}}\mathbb{E}[U(X_T^K)] = V(t,x). \tag{4.11}$$

We shall now that $P(t,x) \leq V(t,x)$
Suppose that $\exists K^* \in \mathcal{A}$ such that $K^*$ attains a supremum in the HJB Equation of Eqn. (4.9). That is,

$$P_t + [(\mu - r)K^* + rx + a]\, P_x + \frac{1}{2}\left[\sigma^2 K^{*2} + b^2 + 2\rho b\sigma K^*\right] P_{xx} + \lambda \int_0^\infty [P(t, x-y) - P(t,x)]\, G(dy) = 0. \quad (4.12)$$

By applying Itô formula,

$$\begin{aligned} P(T, X_T^{K^*}) = P(t,x) &+ \int_t^T P_t(s, X_s^{K^*})ds + \int_t^T \left[(\mu - r)K_s^* + rX_{s-}^{K^*} + a\right] P_x ds \\ &+ \int_t^T \sigma K_{s-}^* P_x dW_s^1 + \int_t^T b dW_s^2 + \frac{1}{2}\int_t^T \left[\sigma^2 K_{s-}^{*^2} + b^2 + 2\rho b\sigma K_{s-}^*\right] P_{xx} ds \\ &+ \int_t^T \int_0^\infty \left[P(t, X_{s-}^{K^*} - y) - P(t, X_{s-}^{K^*})\right] \tilde{M}(ds, dy), \end{aligned}$$

thus,

$$\begin{aligned} P(T, X_T^{K^*}) = P(t,x) &+ \int_t^T \left(\mathcal{L}^{K^*} P\right)(s, X_{s-}^{K^*})ds + \int_t^T \sigma K_{s-}^* P_x dW_s^1 + \int_t^T b dW_s^2 \\ &+ \int_t^T \int_0^\infty \left[P(t, X_{s-}^{K^*} - y) - P(t, X_{s-}^{K^*})\right] \left(\tilde{M}(ds, dy) - \lambda G(dy)dt\right) \\ &+ \lambda \int_t^T \int_0^\infty \left[P(t, X_{s-}^{K^*} - y) - P(t, X_{s-}^{K^*})\right] G(dy)dt. \end{aligned}$$

Taking expected values of both sides gives

$$\begin{aligned} &\mathbb{E}\left[P(T, X_T^{K^*})\right] = P(t,x) + \\ &\mathbb{E}\left[\int_t^T \left(\mathcal{L}^k P\right)(t, X_{s-}^K)ds + \int_t^T \lambda \int_0^\infty \left[P(t, X_{s-}^{K^*} - y) - P(t, X_{s-}^K)\right] G(dy)dt\right]. \end{aligned} \quad (4.13)$$

From Eqn. (4.12), Eqn. (4.13) becomes $\mathbb{E}\left[P(T, X_T^{K^*})\right] = P(t,x)$,
Hence,

$$P(t,x) = \mathbb{E}\left[P(T, X_T^{K^*})\right] \leq \sup_{K\in\mathcal{A}} \mathbb{E}\left[P(T, X_T^{K^*})\right] = \sup_{K\in\mathcal{A}} \mathbb{E}\left[U(X_T^{K^*})\right] = V(t,x).$$

Therefore,

$$P(t,x) \leq V(t,x) \quad (4.14)$$

Combining Eqns. (4.11) and (4.14), we get

$$P(t,x) \geq V(t,x) = \sup \mathbb{E}\left[U(X_T^{K^*})\right] \geq P(t,x).$$

That is,

$$P(t,x) = V(t,x) = \mathbb{E}\left[U(X_T^{K^*})\right].$$

$\square$

So far, we have described the problem, given the associated HJB equation and proved the verification theorem. Now, we solve for the optimal value function and the optimal investment strategy. First, we give the result in the theorem below and outline the solution process in the proof.

**Theorem 4.5.** *Consider the wealth process in Eqn.* (4.8) *and the utility function defined by*

$$U(x) = m - \frac{\alpha}{\eta} e^{-\eta x},$$

*with objective to maximize the expected utility of terminal wealth, the optimal investment policy is given by*

$$K^* = \frac{\mu - r}{\eta \sigma^2} e^{-r(T-t)} - \frac{\rho b}{\sigma},$$

*and the optimal value function is given by*

$$V(t,x) = m - \frac{\alpha}{\eta} \mathit{exp} \left( -\eta x e^{r(T-t)} - \frac{1}{2} \left( \frac{\mu - r}{\sigma} \right)^2 (T-t) + n(T-t) \right),$$

*where $n(T-t)$ satisfies the ODE*

$$n'(T-t) = -\eta e^{r(T-t)} \left[ a - \rho b \left( \frac{\mu - r}{\sigma} \right) \right] + \frac{1}{2} b^2 (1-\rho)^2 \eta^2 e^{2r(T-t)} + \lambda \int_0^\infty \left[ \eta y e^{r(T-t)} - 1 \right] G(dy),$$

*with the initial value*

$$n(0) = 0.$$

*Proof.* **Step 1:** Obtain the candidate optimal investment strategy $K^*$ and substitute into HJB (4.9).

Let us assume that the HJB Equation of Eqn. (4.9) has a classical solution satisfying $V_x > 0$ and $V_{xx} < 0$. Then we differentiate the HJB Equation with respect to $K$ to obtain the candidate optimal $K^*$ via

$$(\mu - r) V_x + \sigma^2 K^* + \rho b \sigma V_{xx},$$

namely,

$$K^* = \frac{(\mu - r) V_x}{\sigma^2 V_x x} - \frac{\rho b}{\sigma}. \tag{4.15}$$

We substitute Eqn. (4.15) into the HJB Equation of Eqn. (4.9) to obtain

$$
\begin{aligned}
V_t &+ \left[\frac{(\mu-r)^2 V_x}{\sigma^2} V_{xx} - \frac{(\mu-r)\rho b}{\sigma} + rx + a\right] V_x \\
&\quad + \frac{1}{2}\left[\sigma^2\left(\frac{(\mu-r)V_x}{\sigma^2 V_x x} - \frac{\rho b}{\sigma}\right)^2 + b^2 + 2\rho b\sigma\left(\frac{(\mu-r)V_x}{\sigma^2 V_x x} - \frac{\rho b}{\sigma}\right)\right] \\
&\quad + \lambda\int_0^\infty \left[V(t,x-y) - V(t,x)\right] G(dy) = 0,
\end{aligned}
$$

i.e.,

$$
\begin{aligned}
V_t &+ \left[rx + a - \frac{(\mu-r)\rho b}{\sigma}\right] V_x - \frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2 \frac{V_x^2}{V_{xx}} + \frac{1}{2} b^2 (1-\rho)^2 V_{xx} \\
&\quad + \lambda\int_0^\infty \left[V(t,x-y) - V(t,x)\right] G(dy) = 0. \qquad (4.16)
\end{aligned}
$$

**Step 2:** Guess the value function that solves Eqn. (4.9) and take the partial derivatives of the value function.
We guess a solution to Eqn. (4.9) of the form

$$
V(t,x) = m - \frac{\alpha}{\eta}\text{exp}\left(-\eta x e^{r(T-t)} - \frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2 (T-t) + n(T-t)\right), \qquad (4.17)
$$

where $n(\cdot)$ is a function such that $V(t,x)$ solves Eqn. (4.14) and the terminal condition in Eqn. (4.10) implies that $n(0) = 0$. From Eqn. (4.17), we obtain

$$
\begin{aligned}
V_t(t,x) = -\frac{\alpha}{\eta}&\left[\eta x r e^{r(T-t)} + \frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2 - n'((T-t)\right] \\
&\text{exp}\left\{-\eta x e^{r(T-t)} - \frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2 (T-t) + n(T-t)\right\},
\end{aligned}
$$

hence,

$$
V_t(t,x) = -\frac{\alpha}{\eta}\left[\eta x r e^{r(T-t)} + \frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2 - n'((T-t)\right] \times \left[-\frac{\eta}{\alpha}(V(t,x) - m)\right].
$$

Therefore,

$$
V_t(t,x) = \left[V(t,x) - m\right]\left[\eta x r e^{r(T-t)} + \frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2 - n'(T-t)\right], \qquad (4.18)
$$

and

$$
V_x(t,x) = \alpha e^{r(T-t)}\text{exp}\left(-\eta x e^{r(T-t)} - \frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2 (T-t) + n(T-t)\right),
$$

thus,

$$V_x(t,x) = \alpha e^{r(T-t)} \left[ -\frac{\eta}{\alpha}(V(t,x)-m) \right].$$

Therefore,

$$V_x(t,x) = \left[ -\eta e^{r(T-t)} \right] [V(t,x)-m], \tag{4.19}$$

$$V_{xx}(t,x) = -\alpha\eta e^{2r(T-t)} \text{exp} \left( -\eta x e^{r(T-t)} - \frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2 (T-t) + n(T-t) \right),$$

$$V_{xx}(t,x) = \left[ -\alpha\eta e^{2r(T-t)} \left[ -\frac{\eta}{\alpha}(V(t,x)-m) \right] \right].$$

Therefore, we get

$$V_{xx}(t,x) = \left[ \eta^2 e^{2r(T-t)} \right] [V(t,x)-m]. \tag{4.20}$$

**Step 3:** Simplify the expression $\int_0^\infty [V(t,x-y)-V(t,x)]\, G(dy)$ that is found in the HJB Eqn. (4.16).

$$\int_0^\infty [V(t,x-y)-V(t,x)]\, G(dy) =$$
$$\int_0^\infty \left[ \left( m - \frac{\alpha}{\eta} \text{exp} \left\{ -\eta(x-y) e^{r(T-t)} - \frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2 (T-t) + n(T-t) \right\} \right) \right] G(dy)$$
$$-\int_0^\infty \left[ \left( m - \frac{\alpha}{\eta} \text{exp} \left\{ -\eta x e^{r(T-t)} - \frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2 (T-t) + n(T-t) \right\} \right) \right] G(dy).$$

This becomes

$$\int_0^\infty [V(t,x-y)-V(t,x)]\, G(dy) =$$
$$\int_0^\infty -\frac{\alpha}{\eta} \left[ \left( \text{exp} \left\{ -\eta(x-y) e^{r(T-t)} - \frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2 (T-t) + n(T-t) \right\} \right) \left( \eta y e^{r(T-t)} - 1 \right) \right] G(dy).$$

Hence,

$$\begin{aligned} \int_0^\infty [V(t,x-y)-V(t,x)]\, G(dy) &= \left[ \frac{-\alpha}{\eta} \cdot \frac{-\eta}{\alpha} (V(t,x)-m)(\eta y e^{r(T-t)} - 1) \right] G(dy) \\ &= \int_0^\infty [V(t,x)-m][\eta y e^{r(T-t)} - 1] G(dy), \end{aligned}$$

and, so,

$$\int_0^\infty [V(t,x-y)-V(t,x)]\, G(dy) = [V(t,x)-m] \int_0^\infty [\eta y e^{r(T-t)} - 1] G(dy). \tag{4.21}$$

**Step 4:** Substitute the candidate optimal investment strategy and the partial derivatives of the value function into HJB Eqn. (4.16).

Substituting Eqn. (4.19) and Eqn. (4.20) into Eqn. (4.15), we get that the supremum is achieved at $K^*$ and so the optimal investment strategy is given by:

$$K^* = \frac{[\mu - r][-\eta e^{r(T-t)}][V(t,x) - m]}{\sigma^2[\eta^2 e^{2r(T-t)}][V(t,x) - m]} - \frac{\rho b}{\sigma}.$$

Thus,

$$K^* = \frac{\mu - r}{\eta\sigma^2} e^{-r(T-t)} - \frac{\rho b}{\sigma}$$

We substitute Eqns. (4.18) - (4.21) into Eqn. (4.16) to get

$$\begin{aligned}
[V(t,x) - m] \cdot \left[\eta x r e^{r(T-t)} + \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 - n'(T-t)\right] \\
+ \left[rx + a - \frac{(\mu - r)\rho b}{\sigma}\right] \cdot \left[-\eta e^{r(T-t)}\right] [V(t,x) - m] \\
- \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 \frac{\eta^2 e^{2r(T-t)} [V(t,x) - m]^2}{\left[\eta^2 e^{2r(T-t)}\right][V(t,x) - m]} \\
+ \frac{1}{2} b^2 (1-\rho)^2 \left[\eta^2 e^{2r(T-t)}\right][V(t,x) - m] \\
+ \lambda [V(t,x) - m] \int_0^\infty [\eta y e^{r(T-t)} - 1] G(dy) = 0,
\end{aligned}$$

hence,

$$\begin{aligned}
\frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 - n'(T-t) - \left[a - \frac{(\mu - r)\rho b}{\sigma}\right]\left[\eta e^{r(T-t)}\right] - \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 \\
+ \frac{1}{2} b^2 (1-\rho)^2 \left[\eta^2 e^{2r(T-t)}\right] + \lambda \int_0^\infty [\eta y e^{r(T-t)} - 1] G(dy) = 0
\end{aligned}$$

and, thus,

$$\begin{aligned}
n'(T-t) = -\eta e^{r(T-t)} \left[a - \rho b \left(\frac{\mu - r}{\sigma}\right)\right] + \frac{1}{2} b^2 (1-\rho)^2 \eta^2 e^{2r(T-t)} \\
+ \lambda \int_0^\infty \left[\eta y e^{r(T-t)} - 1\right] G(dy), \quad (4.22)
\end{aligned}$$

with the initial value

$$n(0) = 0.$$

If the distribution of the jump $y$ is known, one can find a closed form expression of $n(\cdot)$. Hence, we can say that the value function satisfies the HJB Eqn. in Eqn. (4.9). Notice that by substituting the optimal investment into the wealth process, the resulting optimal wealth process can be rewritten as a linear SDE plus a jump process as outlined below:

$$dX_t^{K*} = \left[(\mu - r)\left[\frac{(\mu - r)e^{-r(T-t)}}{\eta\sigma^2} - \frac{\rho b}{\sigma}\right] + rX_t^* + a\right] dt \\ + \sigma\left[\frac{(\mu - r)e^{-r(T-t)}}{\eta\sigma^2} - \frac{\rho b}{\sigma}\right] dW_t^1 + bW_t^2 - dR_t,$$

hence,

$$dX_t^{K*} = \left[\frac{(\mu - r)^2 e^{-r(T-t)}}{\eta\sigma^2} - \frac{(\mu - r)\rho b}{\sigma} + rX_t^* + a\right] dt + \\ \sqrt{\left(\frac{\mu - r}{\eta\rho}e^{-r(T-t)}\right)^2 + b^2(1-\rho^2)}dW_t - dR_t,$$

where

$$W_t = \frac{\frac{(\mu-r)e^{-r(T-t)}}{\eta\sigma} - \rho b}{\sqrt{\left(\frac{\mu-r}{\eta\rho}e^{-r(T-t)}\right)^2 + b^2(1-\rho^2)}}$$

is another standard Brownian motion. It is known that the linear SDE admits a unique strong solution. Therefore, since the value function is twice continuously differentiable, the conditions of the verification theorem is satisfied. □

## 4.3 Optimal Investment and Reinsurance for Insurers with Jump-Diffusion Risk Processes

This application is devoted to the study of the optimal investment and reinsurance strategy that maximizes the expected utility of terminal wealth of an insurer. This application follows the problem put forward by Lin & Yang in [20]. As in the previous application, the risk process of the insurer is modelled by a compound Poisson process. To reduce risk, the insurer is allowed to invest in a financial market with one risk-free and one risky asset and purchase proportional reinsurance. A closed form solution is obtained for the optimal strategy when the utility preference of the insurer is exponential.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space and $W_t^1$ and $W_t^2$ are standard Brownian motions adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$. We are concerned with modelling the surplus process given by

$$\text{Surplus} = \text{Initial Capital} + \text{Income} - \text{Outflow}.$$

The surplus process of the insurer is also affected by the presence of a stochastic cash flow and so the surplus of the insurer at time $t$ is denoted by $(M_t : t \geq 0)$ modelled as:

$$dM_t = adt + bdW_t^1 - dR_t$$

where $M_0 > 0$ denotes the initial capital; $a > 0$ denotes the premium rate per unit time; $R_t = \sum_{j=1}^{N_t^1} Y_j$ as defined in Definition 2.14 is a compound Poisson process which denotes cumulated claims with intensity $\lambda_1 > 0$ and jump size distribution $G$ with $G(0) = 0$. Here, $(N_t^1 : t \geq 0)$ is a Poisson process with parameter $\lambda_1$ which denotes the total number of claims up to time $t$ and $Y_j$ is the size of *j*th claim and is independent of the claim number process with density function g with first and second moment $\mathbb{E}[Y] = \mu_1$ and $\mathbb{E}[Y^2] = \mu_2$, respectively. To share risk, the insurer uses reinsurance so that the reinsurer can cover some parts of the claim. The amount of claim to be paid by the primary insurer or cedent is the retention level $c$ of insurance acquired by the insurer. In this problem, the part of the claim $Y$ the cedent has to pay is given by $cY$. Hence, the reinsurer pays the remaining $(1-c)Y$ of the claim. Here, $c \in (0,1)$ corresponds to the proportional reinsurance. The insurer uses the variance premium principle so the insurance pays a premium of

$$(1-c)\lambda_1\mu_1 + \alpha(1-c)^2\lambda_1\mu_2,$$

where

$$a \leq (\mu_1 + \alpha\mu_2), \ \alpha > 0.$$

After reinsurance, the surplus process becomes

$$dM_t = \left[a - (1-c)\lambda_1\mu_1 - \alpha(1-c)^2\lambda_1\mu_2\right] t + bdW_t^1 - cdR_t \tag{4.23}$$

The market consists of two assets: a risk-free asset that follows the process

$$dS_t^0 = rS_t^0 dt, \tag{4.24}$$

where $r > 0$ is the risk-free interest rate and a risky asset that satisfies the SDE with jumps

$$dS_t = S_t\left(\mu dt + \sigma dW_t^2 + \int_{\mathbb{R}} zN(dt, dz)\right). \tag{4.25}$$

Here, $\mu > 0$, $\sigma > 0$ are constants that represent the rate of return and volatility of the risky asset respectively and $\int_0^t \int_{\mathbb{R}} zN(dt, dz) := \sum_{j=1}^{N_t^2} Z_j$ is a compound Poisson process with parameter $\lambda_2 > 0$. Also, $Z_j$ are i.i.d. random variables with distribution $H$. We shall assume that $(W_t^1 : t \geq 0)$ and $(W_t^2 : t \geq 0)$ are two correlated Brownian motions with correlation coefficient $\rho \in [-1, 1]$. We also assume that $\mu \geq r$ and that continuous trading is allowed, no transaction cost/tax and all assets are infinitely divisible.

Let $K_t$ be the amount invested in risky asset at time $t$ and $X_t - K_t$ be the amount invested in the risk-free asset at time $t$. Here, $X_t$ is the insurer's total wealth. The wealth process of the insurer evolves according to

$$dX_t^u = K_t\frac{dS_t}{S_t} + (X_t - K_t)\frac{dS_t^0}{S_t^0} + dM_t. \tag{4.26}$$

If we insert Eqns. (4.23) , (4.24) and (4.25) into Eqn. (4.26), we obtain:

$$\begin{aligned} dX_t^u = K_t &\left[\mu dt + \sigma dW_t^2 + \int_{\mathbb{R}} zN(dt, dz)\right] + (X_t - K_t)\, rdt \\ &+ \left[a - (1-c_t)\lambda_1\mu_1 - \alpha(1-c_t)^2\lambda_1\mu_2\right] dt - c_t dR_t + bW_t^1, \end{aligned}$$

i.e.,

$$dX_t^u = \left[(\mu - r)K_t + rX_t + a - (1-c_t)\lambda_1\mu_1 - \alpha(1-c_t)^2\lambda_1\mu_2\right] dt \\ + \sigma K_t dW_t^2 + bW_t^1 - c_t dR_t + K_t \int_{\mathbb{R}} zN(dt, dz). \quad (4.27)$$

The initial wealth of the insurer is given by $X_0 = x > 0$. The wealth process is influenced by the control policy $u_t = (K_t, c_t)$ with $c_t \in [0, 1]$ and $\int_0^T |K_t|^2 dt < \infty$, for any $T < \infty$. Here, $(K_t)$, $(c_t)$ are adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$. The control policy $(u_t)$, where $u_t \in \mathcal{A}$ is admissible. Here, $\mathcal{A}$ is the set of all admissible strategies.

Let $\varphi_Y(r) = \mathbb{E}[e^{rY}] = \int_0^\infty e^{rY} \varphi G(y)$ be the moment generating function of $Y$. Suppose that there exists a $\theta$ such that $\varphi_Y(r) \uparrow \theta$. Then, $\int_0^\infty dG(y) = 1$ and $d$ is increasing, convex and continuous on $[0, \theta)$.
Suppose the utility preference of the insurer is exponential. That is, the utility function is defined by:

$$U(x) = m - \frac{\alpha}{\eta} e^{-\eta x}, \ \alpha > 0, \ \eta > 0$$

where $\eta$ represents the coefficient of absolute risk aversion, i.e., $\eta = -\frac{U_{xx}(x)}{U_x(x)}$.

**Objective:** The objective of the insurer is to find an optimal investment/reinsurance strategy that will maximize the expected utility of terminal wealth

$$V(t, x) := \text{sup}_{u\in\mathcal{A}} \mathbb{E}\left[U(X_T^u) | X_t^u = x\right],$$

where $V(\cdot)$ is the value function, $(X_t^u : t \geq 0)$ is the wealth process influenced by the control policy $u$. Since $U(\cdot)$ is an increasing concave function, there exists a unique optimal control policy $\hat{u} = \left(\hat{K}, \hat{c}\right)$ such that the value function attains a maximum.

We use the Dynamic Programming principle to solve the problem posed above. The following theorem gives the corresponding HJB equation to the value function $V(t, x)$.

**Theorem 4.6.** ***(Hamilton-Jacobi-Bellman Equation)*** *Let* $(X_t^u : t \geq 0)$ *satisfy the dynamics of the wealth process in Eqn.* (4.27) *and G, H are jump-size distributions of the compound Poisson process* $R_t$ *and the Poisson process* $Z_j$*, respectively. Assume* $V(t, x)$ *and its partial derivatives* $V_t$*,* $V_x$ *and* $V_{xx}$ *are continuous on* $[0, T] \times \mathbb{R}$*, then* $V(t, x)$ *satisfies the HJB Equation*

$$0 = \textit{sup}_{u\in\mathcal{A}} \Big\{V_t + \left[K[\mu - r] + rx + a - (1-c)\lambda_1\mu_1 - \alpha(1-c)^2\lambda_1\mu_2\right] V_x \\ + \frac{1}{2}\left[K^2\sigma^2 + b^2 + 2k\sigma b\rho\right] V_{xx} + \lambda_1 \int_0^\infty \left[V(t, x - cy) - V(t, x)\right] G(dy) \\ \lambda_2 \int_{-\infty}^\infty \left[V(t, x + Kz) - V(t, x)\right] H(dz)\Big\} \quad (4.28)$$

*for* $(t, x) \in [0, T) \times \mathbb{R}$ *with the boundary condition*

$$V(T, x) = U(x) = m - \frac{\alpha}{\eta} e^{-\eta x}. \quad (4.29)$$

*Proof.* The proof follows from the Theorems 2.4 and 3.1. $\square$

The Verification Theorem below allows us to derive an optimal policy for the problem from a smooth solution to the HJB equation in Eqn. (4.28).

**Theorem 4.7.** ***(Verification Theorem)*** *Assume that the HJB Equation in Eqn.* (4.28) *with terminal condition in Eqn.* (4.29) *has a solution $P(t,x)$ which is continuously differentiable on $t \in [0,T]$ and twice continuously differentiable on $x \in \mathbb{R}$. Then, subject to boundary conditions, the value function is $V(t,x) = P(t,x)$. Now suppose that there exists admissible control policy $\hat{u} = (\hat{K}, \hat{c}) \in \mathcal{A}$ such that for each $(t,x)$, $P(t,x)$ attains the supremum in the HJB Equation then $\hat{u} = (\hat{K}, \hat{c})$ is the optimal control policy and $P = V$ is the optimal value function.*

*Proof.* The proof is similar to that of Theorem 4.4. $\square$

### Closed form solution for optimal control policy

Thus far, we have described the problem of the insurer, given the corresponding HJB equation to the value function $V(t,x)$ and the verification theorem. Now, we solve for the optimal value function and the optimal reinsurance-investment strategy. First, we give the result in the theorem below and outline the solution process in the proof.

**Theorem 4.8.** *Consider the wealth process of Eqn.* (4.27) *and the utility function defined by*

$$U(x) = m - \frac{\alpha}{\eta} e^{-\eta x},$$

*with the objective to maximize expected utility of terminal wealth, the optimal investment strategy satisfies*

$$\mu - r - \eta\sigma^2 K e^{r(T-t)} + \lambda_2 \int_{-\infty}^{\infty} -\eta z e^{r(T-t)} \textit{exp}\left(-\eta K z e^{r(T-t)}\right) H(dz) = 0,$$

*the optimal reinsurance strategy $\hat{c}$ satisfies*

$$\mu_1 + 2(1-c)\alpha\mu_2 - \int_0^{\infty} y \exp\left(\eta c y e^{r(T-t)}\right) G(dy) = 0$$

*and the optimal value function of the insurer is given by*

$$V(t,x) = m - \frac{\alpha}{\eta} \exp\left(-\eta x e^{r(T-t)} + n(T-t)\right)$$

*where $n(T-t)$ satisfies the following ODE:*

$$\begin{aligned} n'(T-t) = &-\eta e^{r(T-t)} \left[K[\mu - r] + rx + a - (1-\hat{c})\lambda_1\mu_1 - \alpha(1-\hat{c})^2\lambda_1\mu_2\right] \\ &+ \frac{1}{2}\eta^2 e^{2r(T-t)} \left[\hat{K}^2\sigma^2 + b^2 + 2\hat{K}\sigma b\rho\right] + \lambda_1 \int_0^{\infty} \textit{exp}\left(\eta\hat{c} y e^{r(T-t)}\right) G(dy) \\ &+ \lambda_2 \int_{-\infty}^{\infty} \textit{exp}\left(-\eta\hat{K} z e^{r(T-t)}\right) H(dz) \end{aligned}$$

*with the initial condition* $n(0) = 0$.

*Proof.* **Step 1:** Guess the value function that satisfies the HJB equation in Eqn. (4.28) and take the partial derivatives of the value function.

We make a guess about the optimal value function $V$ of the form

$$V(t,x) = m - \frac{\alpha}{\eta} \exp\left\{-\eta x e^{r(T-t)} + n(T-t)\right\}, \tag{4.30}$$

where $n(\cdot)$ is a function such that $V(t,x)$ solves the HJB Equation in Eqn. (4.28) and its initial value is $n(0) = 0$ from Eqn. (4.29).
We proceed to obtain the partial derivatives

$$\begin{aligned} V_t(t,x) &= -\frac{\alpha}{\eta}\left[\eta x r e^{r(T-t)} - n'(T-t)\right] \exp\left\{-\eta x e^{r(T-t)} + n(T-t)\right\} \\ &= -\frac{\alpha}{\eta}\left[\eta x r e^{r(T-t)} - n'(T-t)\right]\left[-\frac{\eta}{\alpha}(V(t,x) - m)\right], \end{aligned}$$

i.e.,

$$V_t(t,x) = [V(t,x) - m]\left[\eta x r e^{r(T-t)} - n'(T-t)\right] \tag{4.31}$$

and

$$\begin{aligned} V_x(t,x) &= \alpha e^{r(T-t)} \exp\left\{-\eta x e^{r(T-t)} + n(T-t)\right\} \\ &= \alpha e^{r(T-t)}\left[-\frac{\eta}{\alpha}(V(t,x) - m)\right]. \end{aligned}$$

Therefore,

$$V_x(t,x) = -\eta e^{r(T-t)}\left[V(t,x) - m\right], \tag{4.32}$$

$$\begin{aligned} V_{xx}(t,x) &= -\alpha\eta e^{2r(T-t)} \exp\left(-\eta x e^{r(T-t)} + n(T-t)\right) \\ &= -\alpha\eta e^{2r(T-t)}\left[-\frac{\eta}{\alpha}(V(t,x) - m)\right]. \end{aligned}$$

Therefore,

$$V_{xx}(t,x) = \eta^2 e^{2r(T-t)}\left[V(t,x) - m\right]. \tag{4.33}$$

**Step 2:** Simplify the expressions $\int_0^\infty [V(t, x-cy) - V(t,x)]\, G(dy)$ and $\int_{-\infty}^\infty [V(t, x+Kz) - V(t,x)]\, H(dz)$ that are found in the HJB Eqn. (4.28).

$$\begin{aligned} &\int_0^\infty [V(t,x-cy) - V(t,x)]\, G(dy) = \\ &\qquad \int_0^\infty \left[m - \frac{\alpha}{\eta}\exp\left\{-\eta(x-cy)e^{r(T-t)} + n(T-t)\right\}\right] G(dy) \\ &\qquad\qquad - \int_0^\infty \left[m - \frac{\alpha}{\eta}\exp\left(-\eta x e^{r(T-t)} + n(T-t)\right)\right] G(dy), \end{aligned}$$

thus,

$$\int_0^\infty \left[V(t,x-cy)-V(t,x)\right]G(dy) = \\ \int_0^\infty -\frac{\alpha}{\eta}\text{exp}\left\{-\eta x e^{r(T-t)}+n(T-t)\right\}\text{exp}\left(\eta c y e^{r(T-t)-1}\right)G(dy).$$

Hence,

$$\int_0^\infty \left[V(t,x-y)-V(t,x)\right]G(dy) = \left[V(t,x)-m\right]\int_0^\infty \text{exp}\left\{\eta c y e^{r(T-t)}\right\}G(dy), \tag{4.34}$$

$$\int_{-\infty}^\infty \left[V(t,x+Kz)-V(t,x)\right]H(dz) = \\ \int_{-\infty}^\infty \left[m-\frac{\alpha}{\eta}\text{exp}\left\{-\eta(x+Kz)e^{r(T-t)}+n(T-t)\right\}\right]H(dz) \\ -\int_{-\infty}^\infty \left[m-\frac{\alpha}{\eta}\text{exp}\left(-\eta x e^{r(T-t)}+n(T-t)\right)\right]H(dz),$$

thus,

$$\int_{-\infty}^\infty \left[V(t,x+Kz)-V(t,x)\right]H(dz) = \\ \int_{-\infty}^\infty -\frac{\alpha}{\eta}\text{exp}\left\{-\eta x e^{r(T-t)}+n(T-t)\right\}\text{exp}\left\{-\eta K z e^{r(T-t)-1}\right\}H(dz).$$

Herewith,

$$\int_{-\infty}^\infty \left[V(t,x+Kz)-V(t,x)\right]H(dz) = \left[V(t,x)-m\right]\int_{-\infty}^\infty \text{exp}\left\{-\eta K z e^{r(T-t)}\right\}H(dz). \tag{4.35}$$

**Step 3:** Substitute the partial derivatives of the value function into HJB Eqn. (4.28) to check if it solves the equation and to obtain the optimal investment and reinsurance strategy.

Substituting Eqns. (4.31) to (4.35) into the HJB Equation (4.28), we obtain

$$0 = \text{sup}_{u\in\mathcal{A}}\Big(-\eta e^{r(T-t)}\left[V(t,x)-m\right]\left[K[\mu-r]+rx+a-(1-c)\lambda_1\mu_1-\alpha(1-c)^2\lambda_1\mu_2\right] \\ +\left[V(t,x)-m\right]\left[\eta x r e^{r(T-t)}-n'(T-t)\right] \\ +\frac{1}{2}\eta^2 e^{2r(T-t)}\left[V(t,x)-m\right]\left[K^2\sigma^2+b^2+2k\sigma b\rho\right] \\ +\lambda_1[V(t,x)-m]\int_0^\infty \text{exp}\left\{\eta c y e^{r(T-t)}\right\}G(dy) \\ +\lambda_2\left[V(t,x)-m\right]\int_{-\infty}^\infty \text{exp}\left\{\eta K z e^{r(T-t)}\right\}H(dz)\Big),$$

i.e.,

$$0 = \sup_{u\in\mathcal{A}} \Big(-\eta e^{r(T-t)} \left[K[\mu - r] + rx + a - (1-c)\lambda_1\mu_1 - \alpha(1-c)^2\lambda_1\mu_2\right] - n'(T-t)$$
$$+ \frac{1}{2}\eta^2 e^{2r(T-t)} \left[K^2\sigma^2 + b^2 + 2k\sigma b\rho\right] + \lambda_1 \int_0^\infty \exp\left\{\eta c y e^{r(T-t)}\right\} G(dy)$$
$$+ \lambda_2 \int_{-\infty}^\infty \exp\left\{-\eta K z e^{r(T-t)}\right\} H(dz)\Big). \tag{4.36}$$

Now, we shall differentiate the left-hand side of Eqn. (4.36) with respect to $K$ and $c$ to find the critical points where the maximum is attained.
First, we differentiate with respect to $K$ and equate to zero:

$$-(\mu - r)e^{r(T-t)} + \eta^2 e^{2r(T-t)} \left[K\sigma^2 + \sigma b\rho\right]$$
$$+ \lambda_2 \int_{-\infty}^\infty -\eta z e^{r(T-t)} \exp\left\{-\eta K z e^{r(T-t)}\right\} H(dz) = 0.$$

Thus,

$$\mu - r - \eta\sigma^2 K e^{r(T-t)} + \lambda_2 \int_{-\infty}^\infty -\eta z e^{r(T-t)} \exp\left\{-\eta K z e^{r(T-t)}\right\} H(dz) = 0.$$

This implies that the optimal investment in risky asset $\hat{K}$ satisfies $L(K) = 0$, where

$$L(K) := \mu - r - \eta\sigma^2 K e^{r(T-t)} + \lambda_2 \int_{-\infty}^\infty -\eta z e^{r(T-t)} \exp\left\{-\eta K z e^{r(T-t)}\right\} H(dz).$$

We notice that

$$L'(K) = -\eta\sigma^2 e^{r(T-t)} - \lambda_2 \eta e^{r(T-t)} \int_{-\infty}^\infty \eta^2 \exp\left(-\eta K z e^{r(T-t)}\right) H(dz) < 0.$$

So, $L(K)$ is a decreasing function. Also $\lim_{K\to-\infty} L(K) > 0$ and $\lim_{K\to\infty} L(K) < 0$
Therefore, there exists $\hat{K}$ such that $L(K) = 0$.
Second, we differentiate with respect to $c$ and equate to zero:

$$-\eta e^{r(T-t)}\lambda_1\mu_1 - 2\eta e^{r(T-t)}(1-c)\alpha\lambda_1\mu_2 + \lambda_1 \eta e^{r(T-t)} \int_0^\infty \exp\left\{\eta c y e^{r(T-t)}\right\} G(dy) = 0.$$

Thus,

$$\mu_1 + 2(1-c)\alpha\mu_2 - \int_0^\infty y \exp\left(\eta c y e^{r(T-t)}\right) G(dy) = 0.$$

Denote the left-hand side as the function $J(c)$. That is,

$$J(c) := \mu_1 + 2(1-c)\alpha\mu_2 - \int_0^\infty y \exp\left(\eta c y e^{r(T-t)}\right) G(dy).$$

We notice that

$$J'(c) = -2\alpha\mu_2 - \int_0^\infty y^2 \eta e^{r(T-t)} \exp\left(\eta c y e^{r(T-t)}\right) G(dy) < 0$$

and

$$J''(c) = -\int_0^\infty y^3 \eta^2 e^{2r(T-t)} \text{exp}\left(\eta c y e^{r(T-t)}\right) G(dy) < 0.$$

So, $J(c)$ is a decreasing and concave function. Also, $J(0) = 2\alpha\mu_2 > 0$ and $J(1) = \mu_1 - \int_0^\infty y\text{exp}\left\{\eta y e^{r(T-t)}\right\} G(dy) < 0$. Therefore, there exists $\hat{c} \in (0,1)$ such that $J(c) = 0$ and $\hat{c}$ is the optimal reinsurance strategy. Substituting $\hat{K}$ and $\hat{c}$ into the HJB Equation in Eqn. (4.36), we get that

$$\begin{aligned} n'(T-t) = & -\eta e^{r(T-t)} \left[K[\mu - r] + rx + a - (1-\hat{c})\lambda_1\mu_1 - \alpha(1-\hat{c})^2\lambda_1\mu_2\right] \\ & + \frac{1}{2}\eta^2 e^{2r(T-t)} \left[\hat{K}^2\sigma^2 + b^2 + 2\hat{K}\sigma b\rho\right] + \lambda_1 \int_0^\infty \text{exp}\left(\eta\hat{c}ye^{r(T-t)}\right) G(dy) \\ & + \lambda_2 \int_{-\infty}^\infty \text{exp}\left(-\eta\hat{K}ze^{r(T-t)}\right) H(dz) \end{aligned} \quad (4.37)$$

with the initial condition $n(0) = 0$. If distributions of claim size $Y$ and jump size $Z$ is known, one can find a closed form expression for $n(\cdot)$. Hence, we can say that the value function satisfies the HJB Eqn. in Eqn. (4.28). $\square$

# CHAPTER 5

# CONCLUSION AND OUTLOOK

The aim of the thesis was to review optimal control problems through their applications in finance and insurance. We mainly focused on obtaining closed form solutions to the optimization problems. Optimal control methods is one of the methods used to handle portfolio optimization problems. An optimal control problem typically consists of a state process $X$, a control process $u$ and a cost functional $J(u)$. The objective in optimal control problems is to obtain a control process $u$ that optimizes the cost functional $J(u)$ over all admissible controls. In this thesis, we used the Dynamic Programming (DP) approach to solve optimal control problems. DP principle reduces the optimal control problem to the problem of solving the Hamilton-Jacobi-Bellman (HJB) Equation.

First of all, we studied important mathematical results needed to solve optimization problems. Major results in diffusion models and jump models were outlined. In addition, we provided the general class of optimal control problem and defined the objective function. The control problem is a minimization problem if the objective function is a cost functional. On the other hand, the control problem is a maximization problem if the objective function consists of utility functions. Further, the DP principle was described. The HJB equation and the Verification Theorem were also derived. We closely followed [6] and [29].

Optimal control problems have applications in finance, insurance, engineering, etc. In controlled diffusion processes, we discussed an application of stochastic optimal control in engineering followed by applications in finance. In the engineering application, called the Linear Quadratic Regulator, we find the optimal control process that keeps an initially excited system close to its equilibrium position. In the first financial applications, we studied the investment-consumption problem of an investor who has an initial endowment and is allowed to consume and invest in a financial market with a risk-free asset and a risky asset in two cases. In the second application, we considered a problem where the aim is to maximize expected utility of terminal wealth of an investor over finite time horizon.

Beyond controlled diffusion processes, we considered three applications of optimal control for jump processes. The first application is in finance and the other two are in insurance. The first application is an investment-consumption problem similar to the application treated in the controlled diffusion case. The difference in the two applications is that, in the latter problem, the dynamics of the risky asset is represented with a jump-diffusion process. This representation was done to account for the rare cases where there are large movements and sudden breaks in the risky asset price. The objective was to find a closed-form solution for the optimal investment and consumption strategy that will maximize the utility of the investor over an infinite time horizon. In the second application, a closed-form solution for the investment strategy that maximizes the expected exponential utility of terminal wealth of an insurer was obtained. The wealth process of this insurer is affected by the risk process which involves the cash inflow of premium income and cash outflow of insurance claims plus the initial capital. The risk process is modelled by a jump-diffusion process. In the third application, the insurer is allowed to invest in a financial market and purchase proportional reinsurance to reduce and share risk. The optimal investment and reinsurance strategy that maximizes the expected exponential utility of terminal wealth of an insurer was found.

This thesis can be extended for theoretical and practical purposes. In all the applications treated herein, closed form solutions were obtained. In general, however, the range of optimal control problems for which analytical solutions can be obtained is small. Hence, numerical methods are recommended to approximate optimal control processes. Kushner & Dupuis [17] provide numerical methods that could be used to model a wide range of stochastic control problems for diffusion and jump-diffusion processes. In the optimal control insurance applications that we studied, we obtained closed form solutions for cases where the jump size distribution is unknown. Analytical and computational methods could be used to solve or approximate the optimal value function of the insurer for specific jump size distributions. Baltas, Frangos & Yannacopoulos [4] consider the optimal investment and reinsurance in the presence of insider information in a Black-Scholes financial market. The problems could also be extended to where the surplus process of an insurer is modeled by a controlled regime-switching diffusion. Optimal control of stochastic hybrid systems can be used to formulate more practical problems. In such problems, the dynamics of the state process takes jumps and regime switches into consideration. The state process could be assumed to switch jump-diffusion paths between jumps. See Azevedo, Pinheiro & Weber [3] for an application in this setting. Temoçin & Weber in [33] provide an alternative way that uses a numerical discretization scheme to solve optimal control of stochastic hybrid system with jumps.

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