EXPLORING MIDDLE SCHOOL MATHEMATICS TEACHERS' TREATMENT OF RATIONAL NUMBER EXAMPLES IN THEIR CLASSROOMS: A MULTIPLE CASE STUDY

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ABSTRACT

EXPLORING MIDDLE SCHOOL MATHEMATICS TEACHERS' TREATMENT OF RATIONAL NUMBER EXAMPLES IN THEIR CLASSROOMS: A MULTIPLE CASE STUDY

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The purpose of this study was to explore middle school mathematics teachers' treatment of rational number examples in their seventh grade classrooms. The data were collected from four middle school mathematics teachers who were teaching in different public schools in Aksaray during the fall semester of 2013-2014 education year. Data were mainly based on classroom observations, post lessons interviews, the student textbook and the middle school mathematics curriculum.

The analysis of data revealed that teachers used 704 mathematically correct and 14 mathematically incorrect examples during the teaching of rational number concepts. Among the correct examples, 361 of them were spontaneous and 343 of them were pre-planned. Besides, teachers used 9 non-examples and 5 counter-examples. More importantly, findings showed that teachers employed the following principles or considerations when choosing or using rational number examples: starting with a simple or familiar case; drawing attention to students' difficulty, error or misconception; keeping unnecessary work to minimum; taking account of examinations; including uncommon cases; and drawing attention to relevant features. Finally, this study revealed teachers' three different poor choices of examples as mathematically incorrect examples, examples with improper language or terminology, and examples that are to be avoided in the teaching of rational number concepts.

The findings of the study suggested that mathematics teachers could be provided information or training with different uses of examples in the mathematics classroom in order to enhance students' learning experiences. The effects of national policies were also discussed.

Keywords: Middle School Mathematics Teachers, Rational Number Concepts, Mathematical Examples, Teacher Considerations or Principles for Choosing Examples

ÖZ

ORTAOKUL MATEMATİK ÖĞRETMENLERİNİN RASYONEL SAYI ÖRNEKLERİNİ SINIF ORTAMINDA ELE ALIŞ BİÇİMLERİNİN İNCELENMESİ: ÇOKLU DURUM ÇALIŞMASI

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Bu çalışmanın amacı ortaokul matematik öğretmenlerinin rasyonel sayı örneklerini sınıf ortamında nasıl ele aldıklarını incelemektir. Çalışmanın verileri 2013-2014 eğitim öğretim yılının güz döneminde Aksaray ilindeki farklı devlet okullarında görev yapmakta olan dört matematik öğretmeninden toplanmıştır. Çalışmanın veri kaynaklarını sınıf içi gözlemler, yarı yapılandırılmış görüşmeler, öğrenci ders kitabı ve ortaokul matematik öğretim programı oluşturmuştur.

Toplanan verilerin analizleri öğretmenlerin rasyonel sayı kavramlarını öğretirken matematiksel olarak doğru olan 704 örnek, matematiksel olarak doğru olmayan 14 örnek kullandıklarını göstermiştir. Matematiksel olarak doğru olan örneklerin 361 tanesinin spontane (anlık) örnek olarak kullanıldığı 343 tanesinin ise planlanmış örnek olarak kullanıldığı görülmüştür. Ayrıca, öğretmenler rasyonel sayı kavramlarının öğretiminde 9 örnek olmayan ve 5 karşıt örnek kullanmıştır. Matematik öğretmenleri rasyonel sayı örneklerini seçerken veya kullanırken altı farklı prensibi/hususu göz önünde bulundurmuşlardır. Bunlar, kolay ya da bilinen örneklerden başlama; yaygın öğrenci güçlüklerine, hatalarına ya da kavram yanılgılarına dikkat çekme; gereksiz iş yükünü en aza indirme; sınavları dikkate alma; yaygın olmayan örnekleri sınıf ortamına dâhil etme ve örneklerin kritik özelliklerine dikkat çekme şeklinde olmuştur. Son olarak bu çalışmada öğretmenlerin matematiksel olarak hatalı örnekler, kullanılan dil ve terminoloji açısından uygun olmayan örnekler ve pedagojik açıdan kaçınılması gereken örnekler şeklinde üç tür uygun olmayan örnek kullandıkları ortaya çıkmıştır.

Çalışmanın sonuçları matematik öğretmenlerinin matematik dersinde öğrencilerin öğrenmelerini zenginleştirmek için örneklerin farklı kullanımları hakkında bilgilendirilmelerinin ya da eğitim almalarının yerinde olabileceğini ortaya çıkarmıştır. Eğitim politikalarının etkileri de tartışılmıştır.

Anahtar Kelimeler: Ortaokul Matematik Öğretmenleri, Rasyonel Sayı Kavramları, Matematiksel Örnekler, Öğretmenlerin Örnek Seçimlerine Yönelik Prensipleri

To my parents
For their care, support, and encouragement
To my wife, Seher
For her love, patience, and understanding

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LIST OF ABBREVIATIONS

MoNE: Ministry of National Education

CCSSM: Common Core State Standards for Mathematics

NCTM: National Council of Teachers of Mathematics

CHAPTER I

INTRODUCTION

Examples play a central role in mathematics education (Antonini, 2011; Goldenberg & Mason, 2008; Rowland, 2008; Zaslavsky & Zodik, 2007; Zaslavsky, 2010; Zazkis & Chernoff, 2008; Zazkis & Leikin, 2007; Zazkis & Leikin, 2008; Zodik & Zaslavsky, 2008). Examples are intensely used in the teaching and learning of mathematics, in designing curriculum and in inventing (Zazkis & Chernoff, 2008). More specifically, they are essential for conceptualization, generalization, abstraction, argumentation, and analogical reasoning (Zaslavsky & Zodik, 2007). The importance of examples and exemplification in mathematical thinking, learning, and teaching is well recognized not only by mathematics educators but also by mathematicians and epistemologists (Antonini, Presmeg, Mariotti & Zaslavsky, 2011). For instance, Polya (1945) explicitly suggested learners to generate examples in problem solving and similarly, a renowned mathematician, Halmos (1983) emphasized that "A good stock of examples, as large as possible, is indispensable for a thorough understanding of any concept, and when I want to learn something new, I make it my first job to build one…" (p. 63). From an epistemological standpoint, Lakatos (1976) claimed that the generation and analysis of examples can be regarded as one of the most prominent activities in the development of mathematics as a science.

Examples are fundamental tools that are used for illustrating and communicating concepts between teachers and learners (Bills, Mason, Watson & Zaslavsky, 2006). In addition, they play a crucial role as a communication tool intrinsic to explanations and mathematical discourse (Leinhardt, 2001). As stated by Leinhardt, Zaslavsky and Stein (1990), constructing explanations in the course of teaching is a very difficult work that depends on the specific choice of examples to a great extent. They further noted that:

> "Explanations consist of the orchestrations of demonstrations, analogical representations, and examples… A primary feature of explanations is the use of well-constructed examples, examples that make the point but limit the generalization, examples that are balanced by non- or counter-cases" (p. 6).

Teachers may use examples in the teaching of mathematics for two different purposes (Rowland, Turner, Thwaites & Huckstep, 2009; Zodik & Zaslavsky, 2008). First, they may use an example of a concept or procedure as a particular instance of generality, that is to say, as an example of something (Mason & Pimm, 1984; Rowland et al., 2009; Watson & Mason, 2005; Zodik & Zaslavsky, 2008). Here, the 'something' is typically general such as the notion of a rational number or the procedure for converting repeating decimals into common fractions. Hence, teachers customarily use examples for representing abstract mathematical concepts or illustrating general procedures as a pedagogical practice (Rowland, 2008). Second, teachers may use examples as an example for 'something' and examples in this case are usually called exercises (Rowland, 2008; Watson, Mason, 2005). Exercise examples are often chosen from a large number of possible examples and are primarily used to support retention of a procedure by repeating and to gain fluency with it (Rowland et al., 2009).

In mathematics, there are other kinds of examples such as non-examples and counter-examples (Watson & Mason, 2005). Non-examples are related with conceptualization and definitions, and draw attention to critical attributes of mathematical concepts (Zodik & Zaslavsky, 2008). Besides, they show the boundaries or necessary conditions of a concept (Watson & Mason, 2005). In other words, they "serve to clarify boundaries" of a concept (Bills et al., 2006, p. 127). Thus, by their very nature, non-examples are intrinsic to concept formation (Tsamir, Tirosh & Levenson, 2008). Counter-examples are related with claims and their rebuttals (Zodik & Zaslavsky, 2008). Simply put, they show that a statement is not true and sharpen the distinctions between mathematical concepts (Michener, 1978).

Bills et al. (2006) stated that in order for a mathematical example to be pedagogically useful, it should possess two main features as transparency and generalizability. Meanwhile, they described the two terms as:

> "Transparency: making it relatively easy to direct the attention of the target audience to the features that make it exemplary. Generalizability: the scope

> for generalization afforded by the example or set of examples, in terms of what is necessary to be an example, and what is arbitrary and changeable" (ibid, p.135).

The transparency of an example, its interpretation and the features noticed by a learner are subjective and context dependent and thus, teachers play an important role in introducing a wide range of pedagogically useful examples to their students in order to deal with different needs and characteristics of learners (Zaslavsky, 2010). However, choosing examples is not a simple task and it involves numerous complex considerations some of which can be done beforehand with careful planning while some others can only be done in the course of actual classroom practice (Zodik & Zaslavsky, 2008).

According to Bills and Rowland (1999) examples may not always achieve their intended purposes. Similarly, Mason and Pimm (1984) asserted that there may be a mismatch between teacher intention and what students pay attention to. This may, to some extent, have to do with the irrelevant information carried by examples in addition to their relevant attributes (Zaslavsky & Zodik, 2014). Skemp (1987) used the term 'noise' for this irrelevant information carried by the examples. Skemp (1971) claimed that if the noise in an example increases, then it becomes more difficult to form a concept. Thus, students may focus on irrelevant aspects of examples although teachers may try to instantiate certain mathematical ideas from his/her own perspective (Zaslavsky & Zodik, 2007). Hence, "the examples provided by a teacher ought, ideally, to be the outcome of a careful process of choice, a deliberate and informed selection, because some are simply better than others" (Rowland, 2014, p. 98).

Inspired by Marton and Booth's (1997) notion of 'dimensions of variation', Watson and Mason (2006) derived the notions of 'dimensions of possible variation' and 'range of permissible change' to gain insights into the pedagogical role of examples. They assumed that discerning variations within any mathematical object is a starting point for making sense of it. Besides, they suggested that teachers can uncover the mathematical structure of any object by varying some of its features while keeping other features constant. Mathematical structure means "the identification of general properties which are instantiated in particular situations such as relationships between elements" (Mason, Stephens & Watson, 2009, p. 10). As a consequence, learners are compelled to discern the structure and generalize because "learners cannot resist looking for, or imposing pattern, and hence creating generalizations, even if these are not expressed or recognized" (Watson and Mason, 2006, p. 95). This suggestion is of particular importance since it can be used to emphasize and distinguish critical and non-critical attributes of mathematical objects.

As mentioned before, choice and use of examples is an important and complex domain (Zaslavsky, 2010). Thus, teachers need to take up the challenge of choosing judicious examples since their choices of examples have the potential to support or impede mathematical learning (Zaslavsky & Zodik, 2007). In this sense, it can be said that teachers' choice and use of examples influence and give shape to students' learning process. Hence, in-depth exploration of the quality of mathematical examples employed by the teachers might give some insights into the quality of actual classroom practices. In addition to this, the selection and use of examples may present the teachers with actual classroom events that constitute learning opportunities for them and that would affect their future choice and use of examples (Zaslavsky & Zodik, 2007; Zodik & Zaslavsky, 2009).

Teachers continuously respond to their students' interests and inquiries as part of the ongoing classroom interaction and it is not possible for teachers to know how each student will react or respond to any situation, therefore teachers may quite often need to make split-second decisions in the course of lessons (Rowland et al., 2009). The immediate actions on the part of teachers in such classroom situations reflect teachers' ability to think 'on their feet' (Schon, 1987). While Mason and Spence (1999) coined the term 'knowing-to act in the moment' for this type of decision making, Rowland, Huckstep and Thwaites (2005) dealt with such in-the-moment actions by means of the contingency dimension of the Knowledge Quartet. Selecting or constructing mathematics examples for teaching usually entails in-the-moment decisions in return for classroom interactions and it is closely associated with teachers' increasing awareness and ongoing reflection (Zodik & Zaslavsky, 2008). Thus, this study sought to determine spontaneous and pre-planned examples generated or selected by the middle school mathematics teachers.

Teachers' choices of pre-planned and spontaneous examples reflect their underlying considerations or principles in choosing those examples and enable them to become more aware of their planning and in-the moment actions (Zaslavsky & Zodik, 2008). Thus, the current study also sought to examine middle school mathematics teachers' considerations or principles that guide them in selecting or constructing examples.

Despite the fact that examples play a crucial role in the teaching and learning of mathematics (Zaslavsky, 2010), there are some common pitfalls in the selection of examples (Rowland, 2008). According to Rowland, Thwaites and Huckstep (2003) there are three kinds of examples that should be avoided in the teaching of mathematics:

> "examples that obscure the role of variables within it; examples intended to illustrate a particular procedure, for which another procedure would be more sensible; and examples for instruction (as opposed to exercise examples) being randomly generated, typically by dice, at a point when it would be preferable for the teacher to be making careful choices" (p. 245).

In this study, not only middle school mathematics teachers' well chosen-examples but also their poor choices of examples were taken into consideration. Zaslavsky and Zodik (2007) argued that teachers' poor choices of examples might be deliberately incorporated into the classroom as part of learning so as to question students' mathematical thinking. Furthermore, Zodik and Zaslavsky (2008) suggested that classroom events that include both good and poor examples might serve for teacher education programs and professional development activities. Thus, the findings of this study may be helpful for pre-service and in-service teachers in gaining practical knowledge about treatment of mathematical examples in their classrooms.

1.1. Rational Number Concepts in Turkish School Mathematics Curricula

Rational number concepts are among the most important mathematical ideas students encounter in their school years (Alacacı, 2009; Behr, Lesh, Post, Silver, 1983; Behr, Wachsmuth, Post & Lesh, 1984; Yanık, 2013). They are important for the following reasons: from a mathematical standpoint, they form the basis of elementary algebraic operations; from a practical standpoint, they develop students' ability to cope with real world problems; and finally from a psychological standpoint, they help students develop and extend mental structures required for continuous intellectual development (Behr et al., 1983).

Due to their importance, Turkish elementary and middle school mathematics programs (Ministry of National Education [MoNE], 2009a, 2009b) also give considerable emphasis on rational number concepts. In grade 1, students learn how to partition physical objects into two equal parts and explain the relationship between one half and a whole. In grade 2, students explain the relationship among one half, one quarter and a whole. In grade 3, students learn how to partition a whole into equal parts and know that each part is a unit fraction; learn proper fractions that include at most two digit numbers as denominators; learn how to compare and order at most three fractions that include at most two digit numbers as denominators and learn how to find unit fractions of given quantities. In grade 4, students learn how to obtain fractions with at most two digits numerators and denominators by using unit fractions; locate fractions with at most two digits numerators and denominators on a number line, compare fractions; order at most four fractions with same denominators; order at most four fractions with same numerators; find unit fractions of given quantities; add fractions with same denominators; subtract fractions with same denominators; and finally pose and solve problems related with addition and subtraction of fractions. In grade 5, students learn how to convert among mixed numbers and improper fractions; compare a whole number with a fraction; compare and order fractions and locate them on a number line; find equivalent fractions of a given fraction; find whole quantity by means of its fractional amount; explain the relationship between a fraction and a division operation; add fractions with same denominators; add a whole number and a fraction; subtract fractions with same denominators; subtract a fraction from a whole number; pose and solve problems related with addition and subtraction of fractions and finally, they learn how to find a fraction of another fraction (MoNE, 2009a).

In grade 6, students learn how to compare, order and locate fractions on a number line; add and subtract fractions; multiply and divide fractions; estimate fraction operations by using a relevant strategy and finally pose and solve problems related with fractions. In grade 7, students apply and extend their previous understandings about fraction concepts and operations to rational number concepts and operations. Namely, students learn how to explain and locate rational numbers on a number line; express rational numbers in different forms; compare and order rational numbers; add or subtract rational numbers; multiply or divide rational numbers; perform multi-step operations with rational numbers and finally pose and solve rational number problems (MoNE, 2009b).

Although students are introduced rational numbers, in particular fractions, at all grade levels, rational numbers are infamous for the difficulty encountered not only by elementary school students (e.g., Bright, Behr, Post & Wachsmuth, 1988; Haser & Ubuz, 2003; Lesh, Behr & Post, 1987; Ni, 2001; Vamvakoussi & Vosniadou, 2010) but also by middle school students (e.g., Birgin & Gürbüz, 2009; Lamon, 2007). As Lamon (2007) expressed, rational numbers:

> "arguably hold the distinction of being the most protracted in terms of development, the most difficult to teach, the most mathematically complex, the most cognitively challenging, the most essential to success in higher mathematics and science, and one of the most compelling research sites" (p. 629).

Rational number concepts are even very challenging for elementary school teachers (An, Kulm, & Wu, 2004, Graeber, Tirosh, & Glover, 1989; Izsak, 2008; Ma, 1999; Tirosh, 2000). There are many teachers who have procedural understanding of rational numbers (Ball, 1990a, 1990b) but many of them experience difficulties with fraction concepts such as equivalent fractions (Cramer & Lesh, 1988).

The difficulties encountered by students about rational number concepts mainly stem from two factors: interference of natural number knowledge to rational numbers and problems with notation of rational numbers (Moss, 2005; Ni & Zhou, 2005; Smith, Solomon & Carey, 2005). For instance, students misinterpret the symbol $\frac{a}{b}$ by thinking a and b as two unrelated numbers, think that a and b are additively related, or think that rational numbers with large numerators and denominators are greater than rational numbers with small numerators and denominators (Lamon, 2012; Moskal & Magone, 2000; Moss, 2005; Stafylidou & Vosniadou, 2004).

To help teachers lessen the difficulties experienced by their students about rational numbers, Greer (1987) attempted to identify students' common misconceptions about rational numbers such as 'multiplication makes bigger, division makes smaller'. Moss and Case (1999) proposed a new curricular approach and tested it in a study involving 5^{th} and 6^{th} grade students. Moreover, National Council of Teachers of Mathematics [NCTM] (2000) emphasized using standard documents to develop elementary and middle school students' rational number reasoning.

Despite the emphasis on enhancing students' rational number understanding, student difficulties about rational numbers still persist (Wilson, Mojica & Confrey, 2013). Besides, many of the elementary and middle school mathematics topics involve rational number concepts and large scale international studies such as Programme for International Assessment (PISA) (OECD, 2010) and Trends in International Mathematics and Science Study (TIMSS) (Mullis, Martin & Foy, 2008) document low mathematics performance of Turkish students. Morrison (2013) attributed students' poor performance in mathematics to "poor sequencing of examples, limited ranges of examples in the low rates of task completion within and across lessons and to more general slow pacing (p.97). Thus, it is significant to explore the quality of rational number examples used by middle school mathematics teachers in actual classroom practices in order to improve students' learning.

1.2. Purpose of the Study and Research Questions

Teachers' choice of examples depends on factors such as knowledge competency, teaching goals, teachers' awareness of their students' misconceptions and dispositions and the like (Bills et al., 2006). These factors refer explicitly to the domain of pedagogical content knowledge of teachers and in particular to the sub-domain of knowledge of content and students theoretically defined by Ball, Thames and Phelps (2008). Knowledge about mathematics examples is a part of teachers' specialized content knowledge as well (Mohamed & Sulaiman, 2010). Specialized content knowledge is a mathematical knowledge that is unique to teaching and is a subset of subject matter knowledge described by Ball et al. (2008). Briefly, teachers' examples may reflect both their mathematical and pedagogical knowledge (Zazkis & Leikin, 2007). More importantly, the knowledge about mathematical examples is acquired through teaching experience and hence can be considered craft knowledge (Kennedy, 2002; Leinhardt, 1990). To be more precise, teachers' purposes for selecting, their design or effective treatment of examples are mostly constructed through their teaching experience (Rowland, 2008; Zaslavsky & Zodik, 2007). Thus, it can be suggested that examples are an important component of expert knowledge (Michener, 1978).

A few researchers have recently concentrated on teachers' choice and use of examples in mathematics classrooms (e.g., Rowland & Zazkis, 2013; Rowland, 2008; Rowland, 2014; Watson & Mason, 2005; Zaslavsky & Zodik, 2007; Zaslavsky & Zodik, 2014; Zaslavsky, 2010; Zodik & Zaslavsky, 2008). Thus, the role of examples in the teaching of mathematics is notably absent from teacher education literature not only in Turkey but also in other countries almost all over the world (Rowland, 2008). Therefore, further studies are needed to explore the examples chosen or used by teachers in their actual classroom practices.

The purpose of this study was to explore how middle school mathematics teachers treated rational number examples in their seventh grade classrooms. More specifically, this study aimed to investigate overall characteristics of teachers' rational number examples, the principles or considerations used by teachers while choosing or using rational number examples and the potential shortcomings of the examples used by the teachers. Through this purpose, the following major questions and sub-questions were formulated:

1. What are the overall characteristics of examples used by middle school mathematics teachers in the teaching of rational numbers in their seventh grade classrooms?

   a. What are the ideas emphasized in the rational number examples used by the teachers?
   b. To what extend do teachers use specific examples in the teaching of rational numbers?

c. To what extend do teachers use non-examples and counter-examples in the teaching of rational numbers?
d. To what extend do teachers use pre-planned and spontaneous examples in the teaching of rational numbers?
e. Which sources do teachers use while choosing pre-planned examples in the teaching of rational numbers?

2. What are the underlying principles or considerations that guide middle school mathematics teachers in choosing or generating examples?

3. What mathematical or pedagogical shortcomings do the examples used by the teachers in the teaching of rational numbers have?

a. What are the mathematically incorrect examples used by the teachers during the teaching of rational numbers?
b. What are the pedagogically improper examples used by the teachers during the teaching of rational numbers?

In this study, I used the following theoretical frameworks to give a comprehensive explanation of how middle school mathematics teachers treat rational number examples in their classrooms: Marton and Booth's (1997) variation theory; Zodik and Zaslavky's (2008) dynamic framework for explaining teachers' choices and generation of examples during the lesson, and finally Rowland et al.'s (2005) the Knowledge Quartet Framework for making sense of teachers' choice and use of examples. These frameworks are explained in detail in the literature review chapter.

1.3. Definitions of Important Terms

The research question consists of several terms that need to be clearly defined. These terms are defined either constitutively or operationally in the following way:

Example

As mentioned before, Watson and Mason (2005) defined examples as "illustrations of concepts and principles, placeholders used instead of general definitions and theorems, worked examples, exercises, representatives of classes used as raw material for inductive mathematical reasoning, specific contextual situations that can be treated as cases to motivate mathematics" (p. 3).

In this study, I examined worked-out examples and exercise examples that were used by the teachers or included in the student textbook for teaching rational number concepts. Worked examples referred to examples that were worked through by the middle school mathematics teachers in the course of teaching rational number concepts and by the student textbook in order to explain a rational number topic. Exercise examples referred to examples that were worked through by the teachers after introducing rational number concepts so as to develop fluency and to the textbook examples that were left to the students for practicing a specific technique.

Specific example

Mason and Pimm (1984) defined specific examples as examples that are used to represent a whole class of an object. Edwards (2011) defined a specific example as "a one-off situation that may or may not be general" (p. 19). In this study, a specific example referred to one of the possible examples through which rational number concepts were expressed.

Non-example

Non-examples are examples used to show the boundaries or necessary conditions of a concept (Watson & Mason, 2005). In this study, non-examples referred to the examples that were used by the teachers in order to show that not all numbers are rational.

Counter-example

Counter examples are examples which demonstrate that a certain conjecture is invalid (Watson & Mason, 2005). In this study, counter-examples referred to the examples used by the teachers to demonstrate the falsity of a student conjecture related with a rational concept or procedure were treated as counter-examples.

Teachers' considerations or principles

In this study, teachers' considerations referred to teachers' intentions or aims for selecting or using each example during the teaching of rational number concepts. Similarly, teachers' principles referred to teachers' use of pedagogical approaches such as pattern breaking (Watson & Mason, 2005) and structured variation (Watson

& Mason, 2005) when demonstrating a rational number concept or procedure by means of an example or a set of examples during actual classroom practices.

Mathematical shortcoming

In this study, mathematical shortcoming referred to the mathematical incorrectness of an example generated by the teachers in the course of teaching rational number concepts.

Pedagogical shortcoming

In this study, pedagogical shortcoming referred to the inappropriateness of an example generated by the teachers in the course of teaching rational number concepts. In more detail, examples that included improper language or terminology, examples that obscured the role of variables and examples which called for more sensible procedures were treated as examples that included pedagogical shortcomings.

1.4. Significance of the Study

As evidenced from earliest records to modern sources, the use of examples in mathematics education has a long history (Bills et al., 2006; Rowland, 2008; Sinclair, Watson, Zazkis & Mason, 2011) and it still continues to receive increasing attention in mathematics education research (Antonini et al., 2011; Bills & Watson, 2008; Sinclair et al., 2011). In the last ten years, a great deal of research papers have been published and some working groups have focused on examples (e.g., special issue of ZDM entitled ‘Examples in Mathematical Thinking and Learning from an Educational Perspective’, Volume 43, Issue 2, May 2011; special issue of Educational Studies in Mathematics entitled ‘The Role and Use of Examples in Mathematics Education’, Volume 69, Issue 2, October, 2008 and the research forum entitled ‘Exemplification in Mathematics Education’ at PME 30 by Bills et al., 2006).

Examples are used comprehensively in the acquisition of various mathematical domains such as proof (e.g., Alcock & Inglis, 2008; Buchbinder & Zaslavsky, 2011; Iannone Inglis, Mejia-Ramos, Simpson & Weber, 2011; Komatsu, 2010; Leung & Lew, 2013; Sandefur, Mason, Stylianides & Watson, 2013;

Pedemonte & Buchbinder, 2011; Zazkis & Chernoff, 2008), geometry (e.g., Guo, Pang, Yang & Ding, 2012; Tsamir, Tirosh & Levenson, 2008; Zaslavsky, 2008; Zaslavsky, 2010; Zazkis & Leikin, 2008), elementary number theory (e.g., Goldenberg & Mason, 2008; Rowland, 2008) advanced mathematics (e.g., Antonini, 2011; Arzarello, Ascari & Sabena, 2011; Mason, 2011; Watson & Chick, 2011), patterns and generalizations (e.g., Sinclair et al., 2011; Zazkis, Liljedahl & Chernoff, 2007) and the like. Antonini et al. (2011) also emphasized the same point that examples pervade concept formation (Dahlberg & Housman, 1997), generalization from particular to general (Mason & Pimm, 1984), concept definition and concept image (Tall & Vinner, 1981).

Examples serve many purposes in mathematics education. For instance, example generation can be used as a tool for diagnosing some components of students' conceptions (Bratina, 1986). Zazkis and Leikin (2007) suggest that asking learners to generate examples provides a 'window' into their mind since the examples generated by them "mirror their conceptions of mathematical objects involved in an example generation task, their pedagogical repertoire, their difficulties and possible inadequacies in their perceptions" (p. 15). Goldenberg and Mason (2008) further claim that

> "Examples can usefully be seen as cultural mediating tools between learners and mathematical concepts, theorems, and techniques. They are a major means for 'making contact' with abstract ideas and a major means of mathematical communication, whether 'with oneself', or with others. Examples can also provide context, while the variation in examples can help learners distinguish essential from incidental features and, if well selected, the range over which that variation is permitted" (p. 184).

Despite being essential in a classroom environment, generating examples of mathematical objects can be a complicated work for teachers (Bills et al., 2006; Zaslavsky & Peled, 1996). Besides, it entails many circumstances that should be considered (Antonini et al., 2011; Zodik & Zaslavsky, 2008). From this point of view, it can be said that teachers' choice of examples may either promote or hinder students' learning. Although teachers' choice of examples play a substantial role in student learning, a large proportion of mathematics teacher education programs do not overtly speak to this issue and do not systematically train pre-service teachers to cope with examples in an educated way (Zaslavsky & Zodik, 2007). Thus, it can be suggested that teachers' ability to generate effective examples develop through their teaching experience and thus constitutes their craft knowledge (Kennedy 2002; Leinhardt 1990). In-depth exploration of teachers' craft knowledge regarding treatment of examples may give us the opportunity to gain entry into their specific aspects of knowledge and use it as groundwork for devising professional development programs or courses that may foster teachers' building up of systematic knowledge (Zaslavsky, 2008; Zaslavsky & Zodik, 2007).

Despite the centrality of examples in developing conceptual understanding of mathematics (Watson & Mason, 2002), only a few researchers focused on teachers' choice and use of examples in their classrooms (e.g., Rowland 2008; Watson & Mason 2005; Zodik & Zaslavsky, 2008). Besides, these researchers examined examples used by the teachers for teaching different mathematical concepts in a more superficial sense (e.g., Rowland 2014; Rowland, 2008; Zaslavsky & Zodik, 2007; Zodik & Zaslavksy, 2008). Therefore, there is a need for studies that explore examples used by the teachers' in the teaching of specific mathematical concepts in greater depth. Furthermore, different education systems in different countries may influence teachers' choice and use of examples in their classrooms and thus, the quality and quantity of examples used by the teachers for teaching a specific mathematical concept may differ from one country to another.

As suggested by Bills et al. (2006), there is a scarcity of research on teachers' choice and use of examples related with certain mathematical concepts. Therefore, I want to go further in this direction and attempt to fill this gap by examining middle school mathematics teachers' treatment of rational number examples in their classrooms in a national context. It is significant to explore teachers' treatment of rational number examples for several reasons. First, rational number concepts are among the most important mathematical concepts students experience in their school years (Alacacı, 2009; Yanık, 2013). Second, although students are introduced to rational numbers at all grade levels; they experience difficulties in understanding them due to their complexity (Haser & Ubuz, 2003; Lamon, 2007; Vamvakoussi & Vosniadou, 2010). Thus, exploration of teachers' choice and use of rational number examples might help teachers improve the quality and quantity of examples used in the teaching of rational number concepts and might be particularly helpful for teachers in overcoming their students' difficulties in these concepts and operations.

Exploration of teachers' treatment of rational number examples might be used in the development of a possible framework that might be used to capture middle school mathematics teachers' generation and choice of rational number examples in their classrooms. Similarly, exploration of teachers' considerations or principles in choosing or using rational number examples might be used in the development of a possible framework that might be used to examine middle school teachers' principles or considerations in selecting or generating rational number examples in their classrooms. Future studies in different education systems might provide empirical support to the development of a possible framework for analyzing teachers' considerations in choosing and using rational number examples.

In a broader sense, it is anticipated that investigation of teachers' treatment of examples might help teachers raise their awareness in choosing or using appropriate examples during the teaching of mathematics and consequently improve the quality of their teaching and foster student learning.

1.5. My Motivation for the Study

Before I began to explore middle school mathematics teachers' treatment of examples in their own classrooms, I had participated in Special Teaching Method Courses implemented by a member of my own department. As I observed pre-service middle school mathematics teachers' selection and use of examples for teaching various mathematical concepts, I noticed that some examples were generated by them randomly without any thinking in-advance about negative influences of examples in learning these mathematical concepts. Besides, the pre-service teachers did not seem to give enough importance to the careful selection of initial examples when starting to teach novel concepts that the students have not experienced before. In my opinion, it is important to introduce examples that recall prior knowledge of students before teaching a novel concept. For instance, it is crucial to introduce fraction or integer examples to the students before teaching rational number concepts. Thus, pre-service middle school mathematics teachers' treatment of examples in Special Teaching Method Courses initially prompted me to carry out a study in the area of exemplification.

Another factor that encouraged me to conduct this study was the low mathematics performance of middle school students that were reported both in national high-stakes exams such as SBS or TEOG and international student assessment programs such as TIMSS and PISA. Based on my own experience, I thought that the low performance of middle school students in mathematics might be associated with their teachers' way of using examples in teaching mathematical topics. In particular, I thought that the quality and quantity of examples used by the mathematics teachers might give some clues about the quality of their teaching practices and consequently might reflect student achievement in mathematics.

Finally, I thought it would be crucial to convey mathematics teaching experiences of in-service teachers to pre-service teachers enrolled in teacher education programs since it takes considerable time for pre-service teachers to gain craft knowledge about teaching particular mathematical topics. More specifically, it is important to inform pre-service teachers about in-service teachers' principles or considerations in selecting or using certain examples in the teaching of mathematics so that they will benefit from in-the-moment decisions that teachers make. In short, this study might play an important role in bridging between in-service teachers' craft knowledge of mathematical examples and pre-service teachers' initial teaching experiences.

CHAPTER II

LITERATURE REVIEW

The goal of this study was to explore middle school mathematics teachers' treatment of rational number examples in their mathematics classrooms. More specifically, this study aimed to answer the following research questions:

1. What are the overall characteristics of examples used by middle school mathematics teachers in the teaching of rational numbers in their seventh grade classrooms?

   a. What are the ideas emphasized in the rational number examples used by the teachers?
   b. To what extend do teachers use specific examples in the teaching of rational numbers?
   c. To what extend do teachers use non-examples and counter-examples in the teaching of rational numbers?
   d. To what extend do teachers use pre-planned and spontaneous examples in the teaching of rational numbers?
   e. Which sources do teachers use while choosing pre-planned examples in the teaching of rational numbers?

2. What are the underlying principles or considerations that guide middle school mathematics teachers in choosing or generating examples?

3. What mathematical or pedagogical shortcomings do the examples used by the teachers in the teaching of rational numbers have?

   a. What are the mathematically incorrect examples used by the teachers during the teaching of rational numbers?
   b. What are the pedagogically improper examples used by the teachers during the teaching of rational numbers?

In the light of these research questions, this chapter elaborated on various theoretical constructs related with examples and provided the theoretical frameworks that were used in this study. Finally, relevant studies on teachers' treatment of mathematical examples were reviewed. The following section sought to describe what a mathematical example is.

2.1. What is a Mathematical Example?

Given that examples have a wide variety of educational uses (Bills et al., 2006), it is important to shed some light into what constitutes an example. The notion of 'example' has several different meanings. Michener (1978) described examples as illustrative material and underlined the dual relations among examples, results and concepts. That is, she emphasized that examples can be constructed from results and concepts and alternately they can motivate concepts and results. In her subsequent study, Michener (1991) pointed out that an example can be viewed as "a set of facts or features viewed through a certain lens" (p. 190). Mason and Pimm (1984) stressed the generality aspect of examples and announced that the ability to perceive the general by means of the particular is at the core of the exemplification. In a similar way, Zodik and Zaslavsky (2008) meant that "examples are a particular case of a larger class, from which one can reason and generalize" (p. 165). In the meantime, Zazkis and Leikin (2008) viewed examples as "illustrations of concepts and principles" (p. 131). In a recent study elaborating on the notion of personal example spaces, Sinclair et al. (2011) explained that "an example refers to a specific instantiation of a more general notion" and further described a mathematical example as "an instance of a mathematical class with specified properties, a worked solution to a problem, an instance of a theorem or method of reasoning" (p. 292).

Similarly, Yopp (2014) referred to an example as "any mathematical object used to instantiate properties or concepts involved in a mathematical task" (p. 182). Furthermore, he attempted to be cautious about the distinction between example generation and example use. He broadly defined example generation as building or extending learners' example space and included learner-generated examples, examples built by modification of pre-existing examples and examples obtained from other people or sources such as friends or software into example generation process. However, he defined example use as using an example from an example space irrespective of when or how that example was obtained. As can be seen, the aforementioned definitions of ‘example’ have left the learners out of the picture and referred only to a mathematical requirement.

Alcock and Weber (2010) described examples in a more restricted sense and meant that an example is “a mathematical object satisfying the definition of some concept” (p. 4) and they further added that

> “6 is an examples of an even number and $f(x) = x^2$ is an example of a continuous real-valued function. The latter could, of course, be represented graphically rather than via a formula, and we consider such a graph to be an example too” (p. 4).

Similar to Alcock and Weber (2010), Fukawa-Connelly and Newton (2014) regarded it pedagogically important to differentiate between examples of a concept and examples of a process and drew upon only concept examples and adopted the following definition: “a mathematical object satisfying the definition of some concept” (p. 325). Mills (2014) considered that for a mathematical object to be an example it should satisfy two properties as specificity and concreteness. She explained that an example first should be specific and concrete in contrast to being general and abstract. Besides, she added that specificity is a mathematical necessity and concreteness has to do with accessibility of the mathematical object to the learners. Finally, she defined an example as “a specific, concrete representative of a class of mathematical objects, where the class is defined by a set of criteria” (p. 107).

Watson and Mason (2005) took a much broader view of what constituted an example and used it to represent anything from which a learner might generalize. Hence, an example referred to:

> “Illustrations of concepts and principles, such as a specific equation that illustrates linear equations or two fractions that demonstrate the equivalence of fractions; placeholders used instead of general definitions and theorems, such as using a dynamic image of an angle whose vertex is moving around the circumference of a circle to indicate that angles in the same segment are equal; questions worked through in textbooks or by teachers as a means of demonstrating the use of specific techniques, which are commonly called worked examples; questions to be worked on by students as a means of learning to use, apply, and gain fluency with specific techniques, which are usually called exercises; representatives of classes used as raw material for inductive mathematical reasoning, such as numbers generated by special

> cases of a situation and then examined for patterns; specific contextual situations that can be treated as cases to motivate mathematics" (p. 3).

Watson and Mason's (2005) use of examples is learner-dependent. That is, their use of examples permits the learners to generate examples which may not be mathematically correct. Similar to Watson and Mason (2005), Bills et al. (2006) defined the term example as any object employed as a raw material for generalization; illustrating concepts and procedures; representing a larger class; motivating; disclosing possible variation; and finally exercising a technique.

Kamin (2010) stressed that examples are in part vague entities and they may be composed of either simple expressions or complex multi-step problems. He further noted that different meanings attributed to the notion of example stems from the fact that researchers, mathematics educators and mathematics teachers all have different perspectives. Finally, he explained that examples may exist as isolated objects or may be used to define, characterize or illustrate mathematical concepts.

2.2. Classification of Examples

Several researchers categorized examples with respect to their particular use in mathematics or in the teaching of mathematics (e.g., Mason & Pimm, 1984; Mason & Watson, 2005; Mischener, 1978; Peled & Zaslavsky, 1997; Rowland, Turner, Thwaites & Huckstep, 2009; Zazkis & Chernoff, 2008; Zodik & Zaslavsky, 2008). The definitions and explanations of different types of mathematical examples are presented below.

2.2.1. Start-up examples

Michener (1978) analyzed acquisition of mathematical knowledge from an epistemological perspective and distinguished three main categories of items as results (traditional logical deductive elements of mathematics), examples (illustrative material) and concepts (mathematical definitions and heuristic notions and advice). She indicated that examples-space, results-space and concepts-space are three representation spaces for a mathematical theory and introduced start-up examples as one category of epistemological classes of the example-space. She defined start-up examples as examples that help motivate essential definitions and results and initiate into a topic and stated that a start-up example should have the following properties: it should motivate fundamental concepts, it should be understood by itself, it should be projective and finally it should provide a simple and evocative picture.

2.2.2. Reference examples

Michener (1978) introduced reference examples as another category of epistemological classes of the example-space. She defined reference examples as examples that are illustrations of concepts, results, models, and counter examples and as examples that are recurrently used in the development of theories. Watson and Mason (2005) defined reference examples as typical cases which are to a large extent applicable and may be linked various concepts and results. Besides, they suggested using R^2 to make sense of how things function in real analysis as a reference example.

2.2.3. Specific examples

Mason and Pimm (1984) defined specific examples as examples that are used to represent a whole class of an object. By using the same examples used by Mason and Pimm (1984), Edwards (2011) defined a specific example as “a one-off situation that may or may not be general” (p. 19). Edwards (2011) introduced “THE even number 6” as a specific example and further explained that “the existence of such an object is the important point rather than necessarily the representation of a wider collection of objects. In this sense counterexamples to theorems are specific” (p. 19). Peled and Zaslavsky (1997) categorized examples with respect to their explanatory power and explained that specific examples have weaker explanatory power. In a more recent study, Zaslavsky (2010) illustrated specific examples by the use of rectangle pairs as given in Figure 2.1.

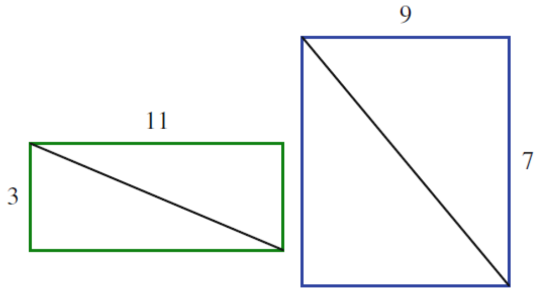


Figure 2.1. A specific example of a pair of distinct rectangles with the same diagonal (Zaslavsky, 2010, p. 109).

2.2.4. Particular examples

In order to clarify some of the uncertainties related with various example types, Mason and Pimm (1984) introduced particular examples as one category of examples. They explained that a lecturer might present $x \rightarrow |x|$ as an example of a continuous but non-differentiable function presented to the students and added that the lecturer might see this example as a generic example that indicates a whole class of functions (i.e., $x \rightarrow k|x|+C$), but the students might concentrate on the particular example and see a single function instead of whole class of functions. Similarly, Edwards (2011) defined a particular example as "using a general example in a specific situation or argument" (p. 19) and introduced "2N is even, 2N + 2N = 4N so 4N is also even" as a particular example and further indicated that "each 2N implicitly refers to the same number, so although N in isolation is a general example, when used in this context 2N is a particular example" (p.19). Finally, Edwards (2011) noted that the distinction between specific examples and particular examples is subtle.

2.2.5. Generic examples

Michener (1978) defined model or generic examples as examples that summarize assumptions about results and concepts and can be used to construct particular instances. She added that model examples should be flexible and manipulatable and should be adjusted finely to satisfy the specifics of a problem.

Mason and Pimm (1984) defined generic examples as “an actual example, but one presented in such a way as to bring out its intended role as the carrier of the general” (p. 287). Edwards (2011) defined general examples as “using an example to represent a class of examples with a similar property” (p. 19), introduced “AN even number such as 6” as a generic example and explained that “the example is used to represent other objects, but there is no intention to represent a complete class of objects” (p. 19). Similar to Mason and Pimm (1984), Bogomolny (2006) described a generic example as an actual example that is introduced in such a way that it uncovers the intended role as the carrier of the general. She added that general examples are presented by means of particular numbers but generic proof is never dependent on the specific properties of those numbers. Besides, Rowland (1998) suggested that generic examples may be used in proofs related with number theory theorems and added that generic examples help students better understand the mathematical topic when compared to the formal proofs. Rowland (2014) stressed that the standard procedure for verifying mathematical truths is by general proof, however gaining insights into such proofs might usually be attained via well-structured arguments on the basis of generic examples. The generic example provided by Rowland et al. (2009) to prove the conjecture that ‘the sum of 1+3+5+… up to any odd number is always a square number’ is presented in Figure 2.2.

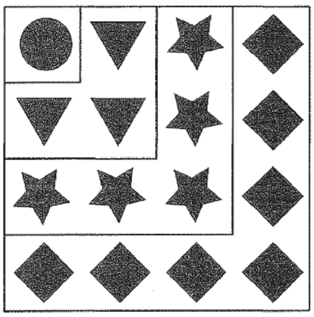


Figure 2.2. A generic example that shows $1+3+5+7 = 4^2$ (Rowland et al., 2009, p.98).

As can be seen, the figure started with one circle at the top left. The second, third and fourth layers are shown with triangles, stars and squares respectively. In the first stage, there is one circle and can be represented as $1= 1^2$. In the second stage, there is one circle and 3 triangles and the total number of shapes can be represented as 1+3 =

2^2. In the third stage, there is one circle, three triangles and five stars and the total number of shapes can be represented as 1+3+5 = 3^2. In the fourth stage, there is one circle, three triangles, five stars and seven squares and the total number of shapes can be represented as 1+3+5+7 = 7^2. Thus, the above given figure is a generic example since it is apparent that addition of each posterior odd number conserves the square array.

2.2.6. General examples

Mason and Pimm (1984) also distinguished between generic examples and general examples. They defined general examples as examples that represent whole class of mathematical objects. Similarly, Edwards (2011) described general examples "as using an example to represent an operation on a wider class" (p. 19), introduced "ANY even number like 6" as a general example and added that "the extent of the class to that the example refers to is known, or implied" (p.19). Peled and Zaslavsky (1997) stressed that general examples are more advantageous than specific examples with regards to their generality and explanatory power. In a similar way, Zaslavsky (2010) indicated that general examples "offer explanation and provide insight about a certain phenomenon as well as ideas about how to generate more examples of this phenomenon" (p. 108). Further, she illustrated general examples by the use of a pair of rectangles as given in Figure 2.3.

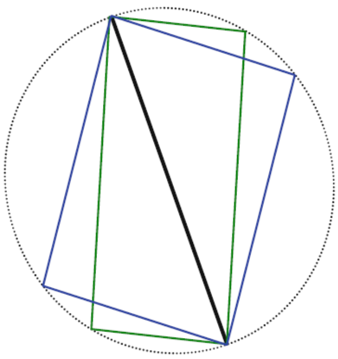


Figure 2.3. A general example of a pair of distinct rectangles with the same diagonal (Zaslavsky, 2010, p. 109).

2.2.7. Boundary examples

Askew and William (1995) referred to two kinds of examples as only just examples and very nearly examples. They defined that an example is an only just example if any change in that example turns it into a non-example and a nearly example needs one more modification in order for it to become an example. Mason and Watson (2001) preferred to use the notion of boundary examples instead of only just examples to distinguish between examples which have a certain property and examples which do not have that certain property. They asserted that if a learner cannot generate boundary examples for a technique or theorem, then they cannot fully appreciate or comprehend it. Moreover, they emphasized that

> "By constructing a boundary example students are forced to extend their example-space in order to complete the task. So one effect is that students become more aware of the range of possibilities from which they are choosing when they select an example, and this is a precursor to expressing generality" (p. 11).

According to Watson and Mason (2005), boundary examples are rather extreme to represent the whole classes but they do display what happens at the 'edges' of those classes. Besides, they emphasized that unless there is access to extreme examples, then there is the probability of being misled in appreciating the scope of the related concept. Further, they exemplified the affordances of extreme examples as follows:

> "not all fractions have terminating decimals; subtraction and division can make larger; triangles are limiting cases of trapezia, squares are also rectangles, trapezia and parallelograms; multiplying zero offers a counter-example to the belief that division always undoes multiplication" (p. 100).

2.2.8. Pivotal and bridging examples

Zazkis and Chernoff (2008) expressed that counter-examples may serve to falsify a conjecture from a mathematical standpoint, however they may not have enough power to convince the learner to abandon his/her previously made generalization. More precisely, they indicated that counter-examples may not create a cognitive conflict and the learner may simply dismiss or treat that counter-example as an exception. Thus, the researchers introduced the notions of pivotal example and bridging example and noted that while a counter-example is a mathematical concept, pivotal or bridging examples are pedagogical concepts. They explained that pivotal examples serve to create a cognitive conflict while bridging examples assist in conflict resolution and they defined pivotal examples as examples that help learners achieve ‘conceptual change’ (Tirosh & Tsamir, 2004; Vosniadou & Verschaffel, 2004). Besides, they noted that counter-examples may be determined universally and in advance whereas it is not possible to determine whether an example works as a pivotal or a bridging example for a student cannot be determined before the instructional implementation and that may be totally identified only after that implementation.

Zazkis and Chernoff (2008) described and analyzed two episodes to illustrate the notions of pivotal and bridging examples. One of the episodes was about prime numbers and the researchers asked Selina, a prospective elementary school teacher, to simplify the following expression: $\frac{13\times 17}{19\times 23}$. Selina started working on the task by multiplying the numbers included in the numerator and denominator of the expression. She wrote the expression as $\frac{221}{437}$ and started checking whether 221 and 437 are both divisible by 2, 3 and 5. She realized that 221 and 437 are not divisible by 2, 3 and 5 and she came up with a conjecture that 437 is a prime number. Nonetheless, she kept checking whether 437 is divisible by 7, 13 and 17. After trying 19, she confirmed that 19 was in the original expression and at that moment she admitted that 19 and 23 were prime but she concluded that “two prime numbers multiplied by each other are prime.” Selina’s such inference was described as ‘intuitive tendency towards closure’ (Zazkis & Liljedahl, 2004). Just then, Selina was asked to identify 15 is a prime number. This strategic example invoked a cognitive conflict and caused Selina to question her initial ideas. She realized that 15 is not a prime number, despite it is equal to the multiplication of the two prime numbers as 3 and 5. Thus, she refuted her initial conjecture that the product of two prime numbers is also a prime number. Previously, Zazkis and Liljedahl (2004) identified that students tended to determine a number’s primality by checking whether that number was divisible by small primes. By making reference to this study, Zazkis and

Chernoff (2008) pointed out that Selina's list of small primes was limited to 2, 3 and 5.

After presentation of 15 as a pivotal example, 77 was introduced to Selina as another example to establish the strength of her belief about primality. Here, 77 served as a bridging example for Selina since it helped her resolve the conflict about primality. From a mathematical standpoint, 15, 77, 221 and 437 are all similar to each other in terms of their prime decomposition structure. However, from a pedagogical standpoint, 77 is small enough to 15 since its factors are easily noticeable. However, it is not comprised of 2, 3 or 5, the number which Selina named as building blocks. The example 77 led Selina to change her initial thinking about primality and guided her towards the following correct conjecture: prime numbers do not have to be closed under multiplication.

2.2.9. Pre-planned and spontaneous examples

In an attempt to describe teachers' choice of examples in and for the mathematics classroom, Zodik and Zaslavsky (2008) distinguished between pre-planned and spontaneous examples. The researchers investigated the examples used by secondary school mathematics teachers in the classroom with regards to the amount of pre-planning underlying their choices. They described pre-planned examples as examples which teachers think in advance and intend to use them in the lesson and they added that pre-planned examples might appear in teachers' planning notes, worksheets prepared for students, textbooks used for structuring the lesson or might be inferred from teacher expressions and actions. When there was not enough evidence for determining whether an example was pre-planned or not, the researchers conducted interviews with the teachers and asked them to explain how they got access to the examples they employed.

If the chosen examples involved in-the-moment decision making to a certain extent, then the researchers considered them to be spontaneous examples. When deciding whether an example is spontaneous, researchers took account of time allocated by teachers for generating the example, teachers' degree of certainty when generating the example and finally, teachers' gestures and facial expressions when generating the example. For instance, researchers determined an example to be spontaneous when teachers used one of the following expressions: “I’m trying to construct a simple example but it is not working” or “I just chose these numbers now without giving them more thought” or when they confirmed a student query such as “Are you inventing the example right now?” (p. 172).

The researchers found out that secondary school teachers generated spontaneous examples mainly as a response to student queries and conjectures. Besides, they observed that teachers generated not only spontaneous examples but also spontaneous counter-examples. For instance, one of the teachers wanted to demonstrate that complex fractions might be equal to a number such as $\frac{1}{3}$. The teacher started constructing the example on her feet in front of the classroom, from time to time she erased some parts of the example and corrected it and she continued this iteration until the example fit her intended purpose. Finally, she generated the following spontaneous example: $\frac{3a^4b^2c^3 \cdot 4a^3b}{36a^7b^3c^3}$. To give another example, one of the teachers generated the spontaneous counter-example in Figure 2.4 as a response to a student’s invalid conjecture that “if in a quadrangle there are two opposite right angles it is a kite” (p. 172):

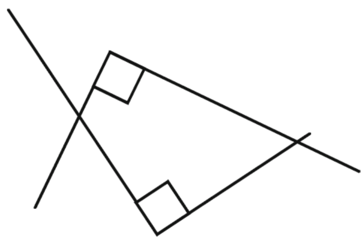


Figure 2.4. A spontaneous counter-example of a quadrangle with two opposite right angles that is not kite (Zodik & Zaslavsky, 2008, p. 172).

In order to convince the student that her conjecture is invalid, the teacher constructed on her feet a dynamic counter-example by constructing two right angles first and then positioning them in a way that intersect with each other as depicted in Figure 2.4.

Mason and Spence (1999) coined the term ‘knowing to act in the moment’ for teachers’ ability to think ‘on their feet’. Similarly, Rowland et al. (2005) dealt with such in-the-moment actions by means of the contingency dimension of their Knowledge Quartet Framework. Rowland et al. (2009) stressed that teachers continuously respond to their students’ interests and inquiries as part of the ongoing classroom interaction and they added that it is not possible for the teacher to know how each student will react or respond to any situation, therefore teachers may quite often need to make split-second decisions in the course of mathematics lessons. Rowland and Zazkis (2013) also made the same point that teaching not only includes paying attention to pre-determined sequence of events and providing the pre-determined curriculum but it also has to do with paying attention to “students’ questions, anticipating some difficulties and dealing with unexpected ones, taking advantage of opportunities, making connections, and extending students’ horizons beyond the immediate tasks” (p. 138). To conclude, the act of teaching entails the ability to handle unpredictable or contingent events in the classroom and this ability is associated with classroom events that fall outside a teacher’s own lesson image (Rowland & Zazkis, 2013). In his ‘theory of teaching-in-context’ Schoenfeld (1998) described the term lesson image as follows:

> “The teacher’s lesson image includes knowledge of his or her students and how they may react to parts of the planned lesson; it includes a sense of what students are likely to be confused about, and how the teacher might deal with that confusion; and more… I can tell you, before the class starts, how things are likely to unfold… there are many branch points and contingencies. However, I know what most of them are likely to be. And, there are few surprises” (p. 17-18).

Schoenfeld’s (1998) description of lesson image implies that teachers with more teaching experience might better predict what would happen in the classroom and might confront with less surprising events in the course of teaching.

2.2.10. Examples of and examples for mathematical concepts or procedures

By pondering how students can benefit from examples, Rowland et al. (2009) distinguished between different uses of examples in the teaching of mathematics. First, a teacher might use an example for teaching concepts and procedures as a particular instance of a generality (Rowland, et al., 2003). This way of using examples is inductive. That is, when teaching concepts or procedures, teachers provide or motivate students to provide examples of 'something' (Mason & Pimm, 1984; Rowland, 2008). The 'something' is general in character such as the notion of a rational number or the traditional algorithm for subtracting rational numbers and examples and the purpose for using examples is to represent abstract mathematical concepts and to exemplify general procedures (Rowland, et al., 2003). For instance, the rectangle example given in Figure 2.5 is an example of teaching a concept.

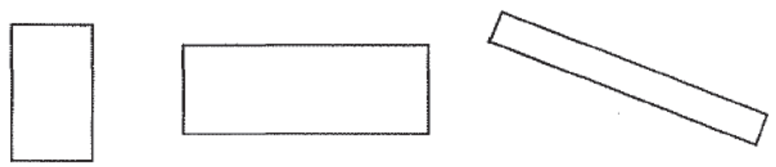


Figure 2.5. An example of a rectangle, and another, and another (Rowland et al., 2009, p.69)

Here, the notion of a rectangle is mathematically abstract and is encapsulated by a definition such as 'a shape with four sides and four right angles' (Rowland et al., 2009).

Similar to the teaching of concepts, teachers teach general procedures by particular demonstrations of those procedures (Rowland et al, 2003). To illustrate, if a teacher aims to teach grid method for $\mathrm{TU} \times \mathrm{U}$ multiplication (i.e., a two-digit by one-digit multiplication) he might select a two-digit number and a one-digit number and then multiply them by using the grid method (Rowland, 2008). Namely, a teacher may demonstrate the (general) grid method by means of the particular example of this procedure given in Figure 2.6.

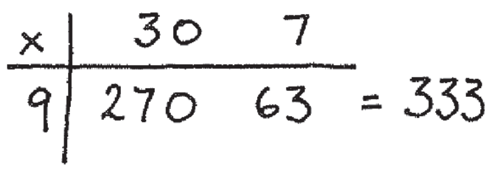


Figure 2.6. Finding 37×9 by using the grid method (Rowland, 2009, p. 119)

However, it is worthy of note that the teacher needs to select the numbers 3, 7 and 9 with some care and thus, the examples introduced by the teacher should preferably be "the outcome of a reflective process of choice, a deliberate and informed selection from the available options" (Rowland, 2008, p.151). Probably, it would not be judicious for the teacher to demonstrate, for example, 10×9 by using the grid method since it is more sensible to work out this example by using a mental calculation strategy.

The second use of examples in the teaching of mathematics is about familiarization and practice after the teaching of a new concept or procedure (Rowland, 2009). Examples used for this purpose are usually called exercise examples and rather than being inductive as in the case of concept or procedure examples, they are illustrative and practice-oriented (Rowland, 2008). Exercise examples are often chosen from a large number of examples (Rowland et al., 2003). For instance, after teaching the grid method for $\mathrm{TU} \times \mathrm{U}$ multiplication procedure by the aforementioned example (i.e, by 37×9), a teacher may well ask his/her students to do several more exercise examples as a group work or as a homework to promote retention of the procedure by repeating and to gain fluency with it (Rowland et al., 2009). From a teacher's point of view, an exercise example may also be used as an instrument for assessment and such practice might result in various types of awareness and understandings (Rowland et al., 2003). Besides, exercise examples do not need to impose too much burden on students since they may also give rise to different types of awareness and understandings (Rowland et al., 2009). As for concept or procedure examples, exercise examples need to be chosen with some care since teachers' generation or selection of such examples is neither trivial nor arbitrary (Rowland, 2008).

2.2.11. Counter-examples

Michener (1978) described counter-examples as examples that show that a conjecture is false and that clarify the distinctions between concepts or definitions. Rowland et al. (2009) indicated that counter-examples promote and challenge students' mathematical reasoning remarkably. Similar to Michener (1978), Rowland

(2014) defined a counter-example as an example that is used to show that a conjecture is false and explained this notion by using the following task: "How many ways are there of ascending a flight of stairs if you can take one or two stairs at a time? For three stairs, for example, there are 3 ways: 111, 12 and 21" (p. 103). As described by Rowland (2014), there is one way for ascending one stair, two ways for ascending two stairs, three ways for ascending three stairs and so on. Thus, the sequence proceeds in the following way: 1, 2, 3, 5, 8, 13, 21, 34, … Students may easily recognize the pattern in this sequence (i.e., 3 = 1+2, 5 = 3+2, 8 = 5+3, 13 = 8+5, 21 = 13+8 and 34 =21+13) and they may generate the following conjecture: every term is the sum of the previous two terms. To verify the truth of this conjecture, students need to prove it, and to show that it is incorrect they need a counter-example. For instance, if the number of ways for ascending eight stairs appeared to be 35 instead of 34, then the conjecture "every term is the sum of the previous two" could not be true in general.

Counter-examples are very often used in the teaching of mathematics (Bogomolny, 2006). They "can serve to sharpen distinctions and deepen understanding of mathematical identities" (Zodik & Zaslavsky, 2008, p. 165). Watson and Mason (2005) defined counter-examples as examples which demonstrate that a certain conjecture is invalid. Besides, they indicated that the same example may both be a non-example and a counter-example depending on the context. They illustrated that $\frac{1}{5}$ is a non-example of a fraction that is a repeating decimal and a counter-example to the claim that all fractions with non-repeating decimals include even denominators.

Counter-examples are in a very powerful position when compared to other examples since one counter-example may be sufficient for establishing the invalidity of a claim while using many examples for establishing the truth of a claim may not be sufficient (Bogomolny, 2006). Nevertheless, asking learners to generate counter-examples may be extremely troublesome, particularly if the learners have not generated before (Watson & Mason, 2006). For instance, Zaslavsky and Ron (1998) investigated ninth and tenth grade students' understanding of the role of counter-examples in falsifying mathematical statements, their achievement in generating correct counter-examples and the difficulties experienced by them when generating counter-examples. They found that “students often feel that a counter-example is an exception that does not really refute the statement in question” (p. 231). Besides, the students persistently believed that a counter-example is enough for falsifying a geometric statement than an algebraic statement. Their findings also revealed students’ inability to distinguish between an example that fulfil the necessary conditions of a counter-example and an example that does not fulfil them.

Similarly, Mason and Klymchuk (2009) lamented that students do not attach much importance to counter-examples and regard them as insufficient tools for establishing the invalidity of a given proposition. Instead, students choose to use exemplary illustrations such as rough outlining, rapid calculation or draft arrangement to demonstrate the association among variables (Zaslavsky & Ron, 1998). Besides, although proving true propositions are commonly shown in the teaching of mathematics, refuting of an invalid proposition is usually overlooked and thus, students fall short of training and confidence in falsifying invalid propositions by using counter-examples (Leung & Lew, 2013).

Although counter-examples are not emphasized in Turkish middle school mathematics curriculum, Common Core State Standards for Mathematics [CCSSM] (2010) has recently released new mathematics standards to enhance students’ ability to justify mathematical conjectures and use counter-examples. More specifically, CCSSM (2010) expects students to

> “make conjectures and build a logical progression of statements to explore the truth of their conjectures, analyze situations by breaking them into cases and recognize and use counter-examples; justify their conclusions and communicate them to others and respond to the arguments of others” (p. 6-7).

To achieve this goal, teachers are expected to comprehend their students’ proving and disproving processes clearly and teacher educators are expected to improve pre-service and in-service teachers’ ability to cope with counter-examples in the course of teaching (Leung & Lew, 2013).

2.2.12. Non-examples

Non-examples show the boundaries or necessary conditions of a concept (Watson & Mason, 2005). Similar to counter-examples, they "serve to clarify boundaries" of a concept (Bills et al., 2006, p. 127). Non-examples play a crucial role in promoting high levels of concept attainment (Charles, 1980; Cohen & Carpenter, 1980; Petty & Jansson, 1987; Tsamir et al., 2008). Besides, non-examples give teachers the chance to analyze their students' thinking and are supportive for students in reasoning out loud (Clements, Swaminathan, Hannibal, & Sarama, 1999).

In mathematics education, research related with non-examples mainly focused on acquisition of geometric concepts (e.g., Cohen & Carpenter, 1980; Petty & Johnson, 1987; Wilson, 1986; Tsamir et al., 2008). For instance, Tsamir et al. (2008) differentiated between two types of non-examples as intuitive non-examples and non-intuitive non-examples. The non-examples which were immediately identified by the students as non-examples were named as intuitive non-examples. The non-examples that had notable similarities with the true examples of a geometric concept and that were more often erroneously identified as examples of that concept were named as non-intuitive non-examples. The researchers indicated that not all non-examples encouraged the same type of reasoning. More precisely, they explained that intuitive non-examples (such as, square, hexagon and ellipse) promoted more visual reasoning whereas non-intuitive non-examples (such as, zig-zag triangle, pentagon, open triangle and rounded triangle) promoted analytical thinking based on critical attributes.

Cohen and Carpenter (1980) examined the effectiveness of non-examples in the acquisition of the geometric concept semi-regular polyhedra. The researchers stressed that a sequence of examples and non-examples is superior to a sequence of examples alone in concept acquisition. Besides, the introduction of non-examples in different order (such as, four examples first, four non-examples next versus four different example-non example pairs) did not have any effect on the acquisition of the geometric concept.

In a similar study, Petty and Johnson (1987) pointed to the superiority of a rational sequence of examples and non-examples over a randomly arranged sequence of examples and non-examples on sixth grade students' acquisition of parallelogram. They explained that rational sequence of examples and non-examples help students better identify distinguish between critical and non-critical attributes of the concept of parallelogram. Finally, they suggested that when choosing school geometry textbooks, the priority should be given to textbooks that present examples and non-examples in a rational manner.

Wilson (1986) suggested the use of non-examples so as to diminish the influence of prototype examples. Prototype examples are accepted immediately, intuitively, without thinking the need for any kind of justification (Tsamir et al., 2008). However, prototype examples may lead to cognitive obstacles since they have "coercive impact on our interpretations and reasoning strategies" (Fischbein, 1993, p. 233). Actually, students are inclined to consider prototypical examples as examples of the concept and consider other examples as non-examples of that concept (Hershkowitz 1989; Wilson, 1990). Watson and Mason (2005) made the same point that students generally identify concepts with one or two examples introduced earlier by their teachers and they are often left with incomplete and limited sense of the concept. In order to lessen the influence of prototype examples, the students might be introduced to non-examples with the same non-critical attributes and thus they may start to distinguish between critical and non-critical attributes of the concepts being taught (Wilson, 1986).

2.3. The Notion of Example Spaces

The use of examples is a fundamental and deep-seated aspect in the teaching of mathematics (Atkinson, Derry, Renkl & Wortham, 2000; Mason, 2006). Besides, examples are essential components of explanation as mentioned by Leinhardt et al. (1990):

> "Explanations consist of the orchestrations of demonstrations, analogical representations, and examples… A primary feature of explanations is the use of well-constructed examples, examples that make the point but limit the generalization, examples that are balanced by non- or counter-cases" (p. 6).

In mathematics, an example may be an instantiation of a mathematical class with specific properties, a worked-out solution, an illustration of a theorem or a reasoning method (Sinclair, et al., 2011). However, Watson and Mason (2005) claim that "one special example may not be enough to give learners an idea of the full extent of what is possible, and it may it indeed be misleading in its details" (p.5). Therefore, they developed the notion of example spaces and described it in the following way:

> "Think of an example space as a toolshed containing a variety of tools – examples that can be used to illustrate or describe or as raw material. Some tools are familiar and come to hand whenever the shed is opened, whereas others are more specialised and come to hand only when specifically sought" (p.61).

This description is similar to Tall and Vinner's (1981) construct of concept image. The term concept image has been described as follows:

> "We shall use the term concept image to describe the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes. It is built up over the years through experience of all kinds, changing as the individual meets new stimuli and matures" (p.152).

An example space can be described as the set of or classes of examples that an individual has access to and therefore it can be regarded as a subset of concept image (Edwards, 2011). Similarly, Mason and Watson (2008) stressed that an example space constitutes an individual's important part of his/her concept image and defined it as the collection of examples and non-examples which the learner have access to. In a more recent study, Zaslavsky and Zodik (2014) considered an example space "as the collection of examples one associates with a particular concept at a particular time or context" (p. 527) and emphasized that the notion of example spaces in closely linked to Vinner and Tall's concept image (Vinner 1983; Tall & Vinner 1981).

However, learners can access only to a limited number of examples at any specific moment and this is referred to as evoked example space (Zazkis & Leikin, 2007) or situated (local) personal (individual) example space (Watson & Mason, 2005), or accessible example space (Goldenberg & Mason, 2008). As Watson and Mason (2005) mentioned, examples in this space are not isolated from each other: "Example spaces are not just lists; they have an internal, idiosyncratic structure - in terms of how the members and classes in the space are interrelated - and it is through this structure that examples are produced" (p.51). In addition, example spaces are not static, but rather dynamic and evolving (Goldenberg & Mason, 2008). Likewise, a learner's concept image is not fixed; it might grow or change via experience and different parts of a learner's concept image may develop at different times and in different ways (Tall & Vinner, 1981). It can be suggested that introducing students specific examples of a concept is part of such experience and if it is accepted that students' concept image is affected by examples, then a plausible approach to rebuilding their image is to extend their example space from which they make generalizations (Zazkis et al., 2007). Similarly, Watson and Mason (2005) stress that the extension and exploration of example spaces are essential in learning mathematics:

> "Learning mathematics consists of exploring, rearranging, and extending example spaces and the relationships between and within them. Through developing familiarity with those spaces, learners can gain fluency and facility in associated techniques and discourse. Experiencing extensions of your example spaces (if sensitively guided) contributes to flexibility in thinking not just within mathematics but perhaps even more generally, and it empowers the appreciation and adoption of new concepts" (p. 6).

Generating examples not only enriches learners' example space in terms of its content but also provides opportunities for exploring its structure in terms of the connections among the elements of that space and in turn reveals or changes their sense of generality (Abdul-Rahman, 2005). To describe the structure of example spaces and to encourage learners distinguish varying aspects and structural aspects of mathematical objects, Watson and Mason (2005) extended Marton and Booth's variation theory (Marton & Booth, 1997). Put another way, Watson and Mason (2005) introduced the notions of dimensions of possible variation and the range of permissible change so as to describe the structure of examples spaces.

As mentioned before, an example space is "a set of all examples of a particular mathematical object or concept that an individual is consciously or implicitly aware, together with many associated properties the individual believes the examples possess, and any links the individual has drawn between examples" (Edwards & Alcock, 2010, p. 3). Watson and Mason (2005) also distinguished between several kinds of example spaces and they mentioned the following types:

> "situated (local), personal (individual) example spaces, triggered by a task, cues and environment as well as by recent experience; personal potential example space, from which a local space is drawn, consisting of a person's past experience (even though not explicitly remembered or recalled), and which may not be structured in ways which afford easy access; conventional example space, as generally understood by mathematicians and as displayed in textbooks, into which the teacher hopes to induct his or her students; a collective and situated example space, local to a classroom or other group at a particular time, that acts as a local conventional space" (p. 76).

In the previous sections, the definition of an example, example types, and the notion of example spaces were described in detail. In the following section, learners' difficulties and misconceptions about rational number concepts were described in some detail.

2.4. Learners' Difficulties and Misconceptions about Rational Number Concepts

In mathematics, there is not an agreed upon definition about the notion of a rational number. For instance, Tattersall (2005) and Niven (1990) defined a rational number as any number that can be written in the form of $\frac{a}{b}$ where *a* and *b* are integers and b is not equal to zero. On the other hand, Breuer (2006), Lang (2006) and Sierpinski (1998) defined a rational number as any number expressed in the form of $\frac{a}{b}$ where and b are integers, $b \neq 0$, and *a* and *b* are relatively prime. Yanık (2013) cautioned that most mathematics textbooks do not explicitly express that numerators and denominators are relatively prime in rational numbers. Başkan, Bizim and Cangül (2006) and Çelik, Çelik, Bizim and Öztürk (2013) made the same point and stressed that this restriction (*a* and *b* are relatively prime numbers) is crucial since it is used when proving whether a given number is rational or not. Nevertheless, the Turkish middle school mathematics curriculum (MoNE, 2009b) and the school mathematics textbook prepared by Aydın and Beşer, (2013a) defined the notion of a rational number as:

$\mathbb{Q} = \{\text{Any number in the form of } \frac{a}{b}, a \in \mathbb{Z}, b \in \mathbb{Z}, b \neq 0\}$. As it can be seen, this definition also ignores the relative primeness of *a* and *b*.

Rational number concepts are among the most important mathematical ideas students encounter in their school years (Alacacı, 2009; Behr et al., 1983; Behr et al., 1984; Yanık, 2013). Behr et al. (1983) stressed that rational numbers form the basis of elementary algebraic operations, they develop the ability to cope with real life problems, and help to develop and expand mental structures essential for students' continuous intellectual development. Due to their importance, Turkish mathematics curricula (MoNE, 2009a, 2009b, 2011) also give substantial emphasis on rational number concepts. According to MoNE (2009a), elementary school students (grade 1-5) are expected to learn and develop fluency with fraction concepts and operations. By the way, fractions refer to non-negative rational numbers since "students begin to study fractions long before they are introduced to the integers" Lamon (2012, p. 29).

Similarly, in middle schools (grade 6-8) sixth grade students are expected to understand fraction concepts, procedures and operations in grade 6. In addition, they are expected to apply and extend their previous understandings about fraction concepts and operations to rational number concepts and operations in grade 7. More specifically, the middle school mathematics curriculum expects seventh grade students to learn how to explain and locate rational numbers on a number line, express rational numbers in different forms, compare and order rational numbers, add or subtract rational numbers, multiply or divide rational numbers, perform multi-step operations with rational numbers, and finally pose and solve rational number problems (MoNE, 2009b).

Ultimately, in secondary schools (grade 9-12) ninth grade students are expected to explain the concept of rational number, perform addition, subtraction, multiplication and division operations with rational numbers, express the properties of addition and multiplication operations with rational numbers, order rational numbers and locate them on the number line, demonstrate the density of rational number set and convert rational numbers into their decimal forms (MoNE, 2011).

2.4. 1. Students' difficulties and misconceptions about rational number concepts

Although students are introduced to rational numbers almost at all grade levels, rational numbers are notorious for the difficulty encountered by students (e.g., Haser & Ubuz, 2003; Lamon, 2007; Ni, 2001; Vamvakoussi & Vosniadou, 2010).

Similarly, Yetim and Alkan (2013) stated that student misconceptions are prevalent in the domain of rational numbers. The differences between rational numbers and integers and natural numbers give rise to difficulties in teaching rational number concepts (Siegler, Thompson & Schneider, 2011; Stafylidou & Vosniadou, 2004). One of the factors that gives rise to student difficulties in rational number concepts is students' interference of natural number knowledge to rational numbers (Ni & Zhou, 2005; Streefland, 1991). Similarly, Post, Wachsmuth, Lesh and Behr (1985) claim that "children's understanding about whole numbers often adversely affect their early understandings about fractions. For some children, these misunderstandings persist even after relatively intense instruction based on the use of manipulative aids" (p.33). Van de Walle, Karp and Bay-Williams (2013) explained students' four common misapplications of natural number reasoning to fractions as thinking numerators and denominators as separate entities, thinking the numerator refers to any number of parts rather than the number of equal-sized parts, thinking that the fraction with a larger denominator is larger than the one with a smaller denominator, and overgeneralizing operations with natural numbers to fractions such as $\frac{1}{2}+\frac{3}{5}=\frac{4}{7}$. Similarly, some other researchers pointed out that students misinterpret the symbol $\frac{a}{b}$ by thinking *a* and *b* as two unrelated numbers, think that *a* and *b* are additively related, or think that rational numbers with large numerators and denominators are greater than rational numbers with small numerators and denominators (Moskal & Magone, 2000; Moss, 2005; Lamon, 2012; Stafylidou & Vosniadou, 2004).

Another factor that gives rise to student difficulties in rational number concepts is problems with notation of rational numbers (Moss, 2005; Ni & Zhou, 2005; Smith, Solomon & Carey, 2005). Kilpatrick, Swafford and Findell (2001) noted that there are many different representations and interpretations for rational numbers and these representations and interpretations make rational numbers difficult to understand. Accordingly, the number $\frac{a}{b}$ has five different interpretations. These are part-whole, measurement, division, operator, and ratio (Van de Walle, et al., 2013). For a meaningful understanding of fractions, students need to translate among these representations flexibly (Yetim & Alkan, 2013). However, many researchers reported that students have great difficulty in translating among these representations (e.g., Haser & Ubuz, 2002; Tirosh, Tsamir & Hershkovitz, 2008). This might be due to the emphasis placed on part-whole subconstruct by most textbooks and schools (Clarke, Roche & Mitchell, 2008; Siebert & Gaskin, 2006). Therefore, teachers should provide the students with the opportunity that there are other conceptions of fractions beyond the part-whole subconstruct (Mack, 2001; Steffe & Olive, 2010).

Not exploring fractions with different models might also explain student difficulties or misconceptions about rational number concepts. Indeed, textbooks most often do not use manipulatives and when they do, they are inclined to use only region models (Hodges, Cady & Collins, 2008). Similarly, Sowder (1988) stated that students are model poor and many of them only have circular region as their fraction model and she added that being model poor may lead to additional problems in developing understanding of fractions. Lesh, Post and Behr (1987) proposed a translation model by assuming that elementary mathematical ideas can be represented in five different ideas as real life situations, manipulatives, written symbols, verbal symbols, and pictures. Translating within and among these representations can be considered as key tasks (Ainsworth, Bibby & Wood, 2002) and may help to deepen students' conceptual understanding of fractions (Cramer, 2003). More specifically, Yetim and Alkan (2013) conducted a study to examine seventh grade students' common misconceptions and errors in expressing rational numbers in different forms. They found out that students had difficulty identifying rational numbers, linking rational numbers to decimals, representing rational numbers, performing rational number divisions and understanding how to divide a number by zero or the vice versa. Consequently, they suggested the use of concrete materials and representations in overcoming student difficulties or misconceptions about rational numbers.

Students' difficulty in understanding rational number concepts and operations seems to stem from the fact that students memorize the algorithms and the related formulas rather than understanding the essence (Şiap & Duru, 2004). Student difficulties about rational number concepts and operations can be attributed to traditional education which forces students to rote memorization rather than to conceptual understanding (Moseley, 2005). Van de Walle et al. (2013) supported this idea and expressed that students struggle with fractions since "instruction does not focus on conceptual understanding of fractions" (p. 291).

2.4. 2. Pre-service and in-service teachers' difficulties and misconceptions about rational number concepts

Rational number concepts are also very challenging for pre-service and in-service teachers (An, Kulm, & Wu, 2004; Izsak, 2008; Ma, 1999; Tirosh, 2000). More specifically, the previous research on teachers' knowledge of fractions converged on three main findings: (a) having difficulty in carrying out fraction procedures, (b) having limited understanding of fraction concepts and operations, and (c) holding misconceptions about fractions which are resistant to change (Osana & Royea, 2011). A brief summary of these main findings are presented below.

Teachers experience difficulties when performing four operations with fractions (Newton, 2008; Tirosh, 2000). For instance, Newton (2008) pointed out that pre-service elementary teachers used cross multiplication algorithm when performing multiplication of fractions and they added or subtracted across denominators when performing addition or subtraction of fractions.

There is also considerable evidence to suggest that teachers lack understanding of fraction concepts and operations (Ball, 1990; Ma, 1999; Simon, 1993). They have limited capacities in explaining the product of two rational numbers or decimals (e.g., Armstrong & Bezuk, 1995; Eisenhart et al., 1993). To illustrate, Armstrong and Bezuk (1995) introduced middle school teachers a word problem for which calculating $\frac{1}{3}$ of $\frac{3}{4}$ would be relevant. The middle school teachers realized that the problem entailed rational number multiplication. However, they had great difficulty in explaining their thinking and understanding the relevant unit or the whole for the given problem.

Some other researchers found out that teachers have difficulty realizing which problems or situations entail multiplication of decimal numbers (Graeber & Tirosh, 1988; Graeber, Tirosh & Glover, 1989). For instance, Graeber and Tirosh (1988) found out that teachers often employed inappropriate division operations when solving problems such as “One kilogram of detergent is used in making 15 kilograms of soap. How much soap can be made from 0.75 kilograms of detergent?” (p. 264). Graeber and Tirosh (1988) also found out that teachers erroneously believed that a larger number must be always divided by a smaller one in division problems such as “Twelve friends together bought 5 pounds of cookies. How many pounds did each friend get if they each got the same amount?” (p. 265).

Likewise, Tirosh (2000) reported that pre-service and in-service teachers tend to pose multiplication problems or are not able to pose correct problems for given division operations. Moreover, Tirosh (2000) organized the literature on learners’ mistakes about fraction division into three main categories as algorithmically based mistakes, intuitively based mistakes, and mistakes based on formal knowledge. Algorithmically based mistakes refer to inversion of the dividend in place of the divisor or inversion of both terms before multiplication. Intuitively based mistakes refer to overgeneralization of properties of natural number operations to fraction operations and to the interpretation of division solely as partitive model of division. Finally, mistakes based on formal knowledge occur due to learners’ limited conceptions of fractions. For instance, believing that division operation is commutative may lead to errors such as $1 \div \frac{1}{2} = \frac{1}{2}$ because $1 \div \frac{1}{2} = \frac{1}{2} \div 1 = \frac{1}{2}$. Similar to Tirosh (2000), Ball (1990a) indicated that pre-service teachers successfully performed division operations such as $1\frac{3}{4} \div \frac{1}{2}$ but they could not pose word problems for such division operations.

In another study, Işık and Kar (2012) aimed to make an error analysis of the problems posed by prospective elementary mathematics teachers about division of fractions. They observed seven different types of errors in the problems posed by prospective teachers as confusion in units (E1), assigning natural number meanings to fractional numbers (E2), posing problem using ratio and proportion (E3), not being able to establish part-whole relationships (E4), dividing to the denominator of the divisor (E5), using multiplication operation instead of division operation (E6), and posing problem through inverting and multiplying the divisor fraction (E7). E1 occurred when prospective teachers did not use the units consistently for the rational numbers included in a division operation. E2 was related with assignment of natural number meanings to fractional numbers. E3 referred to the cases in which problems were posed by comparing different units or by comparing two fractions with same units. E4 occurred due to posing division problems which include quotients (the final amount) that are larger than the dividend (initial amount). E5 occurred when prospective teachers attempted to pose problems in a way that entailed division of first rational number (dividend) to the denominator of the second rational number (divisor) instead of the second rational number itself. E6 referred to the cases in which prospective teachers posed problems that required multiplication of the dividend fraction with the divisor fraction. Finally, E7 occurred due to posing problems which entail inverting the divisor fraction and multiplying it by the dividend fraction.

Finally, some researchers reported that pre-service teachers hold several misconceptions about fraction operations (e.g., Newton, 2008; Tirosh & Graeber, 1990). For instance, Newton (2008) extensively analyzed pre-service elementary teachers' knowledge of fraction operations including addition, subtraction, multiplication and division. By including all four operations in her study, Newton (2008) detected a misconception that was very common among pre-service teachers. That is, pre-service teachers erroneously believed that having same denominators necessitates keeping the denominator of the answer the same, while having different denominators necessitates employing the given operation on the denominators. Newton (2008) suggested that pre-service teachers appeared to hold a misconception about the role of denominators unlike young students who add across numerators and denominators due to the overgeneralization of whole number thinking to fraction operations.

Similarly, Tirosh and Graeber (1990) explored the common misconceptions held by pre-service teachers. That is, many of the pre-service teachers erroneously believed that in a division operation the dividend must always be larger than the quotient. The researchers interviewed with the pre-service teachers and found out that they were able to accurately perform division operations in which the divisors are less than 1. However, they also agreed that the quotient must always be less than the dividend. Their interviews revealed that pre-service teachers relied on whole number thinking and their procedural understanding of division algorithm promoted their misconception. The researchers further noted that pre-service teachers lacked measurement interpretation of division and they tended to change procedures to keep their misconceptions.

2.4.3. Why is it important to explore teachers' treatment of rational number examples in their classrooms?

To help teachers diminish the difficulties or misconceptions encountered by students about rational numbers, Greer (1987) determined students' common misconceptions about rational numbers such as 'multiplication makes bigger, division makes smaller'. Moss and Case (1999) proposed a new curricular approach and tested it in a study involving 5^{th} and 6^{th} grade students. Moreover, NCTM (2000) emphasized using standard documents to develop elementary and middle school students' rational number reasoning.

Despite the emphasis on enhancing students' rational number understanding, student difficulties about rational numbers still persist (Wilson et al., 2013). Besides, many of the elementary and middle school mathematics topics involve rational number concepts and large scale international studies such as Programme for International Assessment (PISA) (OECD, 2010) and Trends in International Mathematics and Science Study (TIMSS) (Mullis et al., 2008) documented low mathematics performance of Turkish students. This low performance of Turkish students in mathematically important topics might indicate that less attention has been paid to rational number concepts in Turkish education system (Yetim & Alkan, 2013). Students' low performance in rational number concepts might give some clues about teachers' teaching of rational numbers. Thus, it might be essential to examine teachers' treatment of rational number examples in their classrooms in order to shed some light on this issue. In this study, I take an initial step in this direction and attempt to explore the quality and quantity of examples used by middle school mathematics teachers in the course of teaching rational number concepts. I believe that observing middle school mathematics teachers' actual classroom practices would improve not only the quality of teachers' teaching but also students' learning. The findings of Morrison (2013) strengthen this belief since she attributed students' poor performance in mathematics to "poor sequencing of examples, limited ranges of examples in the low rates of task completion within and across lessons and to more general slow pacing" (p.97).

2.5. Related Studies on Teachers' Treatment of Mathematical Examples

The review of the literature regarding teachers' treatment of examples revealed a few studies and among those while some dealt with pre-service teachers (e.g., Rowland, 2014; Rowland 2008) some others chose to study with in-service teachers (e.g., Morrison, 2013; Zaslavsky & Zodik, 2007; Zaslavsky, 2010; Zodik & Zaslavsky, 2008). The research studies related with teachers' treatment of examples are reviewed in the following section.

In a recent study, Rowland (2014) examined the examples used by two pre-service elementary teachers, why they chose those examples and whether they chose them well. One of the pre-service teachers was in the later stages of his Postgraduate Certificate in Education (PGCE) and he was teaching quadratic equations and finding equivalent expressions by completing the square (CTS) in a secondary mathematics course. The pre-service teacher used six different examples to teach CTS procedure and all of the examples were chosen by him in-advance, since they were listed in his lesson plan. He introduced $x^2+6x+8=(x+3)^2-1$ and worked out this example initially and then wrote on the board the following examples for students to try CTS on their own: $(ii)\,x^2-8x+14, (iii)\,x^2+2x-8, (iv)\,x^2+6x+5, (v)\,x^2+3x-1$ and $(vi)\,2x^2+4x-2.$ After some time, the pre-service teacher solved each example one by one together with his students. Rowland (2014) examined these examples through the lens of variation theory (Marton & Booth, 1997). He indicated that the parameters *a*, *b* and *c* correspond to dimensions of variation in the following quadratic function formula: ax^2+bx+c. The choice of the variable '*a*' affects the complexity of CTS. As can be seen, all of the examples selected by the pre-service teacher have $a=1$, except for the last example. The variable '*a*' can also take negative values and non-integer values but the pre-service teacher preferred to delay them to another lesson. Similarly, the choice of the variable 'b' remarkably influences the complexity of CTS. Especially, when '*b*' is selected to be even, the complexity decreases dramatically. As can be seen, the selected examples all included even values for the variable 'b'. Finally, the choice of 'c' is another dimension of variation and it does not influence the complexity of CTS too much, yet the pre-service teacher included both positive and negative values of 'c'.

Another secondary PGCE participant was reviewing simultaneous equations and she chose to set out by the following example: $2x+3y=16; 2x+5y=20.$ The pre-service anticipated that the students would eliminate *x* by subtraction, however she was surprised when students preferred to eliminate *y* since she did not know why they did so. Later, the pre-service teacher noticed that the sign of the coefficients of *y* included in the examples of her were always explicit whereas the sign of the coefficients of *x* were implicit (e.g., 2*x* instead of +2*x*). Thus, she noticed that limiting the coefficients of *x* to positive values might explain the choice for eliminating *y* even when it is much more easy to eliminate *x*.

Finally, Rowland (2014) recommended that it is much better to have a pre-planned sequence of examples when setting out to teach mathematical concepts or procedures such as solving simultaneous equations by substitution method. By this way, the teachers might introduce examples by gradually increasing their complexities.

In a similar study, Rowland (2008) observed the teaching of twelve prospective elementary teachers during their final school placement to determine for which purposes they used mathematical examples in their teaching. Namely, he examined pre-service teachers' good and poor choices of examples. However, he observed that pre-service teachers' poor choice of examples were more prevalent. He reported teachers' good and poor choices of examples under four categories as: variables, sequencing, representations and learning objectives.

The examples selected by pre-service teachers for the variables category reflected their poor choice of examples. More specifically, when teaching how to add and subtract whole numbers, identify co-ordinates of a point and tell the time, the examples selected by the teachers obscured the role of variables. For instance, for teaching addition and subtraction $9+9=18$ and $4-2=2$ were selected as examples, for teaching how to identify the co-ordinates (1,1) was introduced first and finally when teaching half past, half past six was demonstrated on an analogue clock (Note that both hour and minute hands point to 6 on the analogue clock for half past six).

In the sequencing category, teachers usually generated a set of examples at once and the examples reflected both good and poor choices of teachers. The sequences of examples were generated when practicing number bonds to 10 and number bonds to 100. For instance, when practicing number bonds to 10, the following numbers were selected by a teacher: 8, 5, 7, 4, 10, 8, 2, 1, 7 and 3. According to Rowland, this is a very well chosen sequence for several reasons. First, 8 and 7 are close to 10, so they require little or no counting to reach the answer. 5 puts into play doubling strategy as a key for mental computation. The selection of 4 is more confusing and the selection of 10 is a degenerate case and it does not entail counting but it emphasizes the idea that 0 can be added to 10. Finally, by selecting 8 and 2 successively, the teacher pointed to commutative property of addition.

In representations category, one of the teachers modelled subtraction operation by moving a counter vertically and horizontally on a hundred grid. The first demonstration example was selected to be 70-19. Modelling this subtraction operation on a hundred square is a very complex work since 70 is on the right boundary of hundred grid and after moving the counter two squares upwards, there is not any square on the right side of 50 (i.e., 70-20+1). Thus, it is essential to move down and then to move to the extreme left of the next row. As can be seen, the selected example obscured the general procedure for subtracting on a hundred grid. It is important to note that any of the numbers on the hundred grid except for 20, 30, 40, 50, 60, 70, 80, 90 and 100 would work properly as a minuend when demonstrating subtraction of 19.

In learning objectives category, a pre-service teacher was trying to teach distinguishing features of the concepts of translation and reflection. The teacher randomly selected a circle and a rectangle from a pile of shapes. However, the teacher's selection of shapes was not judicious since both shapes remained invariant not only after translation and but also after reflection.

Ultimately, Rowland (2008) suggested that pre-service teachers should be guided specifically and helped in understanding the various roles of examples in the teaching of mathematics. Besides, they should be informed about the existence of potential dangers or unexpected difficulties in selecting examples.

In another study, Morrison (2013) compared two foundation phase teachers' (Zelda and Deborah teach Grade 1 and Grade 2 respectively) choice and use of examples in the course of teaching number concepts. She analyzed the data from lesson observations by using the analytical framework of Rowland (2008). In her first lesson, Deborah focused on addition and in particular on counting, ordering numbers on a number line and addition on a number line. However, she did not take into account dimensions of possible variation and provided her students with examples that all involve join conception of addition. Besides, she presented those examples by using the 'result unknown' such as $3+5=\square$. She ignored taking account of variables by not using the 'change unknown' and 'start unknown' such as $3+\square=8$ and $\square+5=8$ respectively. Thus, the addition examples used by Deborah did not expose the students to a variety of addition problems that they may confront. To teach addition on a number line, Deborah generated the following sequence of examples: (i) 2+6, (ii) 10+10, (iii) 3+5 and (iv) 10+6. Deborah's second example was more complex than the first one, since it included two-digit numbers whereas the first example included one-digit numbers. However, the third example was less challenging than the second example since children usually learn double number facts such as 10+10 very quickly. Similarly, the fourth example is relatively easy when compared to the second example since it entails the addition of a single digit number to 10. Deborah chose to illustrate addition of numbers on a number line. However, she did not use the number line in a way that provided the students greater access to the concept or procedure being taught for several reasons. First, the range of numbers in the examples was so small that the students did not find it necessary to use the number line when adding. Second, the selection of well-known double (i.e., 10+10) also diminished the need for adding on a number line since it is very easy to add 10 and 10. Finally, her way of demonstrating addition on a number line resulted in a calibrated number line with an irregular scale and this made it procedurally confusing.

Zelda took account of variables when teaching counting. The possible dimensions of variation in her examples included the interval or size of the count (2 and 3); the direction of the count (backwards and forwards) and the start and end points of the counting sequences (counting in 3s by starting from 3, counting in 2s by starting from 6, and counting backwards in 2s by starting from 29). By using variation, Zelda considered the relative complexity of examples since counting from the first number in a counting sequence is easier than counting by starting at a number that is far from the first number such as starting at 6 when counting in 2s. Zelda used a random sequence of numbers when teaching how to order numbers from 0 to 15. By using an activity, Zelda asked the students to fish the numbers out of the pond and the following sequence was generated: 8, 7, 3, 11, 0, 13, 15, 1, 5, 6, 4, 12, 9, 14, 10 and 2. This sequence worked well since it helped the teacher to easily notice the students who had difficulty identifying the symbolic forms of the numbers. Zelda also used several well designed representations in a planned manner. For instance, when counting back mentally from 29 in 2s, the students had great difficulty and the teacher provided the students with a 1-100 wall chart and this representation enhanced students' ability to count backwards. Besides, Zelda gave importance to making connections between representations and she established connections between words, symbols and actions that she used to explain addition on a number line. Ultimately, Morrison (2013) pointed to the link between a higher content knowledge and the extent of a teacher's example space and suggested researchers to further explore this by using a larger sample.

In another study with secondary school teachers, Zaslavsky and Zodik (2007) attempted to explore experienced teachers' treatment of mathematical examples. They analyzed five examples in terms of their strengths and weaknesses and aimed at increasing the awareness of teachers, teacher educators and researchers to possible consequences of particular choices. Example 1 included a gradual sequence of examples that might be used to facilitate the notion of invariance, Example 2 called for sensitivity to students' misconceptions, Example 3 called for increasing students' awareness of overgeneralization, Example 4 called for a more general case and Example 5 included a teacher's poor choice of coefficients as a result of arbitrariness.

In Example 1, a teacher wanted to introduce her students' the area formula of a triangle in an 8^{th} grade pre-algebra classroom. The teacher initially provided a rectangle and its area calculation formula. Next, she introduced a right angle that is clearly half of the afore-given rectangle. Finally, she introduced a more general triangle and kept the length measurements constant. The teacher seemed to use a well-chosen set of examples for some reasons. First, the three examples were provided in a well-connected manner. Second, some features were kept constant while some others varied and this helped students better focus on varying elements such as the type of figure and the link between a side and its corresponding height. Nevertheless, there were several missed opportunities that might have influenced students' comprehension. First, it was not obvious whether the teacher considered the triangle as a general triangle and she did not articulate that in the classroom. The students may easily perceive the triangle as another right triangle that has been turned around. Second, the teacher could have asked for more suggestions on how to divide the base of the triangle into two parts such as 1-5, 2.5-3.5 if the length of the base is 6 units. By this way, she could have clearly demonstrated the idea that the area of a triangle remains invariant even if the location of the point where the height intersects the base is changed.

In Example 2, a teacher wanted to teach the notion of slope to her students. For this aim, she drew a figure on the board and at that time by examining the figure one of the students claimed that 'the first mountain is higher than the second one and thus the first one is steeper than the second one'. As can be seen, the teachers' drawing fostered a common student misconception, confusion of the concept of height with the concept of slope, of which the teacher was not aware. In response to her students' claim, the teacher erased her first drawing and drew another figure in which the two mountains had same heights but different slopes. The students' remark helped the teacher become aware of the limitation of her initial example and led her to modify it.

In Example 3, a teacher wanted to teach the concept of median of a triangle in a geometry lesson. By looking through the teachers' initial example of a median, one of the students overgeneralized that any median is also an angle bisector. Again, the teacher was not aware of the limitation of her initial example and after the student's remark, the teacher recognized that her initial example had some non-critical features that might mislead students in concept acquisition.

In Example 4, a teacher wanted to teach the concept of kite. She first drew the figure of a prototypical kite and then introduced its definition as a quadrilateral that consists of two isosceles triangles sharing the same base. Next, the teacher wanted to provide a non-example by changing the position of one of the isosceles triangles. However, this initial non-example had particular visual inferences that the teacher was not aware. At that moment, one of the students interfered and stated that the teacher drew an equilateral triangle accidentally. The teacher immediately noticed that her initial non-example was perceived as a special kite in which one of the two isosceles triangles is an equilateral triangle. Then, the teacher drew another more persuasive non-example that could be considered as a general non-example.

In Example 5, a teacher chose to use the quadratic equation $2x^2+4x+5=0$ to teach how to use the Viete formula that has to do with the sum and products of the roots of a quadratic equation. As the teacher set out to teach the formula, he noticed that the choice of his quadratic equation was a poor one since it did not have real roots $(\Delta = b^2 - 4ac = 16 - 4.2.5 = -24 < 0)$. Thus, the students were not able to use the Viete formula. Here, the teacher wanted to give a sense of randomness when selecting coefficients of the quadratic equation. However, he did not notice that the coefficients needed to be selected with some care and thus he did not check the necessary conditions. When he became aware of the limitation of his example, he provided another quadratic equation that he knew that there were certainly two real roots.

Zaslavsky and Zodik (2007) suggested teachers to keep in mind that random choice of examples may lead to visual examples that do not exist and added that teachers might better plan the examples they introduce so as to avoid mismatches and misunderstandings.

In a similar study, Zaslavsky (2010) used a number of cases to examine mathematical examples in terms of their explanatory power and the challenge of selecting relevant ones. She discussed the following major themes that are all associated with instructional explanations: (1) conveying generality and invariance, (2) explaining and justifying notations and conventions, (3) establishing the status of pupils' conjectures and assertions, (4) connecting mathematical concepts to real-life experiences and finally, (5) the challenge of constructing examples with given constraints.

Case 1 had to do with introducing the area formula of a triangle by moving from the area calculation of a rectangle, to the area calculation of a right angle and finally to the area calculation of a more general triangle. This case was explained in detail when reviewing the study of Zaslavsky and Zodik (2007).

Case 2 illustrated the potential power of examples in explaining and justifying mathematical notations and conventions. A group of mathematics educators pointed to the necessity of listing the vertices of a polygon systematically in either clockwise direction or in anticlockwise direction to avoid ambiguity related with the notation of it. To show that random use of choice of vertices of a quadrangle may lead to ambiguities, the mathematics educators generated three different quadrangles labelled as ABDCA, ACBDA and ABCDA. When these quadrangles were presented to secondary school mathematics teachers, one of them claimed that the three quadrangles were congruent. In response to the teacher's claim mathematics educators generated three distinct quadrangles with the aforementioned labels. Thus, this case showed that without mathematical convention, ambiguities may occur and this may hamper mathematical communication among learners.

Case 3 had to do with a group of secondary school mathematics teachers' validation of the claim that $\frac{2}{3}$ is in the midst of $\frac{1}{2}$ and $\frac{3}{4}$ since 2 is in the midst of 1 and 3 and 3 is in the midst of 2 and 4. This case portrayed a classroom event that entails in-the-moment decision. The teachers generated twelve examples until they reached the consensus that the claim is not always true. $\frac{2}{3}, \frac{3}{4}$ and $\frac{4}{5}$ were the counter-examples generated by the teachers since these fractions were not between $\frac{1}{2}$ and $\frac{5}{7}$. This case foregrounded the challenge that is encountered by teachers when choosing or generating relevant examples in contingent classroom events.

Case 4 is about an eighth grade teacher's choice of examples when introducing the notion of slope to her students. This case was explained in detail when reviewing the study of Zaslavsky and Zodik (2007).

In Case 5, a group of mathematics educators were asked to generate examples of two non-congruent rectangles that have equal-length diagonals. The mathematics educators proposed several solution strategies. Some of them based their solutions on Pythagoras theorem (i.e., $a^2 + b^2 = m^2 + n^2 \quad a, b, m, n \in \mathbb{N}$) and some others relied on number theory (i.e., $(ac - bd)^2 + (ad + bc)^2 = (ac + bd)^2 + (ad - bc)^2 \quad a, b, c, d \in \mathbb{N}$). This case reflected the challenge of generating examples with specific constraints and was useful in notifying that generation of a relevant mathematical example for a given purpose is an art or a problem solving process.

Zaslavsky (2010) considered the cases she presented as meta-examples and concluded that teachers need to know the critical features of examples they introduced, to be aware of the affordances of the examples generated and to have the skills to improve and extend the examples generated by their own students.

In an attempt to characterize teachers' choice of examples in and for the mathematics classroom, Zodik and Zaslavsky (2008) observed both randomly and carefully selected mathematics lessons of five experienced secondary school teachers that have at least ten years of mathematics teaching. In all their observations, they identified 604 teacher-generated examples and only 35 student generated examples. Of the teacher-generated examples that were observed, 317 of them were pre-planned and 278 of them were spontaneous. More importantly, the researchers shed considerable light into the underlying considerations or principles used by the teachers while selecting or generating mathematical examples. Namely, the teachers employed the following considerations when selecting or generating mathematical examples: starting with a simple or familiar case, attending to students' errors, drawing attention to relevant features, conveying generality by random choice, including uncommon cases and keeping unnecessary work to minimum.

The teachers started teaching concepts or procedures by simple or familiar examples. For instance, a teacher indicated that he started teaching right triangles with bases horizontal first and he introduced tilted right triangles later. Another teacher indicated that he began teaching radicals by simple and familiar examples such as $\sqrt{9}, \sqrt{16}, \sqrt{25}$ and so forth. Besides, some of the teachers generated sequences of examples and they gradually increased the complexity or the difficulty level of the examples included in those sequences. For instance, one of the teachers generated a sequence of systems of equations in the following order: First example included simple expressions in the numerator and it was easy to find the least common multiple of the denominators. Second example was not as simple as the first one since it entailed using the distributive property and finding the least common multiple was more complex. Finally, the third example included fractions that were represented differently and the choice of signs presented another difficulty.

Teachers usually generated mathematical examples by considering the difficulties, errors or misconceptions they know their students make. For instance, one of the teachers articulated that students tend to think that 'when all the variables get simplified in an algebraic expression, the answer will be equal to 0." Thereby, the teacher chose to use the algebraic expression $\frac{4a^4b^3c^2 \cdot 3a^2b^4c}{6a^2b^4c^2 \cdot 2a^4b^3c}$ to help students notice that it is equal to 1, not 0. Another teacher indicated that students tend to have common misconceptions about square roots. For instance, students tend to think that $\sqrt{50} = 2 \cdot \sqrt{25} = 2 \cdot 5 = 10$ even though $\sqrt{50}$ is equal to $\sqrt{50} = \sqrt{25 \cdot 2} = 5\sqrt{2}$ not 10. The teacher also chose to introduce $\sqrt{25}$ to draw students' attention to the fact that $\sqrt{4^2 + 3^2}$ is not equal to $\sqrt{4^2} + \sqrt{3^2}$.

The teachers also deliberately attempted to diminish the noise of examples they introduced. In other words, they tried to avoid cases that might lead to false generalizations. For instance, to teach the Pythagorean Theorem a teacher gave two examples of right triangles in which the length of the perpendicular sides were 3, 4 and 6, 8 respectively. Note that, the second pair is twofold of the first pair. To break this pattern, or in order for students not to make incorrect generalizations, the teacher then introduced 5, 12 as another example. As can be seen, this pair is not a multiple of the previous two pairs. Teachers also used structured variation in order to draw students' attention to relevant features. For instance, one of the teachers initially introduced $\left|x^2 - x\right| < 20$ and then deliberately moved to $\left|x^2 + x\right| < 20$. Similarly, another teacher presented a task that included a sequence of linear functions as follows: $f(x) = -x + 5,\ f(x) = -2x + 5,\ f(x) = -3x + 5$ and then she broke the pattern by changing the degree of polynomial from 1 to 2 and by keeping the free term constant as follows: $f(x) = -x^2 + 5$.

The teachers attempted to convey generality by selecting or generating examples at random. In some cases, random choices of examples were helpful but in some other cases they misled or caused to miss the point. For instance, a teacher wanted to teach exterior angle theorem to his students. To demonstrate that the size of an exterior angle at a vertex of a triangle is equal to the sum of the sizes of the interior angles at the other two vertices of the triangle (remote interior angles), the teacher asked his students to suggest measurements for these two remote interior angles. The students suggested 42° and 73° as the measurements. Next, the teacher wanted them to measure the size of the exterior angle. At that time, the students noticed that the size of the exterior angle was equal to 115° = 42°+73°. However, random choices were sometimes not helpful for the teachers. For instance, a teacher wanted to teach how to use Viete formula to his students $(ax^2 + bx + c = 0,\ x_1 + x_2 = -\frac{b}{a},\ x_1 \cdot x_2 = \frac{c}{a})$. The teacher randomly selected the coefficients of the quadratic equation $2x^2 + 4x + 5 = 0$. This quadratic equation did not have real roots and the students did not have prior knowledge of complex numbers that are not real. Thus, application of the formula by this example was meaningless for the students. Besides, this reflected teacher's poor choice.

Teachers also paid attention to including uncommon cases into their classrooms. That is, cases that were rather exceptional in mathematics or cases that were under-represented in the teaching of mathematics were incorporated into the lessons. For instance, some of the teachers articulated that 0 and 1 are the only numbers that remain invariant under rational number coefficients when teaching square roots such as $\sqrt{0}=0$ and $\sqrt{1}=1$. Besides, one of the teachers paid attention to introducing non-prototypical examples of a concept in addition to prototypical ones. That is, the teacher drew on the board a concave kite instead of a convex kite and asked the students to ponder whether the definition of a kite held for that example.

Teachers attempted to keep unnecessary work to minimum when teaching concepts or procedures. For instance, one of the teachers chose to use $\frac{1}{7}$ in place of $\frac{1}{17}$ or $\frac{1}{19}$ for teaching the period of a number since he considered that $\frac{1}{7}$ had a period that is long enough and there was no need to spend extra time on the technical work of finding the period of $\frac{1}{17}$ or $\frac{1}{19}$ that is much longer. In addition to this, teachers tried to keep unnecessary work to minimum by highlighting the appropriate parts of examples and not going into extra details. For example, a teacher introduced a problem to his/her students and explained how to solve that problem without finishing all the computations.

Zodik and Zaslavsky (2008) emphasized that the wide range of episodes they observed might provide a rich source for adapting them into teacher education programs and they added that this might be very helpful in obtaining systematic knowledge that promotes teachers' theoretical and practical knowledge of treatment of mathematical examples.

Similar to the previous study, Bills and Bills (2005) explored the initial examples used by in-service and pre-service teachers in introducing particular mathematical topics. More specifically, the teachers were asked to consider the examples they might select as the first one to use in introducing the calculation of area of a triangle, addition of fractions and solution of linear equations. Besides, the experienced teachers were asked to articulate their pedagogical intentions in selecting a particular example as an introductory example. When teachers were asked to give the initial example they would use to introduce the calculation of area of a triangle, all but one of the in-service teachers preferred to start with a right-angled triangle, whereas only nearly half of the pre-service teachers preferred to start with a more general triangle that is not right-angled. The in-service teachers gave emphasis on building up from a simple case (from a right-angled triangle) to help students learn how to use the formula for calculating the area of any triangle.

The in-service and pre-service teachers were next asked to give the first example they would use in introducing addition of fractions. Two thirds of the in-service teachers preferred to start with examples that included only halves and/or quarters such as ½ + ½, ¼ + ¼ and ½ + ¼, whereas six out of ten pre-service preferred to start with ½ + ½ and ½ + ¼. The in-service teachers addressed the role of known facts and procedures in developing understanding and again they emphasized starting with a simple case when introducing addition of fractions.

Finally, the in-service and pre-service teachers were asked to give the first example they would use in introducing solution of linear equations. There was a common consensus among in-service teachers on choosing examples where the solution is a positive whole number, where the unknown appears first and where there is a single operation on the unknown such as $x+1=3$ instead of $1+x=3$ or $3=x+1$. However, pre-service teachers' initial example preferences for introducing the solution of linear equations were more varied.

Bills and Bills (2005) identified two themes that emerged from the discussions among in-service and pre-service teachers: simple example as a first step in developing understanding of a mathematical concept and the use of mathematical examples to avoid confusion. However, the researchers alerted that their data analyses was based on in-service and pre-service teachers articulations of their possible choices of examples as initial examples rather than the data derived from observing in-service and pre-service teachers' actual classroom practices.

2.6. Summary of the Related Studies on Teachers' Treatment of Mathematical Examples

Rowland (2008) developed a conceptual framework for analyzing pre-service teachers' choice and use of examples in the course of teaching elementary mathematics concepts such as addition and subtraction of whole numbers and geometric transformations. This framework included four categories of uses of examples as variables, sequencing, representations and learning objectives. Rowland (2008) found out that the examples included under these categories mainly reflected pre-service teachers' poor choices. Morrison (2013) conducted a similar study by using the same framework. However, she selected in-service teachers as the participants of the study and focused on in-service teachers' examples related with number concepts. The findings of Morrison (2013) were similar to that of Rowland (2008). That is, Morrison (2013) pointed out that in-service teachers did not take into account dimensions of possible variation when using examples related with number concepts. In a more recent study, Rowland (2014) focused on only the variables category of his conceptual framework and analyzed pre-service teachers' choice of examples related with quadratic and simultaneous equations. Similar to his previous study, Rowland (2014) suggested that pre-service teachers needed to better plan examples before setting out to teach mathematical concepts or procedures in order to introduce examples by gradually increasing their complexities.

Unlike the previous studies, Zaslavsky (2010) examined the explanatory power of examples used by in-service teachers. She discussed the following themes that were all related with instructional explanations: conveying generality and invariance, explaining and justifying notations and conventions, establishing the status of pupils' conjectures and assertions, connecting mathematical concepts to real life experiences and the challenge of constructing examples with given constraints. Zaslavsky (2010) suggested that teachers needed to know the critical features of examples they introduced, to be aware of the affordances of the examples they generated and to have the skills to improve and extend the examples generated by their own students. Similar to Zaslavsky (2010), Zaslavsky and Zodik (2007) focused on examining strengths and weaknesses of examples generated by in-service teachers. They analyzed almost the same examples included in the study of Zaslavsky (2010). Similar to Rowland (2014), they suggested teachers to plan their examples to avoid mismatches and misunderstandings.

In another study, Bills and Bills (2005) examined experienced teachers' pedagogical intentions in selecting particular examples for introducing the calculation of area of a triangle, addition of fractions and solution of linear equations. They found out that experienced teachers' preferred to use simple examples as a first step in developing understanding of a mathematical concept and to avoid confusion. Similarly, Zodik and Zaslavksy (2008) focused on exploring experienced teachers' considerations or principles in using mathematical examples. However, they not only found that teachers considered to start with simple or familiar examples for introducing mathematical concepts but also they revealed that teachers considered to attend to students' error, draw attention to relevant features, convey generality by random choice, include uncommon cases, and keep unnecessary work to minimum when using examples.

CHAPTER III

METHODOLOGY

The purpose of this study was to explore middle school mathematics teachers' treatment of rational number examples in their seventh grade classrooms. More specifically, this study aimed to shed light on overall characteristics of teachers' rational number examples, the principles or considerations used by teachers while choosing rational number examples and the mathematical and pedagogical shortcomings of the examples used by the teachers. Through this purpose, the following major questions and sub-questions were formulated:

1. What are the overall characteristics of examples used by middle school mathematics teachers in the teaching of rational numbers in their seventh grade classrooms?

    a. What are the ideas emphasized in the rational number examples used by the teachers?
    b. To what extend do teachers use specific examples in the teaching of rational numbers?
    c. To what extend do teachers use non-examples and counter-examples in the teaching of rational numbers?
    d. To what extend do teachers use pre-planned and spontaneous examples in the teaching of rational numbers?
    e. Which sources do teachers use while choosing pre-planned examples in the teaching of rational numbers?

2. What are the underlying principles or considerations that guide middle school mathematics teachers in choosing or generating examples?

3. What mathematical or pedagogical shortcomings do the examples used by the teachers in the teaching of rational numbers have?

    a. What are the mathematically incorrect examples used by the teachers during the teaching of rational numbers?

b. What are the pedagogically improper examples used by the teachers during the teaching of rational numbers?

In this chapter, (a) the overall research design and the selected strategy of inquiry, (b) participants of the study and the contexts, (c) data collection process, (d) data sources, and (e) data analysis procedure were described first. Next, the methods that might be employed to ensure (f) the trustworthiness of the current research and (g) researcher role and bias were explained. Finally, (h) the limitations of the study were discussed.

3.1. Overall Research Design

Creswell (2009) stated that qualitative and quantitative research designs differ from each other basically in terms of using words rather than numbers and using open-ended questions rather than close-ended questions. He added that the philosophical assumptions of researchers, the types of research strategies used in the overall study and the specific methods used to conduct these strategies also provided a more complete way to view the difference between qualitative and quantitative research designs.

Qualitative researchers are interested in understanding people's interpretation of their experiences, their construction of the world and the meaning they give to their experiences (Merriam, 2009). They tend to collect data at the site where participants experience the problem or the phenomena under study (Creswell, 2007) and they do not attempt to manipulate the phenomenon of interest while seeking to understand it (Patton, 2002). Qualitative researchers collect descriptive data and the data collected take the form of words or pictures instead of numbers (Bogdan & Biklen, 2007). They collect data themselves and gather multiple forms of it such as observations, interviews and documents instead of relying on a single data source (Creswell, 2007). They tend to analyze their data inductively rather than finding evidence to prove or disprove hypothesis held before conducting the study (Bogdan & Biklen, 2007).

The primary focus of this study was to make a detailed description of the examples used by middle school mathematics teachers via qualitative research methodologies. This study was conducted in the hope that it would add significant findings to the literature by in depth exploration of the phenomenon of mathematical examples. This study used open-ended research questions (how and what questions) in order to get rich and detailed ideas about middle school mathematics teachers' experiences of using or choosing mathematical examples in their classrooms. In addition, the study limited its participants to four middle school mathematics teachers since qualitative research designs produce much detailed data about a small number of cases. To gain insights about participants' treatment of mathematical examples, data were collected through observations and interviews and the data collected took the form of words or pictures after the transcription process. In this study, rather than using statistics, the words or pictures generated by the teachers were analyzed to describe the central phenomenon under study. Moreover, this study described individuals and identified themes. As a result, a rich and complex picture emerged and by using this complex picture, I tried to make an interpretation of the meaning of the data by reflecting upon the relationship between my findings and the previous research on examples. Finally, when reporting the findings, I reflected my own biases, values, and assumptions and actively wrote them into the current research study. By this way, I discussed my role or position in the research study.

There are several qualitative research methodologies addressed by researchers. The strategy of inquiry selected for the current study was a case study. The rationale for using this strategy is described in details in the following section.

3.2. The Selected Strategy of Inquiry

In order to investigate middle school mathematics teachers' treatment of rational number examples in their classrooms, qualitative case study was used. Broadly speaking, "qualitative case study is characterized by the researcher spending extended time on site, personally in contact with activities and operations of the case, reflecting, and revising descriptions and meanings of what is going on" (Stake, 2005, p.450).

Case study is used by many people in several ways to mean several things (Merriam, 2009). Besides, the definitions of case study provided by educational researchers differ from each other to a certain extent. To give an example, Creswell (2007) viewed case study as a methodology, a type of design in qualitative research, an object of study or a product of the inquiry and defined it in the following way:

> "Case study is a qualitative approach in which the investigator explores a bounded system (a case) or multiple bounded systems (cases) over time, through detailed, in-depth data collection involving multiple sources of information (e.g., observations, interviews, audiovisual material, and documents and reports), and reports a case description and case based themes" (p. 73).

Similarly, Merriam (2009) defined qualitative case study as "an intensive, holistic description and analysis of a bounded phenomenon such as a program, an institution, a person, a process, or a social unit" (p. x). By this definition, Merriam stressed the importance of the case as a single entity or a unit that has boundaries. Stake (1995) pointed out that "case study is the study of the particularity and complexity of a single case, coming to understand its activity within important circumstances" (p. xi).

Yin (2003) distinguished case study from other methods such as experiments, history, and survey by making a comparison of the features of the related methodologies. In addition, he defined case study in two phases in a more technical way when compared with the previous definitions. In the first phase, the context and the phenomenon could be easily noticed and therefore he defined case study as "an empirical inquiry that investigates a contemporary phenomenon within its real-life context, especially when the boundaries between phenomenon and context are not clearly evident" (p. 13). In the second phase, the context and the phenomenon were not always distinguishable, and he indicated that

> "case study inquiry copes with the technically distinctive situation in which there will be many more variables of interest than data points, and as one result relies on multiple sources of evidence, with data needing to converge in a triangulating fashion, and as another result benefits from the prior development of theoretical propositions to guide data collection and analysis" (p. 13-14).

By the help of these different definitions of case study, it can be inferred that the most significant aspect of a case study is the object of the study or the case and its relation with its context. Thus, cases and their contexts should be carefully defined. In addition, it can be suggested that a case is a specific, unique and bounded system. Merriam (2009) mentioned that boundaries of a case play an important role in defining the case. Hence, the purpose of case study is to describe and interpret the case within its boundaries and the context, but not to represent the world (Stake, 2005; Yin, 2003).

The current study might characterize the definitions of Creswell (2007), Merriam (2009), Stake (2005), and Yin (2003). In this study, my aim was to "gain in-depth understanding of the situation and meaning for those who are involved" (Merriam, 1998, p. 19) and I specifically focused on exploring middle school mathematics teachers' treatment of examples in their classrooms.

Education researchers also made different categorizations for case studies (e.g., Creswell, 2007; Merriam, 1998; Stake, 2000; Yin 2003). Yin (2003) mentioned four types of case study designs: single-case design with single unit of analysis (holistic), single-case design with multiple units of analysis (embedded), multiple-case design with single unit of analysis (holistic), and multiple-case design with multiple units of analysis (embedded). In this study, multiple-case design with single unit of analysis was used. This design was modeled by Yin (2003) in the following way in Figure 3.1.

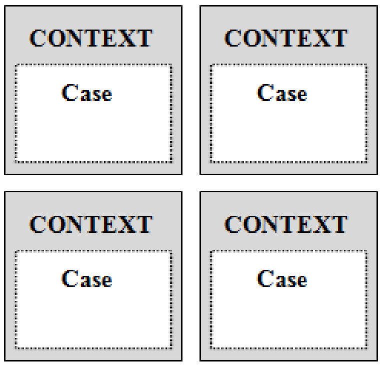


Figure 3.1. Multiple case study design with single unit of analysis (Yin, 2003, p. 40)

The context of this study was rational number instruction and the cases were middle school mathematics teachers with different rational number teaching experiences with the unit of analysis as middle school mathematics teachers' rational number examples. The context, cases and the unit of analysis could not be separated from each other and therefore they were considered all together. The model for rational number instructions, middle school mathematics teachers with different rational number teaching experiences and middle school mathematics teachers' examples is given in Figure 3.2.

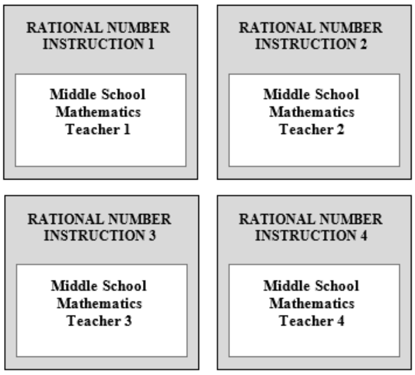


Figure 3.2. Multiple case study design with single unit of analysis (Adapted from, Yin, 2003, p. 40)

In the following section, I will give detailed information about middle school mathematics teachers who participated in the current study.

3.3. Participants of the Study

In this section, middle school mathematics teachers who were the cases of the current study were described. Four middle school mathematics teachers from four different public middle schools located in the Aksaray city center participated in the current study. All of the teachers were teaching rational number concepts to 7th grade students during the data collection process. In the selection of the participant schools, convenience sampling (Fraenkel, Wallen & Hyun, 2012; Gall, Gall & Borg, 2007) was used to ensure feasibility during the data collection process to the most extent possible and to work with teachers with different rational number teaching experience who were teaching rational numbers and who volunteered to be a participant. Merriam (2009) expressed that "purposeful sampling is based on the assumption that the investigator wants to discover, understand, and gain insight and therefore must select a sample from which the most can be learned" (p.77). In this study, it was crucial for me to select information-rich middle school mathematics teachers in order to get in-depth information about their treatment of rational number examples. Thus, rather than using probabilistic sampling, I chose to use purposive sampling.

A variety of purposeful sampling strategies might be used to select cases for a research study such as extreme/deviant case, typical case, maximum (maximal) variation, homogeneous, critical, opportunistic, snowball and so forth (Creswell, 2012; Gall, Gall & Borg, 2007). In the current study, I used maximum variation sampling. Creswell (2007) stated that this purposeful sampling strategy enables researchers to select cases that show different perspectives on the problem, process, or event they might portray. Thus, I was able to select middle school mathematics teachers with different years of rational number teaching experience ranging from two years to fourteen years. In brief, I selected the participant schools and the participant teachers based on the following criteria:

1. Rational number teaching experience of teacher: Because my aim as a researcher was to understand how rational number examples are treated by different teachers, in different settings and with different rational number teaching experience.
2. Convenience of time: Because I was also working as a research assistant at a University, my own work schedule and that of the four teachers had to fit with each other.
3. Convenience of location: Since I visited schools 4 days a week for about 4 months, the schools had to be close to each other and the transportation to these four schools had to be easy.
4. Voluntary participation: Middle school mathematics teachers who were the participants of the study were selected based on voluntariness. Altogether, there were 17 middle school mathematics teachers in these 4 schools. However, not all teachers wanted to participate in the current study.

This study was conducted in four different public middle schools in Aksaray in the first semester of the 2013-2014 school year. Three male and one female middle school mathematics teachers with different teaching experiences took part in the study. The names of the participating teachers were changed to ensure confidentially and pseudonyms were used during the study. The description of each participant is presented below according to their rational number teaching experience in descending order.

3.3.1. Teacher A

Teacher A was 36 years old at the time of data collection and he graduated from the mathematics department of a public university in 2000. During his undergraduate education, he earned a non-thesis master degree in Secondary Education Teacher Graduate Program at a different public university. Since he graduated from mathematics department, this degree is required for being a mathematics teacher at public schools governed by Ministry of National Education in Turkey. He does not hold a graduate degree. Teacher A started teaching mathematics to middle school students soon after he graduated from mathematics department. He has 14 years of mathematics teaching experience and 14 years of rational number teaching experience. He has been a mathematics teacher of 6^{th}, 7^{th}, and 8^{th} grade middle school students for 14 years. In addition, he has been teaching 5^{th} grade students for the last 6 years. He has worked in 3 different cities in Turkey and at 5 different public middle schools throughout his teaching profession. Since 2005, he has been teaching in Aksaray city and he has been teaching for 3 years in the current school that has been observed by the researcher. In his current school, there are five 7^{th} grade classrooms and he is teaching two of those classrooms. Besides, he was also teaching 5^{th}, 6^{th}, and 8^{th} grade students during the implementation of this study.

3.3.2. Teacher B

Teacher B was 36 years old at the time of data collection and he graduated from the mathematics department of a public university in 2000. After his graduation from the mathematics department, he earned his non-thesis master degree in Secondary Education Teacher Graduate Program at the same university. He earned his master's degree in the Department of Mathematics at a public university in 2003. He is currently doing his PhD in the Department of Mathematics at a different public university. Teacher B started his teaching profession as soon as he completed his non-thesis master program. He has 11 years of mathematics teaching experience and 10 years of rational number teaching experience. He first started teaching mathematics at a public secondary school to 9th, 10th, 11th, 12th grade students for one year. For the last 10 years, he has been a teacher in Aksaray city at three different public middle schools. More specifically, he has been teaching mathematics to 5th, 6th, 7th, and 8th grade students for 2, 10, 10, and 8 years respectively. In addition, he has been teaching for 2 years in the current school that I observed. In his current school, there were four 7th grade classrooms and he was teaching mathematics to all these classrooms. In addition to this, he has been conducting mathematical applications courses for 5th grade students during that time.

3.3.3. Teacher C

Teacher C was 31 years old at the time of data collection and he graduated from Elementary Mathematics Education Program of a Department of Elementary Education of a public university in 2005. He does not hold a graduate degree. After his graduation, he immediately started his profession at a public middle school in the middle regions of Turkey. He has 9 years of mathematics teaching experience and 8 years of rational number teaching experience. In more details, he has taught mathematics to 5th, 6th, 7th, and 8th grade students for 1, 9, 8, and 7 years respectively. He has worked in 2 different cities and in 3 different public middle schools since the beginning of his teaching profession. He has been working in Aksaray since 2010 and has been teaching in the current observed school for 2 years. In this school, there were four 7th grade classrooms and he was teaching mathematics to all of these classrooms. In the meantime, he was also a mathematics teacher of 5th and 6th grade students.

3.3.4. Teacher D

Teacher D was 26 years old at the time of data collection and she graduated from Elementary Mathematics Education Program of a Department of Elementary Education of a public University in 2010. She does not hold a graduate degree. Between years 2010 and 2012, she worked at a private studies centre (etüt merkezi) in Aksaray city. During this time, she also worked as a private tutor of mathematics. Two years after her graduation, she started working at a public middle school located in the eastern part of Turkey. She has been a teacher at this middle school for one year and taught mathematics to 5^{th}, 6^{th}, 7^{th}, and 8^{th} grade students during that time. In 2013, she started teaching at a public middle school located in Aksaray city and taught mathematics for 6^{th}, 7^{th}, and 8^{th} grade students. Together with private tutoring, she has 4 years of mathematics teaching experience and 2 years of rational number teaching experience. More precisely, she has been a mathematics teacher of 5^{th}, 6^{th}, 7^{th}, and 8^{th} grade students for 1, 3, 4, and 3 years respectively. She has worked in 2 different cities in 2 different public middle schools since the beginning of her teaching profession. In her current school, there were four 7^{th} grade classrooms and she was a mathematics teacher of three of these classrooms. Apart from these, she was also a teacher of 8^{th} grade students.

To sum up, a brief descriptive demographic account of these four middle school mathematics teachers are presented in Table 3.1.

Table 3.1. Teachers' demographic information for the four classrooms at the time of the study

Description	Teacher A	Teacher B	Teacher C	Teacher D
Gender	Male	Male	Male	Female
Age	36	36	31	26
University	Public	Public	Public	Public
Background	Mathematics	Mathematics	Elementary Mathematics Education	Elementary Mathematics Education
Total years in teaching	14	11	9	4
Total years in the teaching of rational numbers	14	10	8	2
Years in Aksaray	9	10	4	3
Years in current school	3	2	2	1

3.4. The Contexts of the Study

Turkish education system is a centralized system. Thus, all teachers and students follow the same national mathematics curricula. MoNE (2009a, 2009b, 2011) had three different official curriculum guidebooks for elementary (grades 1-5), middle (grades 6-8) and secondary (grades 9-12) levels at the time of the study. These guidebooks represented the intended mathematics curricula by “providing in-depth background information about the philosophy, goals and approaches of the curriculum, content to be covered together with some sample introductory tasks and tips to be used in the classroom” (Ubuz, Erbaş, Çetinkaya & Özgeldi, 2010, p. 484). At the time of the data collection of the study, the seventh grade school mathematics curriculum (MoNE, 2009b) being implemented by the schools provided the learning objectives, sample activities and explanations given in Table 3.2 for teachers to use in their classrooms when teaching rational numbers.

Table 3.2. The learning objectives, sample activities and explanations included in the middle school mathematics curriculum for teaching rational numbers (MoNE, 2009b, p. 224-226) (Translations by the researcher)

Learning Objectives	Sample Activities	Explanations
Explain and locate rational numbers on a number line	□ Let students discuss why there is a need for rational numbers by asking several questions to them. Write natural numbers and several fractions on cards and put them in a bag. Pick up the cards randomly from the bag and locate them on the number line. Next, place a symmetry mirror at the origin of the number line. Determine the symmetries of the points on the number line and emphasize negative numbers and absolute value concept.	[!] Denote rational number set by the symbol Q and define it. [!] Have students examine the relationships among natural numbers, integers and rational numbers. □ Have students search the history of rational numbers.
Express rational numbers in different forms	□ Write rational numbers on cards and put them in a bag. Pick up the cards randomly and find the decimal representations of the rational numbers written on these cards by using a calculator. Finally, classify the decimal representations of the rational numbers and have students interpret the results. $\frac{28}{4} = 7 \quad -\frac{8}{16} = -0.5 \quad \frac{2}{9} \approx 0.\bar{2}$	[!] Have students convert repeating decimals into rational numbers. [!] Demonstrate by using examples that a rational number can also be expressed as an integer, as a natural number, as a terminating or as a repeating decimal.

Table 3.2. (Continued)

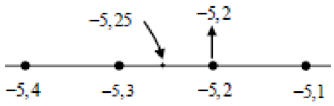
Learning Objectives	Sample Activities	Explanations
Compare and order rational numbers	□ To have students notice the difference between a rational number and an integer, ask them to write the integers between two integers. This activity can be repeated by using rational numbers. Then, the two activities can be compared and discussed. □ Fraction estimation strategies, common denominator approach and conversion to decimals can be used to compare rational numbers. For instance, to compare -5.2 and $-5\frac{1}{4}$ we can convert the negative mixed number into its decimal form as -5.25 and write as $-5.2 >$ -5.25. Thus, $-5.2 > -5\frac{1}{4}$. These two rational numbers can be located on the number line as follows:  	[!] Emphasize that the comparison strategies used for fractions and integers may also be used for comparing rational numbers.
Perform addition and subtraction operations with rational numbers	□ Recall addition and subtraction of fractions and have students participate in activities that include addition and subtraction of rational numbers. □ Have students estimate the addition of rational numbers first, have them perform addition operation next and finally have them compare the estimated answer and the actual answer in the following way: Let's estimate the answer of $\frac{3}{8}+\frac{6}{7}$. The first addend is closer to and the second addend is closer to 1. Thus, $\frac{3}{8}+\frac{6}{7} \approx \frac{1}{2}+1 = 1\frac{1}{2}$. Now, let's find the actual answer: $\frac{3}{8}+\frac{6}{7}=\frac{21}{56}+\frac{48}{56}=\frac{69}{56}=1\frac{13}{56}$	[!] Have students examine the commutative, associative, identity and inverse property of addition of rational numbers and have them write the algebraic representations of these properties. [!] Give students examples related with estimation of addition and subtraction with rational numbers. Use the estimation strategies included in the initial part of the curriculum guidebook.

Table 3.2. (Continued)

Learning Objectives	Sample Activities	Explanations
Perform multiplication and division operations with rational numbers	□ Recall multiplication and division of rational numbers and have students participate in activities related with multiplication and division of rational numbers.	[!] Have students examine the influence of 0, 1 and -1 on multiplication and division operations. [!] Have students examine commutative, associative and zero property of multiplication of rational numbers and have them write the algebraic representations of these properties. [!] Emphasize that if the product of two rational numbers is equal to 1, then these two numbers are multiplicative inverses of each other. [!] Give students examples related with estimation of multiplication and division with rational numbers. Use the estimation strategies included in the initial part of the curriculum guidebook. [!] Have students compute the square and cube of rational numbers.
Solve multi-step operations with rational numbers		[!] Remark that the operations that needs to be done initially in multi-step operations are specified by brackets or parentheses. [!] Emphasize that in complex fractions the order of operations are determined by the main fraction bar.
Pose and solve rational number problems	□ Ask students to read the problem very carefully, restate the problem with their own words, identify the givens in the problem, make a plan for the solution of the problem, carry out the plan, check the solution and discuss the problem.	[!] The explanations included in the introductory part of the curriculum guidebook about problem solving should be taken into consideration.

All of the classrooms I observed used the same mathematics textbook prepared by Aydın and Beşer (2013a). This textbook was prepared by a private publisher in triple sets comprising student textbook, student workbook and teacher guidebook. In Turkey, the textbooks prepared either by MoNE or by the private publishers need to be reviewed and approved by the Turkish Board of Education

(Talim ve Terbiye Kurulu Başkanlığı-TTKB) so that they are used as official textbooks in public schools (Ubuz et al., 2010). Thus, it is believed that the mathematics textbook used by the four classrooms completely portrays the curriculum content that needs to be learnt by the seventh grade students as it was approved by the Board.

The teachers were observed during the unit entitled "Rasyonel Sayılarla Dans Edelim (Let's Dance with the Rational Numbers" in the student textbook. In this unit, rational number concepts were introduced under two main sections. The first section was about explaining and locating rational numbers on a number line, expressing rational numbers in different forms and comparing and ordering rational numbers. The second section was about performing addition and subtraction operations with rational numbers, performing multiplication and division operations with rational numbers, solving complex fractions using four operations and posing and solving rational number problems. In this study, the worked-out examples and exercises that were included in the explanatory part of the textbook and that might be offered by the teachers while teaching rational number concepts were treated as student textbook examples. The illustrative worked-out and exercise examples included in the student textbook for teaching each rational number objective are presented in Table 3.3.

Table 3.3. Illustrative worked-out examples (WE) and exercise examples (EE) included the student textbook for introducing each learning objective (Aydın & Beşer, 2013a, p. 47-77)

Learning Objectives	Example type	Illustrative examples
Explain and locate rational numbers on a number line	WE	$\frac{-26}{-5}$ The numerator is a negative integer; The denominator is a negative integer } $\frac{-26}{-5}=\frac{26}{5}\in Q^{+}$
	EE	Locate $\frac{7}{3}, -\frac{5}{4}, -2, \frac{3}{7}$ on the number line.
Express rational numbers in different forms	WE	$\frac{7}{2}=3.5,\ \frac{7}{3}=2.333...=2.\bar{3}$
	EE	Express the following numbers in different forms: $1.4=...\quad 2=...\quad \frac{7}{8}=...\quad 0.0\bar{5}=...$
Compare and order rational numbers	WE	Let's order $\frac{8}{10}, \frac{11}{9}, \frac{16}{16}$ by benchmarking to 1. $\frac{8}{10}<1,\ \frac{11}{9}>1,\ \frac{16}{16}=1$ Therefore, $\frac{8}{10}<\frac{16}{16}<\frac{11}{9}$
	EE	Order the following numbers from the largest to the smallest and explain your strategy for ordering. $1.9; 1.08; 1\frac{7}{8}\quad -4.45; -5.54; -5.\bar{5}\quad \frac{3}{8}; \frac{5}{12}; \frac{17}{24}\quad -\frac{2}{5}; -\frac{4}{11}; -\frac{8}{19}$
Perform addition and subtraction operations with rational numbers	WE	Let's perform $\frac{4}{5}+\left(-\frac{2}{3}\right)$: $\frac{4}{5}+\left(-\frac{2}{3}\right)=\frac{4.3+5.(-2)}{5.3}=\frac{12+(-10)}{15}=\frac{2}{15}$
	EE	Perform $\frac{2}{3}-\left(-\frac{3}{2}\right)$ and $-\frac{2}{3}+\left(-\frac{3}{2}\right)$
Perform multiplication and division operations with rational numbers	WE	$\left(+\frac{2}{3}\right)\cdot\left(+\frac{3}{4}\right)=\frac{(+2)}{3}\cdot\frac{(+3)}{4}=\frac{+6}{12}=+\frac{6}{12}$; $\left(-\frac{4}{5}\right)\cdot\left(-\frac{2}{3}\right)=\frac{(-4)}{5}\cdot\frac{(-2)}{3}=\frac{+8}{15}=+\frac{8}{15}$ } The multiplication of two rational numbers with same signs yields a positive product
	EE	Estimate the following multiplication operations: $479\cdot 3\frac{1}{18}\quad \left(-24\frac{1}{9}\right):\left(-11\frac{7}{8}\right)\quad 580:\left(19\frac{1}{19}\right)$

Table 3.3. (Continued)

Learning Objectives	Example type	Illustrative examples
Solve multi-step operations with rational numbers	WE	$-\frac{2}{3}-\left(\frac{5}{6}:\frac{1}{4}\right)+\frac{1}{3}$ $=-\frac{2}{3}-\left(\frac{5}{6}\cdot\frac{4}{1}\right)+\frac{1}{3}$ → Perform division operation $=-\frac{2}{3}-\frac{20}{6}+\frac{1}{3}$ → Simplify the answer $=\frac{-2}{3}+\left(\frac{-10}{3}\right)+\frac{1}{3}$ → Determine the numerators of the addends by means of rational number signs $=\frac{-2+(-10)+1}{3}=-\frac{11}{3}$ → The numerators are added to each other and the denominator stays the same
	EE	Perform the following multi-step operations and explain which rule you used in each step. $\frac{1-\frac{3}{5}}{\frac{3}{5}-1}:5$ $\quad \frac{2}{3}-\frac{1}{\frac{1}{3}+\frac{1}{9}}$ $\quad 1+\frac{1}{1+\frac{1}{1+\frac{1}{3}}}$ $\quad \left[3\cdot\left(1+\frac{5}{3}\right):\frac{1}{7}\cdot\left(\frac{1}{4}+\frac{1}{3}\right)\right]:6$
Pose and solve rational number problems	WE	Ahmet initially walked $\frac{1}{3}$ of his route. After some time, he walked $\frac{2}{5}$ of the remaining route and he had to walk 36 meters more to finish his route. Find the total length of his route.
	EE	Fill in the blanks with the numbers relevant to the problem. An athlete each day runs … times as much as the distance she runs the day before. The athlete runs for … days and finishes her training program. If the athlete runs … kilometers in her … day, then how many kilometers does she run on the last day of her training program?

The four public middle schools that were selected as the contexts of this study were located in Aksaray city center. The schools were close to each other and this made it possible for the researcher to commute among them easily. The participant schools and the participant classrooms were named as School A – Classroom A, School B – Classroom B, School C – Classroom C, and School D – Classroom D to protect the privacy of the middle school mathematics teachers.

Teacher A was a teacher in School A. There were 40 teachers (4 of whom were mathematics teachers) and 811 students in this school at the time of the study.

In addition to 24 classrooms, there was a computer laboratory but not a science and technology laboratory or a mathematics laboratory. There was also an auditorium and a library in this school. The students of School A had high socio-economic status. However, this school did not have mathematics clubs for active participation of students.

Teacher B was a teacher in School B. There were 47 teachers and 850 students in the school during the study. There were 4 mathematics teachers in this school. There were a total of 34 classrooms and there was a science and technology laboratory and 2 computer laboratories but not a mathematics laboratory. Besides, there was a library but not an auditorium in this school and the students of School B had medium socio-economic status. It is important to note that this school organized mathematics clubs to support students' mathematical thinking with some activities.

Teacher C was a teacher in School C. There were 35 teachers and 654 students in the school at the time of this study. Four of the teachers were mathematics teachers in this school. There were 15 classrooms and there was a science and technology laboratory and a computer laboratory but not a mathematics laboratory. Moreover, there was a library and an auditorium in this school and the students of School C had high socio-economic status. Lastly, this school also organized mathematics clubs for fostering students' mathematical thinking.

Teacher D was a teacher in School D. There were 31 teachers and 461 students in the school at the time of the current study. There were 5 mathematics teachers in this school. Totally, there were 18 classrooms and there were not any science and technology, computer and mathematics laboratories. Besides, the students of School D had medium socio-economic status and the school did not organize any mathematics clubs.

To summarize, School A and School B were more populated than School C and School D. However, they were more or less similar to each other in terms of their environments, classroom size, laboratories, equipment and so forth. Broadly, the teachers in these schools had more than 10 years of mathematics teaching experience. While School A, School B and School C had four mathematics teachers, School D had five mathematics teachers. Moreover, School B and School C had students actively participate in mathematics clubs, but the other two schools did not have such clubs.

In the next section, the observed classrooms of participant schools were described at length. The observed classrooms all consisted of 7th grade students and these classrooms were of different socio-economic level and achievement level.

3.4.1. Teacher A's classroom

Teacher A's class in which the study was conducted had a total of 32 students including 18 female students and 14 male students. The classroom had a teacher desk, student desks, a large whiteboard, a bulletin board and an overhead projector. The teacher projected the mathematical examples on the white board when necessary. On some of the desks, one student was sitting while on other desks there were two students sitting. The bulletin board was used by the teacher for displaying examination results or by students for hanging activity sheets or drawings. There were 3 columns and 6 rows of desks in the classroom. Teacher A stated that students in this classroom were from high socio-economic status families and their achievement level was average. In all the classrooms, I sat at the backmost desk in order not to interrupt the classroom during my observations. A snapshot of the classroom environment and the seating plan is presented in Figure 3.3.

Figure 3.3. A snapshot of the Classroom A and the seating plan

3.4.2. Teacher B's classroom

Teacher B's class in which the study was conducted had a total of 22 students including 13 female students and 9 male students. The classroom had a teacher desk, student desks, a medium sized whiteboard, a cupboard, a bulletin board, a computer, and an overhead projector. Similar to Teacher A, Teacher B projected the mathematical examples on the white board when necessary. The students were sitting in pairs on all desks. The bulletin board was used for displaying drawings, term projects, activity sheets and examination results. There were 3 columns and 5 rows of desks in the classroom. Teacher B stated that students in this classroom were from medium socio-economic status families, and their achievement level was high. A snapshot of the classroom environment and the seating plan is presented in Figure 3.4.

Figure 3.4. A snapshot of the Classroom B and the seating plan

3.4.3. Teacher C's classroom

Teacher C's class in which the study was conducted had a total of 28 students including 13 female students and 15 male students. The classroom had a teacher desk, student desks, a medium sized blackboard, a cupboard, a bulletin board, and an overhead projector. The projector and the blackboard were not designed in such a way to permit the teacher to project the mathematical examples on the blackboard. Therefore, Teacher C could not use as many examples as the two previous teachers. The students were sitting in pairs on all desks. The bulletin board was used for displaying drawings, term projects, activity sheets and examination results. There were 3 columns and 5 rows of desks in the classroom. Teacher C stated that students in this classroom were from high socio-economic status families and their achievement level was high. A snapshot of the classroom environment and the seating plan is presented in Figure 3.5.

Figure 3.5. A snapshot of the Classroom C and the seating plan

3.4.4. Teacher D's classroom

Teacher D's class in which the study was conducted had a total of 29 students including 12 female students and 17 male students. The classroom had a teacher desk, student desks, a medium sized blackboard, a cupboard, a bulletin board, but not an overhead projector. Similar to Teacher C, Teacher D did not have the opportunity to project the mathematical examples on the blackboard and thus she could not use many examples as during the teaching of rational numbers. A few students were sitting alone on their desks but the rest of them were sitting in pairs. The bulletin board was used for displaying drawings, term projects, activity sheets and examination results. There were 3 columns and 5 rows of desks in the classroom. Teacher D stated that students in this classroom were from medium socio-economic status families and their achievement level was average. A snapshot of the classroom environment and the seating plan is presented in Figure 3.6.

Figure 3.6. A snapshot of the Classroom D and the seating plan

In short, the main participants of the current study were four middle school mathematics teachers with different rational number teaching experiences and they have taught classes of different socio-economic level and achievement level. Two of the classrooms had high socio-economic status while the other two classrooms had medium socio-economic status. Evenly, two classrooms had top level students in terms of achievement and the other two classrooms had average achievement levels. Finally, while there were roughly 30 students in each of the three classrooms, in the fourth one there were nearly 20 students.

3.5. Data Sources

This study aimed to make an in-depth exploration of teacher's treatment of rational number examples in their classrooms. To get rich information from these middle school mathematics teachers, I employed multiple methods for data collection. Creswell (2007) referred to this as multiple sources of information. To be more explicit, he stated that “the data collection in case study research is typically extensive, drawing on multiple sources of information, such as observations, interviews, documents, and audiovisual materials” (p. 75).

By taking these into consideration, several data sources were used in this study. Classroom observations and interviews with the participating teachers were conducted immediately after each observation session. Besides, lesson observations and post lesson interviews were recorded by a videotape and an audiotape respectively. Descriptive and reflective field notes were taken throughout the study. Finally, written materials delivered to students by teachers such as worksheets, homework assigned to students such as textbook exercises, questions asked in the examination and so forth were collected. However, classroom observations and interviews with the participating teachers were the major data sources. Other data sources were used to support findings from observations and interviews.

3.5.1. Classroom observations

I observed each middle school mathematics teacher throughout all mathematics lessons related with rational number concepts. I observed a total of 60 mathematics lessons of four mathematics teachers. The observations were conducted to identify examples used by the teachers during the teaching of rational numbers to 7^{th} grade students. In more details, all teacher actions that took place in the mathematics classroom such as their instructional explanations, their use of worked-out problems, and the mathematical tasks they posed to students became the focus of my observations.

After reviewing the relevant literature about teachers’ purpose, use, and design of examples in the teaching of rational number concepts, I constructed an observation form. Later, a mathematics education researcher reviewed the observation questions. Finally, the supervisor of the researcher examined the questions with respect to their clarity, and content-specificity and the necessary revisions were done thereafter (See Appendix A). This observation form determined the scope of my observations and it helped me to record teacher actions that were related with focus of this study. In light of this focus, I tried to find answers to the following questions during all observations:

1. What type of rational number examples do teachers use in the classroom?
2. How do teachers select rational number examples during the teaching?
3. What principles or considerations guide teachers during choosing or generating rational number examples?
4. How do teachers address rational number examples to students?
    a. How do they organize rational number examples?
    b. How do they convey learning objectives regarding rational numbers?
    c. To what extend do teachers provide mathematically correct or pedagogically appropriate rational number examples?

During the observations, descriptive field notes were taken to describe the classroom environment and to record the rational number examples used by each teacher. In addition, reflective field notes were taken to record my personal thoughts about the rational number examples used by the teachers during the classroom. The field notes were used to make better sense of teachers' rational number examples and consequently to better analyze these examples. In the meantime, I videotaped all my observations. By this way, I had the chance to watch video camera recordings as many times as possible. This also gave me the opportunity to identify what aspects of teacher actions regarding rational number examples I failed to notice during actual observations. More importantly, the video camera recordings provided me with an opportunity to conduct stimulated recall interviews with middle school mathematics teachers when their actions regarding rational number examples were ambiguous. This is explained in more detail in the interview section.

According to Creswell (2009) there are four types of observation: complete participant, observer as participant, participant as observer and complete observer. In the current study, I adopted a complete observer role. To achieve my role, I sat at the back of the classroom and did not interrupt the ongoing dialogue among teachers and their students. The video camera recorded only the teacher examples that were written on the board, so I also took notes about examples generated by the teachers as a result of student query that might not be captured by video camera recordings.

Direct observation has several superiorities when compared to quantitative data gathering techniques (Hiebert et al., 2003). First of all, I had a first-hand experience with middle school mathematics teachers in their own classrooms. If direct observations were not used as a procedure to gather data about teachers' treatment of rational number examples, another possible way to gather data would be constructing questionnaires that test teachers' pedagogical content knowledge of rational numbers. Indeed, Ball et al. (2005) developed a questionnaire to test pedagogical content knowledge of teachers regarding a wide range of mathematical concepts. This questionnaire might give some clues about teachers' pedagogical content knowledge to some extent. However, as emphasized by Rowland, Thwaites and Huckstep (2009), this questionnaire might not reflect how teachers act in practice. Therefore, in order to assess teachers in their actual practice, we need to observe them while they are teaching. To sum up, teacher observations helped me gather crucial information about teachers' treatment of examples during their actual practice.

3.5.2. Post lesson interviews

Before the implementation of the study, I was planning to conduct both pre and post lesson interviews with the teachers to see the examples appearing in their lesson plans. However, the teachers stated that none of them prepared lesson plans in advance. Therefore, it was not possible for me to conduct pre-lesson interviews. Thus, post lesson interviews became another main data source of the current study.

Each interview was conducted immediately after each observation session. That is, I observed two mathematics lessons of two different teachers and in total four mathematics lessons a day. Conducting the interviews just after the observation of each two mathematics lessons was very important for this study. If the interviews were not conducted immediately, mathematics teachers could have forgotten which examples they used, how and why they used those examples during the teaching of rational number concepts. As a consequence, this might have become an obstacle for me in gathering data in line with my research questions. Besides, it was important for this study to determine which examples were pre-planned and which examples were generated spontaneously. Observations alone, gave some clues about teachers' intentions for using these examples. For instance, their utterances in the classroom, the amount of time they spent for generating rational number examples, and their hesitations and body expressions helped me to predict whether the examples being used were spontaneous or not. However, without immediate interviews, it would have been impossible for me to clearly distinguish between pre-planned or spontaneous examples. To summarize, classroom observations and post lesson interviews have been used complementarily to achieve the goals of this study.

Yin (2003) stated that interviews are one of the most important data sources for case studies and he classified interviews under three headings: open ended interviews, focused interviews, and structured interviews. In this study, each post lesson interview was a focused interview in which I interviewed each middle school mathematics teacher for a short period of time, approximately 10-15 minutes, to obtain a more holistic picture of treatment of rational number examples. Focused interviews were conducted by means of a semi-structured interview protocol containing several open-ended questions and they were all recorded by an audio recorder. The interview questions focused on clarifying the considerations employed by each teacher in choosing or generating examples and on resolving questions that arose in the mind of the researcher during the observations (See Appendix B). Through this focus, the following interview questions were asked to the middle school mathematics teachers:

1. Which of the examples you used during the classroom were pre-planned and which of them were spontaneously constructed?
2. What were your purposes for using each example during the teaching of rational number concepts?
3. What considerations did you employ while selecting or generating each rational number example?
4. What do you think about the efficiency of each example you used during the classroom?

a. Are there any examples that you think that they impeded students' understanding of rational number understanding?
b. If you were to revise your examples, which examples would you revise and how would you revise them?

5. Have there been any instances in which you provided mathematically incorrect or inappropriate examples and you noticed it later?
    a. If yes, how would you modify them?

During post lesson-interviews I also took notes. In the meantime, I used an audio recorder to be able to transcribe each interview session later. This also gave me the chance to listen to each interview and make better sense of my data. The interviews were all conducted in a silent room in the schools and I made sure that nobody would interrupt us during the interview. I constructed the semi-structured interview protocol questions with the help of instructional example literature and reviewed the questions with a doctoral student in the field of mathematics education. Also my supervisor examined the interview questions to determine whether they matched with the focus of the study and to eliminate possible biased or leading questions. To test the usability of the semi-structured interview protocol, I piloted it with a middle school mathematics teacher who did not participate in this study. I revised the interview questions in order for them to be more understandable by the participants of my study. Pilot interviews have played an important role for this study since they helped me to find out which questions were confusing, which questions needed rewording and which of them produced data which would not be considered for this study.

In addition to semi-structured post lesson interviews, I also conducted stimulated recall interviews with the middle school mathematics teachers. Calderhead (1981) stated that a stimulated recall interview "involves the use of audiotapes or videotapes of skilled behavior, which are used to aid participant's recall of his thought processes at the time of that behaviour" (p. 212) and he added that the stimulated recall technique might be adopted to examine teachers' thought processes and decision-making in the case of classroom-based research. Similarly, Clark and Peterson (1986) pointed out that this type of interview is a method for investigating teachers' ideas and beliefs about teaching and learning. Shane (2002) expressed that "in the stimulated recall interview, most often a video of the lesson is shown to promote reflection and insight into teacher's thinking" (p. 142). In this study, sometimes teachers had difficulty remembering the examples they used in the classroom. Besides, in some cases it was very difficult for me to identify teachers' purpose, design or use of rational number examples when they acted ambiguously. In such cases, stimulated recall interviews proved to be very useful since I had the chance to gain insights into teachers' purpose for using or generating certain examples. In this study, stimulated recall interviews were conducted by having teachers watch the video camera recordings when they had difficulty remembering the purpose for using a particular rational number example. The stimulated recall interviews have been conducted twice with each middle school mathematics teacher and each interview took about 2 hours.

Ultimately, demographic data about participant teachers, participant classrooms and participant schools were also gathered through interviews.

3.6. Data Collection

Timeline for data collection is presented in the following table.

Table 3.4. Timeline for data collection

Date	**Events**
August 2013	Permissions from Research Center for Applied Ethics and Aksaray Provincial Directorate for National Education
September 2013	Participant schools, classrooms and teachers were determined
September 2013 – November 2013	Pilot observations and interviews
November 2013-December 2013	Actual observations and interviews
November 2013- January 2014	Post observations and interviews
November 2013 - March 2014	Transcription of observation and interview data

This study was conducted during the Fall semester of 2013-2014 education year. Before collecting data, I reviewed the literature regarding mathematical examples and I prepared and an observation form and an interview protocol. I applied to Research Center for Applied Ethics of Middle East Technical University to get the necessary permissions for conducting my study (see Appendix E for approval document). After getting permission from this center, I applied to Aksaray Provincial Directorate for National Education in order to get necessary permissions for conducting my study in particular public middle schools located in Aksaray city center (see Appendix F for permission document). First, I have determined 12 candidate schools in case I may not be allowed to conduct my study in the most convenient schools. I initially visited the school principals and informed them about my study. I explained the purpose of my study and I got in touch with the mathematics teachers after I took school principals' approval for data collection. Similarly, I informed each middle school mathematics teacher about my study and asked them if they would like to voluntarily participate in my study. All middle school mathematics teachers that were volunteered to participate in my study signed the Voluntary Participation Form.

One week before the start of the Fall semester of 2013-2014 education year, I visited the schools to learn about the time table of 7^{th} grade classrooms. This helped me to determine which 7^{th} grade mathematics teachers to observe and to avoid overlapping of lesson hours of different teachers. It was a difficult job for me since I observed four mathematics lessons of four 7^{th} grade middle school mathematics teachers and in total sixteen mathematics lessons each week during the whole semester. After I organized my own time table for lesson observations, I started interviews and pilot observations on September 17^{th}, 2013 and they ended on November 13^{th}, 2013. Through the pilot observations, I became familiar with students, the classroom environment and the mathematics teachers. Pilot observations lasted for 8 weeks and Teacher A, Teacher B, Teacher C and Teacher D were observed for 25, 26, 24, and 19 lesson hours respectively. A total of 94 lesson hours were devoted to pilot observations. After each two hours of observation, post-lesson interviews were conducted with the teachers in order to gain insights into their treatment of examples.

On November 14th, 2013 actual data collection started. In more details, all four teachers were observed and interviewed throughout the unit of rational numbers. Teacher A, Teacher B, Teacher C and Teacher D were observed for 18, 17, 10 and 15 lesson-hours respectively. The actual data collection ended on December 27th, 2013. After the actual data collection process, I continued conducting post observations and interviews with the teachers until the end of the fall semester. Because, I wanted to see whether teachers attempted to change their classroom practices after the end of actual data collection process. During the course of the lesson observations and post-lesson interviews, I also transcribed observation and interview data and the transcription of whole data ended in March 2014.

3.7. Data Analysis Procedure

In this study, major data consisted of videotape recordings of lesson observations and audiotape recordings of post lesson interviews. Descriptive and reflective field notes, written materials delivered to students by teachers such as worksheets, homework assigned to students such as textbook exercises, and questions asked in the examination were other data sources used to support findings from observations and interviews. By using two different strategies that are methodologically connected - observations and interviews - I tried to obtain a holistic analysis of teachers' treatment of rational number examples in mathematics classrooms. After the end of the lesson observations and post-lesson interviews, all videotaped and audiotaped data were transcribed verbatim. This was the first step in data analysis and it took a long time for the researcher to transcribe all data. In all the observations and interviews, the spoken language was Turkish. Therefore, I initially transcribed all data in Turkish and then translated the necessary data into English for use in the results chapter of this study. During the transcription process, I watched the videotapes of the lessons and audiotapes of interviews for several times to engage myself with the data. Besides, I compared the translated data and the original data in terms of their grammatical, syntactic and linguistic aspects to enhance the quality of transcription process.

After transcribing all data, the next step was the identification of the themes, sub-themes and categories used in the study. Merriam (2009) stated that data analysis and data collection are simultaneously done in qualitative studies and added that data analysis is a complex process comprising moving back and forth between concrete bits of data, abstract concepts, and between inductive and deductive reasoning. Bogdan and Biklen (2007) described data analysis process as "systematically searching and arranging the interview transcripts, field notes and other materials that you accumulate to enable you to come up with findings" (p. 159). In a similar way, Creswell (2007) indicated that "data analysis in qualitative research consists of preparing and organizing the data for analysis, then reducing the data into themes through a process of coding and condensing the codes, and finally representing the data in figures, tables, or a discussion" (p. 148). In particular, Yin (2003) stated that data analysis in case studies provide intensive and holistic description of cases and mentioned that analyzing case study data would be especially difficult since there were no well-defined strategies and techniques.

In this study, observations and interviews were conducted with different teachers in different settings and thus multiple cases were chosen. Creswell (2007) suggested that "when multiple cases are chosen, a typical format is to first provide a detailed description of each case and themes within the case called a within-case analysis, followed by a thematic analysis across the cases, called cross-case analysis" (p.75). Similarly, Yin (2003) suggested five analytic techniques for analyzing case study evidence: pattern matching, explanation building, time-series analysis, logic models, and cross-case synthesis. Yin emphasized that although the first four techniques can be used with either single or multiple case studies, cross-case synthesis is especially relevant if a case study consists of at least two cases. Hence, this study analyzed the data obtained from the cases by using Yin's (2003) analytic technique of cross-case synthesis.

Using this technique, I first examined each case independently. That is, I first examined the rational number examples used by one of the teachers and tried to sort out the examples in terms of their similarities. As I repeatedly looked into the data, the categories started to emerge. While some categories were identified by means of pre-existing categories on teacher-generated examples, some others emerged in the current study. In the end, the examples were categorized according to the following ideas: the characteristics of the examples in themselves, the principles or considerations guiding teachers in choosing examples, and the erroneous examples and their potential pitfalls in students' understanding of rational number concepts. After, examining each case independently, I compared the findings of the analysis of each case with other three cases. For instance, if it was evident that the purpose, design or use of certain examples recurred in the classroom of Teacher A, then this recurrence was also searched in the classrooms of Teacher B, Teacher C and Teacher D. After case by case examination of the purpose, design or use of examples in each classroom, I examined all examples from four cases altogether (See Appendix C for sample coding sheet). The categorization of teachers' treatment of examples is presented in Table 3.5.

Table 3.5. Categorization of teachers' treatment of rational number examples

Themes	**Sub-Themes**	**Categories**
Mathematically correct examples	Types of examples	Examples
		Non-examples
		Counter-examples
	Source of examples	Pre-planned examples from textbook
		Pre-planned examples from workbook
		Pre-planned examples from teachers' guidebook
		Pre-planned examples from auxiliary books
		Pre-planned examples from online educational software
		Pre-planned examples from high-stakes examination questions
		Spontaneous examples
	Teachers' considerations in choosing examples	Starting with a simple or familiar case
		Attending to student error/difficulty/misconception
		Drawing attention to relevant features
		Including uncommon cases
		Keeping unnecessary work to minimum
		Taking account of examinations
Mathematically incorrect/pedagogically improper examples	Types of errors	Incorrect example
		Improper language/terminology
		To be avoided examples

In my data, there were examples being indicative of either 'a practice to be aspired to' or a 'pitfall to be avoided'. Therefore, I initially classified rational number examples as being mathematically correct or mathematically incorrect/pedagogically inappropriate. Type of examples, source of examples, and teachers' considerations in choosing examples were the sub-themes of mathematically correct examples. Type of examples was related with the characteristics of examples in themselves and this categorization was mainly based on the work of Watson and Mason (2005). Some examples reflected teachers' careful planning while some others were constructed during the lesson as a response to an entirely new or an unfamiliar classroom situation. This categorization of examples as pre-planned versus spontaneous examples was drawn from Zodik and Zaslavsky (2008). In addition, pre-planned examples were further categorized by taking account of their source. That is, pre-planned examples were categorized as examples from textbook/workbook, examples from auxiliary books and examples from an online content. This categorization was solely based on classroom observations of the researcher.

Understanding middle school mathematics teachers' considerations or underlying principles that guided them in choosing or generating examples were an important component of the study apart from characterizing examples in themselves. Teachers' considerations were categorized under six headings: Starting with a simple or familiar case, attending to student error/difficulty/misconception, drawing attention to relevant features, including uncommon cases, keeping unnecessary work to minimum, and taking account of examinations. These categorizations were drawn from the work of Zodik and Zaslavsky (2008) with minor changes. To be more precise, the data of this study did not provide a category that suggest that middle school mathematics teachers 'convey generality by random choice' while teaching rational number concepts. This might have been due to the fact that middle school mathematics teacher did not find it necessary to make generalizations of examples in middle school mathematics classrooms. On the other hand, the examples used by the teachers suggested that they took account of national exams while choosing or using examples. Thus, the data suggested that it was essential to include the category of 'taking account of examinations' under teacher considerations sub-theme. As a result, this sub-theme was based not only on the literature and but also the lesson observations and the interviews conducted in this study. Finally, it is important to note that the categories were not purely distinct from each other, since one instance of the choice of example could be placed under more than one category.

Middle school mathematics teachers participated in this study occasionally generated or selected examples that were mathematically incorrect or pedagogically improper. These typeS of examples were also analyzed since they were considered to be a potential pitfall for students' understanding of rational number concepts. The analysis of middle school mathematics teachers' examples in terms of their correctness was rather objective although they were context based. Although some examples were entirely incorrect when evaluated from a mathematical standpoint, some others were not totally incorrect but they caused difficulties in communicating about the complicated topic of rational numbers. Mathematically incorrect or pedagogically improper examples were categorized as mathematically incorrect examples, pedagogically improper examples with improper language and terminology and pedagogically improper examples that are to be avoided. For instance, claiming that $\sqrt{2}$ cannot be located on a number line is a mathematically incorrect example.

The teachers often used the word 'fraction' when they intended a 'rational number'. Perhaps, they used these words interchangeably due to carelessness. This type of examples may lead to difficulties in communicating about rational number examples. Similarly, teachers' used the expression 'flipping' instead of the terms 'reciprocal' or 'multiplicative inverse' when finding the multiplicative inverse of a rational number. Cangelosi, Madrid, Cooper, Olson, and Hartter (2013) emphasized that the colloquial use of the term 'flipping' might hinder students' understanding of the concept of multiplicative inverse. Based on the previous literature, examples of this type were grouped under the category of examples with improper language or terminology.

Finally, studying with middle school mathematics teachers has brought to light some type of examples that should be better avoided. In more, details, some of the examples provided by the participants included particular pitfalls that might be an obstacle for students to understand the mathematical object, concept or procedure that they confronted for the first time. This type of examples were categorized as 'the use of to be avoided examples'. This categorization was basically drawn from Rowland et al. (2009) but there were also some contributions from researcher observations and interviews.

In this study, I employed several theoretical frameworks to give a comprehensive explanation of how middle school mathematics teachers treat rational number examples in their classrooms. These frameworks are explained in detail in the following section.

3.7.1. Theoretical frameworks used in this study for analyzing middle school teachers' treatment of rational number examples in their classrooms

In this study, the following theoretical frameworks were used to explore middle school mathematics teachers' treatment of rational number examples in their classrooms: Marton and Booth's (1997) variation theory; Zodik and Zaslavsky's (2008) dynamic framework for explaining teachers' choices and generation of examples during the lesson and finally, Rowland et al.'s (2005) the Knowledge Quartet Framework for making sense of teachers' choose and use of examples. The use of variation theory in mathematics education is explained below.

3.7.1.1. Marton and Booth's (1997) variation theory

Learning takes place through extending awareness of what constitutes an example (Marton & Booth 1997; Marton & Tsui, 2004). That is, discerning or making distinctions by detecting variation is at the core of learning (Marton & Booth 1997; Marton & Trigwell, 2000). Briefly, in variation theory (Marton & Booth 1997), variation is epistemologically essential for learning to occur. Marton and his colleagues detected differences in learning with respect to the nature and range of variation to which learners were exposed and to seize this, they introduced the notion of dimensions of variation (Marton & Booth 1997; Marton, Runesson & Tsui, 2004; Marton & Tsui, 2004). The notion of dimensions of variation refers to "the different parts of an object which can be varied and still that object remains an example of a specified concept" (Mason & Watson, 2008, p. 195). At the level of cognition, an example of a concept is accepted as an example only when certain features are acknowledged as being permitted to change, while some other features remain relatively invariant (Mason, 2006).

Different people may be aware of different dimensions (Goldenberg & Mason, 2008). For instance, teachers and their students may be aware of different dimensions in an example. In particular, novices may not be aware of the richness of all possible variation. Furthermore, the same individual may be aware of different dimensions of variation at different times (Goldenberg & Mason, 2008; Mason, 2006; Mason & Watson, 2008). By considering all these factors, Watson and Mason (2005) extended the notion of dimensions of variation to dimensions of possible variation. Further, they added the notion of range of permissible change and indicated that each dimension of possible variation has an associated range of permissible change which might also not be shared between different individuals such as novices and experts. Briefly, dimensions of possible variation indicate that "different people may be aware of different things that is possible to vary" and range of permissible change indicates that "what can vary may be perceived as varying over different ranges by different people or at different times" (Mason, 2011, p. 195).

The notions of dimensions of possible variation and the associated range of permissible change help learners discern what features of an object is critical and what features of it can be changed in what ways (Goldenberg & Mason, 2008). These two parameters are especially powerful in mathematics in that they help learners appreciate mathematical structure (Mason et al., 2009). Mathematical structure shows itself by means of relations among variance/invariance and similarity/difference (Watson & Shipman, 2008). This structure might help learners detect both critical and uncritical aspects of examples being experienced (Sun, 2011).

According to variation theory, discerning certain critical features of an object is vital for learners owing to the fact that it is essential first to identify critical aspects in order to learn that object (Guo et al., 2012). To discern a particular aspect/feature, learners should experience variation in the related dimension and as a consequence, the aspect that varies while other aspects remain invariant would easily be discerned by those learners (Pang & Marton, 2005). More specifically, Marton et al. (2004) identified four patterns of variation and invariance to assist in discerning critical aspects of mathematical objects as contrast, separation, fusion and generalization. The notion of patterns of variation and invariance can be used in the teaching of certain mathematical concepts as well. Rowland et al. (2009) proposed the following example to illustrate the use of variation theory in mathematics education:

> "understanding of the concept of square is marked by growing awareness of the various ways that squares can vary, and the variants

that do not qualify as squares. These dimensions include: sides – these must have same length, but the length can vary between different squares; angles – these must all be right angles, and there exist rhombuses with equal sides which are not squares; orientation – diamonds with equal sides and angles are squares, rotated from the conventional position on the page; other less overtly geometrical dimensions such as colour, texture and so on, can also vary" (p. 84).

In the current study, the concept of rational numbers involves possible variation in dimensions such as numerator, denominator, proper, improper or mixed, being in lowest terms or not, being positive or negative and so forth. For instance, when teaching subtraction of rational numbers, teachers should be aware of the fact that the minuend and subtrahend can take many different values with respect to their signs and forms. Table 3.6 illustrates how many variations there exist for the minuend (i.e., $\frac{a}{b}$) and the subtrahend (i.e., $\frac{c}{d}$) in the following subtraction operation: $\frac{a}{b}-\frac{c}{d}$.

Table 3.6. The variety of examples for teaching subtraction of rational numbers

				Subtrahend					
				Positive			Negative		
				Proper	Improper	Mixed	Proper	Improper	Mixed
					$\frac{4}{3}$	$1\frac{2}{5}$	$-\frac{1}{2}$	$-\frac{4}{3}$	$-1\frac{2}{5}$
Minuend	Positive	Proper		0	$-\frac{5}{6}$ $\frac{1}{2}$	$-\frac{9}{10}$	1	$\frac{11}{6}$	$\frac{19}{10}$
		Improper	$\frac{4}{3}$	$\frac{5}{6}$ $\frac{1}{2}$	0	$-\frac{1}{15}$	$\frac{11}{6}$	$\frac{8}{3}$	$\frac{41}{15}$
		Mixed	$1\frac{2}{5}$	$\frac{9}{10}$	$\frac{1}{15}$	0	$\frac{19}{10}$	$\frac{41}{15}$	$\frac{14}{5}$
	Negative	Proper	$-\frac{1}{2}$	-1	$-\frac{11}{6}$	$-\frac{19}{10}$	0	$\frac{5}{6}$	$\frac{9}{10}$
		Improper	$-\frac{4}{3}$	$-\frac{11}{6}$	$-\frac{8}{3}$	$-\frac{41}{15}$	$-\frac{5}{6}$	0	$\frac{1}{15}$
		Mixed	$-1\frac{2}{5}$	$-\frac{19}{10}$	$-\frac{41}{15}$	$-\frac{14}{5}$	$-\frac{9}{10}$	$-\frac{1}{15}$	0

3.7.1.2. Zodik and Zaslavsky's (2008) dynamic framework for explaining teachers' choices and generation of examples

Simon (1995) developed Mathematics Teaching Cycle as a model of the relationship among teacher knowledge, thinking, decision making and classroom activity. The Mathematics Teaching Cycle is presented in Figure 3.7.

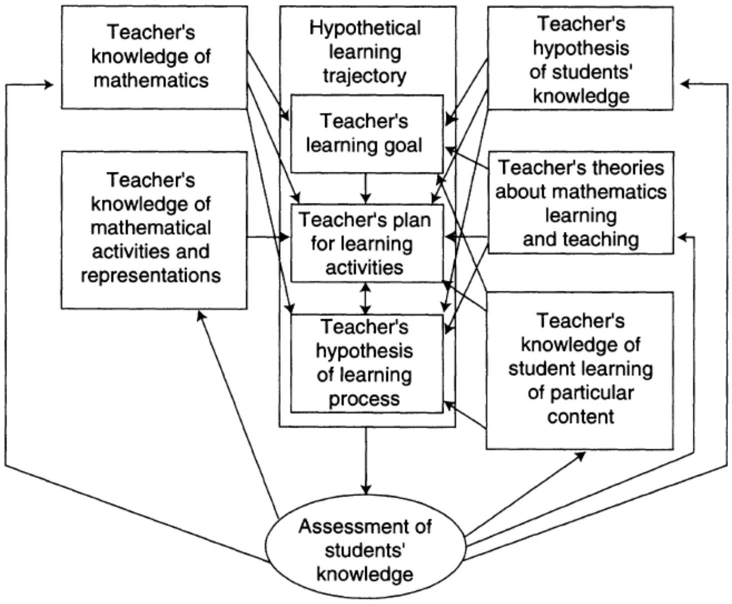


Figure 3.7. Mathematics Teaching Cycle (Simon, 1995, p. 137).

This model demonstrates the relationship among different teacher knowledge domains, the hypothetical learning trajectory and the interactions with students. According to this model, the hypothetical learning trajectory refers to a teacher's development of a plan for classroom activity before incorporating it into the classroom. More precisely, the hypothetical learning trajectory affords teachers the opportunity to put forward a reason for selecting a specific instructional design; hence, help teachers make design decisions on the basis of their predictions about how learning might continue in the classroom. This can be observed both in the thinking and planning prior to the instruction or in the course of the lesson as a spontaneous decision made in response to a student thinking (Simon, 1995). In short, Mathematics Teaching Cycle emphasizes the relationship between teacher knowledge, pre-planning and classroom interactions that involve spontaneous actions.

In their study, Zodik and Zaslavsky (2008) used the abovementioned constructs for exploring secondary school teachers' choices and generation of examples in the course of teaching mathematics. Besides, they examined underlying principles or considerations that guided teachers in choosing or generating examples by focusing on the mathematical knowledge they used and by foregrounding teachers' knowledge in-action and their accessible personal example spaces. Encouraged by the Mathematics Teaching Cycle of Simon (1995), Zodik and Zaslavsky (2008) proposed a dynamic framework for examining teachers' choice and use of examples in the course of teaching mathematics. This theoretical framework is presented in Figure 3.8.

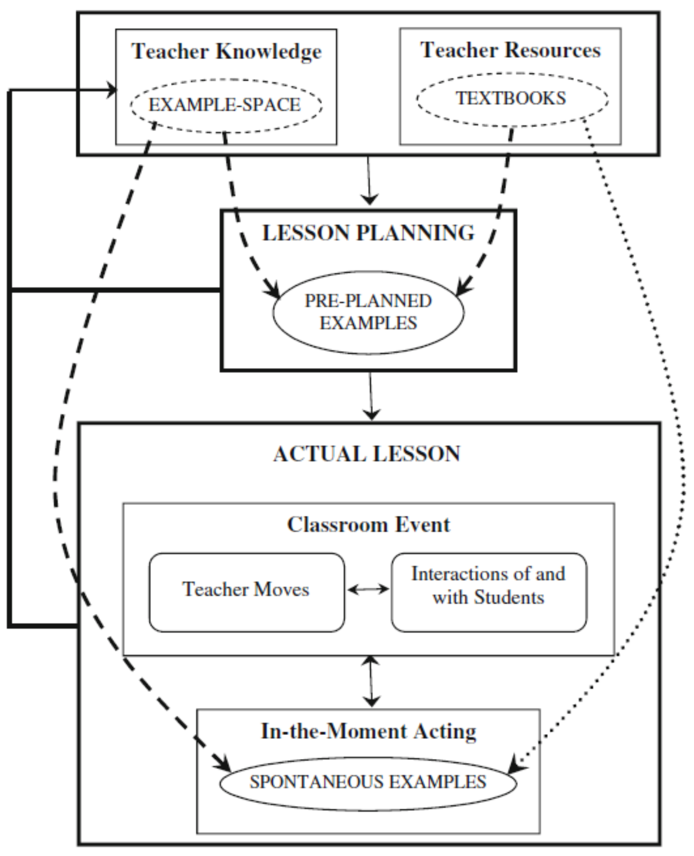


Figure 3.8. Mathematics Example-Related Teaching Cycle (Zodik & Zaslavsky, 2008, p. 179).

As can be seen in Figure 3.8, examples used by a teacher during the teaching of mathematics are located on three different components as teacher knowledge, lesson planning and the actual lesson. The interrelationships among these components are denoted by different arrows. Teachers' example spaces and textbooks are major sources for their choice and use of examples. Moreover, textbooks are mainly used during the lesson planning phase, example spaces are used by teachers both in the planning phase and in the actual lesson implementation phase. Teachers have some underlying principles or considerations that guide them while choosing and using examples and these considerations are influenced by teachers' personal dispositions and evaluations.

The figure also demonstrates that teachers mainly work with choosing or generating examples in the course of planning their lessons. Besides, actual lesson implementation comprises classroom events and in-the-moment actions of teachers. In particular, classroom events include teacher moves and interactions of and with students. The classroom events usually call for teachers to act-in-the moment and provide the relevant example that is needed at that moment. In the study of Zodik and Zaslavsky (2008), spontaneous examples were generated quite immediately by some of the teachers and this indicated their easily accessible example spaces. On the other hand, for some other teachers it took longer to generate examples and these examples indicated remote accessibility to those teachers' example spaces. Such moments were considered as learning opportunities by Zodik and Zaslavsky (2007). Thus, as can be seen in Zodik and Zaslavsky's (2008) Mathematics Example-Related Teaching Cycle, teachers learn through their teaching and in particular they learn through example generation or selection.

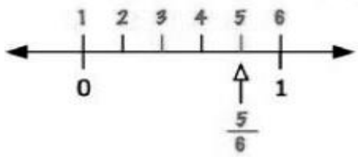
In this study, underlying principles or considerations that guided middle school mathematics teachers in choosing or using rational number examples were examined by the help of the aforementioned framework. This framework consisted of the following six categories: starting with a simple or familiar case, attending to students' errors, drawing attention to relevant features, conveying generality by random choice, including uncommon cases and keeping unnecessary work to minimum. However, the data of the current study did not provide a category which suggested that middle school mathematics ‘conveyed generality by random choice’ while teaching rational number concepts. On the other hand, the rational number examples used by the teachers suggested ‘taking account of examinations’ as a category distinct from the ones included in the framework of Zodik and Zaslavsky (2008). To clarify how I determined the category of each rational number example or sets of rational number examples, I present the following example tasks.

Starting with a simple or familiar case: Middle school mathematics teachers often generated sequences of rational number examples and each rational number example gradually increased in its level or complexity. For instance, to teach how to order rational numbers, one teacher used the following sets of rational number examples:

$$\frac{(-2)}{5}, \frac{(-7)}{5}, 0, \frac{1}{5}, \frac{3}{5}; \quad \frac{1}{3}, \frac{1}{5}, -\frac{1}{7}, -\frac{1}{2}, 0; \quad -\frac{2}{7}, -\frac{2}{13}, 0, \frac{2}{15}, \frac{2}{19}; \quad \frac{2}{19}, \frac{3}{13}, \frac{6}{17}; \quad \frac{1996}{1997}, \frac{1997}{1998}, \frac{1998}{1999}$$

The first sequence is easy to order since there is no need to find the least common multiple of the denominators. In the second sequence, the numerators of the rational numbers are all ‘1’. Thus, this sequence may also be ordered easily by using the same numerator algorithm. In the third sequence, the numerators of the rational numbers are all ‘2’ and this sequence may also be ordered easily by using the same numerator approach. However, the third sequence includes large denominators so it might be difficult for students to order the rational numbers by using common denominator algorithm when they do not think of using the same numerator algorithm. The fourth sequence includes rational numbers with different numerators and denominators. Thus, this sequence is more difficult to order when compared to the previous three sequences. Finally, the last sequence cannot be ordered by using common numerator or denominator approach. The students need to use a more conceptual ordering strategy such as residual thinking. Therefore, the last sequence can be considered the most complex ordering example when compared to the previous sequences.

Attending to students’ errors: Middle school mathematics teachers often built examples according to the errors they knew the students made. For instance, one of the teachers expressed that students erroneously focused on tick-marks rather than equal distances when locating rational numbers on a number line. She drew on the board the following number line to illustrate how students erroneously locate $\frac{5}{6}$ on it:



Drawing attention to relevant features: This consideration had to do with teachers' deliberate attempts to decrease the irrelevant features of specific examples. For instance, Teacher A initially provided $K = \left(8 - \frac{1}{4}\right) \div \frac{1}{8}$ as a multi-step operation example. Next, he omitted the parenthesis in this example and provided $L = 8 - \frac{1}{4} : \frac{1}{8}$ as a second multi-step operation example. Finally, the teacher asked the students to think of whether the two examples were identical. By this way, the teacher checked whether students could recognize which operations to perform first in the two expressions.

Including uncommon cases: This consideration had to do with teachers' attempts to use examples that were rather exceptional or special in mathematics or examples that were under-represented in the teaching of rational numbers. For instance, one teacher focused on $\left(-\frac{2}{3}\right)^0$ when teaching how to perform exponentiation with rational numbers. It is important to note that for this exponential number, the intuitive definition of exponents (i.e., repeated multiplication) does not work. Thus, the teacher treated the case of zero exponent as a special case and explicitly expressed the following utterances: "Raising any nonzero rational number to the power of 0 yields 1. Thus, $\left(-\frac{2}{3}\right)^0$ is equal to 1".

Keeping unnecessary work to minimum: Teachers deliberately attempted to keep unnecessary work to minimum by reducing technical work and focusing on the essence, by highlighting relevant parts of examples and not going into extra details and by using properties of operations to reduce workload. For instance, one of the teachers preferred to use distributive property of multiplication over addition rather than performing several operations for solving the following task: $\frac{3}{7}+\frac{5}{9}+\frac{3}{11}-\left(\frac{5}{9}-\frac{4}{7}+\frac{3}{11}\right)$.

Taking account of examinations: This consideration might be specific to Turkish educational context. Teachers highlighted examples that had the potential to appear in written examinations, practice examinations of private teaching institutions, and high stakes examinations. Besides, they demonstrated their students how to find the answer of multiple choice complex fraction tasks by trial and error of the alternatives and taught shortcut methods for gaining speed in the high stakes examinations. For instance, one of the teachers solved $\frac{2}{\frac{6}{x-1}-5}=-1$ by trial and error of the alternatives. The alternatives were $\frac{1}{4},\frac{1}{2},2$ and 3 respectively. In his third trial, the teacher substituted 3 into complex fraction and reached the correct answer as follows: $\frac{2}{\frac{6}{3-1}-5}=\frac{2}{\frac{6}{2}-5}=\frac{2}{3-5}=\frac{2}{-2}=-1$.

3.7.1.3. Rowland et al.'s (2005) the Knowledge Quartet framework for making sense of teachers' choice and use of examples

It is widely accepted that pupil achievement is dependent to a large extent on the quality of teaching (Stronge, Ward, & Grant, 2011). Besides, mathematical content knowledge of teachers is regarded as an important factor in the teaching and learning of mathematics (Williams, 2008). Nevertheless, researchers identified that teachers had limitations in their mathematical content knowledge (e.g., Ball, 1990a, 1990b; Ma, 1999). Therefore, mathematics educators around the world attempted to develop measures or generate theories for deepening teachers' mathematical content knowledge (e.g., Ball, Hill, & Bass, 2005; Rowland et al., 2005). Ball et al. (2005) developed items to test both common and specialized content knowledge of teachers.

According to Rowland, et al. (2009), Ball et al.'s questionnaire might give some clues about teachers' pedagogical content knowledge but might not reflect how teachers act in practice. Rowland et al. (2009) added that in order to assess teachers in their actual practice, there is a need for observing those teachers while they are teaching. By adopting this idea, Rowland et al. (2005) attempted to generate an empirically-based conceptual framework called the Knowledge Quartet. This framework consisted of four broad categories as foundation, transformation, connection and contingency. More specifically, it included eighteen codes and these codes provided Rowland et al. (2005) with considering and discussing mathematics teaching in practice by focusing on elementary pre-service teachers' mathematical knowledge for teaching. Rowland et al.'s (2005) The Knowledge Quartet Framework is shortly summarized in Table 3.7.

Table 3.7. The Knowledge Quartet Framework (Rowland et al., 2005, p. 265)

Foundation	Propositional knowledge and beliefs concerning: • the meanings and descriptions of relevant mathematical concepts, and of relationships between them; • the multiple factors which research has revealed to be significant in the teaching and learning of mathematics; • the ontological status of mathematics and the purposes of teaching it. Contributory codes: awareness of purpose; identifying errors; overt subject knowledge; theoretical underpinning of pedagogy; use of terminology; use of textbook; reliance on procedures.
Transformation	Knowledge-in-action as revealed in deliberation and choice in planning and teaching. The teacher's own meanings and descriptions are transformed and presented in ways designed to enable students to learn it. These ways include the use of powerful analogies, illustrations, explanations and demonstrations. The choice of examples made by the teacher is especially visible: • for the optimal acquisition of mathematical concepts, procedures or essential vocabulary; • for confronting and resolving common misconceptions; • for the justification (by generic example) or refutation (by counter-example) of mathematical conjectures. Contributory codes: choice of representation; teacher demonstration; choice of examples.
Connection	Knowledge-in-action as revealed in deliberation and choice in planning and teaching. Within a single lesson, or across a series of lessons, the teacher *unifies* the subject matter and draws out *coherence* with respect to: • connections between different meanings and descriptions of particular concepts or between alternative ways of representing concepts and carrying out procedures; • the relative complexity and cognitive demands of mathematical concepts and procedures, by attention to sequencing of the content. Contributory codes: making connections between procedures; making connections between concepts; anticipation of complexity; decisions about sequencing; recognition of conceptual appropriateness

Table 3.7. (Continued)

Contingency	Knowledge-in-interaction as revealed by the ability of the teacher to 'think on her feet' and respond appropriately to the contributions made by her students during a teaching episode. On occasion this can be seen in the teacher's willingness to deviate from her own agenda when to develop a student's unanticipated contribution: • might be of special benefit to that pupil, or • might suggest a particularly fruitful avenue of enquiry for others. Contributory codes: responding to children's ideas; use of opportunities; deviation from agenda

In this study, the focus was on transformation dimension of the Knowledge Quartet since this dimension involves teachers' choice and use of examples in the teaching of mathematics. In particular, the focus was on identifying middle school mathematics teachers' poor choice of examples that were regarded as common pitfalls to be avoided in the selection of examples. As mentioned by Rowland (2008), teachers learn most easily by poor choice of examples and on the contrary good choice of examples is generally so subtle that it may not be readily noticeable by them. Rowland et al. (2003) brought to light three types of examples that would be avoided. Namely, they identified three types of examples that reflect prospective elementary teachers' poor choice:

> "examples that obscure the role of variables in it; examples intended to illustrate a particular procedure, for which another procedure would be more sensible; examples for instruction (as opposed to exercise examples) being randomly generated, typically by dice, at a point when it would preferable for the teacher to be making careful choices" (p.245).

According to Marton and Booth's (1997) notion of dimension of variation, most of the mathematical concepts or procedures include two or more components or variables. When selecting introductory examples for teaching a mathematical concept or procedure, it is often judicious to keep the magnitudes of these variables different from each other (Rowland, 2014). Selecting variables with different values is important from a pedagogical perspective, since it helps students recognize the role of different variables in a concept or procedure. On the other hand, selecting variables with same values makes the distinction among the variables obscure and this leads to the generation of an example that obscure the role of examples. (Rowland et al., 2009). A very striking example that obscured the role of variables was observed by Rowland (2008) in a mathematics lesson of a prospective elementary teacher about Cartesian co-ordinates. In more detail, before teaching how to identify the coordinates of a specific point on a co-ordinate grid, the teacher reminded her students that the x-axis goes first. However, the teacher initially chose to identify the co-ordinates of the point (1,1). As can be seen, this example seems to be completely ineffective in demonstrating the importance of the order of two elements of the ordered pair. Hence, selecting (1,1) as the first point to mark on a co-ordinate grid might give rive rise to the confusion between the notations (*x*,*y*) and (*y*,*x*).

A second category of poor choices of examples occurs due to selecting examples for an intended procedure when in fact another procedure would be more sensible to perform for that selected example (Rowland et al., 2003). For instance, a teacher may choose $49\times 4, 49\times 8 \text{ and } 19\times 4$ as demonstration examples when introducing column multiplication to her students. However, the teacher's choice of examples for teaching column multiplication does not seem to be well-judged since there are more suitable calculation strategies for those examples. For instance, 49×4 would be more efficiently performed by rounding up, multiplication and compensation as $49\times 4=(50-1)\times 4=(50\times 4)-4=200-4=196$. After performing 49×4, the teacher would then introduce her students doubling strategy to find the answer of 49×8 in an easier and more sensible way since $49\times 8=(49\times 4)\times 2$. Similarly, 19×4 could be more sensibly performed by the use of doubling strategy rather than column multiplication since $19\times 4=(19\times 2)\times 2$.

Finally, the third category of poor choices of examples has to do with choosing examples at random generally by using a dice. Rowland (2008) stated that "there is something intuitively attractive about generating examples with dice, possible because the teacher is demonstrating confidence to let go of some aspect of the lesson, perhaps giving it a more democratic feel" (p. 158). However, it is very dangerous to use a dice when selecting a concept or procedure example, despite it might be a useful method for selecting exercise examples (Rowland et al., 2003). For instance, one of the prospective elementary teachers observed by Rowland (2008) was teaching his reception class (students at the age of 4 or 5) how to find a pair of numbers whose sum is equal to 10. However, he asked one of his students to randomly generate a number between 1 and 10 by using a dice. At that moment, the dice generated the numbers 5, 3 and 8 respectively and the teacher wanted other students to find their complements to 10. As it can be seen, the teacher's example generation is in contrast with skillful control of the examples during the teaching of a mathematical concept. Rowland (2008) concluded that most of the prospective elementary teachers confused between choosing examples for teaching new concepts or procedures and choosing examples for convincing the learner about the truth of a principle or the effectiveness of a previously taught procedure. Finally, Rowland et al. (2009) suggested that it is often better for teachers to control (i.e., choose and use carefully) examples that are selected for introducing new concepts or procedures and they added that choosing examples at random is less likely to serve for the intended pedagogical purpose.

This study used the abovementioned framework for analyzing middle school mathematics teachers' poor choice of rational number examples. Teachers' poor choices of rational number examples were correct from a mathematical standpoint but they were inappropriate from a pedagogical standpoint. In particular, the use of this framework revealed two different types of pedagogically improper examples as examples that obscure the role of variables and examples intended to illustrate a procedure, for which another procedure would be more sensible. For instance, some of the teachers initially selected $\frac{10}{3} = 3.333$ when teaching repeating decimals. However, by selecting this example, the teachers made the distinction between the non-repeating digit and the repeating digit obscure. In this example, 3 was made to do the work of two variables. Thus, this example obscured the role of variables. Another pedagogically improper example choice has to do with using relevant strategies for the selected example. For instance, one teacher wanted to teach how to order the following rational numbers: $\frac{13}{12}, \frac{11}{10}, \frac{7}{6}$ and $\frac{5}{4}$. Although these rational numbers lend themselves more readily to residual thinking, the teacher preferred to use common denominator algorithm. As can be seen, it is not sensible to use common denominator algorithm for the selected set of rational numbers.

3.8. Trustworthiness of the Study

Validity and reliability play an important role in designing a study, analyzing the findings and in determining the quality of the study (Patton, 2002; Shenton, 2004). Creswell (2007) stated that the accuracy of the findings and interpreting the data in a correct fashion are the main concerns for qualitative research studies. However, different qualitative researchers have different views about how to determine the quality of a qualitative research study (e.g., Creswell, 2007; Merriam, 2009; Miles & Huberman, 1994; Stake, 2005; Yin, 2003). In addition to these, validity and reliability of qualitative studies are generally not discussed separately as in quantitative research studies and researchers used different terminologies such as 'rigor', 'credibility' or 'trustworthiness' to address both validity and reliability (e.g., Golafshani, 2003; Lincoln & Guba, 1985; Shenton, 2004). Lincoln and Guba (1985) used the term 'trustworthiness' to refer to the validity and reliability of qualitative research studies. In this qualitative case study, I preferred to use the term 'trustworthiness' to address validity and reliability issues. Lincoln and Guba (1985) used the terms credibility, transferability, dependability, and comfirmability as equivalents for internal validity, external validity, reliability, and objectivity to establish the trustworthiness of a qualitative study. In the following section, I try to address the credibility, transferability, dependability, and comfirmability issues of this study respectively.

3.8.1. Credibility

First, credibility corresponds to internal validity in quantitative research studies. According to Merriam (2009) credibility is concerned with finding answers to the questions "How congruent are the findings with reality? Are investigators observing or measuring what they think they are measuring?" (p. 201). In this study, to increase credibility, the following strategies suggested by Shenton (2004) were used: establishing the adoption of research methods, developing an early familiarity with the culture of participating organizations, ensuring honesty in participants, thick description of the phenomenon under scrutiny, and examining the previous research findings. These tactics are used in the following way:

To establish the adoption of research methods, I explained the rationale for using a qualitative research methodology, the reasons for using lesson observations and post lessons interviews for gathering research data, and why these methods were relevant for the purposes of this study. To develop an early familiarity with the culture of participating organizations, I began to observe four middle school classrooms two months before the actual data collection process. Meanwhile, I used a video camera to record the examples that are written on the board by each middle school mathematics teachers. To ensure honesty in participants, I observed only the teachers who volunteered to participate in my study. There were a total of seventeen middle school teachers in four schools but not all of them were willing to participate in my study. Therefore, there were four participant teachers in my study. To present a thick description of the phenomenon under study, I described the characteristics of each participant school, participant classroom and participant teacher as much as I could to portray the actual situations that were explored. Finally, I examined the previous research findings on examples and tried to relate them with the findings of the current study in the discussion chapter.

In addition to the suggestions of Shenton (2004), Creswell (2007) suggested eight different strategies to establish credibility: triangulation, member checking, using thick description, clarifying researcher bias, negative case analysis, spending prolonged time in the field, peer debriefing, and using an external audit.

Creswell and Miller (2000) defined triangulation as "a validity procedure where researchers look for convergence among multiple and different sources of information to form themes or categories in a study" (p. 126). In addition, Stake (2000) pointed out that "triangulation has been generally considered as a process of using multiple perceptions to clarify meaning, verifying the repeatability of an observation or interpretation" (p. 443). There are four different types of triangulation in qualitative research literature: data triangulation, investigator triangulation, methodological triangulation, and theory triangulation (Creswell & Miller, 2000; Creswell, 2007; Patton, 2002). In this study, data triangulation, investigator triangulation and methodological triangulation was used to increase the credibility. That is, there were four different cases as data source (data triangulation), a second coder was used for analyzing the data (investigator triangulation) and different types of data including observations and interviews were gathered (methodological triangulation).

In addition to triangulation, I used member checking after transcribing the observation and interview data. I had the participants' view and read the whole transcription and wanted to see if there were any conflicts between their understandings. Besides, I conducted stimulated interviews with teachers in cases that were ambiguous. As mentioned before, I made thick and rich descriptions to enable the researchers to decide on the applicability to other settings. I clarified researcher bias by acknowledging and describing my entering beliefs and biases about the current study in the following sections. This was explicitly stated in the researcher role and bias section. To build trust and establish rapport with the participants I spent extensive time in four classrooms. That is, I spent 16 lesson hours a week in four classrooms during the whole fall semester. Peer debriefing is defined as "the review of the data and research process by someone who is familiar with the research or the phenomenon being explored" (Creswell & Miller, 2000, p.129). In this study, I had the chance to get feedbacks from a researcher experienced in qualitative research and teacher knowledge.

3.8.2. Transferability

Second, transferability corresponds to external validity in quantitative research studies. Transferability is concerned with the generalizability of the findings of a study. Nevertheless, in qualitative research studies, generalizability does not serve the purpose of making inferences from a small sample to a wider population as in quantitative studies. Shenton (2004) indicated that "since the findings of a qualitative project are specific to a small number of particular environments and individuals, it is impossible to demonstrate that the findings and conclusions are applicable to other situations and populations" (p.69). Nonetheless, Miles and Huberman (1994) suggested researchers to provide "thick descriptions for the readers to assess the potential transferability and appropriateness for their own settings" (p. 279). According to Lincoln and Guba (1985) researchers are responsible from making sure that adequate contextual information about the fieldwork site is provided to the readers so that they transfer the findings to their own contexts. Additionally, Shenton (2004) emphasized the importance of conveying the reader the boundaries of study. Thus, the following contextual information was presented in this study: “the number of organizations taking part in the study and where they are based; any restrictions in the type of people who contributed data; the number of participants involved in the fieldwork; the data collection methods that were employed; the number and length of the data collection sessions; and the time period over which the data was collected” (p. 69).

Furthermore, Yin (2003) stated that transferability is a main problem in case studies. He explained this problem in the following way:

> The external validity problem has been a major barrier in doing case studies. Critics typically state that single cases offer a poor basis for generalizing. However, such critics are implicitly contrasting the situation to survey research, in which a sample readily generalizes to a larger universe. This analogy to samples and universes is incorrect when dialing with case studies. This is because survey research relies on statistical generalization, whereas case studies rely on analytical generalization. In analytical generalization, the investigator is striving to generalize a particular set of results to some broader theory (p.37).

Yin (2003) suggested researchers to use theory in single case studies and to use replication logic in multiple case studies. Since this study was a multiple case study, I tried to address the issue of transferability by using replication logic for each case. To be more precise, I tested the inferences that I drew for a case study by replications of the findings in other three cases. For instance, when I found a pattern in the case of Teacher A, I tried to figure it out in the cases of Teacher B, Teacher C and Teacher D.

3.8.3. Dependability

Third, dependability corresponds to reliability in quantitative research studies. Merriam (2009) explained reliability in qualitative research studies in the following way:

> Reliability refers to the extent to which research findings can be replicated. In other words, if the study is repeated will it yield the same results? Realistically, a qualitative study by its design and its structure cannot be

> replicated largely because human behavior is never static and the phenomenon being studied is assumed to be in flux, multifaceted, and highly contextual” (p.220).

Rather than using the term ‘reliability’, Lincoln and Guba (1985) recommended discussing the ‘dependability’ or ‘consistency’ of the findings garnered from research data. In this way, the researcher should have the outsiders convince that the findings are consistent and dependable. Hence, according to Lincoln and Guba (1985) reliability in a qualitative study is not concerned with finding similar findings, but concerned with the findings that are consistent with the gathered data.

In particular, Yin (2003) stated that the aim of reliability in case studies is to be sure that “if a later investigator followed the same procedures as described by an earlier investigator and conducted the same case study all over again, the later should arrive at same findings and conclusions” (p. 37). Moreover, he added that by addressing the issue of reliability, the researchers try to reduce the errors and biases in their case studies. Shenton (2004) pointed out that in order for the readers to get in-depth understanding of the methods used by a qualitative researcher, the researcher should report “the research design and its implementation, the operational detail of data gathering, and reflective appraisal of the project” (p. 71-72). In this study, the issue of dependability was addressed to a certain extent by describing the research design and its implementation and by in-depth description of data gathering and analyzing procedures.

To establish dependability during the coding process, I and another doctoral student in the field of mathematics education coded the data independently. The second coder was experienced in coding qualitative data and was informed about the purpose and research questions of the study in detail. Besides, I informed her about the coding process before starting the actual coding and hence clarified the focal points of data analysis. Wiersma (2000) claimed that “if two or more researchers independently analyze the same data and arrive at similar conclusions, this is strong evidence for internal consistency” (p. 211). After coding the research data independently, the codings were compared to each other and about 75% agreement was found between the two researchers. Later, I came together for several times with the second coder to discuss and reach an agreement on codings and categories of this study. In each meeting session, the different opinions were further discussed and as a result the conflicts diminished to a lesser extent. Finally, the two coders arrived at an almost full consensus at the end of the meeting sessions and the coding process was finished.

3.8.4. Comfirmability

Finally, comfirmability is the last criterion to establish trustworthiness in qualitative research studies and it corresponds to objectivity in quantitative research methodology. Patton (2002) stated that the power of scientific method comes from objectivity and added that "objective tests gather data through instruments that, in principle, are not dependent on human skill, perception or even presence" (p. 50). However, he acknowledged that instruments were designed by humans and thus they were subject to the researchers' bias. Besides, Shenton (2004) pointed out that to address comfirmability "steps must be taken to help ensure as far as possible that the work's findings are the result of the experiences and ideas of the informants, rather than the characteristics and preferences of the researcher" (p.72). Shenton (2004) also recommended researchers to use triangulation to increase comfirmability. In this study, I tried to establish comfirmability by triangulating observation data and interview data and by in-depth description of the research methodology. Likewise, Miles and Huberman (1994) stressed that comfirmability might be addressed to a certain extent if researchers' acknowledge their own biases. Thus, the following section aims at describing my role and biases as a researcher.

3.9. Researcher Role and Bias

In qualitative studies, researchers are key instruments for gathering and analyzing data (Merriam, 2009). Therefore, subjectivity is one of the main concerns for researchers when considering the validity of the qualitative research. For instance, a researcher might record what she wants to see instead of recording what is really happening and therefore she may not control her bias. Besides, a researcher's views and beliefs might affect his/her interpretations in a qualitative study. In a similar way, I might have distorted my qualitative data due to my biases. The key strategy for understanding the researcher bias is reflexivity. Robson (2011) defined reflexivity as "the process of researchers' reflecting upon their actions and values during research (e.g., in producing accounts and writing accounts), and the effects that they may have" (p.531). In this research, I followed the suggestions of Ahern (1999) to achieve reflective bracketing (i.e., using reflexivity to identify areas of potential bias).

Before observing the middle school teachers with different rational number teaching experience, I clarified my presence at the classroom and explained the purpose of my thesis. I stated explicitly that it was not compulsory to participate in the study and made sure which participants volunteered to participate in my study. I also informed them that the video recordings and the interview transcripts were going to be kept confidential. During data analysis, I did not pay attention to teacher names in order to eliminate bias.

As I was a non-participant observer, I did not interact with the teachers or the students during the classroom practices of teachers. I kept my presence as passive as possible. This policy sometimes limited my observations and my ability to get more detailed information about teachers' intentions for choosing certain examples. However, it also provided a natural setting for my observations. During the classroom practices of teachers, I stayed at the end of the classroom. This vantage point kept me out of students' line of sight and provided me a good view of the teachers and the students. During the first few weeks, the students that were closer to me attempted to ask me for help and I politely replied them that they should ask for help from their friends or the teacher. I tried to keep myself away from offering any help or giving tips to the students about the questions asked by their teachers.

Because I am the primary means of data collection, interpretation and analysis, it is significant to state not only my role in this study, but also my own biases that might influence data analysis and interpretation of data. I firmly believe that teachers should provide well-thought examples to their students when teaching mathematical concepts. I think that this might promote students' understanding of mathematical concepts. Besides, teachers should provide their students with the opportunity to generate mathematical examples themselves. Thus, teachers need to encourage their students to be more active in classroom practices. Besides, students should be introduced to a wide range of examples when learning mathematical concepts and these examples should support not only students' procedural understanding but also their conceptual understanding of mathematics. Besides, I believe that the examples used by the teachers should be the outcome of their reflective process of choices. However, when I was a middle or secondary school student, I became experienced with examples that reinforced mainly mathematical procedures or operations and the teachers did not appear to employ deliberate considerations for choosing well-thought examples. The contradiction between what I experienced and what I think about example generation or selection might have provided me with the impetus for conducting this study.

CHAPTER IV

OVERALL CHARACTERISTICS OF TEACHERS' RATIONAL NUMBER EXAMPLES

The purpose of this study was to explore middle school mathematics teachers' treatment of rational number examples in their seventh grade classrooms. In this chapter, the focus was on describing overall characteristics of teachers' rational number examples. Through this focus, the following research question and sub-questions were formulated:

1. What are the overall characteristics of examples used by middle school mathematics teachers in the teaching of rational numbers in their seventh grade classrooms?

   a. What are the ideas emphasized in the rational number examples used by the teachers?
   b. To what extend do teachers use specific examples in the teaching of rational numbers?
   c. To what extend do teachers use non-examples and counter-examples in the teaching of rational numbers?
   d. To what extend do teachers use pre-planned and spontaneous examples in the teaching of rational numbers?
   e. Which sources do teachers use while choosing pre-planned examples in the teaching of rational numbers?

More specifically, this chapter included two sections as types of examples and sources of examples. Types of examples were reported under three sub-sections as specific examples, non-examples and counter examples. Next, sources of examples were reported under two subsections as spontaneous examples and pre-planned examples.

Mathematical examples that were generated or used by the teachers were checked to determine whether they satisfied their intended requirements to be an example. Mostly, the correctness of examples were investigated by examining whether they satisfied the definition of the concept being illustrated (i.e., concept definition). That is, the examples used by the teachers were investigated to see whether they match with the agreed upon mathematical definition or whether it is the definition held merely by the teachers. In all the observations, I identified 704 mathematically correct examples out of 714 examples that were used by the teachers in 60 hours of classroom observation. It is important to note that the examples reported in this study refer only to the examples generated by the teachers, not by the students. Besides, all of the observed classrooms used the same mathematics textbook prepared by Aydın and Beşer (2013a).

4. 1. Types of Mathematical Examples

In this study, the examination of mathematically correct examples showed that they played different roles in the teaching of rational number concepts. Thus, the examples used by the middle school teachers were categorized as specific examples, non-examples and counter-examples. In the following section, the specific examples provided by the four middle school mathematics teachers and by the followed mathematics textbook were described in detail. It is important to note that in the following sections the word 'textbook' shortly refers to 'the followed mathematics textbook.'

4.1.1. Specific examples

In this study, almost all mathematical examples generated by teachers were classified as specific examples. Almost all worked-out examples and exercise examples included in the explanatory part of the student textbook were also considered to be specific examples. An example that included both the task and its solution was considered to be a worked-out example while an example that did not include its solution was considered to be an exercise example. The number of specific examples included in the textbook and the number of specific examples used by four middle school mathematics teachers with respect to the learning objectives described by the middle school mathematics curriculum were presented in Table 4.1.

Table 4.1. Number of specific examples provided by the student textbook and the teachers

Learning Objectives	Number of specific examples provided by				
	Student Textbook	Teacher A	Teacher B	Teacher C	Teacher D
Explain and locate rational numbers on a number line	18	29	30	82	30
Express rational numbers in different forms	20	25	33	3	22
Compare and order rational numbers	22	36	22	2	14
Perform addition and subtraction operations with rational numbers	41	37	46	31	32
Perform multiplication and division operations with rational numbers	62	54	60	4	32
Solve multi-step operations with rational numbers	15	23	13	6	13
Pose and solve rational number problems	9	6	6	4	9
Total	187	210	210	132	152

Note: Worked-out examples and exercise examples that were included in the explanatory part of the textbook and that might be offered by the teachers within the context of learning rational number concepts were counted as textbook examples.

The number of examples used by the teachers to teach rational number concepts were quite different from each other and from the number of examples suggested by the mathematics textbook. Overall, the number of examples used by Teacher A and Teacher B was more than the number of examples included in the mathematics textbook. On the contrary, the number of examples used by Teacher C and Teacher D were less than that of suggested by the textbook.

More specifically, to 'explain and locate rational numbers on a number line' Teacher C used a great number of examples when compared to the number of examples included in the textbook for this learning objective. Similarly, Teacher A, Teacher B and Teacher D used more number of examples for explaining and locating rational numbers on a number line when compared to the textbook. For teaching how to express rational numbers in different forms, Teacher A, Teacher B and Teaching D used more than number of examples while Teacher C used very few examples than the ones included in the textbook. For teaching how to compare and order rational numbers, only Teacher A used more number of examples than the ones included in the textbook for teaching this objective. Besides, while Teacher B used the same number of examples, Teacher D used less number of examples and Teacher C used very few examples in comparison with textbook examples. For teaching how to add and subtract rational numbers, Teacher B used more number of examples and the other teachers used slightly less number of examples than the ones included in the textbook. The number of examples used by Teacher A and Teacher B to teach multiplication and division of rational numbers was slightly less than the number of textbook examples. Moreover, the number of examples provided by Teacher D for teaching this objective was slightly more than half of the number of examples included in the textbook for teaching this concept. However, the number of examples used by Teacher C for teaching this objective was remarkably less than that of textbook examples. When teaching how to solve multi-step operations with rational numbers, Teacher A used more number of examples while the other teachers used less number of examples than the ones included in the textbook for teaching this objective. Finally, while Teacher D used the same number of problem posing and solving examples, the other teachers used less number of problem posing and solving examples when compared to the ones included in the textbook.

More generally, the middle school mathematics textbook followed by the participating classrooms abounded in examples related with rational number operations and procedures while it included fewer problem posing and solving examples. Similar to this, middle school mathematics teachers gave more emphasis on rational number operations and procedures and thus they used a great number of examples for teaching rational number operations and procedures. In contrast, teachers provided fewer examples for teaching how to pose and solve real life problems regarding rational numbers.

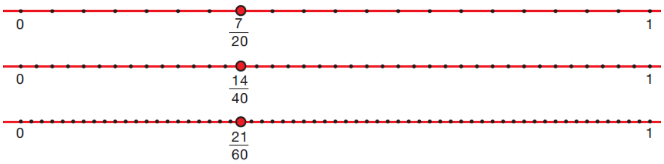
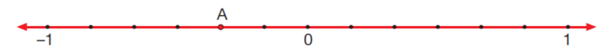
In the following sections, the specific examples included in the middle school mathematics curriculum and in the textbook and those that were used by the middle school mathematics teachers to teach each learning objective were described in detail.

4.1.1.1. Examples used for explaining and locating rational numbers on a number line

Middle school mathematics curriculum emphasized that in order for teachers to teach this objective they need to define rational numbers by using the symbol Q and have students examine the relationship between integers, fractions, and rational numbers. Besides, it was emphasized that students need to realize where rational numbers are used in daily life situations. Finally, the curriculum suggested an activity for locating rational numbers on a number line. In this activity, teachers were suggested to emphasize negative rational numbers by recalling the absolute value concept and by finding the symmetries of positive rational numbers through a symmetry mirror that is placed on the origin of the number line.

In the mathematics textbook followed by the classrooms, the examples related with this objective were presented under the following ideas: finding equivalent classes of a fraction, locating equivalent fractions on a number line, locating rational numbers on a number line, determining the positivity/negativity of rational numbers, and finding the rational value of a point located on a number line. Some illustrative examples included in the textbook for the aforementioned ideas were presented in Table 4.2. It is important to note that only the examples with different mathematical structure were used as illustrative examples. Therefore, the number of examples provided for each idea were also presented.

Table 4.2. Examples included in the textbook for explaining and locating rational numbers on a number line

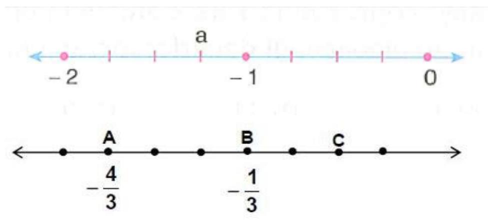
Ideas	Illustrative examples	Number of examples used
Finding equivalent classes of a fraction	$\frac{7}{20} \equiv \frac{14}{40} \equiv \frac{21}{60} \equiv \frac{28}{80} \equiv \frac{35}{100} \equiv \frac{42}{120} \equiv ...$ $-\frac{1}{2} \equiv -\frac{2}{4} \equiv -\frac{3}{6} \equiv -\frac{4}{8} \equiv ...$	4
Locating equivalent fractions on a number line		1
Locating rational numbers on a number line	$\frac{1}{2}$; $\frac{7}{3}$; $1\frac{1}{4}$; $-\frac{2}{3}$; $-\frac{5}{2}$; -2	8
Determining the positivity/negativity of rational numbers	$\frac{2}{5} \in \mathbb{Q}^{+}$; $\frac{4}{-7} = -\frac{4}{7} \in \mathbb{Q}^{-}$; $-\frac{14}{5} \in \mathbb{Q}^{-}$; $\frac{-26}{-5} = \frac{26}{5} \in \mathbb{Q}^{+}$	4
Finding the rational value of a point located on a number line		1

The textbook included 18 examples for explaining and locating rational numbers on a number line. The textbook initially introduced $\frac{7}{20}$ as an example for the concept of fraction. Then, fractions that were equivalent to this fraction were determined and $A = \left\{ \frac{7}{20}, \frac{14}{20}, \frac{21}{60}, \frac{28}{80} \cdots \right\}$ was described as the set of fractions that were equivalent to this fraction. Later, the equivalent fractions were located on three distinct number lines that were one under the other. The number line representation showed that the equivalent fractions located on the same point and thus $\frac{7}{20}$ was named as the identifier of the set. As a result, it was stated that each identifier of a set consisting of equivalent fractions was a rational number. Finally, rational numbers were defined as numbers that can be written in the form of $\frac{a}{b}$, where a and b are integers, $b \neq 0$ and the set of rational numbers was denoted by Q.

After defining rational numbers, the textbook gave examples about locating rational numbers on a number line. The selected rational numbers included several variations in the following dimensions: proper, improper, mixed, positive or negative. Next, to examine positivity/negativity, a rational number with a positive numerator and a positive denominator, a rational number with a positive numerator and a negative denominator, a rational number with a negative numerator and a positive denominator and a rational number with a negative numerator and a negative denominator were presented. Finally, the textbook ended up with a worked example asking students to find the rational value of a point located on a number line between 0 and -1.

The examples used by Teacher A for explaining and locating rational numbers on a number line were classified as follows: identifying whether a given number is rational, locating rational numbers on a number line, finding the rational value of a point located on a number line and examining the location of a minus sign in a negative rational number. Some illustrative examples used by Teacher A for these ideas were presented in Table 4.3. Note that only the examples with different mathematical structure were used as illustrative examples. Therefore, the number of examples provided for each idea were also presented.

Table 4.3. Examples used by Teacher A for explaining and locating rational numbers on a number line

Ideas	Illustrative examples	Number of examples used
Identifying whether a given number is rational	$0 \in \mathbb{Q};\ -\frac{2}{3} \in \mathbb{Q};\ -125 \in \mathbb{Q};\ 0.12 \in \mathbb{Q};\ \frac{1}{0} \notin \mathbb{Q};\ \pi \notin \mathbb{Q}$	17
Locating rational numbers on a number line	$\frac{3}{5};\ -\frac{3}{4};\ -3\frac{2}{5};\ \frac{12}{5}$	6
Finding the rational value of a point located on a number line		2
Examining the location of a minus sign in a negative rational number	$\frac{-1}{2} = -\frac{1}{2} = \frac{1}{-2}$	4

Teacher A used 29 examples for explaining and locating rational numbers on a number line. He started teaching for this objective by having students recall the number sets learned before. That is, he defined counting numbers, natural numbers and integers by using the listing method in the following way: $N = \{0, 1, 2, 3, ..., \infty\}$, $C = \{1, 2, 3, ..., \infty\}$, and $\mathbb{Z} = \{-\infty, -2, -1, 0, +1, +2, ..., \infty\}$ at the beginning of the lesson. Although the rational numbers were defined by the help of equivalent fractions and by locating equivalent fractions on a number line in the textbook, Teacher A did not emphasize these ideas. Instead, he directly defined rational numbers after having students remember counting numbers, natural numbers and integers. He symbolically defined rational numbers as $\mathbb{Q} = \left\{ a, b \in \mathbb{Z}, \text{and } b \neq 0, \frac{a}{b} \right\}$ and he stated that any number that can be written in the form $\frac{a}{b}$, where b is not equal to zero is called a rational number. After this definition, Teacher A wrote several numbers on the board and asked the students to find out which one of them were rational numbers. It is important to note that the teacher selected two numbers from natural number set, two numbers from integer set and finally one number from decimal number set. By this way, he emphasized the relationship among rational numbers, integers, decimals and natural numbers. Finally, he drew a Venn diagram on the board to show the relationship between counting numbers, natural numbers, integers and rational numbers as shown in Figure 4.1.

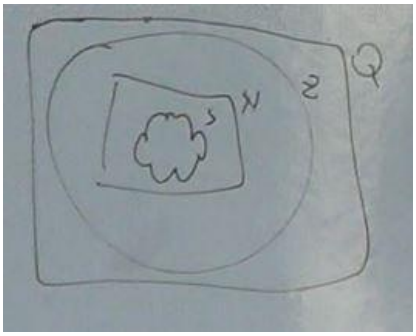


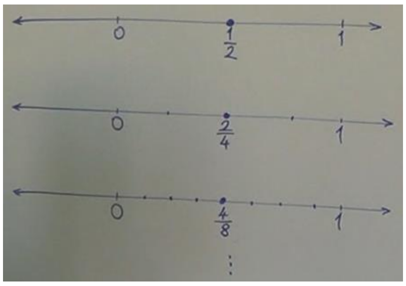
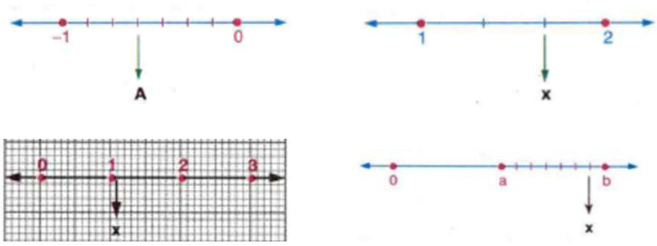
Figure 4.1. A Venn diagram used by Teacher A

Before teaching how to locate rational numbers on a number line, Teacher A had students remember proper fractions, improper fractions and mixed numbers. Then, he selected a set of rational numbers in the following forms to locate on a number line: a positive proper number, a negative proper number, a positive mixed number and a positive improper number respectively. The students were already familiar with locating the first rational number on the number line since they learnt this in their previous years in the topic of fractions. Next, the teacher selected a negative rational number whose numerator was less than its denominator. The third example selected was between 3 and 4 and was a mixed number. Lastly, before locating a positive improper number on the number line, the teacher converted it into a mixed number to determine the whole and fractional parts. The form of rational number examples selected by the teacher were similar to those included in the textbook. That is, both Teacher A and the textbook presented examples in proper, improper, mixed, and negative forms.

After locating several rational numbers on a number line, Teacher A tried to draw students' attention to the idea that the position of minus sign does not change the value of a rational number. To do so, he wrote $-\frac{1}{2}=\frac{-1}{2}=\frac{1}{-2}$ on the board and stressed that $-\frac{1}{2}$, $\frac{-1}{2}$ and $\frac{1}{-2}$ were all the same regardless of the position of minus sign. However, the teacher did not provide examples for inspecting the negativity/positivity of rational numbers although there were examples of this kind in the textbook. Besides, the teacher did not provide examples for finding the rational value of a point located on a number line during the teaching of this objective. However, after teaching the objective 'comparing and ordering rational numbers' he provided two exercise examples of this kind from an auxiliary book.

The examples used by Teacher B for explaining and locating rational numbers on a number line were classified as follows: identifying whether a given number is rational, finding equivalent classes of a fraction, locating equivalent fractions on a number line, locating rational numbers on a number line and finding the rational value of a point located on a number line. Some illustrative examples used by Teacher B for teaching these ideas were presented in Table 4.4. Note that only the examples with different mathematical structure were used as illustrative examples. Therefore, the number of examples provided for each idea were also presented.

Table 4.4. Examples used by Teacher B for explaining and locating rational numbers on a number line

Ideas	Illustrative examples	Number of examples used
Identifying whether a given number is rational	$\frac{1}{2} \in \mathbb{Q};\ \frac{103}{85} \in \mathbb{Q};\ -\frac{5}{8} \in \mathbb{Q};\ \sqrt{5} \notin \mathbb{Q}$	6
Finding equivalent classes of a fraction	$\frac{1}{2} \equiv \frac{2}{4} \equiv \frac{3}{6} \equiv \frac{4}{8} \equiv \ldots;\ -\frac{5}{2} \equiv -\frac{10}{4} \equiv -\frac{-15}{6} \equiv -\frac{20}{8} \equiv \ldots$	4
Locating equivalent fractions on a number line		1
Locating rational numbers on a number line	$-\frac{435}{500};\ -2\frac{1}{5};\ \frac{3}{5};\ 1\frac{2}{3}$	11
Finding the rational value of a point located on a number line		8

Teacher B used 30 examples for explaining and locating rational numbers on a number line. He started the lesson by briefly touching upon previously learnt number sets. He selected several numbers to exemplify counting numbers and added 0 to these numbers to define natural numbers. Next, he selected several negative numbers to recall integers. Similar to Teacher A, Teacher B used a Venn diagram when providing examples for counting numbers, natural numbers and integers. After mentioning about these number sets, the teacher asked students to ponder whether these number sets fill up the number line. During this time, the teacher drew a number line on the board and selected a point between 1 and 2. Then, the teacher wrote $1\frac{1}{2}$ as a corresponding value of this point and asked the students to find out to which number set it belonged. By this way, the teacher had students remember early fraction ideas and feel the need for a new number system. The teacher introduced rational number set as a new number system and denoted it with the symbol *Q*. Then he switched back to the Venn diagram and selected several rational number examples as shown in Figure 4.2.

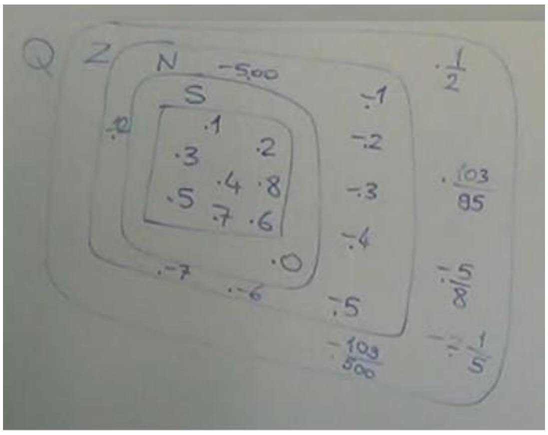


Figure 4.2. A Venn diagram used by Teacher B

The rational numbers selected by Teacher B included some variations such as being a proper number, improper number, mixed number, positive number or negative number. However, the selections constrained rational numbers to the numbers in the form of $\frac{a}{b}$ since Teacher B did not select any natural number, counting number or an integer to exemplify rational numbers.

After explaining rational numbers by the help of different number sets, Teacher B taught students how to locate different rational numbers on a number line. He selected a set of rational numbers in the following forms: a positive mixed number, a negative mixed number, a positive proper number, a positive mixed number and a negative proper number respectively. The sequence of examples used for locating rational numbers on a number line was different from textbook examples since Teacher B neither started locating by an already known proper fraction or nor provided examples for improper fractions.

In addition to defining rational number set by recalling counting numbers, natural numbers and integers, Teacher B also used the definition that was included in the textbook. That is, he located the fractions $\frac{1}{2}, \frac{2}{4}, \text{and} \frac{4}{8}$ on three different number lines that were one under the other and stressed that these fractions corresponded to the same point and thus they were equivalent to each other. Then, he expressed that these equivalent fractions form the set $A = \left\{\frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \frac{4}{8}, \frac{5}{10}, ...\right\}$ and added that the simplest fraction of this set is also the identifier and the rational number denoted by this set. Teacher B also selected a negative proper number and a negative improper number to form sets of equivalent fractions.

To emphasize that the number of rational numbers included in rational number set is infinite, the teacher asked students to think of the number of rational numbers between 0 and -1. In addition, he generated $-\frac{435}{500}$ as an example for a rational number which is fairly close to zero. It is important to note that the teacher selected large numbers for the numerator and the denominator to help students grasp that there are also too many rational numbers between 0 and 1. After this, the teacher wrote the definition of a rational number in the following way: “the simplest fraction of a set of equivalent fractions is a rational number denoted by this set”. Besides, the teacher denoted the rational number set with the symbol *Q*. However, he ignored defining rational numbers by using the symbolic form $\mathbb{Q} = \left\{a, b \in \mathbb{Z}, \text{and } b \neq 0, \frac{a}{b}\right\}$. The teacher ended up explaining rational numbers with the mathematical statement $\mathbb{Q} = \mathbb{Q}^{-} \cup \{0\} \cup \mathbb{Q}^{+}$ but did not give any specific example for determining the positivity/negativity of rational numbers.

To teach how to locate rational numbers on a number line, Teacher B selected rational numbers in different forms. Although the textbook initially provided a proper fraction as a start-up example to this concept, Teacher B preferred to use a negative rational number between -1 and 0. After this, the teacher used several more examples to teach students how to locate a positive mixed number, a positive proper number and a negative mixed number. Nevertheless, Teacher B did not provide any example for locating negative improper numbers on a number line despite there were examples of this kind in the textbook.

In the textbook, one example was provided to illustrate finding the rational value of a point located on a number line. Although Teacher B did not provide any example while teaching this concept, he used several exercise examples of this kind after teaching how to express rational numbers in different forms. The rational numbers that corresponded to each of the specified points on the number lines were $-\frac{5}{2}, -\frac{4}{7}, \frac{6}{7}, \frac{5}{3}, \frac{7}{4}, 1\frac{1}{10}, 1\frac{8}{9}$, and $2\frac{1}{5}$. Here, the first example was between -3 and -2, the second example was between -1 and 0, the third example was between 0 and 1, the fourth, the fifth, the sixth and the seventh examples were between 1 and 2. Finally, the last example was between 2 and 3.

The examples used by Teacher C for explaining and locating rational numbers on a number line were categorized as follows: identifying whether a given number is a rational number, an integer or a natural number, examining the location of a minus sign in a negative rational number, determining the positivity/negativity of rational numbers, locating rational numbers on a number line, finding equivalent classes of a fraction, simplifying fractions and converting among mixed numbers and improper numbers. Some illustrative examples used by Teacher C for teaching these ideas were presented in Table 4.5. Note that only the examples with different mathematical structure were used as illustrative examples. Therefore, the number of examples provided for each idea were also presented.

Table 4.5. Examples used by Teacher C for explaining and locating rational numbers on a number line

Ideas	Illustrative examples	Number of examples used
Identifying whether a given number is a rational number, an integer or a natural number	$-8\frac{1}{5}\in\mathbb{Q};\ -5\in\mathbb{Q};\ -\frac{8}{4}\in\mathbb{Q};-\frac{1}{2}\in\mathbb{Q};\ 0\in\mathbb{Q};\ \frac{3}{7}\in\mathbb{Q};\ \frac{99}{83}\in\mathbb{Q};$ $\frac{10}{2}\in\mathbb{Q};\ 19\in\mathbb{Q};\ \frac{7}{0}\notin\mathbb{Q};\ -8\frac{1}{5}\notin\mathbb{Z};\ -\frac{8}{4}\in\mathbb{Z};\ -1\in\mathbb{Z};\ -\frac{1}{2}\notin\mathbb{Z};$ $0\in\mathbb{Z};\ 0.35\notin\mathbb{Z};\ \frac{1}{2}\notin\mathbb{Z};\ 1\in\mathbb{Z};\ \frac{10}{2}\in\mathbb{Z};-8\frac{1}{5}\notin\mathbb{N};-5\notin\mathbb{N};$ $-\frac{8}{4}\notin\mathbb{N};-\frac{1}{2}\notin\mathbb{N};\ 0\in\mathbb{N};\ 0.35\notin\mathbb{N};\ \frac{3}{7}\notin\mathbb{N};\ \frac{10}{2}\in\mathbb{N};1\in\mathbb{N}$	40
Examining the location of a minus sign in a negative rational number	$\frac{-3}{4}=-\frac{3}{4}=\frac{3}{-4}$	2
Determining the positivity/ negativity of rational numbers	$\frac{-3}{-4}\in\mathbb{Q}^{+},\ \frac{3}{5-x}\in\mathbb{Q}^{+}$	2
Finding equivalent classes of a fraction	$\frac{1}{101}\equiv\frac{2}{202}\equiv\frac{3}{303}\equiv\frac{4}{404}\equiv\frac{5}{505}\equiv\frac{6}{606}$	6
Locating rational numbers on a number line	$\frac{1}{8};\ -\frac{3}{5};\ -\frac{8}{5};\ 1\frac{2}{7};\ -2\frac{3}{5}$	15
Simplifying fractions	$\frac{80}{140}=\frac{4}{7};\ \frac{400}{160}=\frac{5}{2}$	4
Converting among mixed numbers and improper numbers	$2\frac{1}{2}=\frac{5}{2};\ -7\frac{1}{4}=-\frac{29}{4};\ \frac{16}{2}=8;\ \frac{12}{5}=2\frac{2}{5};\ -\frac{25}{6}=-4\frac{1}{6}$	13

Teacher C used 82 examples for explaining and locating rational numbers on a number line. He started teaching this objective by the defining rational numbers as numbers that can be written in the form of $\frac{a}{b}$, where *a* and *b* are integers, $b\neq 0$. The teacher emphasized that the denominator of a rational number cannot be zero by giving $\frac{7}{0}$ as a non-example to rational numbers. The teacher continued with the mathematical statement $\mathbb{Q}=\mathbb{Q}^{-}\cup\{0\}\cup\mathbb{Q}^{+}$ without providing any specific example for determining the negativity/positivity of a rational number. However, he tried to draw students' attention to the neutrality of zero as a rational number. Next, he mentioned about counting numbers, natural numbers and integers and stressed that rational numbers are 'larger' than integers, integers are 'larger' than natural numbers, and natural numbers are 'larger' than counting numbers. Besides, he wrote a mathematical statement on the board as a remark to the relationship among counting numbers, natural numbers, integers and rational numbers. As it can be seen in Figure 4.3, Teacher C used the symbol $\subset$ to indicate the relationship among these number sets.

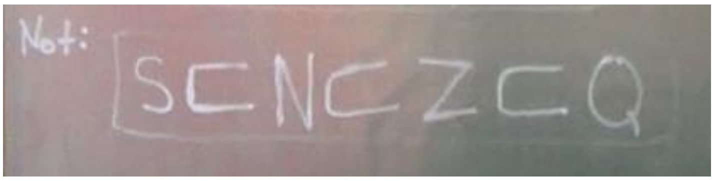


Figure 4.3. A subset notation used by Teacher C

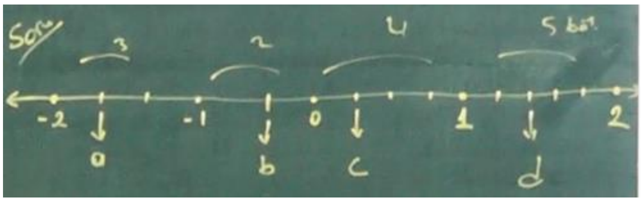
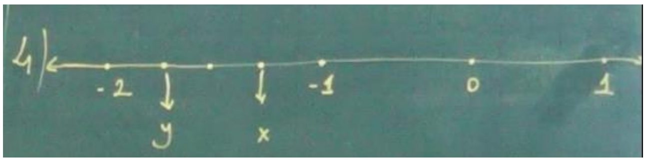
Teacher C provided exhaustive number of examples to help students identify whether the given number is an element of natural number set, integer set or rational number set or an element of all these three sets. After introducing a negative integer as a rational number example, Teacher C emphasized that changing position of minus sign does not alter the negativity and the value of a rational number. He wrote the equality $\frac{-3}{4}=-\frac{3}{4}=\frac{3}{-4}$ on the blackboard as an example for this idea. Similar to the textbook, Teacher C used an example that had a minus sign both in the numerator and the denominator (i.e., $\frac{-3}{-4}$) to help students understand that it is an element of $\mathbb{Q}^{+}$. Although the teacher had students remember the concept of equivalent fractions by providing several examples, he did not use any of these examples as an initial step for introducing the mathematical term 'identifier of a set' and for defining rational number set as done in the textbook.

After providing examples for equivalent fractions, Teacher C selected several rational numbers in the following form to locate on a number line: a positive proper number, a negative mixed number, a positive mixed number and a negative improper number. The sequence of examples used by the teacher was different from the sequence of textbook examples since textbook examples took the form of a positive proper number, a negative proper number, a positive mixed number and a negative improper number respectively. This difference stemmed in part from the fact that the sequence generated by the teacher did not include a negative proper number example. Finally, although the textbook included an example for finding the rational value of a point located on a number line, Teacher C did not provide any example of this kind to this students.

Apart from the examples provided during the teaching of previously mentioned ideas, Teacher C used several exercise examples for simplifying fractions and converting among mixed numbers and improper numbers. More specifically, the selected fractions included large numbers in their numerators and denominators and these fractions were either in proper or improper number form. Similarly, the examples selected by Teacher C for conversion included both positive and negative improper numbers.

Teacher D’s selection of examples for explaining and locating rational numbers on a number line fell under the following categories: feeling the need for positive and negative rational numbers, identifying whether a given number is a rational number, examining the position of the minus sign, determining the positivity/negativity of rational numbers, locating rational numbers on a number line and finding the value of a rational number marked on a number line. The illustrative examples and the total number of examples used for explaining and locating rational numbers on a number line were presented in Table 4.6.

Table 4.6. Examples used by Teacher D for explaining and locating rational numbers on a number line

Ideas	Illustrative examples	Number of examples used
Feeling the need for positive and negative rational numbers	$\frac{1}{4}$ of a cake; $\frac{2}{3}$ meters below sea level; $\frac{2}{3}$ degrees Celsius below zero	3
Identifying whether a given number is a rational number	$-2 \in \mathbb{Q}$; $-1\frac{1}{4} \in \mathbb{Q}$; $-\frac{2}{3} \in \mathbb{Q}$; $0 \in \mathbb{Q}$; $\frac{1}{4} \in \mathbb{Q}$; $0.12 \in \mathbb{Q}$; $\frac{3}{0} \notin \mathbb{Q}$	14
Examining the location of a minus sign in a negative rational number	$\frac{-1}{2} = \frac{1}{-2} = -\frac{1}{2}$	1
Determining the positivity/ negativity of rational numbers	$\frac{-1}{2} \in \mathbb{Q}^-$; $\frac{1}{-2} \in \mathbb{Q}^-$; $\frac{-1}{-2} \in \mathbb{Q}^+$	3
Locating rational numbers on a number line	$\frac{5}{6}$; $-\frac{5}{6}$; $-2\frac{4}{5}$	3
Finding the rational value of a point located on a number line	 	6

Teacher D used 30 examples for explaining and locating rational numbers on a number line. She started the lesson by having students feel the need for positive and negative rational numbers through real-life situations. She first introduced a fraction part model, shaded one-fourth of the fraction pie, emphasized that each piece is equal to each other and represented the shaded region with the fraction $\frac{1}{4}$. In addition, she provided temperature and altitude below sea level examples that modelled negative rational numbers. More specifically, she asked students how to express $\frac{2}{3}$ degrees below 0 on a Celsius temperature scale. Similarly, the teacher asked students to express the altitude of a swimmer that is $\frac{2}{3}$ meters below the sea level. Next, she indicated that integers are signed numbers and fractions may also have signs. At this point, she introduced the term 'rational numbers' and stated that rational numbers are used to express both negative and positive fractions. After that, she recalled natural numbers and integers by using a Venn diagram as done by the previous three teachers. This diagram is presented in Figure 4.4.

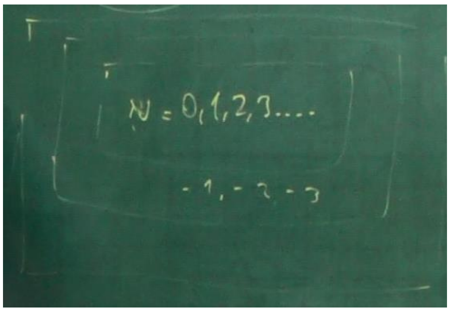


Figure 4.4. A Venn diagram used by Teacher D

Teacher D indicated that the rational number set forms the superset of integer and natural number sets. The teacher provided several examples for rational numbers. These rational numbers were in the following forms: a positive proper number, a positive decimal number, the neutral number zero, a negative proper number, a negative mixed number and a negative integer. Besides, one number was an element of all three sets, one number was an element of integer and rational number sets and the rest were only elements of the rational number set. After providing examples for rational numbers, the teacher defined rational numbers as numbers that can be written in the form of $\frac{a}{b}$, where a and b are integers, $b \neq 0$ and denoted the set of rational numbers by Q. To emphasize that the denominator of a rational number cannot be equal to zero, the teacher provided $\frac{3}{0}$ and $\frac{0}{3}$ successively and pointed out that the former rational number is undefined while the latter is equal to 0. Next, to examine the positivity/negativity of rational numbers, the following forms of examples were used by Teacher D: a negative numerator over a positive denominator, a negative numerator over a negative denominator, and a positive numerator over a negative denominator. Different from textbook examples, these examples had absolute values that were equal to each other. By this way, the teacher hinted at the idea that the position of the minus sign does not alter the value of the rational number. Just after introducing these examples, the teacher wrote on the blackboard the equality $\frac{-1}{2} = \frac{1}{-2} = -\frac{1}{2}$ and explicitly stated that these three different notations of a rational number are equal to each other. Finally, Teacher D ended up explaining rational numbers by introducing the symbolic expression $\mathbb{Q} = \mathbb{Q}^- \cup \{0\} \cup \mathbb{Q}^+$.

To teach how to locate rational numbers on a number line, Teacher D used a sequence of rational numbers in the following forms: a positive proper number, a negative proper number and a negative mixed number respectively. Before locating the positive proper number on the number line, the teacher stated that the students already know how to locate fractions on a number line. More specifically, she divided the numerator of the positive proper number by its denominator, (i.e., $5\overline{)6}$), and showed that the positive proper number is between 0 and 1 and concluded that all proper fractions are located between 0 and 1 on a number line. To have students notice the effect of minus sign on the position of a rational number, Teacher D selected $-\frac{5}{6}$ as the next example and indicated that $-\frac{5}{6}$ is located between 0 and -1. By locating $\frac{5}{6}$ and $-\frac{5}{6}$ on a number line successively, the teacher showed that they are equally distant from 0 on the number line. Besides, she explained that negative rational numbers are located on the left hand side of 0 while positive rational numbers are located on the right hand side. The last example used by the teacher for locating rational numbers was a negative mixed number. While locating the mixed number on a number line, the teacher informed students about a common error they tended to make and warned students to count subintervals instead of tick marks while finding the location of a rational number on a number line. The examples used by the teacher to teach locating were similar to those that were included in the textbook. As mentioned before, the textbook examples included several variations in the following dimensions: proper, improper, mixed, positive or negative. While Teacher D illustrated how to locate a positive proper number, a negative proper number and a negative mixed number respectively, she did not do this for the following number forms: a positive mixed number, a positive improper number or a negative improper number.

In addition to locating rational numbers on a number line, Teacher D provided several examples for finding the rational value of points that are marked on a number line. To be more precise, the teacher marked points on a number line between -2 and -1, -1 and 0, 0 and 1 and finally between 1 and 2 and asked students to find out the rational numbers corresponding to these points. By this way, the teacher selected rational numbers that are in different integer intervals. In addition, each integer interval included different number of subintervals. Thus, the denominator of each rational number was different from each other.

4.1.1.2. Examples used for expressing rational numbers in different forms

In the middle school mathematics curriculum, teachers are suggested to teach this objective by focusing on two main ideas. First, teachers are expected to teach how to express a rational number as an integer, as a natural number, as a terminating decimal number and lastly as a repeating decimal number. Second, they are expected to teach how to convert a repeating decimal number into a rational number.

In the mathematics textbook followed by the classrooms, the examples related with this objective included the following ideas: expressing integers as rational numbers, expressing a rational number as an integer/repeating decimal/terminating decimal, expressing terminating decimals as rational numbers and converting repeating decimals into rational numbers. The illustrative examples and the total number of examples for each category are presented in Table 4.7.

Table 4.7. Examples included in the textbook for expressing rational numbers in different forms

Ideas	Illustrative examples	Number of examples used
Expressing integers as rational numbers	$30 = \frac{30}{1};\ -105 = \frac{-105}{1};\ +15 = \frac{+15}{1}$	5
Expressing a rational number as an integer/repeating decimal/terminating decimal	$\frac{8}{2} = 4;\ \frac{7}{3} = 2.\bar{3};\ -\frac{1}{21} = -0.\overline{047619};\ \frac{7}{8} = 0.875;\ \frac{7}{2} = 3.5$	5
Expressing terminating decimals as rational numbers	$35.5 = \frac{355}{10};\ -89.22 = \frac{-8922}{100}$	5
Converting repeating decimals into rational numbers	$7.\bar{6} = \frac{69}{9};\ 2.0\bar{5} = \frac{185}{90};\ 0.8\overline{75} = \frac{867}{990}$	5

The textbook included 20 examples for expressing rational numbers in different forms. The textbook initially illustrated that integers can be written in the form of rational numbers. Examples of this kind showed that each integer, either positive or negative, can be written in the form of $\frac{a}{b}$ where b is equal to 1. Moreover, the textbook included examples for expressing a rational number as an integer, repeating decimal or terminating decimal. For instance, the equality $\frac{8}{2} = 4$ was included in the textbook for the purpose of expressing a rational number as an integer or a natural number and similarly $\frac{7}{2} = 3.5$ and $\frac{7}{8} = 0.875$ exemplified that rational numbers can be written in the form of terminating decimals. Lastly, the equalities such as $\frac{7}{3} = 2.\bar{3}$ and $-\frac{1}{21} = -0.\overline{047619}$ illustrated that rational numbers can be written in the form of repeating decimals. Besides, the examples such as

$35.5 = \frac{355}{10}$ and $-89.22 = \frac{-8922}{100}$ typified that terminating decimals can be written as rational numbers in the form of $\frac{a}{b}$.

As another category, examples such as $7.\overline{6} = \frac{69}{9}$, $2.0\overline{5} = \frac{185}{90}$ and $0.8\overline{75} = \frac{875}{990}$ were included in the textbook so as to explain the method for converting a repeating decimal into a rational number. The method included in the textbook for converting $7.\overline{6}$ into its rational number is provided in Figure 4.5.

D = 7,$\overline{6}$ = 7, 666... ise 10 . D = 76,666... olur.

$$\begin{array}{lr} & 10 \cdot (7{,}666\ldots) = 76{,}666\ldots \\ - & 1 \cdot (7{,}666\ldots) = 7{,}666\ldots \\ \hline \end{array}$$

$$10 \cdot (7{,}666\ldots) - 1 \cdot (7{,}666\ldots) = 69{,}000$$

$$(10 - 1) \cdot (7{,}666\ldots) = 69$$

$$9 \cdot (7{,}666\ldots) = 69$$

$$\frac{1}{9} \cdot 9 \cdot (7{,}666\ldots) = \frac{1}{9} \cdot 69$$

$$(7{,}666\ldots) = \frac{69}{9} = \frac{23}{3}$$

Figure 4.5. Textbook method for converting $7.\overline{6}$ into its rational number (Aydın & Beşer, 2013a, p. 49)

As it can be seen, in the first example (i.e., $7.\overline{6}$) the repeating pattern begins immediately after the decimal point. In the second and third example, (i.e., $2.0\overline{5} = \frac{185}{90}$ and $0.8\overline{75} = \frac{875}{990}$ respectively) it begins one digit after the decimal point. However, while only one digit repeats in the second example, two digits repeat in the last example.

The examples used by Teacher A for expressing rational numbers in different forms typified the following mathematical ideas: expressing a rational number as a repeating decimal and converting repeating decimals into rational numbers. The illustrative examples and the total number of examples for these two ideas are presented in Table 4.8.

Table 4.8. Examples used by Teacher A for expressing rational numbers in different forms

Ideas	Illustrative examples	Number of examples used
Expressing a rational number as a repeating decimal	$\frac{10}{3} = 3.\bar{3}$; $\frac{1}{10} + \frac{2}{100} + \frac{1}{1000} + \frac{2}{10000} + ... = 0.\overline{12}$	2
Converting repeating decimals into rational numbers	$0.\bar{2} = \frac{2}{9}$; $5.\overline{26} = 5\frac{26}{99}$; $7.\overline{525} = \frac{7525-7}{999}$; $2.6\bar{8} = \frac{268-26}{90}$; $52.7\overline{14} = \frac{52714-527}{990}$; $6.32\overline{84} = \frac{63284-632}{9900}$	19
Expressing terminating decimals as rational numbers	$0.2 = \frac{2}{10}$; $0.34 = \frac{34}{100}$; $3.546 = 3\frac{546}{1000}$; $-2.54 = -2\frac{54}{100}$	4

Teacher A used 25 examples for expressing rational numbers in different forms. Although there were examples in the textbook for expressing integers as rational numbers and for expressing rational numbers as integers and as terminating decimals, the teacher did not provide any example of these kinds during the teaching of the current objective. However, he used several examples for expressing integers as a rational number while teaching the objective 'explain and locate rational numbers on a number line'. More clearly, while explaining rational numbers, he wrote on the board several equalities such as $2 = \frac{2}{1}$ and $0 = \frac{0}{1}$ to show that an integer or a natural number can be written as a rational number.

Teacher A provided one example for teaching how to express a rational number as a repeating decimal number. To teach that $\frac{10}{3}$ is equal to $3.\bar{3}$, the teacher performed a long division algorithm on the board. By the help of this initial example, the teacher explained that each repeating decimal can be expressed as a rational number.

Teacher A provided a large number of examples for converting repeating decimals into a rational number. Meanwhile, he used two different procedures to teach conversion of a repeating decimal into a rational number. The teacher used these two different procedures as a shortcut to the method included in the textbook. The textbook method provided the underlying logic of the conversion. However, Teacher A did not present this method to the students before teaching shortcuts. The teacher used the first procedure for the decimals whose all digits after the decimal point repeated. According to this procedure, repeating digits were written over the main fraction bar as a numerator while 9's as many as the number of repeating digits were written under the main fraction bar as a denominator. For instance, $0.\overline{2} = \frac{2}{9}$ was provided by the teacher to show the conversion of repeating decimals which included only one repeating digit after the decimal point. Similarly, he used $5.\overline{26} = 5\frac{26}{99}$ as an example for converting a decimal with two repeating digits after the decimal point. Moreover, some of the examples provided by the teacher included both non-repeating and repeating digits after the decimal point. To convert these type of repeating decimals into a rational number, Teacher A introduced another procedure for their students. This procedure emphasized adding 9 to the denominator of the rational number as many as the number of repeating digits and adding 0 to the denominator of the rational number as many as the number of non-repeating digits following the decimal point. More specifically, the teacher provided $2.6\overline{8} = \frac{242}{90}$, $52.7\overline{14} = \frac{52714-527}{990}$, $6.3\overline{284} = \frac{63284-63}{9990}$ and $3.7\overline{5} = \frac{375-37}{90}$ as examples for teaching this procedure. As it can be seen, the first and the last example includes one repeating and one non-repeating digit after the decimal point, the second example includes two repeating digits and one non-repeating digit, and the third example includes three repeating digits and one non-repeating digit. When these examples are compared with those of the textbook, it can be seen that Teacher A used a wider variety and more examples to teach conversion of repeating decimals into a rational number.

Finally, Teacher A used several examples for expressing terminating decimals as rational numbers. The teacher used both positive and negative terminating decimals for expressing them as rational numbers. Besides, these terminating decimals included one, two or three digits after the decimal point.

The examples used by Teacher B to teach how to express rational numbers in different forms represented the following ideas: expressing integers as rational numbers, expressing terminating decimals as rational numbers, expressing rational numbers as repeating decimals and converting repeating decimals into rational numbers. The illustrative examples and the total number of examples for each of these ideas are presented in Table 4.9.

Table 4.9. Examples used by Teacher B for expressing rational numbers in different forms

Ideas	Illustrative examples	Number of examples used
Expressing integers as rational numbers	$2=\frac{2}{1};\ -8=\frac{-8}{1}$	3
Expressing terminating decimals as rational numbers	$4.8=\frac{48}{10};\ 1.73=\frac{173}{100};\ -0.04=-\frac{1}{250}$	4
Expressing rational numbers as repeating decimals	$\frac{5}{3}=1.\bar{6}$	2
Converting repeating decimals into rational numbers	$1.\bar{6}=\frac{16-1}{9};\ 2.\overline{15}=\frac{215-2}{99};\ 3.2\bar{4}=\frac{324-32}{90};\ 5.\overline{104}=\frac{5104-5}{999};$ $1.0\overline{45}=\frac{1045-10}{990};\ 0.16\bar{7}=\frac{167-16}{900};\ 3.1\overline{745}=\frac{37145-31}{9990};$ $0.74\overline{19}=\frac{7419-74}{9900}$	24

Teacher B used 33 examples for teaching how to express rational numbers in different forms. The teacher initially provided examples such as $2=\frac{2}{1}$ and $-8=\frac{-8}{1}$ to illustrate that each integer, either positive or negative, can be expressed in the form of $\frac{a}{b}$ where b is equal to 1. Next, he provided examples such as $4.8 = \frac{48}{10}$, $1.73 = \frac{173}{100}$, $-0.04 = -\frac{1}{250}$ to illustrate terminating decimals in the form of rational numbers.

Similar to Teacher A, Teacher B performed a long division algorithm on the board to show that $\frac{5}{3} = 1.666... = 1.\bar{6}$ and thus the teacher illustrated how to convert a rational number into a repeating decimal.

Although the teacher used few examples for converting a rational number into a repeating decimal, he used many examples for converting a repeating decimal into a rational number. Before teaching how to convert a repeating decimal into a rational number, the teacher introduced the textbook method for conversion that is presented in Figure 4.5 and added that this method for conversion is long and time consuming. Hence, he provided a shortcut procedure for converting all types of repeating decimals including decimals with only repeating digits or those with both repeating and non-repeating digits after the decimal point. Teacher B explained the shortcut procedure for converting repeating decimals into a rational number by means of the following steps: (1) write down the repeating decimal without its decimal point; (2) subtract non-repeating part from Step 1; (3) divide the number obtained from Step 2 by the number with 9's and 0's: for every repeating digit write down a 9 and for every non-repeating digit write down a 0 after 9's.

The variety of examples used by the teacher to teach this procedure are presented in Table 4.10.

Table 4.10. A variety of repeating decimals used by Teacher B for conversion

Type of repeating decimal	The number in the denominator	Examples used by the teacher
$a.\overline{b}$	9	$1.\overline{6}$, $7.\overline{6}$, $0.\overline{7}$, $1.\overline{3}$, $2.\overline{7}$
$a.\overline{bc}$	99	$2.\overline{15}$, $15.\overline{91}$, $1.\overline{29}$
$a.b\overline{c}$	90	$3.2\overline{4}$, $1.1\overline{7}$
$a.\overline{bcd}$	999	$5.\overline{104}$, $10.\overline{394}$, $7.\overline{014}$
$a.b\overline{cd}$	990	$1.0\overline{45}$, $0.8\overline{75}$, $3.2\overline{07}$, $4.1\overline{14}$, $2.5\overline{81}$, $4.2\overline{91}$, $5.2\overline{79}$
$a.bc\overline{d}$	900	$0.16\overline{7}$, $0.76\overline{4}$
$a.b\overline{cde}$	9990	$3.1\overline{745}$
$a.bc\overline{de}$	9900	$0.74\overline{19}$

As seen in Table 4.10, Teacher B used five examples for converting repeating decimals with only one repeating digit. For decimals with two repeating digits, he used three examples. For those with one repeating digit and one non-repeating digit, the teacher used two examples. In addition to providing repeating decimals with one or two digits after the decimal point, the teacher also presented repeating decimals with three or four digits after the decimal point. The teacher used three examples for illustrating the conversion of decimals with three repeating digits. The teacher gave more emphasis on the conversion of decimals with two repeating digits and one non-repeating digit and thus provided seven examples of this kind. For decimals with one repeating digit and two non-repeating digits, he used two examples. Finally, he illustrated the conversion of decimals with three repeating digits and one non-repeating digit or with two repeating and two non-repeating digits by giving one example for each type.

Teacher C merely provided examples for converting repeating decimals into rational numbers during the teaching of expressing rational numbers in different forms. He provided three different examples to teach how to convert repeating decimals into rational numbers. Unlike previous teachers, Teacher C taught conversion of a repeating decimal number into a rational number after teaching the objective of ‘comparing and ordering rational number’. Initially, he wrote on the board the shortcut procedure that was also used by Teacher B. He introduced this procedure to his students in the following way:

$$\frac{\text{the repeating decimal without its decimal point} - \text{non repeating part}}{\text{write a 9 for every repeating digit and a 0 for every non repeating digit after 9's}}$$

He initially used $1.\overline{3}$ as an example for teaching the conversion to his students. As it can be seen, this example was a decimal with only one repeating digit after the decimal point. The teacher wrote on the board the equality $1.\overline{3} = \frac{13-1}{9} = \frac{12}{9}$ as an application of the procedure for this repeating decimal. Then, the teacher explained that he selected this example from the workbook so as to have students understand the underlying logic of conversion as emphasized by textbook. More precisely, this repeating decimal was included in the workbook as an exercise example and the students were asked to fill in the blanks with relevant numbers. This example is presented in Figure 4.6.

$$
\begin{aligned}
D = 1{,}3333\ldots \rightarrow 10.D &= \ldots.. \\
- \qquad D &= \ \ 1{,}3333\ldots \\
\hline
10.D - D &= 12{,}0000\ldots \\
\ldots..D &= 12 \\
\frac{\ldots..D}{\ldots..} &= \frac{12}{\ldots..} \\
D = \frac{\ldots..}{\ldots..} &= 1\frac{\ldots..}{\ldots..}
\end{aligned}
$$

Figure 4.6. An example used by Teacher C to teach the logic of conversion (Aydın & Beşer, 2013b, p. 34)

Next, the teacher used $3.\overline{07}$ as another repeating decimal with two repeating digits after the decimal point. This time, the teacher converted this repeating decimal into a rational number by using method depicted above. That is, the teacher used the method included in the textbook. However, he indicated that this way of converting is a long and complicated process. Thus, he switched back to using the shortcut procedure that he wrote on the board at the beginning of the lesson. He converted

$3.\overline{07}$ into a rational number as $3.\overline{07} = \frac{307-3}{99} = \frac{304}{99}$ and emphasized that the students need to convert $\frac{304}{99}$ into a mixed number to finalize the conversion.

As can be understood from the given examples, Teacher C provided only two different types of repeating decimals in the course of teaching the objective ‘express rational numbers in different forms’. However, he provided another different type of repeating decimal for conversion while he was teaching the objective of ‘perform addition and subtraction operations with rational numbers’. Namely, Teacher C converted $24.7\overline{89}$ into a rational number by using the shortcut procedure that he taught at the beginning of the lesson as follows: $24.7\overline{89} = \frac{24789-247}{990} = \frac{24542}{990}$. As it can be seen, $24.7\overline{89}$ is a repeating decimal with one non-repeating digit and two repeating digits after the decimal point. Finally, the teacher asked the students to divide 24542 by 990 with a calculator to have them see that $\frac{24542}{990}$ is equal to 24.789898989…

The examples generated by Teacher D to teach how to express rational numbers in different forms represented the following mathematical ideas: expressing integers as rational numbers, expressing terminating decimals as rational numbers, expressing rational numbers as integers/repeating decimals/terminating decimals and converting repeating decimals into rational numbers. The illustrative examples and the total number of examples for each of these ideas are presented in Table 4.11.

Table 4.11. Examples used by Teacher D for expressing rational numbers in different forms

Ideas	Illustrative examples	Number of examples used
Expressing integers as rational numbers	$3 = \frac{3}{1}$; $-2 = \frac{-2}{1}$	4
Expressing terminating decimals as rational numbers	$0.3 = \frac{3}{10}$; $0.03 = \frac{3}{100}$; $-0.5 = -\frac{5}{10}$; $1.256 = \frac{1256}{1000}$	6
Expressing rational numbers as integers, terminating decimals or repeating decimals	$\frac{15}{5} = 3$; $-\frac{3}{4} = -0.75$; $\frac{10}{3} = 3.\bar{3}$	5
Converting repeating decimals into rational numbers	$0.\bar{3} = \frac{3}{9}$; $2.\bar{5} = 2\frac{5}{9}$	7

Teacher D used 22 examples for teaching how to express rational numbers in different forms. She initially provided examples to illustrate that integers, either positive or negative, can be written in the form of rational numbers. While providing these examples to students, Teacher D stressed that each integer can be written in the form of $\frac{a}{b}$ where b is equal to 1. At the same time, she referred to the term 'hidden denominator' to emphasize the role of 1 in the above mentioned examples.

After providing examples for expressing integers as rational numbers, Teacher D wrote on the board several equalities such as $0.3 = \frac{3}{10}$ and $-0.5 = -\frac{5}{10}$ to illustrate the idea that each terminating decimal can be expressed as a rational number. During this time, she explained that while the decimal number without its decimal point will be the numerator, 10 to the power of the number of digits in the decimal will be the denominator of the rational number. She ended up by stressing that some decimal numbers can be expressed as rational numbers. In doing so, she aimed to draw students' attention to the fact that non-terminating repeating decimals are examples for rational numbers while non-terminating non-repeating decimals are non-examples for rational numbers. She provided 0.25784… and the transcendental number π as non-examples for rational numbers and stressed that these two numbers cannot be written in the form of rational numbers since they go on forever without repeating.

Similar to Teacher A, Teacher D used $\frac{10}{3} = 3.333... = 3.\overline{3}$ as a start-up example for teaching how to convert a rational number into a repeating decimal. Then, she immediately worked backwards to teach converting a repeating decimal into its rational number. In other words, she converted $3.\overline{3}$ into its rational number by using the shortcut procedure that was also preferred by Teacher A, Teacher B and Teacher C as follows: $3.\overline{3} = \frac{33-3}{9} = \frac{30}{9}$. In addition to this shortcut procedure, Teacher D emphasized the use of a more specific procedure that could only be used for converting decimals with a one-digit repetend as follows: $0.\overline{a} = \frac{a}{9}$, such that *a* is a one-digit numeral. Then, Teacher D implemented this more specific procedure by converting several repeating decimals into their rational numbers. Besides, she extended this procedure to repeating decimals in which a whole number preceded the decimal point and wrote on the board a new procedure as follows: $a.\overline{b} = \frac{a}{9}$, such that a and b are both one-digit numerals. She exemplified this procedure by converting several repeating decimals such as $2.\overline{5}$ and $3.\overline{1}$ into their rational numbers. It is important to note that Teacher D converted $2.\overline{5}$ into its rational number by using both the shortcut procedure and the more specific procedure. Subsequently, she stressed that both procedures are applicable for the conversion of decimals with only repeating digit after the decimal point. The teacher ended up the lesson by conveying the idea that $a.\overline{9}$ is equal to $a+1$ by means of the conversion $0.\overline{9} = \frac{9}{9} = 1$. To conclude, all the examples used by Teacher D for conversion were decimals with only one repeating digit after the decimal point, although the textbook included decimals with both non-repeating and repeating digits.

4.1.1.3. Examples used for comparing and ordering rational numbers

The middle school mathematics curriculum suggests that the strategies used for comparing fractions and integers can also be used for comparing rational numbers. More specifically, the curriculum emphasizes the use of benchmarking to 0, ½ and 1 as a mental strategy for comparing rational numbers. In addition to benchmarking, the teachers are recommended to use the following strategies while comparing rational numbers: converting to common denominator, converting to decimals and locating rational numbers on a number line. The curriculum provided one example for comparing. More precisely, this example included a rational number pair as -5.2 and $-5\frac{1}{4}$ and this pair was compared by converting to decimals strategy and locating on a number line strategy. By the use of the former strategy, $-5\frac{1}{4}$ was converted to -5.25 and was compared with -5.2 as $-5.2 > -5.25$ and was concluded that $-5.2 > -5\frac{1}{4}$. This same rational number pair was also compared by using the locating on a number line strategy. In this strategy, learners need to locate each rational number on a number line and then construe that the one on the leftmost side is smaller than the other. This strategy is presented in Figure 4.7.

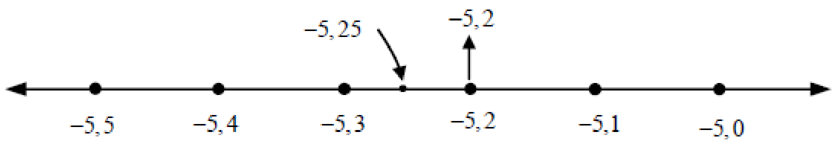
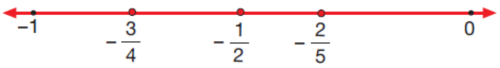


Figure 4.7. Locating on a number line strategy for comparing rational number pairs

In the mathematics textbook followed by the classrooms, the examples selected for teaching the concept of ordering rational numbers represented the following strategies: ordering by locating on a number line, ordering by converting to decimals, ordering by common denominator approach, ordering by benchmarking, ordering by equivalent fractions, and ordering by common numerator approach.

These range of strategies employed in the textbook for ordering rational numbers and the set of rational numbers selected for these strategies are presented in Table 4.12.

Table 4.12. Examples provided by the textbook for ordering rational numbers

Illustrative example	Number of examples used	Strategy	Explanation
$-\frac{3}{4}, -\frac{1}{2}, -\frac{2}{5}$	2	Locating on a number line	 The one on the left is smaller so $-\frac{3}{4} < -\frac{1}{2} < -\frac{2}{5}$
$-\frac{29}{4}, -\frac{184}{25}, -\frac{371}{50}$	2	Converting to decimals	$-\frac{29}{4} = -7.25, -\frac{184}{25} = -7.36, -\frac{371}{50} = -7.42,$ $-7.25 > -7.36 > -7.42,$ Therefore, $-\frac{29}{4} > -\frac{184}{25} > -\frac{371}{50}$
$\frac{3}{8}, \frac{6}{8}, \frac{5}{8}$	4	Common denominator approach	$3 < 5 < 6$ so $\frac{3}{8} < \frac{5}{8} < \frac{6}{8}$
$\frac{8}{10}, \frac{11}{9}, \frac{16}{16}$	1	Benchmarking	$\frac{8}{10} < 1, \frac{11}{9} > 1, \frac{16}{16} = 1$ therefore $\frac{8}{10} < \frac{16}{16} < \frac{11}{9}$
$\frac{2}{3}, \frac{5}{6}, \frac{3}{4}$	1	Equivalent fractions	$\frac{2.4}{3.4} \equiv \frac{8}{12}, \frac{5.2}{6.2} \equiv \frac{10}{12}, \frac{3.3}{4.3} \equiv \frac{9}{12}$ $\frac{8}{12} < \frac{9}{12} < \frac{10}{12}$ therefore $\frac{2}{3} < \frac{3}{4} < \frac{5}{6}$
$\frac{-3}{4}, \frac{-2}{3}, \frac{-1}{7}$	3	Common numerator approach	$\frac{(-3).2}{4.2} = \frac{-6}{8}, \frac{(-2).3}{3.3} = \frac{-6}{9}, \frac{(-1).6}{7.6} = \frac{-6}{42}$ $\frac{-6}{42} > \frac{-6}{9} > \frac{-6}{8}$ therefore $\frac{-1}{7} > \frac{-2}{3} > \frac{-3}{4}$
1.9, 1.08, $1\frac{7}{8}$; -4.45, -5.54, $-5.\overline{5}$	2	-	The students are expected to use a relevant strategy

The textbook included 13 worked-out examples and 2 exercise examples in different types for ordering rational numbers. In the first exercise example, the students are asked to order 1.9, 1.08 and $1\frac{7}{8}$ by using different strategies. As it can be seen, two of these rational numbers are in decimal form while the last one is a mixed number. In the second exercise example, the students are asked to order $-4.45, -5.54$ and $5.\overline{5}$ by employing different strategies. As seen, the first two rational numbers are negative terminating decimals while the last one is a positive repeating decimal.

The explanatory part of the textbook did not include worked examples for comparing rational number pairs. However, there were 7 exercise examples in the textbook and these examples asked students to compare rational number pairs by using relevant strategies. These pairs included rational numbers in different forms. Namely, the pairs entailed the following comparisons: comparing a decimal number with a rational number, comparing a positive number with a negative number, comparing a repeating decimal with a non-repeating decimal and comparing an exponential number with an integer. The first type had to do with comparing a rational number in the form of $\frac{a}{b}$ with a rational number in decimal form such as $\frac{3}{7}, 0.75;$ and $-2.32, -2\frac{1}{3}$. The second type of example included one positive and one negative rational number. More specifically, the students were asked to compare -3 with $\frac{1}{100}$. The comparison of a terminating decimal with the repeating decimal was of the third type and $-4.\overline{37}, -4.37$ was the rational number pair selected for illustrating this type. Finally, for the last type, -2^3 and -6 were selected as a rational number pair. In this pair, the first rational number was an exponential number, while the second rational number was a negative integer.

During the teaching of ordering rational numbers, the examples selected by Teacher A served for the following strategies: ordering by common denominator approach, ordering by common numerator approach, ordering by residual thinking, ordering by benchmarking and ordering by equating the number of decimal digits by adding 0’s. The set of rational numbers selected by Teacher A for these strategies are presented in Table 4.13.

Table 4.13. Examples used by Teacher A for ordering rational numbers

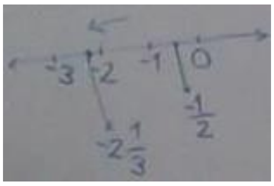
Illustrative example	Number of examples used	Strategy used by Teacher A	Explanation
$\frac{17}{2}, \frac{19}{3}, \frac{35}{6}$	11	Common denominator approach	It is very difficult to equate numerators, but it is easy to equate the denominators. Thus; $\frac{17}{2} = \frac{51}{6}, \frac{19}{3} = \frac{38}{6}, \frac{35}{6} = \frac{35}{6}$; $51 > 38 > 35$ therefore $\frac{17}{2} > \frac{19}{3} > \frac{35}{6}$ (3) (2) (1)
$\frac{2}{19}, \frac{3}{13}, \frac{6}{17}$	7	Common numerator approach	It is very difficult to equate denominators, but it is easy to equate the numerators. Thus; $\frac{2}{19} = \frac{6}{57}, \frac{3}{13} = \frac{6}{26}, \frac{6}{17} = \frac{6}{17}$; $17 < 26 < 57$ therefore $\frac{6}{17} > \frac{3}{13} > \frac{2}{19}$ (3) (2) (1)
$\frac{1996}{1997}, \frac{1997}{1998}, \frac{1998}{1999}$	4	Residual thinking	All rational numbers are very close to 1. However, the first rational number requires $\frac{1}{1997}$ to make the whole, the second rational number requires $\frac{1}{1998}$ to make the whole and the third fraction requires $\frac{1}{1999}$ to make the whole. $\frac{1}{1997} > \frac{1}{1998} > \frac{1}{1999}$ therefore, $\frac{1996}{1997} < \frac{1997}{1998} < \frac{1998}{1999}$
$\frac{1}{7}, \frac{4}{5}, \frac{8}{19}$	5	Benchmarking	$\frac{1}{7}$ is close to 0, $\frac{8}{19}$ is close to $\frac{1}{2}$ and $\frac{4}{5}$ is close to 1, therefore $\frac{1}{7} < \frac{8}{19} < \frac{4}{5}$
$0.21, 0.2001, 0.201$	3	Equating the number of decimal digits by adding 0's	We need to equate the number of digits after the decimal point. $0.21 = 0.2100$, $0.201 = 0.2010$ and $2100 > 2010 > 2001$ therefore $0.21 > 0.201 > 0.2001$

As it can be understood from Table 4.13, Teacher A provided 30 examples related with ordering rational numbers. These examples were generated not only in the course of teaching the concept of rational numbers but also during the provision of exercises. Although the examples used by Teacher A for ordering rational numbers had some similarities with the textbook, the teacher did not provide examples that foster ordering by locating on a number line, ordering by converting to decimals and ordering by equivalent fractions as emphasized by the mathematics textbook. Apart from using examples that promote the use of strategies included in the textbook, Teacher A selected rational number examples that suggested either ordering by residual thinking or ordering by equating the number of decimal digits.

Teacher A provided fewer examples for comparing rational number pairs when compared to the number of examples used by him for ordering rational numbers. More specifically, Teacher A focused on the comparison of the following rational number pairs in different forms: comparing a repeating decimal with a terminating decimal, comparing a decimal number with a rational number in the form of $\frac{a}{b}$, comparing a positive rational number with a negative rational number. Teacher A selected the following rational number pairs to illustrate the above mentioned comparison ideas respectively: $2.\overline{45}, 2.45;\ -2.32, -2\frac{1}{3}$; and $-3, \frac{1}{100}$. In sum, Teacher A used 6 examples for comparing rational numbers.

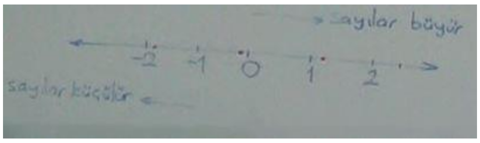
In contrast to the textbook, Teacher B started teaching comparison of rational numbers rather than teaching ordering of rational numbers first. The examples used by Teacher B for comparing rational numbers promoted the use of following strategies: comparing by locating on a number line, comparing by benchmarking, comparing by considering the sign of the rational number and comparing by converting. The rational number pairs selected by Teacher B for employing these strategies are presented in Table 4.14.

Table 4.14. Examples used by Teacher B for comparing rational numbers

Illustrative example	Number of examples used	Strategy used by Teacher B	Explanation
$\frac{-1}{2}, \frac{-7}{3}$	4	Locating on a number line	As we go leftward, the numbers become smaller. 
$\frac{7}{9}, \frac{15}{12}$	3	Benchmarking	$\frac{7}{9} < 1$ and $\frac{15}{12} > 1$ therefore $\frac{7}{9} < \frac{15}{12}$
$\frac{1}{4}, \frac{-1}{3}$	2	Considering number sign	Whatever the magnitude of rational numbers are, a negative rational number is always smaller than 0 and a positive rational number is always larger than 0. $\frac{1}{4} > 0$, $\frac{-1}{3} < 0$ therefore $\frac{1}{4} > \frac{-1}{3}$
$2\frac{1}{4}, \frac{9}{4}$	1	Converting	The two fractions are equivalent since $2\frac{1}{4} = \frac{2 \times 4 + 1}{4} = \frac{9}{4}$

After using 10 examples for comparing rational numbers, Teacher B started teaching how to order rational numbers. The examples used by the teacher for ordering rational numbers focused on the following strategies: ordering by common denominator approach, ordering by locating on a number line and ordering by benchmarking. The set of rational numbers selected by Teacher B for employing these strategies are presented in Table 4.15.

Table 4.15. Examples used by Teacher B for ordering rational numbers

Illustrative example	Number of examples used	Strategy used by Teacher B	Explanation
$\frac{-1}{3}, \frac{7}{4}, \frac{-3}{5}, \frac{-1}{12}$	5	Common denominator approach	$\frac{-1}{3}=\frac{-20}{60}$, $\frac{7}{4}=\frac{105}{60}$, $\frac{-3}{5}=\frac{-36}{60}$, $\frac{-1}{12}=\frac{-5}{60}$ (20) (15) (12) (5) -36<-20<-5<105 therefore $\frac{-3}{5}<\frac{-1}{3}<\frac{-1}{12}<\frac{7}{4}$
$\frac{-1}{7}, 2\frac{4}{9}, 1\frac{1}{4}, -1\frac{5}{6}$	5	Locating on a number line	As we go leftward, the numbers become smaller. On the contrary, if go rightward the numbers become larger.  Therefore, $-1\frac{5}{6}<\frac{-1}{7}<1\frac{1}{4}<2\frac{4}{9}$
$\frac{8}{6}, \frac{9}{19}, \frac{13}{23}$	2	Benchmarking	$\frac{9}{19}<\frac{1}{2}$, $\frac{13}{23}>\frac{1}{2}$, $\frac{8}{6}>1$ therefore, $\frac{9}{12}<\frac{13}{23}<\frac{8}{6}$

As shown in Table 4.15, 12 examples were used by Teacher B for ordering rational numbers. When compared to textbook examples, the examples used by Teacher B for ordering rational numbers focused on fewer strategies. For instance, Teacher B did not provide examples for ordering by converting to decimals, ordering by equivalent fractions and ordering by common numerator approach. Moreover, while the textbook examples asked students to order at most three rational numbers, Teacher B used examples that included ordering of three or four rational numbers. Finally, the examples used by Teacher B had some similarities with those included in the textbook for ordering rational numbers. That is, examples provided both by the teacher and the textbook included positive rational numbers, negative rational numbers, mixed numbers, proper numbers and improper numbers.

Similar to the textbook, Teacher C started the lesson by teaching ordering rational numbers. However, the teacher did not provide any specific example for teaching comparison of rational numbers. Before providing specific examples for ordering rational numbers, he wrote on the board a note that explains how to order rational numbers. According to this note, students needed to equate either the numerators or the denominators of the rational numbers in order to order them correctly. If the rational numbers have same denominators, then the one with a larger numerator will be larger. On the contrary, if the rational numbers have same numerators, then the one with a smaller denominator will be larger. Finally, if the rational numbers were negative then the ordering will be the other way round. In accordance with this explanation, Teacher C provided only two examples for ordering rational numbers and these examples focused on the use of common numerator approach or common denominator approach. Teacher C provided the following rational number sequence for ordering by using common denominator approach: $-\frac{5}{3}, 0, \frac{4}{3}, \frac{5}{12}, \frac{2}{18}$ and $-\frac{2}{6}$. As it can be seen, two of the rational numbers are negative, while three of them are positive. In addition, the neutral number 0 was included to this set. Besides, $-\frac{5}{3}$ and $\frac{4}{3}$ were improper numbers while $\frac{5}{12}, \frac{2}{18}$ and $-\frac{2}{6}$ were proper numbers. However, there was not any rational number in mixed number form in this sequence. As there were 6 rational numbers to compare, it was difficult for students to find the common denominator mentally. Considering this, Teacher C recalled the concept of LCM (Lowest Common Multiple) as a way to find the common denominator of the rational numbers.

The next example used by Teacher C for ordering rational numbers included the following number sequence: $-\frac{3}{9}, -\frac{2}{5}$ and $-\frac{6}{12}$. The teacher ordered these rational numbers by using common numerator approach. The teacher noted that it is more difficult to order this sequence by using common denominator approach since the denominators included big numbers when compared to the numerators. In addition, the rational numbers in this sequence were all negative and proper form.

Like the previous example, this example did not include rational numbers in mixed form and there were three rational numbers to order. Moreover, the number of examples used by Teacher C for ordering rational numbers was less than the number of examples included in the textbook. Finally, the teacher did not provide examples that focus on comparing by locating on a number line, comparing by converting to decimals, comparing by benchmarking and comparing by equivalent fractions.

Teacher D merely provided examples for ordering rational numbers. During the teaching of ordering rational numbers, the examples selected by Teacher D served for the following strategies respectively: ordering by locating on a number line, ordering by converting to decimals, ordering by common numerator approach and ordering by common denominator approach. The set of rational numbers selected by Teacher D for employing these strategies are presented in Table 4.16.

Table 4.16. Examples used by Teacher D for ordering rational numbers

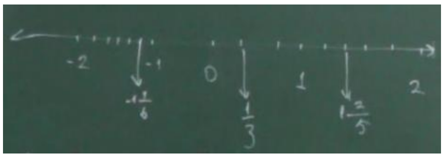
Illustrative example	Number of examples used	Strategy used by Teacher B	Explanation
$\frac{1}{3}, 1\frac{2}{5}, -1\frac{1}{6}$	2	Locating on a number line	On the number line, rational numbers become larger as we go rightward.  Thus, $-1\frac{1}{6} < \frac{1}{3} < 1\frac{2}{5}$
$\frac{5}{2}, \frac{12}{5}, \frac{23}{10}$	2	Converting to decimals	If the denominators of rational numbers are multiples of 10 or if they can be easily enlarged to 10, then it is more relevant to order them by converting to decimals $\frac{5}{2} = \frac{25}{10} = 2.5$, $\frac{12}{5} = \frac{24}{10} = 2.4$, $\frac{23}{10} = \frac{23}{10} = 2.3$ (5) (2) (1) $2.5 > 2.4 > 2.3$ thus $\frac{5}{2} > \frac{12}{5} > \frac{23}{10}$

Table 4.16. (Continued)

Illustrative example	Number of examples used	Strategy used by Teacher B	Explanation
$-\frac{7}{4}, -\frac{7}{2}, -\frac{7}{9}$	6	Common numerator approach	First, assume that rational numbers are positive. In that case, the one with a smaller denominator will be larger since they have same numerators. Thus, $\frac{7}{2} > \frac{7}{4} > \frac{7}{9}$ and $-\frac{7}{2} < -\frac{7}{4} < -\frac{7}{9}$
$-\frac{4}{8}, -\frac{7}{8}, -\frac{15}{8}$	4	Common denominator approach	First, assume that rational numbers are positive. In that case, the one with a larger numerator will be larger since they have same denominators. Thus, $\frac{4}{8} < \frac{7}{8} < \frac{15}{8}$ and $-\frac{4}{8} > -\frac{7}{8} > -\frac{15}{8}$

As presented in Table 4.16, 14 examples were used by Teacher D for ordering rational numbers. When compared to textbook examples, the examples used by Teacher D for ordering rational numbers focused on fewer strategies. For instance, Teacher D did not provide examples for ordering by benchmarking and ordering by equivalent fractions. Identical to the textbook examples, Teacher D asked her students to order three rational numbers. Finally, the examples used by Teacher D had the following dimensions of variation: being positive or negative and being a proper, improper or mixed rational number.

4.1.1.4. Examples used for performing addition and subtraction operations with rational numbers

In the middle school mathematics curriculum, teachers are suggested to start teaching addition and subtraction of rational numbers by having students remember addition and subtraction of fractions. After recalling addition and subtraction of fractions, teachers are suggested to use activities related with addition and subtraction of rational numbers. In addition to this, the curriculum emphasized teaching the properties of addition of rational numbers. Namely, teachers are suggested to give weight to the teaching of commutative property, associative property, identity property and inverse property of addition and also to the algebraic representations of these properties.

Another idea that was emphasized in the curriculum was the use of estimation techniques. The curriculum provided one specific example for the estimation of addition with rational numbers. That is, the estimation of $\frac{3}{8}+\frac{6}{7}$ was explained in the following way: $\frac{3}{8}$ is close to $\frac{1}{2}$ so we can round it to $\frac{1}{2}$. Similarly, $\frac{6}{7}$ is close to 1 so we can round it to 1. Thus, $\frac{3}{8}+\frac{6}{7}\approx\frac{1}{2}+1=\frac{3}{2}$. Finally, the addition operation $\frac{3}{8}+\frac{6}{7}=\frac{21}{56}+\frac{48}{56}=\frac{21+48}{56}=\frac{69}{56}=1\frac{13}{56}$ was presented to compare the estimated answer with the exact answer.

In the mathematics textbook followed by the classrooms, the examples selected for teaching addition and subtraction with rational numbers represented the following ideas respectively: using models for the addition and subtraction of rational numbers, adding and subtracting rational numbers with same denominators, estimating the addition and subtraction of rational numbers, adding and subtracting rational numbers with different denominators and properties of addition of rational numbers. When addition and subtraction examples were examined, it was seen that there were some structural differences in terms of sign and form of terms included in the operations. Some illustrative examples included in the textbook for the above mentioned ideas are presented in Table 4.17. It is important to note that only the examples that have different structural features were used as illustrative examples. Therefore, the number of examples for each ideas were also presented.

Table 4.17. Examples included in the textbook for adding and subtracting rational numbers

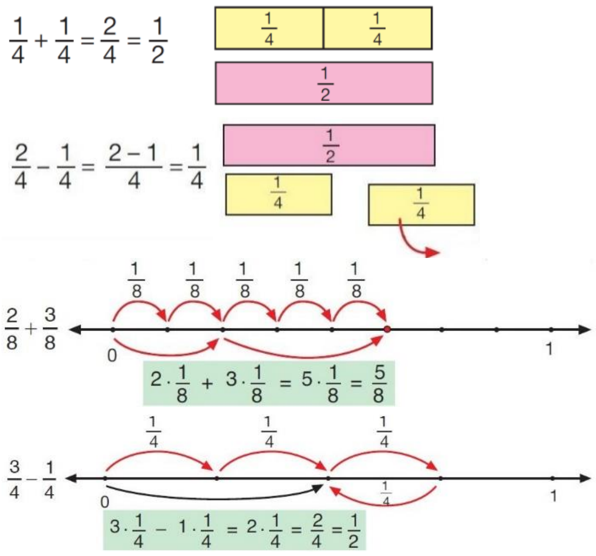
Ideas for adding and subtracting rational numbers	Illustrative examples	Number of examples used
Using models for the addition and subtraction of rational numbers		4
Adding and subtracting rational numbers with same denominators	$\frac{3}{18}+\frac{17}{18}$; $\left(+\frac{7}{3}\right)+\left(-\frac{2}{3}\right)$; $\left(+\frac{3}{5}\right)-\left(+\frac{4}{5}\right)$; $\frac{5}{9}-\frac{2}{9}$	5
Estimating the addition and subtraction of rational numbers	$\frac{3}{18}+\frac{17}{18}\approx 0+1$; $3\frac{1}{26}+3\frac{24}{25}\approx 3+4=7$; $\frac{7}{10}-\frac{2}{5}\approx 1-\frac{1}{2}=\frac{1}{2}$	3
Adding and subtracting rational numbers with different denominators	$\frac{1}{2}+\frac{1}{10}$; $4+\frac{7}{8}$; $\left(+\frac{3}{4}\right)+\left(-\frac{5}{6}\right)$; $\frac{4}{5}+\left(-\frac{2}{3}\right)$; $-\frac{2}{3}+\left(-\frac{3}{2}\right)$; $\frac{1}{4}-\frac{1}{8}$; $\frac{1}{7}-\frac{1}{4}$; $-\frac{13}{5}-\left(-\frac{1}{10}\right)$; $\frac{2}{13}-\left(-\frac{1}{17}\right)$; $-\frac{4}{5}-\frac{5}{4}$; $1-\frac{1}{9}$; $-1-\left(-\frac{1}{11}\right)$	19

Table 4.17. (Continued)

Ideas for adding and subtracting rational numbers	Illustrative examples	Number of examples used
Properties of addition of rational numbers	Commutative property $\left(-\frac{2}{3}\right)+\left(+\frac{1}{4}\right)=\left(+\frac{1}{4}\right)+\left(-\frac{2}{3}\right)$; $\left(-\frac{4}{7}\right)+\left(-\frac{2}{3}\right)=\left(-\frac{2}{3}\right)+x;\ a+\frac{5}{7}=\frac{5}{7}-\frac{2}{3}$ Identity property $+\frac{3}{4}+0=+\frac{3}{4};\ 0+\frac{3}{4}=\frac{3}{4}$; $-\frac{4}{7}+0=-\frac{4}{7};\ 0+\left(-\frac{4}{7}\right)=-\frac{4}{7}$ Associative property $\left[\left(-\frac{2}{3}\right)+\left(+\frac{1}{4}\right)\right]+\left(-\frac{1}{2}\right)=\left(-\frac{2}{3}\right)+\left[\left(+\frac{1}{4}\right)+\left(-\frac{1}{2}\right)\right]$	10

The textbook included 41 examples for adding and subtracting rational numbers. It emphasized modeling of addition and subtraction operations before symbolically expressing them. It provided fraction bars and number lines as two different types of models. The examples provided for modeling addressed students' prior knowledge on fractions. To be more precise, each example used for modeling included terms that are both positive rational numbers. Besides, the examples provided for modeling were only in proper form. However, there were not any examples that modelled addition and subtraction of rational numbers which are greater than 1.

Next, the textbook presented examples for teaching addition and subtraction of rational numbers with same denominators. The addition examples included the following structural properties regarding the sign of their addends: (+, +) and (+, -). However, there were not any examples that included (-,-) as the sign of their addends. Besides, the addends of addition examples were either in proper or improper form but not in mixed number form. When subtraction examples were examined, it was seen that the examples included minuends and subtrahends that were both positive. However, there was not any example including (-,-), (-,+) or (+,-)

as the sign of their minuends and the subtrahends respectively. In some of these examples, the positive signs were omitted and sometimes they preceded the minuend and the subtrahend. Finally, the minuends and subtrahends were all proper numbers.

The textbook gave considerable emphasis on estimating the answer of addition and subtraction operations with rational numbers. By the estimation examples, it was aimed to teach that if the numerator is much less than the half of the denominator, then the rational number is rounded to 0; if the numerator is close to the half of the denominator, then the rational number is rounded to $\frac{1}{2}$ and if the numerator is close to the denominator, then the rational number is rounded to 1. The estimation examples included terms that were all positive rational numbers. In addition, these examples included terms that were either proper or mixed number.

The textbook included 19 examples for teaching the addition or subtraction of rational numbers with different denominators. Addition examples were in the following form in terms of the sign of their addends: (+, +) and (+, -). However, the textbook did not provide examples with addends in the form of (-, +) and (-, -). Besides, the addition examples involved either two addends in proper form, or one natural number and one proper number addend. However, the addition examples did not involve any addends in proper or mixed number form. The examination of subtraction examples showed that they included all possible variations in terms of their signs. That is, subtraction examples included minuends and subtrahends with the following signs respectively: (+, +), (+, -), (-, +) and (-, -). Moreover, the minuends and subtrahends were either in proper or improper form but not in mixed form and there were some examples that illustrated the subtraction of a rational number from an integer. Finally, addition and subtraction examples included two terms and there was not any example that included three or more terms.

The textbook included a few examples for teaching commutative property of addition of rational numbers. These examples included one negative and one positive addend or two negative addends. Besides, all addends were proper numbers. There were also some examples in the textbook for teaching identity property of addition of rational numbers. In these examples, the rational number accompanying zero was either a positive or a negative proper number. Associative property of addition of rational numbers was explained by examples that included negative and positive proper numbers. However, there was not any specific example in the textbook for finding the additive inverse of a rational number.

The examples used by Teacher A for teaching addition and subtraction of rational numbers focused on the following mathematical ideas respectively: adding and subtracting rational numbers with same denominators, adding and subtracting rational numbers with different denominators, performing multi-step operations with rational numbers and teaching properties of addition of rational numbers. Some illustrative examples used by Teacher A for teaching these ideas are presented in Table 4.18. Only the examples that had different structural features were used as illustrative examples. Therefore, the number of examples for each ideas were also presented.

Table 4.18. Examples used by Teacher A for adding and subtracting rational numbers

Ideas for adding and subtracting rational numbers	Illustrative examples	Number of examples used
Adding and subtracting rational numbers with same denominators	$\frac{3}{8}+\frac{2}{8}$; $2\frac{5}{8}+3\frac{3}{8}$; $\left(\frac{-5}{8}\right)-\left(\frac{-2}{8}\right)$; $2\frac{3}{7}-1\frac{1}{7}$; $\frac{(-2)}{3}-\left(\frac{-7}{3}\right)$; $\frac{(-2)}{8}-\left(\frac{3}{8}\right)$	6
Adding and subtracting rational numbers with different denominators	$\frac{2}{3}+\frac{5}{4}$; $3\frac{5}{12}+\frac{5}{6}$; $5+\frac{3}{7}$; $-5+\frac{2}{3}$; $3+\frac{1}{8}$; $\left(+\frac{3}{5}\right)+\left(+\frac{9}{21}\right)$; $2\frac{5}{7}-1\frac{2}{3}$; $\frac{5}{7}-\left(+\frac{8}{5}\right)$; $-\frac{3}{7}-\frac{2}{3}$; $5-\frac{2}{3}$; $-3-\frac{1}{8}$	13

Table 4.18. (Continued)

Ideas for adding and subtracting rational numbers	Illustrative examples	Number of examples used
Multi-step operations with rational numbers	$\frac{(-2)}{5}-\frac{(-2)}{5}-\left(-\frac{2}{5}\right)-\left(+\frac{2}{5}\right); \frac{(-3)}{7}+\frac{(+5)}{7}-\frac{(-2)}{7};$ $\frac{5}{13}-\frac{2}{13}+\frac{7}{13}; \frac{(-1)}{5}-\frac{(-7)}{5}-\frac{(-8)}{5}+\left(-\frac{21}{5}\right)$ $\frac{(-2)}{3}+\frac{(+7)}{2}-\frac{(-1)}{6}; 2\frac{1}{5}-\left(-\frac{3}{7}\right)+\left(+\frac{2}{3}\right);$ $\left(-\frac{3}{8}\right)-\left(-\frac{1}{4}\right)+\Delta=-\frac{1}{48}; \bigcirc+\left(-\frac{1}{18}\right)+\left(-\frac{1}{12}\right)=+\frac{4}{9}$	8
Properties of addition of rational numbers	Commutative property $2+3=3+2; \frac{2}{5}+\frac{3}{5}=\frac{3}{5}+\frac{2}{5}; \left(\frac{-3}{5}\right)+\left(\frac{8}{7}\right)=\Delta+\left(\frac{-3}{5}\right)$ Associative property $(2+3)+5=2+(3+5); \left(\frac{1}{2}+\frac{3}{2}\right)+\frac{7}{2}=\frac{1}{2}+\left(\frac{3}{2}+\frac{7}{2}\right)$ Identity property $2+0=0+2; \frac{3}{4}+0=0+\frac{3}{4}$ Additive inverse property The additive inverse of 3 is (-3); the additive inverse of $\frac{3}{4}$ is $\left(-\frac{3}{4}\right)$; the additive inverse of $\left(-\frac{3}{4}\right)$ is $\frac{3}{4}$	10

Teacher A used 37 examples in the teaching of addition and subtraction operations with rational numbers. Although the textbook suggested the use of models before symbolically expressing addition and subtraction of rational numbers, Teacher A did not use any models. Instead, he started the lesson by providing examples for adding and subtracting rational numbers with the same denominators. The addition examples included two positive rational numbers as their addends. However, Teacher A did not use any examples that included (+, -), (-, +) or (-, -) as the sign of their addends. Moreover, the addends were either in proper or mixed number form. When subtraction examples were examined, it was seen that there was some variability in terms of the signs of the minuends or subtrahends. To be more precise, there were examples that included (+, +), (-, +) and (-,-) as the sign of the minuends and subtrahends respectively. However, there was not any specific example that had a positive minuend and a negative subtrahend. Finally, the subtraction examples included terms that were either in proper, improper or mixed number form.

After providing examples for addition and subtraction of rational numbers with same denominators, Teacher A moved on to the teaching of addition and subtraction of rational numbers with different denominators. Examples used by Teacher A for teaching addition of rational numbers with different denominators included (+, +) and (-, +) but not included (+, -) and (-, -) as the sign of their addends. Besides, the addends of the addition examples were either proper number, integer or mixed number. However, none of the examples included addends in improper form. Several examples were used for subtraction of rational numbers with different denominators. When these subtraction examples were examined, it was seen that the minuends and the subtrahends had the following signs: (+, +), (+, -), and (-, +) but did not have (-, -). Besides, the terms of the subtraction examples were either proper number, improper number, mixed number or integer.

Although estimating the answer of addition and subtraction operations with rational numbers were emphasized in the middle school mathematics curriculum and in the textbook, Teacher A ignored this idea and therefore he did not provide any specific estimation example. However, although not included in the textbook, Teacher A provided several examples that include multi-step operations with rational numbers. Half of these examples included rational numbers with the same denominators and the rest included different denominators. Moreover, the multi-step examples included three or four terms. These examples included terms that were either negative proper number, positive proper number, negative improper number, positive improper number or positive mixed number. Lastly, different from the examples included in the textbook, Teacher A provided multi-step examples that included minus signs both in front of the fraction bar and in the numerator of the terms to emphasize that the location of the minus sign does not alter the value of the rational number.

Teacher A used several examples for teaching properties of addition operation with rational numbers. Teacher A started the teaching of each property by using examples that checked students’ prior knowledge on properties of addition of natural numbers. Other examples used by Teacher A for each property included addends that were either negative proper number, positive proper number or positive improper number. The examples used for commutative and associative property only included rational numbers with same denominators. Moreover, none of the examples included rational numbers in mixed number form. Unlike the textbook, Teacher A used some examples related with inverse property. In more detail, he selected a positive integer, a positive proper number and a negative proper number respectively to find their additive inverses.

The examples used by Teacher B for teaching addition and subtraction of rational numbers represented the following mathematical ideas respectively: using models for the addition and subtraction of rational numbers, adding or subtracting rational numbers with same denominators, finding common multiples of the denominators of rational numbers, adding and subtracting rational numbers with different denominators, performing multi-step operations with rational numbers; and teaching properties of addition of rational numbers. Some illustrative examples used by Teacher B for teaching these ideas are presented in Table 4.19. Only the examples that have different structural features were used as illustrative examples. Therefore, the number of examples for each ideas were also presented.

Table 4.19. Examples used by Teacher B for adding and subtracting rational numbers

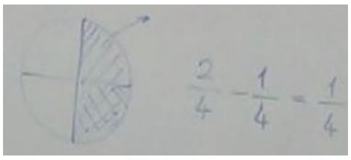
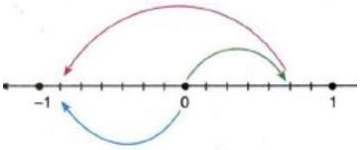
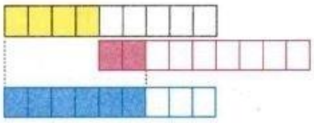
Ideas for adding and subtracting rational numbers	Illustrative examples	Number of examples used
Using models for the addition and subtraction of rational numbers	  $\left(+\frac{5}{7}\right)+\left(-\frac{11}{7}\right)=\left(-\frac{6}{7}\right)$  $0,\bar{4}+0,\bar{2}=0,\bar{6}$	4
Adding or subtracting rational numbers with same denominators	$\left(\frac{-2}{5}\right)+\left(\frac{9}{5}\right)$; $2\frac{1}{4}+\frac{3}{4}$; $\left(\frac{-1}{7}\right)+\left(\frac{-5}{7}\right)$; $5\frac{3}{4}+\left(-1\frac{1}{4}\right)$; $\frac{9}{7}-\left(\frac{-4}{7}\right)$; $\left(-2\frac{9}{11}\right)-\frac{3}{11}$; $\frac{21}{9}-1.\bar{3}$; $\frac{4}{7}+a=\frac{1}{7}$	12
Finding common multiples of the denominators	$\frac{\square}{8},\frac{\square}{5}$; $\frac{\square}{12},\frac{\square}{8}$	3
Adding or subtracting rational numbers with different denominators	$-\frac{2}{3}+\left(-\frac{3}{2}\right)$; $1\frac{2}{4}+\frac{5}{3}$; $\left(-2\frac{1}{3}\right)+\left(-1\frac{1}{2}\right)$; $\left(\frac{-3}{4}\right)+\frac{5}{9}$; $\frac{2}{3}+0.\bar{3}$; $\left(-\frac{2}{4}\right)+\frac{1}{3}$; $\frac{1}{9}-\frac{1}{18}$; $-\frac{4}{5}-\frac{5}{4}$; $-\frac{8}{7}-\left(-\frac{3}{42}\right)$; $1-\frac{1}{9}$; $-1-\frac{1}{11}$; $\frac{2}{3}-\left(-\frac{3}{2}\right)$; $\left(-1\frac{2}{3}\right)-\left(\frac{-3}{5}\right)$	16
Performing multi-step operations with rational numbers	$\left(2\frac{1}{3}\right)-\left(\frac{-2}{5}\right)+\left(2\frac{3}{5}\right)$; $1\frac{1}{4}+\left(\frac{-3}{12}\right)+\frac{5}{3}$	2

Table 4.19. (Continued)

Ideas for adding and subtracting rational numbers	Illustrative examples	Number of examples used
Teaching properties of addition of rational numbers	Commutative property $\frac{1}{3}+A=\left(\frac{-5}{2}\right)+\frac{1}{3}$; $\left(\frac{-2}{13}\right)+x=\frac{5}{4}+\left(\frac{-2}{13}\right)$ Associative property $\left[\frac{3}{8}+\frac{5}{4}\right]+\left(-\frac{1}{6}\right)=\frac{3}{8}+\left[A+\left(-\frac{1}{6}\right)\right]$; $\left(\frac{-5}{6}\right)+\left[B+\frac{9}{11}\right]=\left[\left(\frac{-5}{6}\right)+1\frac{1}{3}\right]+\frac{9}{11}$ Identity property $\frac{5}{2}+0=\frac{5}{2}$; $\frac{2}{3}+0=0+\frac{2}{3}$ Inverse property The additive inverse of $-2\frac{3}{4}$ is $2\frac{3}{4}$; The additive inverse of $\frac{4}{3}$ is $\frac{8}{x}$	9

Teacher B used 46 examples for teaching addition and subtraction of rational numbers. As emphasized by the textbook, Teacher B started the lesson by modeling the operation $\frac{2}{4}-\frac{1}{4}$ before teaching symbolic expressions of addition and subtraction operations with rational numbers. This subtraction operation was modeled by using circular pieces. The minuend and the subtrahend were both positive rational numbers, thus this example addressed students' prior knowledge on subtraction of fractions. In addition to this, Teacher B provided several exercise examples to his students for finding the symbolic expressions of the given models. To be more precise, the number line model referred to the addition of one positive and one negative rational number, while the region model referred to the addition of two positive rational numbers in decimal form.

After modeling the subtraction of two fractions, Teacher B started teaching addition and subtraction of rational numbers with same denominators. The addition examples included all variations in terms of the sign of the addends. That is, Teacher

B used examples that included (+, +), (+, -), (-, +) and (-, -) as the sign of addends. Moreover, the addends were in proper, improper or mixed number form. When examples regarding subtraction of rational numbers with same denominators were examined, it was revealed that there was some variability in terms of the signs of the minuends and subtrahends. To put it differently, the examples selected by Teacher B for the subtraction of rational numbers with same denominators included (+, +), (+, -) and (-, +) but not included (-, -) as the sign of the minuends and subtrahends respectively. Besides, the minuends and subtrahends in each subtraction example took the form of either, proper number, improper number or mixed number.

Unlike the textbook, Teacher B provided examples for finding common multiples of the denominators of the given rational number pairs. Teacher B included pairs that are prime such as 8 and 5 and pairs that have a common divisor such as 8 and 12. But Teacher B did not include pairs in which one is a multiple of the other such as 2 and 8. In the course of teaching how to find common multiples of the denominators, the teacher suggested that it would be much easier to operate with rational numbers if the common multiple was selected to be the smallest one. By this way, he touched upon the concept of LCM (Least common multiples) before teaching the addition or the subtraction of rational numbers with different denominators.

Teacher B provided many examples for the addition and subtraction of rational numbers with different denominators. Examples used by Teacher B for adding rational numbers with different denominators included (+, +), (-, +) and (-, -) as the sign of the first and second addend respectively. However, the teacher did not provide any addition example that included a positive rational number as a first addend and a negative rational number as a second addend. Furthermore, the addends were either in proper, improper or mixed number form. Nevertheless, there was not any specific example that included the addition of a rational number in the form of $\frac{a}{b}$ with an integer or the vice versa. When examples regarding subtraction of rational numbers with different denominators were examined, it was revealed that there was a great variability in terms of the signs and forms of the minuends and subtrahends. To be more precise, examples used by Teacher B for subtraction of rational numbers with different denominators included (+, +), (+, -), (-, +) and (-, -) as the sign of the first and second term respectively. In addition, the first and second term of the subtraction operations were either in proper number, improper number, mixed number or integer form.

When compared to Teacher A, Teacher B provided few examples that included multi-step operations with rational numbers. These examples included rational numbers only with different denominators. In addition, they included three terms. The terms were either negative proper number, positive improper number or positive mixed number. However, none of the examples included a positive proper number, a negative improper number or a negative mixed number.

Teacher B used several examples for teaching properties of addition operation with rational numbers. Unlike the textbook, the examples provided by Teacher B for teaching commutative property and associative property included unknown values as A, B and *x*. The teacher emphasized that by matching the same rational numbers on both sides of the equations included in commutative and associative properties, it is possible to find the values of A, B and *x* without actually computing. The examples used for the commutative property included the addition of a positive proper number with a negative improper number or positive improper number with a negative proper number. Likewise, the associative property examples included the addition of a positive proper number, a positive improper number and a negative proper number or the addition of a negative proper number, a positive mixed number and a positive proper number. The examples used for commutative and associative property included rational numbers both with different and same denominators. In identity property examples, the rational number accompanying zero was either a positive proper number or a positive improper number. Finally, unlike Teacher A, the inverse property examples used by Teacher B entailed finding the additive inverse of a positive improper number and a negative mixed number.

Teacher C provided examples only for the addition of rational numbers and did not give any specific example for the subtraction of rational numbers. The examples used by Teacher C for teaching addition of rational numbers focused on the following mathematical ideas respectively: using models for the addition of rational numbers, adding rational numbers with same denominators, adding rational numbers with different denominators and teaching properties of addition of rational numbers. Some illustrative examples used by Teacher C for teaching these ideas are presented in Table 4.20. Only the examples that have different structural features were used as illustrative examples. Therefore, the number of examples for each idea were also presented.

Table 4.20. Examples used by Teacher C for adding rational numbers

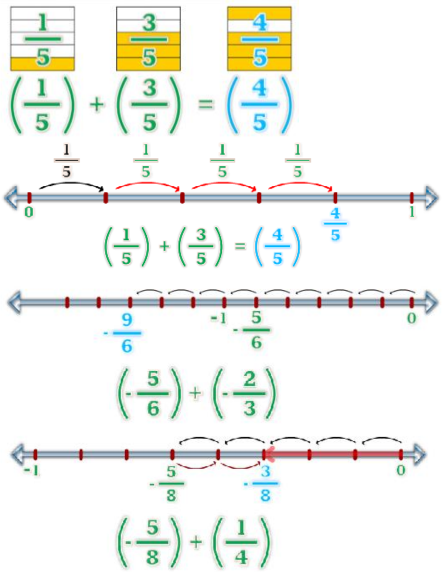
Using models for the addition of rational numbers		10
Adding rational numbers with same denominators	$\frac{1}{5}+\frac{3}{5};\ \frac{4}{4}+\frac{6}{4};\ \frac{7}{4}+\frac{7}{4}$	4
Adding rational numbers with different denominators	$\frac{2}{5}+\frac{3}{8};\ \frac{5}{3}+\frac{7}{5};\ \frac{6}{7}+\frac{8}{6};\ 2+\frac{4}{7};\frac{2}{2}+\frac{6}{8};$ $3\frac{6}{6}+\frac{3}{7};\ \left(-\frac{5}{6}\right)+\left(-\frac{2}{3}\right);\left(-\frac{5}{8}\right)+\frac{1}{4}$	13

Table 4.20. (Continued)

Ideas for adding rational numbers	Illustrative examples	Number of examples used
Teaching properties of addition of rational numbers	Commutative property $\left(+\frac{3}{5}\right)+\left(-\frac{2}{7}\right)=\left(-\frac{2}{7}\right)+\left(+\frac{3}{5}\right)$ Associative property $\left(\frac{1}{2}+\frac{1}{4}\right)+\frac{1}{5}=\frac{1}{2}+\left(\frac{1}{4}+\frac{1}{5}\right)$ Identity property $\frac{1}{2}+0=\frac{1}{2}$; $\left(-\frac{3}{5}\right)+0=\left(-\frac{3}{5}\right)$	4

Teacher C used 31 examples for teaching addition of rational numbers. As emphasized by the textbook, Teacher C started teaching this idea by providing an example that modeled $\frac{1}{5}+\frac{3}{5}$. In this first example, the teacher used a region model. In addition to the region model, Teacher C used several number line models for adding either two positive rational numbers, two negative rational numbers or for adding a negative rational number and a positive rational number. By the number line models, the teacher explained that the sign of the addends showed which way to move and the operation sign meant that the second addend is joined to the end of the first addend.

After modeling addition of rational numbers, Teacher C moved on to teaching addition of rational numbers with same denominators. The examples used by the teacher for this idea included only positive addends. Nevertheless, he did not use examples that included (-, +), (-, -) and (+, -) as the sign of their addends. Moreover, the addition examples included addends either in proper or improper form but not in mixed number form.

Teacher C provided many examples for teaching the addition of rational numbers with different denominators. Examples used by Teacher C for teaching this idea included (+, +), (-, +) and (-, -) as the sign of the first and second addend respectively. Nonetheless, he did not use any example that included a negative rational number as a first addend and a positive rational number as a second addend. Finally, the addends of the addition examples included all possible numbers in different forms. That is, each addend was either a proper number, an improper number, a mixed number or an integer.

Teacher C used a few examples for teaching properties of addition of rational numbers. To be more precise, he used one example for commutative and associative property of addition, two examples for identity property of addition. The example used for commutative property included a positive proper number and a negative proper number as addends of the addition operation. The example used for associative property included three positive proper numbers as addends of addition. In addition, the examples used for commutative property and associative property included rational numbers with different denominators. In identity property examples provided by Teacher C, the rational number accompanying zero was either a positive proper number or a negative proper number. Lastly, the teacher neither mentioned about nor provided any specific example for the inverse property of addition of rational numbers.

The examples used by Teacher D for teaching addition and subtraction of rational numbers focused on the following mathematical ideas: using models for the addition and subtraction of rational numbers, adding and subtracting rational numbers with same denominators, adding and subtracting rational numbers with different denominators, estimating the addition of rational numbers and teaching properties of addition of rational numbers. The examples used by Teacher D for teaching addition and subtraction of rational numbers were all structurally different from each other. Therefore, all of these examples are presented in Table 4.21.

Table 4.21. Examples used by Teacher D for adding and subtracting rational numbers

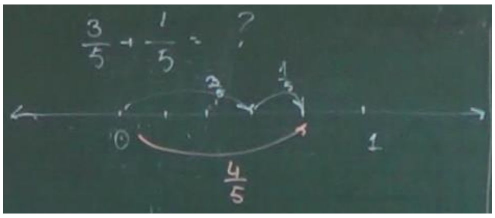
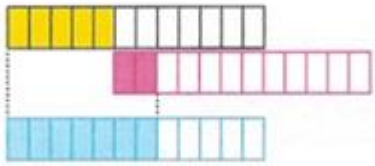
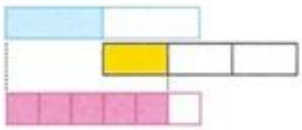
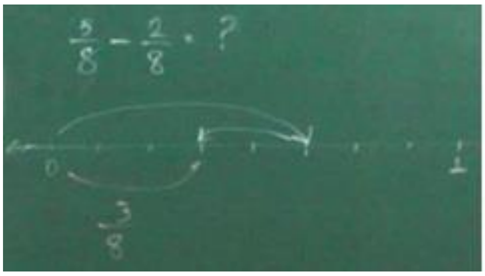
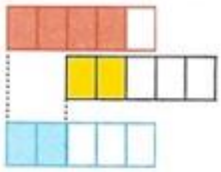
Ideas for adding rational numbers	Examples used by Teacher D	Number of examples used
Using models for the addition and subtraction of rational numbers	    	5
Adding and subtracting rational numbers with same denominators	$\frac{3}{5}+\frac{1}{5}$; $\frac{5}{9}+\left(-\frac{7}{9}\right)$; $\left(\frac{-5}{9}\right)+\left(\frac{-2}{9}\right)$; $\left(\frac{-13}{4}\right)+\frac{7}{4}$; $1\frac{1}{5}+4\frac{3}{5}$; $0.\bar{7}+2.\bar{5}$; $\frac{5}{8}-\frac{2}{8}$; $\frac{35}{9}-3.\bar{7}$; $\left(\frac{-9}{4}\right)-\left(-\frac{3}{4}\right)$	9
Adding and subtracting rational numbers with different denominators	$2+\frac{3}{5}$; $4\frac{1}{2}+\frac{1}{5}$; $\left(-3\frac{1}{4}\right)+\left(-1\frac{1}{2}\right)$; $\frac{1}{3}-\frac{1}{8}$; $2-\frac{3}{7}$; $\frac{1}{2}-4$; $-1-\frac{1}{7}$; $\left(\frac{-2}{3}\right)-\frac{4}{5}$; $0.2-0.12$	9
Estimating the addition of rational numbers	$-2\frac{1}{100}+2\frac{95}{100}$	1

Table 4.21. (Continued)

Ideas for adding rational numbers	Examples used by Teacher D	Number of examples used
Teaching properties of addition of rational numbers	Commutative property $\frac{3}{9}+\frac{4}{9}=\frac{4}{9}+\frac{3}{9}$; $2\frac{1}{3}+\left(-1\frac{1}{7}\right)=\left(-1\frac{1}{7}\right)+\Delta$ Associative property $-\frac{3}{7}+\left(\frac{4}{7}+\frac{5}{7}\right)=\left(-\frac{3}{7}+\frac{4}{7}\right)+\frac{5}{7}$; $\left(-\frac{1}{2}\right)+\left[\left(-\frac{1}{3}+\square\right)\right]=\left[\left(-\frac{1}{2}\right)+\left(-\frac{1}{3}\right)\right]+\frac{5}{6}$ Identity property $\frac{2}{11}+0=\frac{2}{11}$ Inverse property $\frac{4}{5}+\left(-\frac{4}{5}\right)=0$; $\left(-\frac{2}{9}\right)+\square=0$; $\left(+\frac{9}{11}\right)+\Delta=0$	8

Teacher D used 32 examples for teaching addition and subtraction of rational numbers. As emphasized by the textbook, Teacher D started teaching addition and subtraction of rational numbers by using a number line model for $\frac{3}{5}+\frac{1}{5}$ before adding these rational numbers by using same denominators algorithm. In addition to this, she used two different region models as exercise examples and asked her students to find the symbolic expressions of these models. The first model illustrated the addition of two rational numbers with same denominators while the second one illustrated the addition of two rational numbers with different denominators. Unlike the textbook, Teacher D taught subtraction of rational numbers after completely teaching addition of rational numbers. That is, the teacher did not provide addition and subtraction examples concurrently. Similar to the addition of rational numbers, Teacher D started teaching subtraction operation with rational numbers by using a number line model for $\frac{5}{8}-\frac{2}{8}$ before computing it by using same denominators algorithm. Besides, she used a region model as an exercise example and asked the students to find the symbolic expression of the given model. By this model, Teacher D illustrated the subtraction of two rational numbers that have same denominators. Each addition and subtraction model included positive rational numbers that are in proper form. Therefore, these examples also addressed students' prior knowledge on addition and subtraction of fractions.

After modeling addition of rational numbers, Teacher D moved on to teaching addition of rational numbers with same denominators. The addition examples used by the teacher included all possible variations in terms of the sign of the addends. In other words, the examples used by the teacher included (+, +), (+, -), (-, +) and (+, +) as the sign of the addends. In a similar fashion, the addition examples included all possible variations with respect to form of the addends. More precisely, the first and the second addend of the addition operations were either in proper number, improper number or mixed number form. In addition to these examples, the teacher provided an example that included terminating decimals as the first and second addend of the addition operation.

As she did in the teaching of addition of rational numbers, Teacher D started teaching subtraction of rational numbers with same denominators immediately after modeling subtraction of rational numbers. However, the examples used for teaching subtraction of rational numbers with same denominators was relatively few when compared to the number of examples used for addition of rational numbers with same denominators. Besides, when subtraction examples were examined, it was seen that there were some variations with respect to the sign of the terms. Precisely, Teacher D used subtraction examples that included (+, +) or (-, -) as the sign of the minuends and subtrahends respectively. However, she did not provide subtraction examples that included positive minuends and negative subtrahends or negative minuends and positive subtrahends. Similarly, there was some variability in the subtraction examples in terms of the forms of the first and second terms. Namely, the subtraction examples used either proper or improper numbers as the form of the minuends, and proper numbers or a repeating decimal number as the form of the subtrahends.

Soon after teaching the addition of rational numbers with same denominators, Teacher D started to teach properties of addition of rational numbers. The examples provided by the teacher for teaching commutative property, associative property and inverse property included not only terms that are rational numbers but also unknown values. In addition, the examples used for teaching commutative property included the addition of two positive proper numbers or the addition of a positive mixed number and a negative mixed number. Likewise, the associative property examples included the addition of a negative proper number and two positive proper numbers or a positive proper number and two negative proper numbers. Although the teacher did not teach how to add rational numbers with different denominators, the examples used by her for teaching commutative and associative property included not only rational numbers with same denominators but also rational numbers with different denominators. Teacher D used only one example for teaching the identity property of addition. In this example, the rational number accompanying zero was a positive proper number. Finally, to teach inverse property of addition, Teacher D provided three examples. These examples were provided in a way that emphasized the idea that the addition of a rational number with its additive inverse is equal to zero. While one of these examples included two rational numbers, the other two examples included unknown values that corresponded to additive inverses.

After teaching the properties of addition of rational numbers, the teacher moved on to teaching addition of rational numbers with different denominators. The teacher used three different examples for this idea, and these examples included (+, +), (-, +) and (-, -) as the sign of the first and second addend respectively. In addition, these examples illustrated the addition of a positive integer and a positive proper number, two negative mixed numbers or two positive proper numbers. The teacher provided examples for the subtraction of rational numbers with different denominators after teaching the subtraction of rational numbers with same denominators. The subtraction examples of this kind used by Teacher D included (+, +) and (-, +) as the sign of the minuends and subtrahends respectively. Besides, these examples included minuends and subtrahends that were either a proper number, an improper number, or an integer. However, none of the terms of the subtraction examples with different denominators was in mixed number form.

Although estimation of addition and subtraction operations with rational numbers was emphasized in the middle school mathematics curriculum and in the textbook, Teacher D provided only one specific estimation example. In this example, the students were asked to estimate the addition of $-2\frac{1}{100}$ and $2\frac{95}{100}$. Teacher D explained that estimation is synonymous with rounding. Next, she focused on the fractional parts of the rational numbers and indicated that $\frac{1}{100}$ is very close to 0 while $\frac{95}{100}$ is close to 1. Besides, she located these rational numbers on a number line to support her idea. Eventually, she rounded $-2\frac{1}{100}$ to -2 and similarly $2\frac{95}{100}$ to 3. As it can be seen, this example illustrated the estimation of addition of a negative mixed number and a positive mixed number. However, Teacher D did not provide any example for estimating the subtraction of rational numbers.

4.1.1.5. Examples used for performing multiplication and division operations with rational numbers

In the middle school mathematics curriculum, teachers are suggested to start teaching multiplication and division of rational numbers by having students remember multiplication and division of fractions. After recalling addition and subtraction of fractions, teachers are suggested to use activities related with multiplication and division of rational numbers. More importantly, the curriculum explained that middle school mathematics teachers should be careful about several points while teaching multiplication and division of rational numbers. First, teachers were alerted to teach special cases of multiplication and division by 0, 1 and (-1). Second, they were informed to teach the properties of multiplication operation with rational numbers and their algebraic notations. More specifically, they were warned to teach distributive property of multiplication over addition and subtraction and their algebraic notations and emphasize that product of a rational number by its multiplicative inverse is equal to 1. Third, they were alerted to teach estimation of multiplication and division of rational numbers. Finally, they were notified to teach calculation of square and cube of rational numbers.

In the mathematics textbook followed by the classrooms, the examples selected for teaching multiplication and division of rational numbers represented the following ideas: modeling multiplication of rational numbers, multiplication and division of rational numbers, multiplication and division by 0, 1 and (-1), estimation of multiplication and division of rational numbers, modeling and calculating the square and cube of rational numbers, multi-step operations with rational numbers, and properties of multiplication of rational numbers. When examples included in the textbook for teaching each of these ideas were examined, it was seen that there were some structural similarities and differences among them with respect to the sign and form of terms included in the operations. Therefore, only the examples that have different structural features were presented in Table 4.22. Besides, the number of examples for each idea was also presented.

Table 4.22. Examples included in the textbook for multiplication and division of rational numbers

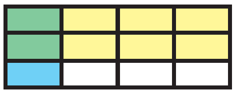
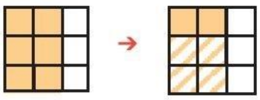
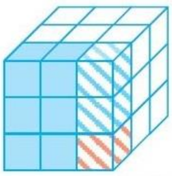
Ideas	Illustrative examples	Number of examples used
Modeling multiplication of rational numbers	 $\frac{1}{4}\cdot\frac{2}{3}$	5
Multiplication and division of rational numbers	$\left(+\frac{2}{3}\right)\cdot\left(+\frac{3}{4}\right)$; $\left(-\frac{4}{5}\right)\cdot\left(-\frac{2}{3}\right)$; $\left(+\frac{2}{3}\right)\cdot\left(-\frac{4}{5}\right)$; $\left(+\frac{5}{8}\right):\left(+\frac{4}{10}\right)$; $\left(-\frac{4}{5}\right):\left(-\frac{2}{3}\right)$; $\left(+\frac{2}{3}\right):\left(-\frac{4}{5}\right)$; $\left(-\frac{2}{9}\right):\left(\frac{1}{18}\right)$; $\left(+1\frac{1}{3}\right):\left(+2\frac{1}{4}\right)$; $\frac{8}{3}:\left(-\frac{6}{7}\right)$; $\frac{18}{25}:(1.4)$	13
Multiplication and division by 0, 1 and (-1)	$\left(-\frac{4}{5}\right)\cdot 0$; $\left(-\frac{4}{5}\right)\cdot 1$; $\left(-\frac{4}{5}\right)\cdot(-1)$; $0:\left(-\frac{2}{3}\right)$; $\left(-\frac{2}{3}\right):0$; $1:\frac{1}{99}$; $1:\left(-\frac{2}{3}\right)$; $\left(-\frac{2}{3}\right):1$; $(-1):\left(-\frac{2}{3}\right)$; $\left(-\frac{2}{3}\right):(-1)$	11
Estimation of multiplication and division of rational numbers	$\frac{1}{7}\cdot 30$; $\frac{7}{8}\cdot\left(-\frac{6}{11}\right)$; $\left(12\frac{8}{9}\right)\cdot\left(4\frac{1}{8}\right)$; $\left(6\frac{5}{12}\cdot 603\right)$; $3\frac{1}{11}:1\frac{4}{9}$; $378:\left(4\frac{1}{9}\right)$; $\left(-24\frac{1}{9}\right):\left(-11\frac{7}{8}\right)$	11
Modeling and calculating the square and cube of rational numbers	 $\frac{2}{3}$ $\left(\frac{2}{3}\right)^2=\frac{2}{3}\cdot\frac{2}{3}=\frac{2.2}{3.3}=\frac{4}{9}$  $\left(\frac{1}{3}\right)^3=\frac{1}{3}\cdot\frac{1}{3}\cdot\frac{1}{3}=\frac{1.1.1}{3.3.3}=\frac{1}{27}$ $\left(\frac{3}{4}\right)^2$; $\left(\frac{2}{3}\right)^3$; $\left(-\frac{2}{5}\right)^2$; $\left(-\frac{2}{3}\right)^3$	8

Table 4.22. (Continued)

Ideas	Illustrative examples	Number of examples used
Multi-step operations with rational numbers	$\frac{3}{5}\cdot\left(\frac{5}{3}+1\right)\cdot\frac{4}{6}\cdot\left(1-\frac{3}{2}\right)$; $\left[\left(\frac{1}{2}+\frac{1}{3}\right)\right]\cdot\left(\frac{3}{4}-\frac{7}{5}\right)$; $\left[\left(-2\frac{1}{3}\right)-\left(-1\frac{1}{4}\right)\right]:\frac{7}{24}$	3
Properties of multiplication of rational numbers	Commutative property $\left(-\frac{2}{3}\right)\cdot\left(+\frac{1}{4}\right)=\left(+\frac{1}{4}\right)\cdot\left(-\frac{2}{3}\right)$; $\frac{2}{13}\cdot\frac{4}{7}\cdot\frac{13}{5}=\frac{2}{\cancel{13}}\cdot\frac{\cancel{13}}{5}\cdot\frac{4}{7}$; $\left(-\frac{1}{21}\right)\cdot\left(-\frac{3}{8}\right)\cdot\left(-\frac{21}{5}\right)=\left(-\frac{3}{8}\right)\cdot\left(-\frac{1}{\cancel{21}}\right)\cdot\left(-\frac{\cancel{21}}{5}\right)$ Associative property $\left(\frac{1}{2}\cdot\frac{3}{4}\right)\cdot\frac{1}{3}=\frac{1}{2}\cdot\left(\frac{3}{4}\cdot\frac{1}{3}\right)$; $\left[\left(-\frac{2}{3}\right)\cdot\left(+\frac{1}{4}\right)\right]\cdot\left(-\frac{1}{2}\right)=\left(-\frac{2}{3}\right)\cdot\left[\left(+\frac{1}{4}\right)\cdot\left(-\frac{1}{2}\right)\right]$ Multiplicative inverse property $\frac{2}{3}\cdot\frac{3}{4}\cdot\frac{3}{2}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{5}{6}\cdot\frac{5}{4}\cdot\frac{6}{5}=$ $\frac{2}{3}\cdot\frac{3}{2}\cdot\frac{3}{4}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{5}{4}\cdot\frac{5}{6}\cdot\frac{6}{5}=\frac{6}{6}\cdot\frac{12}{12}\cdot\frac{20}{20}\cdot\frac{30}{30}=1.1.1.1=1$ Distributive property $\left(-\frac{3}{4}\right)\cdot\left[\left(-\frac{4}{5}\right)+\frac{4}{7}\right]$; $\frac{1}{2}\cdot\left(\frac{3}{4}+x\right)=\frac{1}{2}\cdot y+\frac{1}{2}\cdot\frac{1}{3}$; $\frac{5}{4}\cdot\left(\frac{4}{3}-\frac{2}{5}\right)$; $\left(-\frac{2}{3}\right)\cdot\left[\frac{4}{5}-\frac{3}{20}\right]$; $\frac{2}{99}\cdot\left[-\frac{99}{48}-\left(-\frac{33}{24}\right)\right]$	11

The textbook included 62 examples for multiplication and division of rational numbers. The textbook emphasized modeling of multiplication of rational numbers before introducing the traditional multiplication algorithm. It provided several modeling examples and each model included factors that are positive rational numbers. Actually, these examples addressed students' prior knowledge on modeling multiplication of fractions. Besides, the examples included factors that are in proper form and there was not any modeling example that included factors that are greater than one. Although the textbook emphasized modeling of multiplication of rational numbers, it did not provide any example for modeling the division of rational numbers.

After the provision of modeling examples, the textbook presented examples related with multiplication of rational numbers by using the traditional algorithm. These examples included either two positive factors, two negative factors or one negative and one positive factor. By presenting these examples, the textbook emphasized that the multiplication of two positive or two negative rational numbers will yield a positive product while the multiplication of one positive and one negative rational number will yield a negative product. Nevertheless, the factors included in the multiplication operations were all proper numbers.

The textbook included examples for the division of rational numbers after presenting all ideas related with the multiplication of rational numbers. The textbook did not provide any model for the division operation. Instead, it began with the explanation of invert and multiply algorithm to teach the division of rational numbers and it directly provided examples to illustrate this algorithm. Besides, when examples related with division of rational numbers were examined, it was seen that these examples included more variations in terms of sign and form of terms when compared to the variability in multiplication examples. For instance, the dividends and the divisors existing in the division examples were either positive-positive, positive-negative, negative-positive or negative-negative with respect to their signs. Besides, the dividends were either in proper, improper or mixed number form while the divisors were in proper, mixed or decimal form.

The middle school mathematics curriculum and the textbook explicitly suggested teachers to emphasize the teaching of special cases of multiplication and division of rational numbers by 0, 1 and (-1). The textbook included several examples for teaching multiplication of rational numbers by 0, 1 and (-1). Namely, the textbook presented the multiplication of $\left(-\frac{4}{5}\right)$ by 0, 1 and (-1) respectively and subsequently provided the following explanations: “the multiplication of each rational number by 0 results in a product of 0, the multiplication of each rational number by 1 is equal to the rational number itself and the multiplication of each rational number by (-1) is equal the additive inverse of that rational number". When examples related with division of rational numbers by 0, 1 and (-1) or with the division of 0, 1 and (-1) by any rational number excluding zero were examined, it was seen that there were a lot more examples when compared to the examples related with multiplication by 0, 1 and (-1). To be more specific, the textbook presented the division of $\left(-\frac{2}{3}\right)$ by 0, 1 and (-1) or the division of 0, 1 and (-1) by $\left(-\frac{2}{3}\right)$ and thereafter provided the following explanations: "the division of 0 by any rational number that is different from 0 yields 0, the division of 1 by any rational number that is different from 0 is equal to the multiplicative inverse of that rational number, the division of (-1) by any rational number excluding 0 is equal to the additive inverse of the multiplicative inverse of that rational number, the division of any rational number by 0 is undefined, the division of any rational number by 1 is equal to the rational number itself and finally the division of any rational number by (-1) is equal to the additive inverse of that number".

Similar to the middle school mathematics curriculum, the textbook also emphasized estimation of multiplication and division of rational numbers. The examples presented for the estimation of multiplication included either a proper number, a mixed number, or an integer as the first factor. Identically, those examples included either an integer, a proper number or a mixed number as the second factor of the multiplication operation. Besides, except for one example, all estimation examples regarding multiplication of rational numbers included positive factors. The number of examples provided by the textbook for the estimation of division was similar to the number of examples provided for the estimation of multiplication. In more detail, estimation of division examples included dividends in the form of a mixed number or an integer while the dividends were all in mixed number form. Finally, all but one of the estimation of division examples included dividends and divisors that were both positive and one example included a negative dividend and a negative divisor.

As emphasized by the middle school mathematics curriculum, the textbook provided several examples for modeling and calculating the square and cube of rational numbers. The textbook initially presented modeling examples that included the square and cube of positive rational numbers that are less than 1. Later, it presented examples that showed how to calculate the square and cube of rational numbers that are either positive or negative. However, the bases of the exponents were all in proper number form. The textbook neither provided examples that included a base in improper number form nor a base in mixed number form.

The explanation part of the textbook did not include multi-step operations with rational numbers. However, there were three exercise examples that were left for the students. These examples were all structurally different from each other. More specifically, the first example included four factors. The first and the third factor of this example was a proper number while the second and fourth factor included the addition of a mixed number and an integer and the subtraction of an improper number from an integer respectively. The second example included two factors. The first factor of this example included the addition of two positive proper numbers while the second factor included the subtraction of a positive improper number from a positive proper number. Finally, the third example was a multi-step division example. In this example, the divisor included the subtraction of a negative mixed number from another negative mixed number and the divisor was a positive rational number in proper form.

After the provision of examples which illustrated the traditional multiplication algorithm, the textbook included examples for the properties of multiplication of rational numbers. The textbook initially provided examples for the commutative property of multiplication of rational numbers. The first commutative property example was used as a basis for justifying that the equality holds for every rational number. The other two examples showed how commutative property of rational numbers can be applied to mathematical problems. That is, those two examples paved the way for solving problems easily without actually making computations. Finally, the commutative property examples included factors in the form of a positive proper number, negative proper number, or negative improper number.

The textbook provided two associative property examples. These two examples were used as a basis for justifying that the associative property holds for every rational number. The first example included factors in positive proper number form while the second example included factors as two negative proper numbers and one positive proper number. The example provided for the multiplicative inverse property of rational numbers was not used as a basis for justification. Instead, this example showed how multiplicative inverse property can be applied to mathematical problems so as to solve them easily. Besides, this example merely included factors that were positive proper numbers.

Last, the textbook presented examples for teaching the distributive property of multiplication over addition and subtraction. Three examples were provided in the textbook to illustrate the distributive property of multiplication over subtraction. One of these examples was used as a basis for justifying that the product obtained by using the distributive property is equal to the product obtained by taking account of the order of operations. The other two examples illustrated how to apply distributive property to mathematical problems. Two examples were included in the textbook for teaching distributive property of multiplication over addition. The first example showed how to distribute a negative rational number over a negative and a positive rational number. The second example included unknown values and it illustrated how these unknown values can be found by using distributive property without actually computing. This example included rational numbers in positive proper number form.

The examples used by Teacher A for teaching multiplication and division of rational numbers represented the following ideas: modeling multiplication of rational numbers, multiplication and division of rational numbers, calculating the square and cube of rational numbers, multi-step operations with rational numbers, and properties of multiplication of rational numbers. When examples used by Teacher A for teaching multiplication and division of rational numbers were examined, it was seen that there were some structural similarities and differences among them with respect to the sign and form of terms included in the operations. Therefore, only the examples that have different structural features were presented in Table 4.23. Besides, the number of examples for each idea was also presented.

Table 4.23. Examples used by Teacher A for teaching multiplication and division of rational numbers

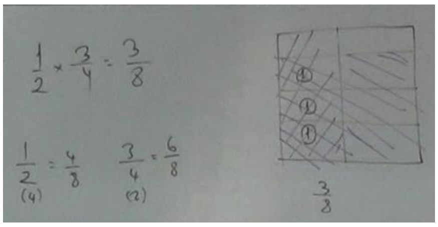
Ideas	Illustrative examples	Number of examples used
Modeling multiplication of rational numbers		3
Multiplication and division of rational numbers	$\frac{3}{5}\cdot\frac{2}{7}$; $\left(-\frac{1}{5}\right)\cdot\left(-\frac{1}{7}\right)$; $\left(-\frac{2}{9}\right)\cdot\frac{3}{2}$; $2\frac{4}{5}\cdot\left(-3\frac{1}{8}\right)$; $\frac{2}{3}\cdot\frac{3}{4}\cdot\frac{4}{5}$; $0.\overline{7}\cdot\frac{9}{14}$; $0.25\cdot(-0.\overline{9})$; $\frac{3}{5}\cdot n=\frac{5}{7}$; $\frac{3}{4}:\frac{5}{6}$; $\left(-\frac{5}{8}\right):\frac{1}{3}$; $-5\frac{2}{3}:2\frac{5}{6}$; $-2\frac{5}{9}:\left(-\frac{2}{7}\right)$; $\frac{\frac{1}{2}}{\frac{5}{3}}$	23
Calculating the square and cube of rational numbers	$\left(\frac{2}{3}\right)^2$; $\left(-\frac{3}{5}\right)^2$; $\left(\frac{3}{5}-\frac{2}{7}\right)^2$; $3-\left(\frac{2}{5}\right)^2$; $\left(-\frac{3}{4}\right)^3$; $\left(-\frac{2^3}{5}\right)$; $\left(-\frac{2}{5^3}\right)$; $\left(-\frac{2}{5}\right)^3$; $\left(\frac{-7^3}{5^2}\right)^2$	11

Table 4.23. (Continued)

Ideas	Illustrative examples	Number of examples used
Multi-step operations with rational numbers	$\left(\frac{2}{3}+\frac{1}{6}\right)\cdot\left(\frac{1}{2}+\frac{2}{5}\right)$; $\left(1-\frac{1}{2}\right)\cdot\left(1+\frac{1}{2}\right)$; $\left(\frac{1}{2}+\frac{1}{3}\right)\cdot\left(\frac{3}{4}-\frac{7}{5}\right)$; $\left(2\frac{1}{2}+1\frac{3}{5}\right)\cdot\left(1\frac{6}{7}+1\right)$; $\left(3\frac{2}{7}-2\frac{3}{4}\right)\cdot\left(4\frac{2}{5}-\frac{9}{10}\right)$; $(-0.35)\cdot 200-(-68)$; $\frac{3}{5}\cdot\left(\frac{5}{3}+1\right)\cdot\frac{4}{6}\cdot\left(1-\frac{3}{2}\right)$; $\left(1+\frac{1}{2}\right)\cdot\left(1+\frac{1}{3}\right)\cdot\left(1+\frac{1}{4}\right)\cdots\left(1+\frac{1}{100}\right)$; $\left[\left(-2\frac{1}{3}\right)-\left(-1\frac{1}{4}\right)\right]:\frac{7}{4}$; $\left\|-\frac{3}{2}\right\|-\frac{(-2)}{3}:\frac{5}{7}+\frac{7}{5}$	10
Properties of multiplication of rational numbers	Multiplicative inverse property The multiplicative inverse of $\frac{19}{7}$ Distributive property $2\cdot(3+5)$; $2\cdot(3-5)$; $2\cdot(\square+5)=\Delta\cdot 7+2\cdot\bigcirc$; $\left(\frac{2}{5}+\frac{1}{10}\right)\cdot\frac{1}{2}$; $\frac{3}{4}\cdot\left(\frac{3}{2}-\frac{6}{5}\right)$; $\frac{2}{99}\cdot\left[\left(-\frac{99}{48}\right)-\left(-\frac{33}{24}\right)\right]$	7

Teacher A used 54 examples for multiplication and division of rational numbers. Although the textbook emphasized modeling multiplication of rational numbers, Teacher A began teaching multiplication of rational numbers by using the traditional multiplication algorithm. In contrast, Teacher A provided several modeling examples at the end of the multiplication of rational numbers just before teaching division of rational numbers. In these models, Teacher A used vertical divisions to show the first factor and horizontal divisions to show the second factor. Meanwhile, he found the common denominator of the two factors in order to divide the whole easily. Finally, he found the product of the multiplication operation by counting the double shaded parts included in the whole. The examples were similar to those included in the textbook. Namely, each model included factors that were positive rational numbers. In fact, these modeling examples addressed students' prior knowledge on modeling multiplication of fractions. Despite this, Teacher A did not provide any modeling example that included factors that are greater than one.

As mentioned above, Teacher A began teaching multiplication of rational numbers by introducing examples which illustrated the use of traditional multiplication algorithm. These examples included either two positive factors, three positive factors, two negative factors or one negative and one positive factor. Besides, the factors included in the multiplication operations were either in proper number, mixed number, repeating decimal or terminating decimal form. Moreover, some other examples included unknowns as the first or the second factor of the multiplication operation. On the whole, by using these examples the teacher emphasized that the multiplication of two positive or two negative rational numbers would yield a positive product while the multiplication of one positive and one negative rational number would yield a negative product. Besides, he explicitly uttered and wrote on the board that the product of two rational numbers with the same signs will be positive and the product of two rational numbers with opposite signs will be negative.

Similar to the textbook, Teacher A generated examples for the division of rational numbers after introducing all ideas related with the multiplication of rational numbers. However, he did not use any model for the division of rational numbers. In place of this, he provided examples for teaching the invert and multiply algorithm for division of rational numbers. When these examples were examined, it was seen that they included some variations with respect to the sign and form of terms. Namely, the sign of the dividends and divisors of the division operations were either (+, +), (-, +), (-, +) or (-, -). Besides, the dividends were in proper or mixed number form and the divisors were in proper, improper and mixed number form. More importantly, the teacher emphasized the use of two different notations for expressing division of rational numbers and provided many examples by using both notations. In the first notation, the teacher showed the division operation by using an obelus in this manner: $\frac{a}{b} \div \frac{c}{d}$. In this notation, the divisor, the dividend and the quotient are all expressed on one line. In the second notation, the teacher showed the division operation by placing the dividend over the divisor with a vinculum between them in this way: $\dfrac{\frac{a}{b}}{\frac{c}{d}}$. In this second notation, the divisor and the dividend are expressed on different lines.

Although suggested by the middle school mathematics curriculum and the textbook, Teacher A did not provide any examples for the following ideas: multiplication and division by 0, 1 and (-1), estimation of multiplication and division of rational numbers and modeling the square and cube of rational numbers. Despite not using any models for the square and cube of rational numbers, Teacher A showed how to calculate them with a broad range of examples. The examples used by the teacher for explaining this idea included much more variability when compared to textbook examples. To be more precise, the examples included in the textbook singly had one rational number as a base. However, the examples provided by the teacher included bases which had an exponent either in the numerator or in the denominator. Besides, some of the examples had bases that included subtraction of rational numbers. In short, the teacher covered this idea in greater depth when compared to the examples included in the textbook.

Although the explanation part of the textbook related with multiplication and division of rational numbers did not provide multi-step operation examples, Teacher A allocated much time for solving multi-step operations with rational numbers. Meanwhile, he used a large number of examples to teach this idea to their students. These examples were very different from each other in terms of their structural components. Most of the multi-step multiplication examples included two factors and these factors were formed either by adding or subtracting two rational numbers. Besides, these factors included the addition or subtraction of rational numbers that were either proper number, improper number, integer or mixed number and all of the rational numbers included in the subtraction and addition operations were positive numbers. Apart from the multi-step multiplication examples with two factors, there was one example with four factors and one example with a continuing pattern. The multi-step multiplication example with a continuing pattern was different in nature from the rest of the multi-step multiplication examples since it could not be solved without cross simplifying the numerators of the antecedent factors with the denominators of the posterior factors.

The number of multi-step division examples provided by Teacher A was very few when compared to the number of multi-step multiplication examples provided by him. That is, Teacher A used only two multi-step division examples and both examples included two terms. The first example had a dividend that included the subtraction of two negative mixed numbers and a positive improper divisor. The second example included division of a negative proper number by a positive proper number, then subtraction of the absolute value of a negative improper number and finally addition of a positive improper number. It is worthy of note that the teacher had his students remember the order of operations before working out this example since it entailed following the rules for the order of operations correctly.

Ultimately, although emphasized by the middle school mathematics curriculum and the textbook, Teacher A did not allocate time for covering properties of multiplication of rational numbers. He taught multiplicative inverse property and distributive property of multiplication as he came across with exercise examples that entailed the use of these properties. He provided only one example for teaching multiplicative inverse property of rational numbers. However, while finding the multiplicative inverse of the selected positive improper number, the teacher used the expression 'flip over' although the middle school mathematics curriculum emphasized that the two rational numbers are multiplicative inverses of each other if their product is equal to one.

Teacher A put more emphasis on the distributive property of multiplication over addition and subtraction. More specifically, the teacher provided three examples for the distributive property of multiplication over addition. In the first example, natural numbers were selected to show the distributive property. Therefore, this example addressed students' prior knowledge on distributive property of multiplication of natural numbers. The second example included unknowns, and the teacher provided this example to show that it was impossible to find the unknowns without using the distributive property of multiplication over addition. Again, natural numbers were used to generate this example. Unlike the previous two examples, the third example was used to show that the multiplication operation is right-distributive over addition operation. This time, the teacher selected positive proper numbers to generate the example. Moreover, Teacher A used two examples to illustrate the distributive property of multiplication over subtraction. In the first example, a positive proper number was multiplied by the subtraction of a positive improper number from another positive improper number. In the second example, a positive proper number was multiplied by the subtraction of a negative improper number from another negative improper number. These two examples illustrated that the multiplication operation was left-distributive over subtraction operation. Finally, Teacher A used these two examples to show how to apply the distributive property to the given numerical expressions.

The examples used by Teacher B for teaching multiplication and division of rational numbers represented the following ideas: multiplication and division of rational numbers, multiplication by 0, 1 and (-1), calculating the square and cube of rational numbers, multi-step operations with rational numbers, and properties of multiplication of rational numbers. When examples used by Teacher B for teaching the above mentioned ideas were examined, it was seen that the components included in most of the examples were structurally different from each other. Therefore, only the examples that have different structural components were presented in Table 4.24. Besides, the number of examples for each ideas were also presented.

Table 4.24. Examples used by Teacher B for teaching multiplication and division of rational numbers

Ideas	Illustrative examples	Number of examples used
Multiplication and division of rational numbers	$\left(+\frac{2}{3}\right)\cdot\left(+\frac{3}{4}\right)$; $\frac{7}{4}\cdot\frac{3}{9}$; $0.6\cdot0.\overline{6}$; $\left(+\frac{2}{3}\right)\cdot\left(-\frac{4}{5}\right)$; $\left(1\frac{2}{3}\right)\cdot\left(\frac{-5}{7}\right)$; $\frac{3}{8}\cdot\left(-\frac{3}{2}\right)$; $(-3)\cdot1\frac{1}{2}$; $(-18)\cdot\frac{11}{24}$; $\left(-\frac{4}{5}\right)\cdot\left(-\frac{2}{3}\right)$; $\left(\frac{-5}{13}\right)\cdot\left(-1\frac{3}{4}\right)$; $\left(\frac{-1}{4}\right)\cdot\left(\frac{-7}{3}\right)$; $(-100)\cdot(-0.3)$; $\left(\frac{-1}{3}\right)\cdot\left(2\frac{1}{2}\right)\cdot(-5)$; $\left(\frac{-5}{9}\right)\cdot\left(\frac{-27}{4}\right)\cdot\left(\frac{-1}{3}\right)$; $\frac{5}{12}:\frac{7}{4}$; $4:\left(-2\frac{3}{7}\right)$; $\frac{5}{9}:(-2)$; $\left(\frac{-3}{6}\right):\frac{5}{4}$; $\left(\frac{-5}{3}\right):2\frac{1}{2}$; $\left(\frac{-2}{9}\right):\left(-\frac{6}{5}\right)$	23
Multiplication by 0, 1 and (-1)	$\left(-\frac{4}{5}\right)\cdot0$; $\left(-\frac{4}{5}\right)\cdot1$; $\left(-\frac{4}{5}\right)\cdot(-1)$	3
Calculating the square and cube of rational numbers	$(+3)^2$; $(-4)^2$; $(-2)^3$; $(-2)^4$; $-\left(2^4\right)$; -2^4; $\left(1\frac{1}{2}\right)^2$; $\left(\frac{1}{2}\right)^3$; $\left(-\frac{3}{5}\right)^3$; $\left(-\frac{3}{2}\right)^3$; $\left(\frac{-1}{5}\right)^4$	19
Multi-step operations with rational numbers	$\left(2\frac{1}{3}-\frac{7}{3}\right):\left(-\frac{1}{2}\right)$; $\left(1-\frac{3}{4}\right):\left(\frac{1}{2}+1\right)$; $\frac{6}{0.\overline{3}}\cdot\frac{0.\overline{2}}{6}$	3

Table 4.24. (Continued)

Ideas	Illustrative examples	Number of examples used
Properties of multiplication of rational numbers	Commutative property $\left(\frac{-5}{8}\right)\cdot\frac{2}{9}=\frac{2}{9}\cdot\left(\frac{-5}{8}\right)$ Associative property $\left(\frac{-6}{12}\right)\cdot\left[B\cdot\left(\frac{5}{6}\right)\right]=\left[\left(\frac{-6}{12}\right)\cdot\left(\frac{4}{3}\right)\right]\cdot\frac{5}{6}$ Identity property $(-7)\cdot 1=(-7)$ Zero property $0\cdot\left(\frac{5}{8}\right)$ Distributive property $4.\left[5+3\right]=4.5+4.3;\ \left(-\frac{7}{10}\right)\cdot\left[\frac{2}{5}+\frac{3}{4}\right];$ $\left(-\frac{5}{4}\right)\cdot\left[\frac{2}{3}+x\right]=\left(-\frac{5}{4}\right)\cdot\frac{2}{3}+\left(-\frac{5}{4}\right)\cdot\left(-\frac{3}{8}\right)$ Multiplicative inverse property The multiplicative inverse of $\left(\frac{-5}{9}\right)$; $\left(\frac{3}{6}\right)$; $\left(1\frac{3}{4}\right)$; and 0.012	12

Teacher B used 60 examples for teaching multiplication and division of rational numbers. Although the textbook emphasized modeling of multiplication of rational numbers before teaching multiplication of rational numbers, Teacher B did not provide any modeling example. In place of this, Teacher B started teaching multiplication of rational numbers by using the traditional multiplication algorithm. More specifically, the teacher initially provided the multiplication of two positive proper numbers and thus, he recalled the multiplication of fractions. Broadly speaking, Teacher B presented many examples for teaching the algorithm for multiplication of rational numbers. Besides, the multiplication examples used by Teacher B included all possible variations in terms of the sign of the factors. The examples with two factors included (+, +), (+, -), (-, +) or (-, -) as the sign of the first and second factors respectively. The teacher also used a few examples with three factors. These examples included either two negative factors and one positive factor or three negative factors. Moreover, the multiplication examples used by the teacher included much more variability in terms of the form of factors when compared to multiplication examples provided by the textbook. More specifically, the factors included in the multiplication examples were generally in proper number, improper number and mixed number form. Integers, repeating decimals and terminating decimals were less frequently used by the teacher as forms of factors included in the multiplication operations.

Unlike the textbook, Teacher B generated examples for the division of rational numbers concurrently with the multiplication of rational numbers. Similar to the textbook, Teacher B did not provide any example for modeling division of rational numbers. Instead of this, the teacher immediately introduced examples that illustrated the use of invert and multiply algorithm for the division of rational numbers. However, the number of examples used by Teacher B for the division of rational numbers was quite few in proportion to the number of multiplication examples. When division examples were examined, it was seen that they included some variations with regards to the sign and form of the terms. That is, the dividends were either in positive proper number, negative proper number, negative improper number or integer form and the divisors were either in positive improper number, negative improper number, positive mixed number, negative mixed number or integer form. Similar to the textbook, Teacher A used only one notation for teaching division of rational numbers. In this notation, the division operation was shown via an obelus as follows: $\frac{a}{b} \div \frac{c}{d}$. Finally, after providing several examples for teaching multiplication and division of rational numbers, Teacher B emphasized that the product/quotient will be negative when the terms have different signs and the product/quotient will be positive when the terms have same signs.

Although Teacher B provided a few examples for teaching the special cases of multiplication by 0, 1 and (-1) at the introductory phase of the lesson, he did not pay attention to this idea for the latter multiplication examples. In fact, the examples related with multiplication by 0, 1 and (-1) were not generated by Teacher B himself. That is, these examples appeared on the initial pages of the textbook and the teacher had students examine these examples by projecting them on the board. In a similar fashion, Teacher B did not use any division example that focused on the special cases of division by 0, 1 and (-1).

There were several examples in the textbook that targeted modeling of the square and cube of rational numbers, however, Teacher B immediately provided examples for calculating the square and cube of rational numbers. To be more precise, Teacher B initially stressed that the power of an exponential number tells how many times the base number is multiplied by itself. Later, he recalled calculating the even and odd powers of integers. In addition to this, he paid attention to the distinction between the integer exponents that have a base inside the parenthesis and those that have a base without parenthesis. After recalling integer exponents, Teacher B provided several examples that showed how to calculate the square and cube of rational numbers. Most of these examples included similar structural components when compared to the examples included in the textbook for teaching this idea. Namely, the bases of the exponents were all positive or negative proper numbers excluding one. Different from the textbook examples, Teacher B generated examples with mixed number bases or examples with bases raised to the power of 4.

Teacher B did not provide any multi-step operation example during the teaching of multiplication and division of rational numbers. However, he provided three multi-step operation examples as he came across with them while working out exercise examples. While the first and the second example were multi-step division examples, the third example was a multi-step multiplication example. In the first example, the dividend included the subtraction of a positive improper number from a positive mixed number and the divisor was a negative proper number. In the second example, the dividend included the subtraction of a positive proper number from a positive integer and the divisor included the addition of a positive proper number and a positive integer. Finally, in the third example, the first factor included the division of a positive integer by a repeating decimal and the second factor included the division of a repeating decimal by a positive integer.

After Teacher B provided examples that illustrated the traditional multiplication algorithm and the invert and multiply algorithm, he moved on to teaching properties of multiplication of rational numbers. Initially, Teacher B provided an example for the commutative property of multiplication of rational numbers. This example included one negative and one positive proper number as factors of the multiplication operation. Next, the teacher provided an example for the associative property of multiplication of rational numbers. In this example, the left hand side of the equality included an unknown value, and the teacher showed the students how to find it without actually doing calculations. Besides, this example included one factor in negative proper number form, one factor in positive improper number form and one factor in positive proper number form. To illustrate the identity property of multiplication of rational numbers, Teacher B demonstrated the multiplication of a negative integer and 1. As a matter of fact, this example was provided by one of the students in the classroom and the teacher did not attempt to rewrite the integer as a rational number. To illustrate the zero property of multiplication, Teacher B selected an example that included a positive proper number as a companion to zero.

Among all properties, Teacher B put more emphasis on the teaching of distributive property and multiplicative inverse property of rational numbers. The teacher provided three examples for teaching the distributive property of multiplication over addition. However, the teacher did not provide any example for teaching the distributive property of multiplication over subtraction. The first distributive property example used by the teacher showed how to distribute a natural number over the addition of two natural numbers. The second example was used to show how to distribute a negative proper number over the addition of two positive proper numbers. The last example included an unknown value and was provided by the teacher to show how to find the unknown value by using the distributive property instead of actually calculating. Finally, this example included rational numbers in positive proper, negative proper and negative improper number form. In comparison with the textbook and Teacher A, Teacher B used a wide variety of examples to teach the multiplicative property of multiplication. To be more specific, Teacher B selected rational numbers in the form of a negative proper number, a positive proper number, a positive mixed number and a decimal number to teach the multiplicative inverses of those rational numbers to his students.

The examples used by Teacher C for teaching multiplication and division of rational numbers were very limited. Teacher C provided only three examples for teaching the traditional algorithm for multiplication of rational numbers. In a similar fashion, Teacher C used only one example to teach the division of rational numbers. On the other hand, the teacher did not provide examples for teaching the following ideas: modeling multiplication of rational numbers, multiplication and division by 0, 1 and (-1), estimation of multiplication and division of rational numbers, modeling and calculating the square and cube of rational numbers, multi-step operations with rational numbers and properties of multiplication of rational numbers.

Before introducing examples related with multiplication of rational numbers, Teacher C explained that there is no need to find the common denominator of the factors included in the multiplication operation. Next, the teacher verbally explained the traditional algorithm for multiplication of rational numbers. Finally, the teacher used $\left(+\frac{1}{3}\right)\cdot\left(-\frac{2}{7}\right)$, $\left(-\frac{1}{3}\right)\cdot\left(-\frac{2}{7}\right)$ and $\left(+\frac{1}{3}\right)\cdot\left(+\frac{2}{7}\right)$ respectively to illustrate multiplication of rational numbers. Although the middle school mathematics curriculum and the textbook emphasized recalling multiplication of fractions, Teacher C started with an example that included a positive proper number as the first factor and a negative proper number as the second factor. In the second example, the teacher used two negative proper numbers as factors of the multiplication operation. In the last example, the teacher used two positive proper numbers as the factors of the multiplication operation. It is important to note that while moving from the first example to the second example and from the second example to the third example, Teacher C changed only the sign of the factors while keeping their magnitudes invariant. Finally, the teacher emphasized that the multiplication of two positive or two negative rational numbers yields a positive product while the multiplication of one positive and one negative rational number gives a negative product.

Similar to the multiplication of rational numbers, Teacher C verbally explained the invert and multiply algorithm for the division of rational numbers before providing any example related with this idea. Then, the teacher provided $\left(-\frac{5}{9}\right):\left(+\frac{7}{8}\right)$ as a specific example for the division of rational numbers. As it can be seen, this example included a negative proper number as the dividend and a positive proper number as the divisor of the division operation. Subsequently, the teacher stressed that division of a negative rational number by a positive rational number yields a negative product. However, Teacher C did not provide further examples for the division of rational numbers and rushed to teaching the objective 'solving multi-step operations with rational numbers' indicating that there is not much time for covering all rational number ideas outlined by the middle school mathematics curriculum.

The examples used by Teacher D for teaching multiplication and division of rational numbers represented the following ideas: modeling multiplication of rational numbers, multiplication and division of rational numbers, calculating the square and cube of rational numbers and properties of multiplication of rational numbers. When examples used by Teacher D for teaching the above mentioned ideas were examined, it was seen that most of the examples included components that were structurally different from each other. These examples with different structural components are presented in Table 4.25. Additionally, the number of examples used by Teacher D was presented to get a better picture about the variability of those examples.

Table 4.25. Examples used by Teacher D for teaching multiplication and division of rational numbers

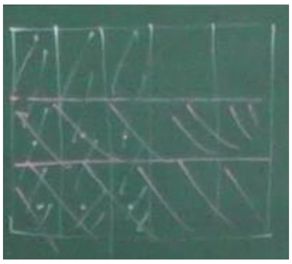
Ideas	Illustrative examples	Number of examples used
Modeling multiplication of rational numbers	 $\frac{3}{5}\cdot\frac{2}{3}$	4
Multiplication and division of rational numbers	$\frac{3}{8}\cdot\frac{7}{14}$; $\left(-\frac{3}{8}\right)\cdot\left(-\frac{7}{14}\right)$; $\left(-\frac{3}{8}\right)\cdot\left(\frac{7}{14}\right)$; $3\frac{1}{2}\cdot\frac{5}{8}$; $2\frac{3}{4}\cdot\left(-\frac{1}{2}\right)$; $4\cdot\frac{1}{2}\overset{?}{=}4\frac{1}{2}$; $3\cdot\frac{11}{8}$; $-4\cdot\frac{1}{7}$; $4\frac{3}{5}\cdot\left(-2\frac{4}{5}\right)$; $\frac{3}{5}:\frac{6}{7}$; $\left(-\frac{3}{5}\right):\frac{6}{7}$; $\left(-\frac{3}{5}\right):\left(-\frac{6}{7}\right)$; $5\frac{1}{2}:\frac{3}{5}$; $0.\bar{4}:1\frac{1}{9}$; $\frac{2.\bar{3}}{0.\bar{3}}$	15
Calculating the square and cube of rational numbers	$\left(\frac{1}{2}\right)^{2}$; $\left(\frac{1}{2}\right)^{3}$; $\left(-\frac{1}{2}\right)^{2}$; $\left(-\frac{1}{2}\right)^{3}$; $\left(-\frac{1}{2}\right)^{-1}$; $\left(-\frac{1}{2}\right)^{-2}$	6

Table 4.25. (Continued)

Ideas	Illustrative examples	Number of examples used
Properties of multiplication of rational numbers	Commutative property $\frac{1}{2}\cdot\frac{3}{5}=\frac{3}{5}\cdot\frac{1}{2}$ Associative property $-\frac{1}{2}\cdot\left[\frac{1}{5}\cdot\frac{2}{3}\right]=\left[-\frac{1}{2}\cdot\frac{1}{5}\right]\cdot\frac{2}{3}$ Zero property $\left(-\frac{1}{2}\right)\cdot 0$ Identity property $\left(+\frac{1}{2}\right)\cdot 1$ Distributive property $\frac{4}{7}\cdot\left[\frac{2}{3}+\frac{1}{5}\right]=\frac{4}{7}\cdot\frac{2}{3}+\frac{4}{7}\cdot\frac{1}{5}$; $-\frac{2}{3}\cdot\left[\frac{1}{4}-\frac{3}{5}\right]=\left(-\frac{2}{3}\right)\cdot\frac{1}{4}-\left(-\frac{2}{3}\right)\cdot\frac{3}{5}$ Multiplicative inverse property The multiplicative inverse of -1.2	7

Teacher D used 32 examples for teaching multiplication and division of rational numbers. Although the textbook emphasized modeling multiplication of rational numbers before introducing the multiplication algorithm, Teacher D started her lesson by providing examples that involved the use of this algorithm and she did not provide any model during the teaching of multiplication of rational numbers. However, she introduced several examples of this kind just before moving on to the teaching of division of rational numbers. More precisely, Teacher D provided her students a few modeling examples after teaching all ideas about multiplication of rational numbers. These examples were similar to those included in the textbook. Namely, each model included factors that are positive proper numbers. Actually, by these examples the teacher recalled modeling multiplication of fractions. Nevertheless, Teacher D did not provide any modeling example that included factors greater than one.

As stated above, Teacher D began teaching multiplication of rational numbers by introducing the traditional multiplication algorithm. When compared to the number of examples included in the textbook for the multiplication of rational numbers, the number of examples used by Teacher D for teaching this idea was approximately the same. However, while the textbook included more examples for the division of rational numbers, Teacher D provided more examples for the multiplication of rational numbers. Furthermore, the examples provided for multiplication included nearly all possible variations in terms of the sign and form of the factors. That is, these examples included either two positive factors, two negative factors or one negative and one positive factor and the factors were either proper numbers, improper numbers, integers or mixed numbers. More importantly, the teacher used $\frac{3}{8}\cdot\frac{7}{14}$; $\left(-\frac{3}{8}\right)\cdot\left(-\frac{7}{14}\right)$ and $\left(-\frac{3}{8}\right)\cdot\left(\frac{7}{14}\right)$ consecutively to draw students' attention to the role of the sign of factors on the sign of the product in multiplication of rational numbers. Besides, Teacher D emphasized that the rules for multiplication of integers are also valid for the multiplication of rational numbers and concluded that the product of two rational numbers with same signs will be positive while the product of two rational numbers with opposite signs will be negative. Finally, all but one of the examples entailed students to multiply the given two factors whereas one example checked whether students hold the misconception that $a\cdot\frac{b}{c}$ is equal to $a\frac{b}{c}$.

Similar to the textbook, Teacher D generated examples for the division of rational numbers after introducing all ideas related with multiplication of rational numbers. Like Teacher A, Teacher B and Teacher C, Teacher D started teaching division of rational numbers without using any models. That is, she immediately started with division examples that included the use of invert and multiply algorithm. Teacher D used less number of division examples when compared to the examples used by her for teaching multiplication of rational numbers. The division examples included some variability with respect to the sign and form of their terms. These examples included dividends that were either positive proper numbers, negative proper numbers, positive mixed numbers or positive repeating decimals. Identically, the divisors were either positive proper numbers, negative proper numbers, positive mixed numbers or positive repeating decimals. The examples used by Teacher D were similar to those included in the textbook for teaching division of rational numbers except for the ones that included repeating decimals as either a dividend or a divisor. More importantly, as she did while teaching multiplication of rational numbers, Teacher D introduced $\frac{3}{5}:\frac{6}{7}$, $\left(-\frac{3}{5}\right):\frac{6}{7}$ and $\left(-\frac{3}{5}\right):\left(-\frac{6}{7}\right)$ to the students respectively to draw their attention to the role of the sign of the dividend or the divisor on the sign of the quotient.

Although emphasized by the textbook, Teacher D did not provide examples for teaching the special cases of multiplication and division by 0, 1 and (-1), estimating multiplication and division of rational numbers, solving multi-step operations with rational numbers and modeling the square and cube of rational numbers. While Teacher D did not provide any example for modeling the square and cube of rational numbers, she provided examples for teaching how to calculate them. Teacher D used examples of this kind after providing examples for the division of rational numbers. The number of examples used by the teacher was less than the number of examples included in the textbook for teaching the square and cube of rational numbers. Besides, these examples included positive proper or negative proper numbers as bases and positive or negative integers as powers of the exponents. Thus, the examples used by Teacher D for teaching the square and cube of rational numbers were structurally similar to those included in the textbook. More specifically, Teacher D used $\left(\frac{1}{2}\right)^2$, $\left(\frac{1}{2}\right)^3$, $\left(-\frac{1}{2}\right)^2$ and $\left(-\frac{1}{2}\right)^3$ respectively to illustrate the second and third power of rational numbers. In the first example, Teacher D selected a positive base and a positive power. In the second example, the teacher kept the base invariant and changed the power. Later, Teacher D compared the two examples to explain the role of power on the magnitude of the exponents. In the third example, Teacher D selected a negative base and a positive power and in the fourth example, the teacher kept the base invariant and changed the power. By providing the third and the fourth examples consecutively, Teacher D stressed the role of odd power and even power on the sign and magnitude of an exponential number with a negative base. Different from the examples included in the textbook, Teacher D used two examples that included negative powers. These examples were generated by the teacher upon student inquiry. Similar to the previous examples, these examples included $\left(-\frac{1}{2}\right)$ as a base and two different negative integers as powers. The teacher pointed out that negative exponents would be covered next year and even so she briefly explained that negative power meant finding the multiplicative inverse of the base and then raising the multiplicative inverse of the base to the power regardless of its minus sign.

Similar to the textbook, Teacher D provided examples for teaching the properties of multiplication of rational numbers, after introducing examples that illustrate the traditional multiplication algorithm. Teacher D initially provided an example for the commutative property of multiplication of rational numbers. In this example, the teacher selected positive proper numbers to illustrate the commutative property. More specifically, the teacher used this example as a basis for justifying that the commutative property holds for every rational number. Next, she used another example to illustrate the associative property of multiplication of rational numbers. This example included a negative proper number, and two positive proper numbers and the teacher calculated both sides of the equality to justify that associative property holds for every rational numbers. To teach zero property of multiplication, Teacher D selected a negative proper number as a companion to 0 and to teach identity property of multiplication she selected a positive proper number as a companion to 1.

Last, Teacher D presented two examples for teaching distributive property of multiplication over addition and subtraction. First, the teacher used an example to teach the distributive property of multiplication over addition. By this example, the teacher showed how to distribute a positive proper number over the addition of two proper numbers. Besides, this example was used as a basis for justifying that the product obtained by using the distributive property is equal to the product obtained by taking account of the order or operations. The other example used by Teacher D illustrated the distributive property of multiplication over subtraction. In this second example, the teacher showed how to distribute a negative proper number over the subtraction of a positive proper number from another positive proper number. Similar to the previous example, the teacher justified that the product obtained by using the distributive property is equal to the product obtained by taking account of the order or operations.

Although emphasized by the middle school mathematics curriculum and the textbook, Teacher D did not provide any example for the multiplicative inverse property in the course of teaching properties of multiplication of rational numbers, she provided one example for this property when working out exercise examples. Unlike the other teachers, she selected a negative decimal number to show the multiplicative inverse of that number. However, when finding the multiplicative inverse used the term ‘flip over’ rather than emphasizing that ‘two numbers whose product is 1 are multiplicative inverses of one another’.

4.1.1.6. Examples used for performing multi-step operations with rational numbers

In the middle school mathematics curriculum, teachers were suggested to use grouping symbols such as parenthesis, brackets and so forth to determine the order of operations included in a mathematical expression. In addition to this, the teachers were suggested to emphasize that the order of operations in complex fractions were determined according to the main fraction bar. However, the middle school mathematics curriculum did not provide any specific example or activity for solving multi-step operations with rational numbers.

In the mathematics textbook followed by the classrooms, the examples provided for teaching multi-step operations with rational numbers were classified as follows: solving multi-step operations that are expressed on one line, solving multi-step operations that are expressed as complex fractions and solving multi-step operations that are expressed as a continuing pattern. The illustrative examples and the total number of examples for each category are presented in Table 4. 26.

Table 4.26. Examples included in the textbook for teaching multi-step operations with rational numbers

Ideas	Illustrative examples	Number of examples used
Multi-step operations that are expressed on one line	$\frac{1}{2}+\frac{1}{3}\cdot\frac{1}{4}-\frac{1}{5}$; $-\frac{2}{3}-\frac{5}{6}:\frac{1}{4}+\frac{1}{3}$; $\left(-\frac{5}{7}\right):\left[\left(1-\frac{6}{7}\right)\cdot\left(1+\frac{6}{7}\right)\right]$; $\frac{1}{3}\cdot\left[\frac{3}{4}+\frac{2}{3}:\left(-\frac{3}{2}\right)\right]$; $\left[3\cdot\left(1+\frac{5}{3}\right):\frac{1}{7}\cdot\left(\frac{1}{4}+\frac{1}{3}\right)\right]:6$	7
Multi-step operations that are expressed as complex fractions	$\frac{\frac{2}{3}+\frac{3}{2}}{\frac{3}{2}-\frac{2}{3}}+\frac{1}{18}$; $\frac{1-\frac{3}{5}}{\frac{3}{5}-1}:5$; $2+\frac{3}{3+\frac{1}{2+\frac{1}{3}}}$; $\frac{2}{3}-\frac{1}{\frac{1}{3}+\frac{1}{9}}$	7
Multi-step operations that are expressed as a continuing pattern	$\left(1+\frac{1}{2}\right)\cdot\left(1+\frac{1}{3}\right)\cdot\left(1+\frac{1}{4}\right)\cdots\left(1+\frac{1}{9}\right)$	1

The textbook included 15 examples for teaching multi-step operations with rational numbers. The first group of multi-step operation examples included in the textbook was formed by expressing rational numbers on one line either by using parentheses and brackets as grouping symbols or without using grouping symbols. In the examples with no grouping symbols, either addition, multiplication and subtraction or subtraction, division and addition occured from left to right. These examples entailed using the correct order of operations in order to find the values of expressions correctly. In addition, these examples were generated mostly by using positive proper numbers and occasionally by integers or negative proper numbers. The textbook included more examples with grouping symbols when compared to those with no grouping symbols. Examples with grouping symbols entailed performing the operations within the grouping symbols first and similar to the examples with no grouping symbols, they were mostly formed by using positive proper numbers and occasionally by using negative improper numbers or integers.

The second group of multi-step operation examples were complex fractions since these kinds of examples included fractions either in the numerator, or in the denominator or both in the numerator and in the denominator. In complex fractions, the main fraction bar was a type of grouping symbol. Therefore, the order of operations in complex fractions was determined by considering the position of the main fraction bar. In this group of examples, the numerators over the main fraction bars were in the following forms: a positive proper number plus a positive improper number, an integer minus a positive proper number, or only an integer. Similarly, the denominators under the main fraction bars were in the following forms: a positive improper number minus a positive proper number, a positive proper number minus an integer and a positive proper number plus a positive proper number. Above all, the number of operations included in the complex fraction examples of the textbook ranged between three and five. This might be indicative of the complexity of the examples included in this category.

Finally, the example with a continuing pattern formed another category of multi-step operations. This example was very different in nature from the other multi-step operation examples, since it entailed discerning the pattern among consecutive factors and performing cross simplifications without necessarily writing down all factors. More specifically, each factor included the addition of one whole with a positive proper number and the difference between the denominators of the two consecutive proper numbers was always equal to 1.

The examples used by Teacher A for teaching multi-step operations with rational numbers were classified as follows: multi-step operations that are expressed on one line, multi-step operations that are expressed as complex fractions and multi-step operations that are expressed as a continuing pattern. The illustrative examples and the total number of examples for each category are presented in Table 4. 27.

Table 4.27. Examples used by Teacher A for teaching multi-step operations with rational numbers

Ideas	Illustrative examples	Number of examples used
Multi-step operations that are expressed on one line	$\left(\frac{2}{3}-\frac{1}{2}\right)+\left(\frac{2}{3}:\frac{1}{2}\right)$; $\left(1-\frac{1}{5}\right)\cdot\left(1-\frac{1}{6}\right)\cdot\left(1-\frac{1}{7}\right)\cdot\left(1-\frac{1}{8}\right)$; $\left(-\frac{3}{4}+2\right)^2:\left(-\frac{2}{3}\right)^0$; $(0.01+0.09)\cdot(0.473+0.527)$; $\frac{1}{17}\cdot\frac{28}{13}-\frac{1}{17}\cdot\frac{11}{13}$; $\left(\frac{1}{2}-\frac{1}{3}\right):\frac{1}{4}+\frac{1}{3}$; $1-\frac{2}{3}\cdot\left(1-\frac{1}{2}\right)$; $K=\left(8-\frac{1}{4}\right):\frac{1}{8}, L=8-\frac{1}{4}:\frac{1}{8}, K-L=?$	8
Multi-step operations that are expressed as complex fractions	$3+\cfrac{1}{1-\cfrac{1}{3}}$; $1-\cfrac{1}{1-\cfrac{1}{1-\cfrac{1}{2}}}$; $\cfrac{3\frac{1}{2}}{1\frac{1}{6}}$; $\cfrac{\frac{1}{2}}{\frac{2}{5}}-\cfrac{\frac{1}{2}}{5}$; $\cfrac{0.2\bar{5}-0.1\bar{4}}{\cfrac{0.02}{0.06}}$; $3-\cfrac{2}{5-\cfrac{1}{x}}=1$; $\frac{0.35}{0.05}+\frac{0.7}{0.0035}-\frac{0.22}{0.0011}$; $A=\frac{1}{2}+\frac{1}{3}+\frac{1}{4}, K=\frac{1}{2}-\frac{1}{3}-\frac{1}{4}, L=\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}, \frac{K+A}{L}=?$	13
Multi-step operations that are expressed as a continuing pattern	$\underbrace{\frac{2}{5}-\frac{1}{2}+\frac{2}{5}-\frac{1}{2}+\cdots+\frac{2}{5}-\frac{1}{2}}_{\text{There are 20 terms}}$; $\frac{1}{10}+\frac{2}{100}+\frac{1}{1000}+\frac{2}{10000}+\cdots$	2

Teacher A used 23 examples for teaching multi-step operations with rational numbers. The first group of multi-step operation examples used by the teacher was formed by expressing rational numbers on one line either by using parentheses as grouping symbols or without using grouping symbols. The examples used by the teacher for this group were almost similar to the ones included in the textbook. Different from the textbook examples, Teacher A used examples that included exponents or decimal numbers as the components of the multi-step operations. The teacher also used several examples with no grouping symbols. In these examples, generally subtraction operation occurred before division or multiplication operations. Thus, these examples required performing operations in the correct order so as to evaluate the expressions accurately. The examples with or without grouping symbols generally included positive proper numbers or integers, and rarely included integers.

The second group of multi-step operation examples used by Teacher A was complex fractions. The complex fractions used by the teacher included more variability when compared to the complex fractions included in the textbook. First, the textbook included complex fraction examples that had either proper number, integer, or mixed number components. However, the complex fractions used by Teacher A also included repeating decimals and terminating decimals either in the numerator or in the denominator. Second, some of the complex fractions used by the teacher included an unknown value either in the numerator or in the denominator and there was not any example of this kind in the textbook. Last, in some of the examples, complex fractions were not explicitly expressed. Instead, these examples entailed constructing complex fractions before calculating them. To conclude, the complex fractions used by Teacher A were more sophisticated than the complex fractions provided by the textbook since teacher generated examples included more number of operations and more variety in number forms.

Finally, Teacher A used two different examples that included a continuing pattern and these examples formed the last category of multi-step operations. In the first example, there was a recurring subtraction operation and this example entailed finding the number of subtraction operations rather than writing down each term to find the answer. In the second example, a repeating decimal was expressed as an infinite series. That is, the repeating decimal was regarded as the sum of an infinite number of rational numbers. This example entailed expressing each rational number as a decimal number and performing column addition to discern the repeating pattern of digits.

The examples used by Teacher B for teaching multi-step operations with rational numbers were classified as follows: multi-step operations that are expressed on one line and multi-step operations that are expressed as complex fractions. The illustrative examples and the total number of examples for each category are presented in Table 4.28.

Table 4.28. Examples used by Teacher B for teaching multi-step operations with rational numbers

Ideas	Illustrative examples	Number of examples used
Multi-step operations that are expressed on one line	$\left(2-\frac{1}{3}\right):\left(3+\frac{1}{4}\right)$; $\frac{1}{8}-\left(\frac{1}{8}-\frac{1}{4}\right)\cdot\frac{1}{2}$; $\frac{3}{7}\cdot\left(7+\frac{1}{3}\right)-3$; $\frac{1}{4}\cdot\frac{1}{9}+\frac{7}{9}:\frac{1}{9}-\left(1-\frac{1}{10}\right)$	4
Multi-step operations that are expressed as complex fractions	$1-\dfrac{2}{3+\dfrac{1}{1+\dfrac{1}{2}}}$; $\dfrac{\frac{1}{2}+3}{3}:\left(\frac{1}{2}-\frac{1}{3}\right)$; $\dfrac{1-\frac{4}{5}}{\dfrac{1-\frac{3}{5}}{2}-\frac{6}{5}}$; $\dfrac{2}{3-\dfrac{7}{x+1}}=1$; $\dfrac{2}{\dfrac{6}{x-1}-5}=-1$; $3+\dfrac{5}{4+\dfrac{6}{x+2}}=4$; $a=\dfrac{4}{1-\frac{2}{3}}, b=\dfrac{\frac{a}{2}+\frac{a}{12}}{7}$ then find b; $A=\dfrac{\frac{1}{2}}{\frac{3}{4}}, B=\dfrac{1}{\dfrac{2}{\frac{3}{4}}}$ then compare A and B	9

Teacher B used 13 examples for teaching multi-step operations with rational numbers. This number is close to the number of examples included in the textbook for teaching multi-step operations. Unlike the textbook, the multi-step operation examples that were expressed on one line were all formed by using parentheses as grouping symbols. This group of examples was formed by using positive proper numbers and positive integers as components of the multi-step operations and the number of operations included in this group of examples ranged between three and five. By using these examples, the teacher emphasized the priority of operations within the grouping symbols and also the priority of multiplication and division operations over addition and subtraction operations. In the meantime, he explicitly explained the rule for the order of operations in order to evaluate the given expressions correctly.

The second group of examples used by Teacher B was formed by expressing multi-step operations in the form of complex fractions. This group of examples used by the teacher included more variations in terms of their structural components when compared to textbook examples. As mentioned before, the complex fractions provided by the textbook included proper number or integer components in the numerators or in the denominators. Thus, these complex fractions included components that were all rational numbers. However, the complex fractions provided by Teacher B included unknown values in the denominators or numerators in addition to complex fractions that are constructed entirely by rational numbers. Besides, the number of operations included in this group of examples provided by the teacher ranged between four and six. Thus, complex fraction examples used by Teacher B were more sophisticated when compared to the same type of examples included in the textbook.

The examples used by Teacher C for teaching multi-step operations with rational numbers were categorized as follows: multi-step operations that are expressed on one line and multi-step operations that are expressed as complex fractions. The illustrative examples and the total number of examples for each category are presented in Table 4. 29.

Table 4.29. Examples used by Teacher C for teaching multi-step operations with rational numbers

Ideas	Illustrative examples	Number of examples used
Multi-step operations that are expressed on one line	$\frac{1}{4}:\left[\frac{1}{2}-\frac{3}{5}\right]+9$	1
Multi-step operations that are expressed as complex fractions	$\frac{\frac{15}{5}+3}{2}+1$; $\frac{\frac{4}{9}+\frac{5}{9}+2}{1+\frac{1}{2}}$; $\frac{2-\frac{1}{1-\frac{3}{2}}}{\frac{1}{2}-\frac{1}{3}}$; $\frac{1}{2}+\frac{\left[\frac{3}{4}-\frac{7}{5}\right]:\frac{4}{5}}{\frac{1}{2}-\frac{5}{8}}$; $\frac{\left(1+\frac{1}{4}\right)\cdot\left(1+\frac{1}{5}\right)\cdot\left(1+\frac{1}{6}\right)}{\left(1-\frac{1}{2}\right)\cdot\left(1-\frac{1}{3}\right)\cdot\left(1-\frac{1}{4}\right)}$	5

Teacher C used 6 examples for teaching multi-step operations with rational numbers. The number of multi-step operation examples provided by Teacher C was very few when compared to those included in the textbook. In addition, the teacher used only one example to illustrate multi-step operations that are expressed on one line. More specifically, Teacher C used a multi-step operation example that was formed by using brackets as a grouping symbol and it included positive proper numbers and a positive integer as components. Besides, in this example, division, subtraction and addition examples occurred from left to right respectively. However, the subtraction operation was within parenthesis. Therefore, this example entailed performing subtraction operation initially. On the other hand, although the textbook provided multi-step operation examples without grouping symbols, Teacher C did not provide any example of this kind.

The second group of examples used by Teacher C was multi-step operations that were in the form of complex fractions. This group of examples used by the teacher was similar to the complex fractions provided by the textbook in terms of the form and sign of the components. That is, the complex fractions used by the teacher were formed by using positive proper numbers, positive improper numbers and positive integers. Thus, these complex fractions used by the teacher included components that were all rational numbers. Additionally, the complex fractions used by Teacher C were more sophisticated when compared to the textbook examples since the number of operations included in Teacher C's examples was greater than the number of operations included in the textbook examples.

The examples used by Teacher D for teaching multi-step operations with rational numbers were categorized as follows: multi-step operations that are expressed on one line, multi-step operations that are expressed as complex fractions and multi-step operations that are expressed as single variable polynomials. The illustrative examples and the total number of examples for each category are presented in Table 4. 30.

Table 4.30. Examples used by Teacher D for teaching multi-step operations with rational numbers

Ideas	Illustrative examples	Number of examples used
Multi-step operations that are expressed on one line	$\left(-\frac{1}{2}\right)^3 : \left(\frac{1}{2}\right)^2 + \frac{1}{2}$	1
Multi-step operations that are expressed as complex fractions	$\frac{\left(1-\frac{3}{2}\right)\cdot\left(1+\frac{1}{3}\right)-1}{2.\frac{2}{3}}$; $\frac{2}{1+\frac{1}{1+\frac{1}{3}}}$; $\frac{2-\frac{1}{2-\frac{1}{2}}}{2+\frac{1}{2+\frac{1}{2}}}$; $\frac{\frac{1-\frac{1}{3}}{2}}{1+\frac{1}{3}}$; $\frac{0.012}{0.3}-\frac{2}{0.8}+\frac{0.4}{0.02}$; $\frac{0.\bar{2}+2}{3-0.\bar{7}}$; $\frac{24}{\frac{x}{8}+6}=4$; $\frac{6}{1+\frac{5}{a}}=1$; $\frac{10}{4-\frac{10}{\frac{x}{5}-4}}=5$	10

Table 4.30. (Continued)

Ideas	Illustrative examples	Number of examples used
Multi-step operations expressed as single variable polynomials	The value of $\dfrac{x^3 - x^2 + 10}{3}$ for $x = -2$; The value of $\dfrac{5x}{3} - x + \dfrac{3}{4}x$ for $x = -\dfrac{3}{4}$	2

Teacher D used 13 examples for teaching multi-step operations with rational numbers. The total number of examples used by Teacher D was close to the number of examples included in the textbook for teaching this idea. However, these examples were rather unevenly distributed among three categories. More specifically, Teacher D used only one example for illustrating multi-step operations that were expressed on one line. This example was constructed without using any grouping symbol and it entailed dividing the cube of a negative proper number by the square of a positive proper number and then adding the same positive proper number. This example was different from the textbook examples that were expressed on one line, since the textbook examples did not include components in exponential form. Last, Teacher D did not provide any multi-step operation example with grouping symbols although there were examples of this kind in the textbook.

The second group of examples used by Teacher D was multi-step operations that were in the form of complex fractions. This group of examples used by the teacher included much more variations in proportion to the textbook examples. More precisely, the textbook examples included positive proper numbers, positive improper numbers and positive integers as components while the examples used by Teacher D included terminating decimals, repeating decimals and unknown values apart from positive proper numbers, positive improper numbers and positive integers. Besides, the number of operations within each complex fraction used by the teacher ranged between three and seven. Thus, the complex fractions used by Teacher D were more sophisticated when compared to the complex fractions included in the textbook.

Unlike the textbook and the previous three teachers, Teacher D used two examples which required substitution of rational numbers into single variable polynomials. Thus, these two examples formed the last category of teaching multi-step operations with rational numbers. When working out the first example of this category, the teacher pointed to a possible student error. That is, she warned her students not to forget enclosing the base (i.e., -2) in parenthesis before calculating the square or cube of it. Similarly, the second example of this category required enclosing the rational number (i.e., $-\frac{3}{4}$) in parenthesis before performing operations in order not to a make an error. In addition, Teacher D explicitly suggested several solution strategies that might be used to keep unnecessary work to minimum when working out the value of the given polynomial in this second example.

4.1.1.7. Examples used for posing and solving rational number problems

According to the middle school mathematics curriculum, students were expected to read the problems very carefully, restate the problems with their own words, identify the givens in a problem, make a plan (deciding on the problem solving strategy), carry out the plan, check the solution and finally discuss the problem with the classmates. Besides, teachers were suggested to pay attention to the explanations that are included at the introductory part of the curriculum book for developing good problem solving skills. However, the middle school mathematics curriculum did not provide any specific example or activity to illustrate how to pose or solve rational number problems.

In the mathematics textbook followed by the classrooms, the examples used for the illustration of posing and solving rational number problems were classified as follows: solving rational number problems with same referent units, solving rational number problems with different referent units and posing rational number problems. The illustrative examples and the total number of examples for each idea are presented in Table 4.31.

Table 4.31. Examples used by the textbook for teaching how to pose and solve rational number problems

Ideas	Illustrative examples	Number of examples used
Solving rational number problems with same referent units	The ratio of the length of a side of one square to that of another square is $\frac{3}{4}$. Then, calculate the ratio of their perimeters and areas.	1
Solving rational number problems with different referent units	Ahmet initially walked $\frac{1}{3}$ of his route. After some time, he walked $\frac{2}{5}$ of the remaining route and he had to walk 36 meters more to finish his route. Find the total length of his route.	5
Posing rational number problems	• Dilek and Tolga dropped a rubber ball from a specific height onto a concrete floor. Each time the ball hit the floor, it bounced back up to a height $\frac{2}{3}$ of the height from which it fell. Pose a rational number problem by using the given data. • Pose a rational number problem by using the words 'farm' and 'hoe' and solve this problem by using the problem solving steps. • An athlete each day runs … times as much as the distance she runs the day before. The athlete runs for … days and finishes her training program. If the athlete runs … kilometers in her … day, then how many kilometers does she run on the last day of her training program? Fill in the blanks with the numbers relevant to the problem.	3

The textbook included 9 examples for teaching problem posing and solving with rational numbers. The textbook included one example that illustrated problem solving with same referent units. In this example, the length of the one side of the larger square corresponded to the referent whole 1 unit, while the length of the one side of the smaller square corresponded to $\frac{3}{4}$ of the same referent whole. Therefore, the numbers 1 and $\frac{3}{4}$ both referred to the same referent unit. The examples included in the textbook to illustrate problem solving with different referent units were more common when compared to the examples included to illustrate problem solving with same referent units. In the second group of rational number problems, the numbers referred to different referent units. For instance, in the route problem presented in Table 4.31, the numbers $\frac{1}{3}$ and 36 referred to the same referent unit, while $\frac{2}{5}$ referred to a different referent unit.

In addition to providing examples regarding problem solving with rational numbers, the textbook included three different problem posing examples. In the rubber ball example, the givens of the problem were explained and the students were expected to pose a problem relevant to the givens. In the farm and hoe example, only the theme of the real life problem was explained and the students were expected to generate the rational numbers themselves and pose a relevant problem by using the generated numbers. Finally, in the athlete example, the whole problem was explained without specifying the numbers and the students were expected to fill in the blanks by using relevant rational numbers. As it can be seen, the problem posing examples included in the textbook were all structurally different from each other.

The examples used by Teacher A for illustrating problem solving with rational numbers were classified into two main ideas as solving rational number problems with same referent units and solving rational number problems with different referent units. The illustrative examples and the total number of examples for these two ideas are presented in Table 4.32.

Table 4.32. Examples used by Teacher A for teaching how to solve rational number problems

Ideas	Illustrative examples	Number of examples used
Solving rational number problems with same referent units	On Monday, Ali spent $\frac{1}{4}$ of his pocket money. The next day, he spent $\frac{2}{3}$ of his pocket money and he had 21 TLs left. How much pocket money did he have at the beginning?	4
Solving rational number problems with different referent units	On Monday, Ali spent $\frac{1}{4}$ of his pocket money. The next day, he spent $\frac{2}{3}$ of his remaining pocket money and he had 21 TLs left. How much pocket money did he have at the beginning?	2

Teacher A used 6 examples for teaching problem solving with rational numbers. Unlike the textbook, the teacher provided more examples with same referent units when compared to the examples with different referent units. In the pocket money example presented above, the numbers $\frac{1}{4}, \frac{2}{3}$ and 21 all referred to the same referent unit. That is, these numbers all referred to the total amount of the pocket money. To have students discern the difference between the problem with same referent units and the problem with different referent units, Teacher A completely used the same pocket money example and added the word 'remaining' to the latter example. More specifically, in this latter example, the numbers $\frac{1}{4}$ and 21 referred to the same referent unit, while $\frac{2}{3}$ referred to a different referent unit.

Finally, although the middle school mathematics curriculum and the textbook included problem posing examples, Teacher A did not provide any example of this kind to his students in the course of teaching this idea or in the course of providing exercise examples.

Similar to the Teacher A, the examples used by Teacher B for illustrating problem solving with rational numbers were classified into two as solving rational number problems with same referent units and solving rational number problems with different referent units. The illustrative examples and the total number of examples for each idea are presented in Table 4.33.

Table 4.33. Examples used by Teacher B for teaching how to solve rational number problems

Ideas	Illustrative examples	Number of examples used
Solving rational number problems with same referent units	A man first travelled $\frac{1}{10}$ of his route. Next, he travelled $\frac{1}{5}$ of his route and thus he travelled a distance of 60 kilometers in total. Then, find the total length of his route.	3
Solving rational number problems with different referent units	One day, Zeynep spent $\frac{1}{5}$ of her money. The other day, she spent $\frac{1}{2}$ of her remaining money and she spent 36 TLs in total. Then, how much money does she still have?	3

Teacher B used 6 examples for teaching problem solving with rational numbers. Unlike the textbook, the examples provided by Teacher B were evenly distributed to the two categories. In the travel example given above, the numbers $\frac{1}{10}, \frac{1}{5}$ and 60 all referred to the same referent unit. As opposed to Teacher A, Teacher B used different problem situations when providing examples for problem solving with same and different referent units. Thus, it was not possible for the students to readily notice the problem structure in two different categories. For instance, Teacher B provided an example with different referent units in the context of money. In this example, $\frac{1}{5}$ and 36 referred to the same referent unit, while $\frac{1}{2}$ referred to a different referent unit.

Finally, although the middle school mathematics curriculum and the textbook included problem posing examples, Teacher B did not provide any problem posing example to his students either in the course of teaching this idea or in the course of providing exercise examples.

Identical to the previous two teachers, the examples provided by Teacher C for teaching how to solve rational number problems were categorized into two as solving rational number problems with same referent units and solving rational number problems with different referent units. The illustrative examples and the total number of examples for each idea are presented in Table 4.34.

Table 4.34. Examples used by Teacher C for teaching how to solve rational number problems

Ideas	Illustrative examples	Number of examples used
Solving rational number problems with same referent units	Kağan travelled $\frac{2}{5}$ of his route and he had 350 meters left. Find the total length of his route in kilometers.	2
Solving rational number problems with different referent units	Ali bought a book with $\frac{1}{2}$ of his money. Next, he spent $\frac{1}{3}$ of his remaining money on cinema tickets. Finally, he bought some snacks with the quarter of the money left over from the book and cinema tickets. After buying snacks, he had 40 TLs left. Then, how much money did he have at the beginning?	2

Teacher C used quite a few examples for teaching problem solving with rational numbers when compared to the number of examples included in the textbook for introducing this idea. More precisely, Teacher C used 4 examples for teaching problem solving with rational numbers. Like Teacher B, Teacher C provided same number of examples for problems with same referent units as she provided for problems with different referent units. In the travel example presented in Table 4.34, the numbers $\frac{2}{5}$ and 350 referred to the same referent unit. Teacher C used different problem contexts for each problem solving example either with same referent units or with different referent units. For instance, one of the examples provided by Teacher C was in the context of money. As given above, this example included different referent units. Namely, in this example, $\frac{1}{2}$ and 40 referred to the same referent whole while $\frac{1}{3}$ and $\frac{1}{4}$ referred to two different referent units. Last, despite it was articulated by the middle school mathematics curriculum and the textbook, Teacher C did not provide any problem posing example to his students.

Same as Teacher A, Teacher B and Teacher C, the examples used by Teacher D for teaching how to solve rational number problems were grouped under two ideas: solving rational number problems with same referent units and solving rational number problems with different referent units. The illustrative examples and the total number of examples for these two ideas are presented in Table 4.35.

Table 4.35. Examples used by Teacher D for teaching how to solve rational number problems

Ideas	Illustrative examples	Number of examples used
Solving rational number problems with same referent units	A man first travelled $\frac{6}{10}$ of his route. If he had travelled 150 meters more, then he would have travelled $\frac{2}{3}$ of the total route. Then, find the initial distance travelled by the man.	5
Solving rational number problems with different referent units	A grocer initially sold $\frac{2}{3}$ of a bag of sugar. Later, he sold $\frac{1}{4}$ of the remaining sugar. Finally, the grocer weighed the rest of the sugar and realized that 12 kilograms of sugar was left over. Then, how many kilograms of sugar did the bag contain at the beginning?	4

The number of examples provided by Teacher D for teaching problem solving with rational numbers was the same as the number of examples included in the textbook for introducing this idea. To be more specific, Teacher D used 9 examples for teaching problem solving with rational numbers. Teacher D provided approximately the same number of examples for teaching problem solving with the same referent units when compared to the number of examples provided by her for teaching problem solving with different referent units. Teacher D initially started teaching problem solving with same referent units. The travel example presented in Table 4.35 was used as a start-up example by her. In this example, the numbers $\frac{6}{10}, \frac{2}{3}$ and 150 all referred to the same referent whole. That is, these numbers all referred to the total distance of the route travelled by the man. Teacher D used different problem contexts for each example either with referent units or with different referent units. For instance, the grocer example was provided by Teacher D to illustrate problem solving with different referent units. In this example, $\frac{2}{3}$ and $\frac{1}{4}$ referred to different referent units.

Finally, like Teacher A, Teacher B and Teacher C, Teacher D did not provide any problem posing example to her students despite it was articulated by the middle school mathematics curriculum and by the textbook.

4.1.2. Non-examples

The rational number examples that were categorized as non-examples were used by the four middle school mathematics teachers to show that not all numbers are rational. These non-examples were mostly generated by the teachers in the course of teaching the objective ‘explain and locate rational numbers on a number line’. The variety of non-examples included in the textbook and the non-examples used by the four middle school mathematics teachers in the course of teaching rational number ideas are presented in Table 4. 36.

Table 4.36. The non-examples provided by the textbook and the teachers for teaching rational number ideas

Form of non-example	Textbook	Teacher A	Teacher B	Teacher C	Teacher D
Ratio of integers to zero	-	$\frac{1}{0}, \frac{2}{0}, \frac{5}{0}$	-	$\frac{7}{0}$	$\frac{3}{0}$
Transcendental numbers	-	π	-	-	π
Radicals	-	-	$\sqrt{5}$	-	-
Infinite non-repeating decimals	-	-	-	-	0.257843…

As it can be seen in Table 4.36, the mathematics textbook followed by the four classrooms did not provide any non-example while explaining or illustrating rational number concepts. In contrast to this, all middle school mathematics teachers provided non-examples for rational numbers. Except for Teacher B, all teachers provided the definition of rational numbers as numbers that can be represented as $\frac{a}{b}$, where *a* is an integer and *b* is a non-zero integer. After providing this definition, the teachers presented rational number examples that were written as a ratio of two integers. In most cases, the examples that included zero as a numerator of the rational number were accompanied with the non-examples that included zero as the denominator. For instance, Teacher A made the following explanation to his classroom to point to the difference between $\frac{0}{1}$ and $\frac{1}{0}$.

> Teacher A: Counting number set begins with 1 and goes to infinity. Similarly, natural number set begins with 0 and goes to infinity. At the beginning of this year, we learnt a new number set. We named this new number set as integers. The set of integers are denoted as $\mathbb{Z} = \{-\infty, ..., -2, -1, 0, +1, +2, ..., +\infty\}$. Integers are formed by whole points on a number line and start with $-\infty$ and go to $+\infty$. We said that numbers between $-\infty$ and 0 are negative integers while numbers between 0 and $+\infty$ are positive integers. We did not say that 0 is negative or positive because it is a neutral number. Besides, 0 has an additive inverse but its multiplicative inverse is undefined. By the way, note that $\frac{0}{1}$ is equal to 0

while $\frac{1}{0}$ is undefined. Now, I will introduce you rational numbers as a new number set. You choose two numbers from integer set as *a* and *b*. However, *b* should be different from 0 since $\frac{1}{0}$ is undefined. Therefore, *b* should not be equal to 0. If $\frac{1}{0}$ is undefined then it cannot be an element of rational number set. Then, numbers in the form of $\frac{a}{b}$ where $b \neq 0$ are elements of rational number set.

This excerpt shows that Teacher A provided $\frac{0}{1}$ as an example for rational numbers. Soon after this, he provided $\frac{1}{0}$ as a non-example for rational numbers. Teacher A provided ratio pairs such as $\frac{0}{1}$ and $\frac{1}{0}$ not only in the course of explaining and locating rational numbers but also during the teaching of other rational number ideas. He did it from time to time to recall that ratios with zero numerators are examples of rational number set while ratios with zero denominators are non-examples of rational number set.

Apart from using ratio representation for providing non-examples for rational numbers, Teacher A and Teacher D presented pi number (π), a specific transcendental number, as another non-example for rational numbers. Teacher A presented this number to his students while reviewing the definition of rational number set that he taught in the previous lessons. The teaching episode of Teacher A regarding this non-example is given below.

Teacher A: In our previous lesson, we defined rational numbers. We denoted this set by the symbol $\mathbb{Q}$. All numbers that can be written as common fractions were called rational numbers. We wrote a note on the board that $\frac{2}{0}$ is undefined while $\frac{0}{-7}$ is equal to 0. Here, we wrote -7 as a denominator of the fraction to show that any integer can be written under the fraction bar except for 0. We defined rational numbers in this way. Well, do you know pi number?

Student 1: I remember it, but I do not exactly know what it is.

Teacher A: Pi number goes to infinity as 3.14… Today, the decimal representation of pi has been computed to include many digits that can wrap the circumference of the earth forty times but it is still being computed. That is, the ratio of a circle's circumference to its diameter goes to infinity and it is called the pi number. What lesson should we take from this? (At this time, the teacher is pointing to a bottle cap that is circular) We can create this bottle cap, but we cannot calculate the ratio of its circumference to its diameter. That is why, I refer to this number as God's number. To repeat again, we can create this bottle cap, but we cannot calculate the ratio of its circumference to its diameter exactly. This ratio proceeds as 3.14… but we cannot express it as a common fraction. Why? Because we do not know its end.

Student 2: That is a repeating decimal!

Teacher A: It is not a repeating decimal. It is something else. If we do not know the final digits of the decimal number, then we cannot write it as a common fraction. Hence, if I cannot write it as a common fraction then it is not an element of rational number set ($\pi \notin \mathbb{Q}$). I introduced you the pi number to illustrate that there are numbers that are not examples of rational number set. I will teach you another mathematical topic involving numbers that are not rational next year. At that time, you will probably remember the above mentioned anecdote.

As can be understood from the episode given above, Teacher A used pi number as a non-example for rational numbers. Besides, he emphasized that the decimal expansion of this number includes infinite number of digits after the decimal point. However, although one of the students suggested that repeating decimals have infinite number of digits after the decimal point, Teacher A did not provide opportunities for students to distinguish infinite repeating decimals that are rational and infinite non-repeating decimals that are irrational.

In addition to using pi number as a non-example for rational numbers, Teacher A indicated that there is also another mathematical topic that includes numbers which are not rational. To elucidate what this mathematical topic is, I conducted post-lesson interviews with the teacher. He explicitly stated that radicals will be taught the next year. Although Teacher A expressed that radicals involve numbers that are not rational, he did not provide any specific non-example related with radicals. Similar to Teacher A, Teacher D used the same transcendental number, pi number, as one kind of non-example for rational numbers. Apart from this, she used an infinite non-repeating decimal number as a non-example for rational numbers. Teacher D provided these two different kinds of non-examples during the teaching of expressing rational numbers in different forms. The verbatim transcripts of this lesson episode are given below.

Teacher D: Each integer can be rewritten as a rational number. For instance, $-2 = \frac{-2}{1}, 3 = \frac{3}{1}, -5 = \frac{-5}{17}$ and $17 = \frac{17}{1}$.Thus, -2, 3, -5 and 17 are all rational numbers. In a similar way, there some other numbers which can be expressed as rational numbers. To give an example, -0.5, 1.25 and 0.15 can be expressed as $-\frac{5}{10}, \frac{125}{100}$ and $\frac{15}{100}$ respectively. Thus, we can say that some decimal numbers are rational numbers.

Student: Teacher, what do you mean by saying ‘some decimal numbers’?

Teacher D: I mean that not all decimal numbers are rational by saying ‘some decimal number’. To be more precise, I mean that not all numbers including infinite number of digits after the decimal point are rational numbers. Thus, infinite decimals can be classified into two as infinite repeating decimals and infinite non-repeating decimals. For instance, 0.257843… is an infinite non-repeating decimal since it does not have a regular repeating pattern. As you can see, there are some decimals which include infinite number of digits but not include a repeating pattern. Since these numbers do not have regular repeating patterns, they cannot be written as common fractions. Finally, since we cannot write them as common fractions, they cannot be accepted as rational numbers.

As the episode given above shows, Teacher D used an infinite non-repeating decimal representation (such as, 0.257843…) as a non-example for rational numbers. Unlike Teacher A, Teacher B and Teacher C, Teacher D generated this non-example as a transparent representation of an irrational number. That is, Teacher D generated this non-example in a way that makes it possible to derive the irrationality of the number from this representation. Apart from this, Teacher D generated another non-example for rational numbers, the pi number, in the course of expressing repeating decimals as common fractions upon student inquiry. This teaching episode is given below.

Teacher D: Each repeating decimal can be expressed as a common fraction. Thus, we can say that each repeating decimal is a rational number. In this case, if I ask you to determine whether 0.3 and $3.\bar{3}$ are rational numbers, how would you respond to me?

Student 1: They are rational numbers.

Teacher: Yeah, they are rational numbers. Because we can express these numbers as common fractions.

Student 2: Well teacher, which numbers were not rational? Hmm, which decimal numbers were not rational?

Teacher: I explained this a few minutes ago. Let me repeat again. Pi number is a non-example for rational numbers since its decimal expansion does not have a regular repeating pattern. As a matter of fact, except for terminating decimals and repeating decimals all decimals are irrational numbers. You learnt terminating decimals and repeating decimals to date. You should know that these decimals are also rational numbers.

As it is depicted in the teaching episode given above, Teacher D used another non-example for rational numbers as a response to student inquiry. That is, like Teacher A, she introduced a transcendental number, pi, to her students as an example for irrational numbers. In brief, Teacher D used three different kinds of non-examples for rational numbers. That is, she expressed these non-examples either as a ratio of an integer to zero, as a transcendental number, or as an infinite non-repeating decimal.

Different from the aforementioned kind of non-examples, Teacher B generated a non-example that was represented algebraically. To put it differently, Teacher B introduced the square root of 5 to his students as a non-example for rational numbers during the teaching of explaining and locating rational numbers on a number line. After locating $-2\frac{1}{5}, -\frac{1}{2}, \frac{4}{5}, 1\frac{1}{2}$ and $2\frac{2}{5}$ on a real number line, Teacher B asked students to ponder whether rational numbers fill up the number line. The dialogue between Teacher B and his students are given below.

> Teacher B: Thus far, we located $-2\frac{1}{5}, -\frac{1}{2}, \frac{4}{5}, 1\frac{1}{2}$ and $2\frac{2}{5}$ on a number line respectively. These rational numbers filled up some portion of the number line. Well, my question is, do all rational numbers fill up the number line when we totally locate them on that number line?
>
> Student 1: Yes!
>
> Student 2: Nooo!
>
> Teacher B: Do they fill the number line or not? Perhaps, you could not understand my question. How many rational numbers are there in this number line?
>
> Students: Infinite
>
> Teacher B: There are infinitely many numbers in natural number set, counting number set, integer set and in rational number set. Do all rational numbers fill up the number line? Raise your hands if you think that rational numbers do not fill up the number line. According to me, rational numbers do not fill up the number line. Why? Because you can also locate some other numbers on a number line apart from rational numbers. Who can give examples to these other numbers? Who knows the mathematical topics that will be taught in grade 8? Who knows radical numbers? For instance, there are numbers such as $\sqrt{5}$. Such numbers are not rational numbers. You will

learn this kind of numbers next year. We call this kind of numbers irrational numbers or numbers that are not rational. This means that we need numbers such as $\sqrt{5}$ in addition to rational numbers in order to fill up the number line completely.

As it can be seen above, Teacher B used an algebraic number to give a non-example for rational numbers. More specifically, he generated $\sqrt{5}$ as a finite opaque representation to illustrate numbers that are not rational. On the other hand, since he did not define rational numbers as the ratio of any integer to any non-zero integer, he did not use any non-example that included zero in the denominator of the ratio. Moreover, he neither used a transcendental number nor a radical in algebraic form to illustrate non-examples of rational numbers.

To summarize, all teachers except for Teacher B generated non-examples for rational numbers by using the zero denominator case after providing the rational number definition as the ratio of any integer to any non-zero integer. In addition to this, Teacher A and Teacher D provided the pi number (π) as a non-example for rational numbers without actually writing down its decimal expansion. Thus, provision of π without introducing of its decimal expansion leaved it opaque that irrational numbers never settle into a permanent repeating pattern. Apart from using the transcendental number π, Teacher D used a non-example that is represented as an infinite non-repeating decimal. More precisely, Teacher D generated 0.257843… as a non-example for rational numbers and this representation pointed to the requirement for regular repeating pattern as a distinguishing feature between rational and irrational numbers. Unlike these three different kinds of non-examples, Teacher B generated a non-example that was in algebraic form. That is, he generated $\sqrt{5}$ as a non-example for rational numbers. This representation also leaved it opaque that irrational numbers do not have a repeating pattern. Finally, while Teacher B and Teacher D explicitly touched upon the concept of irrational number during the provision of non-examples for rational numbers, Teacher A and Teacher D alluded to that concept by indicating that they will teach a topic including numbers that are not rational the next year.

4.1.3. Counter-examples

In this section, an additional use of examples as counter-examples was explained. Although counter-examples are important in the teaching of mathematics, the data of this study suggested that they are less evident in middle school classroom practice. In this study, examples that were used by the middle school mathematics teachers to demonstrate the falsity of a student conjecture were treated as counter-examples. As a result of the observations, in this study I could notice the use of 5 counter-examples. Three of these counter-examples were used by Teacher A while two of them were used by Teacher D and the two teachers treated those counter-examples logically appropriately.

All of the counter-examples were spontaneously generated by Teacher A and Teacher D in response to their students' contingent and invalid conjectures or statements about rational number ideas. Teacher A generated counter-examples in the course of teaching how to order rational numbers, teaching multiplication of rational numbers, and during the teaching of distributive property of multiplication over addition. Teacher D generated counter-examples in the course of teaching multiplication of rational numbers and during the teaching of distributive property of multiplication over addition.

The teaching episode of Teacher A related with ordering rational numbers and the classroom situation that called for a counter-example is given below.

> Teacher A: It seems that children rarely use same numerator approach when ordering rational numbers. Remarkably, most of the ordering problems can be easily solved by using the same numerator approach. In ordering problems, the number '1' is especially selected as the numerators of the rational numbers. This is due to the fact that children do not think of ordering rational numbers by using same numerator approach when the numerators of rational numbers are selected to be '1'. Now, I shall give you an example. Let's order the following set of rational numbers: $+\frac{1}{3}, +\frac{1}{5}, -\frac{1}{7}, -\frac{1}{2}$ and 0. For negative rational numbers, the one which is closer to zero will be larger. Thus, $0 > -\frac{1}{7} > -\frac{1}{2}$ will be the correct ordering.

S1: But teacher, $-\frac{1}{2}$ is closer to zero than $-\frac{1}{7}$.

S2: That is right, teacher. $-\frac{1}{2}$ is closer to zero.

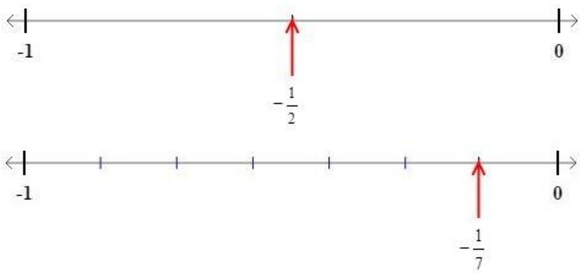
Teacher A: Did I order reversely?

Students: Yes! Yes!

Teacher A: Is $-\frac{1}{7}$ farther to zero than $-\frac{1}{2}$?

Students: Yes!

Teacher A: Then, let's locate these two rational numbers on a number line to see which one is closer to zero. (Teacher A located the two numbers on a number line as follows.)



Teacher A: $-\frac{1}{2}$ is here, $-\frac{1}{7}$ is here. Now, tell me which one is closer to zero.

Students: Aha! $-\frac{1}{7}$ is closer to zero.

Teacher A: You previously claimed that $-\frac{1}{2}$ is closer to zero. However, as you can see, $-\frac{1}{7}$ is closer to zero. Is it alright?

Students: Yes!

As the above teaching episode shows, the students of Teacher A intuitively claimed that a rational number with a smaller denominator will be closer to zero than the one with a larger denominator. To check whether the students are persistent with their claim, the teacher kept asking the same question. Finally, as the students insisted on their conjecture, the teacher decided to locate the two rational numbers on a number line to show that their claim is invalid. When the students examined the number line representation, they were convinced that a rational number with a larger denominator will be closer to zero than the one with a smaller denominator.

Another counter-example was generated both by Teacher A and Teacher D in response to their students' invalid claim that simplification of rational numbers in multiplication can only be done by using the criss-cross method. The teaching episode of Teacher D related with simplification of rational numbers and the classroom situation that called for a counter-example is given below.

Teacher D: How can we simplify rational numbers before multiplying them?

Student: We can simplify by using criss-cross method.

Teacher D: Is it the only way to simplify rational numbers?

Student: Yes.

Teacher D: How about simplifying by using top to bottom method?

Student: I do not think it will work.

Teacher D: We can use both methods for simplifying rational numbers. Let me explain how to simplify $\frac{2}{6} \cdot \frac{3}{6}$ on the board.

$$\frac{\overset{1}{\not 2}}{\underset{3}{\not 6}} \cdot \frac{\overset{1}{\not 3}}{\underset{2}{\not 6}} = \frac{1 \cdot 1}{3.2} = \frac{1}{6}$$

Teacher D: To simplify $\frac{2}{6}$, we divide 2 by 2 and get 1. Similarly, we divide 6 by 2 and get 3. Thus, the simplest form of $\frac{2}{6}$ is $\frac{1}{3}$. To simplify $\frac{3}{6}$, we divide both 3 and 6 by 3 and get 1 and 2 respectively. Thus, we obtain $\frac{1}{3}$ as the simplest form of $\frac{3}{6}$. Now, we multiply 1 by 1 and get 1 as the numerator of the product. Similarly, we multiply 3 by 2 and get 6 as the denominator of the product. Thus, we obtain $\frac{1}{6}$ as the product of this multiplication operation. Now, let's simplify by using the criss-cross method to see whether this method yields the same product as the above mentioned top to bottom method.

$$\frac{\overset{1}{\not 2}}{\underset{2}{\not 6}} \cdot \frac{\overset{1}{\not 3}}{\underset{3}{\not 6}} = \frac{1.1}{2.3} = \frac{1}{6}$$

Teacher D: We simplify 3 and 6 and write 1 and 2 in their place. Similarly, we simplify 2 and 6 and write 1 and 3 in their place. Now, we multiply 1 by 1 and get 1, we multiply 2 by 3 and get 6. Thus, we obtain $\frac{1}{6}$ as the product of this multiplication operation. As you can see the product obtained by using the top to bottom method is equal to the product obtained by using criss-cross method. Therefore, both methods are applicable in rational number multiplication. Is it okay?

Student: Yes!

As the above given episode shows, one of the students of Teacher D claimed that simplification of rational numbers before multiplication can only be done by criss-crossing. As a response to that student's claim, the teacher selected two rational numbers and simplified the rational numbers by using both methods. Finally, the teacher had students compare the products obtained from both methods to demonstrate that both methods yield the same result and to convince that both methods are valid.

Another counter-example about simplification was generated by Teacher A as a response to a student's claim that simplification of rational numbers can always be done after multiplication. This classroom situation is described in the following teaching episode.

Teacher A: Before multiplying rational numbers, you need to check whether the numerators and denominators are evenly divisible by a whole number and if yes you have to simplify them.

Student 1: Must we certainly simplify rational numbers?

Teacher A: Yes you must. As you know, TEOG (A national exam taken by students in order to transit from primary to secondary education) consists of multiple choice questions. In these questions, the alternatives include the simplest form of rational numbers. Thus, you cannot find the answer of the questions in the alternatives unless you simplify the rational numbers.

Student 2: I agree we must simplify the rational numbers. However, we do not have to simplify before multiplying them. I mean, we can always simplify after performing the multiplication operation.

Teacher A: Your friend claims that she can always simplify after finding the product of the multiplication. Now, I will present you a very nice example that refutes her claim. Look at this example;

$$\left(1+\frac{1}{2}\right)\cdot\left(1+\frac{1}{3}\right)\cdot\left(1+\frac{1}{4}\right)...\left(1+\frac{1}{100}\right)$$

In this example, we first need to add the rational numbers inside the parenthesis. Let's do it now.

$$\frac{3}{2}\cdot\frac{4}{3}\cdot\frac{5}{4}\cdots\frac{101}{100}$$

As you can see, it is not possible to perform $3\cdot 4\cdot 5\cdot ...\cdot 101$ and $2\cdot 3\cdot 4\cdot ...\cdot 100$. Therefore, we must simplify before multiplying rational numbers. In this case, we simplify 3 by 3, 4 by 4, 5 by 5 and go on like this until finally simplifying 100 by 100. Let me show this on the board.

$$\frac{\not{3}}{2}\cdot\frac{\not{4}}{\not{3}}\cdot\frac{\not{5}}{\not{4}}\cdots\frac{101}{\not{100}}=\frac{101}{2}$$

As you can see, we cannot always simplify after multiplication. Thus, you should know how to simplify before multiplication. Is this okay?

Student 2: Yes, thank you.

As this episode on simplification of rational numbers shows, Teacher A spontaneously constructed a counter-example in order to convince the student that in some cases simplification of rational numbers before multiplication is compulsory.

Teacher A generated one more counter-example in the course of teaching distributive property of multiplication over addition. The classroom situation that called for this counter-example is presented in the following episode.

Teacher A: Today, I am going to teach you how multiplication by a rational number distributes over addition of two other rational numbers. I will illustrate this property initially by using natural numbers. Let's compute $2\times(3+5)$ by using the distributive property. We multiply each addend by 2 and then add the products. Let me show it on the board.

$$2\times(3+5)=(2\times 3)+(2\times 5)=6+10=16$$

We can also compute $2\times(3+5)$ by using the order of operations rule. According to this rule, we need to perform the operations that are inside the parenthesis first. Thus, we can proceed as follows.

$$2\times(3+5)=2\times 8=16$$

As you can see, the answer obtained by using the distributive property is equal to the answer obtained by using the order of operations rule.

Student: If the results are same, why bother to learn distributive property? According to me, we can solve all the questions by using the order of operations rule. So, I do not think that this property is indispensable for us.

Teacher A: At first glance, what you say seems quite reasonable. However, it is true if the given expression includes all numerical values. That is, if there are unknowns in the expression, then you must use the distributive property in order to find them. In order to have you better understand what I mean, I will write an example on the board.

$$2\times(\square+5)=\Delta\times 7+2\times\bigcirc$$

In this example, you cannot find the unknowns by the using the order of operations rule. You must learn the distributive property in order to find the unknowns. Is it okay?

Student: Yes it is.

As this episode shows, Teacher A initially generated an example that included natural numbers as components. This initial example did not convince one of the students in the classroom of the necessity of learning distributive property. In reply to this, the teacher spontaneously generated a counter-example to negate the student's claim that all mathematical problems that require the use of distributive property can also be solved by using the order of operations rule.

Finally, Teacher D generated one more example while teaching distributive property of multiplication over addition of rational numbers. The classroom situation that called for this counter-example is presented in the following teaching episode.

> Teacher D: Now, I will teach you how to compute, let's say, $\frac{4}{7}\cdot\left[\frac{2}{3}+\frac{1}{5}\right]$ by using the distributive property of multiplication over addition. Normally, you would perform the addition operation inside the brackets first by taking account of the order of operations rule. However, the distributive property spoils the order of operations rule. That is, rather than performing the addition operation first, we distributive the $\frac{4}{7}$ to both the $\frac{2}{3}$ and the $\frac{1}{5}$. Then, we add the products and reach an answer. (Teacher D showed this on the board in the following way.)
>
> $$\frac{4}{7}\cdot\left[\frac{2}{3}+\frac{1}{5}\right]=\frac{4}{7}\cdot\frac{2}{3}+\frac{4}{7}\cdot\frac{1}{5}=\frac{8}{21}+\frac{4}{35}=\frac{40}{105}+\frac{12}{105}=\frac{52}{105}$$
>
> Student: In this example, the distributive property spoils the order of operations rule. So, I do not think we will get the same answer if we compute $\frac{4}{7}\cdot\left[\frac{2}{3}+\frac{1}{5}\right]$ by using the order of operations rule.
>
> Teacher D: No, in contrast to your expectation, the results will be the same. Let me compute this expression by using the order of operations rule. Then, we need to perform the addition operation that is inside the brackets first. (The teacher demonstrated this rule on the board as follows.)
>
> $$\frac{4}{7}\cdot\left[\frac{2}{3}+\frac{1}{5}\right]=\frac{4}{7}\cdot\left[\frac{10}{15}+\frac{3}{15}\right]=\frac{4}{7}\cdot\frac{13}{15}=\frac{52}{102}$$
>
> As you see, we get $\frac{52}{102}$ if we compute by using the order of operations rule. Thus, we may conclude that the distributive property of multiplication over addition and the order of operations rule always yield the same answer. Do you still have any question?
>
> Student: No, thanks.

As this teaching episode reveals, one of the students claimed that computation via distributive property will yield a different product than the one done by using the order of operations rule since Teacher D explained that the order of operations rule is spoiled by the distributive property. Nevertheless, the teacher demonstrated that both ways of computation yield the same product and thus convinced the student that the distributive property always works.

4.2. Sources of Examples

In this section, sources of examples used by middle school mathematics teachers were described in detail. The study revealed two main kinds of teacher-generated examples as spontaneous examples and pre-planned examples. The examples that were actually generated by the teachers during the lesson without any planning in advance or examples that were generated by the teachers as a response to unexpected classroom situations were treated as spontaneous examples. In other words, for an example to be spontaneous, there had to be some evidence that choosing it entailed in-the moment decision making to a certain degree. On the contrary, the examples that were taken from available resources such as textbooks, workbooks and auxiliary books were treated as pre-planned examples. The number of spontaneous and pre-planned examples used by the middle school mathematics teachers with respect to each learning objective is presented in Table 4.37.

Table 4.37. The number of spontaneous examples and pre-planned examples used by the teachers for teaching rational number objectives

Learning Objectives	Number of spontaneous (SP) and pre-planned (PP) examples used by							
	Teacher A		Teacher B		Teacher C		Teacher D	
	SP	PP	SP	PP	SP	PP	SP	PP
Explain and locate rational numbers on a number line	25	4	20	10	18	64	15	15
Express rational numbers in different forms	24	1	30	3	1	2	18	4
Compare and order rational numbers	12	24	13	9	2	-	2	12
Perform addition and subtraction operations with rational numbers	30	7	26	20	4	27	9	23
Perform multiplication and division operations with rational numbers	35	19	37	23	4	-	19	13
Solve multi-step operations with rational numbers	6	17	3	10	2	4	-	13
Pose and solve rational number problems	5	1	-	6	1	3	-	9
Total	137	73	129	81	32	100	63	89

The table shows that middle school mathematics teachers altogether used 361 spontaneous examples and 343 pre-planned examples during the teaching of rational number objectives. This suggests that more than half of the examples used by the teachers were spontaneously generated.

In particular, Teacher A used 137 spontaneous examples and 73 pre-planned examples to teach all rational number objectives. Similar to the overall distribution of examples identified in this study, more than half of the examples used by Teacher A were constructed by him spontaneously. When the number of spontaneous and pre-planned examples used by Teacher A for teaching each rational number objective is examined, it can be seen that the number of spontaneous examples outweighed the number of pre-planned examples except for two objectives. To put it another way, while Teacher A used more number of spontaneous examples than pre-planned examples for teaching how to explain and locate rational numbers on a number line, express rational numbers in different forms, perform addition and subtraction operations with rational numbers, perform multiplication and division operations with rational numbers and pose and solve rational number problems, he used less number of spontaneous examples than pre-planned examples during the teaching of comparing and ordering rational numbers and solving multi-step operations with rational numbers. Finally, it is important to note that the number of spontaneous examples generated by Teacher A for teaching each learning objective was in sharp contrast to the number of pre-planned examples used by him to achieve the same goal.

Similar to Teacher A, Teacher B used 129 spontaneous examples and 73 pre-planned examples for teaching all rational number objectives. This shows that more than half of the examples used by Teacher B were spontaneously generated. This trend on the part of spontaneous examples by Teacher B is in line with the overall distribution of examples identified in this study. In particular, when examples generated by Teacher B for each learning objective was examined it was found that the teacher used spontaneous examples more frequently than pre-planned examples while explaining and locating rational numbers on a number line, expressing rational numbers in different forms, comparing and ordering rational numbers, performing addition and subtraction operations with rational numbers and performing multiplication and division operations with rational numbers. In contrast to this, Teacher B used more number of pre-planned examples than spontaneous examples while solving multi-step operations with rational numbers and posing and solving rational number problems. Ultimately, the magnitude of the differences between the spontaneously generated examples and pre-planned examples of Teacher B for each learning objective were similar to that of Teacher A. That is, in each objective, either spontaneous examples were far more than pre-planned examples or the way other round.

Unlike Teacher A and Teacher B, Teacher C used 32 spontaneous examples and 100 pre-planned examples. This shows that the number of pre-planned examples used by Teacher C is three times more than the number of spontaneous examples. The predominance of pre-planned examples demonstrated that Teacher C drew more on available resources while teaching rational number objectives rather than in-the-moment generations. This trend towards pre-planned examples on the part of the teacher was in contrast with the trend demonstrated by the whole examples identified in this study. More specifically, the examination of the examples generated by Teacher C for each rational number objective revealed that the teacher used far more pre-planned examples than spontaneous examples while explaining and locating rational numbers on a number line and performing addition and subtraction operations with rational numbers. Yet, the number of spontaneous examples were similar to that of pre-planned examples generated by the teacher for expressing rational numbers in different forms, solving multi-step operations with rational numbers and posing and solving rational number problems. Meanwhile, the number of spontaneous and pre-planned examples generated by the teacher for teaching these learning objectives was very few. Furthermore, Teacher C did not use any pre-planned example while comparing and ordering rational numbers and performing multiplication and division operations with rational numbers.

Like Teacher C, Teacher D used 63 spontaneous examples and 89 pre-planned examples. That is, the number of pre-planned examples used by Teacher D was slightly more than the number of spontaneous examples used for teaching all rational number objectives. This tendency on the part of pre-planned examples is in contrast to the tendency of the overall examples generated by four teachers. Moreover, this tendency towards pre-planned examples shows that Teacher D drew more on available resources like Teacher C rather than having recourse to their accessible examples spaces. In more detail, when the examples generated by Teacher D for teaching each rational number objective was examined, it was found that the teacher used much more pre-planned examples than spontaneous examples while comparing and ordering rational numbers and performing addition and subtraction operations with rational numbers. Besides, the teacher did not use any spontaneous example while solving multi-step operations with rational numbers and posing and solving rational number problems. On the other hand, the number of spontaneous examples used by Teacher D for explaining and locating rational numbers on a number line was identical to the number of pre-planned examples used for teaching this objective. Likewise, the number of spontaneous examples used by the teacher for expressing rational numbers in different forms and for performing multiplication and division operations with rational numbers were more than the number of pre-planned examples used for teaching these objectives.

Ultimately, when viewed from a broader perspective, it can be seen that the middle school mathematics teachers all used more number of spontaneous examples than pre-planned examples while performing multiplication and division operations with rational numbers. Besides, excluding Teacher C, all teachers used more number of spontaneous examples than pre-planned examples while explaining and locating rational numbers on a number line and expressing rational numbers in different forms. In contrast, they used less number of spontaneous examples than pre-planned examples while solving multi-step operations with rational numbers. Similarly, excluding Teacher A, all teachers used more number of pre-planned examples than spontaneous examples while posing and solving rational number problems.

In the following parts, the underlying reasons for teachers' use of spontaneous examples and the available sources that were used by the teachers in constructing the pre-planned examples were described at length.

4.2.1. Spontaneous examples

As mentioned above, examples that were generated by the middle school mathematics teachers in the course of the lesson without any planning in advance were treated as spontaneously generated examples. There were two main reasons for teachers' need to construct examples on their feet. That is, teachers generated spontaneous examples either by themselves or as a response to their students' claims or queries. Teacher most often generated spontaneous examples by using their own personal example spaces. In most cases, the examples were generated rather immediately and automatically by the teachers. These examples constituted teachers' easily accessible examples spaces. For instance, while giving examples for numbers that are rational or not rational, Teacher A generated $8, -\frac{2}{3}$, 0, -7, -125, 0.12 and π rather immediately. Similarly, Teacher B generated many repeating decimals such as $0.\overline{7},\ 2.\overline{15},\ 1.0\overline{45},\ 5.\overline{104}$ quite easily. To give another example, Teacher C automatically selected $+\frac{1}{8}, -2\frac{3}{5}, 1\frac{2}{7}$, and $\frac{-5}{3}$ to illustrate how to locate rational numbers in different forms on a number line. Last of all, Teacher D readily generated examples such as $\left(-\frac{3}{8}\right)\cdot\frac{7}{4},\ 3\frac{1}{2}\cdot\frac{5}{8},\ 2\frac{3}{4}\cdot\left(-\frac{1}{2}\right),\ -4\cdot\frac{1}{7}$ to illustrate multiplication of rational numbers in different forms.

Nevertheless, in other cases the time devoted by the teachers to generating spontaneous examples was much longer than the time spent for generating the above mentioned examples. More specifically, generation of some specific examples required a number of iterations until the teachers reached the examples that met their purpose. For instance, in the course of generating a problem solving example, Teacher A initially invented the following story: "On Monday, Ali spent $\frac{1}{4}$ of his pocket money. The next day, he spent $\frac{2}{5}$ of his pocket money and he had 26 TLs left. How much pocket money did he have at the beginning?" After some time, the teacher modified the story in the following way: "On Monday, Ali spent $\frac{1}{4}$ of his pocket money. The next day, he spent $\frac{2}{3}$ of his pocket money and he had 20 TLs left. How much pocket money did he have at the beginning?" The teacher pondered on the example for some time again and apologized that he must write 21 instead of 20 and added that he might modify the problem again to make it suitable for representing it pictorially. Finally, the teacher generated the problem as follows: "On Monday, Ali spent $\frac{1}{4}$ of his pocket money. The next day, he spent $\frac{2}{3}$ of his pocket money and he had 21 TLs left. How much pocket money did he have at the beginning?” However, it took more than one minute for Teacher A to create this example spontaneously. Thus, this spontaneous example indicated remote accessibility to Teacher A’s personal example space.

Apart from the examples that required a number of iterations, the teachers needed to create examples spontaneously when they realized that the examples provided did not satisfy their intended purposes. More precisely, in some cases the teachers had to modify their examples in the course of the lesson when they realized that the examples provided included some limitations or mathematical flaws. For instance, in the following episode on commutative property of addition of rational numbers, Teacher A had to modify his example spontaneously, when he realized that the example did not satisfy his intended purpose.

> Teacher A: … Let me immediately write another example related with commutative property as there is some empty space in this part of the whiteboard.
>
> $$\left(\frac{-3}{5}\right)-\left(\frac{-8}{7}\right)=\Delta-\left(\frac{-3}{5}\right)$$
>
> Teacher A: What should you do primarily in this example to find the value of the triangle?
>
> Student: We need to match the numbers.
>
> Teacher A: You have to change the negative signs into positive signs first. Before doing anything else you have to change the negative signs into positive signs. I am now changing negatives into positives.
>
> $$\left(\frac{-3}{5}\right)+\left(\frac{+8}{7}\right)=\Delta+\left(\frac{+3}{5}\right)$$
>
> Now, I am checking whether there is commutative property. (At this moment, the teacher became aware of the limitation of the example) I am very sorry but, I must change the subtraction sign on the right side of the equality into addition sign. Namely, I must change my example. I failed to notice this, let it be positive.
>
> $$\left(\frac{-3}{5}\right)-\left(\frac{-8}{7}\right)=\Delta+\left(\frac{-3}{5}\right)$$
>
> What shall we do now?
>
> Students: We will match the numbers.

Teacher A: Then, $\left(\frac{-3}{5}\right)$ matches with $\left(\frac{-3}{5}\right)$ and $\left(\frac{-8}{7}\right)$ matches with the triangle. Thus, the triangle is equal to $\left(\frac{-8}{7}\right)$.

As can be seen in the above episode, Teacher A initially provided an example that included subtraction operations on both sides of the equality. After some time, the teacher realized that subtraction operation is not commutative and modified his example by changing the subtraction sign into addition sign. Similar to spontaneous examples generated by the teachers as a result of several iterations, examples of this kind also took more than one minute to generate. Thus, examples of this kind also indicated remote accessibility to teachers' personal examples spaces.

When examples generated by the middle school mathematics teachers as a result of their interactions with the students were examined, it was seen that there are three different types of incidents that give rise to the generation of spontaneous examples. In the first type, the teacher asks students a question and the students react by asking an unexpected question to the teacher. The following episode of Teacher D on expressing rational numbers in different forms illustrates this type of spontaneous example generation.

Teacher D: Are 0.3 and $3.\bar{3}$ rational numbers?

Student 1: Yes!

Teacher D: Yes they are. Because, each repeating decimal and terminating decimal can be expressed as rational numbers in the form of $\frac{a}{b}$.

Student 2: Teacher, may I ask you a question?

Teacher D: Yes you can.

Student 2: Well, I wonder which numbers are not rational.

Teacher D: For instance, the pi number (π) is not rational. It goes on forever as 3.14… and it does not have a regular repeating pattern. Actually, you can think as follows. Excluding repeating decimals and terminating decimals all decimals are irrational numbers.

As this teaching episode shows, Teacher D asked the classroom to indicate whether 0.3 and $3.\bar{3}$ are rational numbers. Being evoked by this question, one of the students in the classroom asked an unexpected question back to the teacher. At that moment, the teacher made a split-second decision and incorporated the student's question into the lesson by generating a spontaneous non-example.

Second type of incident that gave rise to the generation of spontaneous examples occurred when students asked questions to their teachers during classroom conversation. This type of incident is illustrated by the following episode of Teacher B on finding the square and cube of rational numbers.

> Teacher B: …Now, I am switching to an example related with exponents. Let's find the answer of $\left(-\frac{3}{5}\right)^2$. I will ask you a question similar to this one in your third mathematics examination. In the second examination, I asked you to find the answer of $(-6)^3$ and $(-5)^3$. Who knows how to find the answers?
>
> Student 1: Teacher I know, 6 times 6 is 36 and 36 times 6 is 216, thus the answer is minus 216.
>
> Teacher B: What about the other one?
>
> Student 2: Minus 125.
>
> Teacher B: Yes, you are right. In the same manner, we will find the square and cube of rational numbers. What do we need to know for this? We should know that even powers of negative numbers are positive. Let's do it together. 3 times 3 is 9 and 5 times 5 is 25. Therefore, the answer is $+\frac{9}{25}$.
>
> Student 3: Teacher, if we select a number different from 2 as the power of the exponent, how will we find the answer?
>
> Teacher B: For instance, let's find the answer of $\left(-\frac{4}{3}\right)^3$. Here, we must know that odd power of negative numbers are negative. Similar to the previous example, we multiply the numerators and denominators by themselves for 3 times. Then, 4 times 4 is 16 and 16 times 4 is 64. Again, 3 times 3 is 9 and 9 times 3 is 27. Consequently, the answer is $\left(-\frac{64}{27}\right)$. Is that ok?
>
> Student 3: Ok, thanks.

As this episode illustrates, the student interrupted the classroom conversation and asked the teacher a question related with exponentiation. The teacher took account of student's query and provided a spontaneous example as a response to it.

Finally, the last type of incident that led to spontaneous example generation occurred when teachers needed to create counter-examples in response to their students' contingent and invalid conjectures or statements about rational number ideas. These type of examples were described in detail in the aforementioned section entitled counter-examples.

4.2.2. Pre-planned examples

As mentioned earlier, pre-planned examples constituted another main source of examples used by the teachers in the course of the lesson. These type of examples were checked and selected by the teachers in advance and were incorporated into the lesson when needed. The variety of available sources used by the middle school mathematics teachers in choosing the pre-planned examples and the number of pre-planned examples taken from each source is provided in Table 4.38.

Table 4.38. The number of pre-planned examples used by the middle school mathematics teachers during the teaching of rational number concepts

Sources of pre-planned examples	Number of examples used by			
	Teacher A	Teacher B	Teacher C	Teacher D
Student textbook	15	22	-	-
Student workbook	5	-	68	-
Teacher's guidebook	-	-	-	7
High-stakes examination questions	3	-	-	-
Online educational software	-	-	27	-
Auxiliary Book 1	20	-	-	44
Auxiliary Book 2	3	-	-	-
Auxiliary Book 3	8	-	-	4
Auxiliary Book 4	19	-	-	-
Auxiliary Book 5	-	52	-	-

Table 4.38. (Continued)

Sources of pre-planned examples	Number of examples used by			
	Teacher A	Teacher B	Teacher C	Teacher D
Auxiliary Book 6	-	7	-	-
Auxiliary Book 7	-	-	5	-
Auxiliary Book 8	-	-	-	16
Auxiliary Book 9	-	-	-	18
Total	73	81	100	89

As can be seen in Table 4.38, as a whole 14 different resources were used by the middle school mathematics teachers when selecting pre-planned examples prior to the lesson. Besides, all of the middle school classrooms used the same book set prepared by a private publisher for the mathematics lesson. Each book set consisted of a student textbook, a student workbook and a teacher's guidebook. The teachers who used the student textbook as a resource selected examples from this book as a means for introducing or explaining a new rational number topic.

Apart from the official textbooks, the middle school mathematics teachers used auxiliary books prepared by 9 different private publishers. In general, the teachers used these auxiliary books for the purpose of providing exercise examples to their students. In other words, the teachers selected examples from different auxiliary books in order to consolidate the rational number concepts being taught or to promote retention and develop fluency with the procedures related with rational numbers. The selected auxiliary book examples were all multiple-choice questions and were similar to the ones asked in the secondary school entrance examination. This reflected a type of consideration employed by the teachers in choosing or generating rational number examples. The resources used by each teacher for generating pre-planned examples are explained as follows.

Teacher A used 7 different resources when planning which examples to use in the classroom. More specifically, he used 4 different auxiliary books for selecting pre-planned examples. Altogether he selected 73 pre-planned examples from 7 different resources. 50 of the pre-planned examples were selected from auxiliary books, 20 of them were selected from official books and 3 of them were selected from high stakes examination questions (i.e., 2 examples from secondary school entrance examination and 1 example from university entrance examination). He mainly used the student textbook examples and the Auxiliary Book 1 and Auxiliary Book 4 examples in the course of teaching rational number ideas. He less frequently used student workbook examples, high stakes examination questions, Auxiliary Book 2 and Auxiliary Book 4 examples. Yet, he neither considered teacher's guidebook examples, nor online educational software examples while planning which examples to use in the classroom.

Teacher B used student textbook, Auxiliary Book 5 and Auxiliary Book 6 while planning which examples to use in the course of the lesson. Altogether he used 81 pre-planned examples from 3 different sources. He selected 52 examples from Auxiliary Book 5, 22 examples from student textbook, and finally 7 examples from Auxiliary Book 6. Similar to Teacher A, he used student textbook examples during the explanation part of the lessons while he used auxiliary books for providing exercise examples. He selected pre-planned examples mostly from Auxiliary Book 5 and to a lesser extent from student textbook. Besides, he selected only a few pre-planned examples from Auxiliary Book 6. However, he did not take into account student workbook examples, teacher's guidebook examples, high-stakes examination questions and online educational software examples when planning which examples to incorporate into the classroom.

Teacher C used 3 different resources for generating pre-planned examples. Namely, these resources were student workbook, online educational software and Auxiliary Book 7. In total, the teacher used 100 pre-planned examples from these three different resources. 68 of the pre-planned examples were chosen from student workbook, 27 of the pre-planned examples were chosen from an online educational software and finally 5 pre-planned examples were chosen from Auxiliary Book 7.

Teacher C relied heavily on workbook examples and incorporated them into the lesson when compared to the number of examples selected from online educational software and Auxiliary Book 7. Besides, he used far less auxiliary book examples when compared to the previous two teachers. Interestingly, he did not use any textbook example during the teaching of whole rational number ideas. Likewise, he did not use teacher's guidebook examples, high-stakes examination questions and any other auxiliary book examples except for those included in Auxiliary Book 7.

Teacher D used 5 different resources for the purpose of generating pre-planned examples. Namely, the resources used by her were teacher's guidebook, Auxiliary Book 1, Auxiliary Book 3, Auxiliary Book 8 and Auxiliary Book 9. Totally, 89 pre-planned examples were selected from these 5 different resources. The minority of the pre-planned examples were selected from teacher's guidebook (i.e., 7 examples) while the majority of the pre-planned examples were selected from 4 different auxiliary books (i.e., 82 examples). Surprisingly, Teacher D did not incorporate into her lessons any student textbook example, student workbook example, online educational software example or high stakes examination question.

To summarize, while Teacher A and Teacher B incorporated student textbook examples into the classroom while explaining rational number ideas, Teacher C and Teacher D did not. On the other hand, Teacher A and Teacher D used student workbook examples in the classroom while Teacher B and Teacher D did not. The number of student workbook examples used by Teacher C during the lesson was far too much when compared to the examples used by Teacher A. In more detail, Teacher A used student workbook examples as exercise examples, however, Teacher C used them both as teaching examples and exercise examples.

Teacher's guidebook used by the four teachers included student textbook and workbook examples and additional examples apart from these examples. In this study teacher's guidebook examples were treated as the additional examples to be able to distinguish them from student and workbook examples. In this study it was shown that none of the teachers except for Teacher D used teacher's guidebook examples in the course of the lesson. Nonetheless, the number of teacher's guidebook examples used by her was very few. Similarly, despite being very few, only Teacher A brought high stakes examination questions into his classroom. He used these examples to raise his students' awareness about the rational number examples that might be encountered in the secondary school entrance examination. Moreover, only Teacher C incorporated online educational software examples into his classroom. These examples served both as teaching examples and exercise examples. More specifically, Teacher C used the online educational software both as a means for explaining addition and subtraction of rational numbers and for the provision of exercise examples related with addition and subtraction of rational numbers.

When teachers' use of auxiliary book examples were examined comparatively, it was seen that Teacher A and Teacher D used 4 different auxiliary books, Teacher B used 2 different auxiliary books and finally Teacher C used only one auxiliary book. Teacher A used examples from two auxiliary books more frequently when compared to the other auxiliary books he used. Similarly, Teacher B used examples from one auxiliary book more predominantly, while he used the other auxiliary book scarcely. Teacher C used only a single book as an auxiliary book and the number of examples selected by him from this auxiliary book was very few. Rather than using auxiliary book examples, he gave more weight to student workbook examples in his classroom. Teacher D used examples from one auxiliary book more frequently, from two auxiliary books moderately and from one auxiliary book scarcely. Ultimately, two of the auxiliary books used by Teacher D were same as that of auxiliary books used by Teacher A.

Thus far, the overall characteristics of examples used by middle school mathematics teachers in the teaching of rational numbers were described at length. More specifically, in the previous sections the focus was on describing the type of examples used by the teachers in the classroom, the rational number ideas emphasized by the teacher generated examples, the sources of examples as spontaneous and pre-planned examples and the resources teachers resorted to while choosing examples prior to the lesson. In the next chapter, the underlying principles or considerations that guided middle school mathematics teachers in choosing or generating rational number examples were described in detail.

4.3. Summary of Overall Characteristics of Teachers' Rational Number Examples

The findings of this study showed that teachers used specific examples, non-examples and counter-examples as three different types of examples. However, almost all examples used by the teachers were specific examples. Teacher A and Teacher B provided slightly more specific examples than the textbook. However, the number of specific examples used by Teacher C and Teacher D was far less than the number of specific examples included in the textbook. Besides, the number of examples provided for teaching rational number operations was more than half of the total number of examples not only for some of the teachers but also for the textbook. On the other hand, very few examples were provided by the teachers and the textbook for teaching posing and solving rational number problems.

The rational number ideas emphasized by the textbook examples were often emphasized by teacher generated examples as well. In addition, teachers provided examples that emphasized other rational number ideas apart from the textbook. To be more specific, the examples provided by the textbook for explaining and locating rational numbers on a number line involved the following rational number ideas: finding equivalent classes of a fraction, locating equivalent fractions on a number line, locating rational numbers on a number line, determining the positivity/negativity of rational numbers and finding the rational value of a point located on a number line. All teachers provided examples related with identifying whether a given number is rational and locating rational numbers on a number line. However, other rational number ideas about this learning objective (explaining and locating rational number on a number line) were not emphasized by all teachers. Apart from the rational number ideas emphasized by the textbook examples, teachers used examples that emphasized the following ideas: identifying whether a given number is rational, examining the location of a minus sign in a negative rational number, simplifying rational numbers, converting among mixed and improper numbers, and having students feel the need for positive and negative rational numbers.

The examples provided by the textbook for expressing rational numbers in different forms involved the following rational number ideas: expressing integers as rational numbers, expressing rational numbers as integers, repeating decimals or terminating decimals, expressing terminating decimals as rational numbers, and converting repeating decimals into rational numbers. The examples provided by the teachers for expressing rational numbers in different forms did not involve ideas that are different from textbook ideas. However, although all teachers provided examples for converting repeating decimals into rational numbers, not all of them provided examples for introducing other rational number ideas.

The examples provided by the textbook for comparing and ordering rational numbers involved the following rational number ideas: locating on a number line, converting to decimals, common denominator approach, benchmarking, equivalent fractions, and common numerator approach. All teachers provided examples related with common denominator approach. However, other rational number ideas emphasized by the textbook for this learning objective were not emphasized by all teachers. For instance, none of the teachers provided examples for ordering rational numbers by using equivalent fractions. Apart from rational number ideas emphasized by the textbook examples, teachers used examples that emphasized the following ideas: residual thinking, equating the number of decimals by adding 0s, considering number sign, and converting to improper number. Nevertheless, the first two ideas were emphasized only by one of the teachers and similarly the last two ideas were emphasized by another teacher.

The examples provided by the textbook for adding and subtracting rational numbers involved the following rational number ideas: using models for the addition and subtraction of rational numbers, adding and subtracting rational numbers with same denominators, estimating the addition and subtraction of rational numbers, adding and subtracting rational numbers with different denominators and properties of addition of rational numbers. All teachers provided examples related with adding and subtracting rational numbers with same denominators, adding and subtracting rational numbers with different denominators and properties of addition of rational numbers. However, the rest of the rational number ideas about this learning objective were not emphasized by all teachers. For instance, only one teacher provided a single example for teaching estimation of addition and subtraction with rational numbers. Apart from rational number ideas emphasized by the textbook examples, teachers used examples that emphasized the following ideas: performing multi-step operations with rational numbers and finding common denominator of rational numbers. However, these ideas were not emphasized by all teachers. For instance, only one teacher provided examples for finding common denominator of rational numbers.

The examples provided by the textbook for multiplying and dividing rational numbers involved the following rational number ideas: modeling multiplication of rational numbers, multiplication and division of rational numbers, multiplication and division by 0, 1 and (-1), modeling and calculating the square and cube of rational numbers, performing multi-step operations with rational numbers, and properties of multiplication of rational numbers. The examples provided by teachers for teaching this learning objective did not involve ideas that are different from the ideas provided by the textbook examples. However, although all teachers provided examples for teaching the algorithm for multiplying and dividing rational numbers, not all teachers provided examples for introducing the rest of the ideas. More importantly, none of the teachers provided examples for estimating multiplication and division of rational numbers.

The examples provided by the textbook for solving multi-step operations with rational numbers included the following rational number ideas: solving multi-step operations that are expressed on one line, solving multi-step operations that are expressed as complex fractions, and solving multi-step operations that are expressed as a continuing pattern. While all teachers provided examples related with the first two textbook ideas, not all teachers provided examples for the third textbook idea. Apart from rational number ideas emphasized by the textbook examples, one teacher provided examples for solving multi-step operations that are expressed as single variable polynomials.

The examples provided by the textbook for posing and solving rational number problems included the following rational number ideas: solving rational number problems with same referent units, solving rational number problems with different referent units, and posing rational number problems. While all teachers provided examples for solving rational number problems with same and different referent units, none of the teachers provided examples for posing rational number problems. More importantly, the number of examples provided by the textbook and the teachers for posing and solving rational number problems was rather few when compared to the number of examples provided for other rational number objectives.

When non-examples of rational numbers provided by the teachers and the textbook were examined, it was seen that teachers provided four different forms of non-examples while the textbook did not provide any non-example. The teachers provided the following forms of non-examples: ratio of integers to zero, transcendental numbers, radicals, and infinite non-repeating decimals. Teachers more commonly used the ratio of integer to zero representation when providing non-examples of rational numbers. However, non-examples in the form of infinite non-repeating decimals, the only transparent representation of irrational numbers, were only used by one teacher.

Although counter-examples are important in the teaching of mathematics, the findings showed that they are less evident in middle school classroom practice. In this study, only five counter-examples were generated by two teachers to demonstrate the falsity of students' claims. Besides, all counter-examples were generated by the two teachers as a response to contingent classroom situations.

This study revealed two main kinds of teacher-generated examples as spontaneous examples and pre-planned examples. While more than half of the examples used by Teacher A and Teacher B were spontaneously generated, the majority of the examples used by Teacher C and Teacher D were pre-planned. When all examples were considered altogether, it was seen that more than half of them were spontaneously generated by the teachers.

Teachers used several different resources when choosing pre-planned rational number examples. The resources used by the teachers were student textbook, student workbook, teachers' guidebook, high-stakes examination questions, online educational software and nine different auxiliary books. In general, the teachers used the auxiliary books for providing exercise examples to their students. The selected auxiliary book examples were all multiple-choice questions and were similar to the ones asked in the secondary school entrance examination. Teacher's guidebook examples were used only by one teacher. Similarly, high-stakes examination questions were used by one teacher and online educational software examples were used by another teacher. Many different auxiliary books were used by the teachers for selecting pre-planned examples. While two of the auxiliary books were preferred by the same two teachers, the rest of the each auxiliary book was preferred by one teacher.

CHAPTER V

TEACHERS' CONSIDERATIONS IN CHOOSING OR USING EXAMPLES

The purpose of this study was to explore middle school mathematics teachers' treatment of rational number examples in their seventh grade classrooms. In this chapter, the focus was on exploring the principles or considerations used by teachers while choosing or generating rational number examples. Through this focus, the following research question was formulated:

What are the underlying principles or considerations that guide middle school mathematics teachers in choosing or generating examples?

In this chapter, middle school mathematics teachers' considerations in choosing or using rational number examples or the underlying principles that guided them in choosing or using rational numbers were reported on the basis of lesson observations and post lesson interviews. It is important to note that the considerations or principles employed by the teachers were interconnected and they slightly overlapped with each other. Besides, teachers used more than one consideration for several examples. On the contrary, in some cases teachers generated a sequence of examples and for this sequence of examples they employed the same consideration. Therefore, in this part of the study, rather than reporting the number of examples, the different considerations held by the teachers with respect to each category was provided. In the following section, the incidents in which teachers started with a simple or familiar case were described at length.

5.1. Starting with a Simple or Familiar Case

In this category, middle school mathematics teachers most often generated sequences of examples and each example in the sequence gradually increased in its level of complexity or difficulty. The rest of the considerations of this type were employed when teachers generated examples that recalled students' prior knowledge on rational number concepts. The subcategories emerged from this category were (i) considering form of rational numbers, (ii) considering denominators of rational numbers, (iii) considering number of repeating and non-repeating digits of a decimal, (iv) considering number of terms/elements/steps when ordering rational numbers, performing a single operation or multi-step operations with rational numbers, (v) considering increasing complexity of multi-step operations, and of rational number problems by changing their mathematical structure, and finally (vi) recalling prior knowledge on rational number concepts.

5.1.1. Considering form of rational numbers

Teachers selected a sequence of rational numbers in different forms for locating on a number line, for performing four operations and for performing exponentiation. To illustrate, Teacher A selected $\frac{3}{5}, -\frac{3}{4}, 3\frac{2}{5}$ and $\frac{12}{5}$ respectively to locate them on a number line. In this sequence, the first two rational numbers are in proper form, the next rational number is in mixed form and the last rational number is in improper form. While locating these rational numbers on a number line, the teacher used the following expressions to explain why he chose them in that order.

> Teacher A: Proper numbers are fairly easy to locate them on a number line. $\frac{3}{5}$ is between 0 and 1 and $-\frac{3}{4}$ is between 0 and -1… I am skipping improper numbers, because I do not like locating them on a number line. To locate mixed numbers on a number line you should first have a look at the whole part. $3\frac{2}{5}$ has three wholes so it is between 3 and 4… It is more difficult to locate an improper number on a number line. That is why I skipped locating $\frac{12}{5}$ on a number line. To locate $\frac{12}{5}$ on a number line you have to partition each integer interval into 5 and then count from 1 to 12. Other method of locating $\frac{12}{5}$ on a number line is by converting it into a mixed number. This way is easier than the previous way. That is why I said I do not like locating an improper number on a number line.

Another case had to do with the form of terms in a rational number operation. To illustrate, Teacher B performed $\frac{2}{4}-\frac{1}{4}$, $\left(\frac{-2}{5}\right)+\frac{9}{5}$, $\frac{9}{7}-\left(\frac{-4}{7}\right)$, $2\frac{1}{4}+\frac{3}{4}$, $5\frac{3}{4}+\left(-1\frac{1}{4}\right)$ respectively while teaching addition and subtraction of rational numbers. Similarly, he performed $\frac{5}{7}\cdot\frac{4}{9}$, $\frac{3}{8}\cdot\frac{12}{18}$, $\left(\frac{-1}{4}\right)\cdot\left(-\frac{7}{3}\right)$, $\left(\frac{-5}{13}\right)\cdot\left(-1\frac{3}{4}\right)$, $\frac{5}{12}\div\frac{7}{4}$, $\left(-\frac{3}{6}\right)\div\frac{5}{4}$, $\left(-\frac{5}{3}\right)\div 2\frac{1}{2}$ respectively during the teaching of multiplication and division of rational numbers. In these two sets of examples, Teacher B initially selected proper or improper numbers and later he selected mixed numbers as terms of operations. Since the initial examples did not require conversion into proper number numbers, these examples were easier than the latter ones. This consideration was expressed by Teacher B as following:

> Teacher B: Let's start multiplication and division with a few examples. Initially, let me use proper or improper numbers but not mixed numbers. Let's start by using positive ones. We do not have to use parenthesis for positive rational numbers…

Another example of how a teacher takes into account different form of rational numbers was observed in a lesson in which Teacher B introduced exponentiation of rational numbers. More precisely, the teacher provided $\left(\frac{1}{2}\right)^3, \left(\frac{-1}{3}\right)^2, \left(\frac{-2}{3}\right)^3$ and $\left(1\frac{1}{2}\right)^2$ respectively as examples for finding the square and cube of rational numbers. As it can be seen from this sequence, Teacher B ultimately incorporated into the classroom an exponent with a mixed number base. Besides, computation of exponents becomes more complex when proceeded from $\left(\frac{1}{2}\right)^3$ to $\left(1\frac{1}{2}\right)^2$. This type of consideration was expressed by Teacher B as follows:

> Teacher B: You do not have any problem with how to compute $\left(\frac{1}{2}\right)^3$. Similarly, you do not have any problem with how to compute $\left(\frac{-1}{3}\right)^2$. Here you can directly multiply $\left(\frac{-1}{3}\right)$ by $\left(\frac{-1}{3}\right)$. However, computation of $\left(1\frac{1}{2}\right)^2$ is a bit more complex. What should you do to compute exponents with

mixed number bases? You first need to convert $1\frac{1}{2}$ into an improper number.

5.1.2. Considering denominators of rational numbers

Teachers initially used rational numbers with same denominators as members of the sequence when ordering rational numbers or they initially used them during the teaching of addition and subtraction of rational numbers. To illustrate, Teacher A generated the following sequence of examples consecutively when teaching how to order rational numbers:

(1) $\frac{(-2)}{5}, \frac{(-7)}{5}, 0, \frac{1}{5}, \frac{3}{5}$

(2) $\frac{1}{3}, \frac{1}{5}, -\frac{1}{7}, -\frac{1}{2}, 0$

(3) $-\frac{2}{7}, -\frac{2}{13}, 0, \frac{2}{15}, \frac{2}{19}$

(4) $\frac{2}{19}, \frac{3}{13}, \frac{6}{17}$

(5) $\frac{1996}{1997}, \frac{1997}{1998}, \frac{1998}{1999}$

In the first sequence, there was no need to find the least common multiple (LCM) of the denominators since all rational numbers had the same denominators. Besides, the students were already familiar with ordering by using common denominator approach since their primary school years. Therefore, it might be fairly easy for students to order the rational numbers given in the first sequence. In the second sequence, all rational numbers included 1 as a numerator and they can be ordered by using the same numerator approach. However, as indicated by the teacher, same numerator approach did not readily come to students' mind when they saw 1 as the top numbers of all rational numbers. Therefore, ordering the second sequence of rational numbers might be more difficult for students when compared to the first sequence. In the third sequence, rational numbers included 2 as a numerator and they could be ordered by using the same numerator approach. However, this sequence included larger numbers as denominators and thus it might fairly be more difficult for students to order the given rational numbers when they did not think of using the same numerator approach. In the fourth sequence, rational numbers included different numerators and denominators. However, as expressed by the teacher, this sequence included rational numbers that had larger denominators compared to their numerators. Thus, as done by the teacher, it was easier to find the least common multiple of numerators instead of denominators for ordering the given rational numbers. Yet, the students might not readily access to the use of common numerator approach for this sequence when compared to the third sequence. In the last sequence, rational numbers had different numerators and denominators and it was almost impossible for students to order the rational numbers by using common numerator or common denominator approach since all numerators and denominators were very large numbers. Thus, the students needed to use a different approach other than the two approaches such as residual thinking as done by Teacher A. Consequently, ordering the rational numbers included in the last sequence was more difficult than ordering the ones included in the fourth and in the previous sequences.

In another case, Teacher D initially selected rational numbers with same denominators for teaching addition of rational numbers. That is, Teacher D performed the following addition operations consecutively:

$$\frac{3}{5}+\frac{1}{5},\ \frac{5}{9}+\left(\frac{-7}{9}\right),\ \left(\frac{-5}{9}\right)+\left(\frac{-2}{9}\right),\ \left(\frac{-13}{4}\right)+\left(\frac{7}{4}\right),\ 2+\frac{3}{5}, 4\frac{1}{2}+\frac{1}{5},\ 1\frac{1}{5}+4\frac{3}{5}$$

After teaching addition of rational numbers, Teacher D provided her students with subtraction operations and similar to the addition examples, she started with the examples that included same denominators as terms of subtraction operation. She provided the following examples successively:

$$\frac{5}{8}-\frac{2}{8},\ \left(\frac{-9}{4}\right)-\left(-\frac{3}{4}\right),\ \frac{1}{3}-\frac{3}{8},\ 2-\frac{3}{7},\ \frac{1}{2}-4$$

As it can be seen, in the two sets of addition and subtraction examples, Teacher D principally performed addition and subtraction of rational numbers with same denominators and then she moved on to teaching addition and subtraction of rational numbers with different denominators. This shows how she took into consideration the increasing level of complexity in the course of teaching addition and subtraction operations with rational numbers.

5.1.3. Considering number of repeating and non-repeating digits of a decimal

There was a deliberate attempt on the part of teachers to proceed from decimals that included merely repeating digits to decimals that included both repeating and non-repeating digits while teaching conversion of repeating decimals. To give an example, Teacher B used $1.\overline{3}$, $2.\overline{15}$, $5.\overline{104}$, $3.2\overline{4}$ and $1.0\overline{45}$ consecutively in order to convert them into their common fraction forms. As it can be seen, $1.\overline{3}$ includes only one repeating digit and it is fairly easy to convert it into its common fraction form since it includes only one ‘9’ in the denominator. $2.\overline{15}$ includes two repeating digits and conversion of it is a bit more difficult when compared to $1.\overline{3}$ since it requires two ‘9s’ in the denominator. $5.\overline{104}$ includes three repeating digits and conversion of it is more difficult when compared to $2.\overline{15}$ since it entails three ‘9s’ in the denominator. Unlike the previous three repeating decimals, $3.2\overline{4}$ includes one repeating and one non-repeating digit. Converting $3.2\overline{4}$ into its common fraction form is more complex since it entails writing down one ‘9’ for the repeating digit and one ‘0’ for the non-repeating digit after ‘9’ in the denominator. Finally, $1.0\overline{45}$ includes two repeating digits and one non-repeating digit and among all repeating decimals $1.0\overline{45}$ can be regarded as the most complex one since it entails writing down two ‘9s’ and one ‘0’ after ‘9s’ in the denominator. Apart from this, identification of minuends and subtrahends necessary for finding the numerator of each common fraction becomes more complex when moved from $1.\overline{3}$ to $1.0\overline{45}$.

5.1.4. Considering number of terms/elements/steps when ordering rational numbers, performing a single operation or multi-step operations with rational numbers

Teachers gradually increased either the number of terms in an operation, the number of rational numbers selected for ordering in a sequence or the number of steps included in multi-step operations with rational numbers. To illustrate the case of gradually increasing the number of terms in an operation, the examples provided by Teacher B in the course of teaching addition of rational numbers with different denominators are given as follows:

(1) $1\frac{2}{4}+\frac{5}{3}$ (2) $\left(-2\frac{1}{3}\right)+\left(-1\frac{1}{2}\right)$ (3) $\left(\frac{-3}{4}\right)+\frac{5}{9}$ (4) $1\frac{1}{4}+\left(\frac{-3}{12}\right)+\frac{5}{3}$

As it can be seen, the first three addition operations include two terms while the last addition operation includes three terms. These consecutively generated examples showed how Teacher B altered the complexity of operations by increasing the number of terms in the final operation.

Similarly, to illustrate the case of increasing gradually the number of rational numbers in a sequence, the examples provided by Teacher B for teaching comparing and ordering rational numbers can be given. Namely, Teacher B initially generated rational number pairs for comparison as follows:

$\frac{7}{9},\frac{15}{2};\ 1\frac{3}{4},\frac{8}{11};\ \frac{-1}{2},\frac{-7}{3};\ -2\frac{1}{5},\frac{-17}{2};\ \frac{-5}{3},-3\frac{1}{2};\frac{1}{4},\frac{-1}{3};\ 4\frac{2}{3},\frac{12}{19};\ \frac{-7}{15},\frac{17}{18};2\frac{1}{4},\frac{9}{4};\ \frac{-3}{5},\frac{-15}{8}$

Later, he provided the following sequences of rational numbers consecutively for ordering:

$\frac{-1}{7},2\frac{4}{9},1\frac{1}{4},-1\frac{5}{6};\quad \frac{-3}{8},\frac{4}{8},\frac{9}{8},\frac{-15}{8};\quad \frac{-1}{3},\frac{7}{4},\frac{-3}{5},\frac{-1}{12}$

As it can be seen, the examples provided by Teacher B for comparing included two rational numbers while the examples provided for ordering included four different rational numbers. This showed how the teacher applied his principle of going from simple to more complicated by increasing the number of rational numbers included in a sequence for comparing and ordering.

Another case occurred when Teacher A attempted to increase gradually the number of steps included in complex fractions while teaching multi-step operations with rational numbers as follows:

(1) $\dfrac{1}{1-\frac{1}{2}}$ (2) $3+\dfrac{1}{1-\frac{1}{3}}$ (3) $1-\dfrac{1}{1-\dfrac{1}{1-\frac{1}{2}}}$

As the abovementioned examples show, Teacher A first provided his students with a complex fraction that can be solved in 2 steps. The next complex fraction can be solved in 3 steps and finally the third complex fraction can be solved in 5 steps. It appeared that Teacher A increased the number of steps in a complex fraction one or two at a time. Consequently, this showed how Teacher A increased the complexity of complex fractions progressively each time he provided a new complex fraction to his students.

5.1.5. Considering increasing complexity of multi-step operations and of rational number problems by changing their mathematical structure

Teachers considered increasing complexity of multi-step operations and of rational number problems by changing their mathematical structure. To illustrate the case of increasing complexity of multi-step operations, the examples used by Teacher B for teaching multi-step operations with rational numbers are given as follows:

(1) $\left(2-\frac{1}{3}\right)\div\left(3+\frac{1}{4}\right)$ (2) $1-\cfrac{2}{3+\cfrac{1}{1+\cfrac{1}{2}}}$ (3) $3+\cfrac{5}{4+\cfrac{6}{x+2}}=4$

As it can be seen, the first multi-step operation example includes terms that are all expressed on one line and the students are already familiar with this type of example from their early primary school years. The second multi-step operation example is in complex fraction form and this type of example is novel to students when compared to the previous one. Finally, the third multi-step operation example is also in complex fraction form. However, unlike the previous two examples, this example includes an unknown variable and thus it can be considered the most complicated one among three examples. Some of the explicit classroom utterances that support Teacher B's principle of going from simple to more complicated during the provision of abovementioned examples are given as follows:

> Teacher B: The first multi-step operation is fairly easy. I do not think you will have any trouble while solving this problem…The second multi-step operation is in complex fraction form. We also call these fractions as stacked fractions. This problem is a bit troublesome when compared to the previous

one…Look out! The first problem is very easy, the second problem is a bit more difficult and you have great difficulty in the third problem.

Another case occurred when Teacher A attempted to generate rational number problems from simple to more difficult by changing the mathematical structure of each problem gradually. The rational number problems constructed by Teacher A consecutively in the course of the lesson are provided as follows:

(1) Find $\frac{3}{4}$ of 24. (2) On Monday, Ali spent $\frac{1}{4}$ of his pocket money. The next day, he spent $\frac{2}{3}$ of his pocket money and he had 21 TLs left. How much pocket money did he have at the beginning? (3) On Monday, Ali spent $\frac{1}{4}$ of his pocket money. The next day, he spent $\frac{2}{3}$ of his remaining pocket money and he had 21 TLs left. How much pocket money did he have at the beginning?

As it can be seen, the first problem is devoid of real life context and it is fairly easy to solve. The second problem is embedded in a real life context and the referent unit is the same for all rational numbers given in this problem. That is, this problem involves addition and subtraction of given rational numbers. Thus, the second problem is deemed to be more difficult when compared to the first one. Finally, the third problem was generated completely by using the same real life context of the second problem. However, the rational numbers given in this problem all refer to different referent units. Namely, the third problem includes multiplication of rational numbers in addition to addition and subtraction. Thus, the third problem can be accepted as the most complicated one among three problems. Accordingly, the three rational number problems generated by Teacher A manifested how he applied his principle of going from simple to more difficult by incrementally changing the mathematical structure of each problem.

5.1.6. Recalling prior knowledge on rational number concepts

Finally, teachers considered recalling students' prior knowledge on rational number concepts when needed. There were many cases that prompted teachers to check students' prior knowledge on rational number concepts. In one case, teachers recalled natural number set and integer set and gave examples and non-examples for these sets before introducing rational number set. For instance, Teacher C introduced $0, 1, \frac{10}{2}$ and $\frac{100}{2}$ as examples and $-1, -5, -38$ and 0.35 as non-examples for natural number set. Similarly, he introduced -10, -1, 0 and 1 as examples and 0.35 and $\frac{1}{2}$ as non-examples for integer set.

In another case, equivalent fractions were recalled before explaining rational number set. To give an example, Teacher B found the equivalence sets of the $\frac{1}{2}, -\frac{1}{2}$ and $-\frac{5}{2}$ as a means to define rational number set.

Another case was about recalling proper fractions, improper fractions, mixed fractions and locating them on a number line. For example, Teacher A provided $\frac{3}{5}$ as a proper fraction, $3\frac{2}{5}$ as a mixed fraction and $\frac{12}{5}$ as an improper fraction and located them on a number line respectively.

Another manifestation of a teacher's consideration of prior knowledge was observed when Teacher C recalled conversion among mixed fractions and improper fractions. That is to say, Teacher C recalled how to convert $2\frac{1}{2}$ and $20\frac{8}{9}$ into improper fractions before teaching how to convert negative numbers such as $-3\frac{1}{2}, -7\frac{1}{4}$ and $-2\frac{7}{10}$ into their improper forms. Similarly, Teacher C recalled how to convert $\frac{12}{5}, \frac{23}{4}$ and $\frac{17}{16}$ into mixed fractions before teaching how to convert negative numbers such as $-\frac{7}{3}, -\frac{25}{6}, -\frac{100}{3}$ and $-\frac{100}{99}$ into their mixed number forms.

In another case, Teacher A recalled how to find the least common multiple of three natural numbers. More precisely, the teacher was teaching how to order $\frac{2}{5}, \frac{7}{10}, \frac{7}{3}$ and he decided to order them by using common denominator algorithm. At that moment he recalled how to find the least common multiple of 3, 5 and 10. He explained this method step by step as follows:

Teacher A: How can I find the least common multiple of 3, 5 and 10? I first write these three numbers from left to right. I then draw a vertical line to the right hand side of these numbers. Next, I check whether these numbers are divisible by the prime number 2. I write a 2 to the right side of the line. 2 does not divide 3 and 5 so I just bring down 3 and 5 and 2 goes into 10 five times so I write 5 underneath 10. Now, I am left with 3, 5 and 5. I have to repeat the process. 3, 5 and 5 are not divisible by 2, so I check whether they are divisible by the prime number 3. I see that 3 is divisible by 3. So I write 3 to the right side of the line. 3 goes into 3 one time, thus I write 1 under the 3. However, 3 does not go into 5 and 5 so I bring down the two 5's. This time I am left with 1, 5 and 5. Again, I have to repeat the process. The two 5's are divisible by 5 so I write 5 to the right side of the line and then write 1 under each 5. Now, I multiply the prime numbers on the right hand side of the vertical line to get the least common multiple. 2 times 3 times 5 is 30. Thus, the least common multiple of 3, 5 and 10 is 30.

Other consideration of this type occurred when Teacher D recalled how to order integers before ordering rational numbers. More precisely, before ordering $-\frac{1}{8}, -\frac{1}{4}$ and -1, she recalled how to order -4, -3 and -2. She first ordered the integers as if they were positive. That is, she treated negative numbers as if they were positive numbers and arranged them as $4 > 3 > 2$ and then she reversed this arrangement as $-2 > -3 > -4$ so as to order the negative integers. She then expressed that the same reasoning is applicable for ordering rational numbers. Accordingly she arranged the rational numbers as $1 > \frac{1}{4} > \frac{1}{8}$ and reversed this arrangement as $-\frac{1}{8} > -\frac{1}{4} > -1$ so as to order the negative counterparts.

Another example of how a teacher takes into account prior knowledge of students was observed in a lesson in which Teacher A taught addition of rational numbers. In more detail, he computed $2\frac{1}{5} - \left(-\frac{3}{7}\right) + \left(+\frac{2}{7}\right)$ step by step and finally reached the answer $\frac{326}{105}$. As this number included large numerators and denominators the teacher wanted to simplify it. Thus, he asked the students to ponder whether 105 and 326 had a common divisor. As the students invalidly claimed that 105 and 326 were divisible by 3, the teacher felt the need to recall divisibility rules. The teacher explained that a number was divisible by 3 if and only if the sum of its digits was divisible by 3. Finally, he expressed that 326 was not divisible by 3 since the sum of its digits (i.e., 11) was not divisible by 3.

Another consideration related with prior knowledge subcategory occurred when teachers started teaching four operations with rational numbers initially by recalling operations with fractions. For instance, Teacher B expressed this type of consideration by the following utterances:

> Teacher B: Now, let's start addition and subtraction of rational numbers. In grade 5, you learnt how to perform operations with fractions. But, I am not sure whether you did it in grade 4. Did you?
>
> Student 1: Yes, we did.
>
> Teacher B: Ok, we can add fractions, we can subtract fractions. Let me give you an example from fractions. Let's find the answer of $\frac{2}{4}-\frac{1}{4}$ (The teacher drew a region model of fractions to explain the subtraction operation). How do you read $\frac{2}{4}$?
>
> Student 2: It is two-fourths.
>
> Teacher B: We can read $\frac{2}{4}$ as two over four or as two-fourths. Actually, it is one-half. Similarly, we can read $\frac{1}{4}$ as one-fourths and it is also called one-quarter. Now, if we subtract $\frac{1}{4}$ from $\frac{2}{4}$, the remaining part will correspond to $\frac{1}{4}$. How did we perform this operation? Since the two fractions had same denominators, we subtracted their numerators from each other. In fact we do it in this way: 2 minus 1 is equal to 1. Thus, the answer is $\frac{1}{4}$. As I said before, you previously learnt how to add and subtract fractions. Well, how will you add and subtract rational numbers? Is there something new for rational numbers?
>
> Students: No!
>
> Teacher B: What should you attend to when adding or subtracting rational numbers?
>
> Students: To their signs.
>
> Teacher B: Yes, you should attend to the signs of terms.

Teacher B showed the same consideration when teaching multiplication and division of rational numbers. That is, Teacher B recalled multiplication of fractions at the initial phase of the teaching episode related with multiplication of rational numbers. He expressed this type of consideration by the following utterances:

Teacher B: Now, we will have a look at multiplication and division of rational numbers. Let's begin with multiplication of fractions. How were we multiplying fractions?

Student 1: We multiply the numerators and write the answer to the numerator. Then we multiply the denominators and write the answer to the denominator.

Teacher B: Good! Now, we will have a look at the examples included in your mathematics textbook. Initially, we will remember the multiplication and division of fractions. Primarily, we will find the answer of $\frac{2}{3} \cdot \frac{3}{4}$. If you remember from fractions, we multiply the numerators and write the answer to the numerator of the new fraction, similarly we multiply the denominators and write the answer to the denominator of the new fraction. 2 times 3 is 6 and 3 times 4 is 12 and thus the result is $\frac{6}{12}$.

In another case, Teacher A recalled commutative property of addition of integers before teaching the commutative property of addition of rational numbers. Teacher A expressed this type of consideration by the following utterances:

Teacher A: You learned this property earlier in integers. How did we do it in integers? For instance, if 2+3 is equal to 3+2 $\left(\text{i.e., } 2+3 \overset{?}{=} 3+2\right)$, then we say that addition is commutative for integer set. The sum of 2+3 is equal to 5 and the sum of 3+2 is again equal to 5. Thus, addition operation is commutative for the set of integers. Now, let's check if addition is commutative for rational number set. For instance, let's see if $\frac{2}{5}+\frac{3}{5}$ is equal to $\frac{3}{5}+\frac{2}{5}$ $\left(\text{i.e., } \frac{2}{5}+\frac{3}{5} \overset{?}{=} \frac{3}{5}+\frac{2}{5}\right)$. $\frac{2}{5}+\frac{3}{5}$ is equal to $\frac{5}{5}$ and $\frac{3}{5}+\frac{2}{5}$ is again equal to $\frac{5}{5}$. Hence, we can say that addition operation is commutative for rational number set. Is there anything you could not understood?

Students: No!

Teacher A: Ok, now let's move on to another example of commutative property.

Teacher A showed the same consideration when teaching associative property of multiplication of rational numbers. Namely, he recalled associative property of addition of integers before teaching the associative property of addition of rational numbers. This consideration was expressed by Teacher A as follows:

Teacher A: What discriminates associative property from commutative property is that associative property includes three numbers whereas commutative property includes two numbers. If adding the first two numbers initially and later adding the third number yields the same result with adding the second and the third number initially and then adding the first number, then we can say that addition operation is associative for the set of integers. Let me give an example. Let's see whether $(2+3)+5$ is equal to $2+(3+5)$. If we add 2 and 3 we get 5 and if we add 5 and 5 we get 10. Next, if we add 3 and 5 we get 8 and if we add 8 and 2 we again get 10. In this case, $(2+3)+5$ is equal to $2+(3+5)$. Thus, we can say that addition operation is associative for the set of integers. Now, we will follow the same process for rational numbers. Not to spend too much time for finding the common denominator, I want to select rational numbers that have same denominators. Let's see whether $\left(\frac{1}{2}+\frac{3}{2}\right)+\frac{7}{2}$ is equal to $\frac{1}{2}+\left(\frac{3}{2}+\frac{7}{2}\right)$. If we add $\frac{1}{2}$ and $\frac{3}{2}$, we get $\frac{4}{2}$ and if we add $\frac{4}{2}$ and $\frac{7}{2}$ we get $\frac{11}{2}$. Next, if we add $\frac{3}{2}$ and $\frac{7}{2}$, we get $\frac{10}{2}$ and if we add $\frac{1}{2}$ and $\frac{10}{2}$, we get $\frac{11}{2}$. We can see that $\left(\frac{1}{2}+\frac{3}{2}\right)+\frac{7}{2}$ is equal to $\frac{1}{2}+\left(\frac{3}{2}+\frac{7}{2}\right)$. Thus, addition operation is associative for the set of rational numbers.

Teacher A also showed the same consideration when teaching distributive property of multiplication over addition of rational numbers. That is, he first showed how to use distributive property of multiplication over addition of integers and later he carried out the same process for rational numbers. This consideration was expressed by Teacher A as follows:

Teacher: First, I want to teach you distributive property of multiplication over addition by using integers. Let's see how to find the answer of $2\times(3+5)$. Keep in mind that the distributive property spoils the order of operations. That is, if we use the order of operations, we have to perform the addition operation first. But, if we use the distributive property, we multiply first and then add. That is, 2 times 3 is 6 and 2 times 5 is 10 and finally 6 plus 10 is 16. Thus, the answer is 16. Why did I initially use integers instead of rational numbers? If I primarily use rational numbers, you can get confused. Now, I will repeat the same process by using rational numbers. Let's compute $\frac{3}{4}\times\left(\frac{3}{2}-\frac{6}{5}\right)$ by using the distributive property. We multiply $\frac{3}{4}$ by $\frac{3}{2}$ and get $\frac{9}{8}$. Next, we multiply $\frac{3}{4}$ by $\frac{6}{5}$ and get $\frac{18}{20}$. Now we need to perform $\frac{9}{8}-\frac{18}{20}$. We can simplify $\frac{18}{20}$ by dividing both the numerator and the denominator by 2. Thus, we can rewrite the subtraction operation as $\frac{9}{8}-\frac{9}{10}$

…

Finally, the manifestation of this approach was seen when Teacher B recalled how to find the square and cube of integers before teaching the second and third powers of rational numbers. The following excerpt showed how Teacher B took into account this type of consideration:

> Teacher B: In this lesson, I will teach you how to find the second and third powers of rational numbers. But, initially, let's remember how to find the powers of integers. To find the answer of $(+3)^2$ we need to multiply $(+3)$ by itself for two times. 3 times 3 is 9 and my friend's friend is my friend. Thus, $(+3)^2=(+3)\cdot(+3)=(+9)$. Similarly, to find the answer of $(-2)^3$ we need to multiply (-2) by itself for three times. 2 times 2 times 2 is equal to 8. My enemy's enemy is my friend and my friend's enemy is my enemy. Thus, $(-2)^3=(-2)\cdot(-2)\cdot(-2)=(-8)$. Finally, to find the answer of $(-4)^2$ we need to multiply (-4) by itself for two times. 4 times 4 is 16 and my enemy's enemy is my friend. Thus, $(-4)^2=(-4)\cdot(-4)=(+16)$. Now, I will give you some examples from exponents with rational number bases. Is it okay?
>
> Students: Okay!

The incidents in which teachers started with a simple or familiar case were described in detail in this section. In the following section, the cases in which teachers attended to students' errors, misconceptions or difficulties were described thoroughly.

5.2. Attending to Students' Difficulties, Errors or Misconceptions

Middle school mathematics teachers often built examples according to the difficulties they knew their students encountered with, common errors they knew students made or the misconceptions they knew students held. Thus, subcategories emerged from this category were (i) attending to student difficulty, (ii) attending to student error, and finally (iii) attending to student misconception.

5.2.1. Attending to students' difficulties

Teachers expressed that students often had difficulty in (i) understanding the location of a minus sign in a rational number, (ii) dealing with division of a number by zero and division of zero by a number, (iii) understanding that distributive property yields a valid result, (iv) performing subtraction operation with rational numbers, (v) solving complex fractions with unknown values, (vi) ordering rational numbers with the same numerators, (vii) simplifying rational numbers before multiplication, (viii) performing operations including negative rational numbers without parenthesis, and (ix) distinguishing between exponents with a power inside the parenthesis and out outside the parenthesis.

Teachers explicitly stated in the classroom that students often had difficulty in understanding that locating the minus sign either over, in front of or under the main fraction bar does not alter the value of a rational number and they emphasized that the three different representations of the negative rational number mean the same thing. The following episode of Teacher A on teaching rational numbers with same denominators illustrates this consideration:

> Teacher A: Let's find the answer of $\frac{(-3)}{7}+\frac{(+5)}{7}-\left(-\frac{2}{7}\right)$. Before solving this, I want to focus to the following equality: $-\frac{1}{2}=\frac{-1}{2}=\frac{1}{-2}$. You often get confused when you see this equality. This equality means that $-\frac{1}{2}$, $\frac{-1}{2}$ and $\frac{1}{-2}$ are all equal to each other and wherever you put the minus sign, the rational number will always be negative. You can either put it over, under or in front of the fraction bar. By using this equality, we can rewrite $\left(-\frac{2}{7}\right)$ as $\frac{(-2)}{7}$ and aggregate the numerators over one fraction bar in this way: $\frac{(-3)+(+5)-(-2)}{7}$. Finally, we add and subtract rational numbers as we did in integers.

Teachers indicated that students had difficulty in understanding that division of a number by zero was undefined while division of zero by a number was zero. For instance, Teacher A identified in the course of a lesson a student's difficulty in finding the answer of $\frac{0}{5}$ as zero. The expressions he used to explain his consideration is provided by the following teaching episode:

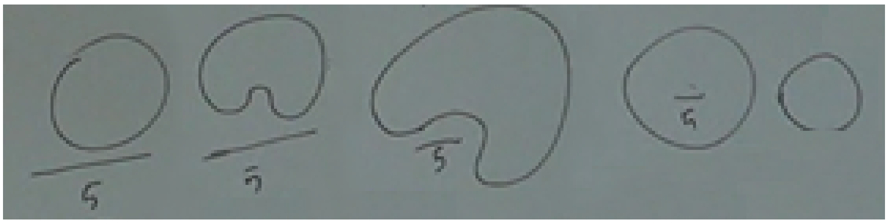
Teacher A: What does $\frac{0}{5}$ equal to? Who wants to tell me the answer?

Student: Zero over five is equal to five.

Teacher A: Good (!) How many times does 5 go into 0? Five times. Now listen to me very carefully. When you were in grade 6, you learnt organelles of a cell in science lessons. Did not you?

Students: Yes, we did.

Teacher: Then, watch me very carefully. I will draw a figure on the board to help you see that $\frac{0}{5}$ is equal to zero. The cells ingest foreign particles by locally infolding their membranes and protruding their cytoplasms around the fold until they surround the particles and engulf them by closing the membrane.



Think as if zero is a cell and 5 is a foreign particle. As you can see, the cell namely zero totally surrounds the particle (i.e., the number 5) as time progresses and at last the cell engulfs the particle. Thus, $\frac{0}{5}$ is equal to zero while $\frac{5}{0}$ is undefined. That is, dividing zero by a number is zero and dividing a number by zero is undefined.

In addition to this, Teacher A expressed in the post-lesson interview that students could not distinguish between $\frac{0}{5}$ and $\frac{5}{0}$. That is, the teacher stated that students had difficulty in understanding how many times 5 goes into 0 or the vice versa.

Teachers expressed their concerns that students did not easily grasp that the result obtained by using distributive property was valid and always the same with the result obtained by following the order of operations. The following teaching episode of Teacher D illustrates this type of consideration:

Teacher D: Normally, you would initially perform addition and then multiplication operation when you come across with $\frac{4}{7}\cdot\left[\frac{2}{3}+\frac{1}{5}\right]$. Another way to solve this task is to distribute $\frac{4}{7}$ over $\frac{2}{3}$ and $\frac{1}{5}$.

Student: Probably, the two methods will not yield the same result, will they?

Teacher D: Good question! At first, it seems as if the results will not be the same and the students often have difficulty grasping that the two methods will always yield the same result. Now, I will solve this task by using the two methods. However, I am sure some of you will still doubt about the validity of the distributive property (The teacher solved the task by using both approaches in the following way).

$$\frac{4}{7}\cdot\left[\frac{2}{3}+\frac{1}{5}\right]=\frac{4}{7}\cdot\frac{2}{3}+\frac{4}{7}\cdot\frac{1}{5}=\frac{8}{21}+\frac{4}{35}=\frac{40}{105}+\frac{12}{105}=\frac{52}{105}$$

$$\frac{4}{7}\cdot\left[\frac{2}{3}+\frac{1}{5}\right]=\frac{4}{7}\cdot\left[\frac{10}{15}+\frac{3}{15}\right]=\frac{4}{7}\cdot\frac{13}{15}=\frac{52}{105}$$

Did the two methods yield the same result?

Students: Yes!

Teacher D: What I am trying to say is that you can distribute $\frac{4}{7}$ over the rational numbers inside the parenthesis. Now, let me give you another example about distributive property of multiplication over addition.

Teacher B drew attention to the difficulty encountered by students in performing subtraction operation with rational numbers when compared to addition. Some of the expressions he used to explain his approach are as follows:

Teacher B: If we know integers well, we can comfortably perform operations with rational numbers. You quite easily add rational numbers. However, subtracting rational numbers is rather troublesome for you. Especially, when subtracting a negative rational number from another negative rational number, you have some difficulties. Let's have a look at the following example: $\left(-1\frac{2}{9}\right)-\left(\frac{-3}{5}\right)$. For instance, you get confused while solving this operation, since there are several minus signs in it. Besides, most of you have difficulty remembering the procedure learnt for subtraction of integers. You ask me to tell which minus sign should be replaced with a positive sign. By the way, let me explain once again. The sign of the first term, the minuend, does not change, we change the sign of the subtrahend. Meanwhile, the subtraction operation turns into addition operation.

Teacher B pointed to a common student difficulty in the course of teaching multi-step operations with rational numbers. He wrote on the board one example for each type and expressed his consideration as follows:

Teacher B: Mainly, there are three different types of multi-step operations. Now, I am going to write them on the board and teach you how to solve each of them (The teacher split the board into three parts and wrote the following examples as the first, second and third type respectively).

$$\left(2-\frac{1}{3}\right)\div\left(3+\frac{1}{4}\right),\ 1-\frac{2}{3+\frac{1}{1+\frac{1}{2}}},\ 3+\frac{5}{4+\frac{6}{x+2}}=4\,.$$

The first multi-step operation is fairly easy. I do not think you will have any trouble while solving this problem…The second multi-step operation is in complex fraction form. We also call these fractions as stacked fractions. This problem is a bit troublesome when compared to the previous one…Look out! The first problem is very easy, the second problem is a bit more difficult and you have great difficulty in the third problem.

Teacher A was teaching how to order rational numbers with same numerators. He wrote on the board the following set of rational numbers for this purpose: $-\frac{2}{7},-\frac{2}{12},0,\frac{2}{15}$ and $\frac{2}{19}$. He deliberately selected large denominators for these rational numbers to encourage the use of common numerator approach. Based on his prior experience, he knew that students tended to use common denominator approach even though the given set of rational numbers had the same numerators. The teacher expressed his consideration of student difficulty in ordering same numerator rational numbers by common denominator algorithm as follows:

Teacher A: You can see that it is very difficult to find the common denominator for these rational numbers. Why is it so difficult to find the common denominator? Because, 7, 13, 15 and 19 are relatively prime numbers. That is, they do not have a common factor. We can order these rational numbers easily by using same numerator approach. Despite this, students always tend to use common denominator approach. They do not think of using same numerator approach although they see same numbers at the top of the rational numbers. Each year, I ask ordering examples of his type and always there are some students who make a great deal of effort while finding the common denominator. So, if possible use common numerator approach for this type of ordering tasks.

Teacher B explicitly expressed his consideration about a difficulty encountered by students in the course of simplifying rational numbers before multiplication. Some of the expressions used by the teacher to explain his approach were as follows:

Teacher B: Today, I am going to teach you multiplication of rational numbers. First, I want to multiply two positive rational numbers in proper form. For example, let's find the answer of $\frac{3}{8}\cdot\frac{12}{18}$. I will perform this operation initially by using long multiplication method as depicted in your mathematics textbook. Later, you can perform it by using short multiplication method.

Student: Is it possible to simplify rational numbers before we multiply them?

Teacher B: You can also do in that way. But, in the previous years, the students had more difficulty when they used that way.

Teacher A was teaching subtraction operation with rational numbers that did not include any parenthesis. Namely, the teacher was performing the following operation: $-\frac{3}{7}-\frac{2}{3}$. During this time, the teacher uttered that most of the students had difficulty understanding operations such as -9-14. The following teaching episode illustrates teacher consideration of this kind:

Teacher A: Well, how shall we perform this operation?

Student: We should find the least common multiple of 3 and 7.

Teacher A: Yes, we initially find the common denominator as your friend indicated. Thus, $-\frac{3}{7}-\frac{2}{3}$ is equal to $-\frac{9}{21}-\frac{14}{21}$ and we aggregate the numerators over one fraction bar as follows: $\frac{-9-14}{21}$. Most of you have problems with performing operations such as the one at the top of this fraction. As you remember, I asked you to perform operations such as $-9-14$ in your previous examination and most of you had trouble with them. $-9-14$ is equal to -23, why? Because if you owe 9 TLs to your friend, and 14 TLs to another friend, you owe 23 TLs to your friends in total.

Finally, Teacher A attended to a difficulty encountered by the students when finding the square and cube of rational numbers. The teacher indicated that students often had difficulty in distinguishing between an exponent with a power inside the parenthesis and an exponent with a power outside the parenthesis. One of the utterances used by the teacher to express this type of concern is given as follows:

Teacher A: Listen to me very carefully. Now, I am going to write on the board two exponential numbers that resemble to each other on the surface but in essence they have nothing to do with each other. Look through $\left(\frac{-2^3}{5}\right)$ and $\left(\frac{-2}{5}\right)^3$ for some time. In the first exponential number the power is inside the parenthesis and in the second one the power is outside the parenthesis. You often have difficulty in deciding which numbers are influenced by 3 in the first and second exponential numbers. In the first exponential number, 3 has impact on only 2. Even, it does not have any impact on the minus sign preceding 2. However, in the second exponential number 3 has impact on 2, 5 and on the minus sign. Thus, $\left(\frac{-2^3}{5}\right)$ is equal to $\left(\frac{-2.2.2}{5}\right)$ and $\left(\frac{-2}{5}\right)^3$ is equal to $\left(\frac{-2}{5}\right)\cdot\left(\frac{-2}{5}\right)\cdot\left(\frac{-2}{5}\right)$. Is that okay?

Students: Yes!

5.2.2. Attending to students' errors

Teachers claimed that students often make the following mathematical errors related with rational number concepts: (i) ignorance of using parenthesis when operating with negative rational numbers, (ii) making sign errors when adding mixed numbers, (iii) making errors when multiplying a rational number and whole number, (iv) using commas instead of greater-than and less-than signs when ordering rational numbers, (v) making notation errors about mixed numbers, (vi) making errors when finding additive inverse of a rational number, (vii) making errors due to not following order of operations, and finally (viii) making notation errors when performing the exponentiation of unknown variables.

Teachers articulated in the lesson that students often made errors due to the ignorance of using parenthesis when performing operations with rational numbers. For instance, Teacher A made up a scenario about the possible error made by the students while solving the following task: $A=\left(\frac{-5}{8}\right),\ B=\left(\frac{-2}{8}\right),\ A-B=?$ Teacher A's consideration of student errors resulting from ignorance of parenthesis is given as follows:

> Teacher A: While I was teaching integers, I warned you to use parenthesis while substituting numbers into the given expressions. Otherwise, your answer will be wrong. Now, I will solve this task like a student. Watch me very carefully. Students often ignore using parenthesis and substitute A and B in this way: $\frac{-5}{8}-\frac{2}{8}$. This was a possible student solution. Now, I will introduce you the teacher solution. We should write $\frac{-5}{8}$ and $\frac{-2}{8}$ in parenthesis. We can write them in parenthesis in the following way:

$\left(\frac{-5}{8}\right)-\left(\frac{-2}{8}\right)$. You may wonder why there are two minus signs. One of them belongs to the subtraction operation and the other belongs to the rational number itself (i.e., the sign of subtrahend). I am sure ninety percent of the students solve this task erroneously because of ignoring the parenthesis when substituting the numbers into the given expression. When expressions include positive rational numbers, you do not make errors too often but when it comes to the expressions with negative rational numbers you make errors very often.

Teachers pointed that their students' sign errors might originate due to not converting mixed numbers into improper numbers before addition. For instance, this type of consideration was expressed by Teacher B after the provision of $5\frac{3}{4}+\left(-1\frac{1}{4}\right)$ as follows:

Teacher B: How shall we perform this operation?

Student 1: Well, we first add the whole parts.

Teacher B: How about the fractional parts?

Student 1: We add 3 and 1 to find the numerator of the fractional part.

Teacher B: Shall we add or subtract?

Student 1: … (No response)

Teacher B: Watch out! We first subtract the whole parts as (5-1). Similarly, we subtract the fractional parts as $\left(\frac{3-1}{4}\right)$ and find the answer as $4\frac{2}{4}$. Did you understand this way of solution?

Students: No!

Teacher B: I do not recommend adding mixed numbers in this way. I am sure most of you will make errors if you add them in this way. To ensure finding a correct answer, you need to convert mixed numbers into improper numbers before adding. On the contrary, you may certainly be mistaken.

Teachers articulated a student error made by the students while performing operations with a rational number and a whole number. Some of the expressions used by Teacher D after the provision of $3\cdot\frac{11}{8}$ is presented below:

Teacher D: Sometimes students feel perplexed when they are asked to multiply a whole number by a rational number. They make errors since 3 is aligned with neither 11 nor 8. Thus, the students often multiply 3 by both the numerator and the denominator as $3\cdot\frac{11}{8}=\frac{3\cdot 11}{3.8}=\frac{33}{24}$ and arrive at a wrong answer.

Teacher B expressed a concern about the error made by the students in the course of finding additive inverse of rational numbers. The classroom utterances expressed by the teacher for this type of consideration is presented as follows:

> Teacher B: Some of your friends still make errors while finding the additive inverse of a rational number. For instance, let me write $-2\frac{3}{7}$ as an example for finding the additive inverse. I am not sure whether you can find the additive inverse correctly. You persistently make errors while finding it. I do not know why but some of you persistently express the answer as either $-2\frac{3}{7}$ or 0.

Teacher B explicitly stated that students erroneously used commas instead of using greater-than or less-than signs when ordering the given numbers. This consideration is expressed by the following teacher utterances:

> Teacher B: Now, let's think of a number line. In a number line, the number on the left is smaller than the one on the right. If you know this, you can locate the rational numbers on the number line and after that you can easily order them. You should absolutely use symbols for ordering. We use symbols when ordering rational numbers. You can either order from least to greatest or from greatest to least. To reiterate, you should certainly use symbols. However, you should not use commas for ordering. In the previous years, there were some students who used commas when ordering rational numbers. If you use symbols, you end up with a correct arrangement, otherwise it will be erroneous.

Teacher A expressed a consideration about a student error resulting from not following mathematical conventions when expressing mixed numbers. By convention, a mixed number needs to be expressed as a whole number and a proper fraction. Teacher A articulated this type of student error in the course of performing the following operation: $3\frac{5}{12}+\frac{5}{6}$. Some of the expressions he used to explain his approach is presented as follows:

> Teacher A: We can perform this operation in two ways. In the first way, we can convert $3\frac{5}{12}$ into an improper number before adding as:
>
> $$3\frac{5}{12}+\frac{5}{6}=\frac{41}{12}+\frac{5}{6}=\frac{41}{12}+\frac{10}{12}=\frac{51}{12}=4\frac{3}{12}.$$
>
> In the second way, we add the whole number parts and fractional parts and add later as:

$$3\frac{5}{12}+\frac{5}{6}=3\frac{5}{12}+\frac{10}{12}=(3+0)\frac{5+10}{12}=3\frac{15}{12}=3+1\frac{3}{12}=4\frac{3}{12}$$

Student: The first way is easier.

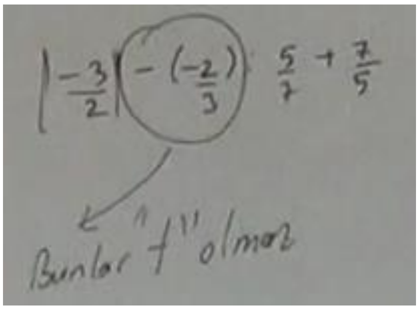
Teacher A: You are absolutely right. I do not advise you to use the second way. Because I am sure you will make a mistake if you use the first way. In the previous years, the students left the operation incomplete by leaving the mixed number in this way: $3\frac{15}{12}$. However, there is no such mixed number in mathematics. Because the fractional part cannot be improper in mixed numbers.

Teacher A paid close attention to a difficulty encountered by the students due to not following the order of operations rule. The teacher was teaching multiplication and division of rational numbers and expressed his consideration explicitly after the provision of $\left|\frac{-3}{2}\right|-\frac{(-2)}{3}\div\frac{5}{7}+\frac{7}{5}$ as follows:

Teacher A: Who wants to explain the step-by-step solution of this task?

Student: Teacher, we will initially perform the subtraction operation in this task, won't we?

Teacher A: Well, you are absolutely wrong. You make the most critical error here. Now, I circle this part of the task and specifically note down a remark as 'the two negatives do not make a positive".



Students: Why?

Teacher A: Because you have to perform the division operation first according to the order of operations rule.

Finally, Teacher D took into account a possible student error that might occur due to using bad notation when performing the exponentiation of unknown variables. The teacher chose to use the following example to express her consideration: find the value of $\frac{x^3-x^2+10}{3}$ where $x=-2$. The convention among mathematicians is to perform x^2 and x^3 by writing -2 inside the parenthesis as $(-2)^2$ and $(-2)^3$

respectively. In the following excerpt, Teacher D manifested her consideration about the error students made when they did not follow the aforementioned convention:

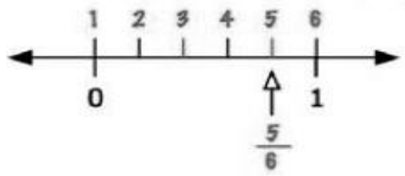
> Teacher D: In this task, you have to substitute -2 into the given single variable expression. You should pay attention to using parenthesis when substituting -2. All you need to do is use parenthesis. Do not forget this! Let me repeat once more. If an expression includes unknown variables such as x^3 and x^2 and if you are asked to substitute a negative number into this expression, you should always put the number in parenthesis while performing exponentiation. If you do not perform in this way, then you will certainly make an error. Namely, you cannot perform the exponentiations as $x^3 = -2^3$ and $x^2 = -2^2$. Is that okay?
>
> Students: Yes!

5.2.3. Attending to students' misconceptions

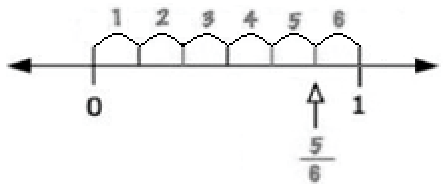
Teachers explicitly uttered that students held the following misconceptions about rational number concepts: (i) counting tick-marks rather than counting equal parts of the line segment when locating a rational number on a number line, (ii) over-generalizing location of positive rational numbers to negative rational numbers, (iii) over-generalizing multiplication and division of rational number algorithms to addition and subtraction of rational numbers, (iv) under-generalizing simplification of rational number multiplication, (v) misapplying multiplication to mixed numbers, (vi) ordering decimals by treating the digits after the decimal points as separate numbers, (vii) performing exponentiation by adding base and power, (viii) performing exponentiation by multiplying base and power, and (ix) believing that a larger number must always be divided by a smaller number.

Teachers expressed that students erroneously focused on tick-marks rather than equal distances when locating rational numbers on a number line. For instance, Teacher D expressed this type of consideration while teaching the location of $\frac{5}{6}$ on a number line as follows:

> Teacher D: When locating rational numbers on a number line, the most salient error you make is counting tick-marks rather than counting equal parts of the line segment. For instance, if I ask you to locate $\frac{5}{6}$ on a number line, never do in the following way:

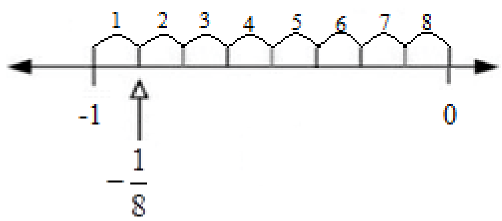


Instead of this, divide the line segment between 0 and 1 into 6 equal parts. Next, count five equal parts by beginning from 0 and mark the end of the fifth equal part as $\frac{5}{6}$ in this way:



Teachers explicitly uttered in the classroom that students over-generalized location of positive rational numbers to negative rational numbers. To illustrate, Teacher C used the following expressions to explain this type of consideration:

> Teacher C: I give considerable emphasis on location of rational numbers on a number line. Students make a lot of mistakes when they try to locate rational numbers on a number line. For instance, when locating a negative rational number they act as if it is a positive rational number. That is, they usually start counting from left to right as they do when locating a positive rational number on a number line. Let's say, we want to locate $-\frac{1}{8}$ on a number line. The student divides the segment between 0 and -1 into eight equal parts. The student needs to start counting the equal parts from 0. However, the students start counting the equal parts from -1 and locate the negative rational number as follows:



> Since the students start counting from the opposite direction, they find the location of $-\frac{7}{8}$ rather than finding the location of $-\frac{1}{8}$.

Teachers articulated in the classroom that students tended to over-generalize multiplication and division of rational number algorithms to algorithms for adding and subtracting rational numbers. The classroom utterances expressed by Teacher A for this type of consideration is given below:

Student: Can I ask you a question?

Teacher A: Yes, you can.

Student: If rational numbers have same numerators and different denominators, can we directly add the numerators?

Teacher A: No, you cannot. Thanks, for your question, you touched upon a good point. When I teach children multiplication and division of rational numbers, they forget how to add and subtract rational numbers. That is, they start to add and subtract rational numbers as if they are multiplying or dividing. For example, when asked to find the answer of $\frac{17}{7}-\frac{8}{7}$, they perform the subtraction operation in this way: $\frac{17}{7}-\frac{8}{7}=\frac{17-8}{7-7}=\frac{9}{0}$. After you learn multiplication and division of rational numbers, never add and subtract rational numbers in this way.

Teachers explicitly stated in the classroom that students under-generalized simplification of rational number multiplication by thinking that simplification can only be done by criss-crossing. The classroom utterances expressed by Teacher A for this type of consideration is given below:

Teacher A: Let's find the answer of $(0.25)\cdot(-0.\bar{9})$. We first convert decimals into rational numbers in this way: $\frac{25}{100}\cdot\left(-\frac{9}{9}\right)$. Now we have to check whether we can simplify rational numbers. 9 divided by 9 is equal to 1.

Student: But, you simplified top to bottom!

Teacher A: In the previous lesson, I emphasized that we can simplify not only by criss-crossing but also by using top to bottom method. However, I still see that there are students who think that they can only simplify rational numbers by criss-crossing. As I stated in the previous lesson, the order of numbers in the numerators and in the denominators is of no importance. What is important is that the numbers that are to be simplified should be at different positions of the main fraction bar. That is, one of them should be over the main fraction bar, the other one should be under the main fraction bar.

Teachers explicitly stated in the classroom that students misinterpreted the meaning of mixed numbers and applied their understanding of whole number multiplication to mixed numbers. The classroom utterances expressed by Teacher D for this type of consideration is given below:

Teacher D: Let's check if $4\cdot\frac{1}{2}$ is equal to $4\frac{1}{2}$ or not.

Students: They are equal! They are equal!

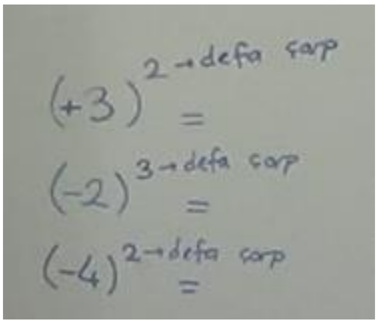
Teacher D: Never think in that way. $4 \cdot \frac{1}{2}$ means that we need to multiply 4 by $\frac{1}{2}$. Thus, $4 \cdot \frac{1}{2}$ is equal to $\frac{4}{1} \cdot \frac{1}{2} = \frac{4 \cdot 1}{1.2} = \frac{4}{2} = 2$. On the other hand, $4\frac{1}{2}$ is a mixed number and it is actually equal to $4 + \frac{1}{2}$. Now, if we convert $4\frac{1}{2}$ into a rational number, we get $4\frac{1}{2} = \frac{4 \times 2 + 1}{2} = \frac{9}{2}$. As you can see $2 \neq \frac{9}{2}$ and thus $4 \cdot \frac{1}{2}$ is not equal to $4\frac{1}{2}$. By asking this problem, I wanted to see whether you can distinguish between mixed numbers and multiplication of a whole number by a rational number. Please, do not confuse them! $4 \cdot \frac{1}{2}$ denotes a multiplication operation, whereas $4\frac{1}{2}$ denotes an addition operation.

Teacher A articulated in the classroom that some students ordered decimals by treating the digits after the decimal points as separate numbers. The classroom utterances expressed by Teacher A for this type of consideration is given below:

Teacher A: Never forget this: if you are ordering decimals that have unequal number of digits after the decimal point, first equate the number of digits after the decimal point. How can you equate the number of digits? You can equate them by adding zeros to the end of the digits. Let me give an example. If you are asked to compare 2.545 and 2.55 you cannot say that 2.545 is larger than 2.55. Because you cannot simply say that 545 > 55 so 2.545>2.55. Then, what should you do? You should first equate the number of digits after the decimal points. 2.545 has three digits after the decimal point and 2.55 has two digits after the decimal point. Therefore, we can write 2.55 as 2.550. Now we can compare the decimal numbers. 550 is larger than 545 so 2.550 is larger than 2.545 and finally 2.55 is larger than 2.545.

Teacher B explicitly stated in the classroom that students erroneously added the base with the power when asked to perform exponentiation of whole numbers. The classroom utterances expressed by Teacher B for this type of consideration is given below:

Teacher B: Well, what does exponentiation mean? It means that we need to MULTIPLY a number with itself as many as the power. I especially wrote “MULTIPLY” by capital letters, since some students erroneously add base and power when performing exponentiations. For instance, let’s perform $(+3)^2, (-2)^3$ and $(-4)^2$ respectively. I want to write “multiply” next to each power in order for you not to make mistakes in this way:



Teacher A articulated in the classroom that students erroneously multiplied the numerator and the denominator of a rational number by the power when asked to perform exponentiation of rational numbers. The classroom utterances expressed by Teacher A for this type of consideration is given below:

> Teacher A: How do you find the answer of $\left(\frac{5}{4}\right)^2$? In the previous years, there were some students who performed exponentiation by multiplying the exponent by the numerator and the denominator in this way: $\left(\frac{5}{4}\right)^2 = \frac{5\times 2}{4\times 2} = \frac{10}{8}$. If you do in this way, your answer will be wrong. $\left(\frac{5}{4}\right)^2$ means that you have to multiply $\frac{5}{4}$ with itself for two times. Thus, $\left(\frac{5}{4}\right)^2$ is equal to $\frac{5}{4}\times\frac{5}{4} = \frac{25}{16}$.

Finally, Teacher D articulated in the classroom that students erroneously had the conception that a larger number must always be divided by a smaller number. Teacher D used the following example for this purpose: "120 bottles of milk with same capacity were evenly filled with 40 liters of milk and there remained 10 liters of milk. Then find the capacity of each milk bottle." The following explanations of Teacher D while solving this task manifested how she took into account student misconception related with whole number thinking:

> Teacher D: If 10 liters of milk remain after filling up all milk bottles then it means that 120 bottles have a total capacity of 30 liters. Now, listen to me very carefully. Most of the students think that they should divide 120 by 30 to find the capacity of each milk. But never forget that there is no rationale for always dividing a larger number by a smaller one. In some cases, you may have to divide a smaller number by a larger one. In this example, you need to divide 30 by 120. Why? Because you have 30 liters of milk and you

> need to distribute them evenly among 120 bottles. You fill each bottle one by one until running out of all of the milk.

In this section, the cases in which teachers attended to student difficulty, error or misconception were described at length. In the following section, the cases in which teachers kept unnecessary work to minimum were described thoroughly.

5.3. Keeping Unnecessary Work to Minimum

Middle school mathematics teachers deliberately attempted to keep unnecessary work to minimum during the provision of rational number examples. Thus, subcategories emerged from this category were (i) reducing technical work by focusing on the essence, (ii) highlighting relevant parts of examples and not going into extra details, and finally (iii) using properties of operations to reduce workload.

5.3.1. Reducing technical work by focusing on the essence

Teachers provided rational number examples in the following way to reduce technical work and focus on the essence: (i) the choice of rational numbers to illustrate repeating decimals, (ii) adding or subtracting whole parts and fractional parts separately when adding or subtracting mixed fractions, (iii) simplifying rational numbers in the course of performing operations, (iv) drawing only the relevant part of a number line when locating rational numbers on it, (v) the choice of relevant strategy when ordering rational numbers, (vi) using LCM method instead of multiplying denominators when finding the common denominator of rational numbers, (vii) not trying to enlarge rational numbers by 1, (viii) using shortcuts for adding and subtracting a whole number and a rational number, (ix) using subtraction formula instead of equating denominators during the subtraction of rational numbers, (x) the choice of same denominator rational numbers when illustrating associative property of addition, (xi) using backwards strategy instead of equating denominators when dealing with complex fractions with unknown values, and (xii) rearranging algebraic expressions for an easier computation.

Teachers attempted to select rational numbers that help students easily notice the repeating pattern when the numerator was divided by the denominator of the rational number. For instance, Teacher D chose $\frac{10}{3}$ to show that it is a repeating decimal. After dividing 10 by 3, the teacher expressed $\frac{10}{3}$ as 3.33333… As it can be seen, only 3 repeats in this decimal and it thus is easy to notice the repeating block in this decimal. The teacher articulated that she deliberately selected $\frac{10}{3}$ since other rational numbers such as $\frac{1}{7}$ needed more technical work to notice that repeating pattern. Indeed to recognize that the repeating block of $\frac{1}{7}$ is 142857, the teacher needs to do extra work for column division of $\frac{1}{7}$.

Teachers took into consideration a shortcut method for adding mixed fractions and suggested their students to add or subtract whole parts and fractional parts separately in order to arrive at the answer in a quicker and shorter way. Some of the expressions used by Teacher D to explain this approach are given as follows:

> Teacher D: There are two methods for computing $1\frac{1}{5}+4\frac{3}{5}$. The first method requires converting mixed fractions into improper numbers as follows: $1\frac{1}{5}+4\frac{3}{5}=\frac{6}{5}+\frac{21}{5}=\frac{26}{5}$. In the second method we need to add whole parts and fractional parts of the mixed numbers separately in this way: $1\frac{1}{5}+4\frac{3}{5}=(1+4)+\left(\frac{1}{5}+\frac{3}{5}\right)=5+\frac{4}{5}=5\frac{4}{5}=\frac{29}{5}$.
>
> Student: The second method is longer.
>
> Teacher D: This is true if you select small mixed numbers. However, if you select large mixed numbers such as $104\frac{2}{8}+99\frac{3}{8}$, then the second way is quite shorter. In this example, it is troublesome to multiply the whole number parts by the denominators and there is the risk of making errors while multiplying them. However, if we use the second method we can easily arrive at the answer in this way:
>
> $104\frac{2}{8}+99\frac{3}{8}=(104+99)+\frac{2+3}{8}=203\frac{5}{8}$.

Teachers recommended students not to simplify rational numbers as a last step since they considered that it was superfluous to work with large numbers after multiplication. For instance, Teacher D uttered the following expressions related with this consideration after the provision of $\frac{0.012}{0.3}-\frac{2}{0.8}+\frac{0.4}{0.02}$ as follows:

> Teacher D: We enlarge the first, second and third term by 1000, 10 and 100 respectively. Thus, we obtain $\frac{12}{300}-\frac{20}{8}+\frac{40}{2}$. It is more convenient to simplify rational numbers now. Because, if you simplify them at the end, you will have to strive for simplifying rational numbers that include very large numerators and denominators. The simplified form of this expression is $\frac{1}{25}-\frac{5}{2}-\frac{20}{1}$ and now we can equate the denominators. The least common multiple of 1, 2 and 25 is 50 so we can rewrite the expression as $\frac{2}{50}-\frac{125}{50}+\frac{1000}{50}$. Finally, this is equal to $\frac{877}{50}$.

Teacher A deliberately attempted to draw only the relevant part of a number line when locating rational numbers on it. That is, when locating $\frac{3}{5}$ on a number line Teacher A drew the interval between 0 and 1. Similarly, when locating $3\frac{2}{5}$ and $\frac{12}{5}$ on a number line, he drew the interval between 0 and 4. The explicit classroom utterances employed by Teacher A for this consideration is provided below:

> Teacher A: You may wonder why I only drew the interval between 0 and 1 when locating $\frac{3}{5}$ on a number line. This is due to the fact that proper fractions are always between 0 and 1. Thus, there is no need to draw a very long number line… $3\frac{2}{5}$ is a positive rational number so there is no need to draw the negative part of the number line… $\frac{12}{5}$ is a positive rational number, so I will draw the interval between 0 and 4.

Teacher B generated a set of rational numbers with large denominators and suggested his students to use benchmark strategies rather than common denominator approach in order not to make an excessive effort for ordering. This consideration was explicitly expressed by the utterances of Teacher B as follows:

> Teacher B: How do you order $\frac{3}{7}, \frac{8}{11}$ and $-\frac{16}{17}$?
>
> Students: By using common denominator approach.

> Teacher B: That is possible. However, you will have to deal with very large numbers if you use common denominator approach. 7, 11 and 17 are prime numbers and thus the least common multiple of these numbers is a very large number. Instead of using this approach, you can use benchmark strategies. Let me explain this: $\frac{3}{7}$ is less than $\frac{1}{2}$ and $\frac{8}{11}$ is greater than $\frac{1}{2}$. Besides, $-\frac{16}{17}$ is the smallest one since it is negative. Thus, we should order the rational numbers as follows: $-\frac{16}{17} < \frac{3}{7} < \frac{8}{11}$.

Teacher B chose to use $\frac{\square}{12} + \frac{\square}{8}$ to help students realize that adding by finding the LCM of the denominators required less technical move when compared to adding by enlarging the first term by the denominator of the second term and the second term by the denominator of the first term. He expressed this consideration explicitly as follows:

> Teacher B: How do you equate the denominators of $\frac{\square}{12}$ and $\frac{\square}{8}$?
>
> Student 1: We multiply 12 by 8 and 8 by 12. Thus, both denominators take the value of 96.
>
> Teacher B: That is true, but the denominators can take a smaller value. How? We should find the least common multiple of 12 and 8. The LCM of 12 and 8 is 24. Thus, we should enlarge $\frac{\square}{12}$ by 2 and $\frac{\square}{8}$ by 3. The more you enlarge the rational numbers with smaller numbers, the less you spend time on finding the answer.

Teacher A indicated that it was unnecessary to enlarge a rational number by 1 when performing the following addition operation: $\frac{(-2)}{3} + \frac{(+7)}{2} - \frac{(-1)}{6}$. The classroom utterances of Teacher A for this type of consideration is provided below:

> Teacher A: We initially need to adjust the signs. Thus, we rearrange the expression as $\frac{(-2)}{3} + \frac{(+7)}{2} + \frac{(+1)}{6}$. Next, we need to equate the denominators of the rational numbers. The least common multiple of 3, 2 and 6 is 6. Thus, we should enlarge $\frac{(-2)}{3}$ by 2, $\frac{(+7)}{2}$ by 3 and $\frac{(+1)}{6}$ by nothing. Please, do not waste your time by enlarging $\frac{(+1)}{6}$ by 1…

Teacher D considered using shortcuts when adding and subtracting a whole number and a rational number. Teacher D expressed this consideration explicitly after the provision of $2+\frac{3}{5}$ as follows:

> Teacher D: There are two ways of adding 2 and $\frac{3}{5}$. In the first way, we add them by using common denominator approach as follows: $2+\frac{3}{5}=\frac{2}{1}+\frac{3}{5}=\frac{10}{5}+\frac{3}{5}=\frac{13}{5}$. This way is quite time consuming and lengthy. I suggest you to use the second way. In the second method, you multiply the whole number by the denominator of the rational number and then add the numerator of the rational number as follows: $2+\frac{3}{5}=\frac{2\times 5+3}{5}=\frac{13}{5}$. Instead of wasting time by using common denominator approach, you can use the second method to find the answer quicker.

Teacher A considered the use of subtraction formula as a shorter and quicker approach to subtracting rational numbers instead of using common denominator approach. The teacher did not write on the board the following formula $\frac{a}{b}-\frac{c}{d}=\frac{a\cdot d-b\cdot c}{bd}$ but verbally explained how to apply it as follows:

> Teacher A: I want to teach you a shortcut method for performing $\frac{2}{5}-\frac{1}{2}$. Listen to me very carefully, I am teaching you how to subtract rational numbers without actually finding the common denominator. We multiply the first numerator by the second denominator and we get $2\times 2=4$. Then, we multiply the first denominator by the second numerator and get $5\times 1=5$. Next, we subtract 5 from 4 and find the numerator of the answer as -1. Finally, we multiply the first denominator by the second denominator and get $5\times 2=10$ as the denominator of the answer. Thus, $\frac{2}{5}-\frac{1}{2}$ is equal to $-\frac{1}{10}$.

Teacher A chose to use the same denominator rational numbers when teaching associative property of addition of rational numbers. The teacher implied that it would bring extra work to select rational numbers with different denominators for teaching this property. Teacher A expressed this type of consideration as follows:

> Teacher A: Previously, we checked whether associative property of addition holds for integers. Now, we do the same thing for rational numbers. I want to select same denominator rational numbers since I do not want to spend time

for the extra work of equating denominators. Let's see whether the following equality holds:

$$\left(\frac{1}{2}+\frac{3}{2}\right)+\frac{7}{2}\overset{?}{=}\frac{1}{2}+\left(\frac{3}{2}+\frac{7}{2}\right)$$

$$\frac{4}{2}+\frac{3}{2}\overset{?}{=}\frac{1}{2}+\frac{10}{2}$$

$$\frac{7}{2}=\frac{11}{2}$$

The left hand side of the equation is equal to the right hand side of it. Then, we can say that associative property of addition holds for rational number set.

Teacher A was teaching how to operate with complex fractions that included unknown values. The teacher used the following example: $3-\dfrac{2}{5-\dfrac{1}{x}}=1$ and alerted students to the use of backwards strategy rather than using common denominator approach for solving it. The following utterances of Teacher A demonstrate how he took into account reducing technical work while solving aforementioned type of complex fractions:

> Teacher A: A normal student would not attempt to use common denominator approach for solving this task. Only an inattentive and mistaken student would use this approach. We can solve this task with less time and effort by using backwards strategy. Otherwise, it would be so hard to arrive at the answer.

Finally, Teacher D pointed to rearranging algebraic expressions for an easier computation. In more detail, the teacher was teaching multi-step operations with rational numbers and she calculated the value of $\frac{5x}{3}-x+\frac{3}{4}x$ for $x=-\frac{3}{4}$ by using two ways. After the explanation of the second way, the teacher suggested her students to use the second way for an easier computation. The explicit classroom utterances employed by Teacher D for this type of consideration is given below:

> Teacher D: We can solve this example in two ways. In the first way, we substitute x into polynomial as follows: $\dfrac{5\cdot\left(-\dfrac{3}{4}\right)}{3}-\left(-\dfrac{3}{4}\right)+\dfrac{3}{4}\cdot\left(-\dfrac{3}{4}\right)$. However, this way imposes more operational burden on you. To reduce this operational burden you can use the second way. In the second way, you need

to rearrange the polynomial as $\frac{5}{3} \cdot x - x + \frac{3}{4} \cdot x$ and then substitute x into this polynomial. We can do it in this way: $\frac{5}{3} \cdot \left(-\frac{3}{4}\right) - \left(-\frac{3}{4}\right) + \frac{3}{4} \cdot \left(-\frac{3}{4}\right)$. As you can see, the second multi-step operation is simpler to calculate. If you do in this way, you will have to work less to calculate it.

5.3.2. Highlighting relevant parts of examples and not going into extra details

Teachers considered highlighting relevant parts of examples and not going into extra details in the following ways: (i) emphasizing important parts of an example and not finishing up all the calculations, (ii) not seeing it essential to perform simplifications in the course of teaching a concept, (iii) not seeing it essential to perform conversions in the course of teaching a concept and finally (iv) not seeing it essential to equate denominators when symbolically expressing the area model of multiplication of rational numbers.

Teachers emphasized important parts of the examples they used and did not find it necessary to finish up all the calculations. For instance, Teacher C provided $\frac{1}{2} + \frac{\left[\frac{3}{4} - \frac{7}{5}\right] : \frac{4}{5}}{\frac{1}{2} - \frac{5}{8}} + 5$ as a multi-step operation example, however he highlighted only the important points of this example and did not finish up the calculation. The explicit classroom utterances employed by Teacher C for this type of consideration is presented below:

> Teacher C: Let me explain briefly how to perform this multi-step operation. Listen to me very carefully. You will first perform $\left[\frac{3}{4} - \frac{7}{5}\right] : \frac{4}{5}$ and find a rational number. Next, you will perform $\frac{1}{2} - \frac{5}{8}$ and find another rational number. Later, you will divide the former rational number to the latter rational number. After this, you will add   to the rational number you obtained as a result of division. Finally, you will $\frac{1}{2}$ add 5 and arrive at the answer.

Teachers did not find it necessary to perform simplifications in the course of teaching a concept. For instance, Teacher D expressed this type of concern when she started teaching division of rational numbers via the following example: $\frac{3}{5} \div \frac{6}{7}$. Some of the expressions she used to explain her approach are as follows:

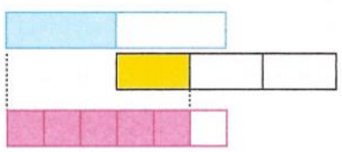
> Teacher D: To perform division operation with rational numbers, you need to write down the first rational number without making any modification, and reverse the second rational number. Then, you need to multiply the rational numbers as follows: $\frac{3}{5} \div \frac{6}{7} = \frac{3}{5} \times \frac{7}{6} = \frac{21}{30}$. In this example, we could have simplified the rational numbers. However, we do not need such work for the time being. Here, knowing how to perform division operation is of prior importance. The rest is extra detail.

Teacher B was teaching the procedure for expressing repeating decimals as rational numbers. The teacher provided the following example for this purpose: $1.\overline{29} = \frac{129-1}{99} = \frac{128}{99}$. As can be seen, the teacher leaved the rational number in improper form and did not find it essential to convert it into a mixed number. The following teaching episode shows how Teacher B took into account this type of consideration:

> Student 1: Shall we convert $\frac{128}{99}$ into a mixed number?
>
> Teacher B: No, there is no need to convert it into a mixed number.
>
> Student 2: Won't we convert it into mixed number?
>
> Teacher B: No, leave it in that form. Do not spend your time for conversion.

Finally, Teacher D was teaching how to express symbolically the area model of multiplication of rational numbers. The teacher did not find it necessary to equate the denominators of the symbolic expression. This consideration was expressed by Teacher D as follows:

> Teacher D: How do you express the following area model symbolically?



> Student: The first shaded region refers to $\frac{1}{2}$, the second shaded region refers to $\frac{1}{3}$. Now, we should add $\frac{1}{2}$ and $\frac{1}{3}$ as follows: $\frac{1}{2} + \frac{1}{3} = \frac{3}{6} + \frac{2}{6} = \frac{5}{6}$.

Teacher D: I see that many of you enlarged $\frac{1}{2}$ by 3 and $\frac{1}{3}$ by 2 to equate the denominators of the rational numbers. However, you do not need to do such work. It is enough to leave the symbolic expression as $\frac{1}{2}+\frac{1}{3}=\frac{5}{6}$.

5.3.3. Using properties of operations to reduce workload

Teachers attempted to use properties of rational number operations to diminish the workload in the following ways: (i) using commutative property of addition operation rather than adding, (ii) using associative property of addition operation rather than adding, (iii) using distributive property of multiplication over addition rather than performing the operation, (iv) using the fact that $\frac{1}{a/b}=\frac{b}{a}$ without actually making computations, (v) using the fact that $\frac{a/b}{a/b}=1$ without actually making computations, (vi) using the fact that $\frac{a}{b}+\left(-\frac{a}{b}\right)=0$ without actually making computations, and finally (vii) enlarging decimal numerators and decimal denominators by multiples of 10 instead of converting into rational numbers when performing multi-step operations.

Teachers deliberately used commutative property of addition rather than performing several operations for solving a mathematical task. By this way, the teachers intended to reduce operational workload needed to solve that task. For instance, Teacher A considered this type of consideration after the provision of $\left(-\frac{3}{5}\right)-\left(-\frac{8}{7}\right)=(\Delta)+\left(-\frac{3}{5}\right)$ as follows:

Teacher A: Using commutative property of addition for finding Δ in this task, will make your work easier and will help you save time. If you do not use commutative property, you need to subtract $\left(-\frac{8}{7}\right)$ from $\left(-\frac{3}{5}\right)$ and then again subtract $\left(-\frac{3}{5}\right)$ to find Δ. That is to say, performing addition and subtraction operations for finding Δ, will make you spend too much time for this task.

Teachers chose to use associative property of addition rather than performing a number of operations for solving a mathematical task. By this way, the teachers intended to reduce operational workload needed to solve that task. For instance, Teacher B considered this type of consideration after the provision of $\left[\frac{3}{8}+\frac{5}{4}\right]+\left(-\frac{1}{6}\right)=\frac{3}{8}+\left[A+\left(-\frac{1}{6}\right)\right]$ as follows:

> Teacher B: You can easily solve this task by using associative property of addition of rational numbers. Never attempt to add or subtract rational numbers to find *A*. If you add or subtract rational numbers, you will certainly waste your time. Besides, there is the risk of making a mistake when performing operations. Thus, let's find *A* by using associative property. $\frac{3}{8}$ matches with the other $\frac{3}{8}$, $\left(-\frac{1}{6}\right)$ matches with the other $\left(-\frac{1}{6}\right)$ and $\frac{5}{4}$ matches with *A*. Thus, *A* is equal to $\frac{5}{4}$.

Teachers preferred to use distributive property of multiplication over addition rather than performing several operations for solving a mathematical task. By this way, the teachers attempted to keep the operational workload to minimum. For instance, Teacher D considered this type of consideration after the provision of $\frac{3}{7}+\frac{5}{9}+\frac{3}{11}-\left(\frac{5}{9}-\frac{4}{7}+\frac{3}{11}\right)$ as follows:

> Teacher D: To solve this task, you should carry out the following steps: First, you should distributive the minus sign to $\frac{5}{9},\frac{4}{7}$ and $\frac{3}{11}$ respectively. Why do we distribute? Otherwise, it would impose too much operational burden on you. As you see, some of the rational numbers have same magnitudes but have opposite signs. In such tasks, use distributive property instead of customarily performing operations. Let me repeat again. Please, do not dare to use common denominator approach for solving this task.

Teachers were teaching complex fractions and they chose to use multiplicative inverse property of rational numbers as a fast solution technique for complex fraction tasks. To be more precise, they used the following equality $\frac{1}{a/b}=\frac{b}{a}$ without actually making computations when solving complex fraction tasks. For instance, Teacher B expressed this type of consideration after the provision of $1-\dfrac{2}{3+\dfrac{1}{\frac{3}{2}}}$ as follows:

> Teacher B: Now, tell me what happens when you divide 1 by any rational number?
>
> Student: The rational number becomes upside down.
>
> Teacher B: Yes, the number becomes upside down. For instance, in this example, 2 and 3 swap places. In a division operation, if the dividend is 1 and if the divisor is any rational number, the quotient will always be multiplicative inverse of the divisor. Thus, $\dfrac{1}{\frac{3}{2}}$ is equal to $\dfrac{2}{3}$ and you can always use this principle and you do not need to perform division operation for such situations.

Teacher D was teaching complex fractions and she chose to use $\dfrac{a/b}{a/b}=1$ as a fast solution technique for complex fraction tasks. Some of the expressions used by Teacher D after the provision of $\dfrac{0.\bar{2}+2}{3-0.\bar{7}}$ as follows:

> Teacher D: In this example, there are repeating decimals. First, we need to convert them into rational numbers. $0.\bar{2}$ can be expressed as $\frac{2}{9}$ and $0.\bar{7}$ can be expressed as $\frac{7}{9}$. Now, we can perform the operations in this way: $\dfrac{0.\bar{2}+2}{3-0.\bar{7}}=\dfrac{\frac{2}{9}+2}{3-\frac{7}{9}}=\dfrac{\frac{20}{9}}{\frac{20}{9}}$. What will be the answer if you divide a rational number by the same rational number? The result will be 1.
>
> Student: Well, won't we invert and multiply when performing division operation?
>
> Teacher: It makes no difference. Let's do it that way: $\dfrac{20}{9}\div\dfrac{20}{9}=\dfrac{\cancel{20}}{\cancel{9}}\times\dfrac{\cancel{9}}{\cancel{20}}=\dfrac{1}{1}=1$. As you can see, the result is 1. Therefore, if you divide a rational number by the same rational number, the result will always be 1. Keep this in your minds and do not waste your time by the lengthy process of performing division operations for such cases. Is that okay?

Students: Yes!

Teacher D was teaching how to perform exponentiation of rational numbers and she considered the use of additive inverse property of rational numbers to reduce operational burdens. In other words, rather than actually making computations, the teacher used the following equality $\frac{a}{b}+\left(-\frac{a}{b}\right)=0$ during the solution of multi-step operation examples. Some of the expressions used by the teacher after the provision of $\left(-\frac{1}{2}\right)^3\div\left(\frac{1}{2}\right)^2+\frac{1}{2}$ is given below:

> Teacher D: In this example, we should initially find the square and cube of rational numbers. Remember how we performed exponentiation with rational numbers. We can find the powers of numerators and denominators separately for faster solution in this way: $\frac{(-1)^3}{2^3}\div\frac{1^2}{2^2}+\frac{1}{2}=-\frac{1}{8}\div\frac{1}{4}+\frac{1}{2}$. Next, we perform division operation as $-\frac{1}{8}\div\frac{1}{4}=-\frac{1}{8}\times\frac{4}{1}=-\frac{1}{2}$. Finally, we should add $-\frac{1}{2}$ and $\frac{1}{2}$. Actually, you do not need to perform addition operation. Why? Because addition of a rational number and its additive inverse will always be equal to 0. Then the answer is 0.

Ultimately, Teacher A was teaching multi-step operations whose terms included decimal numerators and denominators. The teacher considered that the easiest way to perform such multi-step operations was enlarging decimal numerators and denominators by multiples of 10. This consideration was manifested by Teacher A after the provision of $\frac{0.35}{0.05}+\frac{0.7}{0.0035}-\frac{0.22}{0.0011}$ as follows:

> Teacher A: It is not easy to work with decimal terms so we need to enlarge the numerators or denominators by multiples of 10 and get rid of decimal numbers. We enlarge the first term by 100, the second term by 10000 and the third term by 10000 and obtain the following expression: $\frac{35}{5}+\frac{7000}{35}-\frac{2200}{11}$. Now, let's perform division operations. The first term is equal to 7, the second term is equal to 200 and the third term is also equal to 200. Finally 7+200-200 is equal to 7. Thus, the answer is 7. As you can see, this way is quite easy. There is another way of solving this task, but I do not suggest that way. Because it entails a lengthy process such as converting decimals into rational numbers, finding common denominators and so forth.

After all, you can solve this task by using the second way as follows:

$$\frac{\frac{35}{100}}{\frac{5}{100}}+\frac{\frac{7}{10}}{\frac{35}{10000}}-\frac{\frac{22}{100}}{\frac{11}{10000}}.$$

In this section, the cases in which teachers kept unnecessary work to minimum were described at length. In the following section, the cases in which teachers took account of examinations were described in detail.

5.4. Taking Account of Examinations

Middle school mathematics teachers considered examinations during the provision of rational number examples. They manifested this type of consideration in the following cases: (i) highlighting examples that have the potential to appear in written examinations, (ii) highlighting examples that have the potential to appear in practice examinations of private teaching institutions, (iii) highlighting examples that have the potential to appear in high stakes examinations, (iv) explaining the method of scoring for potential written examination questions, (v) incorporating the solution of high-stakes examination examples into the classroom; (vi) expressing the answer of multiple choice questions in their simplest forms in order to find it in the alternatives, (vii) finding the answer of multiple choice complex fraction tasks by trial and error of the alternatives and finally (viii) teaching shortcut methods for gaining speed in the high stakes examinations.

Teachers informed their students about the important rational number concepts and highlighted the examples that had the potential to appear in written examinations. Teacher A expressed the following utterances after providing $-\frac{3}{7}-\frac{2}{3}$ as a subtraction example: “The students have difficulty in performing operations with no parenthesis. However, I am planning to ask such tasks in your written exam”. Teacher B expressed this type of consideration while providing examples related with locating a rational number on a number line, converting repeating decimals into rational numbers, ordering rational numbers, teaching properties of rational number operations, teaching multi-step rational number operations, and exponentiation of rational numbers. For instance, after providing $\left(-\frac{3}{5}\right)^2$ and $\left(-\frac{3}{2}\right)^3$ as examples for exponentiation Teacher B expressed the following utterances: “I will ask you questions of this kind in your written exam. For instance, I will ask one group of the students to find the square of the rational number and ask another group to find the cube of the rational number”. Teacher C expressed this type of consideration after providing the following set of rational numbers for ordering: $-\frac{5}{3}, 0, \frac{4}{3}, \frac{5}{12}, \frac{2}{18}, -\frac{2}{6}$. He expressed this approach by utterances such as: “Next Friday, you will take your second written exam. You will absolutely come across with at least one ordering question. Without a doubt, every year I ask ordering questions to the seventh graders”. Teacher D expressed this type of consideration while providing examples related with locating rational numbers on a number line, expressing rational numbers in different forms, ordering rational numbers, adding rational numbers, modelling of multiplication of rational numbers and performing multi-step rational number operations. For instance, after asking students to find the value of $\frac{5x}{3} - x + \frac{3}{4}x$ for $x = -\frac{3}{4}$, she expressed the following utterances: “Let me inform you that I will ask a question like this one in your written exam”.

Apart from written examinations, teachers highlighted rational number examples that had the potential to come up in practice examinations of private teaching institutions (known as dershane). Teacher B expressed this type of consideration during the provision of examples related with locating a rational number on a number line, adding and subtracting rational numbers and performing complex fraction operations. For instance, he provided $-\frac{8}{7} - \left(-\frac{3}{42}\right)$ as a subtraction example and stated his consideration as follows:

> Teacher B: You often come up with these types of questions in practice examinations of private teaching institutions and you often get confused while solving it. I often denote operation sign larger when compared to the number sign. However, the computers cannot do the same. Namely, the two signs are of the same size in computer print-outs.

Student: You are right teacher, we always mix them up in the practice examinations.

Teacher C articulated this type of consideration during the provision of examples related with explanation and location of rational numbers on a number line. To illustrate, Teacher C was teaching the definition of rational numbers to his students, when he expressed that the rationality of 0 might come up in practice examinations as follows:

Teacher C: Rational numbers include positive rational numbers, negative rational numbers and 0. Zero has a special case, it is neither negative nor positive. Although it does not have any sign, it is a rational number. Please, keep in your mind that practice examinations include questions related with the rationality of 0.

In addition to practice examinations, Teacher A highlighted the examples that had the potential to appear in high stakes examinations such as SBS (Secondary School Entrance Examination for middle school students in Turkey). This consideration was manifested by Teacher A when working out examples related with explanation and location of rational numbers, expressing rational numbers in different forms, comparing and ordering rational numbers, and multiplication of rational numbers. The explicit classroom utterances expressed by the teacher for this type of consideration is given below:

Teacher A: Between 2009 and 2013, three questions have been asked in SBS examinations about number sets. Namely, three questions about number sets have been asked in the last five years. Why? Because, students in general do not pay much attention to number sets… Proper fractions are always between 0 and 1 and negative proper fractions are always between -1 and 0. So far, two questions have been asked in SBS examinations about location of proper fractions. Do not forget this. Let me repeat again, location of proper fractions do appear in SBS examinations. This is due to the fact that proper fractions are the most special ones among fractions... Please note that $5.\overline{9}$ is equal to 6. Why? Because we convert $5.\overline{9}$ into a rational number as $5\frac{9}{9}$ and this mixed number can be expressed as $5+\frac{9}{9}=5+1=6$. Thus, $a.\overline{9}$ is equal to a+1. This was a question similar to the one that came up in SBS examination in the past years… How can we order the following fractions: $\frac{3}{4}$, $\frac{5}{6}$ and $\frac{7}{8}$? The first fraction requires $\frac{1}{4}$ to make 1, the second fraction requires $\frac{1}{6}$ to make 1 and the last fraction requires $\frac{1}{8}$ to make 1. Thus,

$\frac{1}{4} > \frac{1}{6} > \frac{1}{8}$ and $\frac{3}{4} < \frac{5}{6} < \frac{7}{8}$. There are definitely ordering questions of this kind in SBS examination… Let's write a mathematical statement based on the given area model for multiplication of rational numbers. In modelling examples, the double shaded region refers to the product of the multiplication operation. According to me, modelling of multiplication is a very important concept. However, interestingly, there appeared not a single question about modelling of rational number multiplication in SBS examinations till now.

Some of the middle school teachers not only highlighted examples that had the potential to appear in written examinations but also pointed to their method of scoring potential written examination questions. Teacher B expressed this type of consideration in the course of ordering $-\frac{1}{7}, 2\frac{4}{9}, 1\frac{1}{4}$ and $-1\frac{5}{6}$ as follows:

Teacher B: I gave you four rational numbers. Let's order them.

Student: Shall we order from least to greatest?

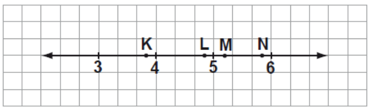
Teacher B: Yes, we may order from least to greatest. If I ask you a question like this in your written examination, you should immediately write the smallest one, then a larger one, again a larger one and finally the largest one. If you order in this way, you get full points. You get full points as long as you correctly order the given rational numbers. If you write one of the rational numbers in the wrong order, you cannot get any points from that question. Since this is an open-ended question, I want you to order all rational numbers correctly. For instance, if you order three of them correctly and one of them incorrectly, I cannot accept your answer. Therefore, please do not make a mistake while ordering rational numbers.

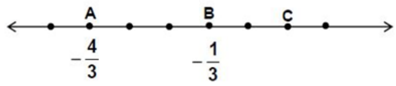
Teacher D expressed this type of consideration while teaching how to locate $-2\frac{4}{5}$ on a number line as follows:

Teacher D: When partitioning the line segment between -2 and -3, please pay attention to counting the intervals rather than counting the tick-marks. If you count the tick-marks, you will be mistaken. If you make such a mistake in your written exam, I will not accept your answers. I am telling this to you again and again in order for you not to make mistakes when locating. Please be careful! Count the intervals rather than the tick-marks. In this example, there are 6 tick-marks and 5 equal intervals between -2 and -3. Suppose you located the number correctly by counting tick-marks, I again do not accept your answer. Why? Because, you should have located the number by counting the equal intervals rather than tick-marks.

In addition to verbally expressing the rational number examples that might appear in written/practice/high stakes examinations, Teacher A attempted to incorporate several SBS and ÖSS (University Entrance Examination for secondary school students in Turkey) examples into the classroom. The teacher explicitly uttered that he would like to present several high stakes examination examples after he finished ordering $\frac{2}{5}, \frac{7}{10}$ and $\frac{7}{3}$ as follows:

> Teacher A: In the previous lesson, we ordered rational numbers that included different numerators and denominators. Now, it is time to have a look at SBS and ÖSS questions that have been asked in the previous years (The teacher sketched the following examples on the board by copying from a booklet).





> Look at these examples. The one on the left have been asked in SBS in 2010 to seventh grade students. In this example, you are asked to determine the point that corresponds to $\frac{21}{4}$. The example on the right have been asked in SBS in 2008 to seventh grade students. In this example, you are asked to determine the number that correspond to point C. Now, let me work out these examples… Now, it is time to order $\frac{1996}{1997}, \frac{1997}{1998}$ and $\frac{1998}{1999}$. This question was asked in ÖSS in 1996. I took this examination and ordered them in that examination. Now it is your turn…

Another way teachers took account of examination was seen in their attempts to express the answer of multiple choice questions in their simplest forms in order to find it in the alternatives. Teacher A manifested this concern when working out examples related with expressing rational numbers in different forms, ordering rational numbers, multiplying rational numbers, teaching distributive property of multiplication over addition and modelling multiplication of rational numbers. For instance, Teacher A expressed this type of consideration after the provision of $\left(\frac{2}{5}+\frac{1}{10}\right)\cdot\frac{1}{2}$ as follows:

> Teacher A: We distributive $\frac{1}{2}$ over $\frac{2}{5}$ and $\frac{1}{10}$ in this way: $\frac{\cancel{2}}{5}\cdot\frac{1}{\cancel{2}}+\frac{1}{10}\cdot\frac{1}{2}=\frac{1}{5}+\frac{1}{20}=\frac{4}{20}+\frac{1}{20}=\frac{5}{20}$. If you leave the answer as $\frac{5}{20}$, you cannot find it in the alternatives. Then, what should you do?
>
> Students: We should simplify it!
>
> Teacher A: By which number should I simplify it?

Students: Five!

Teacher A: We simplify it by 5 as $\frac{5:5}{20:5}=\frac{1}{4}$. This time, the answer is in its simplest form. Now, you can find it in the alternatives.

Teacher B manifested this type of consideration in course of expressing rational numbers in different forms and multiplying and dividing rational numbers. For example, he employed this type of consideration after converting $1.0\overline{45}$ into a rational number as:

Teacher B: We convert this repeating decimal into a rational number as: $1.0\overline{45}=\frac{1045-10}{990}=\frac{1035}{990}$.

Student: Do we have to find the simplest form of $\frac{1035}{990}$?

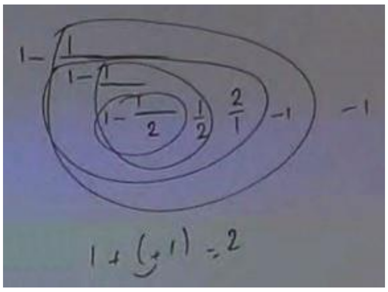
Teacher B: Normally, I do not expect you to simplify it. However, in practice examinations or in high stakes examinations, you cannot find $\frac{1035}{990}$ in the alternatives. In these examinations, the alternatives are given in their simplest forms. Thus, you need to simplify it if you participate in such examinations.

Another case in which teachers took account of examinations occurred when teachers attempted to find the answers of multiple choice complex fraction tasks by trial and error of the alternatives. Teachers used two different methods to solve such tasks. In the first method, teachers used working backwards strategy. In the second method, they substituted each alternative into the given complex fraction task, in order to find the correct answer. After the presentation of $6-\frac{8}{6-\frac{4}{x}}=4$ as a complex fraction example, Teacher A initially found the answer by working backwards. Next, he explicitly expressed the following utterances: "There is another method for finding the answer if the given task is a multiple choice question. That is, you can try each alternative in the given complex fraction task to find the correct answer". Similarly, Teacher B introduced $\frac{2}{\frac{6}{x-1}-5}=-1$ as a complex fraction example and initially solved the task by using working backwards strategy. Next, he solved this task by trial and error of the alternatives. The alternatives were $\frac{1}{4}, \frac{1}{2}, 2$ and 3 respectively. Some of the expressions he used to explain his approach were:

> Teacher B: In addition to using working backward strategy, we can find the answer by trying each alternative. Let's try 2 first. If we substitute 2 into *x*, we find the answer as 2. This is not the correct answer. So, let's try 3 now. If we substitute 3 into *x*, we find the answer as -1. This is the correct answer. You can use this method in practice examinations or in high stakes examinations.

One final consideration of this type was seen in teachers' attempts to teach shortcut methods to their students for gaining speed in the high stakes examinations. Teacher A manifested this concern when working out multi-step operation examples. For instance, after working out $1-\frac{1}{1-\frac{1}{1-\frac{1}{2}}}$ he used the following statements:

> Teacher A: I worked out the first two multi-step operation examples by using a long way. Now, I am switching to a fast solution technique. In this technique, you circle each step and write the answer next to each circle as follows:



> My aim for teaching this technique is to have you save time during examinations. Do not waste your time by writing each step again and again as you proceed towards the result.

Teacher C manifested this concern in the course of expressing rational numbers in different forms and adding and subtracting rational numbers. For instance, he suggested performing $1+\frac{1}{4}$ as $\frac{1\times 4+1}{4}=\frac{5}{4}$ rather than using the common denominator approach. Finally, Teacher D taught shortcut methods to their students for gaining speed in the high stakes examinations in the course of expressing rational numbers in different forms, adding and subtracting rational numbers, performing exponentiation of rational numbers and performing multi-step operations with rational numbers. For instance, after the provision of $\frac{0.012}{0.3}-\frac{2}{0.8}+\frac{0.4}{0.02}$ as a multi-step operation example, Teacher D explicitly uttered the following expressions:

> Teacher D: Normally, you would solve this task by converting decimal numbers into rational numbers and then performing division operations. However, this is a lengthy process. Instead of this, you can enlarge the terms by multiples of 10 and get rid of decimal numbers. You need to gain speed for high stakes examinations. Therefore, you had better use this method when solving these kinds of tasks.

5.5. Including Uncommon Cases

This consideration had to do with middle school mathematics teachers' attempts to choose examples that are rather exceptional or special in mathematics or examples that are under-represented in the teaching of rational number concepts. Thus, subcategories emerged from this category were entitled as (i) exceptional or special cases in the teaching of rational number concepts and (ii) under-represented cases in the teaching of rational number concepts.

5.5.1. Exceptional or special cases in the teaching of rational number concepts

Middle school mathematics teachers chose to use the following exceptional or special cases in the teaching of rational number concepts: (i) multiplying any rational number by 0 yields 0, (ii) multiplying any rational number by 1 yields the rational number itself, (iii) dividing any rational number by 0 is undefined, (iv) dividing any rational number by 1 yields the rational number itself, (v) dividing 0 by any rational number excluding 0 yields 0, (vi) dividing 1 by any rational number excluding 0 yields the multiplicative inverse of that rational number, (vii) dividing -1 by any rational number excluding 0 yields the additive inverse of the multiplicative inverse of that rational number, (viii) raising any nonzero rational number to the power of 0 yields 1, and finally, (ix) raising 1 to any rational number power yields 1.

Teachers pointed to the zero property of multiplication in the course of teaching properties of rational number multiplication. For instance, during the teaching of rational number multiplication, Teacher B initially had students review the examples included in the textbook. Teacher B paid attention to the following example: $\left(-\frac{4}{5}\right)\cdot 0$ and explained that multiplying any rational number by 0 yields 0. In addition, while teaching properties of multiplication of rational numbers he chose to use $0\cdot\frac{5}{8}$ as an example for zero property of multiplication. The explicit classroom utterances expressed by Teacher B for this consideration were provided below:

> Teacher B: What is absorbing element in multiplication?
>
> Students: Zero!
>
> Teacher B: Yes, absorbing element is 0 in multiplication. If you multiply any rational number either from the left side or from the right side by 0, the answer will be 0. Let me give you an example. $0\cdot\frac{5}{8}$ is equal to 0. Why? Because 0 is the absorbing element in multiplication of rational numbers.

While teaching properties of rational number multiplication, some of the teachers pointed out that multiplying any rational number by 1 yields the rational number itself and they recalled 1 as identity property of multiplication operation. For instance, after the provision of $\left(+\frac{1}{2}\right)\cdot 1=\frac{1}{2}$, Teacher D articulated the following consideration:

> Teacher D: Identity element is an element which does not influence the product of multiplication. What is that element? That is 1. If you multiply any rational number by 1 you get the same rational number as a product. If you multiply $\frac{1}{2}$ by 1, you will again get $\frac{1}{2}$. Thus, 1 is the identity element of multiplication of rational numbers.

Apart from teaching the special cases of multiplication of any rational number by 0 or 1, the teachers indicated awareness to teaching the special cases of division of any rational number by 0 or 1. For example, Teacher D defined rational numbers as numbers that can be written in the form $\frac{a}{b}$, where *b* is not equal to zero and provided $\frac{7}{0}$ as a non-example for rational numbers. During this time, Teacher D expressed that division of any rational number by 0 will always be undefined. In another example, Teacher D pointed to the special case of division of any rational number by 1. Namely, she provided $\frac{6}{1+\frac{5}{a}}=1$ as a complex fraction example and while working out this example, she focused on the division of any rational number by 1. Her consideration of this special case is given as follows:

> Teacher D: Here, $1+\frac{5}{a}$ is equal to 6, thus $\frac{5}{a}$ is equal to 5. You divide 5 by such a number that the result will be equal to 5. What is this number?
>
> Student 1: One!
>
> Student 2: Zero!
>
> Teacher D: No! $\frac{5}{0}$ is undefined. Please, do not forget that division of any rational number by 1 will be equal to the rational number itself. Thus, *a* is equal to 1.

A number of teachers also paid attention to teaching the following special cases: dividing 0 by any rational number excluding 0 yields 0 and dividing 1 by any rational number excluding 0 yields the multiplicative inverse of that rational number. To give an example, in the course of teaching multi-step operations with rational numbers, Teacher B provided $\left(2\frac{1}{3}-\frac{7}{3}\right)\div\left(-\frac{1}{2}\right)$ as an example to this idea and subsequently expressed the following consideration:

> Teacher B: If we convert $2\frac{1}{3}$ into an improper number, we get $\frac{7}{3}$. Next, we subtract $\frac{7}{3}$ from $\frac{7}{3}$ and get zero. Now, we should divide 0 by $-\frac{1}{2}$. If we divide 0 by any rational number, the result will be 0. Namely, in a division operation, if 0 is a dividend, then the answer will always be 0. In contrast, if 0 is a divisor in a division operation, then the answer will be undefined.

Moreover, Teacher B provided $1-\frac{2}{3+\frac{1}{1+\frac{1}{2}}}$ as another multi-step operation example and in the course of working out this example, he paid attention to the special case of division of 1 by any rational number excluding 0. Some of the expressions he used to explain his consideration is presented as follows:

Teacher B: This complex fraction is a bit messy. There are five operations in this multi-step operation. We proceed from bottom to top in such multi-step operations. $1+\frac{1}{2}$ is equal to $\frac{3}{2}$. Now, if we divide 1 by any rational number, what happens to that rational number?

Student: It becomes upside down.

Teacher: You are right, the number becomes upside down. Thus, 2 and 3 swap places. In a division operation, if the dividend is 1 and the divisor is any rational number, then the quotient will always be equal to the flipped over version of that rational number. Thus, $\frac{1}{\frac{3}{2}}$ is equal to $\frac{2}{3}$. You do not need to perform any operation for this. However, note that this is true only in cases where the dividend is equal to 1.

In another classroom event, Teacher A drew students' attention to following special case: division of -1 by any rational number excluding 0 yields the additive inverse of the multiplicative inverse of that rational number. Teacher A selected $(-1)\div\left(-\frac{12}{17}\right)$ from the student textbook and thereafter expressed his consideration in the classroom as: "I chose to use this example to have you notice that division of -1 by any rational number except for 0 will be equal to the additive inverse of the multiplicative inverse of that rational number."

Another manifestation of this approach occurred when Teacher A was teaching exponentiation of rational numbers. More specifically, Teacher A used $\left(-\frac{3}{4}+2\right)^2\div\left(-\frac{2}{3}\right)^0$ as a multi-step operation example and focused on $\left(-\frac{2}{3}\right)^0$. It is important to note that for this exponential number, the intuitive definition of exponents (i.e., repeated multiplication) does not work. Thus, the teacher treated the case of zero exponent as a special case and explicitly expressed the following utterances: "Raising any nonzero rational number to the power of 0 yields 1. Thus, $\left(-\frac{2}{3}\right)^0$ is equal to 1".

Finally, Teacher A considered another consideration about exponentiation of rational numbers. That is, while teaching how to find the square and cube of rational numbers, the teacher asked the students to find the answer of 1^{2014}. The teacher chose to use this example to point to the special status of 1 as the number that is invariant under rational number powers. He used the following expressions to explain this type of consideration: “No matter how many times 1 is multiplied by itself, the answer will always be equal to zero. Thus, raising 1 to any rational number power always yields 1”.

5.5.2. Under-represented cases in the teaching of rational number concepts

Middle school mathematics teachers included the following under-represented cases into the teaching of rational number concepts: (i) emphasizing rationality of 0, (ii) including 0 into the sequence of rational numbers when ordering, (iii) adding/subtracting/multiplying/dividing more than two rational numbers, (vi) incorporating equivalent pairs into comparison of rational numbers, (v) incorporating into the classroom ordering examples that entail the use of residual thinking and finally, (vi) estimating the addition/subtraction/multiplication/division of rational numbers.

Teachers provided their students with the definition of rational number set, represented it symbolically as $\mathbb{Q} = \mathbb{Q}^{+} \cup \{0\} \cup \mathbb{Q}^{-}$ and pointed to the rationality of 0. For instance, the consideration employed by Teacher A about the rationality of 0 is presented by the following teaching episode:

> Teacher A: Is 0 a rational number?
>
> Student: No!
>
> Teacher A: Why?
>
> Student: … (No answer)
>
> Teacher A: You often cannot understand that 0 is a rational number. This is probably due to the fact that you confuse neutrality of 0 with rationality of it. Have a look at this: $\mathbb{Q} = \mathbb{Q}^{+} \cup \{0\} \cup \mathbb{Q}^{-}$. This means that rational numbers include positive rational numbers, negative rational numbers and 0. Note that 0 is a rational number but it is neutral.

In another case, some of the teachers deliberately included 0 into the sequence of rational numbers when teaching how to order them. For example, Teacher A used a large number of examples for ordering and $\frac{(-2)}{5}, \frac{(-7)}{5}, 0, \frac{1}{5}, \frac{3}{5}$; $+\frac{1}{3}, +\frac{1}{5}, -\frac{1}{7}, -\frac{1}{2}, 0$; $-\frac{2}{7}, -\frac{2}{13}, 0, \frac{2}{15}, \frac{2}{19}$ were among these examples. Some of the utterances used by Teacher A for this type of consideration while ordering $\frac{(-2)}{5}, \frac{(-7)}{5}, 0, \frac{1}{5}, \frac{3}{5}$ is presented below:

> Teacher A: For positive rational numbers, the one with a larger numerator will be larger. In contrast, the one with a smaller numerator will be larger for negative rational numbers.
>
> Student: Teacher, what are we going to do with 0?
>
> Teacher A: That is easy. Zero is the last thing to consider. I included it into this sequence in order for you to recognize that it is in middle of negative and positive rational numbers. You will better understand when I order them. $\frac{1}{5}$ and $\frac{3}{5}$ are positive numbers so, $\frac{1}{5} < \frac{3}{5}$. Next, $\frac{(-7)}{5}$ and $\frac{(-2)}{5}$ are negative numbers, therefore $\frac{(-7)}{5} < \frac{(-2)}{5}$. We ordered the positive and negative rational numbers separately. Now, we locate 0 in the middle of the positive and negative numbers in this way: $\frac{(-7)}{5} < \frac{(-2)}{5} < 0 < \frac{1}{5} < \frac{3}{5}$. Why do we locate zero in the middle? Because it is a neutral number.

Another way teachers tried to include under-represented cases into the teaching of rational number concepts was seen in their efforts to add/subtract/multiply/divide more than two rational numbers. For instance, Teacher B provided $1\frac{1}{4} + \left(-\frac{3}{12}\right) + \frac{5}{3}$ as an addition operation with three terms and similarly he provided $\left(\frac{-1}{3}\right) \cdot \left(2\frac{1}{2}\right) \cdot (-5)$ as a multiplication operation with three terms. Some of the expressions he used to explain the addition example were:

> Teacher B: In this example, we will add three rational numbers together. You can add them, won't you?
>
> Students: … (No answer)
>
> Teacher B: In the previous examples, we added two rational numbers. Now, we will add three rational numbers. I hope you will not have any trouble with adding these numbers. More precisely, we can add not only three rational numbers but also as many rational numbers as we wish.

Other consideration of this type was manifested when Teacher B incorporated an equivalent pair into the classroom when teaching comparison of rational numbers. More precisely, Teacher B used several examples related with comparison of rational numbers and after the provision of $\left(2\frac{1}{4}, \frac{9}{4}\right)$, he explicitly expressed his consideration by the following utterances:

> Teacher B: We need to convert $2\frac{1}{4}$ into an improper number before comparison. We convert it in this way: $2\frac{1}{4} = \frac{2\times4+1}{4} = \frac{9}{4}$. Thus, $2\frac{1}{4}$ is equal to $\frac{9}{4}$ (i.e., $2\frac{1}{4} = \frac{9}{4}$). I deliberately chose to use this example. I wanted to see who would recognize the equality. So this means that we do not always use greater than (>) or less than sign (<) for comparison. Sometimes, the rational numbers may be equal to each other. As you can see, $2\frac{1}{4}$ is the mixed number form of $\frac{9}{4}$.

Another manifestation of this approach was seen in Teacher A's attempt to incorporate into the classroom ordering examples that entailed the use of more conceptual strategies such as residual thinking. For instance, Teacher A used $\frac{1996}{1997}, \frac{1997}{1998}, \frac{1998}{1999}$ as an example for ordering by residual thinking. Then, he explained how to order the rational numbers in this way:

> Teacher A: All rational numbers are very close to 1. However, the first rational number requires $\frac{1}{1997}$ to make the whole, the second rational number requires $\frac{1}{1998}$ to make the whole and the third fraction requires $\frac{1}{1999}$ to make the whole. $\frac{1}{1997} > \frac{1}{1998} > \frac{1}{1999}$ therefore, $\frac{1996}{1997} < \frac{1997}{1998} < \frac{1998}{1999}$.

It is important to note that algorithmic approaches such as common denominator approach or common numerator approach do not work in the solution of this example.

Finally, Teacher D incorporated into the classroom an example that is often overlooked by the teachers. That is, Teacher D asked her students to estimate the addition of $-2\frac{1}{100}$ and $2\frac{95}{100}$ rather than asking them to find the exact answer.

Then, she explained why she chose to present this example when teaching addition of rational numbers. She wrote on the board the aforementioned estimation example and asked her students whether they could recall the notion of rounding and use it for working out this example.

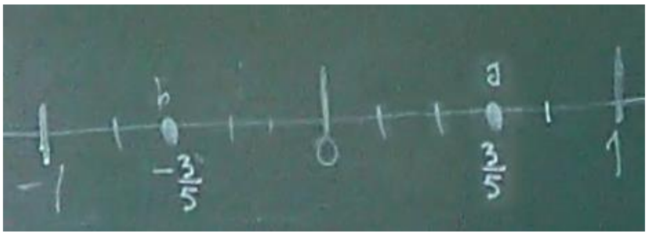
In this section, the incidents in which teachers included uncommon cases were described in detail. In the following section, the cases in which teachers drew attention to relevant features were described thoroughly.

5.6. Drawing Attention to Relevant Features

This consideration had to do with teachers' deliberate attempts to decrease the 'noise' of specific examples. In this study, the irrelevant information carried by the specific examples of middle school mathematics teachers were regarded as noise. The teachers applied some principles in order to prevent students from focusing on irrelevant features of specific examples and to enable them to see the general through the particular. These principles were as follows: (i) locating a positive rational number first, its additive inverse second and then comparing the two locations, (ii) arranging positive rational numbers first, their additive inverses second and then comparing the two arrangements, (iii) performing operations with rational numbers by keeping the magnitude of terms constant and varying one sign at a time, (iv) performing exponentiation by writing the power inside the parenthesis first, by writing the power outside the parenthesis second and then comparing the two results, (v) working out a complex fraction example first, rearranging the same complex fraction by changing the location of the main fraction bar and working out the new complex fraction second, and then comparing the two results, (vi) breaking the pattern when teaching the procedure for converting repeating decimals into rational numbers, (vii) performing a multi-step operation with parenthesis first, omitting the parenthesis of the same multi-step operation and performing second, and then comparing the two results, and finally (viii) solving a rational number problem first, solving another version of the same rational number problem second and then comparing the two rational number problems.

Some of the teachers tried to draw students' attention to relevant features when locating rational numbers on a number line. They initially located a positive rational number on a number line, then they only changed the sign of the rational number into negative and then located it on same the number line. Finally, they compared the locations of the two rational numbers on the number line. For example, Teacher C initially located $\frac{3}{5}$ on a number line, then he located $-\frac{3}{5}$ on a number line and finally compared the locations of $\frac{3}{5}$ and $-\frac{3}{5}$ on the same number line. The following teaching episode illustrates how Teacher C took into account this type of consideration:

> Teacher C: To locate $\frac{3}{5}$ on a number line, to which integer interval should I look for?
>
> Students: Between 0 and 1.
>
> Teacher C: Yes, we should locate $\frac{3}{5}$ between 0 and 1. Thus, we first divide the interval between 0 and 1 into 5 equal pieces and mark the end of the third piece as $\frac{3}{5}$. To locate $-\frac{3}{5}$ on a number line, we have to look for the interval between -1 and 0. Again, we divide the interval between -1 and 0 into 5 equal pieces but start counting from -1 and mark the end of the third piece as $-\frac{3}{5}$. Now, let's draw a number line and locate the rational numbers on it as follows:



> Teacher C: Here, we located two rational numbers that have same numerators and denominators but their signs are opposite of each other. I do not know whether you noticed, but as you can see the two rational numbers have equal distances from 0. However, while $-\frac{3}{5}$ is located on the left side of 0, $\frac{3}{5}$ is located on the right side of it.

Some of the teachers attempted to draw students' attention to relevant features while ordering rational numbers. To do so, they first provided examples related with ordering positive rational numbers. In their next examples, the teachers changed the sign of the rational numbers into negative and then ordered the negative rational numbers. Finally, they compared the two examples to expose the mathematical structure entailed in ordering positive and negative rational numbers. For instance, Teacher D first provided $\frac{7}{4}, \frac{7}{2}$ and $\frac{7}{9}$ and ordered these positive rational numbers as $\frac{7}{9} < \frac{7}{4} < \frac{7}{2}$. In her latter example, she kept the magnitudes of the rational numbers constant and only changed their signs into negative as $-\frac{7}{4}, -\frac{7}{2}$ and $-\frac{7}{9}$. Next, she ordered them as $-\frac{7}{9} > -\frac{7}{4} > -\frac{7}{2}$. Finally, she drew students' attention to the difference between $\frac{7}{9} < \frac{7}{4} < \frac{7}{2}$ and $-\frac{7}{9} > -\frac{7}{4} > -\frac{7}{2}$ and concluded that ordering negative versions of same rational numbers would reverse the order.

Other consideration of this type occurred when teachers provided examples related with addition/subtraction/multiplication/division of rational numbers. That is, when performing such operations, the teachers kept the magnitude of terms constant and varied only one of their signs at a time. To give an example, Teacher C provided the following multiplication examples consecutively to employ this kind of consideration:

(1) $\left(+\frac{1}{3}\right) \times \left(+\frac{2}{7}\right) = \left(+\frac{2}{21}\right)$ (2) $\left(+\frac{1}{3}\right) \times \left(-\frac{2}{7}\right) = \left(-\frac{2}{21}\right)$ (3) $\left(-\frac{1}{3}\right) \times \left(-\frac{2}{7}\right) = \left(+\frac{2}{21}\right)$

As it can be seen, the first multiplication example includes factors that are both positive. In the second example, the teacher kept the first factor entirely constant (i.e., its magnitude and sign), and kept the magnitude of the second factor constant but changed its sign into negative. When proceeded from the second example to the third example, the teacher kept the magnitude and sign of the second factor constant, kept the magnitude of the first factor constant and changed its sign into negative. Finally, the teacher compared the products and expressed the following:

Teacher C: As you see, all multiplication operations have products with same magnitudes. Besides, the rules for multiplying integers are also applicable for rational numbers. That is, when we have a look at the first example we can say that multiplication of two positive rational numbers yields a positive rational number product. In the second example we can see that multiplication of one positive and one negative rational number yields a negative rational number product. Finally, in the third example, the multiplication of two negative rational numbers yield a positive rational number product.

Another manifestation of this approach was seen in teachers' attempts to perform exponentiations by writing the power inside the parenthesis first, by writing the power outside the parenthesis second and finally comparing the similarities and differences of the two exponents. For instance, by using -2, 5 and 3, Teacher A generated $\left(\frac{-2^3}{5}\right)$ as the first exponent, by using the same numbers later he generated $\left(\frac{-2}{5}\right)^3$ as the second exponent. The only difference between the two exponents is the position of the power 3 with respect to the parenthesis. Thus, the teacher tried to expose the mathematical structure of exponents by varying the positon of power and keeping other features invariant. Teacher A worked out the two exponents side by side and manifested his consideration via the following utterances:

Teacher A: Look through $\left(\frac{-2^3}{5}\right)$ and $\left(\frac{-2}{5}\right)^3$ for some time. In the first exponential number the power is inside the parenthesis and in the second one the power is outside the parenthesis. In the first exponential number, 3 has impact on only 2. Even, it does not have any impact on the minus sign preceding 2. However, in the second exponential number 3 has impact on 2, 5 and on the minus sign. Thus, $\left(\frac{-2^3}{5}\right)$ is equal to $\left(\frac{-2.2.2}{5}\right)=\left(-\frac{8}{5}\right)$ and $\left(\frac{-2}{5}\right)^3$ is equal to $\left(\frac{-2}{5}\right)\cdot\left(\frac{-2}{5}\right)\cdot\left(\frac{-2}{5}\right)=\left(-\frac{8}{125}\right)$. As you can see the first exponential number is equal to $\left(-\frac{8}{5}\right)$ while the second one is equal to $\left(-\frac{8}{125}\right)$. The results are different from each other. Is that okay?

Students: Yes!

Another example of how teachers draw students' attention to relevant features of examples was observed when teachers were teaching multi-step operations that are expressed as complex fractions. More specifically, the teachers provided two complex fraction examples that were formed by using same numbers but the locations of main fraction bars included in these two examples were different from each other. For instance, Teacher B provided $\dfrac{\frac{1}{2}}{\frac{3}{4}}$ and $\dfrac{\frac{1}{2}}{\frac{3}{4}}$ simultaneously and wanted students to determine whether they are equal to each other or not. He expressed this type of consideration as follows:

> Teacher B: They look similar to each other. However, in fact, they have nothing to do with each other since the positions of main fraction bars are different from each other. The first complex fraction can be rearranged as $\left(\frac{1}{2} \div 3\right) \div 4$ and similarly the second complex fraction can be rearranged as $1 \div \left(2 \div \frac{3}{4}\right)$. Thus, we solve the first and second complex fractions as: $\left(\frac{1}{2} \div 3\right) \div 4 = \left(\frac{1}{2} \times \frac{1}{3}\right) \div 4 = \frac{1}{6} \div 4 = \frac{1}{6} \times \frac{1}{4} = \frac{1}{24}$, $1 \div \left(2 \div \frac{3}{4}\right) = 1 \div \left(2 \times \frac{4}{3}\right) = 1 \div \frac{8}{3} = 1 \times \frac{3}{8} = \frac{3}{8}$.
>
> One of them is equal to $\frac{1}{24}$, the other one is equal to $\frac{3}{8}$. As you can see, the two complex fractions are visually similar to each other, however they are in no way connected to each other.

In another case, Teacher B attempted to draw attention to relevant features by breaking the pattern of examples used for teaching the procedure for converting repeating decimals into rational numbers. Namely, Teacher B used the following sequence of examples to teach the procedure for converting repeating decimals into rational numbers: $0.\overline{7}$, $1.\overline{3}$, $2.\overline{15}$, $15.\overline{91}$, $3.2\overline{4}$, $1.1\overline{7}$, $1.0\overline{45}$ and $3.2\overline{07}$. As can it be seen, the first two examples include only one repeating digit and hence their common fraction forms entail only one 9 in the denominator. After these two examples, Teacher B broke the pattern by giving two examples that included two 9's in their common fraction forms. He again broke the pattern by converting $3.2\overline{4}$ and $1.1\overline{7}$ into rational numbers since these two examples included one 9 and one 0 in the denominators of their rational number forms. Finally, he broke another pattern by providing the last two examples. Because the rational number forms of these two examples included two 9's and one 0 in their denominators. The teacher deliberately used this sequence of examples and changed the type of repeating decimals after each two examples in order to prevent students from noticing irrelevant patterns and making invalid generalizations about the procedure for conversion.

In another case, Teacher A tried to draw students' attention to relevant features in the course of teaching multi-step operations with rational numbers. In more detail, Teacher A initially provided $K = \left(8 - \frac{1}{4}\right) \div \frac{1}{8}$ as a multi-step operation example. Next, he omitted the parenthesis in this example and provided $L = 8 - \frac{1}{4} : \frac{1}{8}$ as a second multi-step operation example. Finally, the teacher asked the students to think of whether the two examples were identical. By this way, the teacher checked whether students could recognize which operations to perform first in the two expressions. Some of the expressions used by Teacher A to explain his approach were presented in the following teaching episode:

> Teacher A: Is K identical to L?
>
> Student 1: Yes, they are identical.
>
> Student 2: No, they are not!
>
> Student 3: Yes, they are!
>
> Student 4: They are not!
>
> Teacher A: Thank you for all of you, but they are not identical to each other. Because, we first perform division in L whereas we first perform subtraction in K. The first example can be worked out in this way:
>
> $$K = \left(8 - \frac{1}{4}\right) \div \frac{1}{8} = \left(\frac{4 \times 8 - 1}{4}\right) \times \frac{8}{1} = \frac{31}{4} \times \frac{8}{1} = 62 .$$
>
> Let me move on to the solution of next example as:
>
> $$L = 8 - \frac{1}{4} \div \frac{1}{8} = 8 - \left(\frac{1}{4} \div \frac{1}{8}\right) = 8 - \left(\frac{1}{4} \times \frac{8}{1}\right) = 8 - 2 = 6 .$$

> K absolutely has nothing to do with L. Those who said K and L are identical should examine the board very carefully. K is equal to 62, L is equal to 6. Thus, they are not equal to each other.

In the final case, Teacher A attempted to draw students' attention to the relevant features during the provision of rational number problems. That is, Teacher A provided the following two rational numbers consecutively:

> (1) On Monday, Ali spent $\frac{1}{4}$ of his pocket money. The next day, he spent $\frac{2}{3}$ of his pocket money and he had 21 TLs left. How much pocket money did he have at the beginning? (2) On Monday, Ali spent $\frac{1}{4}$ of his pocket money. The next day, he spent $\frac{2}{3}$ of his remaining pocket money and he had 21 TLs left. How much pocket money did he have at the beginning?

As it can be seen, to have students discern the difference between the two problems Teacher A completely used the same context and the numbers. However, he added the word 'remaining' to the latter example. Some of the expressions he used to explain his consideration are presented as follows:

> Teacher A: Now, I will explain the difference between two problems to you. In the first problem, Ali spent $\frac{1}{4}$ of his pocket money first and $\frac{2}{3}$ of his pocket money later. In this problem, Ali spent the two amounts of money separately, but he spent those amounts over the same amount of pocket money. In the second problem, Ali spent $\frac{1}{4}$ of his pocket money first and next he spent $\frac{2}{3}$ of his remaining pocket money. In the second problem, you perform your operations over the remaining pocket money. There is a crucial difference between the two problems. In the first problem, you can directly add the numbers to find the total spent money. However, in the second problem you cannot find the total spent money by directly adding two rational numbers.

5.7. Summary of Teachers' Considerations in Choosing or Using Examples

In this chapter, the focus was on exploring the principles or considerations used by teachers while choosing or generating rational number examples. Through this purpose, the examples that manifested the following teacher considerations were brought to light: starting with a simple or familiar case, attention to students' difficulty, error or misconception, keeping unnecessary work to minimum, taking account of examinations, including uncommon cases, and drawing attention to relevant features.

Teachers manifested their attempts to start with a simple or familiar case through considering form of rational numbers, denominators of rational numbers, number of repeating and non-repeating digits of a decimal, number of terms/elements/steps when ordering rational numbers and when performing a single operation or multi-step operations with rational numbers, increasing complexity of multi-step operations and of rational number problems by changing their mathematical structure; and finally by recalling prior knowledge on rational number concepts. In more detail, teachers selected a sequence of rational numbers in different forms for locating on a number line, for performing four operations or for performing exponentiation by considering their increasing complexities. Teachers initially used rational numbers with same denominators as members of the sequence when ordering rational numbers or when adding or subtracting rational numbers. Teachers considered increasing complexity in converting repeating decimals into rational numbers by proceeding from decimals that included merely repeating digits to decimals that included both repeating and non-repeating digits. Besides, teachers gradually increased either the number of terms in an operation, the number of rational numbers selected for ordering in a sequence or the number of steps included in multi-step operations with rational numbers. Teacher considered increasing complexity of multi-step operations by changing their mathematical structure. Teachers often used multi-step operation examples with terms that are expressed on one line first, complex fractions without unknown values second and complex fractions with unknown values last. Similarly, teachers often attempted to generate rational number problems from simple to more difficult by changing the mathematical structure of each problem gradually. Finally, teachers considered increasing complexity by recalling prior knowledge on rational number concepts such as recalling natural number set and integer set first before introducing rational number set and recalling four operations with fractions before introducing four operations with rational numbers.

Teachers considered their students' difficulties, errors or misconceptions when providing rational number examples. Teachers expressed that students often had difficulty in understanding the location of a minus sign in a rational number, subtraction operation with rational numbers, complex fractions with unknown values, ordering rational numbers with the same numerators, dealing with division of a number by zero and division of zero by a number, simplification of rational numbers before multiplication, performing operations including negative rational numbers without parenthesis, understanding that distributive property yields a valid result and finally distinguishing between exponents with a power inside the parenthesis and out outside the parenthesis. Besides, teachers articulated that students often make the following mathematical errors related with rational number concepts: using commas instead of greater-than and less-than signs when ordering rational numbers, ignoring parenthesis when operating with negative rational numbers, making sign errors when adding mixed numbers, making notation errors about mixed numbers, making errors when multiplying a rational number and whole number, making errors when finding additive inverse of a rational number, making errors due to not following order of operations, and making notation errors when performing the exponentiation of unknown variables. Finally, teachers explicitly uttered that students held the following misconceptions about rational number concepts: counting tick-marks rather than counting equal parts of the line segment when locating a rational number on a number line, over-generalizing location of positive rational numbers to negative rational numbers, ordering decimals by treating the digits after the decimal points as separate numbers, over-generalizing multiplication and division of rational number algorithms to addition and subtraction of rational numbers, under-generalizing simplification of rational number multiplication, misapplication of multiplication to mixed numbers, exponentiation by adding base and power, exponentiation by multiplying base and power, and believing that a larger number must always be divided by a smaller number.

Teachers considered keeping unnecessary work to minimum by reducing technical work and focusing on the essence, highlighting relevant parts of examples and not going into extra details and by using properties of operations. Teachers reduced technical work and focused on the essence in the following ways: drawing only the relevant part of number line when locating rational number on it, choosing certain rational numbers to illustrate repeating decimals, choosing relevant strategy when ordering rational numbers, using LCM method instead of multiplying denominators when finding the common denominator of rational numbers, not trying to enlarge rational numbers by 1, using shortcuts for adding and subtracting a whole number and a rational number, adding or subtracting whole parts and fractional parts separately when adding or subtracting mixed fractions, using subtraction formula instead of equating denominators during the subtraction of rational numbers, choosing same denominator rational numbers when illustrating associative property of addition, simplifying rational numbers in the course of performing operations, using backwards strategy instead of equating denominators when dealing with complex fractions with unknown values and rearranging algebraic expressions for an easier computation. Similarly, teachers highlighted relevant parts of examples and did not go into extra details in the following ways: emphasizing important parts of an example and not finishing up all the calculations, not seeing it essential to perform simplifications in the course of teaching a concept, not seeing it essential to perform conversions in the course of teaching a concept, not seeing it essential to equate denominators when symbolically expressing the area model of multiplication of rational numbers. Finally, teachers reduced workload by using properties of rational number operations as follows: using commutative property of addition operation rather than adding, using associative property of addition operation rather than adding, using distributive property of multiplication over addition rather than performing the operation, using the facts that $1/(a/b) = b/a$, $(a/b)/(a/b) = 1$, $(a/b) + (-a/b) = 0$ without actually making computations and enlarging decimal numerators and decimal denominators by multiples of 10 instead of converting into rational numbers when performing multi-step operations.

Another manifestation of teacher consideration occurred when teachers took account of examinations when using rational number examples. They manifested this type considerations in the following cases: highlighting examples that have the potential to appear in written examinations, highlighting examples that have the potential to appear in practice examinations of private teaching institutions, highlighting examples that have the potential to appear in high stakes examinations, explaining the method of scoring for potential written examination questions, incorporating the solution of high-stakes examination examples into the classroom, expressing the answer of multiple choice questions in their simplest forms in order to find it in the alternatives, finding the answer of multiple choice complex fraction tasks by trial and error of the alternatives and teaching shortcut methods for gaining speed in the high stakes examinations.

Teachers also considered incorporation of uncommon cases into their classrooms either by introducing exceptional or special cases or by introducing under-represented cases. They chose to use the following exceptional or special cases in the teaching of rational number concepts: multiplying any rational number by 0 yields 0, multiplying any rational number by 1 yields the rational number itself, dividing any rational number by 0 is undefined, dividing any rational number by 1 yields the rational number itself, dividing 0 by any rational number excluding 0 yields 0, dividing 1 by any rational number excluding 0 yields the multiplicative inverse of that rational number, dividing -1 by any rational number excluding 0 yields the additive inverse of the multiplicative inverse of that rational number, raising any nonzero rational number to the power of 0 yields 1, and raising 1 to any rational number power yields 1. Besides, the teachers included the following under-represented cases into the teaching of rational number concepts: emphasizing rationality of 0, including 0 into the sequence of rational numbers when ordering, adding/subtracting/multiplying/dividing more than two rational numbers, incorporating equivalent pairs into comparison of rational numbers, incorporating into the classroom ordering examples that entail the use of residual thinking, and estimating the addition, subtraction, multiplication and division of rational numbers.

Ultimately, teachers considered drawing attention to relevant features of rational number concepts by deliberately attempting to reduce irrelevant information carried by specific examples. The teachers applied the following principles to reduce the noise of specific examples: locating a positive rational number first, its additive inverse second and then comparing the two locations, arranging positive rational numbers first, their additive inverses second and then comparing the two arrangements, performing operations with rational numbers by keeping the magnitude of terms constant and varying one sign at a time, performing exponentiation without writing the power inside the parenthesis first, by writing the power outside the parenthesis second and then comparing the two results, working out a complex fraction example first, rearranging the same complex fraction by changing the location of the main fraction bar and working out the new complex fraction second, and then comparing the two results, breaking the pattern when teaching the procedure for converting repeating decimals into rational numbers, performing a multi-step operation with parenthesis first, omitting the parenthesis of the same multi-step operation and performing second, and then comparing the two results, and solving a rational number problem first, solving another version of the same rational number problem second and then comparing the two rational number problems.

CHAPTER VI

INCORRECT OR INAPPROPRIATE EXAMPLES

The purpose of this study was to explore middle school mathematics teachers' treatment of rational number examples in their seventh grade classrooms. In this chapter, the focus was on identifying mathematically incorrect or pedagogically inappropriate rational number examples used by the middle school mathematics teachers. In other words, the focus was on identifying mathematical or pedagogical shortcomings that might be carried by the rational number examples used by the teachers. Through this focus, the following research question was formulated:

1. What mathematical or pedagogical shortcomings do the examples used by the teachers in the teaching of rational numbers have?
    a. What are the mathematically incorrect examples used by the teachers during the teaching of rational numbers?
    b. What are the pedagogically improper examples used by the teachers during the teaching of rational numbers?

More specifically, this chapter was divided into two sections as mathematically incorrect examples and pedagogically improper examples. In the following section mathematically incorrect examples generated by the middle school mathematics teachers were described.

6.1. Mathematically incorrect examples

This section examined middle school mathematics teachers' rational number examples in terms of their mathematical correctness. In this study, some of the examples generated by the teachers were incorrect when evaluated from a mathematical standpoint. However, in some cases the examples provided by them were correct but the instructional explanations related with those examples were not entirely correct. Thus, when determining mathematical correctness of rational number examples, the instructional explanations provided for those examples were also taken into consideration.

Mathematically incorrect examples or explanations provided by the teachers in the course of teaching rational number concepts were discussed through the following cases: (1) explaining that irrational numbers cannot be located on a number line, (2) explaining that rational number set is a subset of irrational number set, (3) explaining that irrational number set includes less number of elements than rational number set, (4) explaining that all numbers in the fraction form are rational numbers, (5) working out an example incorrectly due to the misapplication of absolute value concept, (6) not partitioning the number line into equal distances when locating rational numbers on it, (7) using commutative property of addition when exemplifying associative property of addition, (8) seeing conversion of repeating decimals into rational numbers as being synonymous with rounding, (9) under-generalizing the addition of mixed numbers, and finally (10) using a correct ordering strategy but misnaming it as another strategy.

Teacher A introduced the notion of a rational number and wrote on the whiteboard its definition as $\mathbb{Q}=\left\{\frac{a}{b}, a\in\mathbb{Z}, b\in\mathbb{Z}, b\neq 0\right\}$. He wanted to illustrate the numbers that satisfy this definition and he generated $-\frac{0}{7}$ and $\frac{3}{1}$ as examples for rational number set. Later, he moved on to explaining what pi number (π) is and incorrectly explained that irrational numbers cannot be located on a number line as follows:

> Teacher A: Pi number goes to infinity as 3.14… Today, the decimal representation of pi has been computed to include many digits that can wrap the circumference of the earth forty times but it is still being computed. That is, the ratio of a circle's circumference to its diameter goes to infinity and it is called the pi number… This ratio proceeds as 3.14… but we cannot express it as a common fraction. Why? Because we do not know its end.
>
> Student: That is a repeating decimal!
>
> Teacher A: It is not a repeating decimal. It is something else. If we do not know the final digits of the decimal number, then we cannot write it as a common fraction or locate it on a number line. I introduced you the pi number to illustrate that there are numbers that are not examples of rational number set.

In contrast to Teacher A's claim, pi number in particular and irrational numbers in general can be located on a number line. More importantly, eight grade mathematics curriculum points to the relationship between irrational numbers and radicals and exemplifies how to locate an irrational number on a number line with the help of Pythagorean Theorem. An activity included in the middle school mathematics curriculum for finding the location of $\sqrt{34}$ is presented in Figure 6.1.

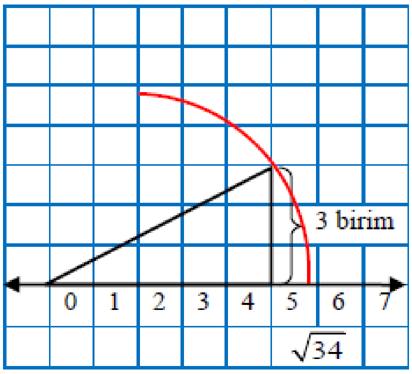


Figure 6.1. The location of $\sqrt{34}$ on the number line (MoNE, 2009b, p.301).

Teacher A could have made a more powerful and correct explanation if he had emphasized that all rational and irrational numbers can be represented by points on a number line and thus the number line is called the real number line.

After introducing rational number set, Teacher C pointed to the relationship between number sets by introducing $C \subset \mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q}$. At that time, one of the students wanted to learn the superset of rational number set. The teacher responded incorrectly and explained that irrational number set is the superset of rational number set. Teacher C's explanations are provided in the following episode:

> Teacher C: You first learnt how to count at school. You started with 1, 2, 3, and kept going. When you were at grade 5, you learnt a new number set as natural number set. You added the number 0 to the counting numbers and got the natural numbers. Natural numbers are denoted by the symbol $\mathbb{N}$. Did you remember?
>
> Students: Yes!
>
> Teacher C: What is the next larger set?
>
> Students: Integer set!

Teacher C: Yes you are right. Integers are denoted by $\mathbb{Z}$. Now, what is the next larger set?

Students: Rational numbers!

Teacher C: Okay, rational numbers are denoted by $\mathbb{Q}$. You do not need to learn the number set that is larger than rational numbers. Therefore, it is enough for you to know these number sets.

Student 1: Are complex numbers the next larger set?

Teacher C: No, irrational number set is the next larger set. You do not need to learn irrational numbers now. I will teach it to you when you are at grade 8. We will denote irrational numbers by the symbol $\mathbb{Q}'$. Again, you do not have to learn it know.

Rational number set is not a subset of irrational number set. The two number sets are disjoint sets and reel number set is the union of these two disjoint sets. The middle school mathematics curriculum suggests teachers to give emphasis on the relationship among different number sets. A sample activity included in the middle school mathematics curriculum for demonstrating the relationship between reel numbers and rational numbers is presented in Figure 6.2

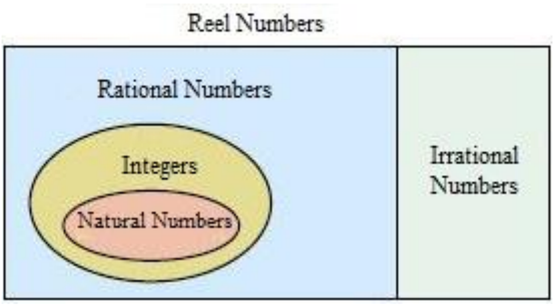


Figure 6.2. The relationship between different number sets (MoNE, 2009b, p.300).

Teacher D was teaching the procedure for converting repeating decimals into rational numbers. After providing $3.\bar{3} = \frac{33-3}{9} = \frac{30}{9}$, she explained that repeating decimals such as $3.\bar{3}$ are all rational numbers. One of the students interrupted and asked the teacher to give examples for numbers that are not rational. As a response to the student query, the teacher incorrectly explained that there are few irrational numbers. The following episode illustrates this incorrect explanation.

Teacher D: We first write the number without its decimal point as 33. Then, we subtract the repeating part from 33 and find 33-3=30. This number is the numerator of the decimal number. Now, we check how many digits repeat in

$3.\bar{3}$. We write 9 to the denominator as many as the number of repeating digits and write 0 after 9 as many as non-repeating digits. As you see, only one digit repeats. Thus, we write 9 as the denominator of the rational number.

Student: Can you please explain ones more?

Teacher D: Do not worry, I will write the formula on the board. As a priority, you should understand that all repeating decimals or terminating decimals are actually rational numbers. Therefore, if I ask you to determine whether $3.\bar{3}$ and 0.3 are rational numbers, how would you respond?

Student: Yes, they are rational numbers.

Teacher D: Yes, because you can express terminating decimals and repeating decimals as rational numbers.

Student: Well teacher, which numbers are not rational?

Teacher D: I previously mentioned that pi number is not rational. It has infinitely many digits but does not have a regular repeating pattern. Actually you can think in this way: Excluding repeating decimals and terminating decimals all numbers are irrational. However, irrational numbers are very rare.

In another case, Teacher D wanted to explain rational numbers and their properties. She initially recalled fractions, natural numbers and integers and gave examples for these number types. Then, she introduced rational numbers and provided $-\frac{2}{3}$ and $-1\frac{1}{4}$ as examples for rational numbers. Finally, she defined rational numbers as numbers that can be written in the form of $\frac{a}{b}$. Note that the teacher incorrectly defined rational numbers due to not restricting *a* and *b* to integers. The following is an excerpt of Teacher D's explanation of rational numbers:

Teacher D: Remember number sets. For instance, do you remember natural number set? It starts with 0,1,2,3 and goes to infinity. Well, which numbers are included in integer set? It includes natural numbers and their negatives. This means that natural number set is a subset of integer set. In a similar way, rational number set is a superset of natural number set and integer set.

Student 1: Well, which number set is a superset of all number sets?

Teacher D: Reel number set. You will learn it at grade 8. I want to say one more thing about rational numbers. You previously learnt fractions. For instance, $\frac{1}{4}$ is a fraction. In fact, it is a number isn't it? Fractions are also rational numbers. So $\frac{1}{4}$ is a rational number. Negative numbers such as

$-\frac{2}{3}$ and $-1\frac{1}{4}$ are also rational numbers. So they are all rational numbers, okay?

Students: Okay!

Teacher D: Actually, we can say that all numbers written in the form of $\frac{a}{b}$ are rational numbers. What does $\frac{a}{b}$ mean? Both *a* and *b* will take numerical values. However, *b* cannot be zero. You already know from fractions that the denominator cannot be equal to zero. For instance, what does $\frac{3}{0}$ mean?

Student: Undefined!

Teacher D: You are right, it is undefined. Thus, all numbers in the form of $\frac{a}{b}$ are rational numbers unless *b* is equal to zero.

For a more precise definition, the teacher should have restricted *a* and *b* to integer values. Since $\frac{\pi}{2}, \frac{\sqrt{5}}{3}, \frac{\sin 64}{4}$ and $\frac{\ln(10)}{7}$ are all rational numbers in the sense that they are written in the form of $\frac{a}{b}$. However, these numbers are actually all irrational numbers.

Other incorrect example was manifested when Teacher A was teaching how to order rational numbers. He wanted to order $-\frac{3}{4}, -\frac{1}{4}, -\frac{9}{4}, -\frac{5}{4}$ and $\frac{1}{4}$ on a number line and find the furthest distance between the two points on the number line. However, he initially arrived at an incorrect answer due to the misapplication of absolute value concept. Fortunately, he recognized his mistake and arrived at the correct answer by using a strategy that do not require the use of absolute value concept straight-forwardly. The following teaching episode illustrates Teacher A's incorrect example generation due to misapplication of absolute value concept:

Teacher A: We first need to find the two points that are furthest to each other. Namely, we need to find the largest and the smallest rational numbers first. What is the largest number?

Students: $\frac{1}{4}$ is the largest!

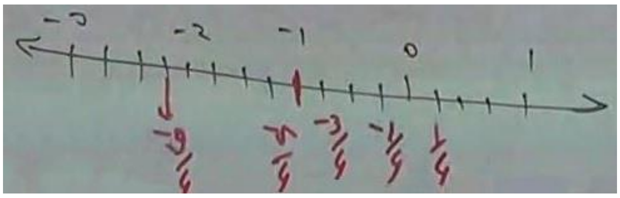
Teacher A: Okay, then which one is the smallest?

Student 1: $-\frac{9}{4}$ must be the smallest one.

Teacher A: Then, we need to find the difference between absolute values of the two rational numbers. Don't we?

Students: … (No answer)

Teacher A: Thus, we find the answer in this way: $\left|-\frac{9}{4}\right|-\left|\frac{1}{4}\right|=\frac{9}{4}-\frac{1}{4}=\frac{8}{4}=2.$ I want to solve this example by locating the rational numbers on a number line. Because, I do not believe that I solved it correctly. So let me draw a number line and locate the rational numbers on it immediately:



The answer is $\frac{10}{4}$ not 2. Where did I make an error? Let's think about elevators. In an apartment, if you move from the 3rd floor to the -5th floor, you first move from the 3rd floor to the ground floor and from the ground floor to the -5th floor. Thus we need to perform addition when finding the distance between these two points not subtraction. Thus, the answer is $\left|-\frac{9}{4}\right|+\left|\frac{1}{4}\right|=\frac{9}{4}+\frac{1}{4}=\frac{10}{4}=2.5$ not 2.

In this example, the teacher erroneously believed that $\left|-\frac{9}{4}\right|-\left|\frac{1}{4}\right|=\left|-\frac{9}{4}-\frac{1}{4}\right|$. The persistence of the teachers' error became apparent when he followed the same reasoning in another example asking to find the distance between $-\frac{4}{3}$ and $-\frac{1}{3}$. The teacher again misapplied the absolute value concept and proceeded as $\left|-\frac{4}{3}\right|-\left|-\frac{1}{3}\right|$. This time he reached a correct answer since $\left|-\frac{4}{3}\right|-\left|-\frac{1}{3}\right|=\left|\left(-\frac{4}{3}\right)-\left(-\frac{1}{3}\right)\right|$. However, he was not able to be aware of his mistake about the application of the absolute value concept since he obtained the correct answer by chance.

Teacher D consistently generated 'non-existing' examples during the teaching of how to locate rational numbers on a number line. Namely, in four out of six examples, she did not partition the number line into equal intervals when locating rational numbers. For instance, when locating $-2\frac{4}{5}$ she partitioned the number line as given in Figure 6.3:

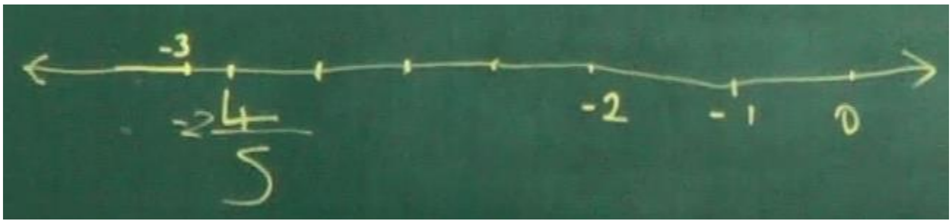


Figure 6.3. Teacher D's location of $-2\frac{4}{5}$ on a number line

As it can be seen, the distance between 0 and -1 or -1 and -2 is not equal to the distance between -2 and -3. None of the students remarked in the classroom that the distances between the integer points were not equal to each other. Thus, the teacher did not have the chance to modify her example. In reality, such an example does not exist and from a mathematical perspective it is problematic.

In another case, Teacher C was teaching associative property of addition of rational numbers. He generated an example to illustrate this property. However, the example was incorrect since he both changed the order of rational numbers and the grouping (i.e., change the position of the parenthesis) although this property does not permit changing the order of numbers. In other words, to demonstrate associative property of rational numbers, Teacher C also used commutative property of addition. He explained this property as follows:

> Teacher C: Rational numbers are associative under addition. Let's show whether associative property holds for $\frac{1}{2}+\frac{1}{4}+\frac{1}{5}$ or not. We will not add these three numbers to each other. We will use them to demonstrate associative property. We group the numbers two by two, first add the two numbers and then add the number that is outside the group. If we get the same answer for each grouping, then we say that associative property holds for $\frac{1}{2}+\frac{1}{4}+\frac{1}{5}$. We first group $\frac{1}{2}$ and $\frac{1}{4}$ as $\left[\frac{1}{2}+\frac{1}{4}\right]+\frac{1}{5}$. Next, we group $\frac{1}{2}$ and $\frac{1}{5}$ as $\left[\frac{1}{2}+\frac{1}{5}\right]+\frac{1}{4}$. Finally, we group $\frac{1}{4}$ and $\frac{1}{5}$ as $\left[\frac{1}{4}+\frac{1}{5}\right]+\frac{1}{2}$. If we

perform addition operations for each grouping, we obtain the same results. That is,

$$\left[\frac{1}{2}+\frac{1}{4}\right]+\frac{1}{5}=\left[\frac{1}{2}+\frac{1}{5}\right]+\frac{1}{4}=\left[\frac{1}{4}+\frac{1}{5}\right]+\frac{1}{2}$$

Thus, we can say that rational numbers are associative under addition operation.

As it can be seen, Teacher C both changed the order and grouping of $\frac{1}{2},\frac{1}{4}$ and $\frac{1}{5}$ by writing $\left[\frac{1}{2}+\frac{1}{5}\right]+\frac{1}{4}$ into the expression. In order for this example to be correct, the teacher needs to express it as $\left(\frac{1}{2}+\frac{1}{4}\right)+\frac{1}{5}=\frac{1}{2}+\left(\frac{1}{4}+\frac{1}{5}\right)$.

Teacher D used $0.\bar{9}=\frac{9}{9}=1$ as a specific example for illustrating how to convert repeating decimals into rational numbers. However, she also focused on an irrelevant feature of this example and incorrectly explained that conversion of repeating decimals into rational numbers are synonymous with the notion of rounding. For this specific example, the teacher's claim appears to be true since $0.\bar{9}=0.999...\cong 1$. However, this is a misleading example because the teacher's claim is not true for all repeating decimals. In short, it is incorrect to make a generalization that conversion is synonymous to rounding by focusing on an irrelevant feature of a particular example. The following excerpt illustrates Teacher D's incorrect explanation about conversion of repeating decimals into rational numbers:

> Teacher D: Normally, what does $0.\bar{9}$ mean to you? In fact, $0.\bar{9}$ goes on in this way: 0.999…To which integer is 0.999… closer to? It is closer to 1. Thus, in fact conversion is synonymous to rounding. In other words, you round to the nearest integer value when converting the repeating decimal into a rational number. Did you understand what I mean?
>
> Students: Yes!
>
> Teacher D: Then, let me teach you how to order rational numbers.

As the above given excerpt shows, Teacher D focused on an irrelevant feature of $0.\bar{9}$ and incorrectly explained that conversion of repeating decimals is synonymous to rounding. However, rounding repeating decimals such as $1.\bar{5}$ to their nearest integers leads to big round-off errors.

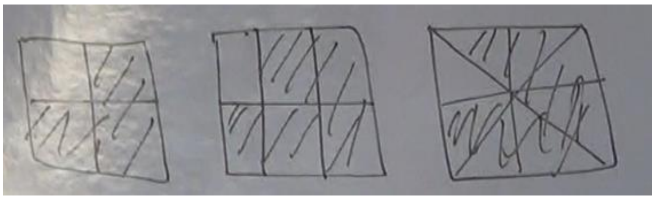
Teacher D made another incorrect explanation when teaching addition of mixed numbers. In more detail, she provided $\left(-3\frac{1}{4}\right)+\left(-1\frac{1}{2}\right)$ as an example for addition of rational numbers and incorrectly stated that whole parts and fractional parts of the mixed numbers cannot be added separately unless they have the same denominators. Thus, she under-generalized adding whole parts and fractional parts separately to mixed numbers with the same denominators. The following excerpt illustrates Teacher D's incorrect explanation about addition of mixed numbers:

> Teacher D: Now, we will add $\left(-3\frac{1}{4}\right)and\left(-1\frac{1}{2}\right)$ together. We cannot add whole parts and fractional parts of these mixed numbers separately since they have different denominators. We can add whole parts and fractional parts separately on condition that mixed numbers have same denominators. Thus, we have to convert mixed numbers into improper rational numbers before adding them. We convert $\left(-3\frac{1}{4}\right)$ into an improper rational number as$\left(-3\frac{1}{4}\right)=-\frac{3\times 4+1}{4}=-\frac{13}{4}$. Similarly, we convert $\left(-1\frac{1}{2}\right)$ into an improper rational number as$\left(-1\frac{1}{2}\right)=-\frac{1\times 2+1}{2}=-\frac{3}{2}$. Now can perform $\left(-\frac{13}{4}\right)+\left(-\frac{3}{2}\right)$ as $\left(-\frac{13}{4}\right)+\left(-\frac{3}{2}\right)=\left(-\frac{13}{4}\right)+\left(-\frac{6}{4}\right)=\left(-\frac{19}{4}\right)$.

In contrast to teacher's explanation, it is possible to add whole parts and fractional parts separately when adding mixed fractions. It can be performed in this way: $\left(-3\frac{1}{4}\right)+\left(-1\frac{1}{2}\right)=-\left(3\frac{1}{4}+1\frac{1}{2}\right)=-\left[(3+1)+\left(\frac{1}{4}+\frac{1}{2}\right)\right]=-\left[4+\frac{3}{4}\right]=-\frac{19}{4}$. Thus, expecting students to use a specific strategy for adding mixed numbers might hamper students' ability to develop their own strategies for adding.

Finally, another incorrect explanation was provided by Teacher A when ordering rational numbers. More precisely, Teacher A ordered several sets of rational numbers by using residual thinking. However, he referred to the ordering strategy as benchmarking to 1 rather than residual thinking. Thus, the teacher provided the examples correctly by residual thinking strategy however, he misnamed it as benchmark strategy. The following teaching episode illustrates this case:

> Teacher A: Now, we will order $\frac{3}{4}, \frac{5}{6}$ and $\frac{7}{8}$ by benchmarking to 1. Here, all rational numbers are very close to 1. For the first rational number, we divide the whole into 4 equal parts and take 3 of them. For the second rational number, we divide the whole into 6 equal parts and take 5 of them. For the third rational number, we divide the whole into 8 equal parts and take 7 of them. As you see, there is only one part left for each of three rational numbers. However, the leftover parts do not have equal sizes. Here are the pictorial representations of these three rational numbers:



> Note that the three wholes have same sizes. Now, the largest leftover part is in the first whole. The leftover part in the second whole is medium sized and the leftover part in the third whole is the smallest. Then, in which whole the largest part is taken? In the third whole the largest part is taken. Next in the second whole and next in the first whole. Thus, we order the rational numbers as $\frac{3}{4} < \frac{5}{6} < \frac{7}{8}$.

As can be seen, the teacher provided a relevant example for ordering rational numbers by residual thinking strategy. Thus, he was able to use his knowledge of the specific teaching strategies to address ordering rational numbers. However, he could not distinguish between benchmarking strategy and residual thinking strategy.

6.2. Pedagogically Improper Examples

This section examined middle school mathematics teachers' pedagogically incorrect examples under two main subsections as examples with improper language or terminology and to be avoided examples. Teachers' rational number examples that included improper language or terminology are explained in the following section.

6.2.1. Examples with improper language or terminology

This section examined middle school mathematics teachers' use of language or terminology for introducing rational number examples. In this study, some of the examples generated by the teachers were correct when evaluated from a mathematical standpoint. However, they were not appropriate from a pedagogical standpoint since they included the use of inappropriate language or terminology.

Examples that included the use of improper language or terminology were described through the following cases: (1) the careless use of the word fraction when rational number is intended, (2) the use of informal language such as opposite, flip and upside down for teaching additive or multiplicative inverses of rational numbers, (3) ill-advised reading of rational numbers, and finally (4) the incorrect use of mathematical symbols in the course of working out rational number examples.

Fractions are non-negative rational numbers. Students start to learn fractions long before they learn integers. Thus, numerators and denominators of fractions are conventionally restricted to whole numbers. Besides, fractions are only a subset of rational number set (Lamon, 2012). However, teachers commonly and carelessly used the word fraction when they intended rational numbers. For instance, Teacher A was teaching how to locate rational numbers on a number line. He first provided examples related with location of proper fractions. Later, he moved on to location of negative rational numbers. He carelessly used the word ‘negative proper fractions’ when locating negative rational numbers into the number line. The following excerpt illustrates Teacher A’s improper use of the word ‘fraction’ instead of the expression ‘negative rational number’:

> Teacher A: Listen to me very carefully. Proper fractions are very special among all fraction types. There are three reasons for this. First, they are only between 0 and 1 on a number line. They never exist in any other part of the number line. Second, proper fractions are commonly used when solving probability problems. In probability, the answers are between 0 and 1. However, they can also be 0 and 1. Third, when we square proper fractions, the result is smaller than the original proper fraction. For instance, the square of $\frac{1}{2}$ is equal to $\frac{1}{4}$ and $\frac{1}{4}$ is smaller than $\frac{1}{2}$. As you see, proper fractions are really very important. Let’s get back to our topic. Proper fractions are between 0 and 1. Thus, if I ask you to locate a proper fraction such as $\frac{3}{5}$ on a number line, you will focus on the interval between 0 and 1. If I ask you to locate a negative fraction such as $-\frac{3}{4}$ on a number line, you will focus on the interval between -1 and 0. Do not forget this. The location of proper fractions have been asked in SBS for two times so far.

As the above given excerpt shows, Teacher A provided $-\frac{3}{4}$ as a negative fraction example. Instead of saying negative fraction, it would be more appropriate to say negative rational number for such examples.

Similarly, Teacher B was teaching how to compare rational numbers. He wrote on the upper part of the board two comparison examples as $\frac{7}{9}, \frac{15}{2}$ and $1\frac{3}{4}, \frac{8}{11}$. Next, he wrote on the lower part of the board two other comparison examples as $-\frac{1}{2}, -\frac{7}{3}$ and $-2\frac{1}{5}, -\frac{17}{2}$. Upon a student's remark, Teacher B indicated that examples in the upper part of the board are fraction examples, while he indicated that examples in the lower part are rational number examples. Thus, Teacher B treated fractions and rational numbers as separate entities from each other. However, as mentioned before, fractions are a subset of rational number set. Thus, it would be better to introduce $\frac{7}{9}, \frac{15}{2}$ and $1\frac{3}{4}, \frac{8}{11}$ as examples for positive rational numbers or fractions, and to introduce $-\frac{1}{2}, -\frac{7}{3}$ and $-2\frac{1}{5}, -\frac{17}{2}$ as examples for negative rational numbers.

In another case, teachers used an informal language such as 'opposite', 'flip', or 'swap places' when teaching how to find additive or multiplicative inverses of rational numbers. For instance, Teacher A asked one of the students to find the multiplicative inverse of $\frac{19}{7}$ after teaching multiplication of rational numbers. Teacher A's use of the colloquial term 'flip' is illustrated by the following teaching episode:

> Teacher A: What is the multiplicative inverse of $\frac{19}{7}$?
>
> Student: … (No answer)
>
> Teacher A: Why don't you say 'we flip it over'?
>
> Student: … (No answer)
>
> Teacher A: Flip $\frac{19}{7}$ over!

> Student: It is $\frac{7}{19}$.
>
> Teacher A: Thank you. To find the multiplicative inverse of a rational number we flip it over.

Teacher A's use of the language in such a way has the potential to hinder the development of students' rational number understanding. Although the middle school mathematics curriculum suggested teachers to emphasize that if the multiplication of two rational numbers is equal to 1, then the rational numbers are multiplicative inverses of each other (i.e., $\frac{a}{b} \cdot \frac{c}{d} = 1$, $\frac{a}{b} \neq 0, \frac{c}{d} \neq 0$), Teacher A preferred to flip over the rational number to find its multiplicative inverse. Instead of this, it would be more appropriate for the teacher to provide $\frac{19}{7} \cdot \frac{a}{b} = 1$ in order to have students understand the notion of multiplicative inverse conceptually.

Teacher B asked his students to perform $\frac{5}{12} \div \frac{7}{4}$ when teaching division of rational numbers. Teacher B's use of the colloquial term 'swap places' during the teaching of rational number division is illustrated by the following excerpt:

> Teacher B: How did we perform division of fractions last year? Let me recall. The first fraction remains the same, division becomes multiplication and the second fraction flips over. That is, 7 and 4 swap places and $\frac{7}{4}$ turns into $\frac{4}{7}$. Thus we multiply $\frac{5}{12}$ by $\frac{4}{7}$ in this way: $\frac{5}{12} \cdot \frac{4}{7} = \frac{20}{84}$.

Teacher B used the following approach for division of fractions: Just change the division sign to multiplication, flip over the second fraction and multiply. This approach to division of fractions might provide students with easy access to procedural understanding but not to relational understanding. Instead of this, it would be more appropriate for the teacher to explain why the division sign is changed into multiplication and why the second fraction is flipped over. Thus, the teacher might explain that division of fractions means multiplying the dividend by the multiplicative inverse of the divisor and might provide the following symbolic expression: $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{1}{\frac{c}{d}} = \frac{a}{b} \cdot \frac{d}{c}$.

Teacher D was teaching how to calculate the square and cube of rational numbers. One of the students asked how to calculate exponents with negative powers. At that moment, Teacher D used the colloquial term ‘flip over’ and explained that the base needs to be flipped over before raising to the desired power. Teacher B’s use of the colloquial term ‘flip over’ during the teaching of rational number exponentiation is illustrated by the following excerpt:

> Student: Teacher, how do we calculate $\left(\frac{4}{5}\right)^{-2}$?
>
> Teacher D: Let me explain what we should do when we come up with a negative power. In this example, you first flip over $\frac{4}{5}$ and then raise the new fraction to the second power.

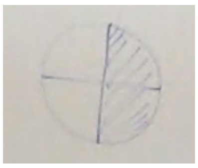
Teacher D’s approach for calculating negative powers of rational numbers also reflects a procedural understanding. Thus, it would provide more relational understanding to students to emphasize that a negative power represents the multiplicative inverse of the base.

Teacher B was teaching additive inverse property of rational numbers. He chose to use $-2\frac{3}{4}$ for teaching how to find the additive inverse. Teacher B used the colloquial term ‘opposite’ and explained that if the rational number is positive then its opposite is negative or if the rational number is negative then its opposite is positive. Teacher B’s use of the colloquial term ‘opposite’ during the teaching of rational number exponentiation is illustrated by the following episode:

> Teacher B: Let’s find the opposites of rational numbers. For instance, tell me the opposite of $-2\frac{3}{4}$.
>
> Student: Shall we first convert it into an improper number?
>
> Teacher B: No, you do not need to convert. Tell me the opposite of $-2\frac{3}{4}$.
>
> Student: Its opposite will be $+2\frac{3}{4}$.
>
> Teacher B: Good! If the rational number is positive, its opposite will be negative and if the rational number is negative its opposite will be positive. Thus, the opposite of $-2\frac{3}{4}$ will be $+2\frac{3}{4}$.

Teacher B's use of the colloquial term 'opposite' might help students gain intuition and provide easy access to procedural understanding, however it might hinder students' conceptual understanding of the additive inverse property. More importantly, the teachers' explanation for finding the additive inverse property as "if the rational number is positive, its opposite will be negative and if the rational number is negative its opposite will be positive" may not help the students notice the structure inherent in rational numbers that are additive inverses of each other. That is, although the middle school middle school mathematics curriculum suggested teachers to emphasize that adding rational numbers that are additive inverses of each other yields 0 (i.e., $\left(\frac{a}{b}\right)+\left(-\frac{a}{b}\right)=0$), Teacher B did not make any explanation in this way. Thus, it would be more appropriate for the teacher to provide $\left(-2\frac{3}{4}\right)+\frac{a}{b}=0$ while teaching additive inverse property of rational numbers.

Another case in which teachers used improper language was observed when they read fractions or rational numbers in an ill-advised manner. For instance, Teacher B started teaching addition and subtraction of rational numbers by initially introducing an example related with fractions. Meanwhile, he recalled how to read fractions. This case is illustrated by the teaching episode of Teacher B as follows:

> Teacher B: Last year we performed operations with fractions, didn't we?
>
> Student: Yes, we did.
>
> Teacher B: We can add and subtract fractions. Let me give an example related with subtraction of fractions.
>
> 
>
> With which fraction do we represent the shaded area?
>
> Student: Two fourths.
>
> Teacher B: Two fourths or two over four. We may also read it as one over two. Now, if we subtract one over four from two over four, then the left over part corresponds to one over four.

As it can be seen, Teacher B used the expression ‘over’ to describe $\frac{2}{4}$. Using such fraction language is somewhat problematic for students since it may obscure the relationship of the parts to the whole and the actions used to operate on fractions.

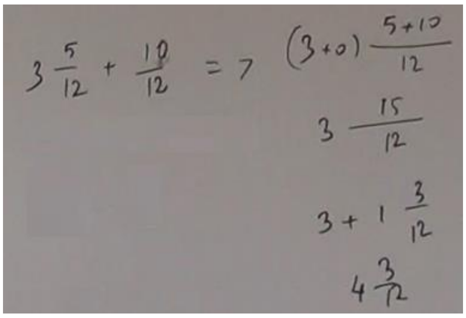
Teachers generated another example with inappropriate language or terminology when they attempted to read rational numbers as fractions. For instance, Teacher B was teaching addition of mixed numbers. After converting mixed numbers into improper rational numbers, he read those rational numbers loudly. However, he used an inappropriate language for reading them. This is illustrated by the teaching episode of Teacher B as follows:

> Teacher B: How do you perform $5\frac{3}{4}+\left(-1\frac{1}{4}\right)$?
>
> Student: We first subtract 1 from 5 and then add $\frac{3}{4}$ and $\frac{1}{4}$.
>
> Teacher B: You will certainly make mistakes if you add whole parts and fractional parts separately. Just to be on the safe side, add the mixed numbers after you convert them into improper rational numbers. Thus, $5\frac{3}{4}$ is equal to $\frac{23}{4}$ and $\left(-1\frac{1}{4}\right)$ is equal to $\left(-\frac{5}{4}\right)$. Now we can add twenty three fourths and negative five fourths to each other…

As this teaching episode shows, Teacher B read $\left(-\frac{5}{4}\right)$ incorrectly as “negative five fourths.” Since rational numbers are also ratios, it would be more precise to read $\left(-\frac{5}{4}\right)$ as ‘negative five to four’ or ‘negative five for four’ (Lamon, 2012).

Finally, teachers presented to their students worked-out examples that included the use of incorrect mathematical symbols. For instance, when teaching addition of $3\frac{5}{12}$ and $\frac{5}{6}$, Teacher A used implication sign instead of equal sign and he did not use equal sign between the expressions he wrote as he proceeded towards the answer. This is illustrated by excerpt given below:

Teacher A: …Another way to add mixed numbers is to add the whole parts first and fractional parts second. We can add the given mixed fractions in this way:

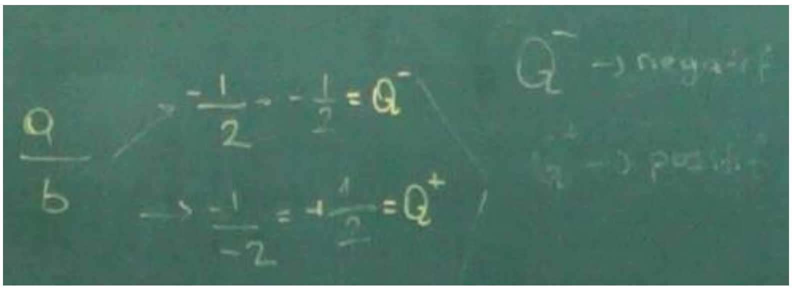


However, I do not suggest you to use this way. Because you certainly make errors when you choose this way.

As can be seen, Teacher A used an implication sign as a stand-in for the equal sign and left equal sign unused in circumstances that called for it. Here, an equal sign would have been the correct symbol to represent the relationship between $3\frac{5}{12}+\frac{5}{6}$ and $(3+0)\frac{5+10}{12}$.

Similarly, Teacher D was explaining rational numbers and she exemplified positive and negative rational numbers by means of $+\frac{1}{2}$ and $-\frac{1}{2}$. However, she incorrectly used equal sign (=) instead of 'is an element of' symbol ($\in$) when demonstrating $+\frac{1}{2}$ as an element of positive rational number set and $-\frac{1}{2}$ as an element of negative rational number set. This is illustrated by the excerpt given below:

Teacher D: Rational numbers that are smaller than zero form negative rational number set. It is denoted by the symbol $\mathbb{Q}^-$. Rational numbers that are larger than zero form positive rational number set and it is denoted by the symbol $\mathbb{Q}^+$. Now, the following examples can be given for these two number sets:



Thus, rational number set is the union of negative rational number set, positive rational number set and zero. We can symbolically express rational number set as $\mathbb{Q} = \mathbb{Q}^{-} \cup \{0\} \cup \mathbb{Q}^{+}$.

6.2.2. To be avoided examples

In-depth exploration of middle school mathematics teachers' choice and use of examples brought to light some examples which should be better avoided in the teaching of rational number concepts. In more detail, some of the examples provided by the teachers included particular pitfalls that might be an obstacle for students to understand the mathematical object, concept or procedure that they confronted for the first time. This type of examples were referred to as 'to be avoided examples' or teachers' poor choice of examples. In this study, two types of middle school mathematics teachers' poor choice of examples were identified. These were examples that 'obscure the role of variables' and 'examples intended to illustrate a particular procedure, for which another procedure would be more sensible.' The first type of teachers' poor choice of examples are presented in the following section.

6.2.2.1. Examples that obscure the role of variables

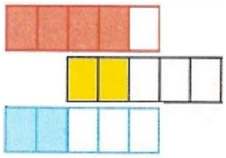
As mentioned in the literature review section, Marton and Booth's (1997) theory of 'dimensions of variation' deals with the idea that most mathematical concepts and procedures and every example of these concepts and processes comprises two or more components or variables. According to this theory, people learn from discerning variation and what varies in people's experience influence what they learn. Thus, the teachers are expected to consider dimensions of variation when providing their students examples about mathematical concepts or procedures.

Particularly, when students encounter with a novel mathematical concept or procedure for the first time, it is helpful to use variables that take different values. This is considered important since it helps learners to distinguish between different variables and the different roles they undertake. For instance, if a teacher wants to teach subtraction of natural numbers to their students, he/she must avoid providing '6-3=3' in her very first example. This is due to the fact that in a subtraction operation there are three different variables as minuend, subtrahend and difference and by selecting same values for the subtrahend and the difference the teacher makes it impossible for the students to distinguish between those two variables. In essence, one value (3, here) is being made to do the work of two variables in subtraction operation. Adjusting this example slightly (e.g., 7-4=3) would resolve the quandary and elucidate the roles of minuend, subtrahend and difference in the subtraction operation.

Examples that obscured the role of the variables were described through the following cases in this study: (1) obscuring the role of repeating and non-repeating digit in the teaching of repeating decimal concept, (2) obscuring the role of subtrahend and difference in teaching the modelling of subtraction of rational numbers, and (3) obscuring the role of interval number and rational number magnitude when locating on a number line.

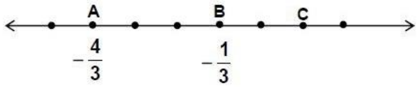
Before teaching the procedure for converting repeating decimals into rational numbers teachers explained the concept of repeating decimals by using a specific example. Teacher A, Teacher B and Teacher D selected the following examples respectively: $\frac{10}{3} = 3.333... = 3.\bar{3}, \frac{5}{3} = 1.333... = 1.\bar{3}, \frac{10}{3} = 3.333... = 3.\bar{3}$. The examples used by Teacher A and Teacher D did not reflect a deliberate and informed selection while the example used by Teacher B reflected a well-chosen example. In a repeating decimal in the form of $a.\bar{b}$, there are two variables as a non-repeating digit (i.e., *a*) and a repeating digit (i.e., *b*). However, by selecting $\frac{10}{3} = 3.333... = 3.\bar{3}$, the teachers made the distinction between non-repeating digit and repeating digit obscure. In this example, 3 was made to do the work of two variables. Thus, the students may hesitate over which 3 to put the vinculum. The point is that by selecting a slightly different example from $\frac{10}{3}$ such as $\frac{7}{3}, \frac{8}{3}$ and $\frac{11}{3}$, it is possible to clarify the role of non-repeating and repeating digits. For instance, $\frac{11}{3} = 3.666... = 3.\bar{6}$ includes 3 as a non-repeating digit and 6 as a repeating digit. Thus, this example would help students distinguish between repeating and non-repeating digits.

Another case occurred when teachers began to teach subtraction of rational numbers. In their first examples, teachers used same rational numbers as subtrahends and differences. For instance, Teacher D obscured the role of subtrahend and difference when modelling of subtraction of rational numbers and selected them to be equal to $\frac{2}{5}$. The following is an excerpt of the teacher explanation:

> Teacher D: In this lesson I am going to teach you how to subtract rational numbers. I will first show you how to express the following area model symbolically.
>
> 
>
> The fraction representing the first region is minuend, the fraction representing the second shaded region is subtrahend, and the fraction representing the third shaded region is difference. In the first whole, 4 parts are shaded so the minuend is equal to $\frac{4}{5}$. In the second whole, 2 parts are shaded so the subtrahend is $\frac{2}{5}$ and in the third whole, again 2 parts are shaded so the difference is $\frac{2}{5}$. Thus, the model can be symbolically expressed as $\frac{4}{5}-\frac{2}{5}=\frac{2}{5}$.

As the above given excerpt shows, the modelling example chosen by Teacher D obscured the role of variables since same number of parts were shaded both in the second and third whole. In this example $\frac{2}{5}$ was made to do the work of both subtrahend and difference. The teacher would resolve this quandary by shading one or three parts in the second whole.

In another case, Teacher A selected an example that obscured the role of interval number and rational number magnitude when locating on a number line. Teacher A's teaching episode related with this case is presented below.

> Teacher A: Find the rational number that corresponds to point C in the following number line.
>
> 
>

Student 1: Is it $+\frac{1}{3}$?

Students: $+\frac{1}{3}$

Teacher A: This is a very good number pattern problem. This pattern proceeds as -4, -3, -2, -1, 0, 1,… Thus, point C corresponds to $\frac{1}{3}$. Well, what if there was no such pattern?

Students: How?

Teacher A: If there was not any pattern, to find the unit distance you would first find the distance between points A and B and then divide the obtained distance by the number of intervals between A and B. Then, the unit distance is equal to $\frac{\left|-\frac{4}{3}\right|-\left|-\frac{1}{3}\right|}{3}=\frac{\frac{3}{3}}{3}=\frac{1}{3}$. Each decrement is equal to $\frac{1}{3}$. Thus, point C corresponds to $\frac{1}{3}$.

Student 2: Teacher, $\frac{3}{3}$ is already equal to one whole, why did we again divide it by 3?

Teacher A: $\frac{3}{3}$ refers to the distance between A and B and 3 refers to the number of intervals between A and B. Do you understand?

Students: No!

As it can be seen, the rational number corresponding to the distance between A and B includes 3 as a numerator and a denominator. Besides, there are 3 equal intervals between A and B. Since all of these numbers were selected to be 3, this example obscured the role of variables. Moreover, the decrement between consecutive points and the corresponding value of point C are equal to $\frac{1}{3}$. Thus, this may also obscure the role of variables. To resolve this quandary, the teacher would change the denominators of point A and point B by another number such as 5.

6.2.2.2. Examples intended to illustrate a procedure, for which another procedure would be more sensible

Mathematics teachers often use examples in the course of teaching a general procedure by a particular demonstration of that procedure. These procedures include the use of several strategies or algorithms. In particular, the teaching of rational number topics entails the use of strategies for comparing and ordering rational numbers, written calculation algorithms for adding, subtracting, multiplying or dividing rational numbers and the use of estimation strategies for adding, subtracting, multiplying or dividing rational numbers. In ordering strategies, estimation strategies, and written calculation algorithms there exists a number of options. However, a rational number example that is selected to demonstrate a particular strategy or algorithm should be relevant for that strategy or algorithm. More precisely, the example selected to illustrate a particular procedure should not call for another more sensible procedure. Otherwise, the students may believe that there is no point in learning that particular procedure.

In this study, there were several instances in which teachers provided examples to illustrate a particular procedure, but the examples called for other procedures that are more sensible. These examples were described through the following cases: (1) not using relevant examples when illustrating the procedure for converting repeating decimals, (2) not using relevant examples when teaching particular strategies for comparing and ordering rational numbers, and finally (3) not using relevant examples when teaching a particular written algorithm for adding rational numbers.

There are two different procedures for converting repeating decimals into rational numbers. The first procedure has to do with repeating decimals that include only repeating digits after the decimal point such as $2.\overline{7}$, $3.\overline{15}$, $4.\overline{245}$ and $5.\overline{3478}$. For these kinds of repeating decimals, it is more sensible to use the following conversion procedures: $a.\overline{b} = a\frac{b}{9}$, $a.\overline{bc} = a\frac{bc}{99}$, $a.\overline{bcd} = a\frac{bcd}{999}$, $a.\overline{bcde} = a\frac{bcde}{9999}$. As it can be seen, this procedure entails writing the decimal digit as the numerator of the rational number and writing down a 9 for every repeating digit as the denominator of the rational number. The second procedure has to do with repeating decimals that include both repeating and nonrepeating digits after the decimal point such as $2.0\overline{5}$, $4.35\overline{8}$ and $3.1\overline{25}$. For these kinds of repeating decimals, it is more appropriate to use the second procedure. This procedure is composed of the following steps: (1)

write down the repeating decimal without its decimal point; (2) subtract non-repeating part from Step 1; (3) divide the number obtained from Step 2 by the number with 9's and 0's: for every repeating digit write down a 9 and for every non-repeating digit write down a 0 after 9's. Thus, the following conversion procedures might be derived by means of the aforementioned steps: $a.b\bar{c} = \frac{abc - ab}{90}$, $a.bc\bar{d} = \frac{abcd - abc}{900}$ and $a.b\overline{cd} = \frac{abcd - ab}{990}$.

Teacher A sensibly used the first procedure as a means for converting repeating decimals that include only repeating digits. For instance, he chose to use the following examples of this kind: $0.\bar{2} = \frac{2}{9}$, $2.\bar{3} = 2\frac{3}{9}$ and $5.\overline{26} = 5\frac{26}{99}$. In a similar fashion, he used the second procedure appropriately for converting repeating decimals with repeating and non-repeating digits. Some of the examples selected by him to illustrate the second procedure were: $2.6\bar{8} = \frac{268 - 26}{90}$, $52.7\overline{14} = \frac{52714 - 527}{990}$ and $6.32\overline{84} = \frac{63284 - 632}{9900}$. However, other teachers used the second procedure for all types of repeating decimals. Although there is no need to deploy the second procedure for repeating decimals with only repeating digits, the teachers did not avoid using it. In such instances, the students quite reasonably might think that there is no point in learning a method for which there seems to be no need. For instance, Teacher B initially selected $0.\bar{7}$, $1.\bar{3}$, $2.\overline{15}$, $5.\overline{104}$, $3.\overline{24}$ and $1.0\overline{45}$ as examples for teaching the second procedure. Demonstrating the second procedure by using $0.\bar{7}$ seems quite problematic for some reasons. First, it does not include any non-repeating digits. Second, it includes 0 before the decimal point and thus the first step of the second procedure (i.e, write down the repeating decimal without its decimal point) makes little sense to the children since they do not come up with a number in the form of 07 during their mathematics lessons. In addition, there is no need to use the second procedure for the latter two examples (i.e., $1.\bar{3}$, $2.\overline{15}$) since they also do not have non-repeating digits. Finally, for the last three examples, it is sensible to use the second procedure since these examples include both repeating and non-repeating digits. Similar to Teacher B, Teacher C only introduced the second procedure to their students. He selected $1.\overline{3}$, $3.\overline{07}$ and $24.7\overline{89}$ as examples for teaching this procedure. As it can be seen, it is not relevant to use the second procedure for the first two examples. However, it is more appropriate to use it for the last example. Finally, Teacher D merely introduced the second procedure to their students as well and selected $0.\overline{3}$, $0.\overline{5}$, $0.\overline{8}$, $0.\overline{9}$ and $2.\overline{5}$ as examples for demonstrating this procedure. Nevertheless, these examples include only one repeating digit. Hence, it would be more sensible to use the first procedure for these kinds of examples.

There are several strategies for comparing and ordering rational numbers. In this study, the following strategies were used by the middle school mathematics teachers in the course of comparing and ordering rational numbers: finding common denominator of rational numbers, finding common numerator of rational numbers, benchmarking, residual thinking, locating rational numbers on a number line and converting rational numbers into decimal numbers. Nonetheless, it is important to know which ordering strategy is more relevant to use for a given set of rational numbers. This is due to the fact that while some set of rational numbers easily lend themselves to a certain strategy, the other set of rational numbers might be more efficiently ordered by another strategy. Thus, teachers play an important role in choosing an appropriate strategy for a given set of rational numbers or in choosing relevant examples for using a specific comparison or ordering strategy.

In this study, middle school mathematics teachers selected certain set of rational numbers and ordered them by using specific strategies. However, in some cases the selected set of rational numbers lent themselves more readily to other strategies which were not used by the teachers. For instance, Teacher B selected an ordering example from an auxiliary book and the example included the following rational numbers for ordering: $\frac{13}{12}, \frac{11}{10}, \frac{7}{6}$ and $\frac{5}{4}$. The teacher suggested his students to order these rational numbers by using common denominator algorithm. However, as can be seen, each rational number includes a numerator that is one more than its denominator. Then, if each rational number is rearranged as $\frac{13}{12} = 1 + \frac{1}{12}$, $\frac{11}{10} = 1 + \frac{1}{10}$,

$\frac{7}{6}=1+\frac{1}{6}$ and $\frac{5}{4}=1+\frac{1}{4}$, it becomes apparent that the use of residual thinking or common numerator algorithm would be more sensible when compared to the use of common denominator algorithm.

To give another example, Teacher D selected the following rational numbers for ordering: $\frac{1}{3}, 1\frac{2}{5}$ and $-1\frac{1}{6}$. However, she intended to order these rational numbers by locating the rational numbers on a number line. For the given set of rational numbers, there is no need to employ such strategy since they can be easily ordered when their magnitudes and directions are taken into consideration. That is, $-1\frac{1}{6}$ is smaller from 0 while $\frac{1}{3}$ and $1\frac{2}{5}$ are larger than 0. Thus, $-1\frac{1}{6}$ is the smallest rational number. In addition, $1\frac{2}{5}$ is larger from 1 and $\frac{1}{3}$ is smaller from 1. Thus, $1\frac{2}{5}$ is the largest rational number. Consequently, the rational numbers can be easily ordered as follows: $-1\frac{1}{6}<\frac{1}{3}<1\frac{2}{5}$. In another example, Teacher D selected the following rational numbers: $\frac{7}{4}, \frac{3}{2}$ and $\frac{8}{5}$. However, she ordered these rational numbers by first using a common denominator algorithm and then converting them to decimals as follows: $\frac{3}{2}=\frac{150}{100}=1.5<\frac{8}{5}=\frac{160}{100}=1.6<\frac{7}{4}=\frac{175}{100}=1.75$. A more sensible strategy for ordering the given set of rational numbers would be common denominator algorithm. By using this strategy the rational numbers can be more easily ordered as follows: $\frac{3}{2}=\frac{30}{20}<\frac{8}{5}=\frac{32}{20}<\frac{7}{4}=\frac{35}{20}$.

There are two different methods for adding mixed numbers. In the first method, the mixed numbers are converted into improper rational numbers before performing the addition algorithm. This first method is more plausible when the numerators and denominators of the mixed numbers are selected to be small numbers. In the second method, there is no need to convert mixed numbers into improper rational numbers before the addition algorithm. That is, the second method entails adding whole parts and fractional parts of mixed numbers separately. The second method is more plausible when the numerators and denominators of the mixed numbers are selected to be large numbers. In this study, Teacher D wanted to demonstrate the second method of adding mixed numbers to their students. However, she selected an example that could be more sensibly worked out by using the first method. The teaching episode of Teacher D illustrates this case as follows:

> Teacher D: Write the following example on your notebooks: $1\frac{1}{5}+4\frac{3}{5}$. Here, I am going to teach you a shortcut procedure for adding mixed numbers. Normally, you used to convert rational numbers into improper rational numbers before adding them in this way: $1\frac{1}{5}+4\frac{3}{5}=\frac{6}{5}+\frac{23}{5}=\frac{29}{5}$. However, you can also perform this operation by adding whole parts and fractional parts of the mixed numbers separately as:
>
> $$1\frac{1}{5}+4\frac{3}{5}=(1+4)+\left(\frac{1}{5}+\frac{3}{5}\right)=5+\frac{4}{5}=5\frac{4}{5}=\frac{29}{5}.$$
>
> Student: But, teacher this way is longer than the first way. We performed more operations!
>
> Teacher D: For this example you are right. However, as the denominators and numerators of the mixed numbers become larger, the second method will be shorter. I am telling the second method in order for you not to make errors. If you use the first method for adding mixed numbers with large denominators and numerators there is the risk of making errors when multiplying numbers.

As remarked by one of the students, the example selected by Teacher D with the intention of demonstrating the second method was more relevant to the first method. To resolve this quandary, the teacher would select and introduce examples such as $102\frac{11}{19}+215\frac{14}{19}$. As it can be seen, it is more sensible to use the second method for working out this example and the first method is a bit risky since there is the risk of making errors when computing the following expressions: $102\times19+11$ and $215\times19+14$.

6.4. Summary of Incorrect or Inappropriate Examples

In this chapter, the focus was on identifying mathematically incorrect or pedagogically inappropriate rational number examples used by teachers. More specifically, teachers' mathematically incorrect or pedagogical inappropriate examples were reported under three main sections as mathematically incorrect examples, examples with improper language or terminology and to be avoided rational number examples.

Teachers provided the following mathematically incorrect examples or explanations in the course of teaching rational number concepts: explaining that irrational numbers cannot be located on a number line, explaining that rational number set is a subset of irrational number set, explaining that irrational number set includes less number of elements than rational number set, explaining that all numbers in the fraction form are rational numbers, working out an example incorrectly due to the misapplication of absolute value concept, not partitioning the number line into equal distances when locating rational numbers on it, using commutative property of addition when exemplifying associative property of addition, seeing conversion of repeating decimals into rational numbers as being synonymous with rounding, under-generalizing the addition of mixed numbers, and finally using a correct ordering strategy but misnaming it as another strategy.

Some of the examples generated by the teachers were correct when evaluated from a mathematical standpoint. However, they were not appropriate from a pedagogical standpoint since they included the use of inappropriate language or terminology. Examples of this type occurred due to careless use of the word fraction when rational number is intended, the use of informal language such as opposite, flip and upside down for teaching additive or multiplicative inverses of rational numbers, ill-advised reading of rational numbers, and finally the incorrect use of mathematical symbols in the course of working out rational number examples.

Finally, exploration of teachers' choice and use of examples brought to light several examples which would be better avoided in the teaching of rational number concepts. These examples included particular pitfalls that might be an obstacle for students to understand the mathematical concept or procedure that they confronted for the first time. Thus, to be avoided examples reflected teachers' poor choice of examples in the teaching of rational number concepts or procedures. In more detail, two types of middle school mathematics teachers' poor choice of examples were identified. These were, examples that obscured the role of variables and examples intended to illustrate a particular procedure, for which another procedure would be more sensible. In this study, examples that obscured the role of the variables were described through the following cases: obscuring the role of repeating and non-repeating digit in the teaching of repeating decimal concept, obscuring the role of subtrahend and difference in teaching the modelling of subtraction of rational numbers, and obscuring the role of interval number and rational number magnitude when locating on a number line. Furthermore, there were several instances in which teachers provided examples to illustrate a particular procedure, but the examples called for another procedure that is more sensible. These types of examples occurred in the following cases: not using relevant examples when illustrating the procedure for converting repeating decimals, not using relevant examples when teaching particular strategies for comparing and ordering rational numbers, and finally not using relevant examples when teaching a particular written algorithm for adding rational numbers.

CHAPTER VII

DISCUSSION, IMPLICATIONS AND RECOMMENDATIONS

The current study explored middle school mathematics teachers' treatment of rational number examples in their seventh grade classrooms. The findings of the study were reported under three main chapters based on the research questions. In the first chapter, the focus was on describing overall characteristics of teachers' rational number examples. Through this focus the rational number ideas that were emphasized by the teacher generated examples, the type of teacher generated examples, the way teachers chose rational number examples, and the resources used by the teachers when choosing rational number examples were described at length. In the second chapter, the focus was on exploring the principles or considerations used by teachers while choosing or generating rational number examples. Through this purpose, the examples that manifested the following teacher considerations were brought to light: starting with a simple or familiar case, drawing attention to students' difficulty, error or misconception, keeping unnecessary work to minimum, taking account of examinations, including uncommon cases, and finally drawing attention to relevant features. In the third chapter, the focus was on identifying mathematically incorrect or pedagogically inappropriate rational number examples used by the teachers.

In this chapter, discussion of the research findings were presented first. Next, implications and recommendations for future research studies were presented. The research findings were discussed under three main sections by depending upon the research questions. In these sections, findings about overall characteristics of teachers' rational number examples, findings regarding teachers' considerations in choosing examples, and findings regarding teachers' mathematically incorrect or pedagogically inappropriate examples were discussed respectively. Finally, two empirically based conceptual frameworks that might be used to examine middle school teachers' choice of examples and considerations for choosing these examples were proposed.

7.1. Overall Characteristics of Teachers' Rational Number Examples

This study revealed that although middle school mathematics teachers used three different types of examples as specific examples, non-examples and counter-examples, they mainly used specific examples for teaching rational number concepts or procedures. As emphasized by Zazkis (2005), it is difficult to think learning mathematics without considering specific examples. Specific examples are important because they help in understanding general (Feynman, 1985). In this study, the quality and quantity of teachers' rational number examples were explored in comparison with the specific examples included in the followed mathematics textbook. When the number of specific examples provided by the teachers were examined, it was seen that teachers with greater years of rational number teaching experience exposed their students to a more number and variety of rational number examples. This finding provides insights into teachers' craft knowledge. Kennedy (2002) was particularly interested in the nature of knowledge emanating from the experience of teaching and referred to it as craft knowledge. This knowledge type is kinesthetic and develops from repeated experiences through working with a specific material and is learned from experience and guidance from a master, but not learned by reading books (Kennedy, 1999). Craft knowledge is one form of professional expertise and it is not a technical skill or an ability to conduct critical analysis; but rather, it represents the building of situated, learner-oriented pedagogical knowledge focusing on procedures and content through purposeful action (Kennedy, 1987).

On the other hand, very few examples were provided by the teachers and the textbook for posing and solving rational number problems when compared to examples provided for teaching four operations with rational numbers. This reflects the emphasis given by the middle school mathematics curriculum on rational number operations or procedures. More specifically, MoNE (2009b) suggests teachers to allocate three lesson hours for teaching problem posing and solving with rational numbers, whereas it suggests teachers to allocate nine lesson hours for teaching operations and procedures with rational numbers. Since more than half of the examples used by middle school mathematics teachers were related with rational number operations, it is natural to expect teachers to be proficient with algorithms for performing rational number operations. Indeed, Izsak, Orrill, Cohen and Brown (2010) pointed out that most of the teachers can multiply or divide rational numbers but many of them have limited capacity to reason about products and quotients when they are included in problem contexts. Similarly, several other studies found out that pre-service and in-service teachers lack performance when explaining multiplication and division of rational numbers (Armstrong & Bezuk, 1995; Tirosh, 2000). Middle school mathematics curriculum might play an important role for teachers in deciding which examples to select or how many examples to use in teaching mathematics topics. Thus, it is significant to revise the middle school mathematics curriculum by increasing the number of lessons devoted to problem posing and solving with rational numbers or by integrating problem solving approach into other learning objectives related with rational numbers. By this way, the number of examples used by teachers for teaching rational number operations and problems would be more balanced.

The teachers in this study relied to some extent on the mathematics textbook and the middle school mathematics curriculum when teaching rational number ideas. The rational number ideas such as problem posing and estimation were emphasized by the textbook examples but were ignored by the teachers. On the other hand, teachers sometimes provided examples that emphasized other rational number ideas apart from the textbook such as identifying whether a given number is rational or not and ordering rational numbers by using residual thinking strategy. This way of teaching was sometimes beneficial to students' understanding of rational numbers and sometimes led students to incomplete understandings about rational number concepts.

One advantage of not strictly relying on textbooks was related with having students identify whether a given number is rational or not. When specific examples provided by the textbook for explaining and locating rational numbers on a number line were examined, it was seen that there was not any example related with identifying whether a given number is rational or not. However, all teachers gave particular importance to teaching this idea. In more detail, teachers asked students to determine whether negative and positive integers, mixed, proper and improper numbers, decimal numbers or radical numbers, pi number and ratio of a number to zero are examples of rational numbers. Providing students with different types and forms of rational number examples is important since it may help students enrich their understanding of rational number concept (Zazkis, 2005).

Another advantage was seen in teachers' attempts to draw students' attention to location of minus sign. Although examples reflecting this idea did not appear in the textbook, teachers' paid attention to using such examples probably based on their previous teaching experiences.

One final advantage was seen in a teacher's attempt to incorporate into the classroom ordering and comparing examples that entailed using conceptual strategies such as residual thinking. Examples that require the use of residual thinking strategy were not provided by the textbook. The term residual refers to the amount needed to make a whole. Clarke and Roche (2009) indicated that residual thinking is a specific strategy that is unlikely to be taught by the teachers and they further argued that providing students with such strategies has the potential to promote student performance and understand relative size of relevant parts in fractions. Similarly, Post and Cramer (2002) claimed that the use of residual thinking helps students successfully compare fraction pairs. As discussed by Clarke and Roche (2009), residual thinking is a strategy that seems to be used by students exhibiting a more conceptual understanding of the size of the fractions. However, the use of this strategy did not seem to be commonly used by the teachers in middle school classrooms. This study revealed that only one of the teachers used ordering examples that entailed residual thinking, supporting Clarke and Roche's (2009) argument about teachers' use of this strategy.

As mentioned before, teachers sometimes ignored the ideas emphasized by the textbook examples when teaching rational number concepts and this led students to an incomplete understanding of rational number concepts. To illustrate, none of the teachers provided students with examples related with estimation of multiplication and division with rational numbers although the middle school mathematics curriculum placed considerable emphasis on this notion. Van de Walle, Karp and Bay-Williams (2013) stressed that the aim of estimation is being able to obtain a rough result that will function for the situation and give a sense of rationality. They further added that the ability to estimate is worthwhile in daily life since in many circumstances there is no need to know the exact answer. Clarke and Roche (2009) suggested teachers to provide their students with greater opportunities and approximation since they aid in developing number sense. National Council of Teachers of Mathematics [NCTM] (2000) emphasized that "teachers should help students learn how to decide when an exact answer or an estimate would be more appropriate, how to choose the computational methods that would be best to use, and how to evaluate the reasonableness of answers to computations" (p. 220). In this study, none of the teachers attempted to provide estimation examples to their students and thus, they did not help student learn these complex considerations. Siegler and Booth (2005) argued that students are better at obtaining exact results than estimating results and they find it difficult to do computational estimation. The students of the participating teachers might also encounter similar difficulties about estimation since teachers omitted using estimation examples in their classrooms. Based on this conjecture, it could be implied that in-service teachers should be provided the opportunity to participate in professional development activities that emphasize the role of estimation on deeper and meaningful understanding of mathematics. Similarly, it could be implied that teacher education programs should provide systematic learning opportunities to pre-service teachers about which computational strategies would work best and how to judge the rationality of answers to computations.

Examples related with posing rational number problems are provided by the textbook and are explicitly emphasized in the middle school mathematics curriculum. However, although teachers provided a few problem solving examples, they ignored providing examples related with problem posing. Despite the fact that problem posing have been accepted as an important part of scientific work among mathematics education researchers and mathematics educators (Stoyanova, 2003), emphasis has been primarily put on problem solving rather than problem solving (Cankoy, 2014). The emphasis placed on problem solving rather than problem posing is also true for the middle school mathematics curriculum released by MoNE (2009b). Mathematical problem posing can be defined as creation of a novel problem or reformulation of a previously existing problem (Silver, 1993). Problem posing can also be regarded as a process ending up with a problem that needs to be solved (Dillon, 1982). In the last twenty years, mathematics education researchers and educators have particularly began to notice the potential and significance of problem posing in the teaching and learning of mathematics (e.g., Chang, 2007; Lowrie, 2002; Silver, 1995). Therefore, there have been many educational attempts to include problem posing activities into mathematics lessons (Knott, 2010; Stoyanova, 2003). Similarly, NCTM (2000) emphasized the need for providing students with essential knowledge about gaining experience, becoming aware and constructing their own problems and added that problem posing is at the centre of doing mathematics. Thus, middle school mathematics teachers are expected to integrate rational number problem posing examples into their classrooms and become aware of the fact that students' problem posing experiences might help them promote mathematical thinking and understanding mathematical concepts in a deeper sense (Mestre, 2002).

As mentioned before, middle school mathematics teachers used non-examples of rational numbers in addition to specific examples. Non-examples show the boundaries or necessary conditions of a concept (Watson & Mason, 2005). Shortly, they "serve to clarify boundaries" of a concept (Bills et al., 2006, p. 127). Non-examples play a crucial role in promoting high levels of concept attainment (Charles, 1980; Cohen & Carpenter, 1980; Cook, 1981; Petty & Johnson, 1987; Tsamir et al., 2008). Besides, non-examples give teachers the chance to analyze their students' thinking and are supportive for students in reasoning out loud (Clements et al., 1999). In this study, teachers, in general, used four different forms of non-examples as ratio of an integer to zero (e.g., $\frac{2}{0}$), transcendental number (e.g., π), radical (e.g., $\sqrt{5}$), and infinite non-repeating decimal (e.g., 0.257843…). This seems to be an advantage on the part of students since the followed mathematics textbook of the classrooms did not provide any non-example for rational numbers. However, as stressed by Sirotic and Zazkis (2007), teachers missed the pedagogical opportunity to open students' minds at least to a variety of irrational number examples (i.e., non-examples of rational numbers) beyond radicals such as $\sqrt{2}$ and transcendental numbers such as π. Besides, Sirotic and Zazkis (2007) indicated that $\sqrt{2}$ and π are generic examples for irrational numbers and prospective secondary mathematics teachers might not be aware of the existence of irrational numbers beyond pi number, Euler's number and some commonly used square roots. Similarly, Zazkis and Leikin (2007) reported that pre-service teachers' personal example space of irrational numbers is limited to π and $\sqrt{2}$. Thus, the middle school mathematics teachers in this study might also have limited example spaces about irrationality. Nevertheless, the maxim 'absence of evidence is not evidence of absence' might not apply to our understanding of teachers' examples spaces (Zazkis & Leikin 2007). Thus, teachers' major use of square roots or the transcendental number π might not mean that their examples spaces of irrational numbers are limited to these numbers. It might simply mean that teachers had access to π and $\sqrt{2}$ as non-examples for rational numbers in that situation and at that time. The collection of examples that the middle school mathematics teachers had access to at that moment referred to teachers' accessible examples that are dependent on many factors such as the context, the trigger and the state of teachers (Goldenberg & Mason, 2008). After all, middle school mathematics teachers can provide their students genuine opportunities by exposing them to non-examples apart from the generic ones such as sin68° and ln15. Thus, mathematics educators play an important role in extending pre-service and in-service teachers examples spaces about non-examples of rational numbers. More importantly, merely one teacher preferred to use infinite non-repeating decimal representation (such as, 0.257843…) as a non-example for rational numbers. Zazkis and Sirotic (2010) suggested that only infinite non-repeating decimal representations are transparent representations of irrational numbers while other forms are opaque representations for irrational numbers. That is, only infinite non-repeating decimal representations of irrational numbers can make it possible for the students to derive the irrationality of numbers. Thus, teachers need to pay more attention to transparent representations when providing non-examples for rational numbers.

Apart from non-examples, teachers also used counter-examples in the teaching of rational number ideas. Similar to non-examples, counter-examples "can serve to sharpen distinctions and deepen understanding of mathematical identities" (Zodik & Zaslavsky, 2008, p. 165). Counter-examples are in a very powerful position when compared to other examples since one counter-example may be sufficient for establishing the invalidity of a claim while using many examples for establishing the truth of a claim may not be sufficient (Bogomolny, 2006). However, although counter-examples are important in the teaching of mathematics, the findings showed that they are less evident in middle school classroom practice. In this study, all of the counter-examples were generated by the teachers as a response to contingent classroom situations such as students' invalid conjectures or students' queries. However, only five counter-examples were generated by the teachers to demonstrate falsity of students' claims. This finding is in line with the findings of previous studies. For instance, Rowland et al. (2009) indicated that counter-examples are important mathematical ideas but their data suggested that they were less evident in primary classroom practice. Similarly, Zodik and Zaslavsky (2008) found out that secondary school mathematics teachers altogether used eighteen counter-examples during 54 lesson hours. This might be due to middle school mathematics teachers' views that mathematics they teach entails less higher-order thinking skills and less attention might have been given by the teachers to the use of counter-examples for disproving mathematical conjectures. Indeed, Zodik and Zaslavsky (2008) found that none of the teachers participated in their study pre-planned to intentionally use a counter-example in the lesson. Similarly, none of the middle school mathematics teachers participated in this study made a deliberate attempt to plan which counter-examples to use in the classroom. Teachers' scarce use of counter-examples might also have stemmed from the fact that the observed classes were rather teacher-centered where middle school mathematics teachers were more engaged in example generation process and the students were less active when compared to a student-centered classroom.

This study revealed two main sources of teacher-generated examples as spontaneous examples and pre-planned examples. The examples that were actually generated by the teachers during the lesson without any planning in advance or examples that were generated by the teachers as a response to unexpected classroom situations were treated as spontaneous examples. In other words, for an example to be spontaneous, there had to be some evidence that choosing it entailed in-the moment decision making to a certain degree. On the contrary, the examples that were taken from available resources such as textbooks, workbooks and auxiliary books were treated as pre-planned examples. Middle school mathematics teachers altogether used 361 spontaneous examples and 343 pre-planned examples during the teaching of rational numbers. This suggests that more than half of the examples used by the teachers were spontaneously generated, although the numbers are close. Teachers in Zodik and Zaslavsky's (2008) study also used close numbers of spontaneous and pre-planned examples, where the number of pre-planned examples was more. It was very difficult to clearly differentiate between pre-planned and spontaneous examples during the research. However, I believe that this distinction would be helpful in making sense of teachers' choice or use of examples.

The magnitude of the difference between teachers' spontaneous and pre-planned examples cannot say much unless there is access to their underlying principles or considerations that lead them to choose or generate rational number examples. Yet, it was possible to observe some tendencies on the part of teachers in terms of generating or selecting spontaneous and pre-planned examples. A closer examination of the number of examples showed that teachers with higher years of rational number teaching experience seemed to use more spontaneous examples than pre-planned examples, whereas teachers with less years of rational number teaching experience appeared to use more pre-planned examples than spontaneous examples. This might be because spontaneous examples tend to depend more on teachers' accessible example spaces (Watson & Mason, 2005). Thus, the increase in teachers' rational number teaching experience might have played an important role in generating spontaneous examples that are more immediate and automatic. In contrast, the decrease in teachers' rational number teaching experience might have led teachers to generate spontaneous examples after much longer time as a result of analytical thinking and self-monitoring (Zodik & Zaslavsky, 2008).

Teachers' mathematics background might also explain their tendency to generate more spontaneous examples than pre-planned examples. Namely, the teachers with mathematics background in the study generated more spontaneous examples while the teachers with elementary mathematics teacher education background used more pre-planned examples in the teaching of rational number ideas. Thus, teachers' way of selecting examples might be associated with their subject matter knowledge to some extent (Shulman, 1986). According to Rowland et al. (2009) subject matter knowledge consists of substantive and syntactic knowledge. They indicated that substantive knowledge refers to "the facts, concepts and processes of mathematics and the links between them" while syntactic knowledge refers to "knowing how mathematical truths are established" (p. 20-21). Therefore, in-service and pre-service middle school mathematics teachers may need to make greater efforts to consolidate their substantive and syntactic knowledge necessary for generation of rational number examples.

Teachers used several resources when choosing pre-planned rational number examples. The resources used were student textbook, student workbook, teachers' guidebook, high-stakes examination questions, online educational software and a wide variety of auxiliary books. In general, the teachers used auxiliary books for providing exercise examples to their students. Teachers mainly selected multiple choice question examples from auxiliary books, high-stakes examinations (SBS and ÖSS questions), and from online educational software. Their recourse to several resources that mainly included multiple choice questions reflected their consideration of Secondary School Entrance Examination (known as TEOG) taken by middle school students. Their use of many different auxiliary books also reflected their consideration of providing students with a wide variety of rational number examples and extending their examples spaces about rational number concepts. Thus, this may explain teachers' provision of examples that involve rational number ideas distinct from the ones included in the student textbook.

In this study, teachers' rational number examples were examined in greater depth. This exploration is summarized in Table 7.1 below. This table might be used in the development of a possible framework that might be used to capture middle school mathematics teachers' generation and choice of rational number examples in their classrooms. Future studies in different education systems might enhance this table and provide empirical support to the development of a possible conceptual framework for analyzing teachers' treatment of rational numbers.

Table 7.1. The summary of teachers' treatment of rational number examples

Explaining and locating rational numbers on a number line *Explaining rational numbers* Examples that demonstrate; • the need for positive and negative rational numbers • equivalent classes of a fraction • location of equivalent fractions on a number line • whether a given number is rational • the positivity and negative of a rational number • the location of a minus sign in a negative rational number • simplification of rational numbers • conversion among mixed and improper numbers *Locating rational numbers on a number line* Examples that demonstrate; • location of a rational number on a number line • finding the rational value of a point located on a number line	**Comparing and ordering rational numbers** *Comparing rational numbers* Examples that demonstrate comparing; • by locating on a number line • by benchmarking • by considering rational number sign • by converting a mixed number into an improper number *Ordering rational numbers* Examples that demonstrate ordering; • by locating on a number line • by converting rational numbers into decimal numbers • by common denominator algorithm • by common numerator algorithm • by benchmarking • by equivalent fractions • by residual thinking • by equating the number of decimal digits by adding 0s
Expressing rational numbers in different forms Examples that demonstrate; • expression of integers as rational numbers • expression of rational numbers as integers • expression of rational numbers as terminating decimals • expression of rational numbers as repeating decimals • expression of terminating decimals as rational numbers • conversion of repeating decimals into rational numbers	**Adding and subtracting rational numbers** Examples that demonstrate; • using models for the addition and subtraction of rational numbers • finding common multiples of the denominators of rational numbers • adding and subtracting rational numbers with same denominators • estimating the addition and subtraction of rational numbers • adding and subtracting rational numbers with different denominators • properties of addition of rational numbers • multi-step operations with rational numbers
Multiplying and dividing rational numbers Examples that demonstrate; • modeling multiplication of rational numbers • multiplication and division of rational numbers • multiplication and division by 0, 1 and (-1) • estimation of multiplication and division of rational numbers • modeling and calculating the square and cube of rational numbers • performing multi-step operations with rational numbers • properties of multiplication of rational numbers	**Performing multi-step operations with rational numbers** Examples that demonstrate solution of multi-step operations that are expressed; • on one line • as complex fractions • as a continuing pattern • as single variable polynomials
	Posing and solving rational number problems • solving rational number problems with same referent units • solving rational number problems with different referent units • posing rational number problems

7.2. Teachers' Considerations in Choosing Rational Number Examples

The selection of examples in the teaching of mathematics is extremely complicated and involves a wide variety of considerations (Zaslavsky & Lavie, 2005). The certain choice of examples may either promote or hinder students' understanding, thus teachers need to select examples with some care (Zaslavsky & Zodik, 2007). However, neither professional development programs nor teacher education programs in Turkey do not overtly address this issue and do not provide pre-service and in-service teachers with systematic knowledge about treatment of mathematical examples. Thus, it can be suggested that the skills necessary for powerful treatment of examples are crafted mainly by means of teachers' own teaching experiences (Leinhardt, 1990). Kennedy (2002) coined the term craft knowledge for the knowledge that emanated from the experience of teaching and summarized its role in teaching as follows:

> "Craft knowledge derives mainly from experience, but can derive from numerous other sources such as newspapers and magazines, advice from colleagues and friends, etc.; craft knowledge mainly helps teachers address concerns about student willingness to participate and orderly task progress; acquisition of craft knowledge is motivated largely by dissatisfaction with events and a desire to not repeat the same mistakes again;…" (p. 362).

It follows that much can be learnt from the experience of middle school mathematics teachers. Hence, inspired by the work of Zodik and Zaslavsky (2008), this study attempted to explore the principles or considerations used by middle school mathematics teachers while choosing or generating rational number examples. Through this purpose, the rational number examples that manifested the following teacher considerations were brought to light: starting with a simple or familiar case, drawing attention to students' difficulty, error or misconception, keeping unnecessary work to minimum, taking account of examinations, including uncommon cases, and finally drawing attention to relevant features. The experienced secondary school mathematics teachers observed by Zodik and Zaslavsky (2008) taught several different mathematical concepts to their students and employed the same considerations with one exception. This exception was teachers' consideration of examinations in Turkish middle school classrooms.

The middle school mathematics teachers attempted to start with simple or familiar cases when teaching rational number concepts. In the study of Zodik and Zaslavsky (2008), teachers considered sequences of examples and constructed these examples by gradually increasing their complexity or difficulty levels. The same consideration was also employed by the middle school mathematics teachers when teaching ordering or adding/subtracting rational numbers. Namely, teachers first provided rational numbers with same denominators and then rational numbers with different denominators while ordering or adding/subtracting rational numbers. Similarly, Bills and Bills (2005) found that the experienced teachers in their study chose to use simple examples as a first stage in developing students' understanding of mathematical procedures. It is natural for teachers to start with simple examples when introducing mathematical concepts or procedures. Because experienced teachers might certainly be aware that it is not reasonable for students to understand complex examples before encountering simpler ones.

Middle school mathematics teachers also drew students' attention to common difficulties, errors or misconceptions held by them about rational number concepts. For instance, when locating rational numbers on a number line, teachers explicitly warned their students to count equal parts of the line segment instead of counting tick-marks. This type of consideration is strongly related with teachers' pedagogical content knowledge (Shulman, 1986) and in particular with their knowledge of content and students (Ball et al., 2008). According to Shulman (1986), pedagogical content knowledge includes "the conceptions and preconceptions that students of different ages and backgrounds bring with them to the learning of those most frequently taught topics and lessons" (p.9). Similarly, Ball et al. (2008) described knowledge of content and students as "knowledge that combines knowing about students and knowing about mathematics" (p. 401). More precisely, knowledge of content and students is the knowledge of how students learn specific topics, the knowledge of the likely misconceptions students have or which topics might be problematic for students to understand and why (Hill, Ball & Schilling, 2008). This showed that teachers' consideration of their students' errors reflected not only their subject matter knowledge but also their pedagogical content knowledge.

Middle school mathematics teachers deliberately attempted to keep unnecessary work to minimum during the provision of rational number examples by reducing technical work and focusing on the essence, highlighting relevant parts of examples and not going into extra details, and by using properties of operations. For instance, teachers' specific choice of examples for illustrating repeating decimals manifested their attempts to keep unnecessary work to minimum. That is, teachers chose repeating decimals in which the repeating blocks were fairly easy to be noticed by the students. This case also occurred when secondary school mathematics teachers in the study of Zodik and Zaslavsky (2008) attempted to illustrate the period of rational numbers by selecting examples that had periods long enough to be noticed by the students. It is thought that this consideration helped teachers teach rational number concepts in a shorter period of time and helped the students learn the key components of the concepts rather than being bogged down with unnecessary work.

Teachers also considered incorporation of uncommon cases into their classrooms either by introducing exceptional or special cases or by introducing under-represented cases. It is believed that teachers' inclusion of uncommon cases to their teaching helped students to gain a complete understanding of rational number concepts. Teachers' this type of consideration might be explained by the emphasis placed by the middle school mathematics curriculum on the teaching of rational number concepts by using special cases such as having students notice the influence of 0, 1 and -1 in multiplication and division operations. The use of non-prototypical examples as uncommon cases was not evident in this study. However, Zodik and Zaslavsky (2008) indicated that secondary school teachers manifested this type of consideration in their study. It was a missed opportunity for middle school mathematics teachers not to incorporate non-prototypical examples related to rational number concepts into their classrooms. Actually, students are inclined to consider prototypical examples as examples of the concept and consider other examples as non-examples of that concept (Hershkowitz, 1989; Wilson, 1990).

Watson and Mason (2005) made the same point that students generally identify concepts with one or two examples introduced earlier by their teachers and they are often left with incomplete and limited sense of the concept. Thus, in order to lessen the influence of prototype examples, the students might have also been introduced to non-prototypical ones.

Another manifestation of teacher consideration had to do with drawing attention to relevant features of rational number concepts by deliberately attempting to reduce irrelevant information carried by specific examples. Skemp (1971) referred to the irrelevant information carried by examples as noise and he point out that if the noise of an example increases, then it becomes more difficult for students to form the concept demonstrated by that example. Thus, middle school mathematics teachers attempted to diminish the noise of the rational number examples by using pattern breaking strategy (Watson & Mason, 2005) and by using the structured variation principle (Watson & Mason, 2006). The findings are concurrent with those of Zodik and Zaslavsky (2008) although the examples provided by the middle school mathematics teachers and the secondary school mathematics teachers served for teaching different mathematical concepts. For instance, the middle school mathematics teachers in this study used pattern breaking strategy when teaching the procedure for converting repeating decimals into rational numbers whereas secondary school teachers in the study of Zodik and Zaslavsky (2008) used that strategy to teach Pythagorean Theorem to their students. In a similar way, the teachers in this study used the structured variation principle to teach four operations with rational numbers whereas the teachers of Zodik and Zaslavsky (2008) used it for teaching inequalities and linear functions. In this study, the use of pattern breaking strategy and structured variation principle as pedagogical strategies were seen to be beneficial for students' understanding of rational number concepts. Watson and Mason (2006) supported this practice of teachers and explained the role of structured variation as follows:

> "Our conclusions after 3 years of work in a range of natural settings are that control of dimensions of variation and ranges of change is a powerful design strategy for producing exercises that encourage learners to engage with

> mathematical structure, to generalize and to conceptualize even when doing apparently mundane questions. This power is easily recognized by teachers, teacher educators and other professionals in mathematics education" (p. 108).

Ultimately, teachers took account of examinations when using rational number examples. This consideration was not adopted by teachers participated in other studies such as Zodik and Zaslavsky (2008). This consideration might be specific to Turkish educational context. In Turkey, middle school students compete with each other to study in well-qualified secondary schools. To enter these well-qualified secondary schools, students need to have high grade point averages in 6^{th}, 7^{th} and 8^{th} grade levels. Besides, they have to take several national examinations called TEOG (Transition from Primary to Secondary Education Examination) in grade 8. Therefore, middle school mathematics teachers spend considerable efforts to help their students enter well-qualified secondary schools and consequently select their examples to serve for their intended purpose. That is, teachers bring to the classroom or generate in the classroom examples that are similar to the questions included in the examinations. Besides, they strive for incorporating high-stakes examination questions that were asked in the previous years into the classroom hoping that similar questions might be asked in the future examinations. These examinations include questions that are all in multiple-choice format. Therefore, teachers help students develop strategies for solving multiple choice questions. As mentioned before, teachers used many different auxiliary books that mainly included multiple choice questions. While solving these multiple-choice questions in the classroom, teachers attempted to give clues to their students about how to find the answer of each question by trial and error of the alternatives. In addition, the teachers aimed to teach shortcut methods to their students for gaining speed in the high stakes examinations.

In this study, teachers' considerations in choosing and using rational number examples were examined in greater depth. This exploration is summarized in Table 7.2 below. This table might be used in the development of a possible framework that might be used to examine middle school mathematics teachers' principles or considerations in selecting or generating rational number examples in their classrooms. Future studies in different education systems might enhance this table and provide empirical support to the development of a possible conceptual framework for analyzing teachers' considerations in choosing and using rational number examples.

Table 7.2. The summary of teachers' considerations in selecting or generating rational number examples

Starting with a Simple or Familiar Case • Considering form of rational numbers • Considering denominators of rational numbers • Considering number of repeating and non-repeating digits of a decimal • Considering number of terms/elements/steps when ordering rational numbers, performing a single operation or multi-step operations with rational numbers • Considering increasing complexity of multi-step operations, and of rational number problems by changing their mathematical structure • Recalling prior knowledge on rational number concepts **Including Uncommon Cases** *Exceptional or special cases in the teaching of rational number concepts* • Multiplying any rational number by 0 yields 0 • Multiplying any rational number by 1 yields the rational number itself • Dividing any rational number by 0 is undefined • Dividing any rational number by 1 yields the rational number itself • Dividing 0 by any rational number excluding 0 yields 0 • Dividing 1 by any rational number excluding 0 yields the multiplicative inverse of that rational number • Dividing -1 by any rational number excluding 0 yields the additive inverse of the multiplicative inverse of that rational number • Raising any nonzero rational number to the power of 0 yields 1 • Raising 1 to any rational number power yields 1 *Under-represented or ignored cases in the teaching of rational number concepts* • Emphasizing rationality of 0 • Including 0 into the sequence of rational numbers when ordering • Adding/subtracting/multiplying/dividing more than two rational numbers • Incorporating equivalent pairs into comparison of rational numbers • Incorporating into the classroom ordering examples that entail the use of residual thinking • Estimating the addition/subtraction/multiplication/division of rational numbers	**Drawing Attention to Relevant Features** • Locating a positive rational number first, its additive inverse second and then comparing the two locations • Arranging positive rational numbers first, their additive inverses second and then comparing the two arrangements • Performing operations with rational numbers by keeping the magnitude of terms constant and varying one sign at a time • Performing exponentiation without writing the power inside the parenthesis first, by writing the power outside the parenthesis second and then comparing the two results • Working out a complex fraction example first, rearranging the same complex fraction by changing the location of the main fraction bar and working out the new complex fraction second, and then comparing the two results • Breaking the pattern when teaching the procedure for converting repeating decimals into rational numbers • Performing a multi-step operation with parenthesis first, omitting the parenthesis of the same multi-step operation and performing second, and then comparing the two results • Solving a rational number problem first, solving another version of the same rational number problem second and then comparing the two rational number problems **Taking Account of Examinations** • Highlighting examples that have the potential to appear in written examinations • Highlighting examples that have the potential to appear in practice examinations of private teaching institutions • Highlighting examples that have the potential to appear in high stakes examinations • Explaining the method of scoring for potential written examination questions • Incorporating the solution of high-stakes examination examples into the classroom • Expressing the answer of multiple choice questions in their simplest forms in order to find it in the alternatives • Finding the answer of multiple choice complex fraction tasks by trial and error of the alternatives • Teaching shortcut methods for gaining speed in the high stakes examinations

Table 7.2. (Continued)

Keeping Unnecessary Work to Minimum	**Attending to Students' Difficulties, Errors or Misconceptions**
Reducing technical work by focusing on the essence • Drawing only the relevant part of number line when locating rational number on it • The choice of rational numbers to illustrate repeating decimals • The choice of relevant strategy when ordering rational numbers • Using LCM method instead of multiplying denominators when finding the common denominator of rational numbers • Not trying to enlarge rational numbers by 1 • Using shortcuts for adding and subtracting a whole number and a rational number • Adding or subtracting whole parts and fractional parts separately when adding or subtracting mixed fractions • Using subtraction formula instead of equating denominators during the subtraction of rational numbers • The choice of same denominator rational numbers when illustrating associative property of addition • Simplifying rational numbers in the course of performing operations • Using backwards strategy instead of equating denominators when dealing with complex fractions with unknown values • Rearranging algebraic expressions for an easier computation *Highlighting relevant parts of examples and not going into extra details* • Emphasizing important parts of an example and not finishing up all the calculations • Not seeing it essential to perform simplifications in the course of teaching a concept • Not seeing it essential to perform conversions in the course of teaching a concept • Not seeing it essential to equate denominators when symbolically expressing the area model of multiplication of rational numbers *Using properties of operations to reduce workload* • Using commutative property of addition operation rather than adding • Using associative property of addition operation rather than adding • Using distributive property of multiplication over addition rather than performing the operation • Using the fact that $1/(a/b)=b/a$ without actually making computations • Using the fact that $(a/b)/(a/b)=1$ without actually making computations • Using the fact that $(a/b)+(-a/b)=0$ without actually making computations • Enlarging decimal numerators and decimal denominators by multiples of 10 instead of converting into rational numbers when performing multi-step operations	*Attending to student difficulties* • Understanding the location of a minus sign in a rational number • Subtraction operation with rational numbers • Complex fractions with unknown values • Ordering rational numbers with the same numerators • Dealing with division of a number by zero and division of zero by a number • Simplification of rational numbers before multiplication • Performing operations including negative rational numbers without parenthesis • Understanding that distributive property yields a valid result • Distinguishing between exponents with a power inside the parenthesis and out outside the parenthesis *Attending to student errors* • Using commas instead of greater-than and less-than signs when ordering rational numbers • Ignorance of using parenthesis when operating with negative rational numbers • Making sign errors when adding mixed numbers • Making notation errors about mixed numbers • Making errors when multiplying a rational number and whole number • Making errors when finding additive inverse of a rational number • Making errors due to not following order of operations and finally • Making notation errors when performing the exponentiation of unknown variables *Attending to student misconceptions* • Counting tick-marks rather than counting equal parts of the line segment when locating a rational number on a number line • Over-generalizing location of positive rational numbers to negative rational numbers • Ordering decimals by treating the digits after the decimal points as separate numbers • Over-generalizing multiplication and division of rational number algorithms to addition and subtraction of rational numbers • Under-generalizing simplification of rational number multiplication • Misapplication of multiplication to mixed numbers • Exponentiation by adding base and power • Exponentiation by multiplying base and power • Believing that a larger number must always be divided by a smaller number

7.3. Teachers' Mathematically Incorrect or Pedagogically Inappropriate Rational Number Examples

Another focus of this study was to identify mathematically incorrect or pedagogically inappropriate rational number examples used by the middle school mathematics teachers. The findings revealed that teachers used three poor choices of rational number examples. These were mathematically incorrect examples, examples with improper language or terminology, and examples that are to be avoided in the teaching of rational number concepts.

One type of mathematical incorrectness was related with one participant teacher's consistent generation of non-existing number line examples. That is, in most of her number lines, she did not partition them into equal intervals. Although the teacher generated number lines were incorrect from a mathematical standpoint, it was unlikely that the teacher lacked the subject matter knowledge about location of rational numbers on a number line. It seems that the teacher did not give sufficient importance to accurately generating number lines. Meanwhile, she might not have been aware that her inaccurate number lines might mislead students' concept formation about number lines. This finding is in parallel with the findings of Zaslavsky and Zodik (2007). They also reported that teachers generated examples that included specific visual entailments or examples that did not actually exist. Similarly, Zodik and Zaslavsky (2008) pointed out that secondary school mathematics teachers generated non-existing examples and considered them as one type of mathematical incorrectness. More importantly, there is the possibility that the participating teacher in this study might have generated such non-existing number lines deliberately. For instance, when the teacher asked the students to locate $-2\frac{4}{5}$, she drew the interval between -2 and -3 longer than the intervals between other consecutive integers. She might have acted in this way to have students locate the given rational number more readily. However, she appeared to be unaware of the possible mismatch between her intentions and what their students would actually attend to.

Another mathematical incorrectness had to do with teachers' incorrect explanations about irrational numbers. More specifically, teachers articulated the following incorrect explanations about irrational numbers: irrational numbers cannot be located on a number line, irrational number set is a superset of rational number set, and irrational number set includes less number of elements than rational number set. Teachers' erroneous knowledge about irrational numbers was also reported by other studies (e.g., Fischbein, Jehiam & Cohen, 1995; Güven, Çekmez & Karataş, 2011; Sirotic & Zaskis, 2007). For instance, Sirotic and Zaskis (2007) asked pre-service secondary school students to find the exact location of $\sqrt{5}$ on a number line and some of the participants did not believe that it was possible to find the exact location of $\sqrt{5}$. Sirotic and Zaskis (2007) inferred that "one may find this difficult to believe if one has never seen an irrational point located on the number line, especially considering the fact that the number line is everywhere dense with rational numbers" (p. 478). This might also be true for the middle school mathematics teachers that participated in this study. That is, middle school mathematics teachers in this study might be lacking of the subject matter knowledge necessary for understanding irrational numbers. Indeed, it is reported that many in-service teachers could not even distinguish between rational numbers and irrational numbers (Arcavi, Bruckheimer & Ben-Zvi, 1987). Therefore, in order to promote pre-service and in-service teachers' understanding of irrational numbers, some modifications to the content courses that cover irrational numbers seems indispensable (Güven et al., 2011).

Some of the examples generated by the teachers were correct when evaluated from a mathematical standpoint. However, they were not appropriate from a pedagogical standpoint since they included the use of inappropriate language or terminology. Examples of this type occurred due to careless use of the word fraction when rational number is intended, the use of informal language such as opposite, flip and upside down for teaching additive or multiplicative inverses of rational numbers, ill-advised reading of fractions, and finally the incorrect use of mathematical symbols in the course of working out rational number examples.

Lamon (2012) claimed that many people carelessly use the word fraction when they intend to mean rational number. She further claimed that the use of such inappropriate terminology may lead to extra difficulties in communicating about the complex topics of fractions and rational numbers. Thus, it is believed that middle school mathematics teachers need to use mathematical terminology more carefully in order to avoid confusion or miscommunication among students and teachers. Teachers in this study also used inappropriate language when reading fractions. However, the language used for labelling fractions might impede students' understanding (Clarke & Roche, 2009). For instance, participant teachers often read fractions like $\frac{2}{5}$ as 'two out of five.' Reading fractions in this way may not help students notice the relative size in fractions. On the other hand, students more likely to grasp relative size and arrive at correct solutions when they read fractions like $\frac{2}{5}$ as 'two-fifths'. Van de Walle et al. (2013) also argued that fractions should be read in a way that supports students' understanding and further stated that reading $\frac{2}{10}$ as 'two-tenth' rather than 'two out of ten' would provide students with the opportunity to see the connections between decimals and fractions. It seems that the middle school teachers in this study were not aware of the danger that may occur as a result of using improper language when reading or saying fractions.

Another salient informal language use occurred when teachers used words like 'opposite', 'flip' or 'upside down' when teaching additive or multiplicative inverses of rational numbers. The ambiguity that is intrinsic to spoken language has the potential to interrupt learners' mathematical understanding (Matz, 1980). Thus, teachers' use of the informal term 'opposite' instead of additive inverse or 'flip' instead of multiplicative inverse has the potential to impede students' conceptual understanding of rational number concepts (Cangelosi et al., 2013). This implies that language and notation may play a crucial role in fostering learners' conceptual understanding. Thus, although middle school mathematics teachers seemed reckless about language and terminology when teaching additive and multiplicative inverses, this might be a potential obstacle for students' further mathematical development.

Moreover, not using proper terminology is likely to cause difficulty for students in understanding the nature of additive and multiplicative inverses (Cangelosi et al., 2013).

Finally, exploration of teachers' choice and use of examples brought to light several examples which would be better avoided in the teaching of rational number concepts. These examples included particular pitfalls that might be an obstacle for students to understand the mathematical concept or procedure that they confronted for the first time. In more detail, middle school mathematics teachers used two different types of to be avoided examples. These were, examples that obscured the role of variables and examples intended to illustrate a particular procedure, for which another procedure would be more sensible.

In this study, the rational number examples selected by the middle school mathematics teachers obscured the role of the variables in the following cases: selecting same values for the repeating and non-repeating digits when illustrating repeating decimal concept, selecting same values for the subtrahend and difference when teaching the modelling of subtraction of rational numbers, and obscuring the role of interval number and rational number magnitude when locating on a number line. These findings were similar with the previous studies (e.g., Rowland et al., 2003; Rowland et al., 2009; Rowland, 2008). Rowland et al. (2003) reported that pre-service teachers' following choice of examples obscured the role of variables: to start teaching half past with half past six with analogue clocks, to start teaching co-ordinates of points by (1,1), and to start teaching adding by 9+9. Similarly, Rowland (2008) observed that a pre-service teacher's first example for teaching subtraction was 4-2=2 and this example obscured the role of variables since the pre-service teacher selected same values for subtrahend and difference. As these results suggests, some of the examples selected by the middle school mathematics teachers were somewhat similar to the ones selected by the pre-service teachers. Rowland (2008) concluded that "novice teachers need specific guidance and help in appreciating the different roles of examples in mathematics teaching, and the existence of some common pitfalls in the selection of examples" (p. 161). In a similar fashion, middle school mathematics teachers may also need some guidance for judicious selection of examples.

Another type of to be avoided examples occurred in cases where teachers provided examples to illustrate a particular procedure, but the examples called for another procedure that is more sensible. Middle school mathematics teachers used these types of to be avoided examples in the following cases: not using relevant examples when illustrating the procedure for converting repeating decimals, not using relevant examples when teaching particular strategies for comparing and ordering rational numbers, and not using relevant examples when teaching a particular written algorithm for adding rational numbers. These findings also concurred with the findings of previous studies (e.g., Rowland et al., 2003; Rowland et al., 2009; Rowland, 2008). Rowland et al. (2003) reported that pre-service teachers' used the following examples to teach particular procedures but the examples called for other more sensible procedures: selecting 11-10 for teaching counting on strategy and selecting $49\times 4, 49\times 8$ and 19×4 for teaching column multiplication. It can be concluded that performing these computations by taking account of teachers' intended strategies disregards the idea of selecting sensible strategies.

7.4. Implications

Based on the findings of the current study and with respect to the current related literature, this section presented possible implications for pre-service and in-service teachers, mathematics education researchers, mathematics teacher educators, textbook authors, and curriculum developers.

Studies have shown that specific choice and use of mathematical examples may promote or hinder learners' understanding. Thus, it confronts mathematics teachers with a challenge, and provoking numerous considerations to be weighed. Nevertheless, mathematics teacher education programs in Turkey do not overtly speak to this issue and do not provide systematical training for pre-service teachers to enable them select or generate thoughtful mathematical examples for their students. For this reason, courses that help pre-service teachers gain not only theoretical but also practical knowledge about examples should be designed. More precisely, these courses should help pre-service teachers know what a mathematical example is, notice the role and power of examples in teaching mathematics, and develop skills in constructing good mathematical examples not only for the teaching of rational number concepts but also for other concepts in school mathematics. By this way, pre-service teachers may develop awareness about good and poor choice of examples. Besides, other pedagogical content knowledge courses in mathematics education program should give more weight to well-thought example selection or generation. In these courses, pre-service teachers might be provided the opportunity to watch video recordings of experienced and novice teachers' teaching episodes and to contemplate on experienced and novice teachers' good and poor choices of examples. Rowland (2008) suggested that pre-service teachers notice and learn more efficiently from poor examples when compared to good ones. He further added that pre-service teachers notice and learn from poor examples more efficiently since good examples are so subtle that they are not visible to the novice observers. Based on this suggestion, exemplification courses might be designed in a way that give more chance to pre-service teachers' exploration of and reflection on poor examples as well as constructing effective ones.

Similarly, in-service teachers' awareness of choosing examples can be enhanced by activities organized by teacher training programs. This would especially valuable for novice in-service teachers since such teacher training activities would make it possible to convey experienced teachers' craft knowledge regarding treatment of examples to the novice ones. This study in particular focused on identifying considerations employed by teachers in selecting rational number examples. Thus, it is expected that these considerations might help teachers improve their own teaching practices. In addition, these considerations might be adapted to the teaching of other mathematical topics.

The findings of this study might also contribute to mathematics education researchers who are interested in the area of exemplification and especially in teachers' treatment of examples. More importantly, this study attempted to fill the void in mathematics education literature by exploring teachers' treatment of rational number examples which have not been addressed before. Eventually, two different sets of summaries explaining overall characteristics of rational number examples and teacher considerations in selecting these rational number examples were developed. It is expected that these summaries might help mathematics education researchers design their own research studies and subsequently analyze their own research findings.

This study might also help mathematics teacher educators to increase the quality of pedagogical content knowledge courses in mathematics education. Turkish pre-service teachers attend practice teaching courses when they become seniors. Practice teaching courses provide pre-service teachers the opportunity to participate actively in educational activities in a selected cooperating school. Thus, the summaries developed in this study would be particularly useful for mathematics teacher educators in evaluating pre-service teachers' teaching of rational number concepts in particular and other mathematical concepts in general.

Textbook examples play an important role in teaching practices of mathematics teachers. Thus, it is important to include well-constructed examples in mathematics textbooks to help students develop more sophisticated understanding of mathematics. At this point, textbook authors should take an active role in preparing textbooks that include carefully selected examples. However, as indicated by Watson and Mason (2006) textbooks generally offer examples with random variation. Thus, textbook authors are expected to be more aware about the pedagogical role of examples. For instance, by considering structured variation principle, they may construct examples in which the dimensions of variation are carefully controlled. That is, they may construct sequences of examples by selecting one variable to be held constant and change other variables systematically when moving from one example to another. Hence, they may help students pay attention to relevant features of examples and help to reduce the noise carried by specific examples.

Finally, this study is expected to inform curriculum developers about the potential role of examples in the teaching of mathematics. Consequently, this study is expected to help them revise and create examples that are included school mathematics curriculum in accordance with pedagogical principles such as pattern breaking and structured variation, and with teacher considerations such as including uncommon cases or drawing attention to relevant features in order to better expose mathematical structure of examples to the students.

7.5. Recommendations for Future Research

The current study explored middle school mathematics teachers' treatment of rational number examples in their seventh grade classrooms. More specifically, overall characteristics of rational number examples used by teachers, the considerations employed by them in choosing or generating rational number examples, and the mathematically incorrect or pedagogically inappropriate examples used by them in teaching of rational number concepts were examined in-depth. In the view of findings, some recommendations are offered for future research studies in the following paragraphs.

This study was carried out with middle school mathematics teachers whose rational number teaching experience varied between 2 to 14 years. A further research with middle school mathematics teachers who are in their early years of teaching might be conducted to see how their treatment of examples evolves as they teach. Besides, pre-service mathematics teachers' treatment of examples might be explored to compare and contrast their selection of examples with in-service mathematics teachers. Even, mathematics teacher educators' treatment of examples might be explored to see how well they select examples during the teaching of pre-service teachers. This is very important since only "well prepared mathematics teacher educators are available to furnish opportunities for teachers to develop in ways that will enable them to enhance the recommended changes" (Zaslavsky & Leikin, 2004, p. 5).

In this study, all of the participating middle school teachers taught seventh grade students. A further research might be conducted with teachers who teach either elementary school students (between 1st grade and 4th grade), middle school students (between 5th grade and 8th grade) and secondary school students (between 9th grade and 12th grade) to see how teachers' treatment of examples change with respect to students' grade levels in certain content areas in mathematics.

This study focused on middle school mathematics teachers' rational number examples as the unit of analysis. Teachers' treatment of examples related with other topics of school mathematics might be explored to see how their example choices and considerations for their choices of examples differ with respect to the nature of mathematical concept.

In order to investigate middle school mathematics teachers' treatment of rational number examples in their classrooms, qualitative case study was employed. Further quantitative research studies might be used to examine pre-service and in-service mathematics teachers' treatment of not only rational number examples but also their examples related with other mathematical concepts. In particular, studies might be conducted to examine whether there 'good' examples and 'poor' examples result in significant differences in students' achievement. By means of these quantitative research studies, researchers might have the chance to generalize their findings to a broader context possessing similar characteristics.

Finally, the results of this study were limited to the data that were gathered from four public middle schools located in the Aksaray city centre. A further research may be conducted to investigate private school teachers' treatment of examples in their classrooms. This might give some clues to the researchers about the possible influences of different schools on the quality and quantity of examples being selected by the teachers.

7.6. Limitations of the Study

The limitations that should be considered while interpreting the findings of this study are explained below.

The number of middle school mathematics teachers I observed was limited to four teachers in this study. Moreover, the results of this data were limited to the data gathered from public middle schools that were at the center of Aksaray and private school mathematics classrooms were not observed in this study. Therefore, findings should be evaluated by considering the specified classrooms and school contexts.

In addition, I observed each classroom as a complete observer. My existence in the classrooms might have influenced teachers' and students' actions or behaviors.

For instance, teachers might have made greater effort to teach rational number concepts due to my existence. To reduce this influence, I started conducting pilot observations and interviews with the teachers eight weeks before the actual data collection process and I videotaped teachers' classroom practices during this time period. Besides, I continued conducting post observations and interviews with the teachers after the actual data collection process until the end of the fall semester. By this way, I wanted to make sure that teachers did not attempt to change their classroom practices after the end of the actual data collection process.

The data of this study were limited to the lesson observations and to the questions included in the observation form. Before the implementation of the study, I was planning to conduct both pre and post lesson interviews with the teachers to see the examples appearing in their lesson plans. However, none of the teachers prepared lesson plans in advance. Therefore, it was not possible for me to conduct pre-lesson interviews. Thus, interview data were limited to the post lesson interviews and to the questions included in the interview protocol. Finally, the data obtained from student textbook were limited to the worked-out examples and exercise examples that were included in the explanatory part of the textbook for introducing rational number concepts.

According to Watson and Mason's (2005) example definition, representations can be also be regarded as mathematical examples. However, representations used by the teachers were not examined in this study. Thus, examples used by the teachers were limited to worked-out examples and exercise examples of teachers and the student textbook.

The rational number teaching experience of participant teachers, ranged between 2 and 14 years. Therefore, teachers with more than 14 years of rational number teaching experience or teachers that have just started teaching rational numbers were not observed in this study. Besides, the teachers that participated in this study taught only to 7^{th} grade students. Therefore, teachers' treatment of mathematical examples was limited to 7^{th} grade level. Likewise, the participants of this study were middle school mathematics teachers and primary school teachers or secondary school mathematics teachers were not observed in this study.

7.7. Implications for my future career

As a mathematics education researcher, this study had crucial impact on my own practice. At the beginning of this study, I could only speak in a general way about the importance of examples. But now, I am able to give a more analytical account of teachers' treatment of examples in the teaching and learning of mathematics.

Conducting research on this topic provided me with ideas about my future teaching. When I become faculty member, the first thing that I will do will be to observe pre-service middle school mathematics teachers' teaching practices in their school practice courses. More specifically, I will have pre-service teachers prepare lessons plans and will have them include their best examples in these lesson plans for teaching specific mathematical concepts in the selected cooperating middle schools. Besides, I will examine how they act in the moment when they come up with contingent classroom situations and the examples they select or use to handle these situations. By this way, I will monitor their improvement in selecting or using powerful instructional examples when teaching mathematical concepts to their students during a semester period. Meanwhile, I will discuss the potential pitfalls included in the examples of pre-service middle school mathematics teachers when they join teacher training courses related with school practice. At the end of the school practice courses, I hope the pre-service middle school mathematics teachers will become more aware of the role of careful selection or use of examples in the teaching of mathematics.

Apart from examining pre-service teachers' choice and use of mathematical examples in their school practice courses, I will give more weight to their treatment of examples during courses about teaching methods. I think this will play an important role in increasing the quality of pedagogical content knowledge courses in mathematics education.

I am also planning to carry out projects about in-service middle school mathematics teachers' treatment of examples in their own classrooms. More specifically, I am planning to devise professional development activities and have in-service middle school mathematics teachers participate in these activities and discuss the examples used by them in teaching particular mathematical topics. By this way, I anticipate that the teachers will be provided the opportunity to develop their experiences about exemplification in the teaching of mathematics.

Ultimately, after I gain sufficient experience about pre-service and in-service mathematics teachers' treatment of examples in their actual classroom practices, I am planning to write mathematics textbooks for middle school students. These textbooks will be prepared in accordance with the pedagogical principals existing in the exemplification literatute and thus they will better expose the mathematical structure inherent in the examples to the students.

REFERENCES

Abdul-Rahman, S. (2005). Learning with examples and students' understanding of integration. In A. Rogerson (Ed.), *Proceedings of the eighth international conference of mathematics education into the 21st century project on "Reform, Revolution and Paradigm Shifts in Mathematics Education"* Johor Bahru, Malaysia.

Ahern, K. J. (1999). Ten tips for reflexive bracketing. *Qualitative Health Research, 9,* 407-411.

Ainsworth, S., Bibby, P., & Wood, D. (2002). Examining the effects of different multiple representational systems in learning primary mathematics. *Journal of the Learning Sciences, 11*(1), 25-61.

Alacacı, C. (2009). Öğrencilerin kesirler konusundaki kavram yanılgıları. E. Bingölbali ve M. F. Özmantar (Eds.), *İlköğretimde karşılaşılan matematiksel zorluklar ve çözüm önerileri* (s. 63-95). Ankara: Pegema Yayıncılık.

Alcock, L. & Inglis, M. (2008). Doctoral students' use of examples in evaluating and proving conjectures. *Educational Studies in Mathematics, 69,* 111-129.

Alcock, L. & Weber, K. (2010). Undergraduates' example use in proof construction: purposes and effectiveness. *Investigations in Mathematics Learning, 3*(1), 1-22.

An, S., Kulm, G., & Wu, Z. (2004). The pedagogical content knowledge of middle school mathematics teachers in China and the U.S. *Journal of Mathematics Teacher Education, 7,* 145-172.

Antonini, S. (2011). Generating examples: focus on processes. *ZDM The International Journal on Mathematics Education, 43,* 205-217.

Antonini, S., Presmeg, N., Mariotti, M. A., & Zaslavsky, O. (2011). On examples in mathematical thinking and learning. *ZDM The International Journal on Mathematics Education, 43,* 191-194.

Arcavi, A., Bruckheimer, M., & Ben-Zvi., R. (1987). History of mathematics for teachers: The case of irrational numbers. *For the Learning of Mathematics, 7*(2), 18-23.

Armstrong, B. & Bezuk, N. (1995). Multiplication and division of fractions: The search for meaning. In J. T. Sowder & B. P. Schappelle (Eds.), *Providing a foundation for teaching mathematics in the middle grades* (pp. 85–119). Albany: State University of New York Press.

Arzarello, F., Ascari, M., & Sabena, C. (2011). A model for developing students' example spaces: the key role of the teacher. *ZDM The International Journal on Mathematics Education, 43,* 295-306.

Askew, M. & Wiliam, D. (1995). *Recent research in mathematics education 5-16*, London: Her Majesty's Stationery Office.

Atkinson, R. K., Derry, S. J., Renkl, A., & Wortham, D. (2000). Learning from examples: Instructional principles from the worked examples research. Review of Educational Research, *70*(2), 181-214.

Aydın, N. ve Beşer, Ş. (2013a). *İlköğretim matematik 7 ders kitabı.* Ankara: Aydın Yayıncılık ve Eğitim Hizmetleri Ltd. Şti.

Aydın, N. ve Beşer, Ş. (2013b). *İlköğretim matematik 7 çalışma kitabı.* Ankara: Aydın Yayıncılık ve Eğitim Hizmetleri Ltd. Şti.

Ball, D. L. (1990a). Prospective elementary and secondary teachers' understanding of division. *Journal for Research in Mathematics Education, 21*(2), 132-144.

Ball, D. L. (1990b). The mathematical understandings that prospective teachers bring to teacher education. *Elementary School Journal, 90*(4), 449-466.

Ball, D. L., Hill, H. C., & Bass, H. (2005). Knowing mathematics for teaching: Who knows mathematics well enough to teach third grade, and how can we decide? *American Educator, 29*(3), 14-17, 20-22, 43-46.

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education, 59*(5), 389-407.

Başkan, T., Bizim, O., & Cangül, İ. N. (2006). *Metrik uzaylar ve topolojiye giriş [Metric spaces and introduction to topology]*. Ankara: Nobel Press.

Behr, M. J., Lesh, R., Post, T. R., & Silver E. A. (1983). Rational number concepts. In R. Lesh & M. Landau (Eds.), *Acquisition of mathematics concepts and processes* (pp. 91-126). New York: Academic Press.

Behr, M., Wachsmuth, I., Post T., & Lesh R. (1984). Order and equivalence of rational Numbers: A clinical teaching experiment. *Journal for Research in Mathematics Education, 15*(5), 323-341.

Bills, C. & Bills, L. (2005). Experienced and novice teachers' choice of examples. In P. Clarkson, A. Downton, D. Gronn, M. Horne, A. McDonough, R. Pierce, & A. Roche (Eds.), *Proceedings of the 28th annual conference of the Mathematics Education Research Group of Australasia* (Vol. 1, pp. 146–153). MERGA Inc., Melbourne.

Bills, L. & Rowland, T. (1999). Examples, generalisation and proof. In L. Brown (Ed.), *Making meaning in mathematics: Visions of mathematics 2.* Advances in mathematics education (Vol. 1, pp. 103-116). York, UK: QED Press.

Bills, L. & Watson, A. (2008). Editorial introduction. Special issue: The role and use of examples in mathematics education. *Educational Studies in Mathematics, 69,* 77-79.

Bills, L., Mason, J., Watson, A., & Zaslavsky, O. (2006). Research Forum 02. Exemplification: The use of examples in teaching and learning mathematics. In J. Novotna, H. Moraova, M. Kratka, & N. Stehlikova (Eds.), *Proceedings of the 30th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 125–154). Prague: Charles University.

Birgin, O. & Gürbüz, R. (2009). İlköğretim ikinci kademe öğrencilerinin rasyonel sayılar konusundaki işlemsel ve kavramsal bilgi düzeylerinin incelenmesi. *Uludağ Üniversitesi Eğitim Fakültesi Dergisi, 22*(2), 529-550.

Bogdan, R. C. & Biklen, S. K. (2007). *Qualitative research methods for education: An introduction to theory and methods* (5th ed.). Needham Heights, MA: Allyn and Bacon.

Bogomolny, M. (2006). *The role of example-generation tasks in students' understanding of linear algebra*. Unpublished doctoral dissertation, Simon Fraser University, Burnaby, Canada.

Bratina, T. A. (1986). Can your students give examples? *Mathematics Teacher, 79,* 524-526.

Breuer, J. (2006). *Introduction to the theory of sets.* New York: Dover Publications.

Bright, G., Behr, M., Post, T., & Wachsmuth, I. (1988). Identifying fractions on number lines. *Journal for Research in Mathematics Education, 19*(3), 215-232.

Buchbinder, O. & Zaslavsky, O. (2011). Is this a coincidence? The role of examples in fostering a need for proof. *ZDM The International Journal on Mathematics Education, 43,* 269-281.

Calderhead, J. (1981). Stimulated recall: A method for research on teaching. *British Journal of Educational Psychology, 51*(2), 211-217.

Cangelosi, R., Madrid, S., Cooper, S., Olson, J., & Hartter, B. (2013). The negative sign and exponential expressions: Unveiling students' persistent errors and misconceptions. *The Journal of Mathematical Behavior, 32,* 69-82.

Cankoy, O. (2014) Interlocked problem posing and children's problem posing performance in free structured situations *International Journal of Science and Mathematics Education, 12*, 219-238.

Çelik, B., Çelik, N., Bizim, O., & Öztürk, M. (2013). *Temel matematik [Basic mathematics]* (6th ed.). Bursa: Dora Press.

Chang, N. (2007). Responsibilities of a teacher in a harmonic cycle of problem solving and problem posing. *Early Childhood Education Journal, 34*(4), 265-271.

Charles, R. I. (1980). Exemplification and characterization moves in the classroom teaching of geometry concepts. *Journal for Research in Mathematics Education, 11*(1), 10-21.

Clark, C. M. & Peterson, P. L. (1986). Teachers' thought processes. In M.C. Wittrock (Ed.), *Handbook of research on teaching* (3rd ed., pp. 255-296). New York: Macmillan.

Clarke, D. M. & Roche, A. (2009). Students' fraction comparison strategies as a window into robust understanding and possible pointers for instruction. *Educational Studies in Mathematics, 72,* 127-138.

Clarke, D. M., Roche, A., & Mitchell, A. (2008). Ten practical tips for making fractions come alive and make sense. *Mathematics Teaching in the Middle School, 13*(7), 373-380.

Clements, D., Swaminathan, S., Hannibal, M., & Sarama, J. (1999). Young children's concepts of shape. *Journal for Research in Mathematics Education, 30*(2), 192-212.

Cohen, M. & Carpenter, J. (1980). The effects of non-examples in geometrical concept acquisition. *International Journal of Mathematical Education in Science and Technology, 11*(2), 259–263.

Common Core State Standards Initiative (2010). *Common core state standards for mathematics.* Washington: National Governors Association Center for Best Practices and the Council of Chief State School Officers.

Cramer, K. (2003). Using a translation model for curriculum development and classroom instruction. In R. Lesh & H. M. Doerr (Eds.), *Beyond constructivism: Models and modeling perspectives on mathematics teaching, learning, and problem solving* (pp. 449-463). Mahwah, NJ: Lawrence Erlbaum Associates, Inc.

Cramer, K., & Lesh, R. (1988). Rational number knowledge of pre-service elementary education teachers. In M. Behr (Ed.), *Proceedings of the Tenth Annual Meeting of the North American Chapter of the International Group For the Psychology of Mathematics Education* (pp. 425-431). Dekalb, Illinois: PME.

Creswell, J. W. (2007). *Qualitative inquiry and research design: Choosing among five traditions* (2nd ed.). Thousand Oaks, CA: Sage Publications.

Creswell, J. W. (2009). *Research design: Qualitative, quantitative, and mixed methods approaches* (3rd ed.). Thousand Oaks, CA: Sage Publications.

Creswell, J. W. (2012). *Educational research: Planning, conducting, and evaluating quantitative and qualitative research* (4th ed.). Boston: Pearson.

Creswell, J. W., & Miller, D. L. (2000). Determining validity in qualitative inquiry. *Theory into Practice, 39*(3), 124-130.

Dahlberg, R. P. & Housman, D. L. (1997). Facilitating learning events through example generation. *Educational Studies in Mathematics, 33*(3), 283-299.

Denzin, N. K. & Lincoln, Y. S. (2005). *The SAGE handbook of qualitative research* (3rd ed.). Thousand Oaks, CA: Sage Publications.

Dillon, J. T. (1982). Problem finding and solving. *Journal of Creative Behaviour*, *16,* 97-111.

Edwards, A. & Alcock, L. (2010). How do undergraduate students navigate their example spaces? *Proceedings of the 13th Annual Conference on Research in Undergraduate Mathematics Education.* Raleigh, North California, USA.

Edwards, A. W. (2011). *Using example generation to explore undergraduates' conceptions of real sequences: A phenomenographic study*. Unpublished doctoral dissertation, Loughborough University, Leicestershire, UK.

Eisenhart, M., Borko, H., Underhill, R., Brown, C., Jones, D., & Agard, P. (1993). Conceptual knowledge falls through the cracks: Complexities of learning to teach mathematics for understanding. *Journal for Research in Mathematics Education, 24*(1), 8-40.

Feynman, R. (1985). *"Surely you're joking, Mr. Feynman!": Adventures of a curious character.* New York: Norton.

Fischbein, E. (1993). The interaction between the formal, the algorithmic and the intuitive components in a mathematical activity. In R. Biehler, R. Scholz, R. Strässer, & B. Winkelmann (Eds.), *Didactics of mathematics as a scientific discipline* (pp. 231–245). Dordrecht, The Netherlands: Kluwer.

Fischbein, E., Jehiam, R., & Cohen, D. (1995). The concept of irrational numbers in high school students and prospective teachers. *Educational Studies in Mathematics, 29*(1), 29-44.

Fraenkel, J., Wallen, N., & Hyun, H. (2012*). How to design and evaluate research in education* (8th ed). New York: McGraw-Hill.

Fukawa-Connelly, T. P. & Newton, C. (2014). Analyzing the teaching of advanced mathematics courses via the enacted example space. *Educational Studies in Mathematics, 87*(3), 323-349.

Gall, M., Gall, J. & Borg, W. (2007). *Educational research: An introduction* (8th ed). Boston: Allyn and Bacon.

Golafshani, N. (2003). Understanding reliability and validity in qualitative research. The Qualitative Report, 8(4), 597-607. Retrieved June 30, 2014, from http://www.nova.edu/ssss/QR/QR8-4/golafshani.pdf

Goldenberg, P. & Mason, J. (2008). Shedding light on and with example spaces. *Educational Studies in Mathematics, 69,* 183-194.

Graeber, A. & Tirosh, D. (1988). Multiplication and division involving decimals: Preservice elementary teachers' performance and beliefs. *Journal of Mathematical Behavior, 7*, 263-280.

Graeber, A., Tirosh, D., & Glover, R. (1989). Preservice teachers' misconceptions in solving verbal problems in multiplication and division. *Journal for Research in Mathematics Education,* 20(1), 95-102.

Greer, B. (1987). Nonconservation of multiplication and division involving decimals. *Journal for Research in Mathematics Education, 18*(1), 37-45.

Guo, J. P., Pang, M. F., Yang, L. Y., & Ding, Y. (2012). Learning from comparing multiple examples: On the dilemma of similar or different. *Educational Psychology Review, 24,* 251-269.

Güven, B., Çekmez, E. & Karataş, İ. (2011) examining preservice elementary mathematics teachers' understandings about irrational numbers. *PRIMUS:*

Problems, Resources, and Issues in Mathematics Undergraduate Studies, 21(5), 401-416.

Halmos, P. R. (1983). *Selecta: Expository writing.* New York: Springer Verlag.

Haser, Ç. & Ubuz, B. (2003). Students' conception of fractions: A study of 5th grade students. *Hacettepe University Journal of Education, 24,* 64-69.

Haser, Ç., & Ubuz, B. (2002). Kesirlerde kavramsal ve işlemsel performans [The conceptual and procedural performance in fractions]. *Education and Science, 27*(126): 53-61.

Hershkowitz, R. (1989). Visualization in geometry-two sides of the coin. *Focus on Learning Problems in Mathematics, 11*(1), 61-76.

Hiebert, J., Gallimore, R., Garnier, H., Givvin, K. B., Hollingsworth, H., Jacobs, J., Chiu, A. M.-Y.,Wearne, D., Smith, M., Kersting, N., Manaster, A., Tseng, E., Etterbeek,W., Manaster, C., Gonzales, P., & Stigler, J. (2003). *Teaching mathematics in seven countries: Results from the TIMSS 1999 video study* (NCES 2003-013). U.S. Department of Education. Washington, DC: National Center for Education Statistics.

Hill, H., Ball, D. L., & Schilling, S. G. (2008). Unpacking pedagogical content knowledge: Conceptualizing and measuring teacher's topic-specific knowledge of students. *Journal for Research in Mathematics Education, 39,* 372-400.

Hodges, T. E., Cady, J., & Collins, R. L. (2008). Fraction representation: The not-so-common denominator among textbooks. *Mathematics Teaching in the Middle School, 14*(2), 78-84.

Iannone, P., Inglis, M., Mejia-Ramos, J. P., Simpson, A., & Weber, K. (2011). Does generating examples aid proof production? *Educational Studies in Mathematics, 77,* 1-14.

Işık, C. & Kar, T. (2012). İlköğretim matematik öğretmeni adaylarının kesirlerde bölmeye yönelik kurdukları problemlerde hata analizi [An error analysis in division problems in fractions posed by pre-service elementary mathematics teachers]. *Educational Sciences: Theory & Practice, 1*2(3), 2289-2309.

Izsak, A. (2008). Mathematical knowledge for teaching fraction multiplication. *Cognition and Instruction, 26,* 95-143.

Izsak, A., Orrill, C. H., Cohen, A. S., & Brown, R. E. (2010). Measuring middle grades teachers' understanding of rational numbers with the mixture rasch model. *The Elementary School Journal, 110*(3), 279-300.

Kamin, D. (2010). *The use of tasks and examples in a high school mathematics classroom: variance of purpose and deployment.* Unpublished master's thesis, Simon Fraser University, Burnaby, Canada.

Kennedy, M. M. (1987). Inexact sciences: Professional education and the development of expertise. In E. Z. Rothkopf (Ed.), *Review of research in education* (Vol. 13, pp. 133-167). Washington, DC: American Educational Re-search Association.

Kennedy, M. M. (1999). Ed schools and the problem of knowledge. In J. D. Raths & A. C. McAninch (Eds.), *Advances in teacher education Volume 5, What counts as knowledge in teacher education?* (pp. 29-45). Stamford CT: Ablex Publishing Corporation.

Kennedy, M. M. (2002). Knowledge and teaching. *Teachers and Teaching: Theory and Practice, 8*(3/4), 355-370.

Kilpatrick, J., Swafford, J., & Findell, B. (Eds.) (2001). *Adding it up: Helping children learn mathematics.* Washington, DC: National Academy Press.

Knott, L. (2010). Problem posing from the foundations of mathematics. *Montana Mathematics Enthusiast, 7*(2&3), 413-432.

Komatsu, K. (2010). Counter-examples for refinement of conjectures and proofs in primary school mathematics. *The Journal of Mathematical Behavior, 29,* 1-10.

Kurt, G. & Çakıroğlu, E. (2009). Middle grade students' performances in translating among representations of fractions: A Turkish perspective, *Learning and Individual Differences, 19,* 404-410.

Lakatos, I. (1976). *Proofs and refutations: the logic of mathematical discovery.* Cambridge: Cambridge University Press.

Lamon, S. J. (2007). Rational numbers and proportional reasoning. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (Vol. 1, pp. 629-667). Charlotte, NC: Information Age Publishing.

Lamon, S. J. (2012). *Teaching fractions and ratios for understanding: Essential content knowledge and instructional strategies for teachers* (3rd ed.). Routledge: New York.

Lang, S. (2005). *Undergraduate algebra* (3rd ed.). New York: Springer Science & Business Media.

Leinhardt, G. (1990). Capturing craft knowledge in teaching. *Educational Researcher, 19*(2), 18-25.

Leinhardt, G. (2001). Instructional explanations: A commonplace for teaching and location for contrast. In V. Richardson (Ed.), *Handbook of research on teaching* (4th edition, pp. 333-357). Washington DC, USA: American Educational Research Association.

Leinhardt, G., Zaslavsky, O., & Stein, M. K. (1990). Functions, graphs, and graphing: Tasks, learning, and teaching. *Review of Educational Research, 60*(1), 1-64.

Lesh, R., Behr, M., & Post, T. (1987). Rational number relations and proportions. In C. Janiver (Ed.), *Problems of representations in the teaching and learning of mathematics* (pp. 41-58). Hillsdale, NJ: Lawrence Erlbaum.

Lesh, R., Post, T., & Behr, M. (1987). Representations and translations among representations in mathematics learning and problem solving. In C. Janvier (Ed.), *Problems of representations in the teaching and learning of mathematics* (pp. 33–40). Hillsdale, New Jersey: Lawrence Erlbaum Associates.

Leung, I. K. C. & Lew, H. (2013). The ability of students and teachers to use counter-examples to justify mathematical propositions: A pilot study in South Korea and Hong Kong. *ZDM The International Journal on Mathematics Education, 45*(1), 91-105.

Lincoln, Y. S. & Guba, E. G. (1985). *Naturalistic inquiry*. Thousand Oaks, California: Sage Publications.

Lowrie, T. (2002). Young children posing problems: The influence of teacher intervention on the type of problems children pose. *Mathematics Education Research Journal, 14*(2), 87-98.

Ma, L. (1999). *Knowing and teaching elementary mathematics: Teachers' understanding of fundamental mathematics in China and the United States.* Mahwah, NJ: Lawrence Erlbaum Associates.

Mack, N. K. (2001). Building on informal knowledge through instruction in a complex content domain: Partitioning, units, and understanding multiplication of fractions. *Journal for Research in Mathematics Education, 32*(3), 267-296.

Marton, F. & Booth, S. (1997). *Learning and awareness.* Mahwah NJ: Lawrence Erlbaum.

Marton, F. & Tsui, A. B. M. (2004). *Classroom discourse and the space of learning*. Mahwah, NJ: Lawrence Erlbaum Associates.

Marton, F., & Trigwell, K. (2000). Varatio est mater studiorum [Variation is the mother of learning]. *Higher Education Research & Development, 19*(3), 381-395.

Marton, F., Runesson, U., & Tsui, A. (2004). The space of learning. In F. Marton & A. Tsui (Eds.), *Classroom discourse and the space of learning* (pp. 3-40). Mahwah, NJ: Lawrence Erlbaum.

Mason, J. & Klymchuk, S. (2009). *Counter examples in calculus*. London: Imperial College Press.

Mason, J. & Pimm, D. (1984). Generic examples: Seeing the general in the particular. *Educational Studies in Mathematics, 15,* 277-289.

Mason, J. & Spence, M. (1999). Beyond mere knowledge of mathematics: The importance of knowing-to act in the moment. *Educational Studies in Mathematics, 38*, 135-161.

Mason, J. & Watson, A. (2001). Getting students to create boundary examples. *MSOR Connections, 1*(1), 9-11.

Mason, J. & Watson, A. (2008). Mathematics as a constructive activity: Exploiting dimensions of possible variation. In M. P. Carlson & C. Rasmussen (Eds.), *Making the connection: Research and practice in undergraduate mathematics* (pp. 189-202). Washington: MAA.

Mason, J. (2006). What makes an example exemplary: Pedagogical and didactical issues in appreciating multiplicative structures. In R. Zazkis, & S. R. Campbell (Eds.), *Number theory in mathematics education: Perspectives and prospects* (pp. 41–68). Hillsdale, NJ: Erlbaum.

Mason, J. (2011). Phenomenology of example construction. *ZDM The International Journal on Mathematics Education, 43,* 195-204.

Mason, J., Stephens, M., & Watson, A. (2009). Appreciating mathematical structure for all. *Mathematics Education Research Journal, 21*(2), 10-32.

Matz, M. (1980). Towards a computational theory of algebraic competence. *Journal of Mathematical Behavior, 3*(1), 93-166.

Merriam, S. B. (1998). *Qualitative research and case study applications in education* (Rev. ed.). San Francisco, CA: Jossey-Bass.

Merriam, S. B. (2009). *Qualitative research: A guide to design and implementation*. San Francisco, CA: John Wiley & Sons.

Mestre, P. J. (2002). Probing adults' conceptual understanding and transfer of learning via problem posing. *Applied Developmental Psychology, 23,* 9-50.

Michener, E. R. (1978). Understanding understanding mathematics. *Cognitive Science, 2,* 361-383.

Michener, E. R. (1991). Example-based reasoning. In J. F. Voss, D. N. Parkins, & J. W. Segal (Eds.), *Informal reasoning in education* (pp. 187–208). Hillsdale, NJ: Lawrence Erlbaum.

Miles, M. B. & Huberman, A. M. (1994). *Qualitative data analysis: An expanded sourcebook* (2nd ed.). Newburg Park, CA: Sage.

Mills, M. (2014). A framework for example usage in proof presentations. *Journal of Mathematical Behavior,* 33, 106-118.

Ministry of National Education (2009a). *İlköğretim matematik dersi 1-5.sınıflar öğretim programı [Elementary school mathematics curriculum: Grades 1-5].* Ankara: Devlet Kitapları Müdürlüğü.

Ministry of National Education (2009b). *İlköğretim matematik dersi 6-8.sınıflar öğretim programı [Middle school mathematics curriculum: Grades 6-8].* Ankara: Devlet Kitapları Müdürlüğü.

Ministry of National Education (2011). *Lise matematik dersi 9-12.sınıflar öğretim programı [Secondary school mathematics curriculum: Grades 9-12].* Ankara: Devlet Kitapları Müdürlüğü.

Mohamed, M. & Sulaiman, F. (2010). Framework of knowledge and teaching skills of expert mathematics teachers in using mathematical examples. *Procedia Social and Behavioral Sciences, 8,* 325-331.

Morrison, S. (2013). Exploring links between foundation phase teachers' content knowledge and their example spaces. *South African Journal of Childhood Education,* 3(2), 96-111.

Moseley, B. (2005). Students' early mathematical representation knowledge: The effects of emphasizing single or multiple perspectives of the rational number domain in problem solving. *Educational Studies in Mathematics, 60*(1), 37-69.

Moskal, B. M. & Magone, M. E. (2000). Making sense of what students know: Examining the referents, relationships and modes students displayed in response to a decimal task. *Educational Studies in Mathematics, 43*(3), 313-335.

Moss, J. & Case, R. (1999). Developing children's understanding of the rational numbers: A new model and an experimental curriculum. *Journal for Research in Mathematics Education, 30*(2), 122-147.

Moss, J. (2005). Pipes, tubes, and beakers: New approaches to teaching the rational-number system. In M. S. Donovan, & J. D. Bransford (Eds.), *How students* *learn: Mathematics in the classroom* (pp. 121–162). Washington, DC: National Academic Press.

Mullis, I. V. S., Martin, M. O., & Foy, P. (2008). *TIMSS 2007 international mathematics report: Findings from IEA's Trends in International Mathematics and Science Study at the fourth and eighth grades.* Chestnut Hill, MA: IEA TIMSS & PIRLS International Study Center.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics.* Reston, VA: NCTM.

Newton, K. J. (2008). An extensive analysis of preservice elementary teachers' knowledge of fractions. *American Educational Research Journal, 45,* 1080-1110.

Ni, Y. J. (2001). Semantic domains of rational numbers and the acquisition of fraction equivalence. *Contemporary Educational Psychology, 26,* 400-417.

Ni, Y., & Zhou, Y-D. (2005). Teaching and learning fraction and rational numbers: The origins and implications of whole number bias. *Educational Psychologist, 40*(1), 27–52.

Niven, I. (1961). *Numbers: Rational and irrational.* New York: Random House and the L. W. Singer Company.

OECD (2010). *PISA 2009 results: What students know and can do - Student performance in reading, mathematics and science* (Volume I). PISA: OECD Publishing.

Osana, H. P. & Royea, D. A. (2011). Obstacles and challenges in preservice teachers' explorations with fractions: A view from a small-scale intervention study. *The Journal of Mathematical Behavior, 30*(4), 333-352.

Pang, M. F. & Marton, F. (2005). Learning theory as teaching resource: Enhancing students' understanding of economic concepts. *Instructional Science, 33*(2), 159-191.

Patton, M. Q. (2002). *Qualitative research and evaluation methods* (3rd ed.). Thousand Oaks, CA: Sage Publications.

Pedemonte, B. & Buchbinder, O. (2011). Examining the role of examples in proving process through a cognitive lens: The case of triangular numbers. *ZDM The International Journal on Mathematics Education, 43*, 257-267.

Peled, I. & Zaslavsky, O. (1997). Counter-examples that (only) prove and counter-examples that (also) explain. *Focus on Learning Problems in Mathematics, 19*(3), 49-61.

Petty, O. & Jansson, L. (1987). Sequencing examples and nonexamples to facilitate concept attainment. *Journal for Research in Mathematics Education, 18*(2), 112-125.

Polya, G. (1945). *How to solve it.* Princeton: Princeton University Press.

Post, T., & Cramer, K. (2002). Children's strategies in ordering rational numbers. In D. Chambers (Ed.), *Putting research into practice in the elementary grades* (Readings from Journals of the National Council of Teachers of Mathematics, pp. 141–144). Reston, VA: NCTM.

Post, T., Wachsmuth, I., Lesh, R., & Behr, M. (1985). Order and equivalence of rational numbers: A cognitive analysis. *Journal for Research in Mathematics Education, 16*(1), 18-36.

Richards, L. & Morse, J. M. (2007). *README FIRST for a user's guide to qualitative methods* (2nd ed.). Thousand Oaks, CA: Sage Publications.

Robson, C. (2011), *Real world research: A resource for users of social research methods in applied settings* (3rd ed.). Chichester, West Sussex: Wiley.

Rowland, T. & Zazkis, R. (2013). Contingency in the mathematics classroom: Opportunities taken and opportunities missed. *Canadian Journal of Science, Mathematics and Technology Education, 13*(2), 137-153.

Rowland, T. (1998). Conviction, explanation and generic examples. In A. Olivier & K. Newstead (Eds.), *Proceedings of the 22nd Annual Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 65-72). Stellenbosch, South Africa: University of Stellenbosch.

Rowland, T. (2008). The purpose, design and use of examples in the teaching of elementary mathematics. *Educational Studies in Mathematics, 69*, 149-163.

Rowland, T. (2014). The role of examples in mathematics teaching. In D. Leslie & H. Mendick (Eds.), Debates in mathematics education (pp. 96 – 105). London: Routledge.

Rowland, T., Huckstep, P., & Thwaites, A. (2005). Elementary teachers' mathematics subject knowledge: The Knowledge Quartet and the case of Naomi. *Journal of Mathematics Teacher Education, 8,* 255-281.

Rowland, T., Thwaites, A., & Huckstep, P. (2003). Novices' choice of examples in the teaching of elementary mathematics. In A. Rogerson (Ed.), *Proceedings of the International Conference on the Decidable and the Undecidable in Mathematics Education* (pp. 242–245). Brno, Czech Republic.

Rowland, T., Turner, F., Thwaites, A., & Huckstep, P. (2009). *Developing primary mathematics teaching: Reflecting on practice with the Knowledge Quartet*. London: Sage.

Sandefur, J., Mason, J., Stylianides, G. J., & Watson, A. (2013). Generating and using examples in the proving process. *Educational Studies in Mathematics,* 83(3), 323-340.

Schoenfeld, A. H. (1998). Toward a theory of teaching-in-context. *Issues in Education, 4*(1), 1-94.

Schon, D. A. (1987). *Educating the reflective practitioner.* San Francisco, CA: Jossey-Bass.

Shane, R. (2002). Context and content: What are student teachers learning about teaching mathematics? In S. Goodchild & L. English (Eds.), *Researching mathematics classrooms: A critical examination of methodology* (pp. 119-153). Westport, CT: Praeger.

Shenton, A. K. (2004). Strategies for ensuring trustworthiness in qualitative research projects. *Education for Information, 22,* 63-75.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher, 15*(2), 4-14.

Şiap, İ., & Duru, A. (2004). Kesirlerde geometriksel modelleri kullanabilme becerisi [Skills of using geometrical models in fractions]. *Journal of Kastamonu Education, 12*(1), 89-96.

Siebert, D., & Gaskin, N. (2006). Creating, naming, and justifying fractions. *Teaching Children Mathematics*, *12*(8), 394-400.

Siegler, R. S. & Booth, J. L. (2005). Development of numerical estimation: A review. In J. I. D. Campbell (Ed.), *Handbook of mathematical cognition* (pp. 197–212). New York Psychology Press.

Siegler, R. S., Thompson, C. A., & Schneider, M. (2011). An integrated theory of whole number and fraction development. *Cognitive psychology, 62,* 273-296.

Sierpinski, W. (1998). *Elementary theory of numbers* (2nd ed.). The Netherlands: Elservier Science Publishers B.V.

Silver, E. (1993). On mathematical problem posing. In I. Hirabayashi, N. Nohda, K. Shigematsu, & F. L. Lin (Eds.), *Proceedings of the Seventeenth International Conference for the Psychology of Mathematics Education* (Vol. 1, pp. 66-85). Tsukuba (Japan): International Group for the Psychology in Mathematics Education.

Silver, E. A. (1995). The nature and use of open problems in mathematics education: Mathematical and pedagogical perspectives. *International Reviews on Mathematical Education*, *27*, 67-72.

Simon, A. M. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education, 26*(2), 114-145.

Simon, M. A. (1993). Prospective elementary teachers' knowledge of division. *Journal for Research in Mathematics Education*, *24*(3), 233-254.

Sinclair, N., Watson, A., Zazkis, R. & Mason, J. (2011). The structuring of personal example spaces. *The Journal of Mathematical Behavior, 30,* 291-303.

Sirotic, N. & Zazkis, R. (2007). Irrational numbers on the number line-where are they? *International Journal of Mathematical Education in Science and Technology, 38*(4), 477-488.

Skemp, R. R. (1971). *The psychology of learning mathematics.* Harmonsdsworth, UK: Penguin Books.

Skemp, R. R. (1987). *The psychology of learning mathematics: Expanded American edition.* Hillsdale: Lawrence Erlbaum Associates.

Smith, C. L., Solomon, G. E. A., & Carey, S. (2005). Never getting to zero: Elementary school students' understanding of the infinite divisibility of number and matter. *Cognitive Psychology, 51*(2), 101-140.

Sowder, J. T. (1988). Mental computation and number comparisons: The role in development of number sense and computational estimation. In J. Hiebert & M. Behr (Eds.), *Number concepts and operations in the middle grades* (pp. 182-197). Reston, VA: Lawrence Erlbaum and National Council of Teachers of Mathematics.

Stafylidou, S. & Vosniadou, S. (2004). The development of students' understanding of the numerical value of fractions. *Learning and Instruction, 14,* 503‑518.

Stake, R. E. (1995). *The art of case study research.* Thousand Oaks, CA: Sage.

Stake, R. E. (2000). Case Studies. In N. K. Denzin, & Y. S. Lincoln (Eds.), *Handbook of qualitative research* (2nd ed.) (pp.435-454). Thousand Oaks, California: Sage Publications.

Stake, R. E. (2005). Qualitative case studies. In N. K. Denzin, & Y.S. Lincoln (Eds.), *The SAGE handbook of qualitative research* (3rd ed.) (pp. 443-466). Thousand Oaks, CA: Sage.

Steffe, L. P. & Olive, J. (Eds.). (2010). *Children's fractional knowledge.* New York: Springer.

Stoyanova, E. (2003). Extending students' understanding of mathematics via problem posing. *Australian Mathematics Teacher, 59*(2), 32-40.

Streefland, L. (1991). *Fractions in realistic mathematics education: A paradigm of developmental research.* Dordrecht, The Netherlands: Kluwer Academic Publications.

Stronge, J. T., Ward, T., & Grant, L. (2011). What makes good teachers good? A cross-case analysis of the connection between teacher effectiveness and pupil achievement. *Journal of Teacher Education, 62*, 339-347.

Sun, X. (2011). Variation problems and their roles in the topic of fraction division in Chinese mathematics textbook examples. *Educational Studies in Mathematics, 76*, 65-85.

Tall, D. O. & Vinner, S. (1981). Concept image and concept definition in mathematics with special reference to limits and continuity. *Educational Studies in Mathematics, 12*(2), 151-169.

Tattersall, J. J. (2005). *Elementary number theory in nine chapters* (2nd ed.). New York: Cambridge University Press.

Tirosh, D. & Graeber, A. (1990). Evoking cognitive conflict to explore preservice teachers' thinking about division. *Journal for Research in Mathematics Education, 21*(2), 98-108.

Tirosh, D. & Tsamir, P. (2004). What can mathematics education gain from the conceptual change approach? And what can the conceptual change approach gain from its application to mathematics education? *Learning and Instruction, 14,* 535-540.

Tirosh, D. (2000). Enhancing prospective teachers' knowledge of children's conceptions: The case of division of fractions. *Journal for Research in Mathematics Education, 31*(1), 5-25.

Tirosh, D., Tsamir, P., & Hershkovitz, S. (2008). Insights into children's intuitions of addition, subtraction, multiplication, and division. In A. Cockburn & G. Littler (Eds.), *Mathematical misconceptions.* New Delhi: Sage Publications.

Tsamir, P., Tirosh, D., & Levenson, E. (2008). Intuitive nonexamples: The case of triangles. *Educational Studies in Mathematics, 69*(2), 81-95.

Ubuz, B., Erbaş, A. K., Çetinkaya, B., & Özgeldi, M. (2010). *ZDM Mathematics Education, 42*, 483-491.

Vamvakoussi, X. & Vosniadou, S. (2010). How many decimals are there between two fractions? Aspects of secondary school students' understanding about rational numbers and their notation. *Cognition and Instruction, 28*(2), 181-209.

Van de Walle, J. A., Karp, K. S., & Bay-Williams, J. M. (2013). *Elementary and middle school mathematics: Teaching developmentally (8th ed.).* New York, NY: Pearson Education.

Vinner, S. (1983). Concept definition, concept image and the notion of function. *International Journal of Mathematical Education in Science and Technology,* 14, 293-305.

Vosniadou, S. & Verschaffel, L. (2004). Extending the conceptual change approach to mathematics learning and teaching. *Learning and Instruction, 14*, 445-451.

Watson, A. & Chick, H. (2011). Qualities of examples in learning and teaching. *ZDM The International Journal on Mathematics Education, 43,* 283-294.

Watson, A. & Mason, A. (2006). Seeing an exercise as a single mathematical object: Using variation to structure sense-making. *Mathematical Thinking and Learning, 8*(2), 91-111.

Watson, A. & Mason, J. (2002). Student-generated examples in the learning of mathematics. *Canadian Journal of Science, Mathematics and Technology Education, 2*(2), 237-249.

Watson, A. & Mason, J. (2005). *Mathematics as a constructive activity: Learners generating examples.* Mahwah, NJ: Erlbaum.

Watson, A. & Shipman, S. (2008). Using learner generated examples to introduce new concepts. *Educational Studies in Mathematics, 69*, 97-109.

Weis, L. & Fine, M. (2000). *Speed bumps: A study-friendly guide to qualitative research.* New York: Teachers College Press.

Wiersma, W. (2000). *Research methods in education: An introduction* (7th ed.). Allyn & Bacon.

Williams, P. (2008). *Independent review of mathematics teaching in early years settings and primary schools.* London: HMSO.

Wilson, P. H., Mojicab, G. F., & Confrey, J. (2013). Learning trajectories in teacher education: Supporting teachers' understandings of students' mathematical thinking. *Journal of Mathematical Behavior, 32*(2), 103-121.

Wilson, S. (1986). Feature frequency and the use of negative instances in a geometric task. *Journal for Research in Mathematics Education, 17,* 130-139.

Wilson, S. (1990). Inconsistent ideas related to definitions and examples. *Focus on Learning Problems in Mathematics, 12,* 31-47.

Yanık, B. (2013). Rasyonel sayılar. İ. Ö. Zembat, M. F. Özmantar, E. Bingölbali, H. Şandır ve A. Delice (Eds.), *Tanımları ve tarihsel gelişimleriyle mathematiksel kavramlar* (s. 95-110). Ankara: Pegema Yayıncılık.

Yetim, S. & Alkan, R. (2013). How middle school students deal with rational numbers? A mixed methods research study. *Eurasia Journal of Mathematics, Science & Technology Education, 9*(2), 213-221.

Yin, R. K. (2003). *Case study research: Design and methods* (3rd ed.). Thousand Oaks, California: Sage Publications.

Yopp, D. A. (2014). Undergraduates' use of examples in online discussions. *Journal of Mathematical Behavior 33,* 180-191.

Zaslavsky, O. & Lavie, O. (2005). Teachers' use of instructional examples. *Paper presented at ICMI-Study15: The Professional Education and Development of Teachers of Mathematics*, Brazil, 15-21 May 2005.

Zaslavsky, O. & Leikin, R. (2004). Professional development of mathematics teacher educators: Growth through practice. *Journal of Mathematics Teacher Education, 7*, 5-32.

Zaslavsky, O. & Ron, G. (1998). Students' understandings of the role of counter-examples. In A. Olivier & K. Newstead (Eds.), *Proceedings of the 22nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 225-233). Stellenbosch, South Africa: University of Stellenbosch.

Zaslavsky, O. & Zodik, I. (2007). Mathematics teachers' choices of examples that potentially support or impede learning. *Research in Mathematics Education, 9*(1), 143-155.

Zaslavsky, O. & Zodik, I. (2014). Example-generation as indicator and catalyst of mathematical and pedagogical understandings. In Y. Li, E. A. Silver, & S. Li (Eds.), *Transforming mathematics instruction: Multiple approaches and practices* (pp. 525-546). Switzerland: Springer International Publishing.

Zaslavsky, O. (2008). What knowledge is involved in choosing and generating useful instructional examples? *Paper presented at the Symposium on the Occasion of the 100th Anniversary of ICMI.* 5-8 March, Rome.

Zaslavsky, O. (2010). The explanatory power of examples in mathematics: Challenges for teaching. In M. K. Stein, L. Kucan (Eds.), *Instructional explanations in the disciplines* (pp. 107-128). New York: Springer.

Zaslavsky, O., & Peled, I. (1996). Inhibiting factors in generating examples by mathematics teachers and student-teachers: The case of binary operation. *Journal for Research in Mathematics Education, 27*(1), 67-78.

Zazkis, R & Sirotic, N. (2010). Representing and defining irrational numbers: Exposing the missing link. *Research in Collegiate Mathematics Education,* 7, 1-27.

Zazkis, R. & Chernoff, E. J. (2008). What makes a counter example exemplary? *Educational Studies in Mathematics, 68,* 195-208.

Zazkis, R. & Leikin, R. (2007). Generating examples: From pedagogical tool to a research tool. *For the Learning of Mathematics, 27*(2), 15-21.

Zazkis, R. & Leikin, R. (2008). Exemplifying definitions: A case of a square. *Educational Studies in Mathematics, 69,* 131-148.

Zazkis, R. & Liljedahl, P. (2004). Understanding primes: The role of representation. *Journal for Research in Mathematics Education, 35*(3), 164-186.

Zazkis, R. (2005). Representing numbers: prime and irrational. *International Journal of Mathematical Education in Science and Technology,* 36, 207-218.

Zazkis, R., Liljedahl, P., & Chernoff, E. G. (2007). The role of examples in forming and refuting generalizations. *ZDM The International Journal on Mathematics Education, 40*(1), 131-141.

Zodik, I. & Zaslavsky, O. (2008). Characteristics of teachers' choice of examples in and for the mathematics classroom. *Educational Studies in Mathematics, 69*, 165-182.

Zodik, I. & Zaslavsky, O. (2009). Teachers' treatment of examples as learning opportunities. In M. Tzekaki, M. Kaldrimidou, & C. Sakonidis (Eds.), *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 5, pp. 425-432). Thessaloniki: PME.

APPENDICES

APPENDIX A

OBSERVATION FORM

Gözlemin Amacı: Bu gözlemin amacı ortaokul matematik öğretmeninin ders esnasında kullanmış olduğu örnekleri nasıl ele aldığını ortaya koymaktır.

İlgili Gözlem Soruları:

1. Ortaokul matematik öğretmenleri ders anlatımı sırasında matematiksel örnekleri nasıl ele almaktadırlar?
   a. Öğretmenler sınıfta ne tür örneklerden yararlanmaktadırlar? (örnekler, örnek olmayanlar, karşıt örnekler, vb.)
   b. Öğretmenler örnekleri nasıl seçmektedirler? (planlayarak/ders esnasında anlık olarak)
   c. Öğretmenler örnekleri hangi amaçlar için seçmektedirler? (konuya ilgi çekmek için, konu anlatımı için, alıştırma yapmak için, vb.)
   d. Öğretmenler örnekleri hangi amaçlar için kullanmaktadırlar? (Hangi durumlarda seçilen örnekler kullanılan örneklerden farklılık göstermiştir?)
2. Öğretmenler derste kullanmış oldukları örnekleri öğrencilere nasıl sunmaktadırlar?
   a. Öğretmenler örnek kullanırken ne tür gösterimlerden faydalanmaktadırlar?
   b. Öğretmenler örnekte kullandıkları şekil/sayı/uzunlukları nasıl seçmektedirler?
   c. Öğretmenlerin vermiş oldukları örnekler sonrasında öğrenciler kavramları nasıl algılamaktadır? (Aşırı genelleştirme ya da aşırı özelleştirmeye neden olan örnekler var mı?)
   d. Öğretmenler art arda verdikleri örneklerde nasıl bir sıralama ve organizasyon vardır?

   e. Öğretmenler örnekleri matematiksel olarak ne derece doğru kullanmaktadırlar?
3. Öğretmenlerin örnek seçimlerinde uyguladıkları belirli kurallar ya da prensipler var mı? Varsa nasıl?
   a. Basit ve bilinen bir örnekle başlıyor mu?
   b. Öğrencilerin hatalarını dikkate alıyor mu?
   c. Kavramların gerekli özelliklerine dikkat çekiyor mu?
   d. Rastgele sayı/şekil/uzunluk kullanarak genelleştirmeler yapıyor mu?
   e. Yaygın olmayan durumları örneklerine dâhil ediyor mu?
   f. Gereksiz iş yükünün en aza indiriyor mu?

Veri Toplama

Ortaokul matematik öğretmeninin ders esnasında kullanmış olduğu örnekleri nasıl ele aldığını ortaya koymak amacıyla farklı devlet okullarında görev yapan dört yedinci sınıf matematik öğretmeninin dersleri bir eğitim öğretim dönemi boyunca gözlemlenecektir. Gözlem sürecinde aşağıda belirlenen boyutlara odaklanılacaktır. Gözlem boyunca yukarıda belirtilen gözlem sorularıyla ilgili her şey tanımlayıcı notlar alınarak kaydedilecektir. Gerektiği durumlarda tanımlayıcı notlardan ayrı olarak yoruma veya çıkarıma dayalı notlar alınacaktır. Bu notlara ek olarak sınıf içi diyaloglar ses kayıt cihazı ile kaydedilecektir.

Gözlemin Boyutları:

Ortam: Sınıfın fiziksel durumu, teknolojik destek, araç gereçler.

Öğretmenin kullandığı örnekler: Sözel olarak ifade edilenler, tahtaya yazılanlar, yazılı kaynaklardan alınanlar.

Örneklerin içeriği: Seçilen sayılar**,** uzunluklar, objeler, gösterimler, materyaller.

Öğretmenin örnek seçimi: Öğretmenin çözdüğü örnekler, öğrencinin çözdüğü örnekler, ödev olarak bırakılanlar.

Sınıf içi diyaloglar: Örnekler üzerinden geçen öğretmen-öğrenci, öğrenci-öğrenci diyalogları.

APPENDIX B

INTERVIEW PROTOCOL

Tarih: Saat: Yer: Katılımcı:

GİRİŞ

Merhaba, benim adım Ramazan Avcu. Orta Doğu Teknik Üniversitesi Eğitim Fakültesi İlköğretim Bölümünde doktora öğrencisiyim. Bu çalışmanın amacı ortaokul matematik öğretmenlerinin derslerinde kullandıkları örnekleri incelemektir. Derslerinizi gözledikten sonra kullanmış olduğunuz örneklerle ilgili birtakım sorulara cevap aramak amacıyla ders sonlarında sizle görüşme yapmak istiyorum. Görüşmeler esnasında herhangi bir nedenden ötürü kendinizi rahatsız hissederseniz görüşmeyi sona erdirmede serbestsiniz. Her bir mülakat yaklaşık 20-30 dakika sürecektir. Sizin için bir sorun teşkil etmiyorsa görüşmeleri kayıt etmek istiyorum. Katılımınız için şimdiden çok teşekkür ederim.

SORULAR

1. Derste kullanmış olduğunuz örneklerden hangilerini önceden planladınız, hangilerini ders anlatımı esnasında oluşturdunuz?
2. Dersteki her bir örneği ne amaçla kullandınız?
3. Derste kullanacağınız örnekleri seçerken veya örnek oluştururken neleri göz önünde bulundurdunuz?
4. Örnekleri seçerken kendinize özgü prensipleriniz ya da temel kurallarınız var mı?
   - Varsa örnek verir misiniz?
5. Dersten önce planlamış olduğunuz örnekleri ders esnasında kullanmadığınız oldu mu?

- Kullanmadıysanız buna neler sebep oldu?

6. Kendi çözdüğünüz sorulara ve öğrencilerin çözmesini istediğiniz sorulara karar verirken neleri göz önünde bulundurdunuz? Örnek verir misiniz?
7. Derste kullanmış olduğunuz örneklerin etkililiği hakkında ne düşünüyorsunuz?
   - Derste kullanmış olduğunuz örneklerden dersin anlaşılmasını olumsuz etkilediğini düşündüğünüz örnekler var mı?
   - Varsa hangileri olumsuz etkiledi? Neden?
8. Örneklerde değişiklik yapmak isteseniz, neleri, nasıl değiştirirsiniz?
9. Ders esnasında kullandığınız örneklerden matematiksel olarak hatalı ya da eksik olduğunu fark ettiğiniz oldu mu?
   - Varsa bunu nasıl düzeltirsiniz/düzelttiniz?

APPENDIX C

SAMPLE CODING SHEET

GEREKSİZ İŞ YÜKÜNÜ EN AZA İNDİRME				
	A	**B**	**C**	**D**
Teknik iş yükünü azaltma				
Devirli ondalık sayıların öğretilmesinde devreden kısmın daha az adımda görülebilmesi için uygun sayının seçimi	4	1		
r.s. da toplama işleminin birleşme özelliğini öğretirken paydaları eşit r.s. ın seçimi	14			
r.s. sıralanmasında payda eşitlemeye gerek kalmadan benchmark kullanılması		4,5		
r.s. sıralanmasında paylar eşitken payda eşitleme ile uğraşmama	6			
r.s. sıralanmasında paylar farklı paydalar farklı olduğu durumda hangisini eşitlemen kolaysa onu eşitlemek	7,8,9			
Payda eşitlerken 1 ile genişletileni 1 ile çarpmama durumu	10,15			
Payda eşitlerken EKOK un kullanılması, paydaları birbiriyle çarpmak yerine		6,7,8,19		
İşlem yaparken sadeleştirerek devam etmek kolaylık sağlar	17	16, 17	2	4,5,6,9, 20,21
Bir tam sayı ile bir rasyonel sayının toplanmasında ve çıkarılmasında payda eşitleme yerine kısa yol kullanılması (çarp- çıkar, çarp-topla)	11,12,15,16, 18,19,21,22, 23,24,25,26, 28,31,32,34, 35			12,14,15,16
İki bileşik kesir toplanırken tamların kendi aralarında kesirli kısımların da kendi aralarında toplanması-çıkarılması	33	9		13
Sayı doğrusunu yetecek kadar kısa çizme	1,2,3			
Merdivenli işlemlerde bilinmeyen x varsa payda eşitlemek yerine geriye doğru çalışma stratejisinin kullanılması	27			
Uzun uzun payda eşitleme yerine çıkarmaya yönelik formül kullanma	29			
Cebirsel ifadelerin kolay çözüm için yeniden düzenlenmesi				8

Kavramın özüne odaklanma				
Önemli kısmı vurgulayıp işlemin hepsini tamamlamama	5	13	1	10
Sadeleştirmeye gerek yok böyle kalsın		2		17
Tama çevirmene gerek yok böyle kalsın		3		
Bileşik kesre çevirmene gerek yok		14		
Modellenen toplama işlemi sembolik hali istendiğinde payda eşitliğine gerek yok				11
Rasyonel sayılarda kuvvet alırken pay ve paydanın ayrı ayrı kuvvetini alırsan hız kazanırsın.				18,19
İşlem özelliklerinin kullanılması				
İşlem yapmadan değişme özelliği kullanılarak sorunun çözülmesi	13	10		2
İşlem yapmadan birleşme özelliği kullanılarak sorunun çözülmesi		11		3
İşlem yapmadan dağılma özelliğini kullanarak sadeleşecek durumlar yaratma, uzun uzun hesaplamadan		12,15,20		1
İşlem yapmadan 1/(a/b)=b/a nın kullanılması	24,25	18		
İşlem yapmadan (a/b) : (a/b) = 1 nın kullanılması				7
İşlem yapmadan (a/b) + (-a/b) = 0 nın kullanılması				19
Ortak paranteze alma özelliğini kullanarak işlemi kolaylaştırma	30			
Ondalık sayılarla işlemlerde rasyonel hale getirmek yerine sayının 10 un katları ile genişletilerek işlemin kolaylaştırılması	36,37,38			

APPENDIX D

CONSENT FORM

Bu çalışma, ODTÜ Eğitim Fakültesi İlköğretim Bölümünde doktora yapmakta olan Ramazan AVCU tarafından Türkiye'de yürütülen bir doktora tez çalışmasıdır. Bu çalışma, ortaokul matematik öğretmenlerinin ders anlatımı esnasında kullanmış oldukları örnekleri derinlemesine incelemeyi amaçlamaktadır. Çalışmaya katılımda gönüllülük esastır. Katılımınız araştırmacının ders esnasında sizi gözlemlemesiyle ve ders sonrasında sizle mülakat yapmasıyla sağlanacaktır. Gözlem ve mülakatlar 2013-2014 eğitim öğretim yılı boyunca devam edecektir. Gözlem ve mülakat yoluyla elde edilen veriler tamamıyla gizli tutulacak ve sadece araştırmacılar tarafından değerlendirilecektir; elde edilecek bilgiler bilimsel yayımlarda kullanılacaktır.

Araştırmacı gözlem yaparken sınıf ortamına herhangi bir müdahalede bulunmayacaktır. Bu sebeple öğrencilerle iletişiminizi engelleyecek herhangi bir olumsuz durum söz konusu olmayacaktır. Mülakatlar, gözlem sırasında kullanmış olduğunuz örneklere yönelik araştırmacının zihninde oluşan sorulara ışık tutması amacıyla yapılacaktır. Gözlem ve mülakatlar kişisel rahatsızlık verecek hiçbir duruma neden olmayacaktır. Ancak, katılım sırasında herhangi bir nedenden ötürü kendinizi rahatsız hissederseniz mülakat ya da gözlemi sona erdirmede serbestsiniz. Böyle bir durumda gözlem ve mülakatları yapan araştırmacıya çalışmaya devam etmek istemediğinizi söylemeniz yeterli olacaktır. Gözlem ve mülakatlar sonrasında, bu çalışmayla ilgili sorularınız cevaplanacaktır. Bu çalışmaya katıldığınız için şimdiden teşekkür ederiz. Çalışma hakkında daha fazla bilgi almak için Ramazan AVCU (Aksaray Üniversitesi, Eğitim Fakültesi İlköğretim Bölümü Matematik Eğitimi Anabilim Dalı; Tel: 0 382 288 22 33; E-posta: ramazan.avcu@metu.edu.tr) ya da öğretim üyelerinden Yrd. Doç. Dr. Çiğdem HASER (ODTÜ Eğitim Fakültesi, İlköğretim Bölümü No: 105; Tel: 0 312 210 64 15; E-posta: chaser@metu.edu.tr) ile iletişim kurabilirsiniz.

Bu çalışmaya tamamen gönüllü olarak katılıyorum ve istediğim zaman yarıda kesip çıkabileceğimi biliyorum. Verdiğim bilgilerin bilimsel amaçlı yayımlarda kullanılmasını kabul ediyorum. (Formu doldurup imzaladıktan sonra uygulayıcıya geri veriniz).

Adı Soyadı	Tarih	İmza
	----/----/-----	

APPENDIX E

APPROVAL OF THE ETHICS COMMITE OF METU RESEARCH CENTER FOR APPLIED ETHICS

UYGULAMALI ETİK ARAŞTIRMA MERKEZİ
APPLIED ETHICS RESEARCH CENTER

ORTA DOĞU TEKNİK ÜNİVERSİTESİ
MIDDLE EAST TECHNICAL UNIVERSITY

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Sayı: 28620816/282 - 310

24.10 2013

Gönderilen : Y.Doç. Dr. Çiğdem Haser
İlköğretim Bölümü

Gönderen : Prof. Dr. Canan Özgen
IAK Başkanı

İlgi : Etik Onayı

Danışmanlığını yapmış olduğunuz İlköğretim Bölümü öğrencisi Ramazan Avcu'nun "Ortaokul Matematik Öğretmenlerinin Matematiksel Örnekleri Ele Alış Biçimlerinin İncelenmesi: Bir Çoklu Durum Çalışması" isimli araştırması "İnsan Araştırmaları Komitesi" tarafından uygun görülerek gerekli onay verilmiştir.

Bilgilerinize saygılarımla sunarım.

Etik Komite Onayı

Uygundur

24/10/2013

Prof.Dr. Canan Özgen
Uygulamalı Etik Araştırma Merkezi
(UEAM) Başkanı
ODTÜ 06531 ANKARA

APPENDIX F

T.C.
AKSARAY VALİLİĞİ
İl Milli Eğitim Müdürlüğü

Sayı : 85705372/44/1365950 14/06/2013
Konu: Bilimsel Araştıtma.

VALİLİK MAKAMINA

İlgi : a) Millî Eğitim Bakanlığı Yenilik ve Eğitim Teknolojileri Genel Müdürlüğünün 07/03/2012 tarih ve 3616 sayılı yazısı. (2012/13 Nolu Genelgesi)
b)Aksaray Üniversitesi Eğitim Fakültesi araştırma Görevlisi Ramazan AVCU' nun 03.06.2013 tarihli dilekçesi.

Aksaray Üniversitesi Eğitim Fakültesi Araştırma Görevlisi Ramazan AVCU, Aksaray İl merkezindeki okullarda görev yapan farklı deneyimlere sahip matematik öğretmenlerinin ders anlatımı sırasında kullanmış oldukları örnekleri, bilimsel araştırma kapsamında incelemek istemektedir. Cahit Zarifoğlu Ortaokulu,23 Nisan Ortaokulu ve Piri Mehmet Paşa Ortaokulu matematik öğretmenleri ve okul yönetimi, video-kamera kayıt sistemiyle yapılmak istenen Bilimsel Araştırmanın okullarında yapılabileceği hususunda yazılı olarak muvafakat etmektedirler.

İlgi (a) genelgede "Araştırma önerisi ve veri toplama araçları Anayasa, Millî Eğitim Temel Kanunu ve Türk Millî Eğitiminin genel amaçlarına uygun olacak; millî ve manevi değerlere aykırı, kişilik haklarını ihlal eden; cinsiyet, din, dil, ırk gibi farklılıkları istismar eden, İnsan Hakları Evrensel Beyannamesi ve uluslar arası bağlayıcılığı olan diğer belgelerce suç kabul edilen hususları içeren, kişisel ve ailevi mahremiyeti ifşa eden soru, ifade, resim ve simgeler yer almayacaktır. Veri toplama araçlarında kişi, kurum ve kuruluşların reklâmını veya tanıtımını yapan ifade ve öğeler bulunmayacaktır." denilmektedir.

Bu nedenle; ilgi (b) dilekçeyle Bilimsel Araştırma yapma isteğinde bulunan Ramazan AVCU'nun, Aksaray İl merkezindeki okullarda görev yapan farklı deneyimlere sahip matematik öğretmenlerinin ders anlatımı sırasında kullanmış oldukları örnekleri, video kamera kayıt sistemiyle bilimsel araştırma kapsamında inceleme isteği, yukarıda belirtilen okullarımızın matematik öğretmenleri ile okul yönetimleri kabul etmiş ise de; uygulamanın video kamera kayıt sistemi ile yapılması halinde, öğrenci ve öğrenci velileri üzerinde olumsuz düşünce ve davranış biçimleri ortaya çıkarabileceği konusunda Müdürlüğümüzün çekinceleri bulunmaktadır. Bu nedenle; Ramazan AVCU'nun yukarıda belirtilen bilimsel araştırmasını video ve kamera cihazları kullanmaksızın eğitim öğretim faaliyetlerini aksatmamak kaydıyla yapması Müdürlüğümüzce uygun görülmektedir.

Makamlarınızca da uygun görüldüğü takdirde, olurlarınızı arz ederim.

Lütfiye DENERİ
İl Millî Eğitim Müdürü

OLUR
14/06/2013
Kubilay ANT
Vali a.
Vali Yardımcısı

Bu belge, 5070 sayılı Elektronik İmza Kanununun 5 inci maddesi gereğince güvenli elektronik imza ile imzalanmıştır

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e-posta: aksaraymem@meb.gov.tr
Ayrıntılı bilgi için: H.YALÇIN
Tel: 0 (382) 213 68 40
Faks: 0 382 213 68 14

APPENDIX G

TURKISH SUMMARY

ORTAOKUL MATEMATİK ÖĞRETMENLERİNİN RASYONEL SAYI ÖRNEKLERİNİ SINIF ORTAMINDA ELE ALIŞ BİÇİMLERİNİN İNCELENMESİ: ÇOKLU DURUM ÇALIŞMASI

Örnekler matematik eğitiminde önemli bir rol oynamaktadır. (Rowland, 2008; Zazkis ve Leikin, 2008; Zodik ve Zaslavsky, 2008). Örnekler özellikle kavramsallaştırmada, genelleştirmede, soyutlamada, argümantasyon ve analojik akıl yürütme sürecinde önemli bir yere sahiptir (Zaslavsky ve Zodik, 2007).

Örnekler matematik eğitiminde iki farklı amaçla kullanılmaktadır (Rowland, Turner, Thwaites ve Huckstep, 2009; Zodik ve Zaslavsky, 2008). İlk olarak, örnekler matematiksel bir kavramı ya da yöntemi örneklendirmede kullanılırlar (Mason ve Pimm, 1984; Rowland vd., 2009; Watson ve Mason, 2005; Zodik ve Zaslavsky, 2008). İkinci olarak ise matematiksel bir kavramın ya da yöntemin pekiştirilmesinde kullanılırlar (Rowland vd., 2009; Rowland, 2008; Watson, Mason, 2005). Örnek olmayanlar ve karşıt örnekler, matematik eğitimindeki diğer örnek türleri arasında yer almaktadır (Watson ve Mason, 2005). Örnek olmayan örnekler, kavramsallaştırma ve tanımlarla ilgilidir ve matematiksel kavramların kritik özniteliklerine dikkat çekerler (Zodik ve Zaslavsky, 2008). Karşıt örnekler matematiksel iddialarla ve bu iddiaların çürütülmesi ile ilgilidir (Zodik ve Zaslavsky, 2008). Kısacası, karşıt örnekler matematiksel bir ifadenin doğru olmadığını göstermede ve matematiksel kavramlar arasındaki ayırt edici özelliklerin netleştirilmesinde önemli bir rol oynamaktadırlar.

Bills ve diğerleri (2006), matematiksel bir örneğin pedagojik olarak yararlı olabilmesi için ‘şeffaflık’ ve ‘genelleştirilebilirlik’ şeklinde iki temel özelliğe sahip olması gerektiğini belirtmişlerdir. Bir örneğin şeffaflığı bireyin örneğin nasıl yorumlandığına ve örneğin özelliklerini nasıl algıladığına bağlıdır ve dolayısıyla bağlam bağımlıdır. Öğretmenler, öğrencilerine pedagojik olarak yararlı olan çok sayıda örneğin temin edilmesinde önemli bir rol oynamaktadırlar (Zaslavsky, 2010).

Fakat uygun örnek seçimi kolay bir iş değildir ve önceden planlanması mümkün olmayan karmaşık birçok düşüncenin göz önünde bulundurulmasını gerektirir (Zodik ve Zaslavsky, 2008). Rowland'e (2014) göre, örnekler dikkatli bir seçim sürecinin ürünü olmalıdır ve bilinçli bir seçim yapmayı gerektirmelidirler. Çünkü bir örnek diğerine göre daha iyi veya uygun olabilir. Ayrıca, örnek seçimi öğrencilerin öğrenmelerini hem olumlu hem de olumsuz olarak etkileyebilir (Zaslavsky ve Zodik, 2007). Bu bağlamda, öğretmenlerin örnek seçiminin ve kullanımının öğrencilerin öğrenme sürecine şekil verebileceği söylenebilir.

Örneklerin seçilmesi ya da oluşturulması öğretmenlerin genelde anlık kararlar vermesini gerektirir. (Zodik ve Zaslavsky, 2008). Bu düşünceden yola çıkarak, bu araştırmada ortaokul matematik öğretmenlerinin rasyonel sayı kavramlarının öğretiminde kullandıkları anlık ve planlı örneklerin belirlenmesi amaçlanmıştır. Öğretmenlerin anlık ve planlı örnek seçiminde kullandıkları prensipler ya da göz önünde bulundurdukları düşünceler, onların planlama sürecinde veya anlık eylemlerinde daha bilinçli olmalarını sağlamaktadır (Zaslavsky ve Zodik, 2008). Bu nedenle bu çalışmanın diğer bir amacı, ortaokul matematik öğretmenlerinin örnek seçerken veya kullanırken göz önünde bulundurdukları prensipleri veya düşünceleri belirlemek olmuştur.

Örnekler, matematiğin öğretilmesinde veya öğrenilmesinde önemli bir yere sahip olmalarına rağmen (Zaslavsky, 2010), örnek seçimi bazı güçlüklere ya da sıkıntılara neden olabilmektedir (Rowland, 2008). Matematik öğretiminde öğretmenlerin şu üç tür örnekten kaçınmaları gerekmektedir: değişkenlerin rolünü anlaşılmaz hale getiren örnekler, matematiksel bir yöntemin öğretilmesinde kullanılan fakat başka bir yöntemin öğretilmesi için daha uygun olan örnekler ve dikkatli seçim yapmayı gerektirdiği halde genelde zar atılarak rasgele seçilen örneklerdir (Rowland, Thwaites ve Huckstep, 2003). Bu çalışmada ortaokul matematik öğretmenlerinin hem iyi örnek seçimleri hem de kötü örnek seçimleri incelenmiştir. Öğretmenler kötü örnekleri sınıf ortamına dâhil ederek öğrencilerin bu örnekleri sorgulamalarını sağlayarak onların matematiksel düşüncelerinin gelişimine olumlu etkide bulunabilir (Zaslavsky ve Zodik, 2007). Buna ek olarak, öğretmenlerin iyi ve kötü örnek kullanımlarını içeren sınıf içi durumlar öğretmen eğitimi programlarında ve mesleki gelişim etkinliklerinin düzenlenmesinde etkili bir şekilde kullanılabilir (Zodik ve Zaslavsky, 2008). Bu nedenle, bu çalışmanın bulguları öğretmen adaylarının sınıf ortamında matematiksel örnekleri nasıl ele almaları gerektiği konusunda pratik anlamda bilgi sahibi olmalarına yardımcı olabilir.

1.1. İlkokul ve Ortaokul Matematiğinde Rasyonel Sayı Kavramları

Rasyonel sayı kavramları okul yıllarında öğrencilerin karşılaşmış oldukları en önemli matematiksel kavramlar arasında yer almaktadır (Alacacı, 2009; Behr, Lesh, Post ve Silver, 1983; Behr, Wachsmuth, Post ve Lesh, 1984; Yanık, 2013). Bu öneminden dolayı ülkemizde rasyonel sayı kavramlarına ilkokul birinci sınıftan ortaokul sekizinci sınıfa kadar her öğrenim seviyesinde yer verilmektedir.

Öğrenciler, rasyonel sayı kavramları ile tüm sınıf seviyelerinde karşılaşmalarına rağmen, bu kavramların anlaşılmasında ilkokul öğrencilerinin yanı sıra (Haser ve Ubuz, 2003; Lesh, Behr ve Post, 1987; Ni, 2001) ortaokul öğrencilerinin oldukça güçlük çektiği sıklıkla dile getirilmektedir (Birgin ve Gürbüz, 2009; Lamon, 2007). Rasyonel sayı kavramlarının anlaşılması ilkokul öğretmenlerine bile zor gelmektedir (Graeber, Tirosh ve Glover, 1989; Izsak, 2008; Tirosh, 2000). Ball (1990a, 1990b), birçok öğretmenin rasyonel sayılarla ilgili yalnızca işlemsel bilgiye sahip oluğunu belirtmiştir. Ni ve Zhou (2005) öğrencilerin rasyonel sayılarla ilgili yaşadıkları güçlükleri iki temel etkene bağlamıştır. Birincisi, doğal sayı bilgisinin rasyonel sayılara genellenmesiyle ilgilidir. İkincisi ise rasyonel sayıların gösteriminde yaşanan problemlerle ilişkilidir.

Öğrencilerin rasyonel sayılarla ilgili güçlüklerini azaltmak için Greer (1987) öğrencilerin rasyonel sayılarla ilgili yaygın kavram yanılgılarını ortaya çıkarmış, Moss ve Case (1999) yeni bir program geliştirmiş ve Amerikan Ulusal Matematik Öğretmenleri Konseyi (2000) stardartlaştırılmış döküman kullanmanı önemi üzerinde durmuştur. Öğrencilerin rasyonel sayı kavramlarını anlamalarını artırmak için birçok çalışma yapılmış olmasına rağmen öğrenci güçlükleri hala devam etmektedir (Wilson, Mojica ve Confrey, 2013). Morrison (2013) öğrencilerin başarı düşüklüğünü örneklerin iyi sıralanmamasından, yeterli çeşitlilikte olmayan ve düşük bilişsel beceri gerektiren örneklerin kullanımından kaynaklandığını vurgulamıştır. Bu nedenle, öğretmenlerin sınıf ortamında kullandıkları rasyonel sayı örneklerinin niteliğinin incelenmesi öğrencilerin öğrenmelerini artırmada önemli bir rol oynayabilir.

1.2. Araştırmanın Amacı ve Araştırma Soruları

Bu çalışmanın amacı matematik öğretmenlerinin yedinci sınıf ortamında rasyonel sayı örneklerini nasıl ele aldıklarını incelemektir. Özel olarak bu çalışmada öğretmenlerin kullandıkları rasyonel sayı örneklerinin karakteristik özelliklerine, öğretmenlerin bu örnekleri kullanırken göz önünde bulundurdukları prensiplere ve kullanılan örneklerde yer alan olası hata veya yetersizliklere odaklanılmıştır. Bu amaçla bu çalışmada aşağıdaki araştırma sorularına cevap aranmıştır:

1. Ortaokul matematik öğretmenlerinin yedinci sınıf ortamında kullandıkları rasyonel sayı örneklerinin karakteristik özellikleri nelerdir?

a. Öğretmenler tarafından kullanılan rasyonel sayı örnekleri hangi fikirleri vurgulamaktadır?

b. Öğretmenler rasyonel sayı öğretiminde ne oranda özel örnek kullanmaktadır?

c. Öğretmenler rasyonel sayı öğretiminde ne oranda örnek olmayan ve karşıt-örnek kullanmaktadır?

d. Öğretmenler rasyonel sayı öğretiminde ne oranda planlı örnek ve anlık örnek kullanmaktadır?

e. Öğretmenler rasyonel sayı öğretiminde hangi kaynakları kullanmaktadır?

2. Öğretmenler örnek seçerken veya oluştururken hangi prensipleri veya düşünceleri göz önünde bulundurmaktadır?

3. Öğretmenlerin rasyonel sayı öğretiminde kullandıkları örnekler hangi matematiksel veya pedagojik yetersizlikler içermektedir?

a. Öğretmenler rasyonel sayı öğretiminde matematiksel olarak hatalı hangi örnekleri kullanmaktadırlar?

b. Öğretmenlerin rasyonel sayı öğretiminde pedagojik olarak uygun olmayan hangi örnekleri kullanmaktadırlar?

1.3. Çalışmanın Önemi

Matematik eğitiminde örnek kullanımı uzun bir geçmişe sahiptir (Bills vd., 2006; Rowland, 2008) ve matematik eğitimi araştırmalarında giderek artan bir ilgiye sahiptir (Antonini vd., 2011; Bills ve Watson, 2008). Son on yılda matematiksel örneklerle ilgili birçok araştırma makalesi yayınlanmıştır ve birkaç çalışma grubu matematiksel örnekler üzerine odaklanmıştır. Örnek kullanımı sınıf ortamında gerekli bir şey olmasına rağmen öğretmenler için karmaşık bir iş olabilir (Bills vd., 2006; Zaslavsky ve Peled, 1996). Buna ek olarak, örnek kullanımı birçok koşulu göz önünde bulundurmayı gerektirir. (Antonini vd., 2011; Zodik ve Zaslavsky, 2008). Bu açıdan bakıldığında, öğretmenlerin örnek seçimi öğrencilerin öğrenmelerini hem destekleyebilir hem de engelleyebilir. Öğretmenlerin örnek kullanımı öğrencilerin öğrenmeleri üzerinde önemli bir role sahip olmasına rağmen, matematik öğretmenliği programları bu konuya açık bir şekilde yer vermemektedir ve öğretmen adaylarının örnekleri nasıl kullanacaklarına ilişkin herhangi bir öğretim yapılmamaktadır (Zaslavsky ve Zodik, 2007). Bu sebeple, öğretmenlerin örnekleri etkili bir şekilde kullanmaları matematik öğretim deneyimleri neticesinde gelişim sağlar ve bu bilgiye mesleki beceri bilgisi adı verilir (Kennedy 2002; Leinhardt 1990). Öğretmenlerin örnek kullanımlarına yönelik mesleki beceri bilgilerinin incelenmesi, onların matematiksel bilgileri hakkında ipucu verir ve öğretmenlerin sistematik bilgilerinin artırılmasına imkân sunan mesleki gelişim programlarının ya da kurslarının tasarlanmasına zemin hazırlayabilir (Zaslavsky, 2008; Zaslavsky ve Zodik, 2007).

Kavramsal öğrenmede örnek kullanımı merkezi bir konumda bulunmasına rağmen (Watson ve Mason, 2002), öğretmenlerin sınıf ortamında kullandıkları ya da seçtikleri matematiksel örnekleri inceleyen çok az çalışma bulunmaktadır (Rowland 2008; Zodik ve Zaslavsky, 2008). Ayrıca, ulaşılabilir alanyazında öğretmenlerin örnekleri sınıf ortamında nasıl ele aldıklarıyla ilgili Türkiye’de gerçekleştirilmiş çalışmalara rastlanmamıştır. Bu sebeple, bu çalışmada ortaokul matematik öğretmenlerinin rasyonel sayı örneklerini nasıl ele aldıkları ulusal bir bağlamda incelenecektir.

Kısacası bu çalışma, öğretmenlerin rasyonel sayıları nasıl ele aldıklarını incelemeye yardımcı olacak bir kavramsal çerçevenin gelişimine ön ayak olabilir. Ayrıca, bu çalışma öğretmenlerin matematik öğretimi esnasında örnekleri nasıl seçeceklerine yönelik bir farkındalık kazanmalarını sağlayabilir. Bu farkındalık sayesinde öğretimin niteliğinin artması ve öğrencilerin öğrenmelerinin teşvik edilmesi beklenmektedir.

ALANYAZIN TARAMASI

2.1. Matematiksel Örnek Nedir?

Örnek kavramının farklı anlamları bulunmaktadır. Zodik ve Zaslavsky (2008), örnek kavramını daha geniş bir sınıfın özel bir durumu olarak tanımlamıştır ve örnekler aracılığıyla akıl yürütme ve genelleme yapılacağını belirtmişlerdir. Aynı şekilde, Zazkis ve Leikin (2008) örneklerin matematiksel kavram ve kuralların açıklanmasında kullanıldığını belirtmiştir. Diğer bir çalışmada Sinclair ve diğerleri (2011), örnek vermeyi daha genel bir kavramı daha özel bir durumla resmetme olarak nitelendirmiştir. Benzer şekilde örnek kavramını Yopp (2014) matematiksel bir görevle ilgili özelliklerin ya da kavramların gösterilmesinde kullanılan herhangi bir matematiksel nesne olarak tanımlamıştır. Watson ve Mason (2005) örnek kavramını daha geniş bir perspektiften ele almış ve öğrencilerin örnekleri herhangi bir şeyi temsil etmek için kullanabileceğini ve bu nesneden yola çıkarak genelleme yapabileceklerini belirtmiştir.

2.2. Matematilsel Örnek Türleri

Birkaç araştırmacı matematiğin öğretilmesinde kullanılan örnekleri göz önünde bulundurarak bu örnekleri sınıflandırma yoluna gitmiştir. Michener (1978) örnekleri dört farklı gruba ayırmıştır. Bunlar başlangıç örnekleri, referans örnekleri, genel örnekler ve karşıt örneklerdir. Benzer şekilde Mason ve Pimm (1984) örnekleri dört sınıfa ayırmıştır. Bunlar ‘specific’, ‘particular’, ‘generic’ ve ‘general’ örneklerdir. Peled ve Zaslavsky (1997) örnekleri bir kavramı ya da kuralı açıklayabilme güçlerine göre üç sınıfa ayırmıştır. Bunlar özel örnekler, yarı genel örnekler ve genel örneklerdir. Askew ve William (1995) ‘only just’ örnekler ve ‘very nearly’ örnekler olmak üzere iki tür örnekten bahsetmiştir. Mason ve Watson (2011) ‘only just’ örneğinin yerine uç örnek kavramını kullanmayı tercih etmiştir. Mason ve Watson (2011) uç örnek üretemeyen öğrencilerin ilgili teknik ya da teoremi tamamıyla anlayamayacağından söz etmiştir. Zazkis ve Chernoff (2008) karşıt-örneklerin matematiksel olarak bir çıkarımı çürütmeye yaradığını fakat öğrencilerin yanlış çıkarımlarından vazgeçmelerinde derecede karşıt-örneklerin yeterli ikna edici rol oynamayabileceğinden bahsetmiştir. Bu nedenle matematiksel bir kavram olan karşıt-örnekler yerine pedagojik kavramlar olan merkezi örnekler ve köprüyelici örneklerden söz etmiştir. Ayrıca, Zazkis ve Chernoff (2008) merkezi örneklerin bilişsel çatışmada işe yaradığını, köprüleyici örneklerin ise bilişsel çatışmanın çözümünde işe yaradığını ifade etmiştir. Zodik ve Zaslavsky (2008) öğretmenlerin bir kavramı ya da kuralı öğretirken kullandıkları örneklerin ya anlık olarak sınıf ortamında üretildiğini ya da derse gelmeden önce öğretmenler tarafından önceden planlandığını belirtmiştir. Yani, Zodik ve Zaslavsky (2008) kullanılma veya üretilme zamanına göre örnekleri planlı örnekler ve anlık örnekler şeklinde ikiye ayırmıştır. Son olarak Rowland, Turner, Thwaites ve Huckstep (2009**)** örnekleri kavram veya kural örnekleri ve alıştırma örnekleri şeklinde ikiye ayırmıştır. Kavram veya kural örnekleri bir kavramın veya kuralın öğretilmesinde kullanılırken alıştırma örnekleri kavram veya konunun tekrarını sağlamak amacıyla kullanılmaktadır. Rowland ve diğerlerine (2009) göre örneklerin iki farklı kullanımı daha mevcuttur. Bunlar karşıt örnekler ve genel örneklerdir. Araştırmacılara göre karşıt örnekler bir ifadenin yanlış olduğunu göstermede kullanılırken genel örnekler konuya açıklık getirmeyi amaçlayan, bir sınıf nesnenin ayırt edici bir özelliği üzerinde yapılan işlemler yoluyla bir iddianın doğruluk sebeplerini açığa çıkaran örneklerdir.

2.3. Örnekler Uzayı (Example Space) Kavramı

Watson ve Mason’ın (2005) iddia ettiğine göre tek bir örnek bir fikrin öğrenciler tarafından tamamen anlaşılmasında yeterli olmayabilir ve öğrencilerin yanlış genellemeler yapmalarına neden olabilir. Bu düşünceden yola çıkarak Watson ve Mason (2005) örnekler uzayı kavramını ortaya atmışlardır. Araştırmacılar, örnekler uzayını şu şekilde izah etmişlerdir: “Örnekler uzayını birçok araç gereç içeren alet edevat dolabı olarak düşünün. Bazı araç gereçler daha bilindiktir ve dolap açıldığında hemen ele gelecek türdendir. Öte yandan, diğer araç gereçler daha geridedir ele gelmeleri özel uğraş gerektirir" (s. 61).

Örnekler uzayı kavramı Tall ve Vinner'ın (1981) kavram imgesi olarak adlandırdığı bilişsel yapı ile yakından ilgilidir. Kavram imgesi, bir kavramla ilgili bilişsel yapının tamamıdır ki bu zihinde o kavramla ilgili bütün resimleri, özellikleri ve işlemleri kapsar (Tall ve Vinner, 1981). Edwards (2011) örnekler uzayını bireyin ulaşabileceği örnekler sınıfı ya da kümesi olarak tanımlamıştır ve örnekler uzayının kavram imgesi kavramının bir alt kümesi olarak düşünülebileceğini belirtmiştir. Benzer şekilde, Mason ve Watson (2008) örnekler uzayının bireyin kavram imgesinin önemli bir kısmını oluşturduğunu ifade etmiş ve örnekler uzayını bireyin ulaşabildiği örneklerin ve örnek olmayanların tümü olarak tanımlamıştır. Zaslavsky ve Zodik (2014) örnekler uzayını "bireyin belirli bir kavramla belirli bir zaman ve bağlamda ilişkilendirdiği örnekler bütünü" (p. 527) olarak tanımlamıştır ve örnekler uzayının Tall ve Vinner'ın kavram imgesi yapısıyla yakından ilişkili olduğunu belirtmiştir.

2.4. Öğretmenlerin Sınıf Ortamında Matematiksel Örnekleri Ele Alış Biçimlerini İnceleyen Çalışmalar

Rowland (2008), öğretmen adaylarının sınıf ortamında kullandıkları örnekleri incelemek amacıyla kavramsal bir çerçeve ortaya atmıştır. Bu çalışmada doğal sayıların toplanması ve çıkarılması, geometrik dönüşümler gibi ilköğretim düzeyindeki matematik konularına yönelik örnekler incelenmiştir. Rowland'ın (2008) kavramsal çerçevesi dört kategoriden oluşmaktadır. Bunlar, değişkenler, sıralama, gösterimler ve kazanımlardır. Rowland (2008) çalışmasında çoğunlukla öğretmen adaylarının kullandıkları uygun olmayan örnekler üzerine yoğunlaşmıştır.

Morrison (2013), Rowland'in (2008) kavramsal çerçevesini kullanarak benzer bir çalışma yapmıştır. Yalnız, Rowland'dan (2008) farklı olarak, Morrison (2013) çalışmasında iki okul öncesi öğretmeninin sayı kavramlarının öğretiminde kullandıkları örnekleri incelemiştir. Morrison (2013) çalışmasında Rowland'ın

(2008) çalışmasıyla benzer sonuçlar elde etmiştir. Daha açıkçası, Morrison (2013) okul öncesi öğretmenlerinin örnek çeşitliliğinin boyutuna dikkat etmediklerini ortaya koymuştur.

Rowland (2014) yakın zamanda yapmış olduğu çalışmasında 2008 yılında geliştirdiği kavramsal çerçevenin sadece değişkenler boyutuna odaklanmıştır ve öğretmen adaylarının ikinci dereceden bir bilinmeyenli denklemlerin ve birinci dereceden iki bilinmeyenli denklemlerin öğretiminde kullandıkları örneklerin ne oranda uygun olduğunu araştırmıştır. Rowland (2014), 2008 yılında yapmış olduğu çalışmaya benzer şekilde öğretmen adaylarının matematiksel kavram ve kuralları öğretmeye başlamadan önce kullanacağı örnekleri planlamalarını tavsiye etmiştir. Buna ek olarak, planlı örneklerin öğretmenlere örnekleri basitten karmaşığa doğru sunma imkânı vereceğini belirtmiştir.

Önceki çalışmalardan farklı olarak Zaslavsky (2010), öğretmenlerin sınıf ortamında kullanmış oldukları örneklerin açıklayıcı gücünü incelemiştir. Zaslavsky (2010) örneklerin öğretimsel açıklamalarıyla ilgili olarak şu temalara değinmiştir: genellemenin ve değişmezliğin aktarılması, notasyonların ve konvensiyonların doğrulanması, öğrencilerin çıkarımlarının ve iddialarının doğruluğunun veya yanlışlığının ortaya konması, matematiksel kavramların gündelik hayatla ilişkilendirilmesi ve istenilen kısıtlılıkta örneklerin üretilmesidir. Zaslavsky (2010) yaptığı araştırmanın sonucunda öğretmenlerin öğrencilere sunduğu örneklerin kritik özelliklerini bilmeleri gerektiğini ve öğrenciler tarafından üretilen örnekleri geliştirecek yeteneklere sahip olmalarını gerektiğini belirtmiştir. Zaslavsky'nin (2010) çalışmasına benzer şekilde, Zaslavsky ve Zodik (2007) öğretmenlerin sınıf içinde kullandıkları örneklerin güçlü ve zayıf yönlerine odaklanmıştır. Zaslavsky ve Zodik'in (2007) incelemiş olduğu örnekler, Zaslavsky'nin (2010) çalışmasındaki örneklerle büyük oranda benzerlik göstermiştir. Zaslavsky ve Zodik (2007), Rowland'e (2014) benzer şekilde öğretmenlerin sınıf ortamında kullanacağı örnekleri planlamalarını tavsiye etmiştir.

Diğer bir araştırmada Bills ve Bills (2005), deneyimli öğretmenlerin üçgeninin alanını, kesirlerin toplanmasını ve doğrusal denklemlerin çözülmesini öğretirken kullandıkları örneklerde pedagojik olarak neyi hedeflediklerini ortaya koymaya çalışmıştır. Bills ve Bills (2005) matematiksel bir kavramın anlaşılmasını sağlamak için ve karışıklığı engellemek için deneyimli öğretmenlerin ilk olarak basit örnekleri tercih ettiklerini ortaya koymuştur. Benzer şekilde Zodik ve Zaslavsky (2008), deneyimli lise matematik öğretmenlerinin matematiksel örnek seçiminde kullandığı pedagojik prensiplerini ve düşüncelerini araştırmıştır. Zodik ve Zaslavsky (2008) çalışmasında öğretmenlerin örnek seçerken veya kullanırken şu prensipleri kullandıklarını belirtmiştir: kolay ve bilinen örneklerle başlama, öğrenci hatalarına dikkat etme, örneklerin kritik özelliklerini ön plana çıkarma, rasgele örnek seçerek genellemelere ulaşmayı sağlama, yaygın olmayan örnekleri sınıf ortamına dâhil etme ve gereksiz iş yükünü en aza indirmedir.

YÖNTEM

3.1. Araştırmanın Deseni

Ortaokul matematik öğretmenlerin yedinci sınıf ortamında kullandıkları rasyonel sayı örneklerinin incelendiği bu araştırmada nitel araştırma yöntemi kullanılmıştır. Daha özel olarak bu çalışmada durum çalışması deseni kullanılmıştır. Yin (2003) dört tür durum çalışması deseninden söz etmiştir. Bunlar bütüncül tek durum deseni, bütüncül çoklu durum deseni, iç içe geçmiş tek durum deseni ve iç içe geçmiş çoklu durum desenidir. Bu çalışmada durum çalışması türlerinden bütüncül çoklu durum deseni kullanılmıştır.

3.2. Çalışmanın Katılımcıları

Bu çalışmaya Aksaray il merkezindeki farklı devlet okullarında görev yapmakta olan dört ortaokul matematik öğretmeni katılmıştır. Veri toplama sürecinde her bir öğretmen yedinci sınıf öğrencilerine rasyonel sayı kavramlarını öğretmiştir. Katılımcı okulların seçilmesinde uygun örnekleme yöntemi kullanılmıştır (Fraenkel, Wallen ve Hyun, 2012). Bu çalışmada zengin veri toplamak önemli bir husus olduğu için çalışmanın katılımcıları amaçlı örnekleme yöntemi kullanılarak belirlenmiştir. Rasyonel sayıların öğretilmesinde özellikle farklı deneyimlere sahip olan öğretmenler katılımcı olarak belirlendiği için bu çalışmada amaçlı örnekleme yöntemlerinden maksimum çeşitlilik örneklemesi kullanılmıştır

(Creswell, 2012). Çalışmanın katılımcılarına ait demografik veriler Tablo 3.1'de verilmiştir.

Tablo 3.1. Çalışmanın katılımcılarına ait demografik veriler

Açıklama	Öğretmen A	Öğretmen B	Öğretmen C	Öğretmen D
Cinsiyet	Erkek	Erkek	Erkek	Kadın
Yaş	36	36	31	26
Üniversite	Devlet	Devlet	Devlet	Devlet
Mezuniyet	Matematik bölümü	Matematik bölümü	İlköğretim Matematik Öğretmenliği	İlköğretim Matematik Öğretmenliği
Öğretmenlik deneyim yılı	14	11	9	4
Rasyonel sayılar öğretiminde deneyim yılı	14	10	8	2
Aksaray ilindeki deneyim yılı	9	10	4	3
Bulunduğu okuldaki deneyim yılı	3	2	2	1

3.3. Veri Toplama Araçları

Bu çalışma ortaokul matematik öğretmenlerinin rasyonel sayı örneklerini nasıl ele aldıklarını derinlemesine araştırmayı amaçlamıştır. Bu öğretmenlerden zengin veri elde edebilmek amacıyla 'çoklu veri toplama araçları' kullanılmıştır (Creswell, 2007). Çalışmanın temel verilerini sınıf içi gözlemler ve gözlem sonrası öğretmenlerle yapılan görüşmeler oluşturmuştur. Sınıf içi gözlemler video kamera ile görüşmeler ses kayıt cihazı ile kayıt edilmiştir. Ayrıca ders gözlemleri ve görüşmeler esnasında alan notları tutulmuştur. Son olarak, öğretmenlerin rasyonel sayı kavramlarının öğretimi esnasında öğrencilerine dağıttıkları yazılı materyaller (örneğin çalışma yaprakları), ödevler ve rasyonel sayı kavramları ile ilgili öğrencilere yöneltilen yazılı soruları çalışmanın ikincil veri kaynaklarını oluşturmuştur.

3.4. Veri Toplama Süreci

Çalışma verilerinin toplanmasına yönelik zaman çizelgesi Tablo 3.2'de verilmiştir.

Tablo 3.2. Veri toplama sürecine yönelik zaman çizelgesi

Tarih	Süreçler
Ağustos 2013	ODTÜ Uygulamalı Etik Araştırma Merkezinden etik onayının ve Aksaray Valiliği İl Milli Eğitim Müdürlüğünden çalışma izninin alınması
Eylül 2013	Katılımcı okul, sınıf ve öğretmenlerin belirlenmesi
Eylül 2013 – Kasım 2013	Veri toplama öncesi gözlem ve görüşmeler
Kasım 2013 – Aralık 2013	Veri toplama sürecinde gözlem ve görüşmeler
Kasım 2013- Ocak 2014	Veri toplama sonrasında gözlem ve görüşmeler
Kasım 2013 - Mart 2014	Gözlem ve görüşme verilerinin transkript edilmesi

3.5. Verilerin analizi

Bu çalışmada gözlem ve görüşmeler farklı öğretmenlerle farklı ortamlarda yürütülmüştür. Dolayısıyla çoklu durumlar seçilmiştir. Creswell (2007) bir araştırmada birden fazla duruma odaklanıldığında öncelikle her bir durumun ve temanın ayrıntılı bir şekilde betimlenmesini önermiştir ve buna durum-içi analiz ismini vermiştir. Creswell (2007) durum-içi analiz sonrasında durumlar arası tematik analiz yapılmasını önermiş ve buna karşılaştırmalı durum analizi adını vermiştir. Benzer şekilde Yin (2003) durum çalışmalarının analiz edilmesinde kullanılan beş analitik teknikten bahsetmiştir. Bunlar örüntü eşleştirme, açıklama oluşturma, zaman serisi analizi, mantık modelleri ve karşılaştırmalı durum sentezidir. Yin (2003) ilk dört tekniğin tekli veya çoklu durum çalışmalarında kullanılabileceğini fakat karşılaştırmalı durum sentezinin iki veya ikiden fazla durum içeren çalışmalarda özellikle kullanılması gerektiğini belirtmiştir. Bundan dolayı, bu çalışmanın verileri karşılaştırmalı durum sentezi tekniği kullanılarak analiz edilmiştir.

Karşılaştırmalı durum sentezi tekniği kullanılarak öncelikle her bir durum birbirinden bağımsız bir şekilde analiz edilmiştir. Diğer bir deyişle, her bir öğretmenin kullanmış olduğu rasyonel sayı örnekleri kendi içlerinde sınıflandırılmıştır. Veriler tekrar tekrar incelendikçe, her bir öğretmenin rasyonel sayı örneklerini kullanımlarına yönelik kategoriler belirginleşmeye başlamıştır. Bazı kategoriler, alanyazında yer alan kategoriler yardımıyla belirlenmiştir. Bazıları ise bu çalışma sonucunda ortaya çıkmıştır. Her bir öğretmenin kullandığı örnekler ayrı ayrı incelendikten sonra, örneklerin diğer öğretmenler tarafından da kullanılıp kullanılmadığı belirlenerek kodlama işlemi yapılmıştır. Analiz sonrasında öğretmenlerin kullandıkları rasyonel sayı örnekleri şu fikirlere göre kategorilere ayrılmıştır: örneklerin genel özellikleri, öğretmenlerin örnek seçiminde benimsedikleri prensipler, öğretmenlerin örneklerinde bulunan matematiksel ve pedagojik yetersizlikler. Öğretmenlerin rasyonel sayı örneklerini ele alış biçimlerine yönelik sınıflandırma Tablo 3.3'te verilmiştir.

Tablo 3.3. Öğretmenlerin rasyonel sayı örneklerini ele alış biçimlerine ilişkin sınıflandırma

Tema	Alt temalar	Kategoriler
Matematiksel olarak doğru olan örnekler	Örnek türü	Özel örnekler
		Örnek olmayanlar
		Karşıt-örnekler
	Örneklerin kaynağı	Öğrenci ders kitabından planlı örnekler
		Öğrenci çalışma kitabından planlı örnekler
		Öğretmen kılavuz kitabından planlı örnekler
		Yardımcı kaynaklardan planlı örnekler
		Çevirim içi eğitim yazılımından planlı örnekler
		ÖSS, SBS sınav sorularının oluşturduğu planlı örnekler
		Anlık örnekler
	Öğretmenlerin dikkat ettikleri hususlar ya da benimsedikleri prensipler	Kolay ya da bilinen örneklerle öğretime başlama
		Öğrencilerin güçlüklerine/hatalarına/kavram yanılgılarına dikkat etme
		Örneklerin kritik özelliklerini ön plana çıkarma
		Yaygın olmayan (alışılmadık) örnekleri sınıf ortamına dâhil etme
		Gereksiz iş yükünü en aza indirme
		Sınavları göz önünde bulundurma
Matematiksel olarak doğru olmayan örnekler /pedagojik olarak uygun olmayan örnekler	Hata türü	Matematikse olarak yanlış olan örnekler
		Pedagojik olarak uygun olmayan dil ya da terminoloji bulunduran örnekler
		Pedagojik olarak kaçınılması gereken örnekler

Bu çalışmada ortaokul matematik öğretmenlerinin kullanmış oldukları incelemek amacıyla şu kavramsal çerçeveler kullanılmıştır: Marton ve Booth'un (1997) varyasyon teorisi, Zodik and Zaslavsky'nin (2008) öğretmenlerin ders esnasında seçtikleri ve oluşturdukları örnekleri açıklayan dinamik çerçevesi ve öğretmen adaylarının örnek seçimini ve kullanımını değerlendirmek amacıyla Rowland ve diğerlerinin (2005) geliştirdiği Dörtlü Bilgi Modeli kullanılmıştır.

3.5.1. Marton ve Booth'un (1997) varyasyon teorisi

Öğrenme neyin örnek olduğunun farkında olunmasıyla gerçekleşir (Marton ve Booth, 1997). Bu teorinin merkezinde öğrenmenin varyasyonun saptanmasıyla olduğu görüşü bulunmaktadır (Marton ve Trigwell, 2000). Bilişsel olarak bir kavrama ait örnek ancak bazı özelliklerinin değişebildiğinin bazılarının ise invaryant kaldığının kabul edilmesi neticesinde örnek olabilir (Mason, 2006).

Farklı kişiler bir örneğin farklı boyutlarından haberdar olabilirler (Goldenberg ve Mason, 2008). Örneğin, bir öğretmen bir örneğin farklı boyutlarından haberdar olabilir. Özel olarak, mesleğe yeni başlayan öğretmenler bir kavramın değişebilecek özelliklerinin tümü hakkında bilgi sahibi olmayabilirler. Ayrıca, bir birey herhangi bir örneğin farklı zamanlarda farklı boyutlarına odaklanabilir (Goldenberg ve Mason, 2008; Mason, 2006). Tüm bu faktörleri göz önünde bulundurarak Watson ve Mason (2005) olası varyasyon boyutları ve izin verilebilir değişim çeşitliliği kavramlarını ortaya atmıştır. Kısacası, 'olası varyasyon boyutları' kavramı farklı kişilerin değişme ihtimali olan farklı şeylerden haberdar olması ile ilgiliyken 'izin verilebilir değişim çeşitliliği' kavramı ise farklı kişilerin farklı zamanlarda değişen şeyin farklı aralıklarda olduğunu algılaması ile ilgilidir (Mason, 2011). Bu kavramlar öğrencilerin matematiksel bir nesnenin hangi özelliklerinin kritik olduğunu anlamalarına yardımcı olur (Goldenberg ve Mason, 2008). Bu iki parametre özellikle matematikte çok etkilidir çünkü bu kavramlar sayesinde öğrenciler matematiksel yapının farkına varırlar (Mason vd., 2009). Matematiksel yapı varyans/invaryans ile benzerlik/farklılık arasındaki ilişki ile aracılığı ile ortaya çıkar (Watson ve Shipman, 2008). Bu yapı deneyim edilen örneklerin kritik olan ve olmayan yönlerinin fark edilmesine yardımcı olur (Sun, 2011). Varyasyon teorisine göre, bir nesnenin belirli bazı kritik özelliklerinin bilinmesi öğrenciler açısından çok büyük önem taşır. Çünkü, bir nesneyi öğrenebilmek için öncelikle o nesnenin kritik özelliklerinin bilinmesi gerekmektedir (Guo vd., 2012). Belirli bir özelliğin öğrenilebilmesi için öğrencilerin ilgili boyuttaki varyasyonu deneyim edinmeleri gerekir ve bir özellik değişirken sabit kalan diğer özelliklerin öğrenciler tarafından kolayca fark edilebilmesi gerekir (Pang ve Marton, 2005).

3.5.2. Zodik ve Zaslavsky'nin (2008) öğretmenlerin ders esnasında seçtikleri ve oluşturdukları örnekleri açıklayan dinamik çerçevesi

Simon (1995) öğretmenlerin bilgisi, düşünmesi, karar vermesi ve sınıf etkinlikleri arasındaki ilişkiyi ortaya koymak amacıyla Matematik Öğretim Döngüsü adını verdiği bir model geliştirmiştir. Zodik ve Zaslavsky (2008) bu modeli kullanarak lise öğretmenlerinin matematik öğretimi sırasında seçtikleri ve kullandıkları örnekleri derinlemesine incelemeyi hedeflemiştir. Zodik ve Zaslavsky (2008) ayrıca öğretmenlerin örnek seçiminde dikkat ettikleri hususları veya benimsedikleri prensipleri belirlemeyi amaçlamışlardır. Bu amaçla Zodik ve Zaslavsky (2008) öğretmenlerin matematiksel bilgilerine, öğretim esnasında matematiksel bilgiyi ve ulaşılabilir kişisel örnekler uzayını nasıl kullandıklarına odaklanmışlardır. Zodik ve Zaslavsky (2008), Simon'un (1995) Matematik Öğretim Döngüsü modelinden hareketle öğretmenlerin ders esnasında seçtikleri ve oluşturdukları örnekleri açıklayan dinamik bir çerçeve ortaya atmıştır ve bu çerçeveye Matematiksel Örnek İlintili Öğretim Döngüsü adını vermiştir. Bu çerçeveye göre matematik öğretimi esnasında öğretmenler tarafından kullanılan örnekler öğretmen bilgisi, ders planlama ve ders ortamı şeklinde üç bileşen altında toplanmıştır. Bu bileşenler arası karşılıklı ilişkiler farklı oklarla gösterilmiştir. Örnekler uzayı ve ders kitapları öğretmenlerin örnek seçiminde kullandıkları başlıca kaynaklar arasındadır. Ayrıca, ders kitapları çoğunlukla ders planlama safhasında kullanılırken, örnekler uzayı hem ders planlama safhasında hem de ders ortamında öğretim yapılırken kullanılmaktadır. Öğretmenlerin örnek seçiminde ve kullanımında kendilerini yönlendiren birtakım prensipler ve hususlar vardır. Bu prensipler veya hususlar öğretmenlerin kişisel eğilimlerinden ve değerlendirmelerinden büyük oranda etkilenmektedir.

Bu çerçeveye göre öğretmenler ders planlama safhasında çoğunlukla örnek seçme veya örnek üretme ile meşgul olmaktadırlar. Ayrıca ders işleme safhası sınıf içi olaylar ve öğretmenlerin anlık eylemlerinden oluşmaktadır. Daha özel olarak, sınıf içi olaylar öğretmenlerin eylemlerini ve öğrencilerle olan etkileşimlerini içermektedir. Sınıf içi olaylar genelde öğretmenlerin anlık eylemlerde bulunmalarını ve o esnada gerekli olan uygun örnekleri temin etmelerini gerektirir. Zodik ve

Zaslavsky (2008) anlık örneklerin bazı öğretmenler tarafından anında üretildiğini ve bunun kolay ulaşılabilir örnekler uzayını belirttiğini ifade etmiştir. Öte yandan, bazı anlık örneklerin üretilmesi öğretmenlerin oldukça fazla zamanını almıştır ve bu türde örnekler öğretmenlerin uzak ulaşılabilir örnekler uzayına karşılık gelmiştir. Zodik ve Zaslavsky (2007), öğretmenlerin uzak ulaşılabilir örnekler uzayından yararlanarak örnek ürettikleri durumları öğrenme fırsatı olarak nitelendirmiştir. Kısacası, öğretmenler kendi öğretimlerinden öğrenirler ve özel olarak da örnek üretme ya da örnek seçme yoluyla öğrenme deneyimi elde ederler.

3.5.3. Rowland, Turner, Thwaites ve Huckstep'in (2005) dörtlü bilgi modeli

Rowland ve diğerleri (2005), Ball, Hill ve Bass (2005) tarafından geliştirilen ölçme araçlarının öğretmenlerin pedagojik alan bilgileri hakkında ipuçları verebileceğini fakat öğretmenlerin sınıf ortamında nasıl bir öğretim sergilediklerini yansıtmayacağını belirtmiştir. Rowland ve diğerleri (2005) öğretmenlerin nasıl bir öğretim sergilediklerini değerlendirebilmek için onları sınıf ortamında gözlemlenmelerinin gerektiğini belirtmiştir. Bu düşünceden yola çıkarak Rowland ve diğerleri (2005) Dörtlü Bilgi Modeli adını verdikleri bir kavramsal çerçeve geliştirmişlerdir. Dörtlü Bilgi Modeli dört birimden oluşmaktadır. Bunlar temel bilgi, dönüşüm bilgisi, ilişki kurma bilgisi ve beklenmeyen olaylar bilgisidir. Bu çalışmada Dörtlü Bilgi Modelinin dönüşüm birimine odaklanılmıştır. Çünkü bu birim öğretmenlerin matematik öğretiminde seçtiği ve kullandığı örnekleri analiz etmeye yardımcı olmaktadır. Daha özel olarak bu çalışmada dönüşüm birimi aracılığıyla ortaokul matematik öğretmenlerinin rasyonel sayıları öğretirken kullandıkları örneklerden matematiksel veya pedagojik açıdan sıkıntılı olanları belirlenmeye çalışılmıştır. Rowland'a (2008) göre öğretmenler kötü örneklerden iyi örneklere göre daha kolay öğrendiklerini belirtmiştir. Rowland ve diğerleri (2003) kavram veya kural öğretimi sırasında öğretmenlerin kaçınması gereken üç tür örnekten bahsetmiştir. Bunlar, değişkenlerin rolünü belirsizleştiren örnekler, daha uygun strateji kullanımını gerektiren örnekler ve zar atarak rasgele üretilen örnekler yerine dikkatli seçim yapmayı gerektiren örneklerdir. Bu araştırmada bu üç tür örneğin ortaokul matematik öğretmenleri tarafından kullanılıp kullanılmadığı da araştırılmıştır.

BULGULAR

Bu araştırmanın bulguları öğretmenlerin kullandıkları rasyonel sayı örneklerinin genel özellikleri, öğretmenlerin rasyonel sayı örneklerini kullanırken dikkat ettiği hususlar ve öğretmenlerin matematiksel olarak hatalı veya pedagojik olarak uygun olmayan rasyonel sayı örnekleri başlıkları altında üç ayrı bölümde incelenmiştir.

4.1. Öğretmenlerin Kullandıkları Rasyonel Sayı Örneklerinin Genel Özellikleri

Bu araştırmanın bulguları ortaokul matematik öğretmenlerinin rasyonel sayı kavramlarının öğretiminde özel örnekler, örnek olmayanlar ve karşıt örnekler şeklinde üç tür örnek kullandıklarını göstermiştir. Öğretmen A ve Öğretmen B öğrenci ders kitabına nazaran daha fazla özel örnek kullanırken Öğretmen C ve Öğretmen D öğrenci ders kitabına göre daha az özel örnek kullanmıştır. Ayrıca, ders kitabında rasyonel sayı işlemleriyle ilgili örnek sayısının rasyonel sayılarla ilgili bütün örneklerin yarısından fazla olduğu ortaya çıkmıştır. Benzer şekilde, öğretmenlerin de rasyonel sayı işlemlerine yönelik kullandıkları örnekler kullandıkları tüm rasyonel sayı örneklerinin yarısından fazla olmuştur. Öte yandan, hem öğretmenler hem de öğrenci ders kitabı rasyonel sayı problemlerinin kurulmasına ve çözülmesine yönelik çok az sayıda örneğe yer vermişlerdir.

Ders kitabında yer alan örnekler tarafından vurgulanan rasyonel sayı fikirlerini genelde öğretmenlerin kullandığı örnekler de vurgulanmıştır. Ayrıca, öğretmenler ders kitabındaki örnekler tarafından vurgulanan rasyonel sayı fikirleri dışında farklı fikirleri vurgulayan rasyonel sayı örnekleri de kullanmışlardır. Özel olarak, rasyonel sayıların açıklanması ve sayı doğrusu üzerinde gösterilmesiyle ilgili ders kitabında yer alan örnekler şu fikirleri vurgulamıştır: bir kesrin denklik sınıfının bulunması, denk kesirlerin sayı doğrusu üzerinde gösterilmesi, rasyonel sayıların pozitif veya negatif olma durumlarının belirlenmesi ve sayı doğrusu üzerinde yer alan bir noktaya karşılık gelen rasyonel sayının belirlenmesi. Tüm öğretmenler rasyonel sayıların sayı doğrusu üzerinde gösterilmesi ile ilgili örnekler kullanmışlardır. Fakat bu kazanımın öğretilmesinde kullanılan diğer rasyonel sayı fikirleri tüm öğretmenler tarafından vurgulanmamıştır. Ders kitabındaki örneklerin içerdiği fikirler dışında öğretmenler tarafından kullanılan örnekler şu fikirleri de içermiştir: negatif bir rasyonel sayıda eksi işaretinin konumunun incelenmesi, rasyonel sayıların sadeleştirilmesi, bileşik ve tam sayılı rasyonel sayıların birbirine çevrilmesi ve pozitif ve negatif rasyonel sayıların ihtiyaçlarının öğrencilere hissettirilmesi.

Ders kitabının rasyonel sayıların farklı biçimlerde gösterilmesine yönelik sunduğu örnekler şu rasyonel sayı fikirlerini içermiştir: tam sayıların rasyonel sayı olarak gösterilmesi, rasyonel sayıların tam sayı, devirli ondalık sayı ve devirsiz ondalık sayı olarak gösterilmesi, devirsiz ondalık sayıların rasyonel sayı olarak gösterilmesi ve devirli ondalık sayıların rasyonel hale getirilmesi. Öğretmenlerin rasyonel sayıların farklı biçimlerde gösterilmesine yönelik sundukları örneklerin içerdiği fikirler ders kitabındaki örneklerin içerdiği fikirlerle birebir örtüşmüştür. Fakat tüm öğretmenler devirli ondalık sayıların rasyonel hale getirilmesiyle ilgili örnekler kullanırken bu kazanıma yönelik diğer fikirleri içeren örneklerin tümünü kullanmamışlardır.

Ders kitabının rasyonel sayıların karşılaştırılması ve sıralanmasına yönelik sunduğu örnekler şu rasyonel sayı fikirlerini içermiştir: rasyonel sayıların sayı doğrusu üzerinde gösterilmesi, rasyonel sayının ondalık sayıya çevrilmesi, ortak payda algoritmasının kullanılması, ortak pay algoritmasının kullanılması, referans noktası kullanımı ve denk kesirler yardımıyla sıralama. Tüm öğretmenler ortak payda algoritması yardımıyla rasyonel sayıların sıralanmasına yönelik örnekler vermiştir. Fakat bu kazanımla ilgili ders kitabında bulunan diğer rasyonel sayı fikirlerine yönelik örneklere benzer örnekler tüm öğretmenler tarafından kullanılmamıştır. Mesela, hiçbir öğretmen denk kesirler yardımıyla rasyonel sayıların sıralanmasına yönelik örnek kullanmamıştır. Ayrıca, ders kitabında yer alan örneklerden farklı olarak öğretmenler şu fikirleri içeren rasyonel sayı örnekleri kullanmışlardır: artakalan miktarı düşünerek sıralama, ondalık sayıların virgülden sonraki kısımlarına 0 ekleyerek sıralama, rasyonel sayıların işaretlerini göz önünde bulundurarak karşılaştırma ve bileşik rasyonel sayıya çevirerek karşılaştırma. Bununla birlikte, yukarıda belirtilen ilk iki fikir sadece bir öğretmen tarafından vurgulanırken son iki fikir de başka bir öğretmen tarafından vurgulanmıştır.

Ders kitabının rasyonel sayıların toplanması ve çıkarılmasına yönelik sunduğu örnekler şu rasyonel sayı fikirlerini içermiştir: rasyonel sayılarda toplama-çıkarma işlemlerinin modellenmesi, paydaları aynı olan rasyonel sayıların toplanması-çıkarılması, rasyonel sayılarda toplama-çıkarma işlemlerinin sonucunun tahmin edilmesi, paydaları farklı rasyonel sayıların toplanması-çıkarılması ve rasyonel sayılarda toplama işleminin özellikleri. Tüm öğretmenler şu fikirlerle ilgili örnekleri kullanmışlardır: paydaları aynı olan rasyonel sayıların toplanması-çıkarılması paydaları farklı rasyonel sayıların toplanması-çıkarılması ve rasyonel sayılarda toplama işleminin özellikleri. Fakat bu kazanımla ilgili diğer fikirler tüm öğretmenler tarafından vurgulanmamıştır. Mesela, yalnızca bir öğretmen rasyonel sayılarda toplama-çıkarma işlemlerinin sonucunun tahmin edilmesiyle ilgili tek bir örnek kullanmıştır. Öğretmenler ders kitabında yer alan örneklerin dışında şu fikirleri içeren örnekler kullanmışlardır: rasyonel sayılarla çok adımlı işlemler yapma ve rasyonel sayıların ortak paydalarının bulunması. Fakat bu fikirler tüm öğretmenler tarafından vurgulanmamıştır. Örneğin, sadece bir öğretmen rasyonel sayıların ortak paydalarının bulunmasıyla ilgili örnekler çözmüştür.

Ders kitabının rasyonel sayıların çarpılması-bölünmesine yönelik sunduğu örnekler şu fikirleri içermiştir: rasyonel sayılarda çarpma işleminin modellenmesi, rasyonel sayıların çarpılması-bölünmesi, 0, 1 ve (-1) ile çarpma-bölme, rasyonel sayıların karesinin ve küpünün modellenmesi ve hesaplanması, rasyonel sayılarla çok adımlı işlemler ve rasyonel sayılarda çarpma işleminin özellikleri. Öğretmenlerin bu kazanım için kullanmış olduğu örnekler ders kitabında yer alan örneklerle örtüşmüştür ve öğretmen farklı fikirlerin yer aldığı örnekler kullanmamışlardır. Öte yandan, tüm öğretmenler rasyonel sayılarda çarpma-bölme işlemleri ile ilgili örnek kullanırken, diğer fikirleri içeren örnekler tüm öğretmenler tarafından kullanılmamıştır. Daha da önemlisi, hiçbir öğretmen rasyonel sayılarda çarpma-bölme işlemlerinin sonuçlarının tahmin edilmesiyle ilgili örneklerden yararlanmamıştır.

Ders kitabının rasyonel sayılarda çok adımlı işlemlerle ilgili sunduğu örnekler şu rasyonel sayı fikirlerini içermiştir: bileşenleri aynı satırda bulunan çok adımlı işlemler, merdiven biçiminde ifade edilmiş çok adımlı işlemler ve içerisinde örüntü bulunduran çok adımlı işlemler. Tüm öğretmenler ilk iki fikre yönelik örnek kullanırken üçüncü fikirle ilgili örnekleri tümü kullanılmamıştır. Son olarak, bir öğretmen ders kitabında yer alan çok adımlı işlem örneklerine ek olarak tek değişkenli polinom biçiminde ifade edilmiş rasyonel sayı örnekleri kullanmıştır.

Ders kitabının rasyonel sayılarda problem çözme ve kurma ile ilgili sunduğu örnekler şu rasyonel sayı fikirlerini içermiştir: aynı birim üzerinden işlem yapmayı gerektiren rasyonel sayı problemlerinin çözümü, farklı birimler üzerinden işlem yapmayı gerektiren rasyonel sayı problemlerinin çözümü ve rasyonel sayı problemlerinin kurulması. Tüm öğretmenler ilk iki fikre yönelik örnekler kullanırken hiçbir öğretmen rasyonel sayı problemlerinin kurulmasına yönelik örnek kullanmamıştır. Daha da önemlisi, ders kitabında yer alan ve öğretmenler tarafından kullanılan bu kazanımla ilgili örneklerin sayısı diğer rasyonel sayı kazanımları için sunulan veya kullanılan örnek sayısından çok daha az olmuştur.

Rasyonel sayıların öğretiminde kullanılan örnek olmayanlar incelendiğinde öğretmenlerin dört farklı türde örnek olmayan kullandığı ve ders kitabında rasyonel sayılarla ilgili herhangi bir örnek olmayana yer vermediği görülmüştür. Öğretmenler bir tam sayının sıfıra oranı biçiminde, aşkın sayı biçiminde, köklü sayı biçiminde ve devirsiz sonlu olmayan ondalık sayı biçiminde örnek olmayanlar kullanmıştır. Öğretmenler genelde bir tam sayının sıfıra oranı biçimindeki örnek olmayanları kullanmışlardır. Fakat bunlardan yalnızca devirsiz sonlu olmayan ondalık sayılar irrasyonel sayıları saydam olarak temsil ederler. Buna rağmen bu gösterim türü sadece bir öğretmen tarafından kullanılmıştır.

Bu araştırmada karşıt-örneklere ortaokul sınıf uygulamalarında çok fazla yer verilmediği görülmüştür. Ayrıca, bu araştırmada öğretmenlerin yalnızca beş karşıt-örnek kullandığı görülmüştür. Bu örneklerin hepsi öğrencilerin iddialarının yanlışlığını göstermek için üretilmiştir ve her biri beklenmedik olayları örneklemektedir.

Bu çalışmada örneklerin üretilme zamanlarına göre anlık ve planlanmış örnek şeklinde iki tür örnekten bahsedilebilir. Öğretmen A ve Öğretmen B'nin örneklerinin yarısından fazlası anlık olarak üretilirken Öğretmen C ve Öğretmen D tarafından üretilen örneklerin çoğu planlı örnek olmuştur. Öğretmenler tarafından kullanılan örnekler genel olarak değerlendirildiğinde yarıdan fazlasının anlık olarak üretildiği görülmüştür.

Öğretmenler planlı örnekleri seçerken çeşitli kaynaklara başvurmuşlardır. Bunlar öğrenci ders kitabı, öğrenci çalışma kitabı, öğretmen kılavuz kitabı, çıkmış sınav soruları, çevirim içi eğitim yazılımı ve yardımcı kaynaklar olmuştur. Genel olarak yardımcı kaynaklar alıştırma sorularının çözümünde kullanılmıştır. Yardımcı kaynak örnekleri çoktan seçmeli soru türünde olmuştur ve sınav sorularına benzer türde sorulardır. Öğretmen kılavuz kitabı örnekleri sadece bir öğretmen tarafından kullanılmıştır ve bunlar ders/çalışma kitabında yer almayan örneklerdir. Benzer şekilde, çıkmış sınav soruları ve çevirim içi eğitim yazılımı birer öğretmen tarafından kullanılmıştır. İki yardımcı kaynak iki öğretmen tarafından ortak olarak kullanılırken geriye kalan yardımcı kaynaklar farklı öğretmenler tarafından kullanılmıştır.

4.2. Öğretmenlerin Örnek Kullanırken Dikkat Ettiği Hususlar

Bu araştırmada öğretmenler rasyonel sayı örneklerini kullanırken şu hususlara dikkat etmişlerdir: kolay ve bilinen örneklerle öğretime başlama, öğrencilerin hatalarına/güçlüklerine/kavram yanılgılarına dikkat etme, gereksiz iş yükünü en aza indirme, sınavları göz önünde bulundurma, yaygın olmayan örneklere yer verme ve örneklerin kritik özelliklerini ön plana çıkarma.

Öğretmenler şu durumlarda kolay veya bilinen rasyonel sayı örnekleri ile öğretime başlamıştır: sıralama ve dört işlem yaparken rasyonel sayıların formatının göz önünde bulundurulması, sıralama ve toplama çıkarma yaparken rasyonel sayıların paydalarının göz önünde bulundurulması, devirli ondalık sayıda devreden ve devretmeyen kısımların göz önünde bulundurulması, rasyonel sayıların sıralanmasında terim sayısının göz önünde bulundurulması, rasyonel sayılarla dört işlem yaparken eleman sayısının göz önünde bulundurulması, çok adımlı işlemlerde adım sayısının göz önünde bulundurulması, rasyonel sayı problemlerinde matematiksel yapının değiştirilerek problemin giderek güçleştirilmesi ve son olarak rasyonel sayı kavramlarının öğretilmesinden önce bu kavramlarla ilgili ön bilgilerin hatırlatılması.

Öğretmenler rasyonel sayıların öğretiminde öğrencilerin güçlüklerine, hatalarına ve kavram yanılgılarına dikkat etmişlerdir. Öğretmenler öğrencilerin şu durumlarda güçlük çektiğini dile getirmişlerdir: rasyonel sayıda eksinin konumunun anlaşılması, negatif rasyonel sayılarda çıkarma işleminin yapılması, çok adımlı işlemlerde bilinmeyen değerli örneklerin çözülmesi, sıfırın bir tam sayıya ve bir tam sayının sıfıra bölünmesiyle elde edilen sonuçların ayırt edilmesi, çarpma işleminden önce sadeleştirme işleminin yapılması, negatif rasyonel sayıların parantez içine alınmadığı örneklerin çözümü, dağılma özelliğinin her zaman doğru sonuç verdiğini anlama ve üslü rasyonel sayılarda kuvvetin parantez içinde ve dışında olmasının sonucu nasıl etkilediğini anlama. Öğretmenler öğrencilerin şu durumlarda hata yaptıklarını dile getirmişlerdir: sıralamada büyüktür/küçüktür sembolü yerine virgül kullanma, negatif rasyonel sayılarla işlem yaparken parantezlerin ihmal edilmesi, tam sayılı rasyonel sayıları toplarken işaret hatalarının yapılması, bir rasyonel sayıyla bir tam sayıyı toplarken hata yapılması, rasyonel sayının toplamaya göre tersini bulurken hata yapılması, çok adımlı işlemlerde işlem önceliğine uyulmamasından dolayı hatalı sonuç bulma, bilinmeyen değerlerin bulunduğu üslü rasyonel sayılarda notasyon hatasının yapılması. Son olarak, öğretmenler öğrencilerin şu tip kavram yanılgılarına sahip olduklarını dile getirmişlerdir: rasyonel sayıları sayı doğrusunda gösterirken aralık yerine çentiklerin sayılması, negatif rasyonel sayıların pozitif rasyonel sayılarda olduğu gibi sayı doğrunda gösterilmesi, ondalık sayıları sıralarken sadece virgülden sonraki kısımların karşılaştırılması, rasyonel sayılarda toplama/çıkarma işlemlerinin çarpma/bölme gibi yapılması, sadeleştirme işleminin sadece çapraz olarak yapılacağının düşünülmesi, doğal sayılarda çarpma işleminin tam sayılı rasyonel sayılara yanlış uygulanması, üslü sayının değerini bulurken taban ve kuvvetin toplanması/çarpılması ve her zaman büyük bir sayının küçük bir sayıya bölünebileceği düşüncesi.

Öğretmenler gereksiz iş yükünü teknik iş yükünü azaltıp işin özüne odaklanarak, örneklerin yalnızca önemli kısımları üzerinde durarak ve işlem özelliklerini kullanarak en aza indirmişlerdir. Öğretmenler şu durumlarda teknik iş yükünü azaltıp işin özüne odaklanmıştır: sayı doğrusunun sadece gerekli kısmını çizme, devirli ondalık sayılara örnek verirken periyodu kısa olan rasyonel sayıların seçilmesi, sıralama yaparken uygun stratejinin seçilmesi, bir rasyonel sayıyla tam sayının toplanmasında/çıkarılmasında kısa yol tercih edilmesi, çıkarma işlemi yaparken payda eşitlemek yerine formül kullanma, birleşme özelliğini paydaları aynı olan rasyonel sayılarla anlatma, içinde bilinmeyen bulunan çok adımlı işlemleri geriye doğru gitme stratejisi ile çözme. Benzer şekilde, öğretmenler şu durumlarda örneklerin önemli kısımları üzerinde durmuş ve ekstra detaylara girmemiştir: bir örneğin önemli kısmını vurgulama ve işlemi tamamlamama, bir kavramı öğretirken sadeleştirmeyi ve sayı formlarını birbirine çevirmeyi gerekli görmeme ve bir toplama modelini sembolik olarak ifade ederken terimlerin paydalarının eşitlenmesini gerekli görmeme. Son olarak, öğretmenler şu durumlarda işlemlerin özelliklerini kullanmışlardır: toplama yerine toplama işleminin değişme veya birleşme özelliğini kullanma, işlem yapmak yerine dağılma özelliğini kullanma, uzun uzun işlem yapmak yerine $1/(a/b)=b/a,\ (a/b)/(a/b)=1$ ve $(a/b)+(-a/b)=0$ özelliklerini kullanma ve pay ve paydadaki ondalık sayıları rasyonel hale getirmek yerine bu sayıları virgülden kurtarma.

Öğretmenlerin rasyonel sayıları öğretirken dikkat ettiği bir diğer husus sınavları göz önünde bulundurmak olmuştur. Öğretmenler şu durumlarda sınavlarla ilgili düşüncelerini dile getirmişlerdir: yazılı/deneme/TEOG sınavlarında çıkabilecek örnekleri açıkça dile getirme, sınıfta çıkmış sınav soruları çözme, bulunan sonuçları en sade haline getirme, çoktan seçmeli soruların çözümlerini deneme yanılma yoluyla çözme ve sınavlarda hız kazanmak için kısa yollara başvurma.

Öğretmenler sınıflarına yaygın olmayan örnekler getirmeyi de göz önünde bulundurmuşlardır. Öğretmenler bu düşüncelerini istisnai/özel durumları ve az temsil edilen örnekleri sınıfa getirerek gerçekleştirmişlerdir. Öğretmenler istisnai/özel durumlar için şu örnekleri kullanmışlardır: bir rasyonel sayının 0'la çarpımı 0'dır, bir rasyonel sayının 1'le çarpımı yine o sayının kendisine eşittir, 0'ın 0'dan farklı bir rasyonel sayıya bölümü yine 0'dır, 1'in 0'dan farklı bir rasyonel sayıya bölümü o rasyonel sayının çarpmaya göre tersine eşittir, -1'in 0'dan farklı bir rasyonel sayıya bölümü o rasyonel sayının çarpmaya göre tersinin toplamaya göre tersine eşittir, 0'dan farklı bir rasyonel sayının sıfırıncı kuvveti 1'e eşittir, 1'in rasyonel üslü kuvvetleri yine 1'e eşittir. Öğretmenler az temsil edilen durumlar için şu örnekleri kullanmışlardır: 0'ın rasyonel olduğunu vurgulama, rasyonel sayılarda sıralamaya 0'ın dâhil edilmesi, ikiden fazla rasyonel sayının toplanması, çıkarılması, çarpılması, bölünmesi, karşılaştırma örneklerine eşit rasyonel sayıların dahil edilmesi ve rasyonel sayılarda işlem sonuçlarının tahmin edilmesine yönelik örnek kullanımı.

Son olarak, öğretmenler örneklerin kritik özelliklerini ön plana çıkarmayı göz önünde bulundurmuşlardır. Bu düşünceyi şu şekilde gerçekleştirmişlerdir: önce pozitif rasyonel sayının, sonra aynı sayının negatifinin sayı doğrusuna yerleştirilmesi ve iki rasyonel sayının konumunun karşılaştırılması, rasyonel sayıların büyüklüklerini sabit tutarak ve her seferinde birinin işaretini değiştirerek dört işlemin yapılması, parantez bulunmayan bir üslü rasyonel sayının hesaplanması, aynı sayının parantezli halinin hesaplanması ve sonuçların karşılaştırılması, aynı çok adımlı işlemin parantezli ve parantezsiz hallerinin hesaplanıp karşılaştırılması, devirli ondalık sayıların rasyonel hale getirilmesini anlatırken örüntünün kırılması, bir rasyonel sayı probleminin çözülmesi, aynı bağlam ve sayıların bulunduğu ikinci bir problemin çözülmesi ve her ikisinin karşılaştırılması.

4.3. Öğretmenlerin Hatalı veya Uygun Olmayan Örnekleri

Öğretmenler şu durumlarda matematiksel olarak hatalı örnekleri kullanmışlardır: irrasyonel sayılar sayı doğru doğrusunda gösterilemez, rasyonel sayı kümesi irrasyonel sayı kümesinin alt kümesidir, rasyonel sayı kümesi irrasyonel sayı kümesinden daha yoğundur, kesir formatındaki tüm sayılar rasyoneldir, mutlak değer kavramının yanlış uygulanmasından dolayı yanlış sonuca ulaşılması, rasyonel sayıları sayı doğrusunda gösterirken aralıkları eşit çizmeme, toplama işleminin birleşme özelliğini anlatırken değişme özelliğinin kullanılması, devirli ondalık sayıların rasyonel hale getirilmesinin yuvarlama ile aynı olduğunu düşünme ve sıralamayı doğru yapma fakat kullanılan stratejinin adını yanlış bilme.

Öğretmenlerin pedagojik olarak uygun olmayan örnekleri üç başlık altında toplanmıştır. Bunlar uygun olmayan dil ve terminoloji içeren örnekler, değişkenlerin rolünü belirsizleştiren örnekler ve başka bir strateji kullanımının daha uygun olduğu örneklerdir. Öğretmenler şu durumlarda uygun olmayan dil ve terminoloji içeren örnekler kullanmıştır: rasyonel sayı yerine kesir kelimesini kullanmak, rasyonel sayıların toplama ve çarpmaya göre tersini bulurken 'ters çevir', 'takla attır' gibi gündelik dil kullanımı, rasyonel sayıların yanlış okunması ve işlemlerde sembollerin yanlış kullanılması. Öğretmenler şu durumlarda değişkenlerin rolünü belirsizleştiren örnekler kullanmışlardır: devirli ondalık sayı kavramını öğretirken devreden ve devretmeyen basamakların rolünü belirsizleştirme, çıkarma işlemini modellerken çıkan ve farkın rolünü belirsizleştirme ve rasyonel sayıları sayı doğrusuna yerleştirirken aralık sayısının ve rasyonel sayının büyüklüğünün rolünü belirsizleştirme. Son olarak, öğretmenler şu durumlarda başka bir strateji kullanımının daha uygun olabileceği örnekler kullanmıştır: devirli ondalık sayıların rasyonel hale getirilmesinde kullanılan örneğin seçilen stratejiye uygun olmaması, rasyonel sayıların sıralarken uygun stratejinin kullanılmaması ve rasyonel sayılarda toplama işleminin öğretilmesinde uygun algoritmanın kullanılmaması.

TARTIŞMA

5.1. Kullanılan Örneklerin Genel Özellikleri

Öğretmenler tarafından kullanılan örnekler incelendiğinde rasyonel sayı öğretim deneyimi daha fazla olan öğretmenlerin öğrencilerine daha fazla sayıda ve daha çeşitli örnekler sundukları görülmüştür. Bu bulgu öğretmenlerin öğretim deneyim bilgisine ışık tutmaktadır. Kennedy'e (1987) göre bu bilgi türü teknik bir beceriyi değil, amaçlı eylemlerle kural ve içeriğe odaklanan pedagojik alan bilgisini temsil etmektedir.

Öğretmenler ve ders kitabı tarafından öğrencilere rasyonel sayılarla problem çözme ve kurmayla ilgili çok az örnek sunulmuştur. Bu, ortaokul matematik öğretim programının rasyonel sayı işlemlerine ve kurallarına daha fazla ağırlık verdiğini göstermektedir. Bu nedenle, ortaokul matematik öğretim programının öğretmenlerin problem çözmeye ve kurmaya daha fazla zaman ayıracak şekilde yeniden düzenlenmesi yerinde olacaktır.

Öğretmenler ders kitabında yer almayan örnekler de vermişlerdir. Bu türden örneklerin kullanımı bazı durumlarda öğrencilerin rasyonel sayıları anlamalarına yardımcı olmuş, bazı durumlarda ise olumsuz etkide bulunmuştur. Verilen sayının rasyonel olup olmadığıyla ilgili örnekler, öğrencilerin dikkatini eksi işaretinin konumuna çeken örnekler ve rasyonel sayıların artakalan miktar göz önünde bulundurularak sıralanmasına yönelik örnekler öğretmenleri ders kitabına göre avantajlı duruma getirmiştir. Öte yandan, ders kitabında yer alan bazı örnek türlerinin kullanılması öğrencilerin rasyonel sayıları yeterli düzeyde anlamamalarına neden olmuştur. Örneğin öğretmenler rasyonel sayılarda çarpma/bölme işlemlerinin tahmin edilmesine yönelik örnek kullanmamıştır. Benzer şekilde, rasyonel sayılarda problem kurmayla ilgili örnekler ders kitabında ve ortaokul matematik programında vurgulanmasına rağmen öğretmenler bu türden örnek kullanmamışlardır. NCTM (2000) problem kurmanın matematik yapabilmede merkezi rol oynadığını belirtmiştir. Bu nedenle matematik öğretmenlerinin rasyonel sayılarla problem kurmaya yönelik örnekleri sınıf ortamına dâhil etmeleri beklenmektedir.

5.2. Öğretmenlerin Örnek Seçiminde Dikkat Ettikleri Hususlar

Öğretmenler rasyonel sayıları öğretirken öncelikle kolay ve alışılmış örnekleri kullanmışlardır. Bu bulgu Zodik ve Zaslavsky'nin (2008) bulguları ile benzerlik göstermektedir. Benzer şekilde, Bills ve Bills (2005) öğrencilerin matematiksel kuralları anlayabilmeleri için deneyimli öğretmenlerin basit örneklerle öğretime başladıklarını belirtmiştir. Öğretmenler öğrencilerin yaygın hatalarına, güçlüklerine ve kavram yanılgılarına da dikkat çekmişlerdir. Bu husus, öğretmenlerin pedagojik alan bilgileri ile yakından ilişkilidir (Shulman, 1986). Öğrenci hatalarına odaklanma öğretmenlerin sadece konu alanı bilgisini değil aynı zamanda pedagojik alan bilgisini de yansıtmaktadır. Öğretmenler teknik iş yükünü azaltarak, örneklerin önemli kısımları üzerinde durarak ve işlem özelliklerini kullanarak gereksiz iş yükünü en aza indirmişlerdir. Bu hususa Zodik ve Zaslavsky'nin (2008) çalışmasında yer alan öğretmenler de dikkate etmişlerdir. Öğretmenler istisnai/özel/yaygın olmayan örnekler kullanarak alışılmadık örnekleri sınıfa dâhil etmeyi göz önünde bulundurmuşlardır. Bu türden hususlar ortaokul matematik öğretimi programının rasyonel sayıların öğretilmesinde vurguladığı özel durumlarla açıklanabilir. Öğretmenlerin dikkat ettiği bir diğer husus örneklerin kritik özelliklerine dikkat çekme olmuştur. Skemp (1971) örneklerin taşıdığı kritik olmayan bilgiye 'gürültü' adını vermiştir. Bu çalışmada öğretmenler örüntü kırma stratejisini ve yapılandırılmış varyasyon prensibi kullanarak örneklerdeki gürültüyü azaltmışlardır. Son olarak, öğretmenler rasyonel sayıları öğretirken sınavlarda çıkan örnekleri göz önünde bulundurmuşlardır. Bu husus Türk eğitim sistemine özgü olabilir. Çünkü Türkiye'de kaliteli eğitim verilen liselerde öğrenim görebilmek için öğrencilerin TEOG sınavlarında başarılı olmaları gerekmektedir.

5.3. Hatalı veya Pedagojik Olarak Uygun Olmayan Örnekler

Öğretmenler üç tür zayıf örnek kullanmıştır: hatalı örnekler, uygun olmayan dil ve terminoloji içeren örnekler ve kaçınılması gereken örnekler. Gerçekte var olmayan sayı doğrularının çizimiyle ilgili örnekler ve irrasyonel sayılarla ilgili bazı örnekler öğretmenlerin kullandığı başlıca hatalı örnekler arasında yer almıştır. Bu türden örneklerin kullanımı öğretmenlerin irrasyonel sayılarla ilgili alan bilgilerinin yetersizliğine işaret edebilir.

Pedagojik olarak uygun olmayan dil ve terminoloji içeren örnekler öğretmen-öğrenci iletişiminde güçlüklere neden olabilir (Lamon, 2012). Ayrıca, matematiksel kavramların öğretilmesinde günlük hayat dilinin kullanılması kavramsal öğrenmeye olumsuz etkide bulunabilir (Cangelosi vd., 2013). Özellikle, rasyonel sayıların toplama/çarpma işlemlerine göre tersi öğretilirken öğretmenler uygun matematiksel dil ve terminoloji kullanmaya özen göstermelidirler.

Son olarak öğretmenler iki tür kaçınılması gereken örnek kullanmışlardır: değişkenlerin rolünü belirsizleştiren örnekler ve daha uygun strateji kullanımı gerektiren örnekler. Rowland (2008) öğretmen adaylarının bilinçli örnek seçebilmeleri için yönlendirilmeleri gerektiğini dile getirmiştir. Benzer şekilde bu çalışmadaki öğretmenlerin örneklerin olası tehlikeleriyle ilgili bilinçlendirilmeleri gerekmektedir.

APPENDIX H

CURRICULUM VITAE

PERSONAL INFORMATION

Surname, Name: Avcu, Ramazan

Nationality: Turkish (TC)

Date and Place of Birth: 16 July 1984, Konya-Seydişehir

Marital Status: Married

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EDUCATION

Degree	Institution	Year of Graduation
Doctor of Philosophy	Elementary Education, Department of Elementary Education, Middle East Technical University, Ankara.	2014
Master of Science	Elementary Mathematics Education Programme, Institute of Education, Selçuk University, Konya.	2010
Bachelor of Science	Elementary Mathematics Education, Department of Elementary Education, Selçuk University, Konya.	2008

WORKING EXPERIENCE

Year	Place	Enrollment
2008-2009	Köprü Elementary School, Isparta-Şarkikarağaç	Mathematics Teacher
2009-2014	Department of Elementary Education, Faculty of Education, Aksaray University	Research Assistant

PUBLICATIONS

Articles published in International Journals

Avcu, R. & Avcu, S. (In press). Turkish adaptation of Utley geometry attitude scale: A validity and reliability study. *Eurasian Journal of Educational Research.*

Avcu, R. & Doğan, M. (2014). Yedinci sınıf öğrencileri orantısal akıl yürütme gerektiren problemlerin çözümünde hangi stratejileri kullanıyorlar? [What are the strategies used by seventh grade students while solving proportional reasoning problems?] *International Journal of Educational Studies in Mathematics, 1*(2), 34-55.

Avcu, R. (2010). Eight graders' capabilities in exponents: Making mental comparisons. *Practice and Theory in Systems of Education, 5*(1), 39-48.

Presentations at the International Professional Meetings

Avcu, R. & Avcu, S. (2014). Reflecting on the teaching of integer addition and subtraction via knowledge quartet: The case of Mr. Tunç. European Conference on Educational Research (ECER), University of Porto, Porto, Portugal.

Avcu, R. & Avcu, S. (2013).Exploring prospective primary teachers' understanding of proportions through double number lines. International Symposium on Changes and New Trends in Education, Konya, Turkey.

Avcu, R. & Avcu, S. (2013). Exploring middle school students' conceptual knowledge of proportions. European Conference on Educational Research (ECER), Bahçeşehir University, İstanbul, Turkey.

Avcu, R. (2011). An investigation into prospective teachers' attitudes towards computer assisted instruction. World Conference on Educational Technology Researches, Nicosia-Cyprus.

Tunç, T. & Avcu, R. (2010). Turkish teacher candidates' graphing skills. 3rd International Conference for Theory and Practice in Education Intercultural Communication, Multicultural Education, Békéscsaba.

Şar, A. H., Avcu, R., & Işıklar, A. (2010). Analyzing undergraduate students' self confidence levels in terms of some variables. Procedia-Social and Behavioral Sciences, 5, 1205-1209.

Avcu, R. & Avcu, S. (2010). 6th grade students' use of different strategies in solving ratio and proportion problems. Procedia-Social and Behavioral Sciences, 9, 1277-1281.

Avcu, S. & Avcu, R. (2010). Pre-service elementary mathematics teachers' use of strategies in mathematical problem solving. Procedia-Social and Behavioral Sciences, 9, 1282-1286.

Ünlü, M., Avcu, S., & Avcu, R. (2010). The relationship between geometry attitudes and self-efficacy beliefs towards geometry. Procedia-Social and Behavioral Sciences, 9, 1325-1329.

Tunç, T. & Avcu, R. (2010). Use of measurement and assessment techniques in Turkish sixth grade science and technology textbooks and workbooks. 14th IOSTE Symposium.

Presentations at the National Professional Meetings

Avcu, R. & Avcu, S. (2014). Bir matematik öğretmeninin tam sayılarda toplama ve çıkarma işlemlerinin öğretimine yönelik dönüşüm bilgisinin incelenmesi. XI. Ulusal Fen Bilimleri ve Matematik Eğitimi Kongresi, Adana.

Avcu, R. & Avcu, S. (2012). Utley Geometri Tutum Ölçeğinin Türkçe Uyarlaması: Geçerlik ve Güvenirlik Çalışması. X. Ulusal Fen Bilimleri ve Matematik Eğitimi Kongresi, Niğde.

Avcu, R. & Avcu, S. (2011). Grafiksel gösterimlerle orantısal durumların ilişkilendirilmesi: Bir durum çalışması. 20. Ulusal Eğitim Bilimleri Kurultayı, Burdur.

Avcu, R. & Doğan, M. (2010). İlköğretim 7. sınıf öğrencilerinin oran-orantı problemlerinde kullandıkları çözüm stratejilerinin belirlenmesi. IX. Ulusal Fen Bilimleri ve Matematik Eğitimi Kongresi, İzmir.

Ünlü, M., Alanyalı, S., & Avcu, R. (2010). Üniversite öğrencilerinin geometriye yönelik öz yeterliklerinin belirlenmesi. IX. Ulusal Fen Bilimleri ve Matematik Eğitimi Kongresi, İzmir.

APPENDIX I

TEZ FOTOKOPİSİ İZİN FORMU

ENSTİTÜ

Fen Bilimleri Enstitüsü

Sosyal Bilimler Enstitüsü

Uygulamalı Matematik Enstitüsü

Enformatik Enstitüsü

Deniz Bilimleri Enstitüsü

YAZARIN

Soyadı: Avcu
Adı: Ramazan
Bölümü: İlköğretim

TEZİN ADI (İngilizce) : Exploring middle school mathematics teachers' treatment of rational number examples in their classrooms: A multiple case study

TEZİN TÜRÜ: Yüksek Lisans Doktora

1. Tezimin tamamından kaynak gösterilmek şartıyla fotokopi alınabilir.
2. Tezimin içindekiler sayfası, özet, indeks sayfalarından ve/veya bir bölümünden kaynak gösterilmek şartıyla fotokopi alınabilir.
3. Tezimden bir (1) yıl süreyle fotokopi alınamaz.

TEZİN KÜTÜPHANEYE TESLİM TARİHİ: