ON GENERALIZED INTEGRAL INEQUALITIES WITH APPLICATIONS IN BIO-MATHEMATICS AND PHYSICAL SCIENCES

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ABSTRACT

ON GENERALIZED INTEGRAL INEQUALITIES WITH APPLICATIONS IN BIO-MATHEMATICS AND PHYSICAL SCIENCES

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In this thesis, applications of generalized integral inequalities especially on biomathematics and physics are studied. Application on Biomathematics is about the predator-prey dynamic systems with Beddington DeAngelis type functional response and application on physics is about water percolation equation.

This thesis consists 6 chapters. Chapter 1 is introductory and contains the thesis structure. Chapter 2 is about under which conditions the two dimensional predator-prey dynamic system with Beddington DeAngelis type functional response is permenent and globally attractive. Chapter 3 is about the same type dynamic system but with impulses. In that chapter under which conditions the dynamic system has at least one periodic solution is investigated. To get the result we use Continuation Theorem. Using impulse on this type of dynamic system is also important. Because we can model the real life much better by this way. In Chapter 4, the predator-prey dynamic system with Beddington DeAngelis type functional response on periodic time scales in shifts is studied. In this chapter, first we prove which kind of periodic time scales in shifts should be used to find there is at least one $\delta_{\pm}$-periodic solution for the given system. Then again by using Continuation Theorem we get the desired result. In Chapter 5, first we generalize the Constantin's Inequality on Nabla and Diamond-α calculus on time scales. Then by using a topological transversality theorem and using the generalization of Constantin's Inequality on Nabla Calculus, we have showed that the water percolation equation on nabla time scales calculus has solution. This solution is unique and bounded. The last chapter is the summary of what we have done in this thesis.

As a result, since this study is on time scales, the findings are also important on the discrete and continuous case.

Keywords: Generalization of Integral Inequalities, Time Scales Calculus, Predator-Prey Dynamic Systems, Beddington DeAngelis Type Functional Response, Impulses, Constantin's Inequality, Water Percolation Equation

ÖZ

BİOLOJİK MATEMATİK VE FİZİKSEL BİLİMLERE UYGULAMALARIYLA BİRLİKTE GENELLEŞTİRİLMİŞ İNTEGRAL EŞİTSİZLİKLERİ ÜZERİNE

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Bu tezde genelleştirilmiş integral eşitsizliklerinin özellikle biomatematiğe ve fiziğe uygulamaları incelenmiştir. Biomatematik üzerindeki uygulama Beddington DeAngelis tipi fonksiyonel cevabı içinde barındıran iki boyutlu bir av-avcı dinamik sistemi üzerindedir ve fizikle ilgili olan araştırmada su sızdırma denklemiyle ilgilidir.

Bu tez 6 bölümden oluşmaktadır. Birinci bölüm giriş niteliğindedir ve tez yapısı hakkında bilgi vermektedir. İkinci bölümdeyse Beddington DeAngelis tipi fonksiyonel cevabı içinde barındıran iki boyutlu bir av-avcı dinamik sisteminin hangi şartlar altında permenent ve global atraktiv çözümlerinin var olduğu incelenmiştir. Üçüncü bölümde Süreklilik Teoremi kullanılarak aynı tip bir dinamik sistem için impuls verilerek hangi şartlar altında en az bir tane periodik çözümünün var olacağı incelendi. Ayrıca bu bölümde sistemin impuls verilmiş halini inceliyoruz çünkü bu bize doğal yaşamın daha iyi bir modellemesini veriyor. Dördüncü Bölümde aynı tip bir dinamik sistemin hangi zaman ötelemelerine göre periodik olan zaman skaları üzerinde en az bir tane $\delta_{\pm}$-periodik çözümü vardır diye bakıyoruz ve sonuca ulaşmak için tekrar Süreklilik Teoremini kullanıyoruz. Beşinci Bölümde ise Constantin Eşitsizliğinin Nabla ve Diamond-α analize genelleştirilmesini inceliyoruz ve Nabla analize uygun olarak verilmiş su sızdırma denkleminin çözümünün var ve tek olduğunu ve sınırlı olduğunu bir topolojik transversal teoremi ve Constantin Eşitsizliğinin nabla analize genelleştilmiş halini kullanarak gösterebiliyoruz. Son bölüm ise tezde yapmış olduklarımızın özeti niteliğindedir.

Sonuç olarak çalışmalarımız zaman skalsı üzerinde yapılmış olduğu için sonuçlar bize ayrık ve sürekli sistemlerle ilgili de fikir vermektedir.

Anahtar Kelimeler: Genelleştirilmiş İntegral Eşitsizlikleri, Zaman Skalası Analizi, Av-Avcı Dinamik Sistemleri, Beddington DeAngelis Tipi Fonksiyonel Cevaplar, İmpulslar, Constantin Eşitsizliği, Su Sızdırma Denklemi

To my grandparents, parents, brother, husband and son

Hüseyin Karaca, Nurullah Aykır, Şerife Aykır, Naile Karaca, Halil Aykır, Mediha Aykır, Ömer Faruk Aykır, Rumi Melih Pelen, Abdullah Yusuf Pelen

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CHAPTER 1

INTRODUCTION

In this thesis, unification of continuous and discrete analysis is significant. For the unification of the differential and difference equations, Stephan Hilger [31] come up with a theory which is known as the theory of Time Scales Calculus.

Another important subject in this thesis is mathematical ecology in biomathematics which can be defined as the relationships between species and the outer environment. The predator-prey dynamic systems can be seen as the subsection of the ecology in biomathematics and can be described as the mathematical model of the connections between different species. For this type of dynamic systems global attractivity and permanence are very important issues. In [13], [25], [52], there are some results about the global attractivity and permanence in different predator- prey dynamic systems. Various type of functional responses such as semi-ratio dependent and Holling-type functional responses in predator-prey dynamic systems have been investigated in several studies like [49], [50]. In this thesis, we consider the predator-prey systems with Beddington DeAngelis type functional response. This type of functional response was first appeared in [6] and [18]. From these studies we know that at low densities this type of functional response can avoid some of the singular behavior of ratio-dependent models. Also predator feeding can be described much better over a range of predator-prey abundances by using this functional response. For such kind of system, boundedness of solution, permanence and global attractivity are also several important topics.

In [25] for discrete predator-prey system with Beddington-DeAngelis type functional response permanence and global attractivity was studied. Also in [21] for continuous predator-prey system with Beddington-DeAngelis type functional response permanence and global attractivity was studied. Therefore in the second chapter of this thesis we try to investigate the permanence and global attractivity of the solutions of predator-prey dynamic systems with Beddington-DeAnglis type functional response in a general time scale.

In the third chapter of this thesis periodic environment becomes important. On the other hand, in a periodic environment significant problem in population growth model is the global existence and stability of a positive periodic solution. This plays a similar role as a globally stable equilibrium in an autonomous model. Therefore, it is important to consider under which conditions the resulting periodic nonautonomous system would have a positive periodic solution that is globally asymptotically stable. And in the third chapter we deal with when we can be able to find a globally asymptotically stable periodic solution for the given dynamic system in the continuous case. For nonautonomous case there are many studies about the existence of periodic solutions of predator-prey systems in continuous and discrete models based on the coincidence theory such as [20], [22], [23], [24], [25], [33], [37], [49],[53]. Also in [8], [26] unification of the existence of periodic solutions of continuous population models i.e. population model for ordinary differential equations and discrete population models i.e. population model in the form of difference equations and generalization of these results to more general time scales is studied.

Impulsive dynamic systems are also important in the third chapter and we try to give some information about this area. Impulsive differential equations used for describing systems with short-term perturbations. Its theory is explained in [5], [45], [36] for continuous case and also for discrete case there are some studies such as [48]. Impulsive differential equations are widely used in many different areas such as physics, ecology, pest control, population ecology, chemotherapeutic treatment of disease and impulsive birth. Most of them uses impulses at fixed time as it is studied in [47] and [52]. By using constant coefficient functions some properties of the solution of predator-prey system with Beddington-DeAnglis type functional response and impulse impact is studied in [51] for continuous case.

In the fourth chapter, we again consider the predator-prey dynamic system with Beddington DeAngelis type functional response on periodic time scales in shifts. Any periodic time scales in shifts is not appropriate to get a $\delta_{\pm}$-periodic solution. Therefore, in this chapter first we determine which kind of periodic time scales in shifts is used. Then again by using Continuation Theorem we try to obtain under which conditions there is a $\delta_{\pm}$-periodic solution for the given dynamic system. And we start to think about this issue after the paper of Murat Adıvar[1].

The main contribution of this thesis in these chapters is for the species that have unusual life cycle. Specifically this contribution is about to be able to find conditions when the given species are extinct or they will able to save their generation and can be able to obtain the equilibria after enough time passes or the life cycle of them are periodic under impulses.

In the fifth chapter, the generalization of Constantin's inequality is given and by using this generalization and a topological transversality theorem; the existence, uniqueness and boundedness of the solutions of the Water Percolation Equation on nabla time scales calculus is investigated.

To study the boundedness of solutions for some nonautonomuous second order linear differential equations Ou-Iang [42] used a nonlinear integral inequality. To show global existence, uniqueness and stability properties of various nonlinear differential equations using Ou-Iang type integral inequality is also significant. Pachpatte [43] gave the generalized Ou-Iang type integral inequality. In 1996, for the generalization of these integral inequalities Constantin [14] get the interesting alternative result ; which is known as Constantin's inequality now. Then researchers continue to work on this inequality. Afterwards, Yang and Tan [54] generalized the inequality and gives the discrete form of the inequality. Then, Ferriara [27] generalized the inequality on delta time scales calculus. And our study in chapter five is continuation of above studies.

CHAPTER 2

SOME RESULTS ON PREDATOR-PREY DYNAMIC SYSTEMS WITH BEDDINGTON DEANGELIS TYPE FUNCTIONAL RESPONSE ABOUT THE PERMANENCE AND GLOBAL ATTRACTIVITY OF THE SOLUTIONS

2.1 Preliminaries

[8]The theory of time scales calculus is the unification of continuous and discrete analysis and by a time scale which is denoted as $\mathbb{T}$, we mean a nonempty closed subset of $\mathbb{R}$.

Definition 2.1.1 *[8],[35] The set* $\mathbb{T}^{\kappa}$ *is defined by* $\mathbb{T}^{\kappa} = \mathbb{T}/(\rho(sup\mathbb{T}), sup\mathbb{T}]$ *and the set* $\mathbb{T}_{\kappa}$ *is defined by* $\mathbb{T}_{\kappa} = \mathbb{T}/[inf\mathbb{T}, \sigma(inf\mathbb{T}))$. *The forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ *is defined by* $\sigma(t) := inf(t, \infty)_{\mathbb{T}}, for\ t \in \mathbb{T}$. *The forward graininess function* $\mu : \mathbb{T} \to \mathbb{R}_0^+$ *is defined by* $\mu(t) := \sigma(t) - t, for\ t \in \mathbb{T}$. *Here it is assumed that* $inf\emptyset = sup\mathbb{T}$ *and* $sup\emptyset = inf\mathbb{T}$.

Definition 2.1.2 *[8], [35] A function* $f : \mathbb{T} \to \mathbb{R}$ *is rd-continuous if it is continuous at right dense points in* $\mathbb{T}$ *and its left-sided limits exist at left-dense points in* $\mathbb{T}$. *The class of real rd-continuous functions defined on a time scale* $\mathbb{T}$ *is denoted by* $C_{rd}(\mathbb{T}, \mathbb{R})$. *If* $f \in C_{rd}(\mathbb{T}, \mathbb{R})$ *then there exists a function* $F(t)$ *such that* $F^{\Delta}(t) = f(t)$. *The delta integral is defined by* $\int_a^b f(x)\Delta x = F(b) - F(a)$. *If* $f \in C_{rd}^1(\mathbb{T}, \mathbb{R})$ *then* $f \in C_{rd}(\mathbb{T}, \mathbb{R})$ *and* $f^{\Delta} \in C_{rd}(\mathbb{T}, \mathbb{R})$.

The definitions related with diamond-alpha and nabla derivatives are can be found in section 5.1.

Theorem 2.1.1 *[39](Diamond-alpha Lagrange's mean value theorem) Suppose f be a delta and nabla differentiable function on (a, b) and continuous function on $[a, b]$. Then there exists $\alpha \in [0, 1]$ and $c \in (a, b)$ such that the following equality hold true.*

$$f^{\diamond_\alpha}(c) = \frac{f(b) - f(a)}{b - a}.$$

2.2 Predator-Prey Dynamic System with Beddington DeAngelis Type functional Response

The equation that was investigated in several studies such as in [10]

$$\begin{aligned} x^{\Delta}(t) &= a(t) - b(t)exp(x(t)) - \tfrac{c(t)exp(y(t))}{\alpha(t)+\beta(t)exp(x(t))+m(t)exp(y(t))}, \\ y^{\Delta}(t) &= -d(t) + \tfrac{f(t)exp(x(t))}{\alpha(t)+\beta(t)exp(x(t))+m(t)exp(y(t))}, \end{aligned} \tag{2.1}$$

In this system, from [50];

1. $a(t) - b(t)exp(x(t))$ is the specific growth rate of the prey in the absence of predator,
2. $d(t)$ is the death rate of predator,
3. $-\frac{c(t)exp(y(t))}{\alpha(t)+\beta(t)exp(x(t))+m(t)exp(y(t))}$ is the Beddington DeAngelis type effect of predator on prey,
4. $\frac{f(t)exp(x(t))}{\alpha(t)+\beta(t)exp(x(t))+m(t)exp(y(t))}$ is the Beddington DeAngelis type effect of prey on predator.

2.3 Permenance

Taking $\tilde{x}(t) = exp(x(t))$ and $\tilde{y}(t) = exp(y(t))$ in (2.1), then we get

$$\begin{aligned} (ln(\tilde{x}(t)))^{\Delta} &= a(t) - b(t)\tilde{x}(t) - \tfrac{c(t)\tilde{y}(t)}{\alpha(t)+\beta(t)\tilde{x}(t)+m(t)\tilde{y}(t)}, \\ (ln(\tilde{y}(t)))^{\Delta} &= -d(t) + \tfrac{f(t)\tilde{x}(t)}{\alpha(t)+\beta(t)\tilde{x}(t)+m(t)\tilde{y}(t)}. \end{aligned} \tag{2.2}$$

Assume $a(t)$, $b(t)$, $c(t)$, $d(t)$, $f(t)$, $\beta(t)$, $m(t) > 0$. $\alpha(t) \geq 0$ and bounded above in (2.2). Each functions are from $C_{rd}(\mathbb{T}, \mathbb{R})$. $x(t), y(t) \in C_{rd}(\mathbb{T}, \mathbb{R})$. $sup_{t \in \mathbb{T}} a(t) = a^u$, $inf_{t \in \mathbb{T}} a(t) = a^l$. Similar for the other functions.

Definition 2.3.1 *[19] We call that the solution* $\begin{bmatrix} x \\ y \end{bmatrix}$ *of the system* (2.2) *is positive solution if it is positive on the time domain of the given system.*

Definition 2.3.2 *[44] The solution* $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ *of the system* (2.2) *is oscillatory about* $\begin{bmatrix} J_1 \\ J_2 \end{bmatrix}$ *if* $\begin{bmatrix} x(t) - J_1 \\ y(t) - J_2 \end{bmatrix}$ *has arbitrarily large zeros.*

Definition 2.3.3 *[25] The system* (2.2) *is called permanent if there exist positive constants* r_1, r_2, R_1, *and* R_2 *such that solution* $(x(t), y(t))$ *of system* (2.2) *satisfies*

$$r_1 \leq lim_{t \to \infty} inf \ \tilde{x}(t) \leq lim_{t \to \infty} sup \ \tilde{x}(t) \leq R_1,$$

$$r_2 \leq lim_{t \to \infty} inf \ \tilde{y}(t) \leq lim_{t \to \infty} sup \ \tilde{y}(t) \leq R_2.$$

Lemma 2.3.1 *In system* (2.1) *if there exists* $T \in \mathbb{T}$ *such that for each* $w \geq T$ *following inequality*

$$-\int_0^w d(t) \Delta t + \int_0^w \frac{f(t)}{\beta(t)} \Delta t < 0$$

holds true, then all positive solutions $y(t)$ *tends to minus infinity as t tends to infinity, eqivalently* $exp(y(t))$ *tends to 0 as t tends to infinity.*

Proof. $y^{\Delta}(t) = -d(t) + \frac{f(t) exp(x(t))}{\alpha(t) + \beta(t) exp(x(t)) + m(t) exp(y(t))} \leq -d(t) + \frac{f(t)}{\beta(t)}$. If we integrate this inequality we obtain,

$$y(t) \leq y(0) + \int_0^t -d(s) + \frac{f(s)}{\beta(s)} \Delta s,$$

$$exp(y(t)) \leq exp(y(0)) exp(\int_0^t -d(s) + \frac{f(s)}{\beta(s)} \Delta s).$$

Since $\int_0^w -d(t)\Delta t + \int_0^w \frac{f(t)}{\beta(t)} \Delta t < 0$, then $lim_{t \to \infty} exp(\int_0^t -d(s) + \frac{f(s)}{\beta(s)} \Delta s) = 0$. Hence $lim_{t \to \infty} exp(y(t)) = 0$.

□

Remark 2.3.1 *In system* (2.2) *if* $a(t) < 0$, *then automatically all positive solutions* $\tilde{x}(t) = exp(x(t))$ *tends to 0 as t tends to infinity*

Remark 2.3.2 *If in sytem* (2.2) $a(t) < 0$, *or the conditions of Lemma 2.3.1 is satisfied then system* (2.2) *can not be permanent by Definition 2.3.3.*

The following results are true for the time scales whose graininess function is bounded over this Time scales and $\mu = max_{t\in\mathbb{T}}\mu(t)$.

Lemma 2.3.2 *For the system* (2.2) *if conditions for the coefficient functions are satisfied, then*

$$\tilde{x}(t) \leq \frac{a^u}{b^l} exp(\mu a^u) = G_1,$$

if $-d^l + \frac{f^u}{\beta^l} \geq 0$ *then* $\tilde{y}(t) \leq \frac{f^u G_1}{d^l m^l} exp\left(\mu(-d^l + \frac{f^u}{\beta^l})\right) = G_2.$

Proof. Let us start with the first equation of (2.2),

$$\begin{array}{rcl} (ln(\tilde{x}(t)))^{\Delta} & = & a(t) - b(t)\tilde{x}(t) - \frac{c(t)\tilde{y}(t)}{\alpha(t)+\beta(t)\tilde{x}(t)+m(t)\tilde{y}(t)} \\ & \leq & a(t) - b(t)\tilde{x}(t) \leq a^u - b^l\tilde{x}(t). \end{array} \tag{2.3}$$

Taking $M_1 = \frac{a^u}{b^l}(k+1)$, where $0 < k < exp\{\mu a^u\} - 1$. If $\tilde{x}(t)$ is not oscillatory about M_1, there exists $T_1 > 0$ such that $\tilde{x}(t) > M_1$ for $t > T_1$ or $\tilde{x}(t) < M_1$ for $t > T_1$.

If $\tilde{x}(t) < M_1$ for $t > T_1$, then $\tilde{x}(t) \leq \frac{a^u}{b^l} exp\{\mu a^u\}$. If $\tilde{x}(t) > M_1$ for $t > T_1$,

then $(ln(\tilde{x}(t)))^{\Delta} \leq -ka^u$, hence there exists $T_2 \in \mathbb{T}$ such that $T_1 < T_2$ and for $t > T_2$, $\tilde{x}(t) < M_1$, which is a contradiction.

If $\tilde{x}(t)$ is oscillatory about M_1 for $t > T_1$ and $\sigma(\tilde{t})$ be an arbitrary local maximum of $ln(\tilde{x}(t))$, then

$$\begin{array}{rcl} 0 \leq (ln(\tilde{x}(\tilde{t})))^{\Delta} & = & a(\tilde{t}) - b(\tilde{t})\tilde{x}(\tilde{t}) - \frac{c(\tilde{t})\tilde{y}(\tilde{t})}{\alpha(\tilde{t})+\beta(\tilde{t})\tilde{x}(\tilde{t})+m(\tilde{t})\tilde{y}(\tilde{t})} \\ & \leq & a(\tilde{t}) - b(t)\tilde{x}(\tilde{t}) \leq a^u. \end{array}$$

Therefore $\tilde{x}(\tilde{t}) \leq \frac{a(\tilde{t})}{b(\tilde{t})}$. If $\tilde{t}$ is right dense then $\tilde{x}(\sigma(\tilde{t})) \leq \frac{a(\tilde{t})}{b(\tilde{t})}$. If $\tilde{t}$ is right scattered, then integrating first equation of (2.2) from $\tilde{t}$ to $\sigma(\tilde{t})$ and using (2.3) we obtain

$$\begin{aligned}\int_{\tilde{t}}^{\sigma(\tilde{t})}(ln(\tilde{x}(s)))^{\Delta}\Delta s &= \int_{\tilde{t}}^{\sigma(\tilde{t})} a(s) - b(s)\tilde{x}(s) - \frac{c(s)\tilde{y}(s)}{\alpha(s)+\beta(s)\tilde{x}(s)+m(s)\tilde{y}(s)}\Delta s \\ &\leq \mu a^u.\end{aligned}$$

$$\tilde{x}(\sigma(\tilde{t}))) \leq \frac{a^u}{b^l}exp(\mu a^u) = G_1 \tag{2.4}$$

Since $\sigma(\tilde{t})$ be an arbitrary local maximum of $ln(\tilde{x}(t))$ then $lim_{t\to\infty}sup\ \tilde{x}(t) \leq G_1$. Hence $lim_{t\to\infty}sup\ x(t) \leq R_1$.

Consider the second equation of (2.2) and get

$$(ln(\tilde{y}(t)))^{\Delta} = -d(t) + \frac{f(t)\tilde{x}(t)}{\alpha(t)+\beta(t)\tilde{x}(t)+m(t)\tilde{y}(t)} \leq -d(t) + \frac{f(t)}{\beta(t)} \leq -d^l + \frac{f^u}{\beta^l}. \tag{2.5}$$

Taking $M_2 = \frac{f^u G_1}{d^l m^l}(k+1)$, where $0 < k < exp\left(\mu(-d^l + \frac{f^u}{\beta^l})\right) - 1$. If $\tilde{y}(t)$ is not oscillatory about M_2, there exists $T_3 > 0$ such that $\tilde{y}(t) > M_2$ for $t > T_3$ or $\tilde{y}(t) < M_2$ for $t > T_3$.

If $\tilde{y}(t) < M_2$ for $t > T_3$, then $\tilde{y}(t) < \frac{f^u G_1}{d^l m^l}(k+1)$. If $\tilde{y}(t) > M_2$ for $t > T_3$, then $(ln(\tilde{y}(t)))^{\Delta} \leq -k\frac{f^u G_1}{\alpha^u+\beta^u G_1+m^u M_2}$, hence there exists $T_4 \in \mathbb{T}$ such that $T_4 > T_3$ and for $t > T_4$, $\tilde{y}(t) < M_2$, which is a contradiction.

If $\tilde{y}(t)$ is oscillatory about M_2 for $t > T_3$, let $\sigma(\hat{t})$ be an arbitrary local maximum of $ln(\tilde{y}(t))$, then by using second equation of (2.2), we can conclude that

$$\begin{aligned}0 \leq (ln(\tilde{y}(\hat{t})))^{\Delta} &= -d(\hat{t}) + \frac{f(\hat{t})\tilde{x}(\hat{t})}{\alpha(\hat{t})+\beta(\hat{t})\tilde{x}(\hat{t})+m(\hat{t})\tilde{y}(\hat{t})} \\ &= \frac{-d(\hat{t})\alpha(\hat{t})-d(\hat{t})\beta(\hat{t})\tilde{x}(\hat{t})-d(\hat{t})m(\hat{t})\tilde{y}(\hat{t})+f(\hat{t})\tilde{x}(\hat{t})}{\alpha(\hat{t})+\beta(\hat{t})\tilde{x}(\hat{t})+m(\hat{t})\tilde{y}(\hat{t})} \\ &\leq \frac{-d(\hat{t})m(\hat{t})\tilde{y}(\hat{t})+f(\hat{t})G_1}{\beta(\hat{t})\tilde{x}(\hat{t})}.\end{aligned}$$

Therefore, $\tilde{y}(\hat{t}) \leq \frac{f(\hat{t})G_1}{d(\hat{t})m(\hat{t})}$. If $\hat{t}$ is right dense then $\tilde{y}(\sigma(\hat{t})) \leq \frac{f(\hat{t})G_1}{d(\hat{t})m(\hat{t})}$.

If $\hat{t}$ is right scattered, integrate (2.5) from $\hat{t}$ to $\sigma(\hat{t})$ for the same w above and we obtain

$$\begin{aligned}\int_{\hat{t}}^{\sigma(\hat{t})}(ln(\tilde{y}(s)))^{\Delta}\Delta s &= \int_{\hat{t}}^{\sigma(\hat{t})} -d(s) + \frac{f(s)\tilde{x}(s)}{\alpha(s)+\beta(s)\tilde{x}(s)+m(s)\tilde{y}(s)}\Delta s \\ &\leq \mu(-d^l + \frac{f^u}{\beta^l}).\end{aligned}$$

$$\tilde{y}(\sigma(\hat{t}))) \leq \frac{f^u G_1}{d^l m^l} exp\Bigg(\mu(-d^l + \frac{f^u}{\beta^l}) \Bigg) = G_2. \tag{2.6}$$

Since $\sigma(\hat{t})$ be an arbitrary local maximum of $ln(\tilde{y}(t))$ then $lim_{t\to\infty} sup\ \tilde{y}(t) \leq G_2$. Hence $lim_{t\to\infty} sup\ y(t) \leq R_2$. □

Remark 2.3.3 *If for the system* (2.1) *all the solutions of* $exp(y(t))$ *does not tend to 0 as t tends to infinity (by Lemma 2.3.1) then Lemma 2.3.2 is otomatically true.*

Lemma 2.3.3 *For* (2.2) *when* $\tilde{x}(t) \leq G_1$, $a^l - b^u G_1 - \frac{c^u}{m^l} \leq 0$, $a^l - \frac{c^u}{m^l} \geq 0$ *is satisfied then*

$$\tilde{x}(t) \geq \frac{1}{b^u}(a^l - \frac{c^u}{m^l}) exp\Bigg(\mu \Bigg(a^l - b^u G_1 - \frac{c^u}{m^l} \Bigg) \Bigg) = \tilde{g}_1,$$

when $\tilde{y}(t) \leq G_2$, $-d^u \alpha^u - (d^u \beta^u - f^l)\tilde{g}_1 \geq 0$,

$-d^u + \frac{f^l \tilde{g}_1}{\alpha^u + \beta^u \tilde{g}_1 + m^u G_2} \leq 0$ *is satisfied then*

$$\tilde{y}(t) \geq \frac{1}{d^u m^u}(-d^u \alpha^u - (d^u \beta^u - f^l)\tilde{g}_1) exp\Bigg(\mu \Bigg(-d^u + \frac{f^l \tilde{g}_1}{\alpha^u + \beta^u \tilde{g}_1 + m^u G_2} \Bigg) \Bigg).$$

Proof. Consider the first equation of (2.2)

$$\begin{array}{rcl} (ln(\tilde{x}(t)))^{\Delta} & = & a(t) - b(t)\tilde{x}(t) - \frac{c(t)\tilde{y}(t)}{\alpha(t) + \beta(t)\tilde{x}(t) + m(t)\tilde{y}(t)} \\ & \geq & a(t) - b(t)\tilde{x}(t) - \frac{c(t)}{m(t)} \\ & \geq & a^l - b^u \tilde{x}(t) - \frac{c^u}{m^l} \\ & \geq & a^l - b^u G_1 - \frac{c^u}{m^l}. \end{array} \tag{2.7}$$

If $a^l - b^u G_1 - \frac{c^u}{m^l} > 0$, then there exist $\tilde{T}$ such that $t > \tilde{T}$, $\tilde{x}(t) > G_1$, for $t > \tilde{T}$. So there is a contradiction. Therefore $a^l - b^u G_1 - \frac{c^u}{m^l} \leq 0$.

Take $N_1 = \frac{1}{b^u}(a^l - \frac{c^u}{m^l})(1 - \tilde{q})$, where $\tilde{q} = 1 - exp\Bigg(w \Bigg(a^l - \frac{c^u}{m^l} - b^u G_1 \Bigg) \Bigg)$. Suppose that $\tilde{x}(t)$ is not oscillatory around N_1. Then there exists T_5, such that $\tilde{x}(t) > N_1$ for $t > T_5$ or $\tilde{x}(t) < N_1$ for $t > T_5$. If $\tilde{x}(t) > N_1$ for $t > T_5$, then $\tilde{x}(t)$ satisfies the desired result. Since $\tilde{x} \geq 0$, then the condition $a^l - \frac{c^u}{m^l} \geq 0$ must be satisfied. If $\tilde{x}(t) < N_1$ for $t > T_5$, then $(ln(\tilde{x}(t)))^{\Delta} \geq (a^l - \frac{c^u}{m^l})\tilde{q}$. Since $(a^l - \frac{c^u}{m^l})\tilde{q} > 0$, then there exists T_6 such that for $t > T_6$ $\tilde{x}(t) > N_1$ which is a contradiction. Suppose that $\tilde{x}(t)$ is oscillatory around N_1 and $\sigma(t_1)$ be an arbitrary local minimum of $ln(\tilde{x}(t))$, thus

$$\begin{array}{rcl} 0 \geq (ln(\tilde{x}(t_1)))^{\Delta} & = & a(t_1) - b(t_1)\tilde{x}(t_1) - \frac{c(t_1)\tilde{y}(t_1)}{\alpha(t_1)+\beta(t_1)\tilde{x}(t_1)+m(t_1)\tilde{y}(t_1)} \\ & \geq & a(t_1) - b(t_1)\tilde{x}(t_1) - \frac{c(t_1)}{m(t_1)}. \end{array}$$

Then we have,

$$\frac{1}{b^u}(a^l - \frac{c^u}{m^l}) \leq \frac{1}{b(t_1)}(a(t_1) - \frac{c(t_1)}{m(t_1)}) \leq \tilde{x}(t_1)$$

If t_1 is right dense then also $\tilde{x}(\sigma(t_1)) \geq \frac{1}{b^u}(a^l - \frac{c^u}{m^l})$. If t_1 is right scattered, then integrate (2.7) from t_1 to $\sigma(t_1)$, we obtain

$$\begin{array}{rcl} \int_{t_1}^{\sigma(t_1)}(ln(\tilde{x}(t)))^{\Delta}\Delta t & = & \int_{t_1}^{\sigma(t_1)} a(t) - b(t)\tilde{x}(t) - \frac{c(t)\tilde{y}(t)}{\alpha(t)+\beta(t)\tilde{x}(t)+m(t)\tilde{y}(t)}\Delta t \\ & \geq & \mu\left(a(t_1) - b(t_1)G_1 - \frac{c(t_1)}{m(t_1)}\Delta t\right). \end{array}$$

$$\tilde{x}(\sigma(t_1)) \geq \frac{1}{b^u}(a^l - \frac{c^u}{m^l})exp\left(\mu\left(a^l - b^u G_1 - \frac{c^u}{m^l}\right)\right) = \tilde{g}_1.$$

Since $\tilde{x}(\sigma(t_1))$ is the arbitrary local minimum, then $lim_{t\to\infty}inf\ \tilde{x}(t) \leq \tilde{g}_1$,

i.e. $lim_{t\to\infty}sup\ x(t) \leq r_1$.

Consider the second equation of (2.2)

$$\begin{array}{rcl} (ln(\tilde{y}(t)))^{\Delta} & = & -d(t) + \frac{f(t)\tilde{x}(t)}{\alpha(t)+\beta(t)\tilde{x}(t)+m(t)\tilde{y}(t)} \\ & \geq & -d^u + \frac{f^l\tilde{g}_1}{\alpha^u+\beta^u\tilde{g}_1+m^u G_2} \end{array} \quad (2.8)$$

If $-d^u + \frac{f^l\tilde{g}_1}{\alpha^u+\beta^u\tilde{g}_1+m^u G_2} > 0$, then there exits T_7 such that for $t > T_7$, $\tilde{y}(t) > G_2$ which is a contradiction. Thus $-d^u + \frac{f^l\tilde{g}_1}{\alpha^u+\beta^u\tilde{g}_1+m^u G_2} \leq 0$. Let us take N_2 such that

$$N_2 = \frac{1}{d^u m^u}(-d^u\alpha^u - d^u\beta^u\tilde{g}_1 + f^l\tilde{g}_1)(1-r)$$

where

$r = 1 - exp\left(\mu\left(-d^u + \frac{f^l\tilde{g}_1}{\alpha^u+\beta^u\tilde{g}_1+m^uG_2}\right)\right)$. Assume $\tilde{y}(t)$ is not oscillatory around N_2. Then there exists T_8, such that for $t > T_8$ $\tilde{y}(t) > N_2$ or $\tilde{y}(t) < N_2$.

For the first case $\tilde{y}(t) > \frac{1}{d^um^u}(-d^u\alpha^u - d^u\beta^u\tilde{g}_1 + f^l\tilde{g}_1)exp\left(\mu\left(-d^u + \frac{f^l\tilde{g}_1}{\alpha^u+\beta^u\tilde{g}_1+m^uG_2}\right)\right)$. Since $\tilde{y} \geq 0$, then $-d^u\alpha^u - d^u\beta^u\tilde{g}_1 + f^l\tilde{g}_1 \geq 0$.

For the second case $(ln(\tilde{y}(t)))^\Delta > (\frac{1}{d^um^u(\alpha^u+\beta^u\tilde{g}_1+m^uG_2)}(-d^u\alpha^u - d^u\beta^u\tilde{g}_1 + f^l\tilde{g}_1))r$.

Since $-d^u\alpha^u - d^u\beta^u\tilde{g}_1 + f^l\tilde{g}_1 > 0$ there exists T_9, such that for $t > T_9$ $\tilde{y}(t) > N_2$, which is a contradiction.

Assume $\tilde{y}(t)$ is oscillatory around N_2 and $\sigma(t_2)$ be an arbitrary local minimum of $ln(\tilde{y}(t))$, then by (2.8),we have

$$\begin{aligned} 0 \geq (ln(\tilde{y}(t_2))^\Delta &= \frac{-d(t_2)\alpha(t_2)-d(t_2)\beta(t_2)\tilde{x}(t_2)-d(t_2)m(t_2)\tilde{y}(t_2)+f(t_2)\tilde{x}(t_2)}{\alpha(t_2)+\beta(t_2)\tilde{x}(t_2)+m(t_2)\tilde{y}(t_2)} \\ &\geq \frac{-d(t_2)\alpha(t_2)-d(t_2)\beta(t_2)\tilde{g}_1-d(t_2)m(t_2)\tilde{y}(t_2)+f(t_2)\tilde{g}_1}{\alpha(t_2)+\beta(t_2)\tilde{g}_1+m(t_2)\tilde{y}(t_2)}. \end{aligned}$$

So we get $\tilde{y}(t_2) \geq \frac{1}{d(t_2)m(t_2)}(-d(t_2)\alpha(t_2) - d(t_2)\beta(t_2)\tilde{g}_1 + f(t_2)\tilde{g}_1)$.

Thus $\tilde{y}(t_2) \geq \frac{1}{d^um^u}(-d^u\alpha^u - d^u\beta^u\tilde{g}_1 + f^l\tilde{g}_1)$.

If t_2 is right dense, also $\tilde{y}(\sigma(t_2)) \geq \frac{1}{d^um^u}(-d^u\alpha^u - d^u\beta^u\tilde{g}_1 + f^l\tilde{g}_1)$.

If t_2 is right scattered,we integrate (2.8) from t_2 to $\sigma(t_2)$ we obtain

$$\begin{aligned} \int_{t_2}^{\sigma(t_2)} (ln(\tilde{y}(t)))^\Delta \Delta t &= \int_{t_2}^{\sigma(t_2)} -d(t) + \frac{f(t)\tilde{x}(t)}{\alpha(t)+\beta(t)\tilde{x}(t)+m(t)\tilde{y}(t)}\Delta t \\ &\geq \mu\left(-d^u + \frac{f^l\tilde{g}_1}{\alpha^u+\beta^uG_1+m^uG_2}\right). \end{aligned}$$

$$\tilde{y}(\sigma(t_2)) \geq \frac{1}{d^um^u}(-d^u\alpha^u - d^u\beta^u\tilde{g}_1 + f^l\tilde{g}_1)exp\left(\mu\left(-d^u + \frac{f^l\tilde{g}_1}{\alpha^u+\beta^u\tilde{g}_1+m^uG_2}\right)\right).$$

Since $\tilde{y}(\sigma(t_2))$ is the arbitrary local minimum, then $lim_{t\to\infty}inf\ \tilde{y}(t) \leq \tilde{g}_2$, i.e. $lim_{t\to\infty}sup\, y(t) \leq r_2$. □

If (2.1) satisfies all the conditions of Lemma 2.3.2 and Lemma 2.3.3, then its solution is permanent.

Example 2.3.1 $\mathbb{T} = [2k, 2k+1],\ k \in \mathbb{N}$ *k start with 0.*

$$x^{\Delta}(t) = (2 - \tfrac{1}{2t+2}) - exp(x) - \tfrac{0.5exp(y)}{exp(x)+exp(y)},$$
$$y^{\Delta}(t) = -1 + \tfrac{(3+\frac{1}{t+1})exp(x)}{exp(x)+exp(y)}.$$

Example 2.3.1 satisfies all of the conditions of Lemma 2.3.2 and Lemma 2.3.3, therefore its solutions are permanent.

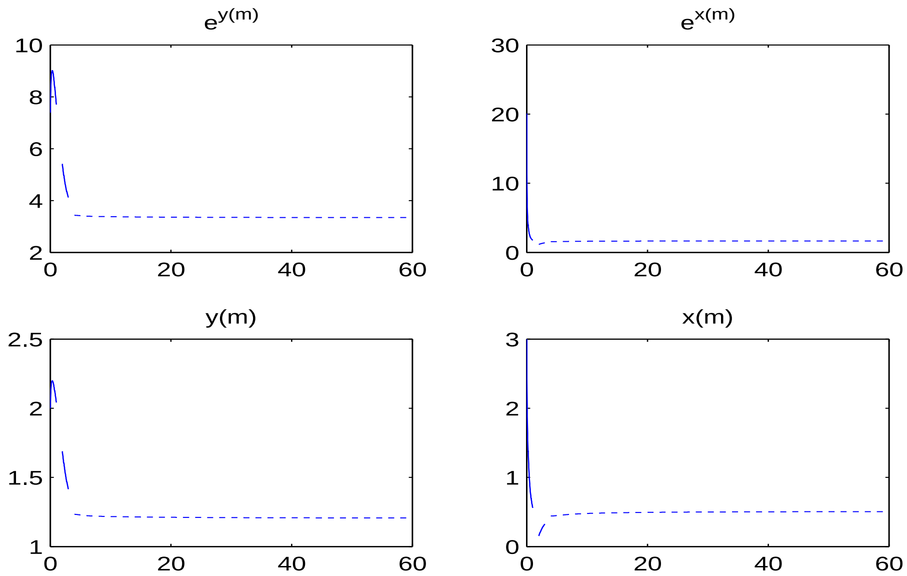

Figure 2.1: Numeric solution of Example 2.3.1 shows the permanence.

2.4 Global Attractivity

Definition 2.4.1 *[25] A positive solution (x*(t),y*(t)) of* (2.2) *is said to be globally attractive if each other positive solution (x(t),y(t)) of* (2.2) *satifies*

$$\lim_{t\to\infty} |x^*(t) - x(t)| = 0, \qquad \lim_{t\to\infty} |y^*(t) - y(t)| = 0.$$

Theorem 2.4.1 *In addition to conditions of Lemma 2.3.2 and Lemma 2.3.3* *if* $a_1, a_2 \in (0,1)$ *and* $\delta > 0$

$$\Bigg(a_1 min\left\{b^l, \tfrac{2}{G_1\mu^u} - b^u\right\} - \Bigg[a_1\tfrac{c^u(\beta^u)^{1/3}G_2^{1/3}}{9(\alpha^l)^{2/3}(m^l)^{2/3}g_1^{2/3}g_2^{1/3}} + a_2\tfrac{f^u(\alpha^u)^{1/3}}{9(\beta^l)^{2/3}(m^l)^{2/3}g_1^{2/3}g_2^{2/3}}$$
$$+a_2\tfrac{f^u(m^u)^{1/3}G_2^{2/3}}{9(\alpha^l)^{2/3}(\beta^l)^{2/3}g_1^{2/3}g_2^{1/3}}\Bigg], a_2 min\left\{\tfrac{f^l g_1 m^l}{(\alpha^u+\beta^u G_1+m^u G_2)^2}, \tfrac{2}{G_2\mu^u} - \tfrac{f^u G_1 m^u}{(\alpha^l+\beta^l g_1+m^l g_2)^2}\right\}$$
$$-\Bigg[a_1\tfrac{c^u(\alpha^u)^{1/3}}{9(\beta^l)^{2/3}(m^l)^{2/3}g_1^{2/3}g_2^{2/3}} + a_1\tfrac{c^u(\beta^u)^{1/3}(t)G_1^{2/3}}{9(\alpha^l)^{2/3}(m^l)^{2/3}g_1^{1/3}g_2^{2/3}}\Bigg]\Bigg) > \delta,$$

then system (2.2) *is globally attractive.*

Proof. For any positive solutions $(x_1(t), y_1(t))$ and $(x_2(t), y_2(t))$ of system (2.1), it follows from Lemma 2.3.2 and Lemma 2.3.3;

$\lim\ inf_{t\to\infty} x_i(t) < g_1$, $\lim\ sup_{t\to\infty} x_i(t) < G_1$, $\lim\ inf_{t\to\infty} y_i(t) < g_2$,

$\lim\ sup_{t\to\infty} y_i(t) < G_2$ for $i = 1, 2$.

Let $V_1(t) = |lnx_1(t) - lnx_2(t)|$, $A = \alpha + \beta x_1(t) + m(t)y_1(t)$, $B = \alpha + \beta x_2(t) + m(t)y_2(t)$. If t is right scattered,

$$\begin{aligned} V_1^{\Delta}(t) &= \tfrac{V_1(\sigma(t)) - V_1(t)}{\mu(t)} \\ &= \tfrac{|lnx_1(\sigma(t)) - lnx_2(\sigma(t))| - |lnx_1(t) - lnx_2(t)|}{\mu(t)} \\ &\le \tfrac{1}{\mu(t)}|lnx_1(t) - lnx_2(t) - \mu(t)b(t)[x_1(t) - x_2(t)]| - \tfrac{1}{\mu(t)}|lnx_1(t) - lnx_2(t)| \\ &+ c(t)\left|\tfrac{\beta(t)y_1(t)[x_1(t)-x_2(t)]}{AB}\right| + c(t)\left|\tfrac{\alpha(t)[y_1(t)-y_2(t)]}{AB}\right| + c(t)\left|\tfrac{\beta(t)x_1(t)[y_1(t)-y_2(t)]}{AB}\right|. \end{aligned}$$

Assume $\xi(t)$ is between $x_1(t)$ and $x_2(t)$, then using Diamond-alpha Lagrange's mean value theorem and taking $\alpha = 1$ we get,

$$\begin{aligned} x_1(t) - x_2(t) &= exp(lnx_1(t)) - exp(lnx_2(t)) \\ &= \xi(t)(lnx_1(t) - lnx_2(t)). \end{aligned} \tag{2.9}$$

Since exp function satisfies all of the conditions of Theorem 2.1.1, we can obtain equality (2.9).

If t is right scattered, using Young's inequality and (2.9) we obtain,

$$V_1^{\Delta}(t) \le -\frac{1}{\mu(t)}\left[\frac{1}{\xi(t)} - \left|\frac{1}{\xi(t)} - \mu(t)b(t)\right|\right]|x_1(t) - x_2(t)| + \left|\frac{c(t)\beta^{1/3}(t)y_1^{1/3}(t)[x_1(t) - x_2(t)]}{9\alpha^{2/3}(t)m^{2/3}(t)x_1^{1/3}(t)x_2^{1/3}(t)y_2^{1/3}(t)}\right|$$

$$+\frac{c(t)\alpha^{1/3}(t)|y_2(t) - y_1(t)|}{9\beta^{2/3}(t)m^{2/3}(t)x_1^{1/3}(t)x_2^{1/3}(t)y_1^{1/3}(t)y_2^{1/3}(t)} + \frac{c(t)\beta^{1/3}(t)x_1^{2/3}(t)|y_2(t) - y_1(t)|}{9\alpha^{2/3}(t)m^{2/3}(t)x_2^{1/3}(t)y_1^{1/3}(t)y_2^{1/3}(t)}$$

$$V_1^{\Delta}(t) \le -min\left\{b^l, \frac{2}{G_1\mu^u} - b^u\right\}|x_1(t)-x_2(t)| + \frac{c^u(\beta^u)^{1/3}G_2^{1/3}|x_1(t)-x_2(t)|}{9(\alpha^l)^{2/3}(m^l)^{2/3}g_1^{2/3}g_2^{1/3}}$$
$$+\frac{c^u(\alpha^u)^{1/3}|y_2(t)-y_1(t)|}{9(\beta^l)^{2/3}(m^l)^{2/3}g_1^{2/3}g_2^{2/3}} + \frac{c^u(\beta^u)^{1/3}G_1^{2/3}|y_2(t)-y_1(t)|}{9(\alpha^l)^{2/3}(m^l)^{2/3}g_1^{1/3}g_2^{2/3}}.$$

If t is right dense, then delta- differentiation becomes equal with the normal differentiation thus we can write,

$$V_1'(t) = sgn(lnx_1(t) - lnx_2(t))\left(\frac{x_1'(t)}{x_1(t)} - \frac{x_2'(t)}{x_2(t)}\right) =$$
$$= sgn(lnx_1(t)-lnx_2(t))\left(-b(t)(x_1(t)-x_2(t)) - \frac{c(t)y_1(t)}{\alpha(t)+\beta(t)x_1(t)+m(t)y_1(t)} + \frac{c(t)y_2(t)}{\alpha(t)+\beta(t)x_2(t)+m(t)y_2(t)}\right)$$

Since $sgn(lnx_1(t) - lnx_2(t)) = sgn(x_1(t) - x_2(t))$, then

$$V_1'(t) \le -b(t)|x_1(t)-x_2(t)| + c(t)\frac{\beta(t)y_1(t)|x_1(t)-x_2(t)|}{AB}$$
$$+c(t)\left|\frac{\alpha(t)[y_1(t)-y_2(t)]}{AB}\right| + c(t)\left|\frac{\beta(t)x_1(t)[y_1(t)-y_2(t)]}{AB}\right|$$
$$\le -b^l|x_1(t)-x_2(t)| + \frac{c(t)\beta^{1/3}(t)y_1^{1/3}(t)|x_1(t)-x_2(t)|}{9\alpha^{2/3}(t)m^{2/3}(t)x_1^{1/3}(t)x_2^{1/3}(t)y_2^{1/3}(t)} +$$
$$+\frac{c(t)\alpha^{1/3}(t)|y_2(t)-y_1(t)|}{9\beta^{2/3}(t)m^{2/3}x_1^{1/3}(t)x_2^{1/3}(t)y_1^{1/3}(t)y_2^{1/3}(t)} + \frac{c(t)\beta^{1/3}(t)x_1^{2/3}(t)|y_2(t)-y_1(t)|}{9\alpha^{2/3}(t)x_2^{1/3}(t)y_1^{1/3}(t)y_2^{1/3}(t)}$$
$$\le -b^l|x_1(t)-x_2(t)| + \frac{c^u(\beta^u)^{1/3}G_2^{1/3}|x_1(t)-x_2(t)|}{9(\alpha^l)^{2/3}(m^l)^{2/3}g_1^{2/3}g_2^{1/3}} +$$
$$+\frac{c^u(\alpha^u)^{1/3}|y_2(t)-y_1(t)|}{9(\beta^l)^{2/3}(m^l)^{2/3}g_1^{2/3}g_2^{2/3}} + \frac{c^u(\beta^u)^{1/3}G_1^{2/3}|y_2(t)-y_1(t)|}{9(\alpha^l)^{2/3}(m^l)^{2/3}g_1^{1/3}g_2^{2/3}}.$$

Therefore

$$V_1^{\Delta}(t) \le -min\left\{b^l, \frac{2}{G_1\mu^u} - b^u\right\}|x_1(t)-x_2(t)| + \frac{c^u(\beta^u)^{1/3}G_2^{1/3}|x_1(t)-x_2(t)|}{9(\alpha^l)^{2/3}(m^l)^{2/3}g_1^{2/3}g_2^{1/3}}$$
$$+\frac{c^u(\alpha^u)^{1/3}|y_2(t)-y_1(t)|}{9(\beta^l)^{2/3}(m^l)^{2/3}g_1^{2/3}g_2^{2/3}} + \frac{c^u(\beta^u)^{1/3}G_1^{2/3}|y_2(t)-y_1(t)|}{9(\alpha^l)^{2/3}(m^l)^{2/3}g_1^{1/3}g_2^{2/3}}.$$

Let $V_2(t) = |lny_1(t) - lny_2(t)|$. Now assume $\xi_2(t)$ lies between $y_1(t)$ and $y_2(t)$, therefore by using Diamond-alpha Lagrange's mean value theorem and taking $\alpha = 1$, we have

$$\begin{aligned} y_1(t) - y_2(t) &= exp(lny_1(t)) - exp(lny_2(t)) \\ &= \xi_2(t)(lny_1(t) - lny_2(t)). \end{aligned} \quad (2.10)$$

Since exp function satisfies all of the conditions of Theorem 2.1.1, we can get equality (2.10).

If t is a right scattered point, by (2.10) we have

$$V_2^{\Delta}(t) \leq -\frac{1}{\mu(t)}\left[\frac{1}{\xi_2(t)} - \left|\frac{1}{\xi_2(t)} - \mu(t)\frac{f(t)x_1(t)m(t)}{AB}\right|\right]|y_1(t)-y_2(t)|$$

$$+\frac{f(t)\alpha^{1/3}(t)|x_1(t)-x_2(t)|}{9\beta^{2/3}(t)m^{2/3}(t)x_1^{1/3}(t)x_2^{1/3}(t)y_1^{1/3}(t)y_2^{1/3}(t)}+\frac{f(t)m^{1/3}(t)y_1^{2/3}(t)|x_1(t)-x_2(t)|}{9\alpha^{2/3}(t)\beta^{2/3}(t)x_1^{1/3}(t)x_2^{1/3}(t)y_2^{1/3}(t)}$$

$$\leq -min\left\{\frac{f^l g_1 m^l}{A^u B^u}, \frac{2}{G_2\mu^u} - \frac{f^u G_1 m^u}{A^l B^l}\right\} + \frac{f^u(\alpha^u)^{1/3}|x_1(t)-x_2(t)|}{9(\beta^l)^{2/3}(m^l)^{2/3}g_1^{2/3}g_2^{2/3}} + \frac{f^u(m^u)^{1/3}G_2^{2/3}|x_1(t)-x_2(t)|}{9(\alpha^l)^{2/3}(\beta^l)^{2/3}g_1^{2/3}g_2^{1/3}}$$

If t is right dense, since delta- differentiation becames equal with the normal differentiation therefore we have

$$V_2'(t) = sgn(lny_1(t) - lny_2(t))\left(\frac{y_1'(t)}{y_1(t)} - \frac{y_2'(t)}{y_2(t)}\right)$$

$$= sgn(lny_1(t) - lny_2(t))\left(\frac{f(t)x_1(t)}{\alpha(t)+\beta(t)x_1(t)+m(t)y_1(t)} - \frac{f(t)x_2(t)}{\alpha(t)+\beta(t)x_2(t)+m(t)y_2(t)}\right)$$

$$\leq -\frac{f(t)x_1(t)m(t)}{AB}|y_1(t)-y_2(t)| + \frac{f(t)\alpha^{1/3}(t)|x_1(t)-x_2(t)|}{9\beta^{2/3}(t)m^{2/3}(t)x_1^{1/3}(t)x_2^{1/3}(t)y_1^{1/3}(t)y_2^{1/3}(t)}$$

$$+\frac{f(t)m^{1/3}(t)y_1^{2/3}(t)|x_1(t)-x_2(t)|}{9\alpha^{2/3}(t)\beta^{2/3}(t)x_1^{1/3}(t)x_2^{1/3}(t)y_2^{1/3}(t)}$$

$$\leq -\frac{f^l g_1 m^l}{A^u B^u}|y_1(t)-y_2(t)| + \frac{f^u(\alpha^u)^{1/3}|x_1(t)-x_2(t)|}{9(\beta^l)^{2/3}(m^l)^{2/3}g_1^{2/3}g_2^{2/3}} + \frac{f^u(m^u)^{1/3}G_2^{2/3}|x_1(t)-x_2(t)|}{9(\alpha^l)^{2/3}(\beta^l)^{2/3}g_1^{2/3}g_2^{1/3}}$$

Thus

$$V_2^{\Delta}(t) \leq -min\left\{\frac{f^l g_1 m^l}{A^u B^u}, \frac{2}{G_2\mu^u} - \frac{f^u G_1 m^u}{A^l B^l}\right\} + \frac{f^u(\alpha^u)^{1/3}|x_1(t)-x_2(t)|}{9(\beta^l)^{2/3}(m^l)^{2/3}g_1^{2/3}g_2^{2/3}} +$$

$$+\frac{f^u(m^u)^{1/3}G_2^{2/3}|x_1(t)-x_2(t)|}{9(\alpha^l)^{2/3}(\beta^l)^{2/3}g_1^{2/3}g_2^{1/3}}.$$

Let us define a Lyapunov function as: $V(t) = a_1V_1(t) + a_2V_2(t)$, $a_1, a_2 \in (0,1)$. $V^{\Delta}(t) = a_1V_1^{\Delta}(t) + a_2V_2^{\Delta}(t)$.

$$V^{\Delta}(t) \leq -\left(a_1 min\left\{b^l, \frac{2}{G_1\mu^u} - b^u\right\} - \left(a_1\frac{c^u(\beta^u)^{1/3}G_2^{1/3}}{9(\alpha^l)^{2/3}(m^l)^{2/3}g_1^{2/3}g_2^{1/3}} + a_2\frac{f^u(\alpha^u)^{1/3}}{9(\beta^l)^{2/3}(m^l)^{2/3}g_1^{2/3}g_2^{2/3}}\right.\right.$$

$$\left.\left.+a_2\frac{f^u(m^u)^{1/3}G_2^{2/3}}{9(\alpha^l)^{2/3}(\beta^l)^{2/3}g_1^{2/3}g_2^{1/3}}\right)\right)|x_1(t)-x_2(t)| - \left(a_2 min\left\{\frac{f^l g_1 m^l}{A^u B^u}, \frac{2}{G_2\mu^u} - \frac{f^u G_1 m^u}{A^l B^l}\right\}\right.$$

$$\left.-\left(a_1\frac{c^u(\alpha^u)^{1/3}}{9(\beta^l)^{2/3}(m^l)^{2/3}g_1^{2/3}g_2^{2/3}} + a_1\frac{c^u(\beta^u)^{1/3}G_1^{2/3}}{9(\alpha^l)^{2/3}(m^l)^{2/3}g_1^{1/3}g_2^{2/3}}\right)\right)|y_2(t)-y_1(t)|.$$

By assumption

$$V^{\Delta}(t) < -\delta[|x_1(t) - x_2(t)| + |y_2(t) - y_1(t)|]$$

Integrating both sides of the above inequality from t_1 to t, we get

$$\int_{t_1}^{t} V^{\Delta}(s)\Delta s < -\delta \int_{t_1}^{t} [|x_1(s) - x_2(s)| + |y_2(s) - y_1(s)|]\Delta s$$

$$\int_{t_1}^{t} [|x_1(s) - x_2(s)| + |y_2(s) - y_1(s)|]\Delta s < \frac{V_1(t_1)}{\delta}$$

Then

$$\int_{t_1}^{t} [|x_1(s) - x_2(s)| + |y_2(s) - y_1(s)|]\Delta s < +\infty$$

$$\lim_{t\to\infty} [|x_1(t) - x_2(t)| + |y_2(t) - y_1(t)|] = 0$$

$\lim_{t\to\infty}[|x_1(t) - x_2(t)|] = 0$ and $\lim_{t\to\infty}[|y_1(t) - y_2(t)|] = 0.$

Hence proof follows.

□

Corollary 2.4.1 *In addition to conditions of Lemma 2.3.2 and Lemma 2.3.3*

if $a_1, a_2 \in (0, 1)$, $\alpha(t) = 0$ *and* $\delta > 0$

$$\Bigg(a_1 min\left\{b^l, \frac{2}{G_1\mu^u} - b^u\right\} - \left[a_1 \frac{c^u G_2^{1/2}}{4m^l g_1 g_2^{1/2}} + a_2 \frac{f^u G_2^{1/2}}{4\beta^l g_1 g_2^{1/2}}\right], \\ a_2 min\left\{\frac{f^l g_1 m^l}{(\beta^u G_1 + m^u G_2)^2}, \frac{2}{G_2\mu^u} - \frac{f^u G_1 m^u}{(\beta^l g_1 + m^l g_2)^2}\right\} - \left[a_1 \frac{c^u G_1^{1/2}}{4m^l g_1^{1/2} g_2}\right]\Bigg) > \delta,$$

then system (2.2) *is globally attractive.*

Example 2.4.1 $\mathbb{T} = [2k, 2k+1],\ k \in \mathbb{N}$ *k start with 0.*

$$x^{\Delta}(t) = (0.5 - \frac{0.1}{t+1}) - exp(x) - \frac{0.01exp(y)}{exp(x)+exp(y)},$$
$$y^{\Delta}(t) = -0.1 + \frac{0.2exp(x)}{exp(x)+exp(y)}.$$

Example 2.4.1 satisfies Corollary 2.4.1, therefore solution of this system is globally attractive.

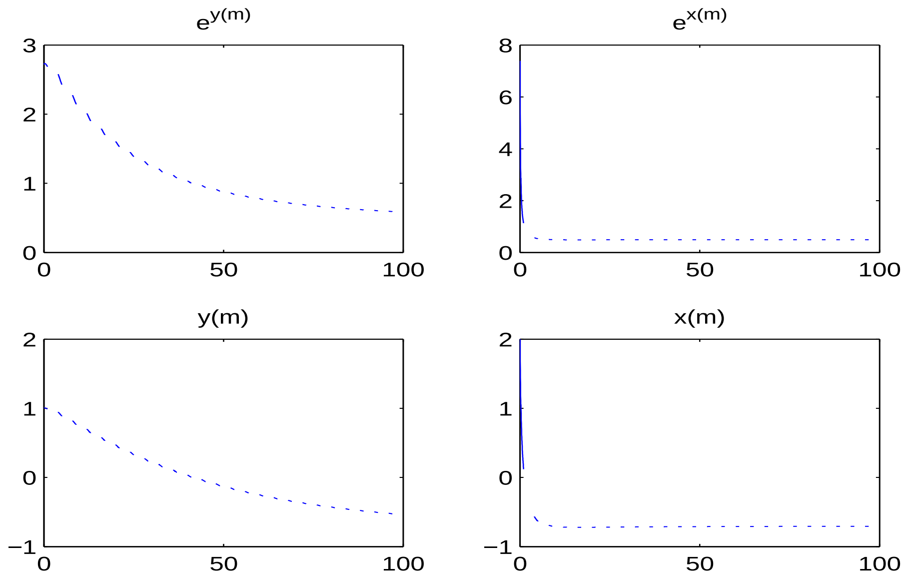

Figure 2.2: Initial conditions in this example is x(0)=3, y(0)=2.

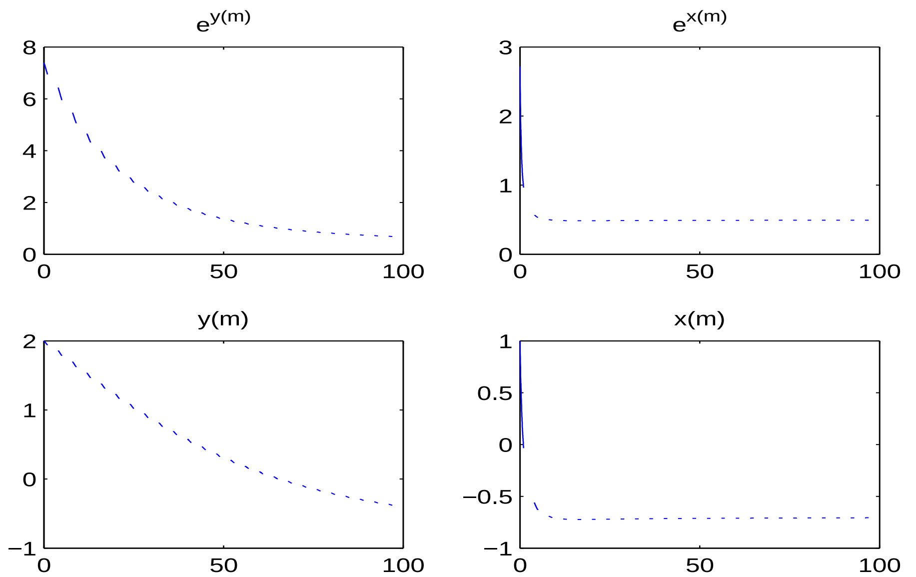

Figure 2.3: Initial conditions in this example is x(0)=2, y(0)=3.

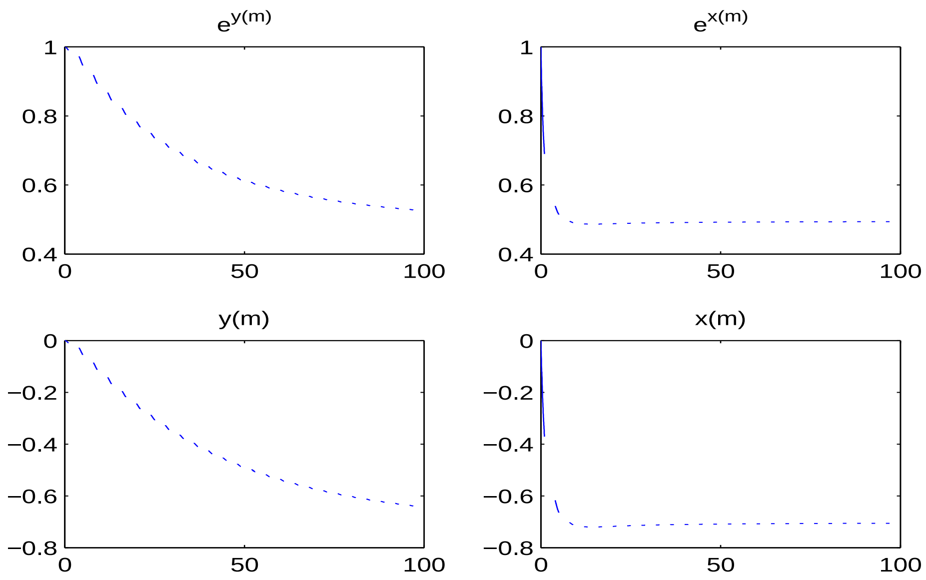

Figure 2.4: Initial conditions in this example is x(0)=0, y(0)=0.

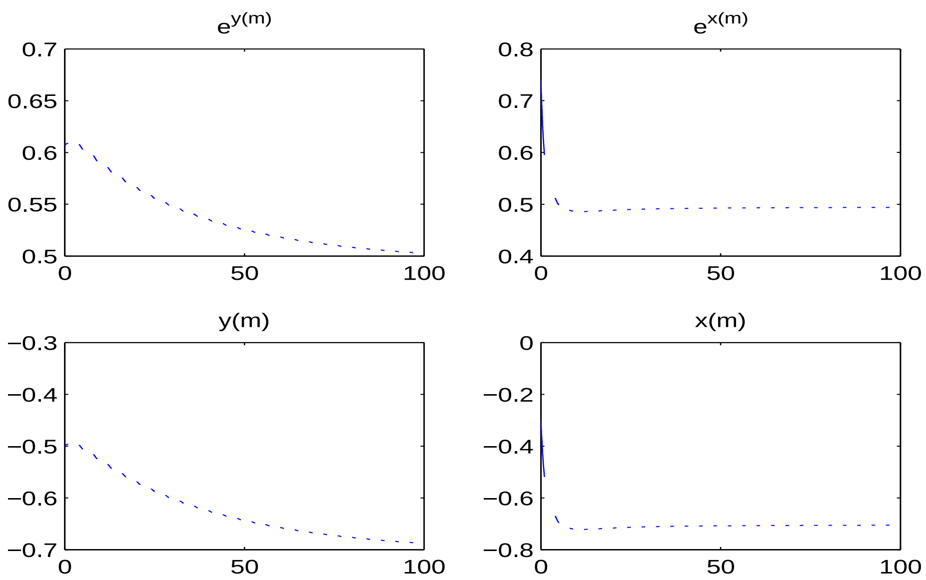

Figure 2.5: Initial conditions in this example is x(0)=-0.3, y(0)=-0.5.

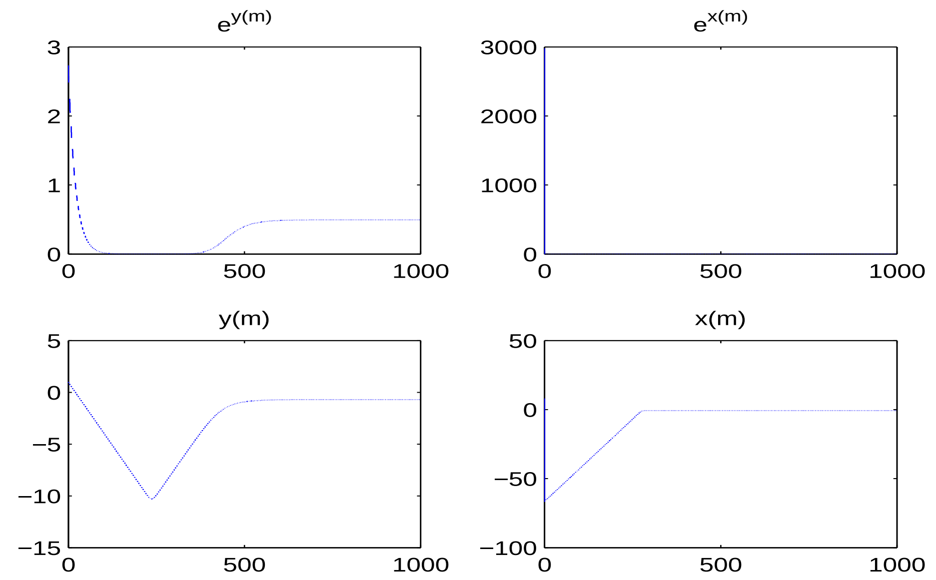

Figure 2.6: Initial conditions in this example is x(0)=8, y(0)=1.

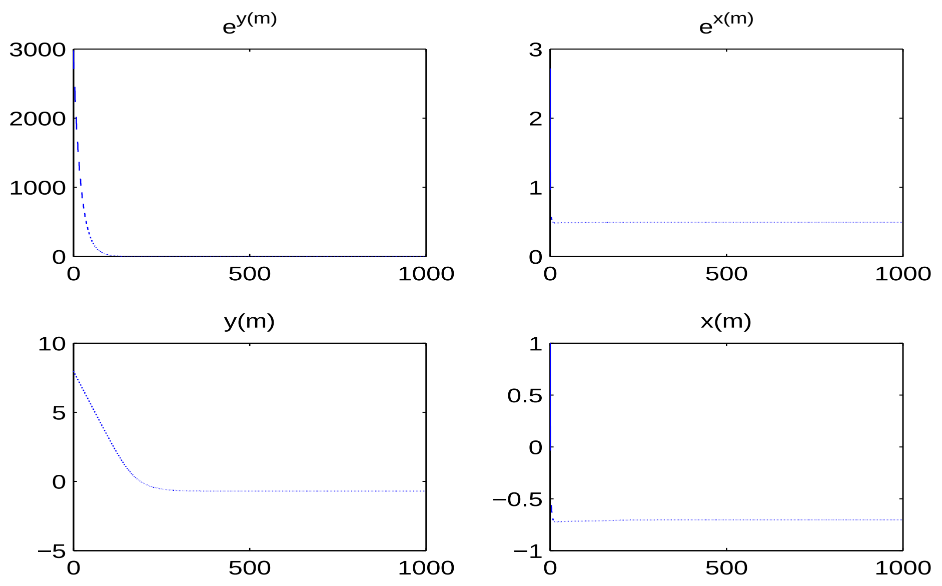

Figure 2.7: Initial conditions in this example is x(0)=1, y(0)=8.

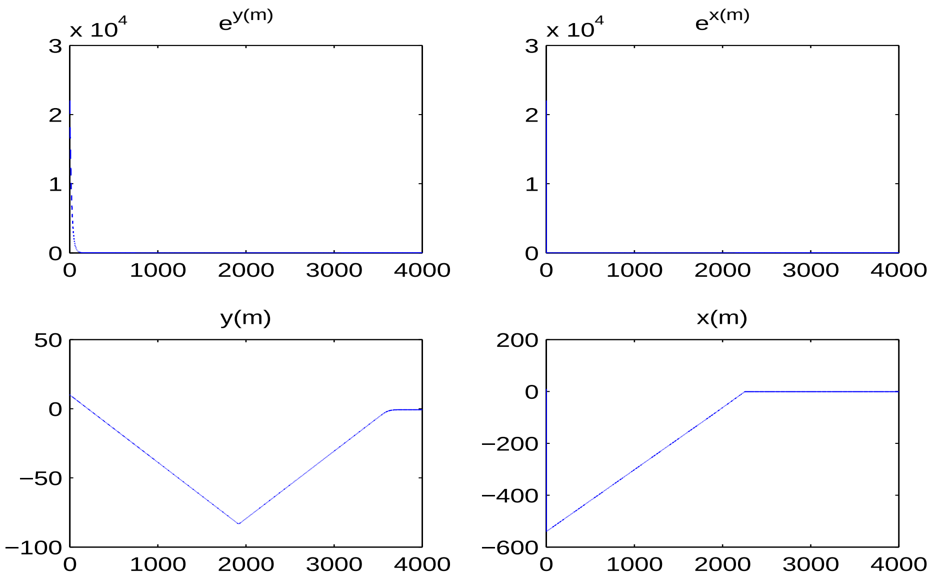

Figure 2.8: Initial conditions in this example is x(0)=10, y(0)=10. Although we take seven different initial conditions, solutions (exp(x(t)),exp(y(t))) approaches to 0.5 in each case. Therefore numeric solution of Example 2.4.1 shows the global attractivity.

CHAPTER 3

PERIODIC SOLUTIONS FOR PREDATOR-PREY DYNAMIC SYSTEMS WITH BEDDINGTON DEANGELIS TYPE FUNCTIONAL RESPONSES AND IMPULSES

3.1 Preliminaries

Below informations are from [10]. Let $L : DomL \subset X \to Y$ be a linear mapping, $C : X \to Y$ be a continuous mapping where X,Y be normed vector spaces. If $dimKerL = codimImL < +\infty$ and ImL is closed in Y, then the mapping L will be called a Fredholm mapping of index zero. There exist continuous projections $U : X \to X$ and $V : Y \to Y$ when L is a Fredholm mapping of index zero such that $ImU = KerL, ImL = KerV = Im(I-V)$, then it follows that $L|_{DomL\cap KerU} : (I-U)X \to ImL$ is invertible. The inverse of that map is denoted as K_U . The mapping C will be called L-compact on Ω if $VC(\Omega)$ is bounded and $K_U(I-V)C : \Omega \to X$ is compact, where Ω is an open bounded subset of X. Since ImV is isomorphic to KerL, the isomorphism $J : ImV \to KerL$ is exist and the above informations are important for the Continuation Theorem that we give below.

Definition 3.1.1 *[11] The codimension (or quotient or factor dimension) of a subspace L of a vector space V is the dimension of the quotient space V/L; it is denoted by $codim_V L$ or simply by $codimL$ and is equal to the dimension of the orthogonal complement of L in V and one has $dimL + codimL = dimV$.*

Theorem 3.1.1 *[10] (Continuation Theorem). Let L be a Fredholm mapping of index zero and C be L-compact on Ω. Assume* *(a) For each* $\lambda \in (0,1)$*, any y satisfying* $Ly = \lambda Cy$ *is not on* $\delta\Omega$*, i.e.*$y \notin \delta\Omega$*;*

(b) For each $y \in \delta\Omega \cap KerL$, $VCy \neq 0$ *and the Brouwer degree*

$deg\{JVC, \delta\Omega \cap KerL, 0\} \neq 0$*. Then* $Ly = Cy$ *has at least one solution lying in* $DomL \cap \delta\Omega$*.*

We will also give the following lemma, which is essential for this paper.

Lemma 3.1.1 *[10] Let* $\tau_1, \tau_2 \in [0,\omega]$ *and* $t \in \mathbb{T}$*. If* $f : \mathbb{T} \to \mathbb{R}$ *is* ω*-periodic, then*

$$f(t) \leq f(\tau_1) + \int_0^\omega |f^\Delta(s)|\Delta s \qquad and \qquad f(t) \geq f(\tau_2) - \int_0^\omega |f^\Delta(s)|\Delta s.$$

Remark 3.1.1 *In the study of [38] for predator-prey dynamic models with several type of functional responses with impulses on time scales is studied and they found a general result. On the other hand on their study only the effect of functional response is seen on the prey, but on predator the effect of the given functional response can not be seen. Therefore our results are also important since the impact of Beddington DeAngelis type functional response is taken into account for both prey and predator.*

Definition 3.1.2 *[32]If a function is regulated and it is rd- continuous at all except possibly at finitely many right dense points* $t \in \mathbb{T}$*, then this function is called piecewise rd-continuous function and the class of piecewise rd-continuous functions denaoted as* $C_{prd}(\mathbb{T}, \mathbb{R})$*.*

3.2 Main Result

The equation that we investigate is:

$$\begin{aligned}
&x^\Delta(t) = a(t) - b(t)exp(x(t)) - \tfrac{c(t)exp(y(t))}{\alpha(t)+\beta(t)exp(x(t))+m(t)exp(y(t))}, t \neq t_k \\
&y^\Delta(t) = -d(t) + \tfrac{f(t)exp(x(t))}{\alpha(t)+\beta(t)exp(x(t))+m(t)exp(y(t))}, t \neq t_k \\
&\Delta x(t_k) = ln(1+g_k) \\
&\Delta y(t_k) = ln(r_k)
\end{aligned} \tag{3.1}$$

where $t_{k+q} = t_k+w, a(t+w) = a(t), b(t+w) = b(t), c(t+w) = c(t), d(t+w) = d(t),$ $f(t+w) = f(t), \alpha(t+w) = \alpha(t), \beta(t+w) = \beta(t), m(t+w) = m(t),\ \forall k, 1 > g_k > -1,$

and $r_k > 0$. Here $\mathbb{T}$ is periodic, i.e if $t \in \mathbb{T}$ then $t + w \in \mathbb{T}$, and $\int_0^w a(t)\Delta t > 0$, $\int_0^w b(t)\Delta t > 0$, $\int_0^w d(t)\Delta t > 0$.

$\beta^l = min_{t\in[0,w]}\beta(t), m^l = min_{t\in[0,w]}m(t), \beta^u = max_{t\in[0,w]}\beta(t), m^u = max_{t\in[0,w]}m(t)$, $m(t) > 0$ and $c(t), f(t) > 0$ $b(t), \alpha(t) \geq 0$, $\beta(t) > 0$. Each coefficient functions are from $C_{rd}(\mathbb{T}, \mathbb{R})$.

Lemma 3.2.1 *If*

$$\int_0^w a(t)\Delta t + ln\prod_{i=1}^{q}(1+g_i) < 0 \; and \; -\int_0^w d(t)\Delta t + ln\prod_{i=1}^{q} r_i + \int_0^w \frac{f(t)}{\beta(t)}\Delta t < 0,$$

then all positive solutions $(exp(x(t)), exp(y(t)))$ *are tends to 0 as t tends to infinity.*

Proof. If we using the first equation of (3.1) we obtain,

$$exp(x(t)) \leq exp(x(0))\prod_{ti<t}(1+g_i)exp(\int_0^t a(s)\Delta s)$$

Since $\int_0^w a(t)\Delta t + ln\prod_{i=1}^{q}(1+g_i) < 0$. Hence $lim_{t\to\infty}exp(x(t)) = 0$

Similarly $lim_{t\to\infty}exp(y(t)) = 0$.

□

Theorem 3.2.1 *In addition to conditions on coefficient functions if*

$$\int_0^w a(t)\Delta t + ln\prod_{i=1}^{q}(1+g_i) - \int_0^w \frac{c(t)}{m(t)}\Delta t > 0$$

and

$$\left(\frac{\int_0^w a(t)\Delta t + ln\prod_{i=1}^{q}(1+g_i) - \int_0^w \frac{c(t)}{m(t)}\Delta t}{\int_0^w b(t)\Delta t}\right)exp\left[-\left(\int_0^w |a(t)|\Delta t + \int_0^w a(t)\Delta t + ln\prod_{i=1}^{q}(1+g_i)\right)\right].$$
$$.(\int_0^w f(t)\Delta t - \beta^u(\int_0^w d(t)\Delta t - ln\prod_{i=1}^{q}(r_i))) - \alpha^u(\int_0^w d(t)\Delta t - ln\prod_{i=1}^{q}(r_i)) > 0$$

then there exist at least a w-periodic solution.

Proof.

$$X := \left\{ \begin{bmatrix} p \\ z \end{bmatrix} \in C_{prd}(\mathbb{T}, \mathbb{R}^2) : p(t+w) = p(t), z(t+w) = z(t) \right\}$$ with the norm:

$$\left\| \begin{bmatrix} p \\ z \end{bmatrix} \right\| = \sup_{t\in[0,w]_{\mathbb{T}}}(|p(t)|, |z(t)|) \text{ and}$$

$Y := \left\{ \left[\begin{bmatrix} p \\ s \end{bmatrix}, \begin{bmatrix} d_1 \\ f_1 \end{bmatrix}, \ldots.., \begin{bmatrix} d_q \\ f_q \end{bmatrix} \right] \in C_{prd}(\mathbb{T}, \mathbb{R}^2) \times (\mathbb{R}^2)^q,\ p(t+w) = p(t), z(t+w) = z(t) \right\}$ with the norm:

$$\left\| \left[\begin{bmatrix} p \\ s \end{bmatrix}, \begin{bmatrix} d_1 \\ f_1 \end{bmatrix}, .., \begin{bmatrix} d_q \\ f_q \end{bmatrix} \right] \right\| = \sup_{t\in[0,w]_{\mathbb{T}}} \left(\left\| \begin{bmatrix} p \\ z \end{bmatrix} \right\|, \left\| \begin{bmatrix} d_1 \\ f_1 \end{bmatrix} \right\|, .., \left\| \begin{bmatrix} d_q \\ f_q \end{bmatrix} \right\| \right).$$

Let us define the mappings L and C by $L : DomL \subset X \to Y$ such that

$$L\left(\begin{bmatrix} p \\ z \end{bmatrix} \right) = \left(\begin{bmatrix} p^{\Delta} \\ z^{\Delta} \end{bmatrix}, \begin{bmatrix} \Delta p(t_1) \\ \Delta z(t_1) \end{bmatrix}, \ldots.., \begin{bmatrix} \Delta p(t_q) \\ \Delta z(t_q) \end{bmatrix} \right)$$

and $C : X \to Y$ such that

$$C\left(\begin{bmatrix} p \\ z \end{bmatrix} \right) =$$

$$\left(\begin{bmatrix} a(t) - b(t)exp(p(t)) - \frac{c(t)exp(z(t))}{\alpha(t)+\beta(t)exp(p(t))+m(t)exp(z(t))} \\ -d(t) + \frac{f(t)exp(p(t))}{\alpha(t)+\beta(t)exp(p(t))+m(t)exp(z(t))} \end{bmatrix}, \begin{bmatrix} ln(1+g_1) \\ ln(p_1) \end{bmatrix}, .., \begin{bmatrix} ln(1+g_q) \\ ln(p_q) \end{bmatrix} \right).$$

Then $KerL = \left\{ \begin{bmatrix} p \\ z \end{bmatrix} : \begin{bmatrix} p \\ z \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \right\}$, c_1 and c_2 are constants.

$$ImL = \left\{ \left[\begin{bmatrix} p \\ z \end{bmatrix}, \begin{bmatrix} d_1 \\ f_1 \end{bmatrix}, \ldots.., \begin{bmatrix} d_q \\ f_q \end{bmatrix} \right] : \begin{bmatrix} \int_0^w p(s)\Delta s + \sum_{i=1}^q d_i \\ \int_0^w z(s)\Delta s + \sum_{i=1}^q f_i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}.$$

ImL is closed in Y and $dimKerL = codimImL = 2$. We can show this as follows. It is obvious that summation of any element from ImL and $KerL$ is in Y. WLOG take $p \in Y$ and $\int_\kappa^{w+\kappa} p(t)\Delta t + \sum_{i=1}^q d_i = I \neq 0$. Let us define a new function

$g = p - \frac{I}{mes(w)}$, where $mes(t)$=$\int_\kappa^{\kappa+t} 1\Delta t$. Then $\frac{I}{mes(w)}$ is constant because $\forall\ \kappa$, $\int_\kappa^{w+\kappa} p(t)\Delta t$ is always same by the definition of periodic time scales and the impulses are constant and there are same number of impulses in the interval $[\kappa, w+\kappa]$, $\forall \kappa$. If we take the integral of g from κ to $w+\kappa$, we get

$$\int_{\kappa}^{w+\kappa} g(t)\Delta t + \sum_{i=1}^{q} d_i = \int_{\kappa}^{w+\kappa} p(t)\Delta t + \sum_{i=1}^{q} d_i - I = 0.$$

Then $p \in Y$ can be written as the summation of $g \in ImL$ and $\frac{I}{mes(w)} \in KerL$. Since $\frac{I}{mes(w)}$ is constant. Similar steps are used for z. $\begin{bmatrix} p \\ z \end{bmatrix} \in Y$ can be written as the summation of an element from Im L and an element from Ker L. Also it is easy to show that any element in Y is uniquely expressed as the summation of an element Ker L and an element from Im L. So $codimImL$ is also 2, we get the desired result. Therefore L is a Fredholm mapping of index zero.

There exist continuous projectors $U : X \to X$ and $V : Y \to Y$ such that

$$U\left(\begin{bmatrix} p \\ z \end{bmatrix}\right) = \frac{1}{mes(w)} \begin{bmatrix} \int_0^w p(s)\Delta s \\ \int_0^w z(s)\Delta s \end{bmatrix}$$

and

$$V\left(\begin{bmatrix} p \\ z \end{bmatrix}, \begin{bmatrix} d_1 \\ f_1 \end{bmatrix}, ..., \begin{bmatrix} d_q \\ f_q \end{bmatrix}\right) = \frac{1}{mes(w)}\left(\begin{bmatrix} \int_0^w p(s)\Delta s + \sum_{i=1}^q d_i \\ \int_0^w z(s)\Delta s + \sum_{i=1}^q f_i \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}, ..., \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right).$$

The generalized inverse $K_U = ImL \to DomL \subset KerP$ is given,

$$K_U\left(\begin{bmatrix} p \\ z \end{bmatrix}, \begin{bmatrix} d_1 \\ f_1 \end{bmatrix}, .., \begin{bmatrix} d_q \\ f_q \end{bmatrix}\right) =$$

$$\begin{bmatrix} \int_0^t p(s)\Delta s + \sum_{t>t_i} d_i - \frac{1}{mes(w)} \int_0^w \int_0^t p(s)\Delta s\Delta t - \sum_{i=1}^q d_i + \frac{1}{mes(w)} \sum_{i=1}^q d_i mes(t_i) \\ \int_0^t z(s)\Delta s + \sum_{t>t_i} f_i - \frac{1}{mes(w)} \int_0^w \int_0^t z(s)\Delta s\Delta t - \sum_{i=1}^q f_i + \frac{1}{mes(w)} \sum_{i=1}^q f_i mes(t_i) \end{bmatrix}.$$

$$VC\left(\begin{bmatrix} p \\ z \end{bmatrix}\right) =$$

$$\frac{1}{mes(w)}\left(\begin{bmatrix} \int_0^w a(s) - b(s)exp(p(s)) - \frac{c(s)exp(z(s))}{\alpha(s)+\beta(s)exp(p(s))+m(s)exp(z(s))}\Delta s + ln\prod_{i=1}^q(1+g_i) \\ \int_0^w -d(s) + \frac{f(s)exp(p(s))}{\alpha(s)+\beta(s)exp(p(s))+m(s)exp(z(s))}\Delta s + ln\prod_{i=1}^q(r_i) \end{bmatrix}, .., \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right).$$

Let

$$a(t) - b(t)exp(p(t)) - \frac{c(t)exp(z(t))}{\alpha(t)+\beta(t)exp(p(t))+m(t)exp(z(t))} = C_1,$$

$$-d(t) + \frac{f(t)exp(p(t))}{\alpha(t)+\beta(t)exp(p(t))+m(t)exp(z(t))} = C_2,$$

$$\frac{1}{mes(w)}\int_0^w a(s)-b(s)exp(p(s))-\frac{c(s)exp(z(s))}{\alpha(s)+\beta(s)exp(p(s))+m(s)exp(z(s))}\Delta s=\bar{C}_1,$$

and

$$\frac{1}{mes(w)}\int_0^w -d(s)+\frac{f(s)exp(p(s))}{\alpha(s)+\beta(s)exp(p(s))+m(s)exp(z(s))}\Delta s=\bar{C}_2.$$

$$K_U(I-V)C\left(\begin{bmatrix} p \\ z \end{bmatrix}\right)=K_U\left(\begin{bmatrix} C_1-\bar{C_1} \\ C_2-\bar{C_2} \end{bmatrix},\begin{bmatrix} ln(1+g_1) \\ ln(p_1) \end{bmatrix},.,\begin{bmatrix} ln(1+g_q) \\ ln(p_q) \end{bmatrix}\right)$$

$$=\begin{bmatrix} \int_0^t C_1(s)-\bar{C_1}(s)\Delta s+ln\prod_{t>t_i}(1+g_i) \\ -\frac{1}{mes(w)}\int_0^w\int_0^t C_1(s)-\bar{C_1}(s)\Delta s\Delta t-ln\prod_{i=1}^q(1+g_i)+\frac{1}{mes(w)}\sum_{i=1}^q ln(1+g_i)t_i \\ \int_0^t C_2(s)-\bar{C_2}(s)\Delta s+ln\prod_{t>t_i} r_i \\ -\frac{1}{mes(w)}\int_0^w\int_0^t C_2(s)-\bar{C_2}(s)\Delta s\Delta t-ln\prod_{i=1}^q r_i+\frac{1}{mes(w)}\sum_{i=1}^q ln(r_i)t_i \end{bmatrix}.$$

Clearly, VC and $K_U(I-V)C$ are continuous. Since X and Y are Banach spaces, then by using Arzela-Ascoli theorem we can find $K_U(I-\bar{V})C(\bar{\Omega})$ is compact for any open bounded set $\Omega\subset X$. Additionally, $VC(\bar{\Omega})$ is bounded. Thus, C is L-compact on $\bar{\Omega}$ with any open bounded set $\Omega\subset X$.

To apply the continuation theorem we investigate the below operator equation.

$$\begin{aligned} x^{\Delta}(t)&=\lambda\left[a(t)-b(t)exp(x(t))-\tfrac{c(t)exp(y(t))}{\alpha(t)+\beta(t)exp(x(t))+m(t)exp(y(t))}\right],t\neq t_k \\ y^{\Delta}(t)&=\lambda\left[-d(t)+\tfrac{f(t)exp(x(t))}{\alpha(t)+\beta(t)exp(x(t))+m(t)exp(y(t))}\right],t\neq t_k \\ \Delta x(t_k)&=\lambda ln(1+g_k) \\ \Delta y(t_k)&=\lambda ln(r_k) \end{aligned}\tag{3.2}$$

Let $\begin{bmatrix} x \\ y \end{bmatrix}\in X$ be any solution of system (3.2). If we integrate both sides of system (3.2) over the interval $[0,w]$ then we have,

$$\begin{cases} \int_0^w a(t)\Delta t+ln\prod_{i=1}^q(1+g_i)=\int_0^w b(t)exp(x(t))+\frac{c(t)exp(y(t))}{\alpha(t)+\beta(t)exp(x(t))+m(t)exp(y(t))}\Delta t & , \\ \int_0^w d(t)\Delta t-ln\prod_{i=1}^q(r_i)=\int_0^w\frac{f(t)exp(x(t))}{\alpha(t)+\beta(t)exp(x(t))+m(t)exp(y(t))}\Delta t & , \end{cases}\tag{3.3}$$

Using (3.2) and (3.3) we obtain,

$$\begin{aligned}\int_0^w |x^\Delta(t)|\Delta t &\le \lambda\left[\int_0^w |a(t)|\Delta t + \int_0^w b(t)exp(x(t)) + \frac{c(t)exp(y(t))}{\alpha(t)+\beta(t)exp(x(t))+m(t)exp(y(t))}\Delta t\right],\\ &\le \lambda\left[\int_0^w |a(t)|\Delta t + \int_0^w a(t)\Delta t + ln\prod_{i=1}^q (1+g_i)\right]\\ &\le M_1;\end{aligned} \tag{3.4}$$

where $M_1 := \int_0^w |a(t)|\Delta t + \int_0^w a(t)\Delta t + ln\prod_{i=1}^q (1+g_i)$.

$$\begin{aligned}\int_0^w |y^\Delta(t)|\Delta t &\le \lambda\left[\int_0^w |d(t)|\Delta t + \int_0^w \frac{f(t)exp(x(t))}{\alpha(t)+\beta(t)exp(x(t))+m(t)exp(y(t))}\Delta t\right]\\ &\le \lambda\left[\int_0^w |d(t)|\Delta t + \int_0^w d(t)\Delta t - ln\prod_{i=1}^q r_i\right]\\ &\le M_2;\end{aligned} \tag{3.5}$$

where $M_2 := \int_0^w |d(t)|\Delta t + \int_0^w d(t)\Delta t - ln\prod_{i=1}^q r_i$

Since $\begin{bmatrix} x \\ y \end{bmatrix} \in X$ and there are q impulses which are constant, then there exist η_i, ξ_i, $i = 1, 2$ such that

$$\begin{aligned} x(\xi_1) &= min\{inf_{t\in[0,t_1]}x(t), inf_{t\in(t_1,t_2]}x(t), ..., inf_{t\in(t_q,w]}x(t)\}\\ x(\eta_1) &= max\{sup_{t\in[0,t_1]}x(t), sup_{t\in(t_1,t_2]}x(t), ..., sup_{t\in(t_q,w]}x(t)\}\end{aligned} \tag{3.6}$$

$$\begin{aligned} y(\xi_2) &= min\{inf_{t\in[0,t_1]}y(t), inf_{t\in(t_1,t_2]}y(t), ..., inf_{t\in(t_q,w]}y(t)\}\\ y(\eta_2) &= max\{sup_{t\in[0,t_1]}y(t), sup_{t\in(t_1,t_2]}y(t), ..., sup_{t\in(t_q,w]}y(t)\}\end{aligned} \tag{3.7}$$

By the second equation of (3.3) and (3.6) and the first assumption of Theorem 3.2.1, we have

$$\begin{aligned}\int_0^w a(t)\Delta t + ln\prod_{i=1}^q (1+g_i) &\le \int_0^w \left[b(t)exp(x(\eta_1)) + \frac{c(t)}{m(t)}\right]\Delta t\\ &= exp(x(\eta_1))\int_0^w b(t)\Delta t + \int_0^w \frac{c(t)}{m(t)}\Delta t\end{aligned}$$

and $x(\eta_1) \ge l_1$; where $l_1 := ln\left(\frac{\int_0^w a(t)\Delta t + ln\prod_{i=1}^q(1+g_i) - \int_0^w \frac{c(t)}{m(t)}\Delta t}{\int_0^w b(t)\Delta t}\right)$.

using the second inequality in Lemma 3.1.1 we have

$$\begin{aligned} x(t) &\ge x(\eta_1) - \int_0^w |x^\Delta(t)|\Delta t \ge x(\eta_1) - \left(\int_0^w |a(t)|\Delta t + \int_0^w a(t)\Delta t + ln\prod_{i=1}^q (1+g_i)\right)\\ &\ge H_1 := l_1 - M_1.\end{aligned} \tag{3.8}$$

By the first equation of (3.3) and (3.6) we get

$x(\xi_1) \leq l_2$, where $l_2 := ln\left(\frac{\int_0^w a(t)\Delta t + ln\prod_{i=1}^q(1+g_i)}{\int_0^w b(t)\Delta t}\right)$.

using the first inequality in Lemma 3.1.1 we have

$$\begin{array}{rcl} x(t) \leq x(\xi_1) + \int_0^w |x^{\Delta}(t)|\Delta t & \leq & x(\xi_1) + \left(\int_0^w |a(t)|\Delta t + \int_0^w a(t)\Delta t + ln\prod_{i=1}^q(1+g_i)\right) \\ & \leq & H_2 := l_2 + M_1. \end{array} \tag{3.9}$$

By (3.8) and (3.9) $max_{t\in[0,w]}|x(t)| \leq B_1 := max\{|H_1|, |H_2|\}$. Using (3.9), second equation of (3.3) and first equation of (3.7), we can derive that

$$\begin{array}{l} \int_0^w d(t)\Delta t - ln\prod_{i=1}^q(r_i) \leq \int_0^w \frac{f(t)exp(x(t))}{\beta^l exp(x(t))+m^l exp(y(t))}\Delta t \leq \int_0^w \frac{f(t)e^{H_2}}{\beta^l e^{H_2}+m^l exp(y(\xi_2))}\Delta t \\ \qquad = \frac{e^{H_2}}{\beta^l e^{H_2}+m^l exp(y(\xi_2))}\int_0^w f(t)\Delta t \end{array}$$

Therefore

$$exp(y(\xi_2)) \leq \frac{1}{m^l}\left(\frac{e^{H_2}\int_0^w f(t)\Delta t}{\int_0^w d(t)\Delta t - ln\prod_{i=1}^q(r_i)} - \beta^l e^{H_2}\right)$$

By the assumption of the Theorem 3.2.1 we can show that

$\int_0^w f(t)\Delta t - \beta^l(\int_0^w d(t)\Delta t - ln\prod_{i=1}^q(r_i)) > 0$ and $y(\xi_2) \leq L_1$

where $L_1 := ln\left(\frac{1}{m^l}\left(\frac{e^{H_2}\int_0^w f(t)\Delta t}{\int_0^w d(t)\Delta t - ln\prod_{i=1}^q(r_i)} - \beta^l e^{H_2}\right)\right)$.

Hence, by using the first inequality in Lemma 3.1.1 and the second equation of (3.3),

$$\begin{array}{rcl} y(t) \leq y(\xi_2) + \int_0^w |y^{\Delta}(t)|\Delta t & \leq & y(\xi_2) + \left(\int_0^w |d(t)|\Delta t + \int_0^w d(t)\Delta t - ln\prod_{i=1}^q(r_i)\right) \\ & \leq & H_3 := L_1 + M_2. \end{array} \tag{3.10}$$

Again by using the second equation of (3.3) we can derive,

$$\begin{array}{rcl} \int_0^w d(t)\Delta t - ln\prod_{i=1}^q(r_i) & \geq & \int_0^w \frac{f(t)exp(x(t))}{\alpha^u+\beta^u exp(x(t))+m^u exp(y(t))}\Delta t \\ & \geq & \int_0^w \frac{f(t)e^{H_1}}{\alpha^u+\beta^u e^{H_1}+m^u exp(y(\eta_2))}\Delta t \\ & = & \frac{e^{H_1}}{\alpha^u+\beta^u e^{H_1}+m^u exp(y(\eta_2))}\int_0^w f(t)\Delta t \end{array}$$

$$exp(y(\eta_2)) \geq \frac{1}{m^u}\left(\frac{e^{H_1}\int_0^w f(t)\Delta t}{\int_0^w d(t)\Delta t - ln\prod_{i=1}^q (r_i)} - \beta^u e^{H_1} - \alpha^u\right).$$

Again using second assumption of Theorem 3.2.1 we obtain

$$e^{H_1}\left(\int_0^w f(t)\Delta t - \beta^u\left(\int_0^w d(t)\Delta t - ln\prod_{i=1}^q (r_i)\right)\right) - \alpha^u\left(\int_0^w d(t)\Delta t - ln\prod_{i=1}^q (r_i)\right) > 0$$

and $y(\eta_2) \geq L_2$ where $L_2 := ln\left(\frac{1}{m^u}\left(\frac{e^{H_1}\int_0^w f(t)\Delta t}{\int_0^w d(t)\Delta t - ln\prod_{i=1}^q (r_i)} - \beta^u e^{H_1} - \alpha^u\right)\right)$.

By using the second inequality in Lemma 3.1.1,

$$\begin{aligned} y(t) &\geq y(\eta_2) - \int_0^w |y^\Delta(t)|\Delta t \\ &\geq y(\eta_2) - \left(\int_0^w |d(t)|\Delta t + \int_0^w d(t)\Delta t - ln\prod_{i=1}^q (r_i)\right) \\ &\geq H_4 := L_2 - M_2. \end{aligned} \tag{3.11}$$

By (3.10) and (3.11) we have $max_{t\in[0,w]}|y(t)| \leq B_2 := max\{|H_3|, |H_4|\}$. Obviously, B_1 and B_2 are both independent of λ. Let $M = B_1 + B_2 + 1$. Then $max_{t\in[0,w]}\left\|\begin{bmatrix} x \\ y \end{bmatrix}\right\| < M$. Let $\Omega = \left\{\left\|\begin{bmatrix} x \\ y \end{bmatrix}\right\| \in X : \left\|\begin{bmatrix} x \\ y \end{bmatrix}\right\| < M\right\}$ and Ω verifies the requirement (a) in Theorem 3.1.1. When $\begin{bmatrix} x \\ y \end{bmatrix} \in KerL \cap \partial\Omega$, $\begin{bmatrix} x \\ y \end{bmatrix}$ is a constant with $\left\|\begin{bmatrix} x \\ y \end{bmatrix}\right\| = M$, then

$$VC\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) =$$

$$\left(\begin{bmatrix} \int_0^w a(s) - b(s)exp(x) - \frac{c(s)exp(y)}{\alpha(s)+\beta(s)exp(x)+m(s)exp(y)}\Delta s + ln\prod_{i=1}^q (1+g_i) \\ \int_0^w -d(s) + \frac{f(s)exp(x)}{\alpha(s)+\beta(s)exp(x)+m(s)exp(y)}\Delta s + ln\prod_{i=1}^q (r_i) \end{bmatrix}, .., \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right)$$

$$\neq \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, ..., \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right).$$

$$JVC\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) =$$

$$\begin{bmatrix} \int_0^w a(s) - b(s)exp(x(s)) - \frac{c(s)exp(y(s))}{\alpha(s)+\beta(s)exp(x(s))+m(s)exp(y(s))}\Delta s + ln\prod_{i=1}^q (1+g_i) \\ \int_0^w -d(s) + \frac{f(s)exp(x(s))}{\alpha(s)+\beta(s)exp(x(s))+m(s)exp(y(s))}\Delta s + ln\prod_{i=1}^q (r_i) \end{bmatrix},$$

where $J : ImV \to KerL$ such that $J\left(\begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}, ..., \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix}$.

Define the homotopy $H_\nu = \nu(JVC) + (1-\nu)G$ where

$$G\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \int_0^w a(s) - b(s)exp(x)\Delta s + ln\prod_{i=1}^{q}(1+g_i) \\ \int_0^w d(s) - \frac{f(s)exp(x)}{\alpha(s)+\beta(s)exp(x)+m(s)exp(y)}\Delta s + ln\prod_{i=1}^{q}(r_i) \end{bmatrix}$$

Take DJ_G as the determinant of the jacobian of G. Since $\begin{bmatrix} x \\ y \end{bmatrix} \in KerL$, then jacobian of G is

$$\begin{bmatrix} -e^x\int_0^w b(s)\Delta s & 0 \\ \int_0^w \frac{-e^x f(s)}{\alpha(s)+\beta(s)e^x+m(s)e^y}\Delta s + \int_0^w \frac{(e^x)^2 f(s)\beta(s)}{(\alpha(s)+\beta(s)e^x+m(s)e^y)^2}\Delta s & -\int_0^w \frac{e^x e^y f(s)m(s)}{(\alpha(s)+\beta(s)e^x+m(s)e^y)^2}\Delta s \end{bmatrix}.$$

All the functions in jacobian of G is positive then $signDJ_G$ is always positive. Hence

$$deg(JC, \Omega\cap KerL, 0) = deg(G, \Omega\cap KerL, 0) = \sum_{\begin{bmatrix} x \\ y \end{bmatrix}\in G^{-1}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right)} signDJ_G\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) \neq 0.$$

Thus all the conditions of Theorem 3.1.1 are satisfied. Therefore system (3.1) has at least a positive w-periodic solution. □

Corollary 3.2.1 *If* $\alpha(t) = 0$ *in the system* (3.1),

$\int_0^w a(t)\Delta t + ln\prod_{i=1}^{q}(1+g_i) - \int_0^w \frac{c(t)}{m(t)}\Delta t > 0$ *and*

$\int_0^w f(t)\Delta t - \beta^u(\int_0^w d(t)\Delta t - ln\prod_{i=1}^{q}(r_i)) > 0$ *is satisfied then the system* (3.1) *has at least one* w*-periodic solution.*

Theorem 3.2.2 *If same conditions are valid for the coefficient functions in system* (3.1)*; except* $\alpha(t)$ *where* $\alpha(t) > 0$ *and*

$$\left(\frac{\int_0^w a(t)\Delta t + ln\prod_{i=1}^{q}(1+g_i)}{\int_0^w b(t)\Delta t}\right)exp\left[-\left(\int_0^w |a(t)|\Delta t + \int_0^w a(t)\Delta t + ln\prod_{i=1}^{q}(1+g_i)\right)\right].$$
$$.(\int_0^w f(t)\Delta t - \beta^u(\int_0^w d(t)\Delta t - ln\prod_{i=1}^{q}(r_i))) - \alpha^u(\int_0^w d(t)\Delta t - ln\prod_{i=1}^{q}(r_i)) > 0$$

is satisfied then there exist at least a w*-periodic solution.*

Proof. First part of the proof is very similar with the proof of Theorem 3.2.1. By (3.2), (3.3) and (3.6)

$$\begin{array}{rcl}\int_0^w d(t)\Delta t - ln\prod_{i=1}^q(r_i) & = & \int_0^w \frac{f(t)exp(x(t))}{\alpha(t)+\beta(t)exp(x(t))+m(t)exp(y(t))}\Delta t \\ & \leq & exp(x(\eta_1))\int_0^w \frac{f(t)}{\alpha(t)}\Delta t\end{array}$$

By (3.3) $\int_0^w d(t)\Delta t - ln\prod_{i=1}^q(r_i) > 0$. Also by the assumption of Theorem 3.2.2 $f(t), \alpha(t) > 0$. Then we get $x(\eta_1) \geq \tilde{l_1}$, $\tilde{l_1} := ln\left(\frac{\int_0^w d(t)\Delta t - ln\prod_{i=1}^q(r_i)}{\int_0^w f(t)/\alpha(t)\Delta t}\right)$.

and using the second inequality in Lemma 3.1.1 we have

$$\begin{array}{rcl}x(t) & \geq & x(\eta_1) - \int_0^w |x^\Delta(t)|\Delta t \\ & \geq & x(\eta_1) - \left(\int_0^w |a(t)|\Delta t + \int_0^w a(t)\Delta t + ln\prod_{i=1}^q(1+g_i)\right) \\ & = & \tilde{H_1} := \tilde{l_1} - \tilde{M_1}.\end{array} \tag{3.12}$$

By the first equation of (3.3) and (3.6)

$$\begin{array}{rcl}\int_0^w a(t)\Delta t + ln\prod_{i=1}^q(1+g_i) & \geq & \int_0^w b(t)exp(x(\xi_1))\Delta t \\ & = & exp(x(\xi_1))\int_0^w b(t)\Delta t\end{array}.$$

Then we get $x(\xi_1) \leq \tilde{l_2}$ where $\tilde{l_2} := ln\left(\frac{\int_0^w a(t)\Delta t + ln\prod_{i=1}^q(1+g_i)}{\int_0^w b(t)\Delta t}\right)$.

Using the first inequality in Lemma 3.1.1 we have

$$\begin{array}{rcl}x(t) & \leq & x(\xi_1) + \int_0^w |x^\Delta(t)|\Delta t \\ & \leq & x(\xi_1) + \left(\int_0^w |a(t)|\Delta t + \int_0^w a(t)\Delta t + ln\prod_{i=1}^q(1+g_i)\right) \\ & \leq & \tilde{H_2} := \tilde{l_2} + \tilde{M_1}.\end{array} \tag{3.13}$$

By (3.12) and (3.13) $max_{t\in[0,w]}|x(t)| \leq \tilde{B_1} := max\{|\tilde{H_1}|, |\tilde{H_2}|\}$. From the second equation of (3.3) and the second equation of (3.7), we can derive that

$$\begin{array}{rcl}\int_0^w d(t)\Delta t - ln\prod_{i=1}^q(r_i) & \leq & \int_0^w \frac{f(t)exp(x(t))}{m(t)exp(y(t))}\Delta t \\ & \leq & \int_0^w \frac{f(t)e^{\tilde{H_2}}}{m(t)exp(y(\xi_2))}\Delta t \\ & = & \frac{e^{\tilde{H_2}}}{exp(y(\xi_2))}\int_0^w f(t)/m(t)\Delta t.\end{array}$$

Therefore

$$exp(y(\xi_2)) \leq e^{\tilde{H_2}}\left(\frac{\int_0^w f(t)/m(t)\Delta t}{\int_0^w d(t)\Delta t - ln\prod_{i=1}^q(r_i)}\right)$$

Since $e^{\tilde{H}_2}, f(t), m(t) > 0$, then $y(\xi_2) \le \tilde{L}_1$

where $\tilde{L}_1 := ln\left(e^{\tilde{H}_2}\left(\frac{\int_0^w f(t)/m(t)\Delta t}{\int_0^w d(t)\Delta t - ln\prod_{i=1}^q (r_i)}\right)\right)$.

Hence, by using the first inequality in Lemma 3.1.1 and the second equation of (3.3),

$$\begin{aligned} y(t) &\le y(\xi_2) + \int_0^w |y^\Delta(t)|\Delta t \\ &\le y(\xi_2) + \left(\int_0^w |d(t)|\Delta t + \int_0^w d(t)\Delta t - ln\prod_{i=1}^q (r_i)\right) \\ &\le \tilde{H}_3 := \tilde{L}_1 + \tilde{M}_2. \end{aligned} \tag{3.14}$$

By the assumption of Theorem 3.2.2 there exists n_0 such that $\forall n \ge n_0$

$$\left(\frac{\int_0^w a(t)\Delta t + ln\prod_{i=1}^q (1+g_i)}{\int_0^w b(t)\Delta t + 1/n\int_0^w c(t)/\alpha(t)\Delta t}\right) exp\left[-\left(\int_0^w |a(t)|\Delta t + \int_0^w a(t)\Delta t + ln\prod_{i=1}^q (1+g_i)\right)\right]$$
$$.(\int_0^w f(t)\Delta t - \beta^u(\int_0^w d(t)\Delta t - ln\prod_{i=1}^q (r_i))) - \alpha^u(\int_0^w d(t)\Delta t - ln\prod_{i=1}^q (r_i)) > 0$$

is true. We need to get $\tilde{H}_4$ such that $\forall\ t \in [0,w]_{\mathbb{T}}\ y(t) \ge \tilde{H}_4$. Let us assume there exists $t, s \in [0,w]_{\mathbb{T}}$ such that $y(s) \ge x(t) - ln(n_0)$. Then by using (3.6) and (3.7) we obtain

$$y(\eta_2) \ge y(s) \ge x(t) - ln(n_0) \ge x(\xi_1) - ln(n_0) \ge \tilde{H}_1 - ln(n_0) := M_4^1.$$

If there does not exists such t,s then $\forall\ t, s \in [0,w]_{\mathbb{T}}$, $y(s) < x(t) - ln(n_0)$. Also from the first equation of (3.3), we have

$$\begin{aligned} \int_0^w a(t)\Delta t + ln\prod_{i=1}^q (1+g_i) &\le exp(x(\eta_1))\int_0^w b(t)\Delta t + exp(y(\eta_2))\int_0^w \frac{c(t)}{\alpha(t)}\Delta t \\ &\le exp(x(\eta_1))\left(\int_0^w b(t)\Delta t + (1/n_0)\int_0^w \frac{c(t)}{\alpha(t)}\Delta t\right). \end{aligned}$$

By using first inequality in Lemma 3.1.1, we have $exp(x(t)) \ge K$, where

$$K := \frac{\int_0^w a(t)\Delta t + ln\prod_{i=1}^q (1+g_i)}{\left(\int_0^w b(t)\Delta t + (1/n_0)\int_0^w \frac{c(t)}{\alpha(t)}\Delta t\right)} exp\left(-\left(\int_0^w |a(t)|\Delta t + \int_0^w a(t)\Delta t + ln\prod_{i=1}^q (1+g_i)\right)\right).$$

Using the second equality in (3.3) and the assumption of the Theorem 3.2.2, we obtain

$$\int_0^w d(t)\Delta t - ln\prod_{i=1}^q (r_i) \ge \frac{K}{\alpha^u + \beta^u K + m^u exp(y(\eta_2))}\int_0^w f(t)\Delta t.$$

This implies $y(\eta_2) \ge M_4^2$, where

$$M_4^2 := ln\left[\left(\frac{\int_0^w a(t)\Delta t + ln\prod_{i=1}^q (1+g_i)}{\int_0^w b(t)\Delta t + (1/n_0)\int_0^w c(t)/\alpha(t)\Delta t}\right) exp\left[-\left(\int_0^w |a(t)|\Delta t + \int_0^w a(t)\Delta t + ln\prod_{i=1}^q (1+g_i)\right)\right]\right.$$
$$\left. .(\int_0^w f(t)\Delta t - \beta^u(\int_0^w d(t)\Delta t - ln\prod_{i=1}^q (r_i))) - \alpha^u(\int_0^w d(t)\Delta t - ln\prod_{i=1}^q (r_i))\right]$$
$$-ln(m^u(\int_0^w d(t)\Delta t - ln\prod_{i=1}^q (r_i))).$$

Thus, according what we have done above $y(\eta_2) \geq M_4 := min\{M_4^1, M_4^2\}$. Using second inequality in Lemma 3.1.1 we have $y(t) \leq \tilde{H}_4$ where

$\tilde{H}_4 := M_4 - (\int_0^w |d(t)|\Delta t + \int_0^w d(t)\Delta t - ln \prod_{i=1}^q (r_i)))$.

Thus $max_{t\in[0,w]_{\mathbb{T}}}|y(t)| \leq \tilde{B}_2 := max\{|\tilde{H}_3|, |\tilde{H}_4|\}$. Obviously, $\tilde{B}_1$ and $\tilde{B}_2$ are both independent of λ. Let $M = \tilde{B}_1 + \tilde{B}_2 + 1$. Then $max_{t\in[0,w]}\left\|\begin{bmatrix} x \\ y \end{bmatrix}\right\| < M$. Let $\Omega = \left\{\left\|\begin{bmatrix} x \\ y \end{bmatrix}\right\| \in X : \left\|\begin{bmatrix} x \\ y \end{bmatrix}\right\| < M\right\}$ then Ω verifies the requirement (a) in Theorem 3.1.1. Rest of the proof is similar to Theorem 3.2.1. □

By using the informations from [10] and [50] we give the model of the example as follows:

Let there are two insect populations, one of them is the predator, the other one is the prey. While in a season, for instance during the six warm months of the year both insects has a continuous life cycle and they die out in winter, while their eggs are incubating or dormant, and then they both hatch in a new season and both of them giving rise to non-overlapping populations. This situation can be modeled using the time scale

$$\mathbb{T} = \bigcup_{k\in\mathbb{Z}} [2k, 2k+1], \quad with \quad w = 2$$

Here impulsive effect of the pest population density is after its partial destruction by catching, poisoning with chemicals used in agriculture and some other negative effects of environment on the prey (can be shown by $-1 < g_k < 0$) and impulsive increase of the predator population density is by artificially breeding the species or releasing some species ($r_k > 0$). In addition to these, if the model assumes a Beddington–DeAngelis functional response as in (3.1) and if the assumptions in Theorem 3.2.1 are satisfied then there exists a 2-periodic solution of (3.1).

Example 3.2.1 $\mathbb{T} = [2k, 2k+1],\ k \in \mathbb{N}$ *k start with 0.*

$x^{\Delta}(t) = (0.2sin(2\pi t) + 0.3) - (0.2sin(2\pi t) + 0.2)exp(x) - \frac{(0.1+0.1cos(2\pi t))exp(y)}{(0.5sin(2\pi t)+0.7)+(1+0.5cos(2\pi t))exp(x)+exp(y)}$,
$y^{\Delta}(t) = -(0.3sin(2\pi t) + 1) + \frac{(4cos(2\pi t)+6.5)exp(x)}{(0.5sin(2\pi t)+0.7)+(1+0.5cos(2\pi t))exp(x)+exp(y)}, t \neq t_k$
$\Delta x(t_k) = ln(1 + g_k)$
$\Delta y(t_k) = ln(r_k)$

Impulse points: $t_1 = 2k + 1/4$, $t_2 = 2k + 3/4$ *and* $q = 2$.

$g_1 = e^{-0.01} - 1$, $g_2 = e^{-0.01} - 1$

$p_1 = e^{0.1}$, $p_2 = e^{0.1}$

Example 3.2.1 satisfies all the conditions of Theorem 3.2.1, thus it has at least one periodic solution.

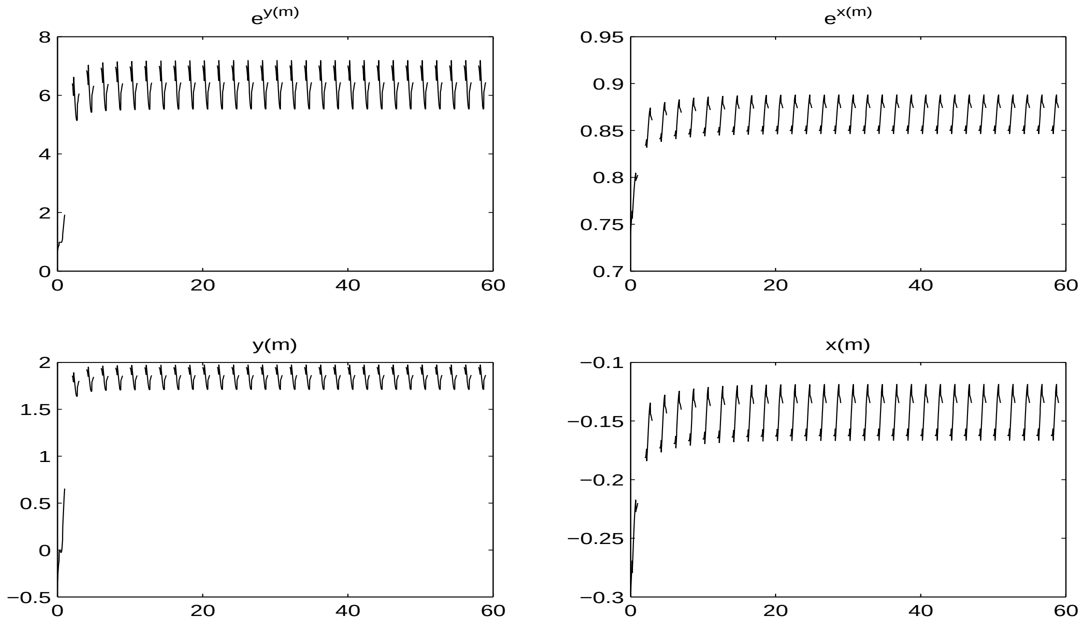

Figure 3.1: Numeric solution of Example 3.2.1 shows the periodicity.

Example 3.2.2 $\mathbb{T} = [2k, 2k+1],\ k \in \mathbb{N}$ *k start with 0.*

$$x^{\Delta}(t) = (0.2sin(2\pi t) + 0.3) - (0.1sin(2\pi t) + 0.2)exp(x) - \frac{(2+cos(2\pi t))exp(y)}{(sin(2\pi t)+2)+(1+0.5cos(2\pi t))exp(x)+4exp(y)},$$
$$y^{\Delta}(t) = -(0.3sin(2\pi t) + 1) + \frac{(4cos(2\pi t)+6.5)exp(x)}{(sin(2\pi t)+2)+(1+0.5cos(2\pi t))exp(x)+4exp(y)}, t \neq t_k$$
$$\Delta x(t_k) = ln(1 + g_k)$$
$$\Delta y(t_k) = ln(r_k)$$

Impulse points: $t_1 = 2k + 1/4$, $t_2 = 2k + 3/4$ *and* $q = 2$.

$g_1 = e^{-0.01} - 1$, $g_2 = e^{-0.01} - 1$

$p_1 = e^{0.1}$, $p_2 = e^{0.1}$

Example 3.2.2 satisfies all the conditions of Theorem 3.2.2, thus it has at least one periodic solution.

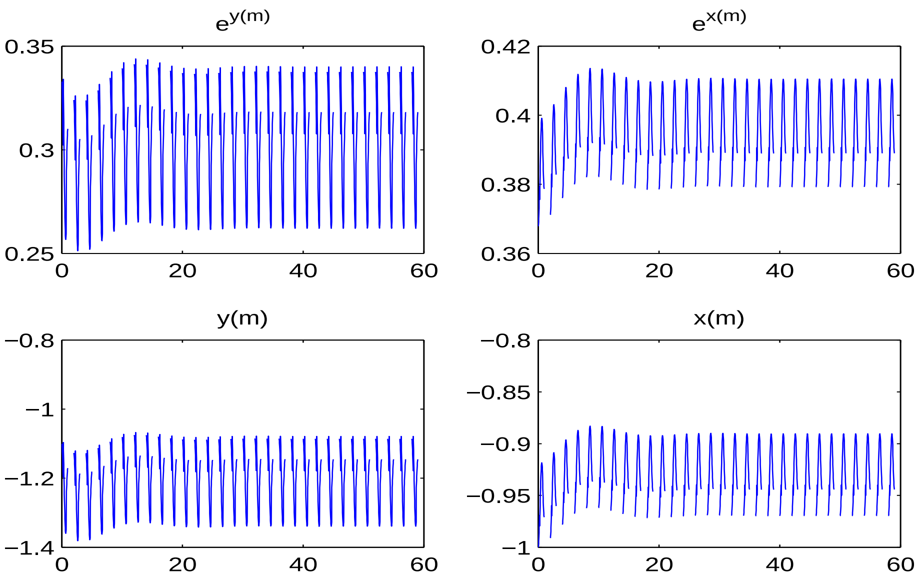

Figure 3.2: Numeric solution of Example 3.2.2 shows the periodicity.

Definition 3.2.1 *[12] Let Ω be a nonempty open subset of $\mathbb{R}$, then the function f is said to be Lipschitz continuous with respect to the first variable if $x \in \Omega$ and if for every $a, b \in \mathbb{T}$ such that $a < b$ and $L \geq 0$ such that*

$$||f(x_1, t) - f(x_2, t)|| \leq L||x_1 - x_2||,$$

for all $x_1, x_2 \in \Omega$ and for all $t \in [a, b)_{\mathbb{T}}$.

Lemma 3.2.2 *Let $x(t),\ u(t) \in C^1_{rd}(\mathbb{T}, \mathbb{R})$. Consider the following inequality and equation $x^{\Delta}(t) \leq (\geq) f(t, x(t)), u^{\Delta}(t) = f(t, u(t))$ if $u(t_0) \geq (\leq) x(t_0)$ and $f(t, v(t))$ is Lipschitz continuous with respect to v(t), then $x(t) \leq (\geq) u(t)$.*

Proof. By contradiction, we get the result. If $x(T) > (<) u(T)$ for some $T > t_0$, then set

$$t_1 = sup\{t:\ t_0 \leq t < T \ and \ x(t) \leq (\geq) u(t)\}.$$

Then $t_0 \leq t_1 < T$, $x(t_1) \leq (\geq) u(t_1)$, and $x(t) > (<) u(t)$ for $t > t_1$.

For $t_1 \leq t \leq T$, $|x(t) - u(t)| = x(t) - u(t)$,($|u(t) - x(t)| = u(t) - x(t)$) so we have

$$(x - u)^{\Delta}(t) \leq f(t, x) - f(t, u) \leq |L||x - u| = |L|(x - u).$$

$$((u - x)^{\Delta}(t) \leq f(t, u) - f(t, x) \leq |L||u - x| = |L|(u - x)).$$

By Gronwall's inequality [2] (applied to $x - u$ on $[t_1, T]$, with $(x - u)(t_1) \leq 0$), $(x - u)(t) \leq 0$ on $[t_1, T]$, since $e_{|L|}(t_1, t) > 0$.

(By Gronwall's inequality [2] (applied to $u - x$ on $[t_1, T]$, with $(u - x)(t_1) \leq 0$), $(u - x)(t) \leq 0$ on $[t_1, T]$, since $e_{|L|}(t_1, t) > 0$.)

Hence there is a contradiction. □

Lemma 3.2.3 *Consider the below equations*

$$\begin{aligned} &x^{\Delta}(t) \leq (\geq) a(t) - b(t) exp(x(t)),\ t \neq t_k \\ &\Delta x(t_k) = ln(1 + g_k) \end{aligned}$$

$$\begin{aligned} &u^{\Delta}(t) = a(t) - b(t) exp(u(t)),\ t \neq t_k \\ &\Delta u(t_k) = ln(1 + g_k) \end{aligned}$$

If $u(t_0) \geq x(t_0)$, then $x(t) \leq (\geq) u(t)$.

Proof. By using induction we try to prove the lemma. First of all since

$x^{\Delta}(t), u^{\Delta}(t) \leq a(t)$, then the solutions $x(t), u(t)$ are bounded on a bounded interval. therefore on the interval $[0, t_1]_{\mathbb{T}}$ exp function becomes Lipschitz continuous and since b(t) is periodic we can apply Lemma 3.2.2. Then $u(t) \geq x(t)$ on $[0, t_1]_{\mathbb{T}}$.

Thus $u(t_1) + ln(1 + g_1) \geq x(t_1) + ln(1 + g_1)$. Similarly in the interval $[t_1, t_2]_{\mathbb{T}}$ since $u(t_1+) \geq x(t_1+)$, then $u(t) \geq x(t)$ on $[t_1, t_2]_{\mathbb{T}}$.

Therefore $u(t_2) + ln(1 + g_2) \geq x(t_2) + ln(1 + g_2)$. Equivalently, $u(t_2+) \geq x(t_2+)$. Let us also assume that $u(t) \geq x(t)$ is true on $[t_{n-1}, t_n]_{\mathbb{T}}$. Then

$u(t_n) + ln(1 + g_n) \geq x(t_n) + ln(1 + g_n)$, In other words $u(t_n+) \geq x(t_n+)$ by the Lemma 3.2.2 $u(t) \geq x(t)$ on $[t_n, t_{n+1}]_{\mathbb{T}}$. Hence $u(t) \geq x(t)$ on $\mathbb{T}$. Other part is similar.

□

Lemma 3.2.4 *If* $\int_0^w a(s) - \frac{c(s)}{m(s)} ds + ln \prod_{i=1}^{q} (1 + g_i) \geq 0$,

$-\int_0^w d(s)ds + ln \prod_{i=1}^{q} (r_i) \geq 0$, $\mathbb{T} = \mathbb{R}$ *then all solutions* $y(t)$ *are bounded below,* $x(t)$ *are bounded above and below.*

Proof. By the first equality of (3.1), we have

$$x'(t) \leq a(t) - b(t)exp(x(t)), \; t \neq t_k$$
$$\Delta x(t_k) = ln(1 + g_k)$$

Then by Lemma 3.2.3 $x(t) \leq u(t)$. Since $\int_0^w a(s)ds + ln \prod_{i=1}^{q} (1 + g_i) > 0$, then $u(t)$ is a periodic solution by Theorem 3.2.1 which is globally attractive by Lemma 2.2 in [50]. Hence $x(t)$ is bounded above.

By the first equality of (3.1) , we obtain

$$x' \geq a(t) - \tfrac{c(t)}{m(t)} - b(t)exp(x(t)), \; t \neq t_k$$
$$\Delta x(t_k) = ln(1 + g_k)$$

Consider the below equality

$$\tilde{u}'(t) = a(t) - \frac{c(t)}{m(t)} - b(t)exp(\tilde{u}(t)), \; t \neq t_k$$

$$\Delta\tilde{u}(t_k) = ln(1 + g_k)$$

By Lemma 3.2.3 $x(t) \geq \tilde{u}(t)$ and since $\int_0^w a(s) - \frac{c(s)}{m(s)} ds + ln \prod_{i=1}^{q}(1+g_i) \geq 0$, then $\tilde{u}(t)$ has a periodic solution which is globally attractive by Lemma 2.2 in [50], then $x(t)$ is bounded below.

By the second equality of (3.1) , we have

$$\begin{aligned} &y'(t) \geq -d(t),\ t \neq t_k \\ &\Delta y(t_k) = ln\ r_k \end{aligned}$$

Then

$$y(t) \geq y(0) - \int_0^t d(s)ds + ln \prod_{0<t_k<t} r_k \geq y(0).$$

Hence y(t) is bounded below.

□

Remark 3.2.1 *For any arbitrary $\mathbb{T}$ if we can find $u(t)$ in Lemma 3.2.3 is globally attractive, then Lemma 3.2.4 will also be true.*

Lemma 3.2.5 *If $\int_0^w a(s) - \frac{c(s)}{m(s)} ds + ln \prod_{i=1}^{q}(1+g_i) \geq 0$, $0 < r_k \leq 1$ for each k and $\mathbb{T} = \mathbb{R}$ then all solutions $(exp(x(t)), exp(y(t)))$ are bounded above and below.*

Proof. By Lemma 3.2.4 $exp(x(t))$ is bounded above and below. By the second equality of (3.1), we have

$$\begin{aligned} &y'(t) \leq \frac{-d(t)exp(y(t)) + f(t)/m(t)exp(x(t))}{exp(y(t))},\ t \neq t_k \\ &\Delta y(t_k) = ln\ r_k \end{aligned}$$

Then

$$\begin{aligned} &[exp(y(t))]' \leq -d(t)exp(y(t)) + f(t)/m(t)exp(x(t)) \\ &exp(y(t_k+)) = r_k exp(y(t_k)). \end{aligned}$$

Since $x(t)$ is bounded above and if $exp(x(t)) \leq S_1$, we obtain

$$\begin{aligned} &[exp(y(t))]' \leq -d(t)exp(y(t)) + f^u/m^l S_1 \\ &exp(y(t_k+)) = r_k exp(y(t_k)). \end{aligned}$$

By [36] Theorem 1.4.1 we have

$$exp(y(t)) \leq exp(y(0)) \prod_{0<t_k<t} r_k exp\left(\int_0^t -d(s)ds\right) + \int_0^t \prod_{s<t_k<t} r_k exp\left(\int_s^t -d(\sigma)d\sigma\right) f^u/m^l S_1 ds.$$

At the beginning we assume that $\int_0^w d(s)ds > 0$ also by [50] Lemma 3.2 since $r_k \leq 1$ for each k and taking $f^u/m^l S_1 = M$, we have

$$\begin{aligned}
exp(y(t)) &\leq exp(y(0)) \prod_{0<t_k<t} r_k\, e^{1+Dw-ct} + Me^{1+Dw} \int_0^t \prod_{s<t_k<t} r_k\, e^{c(s-t)} ds \\
&\leq exp(y(0))\left(e^{1+Dw-ct} + \frac{Me^{1+Dw}}{c}(1-e^{-ct})\right) \\
&\leq exp(y(0))\left(e^{1+Dw} + \frac{Me^{1+Dw}}{c}\right).
\end{aligned}$$

Here $D = \{|d(t)| : t \in [0,w]\}$ and $c = min\{\int_0^w d(s)ds/w, 1/w\}$ therefore apperantly $exp(y(t))$ is bounded above. Hence $y(t)$ is bounded above. $0 \leq exp(y(t))$, then $(exp(x(t)), exp(y(t)))$ are bounded above and below. □

Lemma 3.2.6 *If* $\mathbb{T} = \mathbb{R}$, $\int_0^w a(s) - \frac{c(s)}{m(s)} ds + ln \prod_{i=1}^q (1+g_i) \geq 0$, $0 < r_k \leq 1$ *for each k, and*

$-\int_0^w d(s)ds + ln \prod_{i=1}^q (r_i) + \int_0^w \frac{f(s)s_1}{\alpha(s)+\beta(s)s_1+m(s)S_2} ds \geq 0$, *where* s_1 *is the infimum of the solution*

$$\begin{aligned}
&\tilde{u}' = a(t) - \frac{c(t)}{m(t)} - b(t)exp(\tilde{u}(t)),\ t \neq t_k \\
&\Delta\tilde{u}(t_k) = ln(1+g_k),
\end{aligned}$$

S_2 *is ln of the supremum of the solution*

$$\begin{aligned}
&[exp(v(t))]' = -d(t)exp(v(t)) + S_1 f(t)/m(t) \\
&exp(v(t_k+)) = r_k exp(v(t_k)),
\end{aligned}$$

and S_1 *is exp of the supremum of the solution*

$$\begin{aligned}
&u' = a(t) - b(t)exp(u(t)),\ t \neq t_k \\
&\Delta u(t_k) = ln(1+g_k),
\end{aligned}$$

then all solutions $(exp(x(t)), exp(y(t)))$ *are bounded above and below with positive constants.*

Proof. By the second equality of (3.1), we have

$$\begin{aligned}
&y'(t) \geq -d(t) + \frac{f(t)s_1}{\alpha(t)+\beta(t)s_1+m(t)S_2},\ t \neq t_k \\
&\Delta y(t_k) = ln\, r_k.
\end{aligned}$$

Then by assumption

$y(t) \geq y(0) - \int_0^t d(s) + \frac{f(s)s_1}{\alpha(s)+\beta(s)s_1+m(s)S_2} ds + ln \prod_{0<t_k<t} r_k \geq y(0)$. Hence $exp(y(t))$ is bounded below with a positive constant. By Lemma 3.2.5 we know that $exp(y(t))$ is bounded above with a positive constant. Also $exp(x(t))$ is bounded above and below with a positive constant. □

Remark 3.2.2 *From [29] the definition of globally asymptotically stable solution is same with the globally attractive solution.*

Corollary 3.2.2 *By [50] Theorem 4.4, if the conditions of Lemma 3.2.6 are satisfied, then solution of system* (3.1) *is globally asymptotically stable.*

Lemma 3.2.7 *If* $\alpha(t) = 0$, $\int_0^w a(s) - \frac{c(s)}{m(s)} ds + ln \prod_{i=1}^q (1 + g_i) \geq 0$,

$min_{[0,w]}(-d(t) + f(t)/\beta(t)) \geq 0$,

$\int_0^w \frac{(-d(s)\beta(s)+f(s))s_1}{\beta(s)s_1+m(s)S_2} ds + ln \prod_{i=1}^q \; r_i \geq 0$, *where* s_1, S_2 *are as in Lemma 3.2.6, for each i* $0 < r_i \leq 1$ *and* $\mathbb{T} = \mathbb{R}$ *then all solutions* $(x(t), y(t))$ *are bounded above and below.*

Proof. By the second equality of (3.1), using $min_{[0,w]}(-d(t) + f(t)/\beta(t)) \geq 0$, and by Lemma 3.2.4 since $x(t)$ bounded above and $y(t)$ is bounded above (let say $s_1 \leq x(t) \leq S_1$ and $y(t) \leq S_2$ and s_1, S_1, and S_2 is same as in Lemma 3.2.6), we have

$$\begin{aligned} y'(t) &= -d(t) + \frac{f(t)exp(x(t))}{\beta(t)exp(x(t))+m(t)exp(y(t))}, \; t \neq t_k \\ &= \frac{(-d(t)\beta(t)+f(t))exp(x(t))-d(t)m(t)exp(y(t))}{\beta(t)exp(x(t))+m(t)exp(y(t))}, \; t \neq t_k \\ &\geq \frac{(-d(t)\beta(t)+f(t))exp(x(t))}{\beta(t)exp(x(t))+m(t)exp(y(t))} - \frac{d(t)m(t)exp(y(t))}{\beta(t)exp(x(t))+m(t)exp(y(t))}, \; t \neq t_k \\ &\geq \frac{(-d(t)\beta(t)+f(t))s_1}{\beta(t)s_1+m(t)S_2} - \frac{d(t)m(t)exp(y(t))}{\beta(t)s_1}, \; t \neq t_k \end{aligned}$$

$$\Delta y(t_k) = ln\, r_k$$

Therefore let us consider the equation

$$\check{v}^{\Delta}(t) = \frac{(-d(t)\beta(t)+f(t))s_1}{\beta(t)s_1+m(t)S_2} - \frac{d(t)m(t)exp(\check{v}(t))}{\beta(t)s_1}, \; t \neq t_k$$
$$\Delta\check{v}(t_k) = ln\, r_k.$$

Therefore by Lemma 4 $y(t) \geq \check{v}(t)$ and since

$\int_0^w \frac{(-d(s)\beta(s)+f(s))s_1}{\beta(s)s_1+m(s)S_2}ds + ln\prod_{i=1}^q r_i \geq 0$, then $y(t)$ is bounded below. Other parts are same with Lemma 3.2.4. Hence $(exp(x(t)), exp(y(t)))$ are bounded above and below with positive constants. □

Corollary 3.2.3 *If conditions of Lemma 3.2.7 are satisfied, then automatically Theorem 3.2.1 is satisfied. Also by [50] Theorem 4.4, if the conditions of Lemma 3.2.7 are satisfied, then solution of system* (3.1) *is globally asymptotically stable which means if the conditions of Lemma 3.2.7 are satisfied then there is a globally asymptotically stable periodic solution.*

Corollary 3.2.4 *For the non-impulsive case of system* (3.1) *on* $\mathbb{R}$

if $\alpha(t) = 0$, $\int_0^w a(s) - \frac{c(s)}{m(s)}ds \geq 0$ *and*

$min_{[0,w]}(-d(t) + f(t)/\beta(t)) \geq 0$, *then there exits a periodic solution which is permanent and globally attractive.*

Theorem 3.2.3 *If all the coefficient functions in system* (3.1) *is positive, w-periodic, from* $C_{rd}(\mathbb{T}, \mathbb{R})$ *and impulses are 0; also*

$$(\frac{a^l}{b^u})exp(\mu(a^l - b^u(a^u/b^l(exp(\mu a^u))) - c^u/m^l)).(\int_0^w f(t)\Delta t - \beta^u(\int_0^w d(t)\Delta t)) - \alpha^u(\int_0^w d(t)\Delta t) > 0$$

is satisfied then there exist at least a w*-periodic solution.* $\mu = \sup_{[0,w]_{\mathbb{T}}} \mu(t)$, *for the time scales whose graininess function is bounded over this Time scales.*

Proof. First part of the proof is similar to Theorem 3.2.1, only difference is the zero impulses. If the assumption of Theorem 3.2.3 is true then there exists n_0 such that for all $n \geq n_0$

$$\begin{aligned}(\tfrac{a^l}{b^u} + (1/n_0)\tfrac{c^u}{a^l})exp(\mu(a^l - b^u(a^u/b^l(exp(\mu a^u))) - c^u/m^l))\\ .(\int_0^w f(t)\Delta t - \beta^u(\int_0^w d(t)\Delta t)) - \alpha^u(\int_0^w d(t)\Delta t) > 0\end{aligned}$$

is satisfied. Also let us assume there exist $s, t \in [0, w]_{\mathbb{T}}$ such that $y(s) \geq x(t) - ln(n_0)$. Then similar to proof of Theorem 3.2.2 we can find $\hat{M}_4^1$.

If such s,t does not exist, then we have $y(s) < x(t) - ln(n_0)$. Since $\begin{bmatrix} x \\ y \end{bmatrix} \in X$, then $x(t)$ has a maximum and a minimum. Therefore by using the first equation of (3.1) and assuming $\sigma(\tilde{t})$ is the minimum of $x(t)$. We have

$$0 \geq (x(\tilde{t}))^{\Delta} = a(\tilde{t}) - b(\tilde{t})exp(x(\tilde{t})) - \frac{c(\tilde{t})exp(y(\tilde{t}))}{\alpha(\tilde{t}) + \beta(\tilde{t})exp(x(\tilde{t})) + m(\tilde{t})exp(y(\tilde{t}))}.$$

Thus we get

$$\begin{aligned} a^l &\leq b^u exp(x(\tilde{t})) + \tfrac{c^u exp(y(\tilde{t}))}{\alpha^l} \\ &\leq (b^u + (1/n_0)c^u/\alpha^l)exp(x(\tilde{t})). \end{aligned}$$

Then $exp(x(\tilde{t})) \geq \frac{a^l}{b^u+(1/n_0)c^u/\alpha^l}$.

If $\tilde{t}$ is a right dense point then $exp(x(\sigma(\tilde{t}))) \geq \frac{a^l}{b^u+(1/n_0)c^u/\alpha^l}$. If $\tilde{t}$ is right scattered, we interested with the maximum of the solution. Let $\sigma(\hat{t})$ be the maximum of x(t).

$$\begin{aligned} 0 &\leq (x(\hat{t}))^{\Delta} \\ &= a(\hat{t}) - b(\hat{t})exp(x(\hat{t})) - \tfrac{c(\hat{t})exp(y(\hat{t}))}{\alpha(\hat{t})+\beta(\hat{t})exp(x(\hat{t}))+m(\hat{t})exp(y(\hat{t}))} \\ &\leq a^u. \end{aligned}$$

Then $exp(x(\hat{t})) \leq a^u/b^l$. If $\hat{t} = \sigma(\hat{t})$, then $exp(x(\sigma(\tilde{t}))) \leq a^u/b^l$.

If $\hat{t} \neq \sigma(\hat{t})$,then $exp(x(\sigma(\hat{t}))) \leq a^u/b^l(exp(\mu a^u))$.

Thus

$$\begin{aligned} exp(x(\sigma(\tilde{t}))) &\geq \tfrac{a^l}{b^u+(1/n_0)c^u/\alpha^l} exp(\mu(a^l - b^u(a^u/b^l(exp(\mu a^u))) - c^u/m^l)) \\ &= K_1. \end{aligned}$$

Using (3.3) and (3.7) above results we obtain

$$\int_0^w d(t)\Delta t \geq \frac{K_1}{\alpha^u + \beta^u K_1 + m^u exp(y(\eta_2))} \int_0^w f(t)\Delta t.$$

This implies

$$\begin{aligned} y(\eta_2) \geq ln\Bigg[&\left(\tfrac{a^l}{b^u+(1/n_0)c^u/\alpha^l} exp(\mu(a^l - b^u(a^u/b^l(exp(\mu a^u))) - c^u/m^l))\right) \\ &.(\textstyle\int_0^w f(t)\Delta t - \beta^u \int_0^w d(t)\Delta t) - \alpha^u(\int_0^w d(t)\Delta t)\Bigg] - ln(m^u(\textstyle\int_0^w d(t)\Delta t)) = \hat{M}_4^2. \end{aligned}$$

According to the above findings we have $y(\eta_2) \geq \hat{M}_4 = min\{\hat{M}_4^1, \hat{M}_4^2\}$. Using second inequality in Lemma 3.1.1 we have $y(t) \leq \hat{M}_4 - (2\int_0^w d(t)\Delta t)) = \hat{H}_4$. Thus $max_{t\in[0,w]_{\mathbb{T}}}|y(t)| \leq max\{|\tilde{H}_3|, |\hat{H}_4|\}$. Rest of the proof is similar to Theorem 3.2.1. □

Corollary 3.2.5 *In Theorem 3.2.3 if we take $\mathbb{T}$ as $\mathbb{R}$ then we get Theorem 3 in [26].*

Example 3.2.3 $\mathbb{T} = [2k, 2k+1],\ k \in \mathbb{N}$ *k start with 0.*

$$x^{\Delta}(t) = (0.1sin(2\pi t) + 0.2) - (0.1sin(2\pi t) + 0.25)exp(x) - \frac{(0.1+0.1cos(2\pi t))exp(y)}{(0.2sin(2\pi t)+0.2)+(1+0.5cos(2\pi t))exp(x)+exp(y)},$$
$$y^{\Delta}(t) = -(0.3sin(2\pi t) + 1) + \frac{(4cos(2\pi t)+6.5)exp(x)}{(0.2sin(2\pi t)+0.2)+(1+0.5cos(2\pi t))exp(x)+exp(y)}.$$

Example 3.2.3 satisfies all the conditions of Theorem 4, thus it has at least one periodic solution.

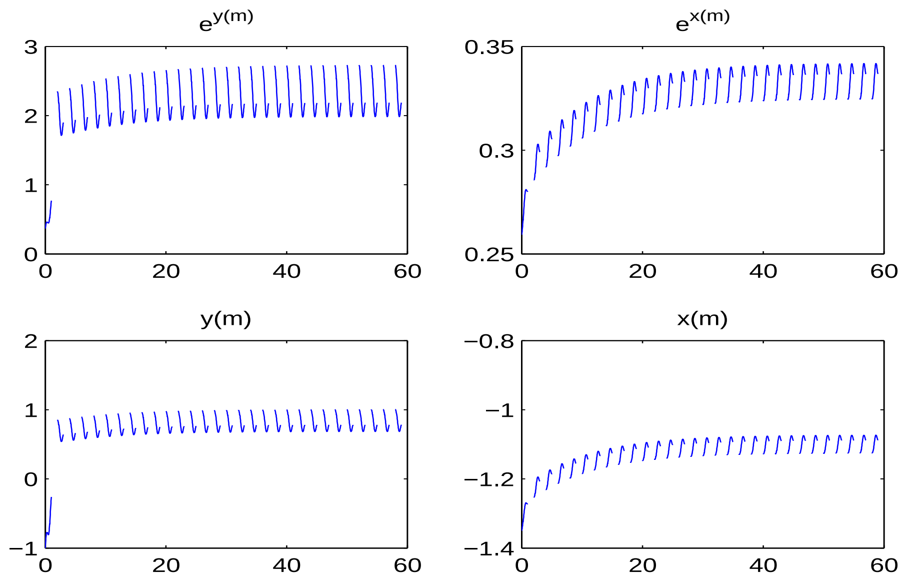

Figure 3.3: Numeric solution of Example 3.2.3 shows the periodicity.

CHAPTER 4

BEHAVIOR OF THE SOLUTIONS FOR PREDATOR-PREY DYNAMIC SYSTEMS WITH BEDDINGTON DEANGELIS TYPE FUNCTIONAL RESPONSE ON PERIODIC TIME SCALES IN SHIFTS

4.1 Preliminaries

Definition 4.1.1 *[1] Let the time scale* $\mathbb{T}$ *including a fixed number* $t_0 \in \mathbb{T}^*$ *where* $\mathbb{T}^*$ *be a non-empty subset of* $\mathbb{T}$, *such that there exist operators* $\delta\pm : [t_0;\infty)_{\mathbb{T}} \times \mathbb{T}^* \to \mathbb{T}^*$ *which satisfies the following properties:*

P.1 With respect to their second arguments the functions $\delta\pm$ *are strictly increasing, i.e., if*

$$(S_0, v), (S_0, s) \in D_{\pm} := \{(u, v) \in [t_0, \infty)_{\mathbb{T}} \times \mathbb{T}^* : \delta\pm(u, v) \in \mathbb{T}^*\},$$

then

$S_0 \leq v < s$ *implies* $\delta\pm(S_0, v) < \delta\pm(S_0, s)$,

P.2 If (S_1, s), $(S_2, s) \in D_-$ *with* $S_1 < S_2$, *then* $\delta_-(S_1, s) > \delta_-(S_2, s)$, *and if* $(S_1, s), (S_2, s) \in D_+$ *with* $S1 < S2$, *then* $\delta_+(S_1, s) < \delta_+(S_2, s)$,

P.3 If $v \in [t_0;\infty)_{\mathbb{T}}$, *then* $(v, t_0) \in D_+$ *and* $\delta_+(v, t_0) = s$. *Moreover, if* $v \in \mathbb{T}^*$, *then* $(t_0, v) \in D_+$ *and* $\delta_+(t_0, v) = v$ *holds,*

P.4 If $(u, v) \in D_{\pm}$, *then* $(u, \delta_{\pm}(u, v)) \in D_{\pm}$ *and* $\delta_{\mp}(u; \delta_{\pm}(u, v)) = v$, *respectively,* *P.5 If* $(u,v)\in D_\pm$ *and* $(s,\delta_\pm(u,v))\in D_\pm$*, then* $(u,\delta_\mp(s,v))\in D_\pm$ *and*

$\delta_\mp(s,\delta_\pm(u,v))=\delta_\pm(u,\delta_\mp(s,v))$*, respectively.*

Then the backward operator is δ_- *and the forward operator is* δ_+ *which are associated with* $t_0\in\mathbb{T}^*$*(called the initial point). Shift size is the variable* $u\in[t_0;\infty)_{\mathbb{T}}$ *in* $\delta_\pm(u,v)$*. The values* $\delta_+(u,v)$ *and* $\delta_+(u,v)$ *in* $\mathbb{T}^*$ *indicate u units translation of the term* $v\in\mathbb{T}^*$ *to the right and left, respectively. The sets* $D_\pm$ *are the domains of the shift operators* $\delta_\pm$*, respectively.*

Definition 4.1.2 *[1] Let* $\mathbb{T}$ *be a time scale with the shift operators* $\delta\pm$ *associated with the initial point* $t_0\in\mathbb{T}^*$ *. The time scale* $\mathbb{T}$ *is said to be periodic in shifts* $\delta\pm$ *if there exists a* $q\in(t_0,\infty)_{\mathbb{T}^*}$ *such that* $(q,t)\in D_\pm$ *for all* $t\in\mathbb{T}^*$*. Furthermore, if*

$$Q:=inf\{q\in(t_0,\infty)_{\mathbb{T}^*}:(q,t)\in D_\pm\ for\ all\ t\in\mathbb{T}^*\}\neq t_0$$

then P is called the period of the time scale $\mathbb{T}$*.*

Definition 4.1.3 *[1] (Periodic function in shifts* δ_+ *and* δ_-*). Let* $\mathbb{T}$ *be a time scale that is periodic in shifts* δ_+ *and* δ_- *with the period Q. We say that a real valued function g defined on* $\mathbb{T}^*$ *is periodic in shifts if there exists a* $\tilde{T}\in[Q,\infty)_{\mathbb{T}^*}$ *such that*

$$g(\delta_\pm(\tilde{T},t))=g(t).$$

The smallest number $\tilde{T}\in[Q,\infty)_{\mathbb{T}^*}$ *such that is called the period of f.*

Definition 4.1.1, Definition 4.1.2 and Definition 4.1.3 are from [1].

Notation 1 $\delta_+^2(T,\kappa)=\delta_+(T,\delta_+(T,\kappa))$,

$\delta_+^3(T,\kappa)=\delta_+(T,\delta_+(T,\delta_+(T,\kappa)))$, ...

$\delta_+^n(T,\kappa)=\delta_+(T,\delta_+(T,\delta_+(T,\delta_+(....))))$.

Lemma 4.1.1 *Let our time scale* $\mathbb{T}$ *be periodic in shifts and for each* $t\in\mathbb{T}^*$, $(\delta_+^n(T,t))^\Delta$ *is constant. Then* $\frac{\int_\kappa^{\delta_+(T,\kappa)}u(t)\Delta t}{mes(\delta_+(T,\kappa))}$ *is also constant* $\forall\kappa\in\mathbb{T}$,

where $\kappa=\delta_\pm^m(T,t_0)$ *for* $m\in\mathbb{N}$ *and* $mes(\delta_+(T,\kappa))=\int_\kappa^{\delta_+(T,\kappa)}1\Delta t$*. Here* $u(t)$ *is a periodic function in shifts.*

Proof. We get the desired result, if we can be able to show that for any

$\kappa_1 \neq \kappa_2$ ($\kappa_1, \kappa_2 \in \mathbb{T}$).

$$\frac{\int_{\kappa_1}^{\delta_+(T,\kappa_1)} u(t)\Delta t}{mes(\delta_+(T,\kappa_1))} = \frac{\int_{\kappa_2}^{\delta_+(T,\kappa_2)} u(t)\Delta t}{mes(\delta_+(T,\kappa_2))}.$$

Since $\mathbb{T}$ is a periodic time scale in shifts (WLOG $\kappa_2 > \kappa_1$) there exits $n \in \mathbb{N}$ such that

$\kappa_2 = \delta_+^n(T,\kappa_1)$. Hence it is also enough to show that

$$\frac{\int_{\kappa_1}^{\delta_+(T,\kappa_1)} u(t)\Delta t}{mes(\delta_+(T,\kappa_1))} = \frac{\int_{\delta_+^n(T,\kappa_1)}^{\delta_+(T,\delta_+^n(T,\kappa_1))} u(t)\Delta t}{mes(\delta_+(T,\delta_+^n(T,\kappa_1))}.$$

Because of the definition of the time scale and u, $u(\kappa_1) = u(\delta_+^n(T,\kappa_1))$,

$u(\delta_+(T,\kappa_1)) = u(\delta_+^{n+1}(T,\kappa_1))$ and for each $t \in [\kappa_1, \delta_+(T,\kappa_1)]$, $u(t) = u(\delta_+^n(T,t))$. By using change of variables we get the result. If $s = \delta_+^n(T,t)$, then by the assumption of the lemma $\Delta s = \tilde{c}\Delta t$. When $s = \delta_+^n(T,\kappa_1)$, then $t = \delta_-^n(T,s) = \kappa_1$ and when $s = \delta_+^{n+1}(T,\kappa_1)$, then $t = \delta_-^n(T,s) = \delta_+(T,\kappa_1)$.

$$\int_{\delta_+^n(T,\kappa_1)}^{\delta_+^{n+1}(T,\kappa_1)} u(s)\Delta s = \tilde{c}\int_{\kappa_1}^{\delta_+(T,\kappa_1)} u(t)\Delta t,$$

$$\int_{\delta_+^n(T,\kappa_1)}^{\delta_+^{n+1}(T,\kappa_1)} 1\Delta t = \tilde{c}\int_{\kappa_1}^{\delta_+(T,\kappa_1)} 1\Delta t,$$

and

$$\frac{\int_{\kappa_1}^{\delta_+(T,\kappa_1)} u(t)\Delta t}{mes(\delta_+(T,\kappa_1))} = \frac{\tilde{c}\int_{\kappa_1}^{\delta_+(T,\kappa_1)} u(t)\Delta t}{\tilde{c}\, mes(\delta_+(T,\kappa_1))}.$$

Hence proof follows.

□

4.2 Main Result

The equation that we investigate is:

$$\begin{aligned} x^{\Delta}(t) &= a(t) - b(t)exp(x(t)) - \frac{c(t)exp(y(t))}{\alpha(t)+\beta(t)exp(x(t))+m(t)exp(y(t))}, \\ y^{\Delta}(t) &= -d(t) + \frac{f(t)exp(x(t))}{\alpha(t)+\beta(t)exp(x(t))+m(t)exp(y(t))}, \end{aligned} \tag{4.1}$$

In equation (4.1), let $a(t) = a(\delta_{\pm}(T,t))$, $b(\delta_{\pm}(T,t)) = b(t)$, $c(\delta_{\pm}(T,t)) = c(t)$, $d(\delta_{\pm}(T,t)) = d(t)$, $f(\delta_{\pm}(T,t)) = f(t)$, $\alpha(\delta_{\pm}(T,t)) = \alpha(t), \beta(\delta_{\pm}(T,t)) = \beta(t)$, $m(\delta_{\pm}(T,t)) = m(t)$,

and $\int_{\kappa}^{\delta_+(T,\kappa)} a(t)\Delta t$, $\int_{\kappa}^{\delta_+(T,\kappa)} b(t)\Delta t$, $\int_{\kappa}^{\delta_+(T,\kappa)} d(t)\Delta t > 0$. $\beta^l = min_{t\in[\kappa,\delta_+(T,\kappa)]}\beta(t)$, $m^l = min_{t\in[\kappa,\delta_+(T,\kappa)]}m(t)$, $\beta^u = max_{t\in[\kappa,\delta_+(T,\kappa)]}\beta(t)$, $m^u = max_{t\in[\kappa,\delta_+(T,\kappa)]}m(t)$, such that $\kappa = \delta_{\pm}^m(T,t_0)$ for $m \in \mathbb{N}$. $m(t) > 0$ and $c(t), f(t), b(t) > 0$ $\alpha(t) \geq 0$, $\beta(t) > 0$. Each functions are from $C_{rd}(\mathbb{T},\mathbb{R})$.

Lemma 4.2.1 *Let $t_1, t_2 \in [\kappa, \delta_+(T,\kappa)]$ and $t \in \mathbb{T}$. κ is defined as in Lemma 1. If $g : \mathbb{T} \to R$ is periodic function in shifts, then*

$$g(t) \leq g(t_1) + \int_{\kappa}^{\delta_+(T,\kappa)} |g^{\Delta}(s)|\Delta s \qquad and \qquad g(t) \geq g(t_2) - \int_{\kappa}^{\delta_+(T,\kappa)} |g^{\Delta}(s)|\Delta s.$$

Proof. We only show the first inequality as the proof of the second inequality is similar to the proof of the other one. Since g is periodic function in shifts , without loss of generality, it suffices to show that the inequality is valid for $t \in [\kappa, \delta_+(T,\kappa)]$. If $t = t_1$ hen the first inequality is obviously true. If $t > t_1$

$$g(t)-g(t_1) \leq |g(t)-g(t_1)| = \left|\int_{t_1}^{t} g^{\Delta}(s)\Delta s\right| \leq \int_{t_1}^{t} |g^{\Delta}(s)|\Delta s \leq \int_{\kappa}^{\delta_+(T,\kappa)} |g^{\Delta}(s)|\Delta s.$$

Therefore $g(t) \leq g(t_1) + \int_{\kappa}^{\delta_+(T,\kappa)} |g^{\Delta}(s)|\Delta s$.

If $t < t_1$

$$g(t_1) - g(t) \geq -|g(t_1) - g(t)| = -\left|\int_{t}^{t_1} g^{\Delta}(s)\Delta s\right| \geq -\int_{t}^{t_1} |g^{\Delta}(s)|\Delta s \leq -\int_{\kappa}^{\delta_+(T,\kappa)} |g^{\Delta}(s)|\Delta s,$$

that gives $g(t) \leq g(t_1) + \int_{\kappa}^{\delta_+(T,\kappa)} |g^{\Delta}(s)|\Delta s$.

The proof is complete. □

Theorem 4.2.1 *In addition to conditions on coefficient functions and*

Lemma 4.1.1 if $\int_{\kappa}^{\delta_+(T,\kappa)} a(t)\Delta t - \int_{\kappa}^{\delta_+(T,\kappa)} \frac{c(t)}{m(t)}\Delta t > 0$ *and*

$$\left(\frac{\int_{\kappa}^{\delta_+(T,\kappa)} a(t)\Delta t - \int_{\kappa}^{\delta_+(T,\kappa)} \frac{c(t)}{m(t)}\Delta t}{\int_{\kappa}^{\delta_+(T,\kappa)} b(t)\Delta t}\right) exp\left[-\left(\int_{\kappa}^{\delta_+(T,\kappa)} |a(t)|\Delta t + \int_{\kappa}^{\delta_+(T,\kappa)} a(t)\Delta t\right)\right]$$
$$.(\int_{\kappa}^{\delta_+(T,\kappa)} f(t)\Delta t - \beta^u(\int_{\kappa}^{\delta_+(T,\kappa)} d(t)\Delta t) - \alpha^u(\int_{\kappa}^{\delta_+(T,\kappa)} d(t))\Delta t > 0$$

are satisfied then there exist at least a $\delta_{\pm}$*-periodic solution.*

Proof. $X := \left\{ \begin{bmatrix} u \\ v \end{bmatrix} \in C(\mathbb{T}, \mathbb{R}^2) : u(\delta_{\pm}(T,t)) = u(t), v(\delta_{\pm}(T,t)) = v(t) \right\}$ with the norm:

$\left\| \begin{bmatrix} u \\ v \end{bmatrix} \right\| = \max_{t\in[t_0,\delta_+(T,t_0)]_{\mathbb{T}}}(|u(t)|, |v(t)|)$

$Y := \left\{ \begin{bmatrix} u \\ v \end{bmatrix} \in C(\mathbb{T}, \mathbb{R}^2) : u(\delta_{\pm}(T,t)) = u(t), v(\delta_{\pm}(T,t)) = v(t) \right\}$ with the norm:

$\left\| \begin{bmatrix} u \\ v \end{bmatrix} \right\| = \max_{t\in[t_0,\delta_+(T,t_0)]_{\mathbb{T}}}(|u(t)|, |v(t)|)$

Let us define the mappings L and C by $L : DomL \subset X \to Y$ such that

$$L\left(\begin{bmatrix} u \\ v \end{bmatrix}\right) = \begin{bmatrix} u^{\Delta} \\ v^{\Delta} \end{bmatrix}$$

and $C : X \to Y$ such that

$$C\left(\begin{bmatrix} u \\ v \end{bmatrix}\right) = \begin{bmatrix} a(t) - b(t)exp(u(t)) - \frac{c(t)exp(v(t))}{\alpha(t)+\beta(t)exp(u(t))+m(t)exp(v(t))} \\ -d(t) + \frac{f(t)exp(u(t))}{\alpha(t)+\beta(t)exp(u(t))+m(t)exp(v(t))} \end{bmatrix}$$

Then $KerL = \left\{ \begin{bmatrix} u \\ v \end{bmatrix} : \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \right\}$, c_1 and c_2 are constants.

$$ImL = \left\{ \begin{bmatrix} u \\ v \end{bmatrix} : \begin{bmatrix} \int_{\kappa}^{\delta_+(T,\kappa)} u(t)\Delta t \\ \int_{\kappa}^{\delta_+(T,\kappa)} v(t)\Delta t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}.$$

ImL is closed in Y. Its obvious that $dimKerL = 2$ To show $dimKerL = codimImL = 2$, we have to prove that $KerL \oplus ImL = Y$. It is obvious that when we take an element from Ker L, an element from Im L; we find an element of Y by summing these two elements. If we take an element $\begin{bmatrix} u \\ v \end{bmatrix} \in Y$, and WLOG taking $u(t)$ we have, $\int_{\kappa}^{\delta_+(T,\kappa)} u(t)\Delta t = I$ where I is a constant. Let us define a new function $g = u - \frac{I}{mes(\delta_+(T,\kappa))}$. Since $\frac{I}{mes(\delta_+(T,\kappa))}$ is constant by Lemma 4.1.1 if we take the integral of g from κ to $\delta_+(T,\kappa)$, we get

$$\int_{\kappa}^{\delta_+(T,\kappa)} g(t)\Delta t = \int_{\kappa}^{\delta_+(T,\kappa)} u(t)\Delta t - I = 0.$$

Similar steps are used for v. $\begin{bmatrix} u \\ v \end{bmatrix} \in Y$ can be written as the summation of an element from Im L and an element from Ker L. Also it is easy to show that any element in Y is uniquely expressed as the summation of an element Ker L and an element from Im L. So $codimImL$ is also 2, we get the desired result. Hence L is a Fredholm mapping of index zero. There exist continuous projectors $U : X \to X$ and $V : Y \to Y$ such that

$$U\left(\begin{bmatrix} u \\ v \end{bmatrix}\right) = \frac{1}{mes(\delta_+(T,\kappa))}\begin{bmatrix} \int_{\kappa}^{\delta_+(T,\kappa)} u(t)\Delta t \\ \int_{\kappa}^{\delta_+(T,\kappa)} v(t)\Delta t \end{bmatrix}$$

and

$$V\left(\begin{bmatrix} u \\ v \end{bmatrix}\right) = \frac{1}{mes(\delta_+(T,\kappa))}\left(\begin{bmatrix} \int_{\kappa}^{\delta_+(T,\kappa)} u(t)\Delta t \\ \int_{\kappa}^{\delta_+(T,\kappa)} v(t)\Delta t \end{bmatrix}\right).$$

The generalized inverse $K_U = ImL \to DomL \cap KerU$ is given,

$$K_U\left(\begin{bmatrix} u \\ v \end{bmatrix}\right) = \begin{bmatrix} \int_{\kappa}^{t} u(s)\Delta s - \frac{1}{mes(\delta_+(T,\kappa))}\int_{\kappa}^{\delta_+(T,\kappa)}\int_{\kappa}^{t} u(s)\Delta s \\ \int_{\kappa}^{t} v(s)\Delta s - \frac{1}{mes(\delta_+(T,\kappa))}\int_{\kappa}^{\delta_+(T,\kappa)}\int_{\kappa}^{t} v(s)\Delta s \end{bmatrix}.$$

$$VC\left(\begin{bmatrix} u \\ v \end{bmatrix}\right) =$$

$$\frac{1}{mes(\delta_+(T,\kappa))}\left(\begin{bmatrix} \int_{\kappa}^{\delta_+(T,\kappa)} a(s) - b(s)exp(u(s)) - \frac{c(s)exp(v(s))}{\alpha(s)+\beta(s)exp(u(s))+m(s)exp(v(s))}\Delta s \\ \int_{\kappa}^{\delta_+(T,\kappa)} -d(s) + \frac{f(s)exp(u(s))}{\alpha(s)+\beta(s)exp(u(s))+m(s)exp(v(s))}\Delta s \end{bmatrix}\right)$$

Let

$$a(t) - b(t)exp(u(t)) - \frac{c(t)exp(v(t))}{\alpha(t) + \beta(t)exp(u(t)) + m(t)exp(v(t))} = C_1$$

$$-d(t) + \frac{f(t)exp(u(t))}{\alpha(t) + \beta(t)exp(u(t)) + m(t)exp(v(t))} = C_2$$

,

$$\frac{1}{mes(\delta_+(T,\kappa))}\int_\kappa^{\delta_+(T,\kappa)} a(s) - b(s)exp(u(s)) - \frac{c(s)exp(v(s))}{\alpha(s) + \beta(s)exp(u(s)) + m(s)exp(v(s))}\Delta s = \bar{C_1}$$

and

$$\frac{1}{mes(\delta_+(T,\kappa))}\int_\kappa^{\delta_+(T,\kappa)} -d(s) + \frac{f(s)exp(u(s))}{\alpha(s) + \beta(s)exp(u(s)) + m(s)exp(v(s))}\Delta s = \bar{C_2}$$

$$K_U(I-V)C\left(\begin{bmatrix} u \\ v \end{bmatrix}\right) = K_U\left(\begin{bmatrix} C_1 - \bar{C_1} \\ C_2 - \bar{C_2} \end{bmatrix}\right)$$

$$= \begin{bmatrix} \int_\kappa^t C_1(s) - \bar{C_1}(s)\Delta s - \frac{1}{mes(\delta_+(T,\kappa))}\int_\kappa^{\delta_+(T,\kappa)}\int_\kappa^t C_1(s) - \bar{C_1}(s)\Delta s \\ \int_\kappa^t C_2(s) - \bar{C_2}(s)\Delta s - \frac{1}{mes(\delta_+(T,\kappa))}\int_\kappa^{\delta_+(T,\kappa)}\int_\kappa^t C_2(s) - \bar{C_2}(s)\Delta s \end{bmatrix}.$$

Clearly, VC and $K_U(I-V)C$ are continuous. Since X and Y are Banach spaces, then by using Arzela-Ascoli theorem we can find $K_U(I-\bar{V})C(\bar{\Omega})$ is compact for any open bounded set $\Omega \subset X$. Additionally, $VC(\bar{\Omega})$ is bounded. Thus, C is L-compact on $\bar{\Omega}$ with any open bounded set $\Omega \subset X$.

To apply the continuation theorem we investigate the below operator equation.

$$\begin{aligned} x^\Delta(t) &= \lambda\left[a(t) - b(t)exp(x(t)) - \tfrac{c(t)exp(y(t))}{\alpha(t)+\beta(t)exp(x(t))+m(t)exp(y(t))}\right] \\ y^\Delta(t) &= \lambda\left[-d(t) + \tfrac{f(t)exp(x(t))}{\alpha(t)+\beta(t)exp(x(t))+m(t)exp(y(t))}\right] \end{aligned} \tag{4.2}$$

Let $\begin{bmatrix} x \\ y \end{bmatrix} \in X$ be any solution of system (4.2). Integrating both sides of system (4.2) over the interval $[0, w]$ we obtain,

$$\begin{cases} \int_\kappa^{\delta_+(T,\kappa)} a(t)\Delta t = \int_\kappa^{\delta_+(T,\kappa)} b(t)exp(x(t)) + \frac{c(t)exp(y(t))}{\alpha(t)+\beta(t)exp(x(t))+m(t)exp(y(t))}\Delta t & , \\ \int_\kappa^{\delta_+(T,\kappa)} d(t)\Delta t = \int_\kappa^{\delta_+(T,\kappa)} \frac{f(t)exp(x(t))}{\alpha(t)+\beta(t)exp(x(t))+m(t)exp(y(t))}\Delta t & , \end{cases} \tag{4.3}$$

From (4.2) and (4.3) we get

$$\begin{aligned}\int_{\kappa}^{\delta_+(T,\kappa)} |x^{\Delta}(t)|\Delta t &\leq \lambda\Bigg[\int_{\kappa}^{\delta_+(T,\kappa)} |a(t)|\Delta t + \int_{\kappa}^{\delta_+(T,\kappa)} b(t)exp(x(t)) + \frac{c(t)exp(y(t))}{\alpha(t)+\beta(t)exp(x(t))+m(t)exp(y(t))}\Delta t\Bigg],\\ &\leq \lambda\Bigg[\int_{\kappa}^{\delta_+(T,\kappa)} |a(t)|\Delta t + \int_{\kappa}^{\delta_+(T,\kappa)} a(t)\Delta t\Bigg]\\ &\leq \int_{\kappa}^{\delta_+(T,\kappa)} |a(t)|\Delta t + \int_{\kappa}^{\delta_+(T,\kappa)} a(t)\Delta t := M_1\end{aligned} \tag{4.4}$$

$$\begin{aligned}\int_{\kappa}^{\delta_+(T,\kappa)} |y^{\Delta}(t)|\Delta t &\leq \lambda\Bigg[\int_{\kappa}^{\delta_+(T,\kappa)} |d(t)|\Delta t + \int_{\kappa}^{\delta_+(T,\kappa)} \frac{f(t)exp(x(t))}{\alpha(t)+\beta(t)exp(x(t))+m(t)exp(y(t))}\Delta t\Bigg]\\ &\leq \lambda\Bigg[\int_{\kappa}^{\delta_+(T,\kappa)} |d(t)|\Delta t + \int_{\kappa}^{\delta_+(T,\kappa)} d(t)\Delta t\Bigg]\\ &\leq \int_{\kappa}^{\delta_+(T,\kappa)} |d(t)|\Delta t + \int_{\kappa}^{\delta_+(T,\kappa)} d(t)\Delta t := M_2\end{aligned} \tag{4.5}$$

Since $\begin{bmatrix} x \\ y \end{bmatrix} \in X$, then there exist $\eta_i, \xi_i,\ i = 1, 2$ such that

$$\begin{aligned}x(\xi_1) = min_{t\in[t\in[\kappa,\delta_+(T,\kappa)]}x(t), x(\eta_1) = max_{t\in[t\in[\kappa,\delta_+(T,\kappa)]}x(t),\\ y(\xi_2) = min_{t\in[t\in[\kappa,\delta_+(T,\kappa)]}y(t), y(\eta_2) = max_{t\in[t\in[\kappa,\delta_+(T,\kappa)]}y(t)\end{aligned} \tag{4.6}$$

If ξ_1 is the minimum point of $x(t)$ on the interval $[\kappa, \delta_+(T,\kappa)]$ because $x(t)$ is a function that is periodic in shifts for any $n \in \mathbb{N}$ on the interval $[\delta_+^n(T,\kappa_1), \delta_+^{n+1}(T,\kappa_1)]$ the minimum point of $x(t)$ is $\delta_+^n(T,\xi_1)$ and $x(\xi_1) = x(\delta_+^n(T,\xi_1))$. We have similar results for the other points for ξ_2, η_1, η_2.

By the first equation of (4.3) and (4.6)

$$\begin{aligned}\int_{\kappa}^{\delta_+(T,\kappa)} a(t)\Delta t &\leq \int_{\kappa}^{\delta_+(T,\kappa)}\Bigg[b(t)exp(x(\eta_1)) + \frac{c(t)}{m(t)}\Delta t\Bigg]\\ &= exp(x(\eta_1))\int_{\kappa}^{\delta_+(T,\kappa)} b(t)\Delta t + \int_{\kappa}^{\delta_+(T,\kappa)} \frac{c(t)}{m(t)}\Delta t.\end{aligned}$$

Since $\int_{\kappa}^{\delta_+(T,\kappa)} b(t)\Delta t > 0$ so we get

$$x(\eta_1) \geq ln\left(\frac{\int_{\kappa}^{\delta_+(T,\kappa)} a(t)\Delta t - \int_{\kappa}^{\delta_+(T,\kappa)} \frac{c(t)}{m(t)}\Delta t}{\int_{\kappa}^{\delta_+(T,\kappa)} b(t)\Delta t}\right) := l_1$$

using the second inequality in Lemma 4.2.1 we have

$$\begin{aligned} x(t) &\geq x(\eta_1) - \int_{\kappa}^{\delta_+(T,\kappa)} |x^{\Delta}(t)|\Delta t \\ &\geq x(\eta_1) - \left(\int_{\kappa}^{\delta_+(T,\kappa)} |a(t)|\Delta t + \int_{\kappa}^{\delta_+(T,\kappa)} a(t)\Delta t \right) \\ &= l_1 - M_1 := H_1 \end{aligned} \tag{4.7}$$

By the first equation of (4.3) and (4.6)

$$\begin{aligned} \int_{\kappa}^{\delta_+(T,\kappa)} a(t)\Delta t &\geq \int_{\kappa}^{\delta_+(T,\kappa)} b(t)exp(x(\xi_1))\Delta t \\ &= exp(x(\xi_1)) \int_{\kappa}^{\delta_+(T,\kappa)} b(t)\Delta t. \end{aligned}$$

Then we get

$$x(\xi_1) \leq ln\left(\frac{\int_{\kappa}^{\delta_+(T,\kappa)} a(t)\Delta t}{\int_{\kappa}^{\delta_+(T,\kappa)} b(t)\Delta t} \right) := l_2$$

using the first inequality in Lemma 4.2.1 we have

$$\begin{aligned} x(t) &\leq x(\xi_1) + \int_{\kappa}^{\delta_+(T,\kappa)} |x^{\Delta}(t)|\Delta t \\ &\leq x(\xi_1) + \left(\int_{\kappa}^{\delta_+(T,\kappa)} |a(t)|\Delta t + \int_{\kappa}^{\delta_+(T,\kappa)} a(t)\Delta t \right) \\ &= l_2 + M_1 := H_2 \end{aligned} \tag{4.8}$$

By (4.7) and (4.8) $max_{t\in[\kappa,\delta_+(T,\kappa)]}|x(t)| \leq max\{|H_1|, |H_2|\} := B_1$. From the second equation of (4.3) and the second equation of (4.7), we can derive that

$$\begin{aligned} \int_{\kappa}^{\delta_+(T,\kappa)} d(t)\Delta t &\leq \int_{\kappa}^{\delta_+(T,\kappa)} \frac{f(t)exp(x(t))}{\beta^l exp(x(t)) + m^l exp(y(t))}\Delta t \\ &\leq \int_{\kappa}^{\delta_+(T,\kappa)} \frac{f(t)e^{H_2}}{\beta^l e^{H_2} + m^l exp(y(\xi_2))}\Delta t \\ &= \frac{e^{H_2}}{\beta^l e^{H_2} + m^l exp(y(\xi_2))} \int_{\kappa}^{\delta_+(T,\kappa)} f(t)\Delta t. \end{aligned}$$

Therefore

$$exp(y(\xi_2)) \leq \frac{1}{m^l}\left(\frac{e^{H_2}\int_{\kappa}^{\delta_+(T,\kappa)} f(t)\Delta t}{\int_{\kappa}^{\delta_+(T,\kappa)} d(t)\Delta t} - \beta^l e^{H_2} \right)$$

By the assumption of the Theorem 4.2.1 we get,

$\int_{\kappa}^{\delta_+(T,\kappa)} f(t)\Delta t - \beta^l(\int_{\kappa}^{\delta_+(T,\kappa)} d(t))\Delta t > 0$ and

$$y(\xi_2) \leq ln\left(\frac{1}{m^l}\left(\frac{e^{H_2}\int_{\kappa}^{\delta_+(T,\kappa)} f(t)\Delta t}{\int_{\kappa}^{\delta_+(T,\kappa)} d(t)\Delta t} - \beta^l e^{H_2} \right) \right) := L_1$$

Hence, by using the first inequality in Lemma 4.2.1 and the second equation of (4.3),

$$\begin{array}{rcc} y(t) & \leq & y(\xi_2) + \int_{\kappa}^{\delta_+(T,\kappa)} |y^{\Delta}(t)|\Delta t \\ & \leq & y(\xi_2) + \left(\int_{\kappa}^{\delta_+(T,\kappa)} |d(t)|\Delta t + \int_{\kappa}^{\delta_+(T,\kappa)} d(t)\Delta t \right) \\ & \leq & L_1 + M_2 := H_3. \end{array} \quad (4.9)$$

Again using the second equation of (4.3) we obtain

$$\begin{array}{rcc} \int_{\kappa}^{\delta_+(T,\kappa)} d(t)\Delta t & \geq & \int_{\kappa}^{\delta_+(T,\kappa)} \frac{f(t)exp(x(t))}{\alpha^u+\beta^u exp(x(t))+m^u exp(y(t))}\Delta t \\ & \geq & \int_{\kappa}^{\delta_+(T,\kappa)} \frac{f(t)e^{H_1}}{\alpha^u+\beta^u e^{H_1}+m^u exp(y(\eta_2))}\Delta t \\ & = & \frac{e^{H_1}}{\alpha^u+\beta^u e^{H_1}+m^u exp(y(\eta_2))} \int_{\kappa}^{\delta_+(T,\kappa)} f(t)\Delta t, \end{array}$$

$$exp(y(\eta_2)) \geq \frac{1}{m^u}\left(\frac{e^{H_1}\int_{\kappa}^{\delta_+(T,\kappa)} f(t)\Delta t}{\int_{\kappa}^{\delta_+(T,\kappa)} d(t)\Delta t} - \beta^u e^{H_1} - \alpha^u \right).$$

Using the assumption of the Theorem 4.2.1 we obtain,

$$e^{H_1}\left(\int_{\kappa}^{\delta_+(T,\kappa)} f(t)\Delta t - \beta^u \left(\int_{\kappa}^{\delta_+(T,\kappa)} d(t)\Delta t \right) \right) - \alpha^u \left(\int_{\kappa}^{\delta_+(T,\kappa)} d(t)\Delta t \right) > 0$$

and

$$y(\eta_2) \geq ln\left(\frac{1}{m^u}\left(\frac{e^{H_1}\int_{\kappa}^{\delta_+(T,\kappa)} f(t)\Delta t}{\int_{\kappa}^{\delta_+(T,\kappa)} d(t)\Delta t} - \beta^u e^{H_1} - \alpha^u \right) \right) := L_2.$$

By using the second inequality in Lemma 4.2.1,

$$\begin{array}{rcc} y(t) & \geq & y(\eta_2) - \int_{\kappa}^{\delta_+(T,\kappa)} |y^{\Delta}(t)|\Delta t \\ & \geq & y(\eta_2) - \left(\int_{\kappa}^{\delta_+(T,\kappa)} |d(t)|\Delta t + \int_{\kappa}^{\delta_+(T,\kappa)} d(t)\Delta t \right) \\ & = & L_2 - M_2 := H_4. \end{array} \quad (4.10)$$

By (4.9) and (4.10) we have $max_{t\in[t_0,\delta_+(T,t_0)]}|y(t)| \leq max\{|H_3|, |H_4|\} := B_2$ Obviously, B_1 and B_2 are both independent of λ. Let $M = B_1 + B_2 + 1$. Then $max_{t\in[t_0,\delta_+(T,t_0)]}\left\| \begin{bmatrix} x \\ y \end{bmatrix} \right\| < M$. Let $\Omega = \left\{ \left\| \begin{bmatrix} x \\ y \end{bmatrix} \right\| \in X : \left\| \begin{bmatrix} x \\ y \end{bmatrix} \right\| < M \right\}$

then Ω verifies the requirement (a) in Theorem 3.1.1. When $\begin{bmatrix} x \\ y \end{bmatrix} \in KerL \cap \partial\Omega$,

$\begin{bmatrix} x \\ y \end{bmatrix}$ is a constant with $\left\| \begin{bmatrix} x \\ y \end{bmatrix} \right\| = M$, then

$$
\begin{aligned}
VC\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) &= \left(\begin{bmatrix} \int_{\kappa}^{\delta_+(T,\kappa)} a(s) - b(s)exp(x) - \frac{c(s)exp(y)}{\alpha(s)+\beta(s)exp(x)+m(s)exp(y)}\Delta t \\ \int_{\kappa}^{\delta_+(T,\kappa)} -d(s) + \frac{f(s)exp(x)}{\alpha(s)+\beta(s)exp(x)+m(s)exp(y)}\Delta t \end{bmatrix}\right) \\
&\neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

$$JVC\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = VC\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$$

where $J : ImV \to KerL$ is the identity operator.

Let us define he homotopy such that: $H_\nu = \nu(JVC) + (1-\nu)G$ where

$$G\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \int_{\kappa}^{\delta_+(T,\kappa)} a(s) - b(s)exp(x)\Delta t \\ \int_{\kappa}^{\delta_+(T,\kappa)} d(s) - \frac{f(s)exp(x)}{\alpha(s)+\beta(s)exp(x)+m(s)exp(y)}\Delta t \end{bmatrix}$$

Take DJ_G as the determinant of the jacobian of G. Since $\begin{bmatrix} x \\ y \end{bmatrix} \in KerL$, then jacobian of G is

$$\begin{bmatrix} -e^x \int_{\kappa}^{\delta_+(T,\kappa)} b(s)\Delta t & 0 \\ \int_{\kappa}^{\delta_+(T,\kappa)} \frac{-e^x f(s)}{\alpha(s)+\beta(s)e^x+m(s)e^y}\Delta t + \int_{\kappa}^{\delta_+(T,\kappa)} \frac{(e^x)^2 f(s)\beta(s)}{(\alpha(s)+\beta(s)e^x+m(s)e^y)^2}\Delta t & -\int_{\kappa}^{\delta_+(T,\kappa)} \frac{e^x e^y f(s)m(s)}{(\alpha(s)+\beta(s)e^x+m(s)e^y)^2}\Delta t \end{bmatrix}$$

All the functions in jacobian of G is positive then $signDJ_G$ is always positive. Hence

$$deg(JVC, \Omega \cap KerL, 0) = deg(G, \Omega \cap KerL, 0) = \sum_{\begin{bmatrix} x \\ y \end{bmatrix} \in G^{-1}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right)} signDJ_G\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) \neq 0.$$

Thus all the conditions of Theorem 3.1.1 are satisfied. Therefore system (4.1) has at least a positive $\delta\pm$-periodic solution. $\square$

Example 4.2.1 *Let* $\mathbb{T} = \{0\} \cup \cup_{n\in\mathbb{Z}}[2^{2n}, 2^{2n+1}]$. $\delta_{\pm}(4,t)$ *is the shift operator and* $t_0 = 1$.

$$x^{\Delta}(t) = (cos(\frac{ln(t)}{ln(1/\sqrt{2})}\pi) + 3) - (cos(\frac{ln|t|}{ln(1/\sqrt{2})}\pi) + 1)exp(x(t)) - \frac{exp(y(t))}{6exp(x(t))+2exp(y(t))},$$
$$y^{\Delta}(t) = -(sin(\frac{ln|t|}{ln(1/\sqrt{2})}\pi) + 0.5) + \frac{(sin(\frac{ln|t|}{ln(1/\sqrt{2})}\pi)+5)\ exp(x(t))}{6exp(x(t))+2exp(y(t))}, \tag{4.11}$$

Each functions in system (11) are $\delta_{\pm}(4,t)$ *periodic and satisfies Theorem 1 then the system has at least one* $\delta_{\pm}(4,t)$ *periodic solution. Here* $mes(\delta_{+}(4,1)) = 1$, $mes(\delta_{+}(4,1.5)) = 2.5$, $mes(\delta_{+}(4,2)) = 4$, $mes(\delta_{+}(4,4)) = 4$, $mes(\delta_{+}(4,8)) = 16$ *... goes like that.*

Example 4.2.2 *Let* $\mathbb{T} = \{0\} \cup q^{\mathbb{Z}}$. $\delta_{\pm}(q,t)$ *is the shift operator and* $t_0 = 1$.

$$x^{\Delta}(t) = ((-1)^{\frac{ln|t|}{ln(q)}} + 4) - ((-1)^{\frac{ln|t|}{ln(q)}} + 0.5)exp(x(t)) - \frac{exp(y(t))}{exp(x(t))+2exp(y(t))},$$
$$y^{\Delta}(t) = -0.3 + \frac{((-1)^{\frac{ln|t|}{ln(q)}}+7)\ exp(x(t))}{exp(x(t))+2exp(y(t))}, \tag{4.12}$$

Each functions in system (12) are $\delta_{\pm}(q^2,t)$ *periodic and satisfies Theorem 1 then the system has at least one* $\delta_{\pm}(q^2,t)$ *periodic solution. Here* $mes(\delta_{+}(q^2,t)) = 2$.

CHAPTER 5

GENERALIZATION OF CONSTANTIN'S INEQUALITY AND ITS APPLICATION ON WATER PERCOLATION EQUATION

5.1 Some Basic Definitions Related to Nabla and Diamond-α Time Scales Calculus

Definition 5.1.1 *[3] For a function $f : \mathbb{T} \to \mathbb{T}$, we define the $\nabla-$derivative of f at $t \in \mathbb{T}_\kappa$, denoted by $f^\nabla(t)$, for all $\epsilon > 0$. There exists a neighborhood $V \subset \mathbb{T}$ of $t \in \mathbb{T}_\kappa$ such that*

$$|f(s) - f(\rho(t)) - f^\nabla(t)(s - \rho(t))| \leq \epsilon|s - \rho(t)|,$$

for all $s \in V$.

Definition 5.1.2 *[41] $\diamond_\alpha$-derivative of f at $t \in \mathbb{T}_\kappa^\kappa$, denoted by $f^{\diamond_\alpha}(t)$ for all $\epsilon > 0$. There is a neighbourhood $U \subset \mathbb{T}$ such that for any $s \in U$*

$$\left|\alpha|f(\sigma(t)) - f(s)||\rho(t) - s| + (1-\alpha)|f(\rho(t)) - f(s)||\sigma(t) - s| - f^{\diamond_\alpha}(t)|\rho(t) - s||\sigma(t) - s|\right|$$
$$\leq \quad \epsilon|\rho(t) - s||\sigma(t) - s|.$$

Definition 5.1.3 *[3] A function $g : \mathbb{T} \to \mathbb{R}$ is ld-continuous if it is continuous at left dense points in $\mathbb{T}$ and its right-sided limits exist at right-dense points in $\mathbb{T}$. The class of real ld-continuous functions defined on a time scale $\mathbb{T}$ is denoted by $C_{ld}(\mathbb{T}, \mathbb{R})$. If $g \in C_{ld}(\mathbb{T}, \mathbb{R})$ then there exists a function $G(t)$ such that $G^\nabla(t) = g(t)$. The nabla integral is defined by $\int_a^b g(x)\nabla x = G(b) - G(a)$. If $g \in C_{ld}^1(\mathbb{T}, \mathbb{R})$ then $g^\nabla(x)$ and $g(x)$ are in $C_{ld}(\mathbb{T}, \mathbb{R})$.*

Definition 5.1.4 *[41] A function $h(t) : \mathbb{T} \to \mathbb{R}$ is C_{rl}-continuous if $h(t) \in C_{rd}(\mathbb{T}, \mathbb{R}) \bigcap C_{ld}(\mathbb{T}, \mathbb{R})$. If $h \in C^1_{rl}(\mathbb{T}, \mathbb{R})$ then $h^{\diamond_\alpha}(x)$ and $h(x)$ are from $C_{rl}(\mathbb{T}, \mathbb{R})$. Fundemental theorem of calculus does not true for $\diamond_\alpha$-derivative. By [46], we know that*

$$\left(\int_a^t f(s) \diamond_\alpha s\right)^{\diamond_\alpha} = (1 - 2\alpha + 2\alpha^2) f(t) + (\alpha - \alpha^2)[f(\rho(t) + f(\sigma(t))].$$

Below formulas are taken from [8], [3] and [17].

The linear combination of $\alpha f + \beta g : \mathbb{T} \to \mathbb{R}$ is delta differentiable at t with

$$(\alpha f + \beta g)^\Delta = \alpha f^\Delta + \beta g^\Delta.$$

The linear combination of $\alpha f + \beta g : \mathbb{T} \to \mathbb{R}$ is nabla differentiable at t with

$$(\alpha f + \beta g)^\nabla = \alpha f^\nabla + \beta g^\nabla.$$

The linear combination of $\alpha f + \beta g : \mathbb{T} \to \mathbb{R}$ is diamond alpha differentiable at t with

$$(\alpha f + \beta g)^{\diamond_\alpha} = \alpha f^{\diamond_\alpha} + \beta g^{\diamond_\alpha}.$$

The linear combination of $\alpha f + \beta g : \mathbb{T} \to \mathbb{R}$ is delta integrable with

$$\int_a^t (\alpha f + \beta g)(s) \Delta s = \alpha \int_a^t f(s) \Delta s + \beta \int_a^t g(s) \Delta s.$$

The linear combination of $\alpha f + \beta g : \mathbb{T} \to \mathbb{R}$ is nabla integrable with

$$\int_a^t (\alpha f + \beta g)(s) \nabla s = \alpha \int_a^t f(s) \nabla s + \beta \int_a^t g(s) \nabla s.$$

The linear combination of $\alpha f + \beta g : \mathbb{T} \to \mathbb{R}$ is diamond alpha integrable with

$$\int_a^t (\alpha f + \beta g)(s) \diamond_\alpha s = \alpha \int_a^t f(s) \diamond_\alpha s + \beta \int_a^t g(s) \diamond_\alpha s.$$

5.2 Literature Review on Constantin's Inequality

Adrian Constantin [14] while studying the integrodifferential equation

$$x'(t) = F\left(t, x(t), \int_0^t K\left[t, s, x(s) ds\right]\right)$$

found the following interesting result in 1996. Applying the below inequality and a topological transversality theorem, he showed that, under some suitable assumptions, the above integrodifferential equation has a solution and gave bounds on that solution.

Theorem 5.2.1 *[14] If for some $k, T > 0$, $u \in C(\mathbb{R}_0^+, \mathbb{R}_0^+)$ satisfies*

$$u^2(t) \leq k^2 + 2\int_0^t \left\{ f(s)u(s) \left[u(s) + \int_0^s g(\tau)w(u(\tau))d\tau \right] + h(s)u(s) \right\} ds$$

$\forall t \in [0, T]$, where $f, g, h \in C(\mathbb{R}_0^+, \mathbb{R}_0^+)$ and w belongs to the continuous nondecreasing functions class on $\mathbb{R}_0^+$ such that $w(r) > 0$ if $r > 0$ and satisfying $\int_1^\infty \frac{ds}{w(s)} = \infty$, then

$$u(t) \leq K(t) + \int_0^t f(s)G^{-1}\left\{ G(K(t)) + \int_0^s [f(\tau) + g(\tau)]d\tau \right\} ds$$

where $K(t) = k + \int_0^t h(s)ds$, $G(r) = \int_1^r \frac{ds}{s+w(s)}, r > 1$. G^{-1} denotes the inverse function of G.

The generalization of this inequality given by Yang and Tan and can be stated as:

Theorem 5.2.2 *[54] Let $u, c \in C(\mathbb{R}_+^0, \mathbb{R}_+^0)$ with c being non-decreasing and*

$\phi \in C^1(\mathbb{R}_+^0, \mathbb{R}_+^0)$ where ϕ' is non-negative and non-decreasing. Let

$f(t, \xi), g(t, \xi), h(t, \xi) \in C(\mathbb{R}_+^0 \times \mathbb{R}_+^0, \mathbb{R}_+^0)$ be nondecreasing in t for every fixed ξ. Further let $w \in C(\mathbb{R}_+^0, \mathbb{R}_+^0)$ be non decreasing, $w(r) > 0$ for $r > 0$, and

$\int_{r_0}^\infty \frac{ds}{w(s)} = \infty$ hold for some number $r_0 > 0$. Then the integral inequality

$$\Phi(u(t)) \leq c(t) + \int_0^t \left\{ f(s)\Phi'(u(s)) \left[u(s) + \int_0^s g(\tau)w(u(\tau))d\tau \right] + h(s)\Phi'(u(s)) \right\} ds$$

implies

$$u(t) \leq K(t) + \int_0^t f(s)G^{-1}\left\{ G(K(t)) + \int_0^s [f(\tau) + g(\tau)]d\tau \right\} ds$$

where

$K(t) = \Phi^{-1}(c(t)) + \int_0^t h(s)ds$

$G(r) = \int_{r_0}^r \frac{ds}{s+w(s)}, r > r_0, \quad 1 > r_0 > 0.$

They also present the discrete analogue of Theorem 5.2.1 which is given by

Theorem 5.2.3 *[54] Let the function $w \in C(\mathbb{R}_+, \mathbb{R}_+)$ be non decreasing,*

$w(r) > 0$ for $r > 0$, $\phi \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ with ϕ' being non negative and non decreasing. $u, c \in C(\mathbb{N}_M, \mathbb{R}_+)$ with $c(n)$ non-decreasing. Further let

$f(n,s), g(n,s), h(n,s) \in C(\mathbb{N}_M \times \mathbb{N}_M, \mathbb{R}_+)$ be non-decreasing with respect to n for every s fixed. Then the discrete inequality

$$\phi[u(n)] \leq c(n) + \sum_{s=0}^{n-1} \{f(n,s)\phi'[u(s)] \times \left[u(s) + \sum_{\xi=0}^{s-1} g(s,\xi) w(u(\xi)\right] + h(n,s)\phi'[u(s)]\},$$

$n \in \mathbb{N}_M$ implies

$$u(n)) \leq L(n) + \sum_{s=0}^{n-1} f(n,s) G^{-1}\Big(G[L(n)] + \sum_{\xi=0}^{s-1} f(n,\xi) + g(n,\xi)\Big),$$

Here $G(r) = \int_{r_0}^{r} \frac{ds}{s + w(s)}$, $r \leq r_0$, $1 > r_0 > 0$, $\lim_{x\to\infty} G(x) = \infty$,

$L(n) = \phi^{-1}[c(n)] + \sum_{s=0}^{n-1} h(n,s)$.

$\mathbb{N}_M = \{n \in \mathbb{N} : n \leq M, M \in \mathbb{N}\}$

Ferriara generalized Constantin's inequality involving delta derivatives on an arbitrary Time Scale.

Theorem 5.2.4 *[27] Assume that $u \in C_{rd}\left([a,b]_{\mathbb{T}}, \mathbb{R}_0^+\right)$, $c \in C_{rd}\left([a,b]_{\mathbb{T}}, \mathbb{R}^+\right)$ is non-decreasing, $\Phi \in C\left(\mathbb{R}_0^+, \mathbb{R}_0^+\right)$ is strictly increasing function such that*

$$\lim_{x\to\infty} \Phi(x) = \infty.$$

Let $f(t,\xi), h(t,\xi) \in C_{rd}\left([a,b]_{\mathbb{T}} \times [a,b]_{\mathbb{T}^\kappa}, \mathbb{R}_0^+\right)$ and $g(t,\xi) \in C_{rd}\left([a,b]_{\mathbb{T}^\kappa} \times [a,b]_{\mathbb{T}^{\kappa^2}}, \mathbb{R}_0^+\right)$ be non-decreasing in for every fixed ξ. Further ,let $w, \phi, \Psi \in C\left(\mathbb{R}_0^+, \mathbb{R}_0^+\right)$ be non-decreasing such that $\{w, \phi, \Psi\}(x) > 0$ for every $x > 0$. Define

$$M(x) = \max\{\phi(x), \Psi(x)\}$$

on $\mathbb{R}_0^+$ and assume that the following function

$$F(x) = \int_{x_0}^{x} \frac{ds}{Mo\Phi^{-1}(s)}$$

with $x > c(a) > x_0 \geq 0$ *if* $\int\limits_0^x \frac{ds}{Mo\Phi^{-1}(s)} < \infty$ *and* $x > c(a) > x_0 > 0$

if $\int\limits_0^x \frac{ds}{Mo\Phi^{-1}(s)} = \infty$; *satisfies* $\lim_{x\to\infty} F(x) = \infty$.

Also assume that the function

$$P(r) = \int\limits_{r_0}^{r} \frac{ds}{w(s)}$$

where $r \geq 0, r_0 \geq 0$ *if* $\int\limits_0^r \frac{ds}{w(s)} < \infty$ *and* $r > 0, r_0 > 0$

if $\int\limits_0^r \frac{ds}{w(s)} = \infty$ *;satisfies* $\lim_{r\to\infty} P(r) = \infty$.

Define $K(t) =: F[c(t)] + \int\limits_a^t h(t,s)\Delta s$ *and* $G(r) = \int\limits_{r_0}^{r} \frac{ds}{w(s)+s}$.

If $\Phi^{-1}[F^{-1}(x)] \leq x$, *for all* $x \geq 0$ *then the inequlity*

$$\Phi[u(t)] \leq c(t) + \int\limits_a^t \left(f(t,s)\phi[u(s)] \left[u(s) + \int\limits_a^s g(s,\xi)w[u(s)]\Delta\xi \right] + h(t,s)\Psi[u(s)] \right) \Delta s$$

for $t \in [a,b]_T$, *implies*

$$u(t) \leq \Phi^{-1}\left[F^{-1}\left(K(t) + \int\limits_a^t \left(f(t,s)G^{-1}\left[G(K(t)) + \int\limits_a^s \max\{f(t,\xi), g(t,\xi)\}\Delta\xi \right] \right) \Delta s \right) \right].$$

5.3 Some New Results

We try to generalize Constantin's inequality containing nabla and diamond-alpha derivatives and present the results we have obtained.

First we look for the discrete analogue of Constantin's inequality involving nabla derivatives. By using mean value theorem and the fundemental theorem of calculus for nabla derivatives we get the below result.

Theorem 5.3.1 *Let the function* $w \in C(\mathbb{R}_+, \mathbb{R}_+)$ *be non decreasing,* $w(r) > 0$ *for* $r > 0$, $\phi \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ *with* ϕ' *being non negative and non-decresing.*

$u, c \in C(\mathbb{N}_M, \mathbb{R}_+)$ *with* $c(n)$ *non-decreasing. Further, let*

$\theta(n,s),\lambda(n,s),\gamma(n,s)\in C(\mathbb{N}_M\times\mathbb{N}_M,\mathbb{R}_+)$ be non-decreasing with respect to n for every s fixed. Then if the following discrete inequality is satisfied

$$\phi[u(n)]\leq c(n)+\sum_{s=1}^{n}\{\theta(n,s)\phi'[u(s)]\times\Big[u(s)+\sum_{\xi=1}^{s}\lambda(s,\xi)w(u(\xi)\Big]+\gamma(n,s)\phi'[u(s)]\},$$

$n\in\mathbb{N}_M$, there exists $k,l>0$ such that the follwing inequality is true

$$u(n))\leq\tilde{L}(n)+\sum_{\xi=1}^{n}k\theta(n,\xi)G^{-1}\Big(G[\tilde{L}(n)]+l\sum_{s=1}^{\xi}k\theta(n,s)+\lambda(n,s)\Big),k,l\geq 1.$$

Here $G(r)=\int_{r_0}^{r}\frac{ds}{s+w(s)}$, $r\leq r_0$, $1>r_0>0$, $\lim_{x\to\infty}G(x)=\infty$,

$\tilde{L}(n)=\phi^{-1}[c(n)]+k\sum_{s=1}^{n}\gamma(n,s)$.

$\mathbb{N}_M=\{n\in N:n\leq M,M\in\mathbb{N}\}$

Proof. Fixing an arbitrary positive integer $m\in(0,M]$. We denote the set

$I=\{0,1,2,...,m\}$ and we define a positive function $v(n)\in K(I,\mathbb{R}_+)$ such that

$$v(n)=c(m)+\sum_{s=1}^{n}\Big\{\theta(m,s)\phi'[u(s)]\Big[u(s)+\sum_{\xi=1}^{s}\lambda(m,\xi)w(u(\xi))\Big]+\gamma(m,s)\phi'[u(s)]\Big\}$$

Then $\phi(u(n))\leq v(n)$, equivalently $u(n)\leq\phi^{-1}[v(n)]$. Taking the nabla derivative of $v(n)$ we get,

$$\begin{aligned}\nabla v(n) &= \theta(m,n)\phi'[u(n)]\Big[u(n)+\textstyle\sum_{s=1}^{n}\lambda(m,s)w(u(s))\Big]+\gamma(m,n)\\ &\leq \theta(m,n)\phi'[\phi^{-1}[v(n)]]\Big[\phi^{-1}[v(n)]+\textstyle\sum_{s=1}^{n}\lambda(m,s)w(\phi^{-1}[v(n)])\Big]+\gamma(m,n)\\ &= \phi'[\phi^{-1}[v(n)]]\Big\{\theta(m,n)\Big[\phi^{-1}[v(n)]+\textstyle\sum_{s=1}^{n}\lambda(m,s)w(\phi^{-1}[v(n)])\Big]+\gamma(m,n)\Big\}.\end{aligned}$$

$$\frac{\nabla v(n)}{\phi'[\phi^{-1}[v(n)]]}\leq\theta(m,n)\Big[\phi^{-1}[v(n)]+\sum_{s=1}^{n}\lambda(m,s)w(\phi^{-1}[v(n)])+\gamma(m,n).\quad(5.1)$$

Since we have finitely many elements in the domain then the value of the functions are bounded and since each function goes to $\mathbb{R}_+$ never takes 0. Then

$$k=max\Big\{\frac{\phi'[\phi^{-1}[v(m)]]}{\phi'[\phi^{-1}[v(m-1)]]},\frac{\phi'[\phi^{-1}[v(m-1)]]}{\phi'[\phi^{-1}[v(m-2)]]},....,\frac{\phi'[\phi^{-1}[v(n)]]}{\phi'[\phi^{-1}[v(n-1)]]},....,\frac{\phi'[\phi^{-1}[v(1)]]}{\phi'[\phi^{-1}[v(0)]]}\Big\}$$

is exist.

If we multiply (5.1) by k, we get

$$\begin{array}{rcl} \frac{\nabla v(n)}{\phi'[\phi^{-1}[v(n-1)]]} & \leq & k\frac{\nabla v(n)}{\phi'[\phi^{-1}[v(n)]]} \\ & \leq & k\theta(m,n)\Big[\phi^{-1}[v(n)] + \sum_{s=1}^{n}\lambda(m,s)w(\phi^{-1}[v(n)])\Big] + k\gamma(m,n). \end{array}$$

By mean value theorem $\nabla\phi^{-1}[v(n)] \leq \frac{\nabla v(n)}{\phi'[\phi^{-1}[v(n-1)]]}$. Therefore

$$\nabla\phi^{-1}[v(n)] \leq k\theta(m,n)\Big[\phi^{-1}[v(n)] + \sum_{s=1}^{n}\lambda(m,s)w(\phi^{-1}[v(n)])\Big] + k\gamma(m,n).$$

Substituting n with ξ in the last assertion and summing over $\xi = 1, ..., n$ we have

$$\phi^{-1}[v(n)] \leq \phi^{-1}[v(0)] + k\sum_{\xi=1}^{m}\gamma(m,\xi) + k\sum_{\xi=1}^{n}\theta(m,\xi)\Big[\phi^{-1}[v(\xi)] + \sum_{s=1}^{\xi}\lambda(m,s)w[\phi^{-1}[v(s)]\Big].$$

Define the right hand side of the last inequality as $y(n)$, then we get

$\phi^{-1}[v(n)] \leq y(n)$ for $n \in I$. Taking the nabla derivtive of $y(n)$, we obtain

$$\begin{array}{rcl} \nabla y(n) & = & k\theta(m,n)\Big[\phi^{-1}[v(n)] + \sum_{s=1}^{n}\lambda(m,s)w[\phi^{-1}[v(s)]\Big] \\ & \leq & k\theta(m,n)\Big[y(n) + \sum_{s=1}^{n}\lambda(m,s)w[y(s)]\Big], n \in I. \end{array}$$

We define $\Omega(n) = y(n) + \sum_{s=1}^{n}\lambda(m,s)w[y(s)]$. Here $y(n) \leq \Omega(n)$. Then we get

$$\nabla y(n) = k\theta(m,n)\Omega(n).$$

Take the nabla derivative of $\Omega(n)$, we get

$$\begin{array}{rcl} \nabla\Omega(n) & = & \nabla y(n) + \lambda(m,n)w(y(n)) \\ & \leq & k\theta(m,n)\Omega(n) + \lambda(m,n)w(\Omega(n)) \\ & \leq & [k\theta(m,n) + \lambda(m,n)][\Omega(n) + w(\Omega(n))]. \end{array}$$

$$\frac{\nabla\Omega(n)}{\Omega(n) + w(\Omega(n))} \leq k\theta(m,n) + \lambda(m,n). \tag{5.2}$$

By mean value theorem

$$\int_{\Omega(n-1)}^{\Omega(n)}\frac{ds}{s + w(s)} \leq \frac{\nabla\Omega(n)}{\Omega(n-1) + w(\Omega(n-1))}.$$

Substituting n with s and summing over $s = 1, 2, ...n$, we have

$$\int_{\Omega(0)}^{\Omega(n)}\frac{ds}{s + w(s)} \leq \sum_{s=1}^{n}\frac{\nabla\Omega(s)}{\Omega(s-1) + w(\Omega(s-1))}.$$

For inequality (5.2) substituting n with s and summing over $s = 1, 2, ...n$, we obtain

$$\sum_{s=1}^{n} \frac{\nabla\Omega(s)}{\Omega(s) + w(\Omega(s)} \leq \sum_{s=1}^{n} k\theta(m, s) + \lambda(m, s). \tag{5.3}$$

Let $l = max_{n\in[0,m]} \frac{\sum_{s=1}^{n} \frac{\nabla\Omega(s)}{\Omega(s-1)+w(\Omega(s-1)}}{\sum_{s=1}^{n} \frac{\nabla\Omega(s)}{\Omega(s)+w(\Omega(s)}}$. Here l is exits again because of the same reason of existence of k. Multiply (5.3) with l, we get

$$\sum_{s=1}^{n} \frac{\nabla\Omega(s)}{\Omega(s-1) + w(\Omega(s-1)} \leq l \sum_{s=1}^{n} k\theta(m, s) + \lambda(m, s).$$

Therefore,

$$\int_{\Omega(0)}^{\Omega(n)} \frac{ds}{s + w(s)} \leq l \sum_{s=1}^{n} k\theta(m, s) + \lambda(m, s).$$

Then

$$G[\Omega(n)] - G[\Omega(0)] \leq l \sum_{s=1}^{n} k\theta(m, s) + \lambda(m, s)$$

and

$$\Omega(n) \leq G^{-1}\Big(G[\Omega(0)] + l \sum_{s=1}^{n} k\theta(m, s) + \lambda(m, s)\Big).$$

Since $\nabla y(n) \leq k\theta(m, n)\Omega(n)$, then

$$\nabla y(n) \leq k\theta(m, n)G^{-1}\Big(G[\Omega(0)] + l \sum_{s=1}^{n} k\theta(m, s) + \lambda(m, s)\Big).$$

Substituting n with ξ and summing over $\xi = 1, 2, ...n$ we get

$$y(n) \leq v(0) + \sum_{\xi=1}^{n} k\theta(m, \xi)G^{-1}\Big(G[\Omega(0)] + l \sum_{s=1}^{\xi} k\theta(m, s) + \lambda(m, s)\Big)$$

Since $u(n) \leq \phi^{-1}[v(n)] \leq y(n)$, we get

$$u(n) \leq \tilde{L}(m) + \sum_{\xi=1}^{n} k\theta(m, \xi)G^{-1}\Big(G[\tilde{L}(m)] + l \sum_{s=1}^{\xi} k\theta(m, s) + \lambda(m, s)\Big).$$

If we set $m = n$ we have

$$u(m) \leq \tilde{L}(m) + \sum_{\xi=1}^{m} k\theta(m, \xi)G^{-1}\Big(G[\tilde{L}(m)] + l \sum_{s=1}^{\xi} k\theta(m, s) + \lambda(m, s)\Big).$$

Hence we get the desired result. □

Example 5.3.1 *Let*

$$u^2(n) = L + 2\sum_{s=1}^{n} P(n-s)u(s). \tag{5.4}$$

Here $n \in [0, M]$. *The unique positive solution for equation* (5.4) *can be obtained by successive substitution. For instance by letting* $n = 0, 1, 2$ *we obtain,*

$u(0) = \sqrt{L}$

$u(1) = \sqrt{L + 2P(0)u(1)}$

$u(2) = \sqrt{L + 2P(1)u(1) + 2P(0)u(2)}$. *By using solution of quadratic equations we can find* $u(0), u(1), u(2), ..., u(M)$ *succesively. If we use the theorem above the bound for* u *will be* $u(n) \leq \sqrt{L} + k\sum_{s=1}^{n} P(n-s)$. *With the help of the proof of Theorem 5.3.1 here*

$$k = max_{n\in[0,M]}\Big\{1 + \tfrac{2P(M-1)u(1)}{L}, 1 + \tfrac{2P(M-2)u(2)}{L+2P(M-1)u(1)}, ...,$$

$$1 + \tfrac{2P(M-n)u(n)}{L+2P(M-1)u(1)+2P(M-2)u(2)....+2P(M-n+1)u(n-1)}, ...,$$

$$1 + \tfrac{2P(0)u(M)}{L+2P(M-1)u(1)+.....+2P(1)u(M-1)},$$

$$1 + \tfrac{2P(M-2)u(1)}{L}, 1 + \tfrac{2P(M-3)u(2)}{L+2P(M-2)u(1)}, ...,$$

$$1 + \tfrac{2P(M-1-n)u(n)}{L+2P(M-2)u(1)+2P(M-3)u(2)....+2P(M-n)u(n-1)}, ...,$$

$$1 + \tfrac{2P(0)u(M-1)}{L+2P(M-2)u(1)+.....+2P(1)u(M-2)}, ...,$$

$$1 + \tfrac{2P(0)u(1)}{L}\Big\}$$

Here $\theta(n, s) = \lambda(n, s) = 0$, $\gamma(n, s) = P(n - s)$, $c(n) = L > 0$, L *is constant,* $u, P \in K(\mathbb{N}_0, \mathbb{R}_+)$, P *is non-decreasing, and* $\phi(x) = x^2$.

If we choose $P(n, s)$, L *good enough* k *does not exceed a number* c *when* M *tends to infinity. For instance, if* $P(n, s) = 1$, $L = 3$ *then* k *does not exceed* 3 *when* M *tends to infinity. Therefore, for the equality* $u^2(n) = 3 + 2\sum_{s=1}^{n} u(s)$, *by using the theorem above we have the bound for*

$$u(n) \leq \sqrt{3} + 3n, n \in \mathbb{N}$$

5.4 Generalization on Nabla Calculus

By using the two lemmas below we proved the generalized version of Theorem 5.3.1 for nabla derivatives.

Lemma 5.4.1 *Let* $f : \mathbb{R} \to \mathbb{R}$ *continuously differentiable and* $g : \mathbb{T} \to \mathbb{R}$ *is nabla differentiable then* fog *is nabla differentiable and the formula is given by*

$$(fog)^{\nabla}(t) = g^{\nabla}(t)\Big[\int_0^1 f'(g(\rho(t)) + h\nu(t)g^{\nabla}(t))dh\Big]$$

1. $\nu(t) = t - \rho(t)$
2. $\mathbb{T}_{\kappa} = \mathbb{T} - \{m\}$ where m is the right scattered minimum.

Proof. Apply ordinary substitution rule from calculus.

$$f(g(s) - f(g(\rho(t))) = \int\limits_{g(\rho(t))}^{g(s)} f'(\tau)d\tau.$$

If we take $\tau = hg(s)+(1-h)g(\rho(t))$, then $d\tau = g(s)-g(\rho(t))dh$. Then our integral becomes

$$f(g(s) - f(g(\rho(t))) = g(s) - g(\rho(t)) \int\limits_0^1 f'(hg(s) - (1-h)g(\rho(t)))dh.$$

Let $\epsilon > 0$ be given and $t \in \mathbb{T}_{\kappa}$. Then there exists neighbourhood U_1 of t, since g is nabla diffrentiable at t such that

$$\left|g(s) - g(\rho(t)) - g^{\nabla}(t)(s - \rho(t))\right| \leq \epsilon^* \left|s - \rho(t)\right|, \forall s \in U_1$$

where $\epsilon^* = \frac{\epsilon}{1+2\int\limits_0^1 |f'(hg(s)-(1-h)g(\rho(t)))|dh}$. Since g is nabla differentiable then g is continuous, then we can obtain $\forall \delta > 0 \ \exists U_2$ neighbourhood of t such that

$|g(s) - g(t)| < \delta, \forall s \in U_2$. Since

$$\begin{aligned} |hg(s) + (1-h)g(\rho(t)) - (hg(t) + (1-h)g(\rho(t)))| &= h\,|g(s) - g(t)| \\ &\leq |g(s) - g(t)| < \delta. \end{aligned}$$

Here f is continuously differentiable function, then f' is continuous on $\mathbb{R}$ and f' is uniformly continuous on closed subsets of $\mathbb{R}$. Therefore we have

$$|f'(h(g(s)+(1-h)g(\rho(t)))-f'(h(g(t)+(1-h)g(\rho(t)))|\leq\frac{\epsilon}{\epsilon^*+|g^{\nabla}(t)|},\forall s\in U_2.$$

Here let us define neighbourhood U of t such that $U=U_1\cap U_2$. Let us take

$\alpha=hg(s)+(1-h)g(\rho(t))$ and $\beta=hg(t)+(1-h)g(\rho(t))$. Then we have

$$\begin{aligned}
&\left|fog(s)-fog(\rho(t))-(s-\rho(t))g^{\nabla}(t)\int_0^1 f'(\beta)dh\right|\\
&=\left|g(s)-g(\rho(t))\int_0^1 f'(\alpha)dh-(s-\rho(t))g^{\nabla}(t)\int_0^1 f'(\beta)dh\right|\\
&=\left|g(s)-g(\rho(t))-(s-\rho(t))g^{\nabla}(t)\int_0^1 f'(\alpha)dh+(s-\rho(t))g^{\nabla}(t)\int_0^1 f'(\alpha)-f'(\beta)dh\right|\\
&\leq\left|g(s)-g(\rho(t))-(s-\rho(t))g^{\nabla}(t)\right|\int_0^1|f'(\alpha)|\,dh+|s-\rho(t)|\left|g^{\nabla}(t)\right|\int_0^1|f'(\alpha)-f'(\beta)|\,dh\\
&\leq\epsilon^*|s-\rho(t)|\int_0^1|f'(\alpha)|\,dh+|s-\rho(t)|\left|g^{\nabla}(t)\right|\int_0^1|f'(\alpha)-f'(\beta)|\,dh\\
&=\epsilon^*|s-\rho(t)|\int_0^1|f'(\alpha)-f'(\beta)+f'(\beta)|\,dh+|s-\rho(t)|\left|g^{\nabla}(t)\right|\int_0^1|f'(\alpha)-f'(\beta)|\,dh\\
&\leq\epsilon^*|s-\rho(t)|\int_0^1|f'(\beta)|\,dh+|s-\rho(t)|\left(\left|g^{\nabla}(t)\right|+\epsilon^*\right)\int_0^1|f'(\alpha)-f'(\beta)|\,dh\\
&\leq\tfrac{\epsilon}{2}|s-\rho(t)|+\tfrac{\epsilon}{2}|s-\rho(t)|=\epsilon|s-\rho(t)|
\end{aligned}$$

Hence fog is nabla differentiable and its derivative is as claimed above. □

Lemma 5.4.1 first occured in the article of F. Atici [4] as

$$(fog)^{\nabla}(t)=g^{\nabla}(t)\Big[\int_0^1 f'(g(t)+h\nu(t)g^{\nabla}(t))dh\Big].$$

With a counter example we can show that their version of the formula is not true.

Example 5.4.1 *Let $g(x):\mathbb{Z}\to\mathbb{R}$ such that $g(n)=\frac{1}{n}$, $f(x):\mathbb{R}\to\mathbb{R}$ such that $f(x)=x^2$. Therefore the first derivative of $f(x)$ is continuous. If we apply the formula in Lemma 5.4.1 to find the nabla derivative of the function $(fog)(t)$ we get*

$$\frac{1-2n}{(n^2)(n-1)^2}=\Big(\frac{1}{n^2}\Big)^{\nabla}\neq\Big(\frac{1}{n}\Big)^{\nabla}\int_0^1\Big[2\Big(\frac{1}{n}\Big)+2h\Big(\frac{1}{n}\Big)^{\nabla}\Big]dh=\frac{3-2n}{(n^2)(n-1)^2}.$$

Lemma 5.4.2 *Let $a,b\in\mathbb{T}$, consider the Time Scale $[a,b]_{\mathbb{T}}$ and a function*

$p\in C^1_{ld}([a,b]_{\mathbb{T}},\mathbb{R})$ with $p^{\nabla}(t)\geq 0$. Suppose that a function $f\in C\left(\mathbb{R}^+_0,\mathbb{R}^+_0\right)$ is positive and non-decreasing on $\mathbb{R}$. Define $F(x)=\int_{x_0}^{x}\frac{ds}{f(s)}$, where $x\geq 0, x_0\geq 0$

if $\int\limits_0^x \frac{ds}{f(s)} < \infty$ *and* $x > 0, x_0 > 0$ *if* $\int\limits_0^x \frac{ds}{f(s)} = \infty$ *. Then for each* $t \in [a, b]_{\mathbb{T}}$ *, we have*

$$F(p(t)) \leq F(p(a)) + \int\limits_a^t \frac{p^{\nabla}(\tau)}{f(p(\rho(\tau)))} \nabla\tau.$$

Proof. Since f is positive and non-decreasing on $(0, \infty)$, we have

$$p(\rho(t)) \leq p(\rho(t)) + h\nu(t)p^{\nabla}(t)$$

$$f\left(p(\rho(t))\right) \leq f\left(p(\rho(t)) + h\nu(t)p^{\nabla}(t)\right)$$

and

$$\frac{1}{f\left(p(\rho(t)) + h\nu(t)p^{\nabla}(t)\right)} \leq \frac{1}{f\left(p(\rho(t))\right)}.$$

If we integrate the last inequality from 0 to 1, we get

$$\int\limits_0^1 \frac{dh}{f\left(p(\rho(t)) + h\nu(t)p^{\nabla}(t)\right)} \leq \int\limits_0^1 \frac{dh}{f\left(p(\rho(t))\right)} = \frac{1}{f\left(p(\rho(t))\right)}.$$

Multiply both sides of the last inequality with $p^{\nabla}(t)$, since $p^{\nabla}(t) \geq 0$ the inequality does not change

$$p^{\nabla}(t) \int\limits_0^1 \frac{dh}{f\left(p(\rho(t)) + h\nu(t)p^{\nabla}(t)\right)} \leq \frac{p^{\nabla}(t)}{f\left(p(\rho(t))\right)}.$$

Here we use Lemma 5.4.1 and we get

$$(Fop)^{\nabla}(t) \leq \frac{p^{\nabla}(t)}{f\left(p(\rho(t))\right)}.$$

If we integrate it from a to t we have

$$F(p(t)) - F(p(a)) \leq \int\limits_a^t \frac{p^{\nabla}(\tau)}{f\left(p(\rho(\tau))\right)} \nabla\tau$$

and

$$F(p(t)) \leq F(p(a)) + \int\limits_a^t \frac{p^{\nabla}(\tau)}{f\left(p(\rho(\tau))\right)} \nabla\tau.$$

Hence we get the desired result. $\square$

The delta derivative version of the above lemma was proved by Torres and Ferriara [28]. By using these results we proved the below theorem.

Theorem 5.4.1 *Assume that* $v \in C_{ld}\left([a,b]_{\mathbb{T}}, \mathbb{R}_0^+\right)$, $c \in C_{ld}\left([a,b]_{\mathbb{T}}, \mathbb{R}^+\right)$ *is non-decreasing,* $\Phi \in C\left(\mathbb{R}_0^+, \mathbb{R}_0^+\right)$ *is strictly increasing function such that*

$$\lim_{x\to\infty} \Phi(x) = \infty.$$

Let $\theta(t,\xi), \gamma(t,\xi) \in C_{ld}\left([a,b]_{\mathbb{T}} \times [a,b]_{\mathbb{T}_\kappa}, \mathbb{R}_0^+\right)$ *and*

$\lambda(t,\xi) \in C_{ld}\left([a,b]_{\mathbb{T}_\kappa} \times [a,b]_{\mathbb{T}_{\kappa^2}}, \mathbb{R}_0^+\right)$ *be non-decreasing in for every fixed* ξ. *Further, let* $w, \phi, \Psi \in C\left(\mathbb{R}_0^+, \mathbb{R}_0^+\right)$ *be non-decreasing such that* $\{w, \phi, \Psi\}(x) > 0$ *for every* $x > 0$. *Define*

$$M(x) = \max\{\phi(x), \Psi(x)\}$$

on $\mathbb{R}_0^+$ *and assume that the following function*

$$\tilde{E}(x) = \int\limits_{x_0}^{x} \frac{ds}{Mo\Phi^{-1}(s)}$$

with $x > c(a) > x_0 \geq 0$ *if* $\int\limits_0^x \frac{ds}{Mo\Phi^{-1}(s)} < \infty$ *and* $x > c(a) > x_0 > 0$

if $\int\limits_0^x \frac{ds}{Mo\Phi^{-1}(s)} = \infty$; *satisfies* $\lim_{x\to\infty} \tilde{E}(x) = \infty$.

Also assume that the function

$$R(r) = \int\limits_{r_0}^{r} \frac{ds}{w(s)}$$

where $r \geq 0, r_0 \geq 0$ *if* $\int\limits_0^r \frac{ds}{w(s)} < \infty$ *and* $r > 0, r_0 > 0$

if $\int\limits_0^r \frac{ds}{w(s)} = \infty$ *;satisfies* $\lim_{r\to\infty} R(r) = \infty$.

Define $\hat{L}(t) := \tilde{E}\left[c(t)\right] + \alpha \int\limits_a^t \gamma(t,s)\nabla s$ *and* $G(r) = \int\limits_{r_0}^{r} \frac{ds}{w(s)+s}$.

If $\Phi^{-1}\left[\tilde{E}^{-1}(x)\right] \leq x$, *for all* $x \geq 0$ *then the inequlity*

$$\Phi\left[v(t)\right] \leq c(t) + \int\limits_a^t \left(\theta(t,s)\phi\left[v(s)\right]\left[v(s) + \int\limits_a^s \lambda(s,\xi) w\left[v(s)\right]\nabla\xi\right] + \gamma(t,s)\Psi\left[v(s)\right]\right)\nabla s$$

for $t \in [a,b]_{\mathbb{T}}$ *, implies*

$v(t) \leq \Phi^{-1}\left[\tilde{E}^{-1}\left(\hat{L}(t)+\alpha\int\limits_a^t\left(\theta(t,s)G^{-1}\left[G(\hat{L}(t))+\beta\int\limits_a^s \max\left\{\alpha\theta(t,\xi),\lambda(t,\xi)\right\}\nabla\xi\right]\right)\nabla s\right)\right]$ *where* $\alpha,\beta \geq 1$ *are constants.*

Proof. If $t=a$ obviously theorem holds. Let us fix an arbitrary number $t_0 \in (a,b]_{\mathbb{T}}$ we define $z(t)$ on $[a,t_0]_{\mathbb{T}}$ such that

$$z(t)=c(t_0)+\int\limits_a^t\left(\theta(t_0,s)\phi(v(s))\left[v(s)+\int\limits_a^s\lambda(t_0,\xi)w(v(\xi))\nabla\xi\right]+\gamma(t_0,s)\Psi\left[v(s)\right]\right)\nabla s.$$

Since $\Phi(v(t)) \leq z(t)$, then $v(t) \leq \Phi^{-1}(z(t))$. Let us differentiate function $z(t)$.

$$\begin{aligned}
z^{\nabla}(t) &= \theta(t_0,t)\phi(v(t))\left[v(t)+\int\limits_a^t\lambda(t_0,\xi)w(v(\xi))\nabla\xi\right]+\gamma(t_0,t)\Psi\left[v(t)\right]\\
&\leq M\left[v(t)\right]\left\{\theta(t_0,t)\left[v(t)+\int\limits_a^t\lambda(t_0,\xi)w\left[v(\xi)\right]\nabla\xi\right]+\gamma(t_0,t)\right\}\\
&\leq M\left[\Phi^{-1}(z(t))\right]\left\{\theta(t_0,t)\left[\Phi^{-1}(z(t))+\int\limits_a^t\lambda(t_0,\xi)w\left[\Phi^{-1}(z(\xi)\right]\nabla\xi\right]+\gamma(t_0,t)\right\}.
\end{aligned}$$

Then we get

$$\frac{z^{\nabla}(t)}{M\left[\Phi^{-1}(z(t))\right]} \leq \theta(t_0,t)\left[\Phi^{-1}(z(t))+\int\limits_a^t\lambda(t_0,\xi)w\left[\Phi^{-1}(z(\xi))\right]\nabla\xi\right]+\gamma(t_0,t). \tag{5.5}$$

Define the function $\check{z}(t)$ on $[a,b]_{\mathbb{T}}$ such that

$$\check{z}(t)=c(b)+\int\limits_a^t\left(\theta(b,s)\phi\left[v(s)\right]\left[v(s)+\int\limits_a^s\lambda(b,\xi)w\left[v(\xi)\right]\nabla\xi\right]+\gamma(b,s)\Psi\left[v(s)\right]\right)\nabla s.$$

It is obvious that $z(t) \leq \check{z}(t)$ on $[a,t_0]_{\mathbb{T}}$. Using $\check{z}(t)$ we define α such that

$$\alpha=\max_{t\in[a,t_0]_T}\frac{Mo\Phi^{-1}o\check{z}(t)}{Mo\Phi^{-1}oz(a)}.$$

Here functions $\check{z}(t)$ and $z(t)$ are C_{ld} continuous because each function that constitude the functions $\check{z}(t)$ and $z(t)$ are continuous or C_{ld} continuous.Here M and Φ^{-1} are continuous functions then $Mo\Phi^{-1}o\check{z}$ and $Mo\Phi^{-1}oz$ are also C_{ld} continuous functions,since each continuous function is C_{ld} continuous. We know that any C_{ld} continuous function is regulated and every regulated function is bounded in a compact interval by theorem from [8]. Therefore the functions $Mo\Phi^{-1}o\check{z}(t)$ and $Mo\Phi^{-1}oz(t)$ are bounded on $[a,b]_{\mathbb{T}}$, and also $z(a)=c(t_0)$ can not be 0 since $c(t)\in C_{ld}\left([a,b],\mathbb{R}^{+}\right)$. Additionally since Φ is strictly increasing $M(x)$ is non decreasing and can not be zero if $x\neq 0$ then $Mo\Phi^{-1}oz(a)\neq 0$ so the constant α exists.Then

$$\alpha=\max_{t\in[a,t_0]_{\mathbb{T}}}\frac{Mo\Phi^{-1}o\check{z}(t)}{Mo\Phi^{-1}oz(a)}\geq\frac{Mo\Phi^{-1}(z(t))}{Mo\Phi^{-1}(z(\rho(t))}$$

for all $t\in[a,t_0]_{\mathbb{T}}$. It is obvious that $\alpha\geq 1$.

If we multiply (5.5) by α , we get

$$\frac{z^{\nabla}(t)}{M\left[\Phi^{-1}(z(\rho(t)))\right]}\leq\alpha\theta(t_0,t)\left[\Phi^{-1}(z(t))+\int\limits_a^t\lambda(t_0,\xi)w\left[\Phi^{-1}(z(\xi))\right]\nabla\xi\right]+\alpha\gamma(t_0,t).$$

By the Lemma 5.4.2 , we get

$$\begin{aligned}\tilde{E}(z(t)) &\leq \tilde{E}(c(t_0)+\alpha\int\limits_a^t\left(\theta(t_0,s)\left[\Phi^{-1}(z(s))+\int\limits_a^s\lambda(t_0,\xi)w\left[\Phi^{-1}(z(\xi))\right]\nabla\xi\right]\right)\nabla s\\ &+ \alpha\int\limits_a^{t_0}\gamma(t_0,s)\nabla s.\end{aligned}$$

$$\begin{aligned}z(t) &\leq \tilde{E}^{-1}\Big[\tilde{E}(c(t_0)+\alpha\int\limits_a^t\left(\theta(t_0,s)\left[\Phi^{-1}(z(s))+\int\limits_a^s\lambda(t_0,\xi)w\left[\Phi^{-1}(z(\xi))\right]\nabla\xi\right]\right)\nabla s\\ &+ \alpha\int\limits_a^{t_0}\gamma(t_0,s)\nabla s\Big].\end{aligned}$$

Let's define $y(t)$ as

$$\begin{aligned}y(t) &= \tilde{E}(c(t_0))+\alpha\int\limits_a^t\left(\theta(t_0,s)\left[\Phi^{-1}(z(s))+\int\limits_a^s\lambda(t_0,\xi)w\left[\Phi^{-1}(z(\xi))\right]\nabla\xi\right]\right)\nabla s\\ &+ \alpha\int\limits_a^{t_0}\gamma(t_0,s)\nabla s.\end{aligned}$$

Then $z(t)\leq\tilde{E}^{-1}(y(t))$. Let us diffrentiate the function $y(t)$,

$$\begin{aligned}y^{\nabla}(t) &= \alpha\theta(t_0,t)\left[\Phi^{-1}(z(t))+\int\limits_a^t\lambda(t_0,\xi)w\left[\Phi^{-1}(z(\xi))\right]\nabla\xi\right]\\ &\leq \alpha\theta(t_0,t)\left[\Phi^{-1}(\tilde{E}^{-1}(y(t)))+\int\limits_a^t\lambda(t_0,\xi)w\left[\Phi^{-1}(\tilde{E}^{-1}(y(\xi)))\right]\nabla\xi\right]\\ &\leq \alpha\theta(t_0,t)\left[y(t)+\int\limits_a^t\lambda(t_0,\xi)w\left[y(\xi)\right]\nabla\xi\right],\ \text{since }\Phi^{-1}(\tilde{E}^{-1}(y(t)))\leq y(t).\end{aligned}$$

Let us define $W(t)$ as $W(t)=y(t)+\int\limits_a^t\lambda(t_0,\xi)w\left[y(\xi)\right]\nabla\xi$, so $W(t)\geq y(t)$. When we differentiate $W(t)$ we get

$$\begin{aligned}W^{\nabla}(t) &= y^{\nabla}(t)+\lambda(t_0,t)w\left[y(t)\right]\\ &\leq \alpha\theta(t_0,t)W(t)+\lambda(t_0,t)w\left[W(t)\right].\end{aligned}$$

Then

$$\frac{W^{\nabla}(t)}{W(t)+w\left[W(t)\right]} \leq \max\left\{\alpha\theta(t_0,t),\lambda(t_0,t)\right\}. \tag{5.6}$$

Let us define $\hat{y}(t)$ and $\hat{W}(t)$ on $[a,b]_{\mathbb{T}}$ such that

$$\hat{y}(t)=\hat{L}(b)+\alpha\int_a^t\left(\theta(b,s)\left[\Phi^{-1}(z(s))+\int_a^s\lambda(b,\xi)w\left[\Phi^{-1}(z(\xi))\nabla\xi\right]\right]\right)\nabla s$$

$$\hat{W}(t)=\hat{y}(t)+\int_a^t\lambda(b,\xi)w\left[\hat{y}(\xi)\right]\nabla\xi.$$

It is obvious that $y(t)\leq\hat{y}(t)$ and $W(t)\leq\hat{W}(t)$ on $[a,t_0]_{\mathbb{T}}$. Using $\hat{W}(t)$ we define β such that

$$\beta=\max_{t\in[a,t_0]_{\mathbb{T}}}\frac{\hat{W}(t)+w(\hat{W}(t))}{W(a)+w(W(a))}\geq 1.$$

Here again $W(t),y(t),\hat{y}(t),\hat{W}(t)$ and $w(\hat{W}(t))$ are C_{ld} continuous. Thus these functions are bounded on $[a,t_0]_{\mathbb{T}}$ [8] and $W(a)+w(W(a))$ can not be zero. We know that $W(a)=y(a)=\tilde{E}(c(t_0))+\alpha\int_a^{t_0}h(t_0,s)\nabla s$. Here also $\tilde{E}(c(t_0))$ can not be zero since $c(t_0)>x_0$, $c(t_0)$ is finite and inside of the integral is never zero in the interval $[x_0,c(t_0)]$ so the constant β exists. Then

$$\beta=\max_{t\in[a,t_0]_T}\frac{\hat{W}(t)+w(\hat{W}(t))}{W(a)+w(W(a))}\geq\frac{W(t)+w(W(t))}{W(\rho(t))+w(W(\rho(t)))}.$$

If we multiply (5.6) by β , then we get

$$\frac{W^{\nabla}(t)}{W(\rho(t))+w(W(\rho(t)))}\leq\beta\max\left\{\alpha\theta(t_0,t),\lambda(t_0,t)\right\}.$$

Again by using the Lemma 5.4.2 we have

$$G(W(t))\leq G(\hat{L}(t_0))+\beta\int_a^t\max\left\{\alpha\theta(t_0,s),\lambda(t_0,s)\right\}\nabla s.$$

Then

$$W(t)=G^{-1}\left[G(\hat{L}(t_0))+\beta\int_a^t\max\left\{\alpha\theta(t_0,s),\lambda(t_0,s)\right\}\nabla s\right].$$

Since $y^{\nabla}(t)\leq\alpha\theta(t_0,t)W(t)$,then $y(t)\leq\hat{L}(t_0)+\alpha\int_a^t\theta(t_0,s)W(s)\nabla s$. Therefore

$$y(t)\leq\hat{L}(t_0)+\alpha\int_a^t\theta(t_0,s)G^{-1}\left[G(\hat{L}(t_0))+\beta\int_a^s\max\left\{\alpha\theta(t_0,\xi),\lambda(t_0,\xi)\right\}\nabla\xi\right].$$

Using above information we have $v(t) \leq \Phi^{-1}(z(t)) \leq \Phi^{-1}(\tilde{E}^{-1}(y(t))$. Hence

$$\begin{aligned} v(t) &\leq \Phi^{-1}\Bigg[\tilde{E}^{-1}\Bigg(\hat{L}(t_0) + \alpha \int\limits_a^t \Bigg(\theta(t_0,s)G^{-1}\Bigg[G(\hat{L}(t_0)) \\ &+ \beta \int\limits_a^s \max\Bigg\{\alpha\theta(t_0,\xi), \lambda(t_0,\xi)\Bigg\}\nabla\xi\Bigg]\Bigg)\nabla s\Bigg)\Bigg]. \end{aligned}$$

Since t_0 is arbitrary we can set $t = t_0$ in the above inequality and we get the desired result.

□

Remark 5.4.1 *The above function $G(r)$ defined above satisfes*

$\lim_{r\to\infty} G(r) = \infty$ *by Constantin*[16] *. This was disgust in* [7], [15], [30]

Example 5.4.2 *If we take $\phi(x) = \psi(x) = \Phi'(x)$ with Φ' is non decreasing then $M(x) = \Phi'(x)$ and this implies*

$$\tilde{E}(x) = \int_{x_0}^{x} \frac{1}{\Phi' o \Phi^{-1}(s)} ds = \Phi^{-1}(x) - \Phi^{-1}(x_0).$$

Choose $x_0 = \Phi(0) \geq 0$. Then $\Phi^{-1}(\Phi(0)) = 0$ hence $\tilde{E}(x) = \Phi^{-1}(x)$. For the particular case $\mathbb{T} = \mathbb{Z}$, an application af Theorem 5.4.1 gives Theorem 5.3.1. For the particular case $\mathbb{T} = \mathbb{R}$ an application of Theorem 5.4.1 gives the generalization of Constantin's inequality done by Yang and Tan. Since $\alpha, \beta \geq 1$,there is no problem.

Example 5.4.3 *Let us take $\mathbb{T} = h\mathbb{Z}$ such that $x, t \in [0, Mh]$, $\gamma(x,t) = P(x-t)$, $\lambda(x,t) = \theta(t,x) = 0$, $\Phi(x) = x^2$, $\phi(x) = 0$, $\psi(x) = \frac{x}{2}$ defined for $x \geq 0$ and $c(n) = L, L \geq 0$ is constant. Then*

$$M(x) = max\Big\{\frac{x}{2}, 0\Big\} = \frac{x}{2}.$$

If we set $x_0 = 0$, we obtain

$$\tilde{E}(x) = \int_0^x \frac{1}{\frac{\sqrt{s}}{2}} ds = 4\sqrt{x}.$$

Thus, $\lim_{x\to\infty} \tilde{E}(x) = \infty$. $F^{-1}(x) = \left(\frac{x}{4}\right)^2$ *and* $\Phi^{-1}(\tilde{E}^{-1}(x)) = \left(\frac{x}{4}\right) \leq x, \forall x \geq 0.$

If we get the equality

$$u(hn)^2 = L + \sum_{t=1}^{n} hP(hn-ht)\frac{u(ht)}{2}.$$

By letting $n = 0, 1, 2,$

$u(0) = \sqrt{L}$

$u(h) = \sqrt{L + hP(0)\frac{u(h)}{2}}$

$u(2h) = \sqrt{L + hP(h)\frac{u(h)}{2} + hP(0)\frac{u(2h)}{2}}.$

If we apply Theorem 5.4.1 we find a upper bound for $u(hn)$ *as*

$$u(hn) \le \sqrt{L} + \alpha\frac{\sum_{t=1}^{n} hP(hn-ht)}{4}.$$

Here

$$\alpha = \max_{n\in[0,M]}\Big\{1 + \frac{P((M-1)h)\frac{u(h)}{2}}{L}, 1 + \frac{P((M-2)h)\frac{u(2h)}{2}}{L+P((M-1)h)\frac{u(h)}{2}}, \ldots\ldots,$$

$$1 + \frac{P((M-n)h)\frac{u(nh)}{2}}{L+P((M-1)h)\frac{u(h)}{2}+P((M-2)h)\frac{u(2h)}{2}....+2P((M-n+1)h)\frac{u((n-1)h)}{2}}, \ldots\ldots\ldots,$$

$$1 + \frac{P(0)\frac{u(Mh)}{2}}{L+P((M-1)h)\frac{u(h)}{2}+.....+P(h)\frac{u((M-1)h)}{2}}, \ldots\ldots\ldots\ldots\ldots\ldots,$$

$$1 + \frac{P((M-2)h)\frac{u(h)}{2}}{L}, 1 + \frac{P((M-3)h)\frac{u(2h)}{2}}{L+P((M-2)h)\frac{u(h)}{2}}, \ldots\ldots,$$

$$1 + \frac{P((M-n-1)h)\frac{u(nh)}{2}}{L+P((M-2)h)\frac{u(h)}{2}+P((M-3)h)\frac{u(2h)}{2}....+2P((M-n)h)\frac{u((n-1)h)}{2}}, \ldots\ldots\ldots,$$

$$1 + \frac{P(0)\frac{u((M-1)h)}{2}}{L+P((M-2)h)\frac{u(h)}{2}+.....+P(h)\frac{u((M-2)h)}{2}}, \ldots\ldots\ldots\ldots\ldots,$$

$$1 + \frac{P(0)\frac{u(h)}{2}}{L}\Big\}.$$

5.5 Generalization on Diamond-α Calculus

We also proved Constantin's inequality for diamond-alpha derivatives. To get the desired result we use the below lemma.

Lemma 5.5.1 *Let* $\mathbb{T}$ *be a regulated Times Scale,* $a, b \in \mathbb{T}$,*and consider the Times Scale* $[a,b]_{\mathbb{T}}$ *such that* $\sigma(a) = a$. *Let* $p \in C^1_{rl}([a,b]_{\mathbb{T}}, \mathbb{R})$ *with*

$p^{\nabla}(t), p^{\Delta}(t) \geq 0$. Suppose that a function $f \in C(\mathbb{R}_0^+, \mathbb{R}_0^+)$ is positive and non-decreasing on $\mathbb{R}$.

Define $F(x) = \int_{x_0}^{x} \frac{ds}{f(s)}$ where $x \geq 0$, $x_0 \geq 0$ if $F(x) = \int_0^x \frac{ds}{f(s)} < \infty$ and $x > 0$, $x_0 > 0$ if $F(x) = \int_0^x \frac{ds}{f(s)} = \infty$. Then for each $t \in [a, b]_{\mathbb{T}}$, we have

$$(Fop)(t) \leq 2(Fop)(a) + 2\int_a^t \frac{p^{\diamond_\alpha}(s)}{f(p(\rho(s)))}.$$

Proof. Since f is positive and non-decreasing on $(0, \infty)$, we have

$$p(\rho(t)) \leq p(\rho(t)) + h\nu(t)p^{\nabla}(t)$$

$$f\left(p(\rho(t))\right) \leq f\left(p(\rho(t)) + h\nu(t)p^{\nabla}(t)\right)$$

and

$$\frac{1}{f\left(p(\rho(t)) + h\nu(t)p^{\nabla}(t)\right)} \leq \frac{1}{f\left(p(\rho(t))\right)}.$$

If we integrate the last in equality from 0 to 1, we get

$$\int_0^1 \frac{dh}{f\left(p(\rho(t)) + h\nu(t)p^{\nabla}(t)\right)} \leq \int_0^1 \frac{dh}{f\left(p(\rho(t))\right)} = \frac{1}{f\left(p(\rho(t))\right)}.$$

Multiply both sides of the last inequality with $p^{\nabla}(t)$, since $p^{\nabla}(t) \geq 0$ the inequality does not change.

$$p^{\nabla}(t)\int_0^1 \frac{dh}{f\left(p(\rho(t)) + h\nu(t)p^{\nabla}(t)\right)} \leq \frac{p^{\nabla}(t)}{f\left(p(\rho(t))\right)}.$$

Again since f is positive and non-decreasing on $(0, \infty)$, we have

$$p(t) \leq p(t) + h\mu(t)p^{\Delta}(t)$$

$$f\left(p(t)\right) \leq f\left(p(t) + h\mu(t)p^{\Delta}(t)\right)$$

and

$$\frac{1}{f\left(p(t) + h\mu(t)p^{\Delta}(t)\right)} \leq \frac{1}{f\left(p(t)\right)}.$$

If we integrate the last in equality from 0 to 1, we get

$$\int_0^1 \frac{dh}{f\left(p(t) + h\mu(t)p^{\Delta}(t)\right)} \le \int_0^1 \frac{dh}{f\left(p(t)\right)} = \frac{1}{f\left(p(t)\right)}.$$

Multiply both sides of the last inequality with $p^{\Delta}(t)$, since $p^{\Delta}(t) \ge 0$ the inequality does not change.

$$p^{\Delta}(t) \int_0^1 \frac{dh}{f\left(p(t) + h\mu(t)p^{\Delta}(t)\right)} \le \frac{p^{\Delta}(t)}{f\left(p(t)\right)}.$$

Since $\alpha \in (0, 1)$, then

$$\alpha p^{\Delta}(t) \int_0^1 \frac{dh}{f\left(p(t) + h\mu(t)p^{\Delta}(t)\right)} \le \alpha \frac{p^{\Delta}(t)}{f\left(p(t)\right)}. \tag{5.7}$$

and

$$(1-\alpha) p^{\nabla}(t) \int_0^1 \frac{dh}{f\left(p(\rho(t)) + h\nu(t)p^{\nabla}(t)\right)} \le (1-\alpha) \frac{p^{\nabla}(t)}{f\left(p(\rho(t))\right)}. \tag{5.8}$$

If we sum the inequalities (5.7) and (5.8) we get

$$\alpha p^{\Delta}(t) \int_0^1 \frac{dh}{f\left(p(t) + h\mu(t)p^{\Delta}(t)\right)} + (1-\alpha) p^{\nabla}(t) \int_0^1 \frac{dh}{f\left(p(\rho(t)) + h\nu(t)p^{\nabla}(t)\right)}$$

$$\le \alpha \frac{p^{\Delta}(t)}{f\left(p(t)\right)} + (1-\alpha) \frac{p^{\nabla}(t)}{f\left(p(\rho(t))\right)}.$$

If we use the chain rule for nabla derivative and chain rule for delta derivative we get

$$\alpha (Fop)^{\Delta}(t) + (1-\alpha)(Fop)^{\nabla}(t) \le \alpha \frac{p^{\Delta}(t)}{f\left(p(t)\right)} + (1-\alpha) \frac{p^{\nabla}(t)}{f\left(p(\rho(t))\right)}.$$

Since $p^{\nabla}(t)$,$p^{\Delta}(t) \ge 0$, then p is non-decreasing. Therefore $p(\rho(t)) \le p(t)$ and we can write above inequality as

$$\alpha (Fop)^{\Delta}(t) + (1-\alpha)(Fop)^{\nabla}(t) \le \frac{\alpha p^{\Delta}(t) + (1-\alpha) p^{\nabla}(t)}{f\left(p(\rho(t))\right)}.$$

In other words,

$$(Fop)^{\diamond_\alpha}(t) \le \frac{p^{\diamond_\alpha}(t)}{f\left(p(\rho(t))\right)}.$$

Now integrate both sides of the inequality from a to t.

$$\int_a^t (Fop)^{\diamond_\alpha}(s)\diamond_\alpha s \leq \int_a^t \frac{p^{\diamond_\alpha}(s)}{f\left(p(\rho(s))\right)}\diamond_\alpha s.$$

$$\int_a^t (Fop)^{\diamond_\alpha}(s)\diamond_\alpha s = \alpha^2 \int_a^t (Fop)^{\Delta}(s)\Delta s + (1-\alpha)^2 \int_a^t (Fop)^{\nabla}(s)\nabla s$$
$$+\alpha(1-\alpha)\int_a^t (Fop)^{\Delta}(s)\nabla s + \alpha(1-\alpha)\int_a^t (Fop)^{\nabla}(s)\Delta s.$$

Since our Times Scales are regulated then by [40] we can use the the condition $(Fop)^{\nabla}(s) = (Fop)^{\Delta}(\rho(s))$ and $(Fop)^{\Delta}(s) = (Fop)^{\nabla}(\sigma(s))$. We have

$$\int_a^t (Fop)^{\diamond_\alpha}(s)\diamond_\alpha s = \alpha^2 \int_a^t (Fop)^{\Delta}(s)\Delta s + (1-\alpha)^2 \int_a^t (Fop)^{\nabla}(s)\nabla s$$
$$+\alpha(1-\alpha)\int_a^t (Fop)^{\nabla}(\sigma(s))\nabla s + \alpha(1-\alpha)\int_a^t (Fop)^{\Delta}(\rho(s))\Delta s.$$

Since a is the initial point then $\rho(a) = a = \sigma(a)$. Therefore we get

$$(1-2\alpha+2\alpha^2)Fop(t) + (\alpha-\alpha^2)Fop(\rho(t)) + (\alpha-\alpha^2)Fop(\sigma(t))$$
$$\leq Fop(a) + \int_a^t \tfrac{p^{\diamond_\alpha}(s)}{f(p(\rho(s)))}\diamond_\alpha s.$$

Since F and p are positive functions then we can write the last inequality as

$$(1-2\alpha+2\alpha^2)Fop(t) \leq Fop(a) + \int_a^t \frac{r^{\diamond_\alpha}(s)}{f\left(p(\rho(s))\right)}\diamond_\alpha s.$$

It is easily seen that $1-2\alpha+2\alpha^2$ takes its minimum value at $\alpha = \frac{1}{2}$ when $\alpha \in (0,1)$ and at that point $1-2\alpha+2\alpha^2 = \frac{1}{2}$.

Hence we get the desired result

$$\frac{1}{2}Fop(t) \leq Fop(a) + \int_a^t \frac{p^{\diamond_\alpha}(s)}{f(p(\rho(s)))}.$$

□

Using Lemma 5.5.1 above we get the following result.

Theorem 5.5.1 $u \in C_{rl}([a.b]_{\mathbb{T}}, \mathbb{R}_+)$ *satisfies for some* $k > 0$ *such that*

$$u^2(t) \leq k^2 + 2\int_a^t \left[f(s)u(s)\left\{u(s) + \int_a^s g(\tau)w(u(\tau))\diamond_\alpha \tau\right\} + h(s)u(s)\right]\diamond_\alpha s$$

Here our Times Scale is regulated and $\sigma(a) = a$. $\forall t \in [a.b]_{\mathbb{T}}$ $f, g, h \in C_{rl}([a.b]_{\mathbb{T}}, \mathbb{R}^0_+)$, $w(t) \in C(\mathbb{R}^0_+, \mathbb{R}^0_+)$ and $w(t)$ is non-decreasing,then

$$\begin{aligned}
u(t) &\leq 4k + 2c\int_a^t (1-2\alpha+2\alpha^2)h(\tau) + (\alpha-\alpha^2)h(\rho(\tau)) + (\alpha-\alpha^2)h(\sigma(\tau)) \diamond_\alpha \tau \\
&+ 2\int_a^t \Bigg[(2c(1-2\alpha+2\alpha^2)^2 + 4c(\alpha-\alpha^2)^2)f(s) + 4c(1-2\alpha+2\alpha^2)(\alpha-\alpha^2)f(\rho(s)) \\
&+ 4c(1-2\alpha+2\alpha^2)(\alpha-\alpha^2)f(\sigma(s)) + 2c(\alpha-\alpha^2)^2 f(\sigma^2(s)) + 2c(\alpha-\alpha^2)^2 f(\rho^2(s))\Bigg] \\
&\times \Bigg[E^{-1}\Bigg\{2E\Big(2k + c\int_a^{t_0}(1-2\alpha+2\alpha^2)h(\tau) + (\alpha-\alpha^2)h(\rho(\tau)) + (\alpha-\alpha^2)h(\sigma(\tau)) \diamond_\alpha \tau\Big) \\
&+ \int_a^s \Bigg[(4mc(1-2\alpha+2\alpha^2)^2 + 8mc(\alpha-\alpha^2)^2)f(\sigma^2(\tau)) + 8mc(1-2\alpha+2\alpha^2)(\alpha-\alpha^2)f(\sigma(\tau)) \\
&+ 8mc(1-2\alpha+2\alpha^2)(\alpha-\alpha^2)f(\sigma^3(\tau)) + 4mc(\alpha-\alpha^2)^2 f(\sigma^4(\tau)) + 4mc(\alpha-\alpha^2)^2 f(\tau)\Bigg] \diamond_\alpha \tau \\
&+ \int_a^s \left[2(1-2\alpha+2\alpha^2)m\, g(\sigma^2(\tau)) + 2(\alpha-\alpha^2)m\, g(\sigma(\tau)) + 2(\alpha-\alpha^2)m\, g(\sigma^3(\tau))\right] \diamond_\alpha \tau\Bigg\}\Bigg] \diamond_\alpha s.
\end{aligned}$$

where $E(r) = \int_1^r \frac{ds}{w(s)+s}$, $r > 0$. Here $m,c \geq 1$.

Proof. If we take $a = t$, then the inequality obviously hols true. Let $t_0 \in (a, b]$ and define $z(t)$ in $[a, t_0]$ such that

$$z(t) = k^2 + 2\int_a^t \left[f(s)u(s)\left\{u(s) + \int_a^s g(\tau)w(u(\tau)) \diamond_\alpha \tau\right\} + h(s)u(s)\right] \diamond_\alpha s.$$

Therefore for $t \in [a, t_0]_{\mathbb{T}}$, $u(t) \leq \sqrt{z(t)}$. By using theorem from Sheng [46] we take the diamond alpha derivative of $z(t)$ and we get

$$\begin{aligned}
z^{\diamond_\alpha}(t) &= 2(1-2\alpha+2\alpha^2)\left[f(t)u(t)\left\{u(t) + \int_a^t g(\tau)w(u(\tau)) \diamond_\alpha \tau\right\} + h(t)u(t)\right] \\
&+ 2(\alpha-\alpha^2)\left[f(\rho(t))u(\rho(t))\left\{u(\rho(t)) + \int_a^{\rho(t)} g(\tau)w(u(\tau)) \diamond_\alpha \tau\right\} + h(\rho(t))u(\rho(t))\right] \\
&+ 2(\alpha-\alpha^2)\left[f(\sigma(t))u(\sigma(t))\left\{u(\sigma(t)) + \int_a^{\sigma(t)} g(\tau)w(u(\tau)) \diamond_\alpha \tau\right\} + h(\sigma(t))u(\sigma(t))\right].
\end{aligned}$$

Since each of the functions constitude $z^{\diamond_\alpha}(t)$ are non-negative, then $z^{\diamond_\alpha}(t) \geq 0$. If we take the delta and nabla derivative of $z(t)$ we also see that $z^{\Delta}(t)$, $z^{\nabla}(t) \geq 0$. Then $z(t)$ is non decreasing, in other words $z(\sigma(t)) \geq z(t) \geq z(\rho(t))$. Since $u(t) \leq \sqrt{z(t)}$ then we have

$$\begin{aligned}
z^{\diamond_\alpha}(t) &\leq 2(1-2\alpha+2\alpha^2)\left[f(t)\sqrt{z(t)}\left\{\sqrt{z(t)} + \int_a^t g(\tau)w(\sqrt{z(\tau)}) \diamond_\alpha \tau\right\} + h(t)\sqrt{z(t)}\right] \\
&+ 2(\alpha-\alpha^2)\left[f(\rho(t))\sqrt{z(\rho(t))}\left\{\sqrt{z(\rho(t))} + \int_a^{\rho(t)} g(\tau)w(\sqrt{z(\tau)}) \diamond_\alpha \tau\right\} + h(\rho(t))\sqrt{z(\rho(t))}\right] \\
&+ 2(\alpha-\alpha^2)\left[f(\sigma(t))\sqrt{z(\sigma(t))}\left\{\sqrt{z(\sigma(t))} + \int_a^{\sigma(t)} g(\tau)w(\sqrt{z(\tau)}) \diamond_\alpha \tau\right\} + h(\sigma(t))\sqrt{z(\sigma(t))}\right].
\end{aligned}$$

Since $z(\sigma(t)) \geq z(t) \geq z(\rho(t))$, we have

$$\begin{aligned}
z^{\diamond_\alpha}(t) &\leq 2\sqrt{z(\sigma(t))}\Bigg[(1-2\alpha+2\alpha^2)\left[f(t)\left\{\sqrt{z(t)} + \int_a^t g(\tau)w(\sqrt{z(\tau)}) \diamond_\alpha \tau\right\} + h(t)\right] \\
&+ (\alpha-\alpha^2)\left[f(\rho(t))\left\{\sqrt{z(\rho(t))} + \int_a^{\rho(t)} g(\tau)w(\sqrt{z(\tau)}) \diamond_\alpha \tau\right\} + h(\rho(t))\right] \\
&+ (\alpha-\alpha^2)\left[f(\sigma(t))\left\{\sqrt{z(\sigma(t))} + \int_a^{\sigma(t)} g(\tau)w(\sqrt{z(\tau)}) \diamond_\alpha \tau\right\} + h(\sigma(t))\right]\Bigg].
\end{aligned}$$

Then we get

$$\begin{aligned}
\frac{z^{\diamond_\alpha}(t)}{2\sqrt{z(\sigma(t))}} &\leq (1-2\alpha+2\alpha^2)\left[f(t)\left\{\sqrt{z(t)}+\int_a^t g(\tau)w(\sqrt{z(\tau)})\diamond_\alpha \tau\right\}+h(t)\right]\\
&+ (\alpha-\alpha^2)\left[f(\rho(t))\left\{\sqrt{z(\rho(t))}+\int_a^{\rho(t)} g(\tau)w(\sqrt{z(\tau)})\diamond_\alpha \tau\right\}+h(\rho(t))\right]\\
&+ (\alpha-\alpha^2)\left[f(\sigma(t))\left\{\sqrt{z(\sigma(t))}+\int_a^{\sigma(t)} g(\tau)w(\sqrt{z(\tau)})\diamond_\alpha \tau\right\}+h(\sigma(t))\right].
\end{aligned} \tag{5.9}$$

Our functions are from $C_{rl}([a.b]_{\mathbb{T}},\mathbb{R}_+^0)$, then they are regulated on $[a.b]_{\mathbb{T}}$, then $z(t)$ is bounded and $z(t)$ never takes zero.

Therefore there exists c such that $c = max_{t\in[a,t_0]_{\mathbb{T}}}\frac{\sqrt{z(\sigma(t))}}{\sqrt{z(\rho(t))}}$. If we multiply (5.9) by c, we obtain

$$\begin{aligned}
\frac{z^{\diamond_\alpha}(t)}{2\sqrt{z(\rho(t))}} &\leq c(1-2\alpha+2\alpha^2)\left[f(t)\left\{\sqrt{z(t)}+\int_a^t g(\tau)w(\sqrt{z(\tau)})\diamond_\alpha \tau\right\}+h(t)\right]\\
&+ c(\alpha-\alpha^2)\left[f(\rho(t))\left\{\sqrt{z(\rho(t))}+\int_a^{\rho(t)} g(\tau)w(\sqrt{z(\tau)})\diamond_\alpha \tau\right\}+h(\rho(t))\right]\\
&+ c(\alpha-\alpha^2)\left[f(\sigma(t))\left\{\sqrt{z(\sigma(t))}+\int_a^{\sigma(t)} g(\tau)w(\sqrt{z(\tau)})\diamond_\alpha \tau\right\}+h(\sigma(t))\right].
\end{aligned}$$

Now we use the Lemma 5.5.1 and we get,

$$\begin{aligned}
\sqrt{z(t)} &\leq 2k+2c(1-2\alpha+2\alpha^2)\int_a^t\left[f(s)\left\{\sqrt{z(s)}+\int_a^s g(\tau)w(\sqrt{z(\tau)})\diamond_\alpha \tau\right\}\right]\diamond_\alpha s\\
&+ 2c(\alpha-\alpha^2)\int_a^t\left[f(\rho(s))\left\{\sqrt{z(\rho(s))}+\int_a^{\rho(s)} g(\tau)w(\sqrt{z(\tau)})\diamond_\alpha \tau\right\}\right]\diamond_\alpha s\\
&+ 2c(\alpha-\alpha^2)\int_a^t\left[f(\sigma(s))\left\{\sqrt{z(\sigma(s))}+\int_a^{\sigma(s)} g(\tau)w(\sqrt{z(\tau)})\diamond_\alpha \tau\right\}\right]\diamond_\alpha s\\
&+ \int_a^{t_0}(1-2\alpha+2\alpha^2)h(s)+(\alpha-\alpha^2)h(\rho(s))+(\alpha-\alpha^2)h(\sigma(s))\diamond_\alpha s.
\end{aligned}$$

Let us say the right hand side of the above inequality $V(t)$, then $\sqrt{z(t)}\leq V(t)$. If we take the diamond alpha derivative of $V(t)$ then we get

$$\begin{aligned}
V^{\diamond_\alpha}(t) &= (2c(1-2\alpha+2\alpha^2)^2+4c(\alpha-\alpha^2)^2)\left[f(t)\left\{\sqrt{z(t)}+\int_a^t g(\tau)w(\sqrt{z(\tau)})\diamond_\alpha \tau\right\}\right]\\
&+ 4c(1-2\alpha+2\alpha^2)(\alpha-\alpha^2)\left[f(\rho(t))\left\{\sqrt{z(\rho(t))}+\int_a^{\rho(t)} g(\tau)w(\sqrt{z(\tau)})\diamond_\alpha \tau\right\}\right]\\
&+ 4c(1-2\alpha+2\alpha^2)(\alpha-\alpha^2)\left[f(\sigma(t))\left\{\sqrt{z(\sigma(t))}+\int_a^{\sigma(t)} g(\tau)w(\sqrt{z(\tau)})\diamond_\alpha \tau\right\}\right]\\
&+ 2c(\alpha-\alpha^2)^2\left[f(\rho^2(t))\left\{\sqrt{z(\rho^2(t))}+\int_a^{\rho^2(t)} g(\tau)w(\sqrt{z(\tau)})\diamond_\alpha \tau\right\}\right]\\
&+ 2c(\alpha-\alpha^2)^2\left[f(\sigma^2(t))\left\{\sqrt{z(\sigma^2(t))}+\int_a^{\sigma^2(t)} g(\tau)w(\sqrt{z(\tau)})\diamond_\alpha \tau\right\}\right].
\end{aligned}$$

Since $\sqrt{z(t)}\leq V(t)$, then we get

$$\begin{aligned}
V^{\diamond_\alpha}(t) &\leq (2c(1-2\alpha+2\alpha^2)^2+4c(\alpha-\alpha^2)^2)\left[f(t)\left\{V(t)+\int_a^t g(\tau)w(V(\tau))\diamond_\alpha \tau\right\}\right]\\
&+ 4c(1-2\alpha+2\alpha^2)(\alpha-\alpha^2)\left[f(\rho(t))\left\{V(\rho(t))+\int_a^{\rho(s)} g(\tau)w(V(\tau))\diamond_\alpha \tau\right\}\right]\\
&+ 4c(1-2\alpha+2\alpha^2)(\alpha-\alpha^2)\left[f(\sigma(t))\left\{V(\sigma(t))+\int_a^{\sigma(t)} g(\tau)w(V(\tau))\diamond_\alpha \tau\right\}\right]\\
&+ 2c(\alpha-\alpha^2)^2\left[f(\rho^2(t))\left\{V(\rho^2(t))+\int_a^{\rho^2(t)} g(\tau)w(V(\tau))\diamond_\alpha \tau\right\}\right]\\
&+ 2c(\alpha-\alpha^2)^2\left[f(\sigma^2(t))\left\{V(\sigma^2(t))+\int_a^{\sigma^2(t)} g(\tau)w(V(\tau))\diamond_\alpha \tau\right\}\right].
\end{aligned}$$

Since each of the functions that constitude $V^{\diamond_\alpha}(t)$ are non-negative then $V^{\diamond_\alpha}(t)\geq 0$. Similarly if we take the delta and nabla derivative of $V(t)$ we also see that $V^{\Delta}(t)$,

$V^{\nabla}(t) \geq 0$. Therefore $V(\sigma^2(t)) \geq V(\sigma(t)) \geq V(t) \geq V(\rho(t)) \geq V(\rho^2(t))$. Then

$$\begin{aligned} V^{\diamond_\alpha}(t) &\leq \Big[(2c(1-2\alpha+2\alpha^2)^2 + 4c(\alpha-\alpha^2)^2)f(t) + 4c(1-2\alpha+2\alpha^2)(\alpha-\alpha^2)f(\rho(t)) \\ &+ 4c(1-2\alpha+2\alpha^2)(\alpha-\alpha^2)f(\sigma(t)) + 2c(\alpha-\alpha^2)^2 f(\sigma^2(t)) + 2c(\alpha-\alpha^2)^2 f(\rho^2(t))\Big] \\ &\times \left\{V(\sigma^2(t)) + \int_a^{\sigma^2(t)} g(\tau)w(V(\tau)) \diamond_\alpha \tau\right\}. \end{aligned}$$

Let us take $V(\sigma^2(t)) + \int_a^{\sigma^2(t)} g(\tau)w(V(\tau)) \diamond_\alpha \tau = \Omega(t)$

$$\begin{aligned} \Omega^{\diamond_\alpha}(t) &= V^{\diamond_\alpha}(\sigma^2(t)) + (1-2\alpha+2\alpha^2)g(\sigma^2(t))w(V(\sigma^2(t))) + (\alpha-\alpha^2)g(\sigma(t))w(V(\sigma(t))) \\ &+ (\alpha-\alpha^2)g(\sigma^3(t))w(V(\sigma^3(t))). \end{aligned}$$

Therefore we get

$$\begin{aligned} \Omega^{\diamond_\alpha}(t) &\leq \Big[(2c(1-2\alpha+2\alpha^2)^2 + 4c(\alpha-\alpha^2)^2)f(\sigma^2(t)) + 4c(1-2\alpha+2\alpha^2)(\alpha-\alpha^2)f(\sigma(t)) \\ &+ 4c(1-2\alpha+2\alpha^2)(\alpha-\alpha^2)f(\sigma^3(t)) + 2c(\alpha-\alpha^2)^2 f(\sigma^4(t)) + 2c(\alpha-\alpha^2)^2 f(t)\Big]\Omega(t) \\ &+ (1-2\alpha+2\alpha^2)g(\sigma^2(t))w(V(\sigma^2(t))) + (\alpha-\alpha^2)g(\sigma(t))w(V(\sigma(t))) \\ &+ (\alpha-\alpha^2)g(\sigma^3(t))w(V(\sigma^3(t))). \end{aligned}$$

Since functions in $\Omega^{\diamond_\alpha}(t)$ non-negative then $\Omega^{\diamond_\alpha}(t) \geq 0$.

Similarly $\Omega^{\Delta}(t)$, $\Omega^{\nabla}(t) \geq 0$. It is obvious that $V(\sigma^2(t)) \leq \Omega(t)$and since w is non-decreasing we also have $w(\Omega(\sigma(t))) \geq w(\Omega(t)) \geq w(\Omega(\rho(t)))$.

Therefore $w(\Omega(\sigma(t))) \geq w(V(\sigma^3(t)))$. Then

$$\begin{aligned} \Omega^{\diamond_\alpha}(t) &\leq \Big[(2c(1-2\alpha+2\alpha^2)^2 + 4c(\alpha-\alpha^2)^2)f(\sigma^2(t)) + 4c(1-2\alpha+2\alpha^2)(\alpha-\alpha^2)f(\sigma(t)) \\ &+ 4c(1-2\alpha+2\alpha^2)(\alpha-\alpha^2)f(\sigma^3(t)) + 2c(\alpha-\alpha^2)^2 f(\sigma^4(t)) + 2c(\alpha-\alpha^2)^2 f(t)\Big]\Omega(t) \\ &+ \left[(1-2\alpha+2\alpha^2)g(\sigma^2(t)) + (\alpha-\alpha^2)g(\sigma(t)) + (\alpha-\alpha^2)g(\sigma^3(t))\right] w(\Omega(\sigma(t))). \end{aligned}$$

$$\begin{aligned} \frac{\Omega^{\diamond_\alpha}(t)}{\Omega(t)+w(\Omega(\sigma(t)))} &\leq \Big[(2c(1-2\alpha+2\alpha^2)^2 + 4c(\alpha-\alpha^2)^2)f(\sigma^2(t)) + 4c(1-2\alpha+2\alpha^2)(\alpha-\alpha^2)f(\sigma(t)) \\ &+ 4c(1-2\alpha+2\alpha^2)(\alpha-\alpha^2)f(\sigma^3(t)) + 2c(\alpha-\alpha^2)^2 f(\sigma^4(t)) + 2c(\alpha-\alpha^2)^2 f(t)\Big] \\ &+ \left[(1-2\alpha+2\alpha^2)g(\sigma^2(t)) + (\alpha-\alpha^2)g(\sigma(t)) + (\alpha-\alpha^2)g(\sigma^3(t))\right]. \end{aligned} \tag{5.10}$$

Since each of the functions in $\Omega(t)$ are from $C_{rl}([a.b]_{\mathbb{T}}, \mathbb{R}_+^0)$, then $\Omega(t)$ is bounded on $[a.b]_{\mathbb{T}}$ and never takes zero.

Therefore there exists m such that $m = max_{t\in[a,t_0]_{\mathbb{T}}} \frac{\Omega(t)+w(\Omega(\sigma(t)))}{\Omega(\rho(t))+w(\Omega(\rho(t)))}$. If we multiply (5.10) by m, we have

$$\begin{aligned}
\frac{\Omega^{\diamond_\alpha}(t)}{\Omega(\rho(t))+w(\Omega(\rho(t)))} \leq & \Big[(2mc(1-2\alpha+2\alpha^2)^2+4mc(\alpha-\alpha^2)^2)f((\sigma^2(t)) \\
+ & 4mc(1-2\alpha+2\alpha^2)(\alpha-\alpha^2)f((\sigma(t)) \\
+ & 4mc(1-2\alpha+2\alpha^2)(\alpha-\alpha^2)f((\sigma^3(t))+2mc(\alpha-\alpha^2)^2f(\sigma^4(t))+2mc(\alpha-\alpha^2)^2f(t)\Big] \\
+ & \left[(1-2\alpha+2\alpha^2)m\,g(\sigma^2(t))+(\alpha-\alpha^2)m\,g(\sigma(t))+(\alpha-\alpha^2)m\,g(\sigma^3(t))\right].
\end{aligned}$$

Again we use Lemma 5.5.1 and get

$$\begin{aligned}
E(\Omega(t)) \leq & \ 2E(\Omega(a))+\int_a^t\Big[(4mc(1-2\alpha+2\alpha^2)^2+8mc(\alpha-\alpha^2)^2)f((\sigma^2(s)) \\
+ & \ 8mc(1-2\alpha+2\alpha^2)(\alpha-\alpha^2)f(\sigma(s))+8mc(1-2\alpha+2\alpha^2)(\alpha-\alpha^2)f((\sigma^3(s)) \\
+ & \ 4mc(\alpha-\alpha^2)^2f(\sigma^4(s))+4mc(\alpha-\alpha^2)^2f(s)\Big]\diamond_\alpha s \\
+ & \ \int_a^t\left[2(1-2\alpha+2\alpha^2)m\,g(\sigma^2(s))+2(\alpha-\alpha^2)m\,g(\sigma(s))+2(\alpha-\alpha^2)m\,g(\sigma^3(s))\right]\diamond_\alpha s. \\
E(\Omega(t)) \leq & \ 2E(2k+c\int_a^{t_0}(1-2\alpha+2\alpha^2)h(s)+(\alpha-\alpha^2)h(\rho(s))+(\alpha-\alpha^2)h(\sigma(s))\diamond_\alpha s) \\
+ & \ \int_a^t\Big[(4mc(1-2\alpha+2\alpha^2)^2+8mc(\alpha-\alpha^2)^2)f((\sigma^2(s)) \\
+ & \ 8mc(1-2\alpha+2\alpha^2)(\alpha-\alpha^2)f(\sigma(s))+8mc(1-2\alpha+2\alpha^2)(\alpha-\alpha^2)f((\sigma^3(s)) \\
+ & \ 4mc(\alpha-\alpha^2)^2f(\sigma^4(s))+4mc(\alpha-\alpha^2)^2f(s)\Big]\diamond_\alpha s \\
+ & \ \int_a^t\left[2(1-2\alpha+2\alpha^2)m\,g(\sigma^2(s))+2(\alpha-\alpha^2)m\,g(\sigma(s))+2(\alpha-\alpha^2)m\,g(\sigma^3(s))\right]\diamond_\alpha s. \\
\Omega(t) \leq & \ E^{-1}\Big\{2E\Big(2k+c\int_a^{t_0}(1-2\alpha+2\alpha^2)h(s)+(\alpha-\alpha^2)h(\rho(s))+(\alpha-\alpha^2)h(\sigma(s))\diamond_\alpha s\Big) \\
+ & \ \int_a^t\Big[(4mc(1-2\alpha+2\alpha^2)^2+8mc(\alpha-\alpha^2)^2)f(\sigma^2(s)) \\
+ & \ 8mc(1-2\alpha+2\alpha^2)(\alpha-\alpha^2)f(\sigma(s))+8mc(1-2\alpha+2\alpha^2)(\alpha-\alpha^2)f(\sigma^3(s)) \\
+ & \ 4mc(\alpha-\alpha^2)^2f(\sigma^4(s)+4mc(\alpha-\alpha^2)^2f(s)\Big]\diamond_\alpha s \\
+ & \ \int_a^t\left[2(1-2\alpha+2\alpha^2)m\,g(\sigma^2(s))+2(\alpha-\alpha^2)m\,g(\sigma(s))+2(\alpha-\alpha^2)m\,g(\sigma^3(s))\right]\diamond_\alpha s\Big\}.
\end{aligned}$$

$$\begin{aligned}
V^{\diamond_\alpha}(t) \leq & \Big[(2c(1-2\alpha+2\alpha^2)^2+4c(\alpha-\alpha^2)^2)f(t)+4c(1-2\alpha+2\alpha^2)(\alpha-\alpha^2)f(\rho(t)) \\
+ & \ 4c(1-2\alpha+2\alpha^2)(\alpha-\alpha^2)f(\sigma(t))+2c(\alpha-\alpha^2)^2f(\sigma^2(t))+2c(\alpha-\alpha^2)^2f(\rho^2(t))\Big] \\
\times & \ E^{-1}\Big\{2E\Big(2k+c\int_a^{t_0}(1-2\alpha+2\alpha^2)h(s)+(\alpha-\alpha^2)h(\rho(s))+(\alpha-\alpha^2)h(\sigma(s))\diamond_\alpha s\Big) \\
+ & \ \int_a^t\Big[(4mc(1-2\alpha+2\alpha^2)^2+8mc(\alpha-\alpha^2)^2)f(\sigma^2(s)) \\
+ & \ 8mc(1-2\alpha+2\alpha^2)(\alpha-\alpha^2)f(\sigma(s))+8mc(1-2\alpha+2\alpha^2)(\alpha-\alpha^2)f(\sigma^3(s)) \\
+ & \ 4mc(\alpha-\alpha^2)^2f(\sigma^4(s)+4mc(\alpha-\alpha^2)^2f(s)\Big]\diamond_\alpha s \\
+ & \ \int_a^t\left[2(1-2\alpha+2\alpha^2)m\,g(\sigma^2(s))+2(\alpha-\alpha^2)m\,g(\sigma(s))+2(\alpha-\alpha^2)m\,g(\sigma^3(s))\right]\diamond_\alpha s\Big\}.
\end{aligned} \tag{5.11}$$

If we integrate $V^{\diamond_\alpha}(t)$ from a to t, we have

$$\int_a^t V^{\diamond_\alpha}(s)\diamond_\alpha s = (1-2\alpha+2\alpha^2)V(t)+(\alpha-\alpha^2)V(\rho(t))+(\alpha-\alpha^2)V(\sigma(t))-V(a).$$

If we integrate (5.11) from a to t, we obtain

$$\begin{aligned}
&(1-2\alpha+2\alpha^2)V(t)+(\alpha-\alpha^2)V(\rho(t))+(\alpha-\alpha^2)V(\sigma(t)) \leq \\
&V(a)+\int_a^t \Bigg[(2c(1-2\alpha+2\alpha^2)^2+4c(\alpha-\alpha^2)^2)f(s)+4c(1-2\alpha+2\alpha^2)(\alpha-\alpha^2)f(\rho(s)) \\
&+4c(1-2\alpha+2\alpha^2)(\alpha-\alpha^2)f(\sigma(s))+2c(\alpha-\alpha^2)^2f(\sigma^2(s))+2c(\alpha-\alpha^2)^2f(\rho^2(s))\Bigg] \\
&\times\Bigg[E^{-1}\Bigg\{2E\Big(2k+c\int_a^{t_0}(1-2\alpha+2\alpha^2)h(\tau)+(\alpha-\alpha^2)h(\rho(\tau))+(\alpha-\alpha^2)h(\sigma(\tau))\diamond_\alpha\tau\Big) \\
&+\int_a^s\Bigg[(4mc(1-2\alpha+2\alpha^2)^2+8mc(\alpha-\alpha^2)^2)f(\sigma^2(\tau)) \\
&+8mc(1-2\alpha+2\alpha^2)(\alpha-\alpha^2)f(\sigma(\tau)) \\
&+8mc(1-2\alpha+2\alpha^2)(\alpha-\alpha^2)f(\sigma^3(\tau))+4mc(\alpha-\alpha^2)^2f(\sigma^4(\tau))+4mc(\alpha-\alpha^2)^2f(\tau)\Bigg]\diamond_\alpha\tau \\
&+\int_a^s\left[2(1-2\alpha+2\alpha^2)m\,g(\sigma^2(\tau))+2(\alpha-\alpha^2)m\,g(\sigma(\tau))+2(\alpha-\alpha^2)m\,g(\sigma^3(\tau))\right]\diamond_\alpha\tau\Bigg\}\Bigg]\diamond_\alpha s.
\end{aligned}$$

Since $V(\rho(t))$,$V(\sigma(t))\geq 0$ and $\alpha\in(0,1)$ we can omit them and get

$$\begin{aligned}
&(1-2\alpha+2\alpha^2)V(t)\leq V(a)+\int_a^t \Bigg[(2c(1-2\alpha+2\alpha^2)^2+4c(\alpha-\alpha^2)^2)f(s) \\
&+4c(1-2\alpha+2\alpha^2)(\alpha-\alpha^2)f(\rho(s)) \\
&+4c(1-2\alpha+2\alpha^2)(\alpha-\alpha^2)f(\sigma(s))+2c(\alpha-\alpha^2)^2f(\sigma^2(s))+2c(\alpha-\alpha^2)^2f(\rho^2(s))\Bigg] \\
&\times\Bigg[E^{-1}\Bigg\{2E\Big(2k+c\int_a^{t_0}(1-2\alpha+2\alpha^2)h(\tau)+(\alpha-\alpha^2)h(\rho(\tau))+(\alpha-\alpha^2)h(\sigma(\tau))\diamond_\alpha\tau\Big) \\
&+\int_a^s\Bigg[(4mc(1-2\alpha+2\alpha^2)^2+8mc(\alpha-\alpha^2)^2)f(\sigma^2(\tau)) \\
&+8mc(1-2\alpha+2\alpha^2)(\alpha-\alpha^2)f(\sigma(\tau)) \\
&+8mc(1-2\alpha+2\alpha^2)(\alpha-\alpha^2)f(\sigma^3(\tau))+4mc(\alpha-\alpha^2)^2f(\sigma^4(\tau))+4mc(\alpha-\alpha^2)^2f(\tau)\Bigg]\diamond_\alpha\tau \\
&+\int_a^s\left[2(1-2\alpha+2\alpha^2)m\,g(\sigma^2(\tau))+2(\alpha-\alpha^2)m\,g(\sigma(\tau))+2(\alpha-\alpha^2)m\,g(\sigma^3(\tau))\right]\diamond_\alpha\tau\Bigg\}\Bigg]\diamond_\alpha s.
\end{aligned}$$

Since $1-2\alpha+2\alpha^2$ can minimum be $\frac{1}{2}$, we have

$$\begin{aligned}
\tfrac{1}{2}V(t) \;\leq\;& 2k + c\textstyle\int_a^{t_0}(1-2\alpha+2\alpha^2)h(\tau)+(\alpha-\alpha^2)h(\rho(\tau))+(\alpha-\alpha^2)h(\sigma(\tau))\diamond_\alpha\tau \\
+\;& \textstyle\int_a^t \Big[(2c(1-2\alpha+2\alpha^2)^2+4c(\alpha-\alpha^2)^2)f(s)+4c(1-2\alpha+2\alpha^2)(\alpha-\alpha^2)f(\rho(s)) \\
+\;& 4c(1-2\alpha+2\alpha^2)(\alpha-\alpha^2)f(\sigma(s))+2c(\alpha-\alpha^2)^2f(\sigma^2(s))+2c(\alpha-\alpha^2)^2f(\rho^2(s))\Big] \\
\times\;& \Big[E^{-1}\Big\{2E\Big(2k+c\textstyle\int_a^{t_0}(1-2\alpha+2\alpha^2)h(\tau)+(\alpha-\alpha^2)h(\rho(\tau))+(\alpha-\alpha^2)h(\sigma(\tau))\diamond_\alpha\tau\Big) \\
+\;& \textstyle\int_a^s \Big[(4mc(1-2\alpha+2\alpha^2)^2+8mc(\alpha-\alpha^2)^2)f(\sigma^2(\tau)) \\
+\;& 8mc(1-2\alpha+2\alpha^2)(\alpha-\alpha^2)f(\sigma(\tau)) \\
+\;& 8mc(1-2\alpha+2\alpha^2)(\alpha-\alpha^2)f(\sigma^3(\tau))+4mc(\alpha-\alpha^2)^2f(\sigma^4(\tau))+4mc(\alpha-\alpha^2)^2f(\tau)\Big]\diamond_\alpha\tau \\
+\;& \textstyle\int_a^s\left[2(1-2\alpha+2\alpha^2)m\,g(\sigma^2(\tau))+2(\alpha-\alpha^2)m\,g(\sigma(\tau))+2(\alpha-\alpha^2)m\,g(\sigma^3(\tau))\right]\diamond_\alpha\tau\Big\}\Big]\diamond_\alpha s.
\end{aligned}$$

Since $u(t)\leq\sqrt{z(t)}\leq V(t)$ we get

$$\begin{aligned}
u(t) \;\leq\;& 4k + 2c\textstyle\int_a^{t_0}(1-2\alpha+2\alpha^2)h(\tau)+(\alpha-\alpha^2)h(\rho(\tau))+(\alpha-\alpha^2)h(\sigma(\tau))\diamond_\alpha\tau \\
+\;& 2\textstyle\int_a^t \Big[(2c(1-2\alpha+2\alpha^2)^2+4c(\alpha-\alpha^2)^2)f(s) \\
+\;& 4c(1-2\alpha+2\alpha^2)(\alpha-\alpha^2)f(\rho(s)) \\
+\;& 4c(1-2\alpha+2\alpha^2)(\alpha-\alpha^2)f(\sigma(s))+2c(\alpha-\alpha^2)^2f(\sigma^2(s))+2c(\alpha-\alpha^2)^2f(\rho^2(s))\Big] \\
\times\;& \Big[E^{-1}\Big\{2E\Big(2k+c\textstyle\int_a^{t_0}(1-2\alpha+2\alpha^2)h(\tau)+(\alpha-\alpha^2)h(\rho(\tau))+(\alpha-\alpha^2)h(\sigma(\tau))\diamond_\alpha\tau\Big) \\
+\;& \textstyle\int_a^s \Big[(4mc(1-2\alpha+2\alpha^2)^2+8mc(\alpha-\alpha^2)^2)f(\sigma^2(\tau)) \\
+\;& 8mc(1-2\alpha+2\alpha^2)(\alpha-\alpha^2)f(\sigma(\tau)) \\
+\;& 8mc(1-2\alpha+2\alpha^2)(\alpha-\alpha^2)f(\sigma^3(\tau))+4mc(\alpha-\alpha^2)^2f(\sigma^4(\tau))+4mc(\alpha-\alpha^2)^2f(\tau)\Big]\diamond_\alpha\tau \\
+\;& \textstyle\int_a^s\left[2(1-2\alpha+2\alpha^2)m\,g(\sigma^2(\tau))+2(\alpha-\alpha^2)m\,g(\sigma(\tau))+2(\alpha-\alpha^2)m\,g(\sigma^3(\tau))\right]\diamond_\alpha\tau\Big\}\Big]\diamond_\alpha s.
\end{aligned}$$

Since we choose t_0 arbitrary then we can take $t=t_0$. Hence we get the desired result.

□

Example 5.5.1 *If we choose $\alpha=1$, $\mathbb{T}=\mathbb{R}$, $m=c=1/2$, then Theorem 5.5.1 becomes Theorem 5.2.1.*

Example 5.5.2 *Let $\mathbb{T}=\{0\}\bigcup\{\frac{1}{n+1}:n\in\mathbb{N}\}\bigcup\mathbb{N}$. Here we assume that $\mathbb{N}$ starts with 1. It is obvious that our time scales is regulated and $0=\sigma(0)$. Let us choose $M\in\mathbb{N}$ and investigate the below equality on the time scales $[0,M]_{\mathbb{T}}$.*

$$u^2(t)=k^2+2\int_0^t h(s)u(s)\diamond_\alpha s$$

$h(t) = 0$ if $t \in \{0\} \bigcup \{\frac{1}{n+1} : n \in \mathbb{N}\}$ and $h(t) = P(t)$ if $t \in \mathbb{N}$. Therefore $u(t) = k$ if $t \in \{0\} \bigcup \{\frac{1}{n+1} : n \in \mathbb{N}\}$, $u^2(1) = k^2 + (1-\alpha)P(1)u(1)$, ...,

$u^2(n) = k^2 + P(1)u(1) + P(2)u(2) + ... + P(n-1)u(n-1) + (1-\alpha)P(n)u(n)$,............,

$u^2(M) = k^2 + P(1)u(1) + P(2)u(2) + ... + P(M-1)u(M-1) + (1-\alpha)P(M)u(M)$

If we apply Theorem 5.5.1 we get the bound for $u(t)$ such that

$$\begin{aligned} u(t) &\leq 4k + 2c\alpha \sum_{\tau=1}^{t-1} (1-2\alpha+2\alpha^2)P(\tau) + (\alpha-\alpha^2)P(\tau-1) + (\alpha-\alpha^2)P(\tau+1) \\ &+ 2c(1-\alpha) \sum_{\tau=1}^{t} (1-2\alpha+2\alpha^2)P(\tau) + (\alpha-\alpha^2)P(\tau-1) + (\alpha-\alpha^2)P(\tau+1) \end{aligned}$$

$$\begin{aligned} u(t) &\leq 4k + 2c \sum_{\tau=1}^{t-1} (1-2\alpha+2\alpha^2)P(\tau) + (\alpha-\alpha^2)P(\tau-1) + (\alpha-\alpha^2)P(\tau+1) \\ &+ 2c(1-\alpha) \left[(1-2\alpha+2\alpha^2)P(t) + (\alpha-\alpha^2)P(t-1) + (\alpha-\alpha^2)P(t+1)\right] \end{aligned}$$

where

$$c = \max_{n\in[1,M]} \left\{ \sqrt{1 + \frac{2(1-\alpha)P(1)u(1)}{k^2}}, \sqrt{1 + \frac{2P(1)u(1)+2(1-\alpha)P(2)u(2)}{k^2}}, ..., \right.$$
$$\sqrt{1 + \frac{2P(n-1)u(n-1)+2(1-\alpha)P(n)u(n)}{k^2+2P(1)u(1)+2P(2)u(2)....+2P(n-2)u(n-2)}}, ..., $$
$$\left. \sqrt{1 + \frac{2P(M-1)u(M-1)+2(1-\alpha)P(M)u(M)}{k^2+2P(1)u(1)+.....+2P(M-2)u(M-2)}} \right\}.$$

If we take $\alpha = \frac{1}{2}$, $k^2 = 2$, then for the equation $u^2(t) = 2 + 2\int_0^t h(s)u(s) \diamond_\alpha s$, where $h(s) = 0$, if $t \in \{0\} \bigcup \{\frac{1}{n+1} : n \in \mathbb{N}\}$ and $h(s) = 1$ if $t \in \mathbb{N}$, then $c = \frac{3}{2}$ and $u(t)$ is bound with $u(t) \leq \sqrt{2}$ if $t \in \{0\} \bigcup \{\frac{1}{n+1} : n \in \mathbb{N}\}$ and $u(n) \leq 4\sqrt{2} + 3n - \frac{3}{2}$, if $n \in \mathbb{N}$.

5.6 An Application on Water Percolation Equation on Nabla Time Scales Calculus

Consider the equality

$$(x(t))^\nabla = F\left(t, x(t), \int_0^t \hat{L}[t, s, x(s)] \nabla s\right)$$

such that $F : C^1_{ld}([0, M]_\mathbb{T}, \mathbb{R}) \to C_{ld}([0, M]_\mathbb{T}, \mathbb{R})$.

Let $x(0) = \sqrt{L}$, $F(t, x, y) = y + \begin{cases} H(\nu(t))\frac{\sqrt{x(t)}}{2}, \; if \; x \geq 0 \\ 0, \; if \; x < 0 \end{cases}$,

$$\hat{L}(t,s,x) = \begin{cases} H^{\nabla}(t-s)\frac{\sqrt{x(s)}}{2},\ if\ x \geq 0 \\ 0,\ if\ x < 0 \end{cases}$$

If we denote $\sqrt{x(t)} = u(t)$, then we obtain

$$(x(t))^{\nabla} = H(\nu(t))\frac{\sqrt{x(t)}}{2} + \int_0^t H^{\nabla}(t-s)\frac{\sqrt{x(s)}}{2},$$

$$(u^2(t))^{\nabla} = \Big(\int_0^t H(t-s)\tfrac{u(s)}{2}\nabla s\Big)^{\nabla}$$

If we take the nabla integral of the last equality we have,

$$u^2(t) = L + \int_0^t H(t-s)\frac{u(s)}{2}\nabla s.$$

Then by Theorem 5.4.1

$$u(t) \leq \sqrt{L} + \alpha\frac{\int_0^t H(t-s)}{4}\nabla s,$$

where α is the constant in the Theorem 5.4.1.

The norm on $C^1_{ld}([0,M]_{\mathbb{T}},\mathbb{R})$ is defined as for $u(t) \in C^1_{ld}([0,M]_{\mathbb{T}},\mathbb{R})$,

$$||u(t)||_1 = max_{[0,M]_{\mathbb{T}}}||u(t)|| + max_{[0,M]_{\mathbb{T}}}||u^{\nabla}(t)||.$$

Define the linear operator $L : C^1_{ld}([0,M]_{\mathbb{T}},\mathbb{R}) \to C_{ld}([0,M]_{\mathbb{T}},\mathbb{R})$ such that

$Lu(t) = u^{\nabla}(t)$. By Theorem 1.65 in [8] L has a bounded inverse L^{-1}.

Let $F_{\lambda} : C^1_{ld}([0,M]_{\mathbb{T}},\mathbb{R}) \to C_{ld}([0,M]_{\mathbb{T}},\mathbb{R})$ and $1 \geq \lambda \geq 0$ such that

$$F_{\lambda}(x(t)) = \lambda F(t, x(t), \int_0^t \hat{L}[t,s,x(s)]\nabla s.$$

and

$$H(x,\lambda) = L^{-1}F_{\lambda}u,\ u \in C^1_{ld}([0,M]_{\mathbb{T}},\mathbb{R}),\ \lambda \in [0,1]$$

This homotopy is compact and is also fixed point free on the boudary of O such that

$$O = \left\{x \in C^1_{ld}([0,M]_{\mathbb{T}} : ||x||_1 \leq 1 + \tilde{M} + sup_{t\in[0,M]_{\mathbb{T}} and |x| \leq \tilde{M}}\left\{\left|\left(F(t,x(t), \int_0^t \hat{L}[t,s,x(s)]\nabla s\right)\right|\right\}\right\},$$

where $\tilde{M} = \sqrt{L} + \alpha\frac{\int_0^t H(t-s)}{4}\nabla s$. Since $H(.,0)$ is the zero map it is essential so $H(.,1)$ is also essential. Then by topological transversality theorem in [14] $H(.,1)$ has a fixed point which is the solution of the above system.

Investigating uniqueness of the solution we will use Gronwall inequality for nabla differentiation. First, let $u(t)$, $v(t)$ be the solutions of the given equation, and let $z(t) = |u(t) - v(t)|$ and $z(0) = 0$, then

$$z^2(t) = \int_0^t H(t-s)\frac{z(s)}{2}\nabla s.$$

Since by Theorem 5.4.1 $u(t), v(t) \leq \sqrt{L}$.

By Theorem 1.65 in [8] $max_{t\in[0,M]}H(t) \leq \beta$. Hence we get

$$2\sqrt{L}z(t) \leq \beta \int_0^t \frac{z(s)}{2}\nabla s.$$

If we take nabla derivative both sides of the inequality we get,

$$z^{\nabla}(t) \leq \frac{\beta}{4\sqrt{L}}z(t).$$

Then by [34] Gronwall inequality for nabla differentiation $z(t) \leq 0$, hence there is unique solution.

CHAPTER 6

CONCLUSION

This thesis is devoted to applications of generalized integral inequalities to specific areas of biological mathematics and physics. The important notion in this thesis is time scales calculus. Especially in population sciences such organisms like insects does not have a continuous life cycle. For instance, they live a season, but the following season because of the weather conditions they could not live. In such a case, time scales calculus becomes a useful tool. Because such a situation can be expressed mathematically by this notion. Therefore in chapter 2, chapter 3 and chapter 4, we have studied some properties of predator-prey dynamical systems on a general time scales and our finding are important for the organisms whose life cycle is unusual. The conditions which are expressed by the generalized integral inequalities in Lemma 2.3.2, Lemma 2.3.3, Theorem 2.4.1 and Theorem 4.2.1 are important to determine the permanence, globally attractivity and periodicity of the solutions the given system. Also in our thesis we have studied the impulsive case of the given predator-prey dynamic system and we have shown how important the impulses are. Lemma 3.1.1 shows apparently the effect of the impulses on prey and predator that can cause extinction or rescue the species from the extiction. In addition to these, because of the impacts of the impulses on the inequalities in Theorem 3.2.1, Theorem 3.2.2, Theorem 3.2.3 and Lemma 3.2.6, impulses have also an important effect on the periodicity, permenance and global attractivity of the solutions. Additionally, if the results that we have find in the third chapter which are related with the globally attractive periodic solutions of the given system for the continuous case can be generalized to any time scales, this will also be an important study.

And in the fifth chapter we have deal with a special inequality, Constantin's Inequality. We have generalized this inequality to the nabla and diamond-alpha calculus. Although we does not have chance to study about the numeric explanation of the diamond alpha Constantin's Inequality, diamond alpha generalization of this inequality is very beneficial to get more accurate results for the numeric solutions of the water percolation equation or such kind of integro-diffrential equations that are convenient to the application of Constantin's inequality. Also in [46] it was shown that solving an integral numerically by using diamond-alpha notion, we can be able to get a more accurate result. Therefore by using the diamond-alpha forms of the equations more accurate numeric results can be obtained and because of this reason our generalizations become significant.

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