PRICING AND RISK MINIMIZING HEDGING STRATEGIES FOR MULTIPLE LIFE UNIT LINKED INSURANCE POLICIES USING CONSTANT PROPORTION PORTFOLIO INSURANCE APPROACH

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ABSTRACT

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A unit-linked life insurance policy (ULIP) is an agreement between an insurer and an insured that the insurance benefits or the obligations of the insurance company depend on the price of some specified stocks. As opposed to the classical life insurance, the payments to be paid at the occurrence of risk or at the end of the period of a unit linked life insurance contract can not be known at the time the policy is sold. Therefore, the benefits are random and unknown in advance, and based on this the obligations of the insurance company are also random. The main purpose in such a situation is to correctly define obligations of the insurance company, and based on these obligations to define the proper hedging approach.

In this thesis, we consider a model which takes into consideration the uncertainty of financial market and portfolio of insured individuals at the same time. It is assumed that financial and the insurance portfolios are stochastically independent and considered to be combined in a common product probability space. For the insurance portfolio under concern, we assume that polices are independent, but lifetimes of insureds in each policy are dependent. For the dependency between the lifetimes of insureds, an appropriate Pseudo-Gompertz distribution is used in the thesis. We investigate two cases for multiple life policies, joint life status and last-survival status. Appropriate obligation equations for both cases are derived and by the Constant Proportion Portfolio Insurance (CPPI) approach optimal portfolio weights are defined. For the solution of optimization problem, mean-variance hedging strategy is used as one of the mostly applied hedging approaches in such situations. The thesis ends with a conclusion and an outlook to future studies.

Keywords : Unit Linked Life Insurance Policies, Constant Proportion Portfolio Insurance with Jump Diffusion, Multiple Life Policies, Mean Variance Hedging, Joint Lifetimes Distribution

ÖZ

ÇOKLU-HAYAT BİRİM BAĞLANTILI SİGORTA POLİÇELERİ İÇİN SABİT ORANLI PORTFÖY SİGORTASI YÖNTEMİ KULLANILARAK FİYATLAMA VE RİSK MİNİMİZASYONU

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Birim Bağlantılı Hayat Sigortası anlaşmalarında sigorta hasar ödemeleri spesifik bir ürünün fiyatına bağlı olmaktadır. Klasik hayat sigortası ürünlerinin aksine bu ürünlerde hasar gerçekleştiği durumda ödenecek tutar sigorta kontratı yapıldığı aşamada belli olmamaktadır. Bu sebepten dolayıdır ki, hem hasar ödemeleri hem de sigorta şirketinin yükümlülükleri belirsizlik teşkil etmektedir. Bu durumda en başlıca amaç gelecekte oluşabilecek hasar ödemelerini, başka bir deyişle sigorta ödemeleli için bugünkü yükümlülükleri doğru belirleyebilmek ve bu yükümlülükler için uygun yatırım ve riskten korunma (hedge) kavramı geliştirmedir.

Tezde, hem finansal piyasalardaki hem de sigorta portföyündeki bireylerden doğacak olan belirsizliği kapsayacak bir modelleme yapılmıştır. İlgili riskler Finansal ve Sigorta portföyleri arasında stokastik bağımsızlık düşünülerek onları ortak olasılık uzayında modellenmiştir. Sigorta portföyü için poliçelerin birbiri arasında bağımsız olduğunu ama poliçeye bağlı yaşamların kendi aralarında bağımlı olduğu gerçeği dikkate alınmıştır. Bu bağımlılık için Pseudo-Gompertz dağılımı kullanılmıştır. Bağımlılık iki durum için incelenmiştir: ”Birlikle Sağ Kalım” ve ”Son Sağ Kalım”. Finansal piyasalardan doğan belirsizliği modellemek için Sıçramalı Difüzyonlu Sabit Oranlı Portföy Sigortası yöntemi kullanılmış ve optimal ağırlıklar belirlenmiştir. Ağırlıkların belirlenmesi için Ortalama-Varians hedge yöntemi kullanılmıştır.

Anahtar Kelimeler : Birim Bağlantılı Hayat Sigortası Poliçeleri, Siçramalı Difüzyonlu Sabit Oranli Portföy Sigortası, Çoklu Hayat Poliçeleri, Ortalama-Varyans Risk Minimizasyon Yöntemi, Ortak Yaşam Süreleri Dağılımları

To My Family

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CHAPTER 1

INTRODUCTION

1.1 Historical Overview and Literature Review

In the classical life insurance agreements, an insurance company guarantees payments to its insureds equal to the promised provisions and hence takes only the investment risk. The company makes its investments, which is the amount of the provisions and decides on the allocation of these provisions to different types of assets. These investment strategies are defined within certain restrictions imposed by the regulator. Typically, the provisions are invested to a large extent in risk-free bonds which are assumed to give rise to a return which is higher than the guaranteed return on the life insurance products.

In unit linked life insurance contracts opposed to classical life insurance agreements the benefit to be paid at the occurrence of risk, since survival or death of insureds, can not be known at the time the agreement made between insurance company and the policyholder. On such type of policies, the insurance installments paid by the insured are invested in some funds made up by the insurer. These funds consist of different type of financial assets. The amount of benefits at time of the occurrence of risk is as much as the value the financial assets have reached, bought by the premiums which have been paid by the insured unit at that time. Therefore, the benefits are random.

Unit linked insurance products carry the financial and insurance risk, that is why should be treated and analyzed from financial and actuarial points. Several papers by Möller [67] deeply investigate life and non-life products from valuation and hedging aspects within the financial and actuarial framework, describing in details historical development of unit-linked life insurance agreements with an analyzes of relevant literature.

Brennan and Schwartz [19, 21, 22] and Boyle and Schwartz [18] were the first authors who studied unit-linked insurance contracts. They used current financial mathematics in their research and defined that for the unit-linked insurance product with guarantee amount the payoff is equal to the payoff of an European call option with the additional payment equaled to guaranteed amount. For removing the uncertainty related with insured lives authors used results from law of large numbers and replaced them with their expected values. Main shortcoming of this method is that it is not considering mortality risk. This approach gives to the authors facility to treat policies with guarantee as contingent claims which carries financial uncertainty in a complete financial market. The standard valuation and hedging techniques which were introduced recently by Black and Scholes [13] and Merton [58] can be applied in this market. Harrison and Pliska [44] and Harrison and Kreps [43] developed in their research martingale based financial approaches which helped Delbaen [29], Bacinello and Ortu [5] and Aase and Persson [1] during the valuation of the unit-linked life insurance agreements. For insurance portfolio these authors also have chosen the same technique for unexpected mortality patterns and replaced mortality by its excepted values. Afterwards, Aase and Persson [2] in their work tried to use different way and use continuous time survival probabilities despite discrete ones which are used in previous papers. In addition to those studies, Bacinello and Ortu [6], Nielsen and Sandmann [72] and Bacinello and Persson [7] as an addition investigated stochastic interest rates within the existing financial models, where all previous authors preferred to use constant interest rates.

Möller [61] was the one who introduced the new approach for combined model. He investigated and developed a new technic in which the unexpected developments of the financial market and the portfolio of insureds are treated by not averaging the mortality. He considered the combined model in his research and according to this research product probability space has been studied as two independent parts. In addition he demonstrated that uncertainty related with the insurance portfolio can turn a complete financial market to incomplete, making riskless hedging as impossible. In mentioned paper the risk-minimization hedging strategies are also derived for a life insurance agreements by taking into account a Black-Scholes financial market. Two different types, such as pure endowment and term life unit linked insurance contracts are investigated in that research. Author applied the risk-minimization hedging theory improved by Föllmer and Sondermann [36]. Results derived from this research is considering both financial and insurance uncertainty. Möller [61] supposes a unit premium paid at policy signing time and assumes that insurance payments are deferred. Here, practically not considering premium payments during the mid-terms is not very limiting for the pure endowment. However, since the benefits are mainly paid just after realization of death for life insurance policies this limits the conditions. For instance, even the annuity payments are paid with intermediate installments. That is why, Möller [64] enlarged the scope of the risk-minimization hedging theory of contingent claims with fixed duration introduced by Föllmer and Sondermann [36] to include the streams of the payment. Insurance payment streams are investigated by taking into account mid-term premium and benefit payments, in his research. In other hand, this theory can be applied only in martingale financial markets taking into consideration that the hedging risk can be interpreted only under the investors individual measure if this is already a martingale measure. In his research Möller [64] furthermore includes the risk-minimization hedging strategies and the related hedging risk for a martingale Black-Scholes financial market and for general unit-linked life insurance agreements, formulated in a multi-state model of Markov along with mid-term installments. In addition to the above listed articles (Möller [61, 64]), the study of Möller [65] analyses the risk-minimization hedging of equity-linked life insurance agreements in a discrete time framework. He applied the financial market model of Cox-Ross-Rubinstein as a case in this analyze. Recently, Möller [66, 68, 69] introduced the indifference pricing approaches which contain the financial variance and standard deviation basis, as an different approach to the valuation and hedging of unit-linked life insurance agreements. On the other hand, Dahl and Möller [28] utilized the risk-minimization hedging theory for payment streams introduced by Möller [64] for the case of classical life insurance with stochastic interest rates and a death intensity of all insureds which was influenced by some stochastic process. The existence of the financial market is obligatory for the treatment of the unit-linked life insurance agreements, since the units are modeled in the financial markets. It can be said that the most of the articles analyzed above consider a Black Scholes financial model of asset returns which follow normal distribution and continuous paths which means that assets prices are following the Brownian motion. Riesner [88] investigate in his research more general Levy-process financial market models. He replaced the Brownian motion by Levy Jump Diffusion process. As a result he got the models with adaptable asset return distributions and path which follows discontinuity. In comparison with other studies, Chan's [24] research can be considered as a crucial investigation, since it proposes general Levy processes to financial market modeling. Nevertheless, a jump-diffusion process has been introduced even by Merton [58] to model asset returns. As the first part of Cont and Tankov [26] states, Levy-process models with jumps normally fit the outcomes of solid financial markets, that are difficult, even not possible to implement in models with diffusion. In his valuable research Hainaut [42] has determined the optimal asset allocation of pure endowments insurance contracts, maximizing the expected utility of a terminal surplus under a budget constraint. The market resulting from the combination of insurance and financial products, is incomplete owing to the un-hedgeable mortality of the insured population, modeled by a Poisson process. For a given equivalent measure, the optimal wealth process is obtained by the method of Lagrange multipliers and the investment strategy replicating at best this process is obtained either by martingale decomposition or either by dynamic programming. They illustrated this method for CARA and CRRA utility functions.

Some later Vandaele [96] in their research disproved the Risner [88]. He proved that the risk-minimization hedging strategy under the proposed martingale measure does not fit the conditions for being the local risk-minimizing strategy under the original measure, which was established by Riesner [88]. Eventually, the correct local risk-minimization hedging strategy was investigated and a link among the several risky assets which held in the suggested portfolio stated in the aforementioned paper and the one suggested here is given.

Bi [11] considered in their research the risk-minimization hedging strategies for unit-linked life insurance policies within a financial market in which stock price process follows shot-noise. Taking into consideration the incompleteness of the financial market, it is impossible to hedge the insurance claims by trading stocks and riskless assets alone. This hedging strategy leaves the part of the risk on the insurer. After a change of measure authors applied the theory of pseudo locally risk minimization.

For two sorts of unit linked agreements such as pure endowment and the term insurance the risk-minimizing trading strategies are investigated and accompanied intrinsic risk processes are established in Qian [84]. In their research they extended the model and analysis done by Vandaele [96].

He described the Levy process parameters which formulates the pattern of risky asset in the financial framework as depending on a finite state Markov chain. Here, the state of the Markov chain can be assumed as the state of the economy. With the regime switching Levy model, the authors got the local risk-minimization hedging strategies for several unit-linked life insurance contracts, among which the pure endowment policy and the term insurance policy are well known. In Wang [97], a general class of stochastic volatility model is considered for modeling risky asset. This class of stochastic volatility model involves nearly all of those without jump component which are mainly utilized in most of the researches. They obtained the minimal martingale measure and local risk minimization hedging strategy in these models, and employed the results to the unit-linked life insurance agreements. Moreover, they also investigate the locally risk minimizing hedging strategy for unit-linked life insurance agreements in a Barndorff Nielsen and Shephard stochastic volatility model. The local risk-minimization approach was investigated by Ceci [23] for a combined financial-insurance model, where there are restrictions on the data accessible to the insurance company. Particularly, we assume that, at any moment, the insurance company may observe the number of deaths from a specific portfolio of insured individuals but not the mortality hazard rate. They considered a financial market which driven by a general semimartingale and they aimed to hedge unit-linked life insurance agreements via the local risk-minimization approach under partial information. The Föllmer-Schweizer decomposition of the insurance claims and explicit formulas for the optimal strategy for pure endowment and term insurance agreements are provided in terms of the projection of the survival process on the information flow. Moreover, in a Markovian framework, they reduce the steps to solve a filtering problem with point process observations.

Nearly all of the articles related to this topic assume as in Möller [61] that, the future lifetimes are independent and identically distributed exponential random variables.

In this thesis we introduce dependence between future lifetimes of insureds and combine it with pricing and risk minimizing strategies.

The premiums are considered to be paid as a single premium and the benefits are postponed to the maturity of the agreement.

In the first and second parts of the thesis, a general information was provided about the financial instruments for risk mitigation and insurance products. Also we investigate unit linked life insurance policies and quadratic hedging approaches, such as local risk minimization and mean variance hedging approach.

In the third part, the model of the thesis is structured in a financial model and insurance portfolio combination. Risk minimizing strategies are investigated for combined obligations and hedging strategy is derived by the help of the mean-variance hedging approaches.

In the forth part, numerical results for the constructed model are provided. Monte-Carlo simulation method is used and calculations are pursued by the help of R software.

1.2 Summary, Contributions and Outline

The main objective of this thesis is to develop a mean-variance hedging strategy for multiple-life unit linked insurance policies, where the unit is modeled by the constant proportion portfolio insurance approach.

Following Möller [61, 64], we consider the financial market and the portfolio of insureds are independent. As well we assume that they are joined in a common product probability space. It gives rise to the concept of modeling the uncertain movement of the asset price and the lives of the policyholders synchronously. For insurance portfolio we do not average away survival probabilities. Which means the uncertain pattern of insured lives is not interchanged by their anticipated developments.

Let us take a person aged x years, with future lifetime denoted by T_x. So, $x + T$ is the age of the person at the end of his life. The same definition for T_y can be used as well.

In general, the future lifetime T is a random variable having a probability distribution function,

$$F_t = Pr(T \leq t),\ t \geq 0.$$

Here, the function $F(t)$ describes the probability with which the one will die during t years, at any fixed t.

We investigate two cases for multiple decrements, as joint life status and last survival models. Joint life considers payments at the first death. Which means, if we have multiple number of beneficiaries for the policy, benefit payment will be made at the first death. For two lives this means,

$$T_{xy} = \min{(T_x, T_y)}.$$

By the same method, we can define last-survival status, and can say this is a status where benefit payments are conditioned to the death of last beneficiary of the policy. The expression of this formula for two lives is,

$$T_{\overline{xy}} = \max{(T_x, T_y)}.$$

Then for modeling the dependency between lifetimes we use Pseudo-Gompertz distribution developed by Yörübulut and Gebizlioglu [100]. The Gompertz distribution is considered as a mostly applied probability distribution for the lifetimes modeling. While modeling the dependent structure of lifetimes through bivariate distributions Pseudo-Gompertz distribution is the most appropriate one for our case.

For financial part we use the well known Constant Proportion Portfolio Insurance approach and define stock price process in jump diffusion modality. As we know, with this strategy investor begins with defining a floor equal to the minimum acceptable value of the portfolio. This helps us to design a portfolio similar to the unit linked life insurance agreement which considers minimum guarantee payment.

Applying the above mentioned two points at the begining we introduce the filtration for the combined portfolio as follows,

$$C = (C_t)_{0 \le t \le T},$$

as,

$$C_t = F_t \vee I_t.$$

We assume that these two filtrations are independent and take,

$$C = F_T \vee I_T,$$

where

$$F_t = \sigma(S_u, u \le t),$$

$$I_t = \left[\sigma(I(T_{x_i} \le t), \quad 0 \le t \le T, \quad i = 1, 2, ..., n) \vee \sigma(I(T_{y_i} \le t), \quad 0 \le t \le T, i = 1, 2, ..., n)\right].$$

Then we derive combined obligations for joint life status,

$$H_T^j = B_T^{-1} \sum_{i=1}^{p} g(T_i, V_{T_{x_i y_i}}) B_{T_i}^{-1} B_T \mathbf{1}_{\{\min(T_{x_i}, T_{y_i}) \le t\}},$$

and for last survival status,

$$H_T^l = B_T^{-1} \sum_{i=1}^{p} g(T_i, V_{T_{x_i y_i}}) B_{T_i}^{-1} B_T \mathbf{1}_{\{\max(T_{x_i}, T_{y_i}) \le t\}}.$$

Then for the combined obligations we define the following optimization problem based on the mean-variance hedging approach:

$$\min_{(Z_0, \nu) \in R \times \Theta} E^Q \left(\tilde{H}_t^M - Z_0 - \int_0^T \nu_u d\tilde{V}_u \right)^2. \tag{1.1}$$

The following equations are obtained as the solutions for the optimal portfolio,

$$Z_0 = E^Q[\tilde{H}_t^M], \quad \nu_t = \frac{\sigma_t \zeta_x(t, V_t) + \zeta(t, V_t + [V_{t^-} - F_t] m_t Y_t) - \zeta(t, V_t) Y_t \lambda_t \psi_t}{\sigma_t + (V_{t^-} - F_t) m_t Y_t^2 \lambda_t \psi_t}. \tag{1.2}$$

As the performance of the numerical calculations, we use R software and provide results one by one for financial portfolio, single life and multiple life for several exposure values and other variables in our model.

CHAPTER 2

BASIC CONCEPTS

2.1 Financial and Insurance Background

A prevailing subject in the finance and insurance areas is the risk of loss. Risk is inherent in all parts of life due to its nature, hence, it cannot be avoided completely. For the mitigation or removal of the risk, several risk management tools and instruments are established.

As we know, the insurance and finance disciplines appeared as distinct fields. At the beginning, the theory of insurance has been generally engaged in the calculation of premiums of the life insurance agreements. As the time past some combined instruments are established and insurance and finance is then nested.

In this section, we provide the preliminary knowledge with regard to the finance and insurance risks.

2.1.1 Classical Insurance Contracts and Their Valuation

As mentioned in the previous paragraph, at the beginning, the main concern of the insurance theory was about the premium calculation of life insurance agreements. Insurance is an equitable transfer of risk in exchange for some specific payment. The ways of transferring or distribution of the insurance risk were expertized by Chinese and Babylonian merchantiles already at the 3rd and 2nd millennia BC.

In general sense, insurance considers collecting premiums from several insureds for paying of the insurance benefits in case of insurance event, differently saying in case of loss. The insureds are protected from risk, with the fee, on other words for risk premium which depends how frequent and how severe the event occurs. In other words the risk have to fit the concrete predefined criteria in order to be considered as an insurable risk. An insurance company can be seen as a business enterprise whose main role is financial intermediacy generally. Insurance company is a main segment of the industry of financial services, but alternatively the different individuals also can make a self insurance independently, via saving money for future loss possibilities. All financially measurable risks are able to be subjected for the insurance. Some particular types of the risk is called perils, which can rise the claims. An insurance company should prepare its policy detailed in order to clarify which types of the perils are included and which types are excluded.

The risks in one or in multiple categories can be covered in a single policy. For instance, both the property risk and the liability risk which arising from a legal claims as a result of traffic accidents can be covered normally by a single motor insurance. Similarly, a home insurance policy covers the sum insured for any loss of house and also the house owners properties, sum legal claims the owner can be met, and also a limited sum for the insurance benefit for the health services of visitors if he got health damage at that house. The insurance of the business usually has a various forms. The most popular forms are different types of professional liability insurance, known as professional indemnity, and the business owners policy which combined variety types of benefit payments to which a business owners can need, in one policy, similarly as home owners policy includes the benefit payment for such things that owner of the house can need.

In actuarial theory, generally, the insurance contracts are divided into two parts as life insurance and non-life insurance contracts. Here, we will share the information about the valuation of classical life insurance policies because of the nature of our research. We mainly use as the references Gerber, Norberg, Möller [39, 76, 63].

Let us consider a portfolio of insured persons which number is equal to n, and ages are equal to y. At time 0 which is the beginning of policy we assume that they have identically and independently distributed future life times $T_1, T_2, \ldots, T_n$ and presume the existence of a continuous mortality rate function μ_{y+t}, with the following survival probability:

$$ {}_tp_y = P(T_1 > t) = \exp\left(-\int_0^t \mu_{y+u} du\right). \tag{2.1} $$

For a pure endowment agreement with sum insured K and maturity equal to T designates that the amount K which is the benefit stemmed form the insurance agreement should be paid at maturity time T. For benefit payment policyholders survival is stated as a condition. Furthermore, let us consider that there is a single premium named as k and that the seller of the contract allocates the premium k in several security which considers a payment as a rate of return $\delta = (\delta_t)_{0\le t\le T}$ during $[0, T]$. Here, the obligation of the insurance company for the ith policy-holder, can be written as in following equation, by the present value which can be formulated as follows:

$$ H_i = \mathbf{1}_{\{}(T_i > t)\} K e^{\int_0^T \delta_t dt}. \tag{2.2} $$

This equation is derived by finding the present value the payment at time T, $\mathbf{1}_{(T_i>t)}K$, utilizing the rate of interest denoted by δ. What the fundamental principle of equivalence says is that, the premiums have to be calculated such that the discounted values of premiums and benefits on average should be balanced. It is assumed that in addition δ and the remaining life times are stochastically independent. Then by the help of the principle of equivalence we can consider that the following equality is correct:

$$ k = E[H_i] =_T p_y K E[e^{\int_0^T \delta_t dt}], \tag{2.3} $$

for the single case. We know from insurance market that life insurance portfolios are big in common, that is why that concept can be proved partially by the help of the law of large numbers.

By the increasing of the size of the insurance portfolio, the associated number of survivals

$$\frac{1}{n}\sum_{i=1}^{n}\mathbf{1}_{\{T_i>t\}},$$

converges a.s. regarding the probability ${}_Tp_y$ of survival to T. This is possible by application of the strong law of large numbers. Because of lifetimes $T_1, T_2, ..., T_n$ are stocastically independent, for significant large value of n, the factual number of survivors

$$\sum_{i=1}^{n}\mathbf{1}_{\{T_i>t\}},$$

will be nearly same as the expected number n_Tp_y. Here, growing the amount nk by rate of interest take us to

$$nke^{\int_0^T \delta_t dt} = n_Tp_yKE[e^{-\int_0^T \delta_t dt}]e^{\int_0^T \delta_t dt} \approx \sum_{i=1}^{n}\mathbf{1}_{(T_i>t)}KE[e^{-\int_0^T \delta_t dt}]e^{\int_0^T \delta_t dt}. \quad (2.4)$$

Peculiarly, when δ is deterministic, policyholders benefit payment can be calculated as the expression on the right side of previous equality. Under the deterministic interest rate, the principle of equivalence is validated by investigation of the law of large numbers. This, actually guaranties the fact that the actualized number of survivals is around the expected number.

The issue is getting the difficult and complicated shape in the real world where δ is following the stochastic process. This follows from last equation that the simple accumulation of the premium k will not in general generate the amount to be paid. That is why, $e^{-\int_0^T \delta_t dt}$ could have different value than its expected value. Another method of approaching that issue involves the replacement of the ”factual” rate of return process δ in Eqn. (2.3) with deterministic rate of return process $\delta^{'}$. According to this approach the single premium nk grew by the real rate of return δ is bigger than K times expected number of survivals with a big probability values. In this case, the insurer has to include that excess into the sum which will be paid to the insured person like bonus [85, 75]. Anyhow, this method creates a question if it is meaningful to consider the presence of any strictly positive and deterministic $\delta^{'}$ which over quite huge time period has the property that it will be higher than the actual rate of return on investments with a very high probability. Taking into the consideration of the fact of low interest rates at the end of 1990s, this matter can be seen as a critically important issue. There can be another method to deal with matter which includes the replacement of δ by the so named short term interest rate and in addition the replacement of the last term in Eqn. (2.3) by the price on the financial market of a financial security, known as a zero coupon bond, which pays one unit at maturity time T. In [7] is derived a general form of Thiele’s differential equation within above mentioned framework.

2.1.2 Financial Instruments and Risk Mitigation

Cash and cash securities will continue to take relatively small part of the investment portfolio of the long term investment institutions such as pension funds and life insurance companies. In contrast to the long term investors non-life insurance companies with their shorter term obligations, should hold a bigger part of assets in cash for being ready for benefit payments in case of claims [14].

Deposits in risk free assets do not carry any risk of market changes and the return earned is mainly linked to the degree of short term interest rates existing in the market. So, while giving the protection for the capital, cash securities are not providing any guarantee that a certain rate of return will be promised within the predefined time horizon. That is why, cash securities can get the shape of risk-fee assets deposits or the shape of tradeable cash securities. The duration of such tradeable assets is usually not long, but the crucial property is the interest rates movement frequency to consider the changes in short-term interest rates.

Cash securities suggest full capital protection from the credit risk. Based on the degree of short term interest rates their return varies. For some inflation protection purposes cash instruments can be used, but this is valid for long term where increase in inflation can tend to allow to higher short term interest rates.

The fixed interest bond is an agreement which provides special predefined rights to the borrower against to the money lender. Here, in particular, a fixed interest asset commonly gives to the holder of the asset the several fixed coupon installments and a principle repayment at a predetermined maturity time.

One of the well known and mostly used financial instruments for risk mitigation are financial derivatives. Financial derivatives are investment securities whose values are linked to the value of an underlying asset. In other words financial derivative is a combination of investments, with same characteristics. Mainly used financial derivatives which play quite important role, while institutional financiers are considered, are options, forwards, futures, and currency swaps.

The risk of investment is stemming from the situation where these investments are not used for legitimate purposes. During the analyzing of the risk of any derivatives position, the essential point is the consideration of the risk within whole portfolio. All risky strategies in derivatives can be balanced by adequate and counter risks in underlying assets - especially during the purchase of the derivatives with the aim of the risk management. The “counterparty risk” is existing as a rule in every derivative agreement. Principally, the counterparty risk looks like to default risk on a corporate bond agreement. Since, this is the risk that the intermediaries with whom the derivative agreement has been entered into defaults and, hence, a loss is made. In order to minimize the counterparty risk, plenty of the institutional arrangements were done. Those arrangements are consisting of the utilization of the intermediaries with high credit rating to implement the derivative deals, and the utilization of “margin accounts” to obstacle the exposures from building up.

In next section we will discuss the structured portfolio management which is one of the effective risk mitigation instruments by the risk free securities and stocks.

2.2 Structured Portfolio Management

As it is well known the main goal of portfolio insurance is to control portfolio returns from unforeseen falls which are expected in the economy. Also this allows risk managers to get benefit from increases in the financial market. That is why, for insured portfolio values at terminal date, there is a guarantee defined before and during the market go up, the portfolio return should also increase at least at a predefined percentage of a determined index return [81].

For this reason, specification of guarantees and portfolio termination date is crucial in portfolio insurance.

There are two most used portfolio insurance approaches. One of these approaches is the Option Based Portfolio Insurance and the another one is the Constant Proportion Portfolio insurance.

In the OBPI approach investor holds in the portfolio a risky asset S covered by a listed put written on it, introduced by Leland and Rubinstein .

The CPPI was introduced by Perold [78] for fixed income instruments and Black and Jones for equity instruments. For allocation of assets dynamically over one of the most useful strategy is this method.

In the following two subsections, we will provide information about these strategies. Especially, CPPI strategy will be investigated in details.

2.2.1 Option Based Portfolio Management

The OBPI, introduced by Leland and Rubinstein [52], consisted of a portfolio invested in a risky asset S covered by a listed put written on it. Not depending on the value S at maturity date T, payoff of the portfolio will be always higher than the strike price K of the put.

The main objective of the OBPI approach is to guarantee a fixed return at maturity of the portfolio. Nevertheless, it is obvious that, the OBPI approach guarantees one to receive portfolio insurance at any time. Although, it has a disadvantage that on the market it is not always easy to find European put with required exercise price and terminal date. In this case for making the synthetic put one should generate dynamic replicating portfolio which is consisting of riskless and risky asset. In this method the manager of the portfolio is expected to make an investment in two basic assets: a money market account in other words riskless asset, denoted by B, and a portfolio of traded assets such as a composite index, denoted by S, which are risky assets. The period of time considered is $[0, T]$. The strategy is self-financing.

The value of the risk-free asset B evolved according to:

$$dB_t = B_t r dt, \tag{2.5}$$

where r is the deterministic interest rate.

The dynamics of the market value of the risky asset S are given by the standard diffusion process:

$$dS_t = S_t[\mu dt + \sigma dW_t], \tag{2.6}$$

where $(W_t)_{t\geq 0}$ is a standard Brownian motion.

The OBPI method consists the purchase of q shares of the asset S and q shares of European put options on S with maturity T and exercise price K.

Thus, the portfolio value V^{OBPI} is given at the terminal date by:

$$V_T^{OBPI} = qS_T + q(K - S_T)^+, \tag{2.7}$$

which is also

$$V_T^{OBPI} = qK + q(K - S_T)^+, \tag{2.8}$$

due to the Put/Call parity. This relation shows that the insured amount at maturity is the exercise price times the quantity q: qK.

The value V_t^{OBPI} of this portfolio at any time t in the period $[0, T]$ is:

$$V_t^{OBPI} = qS_t + qP(t, S_t, K) = qKe^{-r(T-t)} + qC(t, S_t, K), \tag{2.9}$$

where $P(t, S_t, K)$ and $C(t, S_t, K)$ are the Black-Scholes values of the European put and call.

Without loosing of generality and for simple representation, it should be assumed, q is normalized by setting equal to 1. Then,

$$V_T^{OBPI} = S_T + (K - S_T)^+ = K + (S_T - K)^+. \tag{2.10}$$

With respect to the risky asset price S_T at terminal date, the function above is convex and increasing. That is why, common properties of the portfolio payoffs with guarantee constraints is observed. With this strategy one can get positive return from upward directions of market. In case, the price of the risky asset is overlap the exercise price at terminal date the payback can be represented as in following equation:

$$\left(\frac{V_T}{V_0}\right) = \left(\frac{S_T}{S_0}\right) \times \left(\frac{S_0}{S_0 + P_0(K)}\right).$$

With this, the percentage can be presented as in following equality: $\frac{1}{1+P_0(K)/S_0}$.

2.2.2 Constant Proportion Portfolio Management

The CPPI approach dynamic allocation method for assets over time, introduced by Perold. Structure of the strategy is as follows. The portfolio manager chooses the minimum value for floor which is the smallest acceptable value of the portfolio. As a second step he determines cushion which is the excess of the portfolio value over the floor. Then by the predetermined scalar or in other words multiple he defines the units which will be allocated to the risky asset. This can be done through multiplying the cushion by a predetermined scalar. Two external variables are the floor value and multiple which are the functions of portfolio manager which are showing the risk aversion rate. Difference between total fund and risky asset is allocated to riskfree assets and these assets mainly T-bills.

Multiple shows the risk aversion of the investor and the bigger the multiple, the more the investor is willing to invest in risky assets by increasing the share of it. On the other hand, with the decrease in stock prices, the bigger the multiple, the faster the portfolio will tend to drop to the floor. Exposure approaches to zero as the cushion approaches to zero. In continuous time, this preserves the portfolio payoff from dropping under the floor. If the portfolio manager will not has a chance to trade before the significant falls in the market portfolio value can be fall below the floor. We refer mainly to Prigent [81, 78].

Naturally, the CPPI approach is a managing of a dynamic portfolio with the condition to be above a floor F at any time t. Here, the value of the floor gives the dynamical insured amount. The floor is developed as a risk free asset, based on the below equation:

$$dF_t = F_t r dt. \tag{2.11}$$

It is obvious that, the floor value at the beginning F_0 is under the initial portfolio value V_0^{CPPI}. The difference between these two variables $V_0^{CPPI} - F_0$ is denoted by C_0 and named the *cushion*, . The value of the C_t at time t in $[0, T]$ is defined as:

$$C_t = V_0^{CPPI} - F_t. \tag{2.12}$$

Total amount which is invested to the risky asset is denoted by e_t and defined as the *exposure*. In the standard CPPI approach first step is started by setting,

$$e_t = mC_t, \tag{2.13}$$

in which equation m is a constant called the scalar or multiple. One of the most important points for portfolio insurance related with the $m > 1$ is that with the convex payoff function significant percentage of the market rise can be provided.

Assuming the risky asset price process $(S_t)_t$ is following the diffusion with jump:

$$dS_t = S_{t^-} \left[\mu(t, S_t)dt + \sigma(t, S_t)dW_t + \delta(t, S_t)d\gamma \right], \tag{2.14}$$

where $(W_t)_t$ is independent from the Poisson process with measure of jumps γ, a standard Brownian motion.

Summarizing all above stated we can say following:

- The sequence of random times $(T_n)_n$ accompanied with jumps fits the following properties that the inter arrival times $T_{n+1} - T_n$ are independent. It is assumed that they have the same exponential distribution with parameter λ.

- The relative jumps of the risky asset $\frac{\Delta S_{T_n}}{S_{T_n}}$ are equal to $\delta(T_n, S_{T_n})$. They are expected to be strictly higher than -1, in order for the price S to be strictly positive.

The portfolio value and other important equalities will be investigated in third chapter under the financial model.

In following part, OBPI and CPPI methods are compared and main results are stated following the Bertrand and Prigent [82].

It is assumed that the equal amounts of V_0 is invested at initial time 0. Identically, it is assumed that the analogous guarantee K holds at termination. The last assumption was that the risky asset price pursues a geometric Brownian motion.

Results:

1. Payoffs of both portfolios are not exceed one another for all values of the risky asset at the maturity. That is why, the two payoff functions overlap one another.

2. The expected values of both strategies are equal $E\left[R_T^{OBPI}\right] = E\left[R_T^{CPPI}\right]$ and this takes us to a unique value for the scalar. For any fixed guaranteed amount K, the multiple denoted by $m^*(K)$ satisfied the above condition. In the Black-Scholes value of the call option, and x denotes all possible values of the risk free rate.

3. For first order stochastic dominance both strategies are in same level, in other words neither of them stochastically dominating another at first order.

4. In a mean-variance sense, OBPI strategy dominates the CPPI approach at least for one value of m for any parametrization of the financial markets (S_0, K, μ, σ, r).

5. OBPI approach can be observed as a general form of CPPI approach. For a model with the geometric Brownian motion, the OBPI approach is analogous with the CPPI approach. For holding this it is assumed that the multiple is permitted to differ and defined as

$$m^{OBPI}(t, S_t) = \frac{S_t N(d_1(t, S_t))}{C(t, S_t, K)}. \tag{2.15}$$

This factor is the ratio of the delta of the call $N(d_1(t, S_t))$ multiplied by the price of the risky asset S_t, which is equaled to the risk exposure divided by the cushion amount, equaling to the call price $C(t, S_t, K)$.

2.3 Unit Linked Life Insurance Policies

As it is well known the fundamental principle of equivalence says that present value of the premiums and benefits have to be equal on average for any type of insurance contract. That is why classical actuarial valuation theory for life insurance agreements mainly concentrated on calculation of expected values of discounted random cash-flows. Then the corresponding premium is called the *equivalence premium*. Similarly, the expected reserve is calculated as the conditional expectation of difference between all of the discounted future benefits and premiums, with the available information, during the insurance period. Approach to a subject is changed with introduction of the new product, a unit-linked life insurance agreement, in which benefits depend exactly on a specified stock index. With new contract which is quite different from classic one, the insured will receive the highest values of the stock price and some asset value guarantee specified in the agreement, but other dependencies should be well defined.

2.3.1 Description and Main Concepts

The first investigations of unit-linked contracts based on financial theory belong to Brennan and Schwartz [20, 21, 22] and Boyle and Schwartz [18]. They identified that the terminal value of a unit-linked policy which has a predefined guarantee can be linked to the terminal values of certain financial derivatives. By applying the option pricing theory founded by the Black and Scholes [13] and Merton [59] authors got valuable results.

Afterwards, Harrison and Kreps [43] and Harrison and Pliska [44] extended the theory of unit-linked agreements by applying the martingale pricing theory, which is the extension of the classical Black Scholes Merton model. Before the prevalent diffusion of unit-linked agreements (Tiong [95] and Möller [61, 64]; Boyle, Kolkiewicz, and Tan [17]; Bacinello and Persson [4]) the popularity and high preference of participating policies was because of the fact that the policyholders trusted on the traditional and not risk lover portfolio management of life insurance companies. At the initiation unit-linked insurance policies were considered quite risky not depending on the providing of minimum guarantee. This perspective was the same for policyholder and insurer. Policyholder considered without minimum guarantee product risky, while for the insurer with guarantee product carried the risk. Most of the insurers were not ready and willing to give such a guarantee since they did not have any tools for hedging the risk. A crucial milestone in this field was the option pricing theory developed by the Black and Scholes [13] and Merton [59]. By use of this theory Brennan and Schwartz [20] and Boyle and Schwartz [18] derived valuable results for guaranteed unit-linked products.

Different benefits including caps, differently saying, upper limits, have been studied in [32, 74].

Hipp [45] considered in his research yearly minimum guarantees as an addition to a guarantee on termination for unit-linked policies. Persson [79] studied and gave details for a more common unit-linked insurance agreement, which considers more than one lives and covers disability benefits as well.

Most of the literature assumes deterministic interest rates for valuation. First who introduced the stochastic models of interest rates was Bacinello and Ortu [5, 6], Nielsen and Sandmann [72, 73], and Kurz [50]. As it is know in classic theory the rate of interest applied for pricing of life insurance agreements is considered as the insurers return on its investment portfolio. This is the same in a real world and as it is known this rate depends on the preferred investment strategy. This is again depends on the company's approach in regard to financial risk considering the legislation.

It is well known that unlike to most financial products paid by a unit premium at the beginning of the agreement, life insurance policies are usually consider the annual premiums. Delbaen [29], Bacinello and Ortu [5, 6], Nielsen and Sandmann [72, 73] and Kurz [50] in their study analyzed and included the Black Scholes study for periodical premiums, and periodical payments.

In the more traditional way, Aase and Persson [2] investigated periodical premiums. In their study periodical premiums have been constructed by distributing the unit payment over the duration where multiple premiums can be considered.

Different hedging and replicating strategies which the insurance company may use for reduction of the financial risk related with unit linked products as in addition to pricing for unit linked insurance agreements analyzed by Brennan and Schwartz [21], Aase and Persson [2], Hipp [45], and Möller [61].

As an addition in their study Aase and Persson [2] and Möller [61] applied continuous mortality rates and in Aase and Persson [2] a relationship among the famouns Thiele's differential equation of insurance theory and the celebrated equation of Black-Scholes is improved.

Unit Link Insurance Policies (ULIP) are in common a mix of insurance and financial portfolios. Simple saying, a part of the premium collected from insureds is used to guarantee benefit payment to the insured while the latter is valuated in different portfolio combinations.

The funds collected by the insurance company are used to create a portfolio which is used to allocate in different market tools in differentiating weights. Steps are the same as it is done for mutual funds. Based on the risk aversion level policy holders can choose the funds where they want to invest. As it is in mutual funds, in unit-linked agreements policy holder has units which linked to the some assets and each unit has a net asset value. These units net asset value is declared in daily bases. The net accumulated value is the value based on which the net rate of returns on unit-linked agreements are calculated. The net accumulated value is different for each unit-linked agreement based on market movements and conditions and as well based on the portfolio returns. Opposed to the classical insurance agreements, unit-linked contracts have several charges applicable that are deducted from the payable premium.

Most effective ones are the charges for the policy administration, premium allocation, fund switching, mortality, and a policy surrender or withdrawal. Sometimes the insurance companies also charges “Guarantee Charge” as a certain proportion of Fund Value for built in minimum guarantee under the policies.

The policyholder need to deeply understand the risks which he undertakes and his own risk aversion level before deciding to choose unit-linked agreements. Since, contract returns are straightly depends on the market portfolio performance, the insured bears whole investment risk in investment portfolio.

In unit-linked agreements, the investments are subject to risks related with the financial markets and the insured bears this investment risk in investment portfolio. Therefore, one have to choose the portfolio weights after considering risk aversion and needs, while considering the potential loss which could be observed.

Another factor that should be considered is future needs for funds. Unit-linked agreements give the insured a chance to observe his/her portfolio in detail. By the way, those agreements make it flexible to move the capital from one fund to another, with differentiating risk-return structures.

ULIPs can be defined as the best solution especially for the people willing to stay invested in a respectively long period.

As mentioned above unit-linked insurance agreements connect the amount of benefit to a investment portfolio. That portfolio can be consist of several instruments, such as stock, a stock index, a foreign currency and a riskless asset. For simplicity, let us consider that this is a mutual fund, most generally used one. In comparison with the classical insurance products unit-linked agreements fit the both insured and insurers advantages. Insurance company may benefit by giving more competitive investment products while insureds may benefit by getting better returns from upwards directions of economy. Insured has flexibility for choosing the units in his portfolio, by the same flexibility he can rebalance his portfolio. That is why, he can control the financial risk amount of his policies. In comparison with the traditional agreements, mostly known differentiating characteristic of unit-linked agreements is the random payment amount. The principle of equivalence, where the main concept is, the income of the company in other words premiums, and claims must be balanced in the long term, that classical basis for valuation life insurance agreements, are not applicable for random benefits. For valuation such agreements financial and actuarial theories are applied together. This is typical approach for pricing such products. For using both theories combined main assumption should be defined. This assumption is stochastic independence between financial and insurance portfolios and risk neutrality with respect to survivals. For eliminating the mortality risk most useful approach is to increase the number of identically and independently distributed policies in the portfolio.

Based on the investor’s risk aversion and on the investment objectives there are several type of unit-linked agreements. These products mainly invest in riskless assets and that is why carry less risk. On the other hand some type of unit-linked agreements invest in risky assets and carry more risk. If we will summarize, as a conclusion we can say that based on the units chosen unit-linked agreements can be classified differently.

2.4 Quadratic Hedging Approach

In this section we will provide some useful information about the pricing and hedging options by means of a quadratic criterion.

For describing the financial market in continuous time framework, one has to be started with probability space (Ω, F, P), a filtration $\mathbf{F} = (F_t)_{0\leq t\leq T}$ and a time period $T \in (0, \infty)$ and . Intuitively, F_t defines the information available at time t. There are $d+1$ basic assets which are ready for investment with price processes $S^i = (S_t^i)_{0\leq t\leq T}$ for $i = 0, 1, ..., d$. To simplify the presentation, assume that one asset, say S^0, has a strictly positive price. Then one uses S^0 as numeraire and immediately skip to quantities discounted with S^0. This means that security 0 has price 1 at all times and the other assets prices are, $\tilde{S}^i = S^i/S^0$ for $i = 1, 2, ..., d$. Here, unless mentioned otherwise, all subsequently appearing quantities will be stated in discounted units [92, 36].

One of the most interesting research areas of financial mathematics is the hedging and valuation of contingent claims investigating dynamic trading approaches utilizing $\tilde{S}^i$. European type of the call option defined on asset i whose maturity is defined as T and exercise price as K is identified as the one of the most famous and widely used type of the contingent claims.

The net payoff at maturity T to investor is the random amount

$$H = \max(\tilde{S}_T^i - K, 0) = (\tilde{S}_T^i - K)^+. \tag{2.16}$$

In simple way, contingent claim can be defined as an F_T measurable random variable H which shows the net value at maturity T of some financial asset. Although, the contingent claims are defined as a European type which means the value time fixed with maturity, the amount which will be paid at maturity may depend on overall history of $\tilde{S}_t^i$ until maturity T.

Main problem here is in correct definition of price for contingent claim H at time 0.

By considering the dynamic portfolio strategies with the following definitions one can answer above asked questions.

Here, $(\nu, \eta) = (\nu_t, \eta_t)_{0\leq t\leq T}$ in which ν is a n-dimensional predictable process and ν is adapted.

We can say that in so strategy, ν_t^i states for the number of units held in security i at any time t, and η_t states for the number of riskless assets invested at time t. Then, predictability of ν is a mathematical formula of the informational constraint that ν is not permitted to expect the movement of $\tilde{S}_t^i$. For the portfolio value (ν_t, η_t) at any time t following equality holds:

$$V_t = \nu_t^{tr}\tilde{S}_t + \eta_t, \tag{2.17}$$

and the cumulative gains from investment until the time t are

$$G_t(\nu) = \int_0^t \nu_s d\tilde{S}_t. \tag{2.18}$$

For having the last equality well-defined, it is assumed that $\tilde{S}_t^i$ is a semimartingale. This property lets $G(\nu)$ to be the stochastic integral of ν with respect to $\tilde{S}_t$. The cumulative cost until time t derived by utilizing (ν, η) are formulated as follows,

$$C_t = V_t - \int_0^t \nu_s d\tilde{S}_t = V_t - G_t(\nu). \tag{2.19}$$

When the cumulative cost process of the strategy is constant over time a strategy is said *self-financing*.

It is true as well for the case when if its value process fits the following equation

$$V_t = V_0 + \int_0^t \nu_s d\tilde{S}_t = V_0 + G_t(\nu), \tag{2.20}$$

where $V_0 = C_0$ is the initial outlay required for starting the strategy. Following the time 0, this strategy can be defined as self-financing, and any fluctuations in $\tilde{S}_t$ could be eliminated by changing the proportions for ν and η for not having any future gains or losses as a result.

Now let us define a contingent claim H, also consider there is a self-financing strategy (V_0, ν) which has the value V_T and which is equal to H with probability 1 at the maturity. In case of the absence of any arbitrage opportunities in the financial market, apparently, the price of contingent claim H should be given by V_0 where ν allows a hedging strategy for H. It is the essential intuition directing to famous Black-Scholes Option Pricing approach derived by Black/Scholes [13] and Merton [59]. They gave the solution for this problem with the term where $\tilde{S}_t$ is a one-dimensional geometric Brownian motion and $H = (X_T - K)^+$ is a European call option. The mathematical construction of the issue and its relationships to the theory of martingale have been afterwards analyzed and investigated in deep details by J. M. Harrison and D. M. Kreps [43] and by Harrison/Pliska [44]. A contingent claim H is said to be *attainable* while there is a self-financing strategy with value equal to $V_T = H$ P-a.s. at the termination. By Eqn. (2.20), one can consider that H can be formulated as:

$$H = H_0 + \int_0^T \nu_s^H d\tilde{S}_t, \quad P - a.s.. \tag{2.21}$$

In other words, this is the sum of a constant H_0 and a stochastic integral $\tilde{S}_t$. If every contingent claim is attainable we can claim that the market is *complete market*. Remember the point that there is not given exact definition for a rigorous mathematical equations, it is important to be extremely conscientious regarding the integrability conditions, H and ν^H has been exposed.

It is not possible by definition to find a strategy which is self-financing and which final value is equal to the $V_T = H$ for non-attainable contingent claims. One of the most used methods is to preserve the terminal condition $V_T = H$. As, η is defined as an adapted process, this can be achieved by the selection of η_T. In other hand, this strategies cannot be self-financing in general, and for being preferred strategy it should have a small *cost process* C.

First who investigated and measured a riskiness of a strategy by help of quadratic hedging approach were Föllmer and Sondermann [36]. Following these authors Schweizer [34, 35] in his research studied the case where X is a martingale by extending the model to the general semi-martingale case.

This type of local risk minimization strategy could be identified by two properties:

- **1.** the cost process C must be a martingale, in such a way that the strategy is not self-financing anymore, but anyway preserves mean-self-financing property;

  2 and cost process C which is a martingale, must be orthogonal to the martingale part M of the price process $\tilde{S}_t$.

There is a local risk-minimizing strategy for H if and only if H could be written as a decomposition of the following form:

$$H = H_0 + \int_0^T \nu_s^H d\tilde{S}_s + L_T^H, \quad P - a.s., \tag{2.22}$$

where L_T^H is a martingale orthogonal to M.

The decomposition Eqn. (2.22) named as the Föllmer and Schweizer decomposition of H, and it can be accepted as a general form of the semi-martingale case of the traditional Galtchouk Kunita Watanabe decomposition in martingale approach.

This decomposition financially has an importance because it gives chance to define the local risk minimization strategy for H. The integrand ν^H gives the stock component ν and the conditions that the cost process C have to fit to $H_0 + L^H$ determines η. It is also important to know that the special case of Eqn. (2.21) of an attainable claim keeps corresponding to the not existence of the orthogonal term L_T^H. One can derive more precise constructions for the decomposition Eqn. (2.22). In case of the finite discrete time, ν^H and L^H could be derived repeatedly backward in time. In case of the $\tilde{S}_t$ following a continuous path, the decomposition of Föllmer-Schweizer under P can be derived as a decomposition of Galtchouk Kunita Watanabe, calculated by considering the *minimal martingale measure* $\hat{P}$.

2.4.1 Mean Variance Hedging

The difficulties related with the hedging and pricing of contingent claims under the incompleteness conditions of market gave rise to different valuation approaches. Two famous strategies are local risk minimization and mean-variance hedging. Mean-variance hedging strategy is well known classical one. It minimizes the expectation of the square hedging error which is the square difference between the value of the self-financing portfolio and the contingent claim at the maturity, among all self-financing strategies. Föllmer and Sondermann [36] in their study first investigated this method in martingale case.

After Föllmer and Sondermann [36] extensions to the general semi-martingale case were done.

Lets first discuss the mean-variance hedging approach in the simple case where $\tilde{S}_t$ is a local P-martingale.

Let (Ω, F, P^*) be a probability space, and let $(F_t)_{0\leq t\leq T}$ denote a right continuous family of σ-algebras contained in F; F_t is interpreted as the collection of events which are observable up to time t. A stochastic process $Z = (Z_t)_{0\leq t\leq T}$ is given by a measurable function Z on $\Omega \times [0, T]$. Z is called adopted if Z_t is F_t measurable for each $0 \leq t \leq T$; it is called predictable if it is measurable with respect to the σ-algebra **F** on $\Omega \times [0, T]$ which is generated by the adapted processes with left-continuous paths. The evolution of stock prices will be described by a stochastic process $S = (S_t)_{0\leq t\leq T}$ which is adapted and whose paths are right-continuous with limits S_{t^-} from the left. The process $Y = (Y_t)_{0\leq t\leq T}$ of bond prices is fixed to be $Y_t = 1$. It is assumed that P^* is a martingale measure in the sense of Harrison and Kreps [43],

$$E^*[S_T^2] < \infty, \quad S_t = E^*[S_T|F_t], \quad 0 \leq t \leq T, \tag{2.23}$$

where $E^*[.|F_t]$ denotes the conditional expectation under P^* with respect to the σ-algebra F_t. This means that S_t is a square-integrable martingale under P^*. Let $\langle S_t\rangle = (\langle S_t\rangle_t)_{0\leq t\leq T}$ be the corresponding Meyer process, the unique predictable process with $\langle X\rangle_0 = 0$ and right-continuous increasing paths such that $S_t^2 - \langle S_t\rangle$ is a martingale. Let us denote by $P^*_{S_t}$ the finite measure on $(\Omega \times [0, T], \mathbf{F})$ given by

$$P^*_{S_t}[A] = E^*\left[\int_0^T \mathbf{1}_A(t, \omega) d\,\langle S_t\rangle_t(\omega)\right], \tag{2.24}$$

and by $L^2(P_{S_t^*})$ the class of predictable processes Z which, viewed as a **F**-measurable function on $\Omega\times[0, T]$, are square-integrable with respect to $P^*_{S_t}$. Two such processes will be considered as equal if they coincide $P^*_{S_t}$-a.s. .

A trading strategy will be as in the following form $\varphi = (\xi, \eta)$, where $\xi = (\xi_t)_{0\leq t\leq T}$ and $\eta = (\eta_t)_{0\leq t\leq T}$ is showing the amounts allocated into the stock and into the bond. Thus,

$$V_t = \xi_t S_t + \eta_t \tag{2.25}$$

is the value of the portfolio at time t.

As we know from literature [51],

- $\varphi = (\xi, \eta)$ is defined as a strategy if,
  - **(a)** ξ is a predictable process, and $\xi \in L^2(P^*_{S_t})$,
  - **(b)** η is adapted,
  - **(c)** $V = \xi S + \eta$ has right continuous paths and satisfies,

$$\mathrm{V}_t \in L^2(P^*), \quad (0 \leq t \leq T).$$

Condition **(a)** allows to calculate the accumulated gain derived from the stock price fluctuation up to time t as the stochastic integral:

$$\int_0^t \xi_s dS_s, \quad 0 \leq t \leq T. \tag{2.26}$$

For fixed t, the gain has expectation $E^*\left[\int_0^t \xi_s dS_s\right] = 0$ and variance

$$E^*\left[\left(\int_0^t \xi_s dS_s\right)^2\right] = E^*\left[\int_0^t \xi_s^2 d\left\langle S_s\right\rangle_s\right]. \tag{2.27}$$

Viewed as a stochastic process, Eqn. (2.26) defines a square-integrable martingale with right-continuous paths. The accumulated cost of the strategy up to time t can now be defined as

$$C_t = V_t - \int_0^t \xi_s dS_s. \tag{2.28}$$

Here, $V = (V_t)_{0\leq t\leq T}$ and $C = (C_t)_{0\leq t\leq T}$ are adapted processes with right-continuous paths; they are called the *value process* and the *cost process*.

- A strategy $\varphi = (\xi, \eta)$ is called *mean-self financing* if the corresponding cost process $C = (C_t)_{0\leq t\leq T}$ is a martingale.

Remark: [63] A strategy $\varphi = (\xi, \eta)$ is called mean-self financing if the cost process has constant paths, i.e., if

$$C_t \equiv C_0, \quad P^* - a.s., \quad 0 \leq t \leq T. \tag{2.29}$$

Any self-financing strategy is clearly mean-self-financing. For a self-financing strategy, the value process is of the form

$$V_t = C_0 + \int_0^t \xi_s dS_s, \quad 0 \leq t \leq T, \tag{2.30}$$

hence a square-integrable martingale. Self-financing strategy is the key toll in the analysis of option pricing in complete security markets. But in many situations, security markets are incomplete in the sense that there may not be any self-financing strategy which allows to realize a pre-assigned terminal value $V_T = H$. This is the reason why we introduce the broader concept of a mean-self financing strategy. It is stated in the following lemma, the value process of a mean-self financing strategy is again a martingale. but in general we cannot expect that this martingale can be represented as a stochastic integral with respect to S_t as in Eqn. (2.30).

Lemma 2.1. *[92] A strategy is mean-self-financing if and only if its value process is a square-integrable martingale.*

The intrinsic risk of contingent claims

Let us fix a contingent claim $H \in \Psi^2(P^*)$. Here, H could be a call option of the form $H = (S_T - C)^+$.

From different sources we know that, a strategy is called admissible in regard to H if its value process has terminal value

$$V_T = H, \quad P^* - a.s.$$

For any admissible strategy $\varphi = (\xi, \eta)$, the terminal cost is given by

$$C_T = H - \int_0^T \xi_s dS_s.$$

In particular, the expected value,

$$E^*[C_T] = E^*[H], \tag{2.31}$$

does not depend on the specific choice of the strategy as long as it is admissible. Lets determine all admissible strategies which minimize the variance

$$E^*\left[(C_T - E^*[H])^2\right]; \tag{2.32}$$

the second step will consist in replacing Eqn. (2.32) by a sequential procedure. In view of Eqn. (2.32), let us write the claim H in the form

$$H = E^*[H] + \int_0^T \xi_s^* dS_s + H^*, \tag{2.33}$$

with $\xi^* \in L^2(P^*_{S_t})$ where $H^* \in L^2(P^*)$ has expectations zero and is orthogonal to the space $\left\{\int_0^t \xi_s dS_s | \xi \in L^2(P^*_{S_t})\right\}$ of stochastic integrals with respect to S_t for the existence and uniqueness of this representation.

Theorem 2.2. *[92] An admissible strategy $\varphi = (\xi, \eta)$ has minimal variance*

$$E^*\left[(C_T - E^*[H])^2\right] = E^*\left[(H^*)^2\right], \tag{2.34}$$

if and only if $\xi = \xi^*$.

So far one can draw no conclusion concerning the process $\eta = (\eta_t)_{0 \le t \le T}$, except that it must make the strategy admissible, i.e., to put

$$\eta_t = H - \xi_T S_T. \tag{2.35}$$

One can now show that a sharper formulation of problem Eqn. (2.32) determines a unique admissible strategy $\varphi^* = (\xi^*, \eta^*)$ which has minimal risk in a sequential sense. Consider any strategy $\varphi = (\xi, \eta)$. Just before time $t < T$ we have accumulated cost C_{t-}. The strategy tells us how to proceed at and beyond time t. In particular, it fixes the present cost C_t and determines the remaining cost $C_T - C_t$. Let us measure the remaining risk by

$$R_t^\varphi = E^* \left[(C_T - C_t)^2 | F_t \right]. \tag{2.36}$$

In view of Eqn. (2.36), we might want to compare φ to any other strategy $\tilde{\varphi}$ which coincide with φ at all times $< t$ and which leads to the same terminal value V_T. Let us call such a $\tilde{\varphi}$ an *admissible continuation* of φ at time t.

Definition 2.1.

A strategy φ is defined risk-minimizing if φ at any time minimizes the remaining risk, i.e., for any $0 \leq t < T$, we have

$$R_t^\varphi \leq R_t^{\tilde{\varphi}}, \quad P^* - a.s., \tag{2.37}$$

for every admissible continuation $\tilde{\varphi}$ of φ at time t.

Remark:

1 Any self-financing strategy φ can be naturally identified as risk-minimizing since $R_t^\varphi \equiv 0$.

2 Suppose that $\varphi = (\xi, \eta)$ is a risk-minimizing strategy which is also admissible. Then φ is in particular a solution of problem Eqn. (2.32). In fact, Eqn. (2.37) with $t = 0$ implies that φ minimizes

$$E^* \left[(C_T - C_0)^2 \right] = E^* \left[(C_T - E^*[C_T])^2 \right] + (E^*[C_T - C_0])^2. \tag{2.38}$$

Thus, ξ minimizes the variance of C_T and this implies $\xi = \xi^*$. In addition, following condition is obtained by Föllmer [36]:

$$\eta_0 = C_0 - \xi_0^* S_0 = E^*[H] - \xi_0^* S_0. \tag{2.39}$$

Let us denote by $V^* = (V_t^*)_t$ a right-continuous version of the square-integrable martingale

$$V_t^* = E^*[H|F_t], \quad 0 \leq t \leq T. \tag{2.40}$$

To the representation Eqn. (2.33) of the claim H corresponds the following sequential representation of V^*:

$$V_t^* = V_0^* + \int_0^t \xi_s^* dS_s + N_t^*, \tag{2.41}$$

where $N_t^* = E^*[H^*|F_t]$ is a square-integrable martingale with zero expectation which is orthogonal to X in the following sense.

Remark:

Two square-integrable martingales M_1 and M_2 are defined as orthogonal if their product $M_1 M_2$ is again a martingale, and this is equivalent to the condition

$$\langle M_1, M_2 \rangle = \frac{1}{2} \left(\langle M_1 + M_2 \rangle - \langle M_1 \rangle - \langle M_2 \rangle \right) = 0. \tag{2.42}$$

The process $R^* = (R_t^*)$, defined as a right continuous version of

$$R_t^* = E^* \left[(N_T^* - N_t^*)^2 | F_t \right] = E^* \left[\langle N^* \rangle_T | \, F_t \right] - \langle N^* \rangle_t \, , \tag{2.43}$$

will be called the *intrinsic risk process* of the claim H. The expectation $E^*[R_0^*]$ coincides with the minimal variance calculated in Eqn. (2.34); let us call it the *intrinsic risk* of the claim.

Theorem 2.3. *[92] (2) There is a unique admissible strategy φ^* that is risk-minimizing, namely*

$$\varphi^* = (\xi^*, V^* - \xi^* S_t). \tag{2.44}$$

For this strategy, the remaining risk at any time $t \leq T$ is given by

$$R_t^{\phi^*} = R_t^*, \quad P^* - a.s. \tag{2.45}$$

As a special case of above theorem the following characterization of attainable contingent claims is obtained.

[92] **1** The risk minimizing admissible strategy φ^* is self-financing.

2 The intrinsic risk of the contingent claim H is zero.

3 The contingent claim H is attainable, i.e.,

$$H = E^*[H] + \int_0^T \xi_s^* dS_s, \quad P^* - a.s. \tag{2.46}$$

Changing the Measure

In this part, we will see how the risk-minimizing strategy is affected by an absolutely continuous change of the underlying martingale measure. Let P be any martingale measure which is absolutely continuous with respect to P^*. Thus, the process S_t is again a square integrable martingale under P. Also assume that contingent claim $H \in$

$L^2(P^*)$ is again square-integrable under P. Then, the representation Eqn. (2.41) and the Theorem 2.21, applied to P instead of P^*, show that the risk-minimizing strategy under P is given by $\varphi = (\xi, V - \xi V)$, with

$$V_t = E[H|F_t] = V_0 + \int_0^t \xi_s dS_s + N_t. \tag{2.47}$$

In order to simplify the exposition is added the technical assumption

$$\xi^* \in L^2(P_{S_t}).$$

While S_t is again a martingale under P, the martingale property of $(N_t^*)_t$ in Eqn. (2.41) may be lost. In general, we have the Doob decomposition

$$N^* = M + A,$$

where $M = (M_t)_t$ is a martingale under P and $A = (A)_t$ is a predictable process with $A_0 = 0$ and with right-continuous paths of bounded variation. Below introduced the predictable processes ξ^M and ξ^A defined by

$$\langle M, S_t \rangle_t = \int_0^t \xi_s^M d\, \langle S_s \rangle_s, \quad 0 \leq t \leq T, \tag{2.48}$$

and

$$\left\langle M^A, S_t \right\rangle_t = \int_0^t \xi_s^A d\, \langle S_s \rangle_s, \quad 0 \leq t \leq T, \tag{2.49}$$

where M^A denotes a right-continuous version of the martingale

$$M_t^A \equiv E[A_T|F_t], \quad 0 \leq t \leq T. \tag{2.50}$$

Theorem 2.4. *The risk minimizing strategy under P is given by $\varphi = (\xi, V - \xi V)$ with*

$$\xi = \xi^* + \xi^M + \xi^A, V_t = V_t^* + M_t^A - A_t, \quad 0 \leq t \leq T. \tag{2.51}$$

If both M and M^A are orthogonal to S_t, then we have $\xi = \xi^*$.

Remark:

(1) If S_t is a martingale with continuous paths, then $\langle M, S_t \rangle$ can be evaluated pathwise as a quadratic variation and coincides with $\langle N^*, S_t \rangle = 0 \quad P^*$-a.s. This implies $\xi^M = 0 \quad P_{S_t}$-a.s., hence

$$\xi = \xi^* + \xi^A.$$

(2) If P is a martingale measure in the stricter sense that it also preserves the martingale property of N^*, then we have $A = 0$, hence,

$$\xi = \xi^* + \xi^M,$$

and

$$V_t = V_t^*.$$

If S_t has continuous paths then we can conclude, due to remark (1), that the risk minimizing strategy is completely preserved.

(3) $\xi = \xi^*$ may occur even if the martingale property of N^* is lost under P.

2.5 Modeling Dependence with Pseudo-Gompertz Distribution

The most important element in risk modeling in the fields of finance and insurance are the uncertainty related to the future life expectancies and survival rates of policy holders. Based on the future lifetime or the death of the policyholder high risk of non-payment of loans or insurance premiums is the main risk that financial institution carries. The death of the loan owner is a financial loss for the credit organization. Unexpected mortality cuts the payment stream of planned credit installments. Likewise, the unexpected mortality of policyholder in life insurance agreements which do payments in case of death may end up in extra loss if the death actualizes before actuarially planned and waited time. This is true for the case when insurance payment amount at the end of the term is over the allocated reserve amount.

Main tools in such situations for risk diversification or loss preventing are correct hedging strategies or reinsurance agreements. Most famous books which study the above mentioned concepts of risk mitigation for finance and actuarial fields are [57, 9].

One of the key and most important risk reduction approaches in insurance theory is accurate modeling of dependence between lifetimes of insureds. Because, in reality, there is a dependency pattern between life times and this dependence is significant for most of the situations.

Correct and appropriate choice of joint distribution for lifetimes have a wide research field in literature. For modeling dependence we use Pseudo Gompertz distribution which is the one of most preferred distributions for modeling of joint lifetimes, whose details are given in the upcoming subsection.

2.5.1 Description and Main Functions

Among the probability distribution functions which is used for the modeling of lifetimes the Gompertz distribution is a most preferred one. The Gompertz like distributions are used for construction of life tables for human beings. One can say that, for the interpretation of mortality rates mainly used by authors and empirically legit parametric probability distribution is the Gompertz distribution.

For expression of the dependence between X and Y a bivariate distribution function $F(x, y)$ can be used for a pair of random variables (X, Y). When X or Y is related by a real-valued function $\phi(\cdot)$, then the distribution that can be derived from $F(x, y)$ is a pseudo-distribution with $\phi(\cdot)$ which includes set of parameters. Here, $\phi(\cdot)$ have to satisfy all the conditions which pseudo-distribution has to be a probability distribution.

The first researches related to a pseudo-distribution were done in 60th, which considering about reformulating distinct parameters of a probability function. Garsia [38] in his paper investigates The Wishart distribution under singularity. With dealing of the singularity elements pseudo-Wishart distributions are derived as well.

The work of Adham [3] by investigating the bivariate Gompertz distribution is derives a Gompertz sort distribution by the help of the related functions which is linking appropriate random variables among themselves.

A general form of the Gompertz distribution was investigated and introduced in Willemse [98] by some parametrization. This allows to apply several survival models with empirically verifiable hazard schemes. That is why, the usage of the theory and approaches related with the Gompertz distribution was increased and different methods are investigated by adding to the literature valuable researches.

For the cases where an actual distribution cannot be used, Filus [33] analyzed and investigated in his paper another set of pseudo-distributions for linear combinations of random variables for the statistical applications. Afterwards, another pseudo-distributions have been investigated and proposed by the same way. Following, the pseudo distributions obtained by Shahbaz [94, 93], a bivariate-Gompertz distribution is obtained in Yörübulut and Gebizlioğlu [100], as presented below.

The Gompertz distribution with parameters λ and μ_1 for a random variable X has the following density function

$$\begin{gathered} f_X(x;\lambda,\mu_1) = \lambda e^{\mu_1 x} \exp\left[-\tfrac{\lambda}{\mu_1}(e^{\mu_1 x}-1)\right], \\ \mu_1 > 0, \quad \lambda > 0, \quad x > 0. \end{gathered} \tag{2.52}$$

Consider that other random variable Y has a Gompertz distribution with parameters $\phi(x)$ and μ_2 as well, where $\phi(x)$ is a real valued function of the random variable X. For the definition of the density function of Y following equation could be used:

$$\begin{gathered} f_{Y|X=x}(y;\phi(x),\mu_2|x) = \phi(x) e^{\mu_2 y} \exp\left[-\tfrac{\phi(x)}{\mu_2}(e^{\mu_2 y}-1)\right], \\ \mu_2 > 0, \quad \phi(x) > 0, \quad y > 0. \end{gathered} \tag{2.53}$$

With the help of the marginal density defined in the previous two equations, the compound distribution of X and Y with the following density function can be derived which is the bivariate Pseudo Gompertz distribution,

$$\begin{gathered} f(x,y) = \lambda\phi(x) e^{\mu_1 x} e^{\mu_2 y} \exp\left[-\tfrac{\lambda}{\mu_1}(e^{\mu_1 x}-1) - \tfrac{\phi(x)}{\mu_2}(e^{\mu_2 y}-1)\right], \\ \mu_1 > 0, \quad \mu_2 > 0, \quad \lambda > 0, \quad \phi(x) > 0, \quad y > 0, \quad x > 0, \end{gathered} \tag{2.54}$$

which comes from

$$f(x,y) = f_x(x;\lambda,\mu_1) f_{Y|X=x}(y;\phi(x),\mu_2|x). \tag{2.55}$$

From above general form of the density function several distributions can be obtained depending on different choices of $\phi(x)$ function. In the study of Yorubulut and Gebizlioglu [100] the below given form of bivariate Pseudo-Gompertz distribution is derived by Adopting $\phi(x) = e^{\mu_1 x} - 1$:

$$\begin{gathered} f(x,y) = \lambda(e^{\mu_1 x}-1) e^{\mu_1 x} e^{\mu_2 y} \exp\left[-(e^{\mu_1 x}-1)\left(\tfrac{\lambda}{\mu_1} + \tfrac{(e^{\mu_2 y}-1)}{\mu_2}\right)\right], \\ \mu_1, \quad \mu_2, \quad \lambda, \quad y, \quad x > 0. \end{gathered} \tag{2.56}$$

The function $\phi(x)$ could be chosen by the researchers according to the needs of modeling. The condition that $F(x,y) = \int_x \int_y f(x,y)dydx$ have to have to all the properties to be a probability distribution function.

The marginal distributions of X and Y are derived from equation Eqn. (2.54) as

$$f(x) = \lambda e^{\mu_1 x} \exp\left[-\frac{\lambda}{\mu_1}(e^{\mu_1 x} - 1)\right] \tag{2.57}$$

and

$$f(y) = e^{\mu_2 y} \frac{\lambda}{\mu_1} \left(\frac{\lambda}{\mu_1} + \frac{(e^{\mu_2 y} - 1)}{\mu_2}\right)^{-2}. \tag{2.58}$$

The joint distribution function corresponding to $F(x,y)$ in Eqn. (2.54) is

$$F(x,y) = \int_0^y \int_0^y \lambda(e^{\mu_1 x} - 1)e^{\mu_1 x} e^{\mu_2 y} \exp\left[-(e^{\mu_1 x} - 1)\left(\frac{\lambda}{\mu_1} + \frac{(e^{\mu_2 y} - 1)}{\mu_2}\right)\right] dxdy \tag{2.59}$$

$$\begin{aligned} F(x,y) = {} & \tfrac{\lambda}{\mu_1} \frac{\left[\exp\left(\frac{(e^{\mu_2 y}-1)}{\mu_2} - e^{\mu_1 x}\left(\frac{(e^{\mu_2 y}-1)}{\mu_2} + \frac{\lambda}{\mu_1}\right) + \frac{\lambda}{\mu_1}\right) - 1\right]}{\left(\frac{(e^{\mu_2 y}-1)}{\mu_2}\right.} \\ & + \tfrac{\lambda}{\mu_1}) + \left(1 - \exp\left(-\tfrac{\lambda}{\mu_1}(e^{\mu_1 x} - 1)\right)\right). \end{aligned} \tag{2.60}$$

The joint survival function, that follows from Eqn. (2.55) above, is

$$S(x,y) = 1 - F_1(x) - F_2(y) + F(x,y), \tag{2.61}$$

where $F_1(x)$ and $F_2(y)$ are the marginal distribution function of X and Y, respectively. This functions are derived for the bivariate Pseudo Gompertz distribution:

$$F_1(x) = \lim_{y \to \infty} F(x,y) = 1 - \exp\left(-\frac{\lambda}{\mu_1}(e^{\mu_1 x} - 1)\right) \tag{2.62}$$

and

$$F_2(x) = \lim_{x \to \infty} F(x,y) = 1 - \frac{\lambda \mu_2}{\mu_1(e^{\mu_2 y} - 1) + \mu_2 \lambda}. \tag{2.63}$$

The appropriate joint survival function is obtained as [100] for the joint and marginal distribution functions:

$$S(x,y) = \frac{\lambda\mu_2}{\mu_1(e^{\mu_2 y}-1)+\mu_2\lambda}\exp\left[\frac{(e^{\mu_2 y}-1)}{\mu_2} - e^{\mu_1 x}\left(\frac{(e^{\mu_2 y}-1)}{\mu_2}+\frac{\lambda}{\mu_1}\right)+\frac{\lambda}{\mu_1}\right]. \tag{2.64}$$

CHAPTER 3

HEDGING STRATEGIES FOR MULTIPLE LIFE UNIT LINKED INSURANCE POLICIES USING CPPI APPROACH

3.1 The Model

In this section the two main parts of the model, which are the financial market and a portfolio of individuals to be insured, are introduced, starting with the financial market and then defining the insurance portfolio. For financial portfolio we use Constant Proportion Portfolio Insurance approach and assume that stock price process is a Levy Jump diffusion. Then, we define our insurance portfolio for multiple life policies. We prefer to choose joint life and last survival cases. For the financial portfolio we mainly refer to [88, 26, 48, 46, 49, 13, 10, 51, 12, 15, 61, 83], for insurance part we refer to [56, 80, 16, 40] and for risk minimization concept we use following references [31, 99, 92, 87].

3.1.1 The Financial Market: Constant Proportion Portfolio Insurance with Jump Diffusion

As we mentioned in the first chapter in the CPPI strategy an investor invests in a portfolio and wants to protect the portfolio value from falling below a pre-assigned value. The investor shifts his asset allocation over the investment period among a risk-free asset plus a collection of risky assets [99].

Consider the jump-diffusion process with $Y_n > -1$, representing the percentage of jump-size, i.e., $S_{T_n} = S_{T_n^-}(1 + Y_n)$. Between two jumps, we assume that the risky asset model follows the Black and Scholes model. The number of jumps up to time t is a Poisson processes N_t with intensity λ_t. Our model becomes

$$S_t = S_0 \exp\left[\int_0^t (\mu_s - \frac{\sigma_s^2}{2})ds + \int_0^t \sigma_s dW_s + \sum_{n=1}^{N_t} \ln(1 + Y_n)\right]. \qquad (3.1)$$

We usually assume that the $\ln(1 + Y_n)$ are i.i.d. with density function f_Q.

For our jump-diffusion model defined by (3.1), consider a predictable $\mathbb{F}_t$ process ψ_t

such that $\int_0^t \psi_t \lambda_t ds < \infty$. Choose θ_t and ψ_t such that

$$\mu_t + \sigma_t \theta_t + Y_t \psi_t \lambda_t = r_t \quad \psi_t \geq 0. \tag{3.2}$$

Define,

$$Ł_t = \exp\left[\int_0^t \left[(1-\psi_s)\lambda_s - \frac{1}{2}\theta_s^2\right] ds + \int_0^t \theta_s dW_s + \int_0^t \ln \psi_s dN_s\right], \tag{3.3}$$

for $t \in [0, T]$ and a Radon-Nikodym derivative to be

$$\frac{dQ}{dP} = L_T. \tag{3.4}$$

Then, Q is a risk neutral measure or martingale measure, i.e., a measure under which $\tilde{S}_t = \exp\left[-\int_0^t r_s ds\right] S_t$ is a martingale.

The CPPI strategy is based on a dynamic portfolio allocation on two basic assets: a riskless asset and a risky asset. At time t the exposure e_t is equal to the cushion C_t multiplied by the scalar m. The cushion C_t is defined as the difference between the portfolio value V_t and the floor F_t. Here, $F_t = G\exp[-r(T-t)]$, where G is the floor at time T. Because of the existence of jumps, it is possible to have the case that the portfolio value is less than the floor. Then, the cushion will be negative, and so it will be the exposure. That means that short-selling should be allowed. The following proposition describes the portfolio value under this strategy.

Denote portfolio value as V_t. It consists riskless asset $V_t - mC_t$ and risky asset mC_t, i.e., $V_t = mC_t + (V_t - mC_t)$. Let the interest rate be r and the floor at time t be $F_t = F_0 e^{rt} = F_T e^{-r(T-t)}$.

Name	**Notation**
Interest Rate	r
Time	t
Time Period	$[0, T]$
Floor at time t	F_t
Portfolio Value	V_t
Cushion at time t	C_t
Multiple	m
Exposure at time t	e_t
Riskless asset at time t	B_t

Table 3.1: General Notations.

Their relationship are as follows:

$$C_t = V_t - F_t,$$

$$e_t = mC_t,$$

$$B_t = V_t - e_t.$$

Some Useful Results [97]:

1. Result:

The CPPI portfolio value under the jump-diffusion model defined by (2.1) is

$$V_t = C_0 \exp\left[\int_0^t \left(r + m(\mu_s - r) - \frac{m^2\sigma_s^2}{2}\right) ds + \int_0^t m\sigma_s dW_s\right]\left[\prod_{n=1}^{N_t}(1 + mY_n)\right] + F_t, \tag{3.5}$$

where

$$C_0 = (V_0 - Ge^{-rT}),$$

and

$$F_t = G\exp\left(-r(T-t)\right).$$

The expectation and variance of the CPPI portfolio value are deduced in the following two results. These are obviously two important values to describe the CPPI strategy in our jump-diffusion model.

2. Result:

The expected CPPI portfolio value at time t under the jump-diffusion model is

$$E[V_t] = C_0 \exp\left[\int_0^t (r + m(\mu_s - r))ds\right] \sum_{k=1}^{\infty} \frac{e^{-\int_0^t \lambda_s ds}(\int_0^t \lambda_s ds)^k}{k!} E\left[\prod_{n=1}^{k}(1 + mY_n)\right] + F_t. \tag{3.6}$$

3. Result:

The variance of the CPPI portfolio value at time t under jump-diffusion model is,

$$var[V_t] = var[C_t]$$

$$\begin{aligned} &= C_0^2 \exp\left\{\int_0^t 2[r + m(\mu_s - r) + m^2\sigma_s^2]ds\right\} \textstyle\sum_{k=1}^{\infty} E\left[\prod_{n=1}^{k}(1 + mY_n)\right]^2 \times \\ &\times \frac{e^{-\int_0^t \lambda_s ds}(\int_0^t \lambda_s ds)^k}{k!} - \left[\exp\left\{\int_0^t [r + m(\mu_s - r)]\right\} ds \textstyle\sum_{k=1}^{\infty} \frac{e^{-\int_0^t \lambda_s ds}(\int_0^t \lambda_s ds)^k}{k!} \times \right. \\ &\left. \times E\left[\prod_{n=1}^{k}(1 + mY_n)\right]\right]^2. \end{aligned} \tag{3.7}$$

The Time Varying Multiple Case

In previous part we saw the case where the multiple is a function of time. Let m_t be the multiple at time t. The conclusion does not change substantially in comparison with the constant case.

4. Result:

When the multiple is a function of time at time t, the CPPI portfolio value under the jump-diffusion model is

$$V_t = C_0 \exp\left[\int_0^t \left(r + m_s(\mu_s - r) - \frac{m_s^2\sigma_s^2}{2}\right) ds + \int_0^t m_s \sigma_s dW_s\right] \left[\prod_{n=1}^{N_t} (1 + m_n Y_n)\right] + F_t, \tag{3.8}$$

where m_n is obtained from m_t by the formula $m_n = m_{T_n}$, where $T_0 = 0$.

5. Result:

When the multiple is a function of time at time t, the expected CPPI portfolio value under the jump-diffusion model is

$$E[V_t] = C_0 \exp\left[\int_0^t (r + m_s(\mu_s - r))ds \sum_{k=1}^{\infty} \frac{e^{-\int_0^t \lambda_s ds}(\int_0^t \lambda_s ds)^k}{k!} E\left[\prod_{n=1}^{k} (1 + m_n Y_n)\right]\right] + F_t. \tag{3.9}$$

6. Result:

When the multiple is a function of a time at time t, the variance of the CPPI portfolio value under jump-diffusion model is,

$$var[V_t] = var[C_t]$$

$$= C_0^2 \exp\left\{\int_0^t 2[r + m_s(\mu_s - r) + m_s^2\sigma_s^2]ds\right\} \sum_{k=1}^{\infty} E\left[\prod_{n=1}^{k} (1 + m_n Y_n)\right]^2 \frac{e^{-\int_0^t \lambda_s ds}(\int_0^t \lambda_s ds)^k}{k!}$$

$$- \left[\exp\left\{\int_0^t [r + m_s(\mu_s - r)]\right\} ds \sum_{k=1}^{\infty} \frac{e^{-\int_0^t \lambda_s ds}(\int_0^t \lambda_s ds)^k}{k!} E\left[\prod_{n=1}^{k} (1 + m_n Y_n)\right]\right]^2. \tag{3.10}$$

3.1.1.1 CPPI Portfolio as a Hedging Tool

CPPI Portfolio could be used for hedging purposes. Consider that $\eta = g(S_T)$ is a contingent claim that the investor is going to have at the termination date. Question here is, if the CPPI portfolio can be converted into a synthetic derivative with terminal value which can be specified by $\eta = g(S_T)$?

Theorem 3.1. *[97] If $g : R \to R$ is a sufficiently smooth function, there is a unique self-financed $g(S_T)$ hedging CPPI portfolio V, defined by*

$$V_t = \nu(t, S_t), \quad t \in [0, T], \tag{3.11}$$

where $\nu \in C^{1,2}([0,T] \times R)$ is the unique solution of the following partial integro-differential equations:

$$\frac{\partial u}{\partial t}(t,s) + (\mu_t s)\frac{\partial u}{\partial x}(t,s) + \frac{1}{2}(s\delta_t)^2\frac{\partial u^2}{\partial x^2}(t,s) - ru(t,s) = 0, \tag{3.12}$$

$$sz\frac{\partial u}{\partial x}(t,s) = u(t,s+sz) - u(t,s), \tag{3.13}$$

$$u(T,s) = g(s), \quad (t,s) \in [0,T] \times R, \quad u \in C^{1,2}([0,T] \times R). \tag{3.14}$$

Here, $\frac{\partial u}{\partial x}$ is the partial derivative to the second variable. In particular the CPPI portfolio's gearing factor is given by:

$$m_t = \frac{\frac{\partial u}{\partial x}(t,S_t)S_{t-}}{V_{t^-} - F_t}, \quad t \in [0,T]. \tag{3.15}$$

Theorem 3.2. *[97] Under the risk neutral measure Q, the discounted CPPI portfolio's value $V_t \in [0,T]$,*

$$\tilde{V}_t = e^{-rt}V_t, \quad t \in [0,T], \tag{3.16}$$

is a martingale.

Given any claim $\eta = g(V_T)$ which is a function of the terminal portfolio's price, there is a unique self-financed $\eta = g(V_T)$ - hedging strategy.

Theorem 3.3. *[97] Let $g : R \to R$ sufficiently smooth. There is a unique $\eta = g(V_T)$-hedging self financed trading strategy (U,η), defined as*

$$U_t = u(t,V_t), \quad \eta_{t^-} = \frac{\partial u}{\partial x}(t,V_t), \quad t \in [0,T], \tag{3.17}$$

where $u \in C^{1,2}([0,T]XR)$ is the solution of the following partial integro-differential equation:

$$\frac{\partial u}{\partial t}(t,\nu) + r\nu\frac{\partial u}{\partial x}(t,\nu) + \frac{1}{2}m^2\delta_t^2(\nu - f)\frac{\partial^2 u}{\partial x^2}(t,\nu) - ru(t,\nu) = 0, \tag{3.18}$$

$$mz(\nu - f)\frac{\partial u}{\partial x}(t,\nu) = u(t,\nu + m[\nu - f]z) - u(t,\nu), \tag{3.19}$$

with the final condition $u(T,\nu) = g(\nu)$.

3.1.2 The Insurance Portfolio: Multiple Life Contracts

In this section of the thesis a model will be introduced to represent the lifetimes in a group of policyholders. During our research we realized that all of the authors for simplicity considered the mutually independent and identically distributed lifetimes. (list of authors Möller, Riesner, Devolder, etc.) The i.i.d. assumption implies that the policyholders are chosen among a cohort of same age x. The number of persons in the group is denoted by l_x . In particular, mathematically, this is interpreted by representing the policyholders future lifetimes as a sequence $T_1, T_2, ..., T_{l_x}$ of i.i.d. non-negative random variables defined on $(\Omega, G, \mathbb{P})$. For getting the survival function below one should assume that the distribution of T_i is absolutely continuous with hazard rate function μ_{x+t}, ${}_tp_x = P(T_i > t) = \exp\left(-\int_0^t \mu_{x+\tau} d\tau\right)$ .Now, define a univariate process,

$$N = (N_t)_{0 \leq t \leq T}, \tag{3.20}$$

counting the number of death in the group;

$$N_t = \sum_{i=1}^{l_x} \mathbf{1}_{T_i \leq t}, \tag{3.21}$$

and denote by

$$H = (H_t)_{0 \leq t \leq T} \tag{3.22}$$

the natural filtration generated by N, i.e.,

$$H_t = \sigma(N_u, u \leq t). \tag{3.23}$$

By definition, N is right continuous with left limits which means it is cadlag. On the other hand since, the lifetimes T_i are i.i.d., these two conditions implied that the counting process N is an H-Markov process. Here, λ which is defined as the intensity process of the counting process N can be defined by [61]

$$E[dN_t | H_{t^-}] = (l_x - N_{t^-})\mu_{x+t} dt \equiv \lambda_t dt, \tag{3.24}$$

which is the hazard rate function μ_{x+t} times the number of individuals under exposure just before time t. As a result the compensated counting process M presented by

$$M_t = N_t - \int_0^t \lambda_u du, \tag{3.25}$$

is an H-Martingale [61].

We will define two types of dependence for insureds as given in two following subsections.

3.1.2.1 Joint Life and Last Survival Models

Consider the case of two lives with the age of x and y and whose future lifetimes is denoted by T_x and T_y. Then the joint cumulative distribution function is defined as:

$$F_{T_x, T_y}(s, t) = Pr(T_x \leq s, T_y \leq t), \tag{3.26}$$

the joint density function is

$$f_{T_x,T_y}(s,t) = \frac{\partial^2 F_{T_x,T_y}(s,t)}{\partial s \partial t} \tag{3.27}$$

and the joint survival distribution function is

$$S_{T_x,T_y}(s,t) = Pr(T_x > s, T_y > t). \tag{3.28}$$

3.1.2.2 Joint Life Model

In this type of the agreement benefit payment is considered for the first death occurrence. In case of the two lives representation will be as follows:

$$T_{xy} = \min(T_x, T_y). \tag{3.29}$$

The cumulative distribution function for above case will be as follows,

$$F_{T_{xy}}(t) =_t q_{xy} = Pr \min(T_x, T_y \leq t) = 1 - Pr(T_x > t, T_y > t) = 1 - S_{T_x,T_y}(t,t) \tag{3.30}$$

$$F_{T_{xy}}(t) = 1 -_t p_{xy}, \tag{3.31}$$

where

$$_tp_{xy} = Pr(T_x > t, T_y > t) = S_{T_{xy}}(t) \tag{3.32}$$

is the probability that both lives (x) and (y) survive after t years. The force of mortality is given by,

$$\mu_{x+t:y+t} = \frac{f_{T_{xy}(t)}}{1 - F_{T_{xy}(t)}} = \frac{f_{T_{xy}(t)}}{S_{T_{xy}(t)}} = \frac{f_{T_{xy}(t)}}{_tp_{xy}}. \tag{3.33}$$

The density of T_{xy} can be formulated as follows:

$$f_{T_{xy}(t) =_t p_{xy} \mu_{x+t:y+t}}. \tag{3.34}$$

For joint life status the counting process N_t^j is as follows:

$$N_t^j = \sum_{i=1}^{p} \mathbf{1}_{\min(T_{x_i}, T_{y_i}) \leq t} = \sum_{i=1}^{p} \{\mathbf{1}_{T_{x_i y_i}}\}, \tag{3.35}$$

where p states for the number of policies. We assume that policies are independently and identically distributed.

We define $I = (I_t)_{0 \leq t \leq T}$ the natural filtration generated by N^j, N^j is cadlag, and, since the policies are i.i.d., the counting process N^j is an I-Markov process. Also, we can define the expectation under I as follows:

$$E[dN_t^j | I_{t-}] = (p - N_{t-}^j) \mu_{x+t:y+t} dt, \tag{3.36}$$

and the compensated counting process M is defined by

$$M_t^j = N_t^j - \int_0^t \lambda_u^j du, \tag{3.37}$$

which is an I Martingale.

3.1.2.3 Last Survival Model

This model describes and states the condition where the insurance benefits are paying in case of the death of all members and at the last death. In case of the two lives in policy: : $T_{\overline{xy}} = \max(T_x, T_y)$.

The relationship among T_{xy}, $T_{\overline{xy}}$ and T_y, is given by:

$$T_{xy} + T_{\overline{xy}} = T_x + T_y, \tag{3.38}$$

$$T_{xy} T_{\overline{xy}} = T_x T_y. \tag{3.39}$$

For distribution of $T_{\overline{xy}}$ we can use the following relationship:

$$F_{T_{xy}}(t) + F_{T_{\overline{xy}}}(t) = F_x(t) + F_y(t), \tag{3.40}$$

$$S_{T_{xy}}(t) + S_{T_{\overline{xy}}}(t) = S_x(t) + S_y(t), \tag{3.41}$$

$$_tp_{xy} +_t p_{\overline{xy}} =_t p_x +_t p_y, \tag{3.42}$$

$$f_{T_{xy}}(t) + f_{T_{\overline{xy}}}(t) = f_x(t) + f_y(t), \tag{3.43}$$

and

$$F_{T_{\overline{xy}}}(t) = F_x(t) + F_y(t) - F_{T_{xy}}(t) = F_{T_x T_y}(t) = F_{T_x T_y}(t, t), \tag{3.44}$$

From

$$F_{T_{\overline{xy}}}(t) = Pr(T_x \leq t \cap T_x \leq t) \tag{3.45}$$

is follows that $_tp_{xy}$ is the probability that both (x) and (y) will survive after t years, and that

$_tp_{\overline{xy}}$ is the probability that at least one of the (x) and (y)'s will survive after t years.

Force of mortality can be stated as

$$\mu_{\overline{x+t:y+t}} = \frac{f_{T_{\overline{xy}}(t)}}{1 - F_{T_{\overline{xy}}(t)}} = \frac{f_{T_{\overline{xy}}(t)}}{S_{T_{\overline{xy}}(t)}} = \frac{f_{T_{\overline{xy}}(t)}}{_tp_{\overline{xy}}}. \tag{3.46}$$

For last survival status the counting process that we define is as follows:

$$N_t^l = \sum_{i=1}^{p} \mathbf{1}_{\max(T_{x_i}, T_{y_i}) \leq t} = \sum_{i=1}^{p} \mathbf{1}_{T_{\overline{x_i y_i}}}. \tag{3.47}$$

The filtration $I = (I_t)_{0\leq t\leq T}$ could be used also for N^l, and its expectation is given by

$$E[dN^l_t|I_{t^-}] = (p - N^l_{t^-})\mu_{\overline{x+t:y+t}}dt. \tag{3.48}$$

For the last survival status at time t the compensated process of N^l_t is

$$M^l_t = N^l_t - \int_0^t \lambda^l_u du, \tag{3.49}$$

which is an I -martingale.

3.1.3 The Combined Model

In this section we describe the combined portfolio which is defined as the combination of the financial and the insurance portfolios. We introduce the filtration C which is developed by the above mentioned two portfolios,

$$C = (C_t)_{0\leq t\leq T}, \tag{3.50}$$

as

$$C_t = F_t \vee I_t. \tag{3.51}$$

We assume that these two filtrations are independent and take,

$$C = F_T \vee I_T, \tag{3.52}$$

and

$$F_t = \sigma(S_u, u \leq t), \tag{3.53}$$

$$I_t = [\sigma(I(T_{x_i} \leq t)) \vee \sigma(I(T_{y_i} \leq t)), \quad 0 \leq t \leq T, \quad i = 1, 2, ..., n]\,. \tag{3.54}$$

One of the basic insurance contracts which is the term insurance, is chosen to be analyzed in this thesis. The term insurance says that the benefit payments are payable immediately in case of the death occurrence before maturity time T. For the contingent claim for this case, a time dependent contract function $g_t = g(t, V_t)$ is considered. According to the definition of the agreement, the amount can be paid at any moment within $[0, T]$ and liabilities stemmed from such agreements can not formulate T-claims, only by introduction of specific assumptions. In order to transform the liabilities to the T-claim, it is important to consider that all payments were deferred to the term of the agreement and were grown the riskless interest of return which is equal to r. According to the special construction, if the insured passed at time t the beneficiaries of the policy will get the benefit amount equal to $g(t, V_t)B_T B_t^{-1}$ at time T. A different way of accumulation of the deferred payments is the utilization of some deterministic first order interest rate δ or by making investment $g(t, V_t)$ as per the predefined strategy. Especially for the agreements with short time periods, it can be most reasonable to modify the agreements by deferring the payments. However even if the time periods accompanied with traditional life insurance agreements usually prolongs more, it will be considered the benefits are factually deferred to the insurance agreement termination date. So, the insurance companies liabilities with regard to the portfolio of term insurance agreements with payments that are postponed and grew by the utilization of the riskless asset are now shown with the help of the discounted general T-claim.

For the joint-life status it will be as follows:

$$H_T^j = B_T^{-1} \sum_{i=1}^{p} g(T_i, V_{T_{x_i y_i}}) B_{T_i}^{-1} B_T \mathbf{1}_{\{\min(T_{x_i}, T_{y_i}) \leq t\}}, \tag{3.55}$$

and it can be reformulated as an integral with respect to the counting process N^j by

$$H_T^j = \int_0^T g(s, V_s) B_u^{-1} dN_u^j. \tag{3.56}$$

For the last-survival status it will be

$$H_T^l = B_T^{-1} \sum_{i=1}^{p} g(T_i, V_{T_{x_i y_i}}) B_{T_i}^{-1} B_T \mathbf{1}_{\{\max(T_{x_i}, T_{y_i}) \leq t}\}, \tag{3.57}$$

and it can be reformulated as an integral with respect to the counting process N^l:

$$H_T^l = \int_0^T g(s, V_s) B_u^{-1} dN_u^l. \tag{3.58}$$

Premium for one policy at time $t = 0$:

$$E^Q[e^{-rT'} g(T', V_{T'}) \mathbf{1}_{\{T' < T\}}],$$

where T' is the (random) payment date, V_t is the CPPI portfolio value, $g(t, V_t)$ is the payoff function and T is the maturity of the contract. If

$$g(t, V_t) = V_t,$$

then

$$E^Q[e^{-rT'} g(T', V_{T'}) \mathbf{1}_{\{T' < T\}}] = V_0 P(T' < T),$$

since $e^{-rt} V_t$ is a martingale under Q. So the premium is available in closed form.

Due to the gap risk, which is caused by the presence of the jumps, it may be true that $V_t < F_t$ for some $0 < t \leq T$, where F_t denotes the floor. So, it is sensible to consider $g(t, V_t) = \max(V_t, F_t)$ as the claim of the contract. Since,

$$\max(V_t, F_t) = (F_t - V_t)^+ + V_t,$$

the premium for joint life status is

$$E^Q[e^{-rT'} g(T', V_{T'}) \mathbf{1}_{\{\min(T_x, T_y) \leq T'\}}] = E^Q[e^{-rT'} (F_{T'} - V_{T'})^+ \mathbf{1}_{\min(T_x, T_y) \leq t}] + V_0 P(T' < T) \tag{3.59}$$

and, for the last survival,

$$\mathbb{E}^Q[e^{-rT'}g(T', V_{T'})\mathbf{1}_{\{\max(T_x,T_y)\le T'\}}] = E^Q[e^{-rT'}(F_{T'} - V_{T'})^+\mathbf{1}_{\{\max(T_x,T_y)\le t\}}] + V_0 P(T' < T). \tag{3.60}$$

For the calculation of the first term, Monte-Carlo simulation is required. For a small multiplier m, it will be close to zero.

A question related to the gap risk: What happens when $V_t < F_t$? Do we allow shortselling by setting $e_t = m(V_t - F_t) < 0$, or allocate all the investment to the riskless asset by setting $e_t = 0$?

Joint Life and Last Survival

We need to calculate $P(T' < T)$. Let X, Y : be random life times of two lives T': be the time until payment given that the current ages are u and v, i.e., $T' = \min(X - u, Y - v)$.

Let t denote the time of first death and we are interested in distribution of T'. The cdf $F_{T'}(t) = P(T' < t)$ is given by

$$1 - P(X > u + t, Y > v + t | X > u, Y > v) = 1 -_t p_{uv}, \tag{3.61}$$

for joint life and

$$P(X < u + t, Y < v + t)|(X > u, Y > v), \tag{3.62}$$

for last survival.

Above probabilities are available in closed form. The first one is

$${}_tp_{uv} = P(X > u+t, Y > v+t | X > u, Y > v) = \frac{P(X > u + t, Y > v + t)}{P(X > u, Y > v)}, \tag{3.63}$$

$${}_tp_{uv} = \frac{S(u + t, v + t)}{S(u, v)}, \tag{3.64}$$

where $S(u, v)$ is the joint survival function. Also,

$$P(X < u + t, Y < v + t | X > u, Y > v) = \frac{P(u < X < u + t, v < Y < v + t)}{P(X > u, Y > v)}. \tag{3.65}$$

Here,

$$P(u < X < u+t, v < Y < v+t) = F(u+t, v+t) - F(u+t, v) - F(u, v+t) + F(u+t, v+t), \tag{3.66}$$

where $F(u, v)$ is the joint cdf given by Eqn. (10) in Yörubulut and Gebizlioglu [100].

The section is ended with the discussion of selection of martingale measure in the combined model. It can be said that for any I- predictable process h, such that $h > -1$, represent a likelihood process I by [61]:

$$dI_t = I_{t^-} h_t dM_t \tag{3.67}$$

and the initial conditional $I_0 = 1$, granted that $E^p[I_T]$, a new probability measure Q^* can be determined by

$$\frac{dQ^*}{dP} = L_T I_T, \tag{3.68}$$

where L_T is given by Eqn. (3.4). Utilizing the definition of the measure Q^* and the independence between financial and insurance portfolio under P we can notice that S^* which is determined by Eqn. (3.1) is at the same time a Q^* martingale. For $u < t$ we have

$$E^{Q^*}[S_t^*|F_u] = \frac{E[S_t^* U_T L_T|F_u]}{E[U_T L_T|F_u]} = \frac{E[S_t^* U_T|F_u]E[L_T|F_u]}{E[U_T|F_u]E[L_T|F_u]} = E^Q[S_t^*|F_u] = S_u^*, \tag{3.69}$$

using that S^*-martingale, and so each Q^* is an equivalent martingale measure. Because of this non-uniqueness of the equivalent martingale measure, agreements can not generally be priced uniquely by no-arbitrage pricing theory only.

Since financial and insurance portfolios are independent under Q^* and, according to Girsanov theorem, the processes M^{jh} and M^{lh} defined by

$$M_t^{jh} = N_t^j - \int_0^t \lambda_u^j (1 + h_u) du \tag{3.70}$$

and

$$M_t^{lh} = N_t^l - \int_0^t \lambda_u^l (1 + h_u) du, \tag{3.71}$$

for joint life and last-survival policies, respectively are (C, Q^*) martingales.

Specific martingale measure Q defined by Eqn. (3.4) is applied, which is also known as the *minimal martingale measure*, Schweizer [55, 35]. This particular measure is mainly applicable for the pricing of unit linked agreements. The reason for using this measure is insurance companies risk neutrality with regard to mortality [**?**]. So, it is assumed that the probability space (Ω, F, Q) endowed with the filtration F. The filtration F is equivalently created by the Q-martingales S^* and $\bar{M}$.

The martingale measure Q^* could be equally applied to admissible selections of h. Here, an analogous result must be obtained in regard to the hazard rate function $\mu_{x+t:y+t}$ for joint life and $\mu_{\overline{x+t:y+t}}$ for last survival replaced by $\mu_{x+t:y+t}(1+h_t)$ and $\mu_{\overline{x+t:y+t}}(1+h_t)$, and M^j, M^l replaced by M^{jh}, M^{lh}, respectively. Anyhow, there are martingale measures and those measures do not keep independence among financial and insurance portfolios, and such selections of martingale measures will surely complicate investigations in the numerical part.

3.2 Mean Variance Hedging of Combined Obligations for Term Insurance Policies

In this chapter, we mainly investigate the hedging concept with mean-variance quadratic hedging approach. We define H^M as

$$H^M \in \begin{cases} H^j, & \text{if } \textit{Joint life Contract} \\ H^l, & \text{if } \textit{Last Survival Contract}. \end{cases} \tag{3.72}$$

Given a contingent claim H^M which is defined by Eqn. (3.78) and Eqn. (3.58) and if the financial market models do not allow arbitrage opportunities, in a complete market, H^M is attainable, i.e., there is a self-financing strategy with final portfolio value $Z_T = H^M$, P-a.s. However, when in our jump-diffusion model, the market is not complete and then H^M is not attainable [99]. For our contingent claim our payoff at time T is H^M. Our jump-diffusion model of the risky asset price S is a semimartingale under P and the discounted price process $\tilde{S}$ is a martingale under Q. In our case, we consider H^M as a function of V_T and denote $H^M = g(V_T)$. For any martingale measure Q defined in Eqn. (3.4), we have proved that $\tilde{V}_t = e^{-rt}V_t$ is a Q-martingale. Let us denote, $\tilde{H_t^M} = e^{-rt}H^M$. We want to consider the following optimization problem:

$$\min_{(Z_0,\nu)\in R\times\Theta} E^Q\left(\tilde{H^M}_t - Z_0 - \int_0^T \nu_u d\tilde{V}_u\right)^2. \tag{3.73}$$

We extend following Proposition 12.10 from Wang [97] and apply for new contingent claim equations (Eqn. (3.78) and Eqn. (3.58)) for joint-life and last-survival life insurance contracts.

Proposition 3.4. *The solution of the optimization problem* (3.73) *is:*

$$Z_0 = E^Q[\tilde{H_t^M}], \quad \nu_t = \frac{\sigma_t\zeta_x(t,V_t) + \zeta(t, V_t + [V_{t^-} - F_t]m_tY_t) - \zeta(t,V_t)Y_t\lambda_t\psi_t}{\sigma_t + (V_{t^-} - F_t)m_tY_t^2\lambda_t\psi_t}. \tag{3.74}$$

Define, $\zeta(t,x) = e^{rt}E^Q[\tilde{H^M}|V_t = x]$ and $\tilde{\zeta}(t,x) = e^{-rt}\zeta(t,x)$. By construction, $\tilde{\zeta}(t,x)$ is a Q-martingale. It is deduced that

$$dV_t = [rV_{t^-} + (V_{t^-} - F_t)m_t(\mu_t - r)]dt + (V_{t^-} - F_t)m_t\sigma_t dW_t + (V_{t^-} - F_t)m_tY_tdN_t \tag{3.75}$$

and

$$d\tilde{V}_t = e^{-rt}[(V_{t^-} - F_t)m_t\sigma_t dW_t^Q + (V_{t^-} - F_t)m_tY_tdM_t^Q]. \tag{3.76}$$

Proof. **Joint Life Contracts:**

We have,

$$H_T^j = B_T^{-1} \sum_{i=1}^{p} g(T_i, V_{T_{x_i y_i}}) B_{T_i}^{-1} B_T \mathbf{1}_{\{\min(T_{x_i}, T_{y_i}) \le t\}}, \tag{3.77}$$

and

$$H_T^j = \int_0^T g(s, V_s) B_u^{-1} dN_u^j. \tag{3.78}$$

We can write expectation formula under Q as follows:

$$\begin{aligned} E^Q\left(\tilde{H}^j - Z_0 - \textstyle\int_0^T \nu_u d\tilde{V}_u\right)^2 &= E^Q\left(E^Q[\tilde{H}^j] - Z_0 + \tilde{H}^j - E^Q[\tilde{H}^j] - \textstyle\int_0^T \nu_u d\tilde{V}_u\right)^2 \\ &= E^Q\left[\left(E^Q[\tilde{H}^j] - Z_0\right)^2\right] + E^Q\left(\tilde{H}^j - E^Q[\tilde{H}^j] - \textstyle\int_0^T \nu_u d\tilde{V}_u\right)^2. \end{aligned} \tag{3.79}$$

The optimal value for the initial capital is $Z_0 = E^Q[\tilde{H}^j]$.

Define $\zeta(t,x) = e^{rt} E^Q[\tilde{H}^j | V_t = x]$ and $\tilde{\zeta}(t,x) = e^{-rt}\zeta(t,x)$. By construction, $\tilde{\zeta}(t,x)$ is a Q-martingale. We have deduced that,

$$dV_t = [rV_{t^-} + (V_{t^-} - F_t)m_t(\mu_t - r)]dt + (V_{t^-} - F_t)m_t\sigma_t dW_t + (V_{t^-} - F_t)m_t Y_t dN_t, \tag{3.80}$$

and

$$d\tilde{V}_t = e^{-rt}\left[(V_{t^-} - F_t)m_t\sigma_t dW_t^Q + (V_{t^-} - F_t)m_t Y_t dM_t^Q\right]. \tag{3.81}$$

Then by Ito's formula we have

$$\begin{aligned} d\tilde{\zeta}(t,V_t) &= [-re^{-rt}\zeta(t,V_t) + e^{-rt}\zeta_t(t,V_t) + (rV_{t^-} + (V_{t^-} - F_t)m_t(\mu_t - r))e^{-rt}\zeta_x(t,V_t) \\ &\quad + \tfrac{1}{2}(V_{t^-} - F_t)^2 m_t^2 \sigma_t^2 e^{-rt}\zeta_{xx}(t,V_t)]dt + (V_{t^-} - F_t)m_t\sigma_t e^{-rt}\zeta_x(t,V_t)dW_t \\ &\quad + [e^{-rt}\zeta(t, V_t + [V_{t^-} - F_t]m_t Y_t) - e^{-rt}\zeta(t,V_t)]dN_t \\ &= (V_{t^-} - F_t)m_t\sigma_t e^{-rt}\zeta_x(t,V_t)dW_t^Q \\ &\quad + [e^{-rt}\zeta(t, V_t + [V_{t^-} - F_t]m_t Y_t) - e^{-rt}\zeta(t,V_t)]dM_t^Q. \end{aligned} \tag{3.82}$$

Thus we have

$$\begin{aligned} &\tilde{H}^j - E^Q[\tilde{H}^j] - \textstyle\int_0^T \nu_u d\tilde{V}_u \\ &= \tilde{\zeta}(T,V_T) - \tilde{\zeta}(0,V_0) - \textstyle\int_0^T \nu_t e^{-rt}\left[(V_{t^-} - F_t)m_t\sigma_t dW_t^Q + (V_{t^-} - F_t)m_t Y_t dM_t^Q\right] \\ &= e^{-rt}[\textstyle\int_0^T (V_{t^-} - F_t)m_t\sigma_t(\zeta_x(t,V_t) - \nu_t)dW_t^Q \\ &\quad + \textstyle\int_0^T (\zeta(t, V_t + (V_{t^-} - F_t)m_t Y_t) - \zeta(t,V_t) - \nu_t(V_{t^-} - F_t)m_t Y_t)dM_t^Q]. \end{aligned} \tag{3.83}$$

By the Isometry formula, we have,

$$
\begin{aligned}
&E^Q\left(\tilde{H}^j - E^Q[\tilde{H}^j] - \int_0^T \nu_u d\tilde{V}_u\right)^2 \\
&= e^{-2rt}(E^Q\left[\int_0^T [(V_{t^-} - F_t)m_t\sigma_t(\zeta_x(t,V_t) - \nu_t)\right]^2 dt \\
&+E^Q\left[\int_0^T [\zeta(t, V_t + [V_{t^-} - F_t]m_tY_t) - \zeta(t,V_t) - \nu_t(V_{t^-} - F_t)m_tY_t]^2\lambda_r\psi_t dt\right]).
\end{aligned}
\tag{3.84}
$$

This is the minimizing problem with respect to ν_t. Differentiating the above expression with respect to ν_t and letting the first order derivative equal to 0, we have

$$
\begin{aligned}
&(V_{t^-} - F_t)m_t\sigma_t[\zeta_x(t,V_t) - \nu_t] + [\zeta(t, V_t + (V_{t^-} - F_t)m_tY_t) \\
&-\zeta(t,V_t) - \nu_t(V_{t^-} - F_t)m_tY_t](V_{t^-} - F_t)m_tY_t\lambda_t\psi_t = 0,
\end{aligned}
\tag{3.85}
$$

thus,

$$
\nu_t = \frac{\sigma_t\zeta_x(t,V_t) + \zeta(t, V_t + [V_{t^-} - F_t]m_tY_t) - \zeta(t,V_t)Y_t\lambda_t\psi_t}{\sigma_t + (V_{t^-} - F_t)m_tY_t^2\lambda_t\psi_t}. \tag{3.86}
$$

Last Survival Contracts: Exactly the same steps above could be applied in this case.

□

CHAPTER 4

NUMERICAL IMPLEMENTATIONS

In this chapter, we show some numerical calculations for our model. The Monte-Carlo simulation method is used for illustrative purposes.

We start from simulation of stock price process which is stock price process with jump diffusion. In Table 4.1, is provided some simulation results for stock returns. With different negative and positive jumps returns is showing different patterns. In Table 4.2, we introduce portfolio return results for different values of m. As we mentioned in chapter 1, m is a multiplier, which shows the share of the total capital allocated to the risky security. Difference between total capital and risky asset is allocated to the risk-free security. The multiplier is a value (typically between 2 and 4) which is the representative of the investor's risk aversion. We can consider that, the higher the multiplier, the greater the possibility to invest in the risky asset. This multiplier based on the risk aversion level of the investor is exogenously defined by the investor at the start of the investment and stays unchanged during the life of the product. We can see from our results that with the higher value for m, the more the chance to fall under the floor. Changing the weights for portfolio is based on the market movements, such that risky asset share will increase after a rise and decrease after the fall in the market. For continuous trading and where shortselling is allowed by setting borrowing constraint to $p > 0$, we can get results which are provided in Tables 4.3 - 4.6 for a 120 month period and for different values of m. Here, $i \times j$ stands for monthly bases values of portfolio. For the end of the first month, the portfolio value is 180.1, and for end of the $10th$ year the portfolio value is 246.3, for Table 4.3.

For Contingent claim taken as put option, we simulate option on CPPI portfolio value, with risk-neutral parameters for the Merton Jump Diffusion model we use mainly references [37, 90]. Here, the model parameters are given by, $\sigma = 0.126349$, $\lambda = 0.174814$, $a = -0.390078$, $b = 0.338796$ calibrated to a set of 77 mid-prices of European call options on the $S\&P500$ Index at the close of the market on 18 April 2002.

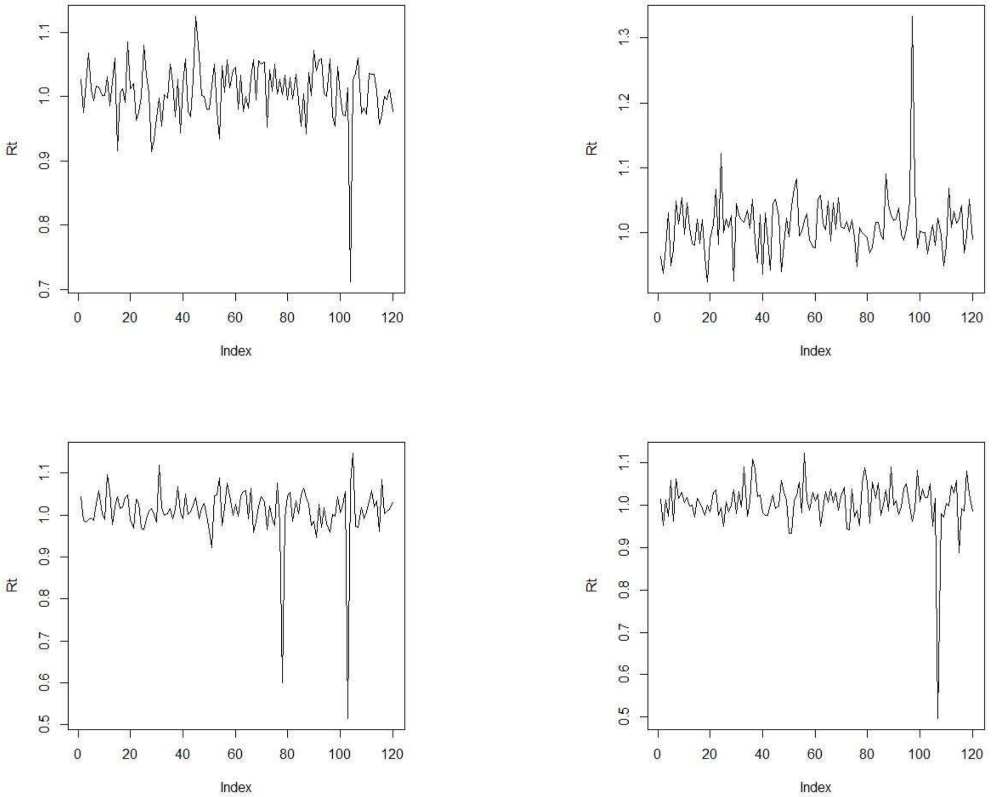
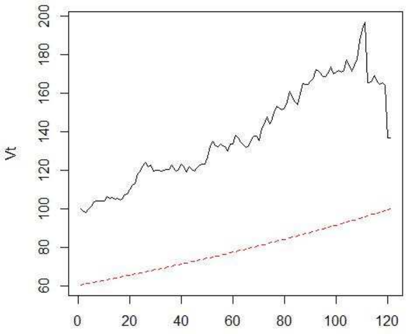
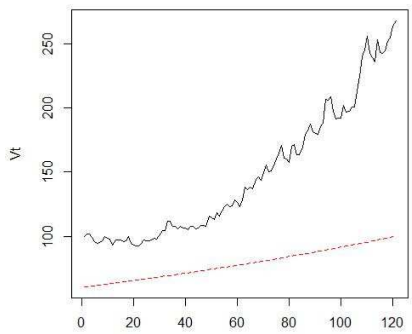


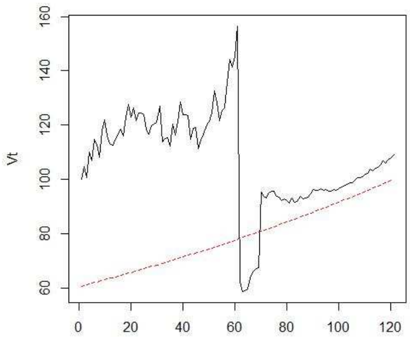
Table 4.1: Return over Time for Stock Price Process with Jump Diffusion.



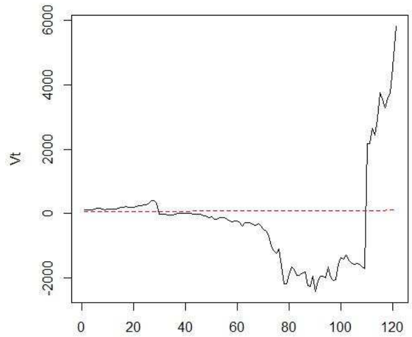
(a) $m = 1$



(b) $m = 2$



(c) $m = 3$



(d) $m = 4$

Table 4.2: Portfolio Values over time for different values of m.

ixj								
180.1	138.7	215.7	245.4	121.4	150.8	200.2	225.6	296.8
159.5	285.9	133.6	161.1	137.5	139.0	155.7	179.4	136.7
125.6	134.1	199.3	161.8	109.6	194.7	116.4	180.9	122.1
125.9	170.7	118.6	208.7	127.1	112.8	160.3	194.4	150.4
108.9	152.3	111.6	106.0	316.9	255.5	159.1	148.4	177.8
254.4	136.3	209.9	161.9	117.1	174.5	254.1	211.3	183.7
132.4	145.0	132.6	123.1	116.9	160.1	161.5	283.0	119.1
179.3	145.8	148.4	156.8	164.7	214.2	128.4	179.4	200.9
233.6	151.5	156.5	210.3	186.0	272.4	125.5	171.3	210.5
169.0	211.8	226.8	123.8	104.9	105.6	123.3	117.5	176.2
112.4	157.1	155.9	133.8	188.0	140.9	112.4	111.7	246.3

Table 4.3: Portfolio Values for $m = 1$.

ixj							
116	132	117	159	100	99	212	165
138	192	100	161	92	182	100	68
55	230	124	465	163	97	87	235
207	99	218	100	285	179	144	230
68	186	92	194	128	245	104	31
103	295	100	126	126	189	176	98
149	36	101	539	102	115	99	110
334	193	174	251	144	289	107	95
123	334	100	98	164	117	837	101
78	103	116	105	93	95	120	104
136	106	109	436	151	92	138	95
83	100	110	390	100	598	89	207

Table 4.4: Portfolio Values for $m = 2$.

ixj						
121	134	161	301	111	-900	133
99	871	92	138	129	100	100
90	159	98	184	59	99	124
115	189	406	71	-287	235	339
283	762	350	103	86	100	54
321	-100	99	204	221	196	101
505	100	95	96	347	72	93
440	100	100	151	145	79	134
119	30	1,166	-116	597	102	361
141	-349	99	586	101	104	127
84	651	109	-10	88	98	152
100	-39	70	131	61	43	-26
90	634	322	182	79	128	355
102	219	153	417	77	356	83

Table 4.5: Portfolio Values for $m = 3$.

ixj					
137	429	99	494	236	77
110	148	110	481	159	84
117	47	542	33	237	-1,418
-1,171	112	101	366	98	33
86	39	99	76	116	-107
312	281	-764	95	86	56
139	-8,543	28	50	-581	715
304	-23	124	1,696	-47	100
126	544	131	50	100	93
127	-345	264	-1,585	112	250
12	-1,010	147	1,064	57	75
113	200	156	228	862	104
94	-284	618	46	142	101
96	95	45	275	59	-748
33	68	470	416	96	66
102	101	-828	105	108	97

Table 4.6: Portfolio Values for $m = 4$.

Put prices based on the results derived below can be calculated as in following Table 4.7.

		Result	**Error Estimate**
m=1	price	0.782	0.06
	nu	-0.523	0.012
m=2	price	0.753	0.046
	nu	-0.473	0.015
m=3	price	1.007	0.133
	nu	-0.415	0.017
m=4	price	2.453	0.267
	nu	-0.486	0.013

Table 4.7: Put Prices.

Until now we have not considered the insurance portfolio for single or for multiple life and just investigated financial portfolio results. Now, we can include life portfolio part but for now just for single life. We will use Gompertz distribution which is one of the most appropriate ones among the lifetime distributions. For its parameters we refer to [53]. They use data for Swedish females in 2010, aged $35 - 100$ years, and the distribution parameters are $\lambda = 0.0002, \mu = 0.116$. For given parameters the histogram is as follows in the following Figure 4.1 :

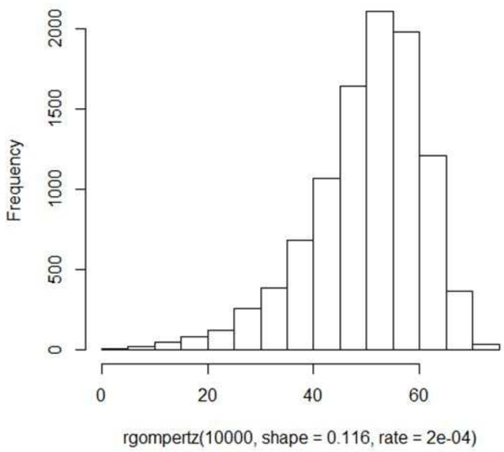


Figure 4.1: Histogram. Swedish females in 2010, aged $35 - 100$ years, and the distribution parameters are $\lambda = 0.0002, \mu = 0.116$

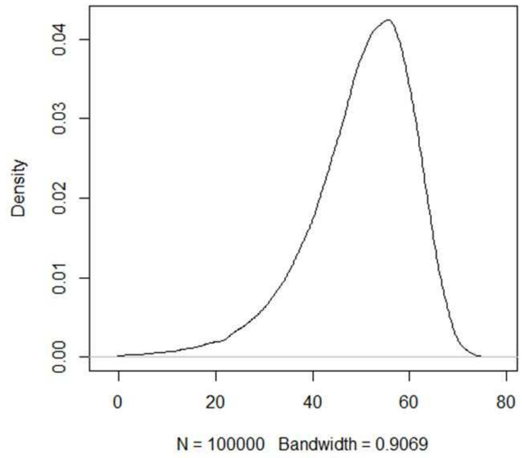


Figure 4.2: Density based on Pseudo Gompertz Distribution.

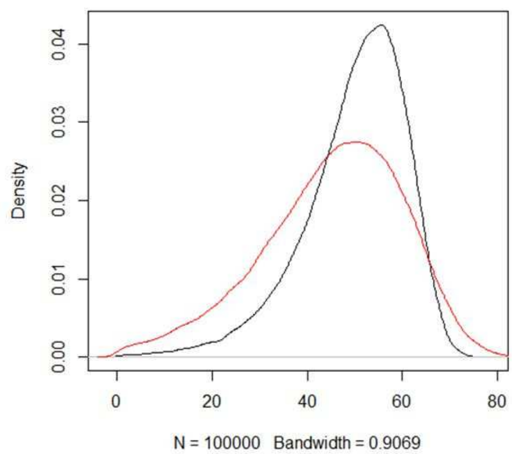


Figure 4.3: Density based on Pseudo Gompertz Distribution for Joint Case.

Now, we can calculate put prices based on single life contract using Gomperz distribution whose outcomes are as in following Table 4.8 for different values of m. It is assumed Term Life Insurance policy which payments linked to the death of the policyholder. Based on the Lenart [53] as the beginning age of the policyholder 35 is considered.

		Result	**Error Estimate**
m=1	price	0	0
	nu	0	0
m=2	price	0.001	0.002
	nu	-0.000	0.000
m=3	price	0.035	0.051
	nu	-0.000	0.000
m=4	price	0.089	0.094
	nu	0.000	0.001

Table 4.8: Put Prices for Unit Linked Life Insurance Policy: Single Life.

For single and for multiple life policies only one policy is considered in the portfolio. For multiple life policies as mentioned before Pseudo Gompertz Distribution is used. The simulation results for multiple lives based on Pseudo Gompertz distribution for x and y are presented in Table 4.9.

X	Y
41.71	31.30
48.57	41.52
51.85	46.27
12.70	16.87
55.04	56.32
35.41	44.78
61.90	65.49
52.16	60.75
57.34	54.18
43.31	39.68

Table 4.9: Values for X and Y simulated using Pseudo-Gompertz Distribution.

Descriptive statistics and Summary statistics for multiple life case using Pseudo Gompertz distribution is as in following Tables, 4.10- 4.11.

X		Y	
Min.	0.00093	Min.	0.00159
1st Qu.	44.23	1st Qu.	46.19
Median	51.73	Median	46.91
Mean	50.00	Mean	45.34
3rd Qu.	57.71	3rd Qu.	56.05
Max.	77.54	Max.	87.30

Table 4.10: Descriptive Statistics based on Pseudo Gompertz Distribution.

	X	Y
Mean	49.95	45.30
Variance	117.86	217.82

Correlation	X	Y
X	1.00	0.70
Y	0.70	1.00

Table 4.11: Summary Statistics based on Pseudo Gompertz Distribution.

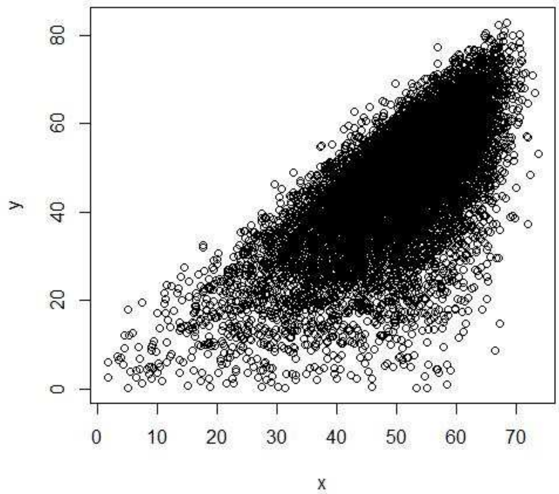


Figure 4.4: Correlation between X and Y.

Now, we can calculate optimal values which are defined in Proposition 3.4. Here, **price** stands for Z_0 and **nu** stands for ν_0. Calculations are done for representatively two cases, for joint-life policies and last-survival policies and results are provided in Tables 4.12 and 4.13. Here, is considered guarantee which is equal to the initial portfolio value. Calculations are investigated for different values of m.

$V(0)$=**100,** G=**100**		**Result**	**Error Estimate**
m=1	price	16.29	0.00
	nu	0.15	0.00
m=2	price	17.56	0.30
	nu	0.16	0.00
m=3	price	29.89	4.50
	nu	0.28	0.04
m=4	price	125.14	73.62
	nu	1.24	0.74

Table 4.12: Continuous Trading with Guarantee and Shortselling for Joint Life Policies.

$V(0)$=**100,** G=**100**		**Result**	**Error Estimate**
m=1	price	4.44	0.00
	nu	0.04	0.00
m=2	price	4.89	0.24
	nu	0.05	0.00
m=3	price	11.28	5.02
	nu	0.11	0.05
m=4	price	99.77	124.73
m=4	price	99.77	124.73
	nu	0.99	1.25

Table 4.13: Continuous Trading with Guarantee and Shortselling for Last Survival Policies.

CHAPTER 5

CONCLUSION

The study was set out to explore and to develop the concept of risk minimization hedging strategies for multiple-life unit linked insurance policies with help of Constant Proportion Portfolio Insurance approach. As it is known from literature and also stated in introduction part of this thesis a Unit-Linked Life Insurance policy is a agreement where the insurance benefits depend on the price of some specific traded security which carry the risk. Because of the inherent nature of the securities the returns are random since the benefits are unknown in advance and based on these, obligations of insurance company are also random. The main purpose in such a situation is to correctly define obligations of company and based on obligations to define the proper hedging concept.

In the thesis, is considered a model describing the uncertainty of the financial market and a portfolio of insured individuals simultaneously. This case investigated in some literature as Möller [61], Riesner [88], etc., but all of these authors considered that insurance portfolio are independent, lifetimes are independently and identically distributed.

The first distinctive point of this thesis is that it is considered dependency between lifetimes of insureds. In other words, it is assumed that policies in insurance portfolio are independent but lifetimes in each policy are dependent. For dependency modeling a Pseudo-Gompertz distribution is used, which is a bivariate distribution in which marginal distributions are Gompertz distribution.

Based on the benefit payment structure different types of schemes can be investigated. In the thesis two different schemes are considered and analyzed. First is joint life status which is considering the benefit payment in case of the occurrence of first death. As opposed to joint life status in last survival status benefit payments are paid in case of the occurrence of last death. Two beneficiaries for each policy are established and dependence between two are modeled with Pseudo-Gompertz distribution function. In other hand the relation between policies is considered as independent and stated that they are independently and identically distributed.

The second distinctive point was the usage of the Constant Proportion Portfolio Insurance approach with jump diffusion for financial portfolio. Standard Black and Scholes model is used nearly in all articles. What about the Black and Scholes model with Jump diffusion, it was firstly investigated in Riesner [88], but for single life policy.

The third distinctive point of the thesis is that, for hedging purposes the mean-variance hedging approach is used instead of the local risk minimization method.

In the numerical result part, tables and figures are provided which help us to describe the dynamics. One can observe from return and portfolio value graphics that effect of jumps is quite high. Here, m is a multiplier, which shows the weight of risky asset in portfolio when the rest of the capital invested to the risk-free asset. The multiplier shows the risk appetite of the investor. The higher the multiplier, the greater the investment in the risky asset. We can observe from tables that when the multiplier is high, portfolio value is too low or negative when jumps occurs. In the first part of numerical results, simulations for CPPI is investigated without considering insurance part. Then simulation for single life and multiple lifetimes are done and provided in related tables. Results for combined model are provided in the last two tables of our numerical results part. The details are as follows.

For single life policy we consider Term Life Insurance policy in which benefit payments are valid in case of the death of policyholder. As a term 10 years is considered with the beginning age of insured equal to 35. For simulation only 1 policy is taken. For single life Gompertz distribution function is used.

When the initial portfolio value is equal to the guarantee given ($G = V_0 = 100TL$) for $r = 0.05$, the results for different m ($m = 1, 2, 3, 4$) values are simulated.

All calculations are done for Joint Life and Last Survival Unit Linked Insurance policies.

We observed that for last survival policies the results are twice lower than for joint life case, and this is expected outcome. Since in the second case benefit payment made in case of both insureds death, this probability is lower than first case.

The numerical results are calculated when contingent claim is a put option. In our tables, **price** stands for Z_0 and **nu** stands for ν_0. So, in this case, Z_0 stands for put price.

For future research, one can investigate dependence also between policies. This dependence could be modeled with the help of copulas. Also, cases of different insurance products like pure endowment could be investigated. Since the mean-variance hedging approach is applicable in the semi-martingale case different portfolio hedging theories could be investigated.

Another extension could be applied to the financial part of the model by adding new tradeable assets to the financial portfolio. Based on our study new extended financial portfolio calculation of optimal weights for assets for multiple life unit linked policies would be an interesting area for further research.

A more interesting but at the same time very useful extension would be the investigation of new martingale measures which do not preserve the independence between financial and insurance portfolios.

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2. Financial Time Series Models (via WinBUGS) (Prof. Dr. Sujit K. Ghosh)
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4. Actuarial Training Program of Azerbaijan, Ministry of Finance of the Republic of Azerbaijan and the United States Agency for International Development (USAID) and Azerbaijan Insurance Association (15.07.2008- 01.09.2008)
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PRESENTATIONS

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