THE RELATIVISTIC BURGERS EQUATION ON A FRIEDMANN–LEMAÎTRE–ROBERTSON–WALKER (FLRW) BACKGROUND AND ITS FINITE VOLUME APPROXIMATION

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ABSTRACT

THE RELATIVISTIC BURGERS EQUATION ON A FRIEDMANN–LEMAÎTRE–ROBERTSON–WALKER (FLRW) BACKGROUND AND ITS FINITE VOLUME APPROXIMATION

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The inviscid Burgers equation is an important model in computational fluid dynamics, and represents one of the simplest example for a nonlinear hyperbolic conservation law. Recently, several relativistic and non-relativistic generalizations of the classical Burgers equation have been introduced by LeFloch and collaborators, by identifying a hyperbolic balance law by the Euler equations of relativistic compressible fluids. Both the model and the classical Burgers equations are obtained from derivation of the Euler system of relativistic compressible fluids by considering vanishing pressure on a curved background. The relativistic generalization of the model is considered on different curved spacetimes, in particular on Minkowski (flat) and Schwarzshild spacetimes. In this thesis we consider the Friedmann–Lemaître–Robertson–Walker (FLRW) background. Since both Schwarzshild and FLRW metric are the solutions of Einstein field equations, this common property inspired us to derive a relativistic version of Burgers equation on FLRW background as it is done on Schwarzshild spacetime. The relativistic Burgers equation on this spacetime is obtained from the Euler system of relativistic compressible fluids. Firstly, Christoffel symbols and tensors for perfect fluids on FLRW background are calculated. Then, by using divergence of the tensors, the Euler equations are obtained. Next, we impose the pressure to be zero on this Euler system as it is carried out in the flat and Schwarzshild spacetimes. Finally, by combining the equations derived by the Euler system, we obtain the desired relativistic Burgers equation on FLRW background. A particular case of the concerning equation gives exactly the classical Burgers equation which is a common property shared by relativistic equations. Different from the flat and Schwarzshild spacetimes cases, some special types of solutions which are spatially homogeneous are found for the concerning equation, which provides an originality to this study. Next, the limiting properties of the relativistic Burgers equation on FLRW background and spatially homogeneous solutions are examined. For the numerical experiments we construct a Godunov type of scheme which solves the Riemann problem on each grid cell. In addition, a well-balanced scheme is constructed for investigating the limiting properties of the homogeneous solutions in detail. The numerical schemes are developed by using the finite volume methodology which are formulated for curved spacetimes in [2], [37] and [38]. These numerical schemes allow to capture discontinuous solutions containing shock waves for the relativistic Burgers equation. Finally, we observe that the proposed scheme is well-balanced, in the sense that it preserves all spatially homogeneous solutions and the numerical experiments illustrate the convergence of the scheme on a FLRW background.

Keywords: the relativistic Burgers equations, the Euler equations, FLRW metric, compressible fluids, finite volume method, well-balanced scheme

ÖZ

FRIEDMANN–LEMAÎTRE–ROBERTSON–WALKER (FLRW) UZAYZAMANINDA RÖLATİVİSTİK BURGERS DENKLEMİ VE SONLU HACİM METODLARI

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Klasik Burgers denklemi hesaplanılabilir sıvılar dinamiğinde önemli bir modeldir ve hiperbolik korunum kanunlarına en basit örneklerden biri olarak gösterilebilir. Yakın zamanda LeFloch ve meslektaşları tarafından, sıkıştırılabilir rölativistik Euler denklemleri kullanılarak hiperbolik denge kanununa uyan rölativistik ve rölativistik olmayan Burgers denklemleri olarak adlandırılan yeni modeller elde edilmiştir. Bu modellerin ve klasik Burgers denkleminin, kıvrık uzayzamanlarında basınç sıfır alınarak sıkıştırılabilir rölativistik Euler denklemlerinden elde edilebilir olduğu da gösterilmiştir. Ayrıca bu modellerin rölativistik genellemeleri, değişik kıvrık uzayzamanlarında (Minkowski ve Schwarzshild uzayzamanlarında) ele alınmıştır. Bu tezde uzayzamanı olarak Friedmann–Lemaître–Robertson–Walker (FLRW) ele alındı. Schwarzshild ve FLRW metriklerinin, Einstein alan denklemlerinin çözümü olmaları, Schwarzshild uzayzamanında olduğu gibi FLRW uzayzamanında da Burgers denklemini elde etmemize ilham kaynağı oldu. Benzer tekniklerin izlenmesi sonucunda FLRW uzayzamanında Burgers denklemi, sıkıştırılabilir sıvılar için Euler sistemleri kullanılarak elde edildi. Bu sebeple ilk olarak Christoffel semboller, sıkıştırılabilir sıvılar için tensörler hesaplandı ve böylece tensörlerin ıraksaması kullanılarak Euler denklemlerine ulaşıldı. Bunu takiben, daha önceki uzayzamanlarında (Minkowski ve Schwarzshild) olduğu gibi Euler sistemi üzerinde basınç sıfır olarak kabul edildi. Son olarak, ba-

sıncın sıfır olduğu kabul edilerek elde edilen bu iki denklemin ortak çözümünden FLRW uzayzamanında rölativistik Burgers denklemi elde edildi. Bütün rölativistik denklemlerde olduğu gibi, söz konusu denklemin de özel bir durumu kullanılarak denklemin rölativistik olmayan durumuna yani klasik Burgers denklemine ulaşıldı ve daha önceki çalışmalardan farklı olarak, FLRW uzayzamanında Burgers denklemine ait, uzay koordinatına bağlı olmayan çözümler elde edildi. FLRW uzayzamanında Burgers denkleminin uzay koordinatına bağlı olmayan çözümlerinin bulunması, bu çalışmayı diğer uzayzamalarında yapılan çalışmalardan ayıran orijinal bir özelliktir. Daha sonra, FLRW uzayzamanında Burgers denkleminin ve uzay koordinatına bağlı olmayan çözümlerinin limit özellikleri değerlendirildi. Nümerik deneylerde kullanılmak üzere, her bir uzay aralığı için Riemann problemini çözen Godunov tipi bir şema oluşturuldu. Buna ek olarak, FLRW uzayzamanında Burgers denkleminin ve uzay koordinatına bağlı olmayan çözümlerinin limit özelliklerini detaylı olarak incelemek üzere 'well-balans' Godunov tipi şemalar oluşturuldu. Bu modelin nümerik şemaları için sonlu hacim metodlar kullanılarak kıvrık uzayzamanlarında [2], [37] ve [38]'de formalize edilen şemalar geliştirildi. Bu şemaların, rölativistik Burgers denkleminin şok dalgaları barındıran sürekli olmayan çözümlerini içerdiği gösterildi. Ayrıca nümerik hesaplamalar neticesinde, önerilen şemaların yakınsadığı ve uzay koordinatına bağlı olmayan çözümleri koruduğu gözlemlendi.

Anahtar Kelimeler: rölativistik Burgers denklemleri, Euler denklemleri, FLRW metriği, sıkıştırılabilir sıvılar, sonlu hacim metodları, 'well-balans' şemalar

To my daughter,

DilŞah

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LIST OF ABBREVIATIONS

δ^i_j	Kronecker's delta function
Λ	cosmological constant
G	gravitational constant
$T^{\alpha\beta}$	energy-momentum tensor for perfect fluids
$\Gamma^{\mu}_{\alpha\beta}$	Christoffel symbols
$\mathcal{R}^{\alpha}_{\beta\mu\nu}$	Riemann curvature tensor
C^{∞}_{c}	compactly supported, infinitely differentiable functions
L^{∞}	L_p spaces with p is ∞
$R_{\alpha\beta}$	Ricci tensor
R	Ricci scalar
$G_{\alpha\beta}$	Einstein tensor
$a(t)$	scale factor
c	light speed
ϵ	inverse of light speed
ρ	density
$p(\rho)$	pressure
div$(\cdot)$	divergence operator
H_t	n dimensional spacelike manifold
H_0	the initial slice of H_t
M	n dimensional manifold
$T(v)$	flux vector field on M
$S(v)$	scalar field on M

CHAPTER 1

INTRODUCTION

The development of the mathematical theory of shock wave solutions to scalar conservation laws defined on manifolds is arised with the progress in compressible fluid dynamics. The shallow water equations of fluid dynamics and the Euler equations in general relativity are important examples where the partial differential equations of interest are posed on a curved spacetime. Scalar conservation laws are simplified mathematical models for nonlinear aspects of shock wave propagation on spacetimes. The theory for nonlinear hyperbolic conservation laws on curved manifolds is established by Artzi and LeFloch [3], and LeFloch and Okutmustur [36], [37], which can be considered as a generalization of fundamental works by Kruzkov [30], Kuznetsov [31], and DiPerna [20] who dealed with equations on the flat Euclidian space. LeFloch and his collaborators developed this idea by generalizing the technique introduced in [14, 18] by Cockburn, Coquel and LeFloch for the Euclidean setting, it is extended to Riemannian manifolds in [1] by Amorim, Artzi, LeFloch and to Lorentzian manifolds in [2] by Amorim, LeFloch and Okutmustur. A generalization of the formulation and convergence of the finite volume method for general conservation law is established in [37] by LeFloch and Okutmustur. Moreover, related results and a comprehensive analysis about the convergence techniques for hyperbolic problems are referred in the articles by Tadmor [54] and Tadmor, Rascle, and Bagneiri [55]. For higher-order schemes, we cite Kröner, Noelle, and Rokyta [29].

Recently, by identifying a hyperbolic balance law via the Euler equations of relativistic compressible fluids, several relativistic and non-relativistic generalizations of the classical Burgers equation have been introduced in [38] by LeFloch, Makhlof and Okutmustur, where a geometric formulation of the finite volume method is constructed. The general formulation of nonlinear hyperbolic balance law is introduced by

$$\operatorname{div}(T(v)) = S(v), \tag{1.1}$$

on $n+1$ dimensional curved spacetime M with boundary where $\operatorname{div}(\cdot)$ represents the divergence operator, $v : M \to \mathbb{R}$ is the unknown function which is a scalar field, $T = T(v)$ is a flux vector field on M and $S = S(v)$ is a scalar field on M. The manifold M is assumed to be foliated by hypersurfaces

$$M = \bigcup_{t \geq 0} H_t, \tag{1.2}$$

such that each slice H_t is an n dimensional spacelike manifold and H_0 is the initial slice. Hence, the class of equations (1.1)–(1.2) provides a scalar model so that one can analyze numerical methods of approximations. In [38] LeFloch and his collaborators derived several models so called the relativistic Burgers equations by considering M to be a curved spcetime and many finite volume approximations are provided for the given models. In the concerning article, the manifold M is taken to be particularly as Minkowski (flat) and Schwarzshild spacetimes. For further details, one can check [38].

In this thesis, we apply the technique used in [38] to derive a new model for the Friedmann–Lemaître–Robertson–Walker (FLRW) background. We proved that the equation (1.1) is constructed to be the desired model so called the **relativistic Burgers equation**

$$a\, v_t + \big(1 - kr^2\big)^{1/2} \partial_r \Big(\frac{v^2}{2}\Big) + v\Big(1 - \frac{v^2}{c^2}\Big)\, a_t = 0,$$

on M where M is a FLRW spacetime, $a = a(t) > 0$ is a scale function, $k \in \{-1, 0, 1\}$ is a discrete parameter, and c is the light speed.

The formulation of finite volume method and numerical experiments are implemented based on this model.

1.1 Structure of the thesis

The thesis consists of five chapters and it is organized as follows:

Chapter 1, is introductory and we give brief information about the whole thesis in this part. Chapter 2 contains a summary of literature review about hyperbolic conservation laws, finite volume methods and spacetimes. In Chapter 3, the main contribution of the thesis is presented. Mainly, we derive the relativistic Burgers equation on FLRW background and examine the limiting properties of the concerning equation in details. In Chapter 4 several numerical experiments about the relativistic Burgers equation on FLRW background are illustrated. A conclusion of theoretical part (Chapter 3) and numerical part (Chapter 4) is presented in Chapter 5.

In the following, we state the content of each chapter.

Chapter 1 is devoted to recent works about the theory of hyperbolic conservation laws, finite volume methods for hyperbolic conservation laws on manifolds and the relativistic Burgers equations on curved spacetimes. References for these recent studies about the theory and numerical constructions are given for further details. Moreover, the organization of the whole thesis is included in this chapter.

The main objective of Chapter 2 is to give literature review of the theory for nonlinear hyperbolic systems of conservation laws, finite volume methods and spacetimes. In the first part, we start by giving definitions of hyperbolicity, conservation laws, weak solutions and shock speeds. In addition, a general formulation of finite volume methods for hyperbolic conservation and balance laws are mentioned. Next, we introduce the Godunov method for numerical experiments in order to solve Riemann problem for our main equation on each grid cell. In the second part of this chapter, we provide some preliminaries for spacetimes. Firstly, we give some basic features for spacetimes, such as events and spacetime intervals which are crucial notions to apprehend the structure of the spacetimes. In addition, in order to clarify the components of the Einstein field equations, definitions of Christoffel symbols and some types of tensors are handled. Finally, we consider basic features of Lorentzian manifold and three particular cases of this manifold; Minkovski, Schwarzschild and FLRW spacetime.

Chapter 3, which is the main part of this thesis, is devoted to obtain the relativistic Burgers equation on FLRW background. A relativistic generalization of the Burgers equations was proposed by LeFloch, Makhlof and Okutmustur [38] and investigated on a flat and Schwarzschild background. The relativistic and classical Burgers equations were obtained from derivation of the Euler system of relativistic compressible fluids by considering vanishing pressure on a curved background. This methodology yields a *geometric relativistic Burgers equation* on the background spacetime under consideration. Since both FLRW and Schwarzshild metrics are solutions for the Einstein field equations, this common property motivated us to derive a relativistic version of the Burgers equation on FLRW background as it is done on flat and Schwarzshild spacetime. Therefore, we extend their analysis to FLRW background by using the same methodology in [38]. The concerning equation on this spacetime is obtained from the Euler system of relativistic compressible fluids. Firstly, Christoffel symbols and tensors for perfect fluids on FLRW background are calculated. Next, by using divergence of the tensors, the Euler equations are obtained and we take the pressure to be zero on this Euler system. Finally, by combining these two equations we obtain the Burgers equation on FLRW background. The proposed relativistic Burgers equation on FLRW background retains several important features of the relativistic Euler equations. The unknown v of our equation lies in the interval $(-c, c)$ limited by the light speed parameter c, same as the velocity component in the Euler system. In the Euler system by sending the light speed to infinity one recovers the classical (non-relativistic) model. Similarly, for a particular case of our relativistic model ($a(t) = 1$ and $k = 0$), we obtain the classical model as c tends to infinity. In addition, we arrive to the classical Burgers equation, by using the special case of the concerning equation ($a(t) = 1$ and $k = 0$) without sending c to infinity. Obtaining the classical (inviscid) Burgers equation in this study is an expected result since the non-relativistic equation is derived from the relativistic one. After deriving the relativistic Burgers model under consideration, we determine its spatially homogeneous solutions. This equation is more challenging due to the existence of non-trivial spatially homogeneous solutions which distinguishes our study from the previous works. In addition, we investigate the limiting properties of the concerning equation and spatially homogeneous solutions in details. Finally, a finite volume methodology which is formulated for curved spacetimes in [37] and [38] is developed for FLRW background spacetime.

In Chapter 4, numerical techniques for solving the Burgers equation on FLRW background is presented. Due to having an extra term named scale factor which depends on time, applying numerical experiments to the concerning equation is harder compared to the Burgers equation on Schwarzshild background. This scale factor appears in the FLRW metric, in the model and in the scheme corresponding to this model. In order to solve Riemann problem between each grid cell, a Godunov type of scheme is constructed for the given model. A parameter, namely k, exists both in the metric and in the model which results exactly three equations to this model. Therefore, numerical experiments are applied for these three equations, separately. The scale factor produces some singularities in the scheme for each particular cases. In addition, the speed term of the concerning equation results some singularities in the scheme for $k = 1$. These drawbacks are solved by making transformations on time and domain. Numerical experiments demonstrate the efficiency of the proposed method to find solutions that may contain shock waves. The solution curves corresponding to the Burgers equation on FLRW background with initial shocks and rarefactions converge for three particular cases. This result shows the efficiency and robustness of our scheme. Moreover, the behaviors of spatially homogeneous solutions and the average of the numerical solutions on space are investigated. In order to make a comparison between spatially homogeneous solutions and numerical solutions for well-balanced schemes, we consider L_1 norm and a proportion of homogeneous solutions and numerical solutions. We observed that the solution curves converge to spatially homogeneous solutions as t tends to infinity. Furthermore, limiting properties of the concerning equation which are revealed in Chapter 3 are also observed by numerical experiments. To sum up, our numerical scheme is based on a finite volume technique, which is well-preserving in the sense that all spatially homogeneous solutions are preserved at the discrete level of approximation.

In Chapter 5, we summarize the results of this thesis theorically and numerically. We derive the relativistic Burgers equation on FLRW background spacetime which satisfies certain properties shared by all relativistic equations, such as starting with the Euler equations and arriving to the classical equations. In addition, we attain the corresponding homogeneous solutions to the concerning equation. The solution curves corresponding to all particular cases of the concerning equation with different initial functions converges, which means that our scheme is efficient, stable and robust. The constructed Godunov scheme is consistent with the conservative form of our model. Therefore, the scheme gives accurate calculations for weak solutions containing shock waves. The concerning scheme is well-balanced, since it preserves all spatially homogeneous solutions which is clearly illustrated in numerical experiments. As a perspective, following the technique used in this work, several versions of the relativistic Burgers equations can be attained for other spacetime backgrounds.

CHAPTER 2

PRELIMINARIES FOR HYPERBOLIC SYSTEMS OF CONSERVATION LAWS, FINITE VOLUME METHODS AND SPACETIMES

The main objective of this chapter is to provide a brief presentation for nonlinear hyperbolic systems of first-order partial differential equations in divergence form which is also called hyperbolic systems of conservation laws (Part I) and basic terms for spacetimes related to this thesis (Part II).

2.1 Part I: Hyperbolic Systems of Conservation Laws, Finite Volume Methods

2.1.1 Hyperbolicity and entropies

To begin with, we consider the systems of n conservation laws in one-space dimension,

$$\partial_t v + \partial_x f(v) = 0, \quad v(x,t) \in \mathcal{V}, \quad x \in \mathbb{R}, \quad t > 0, \tag{2.1}$$

where $f : \mathcal{V} \to \mathbb{R}^n$ is the **flux-function** which is a smooth mapping and $\mathcal{V}$ is an open and convex subset of $\mathbb{R}^n$. In (2.1) the dependent variable v is called the conservative quantity and the independent variables x and t are the spatial and time coordinates, respectively. In order to state the Cauchy problem of (2.1), we assign an initial condition for the conservative quantity,

$$v(x,0) = v_0(x), \quad x \in \mathbb{R} \quad \text{where} \quad v_0(x) : \mathbb{R} \to \mathcal{V}. \tag{2.2}$$

One can also observe that (2.1) is in divergence form. Integrating (2.1) over an interval $[x_1, x_2]$ yields

$$\frac{d}{dt}\int_{x_1}^{x_2} v(x,t)dx = \int_{x_1}^{x_2} v_t(x,t)dx = -\int_{x_1}^{x_2} f(v(x,t))_x dx, \tag{2.3}$$

which is equal to

$$f(v(x_1,t)) - f(v(x_2,t)) = [\text{inflow at } x_1] - [\text{outflow at } x_2]. \tag{2.4}$$

Each component of the vector v represents a quantity which is neither created nor destroyed, i.e. it is conserved. The total amount inside any given interval $[x_1, x_2]$ can change only because of the flow across the boundary points. The next definition takes into account the hyperbolic systems.

Definition 2.1.1 ([34]) *A system defined by* (2.1) *is called a first-order, hyperbolic system of partial differential equations if the corresponding Jacobian matrix* $J(v) := Df(v)$ *has* n *real eigenvalues with the following property*

$$\lambda_1(v) \leq \lambda_2(v) \leq ... \leq \lambda_n(v), \quad v \in \mathcal{V},$$

where the basis of right-eigenvectors is $\{r_j(v)\}_{1\leq j\leq n}$.

We call the eigenvalues as wave speeds or ***characteristic speeds*** *related with* (2.1). *Moreover, this system is said to be strictly hyperbolic if all of the eigenvalues are distinct,*

$$\lambda_1(v) < \lambda_2(v) < ... < \lambda_n(v), \quad v \in \mathcal{V}.$$

Definition implies that $Df(v)r_j(v) = \lambda_j(v)r_j(v)$ where (λ_j, r_j) is the **j-characteristic field**. In addition, $v \mapsto \lambda_j(v)r_j(v)$ is a smooth mapping. In strictly hyperbolic systems we have

$$l_i(v)r_j(v) \equiv \delta_j^i,$$

where δ_j^i is Kronecker's delta function and $\{l_j(v)\}_{1\leq j\leq n}$ is a basis of left-eigenvectors. For $n = 1$, there exists only one eigenvalue $\lambda_1(v) = f'(v)$ and $r_1 = l_1 = 1$.

Definition 2.1.2 ([34]) *A j-characteristic field of* (2.1) *is* ***genuinely nonlinear*** *if the characteristic field satisfies the following condition*

$$\nabla\lambda_j(v).r_j(v) \neq 0, \quad v \in \mathcal{V},$$

*and it is **linearly degenerate** if*

$$\nabla\lambda_j(v).r_j(v) \equiv 0, \quad v \in \mathcal{V},$$

for $j = 1, ..., n$.

In this definition, the equation (2.1) is genuinely nonlinear if and only if $f''(v) \neq 0$ for all v. In addition, it is linearly degenerate if and only if $f''(v) = 0$ for all v.

Definition 2.1.3 ([34]) *If a differentiable solution of* (2.1) *admits the additional conservation law*

$$\partial_t V(v) + \partial_x F(v) = 0,$$

the continuously differentiable function $(V, F) : \mathcal{V} \to \mathbb{R}^2$ *is named as an **entropy pair**. The function* V *is the entropy and the function* F *is the corresponding entropy-flux.*

(V, F) is an entropy pair for (2.1) if and only if

$$\nabla F(v)^T = \nabla U(v)^T Df(v) = 0, \quad v \in \mathcal{V}.$$

After differentiation with respect to v, we attain

$$D^2\, F(v) = D^2\, V(v) D\, f(v) + \nabla V(v)^T D^2 f(u).$$

The matrices $D^2\, F(v)$ and $\nabla V(v)^T D^2 f(u)$ are symmetric and this means that the matrix $D^2\, V(v)D\, f(v)$ has to be also symmetric. This fact yields a useful criterion for the existence of an entropy, given in the following theorem which gives an identification for mathematical entropies.

Theorem 2.1.1 ([34]) *A continuously differentiable function* V *is called an entropy if and only if*

$$D^2\, V(v) D\, f(v) \quad \textit{is an} \quad n \times n \quad \textit{and symmetric matrix,}$$

which gives a second order linear system of $n(n-1)/2$ *partial differential equations.*

We define an entropy pair

$$V(v) = v_j, \quad F(v) = f_j(v), \quad v \in \mathcal{V},$$

where $v = (v_1, v_2, ..., v_n)^T$ and $f(v) = (f_1(v), ..., f_n(v))^T$ for all $j \in \{1, ..., n\}$. $D^2 V(v)$ is a positive definite symmetric matrix for the strictly convex entropies which results to the following identity

$$V(v) - V(u) - \nabla V(u).(v - u) > 0, \quad v \neq u \quad \text{in} \quad \mathcal{V}.$$

We consider a particular case for (2.1); strictly hyperbolic scalar equations in one space dimension ($n = 1$). The following definition states the structure of scalar conservation laws.

Definition 2.1.4 *The partial differential equation*

$$\partial_t v + \partial_x f(v) = 0, \tag{2.5}$$

is a scalar conservation law with the initial condition,

$$v(x, 0) = v_0(x), \quad x \in \mathbb{R},$$

where $v_0 : \mathbb{R} \to \mathcal{V}$ *is given.*

Example 2.1.1 *For a particular case of* (2.1) *with the flux function* $f(v) = v^2/2$ *and* $n = 1$*, we have the inviscid (classical) Burgers equation*

$$\partial_t v + \partial_x (v^2/2) = 0, \tag{2.6}$$

where $v = v(t, x)$*,* $t > 0$ *and* $x \in \mathbb{R}$*.*

The classical Burgers equation is one of the simplest example for nonlinear conservation laws. In general, a scalar conservation law defined in (2.5), that is a conservation law in one variable is always hyperbolic. The classical Burgers equation is a fundamental partial differential equation in fluid mechanics and arises in various areas of applied mathematics, such as modeling of gas dynamics and traffic flow. One can easily observe that the wave speed in (2.6), $f'(v) = v$ directly depends on v. Another observation is that (2.6) is also a basic example for genuinely nonlinear equations.

Definition 2.1.5 *Hyperbolic systems of balance laws under consideration reads*

$$\partial_t v + \partial_x f(v) = S(x, v), \quad v(x,t) \in \mathcal{V}, \quad x \in \mathbb{R}, \quad t > 0, \tag{2.7}$$

*where $f : \mathcal{V} \to \mathbb{R}^n$ is the **flux-function** which is a smooth mapping and $\mathcal{V}$ is an open and convex subset of $\mathbb{R}^n$. In* (2.7) *$S(x, v)$ is the **source term** resulting from geometrical and physical effects and the independent variables x and t are the space and time coordinates, respectively. As in the definition of hyperbolic systems* (2.1.1) *the Jacobian matrix $J(v) := Df(v)$ admits n real eigenvalues*

$$\lambda_1(v) \leq \lambda_2(v) \leq ... \leq \lambda_n(v), \quad v \in \mathcal{V},$$

and the system is said to be strictly hyperbolic if its eigenvalues are distinct,

$$\lambda_1(v) < \lambda_2(v) < ... < \lambda_n(v), \quad v \in \mathcal{V}.$$

In the rest of this part, we investigate the properties of weak solutions for the systems of conservation laws.

2.1.2 Weak solutions and shock speeds

In this subsection, we give the notion of a solution v to a partial differential equation, where v may not even be a differentiable function. This type of solutions are known as weak solutions for partial differential equations. A strong solution does not always exist, or the solutions may not be differentiable, or even continuous. In the following definition, we explicitly give the concept of weak solutions.

Definition 2.1.6 ([34]) *$v \in L^\infty(\mathbb{R} \times \mathbb{R}^+, \mathcal{V})$ is a **weak solution** for the Cauchy problem* (2.1) *and* (2.2) *if the following condition*

$$\int_0^\infty \int_{\mathbb{R}} (v\partial_t\theta + f(v)\partial_x\theta) dx dt + \int_{\mathbb{R}} \theta(0) v_0 dx = 0, \tag{2.8}$$

is satisfied for every function $\theta \in C_c^\infty(\mathbb{R} \times [0, \infty))$ with the initial data $v_0 \in L^\infty(\mathbb{R}, \mathcal{V})$. The function θ is a member of the vector space of real-valued, infinitely differentiable, compactly supported functions.

Weak solutions satisfy both the differential equation where the solution is smooth and the jump conditions at discontinuities. We notice that if v is a strong solution of (2.1) and (2.2), then v is a weak solution of (2.1) and (2.2) but the reverse is obviously not true. It is a known fact that weak solutions for conservation laws are generally not unique. As we noted before, the notion of weak solution permits solutions which even need not be continuous. Nonetheless, weak solutions have some constraints depending on the sort of discontinuities. For instance, assume that v is a weak solution such that it is discontinuous across some curve $x = \xi(t)$, but smooth on both side of this curve. Let $v^-(x,t)$ be the limit of v approaching (x,t) from the left-hand side and let $v^+(x,t)$ be the limit of v approaching (x,t) from the right-hand side. We assert that the curve $x = \xi(t)$ is not random, but there is a relation between $x = \xi(t)$, v^- and v^+. The following theorem explains this phenomena.

Theorem 2.1.2 ***(Rankine-Hugoniot jump condition)*** *A solution v which is discontinuous across the curve $x = \xi(t)$ but smooth on either side of $x = \xi(t)$ satisfies the condition*

$$\frac{f(v^-) - f(v^+)}{v^- - v^+} = \xi'(t), \tag{2.9}$$

across the curve of discontinuity, where $v^-(x,t)$ is the limit of v approaching (x,t) from the left-hand side and $v^+(x,t)$ is the limit of v approaching (x,t) from the right-hand side.

It is concluded that a discontinuity in the weak solutions satisfies the property

$$[f(v)] = s[v], \tag{2.10}$$

where $s = \xi'(t)$ is the **speed** of the discontinuity along a curve $x = \xi(t)$ and

$$[f(v)] = f(v^-) - f(v^+), \quad [v] = v^- - v^+$$

are the **jumps** of $f(v)$ and v across the curve $x = \xi(t)$, respectively. The relation $f'(v^-) > s > f'(v^+)$ is called the **entropy condition** where $f'(v) = dx/dt$ is the speed of a solution v. We say that a curve of discontinuity is a **shock curve** for a solution v if the curve satisfies the Rankine-Hugoniot jump condition and the entropy condition for that solution v. Moreover, if a weak solution satisfies the entropy condition on each discontinuity, then the solution is called **admissible**.

Now, it is time to consider the **Riemann problem** which is an important Cauchy problem of (2.1) and (2.2) with the piecewisely defined initial function,

$$v(x,0) = v_0(x) = \begin{cases} v_l, & x < 0, \\ v_r, & x > 0, \end{cases} \tag{2.11}$$

where $v_l, v_r \in \mathcal{V}$ are given constants. It is a known fact that a rarefaction or shock wave will be generated. For a scalar convex equation with $f'(v)$ is increasing, the solution of the Riemann problem between v_l and v_r is a shock with the speed s if $v_l > v_r$ or a rarefaction if $v_l < v_r$ which is bounded by $f'(v_l)$ on the left-hand side and by $f'(v_r)$ on the right-hand side. The solution along $\frac{x}{t} = 0$ is either v_l a rarefaction or shock moving entirely to right or v_r a rarefaction or shock moving entirely to left. The solution has a different value if $v_l < v_s < v_r$ where v_s is the point satisfying $f'(v_s) = 0$. We call this point as the **sonic** point.

2.2 General formulation of finite volume methods for hyperbolic conservation laws

The finite volume method is based on splitting the spatial domain into **finite volumes** and preserving an approximation to the integral of v, the dependent variable of the system (2.1), over each of these volumes. In every time step these approximated values are updated by using approximations of the flux in the endpoints of the grid cells. The value V_i^n is the approximation of $v(x_i, t^n)$ at time t^n,

$$V_i^n \approx \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} v(x,t^n)dx \equiv \frac{1}{\Delta x} \int_{C_i} v(x,t^n)dx, \tag{2.12}$$

where $C_i = (x_{i-1/2}, x_{i+1/2})$ is the $i'th$ grid cell and $\Delta x = x_{i+1/2} - x_{i-1/2}$. We obtain the integral form of the conservation law (2.1) as

$$\frac{d}{dt} \int_{C_i} v(x,t)dx = f(v(x_{i-1/2},t)) - f(v(x_{i+1/2},t)). \tag{2.13}$$

We object to find V_i^{n+1} by given V_i^n, which is the cell averages at time t^n. After integrating (2.13) we obtain

$$\int_{C_i} v(x,t^{n+1})dx - \int_{C_i} v(x,t^n)dx = \int_{t^n}^{t^{n+1}} f(v(x_{i-1/2},t))dt - \int_{t^n}^{t^{n+1}} f(v(x_{i+1/2},t))dt. \tag{2.14}$$

Dividing (2.14) by Δx and organizing the terms yields

$$\begin{aligned}\frac{1}{\Delta x}\int_{C_i} v(x,t^{n+1})dx =& \frac{1}{\Delta x}\int_{C_i} v(x,t^n)dx \\ &-\frac{1}{\Delta x}\left(\int_{t^n}^{t^{n+1}} f(v(x_{i+1/2},t))dt - \int_{t^n}^{t^{n+1}} f(v(x_{i-1/2},t))dt\right).\end{aligned} \tag{2.15}$$

This is the formulation of updating v in each time step. After introducing the approximation for the flux along $x = x_{i+1/2}$,

$$F^n_{i+1/2} \approx \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} f(v(x_{i+1/2},t))dt, \tag{2.16}$$

we obtain the following numerical approximation

$$V_i^{n+1} = V_i^n - \frac{\Delta t}{\Delta x}(\mathcal{F}(V_i^n, V_{i+1}^n) - \mathcal{F}(V_{i-1}^n, V_i^n)), \tag{2.17}$$

where

$$F^n_{i+1/2} = \mathcal{F}(V_i^n, V_{i+1}^n). \tag{2.18}$$

The particular method depends on the formulation of $\mathcal{F}$ and in this method V_i^{n+1} is obtained by using V_{i-1}^n, V_i^n and V_{i+1}^n from the previous time level. In addition, it can be clearly observed that the method (2.17) is in conservation form, since it comes from the property (2.15) of the exact solution.

2.3 General formulation of finite volume methods for hyperbolic balance laws

Following (2.17), we construct a finite volume formulation by

$$V_i^{n+1} = V_i^n - \frac{\Delta t}{\Delta x}(\mathcal{F}(V_i^n, V_{i+1}^n) - \mathcal{F}(V_{i-1}^n, V_i^n)) + \Delta t S_i^n, \tag{2.19}$$

for the hyperbolic balance law (2.7) that is obtained in a similar way to the previous section. In the method (2.19) the term S_i^n is an approximation for the source term

$$S_i^n \approx \frac{1}{\Delta t \Delta x}\int_{[t^n,t^{n+1}]\times C_i} S(x,v)dtdx, \tag{2.20}$$

where $C_i = (x_{i-1/2}, x_{i+1/2})$ is the $i'th$ grid cell.

In Chapter (3), the main part of this thesis, we will deal with a more challenging finite volume approximation for curved spacetimes and in Chapter (4), a Godunov type of scheme will be used for numerical experiments. Hence, in the following we introduce briefly the Godunov method.

2.4 Godunov method

Godunov method helps to solve the Riemann problem forward in time for each grid cell. Solutions of Riemann problem give important details about the characteristic structure and yield conservative methods since they are themselves exact solutions of the conservation laws. In this method, we use the numerical solution in order to define a piecewise function $\tilde{v}(x, t^n)$ with the value V_i^n on the grid cell $C_i = (x_{i-1/2}, x_{i+1/2})$. The equation (2.1) is solved exactly over the small time interval $t^n \leq t \leq t^{n+1}$ and since the initial data $\tilde{v}(x, t^n)$ is piecewisely constant we obtain a sequence of Riemann problems. The exact solution is attained by simply connecting these Riemann solutions. We define the approximate solution V_i^{n+1} at time t^{n+1} by averaging the exact solution at time t^{n+1},

$$V_i^{n+1} = \frac{1}{\Delta x} \int_{C_i} \tilde{v}(x, t^{n+1}) dx. \tag{2.21}$$

We repeat this process in order to find the new piecewise constant data $\tilde{v}^{n+1}(x, t^{n+1})$ by using (2.21). Since $\tilde{v}^n$ is considered as an exact weak solution it satisfies (2.14) and we have

$$\begin{aligned} \int_{C_i} \tilde{v}^n(x, t^{n+1}) dx = \int_{C_i} \tilde{v}^n(x, t^n) dx + \int_{t^n}^{t^{n+1}} f(\tilde{v}^n(x_{i-1/2}, t)) dt \\ - \int_{t^n}^{t^{n+1}} f(\tilde{v}^n(x_{i+1/2}, t)) dt. \end{aligned} \tag{2.22}$$

By dividing this equation with Δx and using (2.21) we attain

$$V_i^{n+1} = V_i^n - \frac{\Delta t}{\Delta x} (\mathcal{F}(V_i^n, V_{i+1}^n) - \mathcal{F}(V_{i-1}^n, V_i^n)), \tag{2.23}$$

where $\tilde{v}^n(x, t^n) \equiv V_i^n$ over the cell C_i. The flux function $\mathcal{F}$ in (2.23) is given as

$$\mathcal{F}(V_{i-1}^n, V_i^n) = \frac{1}{\Delta x} \int_{t^n}^{t^{n+1}} f(\tilde{v}^n(x_{i-1/2}, t)) dt. \tag{2.24}$$

The integral in (2.24) is easy to compute since $\tilde{v}^n$ is constant at the point $x_{i+1/2}$ over the time interval $t^n \leq t \leq t^{n+1}$. This constant value of $\tilde{v}^n$ along $x = x_{i+1/2}$ depends on both V_i^n and V_{i+1}^n for this Riemann problem. In a different notation, the method (2.23) can be written as

$$V_i^{n+1} = V_i^n - \frac{\Delta t}{\Delta x}(f(u^*(V_i^n, V_{i+1}^n)) - f(u^*(V_{i-1}^n, V_i^n))) \tag{2.25}$$

where the flux reduces to

$$\mathcal{F}(V_{i-1}^n, V_i^n) = f(v^*(V_{i-1}^n, V_i^n)). \tag{2.26}$$

If $V_i^n = V_{i+1}^n = \overline{v}$ then $v^*(V_i^n, V_{i+1}^n) = \overline{v}$ and this property yields that the flux (2.26) is consistent with f.

In general for the convex case, the choice of $v^*(v_l, v_r)$ is formulated by the following

- if $f'(v_l), f'(v_r) \geq 0$ then $v^*(v_l, v_r) = v_l$,
- if $f'(v_l), f'(v_r) \leq 0$ then $v^*(v_l, v_r) = v_r$,
- if $f'(v_l) \geq 0 \geq f'(v_r)$, and $[f]/[v] > 0$ then $v^*(v_l, v_r) = v_l$,
- if $f'(v_l) \geq 0 \geq f'(v_r)$, and $[f]/[v] < 0$ then $v^*(v_l, v_r) = v_r$,
- if $f'(v_l) < 0 < f'(v_r)$ then $v^*(v_l, v_r) = v_s$ where v_s is the sonic point.

Finally, in order to satisfy the stability condition in the method, we require the following condition

$$\left| \frac{\Delta t}{\Delta x} \lambda_j(V_i^n) \right| \leq 1, \tag{2.27}$$

for any eigenvalue λ_j of each V_i^n. The greatest value of this quantity over all the values of v occurring in a particular problem is named as **Cournat Friedrichs Levy (CFL)** condition.

2.5 Part II: Spacetimes

The notion of spacetime gets together 'space' and 'time' to a single abstract universe. In general, an $n+1$ dimensional spacetime requires n space dimensions and one time dimension. An event in a spacetime does not sign just points in space, time is added as another dimension by which one can understand where and when the events occur. In spacetimes, different from usual spatial coordinates there exist restrictions in how measurements are made spatially and temporally. These restrictions yield a special model that differs from the Euclidean space in its evident symmetry.

2.5.1 Basic Concepts

In the following two definitions, we focus on a particular case of $n + 1$ dimensional spacetime, that is $n = 3$.

Definition 2.5.1 *An individual moment in* $3 + 1$ *dimensional spacetime, is described as an* ***event*** *which is uniquely determined by* (t, r, θ, φ) *where* t *and* (r, θ, φ) *are time and spatial coordinates, respectively.*

Compared with the Euclidean space, it is not easy to sketch an event in $n + 1$ dimensional spacetime for $(n \geq 3)$. However, in $2 + 1$ dimension, the structure of spacetime and the sketch of an event to a spacetime is simpler for this case. Figure (2.1) illustrates a simple sketch for an event.

Definition 2.5.2 *In* $3 + 1$ *dimension, a* ***spacetime interval*** *between two events is defined as*

$$ds^2 = -c^2 dt^2 + dr^2 + d\theta^2 + d\varphi^2,$$

where c *is a fixed velocity term between space and time,* t *and* (r, θ, φ) *are again time and spatial coordinates, respectively. In our later discussion* c *will be turn into to the light speed, and this speed is invariant under change of coordinates.*

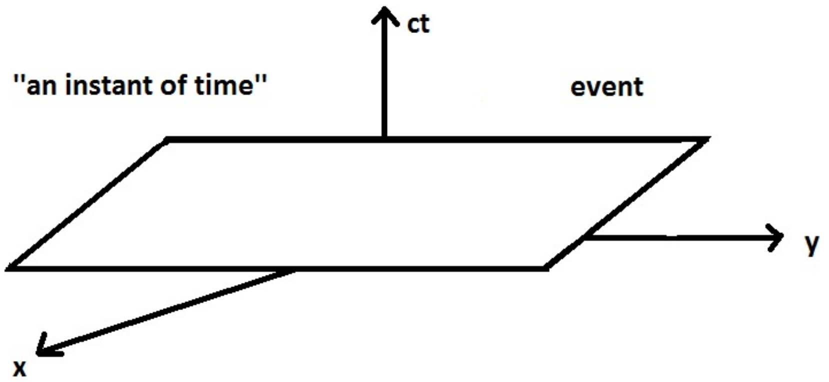

Figure 2.1: A simple sketch of an event on $2 + 1$ dimensional spacetime

2.6 Lorentzian Manifolds

Since our objective is to obtain the relativistic Burgers equation on a special curved spacetime, we need some basic features from general relativity. The spacetime under consideration is a Lorentzian manifold. To distinguish Lorentzian manifolds from n dimensional Euclidean space, we introduce $n + 1$ dimensional (n refers space and 1 refers time) time-oriented, Lorentzian manifold (M, g) where g is a metric with the sign $(-, +, ..., +)$ and we recall that tangent vectors $X \in T_pM$ at a point $p \in M$ can be sorted as

$$\begin{cases} \text{timelike vectors}, \quad g(X, X) < 0, \\ \text{null vectors}, \quad g(X, X) = 0, \\ \text{spacelike vectors}, \; g(X, X) > 0. \end{cases} \tag{2.28}$$

A simple sketch of these vectors are illustrated in Figure (2.2).

In the following, we consider three particular cases of Lorentzian manifolds with $n+1$ dimension, namely Minkowski, Schwarzschild and FLRW backgrounds.

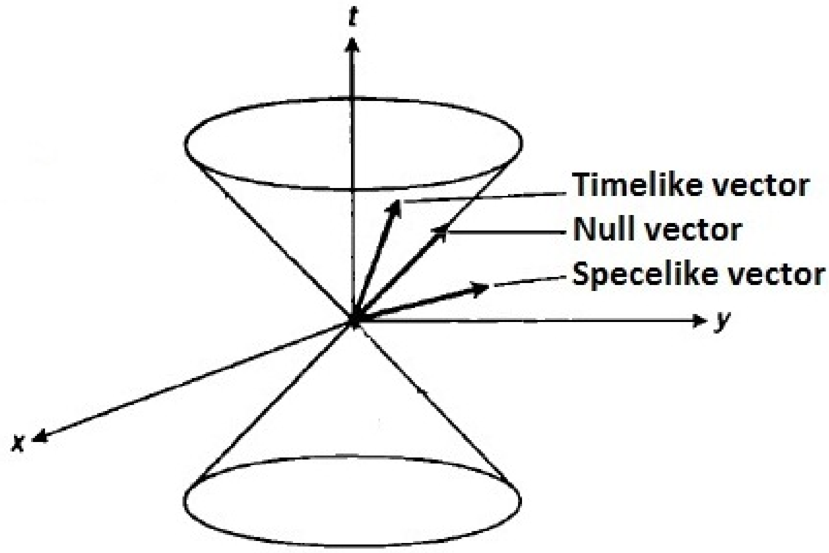

Figure 2.2: Timelike, null and spacelike vectors.

2.6.1 Minkowski Spacetime

A $3+1$ dimensional Minkowski spacetime is nondegenerate and symmetric manifold endowed with a flat metric. The elements of this background are 'events' which are defined as in (2.5.1) and are thought as actual or physically possible point-events. This spacetime is also called as 'flat' spacetime. There exists a special coordinate system containing the whole manifold where the metric is diagonal. The special coordinate system is named as Minkowski coordinate system and is given by

$$x^a = (x^0, x^1, x^2, x^3) = (t, x, y, z).$$

The coordinates $(x^0, x^1, x^2) = (x, y, z)$ and $(x^0) = (t)$ are spatial and time components, respectively, which are supplied by the observer who directs the reference frame.

The line element (metric of Minkowski spacetime) is given as

$$g = -c^2dt^2 + dx^2 + dy^2 + dz^2,$$

and in the matrix representation

$$g = g_{ij}dx^i dx^j = (dt\, dx\, dy\, dz) \begin{pmatrix} -c^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} dt \\ dx \\ dy \\ dz \end{pmatrix},$$

where c is the light speed. Furthermore, in usual spherical coordinates the length element of Minkowski metric is given as

$$g = -c^2 dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2.$$

2.6.2 Schwarzschild geometry

The Schwarzschild background spacetime describes the gravitational field of the Earth and defines a spherically symmetric black hole solution to the Einstein equations in suitably chosen coordinates (ct, r, θ, φ). This spacetime background is represented by Schwarzschild metric

$$g = -\Big(1 - \frac{2m}{r}\Big)c^2 dt^2 + \Big(1 - \frac{2m}{r}\Big)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2),$$

for $t > 0$, c is the light speed, $r > 2m$ and $m > 0$ is the mass parameter. Here r, θ, φ are the usual spherical coordinates. In addition, the metric in matrix form to this background is represented in $3+1$ dimension as

$$g = g_{ij}dx^i dx^j = (dt\, dr\, d\theta\, d\varphi) \begin{pmatrix} -\Big(1 - \frac{2m}{r}\Big)c^2 & 0 & 0 & 0 \\ 0 & \Big(1 - \frac{2m}{r}\Big)^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix} \begin{pmatrix} dt \\ dr \\ d\theta \\ d\varphi \end{pmatrix}.$$

The solution holds for this background outside the body of mass m. There exists a singularity at $r = 2m$ and we can get rid of this singularity by using the Eddington Finkelstein coordinates. It can be clearly observed that, for the case $m = 0$, the Schwarzschild metric reduces to the Minkowski metric. In addition, as $r \to \infty$ Schwarzschild background approaches Minkowski spacetime.

2.6.3 Friedmann–Lemaître–Robertson–Walker background

Friedmann–Lemaitre–Robertson–Walker (FLRW) model is one of the most investigated cosmological model in physics and mathematics. In this model, the metric can be written as

$$g = -c^2dt^2 + a(t)^2\Big(\frac{dr^2}{1-kr^2} + r^2d\theta^2 + r^2\sin^2\theta d\varphi^2\Big), \tag{2.29}$$

where t is the time, c is the light speed, k is the curvature, $a(t)$ is the cosmic expansion factor and (r, θ, φ) are the spherical coordinates. The constant parameter k which takes exactly three values $\{-1, 0, 1\}$, determines the curvature of the spacetime as follows

- $k = -1$, space geometry at constant time is a negatively curved, 3-dimensional ”pseudo-sphere”; the space is infinite.
- $k = 0$, space geometry at constant time is Euclidean,”flat space”; the space is infinite.
- $k = 1$, space geometry at constant time is a 3-sphere, which is positively curved; the total volume of the universe is finite.

The matrix form of this metric for $3 + 1$ dimensional background is given as

$$g = g_{ij}dx^idx^j = (dt\,dr\,d\theta\,d\varphi)\begin{pmatrix} -c^2 & 0 & 0 & 0 \\ 0 & \dfrac{a^2}{1-kr^2} & 0 & 0 \\ 0 & 0 & a^2r^2 & 0 \\ 0 & 0 & 0 & a^2r^2\sin^2\theta \end{pmatrix}\begin{pmatrix} dt \\ dr \\ d\theta \\ d\varphi \end{pmatrix}.$$

As a remark, FLRW metric is a solution for the Einstein equations. If the Einstein equations are solved for the FLRW metric given by (2.29), the system of equations reduces to a double equation, called **Friedmann equations**

$$\begin{aligned} \Big(\frac{\dot{a}(t)}{a(t)}\Big)^2 &= \frac{8\pi G}{3}\rho + \frac{\Lambda}{3} - \frac{k}{a(t)^2}, \\ \Big(\frac{\ddot{a}(t)}{a(t)}\Big)^2 &= \frac{4\pi G}{3}(\rho + 3p), \end{aligned} \tag{2.30}$$

where G is the gravitational constant, ρ is the density, Λ is the cosmological constant, k is the curvature and p is the pressure. In these equations $\dot{a}(t)$ and $\ddot{a}(t)$ are the first and second derivatives of the cosmic expansion factor $a(t)$. The first equation of (2.30) is related with the expansion rate $\dot{a}/a$ and the second one is related with the acceleration $\ddot{a}/a$.

In the rest of this chapter, we give some basic definitions in order to clarify the terms of the Einstein field equations.

Definition 2.6.1 *The **Christoffel symbols** for a given metric are defined by*

$$\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2} g^{\mu\nu}(-\partial_{\nu} g_{\alpha\beta} + \partial_{\beta} g_{\alpha\nu} + \partial_{\alpha} g_{\beta\nu}), \qquad \alpha, \beta, \mu, \nu \in \{0, 1, 2, 3\}, \tag{2.31}$$

where $g^{\mu\nu}$ is the matrix inverse of the inverse metric $g_{\mu\nu}$.

The Christoffel symbols play a great role in differential geometry by explaining how to define parallelism between neighboring points.

Definition 2.6.2 *The **Riemann curvature tensor** is defined by*

$$\mathcal{R}^{\alpha}_{\beta\mu\nu} = \partial_{\mu}\Gamma^{\alpha}_{\beta\nu} - \partial_{\nu}\Gamma^{\alpha}_{\beta\mu} + \Gamma^{\gamma}_{\beta\nu}\Gamma^{\alpha}_{\gamma\mu} - \Gamma^{\gamma}_{\beta\mu}\Gamma^{\alpha}_{\gamma\nu}, \tag{2.32}$$

where $\Gamma^{\alpha}_{\beta\nu}$ is the Christoffel symbols computed by the help of (2.31).

The Riemann curvature tensor plays a crucial role in specifying the geometrical features of a spacetime. It is observed that Riemann curvature tensor is antisymmetric with respect to interchange of the last two lower indices,

$$\mathcal{R}^{\alpha}_{\beta\mu\nu} = \mathcal{R}^{\alpha}_{\beta\nu\mu}. \tag{2.33}$$

Moreover, it is easy to prove that the Riemann curvature tensor satisfies the following property

$$\mathcal{R}^{\alpha}_{\beta\mu\nu} + \mathcal{R}^{\alpha}_{\nu\beta\mu} + \mathcal{R}^{\alpha}_{\mu\nu\beta} = 0. \tag{2.34}$$

Definition 2.6.3 *The **Ricci tensor** is defined by*

$$R_{\alpha\beta} = \mathcal{R}^{\gamma}_{\alpha\gamma\beta}, \tag{2.35}$$

where $\mathcal{R}^{\gamma}_{\alpha\gamma\beta}$ is described in (2.32)*. Furthermore, by using* (2.35) *the formulation of* ***Ricci scalar*** *can be given as*

$$R = g^{\alpha\beta} R_{\alpha\beta}. \tag{2.36}$$

The Ricci scalar and the Ricci tensor is prevalently used in cosmology and general relativity.

Definition 2.6.4 *The* ***Einstein tensor*** *is defined as*

$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R, \tag{2.37}$$

where $R_{\alpha\beta}$ is the Ricci tensor and R is the Ricci scalar defined in (2.35) *and* (2.36)*, respectively.*

The following definition is about the energy momentum tensors of perfect fluids. Perfect fluid is a fluid that has no viscosity or heat conduction. It is fully parameterized by its density ρ and the pressure p.

Definition 2.6.5 *The* ***energy momentum tensors of perfect fluids*** *are calculated by the formula*

$$T^{\alpha\beta} = (\rho c^2 + p)\, u^{\alpha}\, u^{\beta} + p\, g^{\alpha\beta}, \quad \alpha, \beta \in \{0, 1, 2, 3\}, \tag{2.38}$$

where ρ is density, p is pressure, $(u^{\alpha}) = (u^0(x,t), u^1(x,t), u^2(x,t), u^3(x,t))$ is a unit vector, c is the light speed, and $g^{\alpha\beta}$ is determined through the metric.

Finally, we end up this chapter by the definition of the Einstein field equations.

Definition 2.6.6 *The* ***Einstein field equations*** *is given by the relation*

$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R, \tag{2.39}$$

where $G_{\alpha\beta}$ is the Einstein tensor, $R_{\alpha\beta}$ is the Ricci tensor and R is the Ricci scalar which are calculated by the help of the formulas (2.37)*,* (2.35) *and* (2.36)*, respectively.*

CHAPTER 3

THE RELATIVISTIC BURGERS EQUATION ON A FRIEDMANN–LEMAÎTRE–ROBERTSON–WALKER (FLRW) BACKGROUND

3.1 Introduction

The inviscid Burgers equation is an important model in computational fluid dynamics, and represents the simplest (yet challenging) example of a nonlinear hyperbolic conservation law. Recently, several relativistic and non-relativistic generalizations of the classical Burgers equation have been introduced by LeFloch and collaborators [2, 36, 37, 38], which also take into account geometrical effects. In particular, the fundamental *relativistic Burgers equation* was derived by identifying a hyperbolic balance law which satisfies the same Lorentz invariance property as the one satisfied by the Euler equations of relativistic compressible fluids. The relativistic generalization of this model was studied on both a flat background and a Schwarzschild background. A numerical scheme was developed by using the finite volume methodology and allowed to capture discontinuous solutions containing shock waves for the relativistic Burgers equation.

Specifically, we will work on Friedmann–Lemaître–Robertson–Walker (FLRW) background, which is an important solution to Einstein field equations relevant to cosmology. (See for instance [27] for background material.) The main purpose of the thesis is to discuss the relativistic Burgers equation on a FLRW background and to design a finite volume scheme for its approximation by closely following LeFloch, Makhlof, and Okutmustur [38].

In the present study, we continue this analysis and introduce the class of *relativistic Burgers equation on a curved background*, derived as follows. We start from the relativistic Euler equations on a curved background (M, g) (that is, a smooth, time-oriented Lorentzian manifold), which read

$$\begin{aligned} \nabla_\alpha T^{\alpha\beta} &= 0, \\ T^{\alpha\beta} &= (\rho c^2 + p)\, u^\alpha u^\beta + p\, g^{\alpha\beta}, \end{aligned} \tag{3.1}$$

where $T^{\alpha\beta}$ is the so-called energy-momentum tensor for perfect fluids. Here, $\rho \geq 0$ denotes the mass-energy density of the fluid, while the future-oriented, unit timelike vector field $u = (u^\alpha)$ represents the velocity of the fluid and $g_{\alpha\beta}\, u^\alpha u^\beta = -1$.

As usual, the model (3.1) must be supplemented with an equation of state for the pressure $p = p(\rho)$. In the present work, we assume that the fluid is pressureless, that is, $p \equiv 0$, so that the Euler system takes the simpler form

$$\nabla_\alpha \big(\rho\, u^\alpha u^\beta\big) = 0. \tag{3.2}$$

Provided $\rho > 0$ and ρ, u are sufficiently regular and observing that $g_{\alpha\beta} \nabla_\alpha u^\alpha u^\beta = 0$ (that is, u is orthogonal to ∇_u), we arrive at

$$\rho \nabla_\alpha u^\alpha u^\beta + \rho u^\alpha \nabla_\alpha u^\beta + u^\alpha u^\beta \nabla_\alpha \rho = 0.$$

By contracting this equation with the covector u_β, we get

$$u^\alpha \nabla_\alpha \rho = -\rho \nabla_\alpha u^\alpha,$$

which gives us

$$\rho u^\beta \nabla_\alpha u^\alpha + \rho (u^\alpha \nabla_\alpha u^\beta - u^\beta \nabla_\alpha u^\alpha) = 0.$$

Provided $\rho > 0$, it thus follows that

$$u^\alpha \nabla_\alpha u^\beta = 0, \tag{3.3}$$

which is the **geometric relativistic Burgers equation**, that is the focus of this thesis.

3.1.1 The relativistic Burgers equations on curved backgrounds

Let us first summarize the results obtained by LeFloch, Makhlof, and Okutmustur [38] for Minkowski (flat) and Schwarzshild spacetimes. The inviscid Burgers equation is one of the simplest example of nonlinear hyperbolic conservation laws, and reads

$$\partial_t v + \partial_x(v^2/2) = 0, \tag{3.4}$$

where $v = v(t, x)$, $t > 0$ and $x \in \mathbb{R}$. This equation can be formally deduced from the Euler system of compressible fluids

$$\begin{aligned}
&\partial_t \rho + \partial_x(\rho v) = 0, \\
&\partial_t(\rho v) + \partial_x(\rho v^2 + p(\rho)) = 0,
\end{aligned}$$

where $\rho \geq 0$ denotes the density, v the velocity, and $p(\rho)$ the pressure of the fluid. By taking $p(\rho) \equiv 0$ and keeping a suitable combination of the two equations, we find (3.4).

The relativistic Burgers equation on flat spacetime can be derived in several ways, either by imposing the Lorentz invariance property or formally from the Euler system on a curved background. As discussed in [38], the relativistic Burgers equation on flat background is

$$\partial_t v + \partial_r\big(1/\epsilon^2\big(-1 + \sqrt{1 + \epsilon^2 v^2}\big)\big) = 0, \tag{3.5}$$

where ϵ is the inverse of the light speed parameter and the length element of Minkowski metric in spherical coordinates (t, r, θ, φ), is given by

$$g = -c^2 dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2,$$

where c is the light speed. On the other hand, starting from the Euler system for relativistic compressible fluids and imposing vanishing pressure, we arrive at the following version of the non-relativistic and relativistic Burgers equations on Schwarzshild spacetime:

$$\partial_t(r^2 v) + \partial_r\Big(r(r - 2m)\frac{v^2}{2}\Big) = rv^2 - mc^2, \tag{3.6}$$

$$\partial_t(r^2 v) + \partial_r\Big(r(r - 2m)\Big(-1 + \sqrt{1 + v^2}\Big)\Big) = 0, \tag{3.7}$$

where the Schwarzshild metric in coordinates (t, r, θ, φ) is defined by

$$g = -\Big(1 - \frac{2m}{r}\Big)c^2 dt^2 + \Big(1 - \frac{2m}{r}\Big)^{-1} dr^2 + r^2(d\theta^2 + \sin^2 d\varphi^2),$$

so that $m > 0$ is the mass parameter, r is the Schwarzshild radius and $r > 2m$. We refer the reader to [38] for further details. In the present thesis, our main objective is the discussion of yet another generalization, that is the relativistic Burgers equation on a FLRW spacetime.

3.2 FLRW background spacetimes

3.2.1 Motivations from cosmology

Cosmology is based on Einstein's theory of gravity and certain classes of explicit solutions are often considered. (See for instance [27] for the notions in this section.) Recall first that Einstein himself introduced in his field equation the so-called cosmological constant Λ, in order to ensure that static solutions representing a static universe exist. Next, without requiring this cosmological constant, Friedmann discovered solutions to Einstein equations describing an expanding universe. At the same time, Lemaître proposed the "Big Bang model", which describes an expanding universe from a singular state and derived the "distance redshift" relation. This circle of ideas, together with further works by Robertson and Walker, led to a theory based on a family of solutions, now referred as the FLRW spacetimes describing the whole universe evolution.

In short, the cosmological principle states that the universe is *homogeneous* (has spatial translation symmetry) and *isotropic* (has spatial rotation symmetry). According to this principle, the universe may evolve in time, in either a contracting or an expanding direction. Observations indicate that the universe is *expanding*; whereas galaxies, quasars and galaxy clusters evolve with redshift, and the temperature of the cosmic microwave background (a uniform background of radio waves which fill the universe) is decreasing. An important feature in cosmology works is that studies are always done in *comoving coordinates* which expand with the universe. Furthermore, three topologies (positive, negative, or vanishing curvature) are possible and the universe is referred to be closed, open, or flat, respectively.

3.2.2 Expression of the FLRW metric

We will work here with the FLRW metric describing a spatially homogeneous and isotropic three-dimensional space. In term of the proper time t measured by a comoving observer, and by introducing radial r and angular (θ and φ) coordinates in the comoving frame, we can express the metric of such a $3+1$-dimensional spacetime in the form

$$g = -c^2dt^2 + a(t)^2\Big(\frac{dr^2}{1-kr^2} + r^2d\theta^2 + r^2\sin^2\theta d\varphi^2\Big), \tag{3.8}$$

where $k = 0, \pm 1$ and c is the light speed. The variable t is the proper time experienced by comoving observers, who remain at rest in comoving coordinates $dr = d\theta = d\varphi = 0$. The time variable t appearing in the FLRW metric is the time that would be measured by an observer who sees uniform expansion of the surrounding universe; it is named as the *cosmological proper time* or cosmic time.

The function $a = a(t)$ reads

$$a(t) = a_0\Big(\frac{t}{t_0}\Big)^{\alpha}, \tag{3.9}$$

where, for the FLRW metric, $\alpha = \frac{2}{3}$, t_0 is the age of the universe (which is a 'large' number) and $a_0 = 1$ refers to 'today'. In addition, the parameter k, a constant in time and space, is related to the spacetime curvature K by the equation,

$$k = a(t)^2K.$$

We can distinguish between three cases:

$$k = \begin{cases} 1, & \text{sphere (of positive curvature),} \\ 0, & \text{(flat) Euclidean space,} \\ -1, & \text{hyperboloid (of negative curvature).} \end{cases} \tag{3.10}$$

The FLRW metric can also be used to express the line element for homogeneous, isotropic spacetime in matrix form (for $c = 1$) as

$$g = g_{ij}dx^idx^j = (dt\,dr\,d\theta\,d\varphi)\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{a^2}{1-kr^2} & 0 & 0 \\ 0 & 0 & a^2r^2 & 0 \\ 0 & 0 & 0 & a^2r^2\sin^2\theta \end{pmatrix}\begin{pmatrix} dt \\ dr \\ d\theta \\ d\varphi \end{pmatrix}.$$

Thus, the FLRW metric is diagonal with

$$g_{00} = -1,\ g_{11} = \frac{a^2}{1-kr^2},\ g_{22} = a^2r^2,\ g_{33} = a^2r^2\sin^2\theta, \tag{3.11}$$

as its non-zero *covariant* (diagonal entries of the matrix) components, and the corresponding *contravariant* components are

$$g^{00} = -1,\ g^{11} = \frac{1-kr^2}{a^2},\ g^{22} = \frac{1}{a^2r^2},\ g^{33} = \frac{1}{a^2r^2\sin^2\theta}, \tag{3.12}$$

with

$$g^{ik}g_{kj} = \delta^i_j,$$

where δ^i_j is the Kronecker's delta function. The coordinates (r, θ, φ) of the metric are comoving coordinates. In the FLRW metric, as the universe expands the galaxies keep the same coordinates (r, θ, φ) and only the scale factor $a(t)$ changes with time.

3.2.3 Christoffel symbols for FLRW background

We need first to calculate the Christoffel symbols $\Gamma^\mu_{\alpha\beta}$. The metric tensors tell us how to define distance between neighboring points and the connection coefficients tell us how to define parallelism between neighboring points. We calculate the Christoffel symbols by using (3.11) and (3.12) with

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2}g^{\mu\nu}(-\partial_\nu g_{\alpha\beta} + \partial_\beta g_{\alpha\nu} + \partial_\alpha g_{\beta\nu}), \tag{3.13}$$

where $\alpha, \beta, \mu, \nu \in \{0, 1, 2, 3\}$. To begin with, we calculate two typical coefficients by using (3.11) and (3.12), as follows:

$$\Gamma^0_{00} = \frac{1}{2}g^{00}(-\partial_0 g_{00} + \partial_0 g_{00} + \partial_0 g_{00}) = \frac{1}{2}(-1)(0) = 0,$$

and

$$\begin{aligned}\Gamma^0_{11} &= \frac{1}{2}g^{00}(-\partial_0 g_{11} + \partial_1 g_{10} + \partial_0 g_{10}) \\ &= \frac{1}{2}(-1)\big(-\partial_0(\frac{a^2}{(1-kr^2)})\big) = \frac{a\dot{a}}{c(1-kr^2)}.\end{aligned}$$

Similarly, we obtain the other non-vanishing Christoffel symbols as:

$$\begin{aligned}&\Gamma^0_{11} = \frac{a\dot{a}}{c(1-kr^2)}, \quad \Gamma^0_{22} = \frac{a\dot{a}r^2}{c}, \quad \Gamma^0_{33} = \frac{a\dot{a}r^2\sin^2\theta}{c}, \\ &\Gamma^1_{11} = \frac{kr}{1-kr^2}, \quad \Gamma^1_{22} = -r(1-kr^2), \quad \Gamma^1_{33} = -r(1-kr^2)\sin^2\theta, \\ &\Gamma^2_{33} = -\sin\theta\cos\theta, \quad \Gamma^3_{23} = \Gamma^3_{32} = \cot\theta, \\ &\Gamma^2_{12} = \Gamma^2_{21} = \Gamma^3_{31} = \Gamma^3_{13} = \frac{1}{r}, \\ &\Gamma^1_{01} = \Gamma^1_{10} = \Gamma^2_{02} = \Gamma^2_{20} = \Gamma^3_{30} = \Gamma^3_{03} = \frac{\dot{a}}{ca}.\end{aligned} \tag{3.14}$$

Moreover, the zero terms of the Christoffel symbols are calculated as:

$$\begin{aligned}\Gamma^0_{00} &= \Gamma^0_{01} = \Gamma^0_{02} = \Gamma^0_{03} = \Gamma^0_{10} = \Gamma^0_{12} = \Gamma^0_{13} = \Gamma^0_{20} = \Gamma^0_{21} = \Gamma^0_{23}\\ &= \Gamma^0_{30} = \Gamma^0_{31} = \Gamma^0_{32} = \Gamma^1_{00} = \Gamma^1_{02} = \Gamma^1_{03} = \Gamma^1_{12} = \Gamma^1_{13} = \Gamma^1_{20}\\ &= \Gamma^1_{21} = \Gamma^1_{23} = \Gamma^1_{30} = \Gamma^1_{31} = \Gamma^1_{32} = \Gamma^2_{00} = \Gamma^2_{01} = \Gamma^2_{03} = \Gamma^2_{10}\\ &= \Gamma^2_{11} = \Gamma^2_{13} = \Gamma^2_{22} = \Gamma^2_{23} = \Gamma^2_{30} = \Gamma^2_{31} = \Gamma^2_{32} = \Gamma^3_{00} = \Gamma^3_{01}\\ &= \Gamma^3_{02} = \Gamma^3_{10} = \Gamma^3_{11} = \Gamma^3_{12} = \Gamma^3_{20} = \Gamma^3_{21} = \Gamma^3_{22} = \Gamma^3_{33} = 0.\end{aligned}$$

3.3 From the Euler system to the relativistic Burgers equation

3.3.1 The energy-momentum tensor for perfect fluids

We assume that solutions to the Euler equations depend only on the time variable t and the radial variable r, and that the non-radial components of the velocity vanish, that is, $(u^\alpha) = (u^0(t,r), u^1(t,r), 0, 0)$. Since u is unit vector, we have $u^\alpha u_\alpha = -1$ and we can write

$$u^\alpha u_\alpha = u^0 u_0 + u^1 u_1 = g_{00}(u^0)(u^0) + g_{11}(u^1)(u^1),$$

which gives us

$$-1 = g_{00}(u^0)^2 + g_{11}(u^1)^2. \tag{3.15}$$

Plugging the covariant components into this equation, it follows that

$$-1 = -(u^0)^2 + \frac{a(t)^2}{1-kr^2}(u^1)^2. \tag{3.16}$$

By considering the formula for the fluid velocity, which is the proportion of the proper distance of the hypersurface to the elapsed time, it is convenient to introduce the velocity component v. The coordinates are taken to be

$$(x^0, x^1, x^2, x^3) = (ct, r, 0, 0).$$

In order to obtain a relation between u^1 and u^0, we use the identity

$$u^\alpha = \frac{dx^\alpha}{d\tau} = \frac{dx^\alpha}{dt}\frac{dt}{d\tau},$$

$$u^0 = \frac{dx^0}{d\tau} = c\frac{dt}{d\tau},$$

$$u^1 = \frac{dx^1}{d\tau} = \frac{dx^1}{dt}\frac{dt}{d\tau} = \frac{1}{c}v^1 u^0,$$

which gives $u^1/u^0 = v^1/c$, since $dx^1/dt = v^1$.

Next, we introduce the velocity component v as the fraction of the proper distance of the hypersurface and the elapsed time

$$v := \frac{ca(t)}{(1-kr^2)^{1/2}}\frac{u^1}{u^0}. \tag{3.17}$$

By using (3.16) and (3.17) with a simple algebraic manipulation, we obtain the following identities

$$(u^0)^2 = \frac{c^2}{(c^2-v^2)}, \qquad (u^1)^2 = \frac{v^2(1-kr^2)}{a^2(c^2-v^2)}. \tag{3.18}$$

Then, in order to calculate the tensor components, we need to recall the energy momentum tensor of perfect fluids formula,

$$T^{\alpha\beta} = (\rho c^2 + p)\, u^\alpha\, u^\beta + p\, g^{\alpha\beta}. \tag{3.19}$$

By inserting the terms from the relation (3.18) and the contravariant components (3.12) into the formula (3.19), we calculate all the energy momentum tensor components for $\alpha, \beta \in \{0,1,2,3\}$. For example, the first tensor T^{00} can be obtained as

$$T^{00} = (\rho c^2 + p)u^0 u^0 + pg^{00} = \frac{c^2}{c^2-v^2}(\rho c^2 + p) - p = \frac{\rho c^4 + pv^2}{c^2-v^2}.$$

In the same way the other components are found to be

$$\begin{aligned} T^{01} = T^{10} &= \frac{cv(1-kr^2)^{1/2}(\rho c^2+p)}{a(c^2-v^2)}, & T^{11} &= \frac{c^2(1-kr^2)(v^2\rho+p)}{a^2(c^2-v^2)}, \\ T^{22} &= \frac{p}{a^2r^2}, & T^{33} &= \frac{p}{a^2r^2\sin^2\theta}, \end{aligned}$$

while the remaining terms vanish, as shown below

$$T^{02} = T^{03} = T^{12} = T^{13} = T^{20} = T^{21} = T^{23} = T^{30} = T^{31} = T^{32} = 0.$$

3.3.2 The pressureless Euler system on FLRW background

In the previous section, Christoffel symbols and energy momentum tensors for perfect fluids were derived. In this section, we are in a position to derive the Euler system on a FLRW spacetime. We recall the Euler equations $\nabla_\alpha T^{\alpha\beta} = 0$, which can be rewritten as

$$\partial_\alpha T^{\alpha\beta} + \Gamma^\alpha_{\alpha\gamma} T^{\gamma\beta} + \Gamma^\beta_{\alpha\gamma} T^{\alpha\gamma} = 0. \tag{3.20}$$

There are two sets of equations depending on β. Firstly taking $\beta = 0$ in (3.20) yields

$$\partial_\alpha T^{\alpha 0} + \Gamma^\alpha_{\alpha\gamma} T^{\gamma 0} + \Gamma^0_{\alpha\gamma} T^{\alpha\gamma} = 0,$$

which is equivalent to

$$\begin{aligned}&\partial_0 T^{00} + \Gamma^0_{0\gamma} T^{\gamma 0} + \Gamma^0_{\gamma 0} T^{\gamma 0} + \partial_1 T^{10} + \Gamma^1_{1\gamma} T^{\gamma 0} + \Gamma^0_{1\gamma} T^{1\gamma} + \partial_2 T^{20} + \Gamma^2_{2\gamma} T^{\gamma 0}\\ &+ \Gamma^0_{2\gamma} T^{2\gamma} + \partial_3 T^{30} + \Gamma^3_{3\gamma} T^{\gamma 0} + \Gamma^0_{3\gamma} T^{3\gamma} = 0.\end{aligned}$$

Writing this equation in more detail results to the following equation

$$\begin{aligned}&\partial_0 T^{00} + \Gamma^0_{00} T^{00} + \Gamma^0_{00} T^{00} + \Gamma^0_{01} T^{10} + \Gamma^0_{01} T^{01} + \Gamma^0_{02} T^{20} + \Gamma^0_{02} T^{02}\\ &+ \Gamma^0_{03} T^{30} + \Gamma^0_{03} T^{03} + \partial_1 T^{10} + \Gamma^1_{10} T^{00} + \Gamma^0_{10} T^{10} + \Gamma^1_{11} T^{10} + \Gamma^0_{11} T^{11}\\ &+ \Gamma^1_{12} T^{20} + \Gamma^0_{12} T^{12} + \Gamma^1_{13} T^{30} + \Gamma^0_{13} T^{13} + \partial_2 T^{20} + \Gamma^2_{20} T^{00} + \Gamma^0_{20} T^{20}\\ &+ \Gamma^2_{21} T^{10} + \Gamma^0_{21} T^{21} + \Gamma^2_{22} T^{20} + \Gamma^0_{22} T^{22} + \Gamma^2_{23} T^{30} + \Gamma^0_{23} T^{23} + \partial_3 T^{30}\\ &+ \Gamma^3_{30} T^{00} + \Gamma^0_{30} T^{30} + \Gamma^3_{31} T^{10} + \Gamma^0_{31} T^{31} + \Gamma^3_{32} T^{20} + \Gamma^0_{32} T^{32}\\ &+ \Gamma^3_{33} T^{30} + \Gamma^0_{33} T^{33} = 0.\end{aligned}$$

We next consider the exponent $\beta = 1$:

$$\partial_\alpha T^{\alpha 1} + \Gamma^\alpha_{\alpha\gamma} T^{\gamma 1} + \Gamma^1_{\alpha\gamma} T^{\alpha\gamma} = 0,$$

which gives us

$$\begin{aligned}&\partial_0 T^{01} + \Gamma^0_{0\gamma} T^{\gamma 1} + \Gamma^0_{0\gamma} T^{0\gamma} + \partial_1 T^{11} + \Gamma^1_{1\gamma} T^{\gamma 1} + \Gamma^1_{1\gamma} T^{1\gamma} + \partial_2 T^{21} + \Gamma^2_{2\gamma} T^{\gamma 1}\\ &+ \Gamma^1_{2\gamma} T^{2\gamma} + \partial_3 T^{31} + \Gamma^3_{3\gamma} T^{\gamma 1} + \Gamma^1_{3\gamma} T^{3\gamma} = 0.\end{aligned}$$

Again by writing this equation in a detailed form, we attain the following equation

$$\begin{aligned}&\partial_0 T^{01} + \Gamma^0_{00} T^{01} + \Gamma^1_{00} T^{00} + \Gamma^0_{01} T^{11} + \Gamma^1_{01} T^{01} + \Gamma^0_{02} T^{21} + \Gamma^1_{02} T^{02}\\ &+ \Gamma^0_{03} T^{31} + \Gamma^1_{03} T^{03} + \partial_1 T^{11} + \Gamma^1_{10} T^{01} + \Gamma^1_{10} T^{10} + \Gamma^1_{11} T^{11} + \Gamma^1_{11} T^{11}\\ &+ \Gamma^1_{12} T^{21} + \Gamma^1_{12} T^{12} + \Gamma^1_{13} T^{31} + \Gamma^1_{13} T^{13} + \partial_2 T^{21} + \Gamma^2_{20} T^{01} + \Gamma^1_{20} T^{20}\\ &+ \Gamma^2_{21} T^{11} + \Gamma^1_{21} T^{21} + \Gamma^2_{22} T^{21} + \Gamma^1_{22} T^{22} + \Gamma^2_{23} T^{31} + \Gamma^1_{23} T^{23} + \partial_3 T^{31}\\ &+ \Gamma^3_{30} T^{01} + \Gamma^1_{30} T^{30} + \Gamma^3_{31} T^{11} + \Gamma^1_{31} T^{31} + \Gamma^3_{32} T^{21} + \Gamma^1_{32} T^{32}\\ &+ \Gamma^3_{33} T^{31} + \Gamma^1_{33} T^{33} = 0.\end{aligned}$$

Next, by substituting the expression of the Christoffel symbols in the Euler system on a FLRW background, we obtain the simplified system

$$\begin{aligned}
&\partial_0 T^{00} + \partial_1 T^{10} + \frac{3\dot{a}}{ca}T^{00} + \frac{kr}{1-kr^2}T^{10} + \frac{a\dot{a}}{c(1-kr^2)}T^{11} \\
&+ \frac{2}{r}T^{10} + \frac{r^2 a\dot{a}}{c}T^{22} + \frac{a\dot{a}r^2\sin^2\theta}{c}T^{33} = 0, \\
&\partial_0 T^{01} + \partial_1 T^{11} + \frac{4\dot{a}}{ca}T^{01} + \frac{\dot{a}}{ca}T^{10} + \frac{2kr}{(1-kr^2)}T^{11} \\
&+ \frac{1}{r}T^{11} - r(1-kr^2)T^{22} - r(1-kr^2)\sin^2\theta T^{33} = 0.
\end{aligned} \tag{3.21}$$

Finally, using the expressions for perfect fluids into (3.21) and assuming that the pressure p vanishes identically, we obtain the Euler system on a FLRW background:

$$\begin{aligned}
&\partial_0\left(\frac{\rho c^2}{c^2-v^2}\right) + \partial_1\left(\frac{\rho cv(1-kr^2)^{1/2}}{a(c^2-v^2)}\right) + \frac{3\dot{a}\rho c}{a(c^2-v^2)} + \frac{2\rho cv(1-kr^2)^{1/2}}{ra(c^2-v^2)} \\
&\qquad + \frac{kr\rho cv}{a(c^2-v^2)(1-kr^2)^{1/2}} + \frac{\dot{a}v^2\rho}{ca(c^2-v^2)} = 0, \\
&\partial_0\left(\frac{c^2\rho v(1-kr^2)^{1/2}}{a(c^2-v^2)}\right) + \partial_1\left(\frac{cv^2\rho(1-kr^2)}{a^2(c^2-v^2)}\right) + \frac{5\dot{a}\rho vc(1-kr^2)^{1/2}}{a^2(c^2-v^2)} \\
&\qquad + \frac{2krcv^2\rho}{a^2(c^2-v^2)} + \frac{2cv^2\rho(1-kr^2)}{ra^2(c^2-v^2)} = 0.
\end{aligned} \tag{3.22}$$

3.4 The relativistic Burgers equation on a FLRW background

3.4.1 The derivation of the relativistic Burgers equation

It remains now to write the relativistic Burgers equation with the help of the equations (3.22). Namely, we combine the two equations in (3.22) and reduce it to a single equation, that is,

$$a^2\partial_t\left(\frac{v}{a}(1-kr^2)^{1/2}\right) + \partial_r\left(\left(\frac{v^2}{2}\right)(1-kr^2)\right) + v(1-kr^2)^{1/2}a_t\left(2-\frac{v^2}{c^2}\right) + rkv^2 = 0. \tag{3.23}$$

If we take the partial derivatives of the first and the second terms in (3.23), this becomes

$$(av_t - va_t)(1-kr^2)^{1/2} + (1-kr^2)\partial_r\left(\frac{v^2}{2}\right) - rkv^2 + v(1-kr^2)^{1/2}a_t\left(2-\frac{v^2}{c^2}\right) + rkv^2 = 0,$$

after further elementary simplifications, we obtain

$$a v_t (1-kr^2)^{1/2} + (1-kr^2)\partial_r(\frac{v^2}{2}) + v\Big(1-kr^2\Big)^{1/2} a_t (1-\frac{v^2}{c^2}) = 0.$$

Finally, we arrive at the following definition.

Definition 3.4.1 *The* **relativistic Burgers equation on a FLRW background** *is*

$$a\, v_t + \big(1-kr^2\big)^{1/2} \partial_r\Big(\frac{v^2}{2}\Big) + v\Big(1-\frac{v^2}{c^2}\Big)\, a_t = 0, \tag{3.24}$$

in which $a = a(t) > 0$ *is a given function,* $k \in \big\{-1, 0, 1\big\}$ *is a discrete parameter, and the light speed* c *is a positive parameter.*

Writing this equation explicitly for the particular cases depending on the constant parameter k yields

Case 1 **:** $k = -1$

$$a\, v_t + \big(1+r^2\big)^{1/2} \partial_r\Big(\frac{v^2}{2}\Big) + v\Big(1-\frac{v^2}{c^2}\Big)\, a_t = 0, \tag{3.25}$$

Case 2 **:** $k = 0$

$$a\, v_t + \partial_r\Big(\frac{v^2}{2}\Big) + v\Big(1-\frac{v^2}{c^2}\Big)\, a_t = 0, \tag{3.26}$$

Case 3 **:** $k = 1$

$$a\, v_t + \big(1-r^2\big)^{1/2} \partial_r\Big(\frac{v^2}{2}\Big) + v\Big(1-\frac{v^2}{c^2}\Big)\, a_t = 0. \tag{3.27}$$

In the limiting case $c \to +\infty$, the equation (3.24) can be rewritten as

$$\partial_t\Big(\frac{a(t)v}{(1-kr^2)^{1/2}}\Big) + \partial_r\Big(\frac{v^2}{2}\Big) = 0, \tag{3.28}$$

which is a conservation law. Here, we are able to obtain classical model as c tends to infinity with a particular case of our relativistic model ($a(t) = 1$ and $k = 0$). Derivation of the classical (inviscid) Burgers equation from the relativistic one is an expected result since it is a property shared all by relativistic equations.

In order to obtain an analogous equation for (3.28) for finite values c, we propose to rewrite (3.24) as

$$\partial_t\Big(\frac{a(t)v}{(1-kr^2)^{1/2}}\Big) - \frac{v^3}{c^2}\partial_t\Big(\frac{a(t)}{(1-kr^2)^{1/2}}\Big) + \partial_r\Big(\frac{v^2}{2}\Big) = 0. \tag{3.29}$$

Furthermore, in the special case $a(t) \equiv 1$, this latter equation is also a *conservation law*

$$\partial_t\Big(\frac{v}{(1-kr^2)^{1/2}}\Big) + \partial_r\Big(\frac{v^2}{2}\Big) = 0. \tag{3.30}$$

3.4.2 The initial value problem

The equation (3.24) is a nonlinear hyperbolic equation with time- and space-dependent coefficients. The solutions admit jump discontinuities which propagate in time. This equation fits in the general theory of entropy weak solutions to such equations by Kruzkov [30]. The notion of entropy solutions relies on the use of the so-called convex entropy pairs, defined as follows.

Definition 3.4.2 *A pair of Lipschitz continuous functions V, F is a convex entropy-entropy flux pair if $V = V(v)$ is strictly convex and $F' := vV'$ hold almost everywhere. A function $v \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^+)$ is called an entropy solution of* (3.24)*, if for every convex entropy-entropy flux pair (V, F)*

$$\begin{aligned} &av_t + (1-kr^2)^{1/2}\partial_r\big(\frac{v^2}{2}\big) + v\big(1-\frac{v^2}{c^2}\big)a_t = 0, \\ &aV(v)_t + (1-kr^2)^{1/2}\partial_r F(v) + vV'(v)\big(1-\frac{v^2}{c^2}\big)a_t \le 0, \end{aligned} \tag{3.31}$$

hold in the sense of distributions.

In view of the general theory in [30], we obtain the following result.

Theorem 3.4.1 *The equation* (3.24) *admits an entropy weak solution $v \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^+)$ satisfying the conditions* (3.31) *in the sense of Kruzkov's theory.*

Note in passing that, in the particular case $a(t) \equiv 1$ and $k \equiv 0$, we obtain the classical Burgers equation and the approximate solution of this equation satisfies the additional estimate

$$\inf_x v(0,x) \le \inf_x v(t,x) \le \sup_x v(t,x) \le \sup_x v(0,x).$$

This is of course not true in general, and the lack of such properties in one of the challenges in order to numerically cope with discontinuous solutions to (3.31).

3.5 Special solutions and non-relativistic limit

3.5.1 Spatially homogeneous solutions

We look for special classes of explicit solutions to the Burgers equation on a FLRW background (3.24), which involves the variable coefficients $a(t)$ and $a_t(t)$. Due to this t-dependency, it is easily checked that for all three values of k, there does not exist any static solution (except $v \equiv 0$).

On the other hand, in order to find spatially homogeneous solutions of (3.24), we assume that v depends only on t so that the term $\partial_r(\frac{v^2}{2})$ vanishes identically, which means

$$av_t + v(1 - \frac{v^2}{c^2})a_t = 0. \tag{3.32}$$

By changing the notation v_t to v', and a_t to a', we write

$$\frac{v'}{v(1 - \frac{v^2}{c^2})} = -\frac{a'}{a},$$

which is equivalent to

$$\left(\frac{1}{v} + \frac{\frac{v}{c^2}}{1 - \frac{v^2}{c^2}}\right) v' = -(\log a)'.$$

It follows that

$$\frac{\pm v}{\sqrt{1 - v^2/c^2}} = \frac{w}{a}, \qquad \text{where } w \text{ is a constant.}$$

Equivalently, we have

$$\frac{a^2}{w^2}v^2 = 1 - \frac{v^2}{c^2}.$$

Thus the spatially homogeneous homogeneous solutions can be described by the explicit formula

$$v(t) = \frac{\pm c}{\sqrt{1 + \frac{a^2(t)c^2}{w^2}}}, \tag{3.33}$$

where w is a constant parameter. This is obviously true for all $k \in \{-1, 0, 1\}$.

Proposition 3.5.1 *The spatially homogeneous solutions to the relativistic Burgers equation on a FLRW background*

$$v(t) = \frac{w}{\sqrt{a(t)^2 + \frac{w^2}{c^2}}} \in (-c, c) \tag{3.34}$$

are parameterized by a real parameter w (where c is the light speed).

In the relativistic Burgers equation on FLRW background the 'main dependence' is in t rather than r.

3.5.2 Some limit properties of the relativistic Burgers equation

Next, let us consider some limit properties of the equation (3.24) when, for definiteness, $a(t) = a_0\left(\frac{t}{t_0}\right)^\alpha$. Observe in passing that (3.24) is *not linear in terms of the coefficient* $a(t)$ (since the second term in the equation does not include $a(t)$ or $a'(t)$). Recall that the following parameters are relevant:

$$\begin{cases} k \text{ the curvature constant }, \quad k \in [-1, 1] \\ c \text{ the light speed }, \quad c \in (0, \infty) \\ a_0 \text{ the constant in } a(t), \quad a_0 \in (0, \infty) \\ \alpha \text{ the exponent in } a(t), \quad \alpha \in (0, \infty). \end{cases}$$

Two typical ranges of the time variables are relevant here, since shock wave solutions to nonlinear hyperbolic equations are *only defined in a forward time directions:* since at $t = 0$ the equation is singular, we can treat the range $t \in [1, \infty)$ or the range $t \in [-1, 0)$. For $t \in [1, \infty)$ we normalize $a_0 = 1$ and for $t \in [-1, 0)$ we set $a_0 = -1$.

In the case $t > 1$, if we consider the limit $t \to +\infty$, the equation is *expanding toward the future time directions*, while in the case $t < 0$ when $t \to 0$, the equation is *contracting in the future time directions*.

3.5.2.1 Recovering the standard Burgers equation

The special case $a_0 = 1$, $t_0 = 1$, $\alpha = 0$ (which means $a(t) = 1$), with the particular case $k = 0$ for the equation (3.24) leads us to

$$\partial_t v + \partial_r\left(\frac{v^2}{2}\right) = 0, \tag{3.35}$$

which is the classical Burgers equation.

3.5.2.2 The non-relativistic limit

Taking the limit $c \to +\infty$ in the equation (3.24), we obtain

$$\partial_t(av) + (1-kr^2)^{1/2}\partial_r(\frac{v^2}{2}) = 0. \tag{3.36}$$

We can also determine directly the limiting behavior of the spatially homogeneous solutions to (3.24): in view of (3.33), we obtain

$$v(t) = \frac{1}{\sqrt{\frac{1}{c^2} + \frac{a^2(t)}{w^2}}}, \tag{3.37}$$

where w is a constant parameter. Here we have made the following observations:

- For spatially homogeneous solutions, we have $|v| < c$.
- In the expanding direction $t \to +\infty$, we have $v \to 0$.
- In the contraction direction $t \to 0$, we have $v \to c$ since $a(t) \to 0$.
- We have $v \to \frac{w}{a(t)}$ as $c \to +\infty$.

3.6 A shock-capturing, well-balanced, finite volume scheme

3.6.1 Finite volume methodology for geometric balance laws

In this section, we are motivated by the earlier works [48, 49] for nonlinear hyperbolic problems without the relativistic features and [38] concerning the relativistic Burgers equations. In the Burgers equation on a FLRW background, the variable coefficients *depend upon the time variable* t, due to the terms $a(t)$, $a'(t)$ and $k \in \{-1, 0, 1\}$. Hence, the numerical approximation of solutions to the Burgers equation on a FLRW background leads to a new challenge, in comparison with flat or Schwarzschild backgrounds.

As explained earlier, the spacetime of interest is described by a single chart and some coordinates denoted by (t, r). For the discretization, we denote the (constant) time length by Δt and we set $t_n = n\Delta t$, and we introduce equally spaced cells $I_j =$

$[r_{j-1/2}, r_{j+1/2}]$ with (constant) spatial length denoted by $\Delta r = r_{j+1/2} - r_{j-1/2}$. The finite volume method is based on an averaging of the balance law

$$\partial_t(T^0(t,r)) + \partial_r(T^1(t,r)) = S(t,r), \tag{3.38}$$

over each grid cell $[t_n, t_{n+1}] \times I_j$, where $T^\alpha(v) = T^\alpha(t,r)$ and $S(t,r)$ are the flux and source terms, respectively. We thus write the identity

$$\begin{aligned}
&\int_{r_{j-1/2}}^{r_{j+1/2}} (T^0(t_{n+1}, r) - T^0(t_n, r))\, dr \\
&+ \int_{t_n}^{t_{n+1}} (T^1(t, r_{j+1/2}) - T^1(t, r_{j-1/2}))\, dt = \int_{[t_n, t_{n+1}] \times I_j} S(t,r)\, dt\, dr,
\end{aligned}$$

or, by rearranging the terms,

$$\begin{aligned}
\int_{r_{j-1/2}}^{r_{j+1/2}} T^0(t_{n+1}, r)\, dr = \int_{r_{j-1/2}}^{r_{j+1/2}} T^0(t_n, r) dr + \int_{[t_n, t_{n+1}] \times I_j} S(t,r)\, dt\, dr \\
- \int_{t_n}^{t_{n+1}} (T^1(t, r_{j+1/2}) - T^1(t, r_{j-1/2}))\, dt.
\end{aligned} \tag{3.39}$$

We introduce the following approximations

$$\begin{aligned}
&\frac{1}{\Delta r} \int_{r_{j-1/2}}^{r_{j+1/2}} T^0(t_n, r)\, dr \simeq \overline{T}_j^n, \\
&\frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} T^1(t, r_{j\pm1/2})\, dt \simeq \overline{Q}_{j\pm1/2}^n, \\
&\frac{1}{\Delta t \Delta r} \int_{[t_n, t_{n+1}] \times I_j} S(t,r)\, dt\, dr \simeq \overline{S}_j^n.
\end{aligned}$$

so that our scheme take the following finite volume form

$$\overline{T}_j^{n+1} = \overline{T}_j^n - \frac{\Delta t}{\Delta r} (\overline{Q}_{j+1/2}^n - \overline{Q}_{j-1/2}^n) + \Delta t \overline{S}_j^n. \tag{3.40}$$

Keeping in mind the practical implementation of the scheme, we write also $\overline{T}_j^n = \overline{T}(v_j^n)$, where $T = T^0(v)$ is the (invertible) map determined by the equation. The piecewise constant approximations (v_j^n) at the "next" time level are thus given by the formula

$$v_j^{n+1} = \overline{T}^{-1}\Big(\overline{T}(v_j^n) - \frac{\Delta t}{\Delta r} (\overline{Q}_{j+1/2}^n - \overline{Q}_{j-1/2}^n) + \Delta t \overline{S}_j^n\Big). \tag{3.41}$$

For the scheme to be fully specified, we need of course to select a numerical flux and an approximation of the source term.

3.6.2 Well-balanced scheme for the Burgers equation on FLRW spacetime

We now apply the above methodology to the Burgers equation on a FLRW spacetime (3.24). We start from the discrete version (3.40). Since the essential dependence of the source term is with respect to the *time variable* only, we approximate the source-term in the form

$$\overline{S}_j^n = \frac{1}{\Delta\widetilde{V}} \int_{t_n}^{t_{n+1}} S dV_{\widetilde{g}}, \tag{3.42}$$

where $\widetilde{g}$ is the induced metric on a timelike slice. That is, we have

$$g = \widetilde{g} + a(t)^2 \frac{dr^2}{1-kr^2}$$

and we write

$$\widetilde{g} = -c^2 dt^2 = -g_t dt^2, \tag{3.43}$$

while $dV_{\widetilde{g}}$ is the induced volume form on a timelike slice. Thus, we have

$$dV_{\widetilde{g}} := c^2 dt, \qquad \Delta\widetilde{V} := c^2 \Delta t. \tag{3.44}$$

Hence, there is no geometrical effect to be taken into account, as far as an integration in time is concerned.

Now, let us write our equation (3.24) as a balance law (with a source term) and obtain the following equation

$$\partial_t v + \partial_r \Big((1-kr^2)^{1/2} \frac{v^2}{2a(t)} \Big) = -\Big(\frac{krv^2}{2a(t)} (1-kr^2)^{-1/2} + v(1-v^2) \frac{a_t(t)}{a(t)} \Big). \tag{3.45}$$

This is the formulation that we are going to discretize, since it has the advantage that the left-hand side is in a conservative form. In the applications, the dependence in time may be stiff, especially in the contracting directions of the spacetime. For this reason, we built the scheme in order to preserve as much as possible the spatially homogeneous solutions.

The flux terms will be approximated by the Godunov flux defined by solving Riemann problems at each interface. The main difficulty is thus the approximation of the source term in a well balanced way.

We focus first on a solution $v \simeq v(t)$ which is supposed to be nearly spatially homogeneous so that the r-derivative can be neglected. We integrate (3.45) in time and obtain the following expression for the source term:

$$\int_{t_n}^{t_{n+1}} \sqrt{g_t} v(t)\, dt = \int_{t_n}^{t_{n+1}} -\Big(\frac{krv(t)^2}{2a(t)}(1-kr^2)^{-1/2} + v(1-v(t)^2)\frac{a_t(t)}{a(t)}\Big)\sqrt{g_t}dt$$
$$\simeq \Delta\widetilde{V}\overline{S}_j^n. \tag{3.46}$$

Consequently, in view of (3.43), (3.44) and (3.46), we find

$$\overline{S}_j^n \simeq \frac{1}{\Delta\widetilde{V}} \int_{t_n}^{t_{n+1}} \sqrt{g_t}\, v(t)\, dt = \frac{1}{c\,\Delta t} \int_{t_n}^{t_{n+1}} v(t)\, dt \simeq v(t_{n+1/2}),$$

where $t_{n+1/2} = t_n + \Delta t/2$ and we used the trapezoid rule in order to evaluate the latter integral. Here, the function $v = v(t)$ denotes a locally-defined, spatially homogeneous approximation, as defined in Proposition 3.5.1, that is,

$$v(t) = \frac{w}{\sqrt{a(t)^2 + \frac{w^2}{c^2}}},$$

where the value $w = w_j^n$ is defined in each cell from the current state value v_j^n by the formula

$$v_j^n = \frac{w_j^n}{\sqrt{a(t_n)^2 + \frac{(w_n^j)^2}{c^2}}},$$

In other words, we define the approximation of the source term as

$$\begin{aligned}\overline{S}_j^n = v_j^{n+1/2} &= \frac{w_j^n}{\sqrt{a(t_{n+1/2})^2 + \frac{(w_j^n)^2}{c^2}}} \\ &= \frac{v_j^n a(t_n)}{\sqrt{(a(t_{n+1/2}))^2 + \frac{(v_j^n)^2}{c^2}\big((a(t_n))^2 - (a(t_{n+1/2}))^2\big)}}.\end{aligned} \tag{3.47}$$

This formula takes the time variation of the coefficient a into account, and is expected to be robust when approaching the crushing singularity.

In any given computational cell, the left-hand and right-hand numerical flux terms are obtained by Godunov flux formulas, which we write in the abstract form

$$Q_{j+1/2}^n = Q(\overline{T}_{j+1/2-}^n, \overline{T}_{j+1/2+}^n), \qquad Q_{j-1/2}^n = Q(\overline{T}_{j-1/2-}^n, \overline{T}_{j-1/2+}^n),$$

in which the geometry is taken into account by setting

$$\overline{T}_{j+1/2\pm}^n = b_{j+1/2}^n \overline{v}_{j+1/2}^n, \qquad \overline{T}_{j-1/2\pm}^n = b_{j-1/2}^n \overline{v}_{j-1/2}^n,$$

and

$$b_{j+1/2}^n = (1 - k(r_{j+1/2}^n)^2)^{1/2}, \qquad b_{j-1/2}^n = (1 - k(r_{j-1/2}^n)^2)^{1/2}.$$

The numerical tests for this chapter, is conducted in the following chapter. To this aim, a Godunov type of scheme is constructed. In addition, in order to investigate and compare the spatially homogeneous solutions with numerical solutions, we apply a well-balanced Godunov type of scheme.

CHAPTER 4

NUMERICAL EXPERIMENTS

4.1 Introduction

In this chapter we construct a Godunov scheme for the Burgers equation on Friedmann–Lemaître–Robertson–Walker background. First, we apply initial shocks and rarefactions to the concerning model. Next, several numerical tests about the well-balanced scheme and homogeneous solutions are illustrated. Numerical results verify that our well-balanced Godunov scheme preserves homogeneous solutions of the concerning equation.

4.2 Numerical experiments

4.2.1 Godunov scheme for the Burgers equation on a FLRW background

In this section, numerical experiments are illustrated for the model derived on a FLRW spacetime based on the Godunov scheme. Firstly, the behaviors of initial single shocks and rarefactions are examined in the numerical tests depending on three particular cases of constant k. Whereas, in the second part, we investigate the convergence of well-balanced scheme to the homogeneous solution. Analogously, depending on the parameter k in the main equation, we have several illustrations.

We analyze the given model with a single shock and rarefaction for an initial function considering the Godunov scheme with a local Riemann problem for each grid cell. In the experiments for test functions, we choose $a(t) = t^2$ and $r \in [0, 1]$. Since our scheme has singularities at $t = 0$ stemming from the function $a(t)$, we start by taking $t > 1$ for all cases of $k = -1, 0, 1$. In Riemann problem both shocks and rarefaction waves are produced, thus we look for the fastest wave at each grid cell. For the fluxes at the boundary we choose the functions $f_0(u, v)$ and $f_1(u, v)$ for $r = 0$ and $r = 1$, respectively.

After normalization (taking $c = 1$) in the equation (3.24), we obtain the following model

$$v_t + \left(1 - kr^2\right)^{1/2} \frac{1}{a(t)} \partial_r \left(\frac{v^2}{2}\right) + v(1 - v^2)\, \frac{a_t(t)}{a(t)} = 0. \tag{4.1}$$

In order to define this equation in a better form, we rewrite it by

$$\partial_t v + \partial_r \Big((1 - kr^2)^{1/2} \frac{v^2}{2a(t)}\Big) = -\Big(\frac{krv^2}{2a(t)} (1 - kr^2)^{-1/2} + v(1 - v^2) \frac{a_t(t)}{a(t)}\Big), \tag{4.2}$$

and the corresponding finite volume scheme can be written as

$$v_j^{n+1} = v_j^n - \frac{\Delta t}{\Delta r} (b_{j+1/2}^n g_{j+1/2}^n - b_{j-1/2}^n g_{j-1/2}^n) + \Delta t S_j^n, \tag{4.3}$$

where

$$S_j^n = -\Big(\frac{k(r_j^n)(v_j^n)^2}{2a^n} (1 - k(r_j^n)^2)^{-1/2} + (v_j^n)(1 - (v_j^n)^2) \frac{a_t^n}{a^n}\Big),$$

and

$$\begin{aligned} b_{j+1/2}^n &= (1 - k(r_{j+1/2}^n)^2)^{1/2}, \\ g_{j-1/2}^n &= f(v_{j-1}^n, v_j^n), \\ g_{j+1/2}^n &= f(v_j^n, v_{j+1}^n), \end{aligned}$$

with $f(u, v)$ is defined as follows

$$f(u,v) = \begin{cases} \frac{u^2}{2}, & \text{if} \quad u > v \quad \text{and} \quad u + v > 0, \\ \frac{v^2}{2}, & \text{if} \quad u > v \quad \text{and} \quad u + v < 0, \\ \frac{u^2}{2}, & \text{if} \quad u \le v \quad \text{and} \quad u > 0, \\ \frac{v^2}{2}, & \text{if} \quad u \le v \quad \text{and} \quad v < 0, \\ 0, & \text{if} \quad u \le v \quad \text{and} \quad u \le 0 \le v. \end{cases}$$

We use the notation J to indicate the total number of grid cells in space so that v_J^{n+1} reads as the velocity on the right hand side boundary $r = 1$. It follows that

$$v_J^{n+1} = v_J^n - \frac{\Delta t}{\Delta r} (b_{J+1/2}^n\, g_{J+1/2}^n - b_{J-1/2}^n\, g_{J-1/2}^n) + \Delta t S_J^n,$$

$$g^n_{J-1/2} = f(v^n_{J-1}, v^n_J),$$

$$g^n_{J+1/2} = f_1(v^n_J, v^n_{J+1}),$$

where

$$f_1(u,v) = \frac{1}{2}\Big(\max\big(0,u\big)\Big)^2,$$

is the flux function computed at the right hand side boundary $r = 1$.

Next, we use the notation v^{n+1}_1 indicating the velocity on the left hand side boundary $r = 0$. It follows that

$$v^{n+1}_1 = v^n_1 - \frac{\Delta t}{\Delta r}(b^n_{3/2}\, g^n_{3/2} - b^n_{1/2}\, g^n_{1/2}) + \Delta t S^n_1,$$

$$g^n_{1/2} = f_0(v^n_0, v^n_1),$$

$$g^n_{3/2} = f(v^n_1, v^n_2),$$

where

$$f_0(u,v) = \frac{1}{2}\Big(\max\big(0,-v\big)\Big)^2,$$

is the flux function computed at the boundary $r = 0$. In order the stability condition in the scheme to be satisfied, we choose Δt and Δr, so that

$$\frac{\Delta t}{\Delta r}\max_j\Big|\frac{(1-k(r^n_j)^2)^{1/2}v^n_j}{a^n}\Big| \leq 1,$$

where

$$\big(\frac{(1-k(r^n_j)^2)^{1/2}v^n_j}{a^n}\big),$$

is the speed term.

In Figures (4.1) and (4.2), we compare the particular cases $k = 0$ and $k = -1$ for shocks and rarefactions. In addition, in Figure (4.3) we perform a similar comparison by using the initial function

$$v_0(x) = 0.5 + 0.1\sin(24\pi x),$$

for our scheme. From these graphs we observe that the numerical solution for the particular case $k = -1$ which is represented by the red line moves faster than the particular case $k = 0$ represented by the green line. This result can also be verified by plugging $k = -1, 0$ into the speed term given above. We also observe that, for two particular cases of $k = -1$ and $k = 0$, the solution curves converge to zero which shows the efficiency and robustness of the scheme.

4.2.2 Transformation for the particular case $k = 1$

The source term for the particular case $k = 1$ is

$$S_j^n = -\Big(\frac{r_j^n(v_j^n)^2}{2a^n}(1-(r_j^n)^2)^{-1/2} + (v_j^n)(1-(v_j^n)^2)\frac{a_t^n}{a^n}\Big)$$

which produces singularities in the scheme. In order to get rid of this handicap, we modify our scheme by changing the variable $\xi(r) = \arcsin(r)$ so that

$$(1-r^2)^{1/2}\frac{\partial}{\partial r} =: \frac{\partial}{\partial \xi}, \quad d\xi = \frac{dr}{(1-r^2)^{1/2}}, \tag{4.4}$$

which allows us transform the compact domain $[-1, 1]$ to another compact domain $[\frac{-\pi}{2}, \frac{\pi}{2}]$. According to this change of variable, our main equation (3.24) can be written in the related coordinate system as

$$\partial_t(v) + \partial_\xi\big(\frac{v^2}{2a}\big) - v(1-v^2)\frac{a_t}{a} = 0. \tag{4.5}$$

In the scheme we use a flux function on the boundaries $\xi = 0$ and $\xi = \frac{\pi}{2}$ which is similar as the flux used in the previous part for the particular cases $k = -1, 0$. In addition, for stability of the scheme, Δt and $\Delta\xi$ satisfies

$$\frac{\Delta t}{\Delta \xi}\max_j\Big|\frac{v_j^n}{a^n}\Big| \leq 1,$$

where v_j^n/a^n is the speed.

Figures (4.4) and (4.5) represent the numerical solutions for particular case $k = 1$ after the transformation

$$\xi(r) = \arcsin(r),$$

with shocks and rarefactions, respectively. Moreover, in Figure (4.6) we apply the initial function

$$v_0(x) = 0.5 + 0.1\sin(24\pi x),$$

to our scheme. One can observe that the numerical scheme is efficient and robust as the solution curves converge to zero.

4.2.3 Numerical experiments for homogeneous solutions and well-balanced scheme

In this part we investigate the convergence of the well-balanced Godunov scheme. We consider the following quantity

$$\int_0^1 \left|\kappa - \frac{v(t,x)}{v_h(t)}\right| dx, \tag{4.6}$$

where $v(t,x)$ is the numerical solution given by the well-balanced Godunov scheme for three particular cases of k, $v_h(t)$ is the homogeneous solution and κ is a constant close to 1. For the numerical scheme we take the initial function as

$$v_0(x) = 0.5 + 0.1\sin(24\pi x),$$

and the homogeneous solution is

$$v_h(t) = \frac{1}{\sqrt{1+\frac{a^2(t)}{w^2}}},$$

with $w = 0.57$ which is compatible with our initial function. In addition, we choose $a(t) = t^5$ where $t > 1$.

In Figures (4.7), (4.8) and (4.9), the green curve represents the quantity defined in (4.6). One can observe that the numerical solution for the well-balanced Godunov scheme reaches to the homogeneous solution since the curve tends to zero whenever $t \to \infty$. In other words, as

$$v(t,x)/v_h(t) \to \kappa \qquad \text{with} \qquad \kappa \approx 1,$$

it follows that the numerical solution $v(t,x)$ converges to the homogenous solution $v_h(t)$. As a conclusion, our scheme is well-balanced in the sense that it preserves homogeneous solutions.

On the other hand, in Figures (4.10), (4.11) and (4.12), we compare the homogeneous solution $v_h(t)$ (represented by the green line) and the quantity

$$\int_0^1 |v(t,x) - v_h(t)| dx$$

(represented by the red line) where $v(t,x)$ is the numerical solution given by the well-balanced Godunov scheme for the particular cases $k = -1, 0, 1$. In these figures $a(t)$, w and the initial function $v_0(x)$ are chosen to be the same as the choice for the Figures

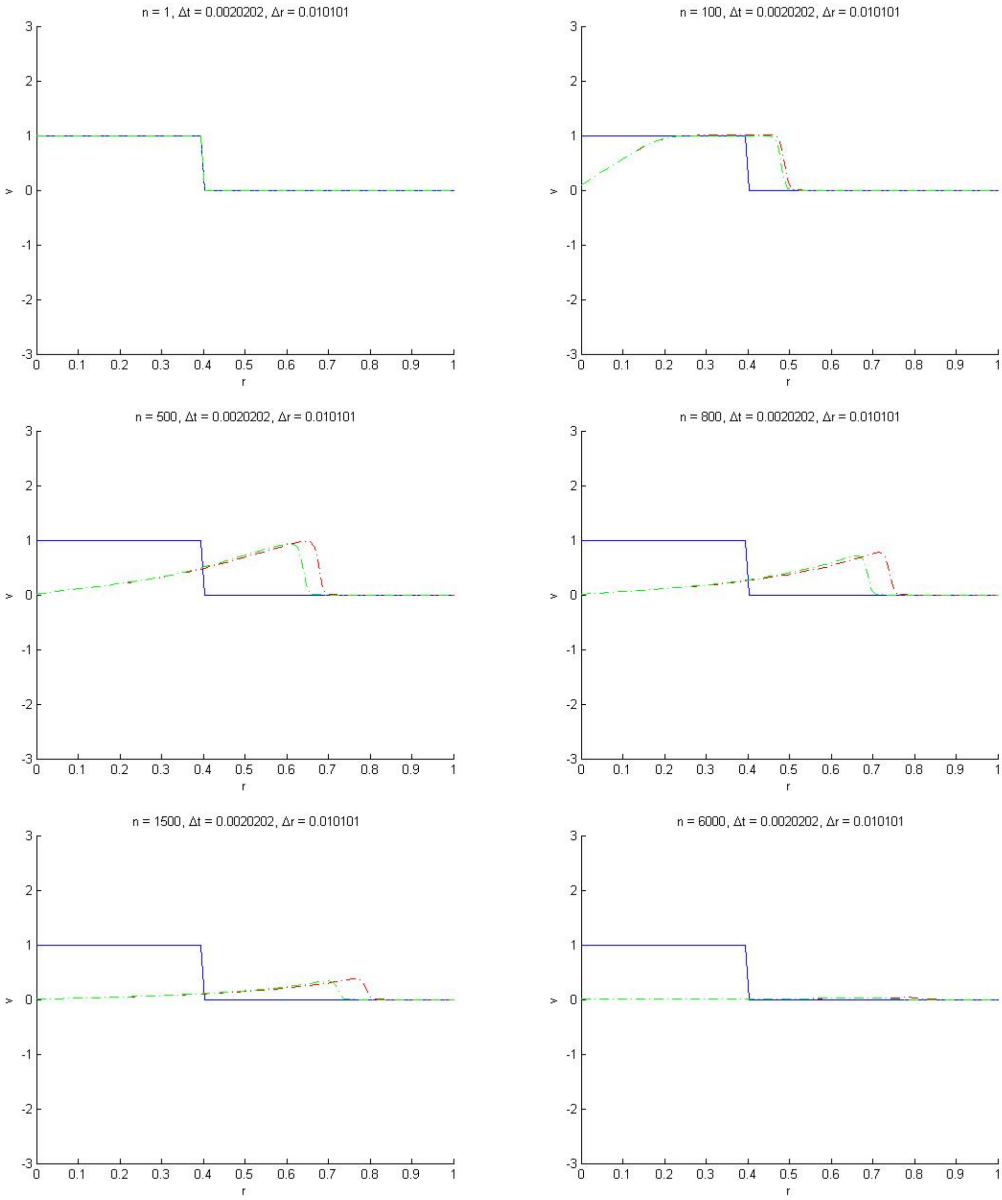

Figure 4.1: The numerical solutions given by the Godunov scheme with a shock for the particular cases $k = 0$ and $k = -1$.

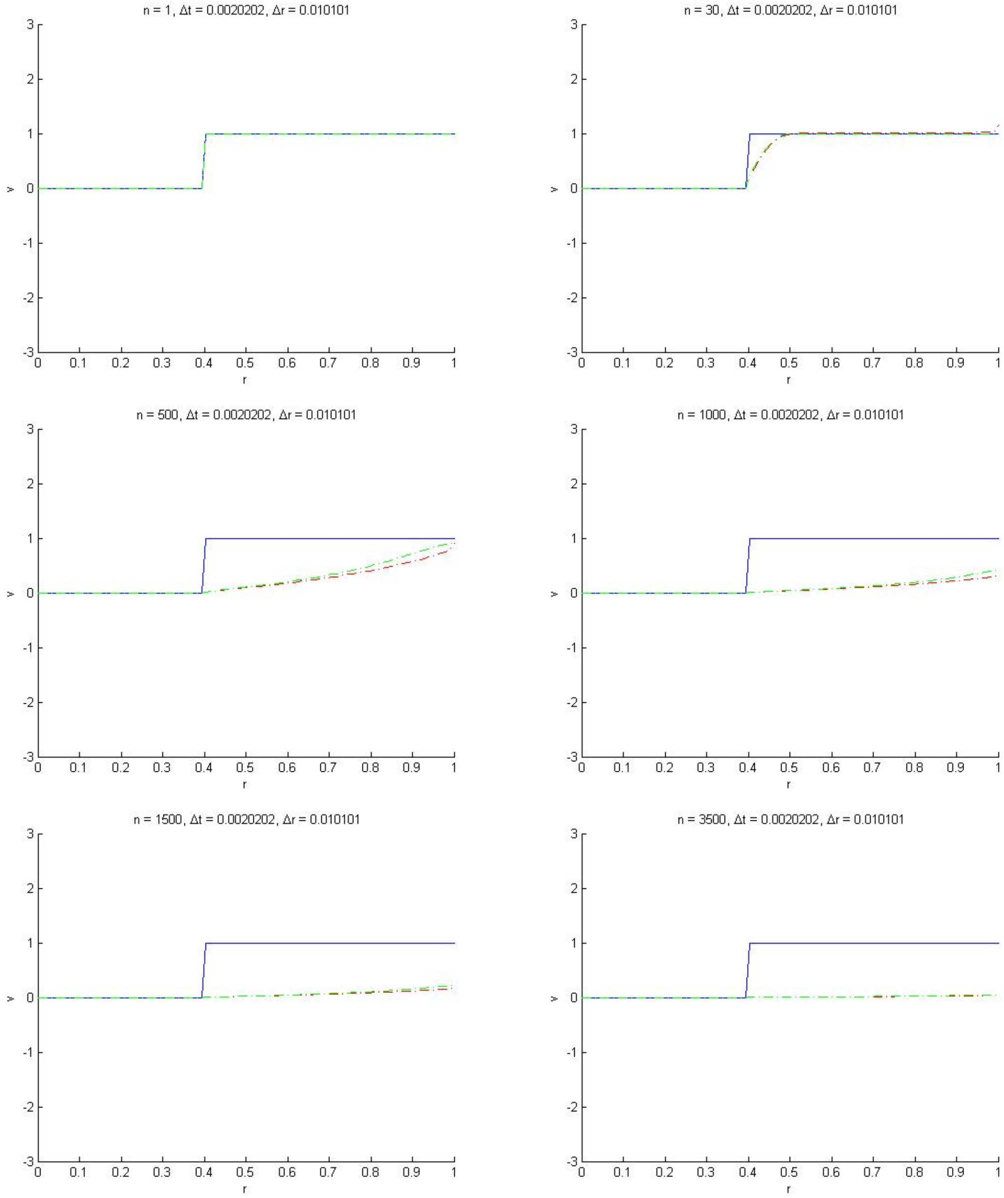

Figure 4.2: The numerical solutions given by the Godunov scheme with a rarefaction for the particular cases $k = 0$ and $k = -1$.

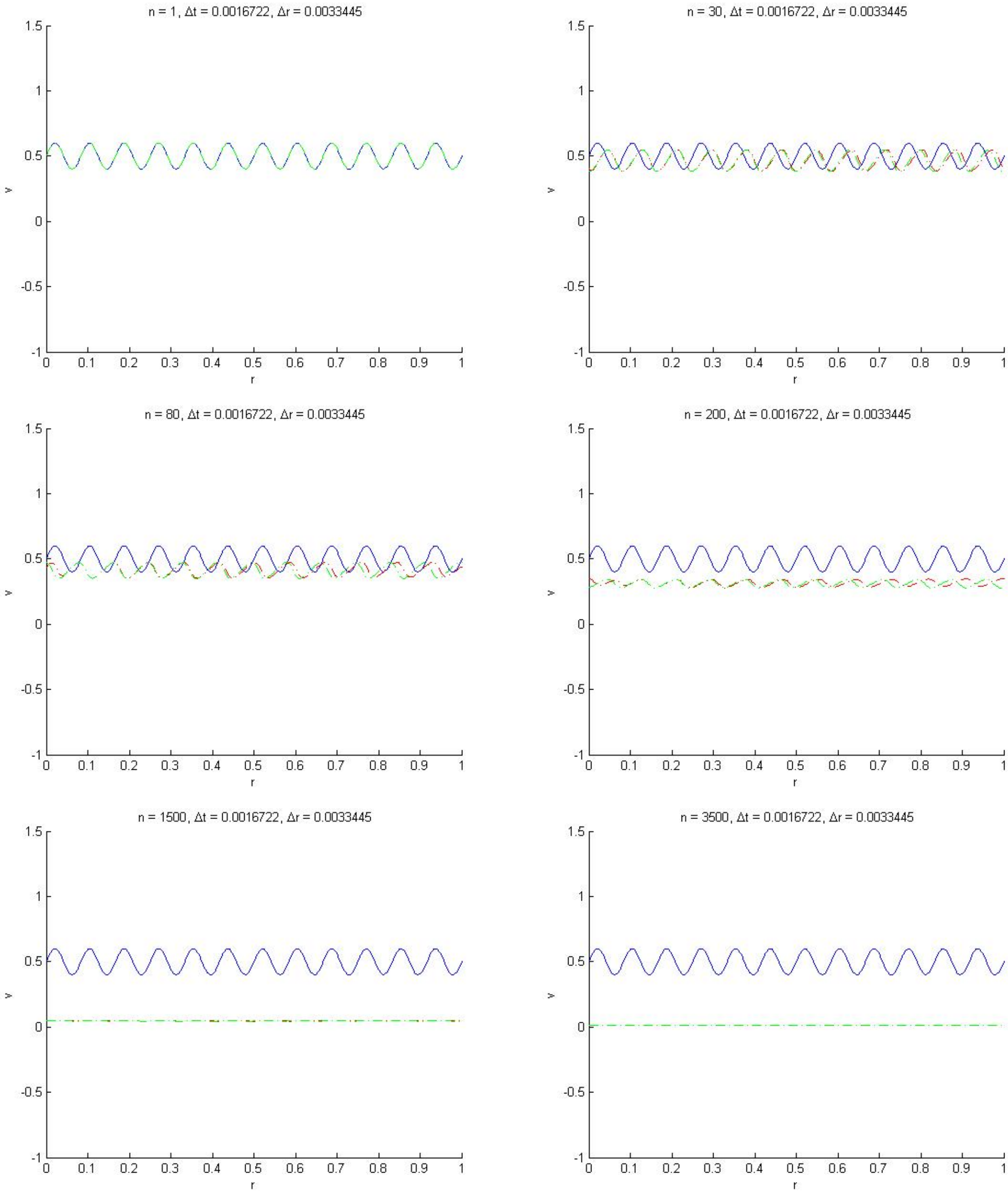

Figure 4.3: The numerical solutions given by the Godunov scheme with the initial function $0.5 + 0.1\sin(24\pi x)$ for the particular case $k = 0$ and $k = -1$.

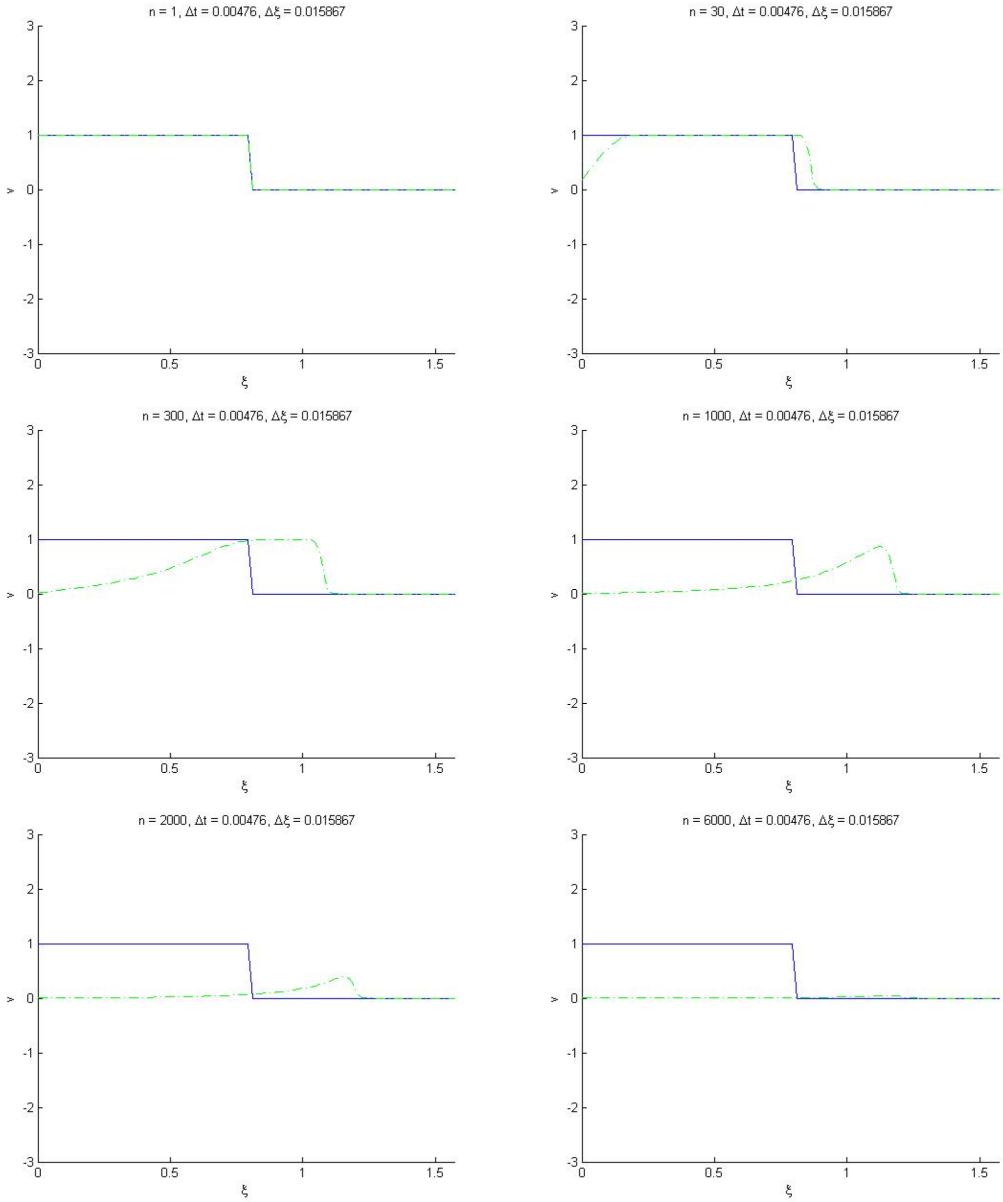

Figure 4.4: The numerical solutions given by the Godunov scheme with a shock for the particular case $k = 1$ with the transformation $\xi(r) = \arcsin(r)$.

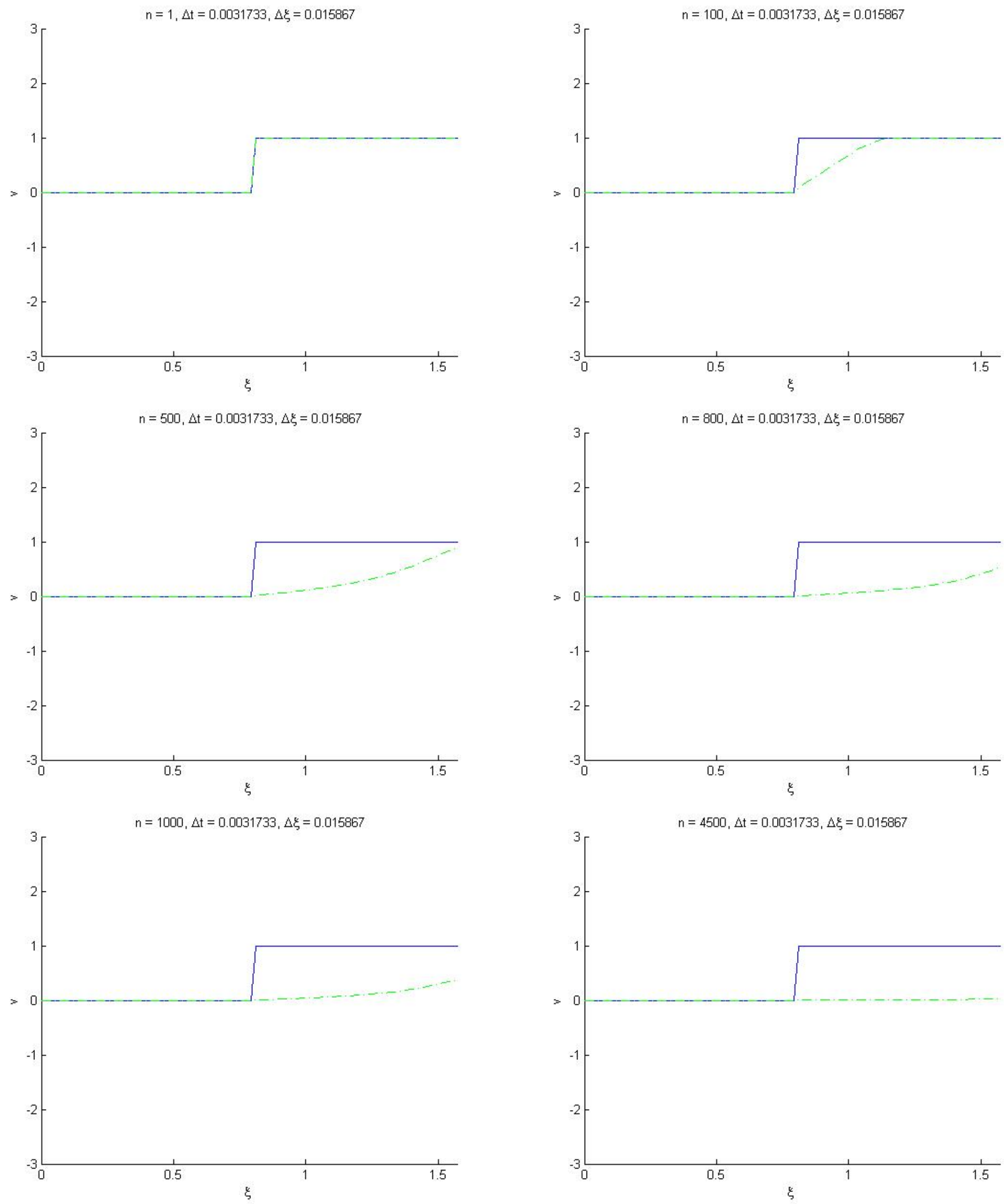

Figure 4.5: The numerical solutions given by the Godunov scheme with a rarefaction for the particular case $k = 1$ with the transformation $\xi(r) = \arcsin(r)$.

(4.7), (4.8) and (4.9). From these graphs one can notice that the numerical solutions converge toward to the homogenous solution by which it can be concluded that the scheme is found to be efficient and well-preserving.

Moreover, in Figures (4.16), (4.17) and (4.18), the green curve represents the quantity

$$\int_0^1 \Big|\kappa - \frac{v_*(t)}{v_h(t)}\Big| dx, \tag{4.7}$$

where $v_*(t)$ is defined as follows

$$v_*(t) = \int_0^1 |v(t,x)| dx. \tag{4.8}$$

The quantity (4.8) is the average of the numerical solution $v(t,x)$ given by the well-balanced Godunov scheme on space and this quantity reaches to zero as time goes to infinity (Figure (4.13), (4.14)and (4.15)). In Figures (4.16), (4.17) and (4.18) again $v_h(t)$ is the homogeneous solution and κ is a constant close to 1. One can observe that the average of numerical solution for the well-balanced Godunov scheme on space reaches to the homogeneous solution since the curve tends to zero whenever $t \to \infty$. In other words, as

$$v_*(t)/v_h(t) \to \kappa \qquad \text{and} \qquad \kappa \approx 1,$$

it follows that the average of the numerical solution on space $v_*(t)$, converges to the homogenous solution $v_h(t)$. To sum up, our scheme is well-balanced in the sense that it preserves homogeneous solutions.

In Figures (4.19), (4.20) and (4.21), we compare the homogeneous solution $v_h(t)$ (represented by the green line) and the quantity

$$\int_0^1 |v_*(t) - v_h(t)| dx,$$

(represented by the red line) where $v_*(t)$ is the average of numerical solution given by the well-balanced Godunov scheme on space. In these figures $a(t)$, w and the initial function $v_0(x)$ are chosen to be the same as the choice for the Figures (4.7), (4.8) and (4.9). In these graphs it is observed that the average of the numerical solutions on space converge toward to the homogenous solution which means the scheme is efficient and well-preserving.

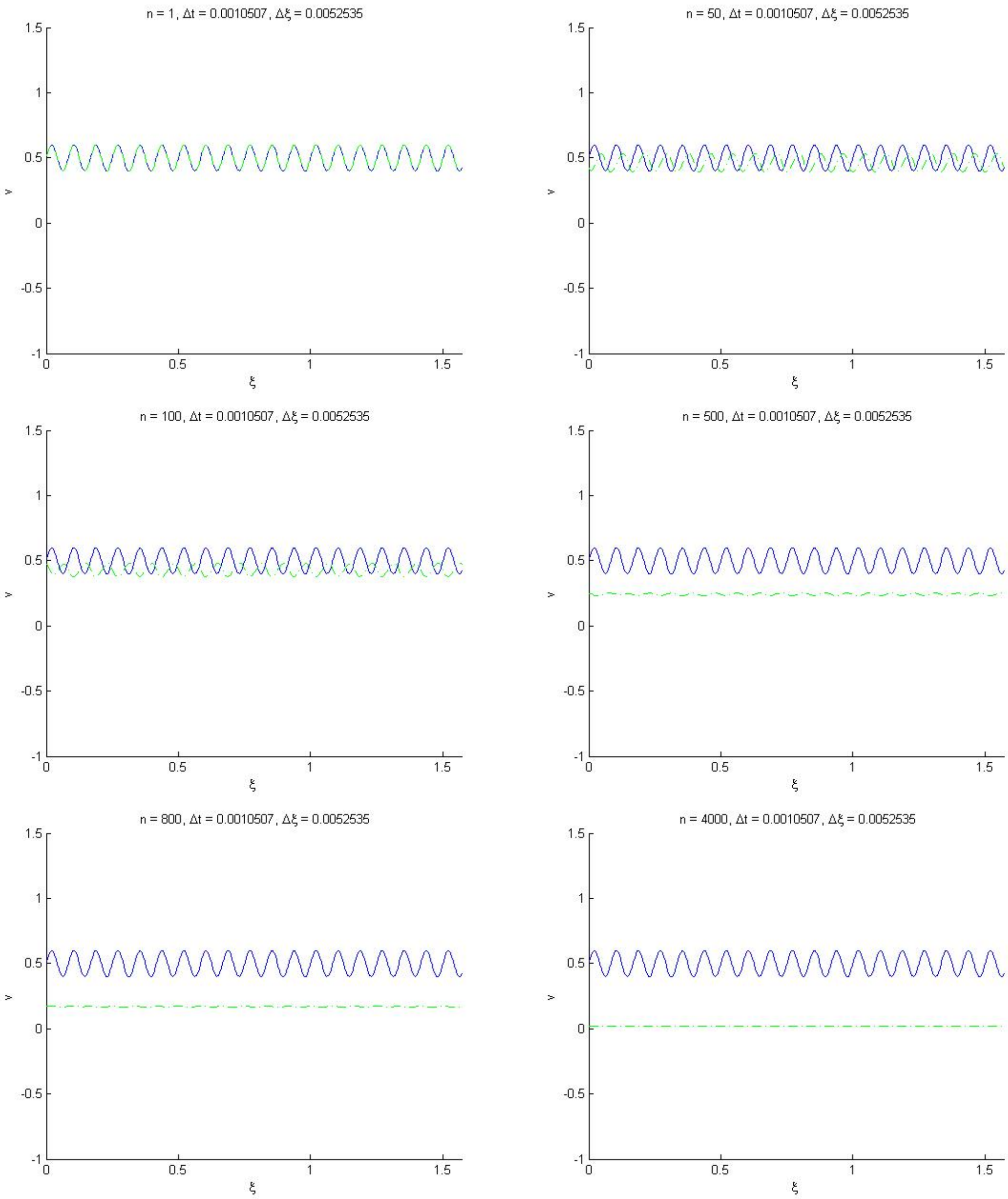

Figure 4.6: The numerical solutions given by the Godunov scheme with the initial function $0.5 + 0.1\sin(24\pi x)$ for the particular case $k = 1$ with the transformation $\xi(r) = \arcsin(r)$.

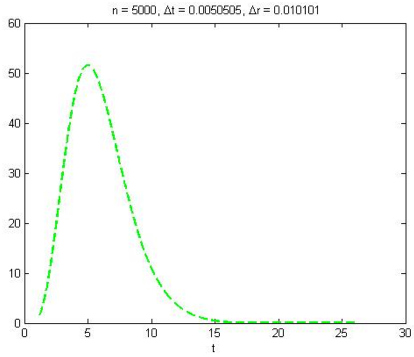

Figure 4.7: The graph of $\int_0^1 |\kappa - \frac{v(t,x)}{v_h(t)}|dx$, $(\kappa \approx 1)$ for the particular case $k = -1$.

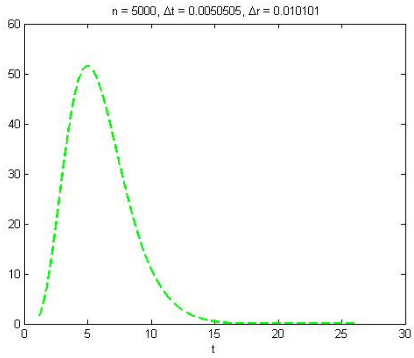

Figure 4.8: The graph of $\int_0^1 |\kappa - \frac{v(t,x)}{v_h(t)}|dx$, $(\kappa \approx 1)$ for the particular case $k = 0$.

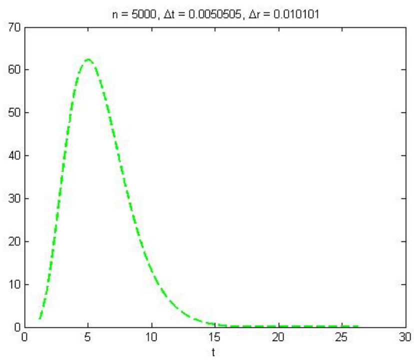

Figure 4.9: The graph of $\int_0^1 |\kappa - \frac{v(t,x)}{v_h(t)}|dx$, $(\kappa \approx 1)$ for the particular case $k = 1$.

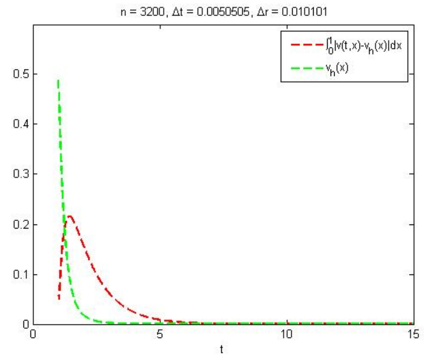

Figure 4.10: The comparison of the homogenous solution and $\int_0^1 |v(t,x) - v_h(t)|dx$ for the particular case $k = -1$.

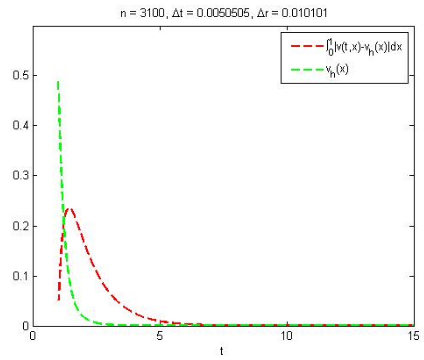

Figure 4.11: The comparison of the homogenous solution and $\int_0^1 |v(t,x) - v_h(t)|dx$ for the particular case $k = 0$.

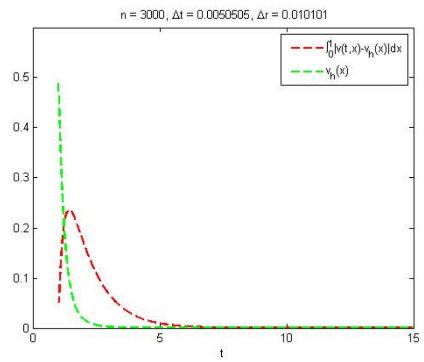

Figure 4.12: The comparison of the homogenous solution and $\int_0^1 |v(t,x) - v_h(t)|dx$ for the particular case $k = 1$.

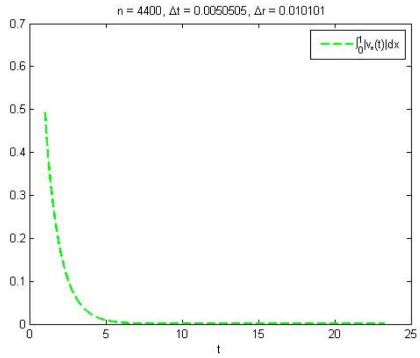

Figure 4.13: The quantity $\int_0^1 |v(t,x)|dx$ for the particular case $k = -1$.

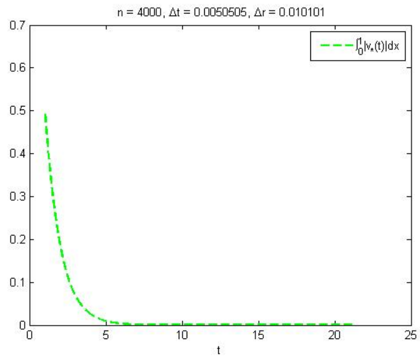

Figure 4.14: The quantity $\int_0^1 |v(t,x)|dx$ for the particular case $k = 0$.

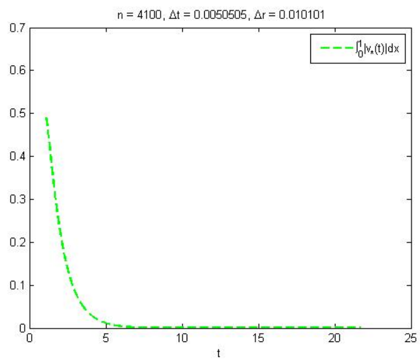

Figure 4.15: The quantity $\int_0^1 |v(t,x)|dx$ for the particular case $k = 1$.

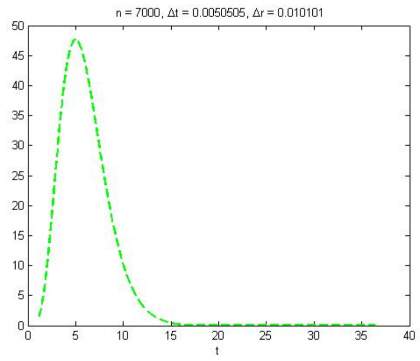

Figure 4.16: The graph of $\int_0^1 |\kappa - \frac{v_*(x)}{v_h(t)}|dx$, ($\kappa \approx 1$) for the particular case $k = -1$.

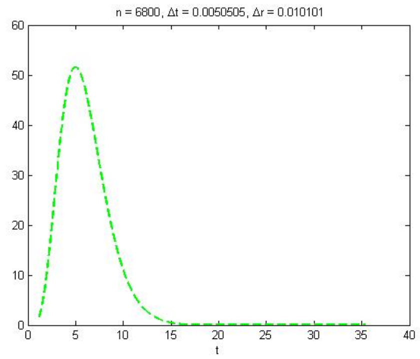

Figure 4.17: The graph of $\int_0^1 |\kappa - \frac{v_*(x)}{v_h(t)}|dx$, ($\kappa \approx 1$) for the particular case $k = 0$.

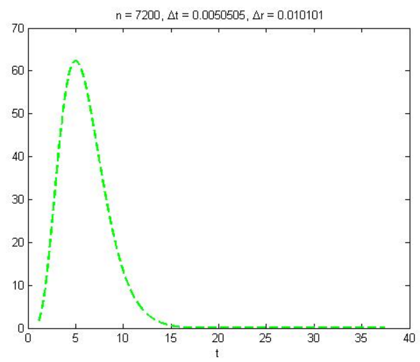

Figure 4.18: The graph of $\int_0^1 |\kappa - \frac{v_*(x)}{v_h(t)}|dx$, ($\kappa \approx 1$) for the particular case $k = 1$.

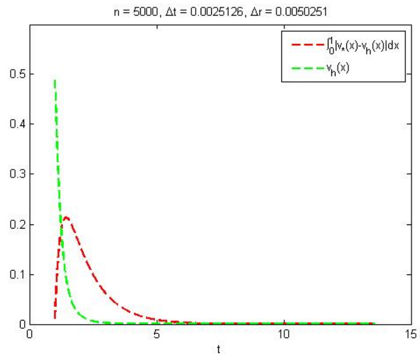

Figure 4.19: The comparison of the homogenous solution and $\int_0^1 |v_*(x) - v_h(t)|dx$ for the particular case $k = -1$.

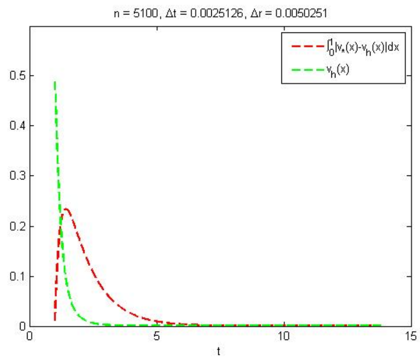

Figure 4.20: The comparison of the homogeneous solution and $\int_0^1 |v_*(x) - v_h(t)|dx$ for the particular case $k = 0$.

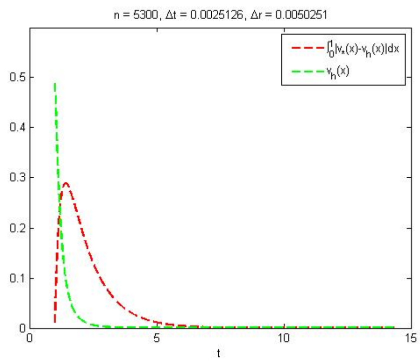

Figure 4.21: The comparison of the homogeneous solution and $\int_0^1 |v_*(x) - v_h(t)|dx$ for the particular case $k = 1$.

CHAPTER 5

CONCLUSION

Concluding remarks and perspectives

In this thesis, we have derived a new nonlinear hyperbolic model which describes the propagation and interactions of shock waves on a Friedmann–Lemaître–Robertson–Walker background spacetime. We started from the relativistic Euler equations on a curved background and we imposed a vanishing pressure in the expression of the energy–momentum tensor for perfect fluids. This led us to a *geometric relativistic Burgers equation* (see (3.3)) on the background spacetime under consideration. On a FLRW spacetime, the equation (3.3) yields the model (3.24) of interest in the present work. The model involves a scale factor $a = a(t)$ which depends on the so-called 'cosmic time' and a constant coefficient k, which can be normalized to take the values ± 1 or 0. We observe that the proposed relativistic Burgers equation on FLRW background shares several important features with the relativistic Euler equations. The unknown v of the model is in the interval $(-c, c)$ limited by the light speed parameter, same as the velocity component in the Euler system. In the Euler system, one recovers the classical (non relativistic) model by sending the light speed to infinity. Analogously, the classical Burgers equation is obtained with the particular case of our relativistic model ($a(t) = 1$ and $k = 0$) whenever c tends to infinity. With the particular case $a(t) = 1$ and vanishing k, the standard Burgers equation is again recovered without sending c to infinity. We have then established various mathematical properties concerning the hyperbolicity, genuine nonlinearity, shock waves, and rarefaction waves, and we studied the class of spatially homogeneous solutions. In the relativistic Burgers equation on a FLRW background, the main dependence is in time rather than space.

We have investigated shock wave solutions to our model for the three possible values of the coefficient k.

- In the case $k = 1$, the source term of the model contains a singularity and we have found it advantageous to introduce a transformation which removes this drawback. Our numerical results demonstrate the convergence of the scheme to shock wave solutions to the model, and we observed the asymptotic property $v \to 0$ as $t \to +\infty$ (Figures (4.4) and (4.5)).
- In addition, we compared numerical solutions for the cases $k = -1$ and $k = 0$, and we found that the solution curve corresponding to $k = -1$ converges faster (to 0) than the solution curve corresponding to $k = 0$ (Figures (4.1) and (4.2)). This can be explained from (3.24) by observing that the characteristic speed $\left(1 - kr^2\right)^{1/2}$ is increased by decreasing k.

Moreover, we compared the numerical solutions of the well-balanced scheme and the spatially homogeneous solution. It is observed by the curve corresponding to the quantity defined in (4.6) that the well-balanced scheme converges to the homogeneous solution (Figure (4.7), (4.8), (4.9), (4.10), (4.11), and (4.12)). Furthermore, in Figures (4.16), (4.17), (4.18), (4.19), (4.20) and (4.21) the average of the numerical solutions on space for all particular cases is compared with homogeneous solution. In these numerical tests the curve which represents the quantity (4.7) proves that our well-balanced Godunov scheme converges to the homogeneous solution.

Our analysis relies on a proposed numerical discretization scheme which applies to discontinuous solutions and is based on the finite volume technique.

- Our scheme is consistent with the conservative form of (the principal part of) our model and therefore correctly compute weak solutions containing shock waves.
- Importantly, the proposed scheme is well-balanced, in the sense that it preserves (at the discrete level of approximation) all spatially homogeneous solutions.

- Our numerical experiments illustrate the convergence of the proposed scheme on a FLRW background.

As a perspective, we emphasize that the proposed methodology leading to a geometric relativistic balance law may be used to derive other relativistic versions of Burgers equations on various classes of spacetimes. The advantages of such simplified nonlinear hyperbolic models is that they allow one to develop and test numerical methods for shock capturing and to reach definite conclusions concerning their convergence, efficiency, etc. Future work may include more singular backgrounds. Depending upon the particular background geometry, different techniques may be required in order to guarantee that certain classes of solutions of particular interest be preserved by the scheme, as we achieved it for time-dependent solutions.

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Visiting Scholar	Pierre et Marie Curie University	2013 Summer
B.S.	Middle East Technical University	2004-2009
High School	Kırıkkale Science High School	2001-2004

PROFESSIONAL EXPERIENCE

Year	Place	Enrollment
2012-present	Central Bank of Turkey	Government Official
2009-2012	METU, Department of Mathematics	Research Assistant

FOREIGN LANGUAGES

English (Advanced), German (Intermediate)

RESEARCH INTEREST

- Partial Differential Equations
- Hyperbolic Balance Laws
- Finite Volume Methods
- Curved Spacetimes

SCHOLARSHIPS

Ph.d. BIDEB 2214-A Scholarship Program, TUBITAK, 2013 Summer

Rectorate of Middle East Technical University, Project BAP 08 11 2013 041

PUBLICATIONS

T. Ceylan and B. Okutmustur, *Derivation of relativistic Burgers equation on de Sitter background*, Proccedings of 19^{th} International Conference on Applied Mathematics, (AMATH'14), Mathematics and Computers in Science and Engineering Series 38, 2014, pp. $41 - 47$.

T. Ceylan and B. Okutmustur, *The relativistic Burgers equation on a de Sitter spacetime. Derivation and finite volume approximation*, submitted for publication.

T. Ceylan, P. G. LeFloch and B. Okutmustur, *The relativistic Burgers equation on a FLRW background and its finite volume approximation*, preparation for publication.

INTERNATIONAL CONFERENCE PRESENTATIONS

Conference on Partial Differential Equations (COPDE 2014), Novacella, Italy, May 28-31, 2014.