MODELLING AND IMPLEMENTATION OF LOCAL VOLATILITY SURFACES

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ABSTRACT

MODELLING AND IMPLEMENTATION OF LOCAL VOLATILITY SURFACES

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In this thesis, Dupire local volatility model is studied in details as a means of modeling the volatility structure of a financial asset. In this respect, several forms of local volatility equations have been derived: Dupire's local volatility, local volatility as conditional expectation, and local volatility as a function of implied volatility. We have proven the main results of local volatility model discussed in the literature in details. In addition, we have also proven the local volatility model under stochastic differential equation of the forward price dynamics of asset prices. Consequently, we have studied the two main approaches to obtaining the local volatility surfaces: parametric methods and non-parametric methods. For the parametric method, we have used Dumas parametrization for the implied volatility function which produces implied volatility surface, which in turn is used in obtaining local volatility surface. While in the non-parametric approach for local volatility surfaces, we have used both implied volatilities and option prices data sets with some numerical techniques that are well-founded in literature. As an outlook, we have also discussed several paths this thesis could take for future studies, one of which is using Tikhonov regularization to obtain solutions of local volatilities by solving a regularized Dupire equation.

Keywords : Dumas parametrization, Dupire local volatility model, implied volatility, local volatility surface, parametric method, non-parametric method, Tikhonov regularization

ÖZ

YEREL VOLATILITE YÜZEYLERININ MODELLENMESI VE UYGULANMASI

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Bu tezde, finansal varlıkların volatilite yapılarının modellenmesi amacıyla Dupire yerel volatilite modeli detaylı olarak çalışılmıştır. Bu yüzden, birçok yerel volatilite denklemi incelenmiş ve türetilmiştir: Dupire yerel volatilite, koşullu beklenen değer olarak yerel volatilite, zımni dalgalanma fonksiyonu. Literatürde detaylı olarak incelenen yerel volatilite modelinin ana sonuçları ispatlanmıştır. Ayrıca varlık fiyatlarının gelecek fiyat (forward price) dinamiklerinin stokastik diferansiyel denklemleri çerçevesinde yerel volatilite modeli kanıtlanmıştır. Sonuç olarak yerel volatilite yüzeylerini elde etmek için parametrik ve parametrik olmayan yöntemler çalışılmıştır. Parametrik yöntemlerde, yerel volatilite yüzeyi elde etmek için kullanılan ve zımni dalgalanma yüzeyi meydana getiren zımni dalgalanma fonksiyonu için Dumas parametrizasyonu kullanılmıştır. Yerel volatilite yüzeyleri için parametrik olmayan yöntemlerde, literatürde sağlam temelleri olan bazı sayısal teknikler ile birlikte zımni dalgalanma fonksiyonları ve opsiyon fiyatları ile ilgili veriler kullanılmıştır. Genel olarak, bu tezin gerçekleştirilebileceği değişik yollar ileriki çalışmalar için tartışılmıştır. Bunlardan biri yerel volatilite çözümleri elde etmek için Dupire denklemi çözülerek elde edilen Tikhonov düzenlemesidir.

Anahtar Kelimeler: Dumas parametrizasyonu, Dupire yerel volatilite modeli, zımni dalgalanma fonksiyonu, yerel volatilite yüzeyi, parametrik yöntem, parametrik olmayan yöntem, Tikhonov düzenlemesi

To My Family

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CHAPTER 1

INTRODUCTION

The financial crisis that caused many economies to go into recession took place in 1987, and ever since continual joint efforts are being made by the practitioners, who are active on the financial market, and the academic researchers, to understand the effective usage of financial instruments including derivatives. In fact, in 1994, Dupire as well as Derman and Kani, and Rubinstein, independently, contributed to the foundation of modeling volatility structure of financial assets by local volatility model. Dupire, Derman and Kani gave proofs based on stochastic calculus while Rubinstein approached it using implied volatility tree. As Gatheral J. duly noted, it was unlikely at the time that Dupire and others ever thought of local volatility as representing a model on how volatilities actually evolve, rather they were likely to view it as a way of finding an average over all instantaneous volatilities [18]. In addition, the financial sectors have increased their level of resources to help improve the pricing of financial instruments and considerate amount of research have been done and still ongoing to develop models that can replicate the market prices accurately.

Specifically, exotic options are being priced with the use of models such as local volatility models, stochastic volatility models as well as local stochastic volatility models and jump-diffusion models. Since this project is mainly concerned with the measure and estimation of volatility parameter, it becomes incumbent to give brief details on it. Volatility of a financial asset is the measure of the market risk as a result of the spread of the outcome of the returns of the asset, which is taken to be a random variable. This volatility which is a form of the risk inherent to the asset is very important when valuating financial derivatives (resp. options) and hence, it becomes necessary to accurately measure it in order to avoid discrepancies between the theoretical prices obtained from the models used and market option prices.

This thesis aims at dealing with local volatility surfaces that exhibit the “smile” effect that are consistent with the market’s underlying asset’s volatility structure. As will be shown and explained later on, Black-Scholes model which assumes a constant volatility in its option valuation does not provide a framework that explains the volatility skew (or sneer) nor does it explain the term structure of volatility that are observed in the market since the financial crash. As a result, there is a need for more realistic models that produce prices that can accurately represent market prices. Consequently, this project aims at presenting the tools of local volatility model and its local volatility surfaces to demonstrate a better performance in exhibiting the “smile effect”.

In the sequel, in Chapter 1, the motivation behind the use of local volatility model will be addressed by examining its crucial model assumptions and Black-Scholes model. Also, some crucial terminologies that help to grasp the tenets of this model will be explained as well as some other concepts and terminologies in financial mathematics that are essential for comprehending this thesis. In Chapter 2, the various theoretical derivations of local volatility will be deeply dealt with, essentially, Dupire local volatility, local volatility as a function of implied volatility, and local volatility as a conditional expectation. Chapter 3 explores various numerical methods that are used in literature to obtain local volatility surfaces: the parametric method and non-parametric methods. The motivation and explanations of novel numerical techniques we have proposed under the above methods will be given. Consequently, the analysis and interpretations of the local volatility surfaces obtained will be discussed in connection with its financial implications. Finally, in Chapter 4, we will explain the foreseeable future work of this thesis, essentially, how to improve the numerical techniques implemented in this thesis.

1.1 Preliminaries

In this part, the mathematical and financial structures that drive the main results as well as some important terminologies that are used in financial mathematics will be explained. The organization of this part begins with explanation of financial terminologies in Section 1.1.1, then in Section 1.1.2, mathematical framework that would help the reader understand the mathematics behind the model derivations will be discussed.

1.1.1 Financial Terminologies

According to Chance M., a *financial derivative* is a contract between two parties providing for a payoff from one party to the other determined by the price of an asset, an exchange rate, a commodity price or an interest rate [5]. In other words, the value of the derivative depends on the value of some other financial asset (underlying asset). Derivatives are used as a way to offset undesirable financial risk an individual is exposed to in the market. As we see above, a derivative depends on the performance of some other financial asset, called the underlying asset, whose value is a random variable. The meaning of an underlying (asset) could be anything from stock, bonds, interest rate to another derivative like forward contracts, futures and it could even be the weather in Ankara, which is not a tradeable asset. There are several types of financial derivatives in the market, one of which is an option. An option is a contract between two parties –a buyer and a seller– that gives the buyer the right but not the obligation, to purchase or sell something at a later date at a price agreed upon today [6]. A call option gives the buyer the right to buy while a put option gives the right to sell. The simplest form of options is the European type which do not allow one to exercise until maturity. However, the American type of option allows the buyer to exercise at anytime between the initiation of the contract till the option's maturity. The above two are referred to as Vanilla options.

Other types of options also exist such as the Bermudan option and the Exotic option. Bermudan options are types of options that can be exercised at some specific points in time from the onset of the contract till maturity while Exotic options are family of options with non-standard payoff structure and exercise possibilities which are different from those of standardized European and American options. These derivatives (resp. options) are traded on a large part in over-the-counter (OTC) market as well as in an organized foreign exchange (FTX) market.

1.1.2 Mathematical Background

In this part, the mathematical tools that will enable easy understanding of the model derivations and basic stochastic calculus concepts will be comprehensively explained. In general, the concepts in Mathematical Finance are defined over a probability space $(\Omega, \mathcal{F}, \mathbb{P})$; Ω denotes the total sample space, $\mathcal{F}_t$ denotes the σ-algebra of all the information known up till time t, and $\mathbb{P}$ denotes the objective probability measure, which guides the probability of events happening in this probability space where the domain of $\mathbb{P}$ is $\mathcal{F}$.

A real valued random variable is defined as a function that assigns real values to the set of outcomes of a probabilistic experiment. Its future values are unknown, hence, they are also referred to as stochastic variables. In addition, the collection X, of $\mathcal{F}_t$-measurable random variables, $\{X_t; 0 \leq t \leq T\}$, is a stochastic process. Further more, a stochastic process X is adapted to the filtration $\{\mathcal{F}_t\}_{0 \leq t \leq T}$ if it behaves such that for every realization X_t is $\mathcal{F}_t$-measurable.

To give an example for practical purpose; let us consider the tossing of a fair coin. The sample space denotes by $\Omega = \{\text{head, tail}\}$. A real-valued random variable $Y(w)$ defined on this space such that it models a payoff of \$1 if a head shows up and \$0, if a tail shows up.

$$Y(\omega) = \begin{cases} 1 & \text{if } \omega = \text{head} \\ 0 & \text{if } \omega = \text{tail} \end{cases}$$

Since the coin is fair, its probability distribution function f_Y is given by

$$f_Y(y) = \begin{cases} \frac{1}{2} & \text{if } y = 1 \\ \frac{1}{2} & \text{if } y = 0 \end{cases}$$

that is, there is equal probability of a head and tail showing up with each toss.

Another important concept is the conditional expectation of a random variable in a given probability space. A conditional expectation is the expected value of a random variable given an amount of information. To give a more mathematical definition, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\mathcal{G}$ be a sub-σ-algebra contained in $\mathcal{F}$. We denote $\mathcal{L}_2(\Omega, \mathcal{G}, \mathbb{P})$ to be the subspace of $\mathcal{L}_2(\Omega, \mathcal{F}, \mathbb{P})$ of equivalent class of $\mathcal{G}$-measurable random variables under the Hilbert space. We identify $\mathbb{P}_{\mathcal{G}}$ to be the restriction of $\mathbb{P}$ to $\mathcal{G}$. Hence,

Definition 1.1 (see [25]). The conditional expectation of a random variable $X \in \mathcal{L}_2(\Omega, \mathcal{F}, \mathbb{P})$ with respect to $\mathcal{G}$ is the projection of X onto $\mathcal{L}_2(\Omega, \mathcal{G}, \mathbb{P})$ or rather any $\mathcal{G}$-measurable random variable belonging to the $\mathbb{P}_{\mathcal{G}}$-equivalent class of this projection. Hence,

$$\forall G \in \mathcal{G}, \quad \int_G \mathbb{E}[X|\mathcal{G}] d\mathbb{P} = \int_G X d\mathbb{P}.$$

Another concept that is very important in studying financial instrument is the *Martingale property*. As in the previous definition, suppose $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space with filtration $\{F_t\}_{t\geq 0}$ under $\mathcal{F}$, then a stochastic process, X_t, under $(\Omega, \mathcal{F}, \mathbb{P})$ is a martingale if,

(*a*) X_t is adapted to the filtration $\{F_t\}_{t\geq 0}$;

(*b*) $\mathbb{E}[|X|] < \infty$;

(*c*) $\mathbb{E}[X_t|\mathcal{F}_s] = X_s$ for $0 \leq s \leq t < \infty$.

It is worth noting that condition (*c*) is typical of martingale processes, because it says the best predictor of the process X_t after time s, given that we have the whole information of X up to time s is X_s. Also note that the condition in (*b*) imposes the integrability of the process. Other related concept worth mentioning are the concepts of *Submartingales* and *Supermartingales*. The definitions are same with the martingale property except in condition (*c*) where *submartingales* satisfies $\mathbb{E}[X_t|\mathcal{F}_s] \geq X_s$ and *supermartingales* satisfies $\mathbb{E}[X_t|\mathcal{F}_s] \leq X_s$.

Note that if X_t is a martingale then each continuous time step is a martingale and we can write:

1. $\mathbb{E}[X_0] = \mathbb{E}[X_1] = \cdots = \mathbb{E}[X_t]$;
2. The sum of two martingales is a martingale.

Another important aspect of financial mathematics is to describe the valuation of options. Hence, in a complete market (that is every contingent claim is attainable by a self-financing portfolio), the value V_t of an option at time t, with payoff function $\phi(S_T)$ at time T, is defined to be the expectation of $\phi(S_T)$ under the risk-neutral probability measure $\mathbb{P}^*$, discounted at the risk-free rate to time t, conditional on the information known upto time t: namely, $V_t = \mathbb{E}^* \left[e^{-\int_t^T r_s ds} \phi(S_T) \mid \mathcal{F}_t \right]$.

Valuation of an option depends on the dynamics of the asset price process under which the option is written. Stochastic differential equations (SDEs) describe the evolution of stochastic processes and they are usually of the form:

$$dX_t = a(t, X_t)dt + b(t, X_t)dW_t, \tag{1.1}$$

with $a(t, X_t)$ and $b(t, X_t)$ $\mathcal{F}_t-$measurable. W_t is standard Brownian motion (also known as a Wiener process). This is a random process that describes a motion beginning at $W_0 = 0$. In each time period $t_2 - t_1$, its increment, $W_{t_2} - W_{t_1}$, is independent of everything that happened before t_1 (stationarity property), and its values are normally distributed with mean 0 and variance $t_2 - t_1$, that is, $W_{t_2} - W_{t_1} \sim \mathcal{N}(0, t_2 - t_1)$.

Another important concept in financial mathematics, is the *Itô formula*, which plays a very important role in stochastic calculus. If a stochastic process X_t follows an SDE of the form in Equation (1.1), then a function $f \in \mathcal{C}^{1,2}$ of this process X_t and time t is described by the *Itô's formula* (refer to Appendix A for details). Asset prices S_t are generally assumed to follow the dynamics in Equation (1.1) under a geometric Brownian motion W_t, and that is why the *Itô formula* plays such an essential role in option pricing.

We now have the necessary tools to highlight another important concept that describes the price of an option, V, over time t, the Black-Scholes partial differential equation (BS-PDE). The PDE arises from the fact that discounted option prices are martingales and subsequent application of *Itô formula* on the discounted prices. The following theorem summarizes the above idea.

Theorem 1.1. *If $X = h(S_T)$ is the terminal payoff of an option, then there exist a function $V : [0, T] \times (0, \infty) \to \mathbb{R}$ such that*

$$V_t = V(t, S_t),$$

which is a solution to the Black-Scholes partial differential equation

$$\frac{\partial V(t,s)}{\partial t} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 V(t,s)}{\partial s^2} + rs\frac{\partial V(t,s)}{\partial s} - rV(t,s) = 0 \tag{1.2}$$

with the terminal condition

$$V(T, s) = h(s)$$

for any $s > 0$ and $t \in [0, T]$. Here, V is the option price, s is the asset price, and r is the risk-free rate.

Proof. The proof to the theorem can be found in introductory stochastic calculus textbooks, for instance [27]. □

1.2 Black-Scholes and Local Volatility Models

In this part, Black-Scholes model as well as local volatility model will be discussed in details. Black-Scholes model will be presented in Section 1.2.1 with its basic assumptions and the extent at which these assumptions hold in reality. In Section 1.2.2, the basic terminologies that help us understand local volatility model will be defined.

1.2.1 Black-Scholes Model

The model has its origin traced back to the original paper "The Pricing of Option and Corporate Liabilities" in the 1973 written by Fischer Black and Myron Scholes.

Robert Merton more or less independently derived the same equation with its current mathematical structure. The first method used by both Black and Scholes to derive the option pricing formula was through a financial approach by the "Capital Asset Pricing model (CAPM)", a well accepted model in finance and the other approach was via stochastic calculus [5]. This model has some main assumptions which are enumerated below.

1. Short selling of the underlying stock is allowed.
2. Trading stocks is continuous.
3. Arbitrage opportunities are not allowed.
4. No transaction costs or taxes.
5. All securities are perfectly divisible (meaning you can sell or buy a fraction).
6. It is possible to borrow or lend cash at a constant risk-free rate.
7. The stock does not pay dividend.
8. Stock prices are random and log-normally distributed.
9. Volatility of the log return on the stock is constant over time.
10. The options are of European type.

Let the price of a stock S be driven by a geometric Brownian motion $(B_t)_{(t\geq 0)}$ under the probability measure $\mathbb{P}$ and follows the stochastic differential equation (SDE):

$$dS_t = \mu_t S_t dt + \sigma S_t dB_t. \tag{1.3}$$

To be able to price an option under the Black-Scholes model, a risk neutral world is needed, hence a risk-neutral probability measure denoted by $\mathbb{P}^*$ is entailed. Firstly, using *Girsanov theorem* (see Appendix A), an SDE driving the stock price is derived with risk-free return driving the interest rate given by

$$dS_t = r_t S_t dt + \sigma S_t dW_t, \tag{1.4}$$

where W_t is a Wiener process under the risk-neutral probability measure $\mathbb{P}^*$. Hence, with the above assumptions, the following theorem holds:

Theorem 1.2 (Black-Scholes Call Price)**.** *Let the asset price S follow the SDE in Equation (1.4) with $\sigma = \sigma_{BS}$ and $r_t = r$, then the price of a call option written on this asset at time t is given by*

$$C_{BS}(t, S_t; T, K, \sigma_{BS}, r) = S_t N(d_1) - Ke^{-r(T-t)}N(d_2) \tag{1.5}$$

with $d_1 = \frac{\ln(\frac{S_t}{K}) + \frac{\sigma_{BS}^2 T}{2}}{\sigma_{BS}\sqrt{T}}$ *and* $d_2 = d_1 - \sigma_{BS}\sqrt{T}$

where S_t is the spot price at time t, K is the strike price, σ_{BS} is annualized volatility of the continuously compounded (log) return on the stock, and r is the continuously compounded risk-free interest rate.

To calibrate the option price of the Black-Scholes model to the market prices, a set of parameters have to be specified in the model to give a theoretical price that conforms to the market. However, all the parameters in Black-Scholes model can be directly observed in the market except the volatility, σ. There are two different approaches to determine the volatility parameter. One approach is to carry out a time series analysis of historical data of asset prices and the other is to invert the Black-Scholes formula for the unique volatility. The volatility in the former approach is called historical volatility while it is called implied volatility in the latter. A problem lies in modeling the volatility structure of financial asset under the Black-Scholes framework because it assumes the volatilities specified in the model have to be the same for options written on the same asset, with the same maturities but different strike prices. This assumption has been falsified with the market data as it can often be found that options written on the same asset, although, with the same maturities, have different volatilities for different strike prices. Indeed, the volatility inherent in pricing these options depend on both strike price and maturity of the options. Practitioners term this dependence as the market phenomenon. The dependence of volatility on strike is called *volatility skew* while on maturity is called *term structure of volatility*. In the next section, this will be elaborated on.

1.2.2 Local Volatility Model

The motivation behind the idea of local volatility is to find a less computationally engaging model unlike stochastic volatility, with more simplifying assumptions such that the market prices are consistent with the prices within the Black-Scholes framework [18]. To better understand why this is necessary for adequate pricing of exotic options, the next section provides explanation for volatility skew and how it affects pricing of options.

Volatility Skew

As explained earlier, the concept of volatility skew (or sneer) started to develop after the 1987 financial crisis. The idea was to build a model that will be able to explain this phenomenon since the Black-Scholes model fails to address this. If the Black-Scholes model was an exact description of reality, it would suggest that every option price written on an underlying asset price, $(S_t)_{(t\geq 0)}$ follows a different price dynamics, which is arguably not the case. It would also suggest that when using binomial trees to build an option price process, a different tree would be needed each time for the asset price process for different options written on the asset. Over the years, different models have been proposed to explain the volatility skew among them are: stochastic volatility, jump diffusion, and local volatility models.

Stochastic volatility was first introduced by Hull and White [24]. The idea behind this is that volatility also has a source of randomness of its own called the volatility of volatility. The most famous stochastic volatility models are Heston model [23] and SABR model [20]. How these models cause the volatility skew is explained in [12, 20].

Another idea was to introduce processes with jumps which emanated from the work of Merton [28]. The basic principle behind this is the assumption that the underlying asset undergoes a jump-diffusion process which is better representation of reality. Sometimes we have sudden price movement due to some events like insider information in practice. How this model performs better in reality is being described in [2]. Lastly, local volatility model also tries to account for the volatility skew by assuming volatility is a deterministic function of time, t, and asset price S_t. Consistent option prices with the market can be built if the asset price follows an SDE of the following nature:

$$dS_t = \mu_t S_t dt + \sigma(t, S_t) S_t dW_t,$$

where $\mu_t = r_t - q_t$, r_t is the continuously compounded interest rate, and q_t is the dividend yield. The above SDE is called the asset price dynamics under local volatility and the risk-neutral probability measure $\mathbb{P}^*$.

The Local Volatility

As mentioned earlier, volatility measures the level of risk inherent in a financial asset, and it is one of the parameters needed in the valuation of option prices. Local volatility is perceived to be an average of all future instantaneous volatilities of the asset from the onset of the option till maturity, provided that the option ends at-the-money (see [4, 18]). As Dupire rightly recognized in his original paper, one of the motivations in developing this method was because local volatility model stays close to Black-Scholes model as option valuation is still under the completion of the market which ensures unique prices [14]. As a result, it gives an advantage over more complex models like stochastic volatility and jump-diffusion models that introduce new sources of randomness into the dynamics of asset prices which are not traded in the market. However, critics have also pointed out the downsides of the model as being too close to Black-Scholes model with some relaxations into the assumption of the volatility [3].

1.3 Numerical Methods for Pricing under Local Volatility

In the literature as well as in practice, several numerical methods have been described for pricing options under local volatility. These methods try to reconcile the option prices obtained theoretically from local volatility model with the observed market prices with the aim of minimizing the difference in measure of those two option prices over a range of strikes and maturities. Depending on the intended purpose, several approaches have been devised in estimating volatility parameter. One approach used frequently is to predict the future movement of asset prices from past behaviors to project expected future volatilities (see [13, 26]). Although this method could be useful, it could be misleading. Another approach is to calibrate the prices of the option model to the market prices of the actively traded derivative securities (options) since the option prices contain information about expected future volatilities [13]. However, the latter is more computationally intensive and could be handled using *Finite Differ-*

ence Method (FDM). For instance, it can account for *Local Volatility Function* (LVF) by adjusting the algorithm [4].

In this thesis, two disparate numerical methods of obtaining local volatility surfaces will be rigorously explored: parametric and non-parametric methods. They volatility surfaces from the methods will also be compared with one anoother in terms of their performance and accuracy in measuring volatilities.

CHAPTER 2

DERIVATION OF LOCAL VOLATILITY MODEL

To be able to price options with local volatility model, an appropriate method to device the volatility function must be determined. In Section 2.1, the derivation of local variance in terms of partial derivatives of actively traded call option prices with respect to strike K and maturity T will be done. In other words, the Dupire's local volatility equation will be derived. Local variance as a conditional expectation will be derived in Section 2.2. For convenient use of Dupire's equation in our numerical computation of local volatility later on, in Section 2.3, Dupire's local volatility function in terms of forward price will be derived. In Section 2.4, the challenges encountered while using Dupire's equation will be explained. Sometimes in practice, the Black-Scholes implied volatilities of the option prices are quoted directly, therefore, its necessary to derive local volatility function in terms of implied volatilities. In Section 2.5, local volatility will be derived as a function of Black-Scholes implied volatility.

2.1 Dupire Local Volatility Model

A stepwise approach in deriving local volatility equation will be used and with this approach, a build up problem solving process will be carried out.

Consider a local volatility model, in which the risky asset price S_t satisfies

$$dS_t = \mu(t)S_t dt + \sigma(t, S_t)S_t dB_t, \tag{2.1}$$

where $B = (B_t)_{t\geq 0}$ is a standard Brownian motion, defined on $(\Omega, \mathcal{A}, \mathbb{P})$, $\mu : \mathbb{R}^+ \to \mathbb{R}$ is a (deterministic) continuous function and $\sigma : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ is a continuous function such that $\forall t \geq 0$, $\forall (x, y) \in \mathbb{R}^2$,

$$|x\sigma(t, x) - y\sigma(t, y)| \leq M\,|x - y|\,,$$

and $\forall (t, x) \in \mathbb{R}_+ \times \mathbb{R}$, $\sigma(t, x) \geq m$, where m and M are positive constants. For simplicity, we assume that the interest rate is null. The natural filtration of $(B_t)_{t\geq 0}$ is denoted by $F = (F_t)_{t\geq 0}$.

Theorem 2.1. *Let S be the dynamics of the asset defined in Equation (2.1) and $C(T, K)$*

be the European call option with strike K and maturity T. Then, we have

$$\sigma^2(T,K) = 2\frac{\frac{\partial C(T,K)}{\partial T}}{K^2\frac{\partial^2 C(T,K)}{\partial K^2}}, \tag{2.2}$$

for $0 < T < \bar{T}$ and $K > 0$.

Remark 2.1. Equation (2.2) is termed as the Dupire local variance equation. Taking the square-root of both sides gives the local volatility.

Proof. The proof to the theorem is long, thus, some parts of the proof will be stated as lemma for easy following.

Step 1. First, show that for every $x \in \mathbb{R}$, Equation (2.1) has a unique solution such that $S_0 = x$:

1. First, is to show that the SDE satisfies the Lipschitz condition. That is,

$$|b(t,x) - b(t,y)| + |\sigma(t,x) - \sigma(t,y)| \leq K\,|x-y|\,,$$

where $K \in \mathbb{R}$ and finite. This leads to

$$|x\mu(t) - y\mu(t)| + |x\sigma(t,x) - y\sigma(t,y)| \leq |\mu(t)|\,|x-y| + M\,|x-y|\,,$$

where $|x\sigma(t,x) - y\sigma(t,y)| \leq M\,|x-y|\,, \quad \forall t \geq 0$ and $\forall (x,y) \in \mathbb{R}^2$. Since $\mu(t)$ is deterministic, let $|\mu(t)| = M_1 < \infty$. Then

$$\begin{aligned} |x\mu(t) - y\mu(t)| + |x\sigma(t,x) - y\sigma(t,y)| &\leq |M_1|\,|x-y| + M\,|x-y| \\ |x\mu(t) - y\mu(t)| + |x\sigma(t,x) - y\sigma(t,y)| &\leq (M_1 + M)\,|x-y| \\ |x\mu(t) - y\mu(t)| + |x\sigma(t,x) - y\sigma(t,y)| &\leq K\,|x-y|\,, \end{aligned} \tag{2.3}$$

where $K = M_1 + M < \infty$. Hence, the Lipschitz condition is satisfied.

2. Next, show that the SDE satisfies polynomial growth condition, that is,

$$|b(t,x)| + |\sigma(t,x)| \leq K(1 + |x|).$$

That is to show that

$$|x\mu(t)| + |x\sigma(t,x)| \leq |x|\,|\mu(t)| + |x\sigma(t,x)| \leq K(1+|x|). \tag{2.4}$$

Note that:

$$\begin{aligned} |x\sigma(t,x) - y\sigma(t,y)| &\leq M\,|x-y|\,, \\ y = 0 \text{ implies } \; |x\sigma(t,x)| &\leq M\,|x|\,. \end{aligned}$$

Substituting this back into Equation (2.4),

$$|x\mu(t)| + |x\sigma(t,x)| \leq |x|\,(M_1 + M),$$

where $M_1 = |\mu(t)|$. This implies that

$$|x\mu(t)| + |x\sigma(t,x)| \leq K\,|x| \leq K(1 + |x|),$$

where the last inequality is true for $K = M + M_1 \in \mathbb{R}^+$. Therefore, the polynomial growth condition is satisfied.

3. And lastly, check if the SDE satisfies $\mathbb{E}[S_0^2] < \infty$ where S_0 is the initial value. In our case, $S_0 = x$ hence, $\mathbb{E}[x^2] < \infty$ is trivially satisfied.

Thus, we conclude that Equation (2.1) has a unique solution.

Step 2. Secondly, show that if S is a solution of Equation (2.1), for $t \geq 0$, then

$$S_t = S_0 \exp\left(\int_0^t \mu(s)ds + \int_0^t \sigma(u, S_u)dB_u - \frac{1}{2}\int_0^t \sigma^2(u, S_u)du\right).$$

To show this, apply the *Itô formula* (see Appendix A) to $f(x) = \ln x$ with $f'(x) = \frac{1}{x}$ and $f''(x) = -\frac{1}{x^2}$. This leads to

$$\begin{aligned}
\ln S_t &= \ln S_0 + \int_0^t \frac{1}{S_s}\left(\mu(s)S_s ds + \sigma(s, S_s)S_s dB_s\right) \\
&\quad - \frac{1}{2}\int_0^t \frac{1}{S_s^2}\sigma^2(s, S_s)S_s^2 ds \\
\ln S_t &= \ln S_0 + \int_0^t \mu(s)ds + \int_0^t \sigma(s, S_s)dB_s - \frac{1}{2}\int_0^t \sigma^2(s, S_s)ds \qquad (2.5)\\
\ln\left(S_t/S_0\right) &= \int_0^t \mu(s)ds + \int_0^t \sigma(s, S_s)dB_s - \frac{1}{2}\int_0^t \sigma^2(s, S_s)ds \\
S_t &= S_0 \exp\left(\int_0^t \mu(s)ds + \int_0^t \sigma(s, S_s)dB_s - \frac{1}{2}\int_0^t \sigma^2(s, S_s)ds\right)
\end{aligned}$$

as desired.

Step 3. Thirdly, show that the natural filtration of the process $(S_t)_{t\geq 0}$ is equal to F (the natural filtration of $(B_t)_{t\geq 0}$).

To show this, write B_t as a stochastic integral with respect to the process $(S_t)_{t\geq 0}$. Let $(F_t)_{(t\geq 0)}$ and $(F'_t)_{(t\geq 0)}$ be the filtration of B_t and S_t respectively. From Equation (2.1), S_t is measurable with respect to the filtration of B_t, that is, $S_t \in F_t$. It remains to show that $B_t \in F'_t$.

From Equation (2.1), we have

$$\sigma(t, S_t)S_t dB_t = \mu(t)S_t dt - dS_t \qquad (2.6)$$

and thence,

$$\begin{aligned}
dB_t &= \frac{1}{\sigma(t, S_t)S_t}\left(\mu(t)S_t dt - dS_t\right) \\
&= \frac{\mu(t)dt}{\sigma(t, S_t)} - \frac{dS_t}{\sigma(t, S_t)S_t} \\
\int_0^t dB_s &= \int_0^t \frac{\mu(s)ds}{\sigma(s, S_s)} - \int_0^t \frac{dS_s}{\sigma(s, S_s)S_s} \qquad (2.7)\\
B_t - B_0 &= \int_0^t \frac{\mu(s)ds}{\sigma(s, S_s)} - \int_0^t \frac{dS_s}{\sigma(s, S_s)S_s}.
\end{aligned}$$

The right hand side is F'_t-measurable, so is the left hand side. Therefore, $B_t \in F'_t$, and the conclusion follows that $F_t = F'_t$.

Lemma 2.2. *Let L be the martingale defined by $L_t = \exp\left(-\int_0^t \theta_u dB_u - \frac{1}{2}\int_0^t \theta_u^2 du\right)$, with $\theta_t = \frac{\mu(t)}{\sigma(t,S_t)}$. Fix the horizon $\bar{T}$ of the model $(0 < \bar{T} < \infty)$ and let $\mathbb{P}^*$ be the probability given by $d\mathbb{P}^*/d\mathbb{P} = L_T$. Given $T \in [0, \bar{T}]$, let $C(T,K)$ be the price of a call option with maturity T and strike price K. The following equations hold:*

1. *$\sigma(t,x) \leq M$ for $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$ and*
$$\mathbb{E}^*[S_t^p] \leq S_0^p \exp\left(\frac{p^2 - p}{2} M^2 t\right)$$
for $p \geq 1$, $0 \leq t \leq \bar{T}$.
2. *$T \longmapsto C(T,K)$ is non-decreasing on $[0,\bar{T}]$ for $K \geq 0$ and $(T,K) \longmapsto C(T,K)$ is continuous on $[0,\bar{T}] \times \mathbb{R}^+$.*
3. *$\mathbb{E}^*[(S_T - K)_+]^2 = 2\int_K^{+\infty} C(T,y)dy$.*

Proof. The proof of the lemma will be shown step-wise.

1. First, show that $\sigma(t,x) \leq M$ for $\sigma(t,x) \in \mathbb{R}_+ \times \mathbb{R}$.
From
$$|x\sigma(t,x) - y\sigma(t,y)| \leq M\,|x - y|\,, \quad \forall t \geq 0,\ \sigma(x,y) \in \mathbb{R}^2,$$
we take $y = 0$, to get
$$\begin{aligned} |x\sigma(t,x)| &\leq M\,|x| \quad \forall x \in \mathbb{R} \\ |x\sigma(t,x)| &= |x|\,|\sigma(t,x)| \leq M\,|x| \\ |\sigma(t,x)| &\leq M, \quad \forall \sigma(t,x) \in \mathbb{R}_+ \times \mathbb{R}. \end{aligned} \tag{2.8}$$
Now apply *Itô formula* to $f(x) = \ln x^p$ with $f'(x) = \frac{p}{x}$ and $f''(x) = -\frac{p}{x^2}$ to attain
$$\begin{aligned} \ln S_t^p &= \ln S_0^p + \int_0^t \frac{p}{S_s}\left(\sigma(s,S_s)S_s dB_s\right) - \frac{1}{2}\int_0^t \frac{p}{S_s^2}\sigma^2(s,S_s)S_s^2 ds \\ \ln S_t^p &= \ln S_0^p + \int_0^t p\sigma(s,S_s)dB_s - \frac{1}{2}\int_0^t p\sigma^2(s,S_s)ds \\ \ln\left(\frac{S_t^p}{S_0^p}\right) &= \int_0^t p\sigma(s,S_s)dB_s - \frac{1}{2}\int_0^t p\sigma^2(s,S_s)ds \\ \frac{S_t^p}{S_0^p} &= \exp\left(\int_0^t p(s,S_s)dB_s - \frac{1}{2}\int_0^t p\sigma^2(s,S_s)ds\right) \\ S_t^p &= S_0^p \exp\left(\int_0^t p\sigma(s,S_s)dB_s - \frac{1}{2}\int_0^t p\sigma^2(s,S_s)ds\right). \end{aligned} \tag{2.9}$$

Using the properties of exponential functions in combination with $\sigma(s, S_s) \leq M$, we have

$$\exp\left(\int_0^t p\sigma(s, S_s)dB_s\right) \leq \exp\left(pM(\int_0^t dB_s)\right),$$

and

$$\exp\left(-\frac{1}{2}\int_0^t p\sigma^2(s, S_s)ds\right) \leq \exp\left(-\frac{1}{2}pM^2\int_0^t ds\right). \tag{2.10}$$

Substitute Equation (2.10) into Equation (2.9) to get

$$\begin{aligned} S_t^p &\leq S_0^p \exp\left[\left(pM\int_0^t dB_s\right) - \left(\frac{1}{2}pM^2\int_0^t ds\right)\right] \\ S_t^p &\leq S_0^p \exp\left[pMB_t - \frac{1}{2}pM^2 t\right] \\ \mathbb{E}^*[S_t^p] &\leq \mathbb{E}^*\left[S_0^p \exp\left(pMB_t - \frac{1}{2}pM^2 t\right)\right] \\ \mathbb{E}^*[S_t^p] &\leq S_0^p\mathbb{E}^*\left[\exp\left(pMB_t - \frac{1}{2}pM^2 t\right)\right]. \end{aligned} \tag{2.11}$$

Since $\frac{1}{2}pM^2t$ is a deterministic function of time, it follows that

$$\mathbb{E}^*[S_t^p] \leq S_0^p \exp\left(-\frac{1}{2}pM^2t\right)\mathbb{E}^*\left[\exp\left(pMB_t\right)\right]. \tag{2.12}$$

Using the moment generating function of a Brownian motion,

$$\mathbb{E}^*\left[\exp(pMB_t)\right] = \exp\left(\frac{p^2M^2t}{2}\right),$$

which leads the inequality in Equation (2.12) to

$$\begin{aligned} \mathbb{E}^*[S_t^p] &\leq S_0^p \exp\left(-(\frac{1}{2}pM^2t)\right)\exp\left(\frac{p^2M^2t}{2}\right) \\ \mathbb{E}^*[S_t^p] &\leq S_0^p \exp\left(\frac{p(p-1)}{2}M^2t\right). \end{aligned} \tag{2.13}$$

2. Let a function f be defined as follows; $f : T \longmapsto C(T, K)$. Then, $f(T_1) \leq f(T_2)$.

   Let $\mathcal{F}_{T_1}$ be the σ-algebra of information up to time T_1. $\tilde{S}$ is a martingale under $\mathbb{P}^*$. Then,

$$\begin{aligned} \mathbb{E}^*\left[\tilde{S}_{T_2} \mid F_1\right] &= \tilde{S}_{T_1} \\ \mathbb{E}^*\left[S_{T_2} \mid F_1\right] &= e^{r(T_2-T_1)}S_{T_1} \geq S_{T_1}, \end{aligned} \tag{2.14}$$

since $T_2 - T_1 > 0$ implies $e^{r(T_2 - T_1)} \geq 1$ with equality when $r = 0$. So,

$$\begin{aligned}
\mathbb{E}^* [S_{T_2} \mid F_1] - K &\geq S_{T_1} - K \\
\mathbb{E}^* [S_{T_2} - K \mid F_1] &\geq S_{T_1} - K \\
\mathbb{E}^* [\mathbb{E}^* [S_{T_2} - K \mid F_1]] &\geq \mathbb{E}^* [S_{T_1} - K] .
\end{aligned} \tag{2.15}$$

By *Tower property* of conditional expectation, we have

$$\begin{aligned}
\mathbb{E}^* [S_{T_2} - K] &\geq \mathbb{E}^* [S_{T_1} - K] \\
\mathbb{E}^* [S_{T_2} - K] &\geq \mathbb{E}^* [S_{T_1} - K] \geq 0 \\
\mathbb{E}^* [S_{T_2} - K]_+ &\geq \mathbb{E}^* [S_{T_1} - K]_+
\end{aligned} \tag{2.16}$$

with equality when $r = 0$. Therefore, $f(T_1) \geq f(T_2)$.

Lemma 2.3. *If $g : (T, K) \longmapsto C(T, K)$ is a mapping of $(T, K) \in \mathbb{R}^+ \times \mathbb{R}^+$ into $C(T, K) \in \mathbb{R}^+$, then g is a continuous function.*

Proof. Show that function g is continuous by ascertaining that it satisfies the three conditions of continuity:

(a) $g(T, K) = C(T, K) = \mathbb{E}^*[S_T - K]_+$ is well defined in domain $g \in \mathbb{R}$, since $\max(0, S_T - K) < \infty$ implies $\mathbb{E}^*[S_T - K]_+ < \infty$.

(b) $\lim_{T \to T_1} g(T, K) = \lim_{T \to T_1} \mathbb{E}^*[S_T - K]_+ = \mathbb{E}^*[S_{T_1} - K]_+$, since S is a continuous variable on $T \in [0, \bar{T}]$.

(c) $\lim_{T \to T_1} g(T, K) = g(T_1, K) = \mathbb{E}^* [S_{T_1} - K]_+$ as consequence of (a) and (b) being true.

Therefore, g is continuous. □

3. Lastly, show that $\mathbb{E}^*[(S_T - K)_+]^2 = 2 \int_K^{+\infty} C(T, y) dy$. To prove this, we know that

$$\mathbb{E}^* \left[(S_T - K)_+^2 \right] = \int_{-\infty}^{\infty} (x - K)_+^2 p(x) dx,$$

where $p(x)$ is the density of S_T. Therefore,

$$\begin{aligned}
\frac{\partial}{\partial K} \mathbb{E}^* \left[(S_T - K)_+^2 \right] &= - \int_{-\infty}^{\infty} 2(x - K)_+ p(x) dx \\
&= -2 \int_{-\infty}^{\infty} (x - K)_+ p(x) dx \\
&= -2 \mathbb{E}^* [(S_T - K)_+] \\
&= -2 C(T, K).
\end{aligned} \tag{2.17}$$

Taking the integral of both sides with respect to K, we get

$$\begin{aligned}
\int_K^{\infty} \frac{\partial}{\partial K} \mathbb{E}^* \left[(S_T - K)_+^2 \right] &= \int_K^{\infty} -2C(T, y) dy \\
\mathbb{E}^* \left[(S_T - \infty)_+^2 \right] - \mathbb{E}^* \left[(S_T - K)_+^2 \right] &= -2 \int_K^{\infty} C(T, y) dy \\
\mathbb{E}^* \left[(S_T - K)_+^2 \right] &= 2 \int_K^{\infty} C(T, y) dy.
\end{aligned} \tag{2.18}$$

Note that $\mathbb{E}^* \left[(S_T - \infty)_+^2\right] = 0$, since the call option will end out-of-the-money when $K \to \infty$.

The proof is completed. □

Lemma 2.4. *Let $f_0(x) = (x_+)^2$ and, for $\epsilon > 0$,*

$$f_\epsilon(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{x}{3\epsilon} & \text{if } x \in [0, \epsilon] \\ x^2 - \epsilon x + \frac{\epsilon^2}{3} & \text{if } x > \epsilon \end{cases}$$

The following hold:

1. *f_ϵ is of class $\mathcal{C}^2$ and $\lim\limits_{\epsilon \mapsto 0} f_\epsilon(x) = f_0(x)$ for every $x \in \mathbb{R}$;*
2. *$0 \leq f_\epsilon(x) \leq f_0(x),\ 0 \leq f_\epsilon{}'(x) \leq 2x,\ 0 \leq f_\epsilon{}''(x) \leq 2$ for every $\forall x \geq 0$ and $\epsilon > 0$;*
3. *$\mathbb{E}^* \left[f_\epsilon(S_T - K)\right] = f_\epsilon(S_0 - K) + \frac{1}{2}\mathbb{E}^* \left[\int\limits_0^T f_\epsilon{}''(S_u - K) S_u^2 \sigma^2(u, S_u) du\right]$ for $K \geq 0$ and $T \in [0, \bar{T}]$.*

Proof. 1. First we obtain that $f_\epsilon \in \mathcal{C}^2\ \ \forall \epsilon > 0$: Taking the 1st and 2nd derivatives of $f_\epsilon(x)$, we obtain

$$f_\epsilon{}'(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{3x^2}{3\epsilon} = \frac{x^2}{\epsilon} & \text{if } x \in [0, \epsilon] \\ 2x - \epsilon & \text{if } x > \epsilon \end{cases}$$

$$f_\epsilon{}''(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{2x}{\epsilon} & \text{if } x \in [0, \epsilon] \\ 2 & \text{if } x > \epsilon \end{cases}$$

Since $f_\epsilon{}''(x)$ exists $\forall x \in \mathbb{R}$, $f_\epsilon \in \mathcal{C}^2$.

Next, we will show that $\lim_{\epsilon \to 0} f_\epsilon(x) = f_0(x)$. To do this, consider a number of cases as shown below:

Case 1 $(x < 0)$:

$\lim_{\epsilon \to 0} f_\epsilon(x) = 0$

Case 2 $(x \in [0, \epsilon))$:

Observe that as $\epsilon \to 0$ and $x \to 0$, x acts like an ϵ. Hence,

$\lim_{\epsilon \to 0} f_\epsilon(x) = \lim_{\epsilon \to 0} \frac{x^3}{3\epsilon} = \lim_{\epsilon \to 0} \frac{\epsilon^2}{3} = 0$

Case 3 $(x > \epsilon)$:

$\lim_{\epsilon \to 0} f_\epsilon(x) = \lim_{\epsilon \to 0} \left(x^2 - \epsilon x + \frac{\epsilon^2}{3}\right) = x^2$

Therefore,

$$\lim_{\epsilon \to 0} f_\epsilon(x) = f_0(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ x^2 & \text{if } x > 0 \end{cases}$$

2. Next, show that $\forall x \geq 0$, we have $0 \leq f_\epsilon(x) \leq f_0(x)$:

   It is trivial that $f_\epsilon(x) \geq 0$, since

$$f_\epsilon(x) = \begin{cases} 0 & \\ \frac{x}{3\epsilon} \geq 0 & \text{since } x \geq 0, \epsilon > 0 \\ x^2 - \epsilon x + \frac{\epsilon^2}{3} > 0 & \text{since } x^2 > \epsilon x \end{cases}$$

   It remains to show that $f_\epsilon(x) \leq f_0(x)$. To show this, compare the terms in both functions.

$$\frac{x^3}{3\epsilon} \leq x^2 \text{ implies } x \leq 3\epsilon \text{ implies } \frac{x}{3} \leq \epsilon.$$

   Similarly,

$$x^2 - \epsilon x + \frac{\epsilon^2}{3} < x^2 \text{ implies } \frac{\epsilon^2}{3} < \epsilon x.$$

   Thence, we have $0 \leq f_\epsilon(x) \leq f_0(x)$.

   Next, show that $0 \leq f_\epsilon{}'(x) \leq 2x$:

   It is trivial that $f_\epsilon{}'(x) \geq 0$, since

$$f_\epsilon{}'(x) = \begin{cases} 0 & \\ \frac{x^2}{\epsilon} \geq 0 & \text{since } x \geq 0, \epsilon > 0 \\ 2x - \epsilon > 0 & \text{since } x > \epsilon \text{ implies} 2x > \epsilon x \end{cases}$$

   It therefore remains to show that $f_\epsilon{}'(x) \leq 2x$. This can be shown however by comparing again the terms in both functions.

$$\frac{x^2}{\epsilon} \leq 2x \text{ implies } x \ \leq 2\epsilon$$

   which is true since $x \leq \epsilon$ implies $x \leq 2\epsilon$. Also,

$$2x - \epsilon < 2x \text{ implies } \epsilon > 0.$$

   Therefore, $0 \leq f_\epsilon{}'(x) \leq 2x$.

   Finally, we show that $0 \leq f_\epsilon{}''(x) \leq 2$.

   Clearly $f_\epsilon{}''(x) \geq 0$, since

$$f_\epsilon{}''(x) = \begin{cases} 0 & \\ \frac{2x}{\epsilon} & \text{since } x \geq 0, \epsilon > 0 \\ 2 & \end{cases}$$

   It remains to show that $f_\epsilon{}''(x) \leq 2$, but this results from

$$\frac{2x}{\epsilon} \leq 2 \text{ implies } 2x \leq 2\epsilon \text{ implies } x \leq \epsilon.$$

   Hence, conclude that $0 \leq f_\epsilon{}''(x) \leq 2$.

3. By *Itô formula*, let $X_t = S_t - K$ and $dX_t = dS_t$.

$$f_\epsilon(S_t - K) = f_\epsilon(S_0 - K) + \int_0^t f_\epsilon{}'(S_s - K)dS_s + \frac{1}{2}\int_0^t f_\epsilon{}''(S_s - K)S_s^2\sigma^2(u, S_u)du. \tag{2.19}$$

$$\mathbb{E}^*\left[f_\epsilon(S_t - K)\right] = \mathbb{E}^*\left[f_\epsilon(S_0 - K)\right] + \mathbb{E}^*\left[\int_0^t f_\epsilon{}'(S_s - K)dS_s + \frac{1}{2}\int_0^t f_\epsilon{}''(S_s - K)S_s^2\sigma^2(u, S_u)du\right]. \tag{2.20}$$

Since $(S_0 - K)$ is constant, $f_\epsilon(S_0 - K)$ is constant, hence,

$$\mathbb{E}^*\left[f_\epsilon(S_0 - K)\right] = f_\epsilon(S_0 - K).$$

Also, S is a martingale under $\mathbb{P}^*$ which means $\left(\int_0^t f_\epsilon{}'(S_s - K)dS_s\right)$ has constant expectation by the property of a martingale. Thus,

$$\mathbb{E}^*\left[\int_0^t f_\epsilon{}'(S_s - K)dS_s\right] = \mathbb{E}^*\left[\int_0^0 f_\epsilon{}'(S_s - K)dS_s\right] = 0. \tag{2.21}$$

Therefore, Equation (2.20) becomes

$$\mathbb{E}^*\left[f_\epsilon(S_t - K)\right] = \mathbb{E}^*\left[f_\epsilon(S_0 - K)\right] + \frac{1}{2}\mathbb{E}^*\left[\int_0^t f_\epsilon{}''(S_s - K)S_s^2\sigma^2(u, S_u)du\right]. \tag{2.22}$$

Hence, the proof is completed. □

Step 4. Let $p(t, \cdot)$ be the density function of the random variable S_T under $\mathbb{P}^*$.

1. We will show that $p(T, K) = \frac{\partial^2 C(T,K)}{\partial K^2}$:

   Knowing that

$$\begin{aligned} C(T, K) &= \mathbb{E}^*[S_T - K]_+ \\ &= \mathbb{E}^*\left[(S_T - K)\mathbb{1}_{(S_T > K)}\right] \\ &= \int_K^{+\infty} (S_T - K)p(T, S)dS; \end{aligned} \tag{2.23}$$

   we take the partial derivative of both sides with respect to K to obtain

$$\frac{\partial C}{\partial K} = \frac{\partial}{\partial K}\int_K^{+\infty} (S_T - K)p(T, S)dS. \tag{2.24}$$

By *Fubini theorem*, we have

$$\begin{aligned}\frac{\partial C}{\partial K} &= \int_K^{+\infty} \frac{\partial}{\partial K}(S_T - K)p(T,S)dS \\ &= -\int_K^{+\infty} p(T,S)dS.\end{aligned} \tag{2.25}$$

Taking the second partial derivative with respect to K now gives

$$\begin{aligned}\frac{\partial^2 C}{\partial K^2} &= -\frac{\partial}{\partial K}(\int_K^{+\infty} p(T,S)dS) = -\left(p(T,S) \mid_{S=K}^{S=+\infty}\right) \\ &= -(0 - p(T,K)) = p(T,K) \\ \frac{\partial^2 C}{\partial K^2} &= p(T,K).\end{aligned} \tag{2.26}$$

For simplicity, we have assumed that $\lim_{S\to\infty} p(T,S) = 0$.

2. Next, we will show that

$$\mathbb{E}^* \left[(S_T - K)_+\right]^2 = \left[(S_0 - K)_+\right]^2 + \int_0^T \left(\int_K^{+\infty} y^2 p(u,y)^2(u,y)dy\right) du.$$

Fortunately, using Equation (2.22) we get

$$\mathbb{E}^* \left[f_\epsilon(S_T - K)\right] = f_\epsilon(S_0 - K) + \frac{1}{2}\mathbb{E}^* \left[\int_0^T f_\epsilon''(S_u - K)S_u^2\sigma^2(u,S_u)du\right].$$

Let $f_\epsilon(S_T, K) = f_\epsilon(S_T - K) = \left[(S_T - K)_+\right]^2$, where $f_\epsilon''(S_T - K) = \frac{\partial^2 f_\epsilon}{\partial S^2} = 2$. Substituting this into Equation (2.22) gives

$$\mathbb{E}^* \left[(S_T - K)_+\right]^2 = \left[(S_0 - K)_+\right]^2 + \frac{1}{2}\mathbb{E}^* \left[\int_0^T 2(\mathbb{1}_{S_u>K})S_u^2\sigma^2(u,S_u)du\right].$$

By *Fubini theorem* again, we have

$$\begin{aligned}\mathbb{E}^* \left[(S_T - K)_+\right]^2 &= \left[(S_0 - K)_+\right]^2 + \int_0^T \left(\mathbb{E}^* \left[(\mathbb{1}_{S_u>K})S_u^2\sigma^2(u,S_u)\right]\right) du \\ &= \left[(S_0 - K)_+\right]^2 + \left[\int_0^T \left(\int_K^{+\infty} S_u^2 p(u,S_u)\sigma^2(u,S_u)dS_u\right) du\right].\end{aligned} \tag{2.27}$$

Taking $S_u = y$, Equation (2.27) becomes

$$\mathbb{E}^* \left[(S_T - K)_+\right]^2 = \left[(S_0 - K)_+\right]^2 + \int_0^T \left(\int_K^{+\infty} y^2 p(u,y)\sigma^2(u,y)dy\right) du. \tag{2.28}$$

3. Lastly, we will deduce the Dupire local variance equation:

From Equation (2.18), we have

$$\mathbb{E}^*\left[(S_T-K)_+\right]^2 = 2\int_K^{+\infty} C(T,y)dy.$$

Equating the right hand side of Equation (2.18) to the right hand side of Equation (2.28), we get

$$2\int_K^{+\infty} C(T,y)dy = \left[(S_0-K)_+\right]^2 + \int_0^T \left(\int_K^{+\infty} y^2 p(u,y)\sigma^2(u,y)dy\right)du. \tag{2.29}$$

Taking the derivative of both sides with respect to T, Equation (2.29) gives

$$\begin{aligned} 2\int_K^{+\infty} \frac{\partial C(T,y)}{\partial T}dy &= \frac{\partial}{\partial T}\left[(S_0-K)_+\right]^2 \\ &+ \frac{\partial}{\partial T}\left[\int_0^T \left(\int_K^{+\infty} y^2 p(u,y)\sigma^2(u,y)dy\right)du\right] \\ &= 0 + \int_K^{+\infty} y^2 p(T,y)\sigma^2(T,y)dy. \end{aligned} \tag{2.30}$$

This leads to

$$\begin{aligned} 2\frac{\partial C(T,y)}{\partial T}\mid_{y=K}^{y=+\infty} &= y^2\sigma^2(T,y)p(T,y)\mid_{y=K}^{y=+\infty} \\ 0 - 2\frac{\partial C(T,K)}{\partial T} &= 0 - K^2\sigma^2(T,K)p(T,K) \\ 2\frac{\partial C(T,K)}{\partial T} &= K^2\sigma^2(T,K)p(T,K) \\ \frac{\partial C(T,K)}{\partial T} &= \frac{1}{2}K^2\sigma^2(T,K)p(T,K). \end{aligned} \tag{2.31}$$

For simplicity, we have assumed

$$\lim_{y\to\infty} 2\frac{\partial C(T,y)}{\partial T} = 0 \text{ and } \lim_{y\to\infty} y^2\sigma^2(T,y)p(T,y) = 0.$$

Therefore,

$$\frac{\partial C(T,K)}{\partial T} = \frac{K^2\sigma^2(T,K)}{2}\frac{\partial^2 C(T,K)}{\partial K^2} for\ 0 < T < \bar{T}\ and\ K > 0. \tag{2.32}$$

for $0 < T < \bar{T}$ and $K > 0$. Thus, we can obtain the local variance function from Equation (2.32) as

$$\sigma^2(T,K) = 2\frac{\frac{\partial C(T,K)}{\partial T}}{K^2\frac{\partial^2 C(T,K)}{\partial K^2}} \tag{2.33}$$

for $0 < T < \bar{T}$ and $K > 0$

This completes the proof of the theorem. $\square$

Equation (2.33) was derived by Derman and Kani [11], although the idea and the method was developed by Dupire [15]. The formula says that at any point in time, the local volatility of the underlying asset can be determined for market option prices across all strike prices and maturities.

Remark 2.2. In the case the interest rate, $r \neq 0$, and the dividend yield, $q \neq 0$, the Dupire local variance equation becomes:

$$\sigma^2(T,K) = 2\frac{\frac{\partial C}{\partial T}(T,K) + (r(T) - q(T))K\frac{\partial C}{\partial K} + q(T)C}{K^2\frac{\partial^2 C}{\partial K^2}(T,K)} \tag{2.34}$$

for $0 < T < \bar{T}$ and $K > 0$ and $C(S_0, T, K)$.

2.2 Local Volatility as a Conditional Expectation

In this section, we will derive local volatility as an expected value.

Theorem 2.5. *Let S be the dynamics of the asset under*

$$dS_t = \mu_t S_t dt + \sigma(t, S_t)S_t dW_t, \tag{2.35}$$

where $\mu_t = r_t - q_t$, and W_t is the Brownian motion under the risk-neutral probability measure $\mathbb{P}^*$. Let $C(T,K) = P(t,T)\mathbb{E}^*(S_T - K)_+$, where $P(t,T) = \exp\left(-\int_t^T r_s ds\right)$. Then,

$$\mathbb{E}^*\left[\sigma_T^2 \mid S_T = K\right] = \frac{\left(2\left[\frac{\partial C}{\partial T} + K(r_T - q_T)\frac{\partial C}{\partial K} + q_T C\right]\right)}{\left(K^2\frac{\partial^2 C}{\partial K^2}\right)}. \tag{2.36}$$

Remark 2.3. Equation (2.36) is termed as the Dupire local variance equation as an expected value. Taking the square-root of both sides gives the local volatility.

Table 2.1: Partial Derivatives

$\frac{\partial}{\partial S}(S-K)_+ = \mathbb{1}_{(S-K)}$	$\frac{\partial}{\partial S}(\mathbb{1}_{(S-K)}) = \delta(S-K)$
$\frac{\partial}{\partial K}(S-K)_+ = -\mathbb{1}_{(S-K)}$	$\frac{\partial}{\partial K}(\mathbb{1}_{(S-K)}) = -\delta(S-K)$
$\frac{\partial C}{\partial K}(S-K)_+ = -P(t,T)\mathbb{E}^*[\mathbb{1}_{(S-K)}]$	$\frac{\partial^2 C}{\partial K^2}(S-K)_+ = P(t,T)\mathbb{E}^*[\delta(S-K)]$

where $\delta(\cdot)$ denotes the Dirac delta function.

Proof. Shown in Table 2.1 are the necessary partial derivatives used in the proof of the theorem.

Let $f : \mathbb{R} \times \mathbb{R} \longmapsto \mathbb{R}$ be a function such that $f(S_T, T) = P(t,T)(S_T - K)_+$. By *Itô formula*,

$$\begin{aligned} f_T &= f_0 + \int_0^T f_u{}' du + \int_0^T f_{S_u}{}' dS_u + \frac{1}{2}\int_0^T f_{S_u}{}'' S_u^2 \sigma^2(u, S_u) du \\ &= f_0 + \int_0^T f_u{}' du + \int_0^T f_{S_u}{}' \left(\mu_u S_u du + \sigma(u, S_u) S_u dB_u\right) \\ &+ \frac{1}{2}\int_0^T f_{S_u}{}'' S_u^2 \sigma^2(u, S_u) du \end{aligned} \tag{2.37}$$

Therefore,

$$df = \left[\frac{\partial f}{\partial T} + \frac{\partial f}{\partial S_T}\mu_T S_T + \frac{1}{2}\sigma_T^2 S_T^2 \frac{\partial^2 f}{\partial S_T^2}\right] dT + \left[\sigma_T S_T \frac{\partial f}{\partial S_T}\right] dB_T, \tag{2.38}$$

where the partial derivatives in Equation (2.38) are given explicitly as:

$$\begin{aligned} \frac{\partial f}{\partial T} &= -r_T P(t,T)(S_T - K)_+ \\ \frac{\partial f}{\partial S_T} &= P(t,T)\mathbb{1}_{(S>K)} \\ \frac{\partial^2 f}{\partial S_T^2} &= P(t,T)\delta(S - K). \end{aligned} \tag{2.39}$$

Substitute Equation (2.39) into Equation (2.38) to obtain

$$\begin{aligned} df &= P(t,T)[-r_T(S_T - K)_+ + \mu_T S_T \mathbb{1}_{(S>K)} + \frac{1}{2}\sigma_T^2 S_T^2 \delta(S-K)]dT \\ &+ P(t,T)\sigma_T S_T \mathbb{1}_{(S>K)} dB_T. \end{aligned} \tag{2.40}$$

For $B_T \sim \mathcal{N}(0, T)$, it follows that

$$\mathbb{E}(B_T) = 0 \text{ implies } d\mathbb{E}(B_T) = 0 \text{ implies } \mathbb{E}(dB_T) = 0.$$

Taking the expectation of both sides of Equation (2.40) yields

$$\mathbb{E}^*(df) = P(t,T)\mathbb{E}^*\left(\left[-r_T(S_T - K)_+ + \mu_T S_T \mathbb{1}_{(S>K)} + \frac{1}{2}\sigma_T^2 S_T^2 \delta(S-K)\right] dT\right), \tag{2.41}$$

and by *Fubini theorem*, we have

$$\mathbb{E}^*(df) = d\mathbb{E}^*(f) = d\left(P(t,T)\mathbb{E}^*(S_T - K)_+\right) = dC. \tag{2.42}$$

Thus,

$$\begin{aligned} dC &= P(t,T)\mathbb{E}^*\left(\left[-r_T - (S_T - K)_+ + \mu_T S_T \mathbb{1}_{(S>K)} + \frac{1}{2}\sigma_T^2 S_T^2 \delta(S-K)\right] dT\right) \\ \frac{\partial C}{\partial T} &= P(t,T)\mathbb{E}^*\left(\left[-r_T(S_T - K)\mathbb{1}_{(S>K)} + \mu_T S_T \mathbb{1}_{(S>K)} + \frac{1}{2}\sigma_T^2 S_T^2 \delta(S-K)\right]\right). \end{aligned} \tag{2.43}$$

Simplifying the first two terms in Equation (2.43) gives

$$\begin{aligned}-r_T(S_T-K)\mathbb{1}_{(S>K)}+\mu_T S_T\mathbb{1}_{(S>K)} &= \mathbb{1}_{(S>K)}\left(-r_T(S_T-K)+(r_T-q_T)S_T\right)\\ &= r_T K\mathbb{1}_{(S_T>K)}-q_T S_T\mathbb{1}_{(S_T>K)}.\end{aligned} \tag{2.44}$$

so that Equation (2.43) becomes

$$\frac{\partial C}{\partial T}=P(t,T)\mathbb{E}^*\left(\left[r_T K\mathbb{1}_{(S_T>K)}-q_T S_T\mathbb{1}_{(S_T>K})++\frac{1}{2}\sigma_T^2 S_T^2\delta(S-K)\right]\right). \tag{2.45}$$

Furthermore, we have

$$\begin{aligned}C &= P(t,T)\mathbb{E}^*\left[(S_T-K)\mathbb{1}_{(S_T>K)}\right]\\ &= P(t,T)\mathbb{E}^*\left[S_T\mathbb{1}_{(S_T>K)}\right]-P(t,T)\mathbb{E}^*[K\mathbb{1}_{(S_T>K)}],\end{aligned} \tag{2.46}$$

which implies that

$$P(t,T)\mathbb{E}^*\left[S_T\mathbb{1}_{(S_T>K)}\right]=C+P(t,T)\mathbb{E}^*\left[K\mathbb{1}_{(S_T>K)}\right]. \tag{2.47}$$

Substitute Equation (2.47) into Equation (2.45) to get

$$\begin{aligned}\frac{\partial C}{\partial T} &= P(t,T)\mathbb{E}^*[r_T K\mathbb{1}_{(S_T>K)}]-q_T\left(C+P(t,T)\mathbb{E}^*[K\mathbb{1}_{(S_T>K)}]\right)\\ &\quad+\frac{1}{2}P(t,T)\mathbb{E}^*[\sigma_T^2 S_T^2\delta(S-K)]\\ \frac{\partial C}{\partial T} &= P(t,T)K(r_T-q_T)\mathbb{E}^*[\mathbb{1}_{(S_T>K)}]-q_T C\\ &\quad+\frac{1}{2}P(t,T)\mathbb{E}^*\left[\sigma_T^2 S_T^2\delta(S-K)\right]\\ \frac{\partial C}{\partial T} &= P(t,T)K(r_T-q_T)\mathbb{E}^*[\mathbb{1}_{(S_T>K)}]-q_T C\\ &\quad+\frac{1}{2}P(t,T)\mathbb{E}^*\left[\sigma_T^2 S_T^2\delta(S-K)\right].\end{aligned} \tag{2.48}$$

By *Markov property* of conditional expectation, the last term in Equation (2.48) can be calculated as

$$\begin{aligned}\frac{1}{2}P(t,T)\mathbb{E}^*\left[\sigma_T^2 S_T^2\delta(S-K)\right] &= \frac{1}{2}P(t,T)\mathbb{E}^*\left[\delta_T^2 S_T^2 \mid S_T=K\right]\mathbb{E}^*\left[\delta(S-K)\right]\\ &= \frac{1}{2}P(t,T)K^2\mathbb{E}^*\left[\delta_T^2 \mid S_T=K\right]\mathbb{E}^*\left[\delta(S-K)\right]\\ &= \frac{1}{2}K^2\mathbb{E}^*\left[\sigma_T^2 \mid S_T=K\right]\frac{\partial^2 C}{\partial K^2},\end{aligned} \tag{2.49}$$

so that Equation $Equation$ (2.48) reads as follows:

$$\frac{\partial C}{\partial T}=-K(r_T-q_T)\frac{\partial C}{\partial K}-q_T C+\frac{1}{2}K^2\mathbb{E}^*\left[\sigma_T^2 \mid S_T=K\right]\frac{\partial^2 C}{\partial K^2}, \tag{2.50}$$

where we have substituted $P(t,T)\mathbb{E}^*[\mathbb{1}_{(S_T>K)}]$ for $\frac{\partial C}{\partial T}$. Hence,

$$\mathbb{E}^*\left[\sigma_T^2 \mid S_T = K\right] = \frac{\left(2\left[\frac{\partial C}{\partial T} + K(r_T - q_T)\frac{\partial C}{\partial K} + q_T C\right]\right)}{\left(K^2\frac{\partial^2 C}{\partial K^2}\right)}.$$

giving us the desired formula. □

Corollary 2.6. *When $r_T = q_T = 0$ in Equation (2.36), we get*

$$\mathbb{E}^*\left[\sigma_T^2 \mid S_T = K\right] = \frac{2\frac{\partial C}{\partial T}(T,K)}{K^2\frac{\partial^2 C}{\partial K^2}(T,K)}.$$

The above equation is equivalent to what we obtained in Equation (2.33).

Comparing Equation (2.36) and Equation (2.33) shows that local volatility can be interpreted as the expected volatility of the asset under the condition that the option ends at-the-money. In other words, it is the average of all instantaneous future spot-volatilities until the maturity of the option. Although, this does not mean that the expected value will be realized, trading of different financial instruments makes it possible to lock into this value at the current time.

2.3 Local Volatility in terms of Forward Price

Let $C(T,K) = C(F_T, T, K, \sigma_{LV}(T,K))$ and $F_t = F(t,T) = S_t e^{\mu_t(T-t)}$, where $\mu_t = r_t - q_t$. Write the SDE in Equation (2.35) under the dynamics of the forward price,

$$\begin{aligned} dF_t &= e^{\mu_t(T-t)}dS_t + S_t e^{\mu_t(T-t)}(-\mu_t)dt \\ &= e^{\mu_t(T-t)}\left(\mu_t S_t dt + \sigma(t,S_t)S_t dB_t\right) - \mu_t F_t dt \\ &= \mu_t F_t dt + \sigma(t,S_t)F_t dB_t - \mu_t F_t dt \\ &= \sigma(t, F_t e^{-\mu_t(T-t)})F_t dB_t \\ &= \tilde{\sigma}(t,F_t)F_t dB_t, \end{aligned} \tag{2.51}$$

where $\tilde{\sigma}(t,x) = \sigma(t, xe^{-\mu_t(T-t)})$. Note that $F_T = S_T$.

Theorem 2.7. *Let F_t be the dynamics of the forward price of the asset in Equation (2.51) and $C(T,K)$ be the European call option with strike K and maturity T. The following equation holds:*

$$\sigma^2(T,K) = \frac{\frac{\partial C(T,K)}{\partial T}}{\frac{1}{2}K^2\frac{\partial^2 C(T,K)}{\partial K^2}}. \tag{2.52}$$

Remark 2.4. Equation (2.52) is termed as the Dupire local variance under the forward price dynamics. Taking the square-root of both sides gives the local volatility.

Proof. The price of a call option under Equation (2.51) is given as

$$C(T,K) = \mathbb{E}^*[S_T - K]_+ = \int_K^\infty (S_T - K)p(T,S_T)dS_T, \tag{2.53}$$

where $p(t,\cdot)$ is the density function of S_T under $\mathbb{P}^*$. Thus,

$$\frac{\partial^2 C}{\partial K^2} = p(T,K).$$

$p(T,\cdot)$ satisfies the forward equation below (see Theorem A.2).

$$\frac{\partial p}{\partial t} = \frac{1}{2}\frac{\partial^2}{\partial x^2}(\sigma^2(t,x)x^2 p). \tag{2.54}$$

Hence,

$$\begin{aligned}\frac{\partial C(T,K)}{\partial T} &= \int_K^\infty (S_T - K)\frac{\partial p(T,S_T)}{\partial T}dS_T \\ &= \frac{1}{2}\int_K^\infty \frac{\partial^2}{\partial S_T^2}\left(\sigma^2(T,S_T)S_T^2 p\right)(S_T - K)dS_T.\end{aligned} \tag{2.55}$$

Using integration by parts twice on the right hand side of Equation (2.55), we obtain

$$\begin{aligned}\frac{\partial C(T,K)}{\partial T} &= -\frac{1}{2}\int_K^\infty \frac{\partial}{\partial S_T}\left(\sigma^2(T,S_T)S_T^2 p\right)dS_T \\ &= \frac{1}{2}\sigma^2(T,K)K^2 p(T,K) \\ &= \frac{1}{2}\sigma^2(T,K)K^2\frac{\partial^2 C}{\partial K^2}.\end{aligned} \tag{2.56}$$

It follows that

$$\sigma^2(T,K) = \frac{\frac{\partial C(T,K)}{\partial T}}{\frac{1}{2}K^2\frac{\partial^2 C(T,K)}{\partial K^2}}$$

holds as desired. □

2.4 Challenges of using Dupire's Local Volatility Equation

With the local volatility derivation in Section 2.1, local volatility surface can be obtained from the option prices observed in the market. A crucial assumption is that the option price belongs to the class of twice continuously differentiable functions $\mathcal{C}^{1,2}$, known over all maturities and strikes. However, even if this assumption holds, there is still a problem with option price function being unknown analytically which makes taking their partial derivatives difficult. Therefore, the partial derivatives of the call function have to be estimated numerically. Due to the imperfect nature of numerical methods, the algorithm used in estimating local volatility function may be unstable. That is, small changes in the input data may result in large error in the function values.

Observe that in the denominator of Equation (2.33), small errors in the partial derivative can be magnified by the square of the strike price. This can lead to negative values of local volatility, which is unacceptable.

Also, the continuity assumption of option prices is unrealistic. In practice, limited number of option values are known for finite number of maturities and strike prices which makes the local volatility equation ill-posed. This leads to a local volatility function that is not unique and stable. To solve this problem, one can smoothen the option price data using Tikhonov regularization [9, 21] or by minimizing the function's entropy. Another viable method is to use smoothing cubic spline interpolation to obtain arbitrage-free option prices [17]. These methods must be able to guarantee the convexity of the option prices in the strike direction which adds extra complexity to the model. Also, the call option function must be monotonically decreasing in strike and increasing in maturity to avoid calender arbitrage. This way, arbitrage-free prices can be ensured.

2.5 Local Volatility as a Function of Implied Volatility

In this section, the relationship between local volatility and Black-Scholes implied volatility will be shown.

Lemma 2.8. *For the Black-Scholes model described in Theorem 1.2, let the Black-Scholes total variance be defined as*

$$w(S_0, T, K) = \sigma_{BS}^2(S_0, T, K)T$$

and the log-strike y be defined as

$$y = \log\left(\frac{K}{F_T}\right),$$

where $F_T = S_0 \exp\left\{\int_0^T r_s ds\right\}$ is the forward price of the stock at time 0.

The Black-Scholes price in terms of w and y is therefore given by

$$C_{BS}(F_T, w, y) = F_T\left(N(d_1) - e^y N(d_2)\right)$$

with $d_1 = \frac{-y}{\sqrt{w}} + \frac{\sqrt{w}}{2}$ and $d_2 = \frac{-y}{\sqrt{w}} + \frac{\sqrt{w}}{2} - \sqrt{w}$.

Proof. Under the forward price dynamics of the asset price, Black-Scholes formula becomes

$$C_{BS}(F_T, T, K) = F_T N(d_1) - K N(d_2) \tag{2.57}$$

with $d_1 = \frac{\ln(\frac{F_T}{K}) + \frac{\sigma_{BS}^2 T}{2}}{\sigma_{BS}\sqrt{T}}$ and $d_2 = d_1 - \sigma_{BS}\sqrt{T}$. Also, we have

$$y = \log\left(\frac{K}{F_T}\right) \text{ implies } K = e^y F_T.$$

Equation (2.57) can be written as

$$C_{BS}(F_T, w, y) = F_T\left(N(d_1) - e^y N(d_2)\right) \tag{2.58}$$

with

$$\begin{aligned} d_1 &= \frac{\ln(\frac{F_T}{K})}{\sigma_{BS}\sqrt{T}} + \frac{\sigma_{BS}^2 T}{2\sigma_{BS}\sqrt{T}} \\ &= \frac{\ln((\frac{K}{F_T})^{-1})}{\sqrt{w}} + \frac{\sigma_{BS} T}{2} \\ &= -\frac{\ln(\frac{K}{F_T})}{\sqrt{w}} + \frac{\sqrt{w}}{2} \\ &= \frac{-y}{\sqrt{w}} + \frac{\sqrt{w}}{2} \end{aligned} \tag{2.59}$$

and

$$\begin{aligned} d_2 &= d_1 - \sigma_{BS}\sqrt{T} \\ &= \frac{-y}{\sqrt{w}} + \frac{\sqrt{w}}{2} - \sqrt{w} \\ &= \frac{-y}{\sqrt{w}} - \frac{\sqrt{w}}{2} \end{aligned} \tag{2.60}$$

as desired. □

Theorem 2.9. *If the call option written on an asset is described by the Black-Scholes formula derived in Lemma 2.8, then the local variance of the asset can be obtained in terms of w and y as*

$$\sigma^2(T, K) = \frac{\frac{\partial w}{\partial T}}{\left[1 - (\frac{y}{w})\frac{\partial w}{\partial y} + \frac{1}{4}\left(-\frac{1}{4} - \frac{1}{w} + \frac{y^2}{w^2}\right)(\frac{\partial w}{\partial y})^2 + \frac{\partial^2 w}{2\partial y^2}\right]}.$$

Proof. First substitute Equation (2.59) and Equation (2.60) into Equation (2.57).

$$C_{BS}(F_T, y, w) = F_T\left\{N\left(\frac{-y}{\sqrt{w}} + \frac{\sqrt{w}}{2}\right) - e^y N\left(\frac{-y}{\sqrt{w}} - \frac{\sqrt{w}}{2}\right)\right\}. \tag{2.61}$$

From Equation (2.34), Dupire's local volatility with $r \neq 0$ and $q = 0$ is given as

$$\frac{\partial C(T, K)}{\partial T} = \frac{\sigma^2(T, K)}{2}K^2\frac{\partial^2 C(T, K)}{\partial K^2} - r_T K\frac{\partial C(T, K)}{\partial K}. \tag{2.62}$$

Calculate the partial derivatives in Equation (2.62):

$$\frac{\partial C(T, K)}{\partial K} = \frac{\partial C(T, y)}{\partial y}\frac{\partial y}{\partial K} = \frac{1}{K}\frac{\partial C(T, y)}{\partial y}. \tag{2.63}$$

Consequently,

$$\frac{\partial^2 C(T,K)}{\partial K^2} = \frac{1}{K}\left(\frac{\partial}{\partial K}\left(\frac{\partial C(T,y)}{\partial y}\right)\right) + \frac{\partial C(T,y)}{\partial y}\left(-\frac{1}{K^2}\right). \tag{2.64}$$

Since the call function is continuous in y and K, we get

$$\begin{aligned}\frac{\partial}{\partial K}\left(\frac{\partial C(T,y)}{\partial y}\right) &= \frac{\partial}{\partial y}\left(\frac{\partial C(T,y)}{\partial K}\right)\\ &= \frac{\partial}{\partial y}\left(\frac{\partial C}{\partial y}\frac{\partial y}{\partial K}\right)\\ &= \frac{\partial}{\partial y}\left(\frac{1}{K}\frac{\partial C}{\partial y}\right) = \frac{1}{K}\frac{\partial^2 C(T,y)}{\partial y^2}.\end{aligned} \tag{2.65}$$

Substitute Equation (2.65) into Equation (2.64) to obtain

$$\frac{\partial^2 C(T,K)}{\partial K^2} = \frac{1}{K^2}\left(\frac{\partial^2 C(T,y)}{\partial y^2} - \frac{\partial C(T,y)}{\partial y}\right). \tag{2.66}$$

Furthermore, we have

$$\begin{aligned}\frac{\partial C(T,K)}{\partial T} &= \frac{\partial C(T,y)}{\partial T} + \frac{\partial C(T,y)}{\partial y}\frac{\partial y}{\partial T}\\ &= \frac{\partial C(T,y)}{\partial T} - r_T\frac{\partial C(T,y)}{\partial y},\end{aligned} \tag{2.67}$$

which leads to

$$\frac{\partial C(T,y)}{\partial T} = \frac{\partial C(T,K)}{\partial T} + r_T\frac{\partial C(T,y)}{\partial y}. \tag{2.68}$$

We should remark that derivatives of y reads as follows:

$$\begin{aligned}\frac{\partial y}{\partial K} &= \frac{\partial}{\partial K}\left(\log\left(\frac{K}{F_T}\right)\right)\\ &= \frac{1}{\left(\frac{K}{F_T}\right)}\left(\frac{1}{F_T}\right) = \frac{1}{K}.\end{aligned} \tag{2.69}$$

and

$$\begin{aligned}\frac{\partial y}{\partial T} &= \frac{\partial}{\partial T}\left(\log\left(\frac{K}{F_T}\right)\right)\\ &= \frac{F_T}{K}K\frac{\partial}{\partial T}\left(\frac{1}{F_T}\right)\\ &= \frac{F_T}{K}K\left(\frac{-r_T}{F_T}\right) = -r_T.\end{aligned} \tag{2.70}$$

Thence, Equation (2.68) becomes

$$\begin{aligned}
\frac{\partial C(T,y)}{\partial T} &= \frac{\partial C(T,K)}{\partial T} + r_T \frac{\partial C(T,y)}{\partial y} \\
&= \frac{\sigma^2 K^2}{2} \frac{\partial^2 C(T,K)}{\partial K^2} - r_T K \frac{\partial C(T,K)}{\partial K} + r_T \frac{\partial C(T,y)}{\partial y} \\
&= \frac{\sigma^2}{2} K^2 \left\{ \frac{1}{K^2} \left(\frac{\partial^2 C(T,y)}{\partial y^2} - \frac{\partial C(T,y)}{\partial y} \right) \right\} \\
&- r_T K \left\{ \frac{1}{K} \frac{\partial C(T,y)}{\partial y} \right\} + r_T \frac{\partial C(T,y)}{\partial y} \\
&= \frac{\sigma^2}{2} \left\{ \frac{\partial^2 C(T,y)}{\partial y^2} - \frac{\partial C(T,y)}{\partial y} \right\}.
\end{aligned} \tag{2.71}$$

Calculate the partial derivatives in Equation (2.71):

$$\frac{\partial C(T,y)}{\partial y} = \frac{\partial C(w,y)}{\partial y} + \frac{\partial C(w,y)}{\partial w} \frac{\partial w}{\partial y}, \tag{2.72}$$

and

$$\begin{aligned}
\frac{\partial^2 C(T,y)}{\partial y^2} &= \frac{\partial^2 C(w,y)}{\partial y^2} + \frac{\partial^2 C(w,y)}{\partial y \partial w} \frac{\partial w}{\partial y} \\
&+ \frac{\partial}{\partial y} \left(\frac{\partial C(w,y)}{\partial w} \frac{\partial w}{\partial y} \right),
\end{aligned} \tag{2.73}$$

where the last term in Equation (2.73) is

$$\begin{aligned}
\frac{\partial}{\partial y} \left(\frac{\partial C(w,y)}{\partial w} \frac{\partial w}{\partial y} \right) &= \left(\frac{\partial w}{\partial y} \right) \left\{ \frac{\partial^2 C(w,y)}{\partial y \partial w} + \frac{\partial^2 C(w,y)}{\partial w^2} \frac{\partial w}{\partial y} \right\} \\
&+ \frac{\partial C(w,y)}{\partial w} \left(\frac{\partial^2 w}{\partial y^2} \right).
\end{aligned} \tag{2.74}$$

Substitute Equation (2.74) into Equation (2.73) to get

$$\begin{aligned}
\frac{\partial^2 C(T,y)}{\partial y^2} &= \frac{\partial^2 C(w,y)}{\partial y^2} + 2 \frac{\partial^2 C(w,y)}{\partial y \partial w} \frac{\partial w}{\partial y} \\
&+ \frac{\partial^2 C(w,y)}{\partial w^2} (\frac{\partial w}{\partial y})^2 + \frac{\partial C(w,y)}{\partial w} \frac{\partial^2 w}{\partial y^2}.
\end{aligned} \tag{2.75}$$

Also, we have

$$\frac{\partial C(T,y)}{\partial T} = \frac{\partial C(w,y)}{\partial T} + \frac{\partial C(w,y)}{\partial T} \frac{\partial w}{\partial T}. \tag{2.76}$$

Then, substituting Equation (2.72), Equation (2.75), and Equation (2.76) into Equation (2.71) yields

$$\frac{\partial C(w,y)}{\partial T} + \frac{\partial C(w,y)}{\partial w}\frac{\partial w}{\partial T} = \frac{\sigma^2}{2}\left\{\frac{\partial^2 C(T,y)}{\partial y^2} - \frac{\partial C(T,y)}{\partial y}\right\}$$
$$= \frac{\sigma^2}{2}\left\{\frac{\partial^2 C(w,y)}{\partial y^2} + 2\frac{\partial^2 C(w,y)}{\partial y \partial w}\frac{\partial w}{\partial y} + \frac{\partial^2 C(w,y)}{\partial w^2}(\frac{\partial w}{\partial y})^2\right\}$$
$$+ \frac{\sigma^2}{2}\left\{\frac{\partial C(w,y)}{\partial w}\frac{\partial^2 w}{\partial y^2} - \frac{\partial C(w,y)}{\partial y} - \frac{\partial C(w,y)}{\partial w}\frac{\partial w}{\partial y}\right\}. \tag{2.77}$$

Calculating what each partial derivative in Equation (2.77) represents under the Black-Scholes formula,

$$\begin{aligned}\frac{\partial C(w,y)}{\partial w} &= F_T\left(\Phi(d_1)\left(\frac{-y}{2\sqrt{w}^3} + \frac{1}{4\sqrt{w}}\right) - e^y\Phi(d_2)\left(\frac{-y}{2\sqrt{w}^3} - \frac{1}{4\sqrt{w}}\right)\right) \\ &= \frac{F_T\Phi(d_1)}{2\sqrt{w}},\end{aligned} \tag{2.78}$$

where Φ is the probability distribution function of standard normal distribution and N is the cumulative distribution function of standard normal distribution. Here also note that

$$\begin{aligned}\Phi(d_2) &= \frac{1}{\sqrt{2\pi}}e^{(-\frac{1}{2}(d_1-\sqrt{w})^2)} \\ &= \frac{1}{\sqrt{2\pi}}e^{(-\frac{1}{2}d_1^2)}e^{(d_1\sqrt{w}-\frac{w}{2})} \\ &= \Phi(d_1)e^{(\frac{-y}{\sqrt{w}}+\frac{\sqrt{w}}{2})\sqrt{w}-\frac{w}{2}} \\ &= \Phi(d_1)e^{-y}.\end{aligned} \tag{2.79}$$

Furthermore,

$$\begin{aligned}\frac{\partial^2 C(w,y)}{\partial w^2} &= \frac{\partial}{\partial w}\left(\frac{F_T\Phi(d_1)}{2\sqrt{w}}\right) \\ &= F_T\frac{\left(2\sqrt{w}\frac{\partial\Phi(d_1)}{\partial w} - \frac{\Phi(d_1)}{\sqrt{w}}\right)}{4w} \\ &= \frac{F_T}{4w}\left(2\sqrt{w}(-d_1\Phi(d_1)\left(\frac{\partial d_1}{\partial w}\right) - \frac{\Phi(d_1)}{\sqrt{w}}\right) \\ &= F_T\Phi(d_1)\left(\frac{-1}{4\sqrt{w}^3} - \frac{1}{2\sqrt{w}}d_1\frac{\partial d_1}{\partial w}\right) \\ &= F_T\Phi(d_1)\left(\frac{-1}{4\sqrt{w}^3} - \frac{1}{2\sqrt{w}}\left(\frac{-y}{\sqrt{w}} + \frac{\sqrt{w}}{2}\right)\left(\frac{y}{2\sqrt{w}^3} + \frac{1}{4\sqrt{w}}\right)\right),\end{aligned}$$

which can be simplified to

$$\begin{aligned}\frac{\partial^2 C(w,y)}{\partial w^2} &= \frac{F_T\Phi(d_1)}{2\sqrt{w}}\left(\frac{-1}{2w}+\left(\frac{y}{\sqrt{w}}-\frac{\sqrt{w}}{2}\right)\left(\frac{y}{2\sqrt{w}^3}+\frac{1}{4\sqrt{w}}\right)\right)\\ &= \frac{\partial C(w,y)}{\partial w}\left(-\frac{1}{2w}+\left(\frac{y^2}{2w^2}+\frac{y}{4w}-\frac{y}{4w}-\frac{1}{8}\right)\right)\\ &= \frac{\partial C(w,y)}{\partial w}\left(-\frac{1}{2w}+\frac{y^2}{2w^2}-\frac{1}{8}\right)\\ &= \frac{\partial C(w,y)}{\partial w}\left(-\frac{1}{8}-\frac{1}{2w}+\frac{y^2}{2w^2}\right);\end{aligned}\tag{2.80}$$

and

$$\begin{aligned}\frac{\partial C(w,y)}{\partial y} &= F_T\left(\left(-\frac{\Phi(d_1)}{\sqrt{w}}\right)-\left(e^yN(d_2)+e^y\Phi(d_2)\left(-\frac{1}{\sqrt{w}}\right)\right)\right)\\ &= F_T\left(-\frac{\Phi(d_1)}{\sqrt{w}}-e^yN(d_2)+e^y\frac{\Phi(d_2)}{\sqrt{w}}\right)\\ &= -F_Te^yN(d_2).\end{aligned}\tag{2.81}$$

while

$$\begin{aligned}\frac{\partial^2 C(w,y)}{\partial y^2} &= \frac{\partial}{\partial y}\left(-F_Te^yN(d_2)\right)\\ &= -F_T\left(N(d_2)e^y+e^y\Phi(d_2)(-\frac{1}{\sqrt{w}})\right)\\ &= -F_Te^yN(d_2)+F_T\frac{\Phi(d_1)}{\sqrt{w}}.\end{aligned}\tag{2.82}$$

Subtracting Equation (2.81) from Equation (2.82), we get

$$\frac{\partial^2 C(w,y)}{\partial y^2}-\frac{\partial C(w,y)}{\partial y}=F_T\frac{\Phi(d_1)}{\sqrt{w}}=2\frac{\partial C(w,y)}{\partial w}.\tag{2.83}$$

Besides, the mixed partial derivative reads as

$$\begin{aligned}\frac{\partial^2 C(w,y)}{\partial y\partial w} &= \frac{\partial}{\partial y}\left(\frac{\partial C(w,y)}{\partial w}\right)\\ &= \frac{\partial}{\partial y}\left(F_T\frac{\Phi d_1}{2\sqrt{w}}\right)\\ &= \frac{F_T}{2\sqrt{w}}\frac{\partial}{\partial y}(\Phi(d_1))\\ &= \frac{F_T}{2\sqrt{w}}\left(-d_1\Phi(d_1)\frac{\partial d_1}{\partial y}\right),\end{aligned}$$

which can be further simplified to

$$\begin{aligned}
\frac{\partial^2 C(w,y)}{\partial y \partial w} &= \frac{F_T}{2\sqrt{w}}\left(-d_1\Phi(d_1)(-\frac{1}{\sqrt{w}})\right) \\
&= \frac{F_T\Phi(d_1)}{2\sqrt{w}}\left(\left(\frac{-y}{\sqrt{w}}+\frac{\sqrt{w}}{2}\right)\left(\frac{1}{\sqrt{w}}\right)\right) \\
&= \frac{\partial C(w,y)}{\partial w}\left(-\frac{y}{w}+\frac{1}{2}\right) \\
&= \frac{\partial C(w,y)}{\partial w}\left(\frac{1}{2}-\frac{y}{w}\right).
\end{aligned} \tag{2.84}$$

The trivial relation

$$\frac{\partial C}{\partial T}(w,y) = 0 \tag{2.85}$$

holds, since the Black-Scholes call function does not depend directly on T anymore but rather depend on T implicitly.

Substitute Equation (2.80), Equation (2.81), Equation (2.82), Equation (2.84), and Equation (2.85) into Equation (2.77) to write

$$\begin{aligned}
\frac{\partial C(w,y)}{\partial w}\frac{\partial w}{\partial T} &= \frac{\sigma^2}{2}\frac{\partial C(w,y)}{\partial w}\left[2+2\left(\frac{1}{2}-\frac{y}{w}\right)\frac{\partial w}{\partial y}+\frac{\partial^2 w}{\partial y^2}-\frac{\partial w}{\partial y}\right] \\
&+\frac{\sigma^2}{2}\frac{\partial C(w,y)}{\partial w}\left[\left(-\frac{1}{8}-\frac{1}{2w}+\frac{y^2}{2w^2}\right)\left(\frac{\partial w}{\partial y}\right)^2\right], \\
\frac{\partial w}{\partial T} &= \sigma^2(T,K)\left[1+\left(\frac{1}{2}-\frac{y}{w}\right)\frac{\partial w}{\partial y}+\left(-\frac{1}{16}-\frac{1}{4w}+\frac{y^2}{4w^2}\right)\left(\frac{\partial w}{\partial y}\right)^2\right] \\
&+\sigma^2(T,K)\left[\frac{\partial^2 w}{2\partial y^2}-\frac{\partial w}{2\partial y}\right] \\
&= \sigma^2(T,K)\left[1-\frac{y}{w}\frac{\partial w}{\partial y}+\frac{1}{4}\left(-\frac{1}{4}-\frac{1}{w}+\frac{y^2}{w^2}\right)\left(\frac{\partial w}{\partial y}\right)^2+\frac{\partial^2 w}{2\partial y^2}\right],
\end{aligned} \tag{2.86}$$

which implies that

$$\sigma^2(T,K) = \frac{\frac{\partial w}{\partial T}}{\left[1-(\frac{y}{w})\frac{\partial w}{\partial y}+\frac{1}{4}\left(-\frac{1}{4}-\frac{1}{w}+\frac{y^2}{w^2}\right)\left(\frac{\partial w}{\partial y}\right)^2+\frac{\partial^2 w}{2\partial y^2}\right]} \tag{2.87}$$

as desired. □

Corollary 2.10. *In the special case where the volatility smile is flat, namely* $\frac{\partial w}{\partial y} = 0$, *for each maturity* T, *Equation (2.87) is simplified to*

$$\sigma^2(T,K) = \frac{\partial w}{\partial T}, \text{ equivalently, } w(T) = \int_0^T \sigma^2 ds.$$

CHAPTER 3

NUMERICAL METHODS OF OBTAINING LOCAL VOLATILITY SURFACES

In this chapter, fundamental concepts of moneyness and cubic spline interpolation will be addressed. Also, different methods of obtaining local volatility surfaces via option price and implied volatility data will be discussed.

In general, we can classify these methods into parametric and non-parametric methods. In the parametric method, an initial implied volatility function is specified and this is used to obtain the implied volatility surface with a regressive algorithm. Consequently, with additional numerical techniques, the local volatility surface can be obtained. On the other hand, the non-parametric method allows the local volatility surface to take any form while an optimization procedure is carried out to determine the particular form of the surface. Each of these methods has its strengths and weaknesses depending on the trade-off involved. The trade-off is usually between fitting the model prices to the observed market prices and attainment of stability of the local volatility function/surface through time.

In Section 3.1, the concept of *moneyness* will be discussed and in Section 3.2, *cubic-spline polynomial interpolation* will be addressed. In Section 3.3, parametric method will be explained as well as how it is used in obtaining local volatility surfaces. Consequently, Section 3.4 will deal with some profound numerical techniques from literature on how to obtain local volatility surfaces via non-parametric methods. In addition, the re-construction of local volatility surfaces using both option prices and implied volatility data via non-parametric method will be carried out.

3.1 Moneyness

Moneyness of an option describes the relative position of the current price or future price of an underlying asset relative to the strike price. It reflects the degree to which an option is in-the-money (*ITM*) or out-of-the-money (*OTM*). Thus, it helps investors evaluate how valuable their options are at any given point in time. Conventionally speaking, moneyness should also depend on time to maturity apart from the strike price as the following example demonstrates: a call option with a strike price of 110 would be classified as deep *OTM* if the current underlying price is 105 and the time to maturity is 1day. However, the same call having a time of maturity of 1year would be reasonably rated at-the-money (*ATM*) since the probability that the underlying price reaches or exceeds the strike price is much higher than it is in the first case [19]. Below is a mathematical definition of moneyness.

Definition 3.1 (Moneyness)**.** Let $m(t, s, K, T, r)$ be a function of time, underlying price, strike price, maturity date, and interest rate. The moneyness M_t at time $t \in [0, T^*]$ is generally defined as

$$M_t = m(t, S_t, K, T, r). \tag{3.1}$$

The function m is referred to as the moneyness function. It is required to be increasing in K.

Definition 3.2 (Valid Moneyness)**.** We call m a valid moneyness function and M_t defined as

$$M_t = m(t, S_t, K, T, r),$$

a valid moneyness for our financial market model if m has the following properties [19]:

1. $m(t, s, K, T, r) \in \mathcal{C}^2([0, T^*] \times \mathbb{R}_{++} \times \mathbb{R}_{++} \times (t, T^*] \times \mathbb{R}_+)$,
2. $\lim_{t\to T} m(t, S_t, K, T, r) < \infty, \quad \mathbb{P} - a.s.$, and
3. $\lim_{t\to T} \frac{\partial m}{\partial t}(t, S_t, K, T, r) < \infty \quad \mathbb{P} - a.s.$, where S_t evolves under the probability measure $\mathbb{P}$.

3.1.1 Moneyness Terms

Definition 3.3 (At the Money)**.** An option is said to be at-the-money (ATM) when the strike price of the underlying asset equals the spot price or futures price [6]. However, options are usually expressed as being near-the-money or close-to-the-money because they are rarely exactly at-the-money.

Definition 3.4 (In the Money)**.** A call (put) option is in-the-money (ITM) when the current price or forward price of the underlying asset exceeds (is less than) the strike price [7].

Definition 3.5 (Out the Money)**.** A call (put) option is out-of-the-money (OTM) when the current price or forward price of the underlying asset is less than (exceeds) the strike price [7].

3.1.2 Choice of Moneyness Measure

Different choices of moneyness measure have been documented in the literature and these choices will be discussed below:

1. **Simple moneyness:** This represents the ratio of strike price to the underlying asset price, i.e., K/S_t, or the inverse of it [19]. Conventionally speaking, the constant term is in the denominator while the term that relatively changes is in the numerator. The choices of numerator and denominator depend on what interpretations one would like to give to the options at hand. Hence, for a specific option, if K is fixed then different spot prices yield different moneyness for that option. This is useful for option pricing and understanding the Black-Scholes model. Conversely, if one has different options at a given point in time, the spot price is fixed and the options have different strike prices, hence, different moneyness.

2. **Log-simple moneyness:** This is a linearized modification of simple moneyness done by taking it's natural logarithm. It is defined to be

$$M_t = \ln\left(\frac{K}{S_t e^{r(T-t)}}\right) = \ln\left(\frac{K}{F_t}\right), \quad \forall t \in [0, T^*]$$

   for $\forall S_t > 0$, $T \in (t, T^*]$, $r \geq 0$, and $\forall K > 0$. As can be verified, simple moneyness as well as the log-simple moneyness are valid moneyness measures.

3. **Time-Dependent moneyness:** As previously argued, moneyness is affected by the maturity date of an option. Hence, a moneyness depending on time to maturity is defined as

$$M_t = \frac{\ln\left(\frac{K}{F_t}\right)}{\sqrt{T-t}}.$$

   Note that $\sqrt{T-t}$ is used in the division to normalize time to maturity since the dispersion of Brownian motion is proportional to square root of time [30]. This measure of moneyness makes the volatility smile to be largely independent of time to maturity.

4. **Standardized moneyness:** Unlike the other parameters, volatility can not be obtained directly from the market data but must be computed from a model. Since dispersion is proportional to volatility, standardizing moneyness with volatility yield gives

$$M_t = \frac{\ln\left(\frac{K}{F_t}\right)}{\sigma\sqrt{T-t}}.$$

   This is termed as the forward standard moneyness and measures moneyness in standard deviation units [31]. Nevertheless, in [31] Tompkins R. uses spot price rather than forward price.

3.2 Interpolation with Spline Functions

In this thesis, *spline interpolation* has been used as part of the numerical techniques in obtaining local volatility surfaces via non-parametric method. Hence, it becomes necessary to explain the fundamentals of this numerical method. In particular, interpolation of given data points using *natural cubic spline function* will be explored in details.

3.2.1 Cubic Spline Interpolation

Firstly, the use of cubic spline functions for interpolation is important because they produce smooth interpolating functions that are twice continuously differentiable. In Dupire's equation, the order of the partial differential equation is two which makes using these functions in interpolating the call option function suitable. The following gives a proper definition for any spline function:

Definition 3.6 (see [8])**.** A function S is called a *spline of degree k* if:

1. The domain of S is an interval $[a, b]$
2. $S, S', S'', S''', \ldots, S^{(k-1)}$ are all continuous functions on $[a, b]$.
3. There are points t_i, the knots of S, such that $a = t_1 < t_2 < t_3 < \cdots < t_n = b$ and such that S is a polynomial of degree $\leq k$ on each sub-interval $[t_i, t_{i+1}]$.

In the case of a cubic spline function, $k = 3$. Assume we want to interpolate by a cubic spline function whose knots coincide with the values of t_i's in the table below:

x	t_1	t_2	$\cdots$	t_n
y	y_1	y_2	$\cdots$	t_n

The t_i's are the knots and are assumed to be arranged in an ascending order. The function S consists of $n - 1$ cubic polynomial pieces, such as

$$S(x) = \begin{cases} S_1(x) & t_1 \leq x \leq t_2 \\ S_2(x) & t_2 \leq x \leq t_3 \\ \vdots & \\ S_{n-1}(x) & t_{n-1} \leq x \leq t_n \end{cases}$$

where S_i denotes the cubic polynomial that will be used on the sub-interval $[t_i, t_{i+1}]$.

The interpolation conditions are

1. $S(t_i) = y_i$ for $1 \leq i \leq n$, and
2. $\lim_{x \to t_i^-} S^{(k)}(t_i) = \lim_{x \to t_i^+} S^{(k)}(t_i)$ for $k = 0, 1, 2$.

   These continuity conditions are imposed only at the interior knots $t_2, t_3, \ldots, t_{n-1}$ because at each of these knots, two different cubic polynomials meets.

Since we have $n-1$ spline functions and there are 4 parameters in each, there are $4(n-1)$ parameters to be determined. From each condition stated above, there are a total of $3(n-2)+n=4n-6$ equations to solve for these parameters. Hence, there are $(4n-4)-(4n-6)=2$ free parameters left to choose. In *natural cubic spline*, these parameters are chosen to satisfy

$$S''(t_1)=S''(t_n)=0.$$

Complete documentation of *cubic spline interpolation* can be found, for instance in [8].

3.3 Numerical Parametric Method

As discussed earlier, parametric method deals with an initial specification of a functional form for the parameter to be estimated and with a regressive algorithm, such function can be estimated for in and out of the sample data points. In other words, it's *a priori* estimation of a parameter since an initial variational functional form is given to the parameter. This section will be organized as follows: Section 3.3.1 will deal with literature survey on *parametric method*. Section 3.3.2 will deal with some of the assumptions that were used in obtaining the local volatility surfaces. Section 3.3.3 will deal with the mathematical structures involved with parametric method via using Dumas parametrization as a choice of parametrizing the implied volatility function. Section 3.3.4 will address the mathematical structures involved with parametrizing the implied volatility function in terms of moneyness rather than strike price. In addition, the detailed algorithm used in obtaining the local volatility surfaces via parametric method will be given. Consequently, analysis of the surfaces obtained will be made. Section 3.3.5 will be about the challenges involved in obtaining local volatility surfaces via parametric method. Suggestions on how to tackle these difficulties will also be proposed.

3.3.1 Dumas Parametric Method

In one of the prolific papers written by Dumas B., Fleming J., and Whaley R.E. [13], several functional forms for a *deterministic volatility function (DVF)* were tested by using "S&P 500 stock index" data. The performance of each of the functional forms were compared against each other and conclusions were drawn.

The following functional forms were focused on

1. Model 0: $\sigma_{imp}(T,K)=b_0$
2. Model 1: $\sigma_{imp}(T,K)=b_0+b_1K+b_2K^2$
3. Model 2: $\sigma_{imp}(T,K)=b_0+b_1K+b_2K^2+b_3T+b_5KT$
4. Model 3: $\sigma_{imp}(T,K)=b_0+b_1K+b_2K^2+b_3T+b_4T^2+b_5KT$

As we can observe from the various specified functions, Model 0 represents the case of Black-Scholes model with constant volatility; Model 1 tries to capture the variation in volatility inherent in the asset price; Models 2 and 3 capture additional variation attributable to maturity [14]. The quadratic forms of DVF (Models 2 & 3) were found to be particularly robust and stable through time, hence, we will base our polynomial parametrization on Model 2. This method of parametrization is widely used to determine the "Practitioner's Black-Scholes" (PBS) prices [14]. PBS involves the use of implied volatility function to determine the corresponding Black-Scholes prices [4]. Calculating PBS prices can be summarized in three steps as follows (see [4]):

1. Invert the Black-Scholes equation for the available data points and obtain the implied volatilities $\sigma_{imp}(T, K)$;
2. The implied volatilities are regressed against a quadratic polynomial;
3. The fitted implied volatilities are then plugged back into the Black-Scholes equation to get the practitioner's price.

To be able to use PBS above in obtaining local volatility surfaces, additional numerical computations are necessary:

1. Take the $1st$ partial derivative of call option function under local volatility with respect to maturity T;
2. Take the $1st$ partial derivative of call option function under local volatility with respect to strike prices K;
3. Take the $2nd$ partial derivative of call option function under local volatility with respect to strike prices K;
4. Plug these derivatives into the Dupire's local volatility equation to obtain the local volatilities.
5. For a mesh grid (Ti, Ki, LV) created, that is, for maturities T, strike prices K, and local volatilities LV respectively, obtain the local volatility surface.

The surface obtained can then be used as a first-hand estimation or in conjunction with other more complex models on pricing exotic options as well as hedging purposes.

3.3.1.1 Alternative Method for Local Volatility Surface using Dumas Parametrization

Another effective method that could be used in obtaining the local volatilities depending on the intent of application involves the use of moneyness in the parametrized implied volatility function instead of strike price. This is the method used in [4]. The choice of moneyness that will be used is

$$M = \frac{\log\left(\frac{K}{F_T}\right)}{\sqrt{T}}.$$

Hence, in the application of PBS, the parameters in Model 2 are determined by regressing the implied volatility data against the choice of quadratic function. Consequently, the practitioner's Black-Scholes prices can then be calculated for the implied volatility surface obtained. This will be implemented in Section 3.3.4 of this thesis.

3.3.2 Proposed Assumptions for Local Volatility Surfaces using Dupire's Equation

In this section, the implied volatility data used in the algorithmic procedures will be described. Also, the main assumptions used in this thesis in calculating the partial derivatives in Dupire's equation as compared to the ones used in [4] will be emphasized.

Firstly, let's describe the data that will be used to test our algorithm. The data was taken from [2], which is a collection of bid-ask spreads of Black-Scholes implied volatilities of European call options on *"S&P 500 stock index"* in April, 1999. The spread consists of 161 observations all together across 18 strikes and 12 maturities. The raw data was refined for effective integration into our algorithm. This was done by averaging the bid and ask implied volatilities for each strike price K and maturity T.

Secondly, let us compare the crucial assumption Cerrato M. used in [4] to obtain local volatility surface and the one proposed in this thesis. He had directly used Black-Scholes call option price function as the call function while taking the partial derivatives in Dupire's equation. However, we have taken the appropriate steps in calculating these partial derivatives by considering that the call option functions of Black-Scholes and local volatility are different but with the same functional values at some data points. To summarize this mathematically, we assume that

$$C(F_T, T, \sigma_{LV}(T, K), K) = C_{BS}(F_T, T, \sigma_{imp}(T, K), K)$$

for the given data samples, where $C(F_T, T, \sigma_{LV}(T, K), K)$ is the call option function from the local volatility model and $C_{BS}(F_T, T, \sigma_{imp}(T, K), K)$ is the call option function from the Black-Scholes model. With this assumption, the partial derivatives of the local volatility call option function with respect to maturity T and strike K are taken by relating the dependence of both call option functions on those independent variables.

To illustrate the importance of this assumption in the one-dimensional case, consider two real valued functions

$$f : \mathbb{R} \to \mathbb{R}, \text{ for } i = 1, 2.$$

such that $f_1(x) = x^2$ and $f_2(x) = 2x$. Although for $x = 2$, $f_1(2) = f_2(2)$, but their derivatives with respect to x at the point $x = 2$ are not equal. That is, ${f_1}'(2) = 4$ and ${f_2}'(x) = 2$. With this in mind, we constructed our method accordingly.

3.3.3 Obtaining Local Volatility Surface using Dupire's Equation

In this part, the explicit mathematical formulations used in the algorithm in obtaining the local volatility surface will be derived. This method involves using the Dupire's equation described in Equation (2.52),

$$\sigma^2(T,K) = \frac{\frac{\partial C(F_T,T,K)}{\partial T}}{\frac{1}{2}K^2\frac{\partial^2 C(F_T,T,K)}{\partial K^2}},$$

where the call option is a function of forward price F_T: $C(F_T, T, \sigma_{LV}(T,K), K)$. Hence, it's necessary to explicitly determine what each partial derivative in Equation (2.52) represents. Since we will use these derivatives in conjunction with Dumas parametric method, it is important to specify our choice of parametric function and write out the various partial derivatives necessary. The choice of Black-Scholes implied volatility function is given below:

$$\sigma_{imp}(T,K) = \sum_{i,j=0}^{n} a_i B_{ij}(T,K) = a_0 + a_1 K + a_2 K^2 + a_3 T + a_4 KT, \tag{3.2}$$

where the a_i's are the unknown parameters to be determined using some numerical techniques with the available data samples and $B_{ij}(T,K)$ are basis functions in terms of the independent variables T and K. This choice of parametrization is due to the assumption that the implied volatility is directly dependent on maturity T and the strike price K.

Lemma 3.1. *If the implied volatility function takes the form of Equation (3.2), then its partial derivatives are*

$$\frac{\partial \sigma_{imp}(T,K)}{\partial K} = a_1 + 2a_2 K + a_4 T.$$

$$\frac{\partial \sigma_{imp}(T,K)}{\partial K^2} = 2a_2.$$

$$\frac{\partial \sigma_{imp}(T,K)}{\partial T} = a_3 + a_4 K.$$

Proof. The proof of the above lemma is easy and straightforward. The reader can verify that the following calculations:

$$\frac{\partial \sigma_{imp}(T,K)}{\partial K} = a_1 + 2a_2 K + a_4 T, \tag{3.3}$$

$$\frac{\partial \sigma_{imp}(T,K)}{\partial K^2} = 2a_2, \tag{3.4}$$

and

$$\frac{\partial \sigma_{imp}(T,K)}{\partial T} = a_3 + a_4 K \tag{3.5}$$

to complete the proof. □

Proposition 3.2. *For local volatility option prices calibrated to Black-Scholes option prices via implied volatilities such that*

$$C(F_T, T, \sigma_{LV}(T,K), K) = C_{BS}(F_T, T, \sigma_{imp}(T,K), K)$$

for some data samples, the following partial derivatives hold:

$$\begin{aligned}
\frac{\partial C}{\partial T} &= \frac{\partial C_{BS}}{\partial T} + \frac{\partial C_{BS}}{\partial \sigma_{imp}}\frac{\partial \sigma_{imp}}{\partial T}, \\
\frac{\partial C}{\partial K} &= \frac{\partial C_{BS}}{\partial K} + \frac{\partial C_{BS}}{\partial \sigma_{imp}}\frac{\partial \sigma_{imp}}{\partial K}, \\
\frac{\partial^2 C}{\partial K^2} &= \frac{\partial^2 C_{BS}}{\partial K^2} + \frac{\partial^2 \sigma_{imp}}{\partial K^2}\left(\frac{\partial C_{BS}}{\partial \sigma_{imp}}\right) + \left(\frac{\partial \sigma_{imp}}{\partial K}\right)^2\left(\frac{\partial^2 C_{BS}}{\partial \sigma_{imp}^2}\right) \\
&\quad + \frac{\partial \sigma_{imp}}{\partial K}\left(\frac{\partial}{\partial \sigma_{imp}}\left(\frac{\partial C_{BS}}{\partial K}\right) + \frac{\partial}{\partial K}\left(\frac{\partial C_{BS}}{\partial \sigma_{imp}}\right)\right),
\end{aligned} \tag{3.6}$$

where F_T is taken to be without dividends, that is, $F_T = S_0 e^{rT}$.

Proof. First, determine the partial derivative of the call option with respect to maturity T.

$$\frac{\partial C}{\partial T} = \frac{\partial C_{BS}}{\partial T} + \frac{\partial C_{BS}}{\partial \sigma_{imp}}\frac{\partial \sigma_{imp}}{\partial T}. \tag{3.7}$$

Each partial derivative can be calculated from the Black-Scholes model and from Equation (3.5).

Next, determine the partial derivative of the call option with respect to strike K.

$$\frac{\partial C}{\partial K} = \frac{\partial C_{BS}}{\partial K} + \frac{\partial C_{BS}}{\partial \sigma_{imp}}\frac{\partial \sigma_{imp}}{\partial K}. \tag{3.8}$$

Each partial derivative term can be calculated from the Black-Scholes model and from Equation (3.3).

Finally, determine the 2nd partial derivative of the call option with respect to strike K.

$$\begin{aligned}
\frac{\partial^2 C}{\partial K^2} &= \frac{\partial}{\partial K}\left(\frac{\partial C_{BS}}{\partial K}\right) + \frac{\partial}{\partial \sigma_{imp}}\left(\frac{\partial C_{BS}}{\partial K}\right)\frac{\partial \sigma_{imp}}{\partial K} \\
&\quad + \frac{\partial \sigma_{imp}}{\partial K}\left(\frac{\partial}{\partial K}\left(\frac{\partial C_{BS}}{\partial \sigma_{imp}}\right) + \frac{\partial^2 C_{BS}}{\partial \sigma_{imp}^2}\frac{\partial \sigma_{imp}}{\partial K}\right) + \frac{\partial C_{BS}}{\partial \sigma_{imp}}\frac{\partial^2 \sigma_{imp}}{\partial K^2} \\
&= \frac{\partial^2 C_{BS}}{\partial K^2} + \frac{\partial^2 \sigma_{imp}}{\partial K^2}\left(\frac{\partial C_{BS}}{\partial \sigma_{imp}}\right) + \left(\frac{\partial \sigma_{imp}}{\partial K}\right)^2\left(\frac{\partial^2 C_{BS}}{\partial \sigma_{imp}^2}\right) \\
&\quad + \frac{\partial \sigma_{imp}}{\partial K}\left(\frac{\partial}{\partial \sigma_{imp}}\left(\frac{\partial C_{BS}}{\partial K}\right) + \frac{\partial}{\partial K}\left(\frac{\partial C_{BS}}{\partial \sigma_{imp}}\right)\right).
\end{aligned} \tag{3.9}$$

This is quite long to calculate in one equation, therefore, each partial derivative will be determined separately and the reader can put them together in the algorithms by defining each as a variable.

Starting with the partial derivative of Black-Scholes call function with respect to K, we get

$$\frac{\partial C_{BS}}{\partial K} = F_T \Phi(d_1)\frac{\partial d_1}{\partial K} - \left(N(d_2) + K\Phi(d_2)\frac{\partial d_2}{\partial K}\right). \tag{3.10}$$

To determine explicitly what $\frac{\partial d_1}{\partial K}$ and $\frac{\partial d_2}{\partial K}$ represent, we have

$$\begin{aligned}\frac{\partial d_1}{\partial K} &= \frac{\sigma_{imp}\sqrt{T}\left(\frac{-1}{K} + \sigma_{imp}T\frac{\partial \sigma_{imp}}{\partial K}\right) - \left(\ln\left(\frac{F_T}{K}\right) + \frac{\sigma_{imp}^2 T}{2}\right)\left(\sqrt{T}\frac{\partial \sigma_{imp}}{\partial K}\right)}{\sigma_{imp}^2 T} \\ &= \frac{\frac{\sigma_{imp}^2 T^{\frac{3}{2}}}{2}\frac{\partial \sigma_{imp}}{\partial K} - \ln\left(\frac{F_T}{K}\right)\sqrt{T}\frac{\partial \sigma_{imp}}{\partial K} - \frac{\sigma_{imp}\sqrt{T}}{K}}{\sigma_{imp}^2 T} \\ &= \frac{\sqrt{T}}{2}\frac{\partial \sigma_{imp}}{\partial K} - \frac{\ln\left(\frac{F_T}{K}\right)}{\sigma_{imp}^2\sqrt{T}}\frac{\partial \sigma_{imp}}{\partial K} - \frac{1}{K\sigma_{imp}\sqrt{T}}\end{aligned} \tag{3.11}$$

and

$$\begin{aligned}\frac{\partial d_2}{\partial K} &= \frac{\partial d_1}{\partial K} - \sqrt{T}\frac{\partial \sigma_{imp}}{\partial K} \\ &= -\left(\frac{\ln\left(\frac{F_T}{K}\right)}{\sigma_{imp}^2\sqrt{T}}\frac{\partial \sigma_{imp}}{\partial K} + \frac{1}{K\sigma_{imp}\sqrt{T}} + \frac{\sqrt{T}}{2}\frac{\partial \sigma_{imp}}{\partial K}\right).\end{aligned} \tag{3.12}$$

Substituting Equation (3.11) and Equation (3.12) into Equation (3.10) and using $\Phi(d_1) = \frac{K}{F_T}\Phi(d_2)$, we obtain

$$\begin{aligned}\frac{\partial C_{BS}}{\partial K} &= F_T\Phi(d_1)\frac{\sqrt{T}}{2}\frac{\partial \sigma_{imp}}{\partial K} - F_T\Phi(d_1)\frac{\ln\left(\frac{F_T}{K}\right)}{\sigma_{imp}^2\sqrt{T}}\frac{\partial \sigma_{imp}}{\partial K} - \frac{F_T\Phi(d_1)}{K\sigma_{imp}\sqrt{T}} \\ &\quad - N(d_2) + \frac{K\Phi(d_2)\ln\left(\frac{F_T}{K}\right)}{\sigma_{imp}^2\sqrt{T}}\frac{\partial \sigma_{imp}}{\partial K} + \frac{K\Phi(d_2)}{K\sigma_{imp}\sqrt{T}} + \frac{K\Phi(d_2)\sqrt{T}}{2}\frac{\partial \sigma_{imp}}{\partial K} \\ &= \frac{\partial \sigma_{imp}}{\partial K}\left(F_T\Phi(d_1)\frac{\sqrt{T}}{2} - F_T\Phi(d_1)\frac{\ln\left(\frac{F_T}{K}\right)}{\sigma_{imp}^2\sqrt{T}}\frac{\partial \sigma_{imp}}{\partial K}\right) \\ &\quad + \frac{\partial \sigma_{imp}}{\partial K}\left(F_T\Phi(d_1)\frac{\ln\left(\frac{F_T}{K}\right)}{\sigma_{imp}^2\sqrt{T}}\frac{\partial \sigma_{imp}}{\partial K}\right) - N(d_2) + \frac{K\Phi(d_2)\sqrt{T}}{2}\frac{\partial \sigma_{imp}}{\partial K} \\ &= \frac{\partial \sigma_{imp}}{\partial K}\left(F_T\Phi(d_1)\sqrt{T}\right) - N(d_2).\end{aligned} \tag{3.13}$$

Taking the partial derivative of Equation (3.13) with respect to strike K gives

$$\frac{\partial^2 C_{BS}}{\partial K^2} = \frac{\partial \sigma_{imp}}{\partial K}F_T\sqrt{T}\frac{\partial \Phi(d_1)}{\partial K} + F_T\Phi(d_1)\sqrt{T}\frac{\partial^2 \sigma_{imp}}{\partial K^2} - \Phi(d_2)\frac{\partial \Phi(d_2)}{\partial K}. \tag{3.14}$$

Thence, to Determine what $\frac{\partial \Phi(d_1)}{\partial K}$ represents, we calculate

$$\begin{aligned}\frac{\partial \Phi(d_1)}{\partial K} &= -\Phi(d_1) d_1 \frac{\partial d_1}{\partial K} \\ &= -\Phi(d_1) d_1 \left(\frac{\sqrt{T}}{2} \frac{\partial \sigma_{imp}}{\partial K} - \frac{\ln\left(\frac{F_T}{K}\right)}{\sigma_{imp}^2 \sqrt{T}} \frac{\partial \sigma_{imp}}{\partial K} - \frac{1}{K \sigma_{imp} \sqrt{T}} \right) \\ &= -\frac{\Phi(d_1) d_1 \sqrt{T}}{2} \frac{\partial \sigma_{imp}}{\partial K} + \frac{\Phi(d_1) d_1 \ln\left(\frac{F_T}{K}\right)}{\sigma_{imp}^2 \sqrt{T}} \frac{\partial \sigma_{imp}}{\partial K} + \frac{\Phi(d_1) d_1}{K \sigma_{imp} \sqrt{T}}.\end{aligned} \tag{3.15}$$

Substituting Equation (3.15) into Equation (3.14) results in

$$\begin{aligned}\frac{\partial^2 C_{BS}}{\partial K^2} &= -\frac{F_T \sqrt{T} d_1 \Phi(d_1)}{2} \left(\frac{\partial \sigma_{imp}}{\partial K} \right)^2 + \frac{F_T \sqrt{T} d_1 \Phi(d_1) \ln\left(\frac{F_T}{K}\right)}{\sigma_{imp}^2} \left(\frac{\partial \sigma_{imp}}{\partial K} \right)^2 \\ &+ \frac{F_T \Phi(d_1) d_1}{K \sigma_{imp}} \frac{\partial \sigma_{imp}}{\partial K} + F_T \sqrt{T} \Phi(d_1) \frac{\partial^2 \sigma_{imp}}{\partial K^2} \\ &+ \frac{\Phi(d_1) d_1 \ln\left(\frac{F_T}{K}\right)}{\sigma_{imp}^2 \sqrt{T}} \frac{\partial \sigma_{imp}}{\partial K} + \frac{\Phi(d_2)}{K \sigma_{imp} \sqrt{T}} + \frac{\Phi(d_2) \sqrt{T}}{2} \frac{\partial \sigma_{imp}}{\partial K}.\end{aligned} \tag{3.16}$$

Note here, that

$$\begin{aligned}\frac{\partial \Phi(d_1)}{\partial K} &= \frac{\partial}{\partial K} \left(\frac{1}{\sqrt{2\pi}} e^{-(\frac{d_1^2}{2})} \right) \\ &= \frac{1}{\sqrt{2\pi}} e^{-(\frac{d_1^2}{2})} \left(\frac{-1}{2} 2 d_1 \right) \frac{\partial d_1}{\partial K} \\ &= -\Phi(d_1) d_1 \frac{\partial d_1}{\partial K}.\end{aligned} \tag{3.17}$$

It remains to calculate the partial derivatives $\frac{\partial C_{BS}}{\partial \sigma_{imp}}$, $\frac{\partial^2 C_{imp}}{\partial \sigma_{imp}^2}$, $\frac{\partial}{\partial K} \left(\frac{\partial C_{BS}}{\partial \sigma_{imp}} \right)$, as well as $\frac{\partial}{\partial \sigma_{imp}} \left(\frac{\partial C_{BS}}{\partial K} \right)$:

$$\frac{\partial C_{BS}}{\partial \sigma_{imp}} = F_T \Phi(d_1) \frac{\partial d_1}{\partial \sigma_{imp}} - K \Phi(d_2) \frac{\partial d_2}{\partial \sigma_{imp}}, \tag{3.18}$$

where

$$\begin{aligned}\frac{\partial d_1}{\partial \sigma_{imp}} &= \frac{\sigma_{imp} \sqrt{T} \left(\sigma_{imp} T \right) - \left(\frac{\sigma_{imp}^2 T}{2} + \ln\left(\frac{F_T}{K}\right) \right) \sqrt{T}}{\sigma_{imp}^2 T} \\ &= \frac{\sigma_{imp}^2 T^{\frac{3}{2}} - \frac{\sigma_{imp}^2 T^{\frac{3}{2}}}{2} - \ln\left(\frac{F_T}{K}\right) \sqrt{T}}{\sigma_{imp}^2 T} \\ &= \frac{\frac{\sigma_{imp}^2 T^{\frac{3}{2}}}{2} - \ln\left(\frac{F_T}{K}\right) \sqrt{T}}{\sigma_{imp}^2 T} \\ &= \frac{\sqrt{T}}{2} - \frac{\ln\left(\frac{F_T}{K}\right)}{\sigma_{imp}^2 \sqrt{T}}\end{aligned} \tag{3.19}$$

and

$$\begin{aligned}\frac{\partial d_2}{\partial \sigma_{imp}} &= \frac{\partial d_1}{\partial \sigma_{imp}} - \sqrt{T} \\ &= \frac{\sqrt{T}}{2} - \frac{\ln\left(\frac{F_T}{K}\right)}{\sigma_{imp}^2\sqrt{T}} - \sqrt{T} \\ &= -\frac{\ln\left(\frac{F_T}{K}\right)}{\sigma_{imp}^2\sqrt{T}} - \frac{\sqrt{T}}{2}.\end{aligned} \tag{3.20}$$

Substitute Equation (3.19) and Equation (3.20) into Equation (3.18) to get

$$\begin{aligned}\frac{\partial C_{BS}}{\partial \sigma_{imp}} &= F_T\Phi(d_1)\left(\frac{\sqrt{T}}{2} - \frac{\ln\left(\frac{F_T}{K}\right)}{\sigma_{imp}^2\sqrt{T}}\right) + K\Phi(d_2)\frac{\ln\left(\frac{F_T}{K}\right)}{\sigma_{imp}^2\sqrt{T}} + \frac{K\Phi(d_2)\sqrt{T}}{2} \\ &= F_T\Phi(d_1)\sqrt{T}.\end{aligned} \tag{3.21}$$

Taking the partial derivative of Equation (3.21) with respect to σ_{imp}, we have

$$\begin{aligned}\frac{\partial^2 C_{BS}}{\partial \sigma_{imp}^2} &= F_T\sqrt{T}\frac{\partial \Phi(d_1)}{\partial \sigma_{imp}} = -F_T\sqrt{T}\Phi(d_1)d_1\frac{\partial d_1}{\partial \sigma_{imp}} \\ &= -F_T\sqrt{T}\Phi(d_1)d_1\left(\frac{\sqrt{T}}{2} - \frac{\ln\left(\frac{F_T}{K}\right)}{\sigma_{imp}^2\sqrt{T}}\right) \\ &= \frac{-F_T T\Phi(d_1)d_1}{2} + \frac{F_T\Phi(d_1)d_1\ln\left(\frac{F_T}{K}\right)}{\sigma_{imp}^2}.\end{aligned} \tag{3.22}$$

Also, taking the partial derivative of Equation (3.21) with respect to K yields

$$\begin{aligned}\frac{\partial}{\partial K}\left(\frac{\partial C_{BS}}{\partial \sigma_{imp}}\right) &= F_T\sqrt{T}\frac{\partial \Phi(d_1)}{\partial K} = -\Phi(d_1)d_1F_T\sqrt{T}\frac{\partial d_1}{\partial K} \\ &= -\Phi(d_1)d_1F_T\sqrt{T}\left(\frac{\sqrt{T}}{2}\frac{\partial \sigma_{imp}}{\partial K} - \frac{\ln\left(\frac{F_T}{K}\right)}{\sigma_{imp}^2\sqrt{T}}\frac{\partial \sigma_{imp}}{\partial K} - \frac{1}{K\sigma_{imp}\sqrt{T}}\right) \\ &= -\frac{\Phi(d_1)d_1F_T T}{2}\frac{\partial \sigma_{imp}}{\partial K} + \frac{\Phi(d_1)d_1F_T}{\sigma_{imp}^2}\ln\left(\frac{F_T}{K}\right)\frac{\partial \sigma_{imp}}{\partial K} \\ &+ \frac{\Phi(d_1)d_1F_T}{K\sigma_{imp}}.\end{aligned} \tag{3.23}$$

Finally,

$$
\begin{aligned}
\frac{\partial}{\partial \sigma_{imp}}\left(\frac{\partial C_{BS}}{\partial K}\right) &= \frac{\partial}{\partial \sigma_{imp}}\left(\frac{\partial \sigma_{imp}}{\partial K}\left(F_T\Phi(d_1)\sqrt{T}\right) - N(d_2)\right) \\
&= F_T\sqrt{T}\frac{\partial \sigma_{imp}}{\partial K}\frac{\partial \Phi(d_1)}{\partial \sigma_{imp}} - \Phi(d_2)\frac{\partial \Phi(d_2)}{\partial \sigma_{imp}} \\
&= F_T\sqrt{T}\frac{\partial \sigma_{imp}}{\partial K}\left(-\Phi(d_1)d_1\frac{\partial d_1}{\partial \sigma_{imp}}\right) - \Phi(d_2)\frac{\partial d_2}{\partial \sigma_{imp}} \\
&= -F_T\sqrt{T}\frac{\partial \sigma_{imp}}{\partial K}\Phi(d_1)d_1\left(\frac{\sqrt{T}}{2} - \frac{\ln\left(\frac{F_T}{K}\right)}{\sigma_{imp}^2\sqrt{T}}\right) \\
&- \Phi(d_2)\left(-\frac{\ln\left(\frac{F_T}{K}\right)}{\sigma_{imp}^2\sqrt{T}} - \frac{\sqrt{T}}{2}\right) \\
&= -\frac{F_Td_1\Phi(d_1)T}{2}\frac{\partial \sigma_{imp}}{\partial K} + \frac{F_Td_1\Phi(d_1)}{\sigma_{imp}^2}\ln\left(\frac{F_T}{K}\right)\frac{\partial \sigma_{imp}}{\partial K} \qquad (3.24) \\
&+ \frac{\Phi(d_2)\ln\left(\frac{F_T}{K}\right)}{\sigma_{imp}^2\sqrt{T}} + \frac{\Phi(d_2)\sqrt{T}}{2} \\
&= \frac{\partial}{\partial K}\left(\frac{\partial C_{BS}}{\partial \sigma_{imp}}\right) + \frac{\Phi(d_2)}{\sigma_{imp}^2\sqrt{T}}\ln\left(\frac{F_T}{K}\right) \\
&+ \frac{\Phi(d_2)\sqrt{T}}{2} - \frac{F_Td_1\Phi(d_1)}{K\sigma_{imp}} \\
&= \frac{\partial}{\partial K}\left(\frac{\partial C_{BS}}{\partial \sigma_{imp}}\right) + \frac{\Phi(d_1)F_T}{\sigma_{imp}^2K\sqrt{T}}\ln\left(\frac{F_T}{K}\right) \\
&+ \frac{F_T\Phi(d_1)\sqrt{T}}{2K} - \frac{F_Td_1\Phi(d_1)}{K\sigma_{imp}}.
\end{aligned}
$$

Plugging all these derivatives into their respective places completes the proof. □

3.3.3.1 Obtaining Local Volatility Surface Using Partial Derivatives of Equation (2.87)

As a way to compare the local volatility surfaces obtained using Equation (2.52) with the one that will be obtained using Equation (2.87), it becomes necessary to also determine the analytic derivations of the partial derivatives in the latter.

Proposition 3.3. *For local volatility option prices calibrated to Black-Scholes prices via implied volatilities such that*

$$C(F_T, T, \sigma_{LV}(T,K), K) = C_{BS}(F_T, T, \sigma_{imp}(T,K), K)$$

for some data samples, the following partial derivatives hold using Equation (2.87):

$$\frac{\partial w}{\partial T} = \sigma_{imp}^2 + 2T\sigma_{imp}\frac{\partial \sigma_{imp}}{\partial T};$$

$$\frac{\partial w}{\partial y} = 2TK\sigma_{imp}\frac{\partial \sigma_{imp}}{\partial K} + 2T(\frac{-1}{r})\frac{\partial \sigma_{imp}}{\partial T} - (\frac{1}{r})\sigma_{imp}^2;$$

$$\begin{aligned}\frac{\partial^2 w}{\partial y^2} &= \frac{\partial \sigma_{imp}}{\partial K}\left(-4K\sigma_{imp}\left(\frac{1}{r}\right)\right) + 2TK^2\left(\frac{\partial \sigma_{imp}}{\partial K}\right)^2 \\ &- 2TK\left(\frac{1}{r}\right)\frac{\partial \sigma_{imp}}{\partial K}\frac{\partial \sigma_{imp}}{\partial T} + 2TK\sigma_{imp}\frac{\partial}{\partial y}\left(\frac{\partial \sigma_{imp}}{\partial K}\right) \\ &- 2T\left(\frac{1}{r}\right)\frac{\partial}{\partial y}\left(\frac{\partial \sigma_{imp}}{\partial T}\right) + \frac{\partial \sigma_{imp}}{\partial T}\left(\frac{2\sigma_{imp}}{r^2} + \frac{2}{r^2}\right),\end{aligned} \tag{3.25}$$

where w is the total variance, y is the log-strike and F_T is taken to be without dividends, that is, $F_T = S_0e^{rT}$.

Proof. First, determine the partial derivative of total variance w with respect to maturity T as

$$\frac{\partial w}{\partial T} = \sigma_{imp}^2 + 2T\sigma_{imp}\frac{\partial \sigma_{imp}}{\partial T}. \tag{3.26}$$

Next, determine the partial derivative of total variance w with respect to log-strike y as

$$\begin{aligned}\frac{\partial w}{\partial y} &= 2T\sigma_{imp}\frac{\partial \sigma_{imp}}{\partial y} + \sigma_{imp}^2\frac{\partial T}{\partial y} \\ &= 2T\sigma_{imp}\left(\frac{\partial \sigma_{imp}}{\partial K}\frac{\partial K}{\partial y} + \frac{\partial \sigma_{imp}}{\partial T}\frac{\partial T}{\partial y}\right) - \frac{1}{r}\sigma_{imp}^2 \\ &= 2TF_Te^y\sigma_{imp}\frac{\partial \sigma_{imp}}{\partial K} + 2T\left(\frac{-1}{r}\right)\frac{\partial \sigma_{imp}}{\partial T} - \frac{1}{r}\sigma_{imp}^2 \\ &= 2TK\sigma_{imp}\frac{\partial \sigma_{imp}}{\partial K} + 2T\left(\frac{-1}{r}\right)\frac{\partial \sigma_{imp}}{\partial T} - \frac{1}{r}\sigma_{imp}^2.\end{aligned} \tag{3.27}$$

Finally, determine the 2nd partial derivative of total variance w with respect to log-strike y as

$$\begin{aligned}\frac{\partial^2 w}{\partial y^2} &= 2TF_Te^y\sigma_{imp}\frac{\partial \sigma_{imp}}{\partial K} + 2F_Te^y\sigma_{imp}\frac{\partial \sigma_{imp}}{\partial K}\frac{\partial T}{\partial y} \\ &+ 2e^yT\sigma_{imp}\frac{\partial \sigma_{imp}}{\partial K}\left(-Ke^{-y}\right) \\ &+ 2TF_Te^y\left(\frac{\partial \sigma_{imp}}{\partial K}\left(\frac{\partial \sigma_{imp}}{\partial K}\frac{\partial K}{\partial y} + \frac{\partial \sigma_{imp}}{\partial T}\frac{\partial T}{\partial y}\right)\right) \\ &+ 2F_TTe^y\left(\sigma_{imp}\frac{\partial}{\partial y}\left(\frac{\partial \sigma_{imp}}{\partial K}\right)\right) + 2\left(\frac{-1}{r}T\right)\frac{\partial}{\partial y}\left(\frac{\partial \sigma_{imp}}{\partial T}\right) \\ &+ 2(\frac{-1}{r})\left(\frac{\partial \sigma_{imp}}{\partial T}\right)(\frac{\partial T}{\partial y}) + 2(\frac{-1}{r})\sigma_{imp}\left(\frac{\partial \sigma_{imp}}{\partial K}\frac{\partial K}{\partial y} + \frac{\partial \sigma_{imp}}{\partial T}\frac{\partial T}{\partial y}\right);\end{aligned}$$

this can further be simplified to give

$$\begin{aligned}
\frac{\partial^2 w}{\partial y^2} &= \frac{\partial \sigma_{imp}}{\partial K}\left(-4F_T e^y \sigma_{imp}\left(\frac{1}{r}\right)\right) + 2TF_T^2 e^{2y}\left(\frac{\partial \sigma_{imp}}{\partial K}\right)^2 \\
&- 2TF_T e^y \left(\frac{1}{r}\right)\frac{\partial \sigma_{imp}}{\partial K}\frac{\partial \sigma_{imp}}{\partial T} + 2TF_T e^y \sigma_{imp}\frac{\partial}{\partial y}\left(\frac{\partial \sigma_{imp}}{\partial K}\right) \\
&- 2T\left(\frac{1}{r}\right)\frac{\partial}{\partial y}\left(\frac{\partial \sigma_{imp}}{\partial T}\right) + \frac{\partial \sigma_{imp}}{\partial T}\left(\frac{2\sigma_{imp}}{r^2} + \frac{2}{r^2}\right) \\
&= \frac{\partial \sigma_{imp}}{\partial K}\left(-4K\sigma_{imp}\left(\frac{1}{r}\right)\right) + 2TK^2\left(\frac{\partial \sigma_{imp}}{\partial K}\right)^2 \\
&- 2TK\left(\frac{1}{r}\right)\frac{\partial \sigma_{imp}}{\partial K}\frac{\partial \sigma_{imp}}{\partial T} + 2TK\sigma_{imp}\frac{\partial}{\partial y}\left(\frac{\partial \sigma_{imp}}{\partial K}\right) \\
&- 2T\left(\frac{1}{r}\right)\frac{\partial}{\partial y}\left(\frac{\partial \sigma_{imp}}{\partial T}\right) + \frac{\partial \sigma_{imp}}{\partial T}\left(\frac{2\sigma_{imp}}{r^2} + \frac{2}{r^2}\right).
\end{aligned} \tag{3.28}$$

Note that the use of $\frac{\partial T}{\partial y} = \frac{-1}{r}$, $\frac{\partial F_T}{\partial y} = -Ke^{-y}$, and $\frac{\partial K}{\partial y} = F_T e^y = K$ completes the proof. □

3.3.3.2 Algorithm

Having found all the necessary partial derivatives in Section 3.3.3, they will be used in the algorithm. In this section, detailed algorithm of how to obtain the implied volatility surface and local volatility surfaces will be given. The algorithm adopted for this method is one originally used in the book of Cerrato M. (see [4]) with modifications on the codes.

The notations for all the algorithms are: T is the time to maturity, K and X represents the strike prices, S is the asset spot price, IV is the implied volatility, r is the risk-free rate, q is the dividend yield, F_T is the forward price, mn is the moneyness, w is the Black-Scholes total variance, and y is the log-strike.

The algorithmic steps are given below:

1. Load data consisting of T, K, S, IV, r, q.
2. Calculate $F_T = Se^{(r-q)T}$ and `mn` $= \frac{\log(K/F_T)}{\sqrt{T}}$ for each T and K.
3. Define a function (`func1(pars`$, X, T, IV$`)`) containing the choice of the parametric function.

   $\sigma_{imp} =$ `pars(1)`$+$`pars(2)`$X+$`pars(3)`X^2+`pars(4)`$T+$`pars(5)`XT

   and returns `e`, the error as the output, which is the difference in the values of σ and IV (i.e. `e=`σ_{imp}`-IV`).

4. Using a starting value, solve (`func1(pars`$, X, T, IV$)) non-linearly in least-squares using the MATLAB built-in function "`lsqnonlin`".

   This step as a whole solves the problem and its outputs are the minimizing parameters, `pars(1),..., pars(5)` with the norm of its residuals.

5. Discretize K into K_i and T into T_i values and create a mesh grid of the discretized values, (i.e. $[K_m, T_m] = \texttt{meshgrid}(K_i, T_i)$).

6. Re-calculate $F_T = Se^{(r-q)T_m}$ and $\texttt{Mn} = \frac{\log{(K_m/F_T)}}{\sqrt{T_m}}$ for the mesh grid created.

7. Define a function (`func2(pars`$, K_m, T_m$)) that contains the choice of parametrization and with the input above, calculates the IV_m for the grid created.

8. Using mesh (T_m, K_m, IV_m), obtain the implied volatility surface.

9. Interpolate q and $r - q$ across all T_i and K_i.

10. Calculate the corresponding Black-Scholes prices from the implied volatilities obtained.

11. Calculate the derivative of local volatility call option function with respect to T as derived in Equation (3.7).

12. Calculate the 1st derivative of local volatility call option function with respect to K as derived in Equation (3.13).

13. Calculate the 2nd derivative of local volatility call option function with respect to K as derived in Equation (3.16).

14. Using the Dupire equation, calculate the local variance and local volatility. Some values of local variance matrix may be negative which makes taking square root impossible. For such values, use either of the following:

    (a) Take the absolute value of the matrix in local variance.

    (b) Make the negative values of the matrix in the local variance zero.

    In this thesis, the two were tested. However, method 2 was preferred for all the plots solely because this only considers the points where the local volatility is defined. In other words, where the surfaces are free of arbitrage opportunities.

15. Using `meshgrid` (T_m, K_m, LV), obtain the local volatility surface for each method described above.

3.3.3.3 Detailed Algorithm for an Alternative Method

Alternatively, the local volatility surface obtained from the above algorithm can also be done using Equation (2.87). Following the determination of the implied volatility surface above, below highlights the additional steps needed to complete the algorithm.

1. Calculate the partial derivative of σ_{imp} with respect to T as in Equation (3.5).
2. Calculate the partial derivative of σ_{imp} with respect to K, as in Equation (3.3).
3. Calculate the 2nd partial derivative of σ_{imp} with respect to K as in Equation (3.4).
4. Calculate the partial derivative of w with respect to T as in Equation (3.26).
5. Calculate the partial derivative of w with respect to y, as in Equation (3.27).
6. Calculate the 2nd partial derivative of w with respect to y as in Equation (3.28).
7. Calculate the local variance and local volatility in Equation (2.87). However, some of the values of local variance may be negative which makes taking square root impossible. For such values, use either of the following:
   (a) Take the absolute value of the variance matrix.
   (b) Make the negative values of the variance matrix zero.
8. Using `meshgrid` (T_m, K_m, LV), obtain the local volatility surface for each method described above.

3.3.3.4 Results and Analysis of the Volatility Surfaces Obtained in Section 3.3.3.2 and Section 3.3.3.3

In this section, the surfaces obtained from implementing the algorithms described in Section 3.3.3.2 and Section 3.3.3.3 will be shown and proper analysis of them will be given. With the data described in Section 3.3.2 and the algorithm in Section 3.3.3.2, the implied volatility surface obtained is shown in Figure 3.1.

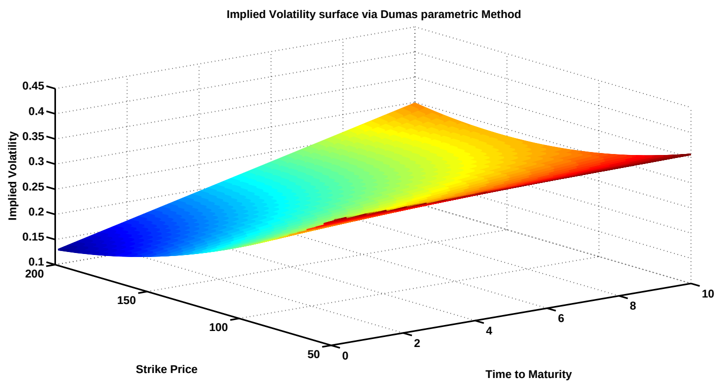

Figure 3.1: Implied Volatility Surface as a Function of Strike and Maturity

The corresponding local volatility surface using the implied volatility surface in Figure 3.1 is shown in Figure 3.2 As we can see from Figure 3.1 with the parametrization choice given, the implied volatility surface exhibits some "smile" effect at lower maturities T and strikes K. This shows that indeed, the volatility function is dependent

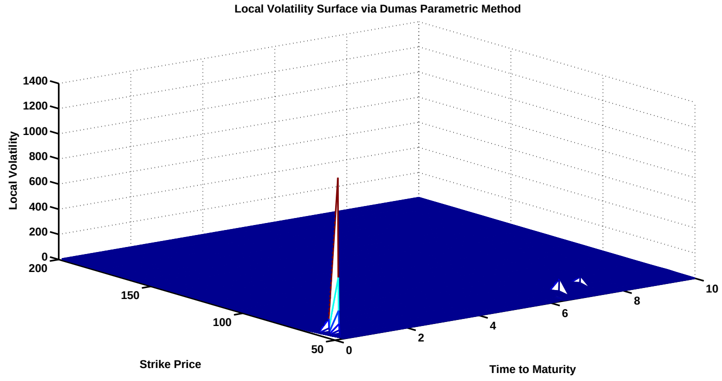

Figure 3.2: Local Volatility Surface under Implied Volatility Surface in Figure 3.1 and Equation (2.52)

on maturity T and strike K. Although, in this case we do not have a well pronounced "smile" in the local volatility function. This is because of the numerical difficulties associated with using Dupire's equation coupled with the fact that obtaining local volatility surface with implied volatilities does not form a good fit with Dupire's equation which is based on call option prices. We would expect a graph of local volatility surface to exhibit better "smile" effect when we use Equation (2.87) because this equation directly links local volatilities with implied volatilities.

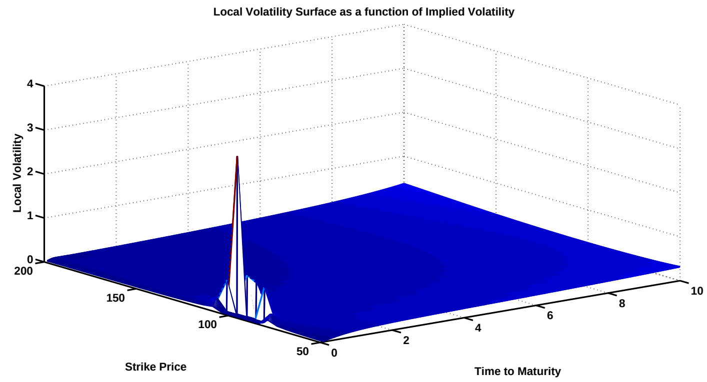

Figure 3.3: Local Volatility Surface under Implied Volatility Surface in Figure 3.1 and Equation (2.87)

As can be seen in Figure 3.3, the local volatility surface exhibits more "smile" effect than in Figure 3.2 especially across lower maturities for all strike prices. Therefore, one would prefer to use Equation (2.87) to Equation (2.52) in obtaining local volatility surfaces since we are dealing directly with Black-Scholes implied volatility data.

3.3.4 Parametrization of Implied Volatility Function using Moneyness

One can also consider the implied volatility to be a function of maturity T and moneyness M. In this case, implied volatility is also indirectly dependent on strike since moneyness is a function of the strike price K. This idea was originally used in [4] which has been adopted in this thesis. It's worth checking how this form of parametrization performs against the parametrization in Equation (3.2). Therefore, let the new choice of parametrization for the implied volatility function be

$$\sigma_{imp}(T, M) = \sum_{i,j=0}^{n} b_i B_{ij}(T, M) = b_0 + b_1 M + b_2 M^2 + b_3 T + b_4 MT. \tag{3.29}$$

Analogously, similar explanations for the parameters in Equation (3.2) can be made for Equation (3.29) except that instead of strike K, moneyness $M = \frac{\log(K/F_T)}{\sqrt{T}}$ is used. Since moneyness is a function of strike and it is directly dependent on strike, one can easily compare the graphs of the implied volatility surfaces obtained using Equation (3.29) and Equation (3.2). Like in Section 3.3.3, we will determine the appropriate partial derivatives of the implied volatility function with respect to maturity T and strike K.

Lemma 3.4. *If the implied volatility function takes the form of Equation (3.29), then the partial derivatives below hold:*

$$\frac{\partial \sigma_{imp}}{\partial M} = b_1 + 2b_2 M + b_4 T;$$

$$\frac{\partial M}{\partial K} = \frac{1}{\sqrt{T}} . \frac{F_T}{K} . \frac{1}{F_T} = \frac{1}{K\sqrt{T}};$$

$$\frac{\partial M}{\partial T} = \frac{-r\sqrt{T} - 0.5M}{T}.$$

Proof. First, determine the partial derivative of σ_{imp} with respect to M.

It is quite straight-forward from Equation (3.29) and this gives

$$\frac{\partial \sigma_{imp}}{\partial M} = b_1 + 2b_2 M + b_4 T. \tag{3.30}$$

Next, determine the partial derivative of M with respect to K.

Given that $M = \frac{\log(K/F_T)}{\sqrt{T}}$, we get

$$\frac{\partial M}{\partial K} = \frac{1}{\sqrt{T}} . \frac{F_T}{K} . \frac{1}{F_T} = \frac{1}{K\sqrt{T}}. \tag{3.31}$$

Finally, determine the partial derivative of M with respect to T :

$$\begin{aligned}
\frac{\partial M}{\partial T} &= \frac{\partial}{\partial T}\left(\frac{\ln\left(\frac{F_T}{K}\right)}{\sqrt{T}}\right) \\
&= \frac{\partial}{\partial T}\left(\frac{\ln(\frac{K}{S}) - rT}{\sqrt{T}}\right) \\
&= \frac{\sqrt{T}(-r) - \ln\left(\frac{F_T}{K}\right)\left(\frac{1}{2\sqrt{T}}\right)}{T} \\
&= \frac{-r\sqrt{T} - 0.5M}{T},
\end{aligned} \tag{3.32}$$

since $M = \ln\left(\frac{F_T}{K}\right)\left(\frac{1}{\sqrt{T}}\right)$ and this completes the proof. □

Proposition 3.5. *If the implied volatility function takes the form of Equation (3.29), then the partial derivatives below hold:*

$$\frac{\partial \sigma_{imp}}{\partial T} = \frac{-(b_1 + 2b_2M + b_4T)(r\sqrt{T} + 0.5M)}{T} + (b_3 + a_4M);$$

$$\frac{\partial \sigma_{imp}}{\partial K} = \frac{b_1 + 2b_2M + b_4T}{K\sqrt{T}};$$

$$\frac{\partial^2 \sigma_{imp}}{\partial K^2} = \frac{2b_2 - ((b_1 + 2b_2M + b_4T)\sqrt{T})}{K^2T}.$$

Proof. First, determine the partial derivative of σ_{imp} with respect to T. From Equation (3.29), we have that

$$\frac{\partial \sigma_{imp}}{\partial T} = (b_3 + a_4M) + \frac{\partial \sigma_{imp}}{\partial M}\frac{\partial M}{\partial T}. \tag{3.33}$$

Using Lemma 3.4, Equation (3.33) turns to be

$$\frac{\partial \sigma_{imp}}{\partial T} = \frac{-(b_1 + 2b_2M + b_4T)(r\sqrt{T} + 0.5M)}{T} + (b_3 + a_4M). \tag{3.34}$$

Next, determine the partial derivative of σ_{imp} with respect to K. From Equation (3.29), we have that

$$\frac{\partial \sigma_{imp}}{\partial K} = \frac{\partial \sigma_{imp}}{\partial M}\frac{\partial M}{\partial K}. \tag{3.35}$$

Hence, using Lemma 3.4, the partial derivative in Equation (3.35) becomes

$$\frac{\partial \sigma_{imp}}{\partial K} = \frac{b_1 + 2b_2M + b_4T}{K\sqrt{T}}. \tag{3.36}$$

Finally, calculate the 2nd partial derivative of σ_{imp} with respect to K :

$$\begin{aligned}\frac{\partial^2\sigma_{imp}}{\partial K^2} &= \frac{\partial}{\partial K}\left(\frac{b_1 + 2b_2M + b_4T}{K\sqrt{T}}\right) \\ &= \frac{K\sqrt{T}\left(2b_2\frac{1}{K\sqrt{T}}\right) - (b_1 + 2b_2M + b_4T)\sqrt{T}}{K^2T} \\ &= \frac{2b_2 - ((b_1 + 2b_2M + b_4T)\sqrt{T})}{K^2T},\end{aligned} \tag{3.37}$$

and this completes the proof. □

Coupling the partial derivatives obtained in Proposition 3.5 with the ones in Section 3.3.3, one would obtain a new sets of local volatility surfaces.

3.3.4.1 Algorithm

The necessary partial derivatives derived in Section 3.3.3 and Section 3.3.4 will be used in this algorithm. The algorithmic steps are given below:

1. Load the data comprising of T, K, P, S, IV, r, q.
2. Calculate $F_T = Se^{(r-q)T}$ and `mn` $= \frac{\log(K/F_T)}{\sqrt{T}}$ for each T and K.
3. Define a function (`func1(pars`$, X, T, IV$`)`) containing the choice of the parametric function:

   $\sigma_{imp} =$ `pars(1)`$+$`pars(2)`$X+$`pars(3)`X^2+`pars(4)`$T+$`pars(5)`XT

   and returns `e`, the error as the output, which is the difference in the values of σ and IV (i.e. `e=`σ_{imp}`-IV`).
4. Using a starting value, solve (`func1(pars`$, X, T, IV$`)`) non-linearly in least-squares using the MATLAB built-in function "`lsqnonlin`". This step as a whole solves the problem and its outputs are the minimizing parameters,

   `pars(1),..., pars(5)`

   with the norm of its residuals.
5. Discretize K into K_i and T into T_i values and create a mesh grid of the discretized values, (i.e. $[K_m, T_m] =$ `meshgrid`(K_i, T_i)).
6. Re-calculate $F_T = Se^{(r-q)T_m}$ and `Mn` $= \frac{\log(K_m/F_T)}{\sqrt{T_m}}$ for the mesh grid created.
7. Define a function (`func2(pars,Mn,`T_m`)`) that contains the choice of the parametrization and with the input above, calculates the IV_m for the grid created.
8. Using mesh (T_m, K_m, IV_m), obtain the implied volatility surface. Notice that `Mn` is directly replaced by K_m in the plotting because M_n is a simple function of strike K.

9. Interpolate q and $r - q$ across all T_i and K_i.
10. Calculate the corresponding Black-Scholes prices from the implied volatilities obtained.
11. Calculate the derivative of local volatility call option function with respect to T as derived in Equation (3.7).
12. Calculate the 1st derivative of local volatility call option function with respect to K as derived in Equation (3.13).
13. Calculate the 2nd derivative of local volatility call option function with respect to K as derived in Equation (3.16).
14. Using the Dupire equation, calculate the local variance and local volatility. Some values of local variance matrix may be negative which makes taking square root impossible. For such values, use either of the following:
    (a) Take the absolute value of the matrix in local variance.
    (b) Make the negative values of the matrix in the local variance zero.
15. Using `meshgrid` (T_m, K_m, LV), obtain the local volatility surface for each method described above.

3.3.4.2 Detailed Algorithm for an Alternative Method

Alternatively, the local volatility surface obtained from the above algorithm can be also done using Equation (2.87). Following the determination of implied volatility surface in the algorithm above, below highlights the additional steps needed to complete the algorithm.

1. Calculate the partial derivative of σ_{imp} with respect to T as in Equation (3.33).
2. Calculate the partial derivative of σ_{imp} with respect to K as in Equation (3.36).
3. Calculate the 2nd partial derivative of σ_{imp} with respect to K as in Equation (3.37).
4. Calculate the partial derivative of w with respect to T as in Equation (3.26).
5. Calculate the partial derivative of w with respect to y as in Equation (3.27).
6. Calculate the 2nd partial derivative of w with respect to y as in Equation (3.28).
7. Calculate the local variance and local volatility as in Equation (2.87). Some values of local variance matrix may be negative which makes taking square root impossible. For such values, use either of the following:
    (a) Take the absolute value of the matrix in the numerator.
    (b) Make the negative values of the matrix in the numerator zero.
8. Using `meshgrid` (T_m, K_m, LV), obtain the local volatility surface for each method described above.

3.3.4.3 Results and Analysis of the Volatility Surfaces Obtained in Section 3.3.4.1 and Section 3.3.4.2

In this section, the surfaces obtained from implementing the algorithm described in Section 3.3.4.1 and Section 3.3.4.2 will be shown and proper analysis will be given. Based on the data described in Section 3.3.2 and the algorithm specified in Section 3.3.4.1, the implied volatility surface is given in Figure 3.4.

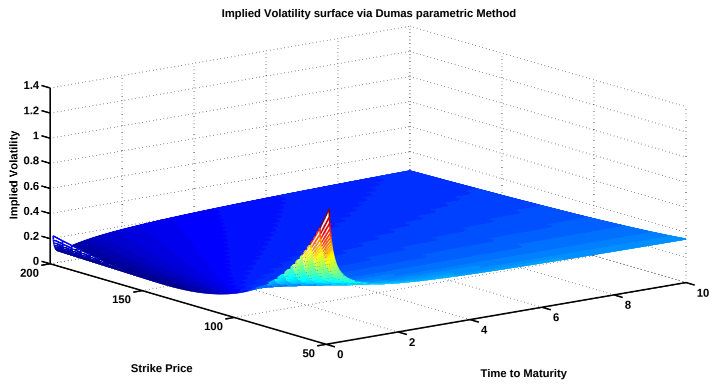

Figure 3.4: Implied Volatility Surface as a Function of Moneyness and Maturity but Obtained with Strike and Maturity

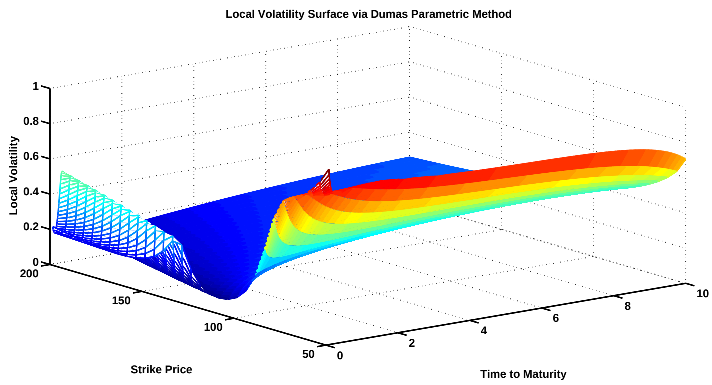

Figure 3.5: Local Volatility Surface under Implied Volatility Surface in Figure 3.4 and Equation (2.52)

The corresponding local volatility surface using the implied volatility surface in Figure 3.4 is shown in Figure 3.5. As can be seen from Figure 3.4 with the parametrization choice given, the implied volatility surface exhibits a well pronounced "smile" at lower maturities T for all strikes K. For higher maturities, the "smile" is not so pronounced except at lower strikes. This shows again that option volatilities are dependent on maturities and strike prices. Also, in this case, we do have a well pronounced "smile" in the local volatility surface. As can be observed, the local volatility surface exhibits less "smile" effect in Figure 3.6 than in Figure 3.5.

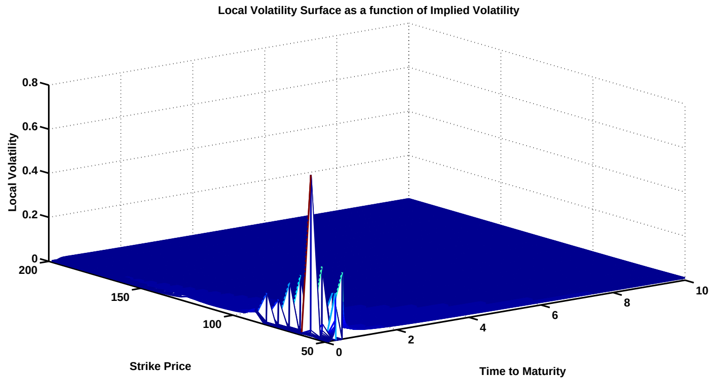

Figure 3.6: Local Volatility Surface under Implied Volatility Surface in Figure 3.4 and Equation (2.87)

3.3.5 Deficiency of the Parametric Methods

Parametric method of obtaining local volatility surfaces poses several challenges in practice. These challenges will be addressed below with various justifications on how we have tackled them.

1. Firstly, we will address the choice of parametric function: The reason for choosing Dumas parametrization for the implied volatility function is based on the empirical studies by Dumas B., Fleming J., and Whaley R. in [13] carried on S&P 500. This parametric function was preferred in this study since the data used is also from the same underlying as the one used in [13]. However, other parametric functions could be a better fit for some other underlying asset. Hence, the objectivity of selecting a parametric function might depend on the underlying. In addition, other complex parametric functions which were not considered in [13] are also possible. Although, this does not in any way nullify the effectiveness of the parametric function chosen in this study. This is because it still captures most of the variations that might be caused by the variables (T and K).

2. Secondly, we will address the difficulty of obtaining local volatility surfaces: There is a difficulty with how to deal with the negative values of local variances that make it impossible to obtain the local volatility surface. There are various ways in literature to tackle this shortcoming which is peculiar to most numerical methods, one of such is *absorption* which involves equating local volatility (LV) process to zero whenever it takes a negative value (i.e. $LV^+ = \max(LV, 0)$). Another way is called *reflection* which involves reflecting the negative values through the origin and continuing from there, (i.e. $|LV|$). There are other ways that are not mentioned in this thesis but can be found in [4]. In our framework, the *absorption* was used because it only accounts for the part of the surface that are positive which are the parts that can be used for pricing and/or hedging purposes.

3.4 Numerical Non-Parametric Method

In this section, the numerical non-parametric methods used in obtaining local volatility surfaces will be discussed in details. In addition, the results and analysis of the surfaces will be made based on the data used.

The section is arranged as follows: Section 3.4.1 deals with the most prominent numerical techniques in the literature on how to obtain local volatility surfaces via non-parametric methods. Section 3.4.2 and Section 3.4.3 involve the description of the data and explanation of the techniques used in obtaining local volatility surfaces via non-parametric methods. Finally, Section 3.4.4 explains the deficiencies of this method and suggestions on possible improvements for further studies.

3.4.1 Literature Survey on Ways to Obtain Local Volatility Surfaces via Non-Parametric Method

In this section, three important prominent approaches of obtaining local volatility surfaces via non-parametric method will be discussed: Lagnado and Osher, Elisabeth R. and Hanke M., and Achdou Y. and Pironneau O. approaches. It's worth noting that these methods are based on obtaining the local volatilities by directly using the market quoted option prices rather than implied volatilities. These approaches are discussed below:

3.4.1.1 Method of Lagnado R. and Osher S.

This method involves solving the parabolic partial differential equation (PDE) associated with arbitrage-free derivative security prices. The local volatility function will be estimated by solving the inverse problem associated with Dupire's equation for some discrete and finite set of observed option prices. The volatility function to be determined appears to be a coefficient of the second-order partial derivative in the pricing PDE. In theory, it can be determined if given enough continuous data of option prices to solve the PDE. However, the market has limited number of option prices available which makes the problem ill-posed and requires a regularization technique to produce a stable and consistent solution through time. Hence, in this methodology, the gradient of local volatility function is minimized in $\mathcal{L}^2$ norm over an appropriate space of smooth functions subject to constraints that ensure the solutions of the pricing PDE match with the observed option prices.

So, if $V(S,t;T,K,\sigma)$ is the option price at time t and $\sigma(t,S)$ is the choice of the volatility function, then the option price V follows the stochastic Black-Scholes-PDE shown below:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma(S,t)^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \tag{3.38}$$

where S is the asset current level, T is the maturity, K is the strike price, r is the continuously compounded risk-free rate, and q is the continuous dividend yield. Hence, if the form $\sigma(t,S)$ is specified, then the option price $V(S_0,0;T,K,\sigma)$ can be uniquely determined by solving Equation (3.38) given the initial and boundary conditions. Since we are dealing with standard European options, the appropriate initial and boundary conditions are;

$$\begin{cases} V(S,T;T,T,\sigma) = \max(S_T - K, 0) & \text{for } S \geq 0 \\ V(0,t;T,K,\sigma) = 0 & \text{for } 0 \leq t \leq T \\ \frac{\partial V}{\partial S}(S,t;T,K,\sigma) = e^{-q(T-t)} \; as \; S \to \infty & \text{for } 0 \leq t \leq T \end{cases} \tag{3.39}$$

In the general context set above, market calibration involves finding a local volatility function σ that solves the PDE in Equation (3.38) such that the obtained option prices fall in between the corresponding bid and ask option prices. That is,

$$V_{ij}^b \leq V(S_0,0;T,K,\sigma) \leq V_{ij}^a$$

for $i = 1,2,3,\ldots,N$ denoting the sets of maturities, T_i's and $j = 1,2,3,...,M$ denoting the sets of strike prices, K_j's for each maturity. Satisfying these inequality constraints, a function $G(\sigma)$ is to be minimized with respect to σ and possibly making it approach zero:

$$G(\sigma) = \sum_{i=1}^{N}\sum_{j=1}^{M_i}[V(S_0,0;T_i,K_{ij},\sigma) - \tilde{V}_{ij}]^2, \tag{3.40}$$

where $\tilde{V}_{ij} = \frac{1}{2}(V_{ij}^a + V_{ij}^b)$ is the average of the bid and ask prices. To this extent, minimizing the function G over a general space of admissible functions is ill-posed, essentially because a finite and discrete number of observable option prices. Hence, the function σ can not be uniquely determined with guaranteed continuous dependence on market option prices. And as a consequence of this, a small perturbation in price data can lead to a large change in the minimizing function. Therefore, a regularization technique (Tykhonov regularization) is necessary to assure the well-posedness of the problem and to obtain an optimal solution which is numerically robust [32]. For full documentation of this method, refer to [26].

3.4.1.2 Method of Achdou Y. and Pironneau O.

This method involves similar setting and arguments in re-construction of the local volatility function as in the previous method. However, it is less computationally intensive compared to the method in Section 3.4.1.1 in that it solves for the option prices only once. Due to the assumption made on $\sigma(t,S)$, that is to be deterministic, one can not find an explicit formula for the option prices as in the case of constant volatility. Thus, this will be done numerically. Just like in Section 3.4.1.1, the aim is to determine a good estimate for σ such that the theoretical prices produced by the BS-PDE matches the market prices. Stating the problem mathematically:

Given a set of times $\{t_i\}$, stock prices $\{S_i\}$, strikes $\{K_j\}$, for $j = 1,2,3,\ldots,M$ for each maturity $\{T_i\}$ and given the market prices $\{C_{ij}\}$ of call options corresponding to these parameters, can one obtain a function $\sigma(t, S)$ such that

$$V(S_i, t_i, T_i, K_j, \sigma) = C_{ij}$$

holds for each $i = 1, 2, 3, \ldots, N$? This is also an ill-posed inverse problem. In this method, the solution of the above problem involves solving the PDE associated with Dupire's equation using *regularized least squares approximation* method. Also, in determining the minimizing volatility parameter, *gradient descent method* was used. The idea behind solving the ill-posed problem proposed by these two authors are basically as the same as that in the problem settings of Section 3.4.1.1 except that they used different numerical techniques. For full documentation of this method, refer to [1].

3.4.1.3 Method of Hanke M. and Elisabeth R.

In this approach, a *natural smoothing cubic spline* is used to interpolate the option prices in strike K and then partial derivative of the option prices with respect to strike K are taken on the interpolated curve. Also, *finite difference method* is used to calculate the partial derivative of the option price with respect to maturity T. Consequently, regularized least squares method is used to solve for the local volatilities in Dupire's equation. The regularization is necessary because of the unstable nature of solving the ill-posed problem as previously explained. This amount to minimizing

$$\|A\mathbf{x} - b\|_2^2 + \lambda \|L\mathbf{x}\|_2^2 \text{ over } \mathbf{x} \in \mathbb{R}^n$$

where A is the denominator in Dupire's equation, $\mathbf{x}$ is the local volatility, b is the numerator in Dupire's equation, λ is the regularization parameter, and L is a differential operator. L could be chosen to be the identity matrix ($\mathcal{I}$) but this does not guarantee a solution that yields non-negative local volatilities [21]. Several methods also exist for determining the regularization parameter λ. Some of these methods are: Discrepancy principle [22], L-criterion (see [16], Section 4.5), and AIC-criterion.

These approaches of obtaining local volatilities are not carried out in this thesis. However, they will be explored in further studies where regularization schemes will be used in solving Dupire's equation. For full documentation of this method, refer to [21].

3.4.2 Non-Parametric Method of Obtaining Local Volatility Surface via Implied Volatility Data

There are two different approaches of obtaining local volatility surfaces via non-parametric method implemented in this study. The two approaches are through the use of implied volatility data and market quoted option prices. In this section, obtaining local volatility surface via the use of implied volatility data will be explored. Consequently, the algorithm used for obtaining the surfaces will be given and analysis of the surfaces will be made.

With the Black-Scholes implied volatility data, a local volatility surface will be constructed. This involves using *cubic spline functions* to interpolate the call option prices in maturity T and strike K. The idea behind cubic spline interpolation in the 1-dimensional case was explained in Section 3.2. However, one should note that this interpolation was done without any constraints that forces the whole interpolated surface to be arbitrage-free. Therefore, there is a possibility of having grid points that exhibit negative local variances. This will be handled by keeping the local variances at those grid points zero.

The following basic steps are used in this method:

1. Calculate the Black-Scholes call options prices of the given implied volatilities.
2. Interpolate the Black-Scholes call option prices as a function of maturity T and strike K.
3. Take the appropriate derivatives of the option price function as derived in Equation (3.7), Equation (3.13), and Equation (3.16).
4. Determine the local volatilities using the Dupire's equation.
5. Obtain the local volatility surface in terms of maturity T and strike K.

It can be easily observed from the above steps that the out-of-sample data points will be accounted for in the interpolated surface. In the algorithms under parametric method, grid points were used when maturities T and strikes K were discretized. However, we will see in Section 3.4.2.2 that the interpolated surface obtained with the use of cubic spline functions does not cover all the grid points. For this reason, the mesh-grid for this section is smaller so that all the sample points will have call option values on the surface.

3.4.2.1 Algorithm

The detailed algorithm on how to implement the steps given in Section 3.4.2 will be carried out here. The algorithmic steps are given below:

1. Load data comprising of T, K, S, IV, r, q.
2. Calculate $F_T = Se^{(r-q)T}$.
3. Calculate the corresponding forward Black-Scholes call option prices, using the call option price function "`blsprice`" on MATLAB.
4. Determine the 2-dimensional cubic spline interpolation of Black-Scholes call price surface in terms of maturity T and strike K. This can be done on the "`cftool`" on MATLAB under *curve fitting*.
5. Discretize T and K into T_i and K_i.
6. Make a meshgrid of T_i and K_i (i.e $[T_m, K_m] = \texttt{meshgrid}(T_i, K_i)$).

7. Calculate the Black-Scholes call option prices of the meshgrid above.

8. Calculate the IV_m of the Blcak-Scholes call option prices at those grids.

9. Calculate the partial derivatives of IV_m with respect to T and K using the *finite (central) difference method*.

10. Calculate the partial derivatives in the Dupire's equation by using Equation (3.7), Equation (3.13), and Equation (3.16).

11. Calculate the local variance and local volatilities from the Dupire's equation.

12. Using `meshgrid` (T_m, K_m, LV), obtain the local volatility surface.

3.4.2.2 Results and Analysis of the Surfaces Obtained from Implementing Section 3.4.2.1

In this section, the surfaces obtained from the implementation of the algorithm in Section 3.4.2.1 will be given as well as their analysis.

The three surfaces obtained after the implementation of the algorithm in Section 3.4.2.1 are: Cubic spline interpolation of the Black-Scholes call option prices, Implied volatility surface via spline interpolation, and Local volatility surface via spline interpolation respectively. The surface of Cubic Spline Interpolation of the Black-Scholes Call Option is given in Figure 3.7. Consequently, the corresponding implied volatility surface for the parts of the interpolated surface that give positive implied volatilities is shown in Figure 3.8.

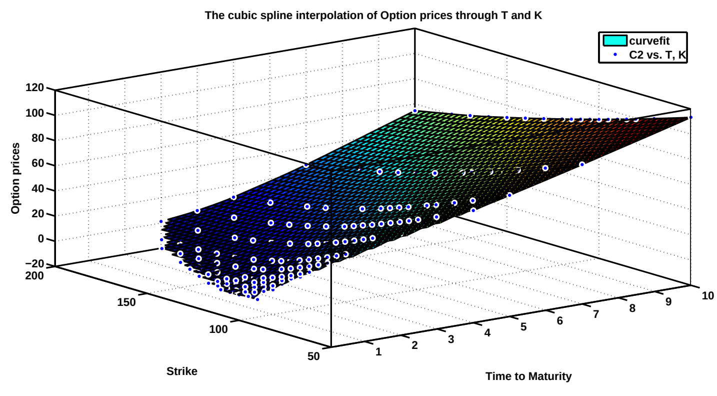

Figure 3.7: Cubic Spline Interpolation of the Black-Scholes Call Option Data Obtained from Implied Volatilities

It can clearly be observed that the implied volatility surface is not available for lower maturities T and strikes K. This is because at those lower values, the implied volatility surface is unrealized due to attainment of negative values at those grid points. However, to have a full picture of the implied volatility surface, cubic spline interpolation of implied volatility data is carried out. This surface is shown in Figure 3.9. As can be

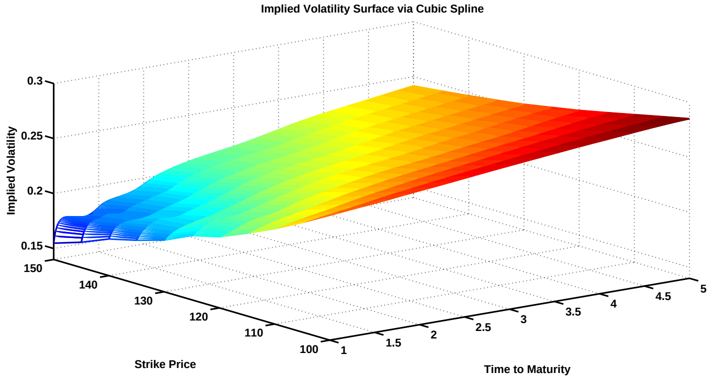

Figure 3.8: Implied Volatility Surface Obtained from Interpolation of Black-Scholes Call Option Data

seen, this surface provides more values for the grid points than shown in Figure 3.8. Also, with the implied volatility surface in Figure 3.8, we do not observe much of a "smile" and this is attributed to the fact that we do not have the surface for lower maturities and lower strikes where the "smile" effect is usually observed. The corresponding local volatility surface for the implied volatility surface in Figure 3.8 is given in Figure 3.10.

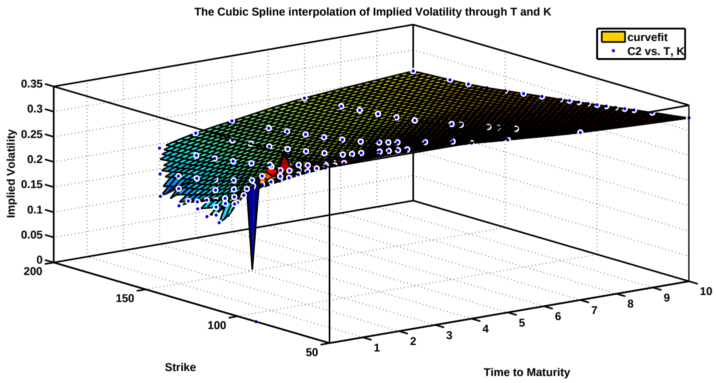

Figure 3.9: Implied Volatility Surface Obtained from Spline Interpolation of Implied Volatility Data

This local volatility surface exhibits some local extrema of volatilities with no obvious "smile" effect. This is also due to the fact that the surface does not include lower maturities T and strikes K.

To enhance this method, it is necessary to first find an interpolating scheme that produces option surfaces that are arbitrage-free. This amounts to solving a constrained cubic spline interpolation of the option prices where the constraints are a combination of linear and non-linear inequalities. This inherently deals with the negative values that we obtained for the implied volatilities at some grid points. This subject is well-dealt with in [17]. This concept of obtaining a whole local volatility surface that is arbitrage-

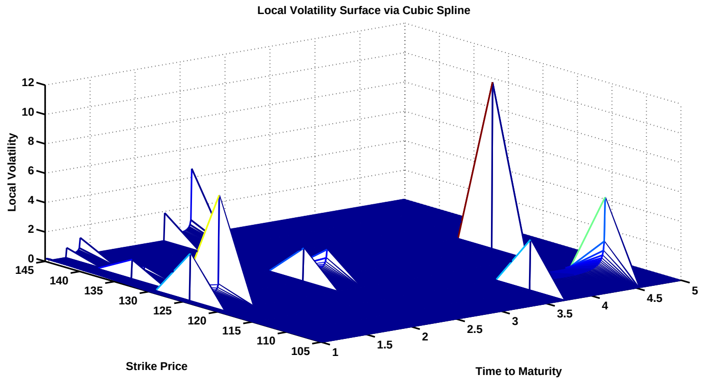

Figure 3.10: Local Volatility Surface via Cubic Spline Interpolation of Black-Scholes Option Prices

free using *arbitrage-free implied volatility surface* will be emphasized in Chapter 4. This will be a continuous work for further studies.

Secondly, another interpolating scheme could have performed better. For example, the use of *Thin-plate splines* produced a call option surface that covers the lower maturities and strikes as depiced in Figure 3.11.

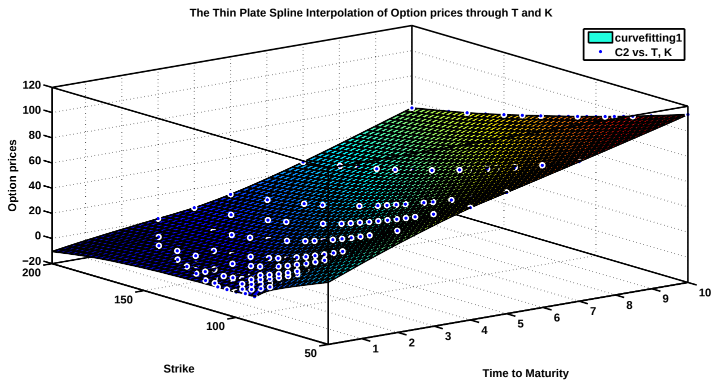

Figure 3.11: Thin-Plate Spline Interpolation of the Black-Scholes Call Option Data Obtained from Implied Volatilities

3.4.3 Non-Parametric Method of Obtaining Local Volatility Surface via Option Price Data

In this section, local volatility surface from option prices data rather than implied volatility data will be obtained. This involves using *natural cubic spline* in the interpolation of market option prices and with the aid of *finite (central) difference method*, the appropriate partial derivatives in the Dupire's equation will be taken on the interpolated surface. The detailed algorithm for obtaining the local volatility surface using market prices will be given in Section 3.4.3.1 and the results and analysis of the surface obtained will be made in light of the new data in Section 3.4.3.2.

Firstly, we will describe the new data obtained. The data was taken from [10]. It's call options of SPX observable on 8th February, 2006. It contains 126 sets of different options observed in the market across all maturities T and strike prices K. The stock had a spot price of $1265.65 and interest rate of 5.25% with no dividends.

3.4.3.1 Algorithm

The algorithm used in constructing the local volatility surface from the market option prices will be given here. The following steps are used in obtaining the local volatility surface.

1. Load data comprising of T, K, S, P, IV, r, q.
2. Determine the *cubic spline interpolation* and *thin-plate spline interpolation* of call prices in terms of T and K. This can be done in the "`cftool`" on MATLAB under *curve fitting*. Let this surface be named "`Surface`".
3. Discretize T and K into T_i and K_i
4. Make a meshgrid of T_i and K_i (i.e $[T_m, K_m] =$ `meshgrid`(T_i, K_i)).
5. Calculate the option prices only at those grid points from the interpolated surface, that is, $C(T, K) =$ `Surface`(T_m, K_m).
6. Using finite (central) difference method, calculate the following:
   (a) The partial derivative of the call option $C(T, K)$ with respect to maturity T.
   (b) The partial derivative of call option $C(T, K)$ with respect to strike K.
   (c) The 2nd partial derivative of $C(T, K)$ with respect to strike K.
7. Using the above derivatives, calculate the local variance in the Dupire equation.
8. Make the grid points with negative local variances zero using `LV(LV<0)=0` since the local volatilities at those points are not feasible.
9. Calculate the local volatilities, $\sigma(T, K)$.
10. Using `meshgrid` (T_m, K_m, LV), obtain the local volatility surface.

3.4.3.2 Results and Analysis of the Surfaces Obtained by Implementing Section 3.4.3.1

Here, two methods of spline interpolations are used; one is based on *cubic spline interpolation* and the other on *thin-plate spline interpolation*. The basic difference between both interpolation methods from the surfaces they produce is that the latter produces a surface for more grid points than the former.

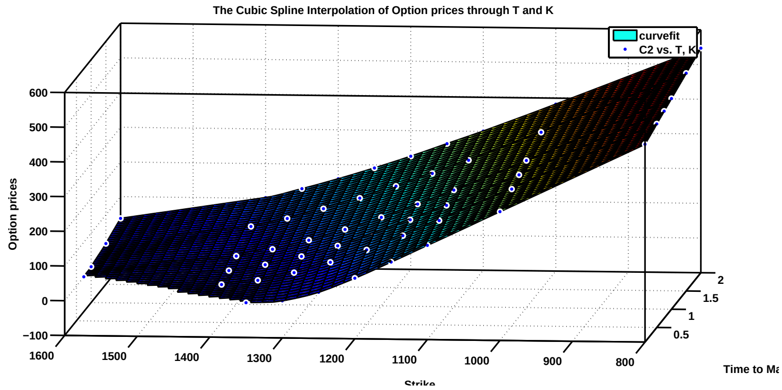

Figure 3.12: Cubic Spline Interpolation of the Market Option Data

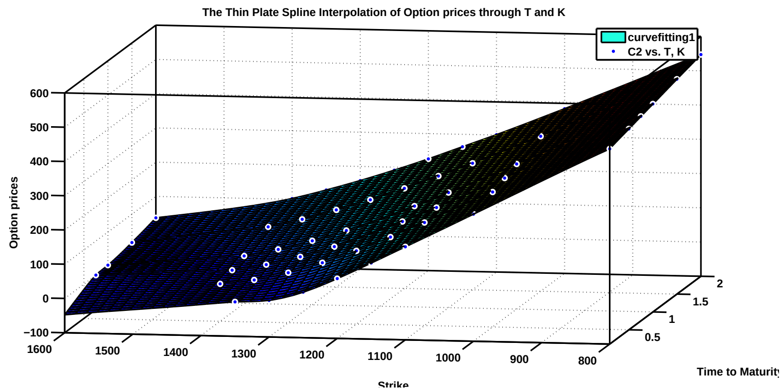

Figure 3.13: Thin Plate Spline Interpolation of the Market Option Data

The cubic spline interpolation surface of the market call option data is shown in Figure 3.12 while that of thin-plate spline interpolation is shown in Figure 3.13. The differences can be easily observed from the two surfaces.

Consequently, the local volatility surfaces of the two spline interpolated surfaces can be viewed in Figure 3.14 and Figure 3.15, where the former is the local volatility surface via thin-plate interpolation of market option data and the latter is via cubic spline interpolation of market option data. These surfaces exhibit some local extrema of high local volatilities and the “smile” effect can be faintly observed along lower maturities for all strikes and lower strikes for all maturities.

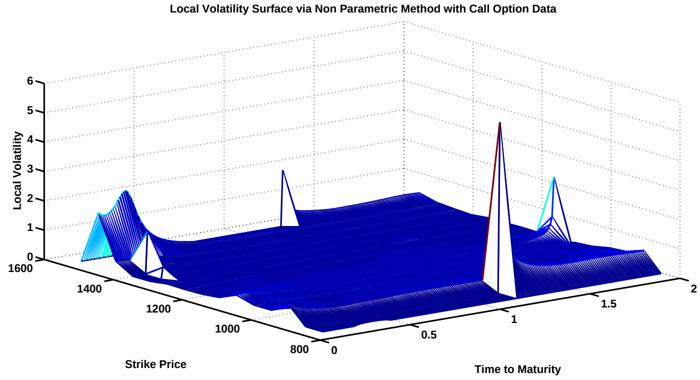

Figure 3.14: Local Volatility Surface via Thin-Plate Interpolation of Market Option Data

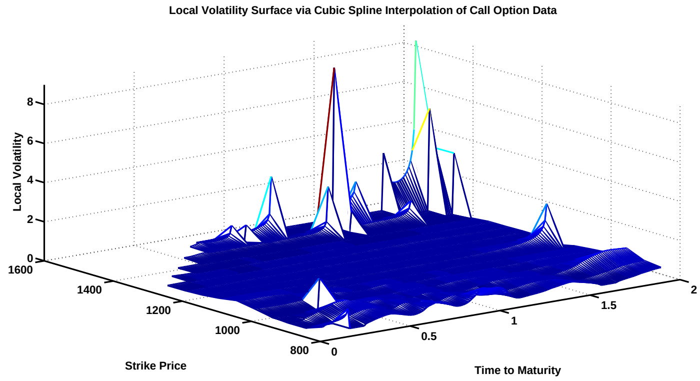

Figure 3.15: Local Volatility Surface via Cubic Spline Interpolation of Market Option Data

3.4.4 Deficiency of Numerical Non-Parametric Method

In this section, some important issues that arise when dealing with non-parametric methods of obtaining local volatility surfaces will be discussed. Firstly, the key important deficiency of using this method to re-construct local volatility function is that this method does not take into account how to make the whole interpolated volatility surfaces arbitrage-free. However, we have considered in this thesis only those points that are arbitrage-free on the local volatility surface that can be used for application purposes. Several suggestions can be given to handle this problem. One of such suggestions involves the non-arbitrage interpolation (cubic spline) of the call options, which in turn gives non-negative values of local variance when the Dupire's equation is solved.

Another suggestion is to solve the Dupire's equation with Tikhnov regularization. The setting involves solving

$$A\mathbf{x} = b,$$

where A is the diagonal matrix consisting of the denominator values in Dupire's equation at the diagonals; **x** is a column matrix consisting of the local variance values to be solved for; and b is a column matrix consisting of the numerator values in the Dupire's equation. In fact, this solves for local variances with non-negative values in the presence of negative values in the matrices A and/or b.

Secondly, there is a question of which spline interpolation function is more suitable for our purpose. We generally use the cubic/thin-plate splines partly because in most studies for instance [21], they were used and also because the Dupire's equation is a differential equation of order two. Moreover, a spline function that is twice continuously differentiable is needed to ensure the smoothness of the call option's interpolated surface.

CHAPTER 4

CONCLUSION

In this concluding chapter, summary of what have been done in this thesis will be highlighted and the outlook for further studies will be discussed.

The aim of this thesis was to emphasize the importance and usefulness of local volatility model in modeling the “volatility smile” of the underlying assets. Broadly speaking, we have investigated the local volatility model in details: from understanding its financial stand point to its mathematical model derivations. Consequently, various numerical techniques used in obtaining local volatility surfaces have been explored. Some as used in the literature and some as proposed and implemented in this thesis.

There are several paths this thesis could take for further studies. First, the volatility surfaces obtained can be used as a first hand estimation in pricing exotic options like barrier, look back, and asian. In fact the surfaces obtained are important because it is crucial for practitioners to have a stable algorithm to determine the volatility structure for the underlying assets that exotic options will be written on. Without a proper pricing mechanism, the market for these options could be raided with arbitrage prices that could lead to mis-pricing of financial derivatives. In addition, the volatility surfaces could also be used to hedge positions in exotic options.

Another important idea whose exploration is very useful is the comparison of the performances of the surfaces obtained against the ones produced from other complex models used in modeling an asset’s volatility structure. These complex models are stochastic volatility, jump-diffusion, local stochastic volatility models, etc. Furthermore, to be able to compare, these complex volatility models need to be studied in details as well. Consequently, it will be a good approach to study a model based on jump diffusion process and one based on stochastic process to be able to compare across disparate models. So far, we have dealt with a deterministic model (local volatility model).

Another rational and well-founded approach is to figure out a stable numerical technique and algorithm that will be able to produce non-arbitrage implied volatility surface. This surface in turn is used to obtain local volatility surface. This approach can be done using either parametric method or non-parametric methods. Under parametric method, obtaining arbitrage-free implied volatility surface involves an algorithm that solves the parametrized implied volatility function in an arbitrage-free way. Therefore, when the Black-Scholes prices are computed for the grid points in maturities T

and strike prices K, non-arbitrage prices will be obtained. This will in turn keep the local volatilities in Dupire's equation positive because arbitrage-free options prices (for fixed t_0 and S_0) conform to the following rule:

1. Monotonically decreasing and convex in strike K;
2. Monotonically increasing in maturity T.

For non-parametric method, consider the method explored in Section 3.4.3. It's possible to write a stable algorithm that will produce the spline interpolation of the option prices in an arbitrage-free way. This is the key idea in the work of Fengler M.R [17].

Another well founded idea is to obtain the non-negative local volatilities by solving the regularized Dupire's equation in the least squares sense over the space of positive real numbers. To explore this idea further, Section 3.4.4 might be of help.

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APPENDIX A

DEFINITIONS AND THEOREMS

In this Appendix, some of the theorems that were used in this thesis are highlighted:

Definition A.1 (Itô process)**.** Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space and $(B_t)_{t\geq 0}$ an $\mathcal{F}_t$-Brownian motion. $(X_t)_{0\leq t\leq T}$ is an $\mathbb{R}$-valued Itô process if it can be written as

$$\mathbb{P}\ a.s.\ \forall t \leq T \ \ X_t = X_0 + \int_0^t K_s ds + \int_0^t H_s dB_s,$$

where

- X_0 is $\mathcal{F}_0$-measurable.
- $(K_t)_{0\leq t\leq T}$ and $(H_t)_{0\leq t\leq T}$ are $\mathcal{F}_t$-adapted processes.
- $\int_0^T |K_s|\, ds < +\infty \ \ \mathbb{P}\ a.s.$
- $\int_0^T |K_s|^2\, ds < +\infty \ \ \mathbb{P}\ a.s.$

Theorem A.1 (Itô formula)**.** *Let $(X_t)_{0\leq t\leq T}$ be an Itô process as in Definition A.1, and f be a twice continuously differentiable function. Then*

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)dX_s + \frac{1}{2}\int_0^t f''(X_s)d < X, X >_s$$

where by definition

$$d\langle X\ ,\ X\rangle_t = \int_0^t H_s^2 ds$$

and

$$\int_0^t f'(X_s)dX_s = \int_0^t f'(X_s)K_s ds + \int_0^t f'(X_s)H_s dB_s.$$

Likewise, if $(t,x) \to f(t,x)$ is a function that is twice differentiable with respect to x and once with respect to t, and if these partial derivatives are continuous with respect to (t,x) (i.e. f is a function of class $\mathcal{C}^{1,2}$), the Itô formula becomes

$$\begin{aligned} f(t,X_t) = f(0,X_0) &+ \int_0^t f'_s(s,X_s)ds \\ &+ \int_0^t f'_x(s,X_s)dX_s + \frac{1}{2}\int_0^t f''_{xx}(s,X_s)d\langle X\ ,\ X\rangle_s . \end{aligned} \tag{A.1}$$

Theorem A.2 (Forward equation)**.** *Let Y_t represents a diffusing particle that follows a stochastic differential equation as in*

$$dY_t = b(t, Y_t)dt + \sigma(t, Y_t)dB_t,$$

then for a fixed x and s, a smooth transition density $p(x, s; .; .)$, of X_t, satisfies the forward equation

$$\partial_t p = \mathcal{A}^*_{y,t} p, \quad t > s$$

where $\mathcal{A}^*_y$ is the adjoint operator defined by

$$\mathcal{A}^*_{y,t} g := \frac{1}{2} \sum_{ij} \frac{\partial^2}{\partial_{y_i} \partial_{y_j}} (a_{ij}(y,t)g) - \bigtriangledown_y . (b(y,t)g)$$

and $a_{ij}(y,t) = \sigma\sigma^T$. This describe the evolution of the density forward in time.

Theorem A.3 (Uniqueness Theorem)**.** *Let Y_t represents a diffusing particle that follows the stochastic differential equation*

$$dY_t = b(t, Y_t)dt + \sigma(t, Y_t)dB_t, \tag{A.2}$$

with $Y_0 = Z$.

If b and σ are continuous functions, and if there exist constant $K < +\infty$ such that

1. $|b(t,x) - b(t,y)| + |\sigma(t,x) - \sigma(t,y)| \leq K |x - y|$,
2. $|b(t,x)| + |\sigma(t,x)| \leq K(1 + |x|)$,
3. $\mathbb{E}(Z^2) < +\infty$,

then for any $T \geq 0$, Equation Equation (A.2) admits a unique solution in the interval $[0, T]$.

Theorem A.4 (Girsanov Theorem)**.** *Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$ be a filtered probability space and $(B_t)_{0 \leq t \leq T}$ an $(\mathcal{F}_t)-$standard Brownian motion. Let $(\theta_t)_{0 \leq t \leq T}$ be an adapted process satisfying $\int_0^T \theta_s^2 ds < \infty$ $a.s$ such that the process $(L_t)_{0 \leq t \leq T}$ defined by*

$$L_t = \exp\left(- \int_0^T \theta_s dB_s - \frac{1}{2} \int_0^T \theta_s^2 ds \right)$$

is a martingale. Then under the probability $\mathbb{P}^{(\mathbb{L})}$ with density $\mathbb{L}_T$ with respect to $\mathbb{P}$, the process $(W_t)_{0 \leq t \leq T}$ defined by $W_t = B_t + \int_0^t \theta_s ds$ is an $(\mathcal{F}_t)-$standard Brownian motion.

APPENDIX B

REGULARIZATION SCHEMES

In this appendix, fundamental concepts of inverse problem as well as regularization schemes will be discussed as a fore study into the numerical techniques that will be used in future work.

B.1 Fundamental Concepts in Inverse Problem

Most of the subjects that will be discussed in this section can be found in [29].

B.1.1 Condition of Function Evaluation

A function to be evaluated at a given point can be well or ill-conditioned.

Definition B.1. An evaluation of a well defined function, $f : \mathbb{R} \to \mathbb{R}$ is well-conditioned if a small error (a ball $B_\delta(x)$ around x, $\forall\delta > 0$) in the point where the function is evaluated does not greatly affect the value of the function.

Definition B.2. An ill-conditioned function is the one in which the evaluation of the function in a neighborhood of a point leads to a large error.

To illustrate these ideas presented above, let's consider a simple example.

Example B.1 (Evaluation of a rational function)**.** Consider the evaluation of the function $f(x) = 1/(1 - x)$. The computation of $f(x)$ is:

1. ill-conditioned, if x is closer to 1 (but differs from 1)
2. well-conditioned, otherwise

To illustrate the two cases:

1. When x is near 1.

Let $x = 1.00049$ and in the computation let's use an approximate value $x^* = 1.0005$. Hence, absolute error of the evaluation is:

$$e_{abs} = f(x^*) - f(x) = -10^3/24.5.$$

While we have an error of $10^{-5} = x^* - x$ in the data that led to an evaluation error of $-10^3/24.5$. This is magnified by the multiplication factor

$$m = \frac{\text{error in the result (of the evaluation)}}{\text{error in the point (of the domain)}} = -\frac{10^3/24.5}{10^{-5}},$$

which in absolute values gives the condition number, $c = |m| > 10^6$.

2. When x is far from 1.

   We observe that when x is far from 1 the previous magnification does not occur. Let's say we take $x = 1998$ and $x^* = 2000$ as an approximation for x. Then, the absolute error in the evaluation is

$$e_{abs} = \frac{1}{1-2000} - \frac{1}{1-1998} = \frac{2}{1999.1997}.$$

   Hence, the magnification factor of the error is $(1999.1997)^{-1} < 10^{-6}$, effectively reducing the error.

Definition B.3. Given $f : \mathcal{D} \subset \mathbb{R} \to \mathbb{R}$ of class $\mathcal{C}^1$, $c_f(x) = |f'(x)|$ is the condition number of the (evaluation) of f at x. We also say that the evaluation of a function of f at x is well-conditioned if $c_f(x) \leq 1$ and ill-conditioned if $c_f(x) > 1$.

NOTE: Consider the quotient below (multiplication factor)

$$m = \frac{\text{error in the result (of the evaluation)}}{\text{error in the point (of the domain)}} = \frac{f(x^*) - f(x)}{x^* - x}. \tag{B.1}$$

Equation (B.1) is *Newton quotient* of f, a primary step in defining the derivative of a function at a point. In the limiting case, as $x^* \to x$, we have that $m \to f'(x)$ and $f'(x)$ is defined to be the *error multiplication factor* of f at point x.

B.1.2 Types of Multiplication Factors

1. $f'(x)$: the (usual) derivative of f at x;
2. $f'(x)/f(x)$: the *logarithmic derivative* of f at x (indeed, it is the derivative of $\ln(f(x))$);
3. $xf'(x)$: derivative (differential operator) without a special name;
4. $xf'(x)/f(x)$: the *elasticity* of f at x (very well used in economics).

B.2 Analysis on the Existence and Uniqueness

Some of the difficulties plaguing inverse problems are related to the available information (data): both quantity (not sufficient or over-abundance data) and quality of information. Let's explain this point with the help of an example.

Suppose that the function that truly generates a phenomenon is

$$f(x) = 2x + 1.$$

In the inverse problem, we assume this function is unknown to us. However, let's suppose we can determine the form to which the function belongs, namely:

$$f_{a,b}(x) = ax + b,$$

where a and b are arbitrary constants. From the available data, we then try to determine what these values of a and b are. We will consider 3 cases each of which depends on the type/nature of data observed or available to us.

B.2.1 Exact Data

1. **Not sufficient data.** Assume we know the point $(1, 3)$ belongs to the graph of f. It is trivial that the datum available is not enough to determine a and b. Hence, we only know that

   $$f(1) = 3 \text{ or } a + b = 3,$$

   It is then impossible to determine the unique values of a and b, which make the model indeterminable.

2. **Sufficient data.** Now, let's say we know $(1, 3)$ and $(2, 5)$. Thus, $a + b = 3$ and $2a + b = 5$, from which we can determine that $a = 2$ and $b = 1$ solving the two equations simultaneously, hence we select the model $f(x) = 2x + 1$.

3. **Too much data.** Suppose now that the points $(0, 1)$, $(1, 3)$, and $(2, 5)$ belong to the graph of f. Then

   $$a = 2 \text{ and } b = 1.$$

   Note that in case of exact data, having too much data does not affect the uniqueness of the chosen model, however, any two of the available data above would have been sufficient to determine the unique function that solves the given inverse problem.

B.2.2 Real Data

In practice, we do not have exact data, due to the methods used in acquiring those data. Hence, these types of data can also be called noisy data because they contain some errors. We still have three possibilities just as discussed above:

1. **Insufficient data.** Datum $(1, 3.1)$ has an error-as we can tell, for our "phenomenon" $f(1) = 3$ and not 3.1. Moreover, this datum is insufficient because we obtain only one equation between a and b,

$$a + b = 3.1,$$

which can not determine a and b uniquely, not even approximately. Additional information must be given to ensure the unique solution of the inverse problem.

2. **Sufficient data.** Suppose we have the following data: $(1, 3.1)$ and $(2, 4.9)$. Then, an approximate values for a and b can be determined by solving the following equations obtained by substituting the above data into the choice of model. This leads to

$$\begin{aligned} a + b &= 3.1, \\ 2a + b &= 4.9. \end{aligned} \tag{B.2}$$

Solving Equation (B.2) gives $a = 1.8$ and $b = 1.3.$

Note: It is not always possible to estimate the parameters by imposing that the model fits or interpolates the data even with a sufficient noisy data. And it is often advised to solve the above problem in least squares sense where the error function is being minimized. As a result, the parameters that minimizes the error function are being determined. In case of the example above, the error function $E(a, b)$ becomes

$$\begin{aligned} E(a, b) &= \frac{1}{2}\left[(f_{a,b}(1) - 3.1)^2 + (f_{a,b}(2) - 4.9)^2\right] \\ &= \frac{1}{2}\left[(a + b - 3.1)^2 + (2a + b - 4.9)^2.\right] \end{aligned} \tag{B.3}$$

The minimum of E is given by its critical points, that is, the points where the gradient of E is null. This involves taking the partial derivative with respect to each variable the error function is dependent on. Therefore, the critical points are determined as follows:

$$\begin{aligned} 0 &= \frac{\partial E}{\partial a} = (a + b - 3.1) + 2(2a + b - 4.9); \\ 0 &= \frac{\partial E}{\partial b} = (a + b - 3.1) + (2a + b - 4.9). \end{aligned} \tag{B.4}$$

Simplifying the equations above, we obtain

$$a + b = 3.1 \text{ and } 2a + b = 4.9.$$

It is coincidental that due to the form of the function $f_{a,b}$, the system in Equation (B.2) and just the one obtained are the same. Sometimes, the system in Equation (B.2) does not have a solution while the system in the second case does. In real inverse problems, solving in the least squares sense is a way to reach a solution.

3. **Too much data.** Assume that

$$(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n) \text{ with } n \geq 3,$$

are several experimental data points that are associated with the "phenomenon" $f(x) = 2x + 1$.
It is inevitable that the data contains errors which makes it impossible to solve for a and b in the system below:

$$\begin{aligned} y_1 - f_{a,b}(x_1) &= y_1 - (ax_1 + b) = 0; \\ y_2 - f_{a,b}(x_2) &= y_2 - (ax_2 + b) = 0; \\ &\vdots \\ y_n - f_{a,b}(x_n) &= y_n - (ax_n + b) = 0. \end{aligned} \tag{B.5}$$

Usually one would say that the system has no solution because there are $n - equations$ and $2 - unknowns$. However, in general, one should say that the given data can not be fitted by the model. Thence, the above system of equations can be re-written in this form:

$$a \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix} + b \begin{Bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{Bmatrix} = \begin{Bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{Bmatrix}$$

We introduce the notation $x = (x_1, x_2, \ldots, x_n)^T$, $\mathbb{1} = (1, 1, \ldots, 1)^T$ and $y = (y_1, y_2, \ldots, y_n)^T$. The above vector equation can be written as

$$ax + b\mathbb{1} = y.$$

Using the method of least squares, define the error of residual vector by

$$r = y - (ax + b\mathbb{1}),$$

given as the difference between the experimental measurements (y) and the predictions of the model with the coefficients a and b (i.e $ax + b\mathbb{1}$). In this method, one tries to choose appropriate a and b such that the functional error

$$E(a, b) = \frac{1}{2}\|r\|^2 = \frac{1}{2}\|y - ax + b\mathbb{1}\| = \frac{1}{2}\sum_{i=1}^{n}(y_i - (ax_i + b))^2,$$

is minimized. This is requiring that the inner product of the vectors x and $\mathbb{1}$ spanning the subspace $\mathbb{R}^2 \subset \mathbb{R}^n$ with the error vector r are 0 (since r is orthogonal to the plane spanned by x and $\mathbb{1}$). Hence,

$$\langle x \, , \, y - ax + b\mathbb{1} \rangle = 0 \text{ and } \langle \mathbb{1} \, , \, y - ax + b\mathbb{1} \rangle = 0.$$

This can be written as

$$x^T(y - ax + b\mathbb{1}) = 0 \text{ and } \mathbb{1}^T(y - ax + b\mathbb{1}) = 0,$$

which leads to

$$\begin{aligned} ax^T x + bx^T \mathbb{1} &= x^T y; \\ a\mathbb{1}^T x + b\mathbb{1}^T \mathbb{1} &= \mathbb{1}^T y. \end{aligned} \tag{B.6}$$

Thence,

$$\begin{pmatrix} x^T x & x^T \mathbb{1} \\ \mathbb{1}^T x & \mathbb{1}^T \mathbb{1} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} x^T y \\ \mathbb{1}^T y \end{pmatrix}.$$

Defining $A = (x, \mathbb{1})$ and an-$n \times 2$ matrix, Equation (B.6) can be re-written as

$$A^T A \begin{pmatrix} a \\ b \end{pmatrix} = A^T y, \tag{B.7}$$

which is usually called the *normal equation*. Note that even if $A^T A$ is not invertible, Equation (B.7) will still have solution which will not be discussed further here. However, if it is invertible, then the solution to the inverse problem can be shown to be equivalent to

$$\begin{pmatrix} a \\ b \end{pmatrix} = (A^T A)^{-1} A^T y. \tag{B.8}$$

This is the solution to the inverse problem given by the *least squares method* which is equivalent to the evaluation of the function

$$y \mapsto (A^T A)^{-1} A^T y.$$

Computing this might be unstable depending on matrix A and also computationally inefficient. Assuming that A is invertible, then, A^T is also invertible since

$$(A^T)^{-1} = (A^{-1})^T,$$

which leads to

$$\begin{pmatrix} a \\ b \end{pmatrix} = A^{-1} (A^T)^{-1} A^T y = A^{-1} y. \tag{B.9}$$

Note that this result obtained in Equation (B.9) is valid if the data is exact or not.

As a closure to this section, the definition of a well-posed problem will be given according to Hadamard

Definition B.4. A well-posed problem is defined as one which has the following properties:

(a) has a unique solution (existence);

(b) the solution is unique (uniqueness);

(c) the solution depends "smoothly" on the given data (regularity).

When any of these is not satisfied we say that the problem is ill-posed.

B.2.3 Spectral Analysis of Inverse Problems

In this section, an example of a problem will be given and its properties will be checked with Hadamard's specifications for well-posedness. Also, Regularization schemes will be discussed (Tikhonov regularization). This section is largely inspired by [29].

Example B.2. Given the matrix $\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{1024} \end{pmatrix}$, and the vector $y = (1, 2^{-10})^T$, it is clear that $x = (1, 1)^T$ is the solution of the system

$$Kx = y. \tag{B.10}$$

Problems like this could be fairly difficult to solve since small perturbation in y could lead to large change in the solution in x. Let's say we perturb y by $p = (0, 2^{-10})^T$. Then, we obtain a solution that differs from x by $r = (0, 1)^T$, given rise to

$$\begin{aligned} K\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) &= K(x + r) = y + p \\ &= \begin{pmatrix} 1 \\ 2^{-10} \end{pmatrix} + \begin{pmatrix} 0 \\ 2^{-10} \end{pmatrix}. \end{aligned} \tag{B.11}$$

Hence, the multiplication factor $m = |r| \,/\, |p| = 1024$ which makes the problem ill-conditioned. This is not true for all perturbations. We observe that if $p = (2^{-10}, 0)^T$, then $r = (0, 2^{-10}, 0)^T$, which in-turn makes the multiplication factor $m = |r| \,/\, |p| = 1$. We realize that evaluations of inverse of K at some points are more sensitive than other points, thus, we change the problem a little bit by solving a perturbed problem of the form:

$$K_\alpha x = y_\alpha, \ \alpha > 0, \tag{B.12}$$

in that Equation (B.12) behaves like Equation (B.10) as much as possible but with additional advantage of being better conditioned to certain small alterations in y. Thus, we call Equation (B.12) as being the regularization of the original problem presented in Equation (B.10). Hence, choose

$$K_\alpha = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{(1-\alpha)^{10}} \frac{1}{2^{10}} \end{pmatrix}, \text{ and } y_\alpha = y, \ 0 < \alpha < 1, \tag{B.13}$$

and note that

$$K_\alpha \to K, \ as \ \alpha \to 0.$$

Choosing an appropriate α, one can solve Equation (B.10) using Equation (B.12) for different perturbations of y.

Definition B.5. The families K_α and b_α, $\alpha > 0$, constitute a *linear regularization scheme* for the linear problem $Kx = y$, if the following conditions hold:

$$K_\alpha \to K, \ |b_\alpha| \searrow 0 \text{ and } \left|K_\alpha^{-1}\right| \nearrow \left|K^{-1}\right|, \text{ as } \alpha \searrow 0. \tag{B.14}$$

Here, α is called the *regularization parameter*. The perturbed problem $K_\alpha x = y + b_\alpha$ is called the *regularized problem*, K_α is the *regularized matrix* and x^α is the *regularized solution*.

Theorem B.1. *The following theorem gives some consequences of a linear regularization scheme.*

1. *If the pair K_α and b_α, constitute a linear regularization scheme, we have*

$$x^\alpha \to x \text{ and } \alpha \to 0. \tag{B.15}$$

2. *The evaluation's condition of the regularized problem solution is less than the evaluation's condition of the unperturbed problem*

The proof of this theorem will be omitted in this thesis. However, interested readers could refer to [29].

B.2.4 Tikhonov Regularization

In this section, we will discuss and analyze a classical regularization scheme, the *Tikhonov regularization scheme*. The analysis here will depend on the classical spectral theory of linear operators in finite dimension vector spaces (this is out of the scope of this thesis, however one can refer to [29] for details.)

We noted earlier that solving Equation (B.10) can be replaced with the problem of minimizing the functional

$$f(x) = \frac{1}{2} \|Kx - y\|^2, \tag{B.16}$$

given K and y.

Theorem B.2. *Let K be an invertible matrix. The following hold:*

1. *x_* is the minimum point of f if and only if x_* is the solution of Equation (B.10);*
2. *The critical point equation of f is $K^T K x = K^T y$;*
3. *The critical point equation of f is equivalent to Equation (B.10).*

Also, the proof of this will not be given here, but can be found in [29]. To avoid amplification of error in the solution of Equation (B.10), a natural notion is to penalize the distance from the solution to a reference value or the norm of the solution vector (the distance with respect to the origin). Here, a reference value means a known approximate solution to the problem denoted by x_r.

In the case of Tikhonov's method, this idea involves a regularization that penalizes the growth of the distance from the reference value. For the problem of the form in Equation (B.10) under consideration, it consists of solving the critical point equation of the function

$$f_\alpha(x) = \frac{1}{2} \|Kx - y\|^2 + \frac{\alpha}{2} \|x - x_r\|^2, \tag{B.17}$$

with $\alpha > 0$, the so-called regularization parameter. The minimum point x^α satisfies the critical point equation

$$\alpha(x^\alpha - x_r) + K^T K x^\alpha = K^T y, \tag{B.18}$$

which can be re-written as

$$\left(\alpha \mathcal{I} + K^T K\right) x^\alpha = K^T y + \alpha x_r. \tag{B.19}$$

We will verify that Equation (B.19) provides a regularization scheme for the normal equation $K^T K x = K^T y$. Care should be taken that the problem that is being regularized is the normal equation above and not Equation (B.10). Hence, let

$$A_\alpha = \alpha \mathcal{I} + K^T K \text{ and } b_\alpha = \alpha x_r, \ \alpha > 0. \tag{B.20}$$

Theorem B.3. *The families A_α, b_α, $\alpha > 0$, are a linear regularization scheme for equation $K^T K x = K^T y$.*

The proof of this is omitted, but can be found in [29] as well.

Remark B.1. The definitions, notions and theorems in this section stated for the linear regularization schemes can also be derived for the non-linear case.