

QUANTUM HALL EFFECT ON GRASSMANN MANIFOLDS

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ABSTRACT

QUANTUM HALL EFFECT ON GRASSMANN MANIFOLDS

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In this work we formulate Quantum Hall Effect(QHE) on Grassmann manifolds $Gr_2(\mathbb{C}^N)$. We, first give a self-contained reviews of integer QHE on \mathbb{R}^2 , $S^2 \equiv \mathbb{C}P^1$ and $\mathbb{C}P^2$ which are oriented towards our purposes. Then, we set up the Landau problem on $Gr_2(\mathbb{C}^N)$ and discuss and formulate the wave functions, energy levels, degeneracy, incompressibility and spatial density properties. Group theoretical techniques are used to explore these properties for both abelian and non-abelian backgrounds and the wave functions are expressed in terms of suitably restricted subset of Wigner- \mathcal{D} functions. For the simplest case of QHE on $Gr_2(\mathbb{C}^4)$ with pure $U(1)$ gauge fields, we introduce Plücker coordinates and express the wave functions and the gauge fields in these coordinates. We calculate the two-point correlation function and deduce the incompressibility of Quantum Hall liquid on $Gr_2(\mathbb{C}^4)$. We indicate how these formulation in local coordinates may be generalized to all $Gr_2(\mathbb{C}^N)$.

Keywords: Hall Effect, Landau Problem, Higher Dimensions, Grassmann Mani-

fold, Abelian charge, Non-Abelian Charge, Plücker, Young Diagram.

ÖZ

GRASSMANN MANİFOLDLARINDA KUANTUM HALL ETKİSİ

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Bu tezde Grassmann manifoldlarında, $Gr_2(\mathbb{C}^N)$, kuantum Hall etkisi formüle edildi. Başlangıç olarak, \mathbb{R}^2 , $S^2 \equiv \mathbb{C}P^1$ ve $\mathbb{C}P^2$ manifoldlarında tam sayı kuantum Hall etkisi, tez konusu olan çalışmaya referans olacak seviyede özetlendi. Akabinde, $Gr_2(\mathbb{C}^4)$ manifoldunda Landau problemi oluşturuldu ve kuantum Hall sistemlerinde önemli olan dalga denklemi, enerji seviyeleri, dejenere durumu, sıkıştırılabilirlik ve yoğunluk gibi özellikler tartışıldı ve formalize edildi. Bu özelliklerin abelyen ve abelyen olmayan ayar alanlarında açıklamak adına grup teori teknikleri kullanıldı ve dalga denklemleri Wigner- \mathcal{D} fonksiyonlarının uygun bir şekilde sınırlandırılmış alt kümeleri cinsinden tariflendi. Grassmann manifoldlarının en basit hali olan $Gr_2(\mathbb{C}^4)$ formunda ve $U(1)$ ayar alanı altında Plücker koordinatları tariflendi ve bu koordinatlar kullanılarak dalga denklemi ve ayar alanı ifade edildi. İki parçacık için korelasyon denklemi hesaplandı ve $Gr_2(\mathbb{C}P^4)$ manifoldunda Kuantum Hall sistemlerinin sıkıştırılmazlık özelliği ortaya kondu.

Anahtar Kelimeler: Hall Etkisi, Landau Problemi, Yüksek Boyutlar, Grassmann Manifoldu, Abelyen Yük, Abelyen olmayan Yük, Plücker, Young Şeması

to

my family

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CHAPTER 1

INTRODUCTION

Quantum Hall effect (QHE) is essentially described as the quantization of the resistivity, R , in 2- D electron systems in low temperatures and under the perpendicular external magnetic field, B_{\perp} . In classical QHE, resistivity R depends on the applied magnetic field linearly as was first discovered by E. Hall in 1879 [1]. In 1980 von Klitzing discovered that at certain intervals of the magnetic field, resistivity value remains the same while it changes discontinuously at the specific values of B_{\perp} . This variation forms a ladder-like shapes on the graph of R versus B_{\perp} that we call Hall plateaux.

The formulation of the QHE on planar systems is given by R. B. Laughlin [2]. He constructed a multi-particle wave function (Laughlin-wave functions) which describe 2 D planar electron systems under the influence of strong perpendicular magnetic field. Laughlin-wave functions propose that QH liquid systems have a incompressibility property due to Pauli exclusion principle. In this system, probability of finding any electron pair in the same locations is zero, which is the main reason for the incompressibility. An important consequence of the Laughlin description is that the wave functions are not invariant under the translation. In 1983 F.D.M Haldane proposed a electron gas system on S^2 which are under the influence on the Dirac magnetic monopole located at the center of S^2 [3]. He obtained the energy spectrum of the system and single- and multi particle wave functions which are translationally invariant.

In 2000, J. Hu and S.C. Zhang constructed the 4-dimensional generalization of the QHE under $SU(2)$ gauge field background [4]. The authors solved the Landau problem for the fermions with additional $SU(2)$ degrees of freedom. They

expressed the spatial density of fermionic particles in the thermodynamic limit. In order to obtain finite spatial density, they attached infinite $SU(2)$ degrees of freedom which gives massless particles with all spin values. They forged a new way to study higher spin massless fields such as photons and gravitons as the edge excitations of QH liquids. However, occurrence of other massless higher spin states is not the desired property of the system. In 2001, D. Nair and V.P. Karabali proposed a new higher dimensional generalization of the QHE [5]. These authors solved the Landau problem on complex projective spaces. They formulate the problem on \mathbb{CP}^1 , which is reformulation of the work of F.D.M Haldane in a mathematical language and adaptable to higher dimension. They solved the Landau problem by taking \mathbb{CP}^N as the coset realization $\mathbb{CP}^N \sim \frac{SU(N+1)}{U(1) \times SU(N)}$. In this realization, wave functions are the suitable subsets of the Wigner- \mathcal{D} functions on $SU(N+1)$ in a manner that certain restrictions are imposed due to abelian $U(1)$, and/or non-abelian $SU(N)$, gauge field backgrounds. In this manner, the wave functions can be thought as the $U(1)$ and $SU(N)$ bundles over \mathbb{CP}^N . D. Nair and V.P. Karabali achieved to express the energy spectrum of the system under the $U(1)$ and/or $SU(N)$ gauge field backgrounds. In this approach, degeneracy of the system is expressed as the dimension of the representation on which restricted Wigner- \mathcal{D} functions belong. An important achievement of their work is that finite spatial density is obtained in the thermodynamic limit without introducing infinite $SU(2)$ degrees of freedom. Another important achievement of this work is to show incompressibility property of the system. For this purpose, they calculated the two-point correlation function that basically gives the probability of the finding any two particles at the given coordinate pairs. Result gives zero when the given coordinates refers to the same point, as expected.

In this thesis we focus our attention on the formulation of QHE on Grassmann manifolds. Building upon the ideas of D. Nair and V.P. Karabali [5], we solve the Landau problem on Grassmanians which are the generalization of \mathbb{CP}^N and defined as the 1-dimensional complex subspaces through origin in \mathbb{C}^N and shown as $Gr_2(\mathbb{C}^N)$. The coset realization may be expressed as $Gr_2(\mathbb{C}^N) \sim \frac{SU(N)}{U(1) \times SU(2) \times SU(N-2)}$ which implies the wave functions may be defined for three different cases: i) single $U(1)$ ii) $SU(2)$ and $U(1)$ iii) $SU(2) \times SU(N-2)$ and $U(1)$ gauge field backgrounds. In chapter 3 we first construct the Landau problem on

the simplest form of the Grassmannians, $Gr_2(\mathbb{C}^4)$. Then, we formulate the wave functions in terms of Wigner- \mathcal{D} functions and give the energy spectrum for these three cases. We also express the two-point correlation function and prove the incompressibility property of the QH system in Grassmannians. In addition we calculate spatial density in thermodynamic limit. We generalize the results for $Gr_2(\mathbb{C}^N)$.

Grassmannian manifold $Gr_2(\mathbb{C}^4)$ may be parametrized by Plücker coordinates. The parametrization can be shown as $\psi : Gr_2(\mathbb{C}^4) \hookrightarrow \mathbf{P}(\wedge^2 \mathbb{C}^N)$ where ψ is in fact an embedding since the image is the totally decomposable vectors in $\wedge^2 \mathbb{C}^N$ [6]. The embedding is given by a homogeneous equation called Klein quadric. We formulate the local form of the wave functions in terms of Plücker coordinates. First, by using the transformation properties of the Wigner- \mathcal{D} functions and gauge field we write the explicit form of the wave functions for the single electron. Then, we express the many-particle wave function by forming the appropriate Slater determinants. In order to analyse the incompressibility property of the system we calculate the two-point correlation function in terms of Plücker coordinates. In the thermodynamic limit, that gives zero if the two particle coordinates are selected as identical. This result is in coherence with incompressibility property of the QH liquid. As a final note, the generalization of the formulation for $Gr_2(\mathbb{C}^N)$ is also possible by defining homogeneous conditions in a proper form for $N > 4$.

CHAPTER 2

INTEGER HALL EFFECT

In this chapter we give a short review of Integer Quantum Hall Effect following the exposition of [7].

2.1 Classical Hall Effect

Hall effect was first observed in 1879 by E. Hall. Some 18 years before the discovery of the electron, E. Hall conducted some experiments to analyse the effect of the magnetic field on the electric current. He observed that a voltage difference occurs on the current carrying plate which is exposed to the external perpendicular magnetic field.

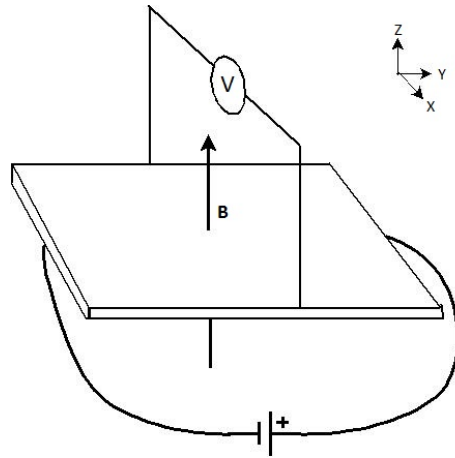


Figure 2.1: Basic Representation of the Hall's Experiment

As can be seen from figure 2.1 a 2D conducting plate is located in xy-plane and

a current is passing through the plate in y -direction. Hall observed that if one applies the magnetic field which has a component perpendicular to the plate, a non-zero potential difference is observed in the voltmeter. This is called the Hall voltage and it is proportional to the applied magnetic field as can be seen from the figure 2.2.

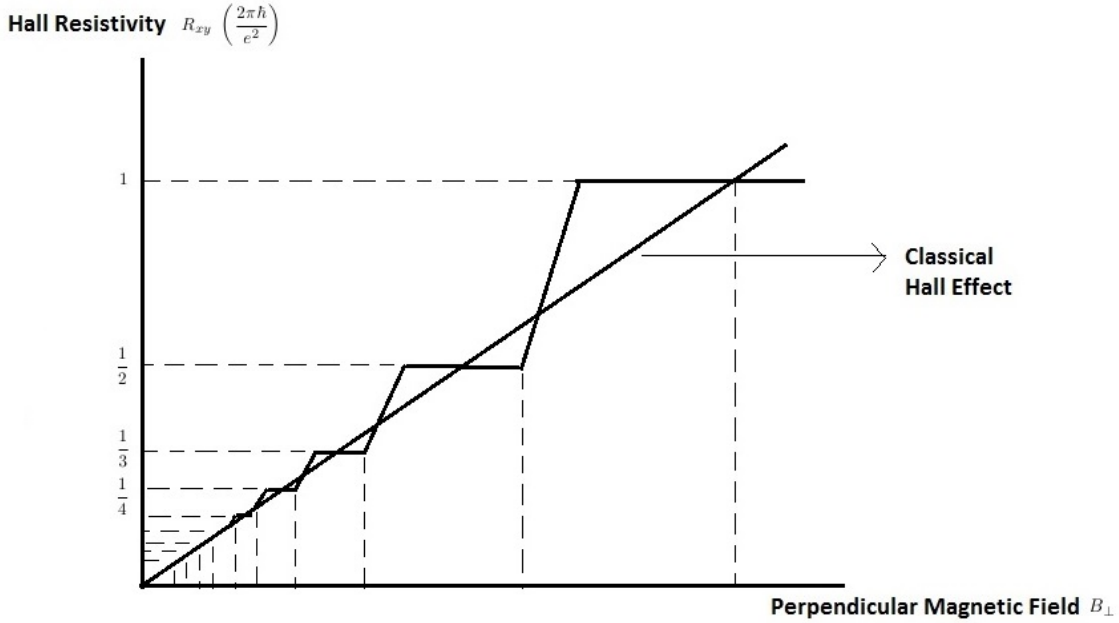


Figure 2.2: Dependence of Hall Resistivity on Perpendicular Magnetic Field regarding Classical Hall Effect versus Integer Quantum Hall Effect

After the discovery of the electron, Hall effect is explained by the Maxwell theory and Lorentz force law. According to the Lorentz force law, a net force acts on a charged particle which is exposed to the external magnetic and electric fields. The law can be expressed in the form

$$\mathbf{F}_{net} = m\dot{\mathbf{v}} = -e(\mathbf{E} + \mathbf{v} \times \mathbf{B}). \quad (2.1)$$

When the external magnetic field is switched on, electrons which move in $+y$ direction will be deflected toward $-x$ direction by the magnetic field. Deflection of the electrons causes a polarization between the two edges of the plate, which causes a electrical field in the $-x$ direction. This electric field gives an electrical force on electrons in the $+x$ direction. After a certain amount of time, these two forces balance each other. This steady-state condition can be expressed as:

$$\mathbf{F}_{net} = 0 = m\dot{\mathbf{v}} = -e(\mathbf{E} + \mathbf{v} \times \mathbf{B}^{ext}), \quad (2.2)$$

where \mathbf{B}^{ext} is in $+z$ direction. For simplicity, we may write $\mathbf{B}^{ext} = B_{\perp} \hat{k}$. (2.2) gives us:

$$v_y = \frac{-E_x}{B_{\perp}}. \quad (2.3)$$

We can relate the current with the charge velocity as

$$\mathbf{J} = e\rho_0 \mathbf{v}, \quad (2.4)$$

where ρ_0 is the charge density of the plate. By using (2.4) and (2.3), we may write

$$\frac{E_x}{J_y} = -\frac{B_{\perp}}{e\rho_0}. \quad (2.5)$$

If we assume that the charge density is constant over the plate, this equation will give:

$$\frac{E_x \ell}{J_y \ell} = \frac{V_x}{I_y} = -\frac{B_{\perp}}{e\rho_0} = R_{yx}, \quad (2.6)$$

where R_{yx} is called Hall resistivity with $R_{yy} = R_{xx} = 0$ and ℓ is the cross-sectional length of the plate k_x, k_y . From (2.6) it is clear that Hall resistivity depends on the external magnetic field linearly and does not depend on the shape of the plate. As will be discussed in the following sections, (2.6) will no longer hold and the relation between Hall resistivity and the magnetic field will be modified.

2.2 Integer Quantum Hall Effect

2.2.1 Landau Problem on the Plane

Landau problem on a plane is described as an electron under the influence of constant magnetic field which is perpendicular to the system. In order to trap electrons in a thin layer, interface between the semiconductors or between semiconductors and insulators can be used. If the interface is located on the xy -plane, then electrons may be considered to be trapped in a potential well in the z -direction, which confines the motion of the electrons to two-dimensions. The interface is thin enough and the system is cooled to almost absolute zero temperature so that the interactions between the electrons can be neglected. Under these conditions an isolated charged particle makes a cyclotron motion.

Hamiltonian for a single electron under the influence of the electromagnetic field [8] is

$$\mathcal{H} = \frac{1}{2M} \left(\mathbf{p} + \frac{e\mathbf{A}}{c} \right)^2 + e\phi, \quad (2.7)$$

where M , \mathbf{p} , \mathbf{A} and c are the electron mass, canonical momentum and magnetic potential and speed of light respectively. If the only external effect is perpendicular magnetic field and we use the unit system in which $c = 1$, \mathcal{H} reads

$$\mathcal{H} = \frac{1}{2M} [(-i\hbar\partial_x + eA_x)^2 + (-i\hbar\partial_y + eA_y)^2] = \frac{1}{2M}(P_x^2 + P_y^2), \quad (2.8)$$

where P_i is called as kinematical momentum with

$$P_x = (-i\hbar\partial_x + eA_x), \quad P_y = (-i\hbar\partial_y + eA_y). \quad (2.9)$$

Even though the components of the canonical momentum commute with each other, i.e., $[p_i, p_j] = 0$, commutator of kinematical momentum gives the non-zero value

$$[P_x, P_y] = i\hbar eB_\perp = i\frac{\hbar^2}{\ell_B^2}, \quad (2.10)$$

where $\mathbf{B} = \nabla \times \mathbf{A}$ and $\ell_B \left(= \sqrt{\frac{\hbar}{eB}} \right)$ is called the magnetic length which is going to be identified as the fundamental scale for the QHE. We may decompose the electron coordinate into two pieces as $\mathbf{x} = \mathbf{R} + \mathbf{X}$ where $\mathbf{X} = (X, Y)$ and $\mathbf{R} = (R_x, R_y)$ are called guiding center and relative coordinate with the form

$$X = x + \frac{P_y}{eB_\perp}, \quad Y = y - \frac{P_x}{eB_\perp}, \quad R_x = -\frac{P_y}{eB_\perp}, \quad R_y = \frac{P_x}{eB_\perp}. \quad (2.11)$$

Heisenberg equation of motion reads

$$\begin{aligned} \frac{dX}{dt} &= \frac{1}{i\hbar}[X, H] = 0, & \frac{dY}{dt} &= \frac{1}{i\hbar}[Y, H] = 0 \\ \frac{dP_x}{dt} &= \frac{1}{i\hbar}[P_x, H] = w_c P_y, & \frac{dP_y}{dt} &= \frac{1}{i\hbar}[P_y, H] = -w_c P_x, \end{aligned} \quad (2.12)$$

where $w_c = \frac{\hbar}{M\ell_B^2}$ is called as cyclotron frequency. By using 2.11 and 2.12 we may write

$$\frac{d}{dt}(R_x + iR_y) = -iw_c(R_x + iR_y). \quad (2.13)$$

Therefore, we obtain

$$R_x = R_x^0 \cos w_c t + R_y^0 \sin w_c t, \quad R_y = -R_x^0 \sin w_c t + R_y^0 \cos w_c t, \quad (2.14)$$

which implies that as a semi-classical approach the system can be seen as an

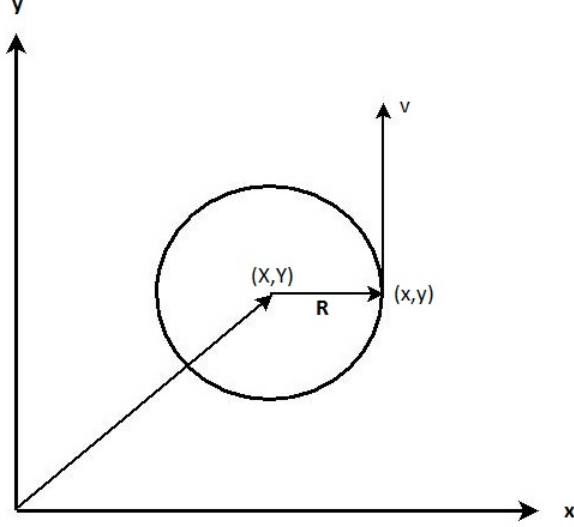


Figure 2.3: Cyclotron Motion of the Electron in an external Magnetic Field

electron which makes a circular motion around the guiding center with a radius R as shown in the figure 2.3. Commutator of the guiding center coordinates gives

$$[X, Y] = -i\ell_B^2, \quad (2.15)$$

and the commutator of the kinematical momentum and guiding center components give

$$[X, P_x] = [X, P_y] = [Y, P_x] = [Y, P_y] = 0. \quad (2.16)$$

Commutation relations, (2.15) and (2.16), allow us to define two sets of uncoupled harmonic oscillator annihilation and creation operators in terms of the kinematical momentum and the guiding center coordinates

$$a = \frac{\ell_B}{\sqrt{2\hbar}}(P_x + iP_y), \quad a^\dagger = \frac{\ell_B}{\sqrt{2\hbar}}(P_x - iP_y) \quad (2.17)$$

$$b = \frac{1}{\sqrt{2\ell_B}}(X - iY), \quad b^\dagger = \frac{1}{\sqrt{2\ell_B}}(X + iY), \quad (2.18)$$

where

$$[a, a^\dagger] = [b, b^\dagger] = 1, \quad [a, b] = 0. \quad (2.19)$$

We can define the number operators

$$\hat{N} = a^\dagger a, \quad \hat{n} = b^\dagger b. \quad (2.20)$$

Since the commutator of these two number operators is zero we can diagonalize them simultaneously and they will span the Fock states labelled by $|N, n\rangle$:

$$\hat{N} |N, n\rangle = N |N, n\rangle, \quad \hat{n} |N, n\rangle = n |N, n\rangle, \quad (2.21)$$

where N and n are the corresponding eigenvalues. The annihilation and creation operator pairs act on the Fock states and give

$$a^\dagger |N, n\rangle = \sqrt{N+1} |N+1, n\rangle, \quad a |N, n\rangle = \sqrt{N} |N-1, n\rangle, \quad (2.22)$$

$$b^\dagger |N, n\rangle = \sqrt{n+1} |N, n+1\rangle, \quad b |N, n\rangle = \sqrt{n} |N, n-1\rangle. \quad (2.23)$$

The ground state, $|0, 0\rangle$, satisfies the condition:

$$a |0, 0\rangle \equiv b |0, 0\rangle \equiv 0. \quad (2.24)$$

By acting creation operators on the ground state we can create the generic state as:

$$|N, n\rangle = \frac{(a^\dagger)^N (b^\dagger)^n}{\sqrt{N!n!}} |0, 0\rangle. \quad (2.25)$$

The Hamiltonian of the system is:

$$\mathcal{H} = (a^\dagger a + a a^\dagger) \frac{\hbar w_c}{2} = (a^\dagger a + \frac{1}{2}) \hbar w_c = (N + \frac{1}{2}) \hbar w_c, \quad (2.26)$$

where w_c is interpreted as the cyclotron frequency as

$$w_c = \frac{e B_\perp}{M}. \quad (2.27)$$

We note that \mathcal{H} depends only on the set (a, a^\dagger) . Energy levels may be written in the form

$$E_N = (N + \frac{1}{2}) \hbar w_c = (N + \frac{1}{2}) \frac{\hbar^2}{M \ell_B^2}. \quad (2.28)$$

These are called the Landau levels.

2.2.2 Density of States

Density of states(DOS) can be thought as the number of states per unit energy per unit volume as:

$$\rho(\epsilon) = \frac{H(\epsilon)}{V}. \quad (2.29)$$

$H(\epsilon)$ is the number of states per unit energy and V is the volume of the system.

We will calculate the DOS with and without external magnetic field.

2.2.2.1 Density of States in the Absence of the Magnetic Field

In the absence of the external magnetic field, the system can be thought as a collection of free particles in an infinite potential well with a square cross-section having side length L . The wave function of an electron may be written as

$$\psi(\mathbf{x}) = \sqrt{\frac{2}{L}} \sin k\mathbf{x}, \quad (2.30)$$

where k is the wave number. It has the energy eigenvalues and wave numbers

$$\epsilon = \frac{p_x^2 + p_y^2}{2m} = \frac{\hbar^2(k_x^2 + k_y^2)}{2m}, \quad (2.31)$$

$$k_x = \frac{2\pi}{L}n_x, \quad k_y = \frac{2\pi}{L}n_y, \quad (2.32)$$

which implies that the degenerate states form a circle with the radius $k = \sqrt{k_x^2 + k_y^2}$ in k -space as shown in the figure (2.4).

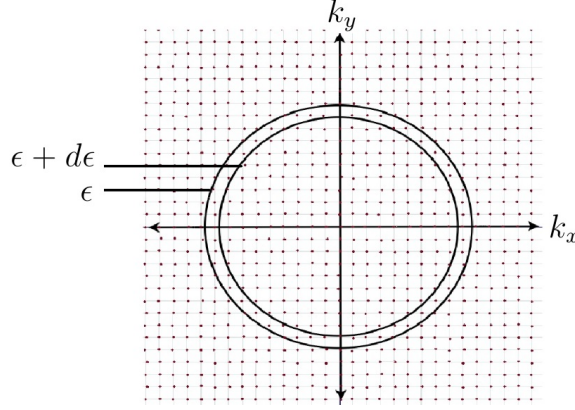


Figure 2.4: States of Electrons at k -space in the absence of the Perpendicular Magnetic Field

From (2.32) we may assume that the side length of each lattice is $\frac{2\pi}{L}$ and then each state is localized in a square lattice having side length $\frac{2\pi}{L}$. Number of the states between the energy levels ϵ and $\epsilon + \Delta\epsilon$ may be expressed in the form

$$H(\epsilon)d\epsilon = \frac{1}{(\frac{2\pi}{L})^2} 2\pi k dk. \quad (2.33)$$

By using (2.31), $H(\epsilon)$ may be written such that

$$\begin{aligned} H(\epsilon) &= \frac{1}{(\frac{2\pi}{L})^2} 2\pi k \frac{dk}{d\epsilon} \\ &= \frac{1}{(\frac{2\pi}{L})^2} 2\pi k \frac{m}{\hbar^2 k} \\ &= \frac{L^2 m}{2\pi \hbar^2}. \end{aligned} \quad (2.34)$$

The electrons are confined in a 2D box with the volume L^2 . DOS is therefore given as

$$\rho_{\Phi} = \frac{m}{2\pi\hbar^2}. \quad (2.35)$$

This result implies that the density of the states is constant independent of every energy level and location.

2.2.2.2 Density of States in the Presence of the Magnetic Field

In the presence of the magnetic field, the Hamiltonian and energy eigenvalues can be read from (2.8) and (2.28), respectively. By using $P_i = \hbar k_i$, these two equations give

$$(k_x^2 + k_y^2)_N = \frac{2N + 1}{\ell_B^2}, \quad (2.36)$$

where subscript N is the Landau level index. The equation implies that the degenerate states of a given Landau level are distributed over a circle in k -space. This can be observed from the figure 2.5. Unlike the free particle case, energy spectrum is discrete and labelled by N . In k -space, area between two successive Landau levels may be written as

$$\pi |\mathbf{k}_N|^2 - \pi |\mathbf{k}_{N-1}|^2 = \frac{2\pi}{\ell_B^2}. \quad (2.37)$$

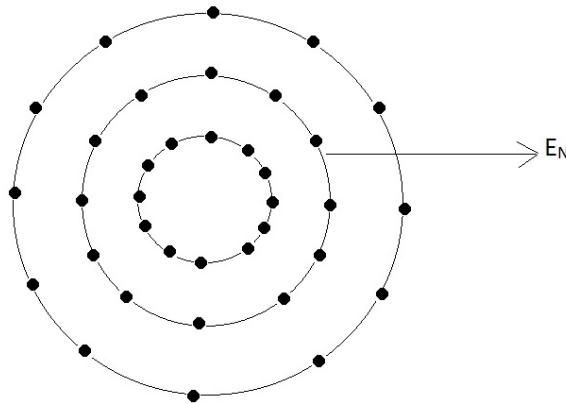


Figure 2.5: States in k -space under the influence of the perpendicular magnetic field

If the number of the particles and volume of the system is kept fixed, a state occupies an area $\left(\frac{2\pi}{L}\right)^2$ in k -space. This gives the number of states per unit

energy $H(\epsilon)$ in the form

$$H(\epsilon) = \frac{1}{\left(\frac{2\pi}{L}\right)^2} \frac{2\pi}{\ell_B^2}, \quad (2.38)$$

Using (2.29), density of states takes the form

$$\rho_\Phi = \frac{1}{2\pi\ell_B^2}. \quad (2.39)$$

This result implies that DOS changes as ℓ_B^2 changes. DOS increases linearly by the applied magnetic field. If we keep increasing the magnetic field, there will be more states available at the lowest Landau level.

2.2.3 Single Electron Wave Function

In the previous sections Landau levels were written in terms of Fock states which are labelled by two quantum number, N and n . Energy eigenvalues are given in terms of the quantum number N and the degeneracy of the states are controlled by the quantum number n . Depending on our gauge choice we can find several different observables apart from $\hat{n} = b^\dagger b$, which commute with the Hamiltonian and therefore simultaneously diagonalized together with the Hamiltonian.

(2.15) implies that we can not diagonalize X and Y simultaneously. This forces us to diagonalize combinations of them. $(X^2 + Y^2)$ is the proper choice if the Hall system has the disk geometry. This operator can be expressed in terms of the operators given in (2.18) and gives eigenvalues as:

$$\begin{aligned} (X^2 + Y^2) |N, n\rangle &= (2b^\dagger b + 1)\ell_B^2 |N, n\rangle \\ &= (2n + 1)\ell_B^2 |N, n\rangle. \end{aligned} \quad (2.40)$$

(2.9) and (2.11) imply that X and Y are gauge dependent, which forces us to make a gauge choice to proceed. The possible choice is the symmetric gauge as

$$\mathbf{A} = \left(\frac{1}{2}B_\perp y, -\frac{1}{2}B_\perp x\right). \quad (2.41)$$

In this gauge, the angular momentum operator can be written as

$$xp_y - yp_x = \frac{eB_\perp}{2}(X^2 + Y^2) - \frac{1}{2eB_\perp}(P_x^2 + P_y^2) = (b^\dagger b - a^\dagger a)\hbar. \quad (2.42)$$

and can be diagonalized simultaneously with the Hamiltonian and gives the eigenvalues

$$(b^\dagger b - a^\dagger a) |N, n\rangle = (n - N) |N, n\rangle . \quad (2.43)$$

We write kinematical momentum and guiding center operators in this gauge in the form

$$\begin{aligned} P_x &= -i\hbar\partial_x + \frac{1}{2}\frac{\hbar}{\ell_B^2}y, & P_y &= -i\hbar\partial_y - \frac{1}{2}\frac{\hbar}{\ell_B^2}x \\ X &= \frac{1}{2}x - i\ell_B^2\partial_y, & Y &= \frac{1}{2}y + i\ell_B^2\partial_x. \end{aligned} \quad (2.44)$$

Defining the dimensionless complex coordinates as

$$z = \frac{1}{2\ell_B}(x+iy), \quad z^* = \frac{1}{2\ell_B}(x-iy), \quad \frac{\partial}{\partial z} = \ell_B\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \quad \frac{\partial}{\partial z^*} = \ell_B\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right). \quad (2.45)$$

we can write the harmonic oscillator operators

$$a \equiv -\frac{i}{\sqrt{2}}\left(z + \frac{\partial}{\partial z^*}\right), \quad a^\dagger \equiv \frac{i}{\sqrt{2}}\left(z^* - \frac{\partial}{\partial z}\right), \quad b \equiv \frac{1}{\sqrt{2}}\left(z^* + \frac{\partial}{\partial z}\right), \quad b^\dagger \equiv \frac{1}{\sqrt{2}}\left(z - \frac{\partial}{\partial z^*}\right). \quad (2.46)$$

Now, we are ready to find the wave function of the system. We start with the lowest Landau level (LLL). In LLL we have

$$a |0\rangle = 0. \quad (2.47)$$

and

$$\begin{aligned} \langle x | a | 0 \rangle &= \int_{-\infty}^{\infty} \langle x | a | x' \rangle \langle x' | 0 \rangle dx' \\ &= a(x) \psi_0(x) \\ &= -\frac{i}{\sqrt{2}}\left(z + \frac{\partial}{\partial z^*}\right) \psi_0(x) = 0, \end{aligned} \quad (2.48)$$

where $\psi_0(x) = \langle x' | 0 \rangle$ is the LLL wave function. Solving this differential equation gives

$$\psi_0(x) = \lambda(z) \exp(-zz^*), \quad (2.49)$$

where $\lambda(z)$ is the holomorphic function of z . In a similar manner we can act annihilation operator b on $\psi_0(x)$. Finally, we obtain the LLL wave function as

$$\psi_0(x) = \frac{1}{\sqrt{2\pi\ell_B^2}} \exp(-zz^*) = \frac{1}{\sqrt{2\pi\ell_B^2}} \exp\left(-\frac{r^2}{4\ell_B^2}\right), \quad (2.50)$$

where $r^2 = x^2 + y^2$. By using (2.25), wave function of a generic state in LLL is found to be

$$\begin{aligned}\psi_n(x) = \langle x|n\rangle &= \frac{1}{\sqrt{n!}}(b^\dagger)^n |0\rangle \\ &= \sqrt{\frac{1}{2^n n!}} \left(z - \frac{\partial}{\partial z^*}\right)^n \exp(-zz^*) \\ &= \sqrt{\frac{2^n}{2\pi\ell_B^2 n!}} z^n \exp(-|z|^2).\end{aligned}\tag{2.51}$$

It is worth to write down the probability density in LLL in order to explore the disk geometry. The probability density is:

$$\begin{aligned}|\psi_n(x)|^2 = \psi_n(x)(\psi_n(x))^* &= \frac{2^n}{2\pi\ell_B^2 n!} |z|^{2n} \exp(-2|z|^2) \\ &\sim r^n \exp\left(\frac{-r^2}{2\ell_B^2}\right),\end{aligned}\tag{2.52}$$

with an extremum at $r_n = \sqrt{2n}\ell_B$. This implies that the most probable points where electrons are confined form a ring structure. According to (2.43), each state is represented by a ring which has angular momentum $n\hbar$. Area between the rings may be written as

$$\Delta S = \pi r_{n+1}^2 - \pi r_n^2 = 2\pi\ell_B^2.\tag{2.53}$$

This means each Landau site occupies an area $2\pi\ell_B^2$.

2.2.4 Incompressibility of the Hall Liquid

One of the most important feature of the quantum Hall systems is the incompressibility. To discuss this property, it is worth to calculate compressibility factor which is the relative change in volume against change in pressure as

$$\kappa = \frac{-1}{V} \frac{\partial V}{\partial P} \Big|_{\mathcal{N}},\tag{2.54}$$

where \mathcal{N} , V and P are the number of particles, volume and pressure of the system, respectively. For an isentropic process, pressure can be thought as the change in internal energy as a response to the change in volume such that

$$P = -\frac{\partial E}{\partial V},\tag{2.55}$$

where E is the internal energy of the system.

QH systems are constructed as a 2-D thin layer between two semiconductors as indicated in previous sections. That means volume of the system refers the area of layer for QH systems so that we should take surface area, S , for the volume, V . Therefore, by combining (2.54) and (2.55) we obtain

$$\kappa^{-1} = -S \frac{\partial}{\partial S} \frac{\partial E}{\partial S} \Big|_N = -S \frac{\partial^2 E}{\partial S^2} \Big|_N. \quad (2.56)$$

We can relate number of the particle, \mathcal{N} , and the surface area of the system, S , as $\rho = \frac{\mathcal{N}}{S}$ where ρ is called as the number density. In the thermodynamic limit energy is an extensive property and can be written in terms of energy density as

$$E = \mathcal{N}\epsilon(\mathcal{N}/S) = \mathcal{N}\epsilon(\rho). \quad (2.57)$$

Now, (2.56) reads:

$$\begin{aligned} \kappa^{-1} &= -S \frac{\partial}{\partial S} \frac{\partial \mathcal{N}\epsilon(\mathcal{N}/S)}{\partial S} \Big|_N \\ &= -S \mathcal{N} \frac{\partial}{\partial S} \left(\frac{-\mathcal{N}}{S^2} \frac{\partial \epsilon(\mathcal{N}/S)}{\partial S} \right) \\ &= \mathcal{N}^2 \left(\frac{2}{S^2} \frac{\partial \epsilon(\mathcal{N}/S)}{\partial S} + \frac{\mathcal{N}}{S^3} \frac{\partial^2 \epsilon(\mathcal{N}/S)}{\partial S^2} \right) \\ &= \rho^2 \frac{d^2(\rho\epsilon)}{d\rho^2}. \end{aligned} \quad (2.58)$$

The chemical potential may be written as

$$\mu = \frac{\partial E}{\partial \mathcal{N}} \Big|_V. \quad (2.59)$$

Since we measure the potential in a volume, the number of the particle is the function of ρ as $\mathcal{N} = \mathcal{N}(\rho)$. By using (2.59) and (2.58), we may write

$$\begin{aligned} \mu &= \frac{1}{S} \frac{\partial}{\partial \rho} (\rho S \epsilon(\rho)) \\ &= \frac{\partial}{\partial \rho} (\rho \epsilon), \end{aligned} \quad (2.60)$$

and

$$\kappa^{-1} = \rho^2 \frac{d\mu}{d\rho}. \quad (2.61)$$

The condition for the incompressibility is $\kappa = 0$, which implies the chemical potential changes discontinuously as a response to change in number density. We may conclude that change in the number of particles would cause a abrupt change in the energy if the QH system is incompressible. However, the classical approach does not allow any discontinuous change on the chemical potential. As an ansatz we may update the resistivity formula (2.6) as

$$R_{xy} = \frac{E_y}{J_x} = \frac{B_{\perp}}{e\rho} = \frac{1}{\nu} \frac{2\pi\hbar}{e^2} = \frac{1}{\nu} \frac{\Phi_D}{e}, \quad \nu = \frac{2\pi\hbar\rho_0}{eB_{\perp}}, \quad (2.62)$$

where ν and Φ_D are called as the filling factor and Dirac flux quantum, respectively. By using (2.39) We may write

$$\nu = \frac{\rho_0}{\rho_{\Phi}} = \frac{N_e}{N_{\Phi}} = \frac{\text{Number of Electrons}}{\text{Number of States}}, \quad (2.63)$$

in which N_e and N_{Φ} are the number of the electrons and the number of the states in the system with area S . As will be explained in the following section, there is a discontinuous jump in the chemical potential if ν is quantized.

2.3 Many Particle Wave Functions

Totally antisymmetric wave function of \mathcal{N} electrons can be represented by the Slater determinant. To eliminate some complexities we assume that all electrons are spin-polarized by Zeeman effect. In other words, we work in a spin frozen system.

Single particle wave function in the quantum Hall sample of disc geometry at the ground state has been given as (2.51)

$$\psi_n = \sqrt{\frac{2^n}{2\pi\ell_B^2 n!}} z^n \exp(-|z|^2). \quad (2.64)$$

If the filling factor ν is 1, say total number of electrons is \mathcal{N} , then each state is occupied by one electron, only. \mathcal{N} -particle wave function is composed of single particle states $|0, n\rangle$, where n goes from 0 to $\mathcal{N} - 1$. Therefore, we may write the \mathcal{N} -particle wave function in the form

$$\Psi(1, 2, \dots, \mathcal{N}) = \sqrt{\frac{1}{\mathcal{N}!}} \begin{vmatrix} \psi_0(1) & \psi_1(1) & \cdots & \psi_{\mathcal{N}-1}(1) \\ \psi_0(2) & \psi_1(2) & \cdots & \psi_{\mathcal{N}-1}(2) \\ \vdots & \vdots & \ddots & \vdots \\ \psi_0(\mathcal{N}) & \psi_1(\mathcal{N}) & \cdots & \psi_{\mathcal{N}-1}(\mathcal{N}) \end{vmatrix} \quad (2.65)$$

By using (2.64), the determinant can be expressed as

$$\Psi(1, 2, \dots, \mathcal{N}) = \sqrt{\frac{1}{\mathcal{N}!}} \rho_0^{\mathcal{N}/2} \begin{vmatrix} 1 & z_1 & \cdots & z_1^{\mathcal{N}-1} \\ 1 & z_2 & \cdots & z_2^{\mathcal{N}-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & z_{\mathcal{N}} & \cdots & z_{\mathcal{N}}^{\mathcal{N}-1} \end{vmatrix} \exp\left(-\sum_{i=1}^{\mathcal{N}} |z_i|^2\right), \quad (2.66)$$

where $\rho_0 = \frac{2^{\mathcal{N}}}{2\pi\ell_B^2\mathcal{N}!}$. We can simplify this expression in the concise form

$$\Psi(1, 2, \dots, \mathcal{N}) = \rho_0^{\mathcal{N}/2} \prod_{i < s} (z_i - z_s) \exp\left(-\sum_{i=1}^{\mathcal{N}} |z_i|^2\right). \quad (2.67)$$

This is the Laughlin-wave function for integer QHE. When we add one electron in this system, electron is placed in higher energy levels, which cause a discontinuous change in chemical potential energy. This implies there is a sudden change in the chemical potential if all states are occupied with the electrons ,i.e, $\nu = 1, 2, 3, \dots$. Therefore, system is incompressible ($\kappa = 0$) if the filling factor, ν , takes integer values.

CHAPTER 3

QHE ON $\mathbb{CP}^1 \cong S^2$ AND \mathbb{CP}^2

3.1 Landau Problem and Haldane Treatment

Momentum operator \mathbf{p} , is the generator of the translations. For a given wave functions $\psi(\mathbf{x}, t)$ we have that [8]

$$\exp\left(-i\frac{\mathbf{p} \cdot \mathbf{a}}{\hbar}\right) \psi(\mathbf{x}, t) = D_a \psi(\mathbf{x}) = \psi(\mathbf{x} + \mathbf{a}, t). \quad (3.1)$$

If $\psi(\mathbf{x}, t)$ are energy eigenstates we further have that

$$\mathcal{H}\psi(\mathbf{x}, t) = E\psi(\mathbf{x}, t). \quad (3.2)$$

Commutator of the \mathcal{H} and D_a is

$$\begin{aligned} [\mathcal{H}, D_a]\psi(\mathbf{x}, t) &= (\mathcal{H}D_a - D_a\mathcal{H})\psi(\mathbf{x}, t) \\ &= \mathcal{H}\psi(\mathbf{x} + \mathbf{a}, t) - E\psi(\mathbf{x} + \mathbf{a}, t). \end{aligned} \quad (3.3)$$

We observe that if the translated wave functions $\psi(\mathbf{x} + \mathbf{a}, t)$ are eigenstates of the Hamiltonian with the same eigenvalue, then the commutator vanishes. As a consequence, the Hamiltonian is translationally invariant. However, the Hamiltonian of the Landau problem (2.8) does not commute with \mathbf{p} , so we conclude that the Hamiltonian is not translationally invariant. We note that although the planar QHE system treated in the previous chapter is translationally invariant, its Hamiltonian given in 2.7 involves the gauge potential \mathbf{A} , which for a uniform and perpendicular magnetic field has the general form $\mathbf{A} = (A_x(y)\hat{i}, A_y(x)\hat{j}, A_z(z)\hat{k})$. From this fact, it can easily be shown that change in the Hamiltonian under translation is equivalent to a gauge transformation [10].

F.D.M Haldane considered QH system on S^2 where electron wave functions are translationally invariant [3]. In this system, electrons are confined to move on S^2 under the influence of the perpendicular magnetic field. On S^2 , the normal vector is radially outward, which implies the perpendicular magnetic field should have only radial component ,i.e, $\mathbf{B} = B\hat{r}$. An obstacle of the Landau problem on S^2 is to obtain magnetic field which has purely radial component. Since the divergence of the magnetic field gives zero ($\nabla \cdot \mathbf{B} = 0$), there is no standard way to write such field. Nevertheless, such a field may be generated by a magnetic monopole proposed by Dirac in 1931. If a magnetic monopole exists in the universe, Dirac quantization rule explains why the electric charge is quantized according to quantization rule,

$$eg = \frac{n}{2} \quad (3.4)$$

in units $c = \hbar = 1$. In this expression, e, g are the electron and magnetic monopole charges and n is an integer.

A magnetic monopole with the charge g creates a magnetic field

$$\mathbf{B} = \frac{g}{R^2}\hat{r} = \frac{n}{2eR^2}\hat{r}. \quad (3.5)$$

The Hamiltonian of the system can be written as a rigid rotor

$$\mathcal{H} = \frac{|\mathbf{\Lambda}|^2}{2mr^2}, \quad (3.6)$$

where r is radius of S^2 and $\mathbf{\Lambda}$ is the dynamical angular momentum

$$\mathbf{\Lambda} = \mathbf{r} \times (-i\hbar\nabla + e\mathbf{A}(\mathbf{r})) = \mathbf{r} \times \mathbf{P}. \quad (3.7)$$

The commutator of the components of the dynamical angular momentum gives

$$[\Lambda^\alpha, \Lambda^\beta] = \epsilon^{\alpha\beta\gamma}i\hbar(\Lambda^\gamma - \hbar\frac{n}{2}r^\gamma), \quad (3.8)$$

which implies that the dynamical angular momentum is not the generator of rotations since it does not satisfy the angular momentum algebra. This leads us to define new operators that satisfy the angular momentum commutation relation.

We can choose the generators as

$$\mathbf{L} = \mathbf{\Lambda} + \hbar\frac{n}{2}\hat{r}, \quad (3.9)$$

and the commutator gives

$$[L^\alpha, X^\beta] = i\hbar\epsilon^{\alpha\beta\gamma}X^\gamma, \quad (3.10)$$

where \mathbf{X} can be chosen as \mathbf{L} , \mathbf{r} and \mathbf{A} . Since the commutation relations satisfy $SU(2)$ algebra, \mathbf{L} can be thought as the generator of the rotations on S^2 . Generators of the $SU(2)$ algebra fulfill $\mathbf{L}^2 = \hbar^2 l(l+1)$. (3.9) implies that \mathbf{L} has both radial and tangential components. Radial component of \mathbf{L} is $L_r = \mathbf{L} \cdot \hat{r} = \hbar \frac{n}{2}$. This suggests that we should have $\ell = q + \frac{n}{2}$ where q is a positive integer. Consequently, the spectrum becomes $\mathbf{L}^2 = \hbar^2(q + \frac{n}{2})(q + \frac{n}{2} + 1)$ which gives

$$\begin{aligned} E_q &= \frac{\Lambda^2}{2mR^2} = \frac{1}{2mR^2} \left(|\mathbf{L}|^2 - \frac{\hbar^2 n^2}{4} \right) \\ &= \frac{\hbar^2}{2mR^2} (q + \frac{n}{2})(q + \frac{n}{2} + 1) \\ &= \frac{e\hbar B}{2m} (2q + 1) + \frac{\hbar^2}{2mR^2} q(q + 1), \end{aligned} \quad (3.11)$$

in which q is the LL index. The degeneracy of a given LL may be expressed as $2\ell + 1 = n + 1 + 2q$. The LLL can be obtained by taking $q = 0$

$$E_{LLL} = \frac{e\hbar B}{2m}. \quad (3.12)$$

Since the ℓ value is composed of two parts, n and q , we can think of it as the total angular momentum. This enables us to establish an analogy between spin-part of the total angular momentum and the radial component of \mathbf{L} . We introduce the radial part with the spinor coordinates as

$$\chi \equiv \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \cos \frac{\theta}{2} \exp(\frac{i\psi}{2}) \\ \sin \frac{\theta}{2} \exp(-\frac{i\psi}{2}) \end{pmatrix}, \quad (3.13)$$

where $r_i = \chi^\dagger \frac{\sigma_i}{2} \chi$ and σ_i are the Pauli matrices.

The gauge potential can be selected as

$$\mathbf{A} = \frac{\hbar S}{eR} \cot \theta \hat{\psi}. \quad (3.14)$$

From Schwinger construction angular momentum operators can be written in terms of the annihilation and creation operators

$$L_i = \frac{\hbar}{2} \begin{pmatrix} a_1^\dagger & a_2^\dagger \end{pmatrix} \sigma_i \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \quad (3.15)$$

where a, b are two uncoupled harmonic oscillation annihilation operators. The algebra has already been explained in (2.19). This enables us to write the operators

in differential form as

$$a_1 \rightarrow \partial_u, \quad a_1^\dagger \rightarrow u \quad (3.16)$$

$$a_2 \rightarrow \partial_v, \quad a_2^\dagger \rightarrow v. \quad (3.17)$$

These give the angular momentum operators

$$L_x = \frac{\hbar}{2}(u\partial_v + v\partial_u), \quad L_y = -i\frac{\hbar}{2}(u\partial_v - v\partial_u), \quad L_z = \frac{\hbar}{2}(u\partial_u - v\partial_v). \quad (3.18)$$

The eigenvalue equation we need to solve is

$$(\hat{r} \cdot \mathbf{L})\Psi(u, v)^{(n)} = \hbar \frac{n}{2} \Psi(u, v)^{(n)}, \quad (3.19)$$

where $\Psi(u, v)^{(n)}$ are the LLL wave functions. Both the operator $(\hat{r} \cdot \mathbf{L})$ and the wave functions are written in terms of the spinor coordinates u and v which transform under the action of spin $\frac{1}{2}$ irreducible representation (IRR) of $SU(2)$.

We may parametrize the spin- $\frac{1}{2}$ IRR of $SU(2)$ as $g = \begin{pmatrix} \alpha^* & \beta^* \\ -\beta & \alpha \end{pmatrix}$, where $\alpha^2 + \beta^2 = 1$

$$\begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} = \begin{pmatrix} \alpha^* & \beta^* \\ -\beta & \alpha \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \quad (3.20)$$

In order to solve 3.19 diagonalize the operator $\hat{r} \cdot \mathbf{L}$ by applying Jacobian transformation

$$\begin{pmatrix} \partial_u \\ \partial_v \end{pmatrix} = \begin{pmatrix} \frac{\partial \gamma_1}{\partial u} & \frac{\partial \gamma_2}{\partial u} \\ \frac{\partial \gamma_1}{\partial v} & \frac{\partial \gamma_2}{\partial v} \end{pmatrix} \begin{pmatrix} \partial_{\gamma_1} \\ \partial_{\gamma_2} \end{pmatrix} = \begin{pmatrix} \alpha^* & -\beta \\ -\beta^* & \alpha \end{pmatrix} \begin{pmatrix} \partial_{\gamma_1} \\ \partial_{\gamma_2} \end{pmatrix}. \quad (3.21)$$

By using (3.18), $\hat{r} \cdot \mathbf{L}$ maybe expressed in the form

$$\hat{r} \cdot \mathbf{L} = \frac{1}{2}(\alpha^* v + \beta^* u) \partial_{\gamma_1} - \frac{1}{2}(\alpha v + \beta u) \partial_{\gamma_2} \quad (3.22)$$

$$= \frac{1}{2} \gamma_1 \partial_{\gamma_1} - \frac{1}{2} \gamma_2 \partial_{\gamma_2}. \quad (3.23)$$

This result implies that the problem gives two different eigenvalue ($+\frac{n}{2}$ and $-\frac{n}{2}$) and the eigenfunction as

$$\Psi_{\alpha\beta}^+(u, v) = \gamma_1^{(n)} = (\alpha^* u + \beta^* v)^n \quad (3.24)$$

$$\Psi_{\alpha\beta}^-(u, v) = \gamma_2^{(n)} = (\alpha u - \beta v)^n. \quad (3.25)$$

That means we have two wave functions which yield different eigenvalues. Desired wave function is $\Psi_{\alpha\beta}^+(u, v)$ which yields eigenvalue $\frac{n}{2}$.

3.2 QHE on S^2 , Treatment of D. Nair and V.P. Karabali

Landau problem on complex spaces has been analysed by D. Karabali, V. P. Nair [5]. They used group theoretical approach to solve the Landau problem on $S^2 \equiv \mathbb{CP}^1$ with the motivation of generalizing this result to all complex projective spaces \mathbb{CP}^N . Before reviewing their approach we will give a brief definitions on complex projective spaces.

Complex projective spaces \mathbb{CP}^N can be thought as the set of all complex lines through origin in \mathbb{C}^{N+1} . From this definition we can say that the set of points $z = (z_1, z_2, \dots, z_{N+1})$ differing by $\lambda = \mathbb{C}$ are identified in the form [11]

$$z = \lambda z \quad \lambda \neq 0, \quad \lambda \in \mathbb{C}. \quad (3.26)$$

We know that \mathbb{C}^{N+1} has $2N + 2$ real dimensions and each complex line in \mathbb{C}^{N+1} through 0 intersects the S^{2N+1} . With this information we express the equivalence relation as

$$z = \lambda z \quad |\lambda| = 1, \quad \lambda \in e^{i\theta}. \quad (3.27)$$

Action of λ on the complex numbers is a $U(1)$ group action.

\mathbb{CP}^1 can be parametrized by two complex coordinates u_1, u_2 with the relations

$$u_\alpha \sim e^{i\theta} u_\alpha, \quad |u_1|^2 + |u_2|^2 = 1, \quad \alpha = 1, 2. \quad (3.28)$$

Semiclassically, Landau problem on \mathbb{CP}^1 can be seen as an electron which moves on S^2 . At the center of the S^2 a magnetic monopole is located with a charge g . Electron is under the influence of this magnetic monopole whose vector potential is

$$A = -inu_\alpha^* du_\alpha, \quad (3.29)$$

where \hbar and c take as unity. Corresponding field strength tensor is

$$\begin{aligned} F_{ij} &= \partial_i A_j - \partial_j A_i \\ \epsilon_{ijk} F_{ij} &= \epsilon_{ijk} \partial_i A_j - \epsilon_{ijk} \partial_j A_i \\ &= 2B_k, \end{aligned} \quad (3.30)$$

where we have used $B_k = \epsilon_{ijk} \partial_j A_k$. We know that the magnetic field is radially outward as given in (3.5)

$$B_r = \frac{1}{2} \epsilon_{ij3} F_{ij} = \frac{1}{2} (F_{12} - F_{21}) = \frac{1}{2} (F_{12} + F_{12}) = F_{\theta\phi}, \quad (3.31)$$

where B_r is at radial direction. Since there is only one component of the magnetic field, we can drop the index in further calculations. The integral of the field strength tensor gives

$$\int_{\Omega} F = \int_{\Omega} dA = (4\pi\ell^2)B = 2\pi n, \quad (3.32)$$

where we have used Stoke's theorem and the integral is taken over $S^2(\equiv \mathbb{CP}^1)$.

The coset realization of \mathbb{CP}^1 is

$$\mathbb{CP}^1 \equiv S^2 = \frac{SU(2)}{U(1)}, \quad (3.33)$$

which implies that the functions on \mathbb{CP}^1 can be thought as the functions on $SU(2)$ which are invariant under the action of $U(1)$. According to Peter-Weyl theorem any function on $SU(2)$ can be written in terms of Wigner- \mathcal{D} functions which may be represented as $\mathcal{D}_{L_3 R_3}^{(j)}(g)$. Trivial right action of $U(1)$ gives spherical harmonics as $\mathcal{D}_{L_3 0}^{(j)}(g) = \sqrt{\frac{4\pi}{2l+1}} Y_{lm}^*(\theta, \phi)$ where $R = 0$.

Let L_i and R_i be the i^{th} component of the left and right invariant vector fields. They satisfy the $SU(2)$ algebra

$$[L_i, L_j] = -\epsilon_{ijk} L_k, \quad [R_i, R_j] = \epsilon_{ijk} R_k. \quad (3.34)$$

From $SU(2)$ algebra we may write

$$\sum_{i=1}^3 R_i^2 = \sum_{i=1}^3 L_i^2 = j(j+1). \quad (3.35)$$

We can parametrize the $SU(2)$ elements in the form

$$g = \begin{pmatrix} u_2^* & u_1 \\ -u_1^* & u_2 \end{pmatrix}, \quad (3.36)$$

and consider the group elements near the identity $e(=I)$ as $(I + \frac{1}{2}\epsilon\sigma_i)$ where ϵ is small and σ_i are Pauli matrices. The right action of the group elements on g will push forward the group elements of the g and induces the right translation vectors as a differential operator [12]. The right action of $(I + \frac{1}{2}\epsilon\sigma_+)$ on g may be expressed as

$$g(I + \frac{1}{2}\epsilon\sigma_+) = \begin{pmatrix} u_2^* & u_1 \\ -u_1^* & u_2 \end{pmatrix} \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \epsilon \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \right] \quad (3.37)$$

$$= \begin{pmatrix} u_2^* - \epsilon u_1 & u_1 \\ -u_1^* - \epsilon u_2 & u_2 \end{pmatrix}. \quad (3.38)$$

Change on the parameters can be shown in the form

$$\delta \begin{pmatrix} u_1 \\ u_2 \\ u_1^* \\ u_2^* \end{pmatrix} = \epsilon \begin{pmatrix} 0 \\ 0 \\ u_2 \\ -u_1 \end{pmatrix}, \quad (3.39)$$

where u_i on the RHS may be thought as the vector components of the $\frac{\partial}{\partial u_i}$ and therefore we can write

$$R_+ = -u_1 \frac{\partial}{\partial u_2^*} + u_2 \frac{\partial}{\partial u_1^*}. \quad (3.40)$$

By applying the same procedure we can also write

$$R_- = u_1^* \frac{\partial}{\partial u_2} - u_2^* \frac{\partial}{\partial u_1}, \quad R_3 = \frac{1}{2} \left(u_1 \frac{\partial}{\partial u_1} + u_2 \frac{\partial}{\partial u_2} - u_1^* \frac{\partial}{\partial u_1^*} - u_2^* \frac{\partial}{\partial u_2^*} \right). \quad (3.41)$$

The left actions of the matrix group elements induce the left rotations. We have left rotation vectors in the form

$$L^i = \frac{1}{2} \left[u_\beta \sigma_{\alpha\beta}^i \frac{\partial}{\partial u_\alpha} - u_\beta^* \sigma_{\alpha\beta}^i \frac{\partial}{\partial u_\alpha^*} \right]. \quad (3.42)$$

We have that $[R_i, L_j] = 0$. As easily noticed left rotations remain invariant under the action of $U(1)$ group, i.e., $u_\alpha \sim e^{i\theta} u_\alpha$, while right invariant vector fields transform with a phase factor ($R_\pm \rightarrow R_\pm e^{\mp 2i\theta}$). We know that the commutator of the covariant derivatives is equal to the field strength tensor which is equal to magnetic field $[D_+, D_-] = B$. We may write the covariant derivatives in the form

$$D_\pm = i \frac{R_\pm}{\ell}. \quad (3.43)$$

The Hamiltonian may be written in the form

$$\mathcal{H} = -\frac{1}{4M} (D_+ D_- + D_- D_+) \quad (3.44)$$

$$= \frac{1}{2m\ell^2} \left(\sum_{i=1}^3 R_i^2 - R_3^2 \right). \quad (3.45)$$

From (3.51) we know that the eigenvalue of the R_3 is $\frac{n}{2}$ which is also a possible value of j . In order to ensure this condition we need to express eigenvalues as $j = \frac{n}{2} + q$ where q is an integer. Therefore, energy eigenvalues read

$$E = \frac{1}{2M\ell^2} \left[\left(\frac{n}{2} + q \right) \left(\frac{n}{2} + q + 1 \right) - \frac{n^2}{4} \right] \quad (3.46)$$

$$= \frac{B}{2M} (2q + 1) + \frac{q(q + 1)}{2M\ell^2}. \quad (3.47)$$

Energy levels can be labelled by q and the LLL energy is obtained by setting $q = 0$ which is equal to $\frac{B}{2M}$.

The left rotations commute with the covariant derivatives, i.e., $[D_i, L_j] = 0$ and therefore with the Hamiltonian. This implies that left quantum number L_3 controls the degeneracy. That means for different L_3 values and fixed R_3 we have same eigenvalue. IRR label j restricts the L_3 as $-j \leq L_3 \leq j$. Therefore, we say L_3 takes $2j + 1$ different values. When we combine this condition with $j = \frac{n}{2} + q$, we obtain $2j + 1 = n + 2q + 1$ which is nothing but the degeneracy of the system. This result is in agreement with F.D.M Haldane.

In the absence of $U(1)$ gauge field background Wigner- \mathcal{D} functions are in the form $\mathcal{D}_{L_3 0}(g)$ which are nothing but the spherical harmonics. In the presence of the magnetic monopole, Wigner- \mathcal{D} functions transform as

$$\begin{aligned}\mathcal{D}(gh) &= \mathcal{D}(g)\mathcal{D}(h) \\ D_{mm'}(gh) &= \mathcal{D}_{mm''}^j(g)\mathcal{D}_{m''m'}^j(h),\end{aligned}\tag{3.48}$$

where $h = e^{i\theta R_3} \in U(1)$. This gives

$$\begin{aligned}\mathcal{D}_{mm'}(gh) &= D_{mm''}^j(g)D_{m''m'}^j(e^{i\theta R_3}) \\ &= D_{mm''}^j(g) \langle m'' | e^{i\theta R_3} | m' \rangle \\ &= D_{mm''}^j(g) e^{i\theta m'} \delta_{m''m'} \\ &= D_{mm'}^j(g) e^{i\theta m'} \\ &= D_{m\frac{n}{2}}^j(g) e^{i\frac{n}{2}\theta}.\end{aligned}\tag{3.49}$$

Gauge transformation of the wave functions may be expressed in the form

$$D(gh) = D(g) e^{i\frac{n}{2}\theta},\tag{3.51}$$

where they are the sections of $U(1)$ bundle over \mathbb{CP}^1 . We noted that LLL is obtained if $j = \frac{n}{2}$. By using (3.36), $R_3 = \frac{n}{2}$ and (3.51) in LLL condition, wave functions for IRR $(\frac{1}{2})$ are in the form

$$\mathcal{D}^{\frac{1}{2}}(g) \sim u_{\alpha_1}.\tag{3.52}$$

In LLL, IRR $(\frac{n}{2})$ is n -fold symmetric tensor product of $\frac{n}{2}$ representation. This can be shown as symmetric tensor product of \square tableau as

$$\square \otimes_S \square \otimes_S \cdots \otimes_S \square \longrightarrow \overbrace{\square \square \square \cdots \square}^n.$$

Local form of the LLL wave functions are

$$\mathcal{D}^{\frac{n}{2}}(g) \sim u_{\alpha_1} \cdots u_{\alpha_n}, \quad (3.53)$$

Degeneracy of LLL is $n+1$ fold. If the filling factor $\nu = 1$, then we have $\mathcal{N} = n+1$ fermions. Slater determinant for the $\mathcal{N} + 1$ particle state is in the form

$$\Psi_{\mathcal{N}}(1, 2, \dots, n+1) = \epsilon^{A_1 A_2 \cdots A_{\mathcal{N}}} \Psi_{A_1}(u^{(1)}) \Psi_{A_2}(u^{(2)}) \cdots \Psi_{A_{\mathcal{N}}}(u^{(\mathcal{N})}). \quad (3.54)$$

The two-point correlation function may be written as ¹

$$\Omega(1, 2) = \int d\mu(3, 4, \dots, \mathcal{N}) \Psi_{\mathcal{N}}^* \Psi_{\mathcal{N}} \sim |\Psi^{(1)}|^2 |\Psi^{(2)}|^2 - |\Psi_A^{(1)} \Psi_A^{*(2)}|^2, \quad (3.55)$$

where the $d\mu$ terms are the measure of the integration. To calculate $\Omega(1, 2)$ we work in a non-homogeneous coordinate system in which

$$u_{\alpha}^{(i)} = \frac{1}{\sqrt{1 + |z^{(i)}|^2}} \begin{pmatrix} 1 \\ z^{(i)} \end{pmatrix}, \quad (3.56)$$

where (i) is the label of i^{th} particle coordinate and $z \in \mathbb{C}$. It is also possible to write LLL wave function in the form

$$\Psi = (u_{\alpha})^n. \quad (3.57)$$

Inserting this equation and 3.56 into (3.55) we may obtain

$$\Omega(1, 2) \sim 1 - \left[\frac{(1 + z^{*(1)} z^{(2)})(1 + z^{*(2)} z^{(1)})}{(1 + |z^{(1)}|^2)(1 + |z^{(2)}|^2)} \right]^n. \quad (3.58)$$

We may write

$$z^{(i)} = \frac{x^{(i)} + iy^{(i)}}{\ell}, \quad (3.59)$$

with $x, y \in \mathbb{R}$. Inserting this into (3.56) yields

$$\begin{aligned} \Omega(1, 2) &\sim 1 - \left[1 - \frac{(x^{(1)} - x^{(2)})^2 + (y^{(1)} - y^{(2)})^2}{\ell^2 (1 + \ell^{-2} |\mathbf{x}^{(1)}|^2 + \ell^{-2} |\mathbf{x}^{(2)}|^2 + \ell^{-4} |\mathbf{x}^{(1)}|^2 |\mathbf{x}^{(2)}|^2)} \right]^n \\ &\sim 1 - \left[1 - \frac{1}{\ell^2} \left(|\mathbf{x}^{(1)} - \mathbf{x}^{(2)}|^2 + \mathcal{O}\left(\frac{1}{\ell^2}\right) \right) \right]^n. \end{aligned} \quad (3.60)$$

In the thermodynamic limit $\ell \rightarrow \infty$ and $n \rightarrow \infty$ we may obtain

$$\lim_{\ell \rightarrow \infty} \left\{ 1 - \left[1 - \frac{|\mathbf{x}^{(1)} - \mathbf{x}^{(2)}|^2}{\ell^2} \right]^n \right\} \longrightarrow 1 - \exp[-2B |\mathbf{x}^{(1)} - \mathbf{x}^{(2)}|^2], \quad (3.61)$$

¹ The details of the calculation and explanation of the terms are given at the section 3 of the next chapter. For the detailed calculation one can refer the part between the equations (4.122) and (4.131).

where we have used $n = 2B\ell^2$. This means if the distance between two particle coordinates is close enough, then the two-point function goes to zero and the probability of locating two particles at the same point approaches to zero, as expected. This shows the incompressibility property of the system on \mathbb{CP}^2 .

As a final discussion, the thermodynamic limit can be obtained by $\mathcal{N} \rightarrow \infty$ and $\ell \rightarrow \infty$. In this limit the number density reads

$$\rho = \frac{\mathcal{N}}{V} = \frac{\mathcal{N}}{4\pi\ell^2} \xrightarrow{\ell \rightarrow \infty, \mathcal{N} \rightarrow \infty} \frac{n}{4\pi\ell^2} = \frac{B}{2\pi}, \quad (3.62)$$

which is finite.

3.3 Landau Problem on \mathbb{CP}^2

In this section, we follow the original work of D. Nair and V.P. Karabali [5].

\mathbb{CP}^2 can be parametrized by three complex coordinates $(u_1, u_2, u_3) \in \mathbb{C}^3$ with the relations

$$u_\alpha \sim e^{i\theta} u_\alpha, \quad |u_1|^2 + |u_2|^2 + |u_3|^2 = 1. \quad (3.63)$$

The coset realization of the \mathbb{CP}^2 is

$$\mathbb{CP}^2 \equiv S^3 = \frac{SU(3)}{U(2)} = \frac{SU(3)}{SU(2) \times U(1)}, \quad (3.64)$$

which implies that the functions on \mathbb{CP}^2 can be thought as the functions on $SU(3)$ which are invariant under the action of $U(1)$ and $SU(2)$. Functions on $SU(3)$ can be written in terms of the Wigner- \mathcal{D} functions in the form $\mathcal{D}_{LL_3L_8;RR_3R_8}^{(p,q)}$ where R_8 is the generator of the $U(1)$ subgroup and its eigenvalue is the $U(1)$ charge. R_i and L_i are right and left generators of $SU(2)$ subgroup. Consequently, they satisfy the $SU(2)$ algebra such as

$$[R_i, R_j] = \epsilon_{ijk} R_k, \quad [L_i, L_j] = \epsilon_{ijk} L_k, \quad (3.65)$$

and we have

$$R_i^2 = L_i^2 = j(j+1) \quad i = 1, 2, 3. \quad (3.66)$$

R_4, R_5, R_6 and R_7 span right invariant tangent vector fields on \mathbb{CP}^2 . As such, they form a basis for derivatives on \mathbb{CP}^2 , which allows us to write Hamiltonian of a

charged particle on \mathbb{CP}^2 as

$$\begin{aligned} H &= \frac{1}{2M\ell^2} \sum_{i=4}^7 R_i^2 \\ &= \frac{1}{2M\ell^2} (\mathcal{C}_2(p, q) - R(R+1) - R_8^2) , \end{aligned} \quad (3.67)$$

where $\mathcal{C}_2(p, q)$ is the quadratic Casimir of $SU(3)$ in the IRR (p, q) with the eigenvalue

$$\mathcal{C}_2(p, q) = \frac{1}{2} [p(p+3) + q(q+3) + pq] . \quad (3.68)$$

We expect that certain restricted subsets of Wigner- \mathcal{D} functions are the wave functions corresponding to this Hamiltonian.

For a pure $U(1)$ gauge field background, wave functions must carry the trivial (spin-0) representation of the $SU(2)$ subgroup (i.e. $R = 0$, $R_i = 0$). wave functions then have the form

$$\mathcal{D}_{LL_3 L_8; 00 R_8}^{(p, q)}(g) .$$

Inspecting the branching of the IRR (p, q) of $SU(3)$ under $SU(2) \times U(1)$ we see that ²

$$\overbrace{\begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array}}^q \cdots \overbrace{\begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array}}^p \cdots \longrightarrow \overbrace{\begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array}}^q \cdots \otimes \overbrace{\begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array}}^p \cdots ,$$

where the Young tableaux on the branching represent $SU(3)$, $SU(2)$ and $U(1)$ IRRs, respectively. When we combine this condition with the standard formula for obtaining $U(1)$ charges of a branching given in terms of a Young tableaux (4.10), we obtain

$$n := q - p, \quad n \in \mathbb{Z} , \quad (3.69)$$

where n is the unnormalized $U(1)$ charge. We need to relate n with R_8 . In order to write the matrix realization of R_8 we choose the fundamental representation $(1, 0)$ of $SU(3)$ with the branching

$$\square \longrightarrow \left(\cdot \otimes \square \right)_{-1} \oplus \left(\square \otimes \cdot \right)_{\frac{1}{2}} , \quad (3.70)$$

² Young tableaux representation and branching rules are explained in Appendix A.

where subscripts stand for the $U(1)$ charges. The dimension of IRR $(1, 0)$ is 3. Matrix realization of R_8 is $R_8 = \frac{1}{2\sqrt{3}} \text{diag}(1, 1, -2)$ and taking the trace normalization as $\text{Tr}(R_\alpha R_\beta) = \frac{1}{2} \delta_{\alpha\beta}$. Corresponding group element is

$$h \equiv \exp(iR_8\theta) = \text{diag}(\exp(\frac{i}{2\sqrt{3}}\theta), \exp(\frac{i}{2\sqrt{3}}\theta), \exp(\frac{-i}{\sqrt{3}}\theta)). \quad (3.71)$$

A wave function in this representation has the property

$$\begin{aligned} D_{LL_3L_8;00R_8}^{(1,0)}(gh) &= \langle LL_3L_8|(gh)|00R_8\rangle \\ &= \langle LL_3L_8|(g)|RR_3R_8\rangle \langle RR_3R_8|(h)|00R_8\rangle \\ &= \langle LL_3L_8|(g)|RR_3R_8\rangle \exp\left(\frac{-i\theta}{\sqrt{3}}(R_8)\right) \delta_{R0}\delta_{R30} \\ &= D_{LL_3L_8;00R_8}^{(1,0)}(g) e^{\left(\frac{-i\theta}{\sqrt{3}}\right)}, \end{aligned} \quad (3.72)$$

which means Wigner- \mathcal{D} functions transform with a phase factor $e^{\frac{-i\theta}{\sqrt{3}}}$ under the action $U(1)$. Now, we can write the relation between the n and R_8

$$R_8 = \frac{n}{\sqrt{3}} = \frac{q-p}{\sqrt{3}}. \quad (3.73)$$

By using (3.67), (3.68) and (3.73), spectrum of the Hamiltonian gives

$$E_{q,n} = \frac{1}{2M\ell^2} (q(q+n+2) + n). \quad (3.74)$$

By setting $q = 0$, the LLL energy eigenvalue may be found as $E_{LLL} = \frac{n}{2M\ell^2}$. Degeneracy is equal to the dimension of the (p, q) representation³ Therefore, the degeneracy may be expressed as

$$\dim(p, q) = \frac{(p+1)(q+1)(p+q+2)}{2}. \quad (3.75)$$

In the LLL the degeneracy is $\dim(n, 0) = \frac{1}{2}(n+1)(n+2)$, which is equal to number of fermions \mathcal{N} at $\nu = 1$. In the thermodynamic limit spatial density is obtained as

$$\rho = \frac{\mathcal{N}}{V} = \frac{\mathcal{N}}{8\pi^2\ell^4} \xrightarrow{\ell \rightarrow \infty, \mathcal{N} \rightarrow \infty} \frac{n^2}{8\pi^2\ell^4} = \left(\frac{B}{2\pi}\right)^2, \quad (3.76)$$

which is finite.

Let us now turn our attention on obtaining the gauge potential

$$\begin{aligned} A &= i \frac{2n}{\sqrt{3}} \text{Tr}(R_8 g^{-1} dg) \\ &= i \frac{2n}{\sqrt{3}} ((R_8)_{\alpha\beta} g_{\beta\gamma}^{-1} dg_{\gamma l}). \end{aligned} \quad (3.77)$$

³ The dimension may be found by Hooks law explained in Appendix A.

By using the matrix realization of R_8 and the identity $Tr(g^{-1}dg) = 0$ we may write

$$\begin{aligned}
A &= i \frac{2n}{\sqrt{3}} ((R_8)_{11} g_{1\gamma}^{-1} dg_{\gamma 1} + (R_8)_{22} g_{2\gamma}^{-1} dg_{\gamma 2} + (R_8)_{33} g_{3\gamma}^{-1} dg_{\gamma 3}) \\
&= \frac{in}{3} (g_{1\gamma}^{-1} dg_{\gamma 1} + g_{2\gamma}^{-1} dg_{\gamma 2} - 2g_{3\gamma}^{-1} dg_{\gamma 3}) \\
&= -in g_{3\gamma}^{-1} dg_{\gamma 3} = -in g_{\gamma 3}^* dg_{\gamma 3} \\
&= -inu_{\alpha}^* du_{\alpha},
\end{aligned} \tag{3.78}$$

where α, β, γ are the matrix indices. In the last line we have used a parametrization of $g \in SU(3)$ in the defining representation with its last column taken as $g_{n3} = u_{\alpha}$. This is related to Berry connection studied in literature [26].

The wave functions under $U(1)$ action transform with a phase as shown in (3.72). Let us show it explicitly for $(1, 0)$ representation. By using the matrix realization of R_8 , we may write

$$g e^{iR_8\theta} = \left(\begin{array}{c|c} \vdots & \vdots \end{array} \middle| \begin{array}{c} u_1 \\ u_2 \\ u_3 \end{array} \right) \begin{pmatrix} e^{\frac{i}{2\sqrt{3}}\theta} & 0 & 0 \\ 0 & e^{\frac{i}{2\sqrt{3}}\theta} & 0 \\ 0 & 0 & e^{-\frac{i}{\sqrt{3}}\theta} \end{pmatrix} \tag{3.79}$$

$$= \left(\begin{array}{c|c} \vdots & \vdots \end{array} \middle| \begin{array}{c} e^{-\frac{i}{\sqrt{3}}\theta} u_1 \\ e^{-\frac{i}{\sqrt{3}}\theta} u_2 \\ e^{-\frac{i}{\sqrt{3}}\theta} u_3 \end{array} \right) \tag{3.80}$$

Local form of the wave functions in LLL for $(1, 0)$ IRR are

$$\mathcal{D}^{(1,0)} \sim g_{n3} = u_{\alpha}. \tag{3.81}$$

In LLL $(p, 0)$ IRR can be thought as the p -fold symmetric tensor product of $(1, 0)$ representation $(p, 0)$.

$$\boxed{} \otimes_S \boxed{} \otimes_S \cdots \otimes_S \boxed{} \longrightarrow \overbrace{\boxed{} \boxed{} \boxed{} \cdots \boxed{}}^p,$$

where \otimes_S stands for the symmetric tensor product. Branching implies LLL wave functions are in the form

$$\begin{aligned}
\mathcal{D}_{LL_3 L_8; 00 \frac{-n}{\sqrt{3}}}^{(p,0)} &\sim g_{i_1 3} g_{i_2 3} \cdots g_{i_p 3} \\
&\sim u_{\alpha_1} u_{\alpha_2} \cdots u_{\alpha_p},
\end{aligned} \tag{3.82}$$

which is holomorphic in the variables u_α as expected.

Under $g \rightarrow ge^{iR_8\theta}$ where $g \in SU(3)$, the gauge potential transforms as

$$\begin{aligned}
A(ge^{iR_8\theta}) &= i\frac{2n}{\sqrt{3}}\text{Tr}(R_8(ge^{iR_8\theta})^{-1}d(ge^{iR_8\theta})) \\
&= i\frac{2n}{\sqrt{3}}\text{Tr}[(R_8e^{-iR_8\theta}g^{-1})(e^{iR_8\theta}dg + gd(e^{iR_8\theta}))] \\
&= A + i\frac{2n}{\sqrt{3}}\text{Tr}(i(R_8)^2d\theta) \\
&= A - d\left(\frac{n\theta}{\sqrt{3}}\right)
\end{aligned} \tag{3.83}$$

.

We may write u_α in non-homogeneous coordinate system as

$$u = \frac{1}{\sqrt{1 + |z_i|^2}} \begin{pmatrix} 1 \\ z_1 \\ z_2 \end{pmatrix}, \tag{3.84}$$

where $z_1, z_2 \in \mathcal{C}$. The corresponding field strength is

$$F = dA = -indu_\alpha^* du_\alpha \tag{3.85}$$

$$= -in \left[\frac{d\bar{z}_i dz_i}{1 + \bar{z} \cdot z} - \frac{d\bar{z} \cdot z \bar{z} \cdot dz}{(1 + z \cdot \bar{z})^2} \right] \tag{3.86}$$

Let us continue to analyse the Landau problem with both $U(1)$ and $SU(2)$ background gauge fields. Relying on the Young tableaux for the branching of $SU(3)$ representation, we now have an interval of possible $SU(2)$ IRs occurring in the branching

$$\begin{aligned}
&\overbrace{\begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array}}^{q+k'} \cdots \overbrace{\begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array}}^{p+k} \cdots \longrightarrow \overbrace{\begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array}}^q \cdots \overbrace{\begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array}}^{k+k'} \cdots \otimes \overbrace{\begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array}}^p \cdots \overbrace{\begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array}}^{k'} \cdots \\
&\longrightarrow \overbrace{\begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array}}^{q+k'-x} \cdots \overbrace{\begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array}}^{k-k'+2x} \cdots \otimes \overbrace{\begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array}}^p \cdots \overbrace{\begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array}}^{k'} \cdots \\
&\longrightarrow \overbrace{\begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array}}^{q+k'} \cdots \overbrace{\begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array}}^{k'-k} \cdots \otimes \overbrace{\begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array}}^p \cdots \overbrace{\begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array}}^{k'} \cdots
\end{aligned}$$

This branching shows that spin R of $SU(2)$ takes values in the interval

$$R = \frac{|k - k'|}{2} \dots \frac{k + k'}{2}, \quad (3.87)$$

where we have assumed $0 < x < k$.

Using (4.10), R_8 may be written in the form

$$R_8 = \frac{1}{2\sqrt{3}}[-2(p - q) + (k - k')] = -\frac{n}{\sqrt{3}}. \quad (3.88)$$

Now, we are ready to write energy eigenvalues (3.67)

$$\begin{aligned} E &= \frac{1}{2M\ell^2}(\mathcal{C}_2(p + k, q + k') - R(R + 1) - R_8^2) \\ &= \frac{1}{2M\ell^2} \left(q^2 + q(2k - m + n + 2) + n(k + 1) \right. \\ &\quad \left. + k^2 + 2k + m^2 - m(k + 1) - R(R + 1) \right), \end{aligned} \quad (3.89)$$

where we have introduced $\frac{k - k'}{2} = m \in \mathbb{Z}$ in order to ensure that n takes only integer values. In LLL, R reads

$$R = \frac{k + k'}{2} = k - m. \quad (3.90)$$

By inserting this condition into (3.89) we obtain the energy spectrum which have maximum R value in the form

$$E = \frac{1}{2M\ell^2} (q^2 + q(2R + n + m + 2) + n(R + m + 1) + (R + m)(m + 1)). \quad (3.91)$$

If we take $q = 0$ and $m = 0$ at fixed background charges n and R , this expression gives the LLL energy eigenvalues

$$E_{LLL} = \frac{1}{2M\ell^2}(nR + R + n). \quad (3.92)$$

The degeneracy of LL is given as $\dim(R + n, R) = \frac{1}{2}(n + R + 1)(R + 1)(n + 2R + 1)$. In order to obtain finite density either n or R should scale like ℓ^2 . In particular, we may choose $n \sim \ell^2$. Then, the spatial density gives

$$\rho \sim \frac{\dim(R + n, R)}{(2R + 1)\ell^2} \xrightarrow{\ell \rightarrow \infty, \mathcal{N} \rightarrow \infty} \frac{n^2}{4\ell^4} \rightarrow \text{finite}, \quad (3.93)$$

in which $SU(2)$ degrees of freedom is finite. In case of pure $SU(2)$ background field, we need to scale $R \sim \ell^2$ to have finite spatial density. The degeneracy is $\dim(R, R) = \frac{1}{2}(R + 1)(R + 1)(2R + 2)$. Then, the spatial density is

$$\rho \sim \frac{\dim(R, R)}{(2R + 1)\ell^2} \xrightarrow{\ell \rightarrow \infty, \mathcal{N} \rightarrow \infty} \frac{R^3}{2R\ell^4} \rightarrow \text{finite} \quad (3.94)$$

The calculation of the correlation function is very similar to the \mathbb{CP}^1 case. We omit the details of the calculation here and give the result as

$$\Omega(1, 2) \sim 1 - \left[1 - \frac{|\mathbf{x}^{(1)} - \mathbf{x}^{(2)}|^2}{\ell^2} \right]^n. \quad (3.95)$$

In the planar limit $\ell \rightarrow \infty$ we obtain

$$\lim_{\ell \rightarrow \infty} \left\{ 1 - \left[1 - \frac{|\mathbf{x}^{(1)} - \mathbf{x}^{(2)}|^2}{\ell^2} \right]^n \right\} \longrightarrow 1 - \exp \left[-2B |\mathbf{x}^{(1)} - \mathbf{x}^{(2)}|^2 \right], \quad (3.96)$$

where the probability of the finding two particle at same point is zero, as expected.

This result shows the incompressibility property of the Hall liquid on \mathbb{CP}^2 .

CHAPTER 4

QUANTUM HALL EFFECT ON GRASSMANN MANIFOLDS

This chapter is the review of [9].

Grassmannians $Gr_k(\mathbb{C}^N)$ are generalizations of the complex projective spaces. Grassmann manifolds $Gr_k(\mathbb{C}^N)$ are the set of all k -dimensional complex subspaces through origin in \mathbb{C}^N . They have the coset realization

$$\mathbf{Gr}_k(\mathbb{C}^N) = \frac{SU(N)}{S[U(N-k) \times U(k)]} \sim \frac{SU(N)}{SU(N-k) \times SU(k) \times U(1)}. \quad (4.1)$$

In this chapter, we will focus on the Landau problem and subsequently QHE on Grassmannians with $k = 2$, i.e. $Gr_2(\mathbb{C}^4)$. We will first analyse the problem on $Gr_2(\mathbb{C}^4)$, which is the simplest of all $Gr_2(\mathbb{C}^N)$, and generalize it to any $N \geq 4$. In section 3 we will give the local forms of the wave functions in terms of the Plücker coordinates.

4.1 Landau Problem on $Gr_2(\mathbb{C}^4)$

In this section we will study the Landau problem on $Gr_2(\mathbb{C}^4)$ by employing and generalizing the group theoretical techniques which were used to analyse the corresponding problems on \mathbb{CP}^1 and \mathbb{CP}^2 in the previous chapter.

$Gr_2(\mathbb{C}^4)$ has the coset realization

$$Gr_2(\mathbb{C}^4) = \frac{SU(4)}{S[U(2) \times U(2)]} \sim \frac{SU(4)}{SU(2) \times SU(2) \times U(1)}. \quad (4.2)$$

This suggests that the functions on $Gr_2(\mathbb{C}^4)$ may be written in terms of a subset of $SU(4)$ Wigner- \mathcal{D} functions which transform trivially under the action of $SU(2) \times$

$SU(2)$ and $U(1)$. General form of the $SU(4)$ Wigner- \mathcal{D} functions may be written as

$$\mathcal{D}(g)^{(p,q,r)}, \quad (4.3)$$

where $g \in SU(4)$ and p, q, r are positive integers labelling the IRRs $SU(4)$. As we did for $SU(3)$, we may label the right and left invariant vector fields on $SU(4)$ with R_α and L_α ($\alpha = 1, \dots, 15$). We may also label the left and right generators of the $SU(2) \times SU(2)$ subgroup with $L_\alpha \equiv (L_i^{(1)}, L_i^{(2)})$ and $R_\alpha \equiv (R_i^{(1)}, R_i^{(2)})$, respectively ($i = 1, 2, 3, \alpha = 1, \dots, 6$). The latter satisfy the commutation relations

$$\left[L_i^{(1)}, L_j^{(1)} \right] = \epsilon_{ijk} L_k^{(1)}, \quad \left[L_i^{(2)}, L_j^{(2)} \right] = \epsilon_{ijk} L_k^{(2)} \quad (4.4)$$

$$\left[R_i^{(1)}, R_j^{(1)} \right] = \epsilon_{ijk} R_k^{(1)}, \quad \left[R_i^{(2)}, R_j^{(2)} \right] = \epsilon_{ijk} R_k^{(2)}. \quad (4.5)$$

and all the other commutators vanish. We write the Wigner- \mathcal{D} functions

$$\mathcal{D}(g)^{(p,q,r)}_{L^{(1)}, L_3^{(1)}, L^{(2)}, L_3^{(2)}, L_{15}; R^{(1)}, R_3^{(1)}, R^{(2)}, R_3^{(2)}, R_{15}}. \quad (4.6)$$

The generators R_α ($\alpha = 7, \dots, 14$) are tangent vector fields on $Gr_2(\mathbb{C}^4)$ and we may write the Hamiltonian in terms of these tangents as

$$\begin{aligned} \mathcal{H} &= \frac{1}{2M\ell^2} \sum_{i=7}^{14} R_i^2 \\ &= \frac{1}{2M\ell^2} \sum_{i=1}^{15} (R_i^2 - R^{(1)}(R^{(1)} + 1) - R^{(2)}(R^{(2)} + 1) - R_{15}^2) \\ &= \frac{1}{2M\ell^2} (C_2(p, q, r) - R^{(1)}(R^{(1)} + 1) - R^{(2)}(R^{(2)} + 1) - R_{15}^2), \end{aligned} \quad (4.7)$$

where $C_2(p, q, r)$ is the quadratic Casimir of $SU(4)$ in the (p, q, r) representation. Now, we will find the spectrum and eigenfunctions of this Hamiltonian in the following three cases:

- i) Pure $U(1)$ background
- ii) Pure $SU(2) \times SU(2)$ background
- iii) Both $U(1)$ and $SU(2) \times SU(2)$ background

Before going into this analysis, let us list a few facts regarding the branching of $SU(N_1 + N_2)$ under $SU(N_1) \times SU(N_2) \times U(1)$ that we will make use of in our discussions:

We may embed $SU(N_1) \times SU(N_2) \times U(1)$ into $SU(N_1 + N_2)$ in the form [15]

$$\begin{pmatrix} e^{iN_2\phi}U_1 & 0 \\ 0 & e^{-iN_1\phi}U_2 \end{pmatrix}, \quad (4.8)$$

where $U_1 \in SU(N_1)$, $U_2 \in SU(N_2)$. The branching of $SU(N_1 + N_2)$ can be written in the compact form as

$$\mathcal{J} = \bigoplus_{\mathcal{J}_1, \mathcal{J}_2} m_{\mathcal{J}_1, \mathcal{J}_2}^{\mathcal{J}} (\mathcal{J}_1, \mathcal{J}_2)_n, \quad (4.9)$$

in which \mathcal{J}_1 , \mathcal{J}_2 and \mathcal{J} represent the IRRs of $SU(N_1)$, $SU(N_2)$, $SU(N_1 + N_2)$, respectively. $m_{\mathcal{J}_1, \mathcal{J}_2}^{\mathcal{J}}$ are the multiplicities of the IRR $(\mathcal{J}_1, \mathcal{J}_2)_n$. We label the $U(1)$ charge with n and it can be found by using

$$n = \frac{1}{N_1 N_2} (N_2 J_1 - N_1 J_2) \quad (4.10)$$

in which J_1 and J_2 represent the number of boxes on IRR of the $SU(N_1)$ and $SU(N_2)$, respectively.

4.1.1 Pure $U(1)$ Gauge Field

In the pure $U(1)$ gauge field background $\mathcal{D}^{(p,q,r)}(g)$ transform as a singlet under the action of $SU(2) \times SU(2)$. To derive the energy spectrum in this background we need trivial right action of $SU(2) \times SU(2)$. For an $SU(2)$ IRR to be trivial, i.e. spin-0 representation, we must have the Young tableau of $SU(2)$

$$\begin{array}{c} \overbrace{\begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array}}^{\alpha} \dots \\ \underbrace{\hspace{1.5cm}}_{\alpha} \end{array}$$

In other words, number of boxes at both rows should be equal to each other. Trivial $SU(2) \times SU(2)$ action on the Wigner- \mathcal{D} functions $\mathcal{D}^{(p,q,r)}(g)$ is possible under the branching

$$\begin{array}{c} \overbrace{\begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array}}^r \dots \overbrace{\begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array}}^q \dots \overbrace{\begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array}}^p \dots \end{array} \rightarrow \begin{array}{c} \overbrace{\begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array}}^{r+q_1} \dots \otimes \overbrace{\begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array}}^{q_2} \dots \overbrace{\begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array}}^r \dots \\ \underbrace{\hspace{1.5cm}}_p \end{array}, \quad (4.11)$$

where RHS is $SU(2) \times SU(2)$ IRR's. Therefore, branching which keeps the $SU(2) \times SU(2)$ in the singlet representation demands that $p = r$. From (4.10) $U(1)$ charge is

$$n = \frac{1}{4}[2(2p + 2q_1) - 2(2q_2 + 2p)] = q_1 - q_2. \quad (4.12)$$

Now, we need to fix relation between R_{15} and n . For this purpose we may choose the $(1, 0, 0), (0, 1, 0)$ and $(0, 0, 1)$ fundamental representation of $SU(4)$. It can be easily seen that we can not obtain trivial representation of $SU(2) \times SU(2)$ in the branching $(1, 0, 0)$ and $(0, 0, 1)$. Branching of the 6-dimensional fundamental IRR $(0, 1, 0)$ of $SU(4)$ under $SU(4) \rightarrow SU(2) \times SU(2) \times U(1)$ reads

$$\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array} \rightarrow \left(\cdot \otimes \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array} \right)_{-1} \oplus \left(\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array} \otimes \cdot \right)_1 \oplus \left(\square \otimes \square \right)_0, \quad (4.13)$$

where subscripts stand for $U(1)$ charges. In this representation, R_{15} is the diagonal matrix in the form $R_{15} = \frac{1}{\sqrt{2}} \text{diag}(0, 0, 0, 0, -1, 1)$. The normalization constant, $\frac{1}{\sqrt{2}}$, can be found by using the formula (C.8) in appendix C. Matrix realization of the corresponding group element has the form

$$\exp(iR_{15}\theta) = \text{diag} \left(0, 0, 0, 0, \exp\left(\frac{-i}{2\sqrt{3}}\theta\right), \exp\left(\frac{i}{2\sqrt{3}}\theta\right) \right). \quad (4.14)$$

Now, we may write the relation between the $U(1)$ charge and R_{15}

$$R_{15} = \frac{n}{\sqrt{2}} = \frac{q_1 - q_2}{\sqrt{2}}. \quad (4.15)$$

In the pure $U(1)$ background, wave functions corresponding to the Hamiltonian (4.7) take the form

$$\mathcal{D}_{L^{(1)}L_3^{(1)}L^{(2)}L_3^{(2)}L_{15}; 0, 0, 0, 0, \frac{n}{\sqrt{2}}}(g) \quad (4.16)$$

and the spectrum of the Hamiltonian is

$$\begin{aligned} \mathcal{H} &= \frac{1}{2M\ell^2} \left(\frac{3}{8}(r^2 + p^2) + \frac{1}{2}q^2 + \frac{1}{8}(2pr + 4pq + 4qr + 12p + 16q + 12r) - R_{15}^2 \right) \\ &= \frac{1}{2M\ell^2} (p^2 + 3p + np + 2q_2^2 + 4q_2 + 2pq_2 + 2n(1 + q_2)), \end{aligned} \quad (4.17)$$

where we have used equation (B.4) for the quadratic Casimir of $SU(4)$ in an IRR labelled by (p, q, r) . The LLL energy can be found by taking $p = q_2 = 0$

$$E_{LLL} = \frac{n}{M\ell^2} = \frac{2B}{M}. \quad (4.18)$$

Spatial density may be written in the form

$$\rho = \frac{\mathcal{N}}{\text{vol}(Gr_2(\mathbb{C}^4))}, \quad (4.19)$$

where $\text{vol}(Gr_2(\mathbb{C}^4)) = \frac{\pi^4 \ell^8}{12}$. If $\nu = 1$, then each state is occupied with one fermion. In the LLL with filling factor $\nu = 1$, number of particles, \mathcal{N} , is equal to degeneracy of the system which is equal to $\dim(0, n, 0) = \frac{1}{12}(n+1)(n+2)^2(n+3)$. With these facts spatial density takes the form

$$\rho = \frac{\dim(0, n, 0)}{\frac{\pi^4 \ell^8}{12}} \quad (4.21)$$

In the thermodynamic limit, $\ell \rightarrow \infty$ and $\mathcal{N} \rightarrow \infty$, it is finite and given as

$$\rho \xrightarrow{\ell \rightarrow \infty, \mathcal{N} \rightarrow \infty} \frac{n^4}{\pi^4 \ell^8} = \left(\frac{2B}{\pi} \right)^4 \quad (4.22)$$

4.1.2 U(1) and Single SU(2) Gauge Field

In case of $U(1)$ and single $SU(2)$ gauge fields, $\mathcal{D}^{(p,q,r)}$ transform as singlet under the action of one of the $SU(2)$. This is possible under the branching

$$\begin{aligned} & \overbrace{\begin{array}{|c|c|} \hline & \\ \hline & \\ \hline & \\ \hline & \\ \hline \end{array}}^r \cdots \overbrace{\begin{array}{|c|c|} \hline & \\ \hline & \\ \hline & \\ \hline & \\ \hline \end{array}}^q \cdots \overbrace{\begin{array}{|c|c|} \hline & \\ \hline & \\ \hline & \\ \hline & \\ \hline \end{array}}^p \cdots \rightarrow \overbrace{\begin{array}{|c|c|} \hline & \\ \hline & \\ \hline & \\ \hline & \\ \hline \end{array}}^{r+q_1} \cdots \otimes \overbrace{\begin{array}{|c|c|} \hline & \\ \hline & \\ \hline & \\ \hline & \\ \hline \end{array}}^{q_2} \cdots \overbrace{\begin{array}{|c|c|} \hline & \\ \hline & \\ \hline & \\ \hline & \\ \hline \end{array}}^{p+r} \cdots \\ & \rightarrow \overbrace{\begin{array}{|c|c|} \hline & \\ \hline & \\ \hline & \\ \hline & \\ \hline \end{array}}^{r+q_1} \cdots \otimes \overbrace{\begin{array}{|c|c|} \hline & \\ \hline & \\ \hline & \\ \hline & \\ \hline \end{array}}^{q_2} \cdots \overbrace{\begin{array}{|c|c|} \hline & \\ \hline & \\ \hline & \\ \hline & \\ \hline \end{array}}^x \cdots \overbrace{\begin{array}{|c|c|} \hline & \\ \hline & \\ \hline & \\ \hline & \\ \hline \end{array}}^{p+r-2x} \cdots \\ & \rightarrow \overbrace{\begin{array}{|c|c|} \hline & \\ \hline & \\ \hline & \\ \hline & \\ \hline \end{array}}^{r+q_1} \cdots \otimes \overbrace{\begin{array}{|c|c|} \hline & \\ \hline & \\ \hline & \\ \hline & \\ \hline \end{array}}^{q_2} \cdots \overbrace{\begin{array}{|c|c|} \hline & \\ \hline & \\ \hline & \\ \hline & \\ \hline \end{array}}^{\underbrace{\quad\quad\quad}_r} \cdots \overbrace{\begin{array}{|c|c|} \hline & \\ \hline & \\ \hline & \\ \hline & \\ \hline \end{array}}^{p-r} \cdots \end{aligned} \quad (4.23)$$

in which we assume $p > r$. On the branching, x stands for expressing the interval of non-trivial $SU(2)$ IRRs where $0 \leq x \leq r$. From this branching, $U(1)$ charge is

¹ As a general formula, the volume of the Grassmannian reads [19]

$$\text{vol}(Gr_k(\mathbb{C}^N)) = \frac{0!1! \cdots (k-1)!}{(N-k)! \cdots (N-2)!(N-1)!} \pi^{k(N-k)}. \quad (4.20)$$

computed as (4.10)

$$n = \frac{1}{4} [2(2r + 2q_1) - (2q_2 + 2x + p + r - 2x)] = \frac{1}{2}(r - p + 2q_1 - 2q_2). \quad (4.24)$$

and using (C.8) yields

$$R_{15} = \frac{1}{\sqrt{2}}n. \quad (4.25)$$

As stated before n should take only integer values, which imposes that $p-r$ should be even numbers. This constraint may be satisfied if we define a parameter m such that

$$m = \frac{p-r}{2}, \quad \left\{ \begin{array}{ll} m = 0, \dots, \frac{p}{2}, & \text{if } p \text{ is even} \\ m = 0, \dots, \frac{p-1}{2}, & \text{if } p \text{ is odd} \end{array} \right\} \quad (4.26)$$

Possible spin values are given as

$$R^{(1)} = 0, \quad R^{(2)} = \frac{p-r}{2} \dots \frac{p+r}{2}. \quad (4.27)$$

Then, Wigner- \mathcal{D} functions take the form

$$\mathcal{D}_{L^{(1)}L_3^{(1)}L^{(2)}L_3^{(2)}L_{15}; 0, 0, R^{(2)}, R_3^{(2)}, \frac{n}{\sqrt{2}}}(g). \quad (4.28)$$

Spectrum of the Hamiltonian (4.7) becomes

$$\begin{aligned} \mathcal{H} &= \frac{1}{2M\ell^2} (C_2(p, q_1 + q_2, p) - R^{(2)}(R^{(2)} + 1) - R_{15}^2) \\ E &= \frac{1}{2M\ell^2} \left(2q_2^2 + 2q_2(n + p + 2) + n(p + 2) + p^2 + 3p + m^2 - m(p + 1) \right. \\ &\quad \left. - R^{(2)}(R^{(2)} + 1) \right), \quad (4.29) \end{aligned}$$

where we have used formulas (B.4), (4.24) and (4.82). The minimum energy eigenvalue may be obtained if we maximize the $R^{(2)}$ value. (4.27) and (4.82) gives $R_{max}^{(2)} = \frac{p+r}{2} = p - m$. Then, we have

$$E = \frac{1}{2M\ell^2} (2q_2^2 + 2q_2(n + R^{(2)} + m + 2) + n(R^{(2)} + m + 2) + (R^{(2)} + m)(2 + m)). \quad (4.30)$$

It is clear that the LLL energy is obtained by taking $q_2 = m = 0$

$$E_{LLL} = \frac{1}{2M\ell^2} (n(R^{(2)} + 2) + 2R^{(2)}). \quad (4.31)$$

In a possible pure $SU(2)$ gauge field background, the degeneracy of the system is

$$\dim(R^{(2)}, 0, R^{(2)}) = \frac{1}{12}(R^{(2)} + 2)^2(2R^{(2)} + 3)(R^{(2)} + 1)^2, \quad (4.32)$$

which is equal to number of fermions, \mathcal{N} , at $\nu = 1$. Then, the spatial density is

$$\rho \sim \frac{\mathcal{N}}{(2R^{(2)} + 1) \text{vol}(Gr_2(\mathbb{C}^4))}. \quad (4.33)$$

When compared with (4.19) we have $2R^{(2)} + 1$ factor on denominator of this equation. This may be seen as the $SU(2)$ degrees of freedom at each point of $Gr_2(\mathbb{C}^4)$. In thermodynamic limit ($\ell \rightarrow \infty$ and $\mathcal{N} \rightarrow \infty$) $R^{(2)}$ should scale like ℓ^2 in order that the density is finite. In this limit, we obtain

$$\rho \sim \frac{\mathcal{N}}{\frac{\pi^4 \ell^8}{12} (2R^{(2)} + 1)} \xrightarrow{\ell \rightarrow \infty, \mathcal{N} \rightarrow \infty} \frac{(R^{(2)})^4}{\pi^4 \ell^8} \rightarrow \text{finite} \quad (4.34)$$

In case of both $U(1)$ and $SU(2)$ gauge fields, the degeneracy of LLL at $\nu = 1$ reads

$$\dim(R^{(2)}, n, R^{(2)}) = \frac{1}{12} (R^{(2)} + n + 2)^2 (2R^{(2)} + n + 3) (R^{(2)} + 1)^2 (n + 1), \quad (4.35)$$

which is equal to number of fermions, \mathcal{N} , at $\nu = 1$. We now have the option to scale n like ℓ^2 and keep R^2 finite. Doing so, we find finite spatial density in the thermodynamic limit as

$$\rho \sim \frac{\mathcal{N}}{\frac{\pi^4 \ell^8}{12} (2R^{(2)} + 1)} \xrightarrow{\ell \rightarrow \infty, \mathcal{N} \rightarrow \infty} \frac{n^4}{2\pi^4 \ell^8 R^{(2)}} \quad (4.36)$$

4.1.3 $SU(2) \times SU(2)$ Gauge Field

In this case, $\mathcal{D}^{(p,q,r)}$ transforms non-trivially under the right action of $SU(2) \times SU(2)$ and $U(1)$. Branching yields non-trivial $SU(2)$ representations and generally non-zero $U(1)$ charge. In this part there are two kind of branchings which differs from each other with respect to $U(1)$ charge.

For $q_2 = 0$, maximal $R^{(2)}$ may be obtained by aligning all boxes at the totally symmetric representation. The corresponding branching may be shown as

$$\begin{array}{c} \overbrace{\begin{array}{|c|c|} \hline & \\ \hline & \\ \hline & \\ \hline & \\ \hline \end{array}}^r \cdots \overbrace{\begin{array}{|c|c|} \hline & \\ \hline & \\ \hline & \\ \hline & \\ \hline \end{array}}^q \cdots \overbrace{\begin{array}{|c|c|} \hline & \\ \hline & \\ \hline & \\ \hline & \\ \hline \end{array}}^p \cdots \rightarrow \overbrace{\begin{array}{|c|c|} \hline & \\ \hline & \\ \hline & \\ \hline & \\ \hline \end{array}}^r \cdots \overbrace{\begin{array}{|c|} \hline \\ \hline \\ \hline \\ \hline \\ \hline \end{array}}^{q_1} \cdots \overbrace{\begin{array}{|c|} \hline \\ \hline \\ \hline \\ \hline \\ \hline \end{array}}^{x+p_1} \cdots \otimes \overbrace{\begin{array}{|c|} \hline \\ \hline \\ \hline \\ \hline \\ \hline \end{array}}^r \cdots \overbrace{\begin{array}{|c|} \hline \\ \hline \\ \hline \\ \hline \\ \hline \end{array}}^{p_2} \cdots \overbrace{\begin{array}{|c|} \hline \\ \hline \\ \hline \\ \hline \\ \hline \end{array}}^x \cdots \end{array} \quad (4.37)$$

This branching reads the $R^{(1)}$, $R^{(2)}$ and the $U(1)$ charge in the form

$$R^{(1)} = \frac{p_1 + x}{2}, \quad 0 \leq x \leq q, \quad 0 \leq p_1 \leq p \quad (4.38)$$

$$R_{max}^{(2)} = \frac{r + p_2 + x}{2} \quad (4.39)$$

$$n = \frac{1}{2}(2q_1 - (p_2 - p_1 - r)). \quad (4.40)$$

We may also align the boxes of second $SU(2)$ IRR with antisymmetry property, which gives us minimum R^2 value in the form

$$R_{min}^{(2)} = \frac{|2\mathcal{M} - S|}{2}, \quad (4.41)$$

where $\mathcal{M} = \max(p_2 + x + r)$ and $S = p_2 + x + r$.

Consider now the case $q_2 \neq 0$. In this case, we should have $p_1 = 0$ due to branching rule limitations. The branching under these conditions may be given as

$$\begin{array}{c} \overbrace{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}}^r \cdots \overbrace{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}}^q \cdots \overbrace{\square}^p \cdots \rightarrow \overbrace{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}}^r \cdots \overbrace{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}}^{q_1} \cdots \overbrace{\square}^x \cdots \otimes \left(\overbrace{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}}^{q_2} \cdots \overbrace{\square}^p \cdots \otimes \overbrace{\square}^{r+x} \cdots \right) \end{array} \quad (4.42)$$

$U(1)$ charge is

$$n = \frac{1}{2}(2(q_1 - q_2) - (p_2 - r)) \quad (4.43)$$

When we combine $q_2 \neq 0$ and $q_2 = 0$ cases, we obtain an interval for $R^{(1)}$ in the form

$$R^{(1)} = \frac{p_1 + x}{2}, \quad 0 \leq x \leq q, \quad 0 \leq p_1 \leq p. \quad (4.44)$$

$R^{(2)}$ interval may be expressed by using (4.39) and (4.41) and by fixing $p = p_2$ as

$$\frac{|2\mathcal{M} - S|}{2} \leq R^{(2)} \leq \frac{S}{2}, \quad S = p_2 + x + r. \quad (4.45)$$

Due to the Young Tableaux branching limitations we have separated the the problem into two parts: $q_2 = 0$ and $q_2 \neq 0$. First, let us find the energy eigenvalues in $q_2 = 0$ case. As we did in the previous subsection, we start with defining a parameter, \tilde{m} , which ensures that n given in (4.40) takes only integer values

$$m = \frac{p_2 - p_1 - r}{2}. \quad (4.46)$$

m can take both positive and negative integer values. We may assume m can take only positive integer values, then we have $p_2 > p_1 + r$, which induces the fact that $R^{(2)} > R^{(1)}$. Using

$$r = R^{(2)} - R^{(1)} - 2m \quad (4.47)$$

$$p_2 = R^{(2)} - R^{(1)} + p_1 + m \quad (4.48)$$

$$q_1 = m + n \quad (4.49)$$

$$x = 2R^{(1)} - p_1. \quad (4.50)$$

we may write the Hamiltonian (4.7) as

$$\begin{aligned} E &= \frac{1}{2M\ell^2} \left(\mathcal{C}_2(R^{(2)} - R^{(1)} + 2p_1 + m, n + m + 2R^{(1)} - p_1, R^{(2)} - R^{(1)} - m) \right. \\ &\quad \left. - R^{(1)}(R^{(1)} + 1) - R^{(2)}(R^{(2)} + 1) - \frac{n^2}{2} \right) \\ &= \frac{1}{2M\ell^2} (p_1^2 + p_1(m + R^{(2)} - R^{(1)} + 1) + m^2 + m(R^{(1)} + R^{(2)} + n + 2) \\ &\quad + n(R^{(1)} + R^{(2)} + 2) + 2R^{(2)}). \end{aligned} \quad (4.51)$$

At fixed $n, R^{(1)}, R^{(2)}$ values, energy spectrum depends on the two integers p and m . The LLL energy value is obtained by taking $p_1 = m = 0$ and gives

$$E_{LLL} = \frac{1}{2M\ell^2} (n(R^{(1)} + R^{(2)} + 2) + 2R^{(2)}). \quad (4.52)$$

For $p_2 < p_1 + r$, we should change the sign of m as $m \rightarrow -m$ in (4.51) to find the correct energy spectrum. If $R^{(1)} = R^{(2)} = R$, then we have $p_1 = p_2 + r$. When we combine this result with the Dirac quantization condition, we obtain

$$\tilde{m} := \frac{p_1 + r - p_2}{2} = r, \quad (4.53)$$

where \tilde{m} takes only positive integer values. As a consequence, energy eigenvalues read

$$\begin{aligned} E &= \frac{1}{2M\ell^2} \left(\mathcal{C}_2(2p_1 - r, n - r + 2R - p_1, r) - 2R(R + 1) - \frac{n^2}{2} \right) \\ &= \frac{1}{2M\ell^2} (2R + p_1(1 + p_1 - \tilde{m}) + (n - \tilde{m})(2 + 2R - \tilde{m})), \end{aligned} \quad (4.54)$$

where p_1 and \tilde{m} appear as the energy level indices. LLL energy eigenvalue may be obtained by taking $p_1 = \tilde{m} = 0$ and the result is the same as obtained from (4.52) when $R := R^{(1)} := R^{(2)}$.

For the case $q_2 \neq 0$, we first ensure that the Dirac quantization condition holds for every type of branching and we define a parameter m as

$$m = \frac{p - r}{2}, \quad (4.55)$$

where we impose the condition that m is an integer to ensure that n is so. We also assume that $p > r$ so that m is positive. In a similar manner to $q_2 = 0$ case we now use,

$$x = 2R^{(1)} \quad (4.56)$$

$$q_1 = n + m + q_2 \quad (4.57)$$

$$r = R^{(2)} - R^{(1)} - m \quad (4.58)$$

$$p = R^{(2)} - R^{(1)} + m. \quad (4.59)$$

We write the energy eigenvalue spectrum as

$$\begin{aligned} E &= \frac{1}{2M\ell^2} \left(C_2 (R^{(2)} - R^{(1)} + m, 2q_2 + 2R^{(1)} + n + m, R^{(2)} - R^{(1)} - m) \right. \\ &\quad \left. - R^{(1)}(R^{(1)} + 1) - R^{(2)}(R^{(2)} + 1) - \frac{n^2}{2} \right) \\ &= \frac{1}{2M\ell^2} (2q_2^2 + 2q_2(n + R^{(1)} + R^{(2)} + m + 2) + n(R^{(1)} + R^{(2)} + 2) + m^2 \\ &\quad + m(R^{(1)} + R^{(2)} + n + 2) + 2R^{(2)}), \quad (4.60) \end{aligned}$$

where q_2 and m appear as the energy level indices. By setting $p_2 = m = 0$, we obtain LLL energy value which is the same as that in (4.52). In case of $p < r$ we may interchange $R^{(1)}$ and $R^{(2)}$ and replace the n with $-n$, which gives

$$E_{LLL} = \frac{1}{2M\ell^2} (-n(R^{(2)} + R^{(1)} + 2) + 2R^{(1)}). \quad (4.61)$$

In case of pure $SU(2) \times SU(2)$ gauge field background (i.e. $n = 0$), the degeneracy of the system at LLL may be expressed as

$$\dim(R^{(2)} - R^{(1)}, 2R^{(1)}, R^{(2)} - R^{(1)}) \sim 4R^{(1)5}R^{(2)}, \quad (4.62)$$

which is equal to number of fermions, \mathcal{N} , at $\nu = 1$. We may define the spatial density as

$$\rho = \frac{\mathcal{N}}{\text{vol}(Gr_2(\mathbb{C}^4))(2R^{(1)} + 1)(2R^{(2)} + 1)} \sim \frac{4R^{(1)5}R^{(2)}}{\pi^4\ell^8(2R^{(1)} + 1)(2R^{(2)} + 1)} \quad (4.63)$$

in which $SU(2)$ degrees of freedom is finite. In the thermodynamic limit, we need to scale both $R^{(1)}$ and $R^{(2)}$ like ℓ^2 to ensure a finite value for ρ ,

$$\rho \sim \frac{4R^{(1)5}R^{(2)}}{\pi^4\ell^8(2R^{(1)}+1)(2R^{(2)}+1)} \xrightarrow{\ell \rightarrow \infty, \mathcal{N} \rightarrow \infty} \text{finite}. \quad (4.64)$$

In case of both $U(1)$ and $SU(2) \times SU(2)$ gauge field backgrounds, the degeneracy may be expressed as

$$\dim(R^{(2)} - R^{(1)}, 2R^{(1)} + n, R^{(2)} - R^{(1)}), \quad (4.65)$$

which is nothing but the number of fermions at $\nu = 1$. In thermodynamic limit, we may scale n like ℓ^2 , while keeping $R^{(1)}$ and $R^{(2)}$ finite. Then, spatial density reads

$$\rho \sim \frac{n^4}{\pi^4\ell^8(2R^{(1)}+1)(2R^{(2)}+1)} \longrightarrow \text{finite}. \quad (4.66)$$

4.2 Landau Problem on $Gr_2(\mathbb{C}^N)$

In this section we will analyse the Landau problem on $Gr_2(\mathbb{C}^N)$. From (4.1), we recall the coset realization of the Grassmannians

$$\mathbf{Gr}_2(\mathbb{C}^N) = \frac{SU(N)}{S[U(N-2) \times U(2)]} \sim \frac{SU(N)}{SU(N-2) \times SU(2) \times U(1)}. \quad (4.67)$$

Functions on $Gr_2(\mathbb{C}^N)$ may be represented as the Wigner- \mathcal{D} functions which transform trivially under the action of $SU(N-2) \times SU(2)$ and $U(1)$. The general form of the Wigner- \mathcal{D} functions on $SU(N)$ is

$$\mathcal{D}^{(P_1, P_2, \dots, P_{N-1})}(g), \quad (4.68)$$

where P_1, P_2, \dots, P_{N-1} are the IRR labels of $SU(N)$ and $g \in SU(N)$. We may label the left- and right-invariant vector fields with L_α and R_α , respectively ($\alpha = 1, 2, \dots, N-1$). In particular, we may label the right- and left- invariant vector fields of $SU(N-2)$ as $L^{SU(N-2)}$ and $R^{SU(N-2)}$, respectively. In this chapter, we will analyse the Landau problem in $U(1)$ and $SU(2)$ field background. We will not be concerned with the $SU(N-2)$ background. As a consequence, the generators which span $SU(N-2)$ may be out of the discussion. The general form of the $SU(N)$ Wigner- \mathcal{D} functions is

$$\mathcal{D}_{L^{SU(N-2)}, L, L_3, L_{N-2-1}, R^{SU(N-2)}, R, R_3, R_{N-2-1}}^{(P_1, P_2, P_3, \dots, P_{N-2}, P_{N-1})}(g), \quad g \in SU(N). \quad (4.69)$$

$L^{SU(N-2)}$ and $R^{SU(N-2)}$ are the suitable sets of the left and right quantum numbers, which we will not need in what follows.

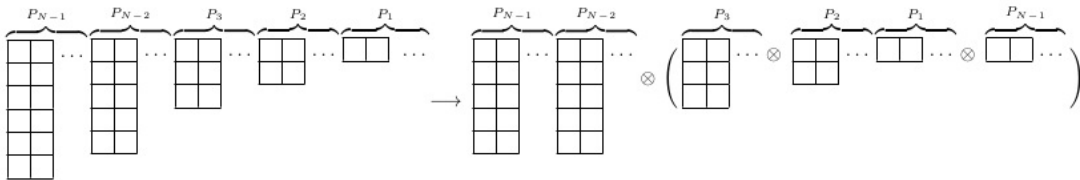
The real dimension of $Gr_2(\mathbb{C}^N)$ is $4N-8$ and the tangent vector fields on $Gr_2(\mathbb{C}^N)$ may be spanned by L_α and R_α , ($\alpha = N^2 - 4N + 7, \dots, 4N - 8$). We may write the Hamiltonian in terms of tangents as

$$\begin{aligned} H &= \frac{1}{2M\ell^2} \sum_{\alpha=N^2-4N+7}^{N^2-2} R_\alpha^2 \\ &= \frac{1}{2M\ell^2} \left(\mathcal{C}_2^{SU(N)} - \mathcal{C}_2^{SU(N-2)} - \mathcal{C}_2^{SU(2)} - R_{N^2-1}^2 \right) . \\ &= \frac{1}{2M\ell^2} \left(\mathcal{C}_2^{SU(N)} - \mathcal{C}_2^{SU(N-2)} - R(R+1) - R_{N^2-1}^2 \right) . \end{aligned} \quad (4.70)$$

The formula for the eigenvalues of the quadratic Casimir of $SU(N)$ ($\mathcal{C}_2^{SU(N)}$) is given in (B.4).

As indicated before, we will deal with $U(1)$ and $SU(2)$ gauge field backgrounds. This means we need to restrict to Wigner- \mathcal{D} functions that transform trivially under the action of $SU(N-2)$. In the language of Young diagram, the branching $SU(N) \rightarrow SU(N-2) \times SU(2) \times U(1)$ should yield the singlet IRR of $SU(N-2)$.

At a first glance, one can see that the singlet representation of $SU(N-2)$ is not possible if any of P_3, P_4 or P_{N-3} are non-zero. To prove it we may use "Reductio ad absurdum" method. Let us assume that we may obtain a trivial $SU(N-2)$ representation if any of P_3, P_4 or P_{N-3} are non-zero. First, let us define the possible trivial representations of $SU(N-2)$ which are $(0, 0, \dots, P_{N-2})$ and $(0, 0, \dots, 0)$. If we want to obtain a trivial $SU(N-2)$ representation, we should move all the boxes of $SU(N)$ labelled by P_3, P_4 or P_{N-3} in its Young diagram to the $SU(2)$ slot in the branching. In particular, we may choose the $(P_1, P_2, P_3, 0, 0, \dots, 0, P_{N-2}, P_{N-1})$. In the language of Young diagrams, this can be shown as

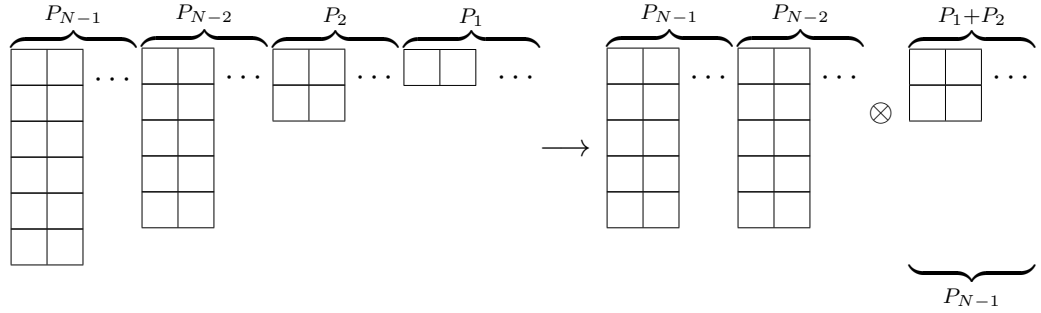


where the first slot in the RHS is for $SU(N-2)$ and the second slot is for

$SU(2)$ representations. P_3 in the $SU(2)$ slot is not allowed since number of boxes on each column cannot exceed 2. Therefore, we may conclude that the singlet representation of $SU(N-2)$ cannot be obtained if any of P_3, P_4 or P_{N-3} are non-zero.

4.2.1 Pure $U(1)$ Gauge Field

Now, we will analyse the Landau problem on $Gr_2(\mathbb{C}^N)$ in pure $U(1)$ gauge field. We need wave functions to be singlet under $SU(N-2)$ and $SU(2)$. Wave functions are singlets if and only if $P_1 = P_{N-1}$. We may show the branching with the help of Young diagrams as



From the branching and (4.10), we may write the $U(1)$ charge as

$$\begin{aligned} n &= \frac{1}{2(N-2)}(2J_{N-2} - (N-2)J_2) \\ &= P_{N-2} - P_2, \end{aligned} \quad (4.71)$$

where J_{N-2} is the number of boxes of $SU(N-2)$ IRR and J_2 is the number of boxes in the $SU(2)$ IRR. The relation between the $U(1)$ charge and the eigenvalues of R_{N^2-1} is derived in appendix C and it is given as

$$R_{N^2-1} = -\sqrt{1 - \frac{2}{N}}n. \quad (4.72)$$

Using (4.72) and $P_1 = P_{N-1}$, the Wigner- \mathcal{D} functions in pure $U(1)$ gauge field backgrounds take the form

$$\mathcal{D}_{L^{SU(N-2)}, L, L_3, L_{N^2-1}, 0, 0, 0, -\sqrt{1-\frac{2}{N}}n}^{(P_1, P_2, 0, \dots, 0, P_{N-2}=P_2+n, P_{N-1}=P_1)}(g) \quad (4.73)$$

and the Hamiltonian given in (4.71) reduces to

$$\mathcal{H} = \frac{1}{2M\ell^2} \left(C_2^{SU(N)} - \left(1 - \frac{2}{N}\right)n^2 \right). \quad (4.74)$$

Using (B.4) to evaluate $\mathcal{C}_2^{(SU(N))}$ in the $(P_1, P_2, 0, \dots, 0, P_{N-2}, P_1)$ representation, the energy spectrum takes the form

$$E = \frac{1}{2M\ell^2} \left(P_1^2 + \left(2 - \frac{4}{N}\right) P_2^2 + (N-1+2n)P_1 + 2\left(n+N-2 + \frac{2}{N}\right)P_2 + 4P_1P_2 + n(N-2) \right). \quad (4.75)$$

In this expression P_1 and P_2 appear as the energy level indices. The LLL energy may be written by taking $P_1 = P_2 = 0$ and it is

$$E_{LLL} = \frac{n(N-2)}{2M\ell^2}. \quad (4.76)$$

The degeneracy of the LLL is

$$\dim(0, \dots, n, 0) = \frac{(n+N-3)!(n+N-4)!(n+N-2)^2(n+N-1)(n+N-3)}{(N+1)!(N-2)!n!(n+1)!}, \quad (4.77)$$

which is equal to the number of fermions \mathcal{N} at $\nu = 1$. The spatial density is

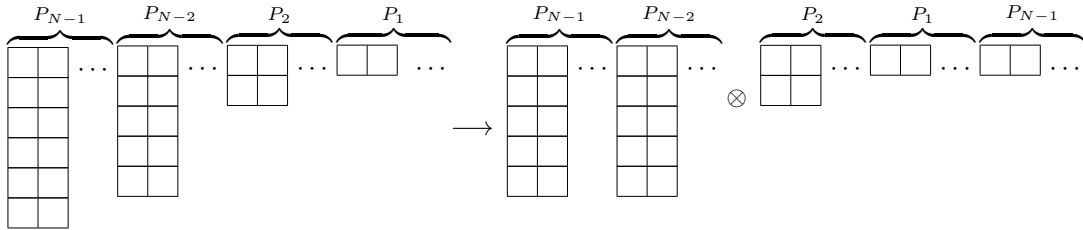
$$\rho = \frac{\mathcal{N}}{\text{vol}(Gr_2(\mathbb{C}^N))} = \frac{\dim(0, 0, \dots, n, 0)}{\frac{\pi^{2(N-2)}}{(N-2)!(N-1)!} \ell^{4N-8}}, \quad (4.78)$$

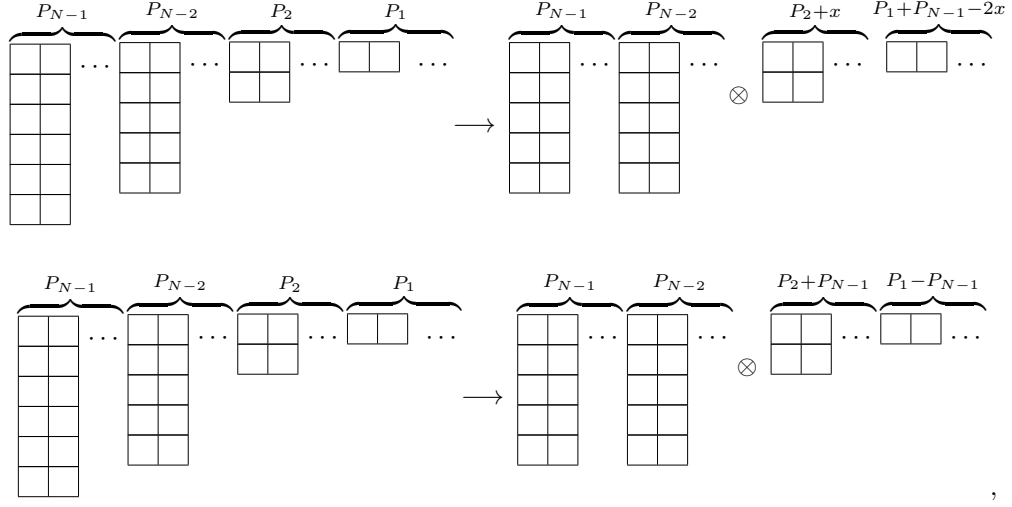
where we have used (4.20). Then, the thermodynamic limit can be obtained by letting $\mathcal{N} \rightarrow \infty$, $\ell \rightarrow \infty$ and the spatial density in the thermodynamic limit is finite:

$$\rho = \frac{\mathcal{N}}{\frac{\pi^{2(N-2)}}{(N-2)!(N-1)!} \ell^{4N-8}} \xrightarrow{\ell \rightarrow \infty, \mathcal{N} \rightarrow \infty} \frac{n^{2N-4}}{\ell^{4N-8}} = \left(\frac{B}{2\pi}\right)^{2N-4}. \quad (4.79)$$

4.2.2 $SU(2)$ Gauge Field

Now, we discuss the Landau problem with $SU(2)$ gauge field background. We will find the branching of $SU(N)$ IRRs where $SU(N-2)$ IRRs is in the singlet while $SU(2)$ IRRs may take on a range of possible values. We have the representation of $SU(N)$ in the form $(P_1, P_2, 0, \dots, 0, P_{N-2}, P_{N-1})$ to ensure that $SU(N-2)$ remains in the singlet representation. The branching of this representation





where $0 \leq x \leq P_{N-1}$ and $P_{N-1} \leq P_1$. The second slot in the product is for the $SU(2)$ IRRs and it implies that there is an interval of possible $SU(2)$ IRRs. These three branchings represent maximum, generic and minimum R values of $SU(2)$, respectively. Therefore, we read that the $SU(2)$ IRRs are in the interval

$$R = \frac{P_1 - P_{N-1}}{2}, \dots, \frac{P_1 + P_{N-1}}{2} \quad (4.80)$$

and the $U(1)$ charge is

$$n = \frac{1}{2}(2P_{N-2} - 2P_2 + P_{N-1} - P_1). \quad (4.81)$$

Due to the Dirac quantization condition, $P_1 - P_{N-1}$ should take only even integer values. We may define the integer m as

$$m := \frac{P_1 - P_{N-1}}{2}, \quad \left\{ \begin{array}{ll} m = 0, \dots, \frac{P_1}{2}, & \text{if } P_1 \text{ is even} \\ m = 0, \dots, \frac{P_1-1}{2}, & \text{if } P_1 \text{ is odd} \end{array} \right\} \quad (4.82)$$

Spectrum of the Hamiltonian (4.71) may be written as

$$E = \frac{1}{2M\ell^2} \left(\mathcal{C}_2^{SU(N)} - R(R+1) - (1 - \frac{2}{N})n^2 \right). \quad (4.83)$$

Using (4.83), (4.82) and (4.81), we may express this in terms of P_1 , P_2 , m and n in the form

$$\begin{aligned} 2M\ell^2 E &= \left(\frac{N-2}{N}\right)P_2^2 + \left(\frac{N-2}{N}\right)(n^2 + m^2 + 2nm + P_2^2 + 2nP_2 + 2mP_2) \\ &+ \left(\frac{N-1}{2N}\right)P_1^2 + \left(\frac{N-1}{2N}\right)(4m^2 + P_1^2 - 4mP_1) + \left(\frac{N-2}{N}\right)P_1P_2 \\ &+ \frac{2}{N}P_1(n+m+P_2) - \frac{1}{N}(2mP_1 - P_1^2)\frac{4}{N}P_2(n+m+P_2) + (N-2)P_2 \\ &- \frac{2}{N}P_2(2m - P_1) - \left(\frac{N-2}{N}\right)(2m - P_1)(n+m+P_2) + \left(\frac{N-1}{2}\right)P_1 \\ &+ (N-2)P_{N-2} + \left(\frac{N-1}{2}\right)(-2m + P_1) - \left(\frac{N-2}{N}\right)n^2 - R(R+1) \end{aligned} \quad (4.84)$$

In order to obtain LLL energy value, we select the maximum value of R as

$$R = \frac{P_{N-1} + P_1}{2} = P_1 - m. \quad (4.85)$$

The reason why we select the maximum value of R is obvious in the equation (4.83). In order to minimize the E value, we select the R_{max} on interval (4.80). Substituting this in (4.84), we obtain

$$\begin{aligned} E &= \frac{1}{2M\ell^2} \left(\frac{N-2}{N} (2P_2^2 + mn + 2nP_2 + 2RP_2 + 2mP_2 + Rn + Rm) \right. \\ &+ \frac{1}{N} (2Rn + 4RP_2 + 2mn + R^2 + m^2 + 2Rm + 4P_2n + 4P_2m + 4P_2^2) \\ &+ \left(\frac{N-1}{2} \right) (2R) + (N-2)(2P_2 + n + m) + \left(\frac{N-1}{2N} \right) (2R^2 + 2m^2) \Big) \\ &- R(R+1), \end{aligned} \quad (4.86)$$

where the energy spectrum is controlled by two integers P_2 and m at fixed R and n . LLL can be found by taking $P_2 = m = 0$ as

$$E_{LLL} = \frac{1}{2M\ell^2} (nR + (N-2)(n+R)). \quad (4.87)$$

The degeneracy in case of pure $SU(2)$ background, i.e. $n = 0$, is

$$\begin{aligned} \dim(R, 0, \dots, 0, R) &= \zeta((R+N-3)!(N-4)!(R+N-3)!(R+N-2) \\ &\times (R+1)(2R+N-1)(N-3)(R+N-2)), \end{aligned} \quad (4.88)$$

where $\zeta = \frac{1}{(N-1)!(N-2)!(N-3)!(R+1)!R!}$. This equation is equal to number of fermions \mathcal{N} at $\nu = 1$. The spatial density in case of pure $SU(2)$ background reads

$$\rho = \frac{\mathcal{N}}{\text{vol}(Gr_2(\mathbb{C}^N))} = \frac{\dim(R, 0, \dots, R)}{\frac{\pi^{2(N-2)}}{(N-2)!(N-1)!} \ell^{4N-8}}. \quad (4.89)$$

In the thermodynamic limit ($\ell \rightarrow \infty, \mathcal{N} \rightarrow \infty$) $R \rho$ is finite if R scales like ℓ^2 :

$$\rho \xrightarrow{\ell \rightarrow \infty, \mathcal{N} \rightarrow \infty} \frac{R^{2N-3}}{k\ell^{4N-8}(2R+1)} \rightarrow \text{finite} \quad (4.90)$$

In case of both $U(1)$ and $SU(2)$ gauge field backgrounds, the degeneracy is equal to $\dim(R, 0, \dots, n, R)$ which is the number of fermions at $\nu = 1$. In the thermodynamic limit, may choose the scaling $n \sim \ell^2$ while R is kept finite. Then, spatial density in the thermodynamic limit is

$$\rho \xrightarrow{\ell \rightarrow \infty, \mathcal{N} \rightarrow \infty} \frac{n^{2N-4}}{k\ell^{4N-8}(2R+1)} \rightarrow \text{finite}, \quad (4.91)$$

where we have $\dim(R, 0, \dots, n, R) \rightarrow n^{2N-4}$ when $\mathcal{N} \rightarrow \infty$.

Due to the pedagogical reasons, we have first analyzed the Landau problem on $Gr_2(\mathbb{C}^4)$. Then, we have generalized our results for $Gr_2(\mathbb{C}^N)$ in this section. For $N = 4$, the results of this section collapse to those of section 1 with the correspondence

$$(p, q = q_1 + q_2, r) \longrightarrow (P_1, P_2 = q_2, 0, \dots, P_{N-2} = q_1, P_{N-1}). \quad (4.92)$$

4.3 Local Form of the Wave Functions and the Gauge Fields

Complex Grassmanians are defined as the set of k -dimensional complex subspaces of N -dimensional complex space. This definition implies that we may parametrize the Grassmanians in the language of projective spaces. This construction was first done by Julius Plücker in late 19th century. Working on real Grassmannians, he introduced a way to define lines on \mathbb{R}^3 with homogeneous coordinates on $R\mathbb{P}^3$ which are called Plücker coordinates [27]. Here we first give a brief exposition of his approach adopted to complex Grassmanians [21]:

Let us pick two generic vectors on \mathbb{C}^4 as $S = S_i \mathbf{e}_i = S_1 \mathbf{e}_1 + S_2 \mathbf{e}_2 + S_3 \mathbf{e}_3 + S_4 \mathbf{e}_4$, $Q = Q_i \mathbf{e}_i = Q_1 \mathbf{e}_1 + Q_2 \mathbf{e}_2 + Q_3 \mathbf{e}_3 + Q_4 \mathbf{e}_4$. The wedge product of the vectors gives

$$\begin{aligned} S \wedge Q = & (S_4 Q_1 - S_1 Q_4) \mathbf{e}_{41} + (S_4 Q_2 - S_2 Q_4) \mathbf{e}_{42} + (S_4 Q_3 - S_3 Q_4) \mathbf{e}_{43} \\ & + (S_2 Q_3 - S_3 Q_2) \mathbf{e}_{23} + (S_3 Q_1 - S_1 Q_3) \mathbf{e}_{31} + (S_1 Q_2 - S_2 Q_1) \mathbf{e}_{12}. \end{aligned} \quad (4.93)$$

Wedge product represents the planes in \mathbb{C}^4 whose normal vector is perpendicular to both S and Q and spans the vector space

$$(\wedge^2 \mathbb{C}^4) := \text{span} \{v_1 \wedge v_2; v_1, v_2 \in \mathbb{C}^4\}. \quad (4.94)$$

Setting $S_4 = Q_4 = 1$ we may consider the vectors as the homogeneous coordinates on \mathbb{C}^3 . Let us impose this on (4.93) and inspect the coefficients of (4.93). We call the 6 components of (4.93) as the Plücker coordinates and write

$$P_{\alpha\beta} = S_\alpha Q_\beta - Q_\alpha S_\beta, \quad \alpha, \beta = 1, 2, 3, 4. \quad (4.95)$$

A complex line L in \mathbb{C}^3 may be represented as $L = \{S - Q; S \times Q\}$. The first three coefficient represent the displacement vector components as

$$P_{41} = Q_1 - S_1, \quad P_{42} = Q_2 - S_2, \quad P_{43} = Q_3 - S_3, \quad (4.96)$$

and the last three components represent the moment bi-vector components in the form

$$P_{23} = S_2Q_3 - S_3Q_2, \quad P_{31} = S_3Q_1 - S_1Q_3, \quad P_{12} = S_1Q_2 - S_2Q_1. \quad (4.97)$$

Therefore, we may conclude that homogeneous form of the lines in \mathbb{C}^3 represents the projective planes in \mathbb{C}^4 . In other words, the coordinates of the 2 dimensional subspaces in \mathbb{C}^4 represent the coordinates of the lines in \mathbb{C}^3 . If we choose different set of points such that

$$S' = \lambda S + (1 - \lambda)Q, \quad Q' = \mu S + (1 - \mu)Q. \quad (4.98)$$

then, we have

$$S' - Q' = (\lambda - \mu)(S - Q), \quad S' \times Q' = (\lambda - \mu)(S \times Q). \quad (4.99)$$

That means the coordinates are unique up to scale factor [22]. This conclusion implies that the Plücker coordinates, $\{P_{12}, P_{13}, P_{14}, P_{23}, P_{24}, P_{34}\}$, represent the homogeneous coordinates on $\mathbb{C}P^5$.

All the discussions above imply that there is a embedding between $Gr_2(\mathbb{C}^4)$ and $\mathbf{P}(\wedge^2 \mathbb{C}^4) \equiv \mathbb{C}P^5$ as

$$\psi : Gr_2(\mathbb{C}^4) \rightarrow \mathbb{C}P^5 \quad (4.100)$$

$$span(v_1, v_2) \hookrightarrow [v_1 \wedge v_2], \quad (4.101)$$

where ψ is called as Plücker embedding. A vector $\mathbf{v} \in \wedge^k \mathbb{R}^N$ is defined to be totally decomposable if it can be written as $x = v_1 \wedge v_2 \wedge \cdots \wedge v_k$ [6]. If x is decomposable, then we have that $\mathbf{v} \wedge \mathbf{v} = 0$. Applying this on (4.93) gives

$$(S \wedge Q) \wedge (S \wedge Q) = (P_{12}P_{34} - P_{13}P_{24} + P_{14}P_{23}) = 0. \quad (4.102)$$

The image of $\psi(Gr_2(\mathbb{C}^4))$ is the projective space of the bivectors spanned by $v_1 \wedge v_2$ which are totally decomposable [6]. That means not all the elements on $\mathbb{C}P^5$ are the image of $\mathbb{C}P^5$, so ψ is an injective map. We have also stated that Plücker coordinates are the homogeneous coordinates on $\mathbb{C}P^5$. However, (4.102) restricts the Plücker coordinates such that they cannot take all the values on $\mathbb{C}P^5$. That means $span(v_1, v_2) \hookrightarrow [v_1 \wedge v_2]$ is an embedding in $\mathbb{C}P^5$. We call the ψ Plücker embedding and algebraic variety specified by the homogeneous condition (4.102) is called as Klein quadric.

We are going to provide expressions for the $U(1)$ background gauge field and wave functions in terms of Plücker coordinates. We may start with defining the matrix realization of the group element $g \in SU(4)$ in the IRR $(0, 1, 0)$. We parametrize the last two columns of the matrix in terms of the Plücker coordinates as $g_{N6} := P_{\alpha\beta}$ and $g_{N5} = \epsilon_{NM} P_M^* = \epsilon_{\alpha\beta\gamma\delta} P_{\gamma\delta}^*$ where $N = (1, \dots, 6)$ and $\alpha\beta = (12, 13, 14, 23, 24, 34)$. We may show the realization in explicit form

$$g := \left(\begin{array}{c|c|c|c|c} & & & & \begin{matrix} P_{34}^* & P_{12} \\ -P_{24}^* & P_{13} \\ P_{23}^* & P_{14} \\ P_{14}^* & P_{23} \\ -P_{13}^* & P_{24} \\ P_{12}^* & P_{34} \end{matrix} \end{array} \right), \quad (4.103)$$

where we have used the fact that the column vectors of unitary matrices are orthogonal to each other and the equation (4.102) in order to express fifth column.

For $U(1)$ gauge field we may write the gauge potential in the form

$$A = \frac{-in}{\sqrt{2}} \text{Tr} (R_{15} g^{-1} dg), \quad (4.104)$$

where $g \in SU(4)$. By using matrix realization of R_{15} and (4.103) we find

$$\begin{aligned} A &= -\frac{in}{\sqrt{2}} (\lambda_{(6)}^{15})_{LM} (g^{-1})_{MN} (dg)_{NL} \\ &= -\frac{in}{2} (-(g^{-1})_{5N} (dg)_{N5} + (g^{-1})_{6N} (dg)_{N6}) \\ &= -\frac{in}{2} (-g_{N5}^* (dg)_{N5} + g_{N6}^* (dg)_{N6}) \\ &= -\frac{in}{2} (-P_N dP_N^* + P_N^* dP_N) \\ &= -in P_N^* dP_N, \end{aligned} \quad (4.105)$$

where we have used $d(P_N^* P_N) = 0$. Under the $U(1)$ gauge transformation $g \rightarrow g e^{iR_{15}\theta}$, the potential transforms as

$$\begin{aligned} A(g e^{iR_{15}\theta}) &= i \frac{-n}{\sqrt{2}} \text{Tr} (R_{15} (g e^{iR_{15}\theta})^{-1} d(g e^{iR_{15}\theta})) \\ &= i \frac{-n}{\sqrt{2}} \text{Tr} [(R_{15} e^{-iR_{15}\theta} g^{-1}) (e^{iR_{15}\theta} dg + g d(e^{iR_{15}\theta}))] \\ &= A + i \frac{-n}{\sqrt{2}} \text{Tr} (i (R_{15})^2 d\theta) \\ &= A + d \left(\frac{n\theta}{\sqrt{2}} \right) \end{aligned} \quad (4.106)$$

. We also noted that the gauge potential may be expressed as Berry connection

$$A = \frac{1}{\sqrt{2}} n \psi_\alpha^* d\psi_\alpha. \quad (4.107)$$

Wave functions transform under the $U(1)$ action as

$$\begin{aligned} \mathcal{D}_{L^{(1)}L_3^{(1)}L^{(2)}L_3^{(2)}L_{15};0,0,0,0,\frac{n}{\sqrt{2}}}^{(p,q,r)}(ge^{iR_{15}\theta}) &= \langle L^{(1)}L_3^{(1)}L^{(2)}L_3^{(2)}L_{15}|e^{iR_{15}\theta}|0,0,0,0,\frac{n}{\sqrt{2}}\rangle \\ &= \langle L^{(1)}L_3^{(1)}L^{(2)}L_3^{(2)}L_{15}|g|R^{(1)}R_3^{(1)}R^{(2)}R_3^{(2)}\frac{n}{\sqrt{2}}\rangle \\ &\quad \times e^{(i\theta R_{15})}\delta_{L^{(1)}0}\delta_{L_3^{(1)}0}\delta_{L^{(2)}0}\delta_{L_3^{(2)}0} \\ &= D_{L^{(1)}L_3^{(1)}L^{(2)}L_3^{(2)}L_{15};0,0,0,0,\frac{n}{\sqrt{2}}}^{(1,0)}(g) \times e^{\frac{in\theta}{\sqrt{2}}} \quad (4.108) \end{aligned}$$

Therefore, we conclude that the wave functions transform with a phase under the $U(1)$ action. Due to pedagogical reasons it would be useful to show transformation properties of the wave functions explicitly for the $(0, 1, 0)$ representation:

$$\begin{aligned} ge^{iR_{15}\theta} &= \begin{pmatrix} & & & & P_{34}^* & P_{12} \\ & & & & -P_{24}^* & P_{13} \\ & & & & P_{23}^* & P_{14} \\ & & & & P_{14}^* & P_{23} \\ & & & & -P_{13}^* & P_{24} \\ & & & & P_{12}^* & P_{34} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{\frac{-i}{\sqrt{2}}\theta} & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{\frac{i}{\sqrt{2}}\theta} \end{pmatrix} \\ &= \begin{pmatrix} & & & & e^{\frac{-i}{\sqrt{2}}\theta}P_{34}^* & e^{\frac{i}{\sqrt{2}}\theta}P_{12} \\ & & & & -e^{\frac{-i}{\sqrt{2}}\theta}P_{24}^* & e^{\frac{i}{\sqrt{2}}\theta}P_{13} \\ & & & & e^{\frac{-i}{\sqrt{2}}\theta}P_{23}^* & e^{\frac{i}{\sqrt{2}}\theta}P_{14} \\ & & & & e^{\frac{-i}{\sqrt{2}}\theta}P_{14}^* & e^{\frac{i}{\sqrt{2}}\theta}P_{23} \\ & & & & -e^{\frac{-i}{\sqrt{2}}\theta}P_{13}^* & e^{\frac{i}{\sqrt{2}}\theta}P_{24} \\ & & & & e^{\frac{-i}{\sqrt{2}}\theta}P_{12}^* & e^{\frac{i}{\sqrt{2}}\theta}P_{34} \end{pmatrix} \quad (4.109) \end{aligned}$$

This transformation shows that $g_{n6}(=P_{\alpha\beta})$ terms satisfy the transformation properties. Therefore, the wave functions are in the form

$$\mathcal{D}^{(0,1,0)}(g) \sim P_{\alpha\beta}. \quad (4.110)$$

The Wigner- \mathcal{D} functions in the $(0, q, 0)$ IRR may be also expressed in terms of Plücker coordinates. $(0, q, 0)$ representation is the symmetric tensor product of

the $(0, 1, 0)$ IRRs

$$\begin{array}{|c|} \hline \\ \hline \end{array} \otimes_S \begin{array}{|c|} \hline \\ \hline \end{array} \otimes_S \cdots \otimes_S \begin{array}{|c|} \hline \\ \hline \end{array} \longrightarrow \overbrace{\begin{array}{|c|c|c|} \hline & & \\ \hline \end{array} \cdots \begin{array}{|c|} \hline \\ \hline \end{array}}^q,$$

where \otimes_S stands for the symmetric tensor product. From (4.15) we observe that $n = q_1 - q_2$. Combining this fact with (4.108) we may write

$$\mathcal{D}^{(0, q_1 + q_2, 0)}(gh) = \mathcal{D}^{(0, q_1 + q_2, 0)}(g) e^{\frac{i(q_1 - q_2)\theta}{\sqrt{2}}}, \quad (4.111)$$

which implies that the wave functions have the local form

$$\mathcal{D}^{(0, q_1 + q_2, 0)}(g) \sim P_{\alpha_1 \beta_1} P_{\alpha_2 \beta_2} \cdots P_{\alpha_{q_1} \beta_{q_1}} P_{\gamma_1 \delta_1}^* P_{\gamma_2 \delta_2}^* \cdots P_{\gamma_{q_2} \delta_{q_2}}^*. \quad (4.112)$$

Taking $q_2 = 0$, LLL wave functions are obtained as

$$\mathcal{D}_{L^{(1)} L_3^{(1)} L^{(2)} L_3^{(2)} L_{15}; 0, 0, 0, 0, \frac{n}{\sqrt{2}}}^{(0, q_1, 0)}(g) \sim P_{\alpha_1 \beta_1} P_{\alpha_2 \beta_2} \cdots P_{\alpha_{q_1} \beta_{q_1}}. \quad (4.113)$$

Casimir operator of $SU(4)$ may be written in the form $\mathcal{C}_2^{SU(4)} = L_k^{SU(4)} L_k^{SU(4)}$ where $L_k^{SU(4)}$ are the left invariant vector fields. From [23] we may write

$$L_k = -v_j(\lambda_k)_{ij} \frac{\partial}{\partial v_i} - w_j(\lambda_k)_{ij} \frac{\partial}{\partial w_i} + v_i^*(\lambda_k)_{ij} \frac{\partial}{\partial v_j^*} + w_i^*(\lambda_k)_{ij} \frac{\partial}{\partial w_j^*}, \quad (4.114)$$

where λ_k is the k^{th} component of the Gell-mann matrices of $SU(4)$ and v_α, w_α are the two sets of complex coordinates, $(\alpha = 1, \dots, 4)$, that is one set for each \mathbb{C}^4 . By choosing complex vectors \mathbf{v} and \mathbf{w} as complex unit vectors and orthonormal to each other, i.e.,

$$v_i w_i^* = 0, \quad |\mathbf{v}|^2 = |\mathbf{w}|^2 = 1, \quad (4.115)$$

and using the identity

$$\sum_{k=1}^{N^2-1} \lambda_{ij}^k \lambda_{mn}^k = \frac{1}{2} \delta_{in} \delta_{jm} - \frac{1}{2N} \delta_{ij} \delta_{mn}. \quad (4.116)$$

for $N = 4$, we may write the $\mathcal{C}_2^{SU(4)}$ in the explicit form

$$\begin{aligned} \mathcal{C}_2^{SU(4)} &= \frac{15}{8} \left(v_i \frac{\partial}{\partial v_i} + w_i \frac{\partial}{\partial w_i} + v_i^* \frac{\partial}{\partial v_i^*} + w_i^* \frac{\partial}{\partial w_i^*} \right) \\ &+ \frac{3}{8} \left(v_i v_j \frac{\partial}{\partial v_i} \frac{\partial}{\partial v_j} + w_i w_j \frac{\partial}{\partial w_i} \frac{\partial}{\partial w_j} + \text{c.c.} \right) \\ &- \frac{2}{8} \left(v_i w_j \frac{\partial}{\partial v_i} \frac{\partial}{\partial w_j} - v_i w_j^* \frac{\partial}{\partial v_i} \frac{\partial}{\partial w_j^*} + \text{c.c.} \right) \\ &+ \frac{1}{8} \left(v_i v_j^* \frac{\partial}{\partial v_i} \frac{\partial}{\partial v_j^*} + w_i w_j^* \frac{\partial}{\partial w_i} \frac{\partial}{\partial w_j^*} + \text{c.c.} \right) \\ &+ v_i w_j \frac{\partial}{\partial v_j} \frac{\partial}{\partial w_i} + v_i^* w_j^* \frac{\partial}{\partial v_j^*} \frac{\partial}{\partial w_i^*} - \frac{\partial}{\partial v_j} \frac{\partial}{\partial v_j^*} - \frac{\partial}{\partial w_j} \frac{\partial}{\partial w_j^*}. \end{aligned} \quad (4.117)$$

A short calculation gives

$$\mathcal{C}_2^{SU(4)} P_{\alpha\beta} = \frac{5}{2} P_{\alpha\beta}, \quad (4.118)$$

where the eigenvalue is equal to that obtained from B.4 for $SU(4)$ in the IRR $(0, 1, 0)$, as expected. We may express the wave function for \mathcal{N} particles as

$$\begin{aligned} \Psi_{MP} &= \frac{1}{\sqrt{\mathcal{N}!}} \begin{vmatrix} \Psi_{\Lambda_1}(P^1) & \cdots & \Psi_{\Lambda_{\mathcal{N}}}(P^1) \\ \Psi_{\Lambda_1}(P^2) & \cdots & \Psi_{\Lambda_{\mathcal{N}}}(P^2) \\ \vdots & \ddots & \vdots \\ \Psi_{\Lambda_1}(P^{\mathcal{N}}) & \cdots & \Psi_{\Lambda_{\mathcal{N}}}(P^{\mathcal{N}}) \end{vmatrix} \\ &= \frac{1}{\sqrt{\mathcal{N}!}} \varepsilon^{\Lambda_1 \Lambda_2 \cdots \Lambda_{\mathcal{N}}} \Psi_{\Lambda_1}(P^{(1)}) \Psi_{\Lambda_2}(P^{(2)}) \cdots \Psi_{\Lambda_{\mathcal{N}}}(P^{(\mathcal{N})}), \end{aligned} \quad (4.119)$$

where $\Psi_{\Lambda_j}(P^{(i)})$ is the wave function of the i^{th} particle at Λ_j one-particle state. From (4.110) one-particle state may be expressed as

$$\Psi_{\Lambda_j}(P^i) = P_{\alpha\beta}^i. \quad (4.120)$$

We may also express the LLL wave function in the form (4.113) as

$$\Psi_{\Lambda_i}(P^i) \sim (P_{\alpha\beta}^i)^n. \quad (4.121)$$

As discussed in the previous chapter we may write the two point correlation function in the form

$$\Omega(1, 2) = \int d\mu(3) d\mu(4) \cdots d\mu(N) \Psi_{MP}^* \Psi_{MP}, \quad (4.122)$$

where $d\mu(i)$ is the measure of integration (or the volume form of the Grassmannian) in terms for the i^{th} particle coordinates. For a short-hand notation, we will sometimes show the integration measure as $d\mu(3, 4, \cdots, N)$. Using (4.119) we may write the two-point function in the form

$$\frac{1}{\mathcal{N}} \int d\mu(3) d\mu(4) \cdots d\mu(N) \zeta \Psi_{i_1}(P^{(1)}) \Psi_{i_2}(P^{(2)}) \cdots \Psi_{i_{\mathcal{N}}}(P^{(\mathcal{N})}) \Psi_{j_1}^*(P^{(1)}) \Psi_{j_2}^*(P^{(2)}) \cdots \Psi_{j_{\mathcal{N}}}^*(P^{(\mathcal{N})}), \quad (4.123)$$

where $\zeta = \varepsilon^{i_1 i_2 \cdots i_{\mathcal{N}}} \varepsilon^{j_1 j_2 \cdots j_{\mathcal{N}}}$. By $e - \delta$ identity [20] we may write

$$\varepsilon^{i_1 i_2 \cdots i_{\mathcal{N}}} \varepsilon^{j_1 j_2 \cdots j_{\mathcal{N}}} = \delta_{j_1 j_2 \cdots j_{\mathcal{N}}}^{i_1 i_2 \cdots i_{\mathcal{N}}}, \quad (4.124)$$

where LHS of the equation is called the generalized Kronecker delta and is given

by the determinant

$$\delta_{j_1 j_2 \dots j_N}^{i_1 i_2 \dots i_N} = \delta' = \det \begin{pmatrix} \delta_{j_1}^{i_1} & \dots & \delta_{j_1}^{i_N} \\ \delta_{j_2}^{i_1} & \dots & \delta_{j_2}^{i_N} \\ \vdots & & \vdots \\ \delta_{j_N}^{i_1} & \dots & \delta_{j_N}^{i_N} \end{pmatrix} \quad (4.125)$$

As an example, we may handle the calculation of two-point function at $N = 3$ as

$$\Omega(1, 2) = \frac{1}{\mathcal{N}} \int d\mu(3) \delta' [\Psi_{i_1}(P^{(1)}) \Psi_{i_2}(P^{(2)}) \Psi_{i_3}(P^{(3)}) \Psi_{j_1}^*(P^{(1)}) \Psi_{j_2}^*(P^{(2)}) \Psi_{j_3}^*(P^{(3)})] . \quad (4.126)$$

The integrand may be expressed as

$$\begin{aligned} \Psi_{MP}^* \Psi_{MP} = \mathcal{A} & \left(\Psi_{i_1}^*(P^{(1)}) \Psi_{i_2}^*(P^{(2)}) \Psi_{i_3}^*(P^{(3)}) - \Psi_{i_1}^*(P^{(1)}) \Psi_{i_3}^*(P^{(2)}) \Psi_{i_2}^*(P^{(3)}) \right. \\ & - \Psi_{i_2}^*(P^{(1)}) \Psi_{i_1}^*(P^{(2)}) \Psi_{i_3}^*(P^{(3)}) + \Psi_{i_2}^*(P^{(1)}) \Psi_{i_3}^*(P^{(2)}) \Psi_{i_1}^*(P^{(3)}) \\ & \left. + \Psi_{i_3}^*(P^{(1)}) \Psi_{i_1}^*(P^{(2)}) \Psi_{i_3}^*(P^{(2)}) - \Psi_{i_3}^*(P^{(1)}) \Psi_{i_2}^*(P^{(2)}) \Psi_{i_1}^*(P^{(3)}) \right) . \end{aligned} \quad (4.127)$$

where $\mathcal{A} = \Psi_{i_1}(P^{(1)}) \Psi_{i_2}(P^{(2)}) \Psi_{i_3}(P^{(3)})$. We may use the notation

$$\begin{aligned} |\Psi(P^{(1)})|^2 &= \Psi_{i_1}(P^{(1)}) \Psi_{i_1}^*(P^{(1)}) \\ &= \Psi_1(P^{(1)}) \Psi_1^*(P^{(1)}) + \Psi_2(P^{(1)}) \Psi_2^*(P^{(1)}) + \Psi_3(P^{(1)}) \Psi_3^*(P^{(1)}) \end{aligned} \quad (4.128)$$

and

$$\Psi_{i_1}(P^{(1)}) \Psi_{i_2}(P^{(2)}) \Psi_{i_2}^*(P^{(1)}) \Psi_{i_1}^*(P^{(2)}) = |\Psi(P^{(1)}) \Psi^*(P^{(2)})|^2 = |\Psi^*(P^{(1)}) \Psi(P^{(2)})|^2 .$$

From the conservation of probability, we have the identity

$$\int d\mu(i) |\Psi^{(i)}|^2 = 1 . \quad (4.129)$$

Using previous three successive equations obtain

$$\int d\mu(3) \Psi_{MP}^* \Psi_{MP} = \int_{\mathcal{M}} |\Psi_{MP}|^2 d\mu(3) = |\Psi^1|^2 |\Psi^2|^2 - |\Psi_{\Lambda}^{*1} \Psi_{\Lambda}^2|^2 . \quad (4.130)$$

We may immediately generalize the result for two-point correlation function for the \mathcal{N} -particle states

$$\Omega(1, 2) = \int_{\mathcal{M}} |\Psi_{MP}|^2 d\mu(3) d\mu(4) \dots d\mu(\mathcal{N}) = |\Psi^1|^2 |\Psi^2|^2 - |\Psi_{\Lambda}^{*1} \Psi_{\Lambda}^2|^2 . \quad (4.131)$$

Let us choose a non-homogeneous coordinate system as

$$\mathcal{P} = \frac{1}{\sqrt{1 + |\gamma_a|^2}} (1, \gamma_1, \dots, \gamma_5)^T := \frac{1}{\sqrt{1 + |\gamma_a|^2}} (1, \gamma), \quad (4.132)$$

where $\gamma_i := \frac{P_{\alpha\beta}}{P_{12}}$ and $P_{12} \neq 0$. Now, the LLL wave functions (4.121) take the form

$$\Psi_\alpha(P^i) \sim (\mathcal{P}_\alpha^i)^n. \quad (4.133)$$

We may use this expression to find (4.131) in the explicit form

$$\begin{aligned} \Omega(1, 2) &\sim 1 - (\mathcal{P}_\alpha^1 \mathcal{P}_\alpha^2)^n \\ &\sim 1 - \left[\frac{(1 - \gamma_a^{*1} \gamma_a^2)(1 - \gamma_b^1 \gamma_b^{*2})}{1 + |\gamma_a^1|^2 + |\gamma_a^2|^2 + |\gamma_a^1|^2 |\gamma_a^2|^2} \right]^n \\ &\sim 1 - \left[1 - \frac{|\gamma^1 - \gamma^2|^2}{1 + |\gamma_a^1|^2 + |\gamma_a^2|^2 + |\gamma_a^1|^2 |\gamma_a^2|^2} \right]^n. \end{aligned} \quad (4.134)$$

By defining a parameter $\mathbf{X} = \gamma\ell$ we may express the (4.134) in the form

$$\begin{aligned} \Omega(1, 2) &= 1 - \left[1 - |\mathbf{X}^1 - \mathbf{X}^2|^2 \left[\ell^2 + |\mathbf{X}^1|^2 + |\mathbf{X}^2|^2 + \ell^{-2} |\mathbf{X}^1|^2 |\mathbf{X}^2|^2 \right]^{-1} \right]^n \\ &\sim 1 - \left[1 - \frac{|\mathbf{X}^1 - \mathbf{X}^2|^2}{\ell^2 (1 + \ell^{-2} |\mathbf{X}^1|^2 + \ell^{-2} |\mathbf{X}^2|^2 + \ell^{-4} |\mathbf{X}^1|^2 |\mathbf{X}^2|^2)} \right]^n \\ &\sim 1 - \left[1 - \frac{1}{\ell^2} \left(|\mathbf{X}^1 - \mathbf{X}^2|^2 + \mathcal{O}\left(\frac{1}{\ell^2}\right) \right) \right]^n \\ &= 1 - \left[1 - \frac{2B}{n} \left(|\mathbf{X}^1 - \mathbf{X}^2|^2 + \mathcal{O}\left(\frac{1}{\ell^2}\right) \right) \right]^n \end{aligned} \quad (4.135)$$

In the thermodynamic limit $\mathcal{N} \rightarrow \infty$ and $n \rightarrow \infty$, this gives

$$\lim_{\ell \rightarrow \infty} \left\{ 1 - \left[1 - \frac{2B |\mathbf{X}^1 - \mathbf{X}^2|^2}{n} \right]^n \right\} \rightarrow 1 - \exp \left[-2B |\mathbf{X}^1 - \mathbf{X}^2|^2 \right] \quad (4.136)$$

as the probability of finding two particles at the positions \mathbf{X}^1 and \mathbf{X}^2 . However, \mathbf{X}^1 and \mathbf{X}^2 are on $\mathbb{C}P^5$, rather being on $Gr_2(\mathbb{C}^4)$. Using (4.139) we may show that the fifth component of the \mathbf{X}^i as

$$X_5^i := \ell(\gamma_2 \gamma_3 - \gamma_1 \gamma_4) := \ell \begin{pmatrix} \gamma_2^i & \gamma_1^i \\ \gamma_4^i & \gamma_3^i \end{pmatrix} := \ell \det \Gamma^i. \quad (4.137)$$

As a consequence, we may write $X_5^1 - X_5^2 = \ell (\det \Gamma^1 - \det \Gamma^2)$ and express (4.136) as

$$1 - e^{-2B |\mathbf{X}^1 - \mathbf{X}^2|^2} = 1 - e^{-2B (\mathbf{x}^1 - \mathbf{x}^2)^2} e^{-2B \ell^2 (\det \Gamma^1 - \det \Gamma^2)^2},$$

where \mathbf{x}^1 and \mathbf{x}^2 are coordinates on $Gr_2(\mathbb{C}^4)$. This result indicates that if the distance between two particles approaches zero, then the two-point function goes to zero as well and the probability of locating two particles at the same point approaches to zero, as expected.

Let us also discuss the $U(1)$ gauge field and the associated field strength in somewhat more detail. We may show the last column of the (4.103) as a column vector as $\tilde{\mathcal{P}} \equiv (P_1, \dots, P_6)^T$. We may also define the non-homogeneous coordinate chart on $Gr_2(\mathbb{C}^4)$ as $\mathcal{Q} \equiv \frac{\tilde{\mathcal{P}}}{P_1}$, $P_1 \neq 0$ in the form

$$\mathcal{Q} = (1, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5), \quad (4.138)$$

which is subject to Klein quadric equation (4.102) that can be written in terms of γ_i as

$$\gamma_5 = \gamma_2\gamma_3 - \gamma_1\gamma_4. \quad (4.139)$$

With these notations have the gauge potential

$$\begin{aligned} A &= -in\mathcal{P}^\dagger d\mathcal{P} \\ &= -inP_1^* \mathcal{Q}^\dagger (\mathcal{Q}dP_1 + P_1d\mathcal{Q}) \\ &= -in(|\mathcal{Q}|^2 P_1^* dP_1 + |P_1|^2 \mathcal{Q}^\dagger d\mathcal{Q}). \end{aligned} \quad (4.140)$$

From $P_1\mathcal{Q} = \mathcal{P}$ we may write $|\mathcal{Q}|^2|P_1|^2 = |\mathcal{P}|^2$. Using the fact that Plücker coordinates is normalized, this equation reduces to $|\mathcal{Q}|^2|P_1|^2 = 1$. By inserting this condition in (4.140) we may write

$$\begin{aligned} A &= -in|\mathcal{Q}|^{-2} \mathcal{Q}^\dagger d\mathcal{Q} - inP_1^* |P_1|^{-2} dP_1 \\ &= -in|\mathcal{Q}|^{-2} \mathcal{Q}^\dagger d\mathcal{Q} - inP_1^{-1} dP_1 \\ &= -in\partial \ln(|\mathcal{Q}|^2) - ind \ln(P_1) \\ &= -in\partial K - ind \ln(P_1), \end{aligned} \quad (4.141)$$

where K is the $\mathbb{C}P^5$ Kähler potential given by

$$K = \ln |\mathcal{Q}|^2 \equiv \ln(1 + |\gamma_i|^2). \quad (4.142)$$

It is worth to note that we may also express the field strength as

$$F = dA = -indP_N^* \wedge dP_N, \quad (4.143)$$

where we have used (4.105). We note that F is an antisymmetric, gauge invariant, closed 2-form on $Gr_2(\mathbb{C}^4)$ and proportional to the Kähler 2-form Ω over $Gr_2(\mathbb{C}^4)$. Using (4.141) we may write the field strength as

$$\begin{aligned} F &= dA = d(-in\partial K) \\ &= (\partial + \partial^*)(-in\partial K) \\ &= in\partial\partial^*K = n\Omega, \end{aligned} \quad (4.144)$$

where ∂ and ∂^* are the Dolbeault operators, $d = \partial + \partial^*$ and Ω is the Kähler two-form over $\mathbb{C}P^5$, which is proportional to F . By using (4.142) we may express the field strength as

$$\begin{aligned} F &= in\partial\partial^*\ln(1 + |\gamma_i|^2) \\ &= in\partial\left(\frac{\gamma_i}{1 + \gamma_i^*\gamma_i}d\gamma_i^*\right) \\ &= -in\left(\frac{d\gamma_i^* \wedge d\gamma_i}{1 + |\gamma|^2} - \frac{\gamma_i d\gamma_i^* \wedge \gamma_j^* d\gamma_j}{(1 + |\gamma|^2)^2}\right), \end{aligned} \quad (4.145)$$

which is subject to Klein quadric equation (4.139). We may rewrite the field strength tensor in the form

$$F = \frac{-in}{(1 + |\gamma|^2)^2} (\delta_{ij}(1 + |\gamma|^2) - \gamma_i^*\gamma_j) d\gamma_i^* \wedge d\gamma_j, \quad (4.146)$$

which allow us to write Kähler two-form as

$$\Omega = -iN_\gamma (\delta_{ij}(1 + |\gamma|^2) - \gamma_i^*\gamma_j) d\gamma_i^* \wedge d\gamma_j, \quad (4.147)$$

where $N_\gamma = (1 + |\gamma|^2)^{-2}$. We can impose the Klein quadric constraint (4.139) on (4.147). For instance, we have

$$\begin{aligned} d\gamma_5 &= (\partial + \partial^*)(\gamma_2\gamma_3 - \gamma_1\gamma_4) = \gamma_2 d\gamma_3 + \gamma_3 d\gamma_2 - \gamma_1 d\gamma_4 - \gamma_4 d\gamma_1 \\ d\gamma_5^* &= (\partial + \partial^*)(\gamma_2^*\gamma_3^* - \gamma_1^*\gamma_4^*) = \gamma_2^* d\gamma_3^* + \gamma_3^* d\gamma_2^* - \gamma_1^* d\gamma_4^* - \gamma_4^* d\gamma_1^* \end{aligned} \quad (4.148)$$

Now, we may write Ω_{11} by considering the contributions from Ω_{15^*} , Ω_{51^*} and Ω_{55^*} . For instance, contributions of Ω_{15^*} is

$$\Omega_{15^*} \rightarrow -iN_\gamma (\gamma_1\gamma_4^*\gamma_2^*\gamma_3^* - |\gamma_1|^2\gamma_4^*\gamma_4^*). \quad (4.149)$$

Summation of three contributions gives

$$\gamma_{11^*} = iN_\gamma [1 + |\gamma_2|^2|\gamma_3|^2 + (1 + |\gamma_4|^2)(|\gamma_2|^2 + |\gamma_3|^2 + |\gamma_4|^2)]. \quad (4.150)$$

Solving for Ω_{ii^*} gives

$$\Omega_{ii^*} = iN_\gamma \left(1 + \prod_{\alpha=1, \alpha \neq i, \hat{i}}^4 |\gamma_\alpha|^2 + (1 + |\gamma_{\hat{i}}|^2) \sum_{\alpha=1, \alpha \neq i}^4 |\gamma_\alpha|^2 \right), \quad (4.151)$$

where $\gamma_{\hat{i}}$ is the second factor which appears in (4.139) in the form $\gamma_i \gamma_{\hat{i}}$. For instance, 1 and 2 are dual to 4 and 3, respectively. We may follow the same steps to find the other components. After straightforward but somewhat long calculations we obtain

$$\begin{aligned} \Omega_{ij^*} &= -iN_\gamma \left(1 + |\gamma_{\hat{i}}|^2 + |\gamma_{\hat{j}}|^2 \right) \left(\gamma_i^* \gamma_j + \gamma_{\hat{i}} \gamma_{\hat{j}}^* \right), \quad i < j, \quad j \neq \hat{i} \\ \Omega_{i\hat{i}^*} &= -iN_\gamma \left(\gamma_i^* \gamma_{\hat{i}} \left(\sum_{\alpha=1}^4 |\gamma_\alpha|^2 - |\gamma_i|^2 - |\gamma_{\hat{i}}|^2 \right) - \frac{1}{2} (\gamma_i^*)^2 \prod_{j \neq i, \hat{i}} \gamma_j \gamma_{\hat{j}} - \frac{1}{2} (\gamma_{\hat{i}})^2 \prod_{j \neq i, \hat{i}} \gamma_j^* \gamma_{\hat{j}}^* \right), \end{aligned} \quad (4.152)$$

where summation over repeated indices is not implied and $i < \hat{i}$ is valid for the second expression.

The integral of F over non-contractible 2-surface Σ on $Gr_2(\mathbb{C}^4)$ gives [25]

$$\int_{\Sigma} \frac{F}{2\pi} = n, \quad (4.153)$$

which is the analogue of the Dirac quantization condition with $\frac{n}{2}$ identified as the magnetic monopole charge. As a final remark, we may discuss the generalization of the results for higher dimensions. For $Gr_2(\mathbb{C}^4)$ case, both vector potential A and field strength F are subject the Plücker relations

$$\gamma_{ik} \gamma_{jl} = \gamma_{ij} \gamma_{kl} - \gamma_{il} \gamma_{kj}, \quad 1 \leq i < k < j < l \leq 2(N-2), \quad (4.154)$$

where γ_i are the non-homogeneous coordinates as $\gamma_{ij} := P_{ij}/P_{12}$.

CHAPTER 5

CONCLUSION

In this thesis we have formulated the QHE on Grassmann manifolds. We have first discussed the QH system in $2D$ spatial geometries and written the single- and multiparticle-wave functions and described the incompressibility property of the QH liquid. Then, we have reviewed the Landau problem on two-sphere [3]. We have summarized the work of D. Nair and V.P. Karabali [5] since their work is the main guide during the preparation process of this thesis. For this purpose, we have given the short review of QHE on \mathbb{CP}^1 and \mathbb{CP}^2 .

In the last chapter we give our analysis results in a detailed manner. First, we have given the energy spectrum. Due to the pedagogical reasons, we first give the detailed analysis of Landau problem on $Gr_2(\mathbb{C}^4)$. In this part, we have given the energy spectrum under the three different gauge field backgrounds. We have defined the wave functions in terms of suitable subsets of the Wigner- \mathcal{D} functions. Next, we have generalized our results for $Gr_2(\mathbb{C}^N)$. Then, we have given the local forms of the wave functions and two-point functions in terms of Plücker coordinates, explicitly. We have shown that finite spatial density in case of $U(1)$ gauge field background is achieved in the thermodynamic limit without introducing infinite $SU(2)$ degrees of freedom. Calculating 2-point correlation function, we have shown that the probability of finding two particles at the same point is zero, which associate the incompressibility of the QH systems.

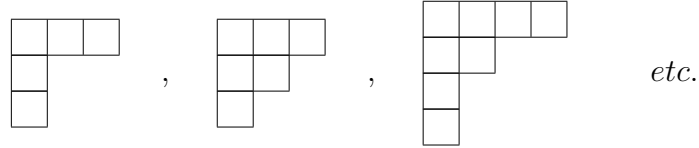
In this thesis we have not considered the formulation of branching rules of $Gr_2(\mathbb{C}^N)$ in case of $SU(N-2) \times SU(2)$ gauge field background. This point is an open question that one can try to formulate.

APPENDIX A

YOUNG TABLEAUX

The young tableaux is a technique to describe irreducible representations(IRR). It was first described by Alfred Young in 1900. Young tableaux can be extended for many kind of groups. However, In this part we will only deal with the $SU(N)$ with English convention.

In this technique every IRR is represented by a series of boxes. Some examples are:



We label each tableau with respect to number of repetitive columns which have same number of boxes. For example, in the first diagram there are 2 columns with 1 box, 0 column with 2 boxes and 1 column with 3 boxes. Then, we can label it by $(2, 0, 1, 0)$, say it is IRR of $SU(5)$ group. The rest of the tableaux can be labeled as $(1, 1, 1, 0)$ and $(2, 1, 0, 1)$.

Young tableaux technique has some common rules:

- I) From right to left, number of boxes on each column should not exceed the the boxes on preceding one. That means When we read the numbers on a label from left to right, each number should be less than or equal to the following one.
- II) For a generic IRR of $SU(N)$ group, the number of boxes on any column should be less than or equal to N .
- III) Dimension of the any IRR can be found from factor over hooks law [13].

The rule gives the dimension as follow: First label the left corner box as N . Then, with each box from left corner to the right increase the number by 1 as $N + 1, N + 2 \dots$. From left corner to the bottom decrease the number by 1 as $N - 1, N - 2 \dots$. Then, starting from the box labeled by $N - 1$ increase the number by 1 from left to right and do it for rest of the rows. Define a number H which is product of all numbers on the boxes. As an example, F for $(1, 1, 1)$ representation is:

$$F = N(N + 1)(N + 2)(N - 1)N(N - 2). \quad (\text{A.1})$$

Second, choose a generic box. Then, note all the boxes which are located on left or below of the generic box. Call the total number of boxes as h (include generic one). Do it for every box and call then as h . Finally, take the product of all h s and label this product as H . As an example, H for $(1, 1, 1)$ representation is 45. Dimension of the IRR $(1, 1, 1)$ may be written as $\frac{F}{H}$. For $(1, 1, 1)$ it is equal to

$$\dim(1, 1, 1) = \frac{F}{H} = \frac{N(N + 1)(N + 2)(N - 1)N(N - 2)}{45}. \quad (\text{A.2})$$

Young tableaux technique allows us to decompose a $SU(N + M)$ into $SU(N) \otimes SU(M)$ [13] [14]. Decomposition rules are based on the symmetry and antisymmetry properties on the Young Tableaux. Any pair of boxes which are located on a row are in symmetry relation relative to each other. If they are located on a columns they have antisymmetry relations. This allow us to represent the young tableaux in a simple tensor form. In this representation we will label the symmetric boxes with a letter. As an example the Young tableau representation of $(2, 2)$ is

$$\begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array} \quad (\text{A.3})$$

The corresponding tensor representation is as $\phi_{(A_1 A_2 A_3 A_4)(B_1 B_2)}$. During braching of the IRR of $SU(N + M)$, we pick a box from the young tableau of $SU(N + M)$ and place it to the young tableau of $SU(M)$ and the rest of the boxes of $SU(N + M)$ represent $SU(N)$. We will label the $SU(N)$ ($SU(M)$) boxes with upper case (roman letters). We can continue to that process until no box is left on $SU(N + M)$. During placement we should notice that any symmetric pair(antisymmetric) of boxes can not be placed in a column(row).

During the branching we write all possible representations and then eliminate the ones that disturbs that general rules and symmetry-antisymmetry properties.

As an example, the all possible branching of the $\phi_{(A_1 A_2 A_3 A_4)(B_1 B_2)}$ under the branching of $SU(4) \rightarrow SU(2) \otimes SU(2)$ is as:

$$\begin{aligned} \phi_{(A_1 A_2 A_3 A_4)(B_1 B_2)} \rightarrow & \phi_{(a_1 a_2 a_3 a_4)(b_1 b_2)} \oplus \phi_{(a_1 a_2 a_3)(\alpha_4)(b_1 b_2)} \oplus \phi_{(a_1 a_2)(\alpha_3 \alpha_4)(b_1 b_2)} \\ & \oplus \phi_{(a_1)(\alpha_2 \alpha_3 \alpha_4)(b_1 b_2)} \oplus \phi_{(\alpha_1 \alpha_2 \alpha_3 \alpha_4)(b_1 b_2)} \oplus \phi_{(a_1 a_2 a_3 a_4)(\beta_1 \beta_2)} \\ & \oplus \phi_{(a_1 a_2 a_3)(\alpha_4)(\beta_1 \beta_2)} \oplus \phi_{(a_1 a_2)(\alpha_3 \alpha_4)(\beta_1 \beta_2)} \oplus \phi_{(a_1)(\alpha_2 \alpha_3 \alpha_4)(\beta_1 \beta_2)} \\ & \oplus \phi_{(\alpha_1 \alpha_2 \alpha_3 \alpha_4)(\beta_1 \beta_2)} \oplus \phi_{(a_1 a_2 a_3 a_4)(\beta_1 \beta_2)} \oplus \phi_{(a_1 a_2 a_3)(\alpha_4)(\beta_1 \beta_2)} \\ & \oplus \phi_{(a_1 a_2)(\alpha_3 \alpha_4)(\beta_1 \beta_2)} \oplus \phi_{(a_1)(\alpha_2 \alpha_3 \alpha_4)(\beta_1 \beta_2)} \oplus \phi_{(\alpha_1 \alpha_2 \alpha_3 \alpha_4)(\beta_1 \beta_2)} \end{aligned}$$

In this branching fourth, fifth and tenth elements from the left disturb the branching rules, so we eliminate them. The branching in terms of the young diagram can be shown as:

$$\begin{aligned} \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & \\ \hline \end{array} \rightarrow & \left(\begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & \\ \hline \end{array} \otimes \cdot \right) \oplus \left(\begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline \end{array} \otimes \begin{array}{|c|} \hline \\ \hline \end{array} \right) \oplus \left(\begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array} \otimes \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array} \right) \\ & \oplus \left(\begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & \\ \hline \end{array} \otimes \begin{array}{|c|} \hline \\ \hline \end{array} \right) \oplus \left(\begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline \end{array} \otimes \begin{array}{|c|} \hline \\ \hline \end{array} \right) \oplus \left(\begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline \end{array} \otimes \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array} \right) \\ & \oplus \left(\begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array} \otimes \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array} \right) \oplus \left(\begin{array}{|c|} \hline \\ \hline \end{array} \otimes \begin{array}{|c|c|c|} \hline & & & \\ \hline & & & \\ \hline \end{array} \right) \oplus \left(\begin{array}{|c|} \hline \\ \hline \end{array} \otimes \begin{array}{|c|c|c|} \hline & & & \\ \hline & & & \\ \hline \end{array} \right) \\ & \oplus \left(\begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & \\ \hline \end{array} \otimes \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array} \right) \oplus \left(\begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline \end{array} \otimes \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array} \right) \oplus \left(\begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array} \otimes \begin{array}{|c|c|c|} \hline & & & \\ \hline & & & \\ \hline \end{array} \right) \\ & \oplus \left(\begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array} \otimes \begin{array}{|c|c|c|} \hline & & & \\ \hline & & & \\ \hline \end{array} \right) \oplus \left(\begin{array}{|c|} \hline \\ \hline \end{array} \otimes \begin{array}{|c|c|c|} \hline & & & \\ \hline & & & \\ \hline \end{array} \right) \\ & \oplus \left(\begin{array}{|c|} \hline \\ \hline \end{array} \otimes \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & \\ \hline \end{array} \right) \oplus \left(\cdot \otimes \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & \\ \hline \end{array} \right), \end{aligned}$$

where the first (second) box stands for $SU(N)$ ($SU(M)$) representation in every paranthesis. By using hooks law we can find the total dimension of the branching. As can be checked easily both the dimension of $SU(N + M)$ and the branching are equal to 126.

APPENDIX B

QUADRATIC CASIMIR C_2 AND DIMENSION OF $SU(N)$

The general form of the eigenvalue of the quadratic Casimir of $SU(N)$ [17] may be written as

$$\langle C_2 \rangle = \sum_{i=1}^N (\ell_i^k - \rho_i^k), \quad \rho_i = n - i, \quad \ell_i = m_i + n - i \quad (\text{B.1})$$

and

$$m_i = \lambda_i - \frac{\lambda}{N}, \quad \lambda = \sum_i^N \lambda_i \quad (\text{B.2})$$

in which λ_i represents number of the boxes at i^{th} row of the young diagram of $SU(N)$ IRR. Our most general $SU(N)$ representation is in the form

$$(P_1, P_2, 0, \dots, 0, P_{N-2}, P_{N-1}). \quad (\text{B.3})$$

When we adopt the labeling of [17] into our convention, we obtain

$$\begin{aligned} \mathcal{C}_2(P_1, P_2, 0, \dots, 0, P_{N-2}, P_{N-1}) &= \left(\frac{N-1}{2N}\right)P_1^2 + \left(\frac{N-2}{N}\right)P_2^2 + \left(\frac{N-2}{N}\right)P_{N-2}^2 \\ &+ \left(\frac{N-1}{2N}\right)P_{N-1}^2 + \left(\frac{N-2}{N}\right)P_1P_2 + \frac{2}{N}P_1P_{N-2} \\ &+ \frac{1}{N}P_1P_{N-1} + \frac{4}{N}P_2P_{N-2} + \frac{2}{N}P_2P_{N-1} \\ &+ \left(\frac{N-2}{N}\right)P_{N-2}P_{N-1} + \left(\frac{N-1}{2}\right)P_1 \quad (\text{B.4}) \\ &+ (N-2)P_{N-2} + \left(\frac{N-1}{2}\right)P_{N-1} + (N-2)P_2. \end{aligned}$$

The (p, q, r) representation of $SU(4)$ would seem not suitable for the general formula. $SU(N)$ representation requires at least 4 parameters to label IRRs. However, we label the IRR of $SU(4)$ with three parameter (p, q, r) . To overcome this mismatching we may split the q parameters into two pieces as

$$(p, q = q_1 + q_2, r) \longleftrightarrow (P_1, q_2, q_1, P_{N-1}), \quad (\text{B.5})$$

where splitting is controlled by the branching of $SU(4)$.

By using hook's law (described on Appendix A), dimension of the IRR given in (B.3) may be written as

$$\begin{aligned}
\dim(P_1, P_2, 0, \dots, P_{N-2}, P_{N-1}) &= \frac{1}{j} ((P_{N-2} + P_{N-1} + N - 3)!(P_{N-2} + N - 4)! \\
&\times (P_1 + P_2 + N - 3)!(P_{N-2} + P_{N-1} + P_2 + N - 2) \\
&\times (P_{N-1} + 1)(P_1 + P_2 + P_{N-2} + P_{N-1} + N - 1) \\
&\times (P_{N-2} + P_2 + N - 3)(P_1 + P_2 + P_{N-2} + N - 2) \\
&\times (P_1 + 1)(P_2 + N - 4)!), \tag{B.6}
\end{aligned}$$

where j is

$$j = (N-1)!(N-2)!(N-3)!(N-4)!P_2!P_{N-2}!(P_{N-2}+P_{N-1}+1)!(P_1+P_2+1)!. \tag{B.7}$$

APPENDIX C

COEFFICIENT OF R_{N^2-1}

For $Gr_2(\mathbb{C}^4)$ and $Gr_2(\mathbb{C}^N)$ we have used the relation between eigenvalues of the generators R_{15} , R_{N^2-1} and $U(1)$ charges. Now, we give a short derivation of this relation. We start by taking the trace normalization as $Tr(R_\alpha R_\beta) = \frac{1}{2}\delta_{\alpha\beta}$ for generators of $SU(N)$ in the N -dimensional defining representation.

It is known that any choice of trace normalization of the generators of $SU(N)$ in an IRR fixes the trace normalization in all other IRR's of $SU(N)$ as [18]

$$Tr(T_a^{(R)} T_b^{(R)}) = \kappa_{ab}, \quad (C.1)$$

where κ_{ab} is a rank-2 tensor invariant under $SU(N)$ transformations. We know that the only rank-2 invariant $SU(N)$ tensor is Kronecker delta, δ_{ab} . Then we may write

$$\kappa_{ab} = X_{(R)} \delta_{ab}, \quad (C.2)$$

where $X_{(R)}$ is the Dynkin index of the representation R of the group $SU(N)$ and given as [18]

$$X_{(R)} = \frac{\dim(R)}{\dim(SU(N))} \mathcal{C}_2(R). \quad (C.3)$$

We have that $\dim(SU(N))$ is equal to $N^2 - 1$ and \mathcal{C}_2^R is the quadratic Casimir of the IRR R given in (B.4). For either $(0, 1, 0, \dots, 0, 0)$ or $(0, 0, \dots, 1, 0)$ of $SU(N)$ IRR this gives

$$X_{(R)} = \frac{N-2}{N}, \quad (C.4)$$

and the (C.1) reads

$$Tr(T_a T_b) = \frac{N-2}{N} \delta_{ab}, \quad (C.5)$$

in either of the $\frac{N(N-1)}{2}$ -dimensional IRR. Our aim is to find the coefficient of R_{N^2-1} in these representations. In terms of the Young diagrams, the branching

of, say, $(0, 1, 0, \dots, 0, 0)$ representation under $SU(N-2) \times SU(2) \times U(1)$ gives

$$\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array} = \left(\cdot \otimes \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array} \right)_{-1} \oplus \left(\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array} \otimes \cdot \right)_{\frac{2}{N-2}} \oplus \left(\square \otimes \square \right)_{\frac{4-N}{2(N-2)}}, \quad (\text{C.6})$$

where the subscripts represent the $U(1)$ charge. Therefore we may write the R_{N^2-1} as

$$R_{N^2-1} = \zeta \text{diag} \left(\underbrace{\frac{N-4}{2(N-2)}, \dots, \frac{N-4}{2(N-2)}}_{2(N-2)}, \underbrace{\frac{-2}{N-2}, \dots, \frac{-2}{N-2}}_{\frac{(N-2)(N-3)}{2}}, 1 \right), \quad (\text{C.7})$$

where ζ represents the coefficient of R_{N^2-1} and the dimensions of the IRR in the branching (C.6) are given in the underbraces. Finally, using (C.7) in ((C.5)) gives

$$\zeta = \sqrt{\frac{N-2}{N}}. \quad (\text{C.8})$$

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