VALUE OF DISRUPTION INFORMATION IN AN EOQ ENVIRONMENT

A THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES
OF
MIDDLE EAST TECHNICAL UNIVERSITY

BY

DENİZ ESİN EMER

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR
THE DEGREE OF MASTER OF SCIENCE
IN
INDUSTRIAL ENGINEERING

SEPTEMBER 2014

Approval of the thesis:

VALUE OF DISRUPTION INFORMATION IN AN EOQ ENVIRONMENT

submitted by **DENİZ ESİN EMER** in partial fulfillment of the requirements for the degree of **Master of Science in Industrial Engineering Department, Middle East Technical University** by,

Prof. Dr. Canan Özgen
Dean, Graduate School of **Natural and Applied Sciences** ______________

Prof. Dr. Murat Köksalan
Head of Department, **Industrial Engineering** ______________

Assoc. Prof. Dr. İsmail Serdar Bakal
Supervisor, **Industrial Engineering Department, METU** ______________

Assoc. Prof. Dr. Zeynep Pelin Bayındır
Co-supervisor, **Industrial Engineering Department, METU** ______________

Examining Committee Members:

Prof. Dr. Yasemin Serin
Industrial Engineering Department, METU ______________

Assoc. Prof. Dr. İsmail Serdar Bakal
Industrial Engineering Department, METU ______________

Assoc. Prof. Dr. Zeynep Pelin Bayındır
Industrial Engineering Department, METU ______________

Assoc. Prof. Dr. Seçil Savaşaneril
Industrial Engineering Department, METU ______________

Assist. Prof. Dr. Benhür Satır
Industrial Engineering Department, Çankaya University ______________

Date: ______________

I hereby declare that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.

Name, Last Name: DENİZ ESİN EMER

Signature :

ABSTRACT

VALUE OF DISRUPTION INFORMATION IN AN EOQ ENVIRONMENT

EMER, DENİZ ESİN
M.S., Department of Industrial Engineering
Supervisor : Assoc. Prof. Dr. İsmail Serdar Bakal
Co-Supervisor : Assoc. Prof. Dr. Zeynep Pelin Bayındır

September 2014, 95 pages

Supply disruptions have important effects on supply chains causing serious financial and intangible damages. In this study, we consider an infinite horizon, continuous review inventory model with deterministic stationary demand where supply is subject to disruption. The supply process alternates between two states randomly: one in which it functions normally ("ON" period) and one in which it is disrupted ("OFF" period). Unsatisfied demand is backordered in off periods. In this setting we seek the value of disruption orders which can be placed at the beginning of OFF-periods with the same fixed cost. Utilizing renewal theory, we derive the total expected cost function and determine the cost minimizing regular order-up-to level and characterize the order-up-to level for disruption orders. We also conduct an extensive numerical analysis and compare the results with the model with no opportunity of disruption orders. We conclude that if the shortage cost is relatively high, placing disruption order always reduces or does not change the expected total cost and if disruption orders are placed then the regular order-up-to levels is generally so close to EQO with backorders.

Keywords: Supply disruption, Value of disruption order, Optimal policy

ÖZ

EKONOMİK SİPARİŞ MİKTARI ORTAMINDA KESİNTİ BİLGİSİNİN DEĞERİ

EMER, DENİZ ESİN
Yüksek Lisans, Endüstri Mühendisliği Bölümü
Tez Yöneticisi : Doç. Dr. İsmail Serdar Bakal
Ortak Tez Yöneticisi : Doç. Dr. Zeynep Pelin Bayındır

Eylül 2014 , 95 sayfa

Tedarik kesintileri ciddi finansal ve soyut sonuçlara yol açtığı için tedarik zincirlerinde önemli etkileri vardır. Bu çalışmada, rassal olmayan sabit talep, kesintiye uğrayan tedarik, sonsuz ufuk ve sürekli kontrol özelliğine sahip bir envanter modeli düşünülmüştür. Tedarik süreci iki durum arasında değişmektedir: ilkinde süreç normal işleyişini sürdürmektedir ("açık" zaman dilimi), diğerinde ise tedarik kesintiye uğramaktadır ("kapalı" zaman dilimi). Kapalı zaman diliminde karşılanamayan talep sonradan karşılanabilmektedir. Bu çalışmada kapalı zaman dilimlerinin başında aynı sabit maliyetle verilen bir kesinti siparişinin değerini araştırıyoruz. Toplam beklenen maliyeti yenileme teorisini kullanarak buluyor ve açık zaman dilimlerindeki işleyişte envanterin çıkarılacağı seviyeyi ve kapalı zaman diliminin başında envanterin kesinti siparişi ile çıkarılacağı seviyeyi maliyeti en azlayan şekilde hesaplıyoruz. Ayrıca geniş nümerik bir analiz yapıyor ve sonuçlarımızı kesinti siparişi içermeyen model sonuçları ile karşılaştırıyoruz. Sonuç olarak kesinti siparişi vermek ardısmarlama maliyetinin fazla olduğu durumlarda beklenen toplam maliyeti her zaman azaltıyor ya da değiştirmiyor ve eğer kesinti siparişi verilirse, açık zaman diliminde verilen siparişler genellikle ardısmarlamalı ekonomik sipariş miktarına çok yakın oluyor.

Anahtar Kelimeler: Tedarik kesintileri, Tedarik kesintisindeki siparişin değeri, Optimum politika

To my mother, father, brother

ACKNOWLEDGMENTS

I would like to thank my supervisor Assoc. Prof. Dr. İsmail Serdar Bakal and my co-supervisor Assoc. Prof. Dr. Zeynep Pelin Bayındır for their supports, and express my gratitude for their understanding and encouraging attitudes. I know "support or understanding" cannot correspond enough to what they did, and I know their tolerant is far more bigger than I have ever hoped for. The most precious thing I have learned from this thesis is an academic perspective which they tried to teach me throughout the study.

Additionally, I would express my gratitude to my parents for their endless love and being supportive everything I came through, I am grateful that they are my parents. I am thankful to my brother, Onur Emer for many things that cannot be sufficiently expressed here, the way who he is is inspired me all the time. Also, I would like to thank my sister-in-law for her love and understanding towards me that she makes me feel the moment we met.

Also, I would like to thank Çiğdem Renkli for her guidance, friendship, understanding; Derya Kılınç for being there whenever I need her, independent of our location on the earth; Derya Dinler for her support at the most needed times, friendship; and Ali Cem Randa for helping me out whenever I have a problem. Also, I am grateful to my colleague, Yasemin Limon for easing my job at the most busy times.

Finally, I would like to thank my students who make my life joyful for the past few years, also for their support for every step of this thesis and afterwards, I cannot forget support of my especially two students.

I would like to thank the examining committee members for their suggestions and contributions to my thesis.

This study was supported by the Scientific and Technological Research Council of Turkey (TÜBİTAK) Grant No: 2210.

TABLE OF CONTENTS

LIST OF TABLES

TABLES

LIST OF FIGURES

FIGURES

CHAPTER 1

INTRODUCTION

Supply uncertainty has a drastic effect on supply chains, causing high operating costs and low customer service level. Forms of supply uncertainty can be categorized as follows: Yield uncertainty (the quantity delivered by a supplier or produced by a manufacturing process is a random variable that depends on the order quantity), capacity uncertainty (the supplier's delivery capacity or the firm's manufacturing capacity is a random variable), lead time uncertainty (the order or processing lead time is stochastic), input cost uncertainty (the procurement costs incurred by the buyer is stochastic) and the supply disruptions that is the focus of this thesis.

Snyder et al. (2012) state that "Disruptions are random events that cause a supplier or other element of the supply chain to stop functioning, either completely or partially, for a (typically random) amount of time". It is commonly known that supply disruptions can be caused by several and serious reasons which Atasoy et al. (2012) categorize under two groups:

- Unpredictable disruptions: natural disasters, terrorist attacks, accidents, labor actions, breakdowns, transportation disruptions, order cancellations.
- Predictable disruptions: price inflation, capacity restrictions and scarcity of some resources at the supplier. In this case supplier may choose to allocate his restricted capacity to other manufacturers or products, or he cannot produce at all, leaving all of his customers not satisfied.

There are numerous examples of supply disruptions causing important effects on the buyer's costs. In some developing countries, the local utilities may cut off the supply at random times for random durations due to excessive demand for electrical energy (Parlar and Perry 1995). Nokia and Ericsson suffered a several-week disruption of computer chips that are used in mobile phones, which was caused by a lightning at one of Philips Electronic facilities in New Mexico. Ericsson lost 1.68 billion dollars for its mobile phone division and retreated from the phone handset production market in January 2001, while Nokia manage to cope with the effects of the disruption (Xia et al., 2004). Apple, Motorola and Sony also make public that they struggled problems with meeting their demands due to shortages of key components caused by unforeseen circumstances in 1995, 1999 and 2000, respectively (Hendricks and Singhal, 2003). Hendricks and Singhal (2005) report over 800 cases of disruptions in supply chains and conclude that "firms suffering from supply chain disruptions experience about 30% lower stock returns than their matched benchmarks".

The mitigation strategies against supply disruptions is similar to the ones against demand uncertainty. Tomlin (2006) discusses three categories of mitigation strategies for supply chain disruptions:

1. Inventory: Inventory control strategies involve ordering and stocking decisions.
2. Sourcing: Sourcing strategies are reactive to an actual shortage which can be categorized as follows:
    - Routine Sourcing: Firms regularly source raw materials from more than one supplier. The order quantities to the non-disrupted suppliers do not change after the disruption.
    - Contingent Rerouting: It is almost the same as routine sourcing. However, in the case of contingent sourcing, if one supplier faces disruption, the order from the non-disrupted suppliers can be changed from the pre-disruption levels.
    - Demand Substitution: If there is a stockout of one product, then the firm can offer superior product with the same cost, or inferior product with provided monetary compensation to the customer.
3. Acceptance: In the case that the cost of mitigating disruptions is too high to do it, firm can accept the risk and the financial consequences.

In this study, we consider an infinite horizon, continuous review inventory system subject to disruptions. Specifically, we model the supply process as a continuous-time Markov Chain alternating between ON and OFF-periods. If the supplier is in "ON" state, the order is shipped by the supplier. If the supplier is in "OFF" state when an order is placed by the buyer, no orders can be processed and the buyer has to wait until the supplier turns to "ON" state. Demand is deterministic; lead time is zero; the replenishments are instantaneous and complete. Planned backorders are not allowed. Only during OFF-periods, unsatisfied demand can be backordered. The cost components we consider are fixed order cost, inventory holding cost and back-order cost. In this setting, we aim to assess the value of an order opportunity that can be placed just before or whenever the supplier is disrupted. For this purpose, we consider an inventory policy consisting of two order-up-to levels: regular order-up-to level, Q, that is the order-up-to level for orders placed when the supplier is ON, and disruption order-up-to level, S, that is the order that may be placed at the beginning of an OFF-period. For ON-periods, planned backorders are not allowed and the buyer increase his inventory level to Q whenever it hits zero. In order to find the value of disruption order opportunity, we compare this model to the model that the buyer does not have any opportunity to place an order in the case of a disruption. This bench-mark model is first studied by Parlar and Berkin (1991) under the assumption that unsatisfied demand is lost. Note that we consider the backordering situation. The un-derlying analysis is corrected later by Berk and Arreola-Risa's erratum (1994). After characterizing the optimal order-up-to levels under two cases, whether the disruption order-up-to level is bigger than the regular order-up-to level, we conduct an extensive numerical analysis in order to find the value of disruption order and make sensitivity analysis0.

The remainder of this thesis is organized as follows: In Chapter 2, literature on supply disruptions and the value of disruption information is presented. No-order-opportunity model is discussed and analyzed in Chapter 3. Chapter 4 includes the analysis of order opportunity model. In Chapter 5, the results of the numerical study for models with and without order opportunity and their conclusions are presented. Finally, summary of the study and future research suggestions are given in Chapter 6.

CHAPTER 2

LITERATURE REVIEW

In this chapter, we review the literature ON inventory models under supply disruptions under three categories: Studies assuming deterministic demand are reviewed in Section 2.1. Then the ones assuming stochastic demand are reviewed in Section 2.2. Finally, in Section 2.3, studies ON the value of information, cooperation, coordination in supply chains with supply disruption are discussed.

Snyder et al. (2012) provide the most extensive review the OR/MS literature ON supply chain disruptions. They provide the forms of supply uncertainty, together with mitigation strategies. The studies in the literature are categorized under six categories: evaluating supply disruptions, strategic decisions, sourcing decisions, contracts and incentives, inventory and facility location. They conclude by specifying some topics that they believe those are important for future research.

The common assumption employed in the all reviewed studies in this chapter are;

- The planning horizon is infinite.
- There can be at most one single outstanding order.
- Replenishments are instantaneous and complete.

2.1 Disruption Models with Deterministic Demand

In this section studies that there is no uncertainty except disruption are reviewed. In all studies the lead time is assumed to be zero. The studies differ in the unsatisfied demand is whether backordered or not, and the cost components, that are taken into consideration. We first review the papers which consider continuous review policies and then the ones that consider periodic review.

The first study in the literature is conducted by Parlar and Berkin (1991). In this study the term EOQ with disruptions (EQOD) is first introduced. They consider an EOQ environment and assume that the unsatisfied demand is lost. The supply alternates between ON and OFF statuses whose durations are governed by general probability distributions. When the supplier is OFF, it is impossible to get an order from the supplier. They use renewal reward theorem to compute the average expected cost per unit time. The renewal cycle can be defined as one ON and one OFF-periods. In their numerical study, they assume that ON and OFF-periods lengths are exponentially distributed.

However, their model was shown to be incorrect by Berk and Arreola-Risa's erratum (1994) in two aspects. First, they point out that Parlar and Berkin (1991) implicitly assume that there is a stockout in every cycle (which is not the case especially if OFF-periods are much shorter than ON-periods). Hence the renewal cycle definition by Parlar and Berkin (1991) is invalid. Second, they pointed out that Parlar and Berkin (1991) derived the expected shortage cost per unit per time which is not appropriate in the case of lost sales. Berk and Arreola-Risa (1994) define a cycle as the time between receipts of successive orders; each cycle begins with exactly the same inventory level. Both ON and OFF-periods are exponentially distributed. Hence, cycles are independent and statistically identical. Therefore, they use Renewal Reward Theorem to derive the expected total cost function. Their expected cost function is quasiconvex, but the closed form formulation for the optimal order quantity cannot be derived. They conclude with sensitivity analysis of cost and noncost parameters and observe the following: *i*) The optimum order size is nondecreasing in the fixed ordering cast, the backorder cost, and demand, *ii*) The optimum order size is nondecreasing in the ratio of exponential ON-period rate over exponential OFF-period, *iii*) The optimum order size appears to exhibit concave behavior in the expected duration of OFF-period.

Later, Snyder (2008) approximates Berk and Arreola-Risa's (1994) cost function assuming that ON-periods last longer on the average than OFF-periods in the same environment. He shows classical EOQ cost function is a lower bound and a cost function under no order policy where every demand is lost is an upper bound. He approximate by ignoring the transient nature of the system. His approximation works under any distribution for ON and OFF-periods lengths as long as the system reaches steady state. So, he can work with the steady state probability of stockout for lost sales. The expected total cost function in the steady state is convex in order quantity and a closed form solution for the optimal order quantity is provided. It is proved that the optimal order quantity and the expected total cost is always higher in EOQD and the expected total cost function in the steady state approaches the classical EOQ cost function in the limit as ON-periods are much longer than OFF-periods.

Qi et al. (2009) consider disruptions at the supplier as well as at the retailer. If disruption occurs at the retailer, then it loses its all ON-hand inventory. They define a four-state continuous-time Markov chain to model ON and OFF states at the retailer and the supplier. The closed form solution for the optimal order quantity cannot be found, they present a computational study. Their results hold for both unit backorder cost occurring and lost sales case. They provide a tight approximation for the expected total cost as Snyder (2008) which is convex in order quantity and a closed form solution for the optimal order quantity is given. They conclude that the disruption at the retailer have bigger impact on the expected total cost than the disruption at the supplier.

It is known that placing an order when the inventory level hits zero (ZIO) is optimal under the classical EOQ environment. Yet, it may not be the case in EOQD. Parlar and Perry (1995) study an EOQD model treating the reorder point as a nonnegative decision variable. They use three decision variables for the reorder point, the order quantity when the supplier's state is ON and how long to wait before the next order if the supplier's state is OFF. ON and OFF-periods are exponentially distributed. They use renewal reward processes to derive average cost function. They deal with two cases; deterministic and random yield (the amount of order which is received) and provide numerical examples. They observe the order-up-to level is always monotone increasing in fixed cost, both unit backorder cost and unit backorder cost per time, the duration of ON-period, and monotone decreasing in holding cost and the expected

OFF-period's length. In addition, they observe that the optimal order-up-to levels are quite insensitive to changes in both unit backorder cost and unit backorder cost per time. Parlar and Perry (1996) continues their studies for single, two and multiple suppliers each having independent and exponentially distributed ON and OFF times. In the case of more than one supplier, buyer can choose any of them and any combination of them to place an order since the cost structure is the same for all suppliers. In the case of two supplier, they construct a four-state continuous time Markov chain and provide numerical examples. They conclude that as the number of supplier increases, the objective function of the multiple supplier converges to the classical EOQ model with no disruption.

Gürler and Parlar (1997) consider a continuous review model with two suppliers where the suppliers' ON-periods follow Erlang distribution and OFF-periods follow a general distribution. They model availability of the suppliers as a semi-Markov process. An order is placed at either of the two suppliers, when the inventory level drops to the reorder point. If none of them is available then the buyer faces backorders and backorder cost is incurred per unit. The other cost components they take into consideration are fixed ordering cost per order and inventory holding cost per unit per time. The resulting process is non-Markovian, so they transform the process into a Markovian one by employing method of stages, in other words they transform a nonexponential random variable into one that is the sum of a finite number of exponential random variables. They conclude that in several examples they conduct, the state-dependent inventory policy dominates (Q,R) policy.

Heimann and Waage (2007) extend the approximate model of Synder (2008) to the case where nonzero reorder points are allowed. They provide a closed form approximation for EOQD with non-zero reorder points and bounds for the optimal order quantity and the optimal reorder point. They conclude that the approximate optimal policy is better than the optimal ZIO policy by an average of 8.5% ON their test instances.

In addition to continuous review inventory models, periodic review models are studied in the literature as well. Moinzadeh and Aggarwal (1997) study the impact of random disruptions in a bottleneck production facility. In their model, production and demand rates are deterministic and identical in each period, ON-period durations have exponential distribution, OFF-periods are constant, there is a setup time and the unsatisfied demand is backordered. They propose an (s, S) policy where S is the order-up-to level and s is the threshold value for reorder. They develop a procedure to find the optimal solution and its cost components are setup cost, holding cost per unit per time and backorder cost per unit. They investigate behavior of the average cost function with respect to changes in the reliability and other parameters. They propose a heuristic method to find a near optimal policy and provide numerical examples. They conclude that the importance of setup cost changes according to reliability of the production system, as the reliability of the system is increasing so as the setup cost. In Just-in-time systems safety stock levels gain more importance in the case of disruption.

Güllü et al. (1997) also consider a periodic review inventory system where demand is deterministic but dynamic over the planning horizon and the unsatisfied demand is backordered. They assume that in any period the supply is either fully available or completely unavailable, demands are nonstationary and independent from one period to another. The supply is assumed to follow a Bernoulli process. They show that an order-up-to level policy is optimal and obtain a newsboy-like formula that determines the optimal order-up-to levels if there is no fixed cost. It turns out that the optimal order-up-to level for a period is the cumulative demand of all the previous periods (including the demand of that period). If there is a fixed cost of ordering, then (s, S) policy is optimal and the optimal S becomes the cumulative demand of previous periods.

Argon et al. (2001) study a periodic review inventory model with deterministic demand but the demand is affected by the previous periods' backorders. In other words; there occurred any backorder in the previous period, demand takes one value, and if the demand is fully satisfied at the previous period, demand takes another value. Supply availability follows a geometric distribution. There is no lead time. Their objective is to maximize the expected profit per period and it turns out that the objective function is neither convex nor concave, so they provide numerical procedure to obtain order-up-to levels.

2.2 Disruption Models with Stochastic Demand

In this section, we review studies assuming stochastic demand and zero lead time unless otherwise is stated. We begin with continuous review models and then consider periodic review models.

Bar-Lev et al. (1993) appear to be the first ones to study stochastic demand under supply disruption. They extent the EOQD model of Parlar and Berkin (1991). The inventry level process is modeled as a Brownian motion with negative drift, that is customers can return items. The system has a finite capacity and when the inventory level reaches that capacity no return is accepted. They derive the expected average cost function by using the renewal theory and they minimize it by a numerical procedure.

Kalpakam and Sapna (1997) study EOQD by adding procurement costs per unit ordered. Demand follows a general distribution, the unsatisfied demands are lost. They define a two state Markov chain for availability of the supplier. The availability of the supplier follows exponential distribution. They derive the expected total cost and propose an algorithm to compute s and S values for an (s, S) policy, and they illustrate their algorithm by providing numerical examples.

Liu and Cao (1999) analyze a similar problem to Moinzadeh and Aggarwal (1997) reviewed in Section 2.1. The demand in a period follows compound Poisson with general demand-size distribution. Unmet demand is backordered. They specify a limit to the production since a storage facility exists in the system, when inventory level hits that limit production stops. So, they defines a decision variable to control a machine's setup. Also, the machine faces random failures which must be repaired to make it operational again. They obtain a cost function using inventory holding cost per time unit, backorder cost per time unit and fixed setup cost as cost components for the case of exponentially distributed demand. Numerical examples are provided to illustrate the relationship between the cost and noncost parameters.

Gupta (1996) considers stochastic demand as well but, unlike Bar-Lev et al. (1993), he relaxes the ZIO assumption. Demand per period has Poisson distribution and ON and OFF-periods are exponential. Unmet demand is lost. He investigates both zero and nonzero lead time cases. The inventory policy is a continuous review (Q, r) policy. He concludes with a numerical analysis indicating that ignoring disruptions when choosing policy parameters can cause high costs, especially in an environment in which disruptions are long and stockout cost is high.

Mohebbi (2003) extends Gupta's model (1996) by considering both compound Poisson demands and Erlang distributed lead times. Unmet demand is lost and ON and OFF-periods are distributed exponentially. He provides the stationary distribution of the inventory level and the cost rate function for the case where demand is exponentially distributed. In a following paper, Mohebbi (2004) relax the assumption that ON and OFF-periods follow exponential distribution and assume that ON-periods are distributed generally and OFF-periods follow a hyperexponential distribution. So, he can assume that availability of the supplier can be modeled as an alternating renewal process. Demands follow again compound Poisson distribution and lead times are hyperexponentially distributed.

Parlar (1997) considers a continuous review stochastic inventory problem with random demand, random lead times and generally distributed ON and OFF-periods. Unmet demand is backordered. The supplier availability is modeled as a semi-Markov process. While the supplier is at its ON period, he uses an approximate version of (Q, r) policy with backorders. When supplier is OFF, then even if the buyer places any order, he still has to wait until the OFF-period is over. By the use of this model, the ordering quantity becomes a random variable, since if an order is placed during an OFF period, then the quantity depends on the inventory level at the end of that OFF-period. He develops the average cost function for k-stage Erlang distributed ON-periods and general OFF-periods by identifying the regenerative cycles and applying the renewal reward theorem. An algorithm is proposed to find the optimal Q and r, and its convergence properties are analyzed. Also, numerical examples are provided to illustrate the results.

Mohebbi and Hao (2006) consider a model under compound Poisson demands and Erlang distributed lead times. Unmet demand is lost. They define a two-state Markov chain to describe the supplier's availability and derive the stationary distribution of the inventory level under (Q, r) policy. Then they find the exact expression for the average cost per unit time and provide numerical examples to illustrate the effects of the expected ON and OFF-periods lengths, changes of lead time and other parameters. They conclude that the average cost and the stockout probability increase with lead time variability. The same problem is also investigated by Mohebbi and Hao (2008). They change the assumption that the supplier definitely produce the order placed at its ON-period, instead they assume the supplier can stop the production if its state turns from ON to OFF-period. In this study, they assumed that the partial production is lost, not preserved until ON-period. Their numerical examples show that there is a significant effect of the supplier's availability and the shape of Erlang lead time distribution ON the expected total cost. The optimal order quantity, average cost, and stockout probability under the optimal order quantity are larger than the ones reported by Mohebbi and Hao (2006).

Arreola-Risa and De-Croix (1998) study a model with Poisson demands, exponential ON and OFF-periods and no lead time. They consider partial backorder situation, a certain fraction of demands that occurred during OFF-period is backordered and the remaining fraction is lost. They consider an (s, S) policy to derive an exact closed form of cost function and propose an algorithm to find the optimal s and S values. Then, they investigate the durations of disruptions and variations of the fractions that are backordered on the expected cost function. They conclude that the order-up-to level move in the same direction as the change in expected cost per demand arriving during a stockout.

There are also studies that consider periodic review models with stochastic demand in the literature. Parlar et al. (1995) consider a finite horizon model with random demand and no lead time where the fulfillment of an order depends on whether the supply was available in the previous period and whether the supply is available or not is assumed to be known by now. ON and OFF-periods are both geometrically distributed. They consider two setup costs; one when an order is placed, and a secondary setup cost when an order is filled. They derive the expected total cost function and besides two setup cost, they use unit purchasing cost, holding cost per unit per time and backorder cost per unit as the cost components. They prove that the (s, S) policy is optimal, where s depends ON the state of the supplier in the last period, while S does not.

Song and Zipkin (1996) study a periodic review inventory model. Lead time changes independently of demands and orders from period to period. The unmet demand is backordered. Although lead times vary within the planning horizon, they assume that the information about the current and future lead times is known. In fact they investigate the importance of this information, which turns out to be valuable. They use dynamic programming in order to find the minimum expected cost. They conclude that if there is no fixed cost, an order-up-to level policy is optimal, otherwise (s, S) policy is optimal, and both policies are state dependent.

Özekici and Parlar (1999) consider infinite planning horizon and periodic-review inventory models with varying demand, supply and cost parameters from period to period. The demand distributions and cost parameters are assumed to be known until end of the planning horizon. If there is no fixed cost then environment-dependent order-up-to level is proved to be optimal. When they analyze the model with fixed ordering cost, the optimal policy is environment-dependent (s, S) policy.

Li et al. (2004) study a periodic review model with identical and independent random demand over periods. ON and OFF-periods have general distribution and availability of the supply is modeled as an alternating renewal process. The buyer knows the state of the supplier at the beginning of each period. They discuss both lost sales and backorder cases. They use purchasing cost per unit ordered, holding cost per unit per time and backorder cost per unit as a cost component. They use dynamic programming to find the minimum expected cost. They let demand can be dependent on the state of the supplier, by relaxing the assumption of independent and identically distributed demand. They show that the optimal ordering policy is a state-dependent order-up-to level policy. They suggest that "end-of-cycle" inventory return contract (which guarantees that the firm can return to the supplier any unwanted inventory at the purchasing cost at the end of an OFF-period) and show that it may be mutually beneficial to both the firm and the supplier.

2.3 Effects of Information/Cooperation/Coordination

Saghafian and Van Oyen (2008) discuss the effect of contracting with a secondary flexible backup supplier in the case supply disruption risk and closely monitoring primary suppliers to get disruption risk information. They consider two products, their demands are stochastic and the unsatisfied demand are lost. They suggest that these strategies can increase flexibility and responsiveness of supply chains in the case of disruptions. Furthermore, monitoring suppliers allows firms to anticipate potential disruptions. They quantify the value of purchasing optimal size of backup capacity, to do that the buyer utilizes a resource option subject to the limited quantity of a capacity reserved a priori via a contract with a flexible backup supplier. This reservation costs to the buyer per unit capacity that reserved. They take into consideration that holding cost, lost sale penalty cost and unit purchasing costs, different for both products, as the cost components. Also they consider the revenue per unit of product sold. They conclude that investing in a secondary flexible backup capacity can be harmful if the current information about the risk of primary suppliers is not perfect. Purchasing flexible backup capacity can be more beneficial if the buyer observes unreliable suppliers to make better risk estimates.

Atasoy et al. (2012) consider a system where the supplier has a nonstationary stochastic availability. However the supplier can predict its near future disruptions based on factors, such as her pipeline stock information, production schedule, seasonality, contractual obligations, and noncontractual preferences regarding other buyers. This information is shared with the buyer. Demand of the buyer is deterministic but nonstationary over time. They investigate two cases, with or without fixed ordering cost. They use dynamic programming to minimize the total cost of the buyer. For the case with no fixed cost, the order-up-to level strategy turns out to be optimal. For the case with fixed cost, state dependent (s, S) policy is optimal but the burden of the dynamic programming is too hard to calculate s, S values. Hence they suggest a heuristic algorithm for finding a good ordering strategy. They conclude by providing numerical examples to illustrate the value of advanced supply information.

There are few papers that models how contractual relationships and risk sharing can mitigate disruption risk. Gurnani and Shi (2006) consider a case where an unreliable supplier estimates his availability to meet the order. At the same time the buyer estimates that as well and he can overtrust or undertrust the supplier. They assume that these estimations can change from period to period. They derive the Nash bargaining solution. They discuss two contracts; the role of using a down-payment or nondelivery penalty in the contract. In the down-payment contract the buyer pays some amount of money to the supplier where the payment is the initial upfront portion of the total amount due. In the nondelivery penalty contract, there is a penalty cost when the supplier cannot deliver the buyer's order. For the case of overtrust, if the supplier's estimate is public, they conclude that a down-payment contract maximizes profits. Otherwise, a nondelivery contract is shown to be efficient and incentive compatible. For the case of undertrust, both the supplier and the buyer do not want to include down-payments in the contract and with the public information, a nondelivery penalty contract maximize profits. If estimates are private information, profits are maximized only if both can estimate accurately.

Yang et al. (2008) study a buyer that faces a supplier with private information about supply disruptions. In the case of disruption, the supplier chooses to either pay a penalty to the buyer for the lacking order or use backup production to fill the buyer's order. The buyer offers to the supplier to choose among them based on its own risk level. They conclude that asymmetric information causes the less reliable supplier to stop using backup production, also the buyer may stop ordering from the less reliable supplier. If the information is shared with the buyer, the value of supplier backup production for the buyer is not necessarily larger. Also, they claim that the value of information increases as suppliers become uniformly more reliable.

CHAPTER 3

NO-ORDER OPPORTUNITY MODEL

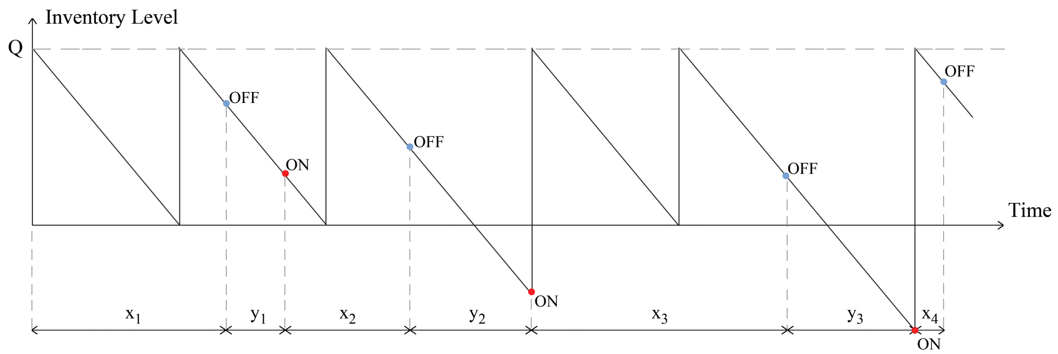
In this chapter, we consider the model where the buyer does not have the opportunity to place an order when the supplier is disrupted. We consider an environment where demand is deterministic, there is no lead time and replenishments are instantaneous and complete. The lengths of ON and OFF-periods are independent and both follow exponential distribution. The supplier alternates between ON and OFF states. The states of the supplier can be modeled as a two-state CTMC. The ordering policy is as follows: When the supplier is ON, the buyer orders Q units whenever the inventory level hits zero and no order can be placed during OFF-periods. If the inventory level is positive when the supplier turns into ON state, the buyer does not place any order. The unsatisfied demand is backordered during OFF-periods. If the inventory level is non-positive when the supplier becomes ON, an order is placed to increase the inventory level up to Q. We consider fixed order cost per order, inventory holding cost per unit per time and backorder cost per unit per time as the cost components. The objective is to find the optimum Q that minimizes the expected total cost per unit time. We derive the expected total cost function per unit time and find the optimal order-up-to level which minimizes the objective function. The notation used is given in Table 3.1.

A sample realization of the inventory process under the given policy can be seen in Figure 3.1. Each order brings inventory level to Q. If a disruption occurs before the inventory level hits zero, the buyer may face backorders depending on the duration of the OFF-period.

Table 3.1: Notations

Q : regular order-up-to level
X : random variable representing the duration of an ON-period, $X \sim Exp(\lambda)$
x_i : i^{th} realization of random variable X
Y : random variable representing the duration of an OFF-period, $Y \sim Exp(\mu)$
y_i : i^{th} realization of random variable Y
$T(Q)$: random variable representing time between receipts of two successive orders for a given $Q \geq 0$
β : probability that a stockout occurs in a cycle
I : Inventory level
D : demand rate
K : fixed order cost per order
h : unit inventory holding cost/unit time
b : unit shortage cost/unit time
$P(Q)$: fixed order cost per cycle for a given $Q \geq 0$
$H(Q)$: inventory holding cost per cycle for a given $Q \geq 0$
$B(Q)$: shortage cost per cycle for a given $Q \geq 0$
$C(Q)$: total cost per cycle for a given $Q \geq 0$
$G(Q)$: total cost per unit time for a given $Q \geq 0$

Figure 3.1 Inventory Level vs. Time: No-order Opportunity Model



In this setting, let a cycle be the time between two successive orders. We will proceed by showing that the cycle-lengths form a sequence of independent random variables with a common distribution. As a result, we will apply Renewal Reward Theorem to obtain the expected total cost per unit time.

3.1 Distribution of Cycle Length

In this section, we derive the distribution of cycle-lengths and observe that they are independent and have a common distribution.

Note that a cycle starts with Q units of inventory. It will end when the inventory level hits zero if the supplier is ON at that time. If the supplier is OFF when the inventory level hits zero, the cycle will continue until the supplier is ON again. By the memoryless property of exponential distribution, this will happen after an exponentially distributed random amount of time. Without loss of generality, assume that a cycle starts at time $t = 0$. Then,

$$T = \begin{cases} \frac{Q}{D} & \text{if the supplier is ON at time } t = Q/D \\ \frac{Q}{D} + Y & \text{if the supplier is OFF at time } t = Q/D \end{cases}$$

Hence, we first obtain the probability that the supplier is OFF at time $\frac{Q}{D}$, β. For this purpose, we observe that the state of the supply process can be modeled as a continuous-time Markov Chain with two states, $Z(t) \in \{0, 1\}$ for $t \geq 0$. If the supplier is ON (OFF) at time t, then $Z(t) = 1$ $(Z(t) = 0)$. From the transient analysis of this

CTMC (Kulkarni, 2011, pg:100), we get;

$$P\{Z(t) = 0|Z(0) = 1\} = \frac{\lambda}{(\lambda + \mu)}\left(1 - e^{-(\lambda+\mu)t}\right).$$

Then,

$$\beta = P\left\{Z\left(\frac{Q}{D}\right) = 0\middle|Z(0) = 1\right\} = \frac{\lambda}{(\lambda + \mu)}\left(1 - e^{-(\lambda+\mu)\frac{Q}{D}}\right).$$

Therefore,

$$P\left(T > \frac{Q}{D}\right) = \beta.$$

To find the cumulative distribution of T we have;

$$P\left\{T > t + \frac{Q}{D}\middle|T > \frac{Q}{D}\right\} = \frac{P\left\{T > t + \frac{Q}{D}\right\}}{P\left\{T > \frac{Q}{D}\right\}} = P\{Y > t\} = e^{-\mu t},$$

$$P\left\{T > t + \frac{Q}{D}\right\} = \beta e^{-\mu t}.$$

Let $t' = t + \frac{Q}{D}$, $t > 0$. Then,

$$\begin{aligned} P\left\{T > t'\right\} &= \beta e^{-\mu\left(t' - \frac{Q}{D}\right)} \\ P\left\{T \leq t'\right\} &= 1 - \beta e^{-\mu\left(t' - \frac{Q}{D}\right)}. \end{aligned}$$

For $t' < \frac{Q}{D}$, we know that $P\left\{T < t'\right\} = 0$ since each cycle will continue at least until Q units of inventory are depleted.

For the case $t' = \frac{Q}{D}$, we know that $P\left\{T = \frac{Q}{D}\right\} = 1 - \beta$ since for the cycle to last exactly $\frac{Q}{D}$ units of time, the supplier's state should be ON at $\frac{Q}{D}$.

Now, we can derive the probability function of T as follows:

$$f_T(t) = \begin{cases} 0 & \text{if } t < Q/D \\ 1-\beta & \text{if } t = Q/D \\ \beta\mu e^{-\mu\left(t-\frac{Q}{D}\right)} & \text{if } t > Q/D \end{cases} \tag{3.1}$$

Note that this function is valid for any cycle. Hence, we can deduce that the cycle-lengths are independent and identically distributed and Renewal Reward Theorem can be applied to find the expected cost per unit time.

3.2 Expected Cost Calculations

By the Renewal Reward Theorem, the expected cost per unit time can be calculated as,

$$E[G(Q)] = \frac{E[C(Q)]}{E[T]}.$$

In order to determine the total expected cost per cycle we need to determine the expected fixed order cost, the expected inventory holding cost and the expected shortage cost over a cycle. Since every cycle the buyer places exactly one order, we have,

$$E[P(Q)] = K.$$

In every cycle, Q units are depleted in $\frac{Q}{D}$ time units independent of everything else since the demand rate is constant (see Figure 3.1). Hence the inventory holding cost per cycle can be characterized as

$$E[H(Q)] = h\frac{Q^2}{2D}.$$

Whether there is a backorder occurrence in a cycle or not depends ON whether there is a disruption in a cycle and if there is, its length. If there is no disruption in a cycle, or if there is a disruption but the supplier's state turns ON again before the inventory level hits zero, the buyer does not face any backorders. ON the other hand, OFF-periods may last long enough to have backorders, as illustrated in the third cycle in Figure 3.1.

Therefore the expected shortage cost per cycle is calculated as follows:

$$\begin{aligned} E\left[B\left(Q\right)\right] &= b\left(1-\beta\right)0 + b\int_{\frac{Q}{D}}^{\infty} \frac{(t-Q/D)(tD-Q)}{2} f_T(t)\mathrm{d}t \\ &= b\int_{\frac{Q}{D}}^{\infty} \frac{(t-Q/D)(tD-Q)}{2} \beta\mu \mathrm{e}^{-\mu(t-\frac{Q}{D})}\mathrm{d}t = b\frac{\beta D}{\mu^2}. \end{aligned} \tag{3.2}$$

Equation 3.2 can be explained as follows: With probability $1-\beta$ there is no stockout in a cycle. ON the other hand, if there is a stockout in a cycle, the stockout begins at $\frac{Q}{D}$. If the duration of the cycle is $t > \frac{Q}{D}$ then the corresponding backorder cost is $b\frac{(t-Q/D)(tD-Q)}{2}$. Considering the distribution of the cycle length (Equation 3.1), we get Equation 3.4. We assume that the unit backorder cost incurs per unit per time, so we calculate the area of the triangle where the inventory level is negative to find the backorder cost incurred per cycle.

So the expected total cost per cycle is;

$$E\left[C\left(Q\right)\right] = K + h\frac{Q^2}{2D} + b\frac{\beta D}{\mu^2}.$$

The expected cycle length is;

$$E[T(Q)] = (1-\beta)\frac{Q}{D} + \beta\int_{\frac{Q}{D}}^{\infty} t\mu \mathrm{e}^{-\mu(t-\frac{Q}{D})}\mathrm{d}t = \frac{Q}{D} + \frac{\beta}{\mu}.$$

Therefore the expected total cost per unit time is;

$$E[G(Q)] = \frac{K + h\frac{Q^2}{2D} + b\frac{\beta D}{\mu^2}}{\frac{Q}{D} + \frac{\beta}{\mu}} = \frac{K + h\frac{Q^2}{2D} + b\frac{\lambda}{\mu^2(\lambda+\mu)}\left(1-\mathrm{e}^{-(\lambda+\mu)\frac{Q}{D}}\right)D}{\frac{Q}{D} + \frac{\lambda}{\mu(\lambda+\mu)}\left(1-\mathrm{e}^{-(\lambda+\mu)\frac{Q}{D}}\right)}. \tag{3.3}$$

$E[G(Q)]$ is similar to the expected cost function in Berk and Arreola-Risa (1994). The only difference is the shortage cost per cycle term: It is $\frac{b\beta D}{\mu^2}$ in our case and $\frac{b\beta D}{\mu}$ in Berk and Arreola-Risa (1994). Since $1/\mu$ is a constant, we can carry out a similar analysis as they did to obtain the optimal order-up-to level.

Proposition 3.2.1 *(a) $E[T(Q)]$ is concave in Q;*
(b) $E[C(Q)]$ is concave in Q for $0 < Q < Q_1$, and convex in Q for $Q > Q_1$, where $Q_1 = \frac{D}{\lambda+\mu} \ln\left[\frac{b\lambda(\lambda+\mu)}{h\mu^2}\right]$

Proof (a)

$$\frac{\mathrm{d}^2 E[T(Q)]}{\mathrm{d}Q^2} = \frac{-\lambda(\lambda+\mu)}{D^2} \mathrm{e}^{-(\lambda+\mu)\frac{Q}{D}} < 0.$$

Hence, E[T(Q)] is concave.

(b)

$$\frac{\mathrm{d}^2 E[C(Q)]}{\mathrm{d}Q^2} = \frac{h}{D} - \frac{b\lambda(\lambda+\mu)}{\mu^2} \mathrm{e}^{-(\lambda+\mu)\frac{Q}{D}}$$

$$\mathrm{e}^{-(\lambda+\mu)\frac{Q}{D}} = \frac{h\mu^2}{b\lambda(\lambda+\mu)}$$

Therefore, $\frac{\mathrm{d}^2 E[C(Q)]}{\mathrm{d}Q^2} < 0$, in other words, E[C(Q)] is concave for $Q < Q_1 = \frac{D}{\lambda+\mu} \ln\left[\frac{b\lambda(\lambda+\mu)}{h\mu^2}\right]$. And, $\frac{\mathrm{d}^2 E[C(Q)]}{\mathrm{d}Q^2} > 0$, in other words, E[C(Q)] is convex for $Q > Q_1 = \frac{D}{\lambda+\mu} \ln\left[\frac{b\lambda(\lambda+\mu)}{h\mu^2}\right]$.

Proposition 3.2.2 *(a) $E[G(Q)]$ is pseudoconvex in Q for $Q > Q_1$;*
(b) $E[G(Q)]$ is unimodal in Q;
(c) $E[G(Q)]$ attains its minimum at $Q = Q^*$, where Q^* solves the following equation:

$$\begin{aligned} &- K\mu + \frac{hQ^2\mu}{2D} + \frac{hQ\lambda}{\lambda+\mu} - \frac{bD\lambda}{\mu(\lambda+\mu)} \\ &+ \left(-K\lambda - \frac{hQ\lambda}{\lambda+\mu} - \frac{hQ^2\lambda}{2D} + \frac{bQ\lambda}{\mu} + \frac{bD\lambda}{\mu(\lambda+\mu)}\right) \mathrm{e}^{-(\lambda+\mu)\frac{Q}{D}} = 0 \end{aligned} \tag{3.4}$$

Proof (a) For $Q > Q_1$, $E[C(Q)]$ is convex, differentiable and positive. Also $E[T(Q)]$ in concave, differentiable and positive. Then $E[G(Q)] = \frac{E[C(Q)]}{E[T(Q)]}$ is pseudoconvex (see Enkhbat, Bazarsad and Enkhbayar, 2011, Lemma 2.3).

(b) Set $E[G(Q)]$ equal to an arbitrary real value, say α. Then;

$$Q\left(\frac{hQ}{2D} - \frac{\alpha}{D}\right) - \mathrm{e}^{-(\lambda+\mu)\frac{Q}{D}}\left(\frac{\alpha}{\mu} - \frac{bD}{\mu^2}\right) = K - \frac{\lambda}{\lambda+\mu}\left(\frac{\alpha}{\mu} - \frac{bD}{\mu^2}\right)$$

Then, it can be noted that, finding the values of Q that satisfy this equation is to finding the intersection points of a quadratic function and a negative exponential function. This eqaution can have at most three roots, two from quadratic equation and one from negative exponential function. Furthermore,

$$\lim_{Q\to 0}\left(\frac{\mathrm{d}E[G(Q)]}{\mathrm{d}Q}\right)$$

$$= \lim_{Q\to 0}\frac{h\left(\frac{Q^2\mu+2QD\frac{\lambda}{\lambda+\mu}\left(1-\mathrm{e}^{-(\lambda+\mu)\frac{Q}{D}}\right)-Q^2\lambda\mathrm{e}^{-(\lambda+\mu)\frac{Q}{D}}}{2D^2\mu}\right) - K\left(\frac{1}{D} + \frac{\lambda\mathrm{e}^{-(\lambda+\mu)\frac{Q}{D}}}{D\mu}\right) + b\left(\frac{Q\lambda\mathrm{e}^{-(\lambda+\mu)\frac{Q}{D}}}{D\mu^2} - \frac{\lambda\left(1-\mathrm{e}^{-(\lambda+\mu)\frac{Q}{D}}\right)}{\mu^2(\lambda+\mu)}\right)}{\frac{Q^2}{D^2} + \frac{2Q\lambda\left(1-\mathrm{e}^{-(\lambda+\mu)\frac{Q}{D}}\right)}{D\mu(\lambda+\mu)} + \left(\frac{\lambda\left(1-\mathrm{e}^{-(\lambda+\mu)\frac{Q}{D}}\right)}{\mu(\lambda+\mu)}\right)^2}$$

$$\to -\infty,$$

and

$$\lim_{Q\to\infty}\left(\frac{\mathrm{d}E[G(Q)]}{\mathrm{d}Q}\right) = \lim_{Q\to 0}$$

$$= \lim_{Q\to 0}\frac{h\left(\frac{Q^2\mu+2QD\frac{\lambda}{\lambda+\mu}\left(1-\mathrm{e}^{-(\lambda+\mu)\frac{Q}{D}}\right)-Q^2\lambda\mathrm{e}^{-(\lambda+\mu)\frac{Q}{D}}}{2D^2\mu}\right) - K\left(\frac{1}{D} + \frac{\lambda\mathrm{e}^{-(\lambda+\mu)\frac{Q}{D}}}{D\mu}\right) + b\left(\frac{Q\lambda\mathrm{e}^{-(\lambda+\mu)\frac{Q}{D}}}{D\mu^2} - \frac{\lambda\left(1-\mathrm{e}^{-(\lambda+\mu)\frac{Q}{D}}\right)}{\mu^2(\lambda+\mu)}\right)}{\frac{Q^2}{D^2} + \frac{2Q\lambda\left(1-\mathrm{e}^{-(\lambda+\mu)\frac{Q}{D}}\right)}{D\mu(\lambda+\mu)} + \left(\frac{\lambda\left(1-\mathrm{e}^{-(\lambda+\mu)\frac{Q}{D}}\right)}{\mu(\lambda+\mu)}\right)^2}$$

$$\to \infty$$

since $Q\mathrm{e}^{-(\lambda+\mu)\frac{Q}{D}} \to 0$ as $Q \to \infty$ by L'Hopial's Rule.

So, $E[G(Q)]$ change direction at most once. Hence, the function assumes the same root at most twice. Therefore, it is unimodal, in other words, there exists Q^* such that $E[G(Q)]$ is monotonically decreasing for $Q < Q^*$ and it is monotonicaaly increasing for $Q > Q^*$.

(c) follows from the unimodality and the first-order condition of $E[G(Q)]$. ∎

Then we know that $E[G(Q)]$ has a global minimum at $\frac{\mathrm{d}E[G(Q)]}{\mathrm{d}Q} = 0$, by the optimality conditions for unconstrained convex function. Then $E[G(Q)]$ can be reduced to Equation 3.4.

We perform complete enumaration with a step size of 0.01 to find Q^*.

CHAPTER 4

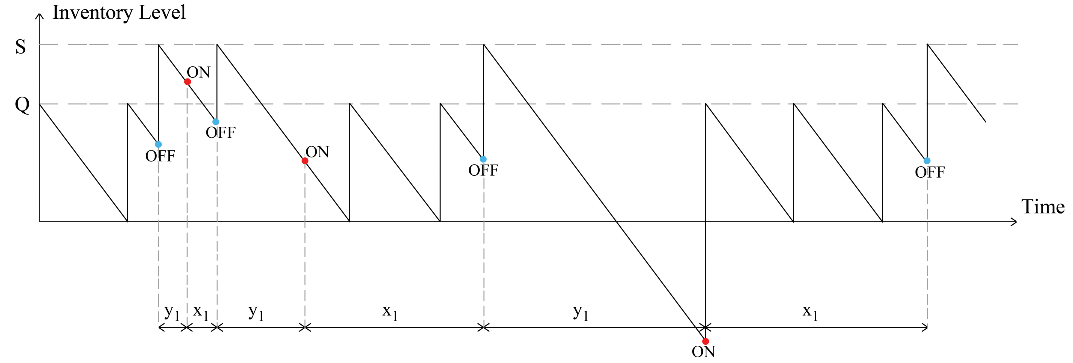
DISRUPTION MODEL WITH AN ORDER OPPORTUNITY

In this chapter, we consider the setting where the buyer has an opportunity to place an order right before or just as the supplier's state turns OFF. We restrict our attention to the following policy: when the inventory level is zero and the supplier is ON, a regular order is placed to raise the inventory level to Q, no backorder is allowed in this case. For OFF-periods we set another order-up-to level, which we call disruption order-up-to level and denote by S. When the supplier turns into an OFF state, if $I < S$ then the buyer places a disruption order to increase I to S.

Our objective is to find Q and S values that minimize the expected total cost per unit time. Cost components are *i*) fixed order costs (identical for regular and disruption orders), *ii*) inventory holding costs, and *iii*) backorder costs. The notation that we use is the same as the one in Chapter 3 with a slight modification: cost components and the cycle length are now functions of Q and S.

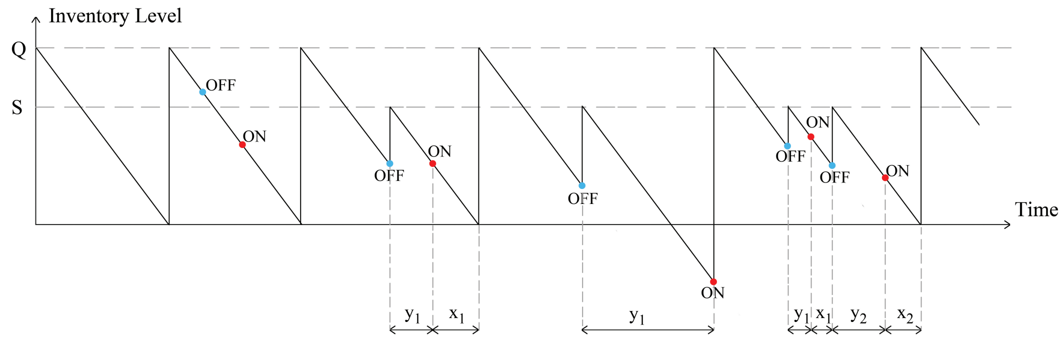
To find the optimal Q and S, we divide the feasible regions for the decision variables into two; *i*) $Q \leq S$ and *ii*) $Q > S$, since the structure of the problem changes significantly depending on the relative magnitudes of Q and S. To illustrate, consider the sample realizations given in Figure 4.1 and Figure 4.2, respectively.

Figure 4.1 Inventory Level vs. Time when $Q \leq S$



When $Q \leq S$ (Figure 4.1), the inventory level is definitely less than S when the supplier is disrupted. Hence, every time the supplier is disrupted, a disruption order is definitely placed.

Figure 4.2 Inventory Level vs. Time when $Q > S$



On the other hand, when $Q > S$ (Figure 4.2), the order opportunity may or may not be utilized, depending on the inventory level at the time that the supplier is disrupted. This distinction is important as "cycles" will be defined accordingly. Hence, we study the two cases separately and then combine the corresponding solutions to get the optimal solution to the original problem. In Section 4.1, we analyze the case where $Q \leq S$ and in Section 4.2, we consider the case where $Q > S$.

4.1 $Q \leq S$

In this case the buyer definitely places an order at the time of disruption or just as the supplier's state turns OFF. We define a cycle as the time between the receipts of two successive disruption orders. In other words, a cycle begins with an OFF-period and continues with an ON-period. That is, each cycle starts with a disruption which brings the inventory level to S. The supplier stays OFF for an exponential amount of time with rate μ. Then, it becomes ON, and after an exponential amount of ON-period with rate λ, another disruption occurs. Therefore, a cycle consists of two independent exponentially distributed random variables:

$$T = X + Y.$$

Recalling that the supplier's state follows a CTMC with two states, we can deduce that all cycles have identical distributions and they are independent. Hence, Renewal Reward Theorem can be applied to derive the expected cost function per unit time.

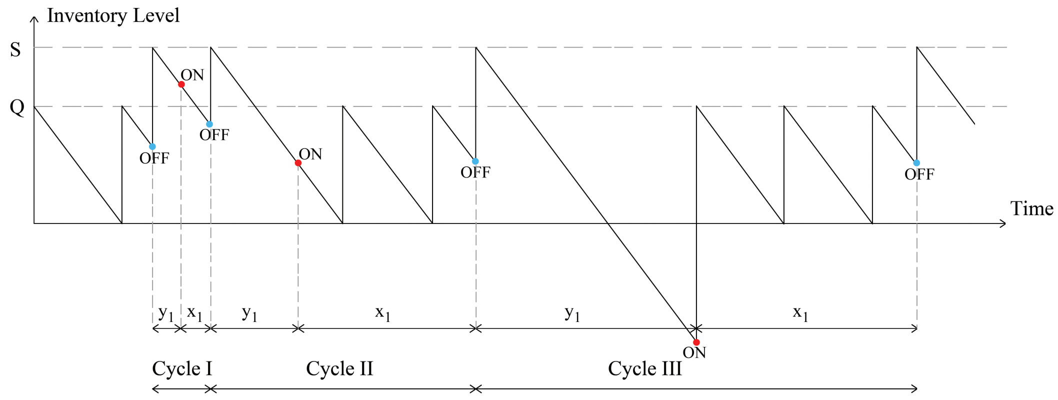
$$E[\mathrm{G(Q,S)}] = \frac{E[\mathrm{C(Q,S)}]}{E[T]} = \frac{E[P(Q,S)] + E[H(Q,S)] + E[B(Q,S)]}{E[T]}$$

and;

$$E[T] = E[X+Y] = E[X] + E[Y] = \frac{1}{\lambda} + \frac{1}{\mu}.$$

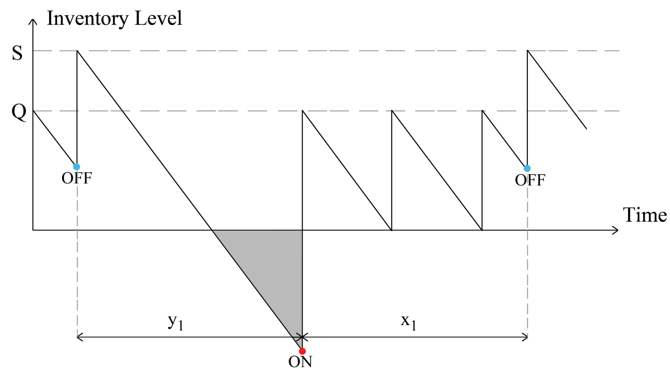
At the beginning of a cycle, the supplier is definitely OFF. Depending on the realization of the OFF and ON-periods' lengths, we may encounter the following types of cycles. Firstly, if the inventory level is positive when the supplier becomes ON and next disruption occurs before inventory level hits zero, the cycle ends without a regular order (see Cycle I in Figure 4.3). Second, if the supplier becomes ON before the inventory level hits zero and stays ON until $I = 0$, a regular order to increase the inventory level to Q is placed and the cycle continues until the next disruption (see Cycle II in Figure 4.3). In this case the number of regular orders may be more than one depending on the ON-period's length. Finally, if the inventory level is negative at time that the supplier's state becomes ON, the buyer faces backorders and a regular order is placed immediately to raise the inventory level to Q and again the cycle continues until the next disruption (see Cycle III in Figure 4.3).

Figure 4.3 Inventory Level vs. Time when $Q \leq S$



We start our analysis by calculating the expected backorder cost per cycle. A sample path where a shortage occurs can be seen at Figure 4.4.

Figure 4.4 Inventory Level vs. Time when $Q \leq S$, $Y > \frac{S}{D}$

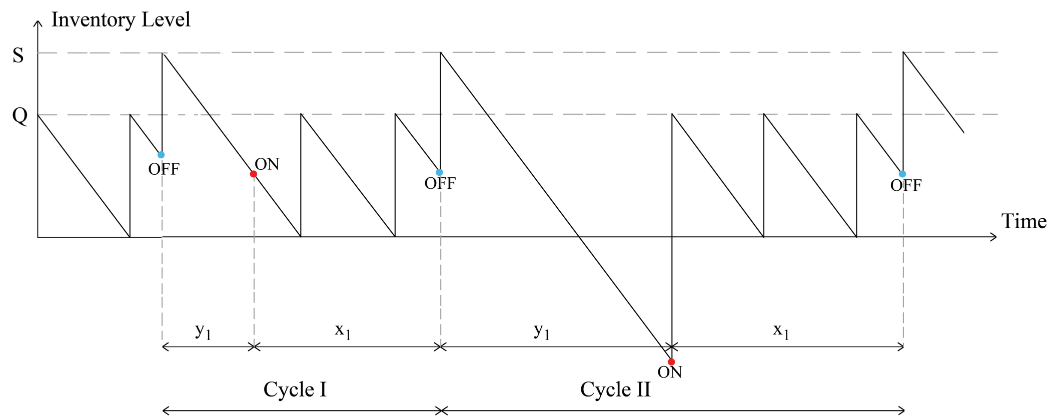


Note that backorders occur only when the duration of the disruption is longer than $\frac{S}{D}$ since the inventory of S is depleted by constant demand rate, D. Considering the shaded region in Figure 4.4, we have;

$$E[\mathrm{B(Q,S)}] = b \int_{\frac{S}{D}}^{\infty} \frac{(yD - S)(y - S/D)}{2} f_Y(y)\mathrm{d}y = b\mathrm{e}^{-\mu \frac{S}{D}} \frac{D}{\mu^2}.$$

We continue with the expected fixed order cost and inventory holding cost per cycle, which depend on the number of regular orders in a cycle. Let $N(Q, S)$ be the random variable representing the number of regular orders in a cycle, for given Q and S values. For the sake of simplicity, we represent $P(Q, S)$, $H(Q, S)$ and $N(Q, S)$ by just P, H and N, respectively, from this point forward. Recall that, we may not have any regular order in a cycle. In such a case, the derivation of fixed order cost and inventory holding cost is trivial. Hence, we focus on the other two cases that we described before, which are illustrated in Figure 4.5. If the OFF-period is shorter than $\frac{S}{D}$, we observe positive inventory when the ON-period starts, and an immediate regular order will not be placed (see Cycle I in Figure 4.5). Otherwise a regular order will be placed immediately when the ON-period starts (see Cycle II in Figure 4.5).

Figure 4.5 Inventory Level vs. Time when $Q \leq S$



Hence, the expected number of regular orders can be calculated as

$$E[N] = E\left[N \middle| Y > \frac{S}{D}\right] P\left\{Y > \frac{S}{D}\right\} + E\left[N \middle| Y \leq \frac{S}{D}\right] P\left\{Y \leq \frac{S}{D}\right\}. \tag{4.1}$$

Since the buyer places a disruption order every cycle, the expected fixed order cost per cycle is;

$$E[P] = K(1 + E[N]).$$

Likewise the expected inventory holding cost per cycle;

$$E[H] = E\left[H \middle| Y > \frac{S}{D}\right] P\left\{Y > \frac{S}{D}\right\} + E\left[H \middle| Y \leq \frac{S}{D}\right] P\left\{Y \leq \frac{S}{D}\right\}. \tag{4.2}$$

As the number of regular orders determine both the fixed order cost and the inventory holding cost in a cycle, we determine its expected value in the following sections.

4.1.1 Positive inventory level at the beginning of an ON-period $\left(Y \leq \frac{S}{D}\right)$

We start with the expected number of regular orders in a cycle where the inventory level is positive when the supplier turns ON. We condition on the total duration of ON and OFF-period, since we may encounter the case that the buyer cannot place any regular order.

$$\begin{aligned} E\left[N \middle| Y \leq \frac{S}{D}\right] = E\left[N \middle| Y \leq \frac{S}{D}, X + Y \leq \frac{S}{D}\right] P\left\{X + Y \leq \frac{S}{D} \middle| Y \leq \frac{S}{D}\right\} \\ + E\left[N \middle| Y \leq \frac{S}{D}, X + Y > \frac{S}{D}\right] P\left\{X + Y > \frac{S}{D} \middle| Y \leq \frac{S}{D}\right\}. \end{aligned} \tag{4.3}$$

We know that if $X + Y \leq \frac{S}{D}$ then the buyer does not place any regular order since S is not depleted before the cycle ends. That is,

$$E\left[N \middle| Y \leq \frac{S}{D}, X + Y \leq \frac{S}{D}\right] = 0. \tag{4.4}$$

By definition;

$$\mathrm{E}\left[N \middle| Y \leq \tfrac{S}{D}, X + Y > \tfrac{S}{D}\right] = \sum_{k=0}^{\infty} kP\left\{N = k \middle| Y \leq \frac{S}{D}, X + Y > \frac{S}{D}\right\}.$$

Let x and y denote realizations of X and Y, respectively. Then, in order to have exactly $k > 0$ regular orders, the duration of the cycle should be large enough to deplete $S + (k-1)Q$ units of inventory, but not longer than $\frac{S+kQ}{D}$ so that $(k+1)^{st}$ regular order is not placed. That is,

$$N = k > 0 \iff \frac{S + (k-1)Q}{D} \leq x + y < \frac{S + kQ}{D}.$$

So, the probability of the number of regular orders being $k \geq 1$ is derived by using the independence of random variables X and Y:

$$P\left\{N = k \middle| Y < \frac{S}{D}, X + Y > \frac{S}{D}\right\} = \frac{P\left\{N = k, Y < \frac{S}{D}, X + Y > \frac{S}{D}\right\}}{P\left\{Y < \frac{S}{D}, X + Y > \frac{S}{D}\right\}}$$

$$= \frac{P\left\{Y < \frac{S}{D}, \frac{S+(k-1)Q}{D} \leq X + Y < \frac{S+kQ}{D}\right\}}{P\left\{Y < \frac{S}{D}, X + Y > \frac{S}{D}\right\}} = \frac{\int_0^{\frac{S}{D}} \int_{\frac{S+(k-1)Q}{D}-v}^{\frac{S+kQ}{D}-v} \lambda \mathrm{e}^{-\lambda u} \mu \mathrm{e}^{-\mu v} \, \mathrm{d}u \mathrm{d}v}{\int_0^{\frac{S}{D}} \int_{\frac{S}{D}-v}^{\infty} \lambda \mathrm{e}^{-\lambda u} \mu \mathrm{e}^{-\mu v} \, \mathrm{d}u \mathrm{d}v}$$

$$= \left(\mathrm{e}^{\lambda \frac{Q}{D}} - 1\right) \mathrm{e}^{-\lambda \frac{kQ}{D}} = \mathrm{e}^{-\lambda \frac{(k-1)Q}{D}} \left(1 - \mathrm{e}^{-\lambda \frac{Q}{D}}\right).$$

Note that this is the probability distribution of a geometric random variable with success probability $\left(1 - \mathrm{e}^{-\lambda \frac{Q}{D}}\right)$, which makes sense due to the memoryless property of exponential distribution. Then, the expected number of regular order per cycle for this case is;

$$E\left[N \middle| Y \leq \frac{S}{D}, X + Y > \frac{S}{D}\right] = \sum_{k=1}^{\infty} k \mathrm{e}^{-\lambda \frac{(k-1)Q}{D}} \left(1 - \mathrm{e}^{-\lambda \frac{Q}{D}}\right) \mathrm{e}^{-\lambda \frac{kQ}{D}} = \frac{1}{\left(1 - \mathrm{e}^{-\lambda \frac{Q}{D}}\right)}. \tag{4.5}$$

Note that;

$$P\left\{X + Y > \frac{S}{D} \middle| Y \leq \frac{S}{D}\right\} = \frac{P\left\{X + Y > \frac{S}{D}, Y \leq \frac{S}{D}\right\}}{P\left\{Y \leq \frac{S}{D}\right\}} = \frac{\mu\left(\mathrm{e}^{-\lambda \frac{S}{D}} - \mathrm{e}^{-\mu \frac{S}{D}}\right)}{\mu - \lambda\left(1 - \mathrm{e}^{-\mu \frac{S}{D}}\right)}. \tag{4.6}$$

Finally, plugging Equations 4.4, 4.5 and 4.6 into Equation 4.3, we get;

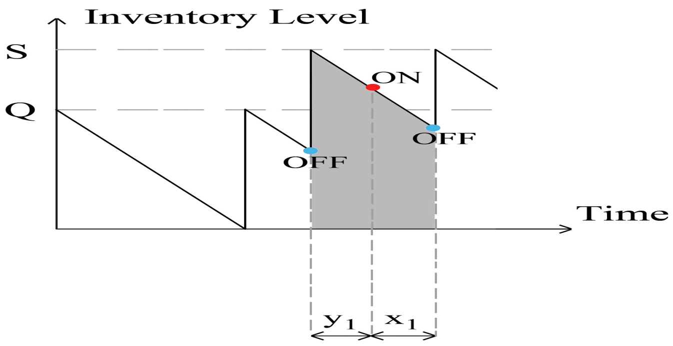
$$E\left[N \middle| Y \leq \frac{S}{D}\right] = \frac{\mu\left(\mathrm{e}^{-\mu \frac{S}{D}} - \mathrm{e}^{-\lambda \frac{S}{D}}\right)}{(\lambda - \mu)\left(1 - \mathrm{e}^{-\mu \frac{S}{D}}\right)\left(1 - \mathrm{e}^{-\lambda \frac{Q}{D}}\right)}. \tag{4.7}$$

To derive the expected inventory holding cost per cycle, we investigate two cases; a cycle with no regular order and a cycle with one or more regular orders. So, we condition on Equation 4.2. Therefore;

$$E\left[H \middle| Y \leq \frac{S}{D}\right] = E\left[H \middle| Y \leq \frac{S}{D}, X+Y \leq \frac{S}{D}\right] P\left\{X+Y \leq \frac{S}{D} \middle| Y \leq \frac{S}{D}\right\} + E\left[H \middle| Y \leq \frac{S}{D}, X+Y > \frac{S}{D}\right] P\left\{X+Y > \frac{S}{D} \middle| Y \leq \frac{S}{D}\right\}. \tag{4.8}$$

Consider the first expectation term in Equation 4.8, which corresponds to a cycle with no regular order, the shaded region in Figure 4.6. At the beginning of this cycle $I = S$. After one OFF and one ON-period, the inventory level decreases to $S - D(X + Y)$.

Figure 4.6 Inventory Level vs. Time when $Q \leq S$, $Y \leq \frac{S}{D}$, No regular order



So, the expected inventory holding cost corresponding to this case can be obtained as follows:

$$\begin{aligned} E\left[H \middle| Y \leq \frac{S}{D}, X+Y \leq \frac{S}{D}\right] &= E\left[H \middle| X+Y \leq \frac{S}{D}\right] \\ &= hE\left[\frac{(2S - D(X+Y))(X+Y)}{2} \middle| X+Y \leq \frac{S}{D}\right] \\ &= hE\left[S(X+Y) - \frac{D}{2}(X+Y)^2 \middle| X+Y \leq \frac{S}{D}\right] \\ &= h\left\{S\,E\left[X+Y \middle| X+Y \leq \frac{S}{D}\right] - \frac{D}{2}E\left[(X+Y)^2 \middle| X+Y \leq \frac{S}{D}\right]\right\}. \end{aligned} \tag{4.9}$$

To compute the expectation in Equation 4.9, we need the distribution of $\left(X+Y \middle| X+Y < \frac{S}{D}\right)$:

$$F_{(X+Y)}\left(z \middle| X+Y \leq \frac{S}{D}\right) = \begin{cases} \dfrac{\mu - \lambda - \mu e^{-\lambda z} + \lambda e^{-\mu z}}{\mu - \lambda - \mu e^{-\lambda \frac{S}{D}} + \lambda e^{-\mu \frac{S}{D}}} & z \leq \frac{S}{D} \\ 1 & \text{otherwise} \end{cases} \tag{4.10}$$

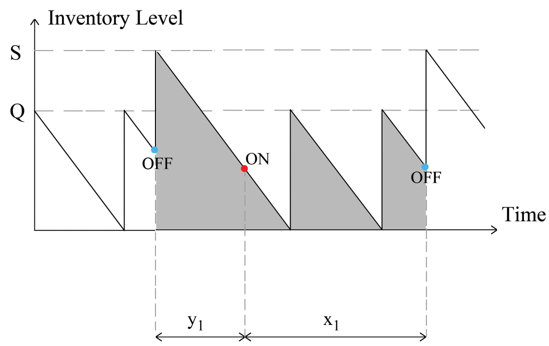
The detailed derivation of Equation 4.10 is given in Appendix A.1.

Rewriting Equation 4.9, we have;

$$E\left[H\middle|Y \le \frac{S}{D}, X+Y \le \frac{S}{D}\right] = h\left(S\,E\left[X+Y\middle|X+Y \le \frac{S}{D}\right] - \frac{D}{2}E\left[(X+Y)^2\middle|X+Y \le \frac{S}{D}\right]\right)$$
$$= h\left(S\int_0^{\frac{S}{D}}\left(1-\frac{\mu-\lambda-\mu e^{-\lambda z}+\lambda e^{-\mu z}}{\mu-\lambda-\mu e^{-\lambda\frac{S}{D}}+\lambda e^{-\mu\frac{S}{D}}}\right)dz - \frac{D}{2}\left(2\int_0^{\frac{S}{D}}\left(z - z\frac{\mu-\lambda-\mu e^{-\lambda z}+\lambda e^{-\mu z}}{\mu-\lambda-\mu e^{-\lambda\frac{S}{D}}+\lambda e^{-\mu\frac{S}{D}}}\right)dz\right)\right)$$
$$= h\left(S\left[\frac{S}{D} - \frac{1}{\mu-\lambda-\mu e^{-\lambda\frac{S}{D}}+\lambda e^{-\mu\frac{S}{D}}}\left(\frac{\mu S}{D} - \frac{\lambda S}{D} + \frac{\mu(e^{-\lambda\frac{S}{D}}-1)}{\lambda} - \frac{\lambda(e^{-\mu\frac{S}{D}}-1)}{\mu}\right)\right]\right.$$
$$- \frac{D}{2}\left[2\left(\frac{S^2}{2D^2} - \frac{1}{\mu-\lambda-\mu e^{-\lambda\frac{S}{D}}+\lambda e^{-\mu\frac{S}{D}}}\left(\frac{\mu S^2}{2D^2} - \frac{\lambda S^2}{2D^2} - \mu\left(-\frac{S}{D\lambda}e^{-\lambda\frac{S}{D}} - \frac{(e^{-\lambda\frac{S}{D}}-1)}{\lambda^2}\right)\right.\right.\right.$$
$$\left.\left.\left.\left. +\lambda\left(-\frac{S}{D\mu}e^{-\mu\frac{S}{D}} - \frac{(e^{-\mu\frac{S}{D}}-1)}{\mu^2}\right)\right)\right)\right]\right)$$
$$= h\left(\frac{S^2}{2D} + \frac{1}{\mu-\lambda-\mu e^{-\lambda\frac{S}{D}}+\lambda e^{-\mu\frac{S}{D}}}\right.$$
$$\left.\left(\frac{S^2(\lambda-\mu)}{2D} + \frac{S(\mu^2-\lambda^2)}{\lambda\mu} + \frac{\mu D}{\lambda^2}(e^{-\lambda\frac{S}{D}}-1) - \frac{\lambda D}{\mu^2}(e^{-\mu\frac{S}{D}}-1)\right)\right). \tag{4.11}$$

Next we consider a cycle with multiple regular orders, that is the second expectation term in Equation 4.8. It corresponds to the shaded region in Figure 4.7.

Figure 4.7 Inventory Level vs. Time when $Q \le S$, $Y \le \frac{S}{D}$, Multiple regular orders



So, the expected inventory holding cost corresponding to this case can be obtained through the derivations in Appendix A.2 as follows;

$$E\left[H \middle| Y \leq \frac{S}{D}, X+Y > \frac{S}{D}\right] = h\left(\frac{S^2}{2D} + \frac{Q^2 e^{-\lambda\frac{Q}{D}}}{2D\left(-e^{\lambda\frac{Q}{D}}-1\right)} + \frac{S^2}{D} + \frac{S}{\lambda} + \frac{SQ}{D\left(1-e^{-\lambda\frac{Q}{D}}\right)}\right.$$
$$+ \frac{Q^2}{D\left(1-e^{-\lambda\frac{Q}{D}}\right)^2} + \frac{Q}{\lambda\left(1-e^{-\lambda\frac{Q}{D}}\right)} - \frac{S^2}{2D} - \frac{S}{\lambda} - \frac{D}{\lambda^2} - \frac{Q^2\left(1+e^{-\lambda\frac{Q}{D}}\right)}{2D\left(-e^{-\lambda\frac{Q}{D}}\right)^2} - \frac{SQ}{\left(1-e^{-\lambda\frac{Q}{D}}\right)}$$
$$\left. - \frac{S^2-Q^2}{2D}\right)$$
$$= h\left(\frac{S^2}{2D} + \frac{Q}{\lambda\left(1+e^{-\lambda\frac{Q}{D}}\right)} - \frac{D}{\lambda^2}\right). \tag{4.12}$$

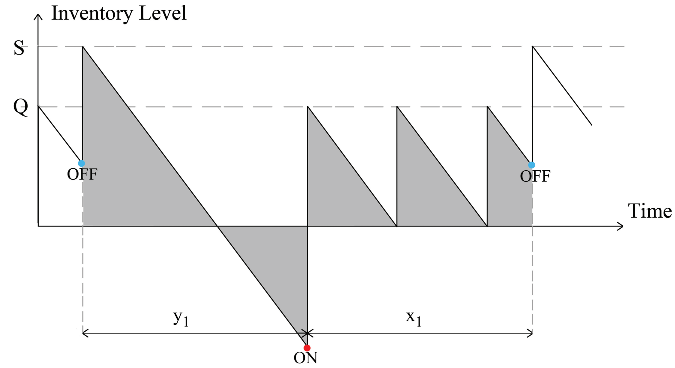
Finally, plugging Equations 4.11 and 4.12 into Equation 4.8, we get;

$$E\left[H \middle| Y \leq \frac{S}{D}\right] = h\left(\frac{S^2\left(\mu-\lambda-\mu e^{-\lambda\frac{S}{D}}+\lambda e^{-\mu\frac{S}{D}}\right)}{2D(\mu-\lambda)\left(1-e^{-\mu\frac{S}{D}}\right)} - \left(\frac{1}{(\mu-\lambda)\left(1-e^{-\mu\frac{S}{D}}\right)}\right.\right.$$
$$\left.\left(\frac{S^2(\lambda-\mu)}{2D} + \frac{S\left(\mu^2-\lambda^2\right)}{\lambda\mu} + \frac{\mu D}{\lambda^2}\left(e^{-\lambda\frac{S}{D}}-1\right) - \frac{\lambda D}{\mu^2}\left(e^{-\mu\frac{S}{D}}-1\right)\right)\right. \tag{4.13}$$
$$\left. + \left(\frac{S^2}{2D} + \frac{Q}{\lambda\left(1+e^{-\lambda\frac{Q}{D}}\right)} - \frac{D}{\lambda^2}\right)\left(\frac{\mu\left(e^{-\lambda\frac{S}{D}}-e^{-\mu\frac{S}{D}}\right)}{(\mu-\lambda)\left(1-e^{-\mu\frac{S}{D}}\right)}\right)\right).$$

4.1.2 Negative inventory level at the beginning of an ON-period $\left(Y > \frac{S}{D}\right)$

Now since the inventory level is negative when the supplier's state turns ON, we know that the OFF-period lasts longer than $\frac{S}{D}$ and the buyer faces backorders. Hence, as soon as disruption is over, the buyer places a regular order to bring its inventory level to Q. This is the main difference from Section 4.1.1. A sample path can be seen in Figure 4.8, the shaded region.

Figure 4.8 Inventory Level vs. Time when $Q \leq S$, $Y > \frac{S}{D}$



Since the buyer definitely places a regular order, $P\left\{N = 0 \middle| Y > \frac{S}{D}\right\} = 0.$

Then in order to have $N = k > 0$, the duration of the ON-period should be such that the inventory consumed during the ON-period should be no less than $(k-1)Q$ and no larger than kQ. That is,

$$N = k > 0 \iff \frac{(k-1)Q}{D} \leq x < \frac{kQ}{D}.$$

Note that the expected number of regular orders depends only the duration of ON-periods. So, the probability of ordering $k \geq 1$ regular orders in a cycle is derived as follows by using the independence of random variables X and Y:

$$\begin{aligned} P\left\{N = k \middle| Y > \frac{S}{D}\right\} &= P\left\{\frac{(k-1)Q}{D} < X \leq \frac{kQ}{D} \middle| Y > \frac{S}{D}\right\} = P\left\{\frac{(k-1)Q}{D} < X \leq \frac{kQ}{D}\right\} \\ &= \mathrm{e}^{-\lambda\frac{(k-1)Q}{D}}\left(1 - \mathrm{e}^{-\lambda\frac{Q}{D}}\right). \end{aligned}$$

Then, the expected number of regular orders per cycle for this case is;

$$E\left[N \middle| Y > \frac{S}{D}\right] = \sum_{k=1}^{\infty} k\left(\mathrm{e}^{-\lambda\frac{kQ}{D}}\left(\mathrm{e}^{\lambda\frac{Q}{D}} - 1\right)\right) = \frac{1}{\left(1 - \mathrm{e}^{-\lambda\frac{Q}{D}}\right)}. \tag{4.14}$$

The expected inventory holding cost per cycle for this case is calculated as follows (see Appendix A.3 for detailed derivation):

$$
\begin{aligned}
E\left[H\middle|Y>\frac{S}{D}\right] &= h\left(\frac{S^2}{2D}+\frac{Q^2}{2D}\left(E\left[N\middle|Y>\frac{S}{D}\right]-1\right)\right.\\
&+\frac{Q^2\mathrm{e}^{-\lambda\frac{Q}{D}}}{D\left(1-\mathrm{e}^{-\lambda\frac{Q}{D}}\right)^2}+\frac{Q}{\lambda\left(1-\mathrm{e}^{-\lambda\frac{Q}{D}}\right)}-\frac{D}{\lambda^2}+\frac{Q^2}{2d}-\left.\frac{Q^2\left(1+\mathrm{e}^{-\lambda\frac{Q}{D}}\right)}{2D\left(1-\mathrm{e}^{-\lambda\frac{Q}{D}}\right)^2}\right)\\
&= h\left(\frac{S^2}{2D}+\frac{Q}{\lambda\left(1-\mathrm{e}^{-\lambda\frac{Q}{D}}\right)}-\frac{D}{\lambda^2}\right).
\end{aligned}
\tag{4.15}
$$

Therefore it can be noted that $E\left[H\middle|Y\le\frac{S}{D},X+Y>\frac{S}{D}\right]$ (Equation 4.12) and $E\left[H\middle|Y>\frac{S}{D}\right]$ (Equation 4.15) turns out to be the same with each other because the memoryless property of the exponential distribution of X. In other words, by the memoryless property of exponential distribution, the remaining time of ON-period after the inventory level hits zero, again follows exponential distribution and the on-hand inventories of both cases are stochastically identical.

4.1.3 Expected Cost Calculations for $Q \le S$

In this section, we combine our findings in Section 4.1.1 and Section 4.1.2. We start with the number of regular orders. Plugging Equation 4.7 and 4.14 into Equation 4.1, we get

$$
E[N]=\frac{1}{\left(1-\mathrm{e}^{-\lambda\frac{Q}{D}}\right)}\mathrm{e}^{-\mu\frac{S}{D}}+0+\frac{\mu\left(\mathrm{e}^{-\mu\frac{S}{D}}-\mathrm{e}^{-\lambda\frac{S}{D}}\right)}{(\lambda-\mu)\left(1-\mathrm{e}^{-\lambda\frac{Q}{D}}\right)}=\frac{\lambda\mathrm{e}^{-\mu\frac{S}{D}}-\mu\mathrm{e}^{-\lambda\frac{S}{D}}}{(\lambda-\mu)\left(1-\mathrm{e}^{-\lambda\frac{Q}{D}}\right)}.
$$

Similarly, to calculate expected inventory holding cost per cycle, we plug Equation 4.13 and 4.15 into Equation 4.2 and obtain,

$$
\begin{aligned}
E[H] &= h\Bigg(\Bigg(\frac{S^2}{2D} + \frac{Q}{\lambda\left(1-\mathrm{e}^{-\lambda\frac{Q}{D}}\right)} - \frac{D}{\lambda^2}\Bigg)\left(\mathrm{e}^{-\mu\frac{S}{D}}\right) \\
&+ \Bigg(\frac{S^2}{2D} + \frac{1}{\mu-\lambda-\mu\mathrm{e}^{-\lambda\frac{S}{D}}+\lambda\mathrm{e}^{-\mu\frac{S}{D}}}\Bigg(\frac{S^2(\lambda-\mu)}{2D} + \frac{S(\mu^2-\lambda^2}{\lambda\mu} + \frac{\mu D}{\lambda^2}(\mathrm{e}^{-\lambda\frac{S}{D}}-1) \\
&-\frac{\lambda D}{\mu^2}(\mathrm{e}^{-\mu\frac{S}{D}}-1)\Bigg)\Bigg) \\
&\Bigg(\frac{\mu-\lambda-\mu\mathrm{e}^{-\lambda\frac{S}{D}}+\lambda\mathrm{e}^{-\mu\frac{S}{D}}}{(\mu-\lambda)\left(1-\mathrm{e}^{-\mu\frac{S}{D}}\right)}\Bigg) + \Bigg(\frac{S^2}{2D} + \frac{Q}{\lambda\left(1-\mathrm{e}^{-\lambda\frac{Q}{D}}\right)} - \frac{D}{\lambda^2}\Bigg)\Bigg(\frac{\mu\left(\mathrm{e}^{-\lambda\frac{S}{D}}-\mathrm{e}^{-\mu\frac{S}{D}}\right)}{(\mu-\lambda)\left(1-\mathrm{e}^{-\mu\frac{S}{D}}\right)}\Bigg)\Bigg) \\
&= h\Bigg(\frac{S\,(\lambda+\mu)}{\lambda\mu} + \frac{Q\left(\mu\mathrm{e}^{-\lambda\frac{S}{D}}-\lambda\mathrm{e}^{-\mu\frac{S}{D}}\right)}{\lambda\,(\mu-\lambda)\left(1-\mathrm{e}^{-\lambda\frac{Q}{D}}\right)} + \frac{D\lambda\mathrm{e}^{-\mu\frac{S}{D}}\,(\lambda+\mu) - D\left(\lambda^2+\mu^2+\lambda\mu\right)}{\lambda^2\mu^2}\Bigg).
\end{aligned}
$$

As a result, the total expected cost per cycle is given by,

$$
\begin{aligned}
E[C(Q,S)] &= K\Bigg(1 + \frac{\lambda\mathrm{e}^{-\mu\frac{S}{D}}-\mu\mathrm{e}^{-\lambda\frac{S}{D}}}{(\lambda-\mu)\left(1-\mathrm{e}^{-\lambda\frac{Q}{D}}\right)}\Bigg) + b\left(\frac{D\mathrm{e}^{-\mu\frac{S}{D}}}{\mu^2}\right) \\
&+ h\Bigg(\frac{S\,(\lambda+\mu)}{\lambda\mu} + \frac{Q\left(\mu\mathrm{e}^{-\lambda\frac{S}{D}}-\lambda\mathrm{e}^{-\mu\frac{S}{D}}\right)}{\lambda\,(\mu-\lambda)\left(1-\mathrm{e}^{-\lambda\frac{Q}{D}}\right)} + \frac{D\lambda\mathrm{e}^{-\mu\frac{S}{D}}\,(\lambda+\mu) - D\left(\lambda^2+\mu^2+\lambda\mu\right)}{\lambda^2\mu^2}\Bigg).
\end{aligned}
$$

Using Renewal Reward Theorem, we get,

$$
\begin{aligned}
E[G(Q,S)] &= \frac{E[C(Q,S)]}{E[T]} \\
&= \frac{\Bigg[K\Bigg(1+\frac{\lambda\mathrm{e}^{-\mu\frac{S}{D}}-\mu\mathrm{e}^{-\lambda\frac{S}{D}}}{(\lambda-\mu)\left(1-\mathrm{e}^{-\lambda\frac{Q}{D}}\right)}\Bigg) + b\left(\frac{D\mathrm{e}^{-\mu\frac{S}{D}}}{\mu^2}\right) + h\Bigg(\frac{S\,(\lambda+\mu)}{\lambda\mu} + \frac{Q\left(\mu\mathrm{e}^{-\lambda\frac{S}{D}}-\lambda\mathrm{e}^{-\mu\frac{S}{D}}\right)}{\lambda(\mu-\lambda)\left(1-\mathrm{e}^{-\lambda\frac{Q}{D}}\right)} + \frac{D\lambda\mathrm{e}^{-\mu\frac{S}{D}}(\lambda+\mu)-D\left(\lambda^2+\mu^2+\lambda\mu\right)}{\lambda^2\mu^2}\Bigg)\Bigg]}{\frac{1}{\lambda}+\frac{1}{\mu}}.
\end{aligned}
\tag{4.16}
$$

Unfortunately, we cannot show that $E[G(Q,S)]$ is convex or not. To find the optimal values of Q and S, we propose a search algorithm.

1. Set the iteration counter i=0.
2. Set Q_0 = EOQ with backorder where EOQ with backorder= $\sqrt{\frac{C_o+C_s}{C_s}}\sqrt{\frac{2C_oD}{C_h}}$.

3. Given Q_0, search for S_0 that minimizes $E[G(Q,S)]$ from Q_0 to $Q_0 + 50000$ with step size 0.01.

4. Set $i = i + 1$.

5. Given S_{i-1} search for Q_i that minimizes $E[G(Q,S)]$ from 0 to S_{i-1} with step size 0.01.

6. Given Q_i search for S_i that minimizes $E[G(Q,S)]$ from Q_i to $Q_i + 50000$ with step size 0.01.

7. Return to Step 4 until $Q_i = Q_{i-1} \pm 0.01$ and $S_{i-1} = S_i \pm 0.01$.

8. Set $Q_i = Q^*$ and $S_i = S^*$.

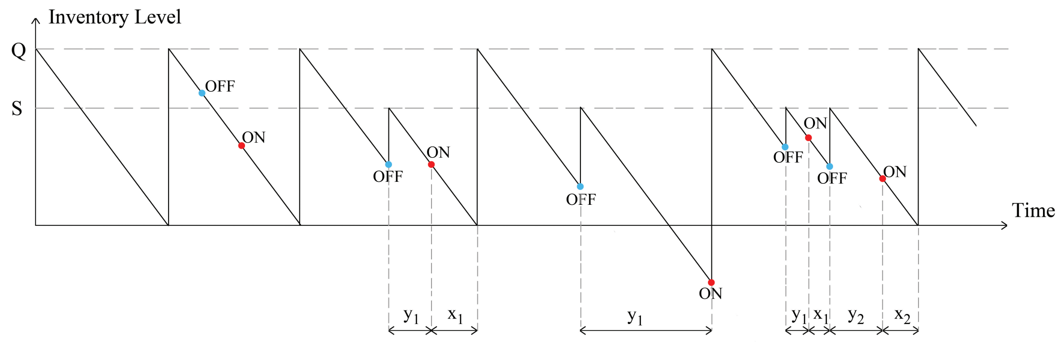
9. Substitute Q^* and S^* to calculate the minimum $E[G(Q,S)]$.

4.2 $Q > S$

When $Q > S$, the buyer may or may not place an order when disruption occurs depending on the inventory level at the time of disruption. That is, when a disruption occurs, if the inventory level is greater than S, the buyer chooses not to place an order; otherwise, the quantity which completes the inventory level up to S is ordered from the supplier. A sample realization of the inventory process can be observed in Figure 4.9.

We define a cycle between two successive receipts of regular orders. Between two regular orders, it is possible that the buyer may not place any disruption orders, because even if the system is disrupted, these disruptions can occur when the inventory level is greater than S. On the other hand, if a disruption occurs when the inventory level is below S, the buyer places a disruption order which brings its inventory level to S. Let M be the number of disruption orders between two successive orders. M can take any integer value from zero to infinity.

Figure 4.9 Inventory Level vs. Time when $Q > S$



Now in order to discuss whether the cycles are regenerative or not, we should investigate the probability distribution of the cycle length. The probability distribution function of cycle length is derived considering two cases; whether the supplier is ON for the first time inventory level hits S or not. We know that, a cycle starts with Q units of inventory, and after $\frac{Q-S}{D}$ time units, $I = S$ for the first time in that cycle. Then depending on the state at time $\frac{Q-S}{D}$, the distribution of cycle length changes. Therefore we condition on the state of the supplier at that point; the first time inventory level hits S. The system state being ON (OFF) when the inventory level hits S for the first time is represented by $Z\left(\frac{Q-S}{D}\right) = 1$ $\left(Z\left(\frac{Q-S}{D}\right) = 0\right)$. While characterizing the conditional distribution of the cycle length, we also compute associated expected costs conditionally. We then get the expected cycle cost as follows:

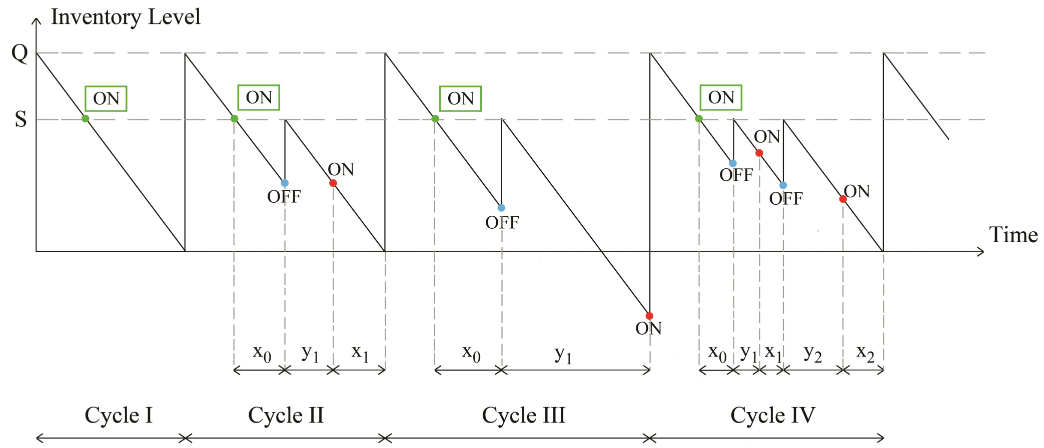
$$E\left[C(Q,S)\right] = E\left[C(Q,S)\middle|Z\left(\frac{Q-S}{D}\right) = 1\right]P\left\{Z\left(\frac{Q-S}{D}\right) = 1\right\}$$
$$+ E\left[C(Q,S)\middle|Z\left(\frac{Q-S}{D}\right) = 0\right]P\left\{Z\left(\frac{Q-S}{D}\right) = 0\right\}. \quad (4.17)$$

4.2.1 The system is ON at the first time the inventory level hits S

We first consider the case where the supplier is ON when the inventory level hits S for the first time. Recall that the buyer may or may not place any disruption order in a cycle. If the system is ON at the first time the inventory level hits S, the buyer does not place any disruption order only if no disruption occurs until the end of cycle. In this case, the buyer does not face any backorders since when the inventory level hits zero, the supplier's state is definitely ON (see Cycle I in Figure 4.10 for illustration). On the other hand, if a disruption occurs when the inventory level is between S and 0, then the buyer definitely places a disruption order. At this point, the number of disruption orders depends on the number of transition of the supply process. If the supplier stays OFF until the inventory level hits zero (no transition), the buyer starts backordering and as soon as the supplier becomes ON, an immediate regular order is placed which ends the cycle with only one disruption order (see Cycle III in Figure 4.10 for illustration). If the supplier becomes ON before $I = 0$ and stays ON until $I = 0$ (only one transition after disruption), then a regular order is placed when $I = 0$ which ends the cycle again (see Cycle II in Figure 4.10 for illustration). If the supplier becomes ON and then back OFF (two transitions) before $I = 0$, then the second disruption order is placed that brings the inventory level back to S. Note that in order to have two transitions after disruption, we need to have the total length of one OFF-period and one ON-period to be less than $\frac{S}{D}$; that is, $X + Y < \frac{S}{D}$. Once disruption order is placed, the above arguments are valid all over again since $I = S$ and the memoryless property of exponential distribution (see Cycle IV in Figure 4.10 for illustration). Hence, in order to have m disruption orders when the supplier is ON for the first time inventory level hits zero; a disruption should occur in $\frac{S}{D}$ time units, then we need to have exactly $(m-1)$ combinations of OFF-periods+ON periods, each taking less than $\frac{S}{D}$ time units, and the next combination should take greater than $\frac{S}{D}$ time units.

Now we are ready to characterize the cycle length under the condition that $Z\left(\frac{Q-S}{D} = 1\right)$. Let X_0 denote the length of the ON-period starting from $t = \frac{Q-S}{D}$. Due to memoryless property of exponential distribution, $X_0 \sim Exp(\lambda)$.

Figure 4.10 Inventory Level vs. Time when $Q > S$, $Z\left(\frac{Q-S}{D}\right) = 1$



Then we have;

$$
\left(T \middle| Z\left(\frac{Q-S}{D}\right) = 1\right) = \begin{cases} \frac{Q}{D} & X_0 > \frac{S}{D} \\ \frac{Q}{D} + X_0 + \sum_{i=1}^{k-1} (X_i + Y_i) & X_i + Y_i \leq \frac{S}{D} \forall i = 1, ..., (k-1), \\ & X_0 \leq \frac{S}{D}, Y_k \leq \frac{S}{D}, \\ & X_k + Y_k > \frac{S}{D} \\ \frac{Q-S}{D} + X_0 + \sum_{i=1}^{k-1} X_i + Y_i + Y_k & X_i + Y_i \leq \frac{S}{D} \forall i = 1, ..., (k-1), \\ & X_0 \leq \frac{S}{D}, Y_k > \frac{S}{D} \end{cases} \tag{4.18}
$$

where k goes from 1 to infinity.

We next consider the expected cycle cost:

$$
E\left[\text{C(Q,S)} \middle| Z\left(\frac{Q-S}{D}\right) = 1\right] = E\left[\text{P(Q,S)} \middle| Z\left(\frac{Q-S}{D}\right) = 1\right] + E\left[\text{H(Q,S)} \middle| Z\left(\frac{Q-S}{D}\right) = 1\right] + E\left[\text{B(Q,S)} \middle| Z\left(\frac{Q-S}{D}\right) = 1\right].
$$

Let m represent a realization of M. Then using Equation 4.18, we get;

$$P\left\{M=m\middle|Z\left(\frac{Q-S}{D}\right)=1\right\}=P\left\{X\leq\frac{S}{D}\middle|Z\left(\frac{Q-S}{D}\right)=1\right\}$$
$$\left(P\left\{X+Y\leq\frac{S}{D}\middle|Z\left(\frac{Q-S}{D}\right)=1\right\}\right)^{m-1}P\left\{X+Y>\frac{S}{D}\middle|Z\left(\frac{Q-S}{D}\right)=1\right\};\forall m\geq 1.$$

Therefore the expected number of disruption orders between receipts of two regular orders can be calculated as

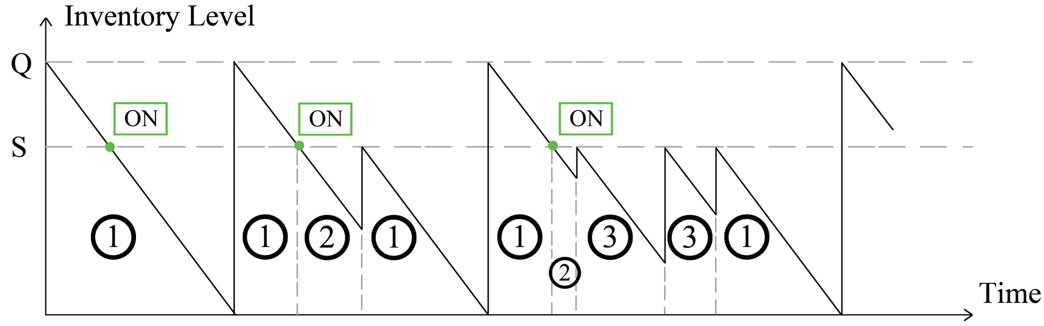
$$\begin{aligned}E\left[M\middle|Z\left(\frac{Q-S}{D}\right)=1\right]&=\sum_{m=1}^{\infty}mP\left\{X\leq\frac{S}{D}\middle|Z\left(\frac{Q-S}{D}\right)=1\right\}\\&\quad\left(P\left\{X+Y\leq\frac{S}{D}\middle|Z\left(\frac{Q-S}{D}\right)=1\right\}\right)^{m-1}P\left\{X+Y>\frac{S}{D}\middle|Z\left(\frac{Q-S}{D}\right)=1\right\}\\&=\sum_{m=0}^{\infty}m\left(1-\mathrm{e}^{-\lambda\frac{S}{D}}\right)\left(1-\frac{\mu\mathrm{e}^{-\lambda\frac{S}{D}}-\lambda\mathrm{e}^{-\mu\frac{S}{D}}}{\mu-\lambda}\right)^{m-1}\left(\frac{\mu\mathrm{e}^{-\lambda\frac{S}{D}}-\lambda\mathrm{e}^{-\mu\frac{S}{D}}}{\mu-\lambda}\right)\\&=\left(1-\mathrm{e}^{-\lambda\frac{S}{D}}\right)\left(\frac{\mu-\lambda}{\mu\mathrm{e}^{-\lambda\frac{S}{D}}-\lambda\mathrm{e}^{-\mu\frac{S}{D}}}\right).\end{aligned}$$

Then the expected fixed order cost per cycle is:

$$\begin{aligned}E\left[P\middle|Z\left(\frac{Q-S}{D}\right)=1\right]&=K\left(E\left[M\middle|Z\left(\frac{Q-S}{D}\right)=1\right]+1\right)\\&=K\left(\left(1-\mathrm{e}^{-\lambda\frac{S}{D}}\right)\left(\frac{\mu-\lambda}{\mu\mathrm{e}^{-\lambda\frac{S}{D}}-\lambda\mathrm{e}^{-\mu\frac{S}{D}}}\right)+1\right).\end{aligned}$$

In order to derive the expected inventory holding cost per cycle, we first note that the buyer depletes the regular order placed at the beginning of the cycle in every cycle independent of the number of disruption orders (this corresponds to regions marked as (1) in Figure 4.11). Next, consider a cycle with $m \geq 1$ disruption orders. In every such cycle, there exists a time period before the first disruption and after the first time $I = S$ (see these regions marked as (2) in Figure 4.11). The lengths of these time periods can be represented by the random variable $\left(X\middle|X\leq\frac{S}{D}\right)$. Finally, consider a cycle with $m \geq 2$ disruption orders. In every such cycle, there exist $(m-1)$ time periods such that each starts with S units of inventory and an OFF-supplier. The lengths of these time periods can be represented by the random variable $\left(X + Y \middle| X + Y \leq \frac{S}{D}\right)$ (see the regions marked as ③ in Figure 4.11). Note that the possible cycle realizations demonstrated in Figure 4.11 do not involve backorders as whether backorders are incurred or not do not change inventory holding cost per cycle.

Figure 4.11 Inventory Level vs. Time when $Q > S$, $Z\left(\frac{Q-S}{D}\right) = 1$



Therefore the inventory holding cost per cycle is derived as follows:

$$
\begin{aligned}
&E\left[H\middle|Z\left(\frac{Q-S}{D}\right)=1\right]=\sum_{m=0}^{\infty}E\left[H\middle|Z\left(\frac{Q-S}{D}\right)=1,M=m\right]P\left\{M=m\middle|Z\left(\frac{Q-S}{D}\right)=1\right\}\\
&=h\left(\frac{Q^2}{2D}+\sum_{m=1}^{\infty}\left(E\left[SX-\frac{D}{2}X^2\middle|X\le\frac{S}{D},Z\left(\frac{Q-S}{D}\right)=1\right]\right.\right.\\
&+(m-1)\,E\left[S\,(X+Y)-\frac{D}{2}\,(X+Y)^2\middle|X+Y\le\frac{S}{D},Z\left(\frac{Q-S}{D}\right)=1\right]\Bigg)\\
&\left(P\left\{X\le\frac{S}{D}\middle|Z\left(\frac{Q-S}{D}\right)=1\right\}P\left\{X+Y\le\frac{S}{D}\middle|Z\left(\frac{Q-S}{D}\right)=1\right\}^{(m-1)}\right.\\
&\left.\left.P\left\{X+Y>\frac{S}{D}\middle|Z\left(\frac{Q-S}{D}\right)=1\right\}\right)\right)\\
&=h\left(\frac{Q^2}{2D}+\left(1-\mathrm{e}^{-\lambda\frac{S}{D}}\right)\left(\frac{Q^2}{2D}+\left(\frac{S}{\lambda\left(1-\mathrm{e}^{-\lambda\frac{S}{D}}\right)}-\frac{S^2\mathrm{e}^{-\lambda\frac{S}{D}}}{2D\left(1-\mathrm{e}^{-\lambda\frac{S}{D}}\right)}-\frac{D}{\lambda^2}\right)\right.\right.\\
&+\left(\frac{S^2}{2D}-\left(\frac{1}{\mu-\lambda-\mu\mathrm{e}^{-\lambda\frac{S}{D}}+\lambda\mathrm{e}^{-\mu\frac{S}{D}}}\right)\left(\frac{S^2(\mu-\lambda)}{2D}-\frac{S\mu}{\lambda}+\frac{S\lambda}{\mu}-\frac{D\mu}{\lambda^2}\left(\mathrm{e}^{-\lambda\frac{S}{D}}-1\right)\right.\right.\\
&\left.\left.\left.\left.\left.+\frac{D\lambda}{\mu^2}\left(\mathrm{e}^{-\mu\frac{S}{D}}-1\right)\right)\right)\left(\frac{\mu-\lambda}{\mu\mathrm{e}^{-\lambda\frac{S}{D}}-\lambda\mathrm{e}^{-\mu\frac{S}{D}}}-1\right)\right)\right)\\
&=h\left(\frac{Q^2-S^2}{2D}+\frac{S}{\lambda}-\frac{D}{\lambda^2}\left(1-\mathrm{e}^{-\lambda\frac{S}{D}}\right)-\frac{1}{\mu\mathrm{e}^{-\lambda\frac{S}{D}}-\lambda\mathrm{e}^{-\mu\frac{S}{D}}}\left(-\frac{S\mu}{\lambda}+\frac{S\lambda}{\mu}-\frac{D\mu}{\lambda^2}\left(\mathrm{e}^{-\lambda\frac{S}{D}}-1\right)\right.\right.\\
&\left.\left.+\frac{D\lambda}{\mu^2}\left(\mathrm{e}^{-\mu\frac{S}{D}}-1\right)\right)\right).
\end{aligned}
\tag{4.19}
$$

The details of the derivations can be found in Appendix A.4.

The third component of our cost function is the shortage cost and in order to derive the expected shortage cost per cycle, we need the probability of stockout occurrence. We denote this probability by β_{ON}. To face backorders in a cycle, we need to have at least one disruption after the inventory level is below S and the last disruption should last for at least $\frac{S}{D}$ time units. Hence, we have;

$$\beta_{ON} = \sum_{k=1}^{\infty} P\left\{X_0 < \frac{S}{D} \middle| Z\left(\frac{Q-S}{D}\right) = 1\right\} P\left\{X + Y \leq \frac{S}{D} \middle| Z\left(\frac{Q-S}{D}\right) = 1\right\}^{k-1}$$
$$P\left\{Y > \frac{S}{D} \middle| Z\left(\frac{Q-S}{D}\right) = 1\right\}$$
$$= \left(1 - \mathrm{e}^{-\lambda\frac{S}{D}}\right)\left(\mathrm{e}^{-\mu\frac{S}{D}}\right)\left(\frac{\mu - \lambda}{\mu \mathrm{e}^{-\lambda\frac{S}{D}} - \lambda \mathrm{e}^{-\mu\frac{S}{D}}}\right).$$

Let W be a random variable representing the duration of shortage in a cycle. Then we have;

$$W = \begin{cases} 0 & \text{with probability } 1 - \beta_{ON} \\ Y & \text{with probability } \beta_{ON} \end{cases}$$

Given that the supplier is OFF when $I = S$, the corresponding backorder cost can be represented as $B(Q, S) = \frac{Y^2 D}{2}$. Hence we have;

$$E\left[B(Q, S) \middle| Z\left(\frac{Q-S}{D}\right) = 1\right] = b\beta_{ON} E\left[\frac{Y^2 D}{2}\right] = b\beta_{ON}\frac{D}{\mu^2}.$$

Then combining all cost components, we get the expected total cost per cycle as:

$$E\left[C\left(Q, S\right) \middle| Z\left(\frac{Q-S}{D}\right) = 1\right] = K\left(\left(1 - \mathrm{e}^{-\lambda\frac{S}{D}}\right)\left(\frac{\mu - \lambda}{\mu \mathrm{e}^{-\lambda\frac{S}{D}} - \lambda \mathrm{e}^{-\mu\frac{S}{D}}}\right) + 1\right)$$
$$+ h\left(\frac{Q^2 - S^2}{2D} + \frac{S}{\lambda} - \frac{D}{\lambda^2}\left(1 - \mathrm{e}^{-\lambda\frac{S}{D}}\right) - \frac{1}{\mu \mathrm{e}^{-\lambda\frac{S}{D}} - \lambda \mathrm{e}^{-\mu\frac{S}{D}}}\left(-\frac{S\mu}{\lambda} + \frac{S\lambda}{\mu} - \frac{D\mu}{\lambda^2}\left(\mathrm{e}^{-\lambda\frac{S}{D}} - 1\right)\right.\right.$$
$$\left.\left.+ \frac{D\lambda}{\mu^2}\left(\mathrm{e}^{-\mu\frac{S}{D}} - 1\right)\right)\right) + b\left(1 - \mathrm{e}^{-\lambda\frac{S}{D}}\right)\left(\mathrm{e}^{-\mu\frac{S}{D}}\right)\left(\frac{\mu - \lambda}{\mu \mathrm{e}^{-\lambda\frac{S}{D}} - \lambda \mathrm{e}^{-\mu\frac{S}{D}}}\right)\frac{D}{\mu^2}. \tag{4.20}$$

Finally, we derive the expected cycle length given that the system is ON at the first time inventory level hits zero using the characterization of T given in Equation 4.18.

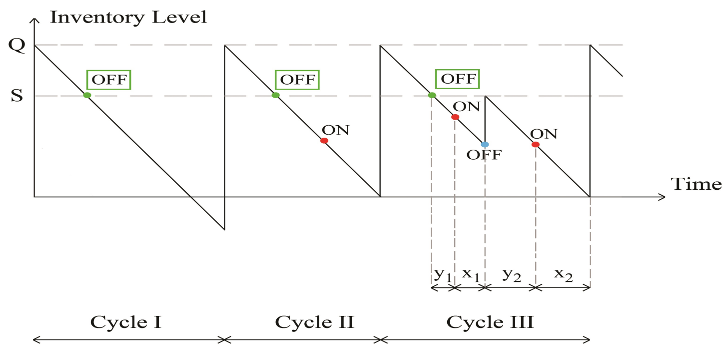
$$
\begin{aligned}
&E\left[T\middle|Z\left(\frac{Q-S}{D}\right)=1\right]=\frac{Q}{D}P\left\{X>\frac{S}{D}\middle|Z\left(\frac{Q-S}{D}\right)=1\right\}\\
&+\sum_{m=1}^{\infty}\left(\frac{Q}{D}+E\left[X|X\leq\frac{S}{D},Z\left(\frac{Q-S}{D}\right)=1\right]\right.\\
&\left.+(m-1)\,E\left[X+Y\middle|X+Y\leq\frac{S}{D},Z\left(\frac{Q-S}{D}\right)=1\right]\right)\\
&\left(P\left\{X\leq\frac{S}{D}\middle|Z\left(\frac{Q-S}{D}\right)=1\right\}P\left\{X+Y\leq\frac{S}{D}\middle|Z\left(\frac{Q-S}{D}\right)=1\right\}^{(m-1)}\right.\\
&\left.P\left\{Y\leq\frac{S}{D},X+Y>\frac{S}{D}\middle|Z\left(\frac{Q-S}{D}\right)=1\right\}\right)\\
&+\sum_{m=1}^{\infty}\left(\frac{Q-S}{D}+E\left[Y\middle|Y>\frac{S}{D}\right]+E\left[X|X\leq\frac{S}{D},Z\left(\frac{Q-S}{D}\right)=1\right]\right.\\
&\left.+(m-1)\,E\left[X+Y\middle|X+Y\leq\frac{S}{D},Z\left(\frac{Q-S}{D}\right)=1\right]\right)\\
&\left(P\left\{X\leq\frac{S}{D}\middle|Z\left(\frac{Q-S}{D}\right)=1\right\}P\left\{X+Y\leq\frac{S}{D}\middle|Z\left(\frac{Q-S}{D}\right)=1\right\}^{(m-1)}\right.\\
&\left.P\left\{Y>\frac{S}{D}\middle|Z\left(\frac{Q-S}{D}\right)=1\right\}\right)\\
&=\frac{Q-S}{D}+\frac{\left(1-e^{-\lambda\frac{S}{D}}\right)}{\lambda}+\frac{e^{-\mu\frac{S}{D}}\left(1-e^{-\lambda\frac{S}{D}}\right)(\mu-\lambda)}{\left(\mu e^{-\lambda\frac{S}{D}}-\lambda e^{-\mu\frac{S}{D}}\right)}\left(\frac{S}{D}+\frac{1}{\mu}\right)\\
&-\frac{S\,e^{-\mu\frac{S}{D}}\left(1-e^{-\lambda\frac{S}{D}}\right)(\mu-\lambda)}{D\left(\mu e^{-\lambda\frac{S}{D}}-\lambda e^{-\mu\frac{S}{D}}\right)}-\left(\frac{\left(1-e^{-\lambda\frac{S}{D}}\right)}{\mu e^{-\lambda\frac{S}{D}}-\lambda e^{-\mu\frac{S}{D}}}\right)\left(\frac{\mu\left(e^{-\lambda\frac{S}{D}}-1\right)}{\lambda}-\frac{\lambda\left(e^{-\mu\frac{S}{D}}-1\right)}{\mu}\right).
\end{aligned}
\tag{4.21}
$$

4.2.2 The system is OFF at the first time the inventory level hits S

If the system is OFF at the first time that the inventory level hits S, then the system behavior is almost the same as the case in Section 4.2.1 with one exception; the "ON" duration represented by X_0 in the Section 4.2.1 does not exist any longer. Recalling the argument for Figure 4.11, the cycles when $Z\left(\frac{Q-S}{D}\right)=0$ consist of regions ① and ③ only are observed in Figure 4.13. If the supplier remains OFF for a period longer than $\frac{S}{D}$, the buyer faces backorders and places a regular order as soon as the supplier becomes ON (see Cycle I in Figure 4.12). Similarly, the supplier may become ON before $I=0$ and remain ON until $I=0$ which indicates a regular order as soon as

$I = 0$. In these cases $\left(Y > \frac{S}{D}, \text{ or } Y \le \frac{S}{D} \text{ and } X + Y > \frac{S}{D}\right)$ there is no disruption order (see Cycle II in Figure 4.12). If the supplier becomes ON and then OFF again before $I = 0$ $\left(X + Y \le \frac{S}{D}\right)$, the buyer places a disruption order which brings the inventory level to S (see Cycle III in Figure 4.12). Noting that the supplier is OFF at this point, we might have infinitely many disruption orders in this manner.

Figure 4.12 Inventory Level vs. Time when $Q > S, Z\left(\frac{Q-S}{D}\right) = 0$



As a result, the cycle length can be characterized as follows:

$$\left(T \middle| Z\left(\frac{Q-S}{D}\right) = 0\right) = \begin{cases} \frac{Q}{D} & Y_1 \le \frac{S}{D}, X_1 + Y_1 > \frac{S}{D} \\ \frac{Q-S}{D} + Y_1 & Y_1 > \frac{S}{D} \\ \frac{Q}{D} + \sum_{i=1}^{k} (X_i + Y_i) & X_i + Y_i \le \frac{S}{D} \forall i = 1, ..., k, \\ & Y_{k+1} \le \frac{S}{D}, X_{k+1} + Y_{k+1} > \frac{S}{D} \\ \frac{Q-S}{D} + \sum_{i=1}^{k} (X_i + Y_i) + Y_{k+1} & X_i + Y_i \le \frac{S}{D} \forall i = 1, ..., k, \\ & Y_{k+1} > \frac{S}{D} \end{cases} \tag{4.22}$$

where k goes from 1 up to infinity.

We next consider the expected cycle cost:

$$E\left[\mathrm{C(Q,S)}\middle|Z\left(\frac{Q-S}{D}\right)=0\right]=E\left[\mathrm{P}\middle|Z\left(\frac{Q-S}{D}\right)=0\right]+E\left[\mathrm{H}\middle|Z\left(\frac{Q-S}{D}\right)=0\right]$$
$$+E\left[\mathrm{B}\middle|Z\left(\frac{Q-S}{D}\right)=0\right].$$

Utilizing Equation 4.22, we get;

$$P\left\{M=m\middle|Z\left(\frac{Q-S}{D}\right)=0\right\}=\left(P\left\{X+Y\leq\frac{S}{D}\middle|Z\left(\frac{Q-S}{D}\right)=0\right\}\right)^{m}$$
$$P\left\{X+Y>\frac{S}{D}\middle|Z\left(\frac{Q-S}{D}\right)=0\right\}.$$

Then, the expected number of disruption orders between receipts of two regular orders can be calculated as:

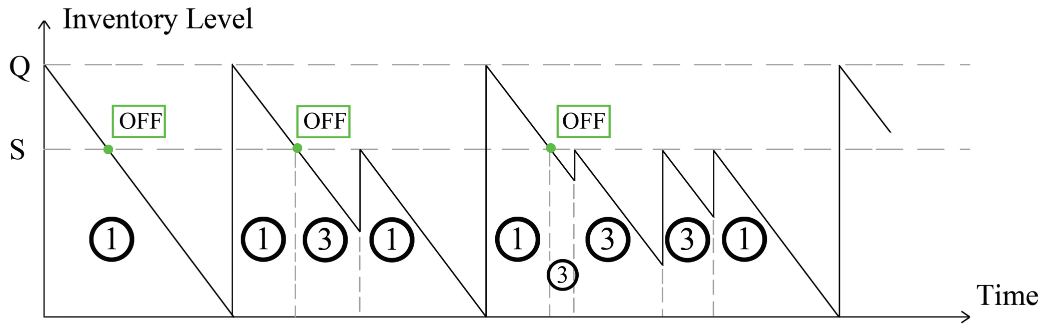
$$E\left[M\middle|Z\left(\frac{Q-S}{D}\right)=0\right]=\sum_{m=0}^{\infty}m\left(P\left\{X+Y\leq\frac{S}{D}\middle|Z\left(\frac{Q-S}{D}\right)=0\right\}\right)^{m}$$
$$P\left\{X+Y>\frac{S}{D}\middle|Z\left(\frac{Q-S}{D}\right)=0\right\}$$
$$=\sum_{m=0}^{\infty}m\left(1-\frac{\mu\mathrm{e}^{-\lambda\frac{S}{D}}-\lambda\mathrm{e}^{-\mu\frac{S}{D}}}{\mu-\lambda}\right)^{m}\left(\frac{\mu\mathrm{e}^{-\lambda\frac{S}{D}}-\lambda\mathrm{e}^{-\mu\frac{S}{D}}}{\mu-\lambda}\right)$$
$$=\left(\frac{\mu-\lambda-\mu\mathrm{e}^{-\lambda\frac{S}{D}}+\lambda\mathrm{e}^{-\mu\frac{S}{D}}}{\mu\mathrm{e}^{-\lambda\frac{S}{D}}-\lambda\mathrm{e}^{-\mu\frac{S}{D}}}\right).$$

Then, fixed order cost per cycle is:

$$E\left[P\middle|Z\left(\frac{Q-S}{D}\right)=0\right]=K\left(E\left[M\middle|Z\left(\frac{Q-S}{D}\right)=0\right]+1\right)$$
$$=K\left(\frac{\mu-\lambda-\mu\mathrm{e}^{-\lambda\frac{S}{D}}+\lambda\mathrm{e}^{-\mu\frac{S}{D}}}{\mu\mathrm{e}^{-\lambda\frac{S}{D}}-\lambda\mathrm{e}^{-\mu\frac{S}{D}}}+1\right).$$

We first note as in Section 4.2.1 that the buyer depletes the regular order placed at the beginning of the cycle in every cycle independent of the number of disruption orders (again this corresponds to regions marked as (1) in Figure 4.13). Then, consider a cycle with $m \geq 1$ disruption orders recalling that the supplier is OFF for the first time that inventory level is S. In every such cycle, there exists m time periods such that each starts with S units of inventory and an OFF-supplier. The lengths of these time periods can be represented by the random variable $\left(X + Y \middle| X + Y \leq \frac{S}{D}\right)$ by the memoryless property of exponential distribution (see the regions marked as ③ in Figure 4.13). Similar to Section 4.2.1, the possible cycle realizations demonstrated in Figure 4.13 do not involve backorders as whether backorders are incurred or not do no change inventory holding cost per cycle.

Figure 4.13 Inventory Level vs. Time when $Q > S$, $Z\left(\frac{Q-S}{D}\right) = 0$



Then, inventory holding cost per cycle is derived as follows:

$$
\begin{aligned}
&E\left[H\middle|Z\left(\frac{Q-S}{D}\right)=0\right]=E\left[H\middle|Z\left(\frac{Q-S}{D}\right)=0,M=m\right]P\left\{M=m\middle|Z\left(\frac{Q-S}{D}\right)=0\right\}\\
&=h\left(\frac{Q^2}{2D}P\left\{X+Y>\frac{S}{D}\middle|Z\left(\frac{Q-S}{D}\right)=0\right\}+\sum_{m=1}^{\infty}\left(\frac{Q^2}{2D}\right.\right.\\
&+mE\left[S\left(X+Y\right)-\frac{D}{2}\left(X+Y\right)^2\middle|X+Y\leq\frac{S}{D},Z\left(\frac{Q-S}{D}\right)=0\right]\Bigg)\\
&P\left\{X+Y\leq\frac{S}{D}\middle|Z\left(\frac{Q-S}{D}\right)=0\right\}^{(m)}P\left\{X+Y>\frac{S}{D}\middle|Z\left(\frac{Q-S}{D}\right)=0\right\}\Bigg)\\
&=h\left(\frac{Q^2}{2D}\left(\frac{\mu e^{-\lambda\frac{S}{D}}-\lambda e^{-\mu\frac{S}{D}}}{\mu-\lambda}\right)+\left(\frac{Q^2}{2D}\left(1-\frac{\mu e^{-\lambda\frac{S}{D}}-\lambda e^{-\mu\frac{S}{D}}}{\mu-\lambda}\right)+\right.\right.\\
&+\left(\frac{S^2}{2D}-\left(\frac{1}{\mu-\lambda-\mu e^{-\lambda\frac{S}{D}}+\lambda e^{-\mu\frac{S}{D}}}\right)\left(\frac{S^2\left(\mu-\lambda\right)}{2D}-\frac{S\mu}{\lambda}+\frac{S\lambda}{\mu}-\frac{D\mu}{\lambda^2}\left(e^{-\lambda\frac{S}{D}}-1\right)\right.\right.\\
&\left.\left.\left.\left.+\frac{D\lambda}{\mu^2}\left(e^{-\mu\frac{S}{D}}-1\right)\right)\left(\frac{\mu-\lambda}{\mu e^{-\lambda\frac{S}{D}}-\lambda e^{-\mu\frac{S}{D}}}-1\right)\right)\right)\right)\\
&=h\left(\frac{Q^2-S^2}{2D}-\frac{1}{\mu e^{-\lambda\frac{S}{D}}-\lambda e^{-\mu\frac{S}{D}}}\left(-\frac{S\mu}{\lambda}+\frac{S\lambda}{\mu}-\frac{D\mu}{\lambda^2}\left(e^{-\lambda\frac{S}{D}}-1\right)\right.\right.\\
&\left.\left.+\frac{D\lambda}{\mu^2}\left(e^{-\mu\frac{S}{D}}-1\right)\right)\right).
\end{aligned}
\tag{4.23}
$$

The third component of our cost function is shortage cost. Utilizing similar arguments to the ones in Section 4.2.1, the stockout probability in a cycle is calculated as:

$$
\begin{aligned}
\beta_{OFF}&=\sum_{k=0}^{\infty}P\left\{X+Y\leq\frac{S}{D}\middle|Z\left(\frac{Q-S}{D}\right)=0\right\}^{k}P\left\{Y>\frac{S}{D}\middle|Z\left(\frac{Q-S}{D}\right)=0\right\}\\
&=\left(e^{-\mu\frac{S}{D}}\right)\left(\frac{\mu-\lambda}{\mu e^{-\lambda\frac{S}{D}}-\lambda e^{-\mu\frac{S}{D}}}\right).
\end{aligned}
$$

Therefore by a similar argument in Section 4.2.1, the expected shortage cost:

$$
E\left[B\middle|Z\left(\frac{Q-S}{D}\right)=0\right]=b\beta_{OFF}E\left[\frac{Y^2D}{2}\right]=b\beta_{OFF}\frac{D}{\mu^2}.
$$

Then combining all cost components, we get the expected total cost per cycle as:

$$
\begin{aligned}
&E\left[C(Q,S)\middle|Z\left(\frac{Q-S}{D}\right)=0\right]=K\left(\frac{\mu-\lambda-\mu \mathrm{e}^{-\lambda\frac{S}{D}}+\lambda \mathrm{e}^{-\mu\frac{S}{D}}}{\mu \mathrm{e}^{-\lambda\frac{S}{D}}-\lambda \mathrm{e}^{-\mu\frac{S}{D}}}+1\right)\\
&+h\left(\frac{Q^2-S^2}{2D}-\frac{1}{\mu \mathrm{e}^{-\lambda\frac{S}{D}}-\lambda \mathrm{e}^{-\mu\frac{S}{D}}}\left(-\frac{S\mu}{\lambda}+\frac{S\lambda}{\mu}-\frac{D\mu}{\lambda^2}\left(\mathrm{e}^{-\lambda\frac{S}{D}}-1\right)\right.\right.\\
&\left.\left.+\frac{D\lambda}{\mu^2}\left(\mathrm{e}^{-\mu\frac{S}{D}}-1\right)\right)\right)+b\left(\mathrm{e}^{-\mu\frac{S}{D}}\right)\left(\frac{\mu-\lambda}{\mu \mathrm{e}^{-\lambda\frac{S}{D}}-\lambda \mathrm{e}^{-\mu\frac{S}{D}}}\right)\frac{D}{\mu^2}.
\end{aligned}
\tag{4.24}
$$

Finally, we derive the expected cycle length given that the system is OFF at the first time inventory level hits zero using the characterization of T given in Equation 4.22;

$$
\begin{aligned}
&E\left[T\middle|Z\left(\frac{Q-S}{D}\right)=0\right]=\frac{Q}{D}P\left\{Y\le\frac{S}{D},X+Y>\frac{S}{D}\middle|Z\left(\frac{Q-S}{D}\right)=0\right\}\\
&+\left(\frac{Q-S}{D}+E\left[Y\middle|Y>\frac{S}{D}\right]\right)P\left\{Y>\frac{S}{D}\middle|Z\left(\frac{Q-S}{D}\right)=0\right\}\\
&+\sum_{m=1}^{\infty}\left(\frac{Q}{D}+mE\left[X+Y\middle|X+Y\le\frac{S}{D},Z\left(\frac{Q-S}{D}\right)=0\right]\right)\\
&\left(P\left\{X+Y\le\frac{S}{D}\middle|Z\left(\frac{Q-S}{D}\right)=0\right\}^m P\left\{Y\le\frac{S}{D},X+Y>\frac{S}{D}\middle|Z\left(\frac{Q-S}{D}\right)=0\right\}\right)\\
&+\sum_{m=1}^{\infty}\left(\frac{Q-S}{D}+E\left[Y\middle|Y>\frac{S}{D}\right]+mE\left[X+Y\middle|X+Y\le\frac{S}{D},Z\left(\frac{Q-S}{D}\right)=0\right]\right)\\
&\left(P\left\{X+Y\le\frac{S}{D}\middle|Z\left(\frac{Q-S}{D}\right)=0\right\}^m P\left\{Y>\frac{S}{D}\middle|Z\left(\frac{Q-S}{D}\right)=0\right\}\right)\\
&=\frac{Q-S}{D}+\frac{\mathrm{e}^{-\mu\frac{S}{D}}(\mu-\lambda)}{\left(\mu \mathrm{e}^{-\lambda\frac{S}{D}}-\lambda \mathrm{e}^{-\mu\frac{S}{D}}\right)}\left(\frac{S}{D}+\frac{1}{\mu}\right)-\frac{S\,\mathrm{e}^{-\mu\frac{S}{D}}(\mu-\lambda)}{D\left(\mu \mathrm{e}^{-\lambda\frac{S}{D}}-\lambda \mathrm{e}^{-\mu\frac{S}{D}}\right)}\\
&-\left(\frac{1}{\mu \mathrm{e}^{-\lambda\frac{S}{D}}-\lambda \mathrm{e}^{-\mu\frac{S}{D}}}\right)\left(\frac{\mu\left(\mathrm{e}^{-\lambda\frac{S}{D}}-1\right)}{\lambda}-\frac{\lambda\left(\mathrm{e}^{-\mu\frac{S}{D}}-1\right)}{\mu}\right).
\end{aligned}
\tag{4.25}
$$

4.2.3 Expected Cost Calculations for $Q > S$

In this section, we combine our findings in Section 4.2.1 and Section 4.2.2. At first, we condition on whether the supplier is ON or OFF for the first time the inventory level hits S to characterize cycle lengths. Then combining them, we get T as a random variable representing the cycle length. We observe that the all T_i's, in other words, all cycles have identical distribution and they are independent. Hence, Renewal Reward Theorem can be applied to derive the expected cost function per unit time.

By using the probability of being ON and OFF at the first time the inventory level hits S, we uncondition the expected costs and the expected cycle length.

$$P\left\{Z\left(\frac{Q-S}{D}\right)=1\right\}=1-\left(\frac{\lambda}{\lambda+\mu}\left(1-\mathrm{e}^{-(\lambda+\mu)\frac{Q-S}{D}}\right)\right) \tag{4.26}$$

$$P\left\{Z\left(\frac{Q-S}{D}\right)=0\right\}=\frac{\lambda}{\lambda+\mu}\left(1-\mathrm{e}^{-(\lambda+\mu)\frac{Q-S}{D}}\right). \tag{4.27}$$

These probabilities are derived by using the transient behavior of our supply process similar in Chapter 3.

Then, the total expected cost per cycle is calculated by substituting Equations 4.20, 4.24 and probabilities given in Equations 4.26, 4.27 into Equation 4.17:

$$\begin{aligned}
E\left[C(Q,S)\right] &= E\left[C(Q,S)\middle|Z\left(\frac{Q-S}{D}\right)=1\right]P\left\{Z\left(\frac{Q-S}{D}\right)=1\right\}\\
&+E\left[C(Q,S)\middle|Z\left(\frac{Q-S}{D}\right)=0\right]P\left\{Z\left(\frac{Q-S}{D}\right)=0\right\}\\
&=\left(\frac{\lambda}{\lambda+\mu}\left(1-\mathrm{e}^{-(\lambda+\mu)\frac{Q-S}{D}}\right)\right)\left[K\left(\frac{\mathrm{e}^{-\lambda\frac{S}{D}}(\mu-\lambda)}{\mu\mathrm{e}^{-\lambda\frac{S}{D}}-\lambda\mathrm{e}^{-\mu\frac{S}{D}}}-1\right)-h\left(\frac{S}{\lambda}-\frac{D}{\lambda^2}\left(1-\mathrm{e}^{-\lambda\frac{S}{D}}\right)\right)\right.\\
&+b\left(\mathrm{e}^{-\lambda\frac{S}{D}}\mathrm{e}^{-\mu\frac{S}{D}}\frac{(\mu-\lambda)}{\mu\mathrm{e}^{-\lambda\frac{S}{D}}-\lambda\mathrm{e}^{-\mu\frac{S}{D}}}\frac{D}{\mu^2}\right)+K\left(\left(1-\mathrm{e}^{-\lambda\frac{S}{D}}\right)\left(\frac{\mu-\lambda}{\mu\mathrm{e}^{-\lambda\frac{S}{D}}-\lambda\mathrm{e}^{-\mu\frac{S}{D}}}\right)+1\right)\\
&+h\left(\frac{Q^2-S^2}{2D}+\frac{S}{\lambda}-\frac{D}{\lambda^2}\left(1-\mathrm{e}^{-\lambda\frac{S}{D}}\right)-\frac{1}{\mu\mathrm{e}^{-\lambda\frac{S}{D}}-\lambda\mathrm{e}^{-\mu\frac{S}{D}}}\left(-\frac{S\mu}{\lambda}+\frac{S\lambda}{\mu}-\frac{D\mu}{\lambda^2}\left(\mathrm{e}^{-\lambda\frac{S}{D}}-1\right)\right.\right.\\
&\left.\left.\left.+\frac{D\lambda}{\mu^2}\left(\mathrm{e}^{-\mu\frac{S}{D}}-1\right)\right)\right)+b\left(1-\mathrm{e}^{-\lambda\frac{S}{D}}\right)\left(\mathrm{e}^{-\mu\frac{S}{D}}\right)\left(\frac{\mu-\lambda}{\mu\mathrm{e}^{-\lambda\frac{S}{D}}-\lambda\mathrm{e}^{-\mu\frac{S}{D}}}\right)\frac{D}{\mu^2}.
\end{aligned}$$

Then the expected length of a cycle is derived using Equation 4.21, 4.25 and probabilities given in Equations 4.26, 4.27 as follows:

$$
\begin{aligned}
E\left[T(Q,S)\right] &= E\left[T(Q,S)\middle| Z\left(\frac{Q-S}{D}\right)=1\right] P\left\{Z\left(\frac{Q-S}{D}\right)=1\right\} \\
&+ E\left[T(Q,S)\middle| Z\left(\frac{Q-S}{D}\right)=0\right] P\left\{Z\left(\frac{Q-S}{D}\right)=0\right\} \\
&= \left(\frac{\lambda}{\lambda+\mu}\left(1-\mathrm{e}^{-(\lambda+\mu)\frac{Q-S}{D}}\right)\right)\left[\frac{\mathrm{e}^{-\mu\frac{S}{D}}\mathrm{e}^{-\lambda\frac{S}{D}}(\mu-\lambda)}{\mu\left(\mu\mathrm{e}^{-\lambda\frac{S}{D}}-\lambda\mathrm{e}^{-\mu\frac{S}{D}}\right)} - \frac{\left(1-\mathrm{e}^{-\lambda\frac{S}{D}}\right)}{\lambda}\right. \\
&\left. - \left(\frac{\mathrm{e}^{-\lambda\frac{S}{D}}}{\mu\mathrm{e}^{-\lambda\frac{S}{D}}-\lambda\mathrm{e}^{-\mu\frac{S}{D}}}\right)\left(\frac{\mu\left(\mathrm{e}^{-\lambda\frac{S}{D}}-1\right)}{\lambda} - \frac{\lambda\left(\mathrm{e}^{-\mu\frac{S}{D}}-1\right)}{\mu}\right)\right] + \frac{Q-S}{D} + \frac{\left(1-\mathrm{e}^{-\lambda\frac{S}{D}}\right)}{\lambda} \\
&+ \frac{\mathrm{e}^{-\mu\frac{S}{D}}\left(1-\mathrm{e}^{-\lambda\frac{S}{D}}\right)(\mu-\lambda)}{\mu\left(\mu\mathrm{e}^{-\lambda\frac{S}{D}}-\lambda\mathrm{e}^{-\mu\frac{S}{D}}\right)} - \left(\frac{\left(1-\mathrm{e}^{-\lambda\frac{S}{D}}\right)}{\mu\mathrm{e}^{-\lambda\frac{S}{D}}-\lambda\mathrm{e}^{-\mu\frac{S}{D}}}\right)\left(\frac{\mu\left(\mathrm{e}^{-\lambda\frac{S}{D}}-1\right)}{\lambda} - \frac{\lambda\left(\mathrm{e}^{-\mu\frac{S}{D}}-1\right)}{\mu}\right).
\end{aligned}
$$

Since cycles are statistically identical and independent of each other, the expected cost per unit time is found by using The Renewal Reward Theorem:

$$
E\left[G(Q,S)\right] = \frac{E\left[C(Q,S)\right]}{E\left[T(Q,S)\right]} \tag{4.28}
$$

Unfortunately, we cannot show that $E\left[G(Q,S)\right]$ is convex or not. To find the optimal values of Q and S, we propose a search algortihm:

1. Set the iteration counter i=0.
2. Set Q_0 = EOQ with backorder where EOQ with backorder= $\sqrt{\frac{C_o+C_s}{C_s}}\sqrt{\frac{2C_oD}{C_h}}$.
3. Given Q_0 search for S_0 that minimizes $E\left[G(Q,S)\right]$ from 0 to $Q_0 - 0.01$ with step size 0.01.
4. Set $i = i + 1$.
5. Given S_{i-1} search for Q_i that minimizes $E\left[G(Q,S)\right]$ from $S_{i-1} + 0.01$ to $S_{i-1} + 50000$ with step size 0.01.
6. Given Q_i search for S_i that minimizes $E\left[G(Q,S)\right]$ from 0 to $Q_i - 0.01$ with step size 0.01.
7. Return to Step 4 until $Q_i = Q_i \pm 0.01$ and $S_i = S_i \pm 0.01$.

8. Set $Q_i = Q^*$ and $S_i = S^*$.

9. Substitute Q^* and S^* to calculate the minimum $E[G(Q,S)]$.

Again because of the complex structure of $G(Q,S)$ in this section, we cannot make any comparison between the total expected cost function derived in Section 4.1 and Section 4.2. The optimal values of Q and S is whichever leads to the minimum total expected cost derived in Section 4.1 and Section 4.2.

CHAPTER 5

COMPUTATIONAL STUDIES

In this chapter, our aim is to assess the value of an order opportunity when the supplier just turns into an OFF state to the buyer. For this purpose, we conduct a numerical study and compare the expected cost with no-order opportunity (Equation 3.3) to the one with order opportunity (min {Equation 4.16, 4.28}). We perform a sensitivity analysis to observe how problem parameters affect optimal decisions and corresponding expected costs. The percentage improvement in expected cost due to disruption order is calculated as;

$$\%\Delta = \frac{z_n - z_o}{z_n} \times 100, \tag{5.1}$$

where z_n is the expected cost with no-order opportunity (Equation 3.3), and z_o is the expected cost with order opportunity ($min\{\text{Equation } 4.16, 4.28\}$).

Recall that the algorithms to find Q^* and S^* are given in Section 4.1.3 and in Section 4.2.3. The algorithms are solved using C.

The rest of this chapter is organized as follows: In Section 5.1, we present results of the computational study that uses Berk and Arreola Risa's (1994) parameter set. In Section 5.2, we construct a base setting and conduct sensitivity analysis for problem parameters.

5.1 Computational Study Using the Parameter Set in Berk and Arreola Risa (1994)

We assess the value of order opportunity by using a parameter set which studied in literature. The parameter set is given in Table 5.1. This parameter set leads to 1120 problem instances. We use their unit shortage cost as our unit backorder cost per unit time and assess the value of disruption order.

Table 5.1: List of Parameter Values Used

Parameter	Symbol	Value
Inventory holding cost	h	1
Fixed order cost	K	0.1, 1, 10, 100
Shortage cost/unit/time	b	0.1, 1, 10, 100
Demand rate	D	100, 1000
OFF period mean	$1/\mu$	10, 1, 0.5, 0.25, 0.1
Ratio of ON and OFF periods' means	λ/μ	1, 0.8, 0.5, 0.25, 0.1, 0.05, 0.01

First of all, note that when there is no disruption order opportunity, the order-up-to levels deviate from EOQ significantly if OFF-period length is significant. However, when there is an order opportunity, regular order-up-to levels are very close to EOQ and disruption risk is handled by significantly large disruption order-up-to levels when necessary. Then, the main advantage of order opportunity is that even if the buyer places more regular orders leading to a larger expected fixed order cost, expected inventory holding costs and expected backorder cost is much more smaller. So, the expected total cost in the order opportunity model is much less than the expected total cost in the no-order opportunity model for this parameter set. We observe that the optimal order-up-to levels of no-order opportunity model and the optimum disruption order-up-to levels get smaller as $1/\lambda$ gets longer if $1/\mu$ is kept constant.

Now, we analyze the percentage improvements for all 1120 instances. The most important observation we make is that, order opportunity model chooses not to place any disruption order in 343 instances, leading $\%\Delta = 0$. In these 343 instances b is generally much smaller than K. Also, as b gets larger and K gets smaller, the order opportunity model chooses $Q \leq S$ as the optimum policy and the percentage improvement gets larger. The order opportunity model never chooses $Q > S > 0$

Table 5.2: An Example of The Results using Berk and Arreola Risa's Parameters

$K = 10, h = 1, b = 10, D = 1000$						
$1/\lambda$	**$1/\mu$**	**No-order Opp with backorders (Q)**	**Order Opp (Q)**	**Order Opp (S)**	**%Δ**	**$z_n - z_o$**
10	10	26605.13	140.09	19435.25	43.75	11797.21
12.5	10	24910.93	140.97	19094.00	46.54	11905.68
20	10	20508.86	140.86	18533.29	52.83	11782.60
40	10	12307.53	141.03	18031.41	61.30	10283.48
100	10	653.29	141.45	17711.19	66.38	6048.09
200	10	198.53	141.22	17603.75	65.44	3178.87
1000	10	148.97	141.41	17516.69	58.85	660.85
1	1	2663.90	137.98	1963.64	42.60	1149.87
1.25	1	2494.96	138.63	1932.84	45.03	1153.49
2	1	2056.71	139.69	1881.73	50.31	1124.15
4	1	1247.43	140.50	1836.38	56.50	951.77
10	1	311.82	141.06	1807.95	56.68	540.60
20	1	188.02	141.17	1798.44	50.54	290.56
100	1	148.19	141.35	1790.47	26.29	61.08
0.5	0.5	1337.07	134.85	994.04	41.01	555.46
0.625	0.5	1253.32	136.22	980.05	43.08	554.03
1	0.5	1037.08	138.02	957.80	47.34	531.83
2	0.5	647.17	139.68	938.08	51.28	437.31
5	0.5	253.00	140.67	925.35	47.66	243.21
10	0.5	179.88	141.05	921.23	38.91	132.13
50	0.5	147.44	141.36	917.61	15.29	28.01
0.25	0.25	678.64	129.12	510.70	36.95	253.70
0.3125	0.25	638.14	131.50	505.40	38.43	250.95
0.5	0.25	535.27	134.98	497.15	40.98	235.26
1	0.25	361.77	138.11	490.14	41.51	185.10
2.5	0.25	208.32	140.02	485.77	33.90	100.49
5	0.25	169.24	140.72	484.23	24.47	55.09
25	0.25	146.21	141.25	482.82	7.41	11.80
0.1	0.1	297.93	114.59	225.64	19.22	57.62
0.125	0.1	284.57	119.05	225.66	20.03	57.69
0.2	0.1	252.85	126.46	226.16	20.67	53.97
0.4	0.1	206.68	133.55	227.15	18.52	40.99
1	0.1	167.95	138.12	227.85	12.10	21.89
2	0.1	154.41	139.76	228.25	7.43	12.10
10	0.1	143.95	141.03	228.38	1.80	2.62

for this parameter set. The optimum policy turns out to be $Q \leq S$ for 777 instances. Among them, we get more than 10% improvement for 604 instances. We get lower improvements especially if b is smaller than both K and h. Also, as $1/\lambda$ gets larger enough, in other words the supplier's state is ON for a long time, the percentage improvements get smaller as expected, even though they seem to get bigger at first. So the change in the percentage improvement is not monotone. In that aspect, we look into the absolute difference between the expected cost of no-order opportunity model and the expected cost of order opportunity model, and observe that the saving is decreasing as $1/\lambda$ gets larger.

In Table 5.3, the descriptive statistics of the improvement percentage is given.

Table 5.3: Descriptive Statistics of Improvement Percentages

N	1120
Mean	26.65
StDev	28.60
Minimum	0
Q1	0
Median	16.28
Q3	55.25
Maximum	90.78

Parameters and expected cost of no-order and order opportunity of maximum improvement are given in Table 5.4. Since demand rate is large, backorders that occur at OFF-periods affect the expected total cost significantly, leading to a large disruption order-up-to level, even though the supplier is at ON state for a very long period of time and occurrence of a disruption is unlikely. For this long period of ON state, the regular order-up-to level of the order opportunity model is very close to order quantity of the basic EOQ model.

Parameters and expected cost of no-order and order opportunity of minimum positive improvement are given in Table 5.5. In this setting, the supplier's OFF state does not last too long and b is too small compared to h. Since $S^* = 23.87$, the expected inventory holding cost is bigger, but the expected backorder cost is smaller in the order opportunity model.

Table 5.4: Maximum Improvement

$K = 0.1, h = 1, b = 100, D = 1000$ $1/\lambda = 1000, 1/\mu = 10$		
EOQ		14.14
No-order Opp	Q	144.00
	$P(Q)$	0.69
	$H(Q)$	71.29
	$B(Q)$	9830.04
	$G(Q)$	9902.02
Order Opp	Q	14.13
	S	33930.17
	$P(Q,S)$	6.84
	$H(Q,S)$	573.63
	$B(Q,S)$	332.74
	$G(Q,S)$	913.21
$\%\Delta$		90.78

Table 5.5: Minimum Positive Improvement

$K = 0.1, h = 1, b = 0.1, D = 1000$ $1/\lambda = 25, 1/\mu = 0.25$		
EOQ		14.14
No-order Opp	Q	14.15
	$P(Q)$	7.00
	$H(Q)$	7.01
	$B(Q)$	0.24
	$G(Q)$	14.25
Order Opp	Q	14.14
	S	23.87
	$P(Q,S)$	7.01
	$H(Q,S)$	7.01
	$B(Q,S)$	0.22
	$G(Q,S)$	14.24
$\%\Delta$		0.02

Now, we move on with sensitivity analysis.

5.2 Sensitivity Analysis

We set our base setting as $K = 10$, $h = 1$, $b = 10$, $D = 100$, $1/\lambda = 4$, $1/\mu = 1$. If there is no disruption, the optimal order quantity in this setting is 44.72.

The results of base setting are given in Table 5.6. Note that, the order opportunity model sets Q^* very close to EOQ. Then, it sets S^* to a higher value to avoid backorders. Although the expected fixed order cost is smaller in the no-order opportunity model, the expected backorder is too high, leading to a significant improvement with the order opportunity model.

Table 5.6: The Results of Base Setting

$K = 10, h = 1, b = 10, D = 100$ $1/\lambda = 4, 1/\mu = 1$		
EOQ		44.72
No-order Opp	Q	137.56
	$P(Q)$	6.49
	$H(Q)$	61.45
	$B(Q)$	106.62
	$G(Q)$	174.56
Order Opp	Q	43.89
	S	192.38
	$P(Q,S)$	16.93
	$H(Q,S)$	49.04
	$B(Q,S)$	29.21
	$G(Q,S)$	95.17
$\%\Delta$		45.48

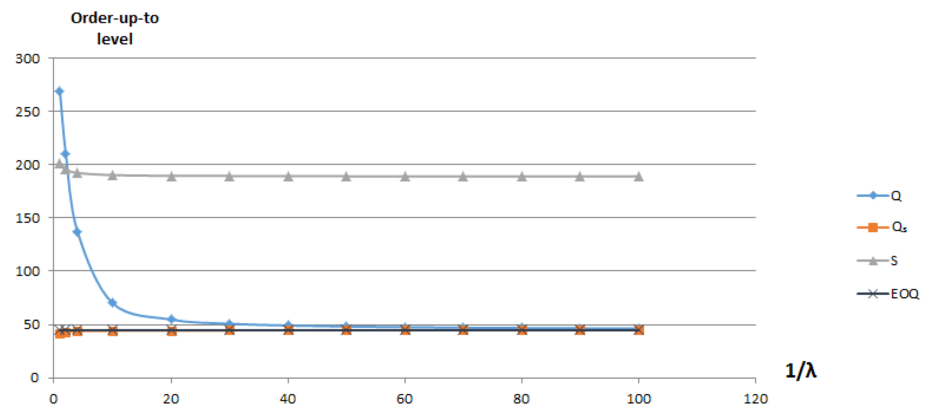
Next, we consider parameters K, h, b, D and λ and conduct sensitivity analysis by changing these parameters one at a time. The values for these parameters are presented in Table 5.6. We provide all the results in Appendix B.

We start with the average length of ON-periods, $1/\lambda$.

Table 5.7: Parameters Used for Sensitivity Analysis

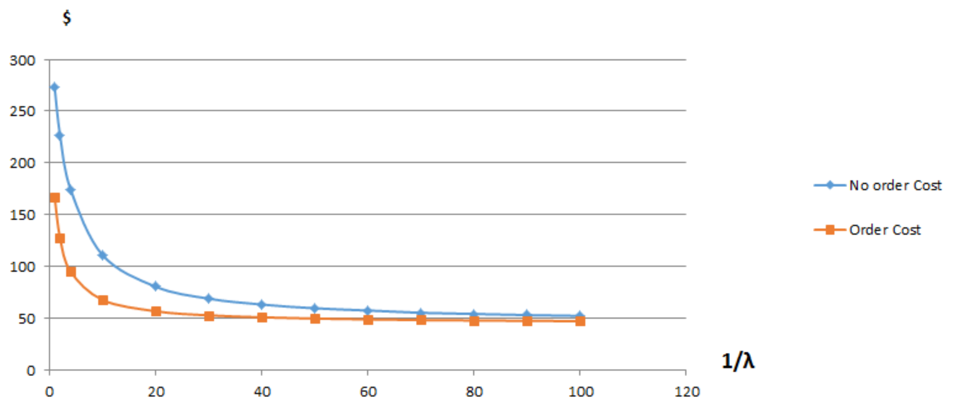
Varying parameter	Value
$1/\lambda$	1, 2, 4, 10, 20, 30, 40, 50, 60, 70, 80, 90, 100
K	5, 10, 15, 20, 25, 30, 35, 40, 45, 50, 55, 60, 65, 70, 75, 80, 85, 90, 95, 100
h	0.1, 0.5, 1, 1.5, 2, 2.5, 3, 3.5, 4, 4.5, 5
b	5, 10, 15, 20, 25, 30, 35, 40, 45, 50, 55, 60, 65, 70, 75, 80, 85, 90, 95, 100

Figure 5.1 Order-up-to Levels vs. Expected ON-period length



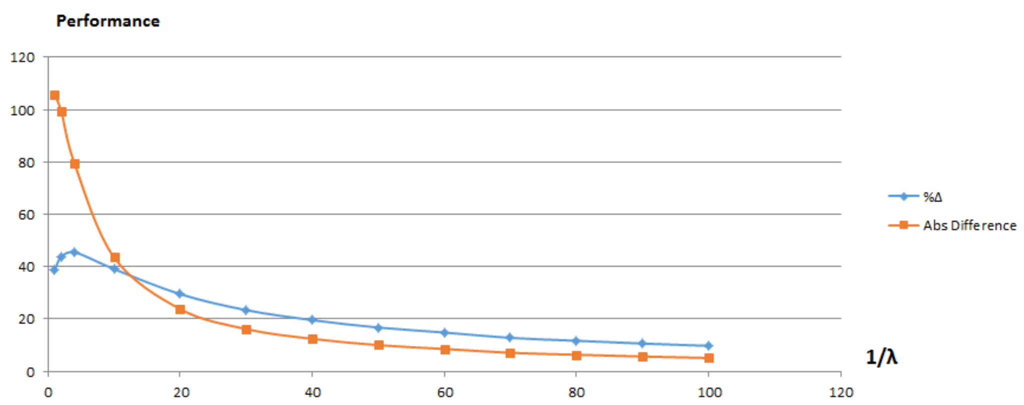
We observe that as the duration of ON-periods gets larger, the order-up-to level in the no-order opportunity model decreases as expected and it converges to EOQ without backorders (see Figure 5.1). In the order opportunity model, Q^* shows a slight increasing behavior and it is usually close to EOQ without backorders. The disruption order-up-to level is twice as much as the average demand during disruption. So it lowers the expected backorders cost significantly compared to the no-order opportunity model.

Figure 5.2 Total Expected Cost vs. Expected ON-period length



We now consider the corresponding costs. Expected costs shows a convex decreasing behavior in both models and they converge to the cost of EOQ setting, which is 44.72 (see Figure 5.2).

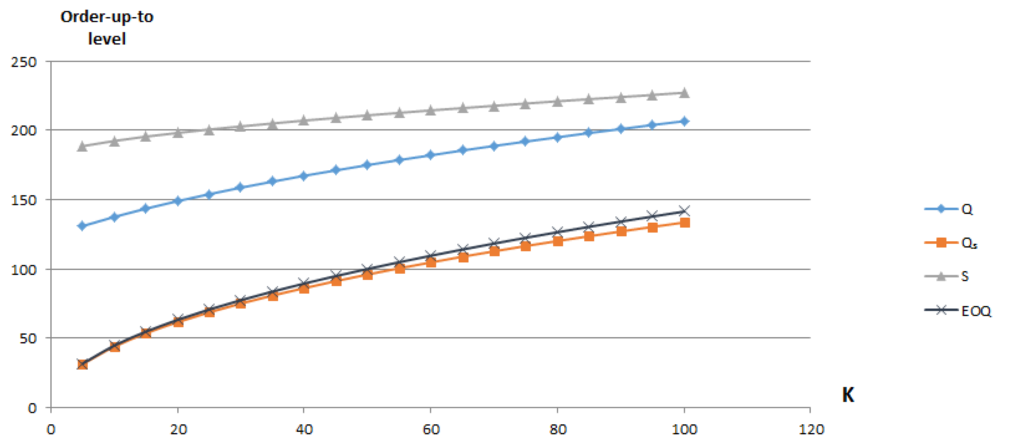
Figure 5.3 Improvement vs. Expected ON-period length



Similar to the discussion in Section 5.1; as $1/\lambda$ gets larger, at first the percentage improvements seem to get bigger. Yet, if $1/\lambda$ gets larger enough, they get smaller as expected. So the change in the percentage improvement is not monotone. When we look into the absolute difference between the expected cost of both models, we observe that the saving is decreasing as $1/\lambda$ gets larger (see Figure 5.3).

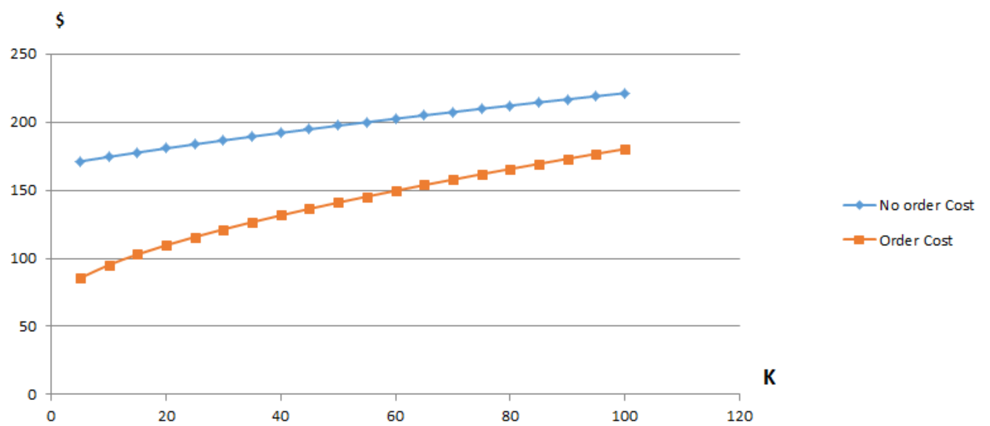
We next consider the effects of fixed order cost. Figure 5.7 depicts the change in optimal ordering decisions.

Figure 5.4 Order-up-to Levels vs. Fixed Order Cost



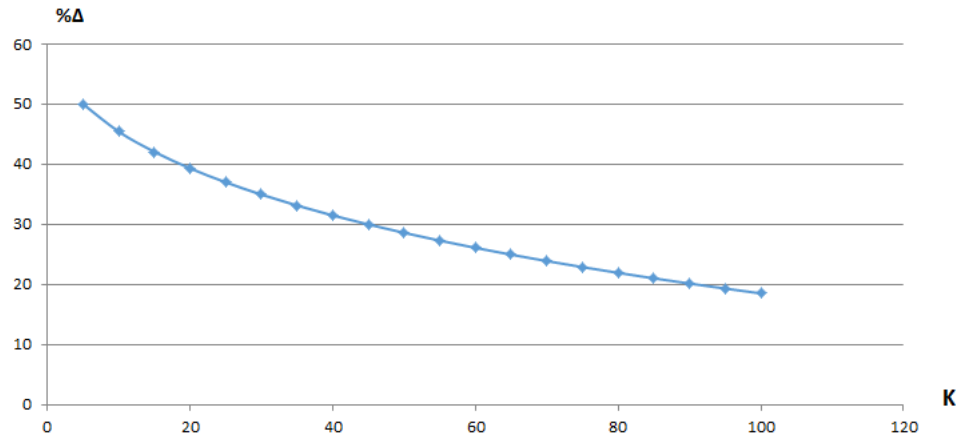
The optimal order-up-to levels for both models increase as expected. Note that the increase in S^* is less steep as a disruption order definitely is placed whenever a disruption occurs if the inventory level below S^*. Hence, increasing S^* will not result in less disruption orders. However, it might result in less regular orders; hence, it increases as well (Figure 5.4).

Figure 5.5 Total Expected Cost vs. Fixed Order Cost



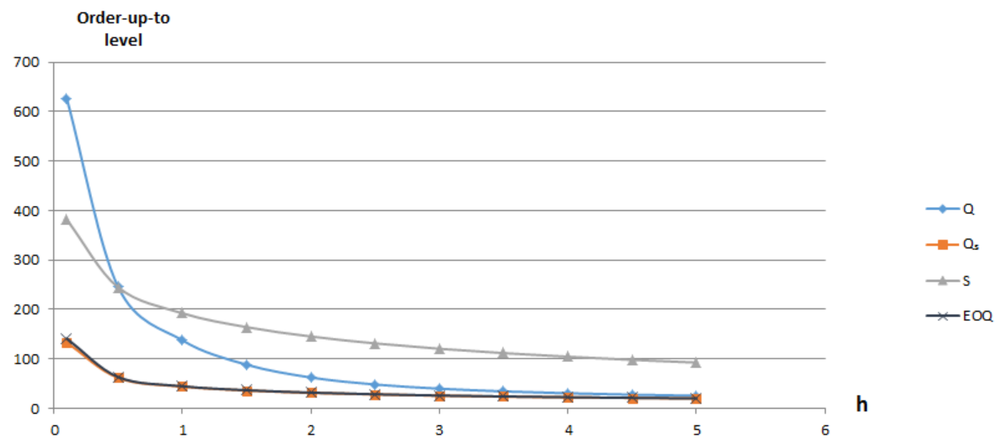
The expected costs increase for both cases (Figure 5.5). However, the gap narrows down as K increases resulting in a decreasing percentage improvement (Figure 5.6). This can be explained as follows: Independent of fixed order cost, the buyer definitely places a disruption order when the supplier becomes OFF. Therefore, the model increases S^* leading more expected fixed order cost per time, but it is still advantageous since its expected inventory holding cost and expected backorder cost is much smaller than no-order opportunity model.

Figure 5.6 Improvement vs. Fixed Order Cost



We now investigate the effect of unit inventory holding cost per unit time.

Figure 5.7 Order-up-to Levels vs. Inventory Holding Cost



When inventory holding cost rate increases, optimal order-up-to levels for both models decrease as expected. The decrease is steeper for the no-order opportunity case as lower inventory holding cost rates are exploited in this case by placing larger orders to avoid backorders (Figure 5.7). Corresponding expected costs are illustrated in Figure 5.8.

Figure 5.8 Total Expected Cost vs. Inventory Holding Cost

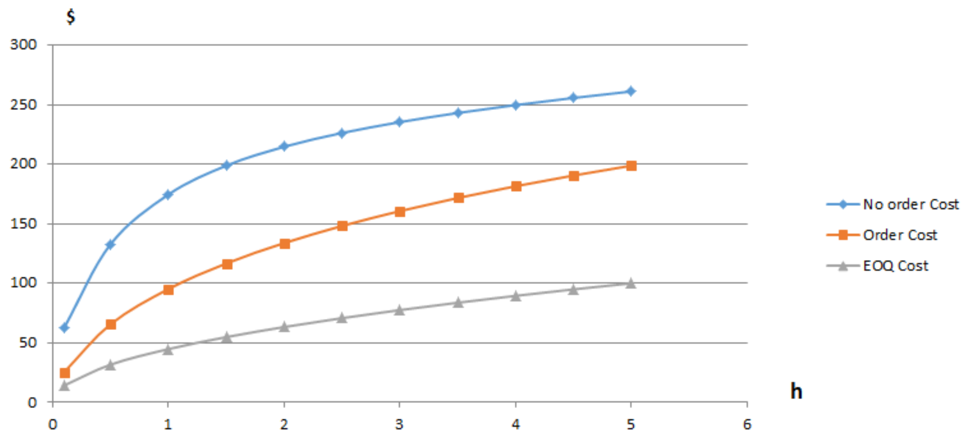
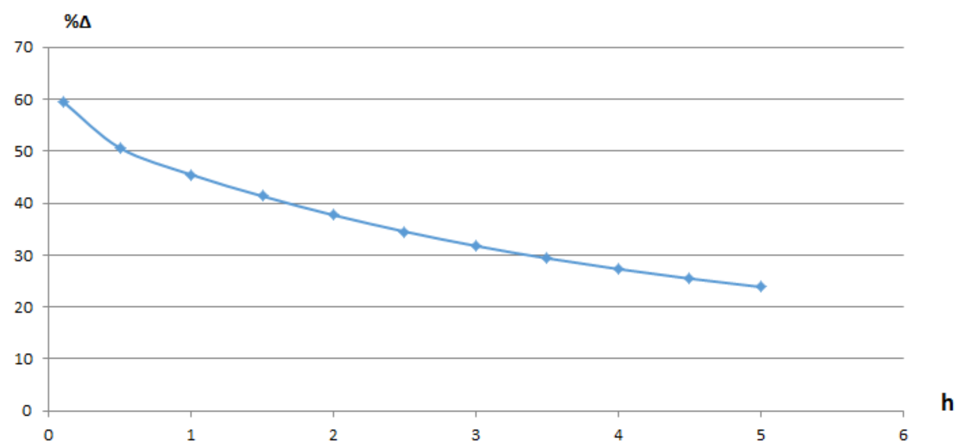


Figure 5.9 Improvement vs. Inventory Holding Cost



The percentage improvement due to disruption order decreases as h increases. This is reasonable as disruption order-up-to level decreases in h; hence, it is not as powerful as before when h increases (Figure 5.9).

Now, we investigate the effects of unit backorder cost per unit time.

Figure 5.10 Order-up-to Levels vs. Shortage Cost

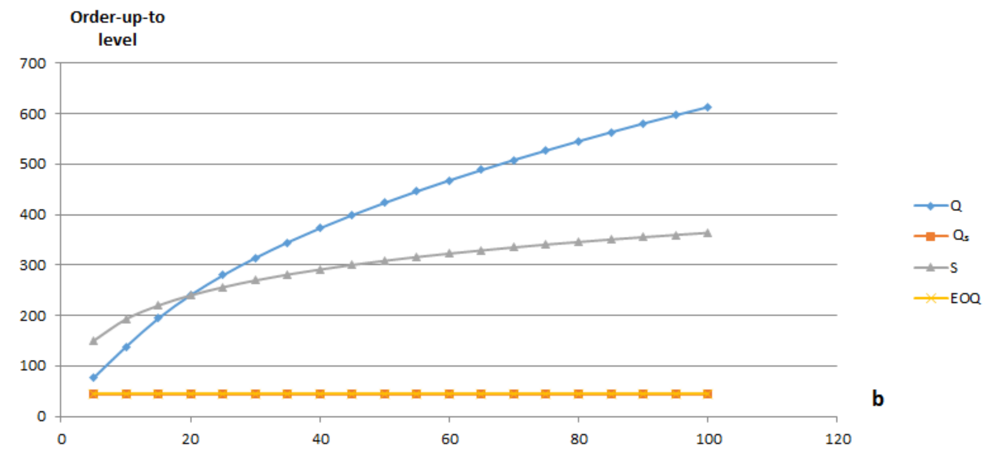
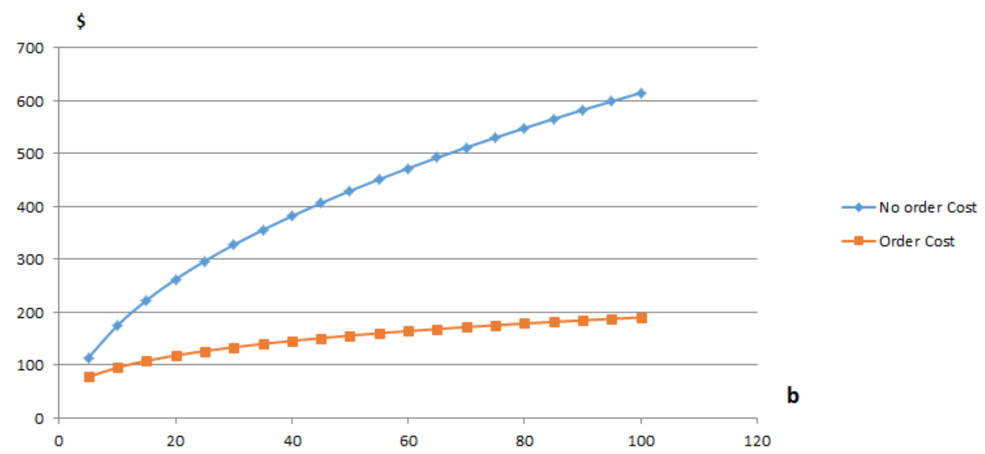


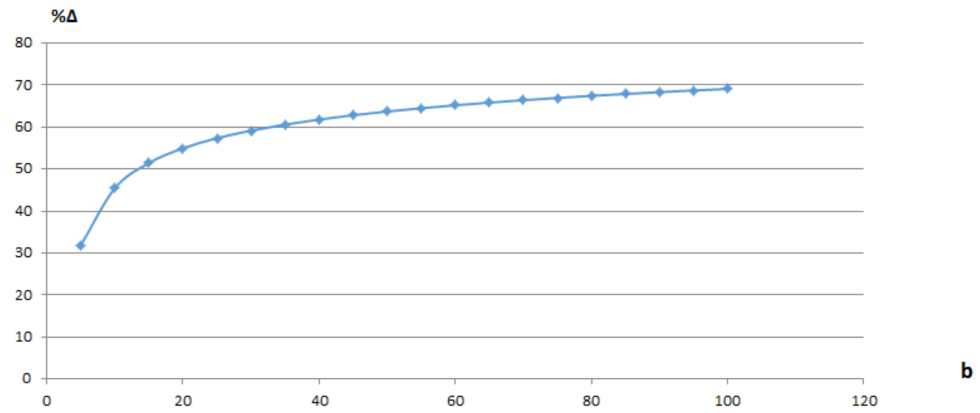
Figure 5.10 demonstrates optimal ordering decisions. Recall that backorders may occur only during disruptions. Since no-order opportunity model does not have a tool against disruption, it increases its regular order-up-to level as b increases, so that it might have sufficient inventory when a disruption hits. When there is an order opportunity, the regular order-up-level is not affected by an increase in b. Only the disruption order-up-to level increases.

Figure 5.11 Total Expected Cost vs. Shortage Cost



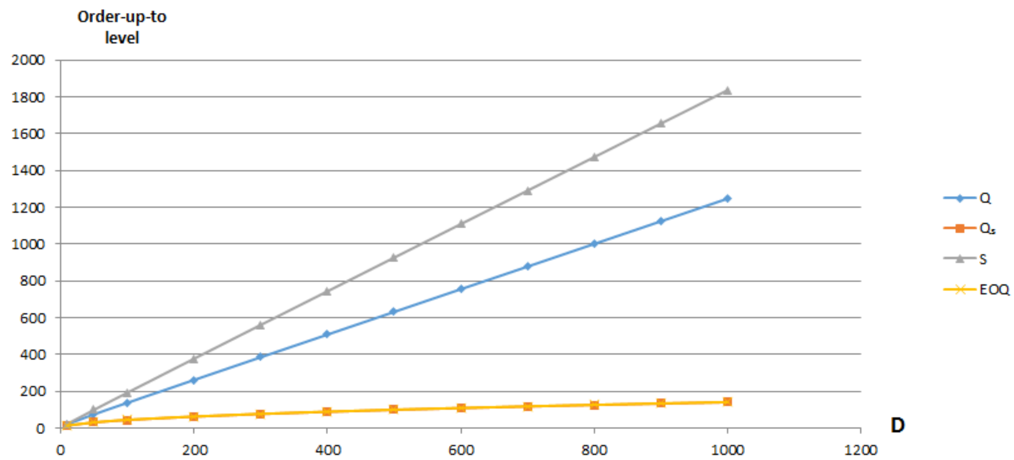
As a result, both expected costs increase in b (Figure 5.11), but the increase is much steeper when there is no order opportunity. Hence, the percentage improvement due to disruption order increases in b (Figure 5.12).

Figure 5.12 Improvement vs. Shortage Cost



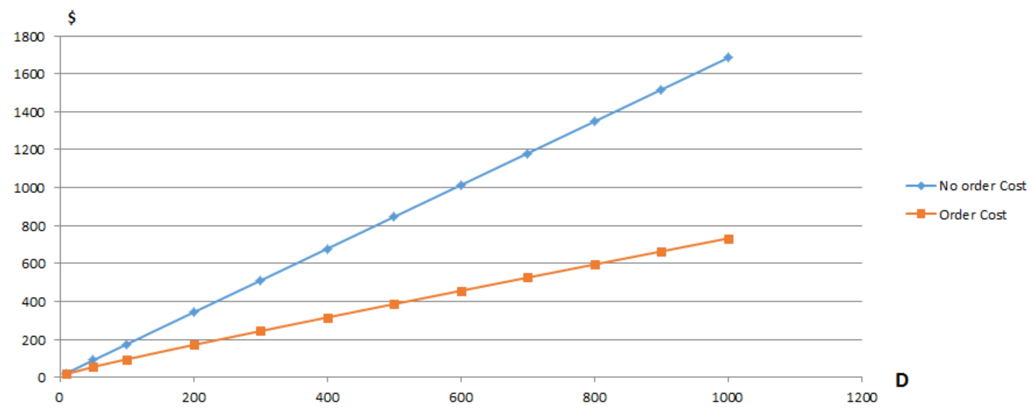
Finally, we investigate the effect of demand.

Figure 5.13 Order-up-to Levels vs. Demand



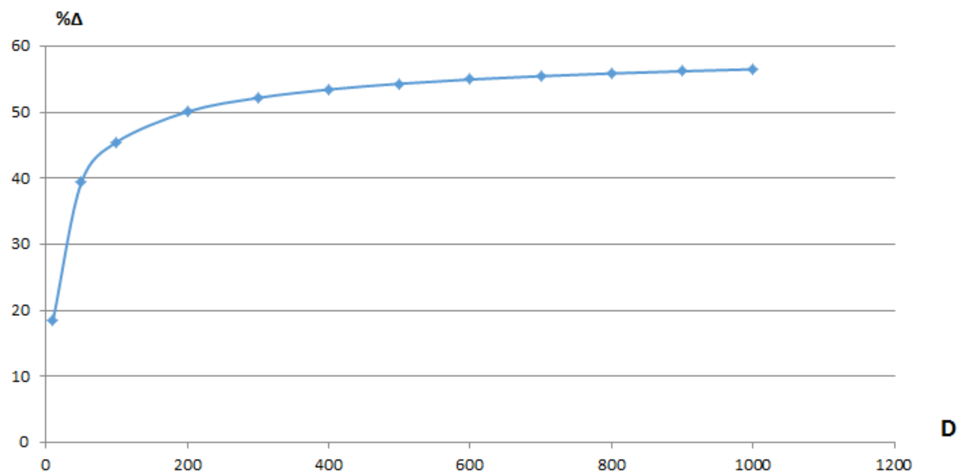
The optimal order-up-to levels for both models increase as expected. Note that the increase in S^* is steeper as if the buyer faces backorders, then the backorder cost can be too high as the demand rate gets larger (Figure 5.13). Also, this increase in S^* is linear as the relationship between S^* and D is directly proportional in the total expected cost function.

Figure 5.14 Total Expected Cost vs. Demand



Both expected costs increase in D (Figure 5.14), but the increase is much steeper when there is no order opportunity. Hence, the percentage improvement due to disruption order increases in D (Figure 5.15).

Figure 5.15 Improvement vs. Demand



CHAPTER 6

CONCLUSION

Supply disruption has become a major topic having drastic effects on supply chains. In this thesis, literature on inventory models under supply disruptions with deterministic and stochastic demand, and on effects of information in inventory models under supply disruptions are reviewed. In order to mitigate disruption risk is, we investigate the value of a disruption order which can be places at the beginning of an OFF-period and so far any order opportunity model have not been studied to our knowledge.

First, we modify Berk and Arreola-Risa's (1994) study by allowing backorders and keeping the assumption that the buyer cannot place any order during disruptions. Then the expected total cost function is derived and its properties are examined.

Then we let the buyer place an order at the beginning of a disruption with the same fixed cost. The main goal is to find the optimal order-up-levels for disruption and regular orders. To achieve that, we study two cases; disruption order-up-to level can be bigger or smaller than regular order-up-to level. For both of these cases total expected cost is derived by using renewal theory. Then, the optimum order up-to levels for both regular and disruption orders are found by the minimum total expected cost of both cases.

Computational study is conducted for above 1120 different parameter sets that used by Berk and Arreola-Risa (1994) to find the value of disruption order. Then we make sensitivity analysis using a base setting parameters. We note that if the backorder cost is smaller compared to fixed order and inventory holding cost, the order opportunity model chooses not to place any disruption orders.

As a future study, the assumption of exponential ON and OFF period can be relaxed and formulation of general distributions can be derived. Also stochastic demand can be used instead of deterministic demand. In addition, whether the information of the supplier's OFF-period length is valuable or not can be investigated.

REFERENCES

[1] Argon, N. T., Güllü, R., Erkip, N. (2001). Analysis of an inventory system under backorder correlated deterministic demand and geometric supply process. *International Journal of Production Economics* 71(1-3), 247-254.

[2] Arreola-Risa, A., DeCroix, G. A. (1998). Inventory management under random supply disruption and partial backorders. *Naval Research Logistics*, 45, 687-703.

[3] Atasoy, B., Güllü, R., Tan, T. (2012). Optimal inventory policies with non-stationary supply disruptions and advance supply information. *Decision Support Systems*, 53, 269–281.

[4] Bar-Lev, S. K., Parlar, M., Perry, D. (1993). Impulse control of a Brownian inventory system with supplier uncertainty. *Journal of Stochastic Analysis and Applications*, 11, 11-27.

[5] Berk, E., Arreola-Risa, A. (1994). Note on "Future supply uncertainty in EOQ models". *Naval Research Logistics*, 41, 129-132.

[6] Enkhbat, R., Bazarsad Ya., Enkhbayar, J. (2011). Convex-concave fractional minimization problem. *Mongolian Mathematical Journal*, 1, 3-10.

[7] Fisher, M., Raman, A. (1996). Reducing the cost of demand uncertainty through accurate response to early sales. *Operations Research*, 44, 87-99.

[8] Gupta, D. (1996). The (Q; r) inventory system with an unreliable supplier. *INFOR*, 34(2), 59-76.

[9] Gurnani, H., Shi, M. (2006). A bargaining model for a first-time interaction under asymmetric beliefs of supply reliability. *Management Science*, 52(6), 865-880.

[10] Güllü, R., Önol, E., Erkip, N. (1997). Analysis of a deterministic demand production/inventory system under nonstationary supply uncertainty. *IIE Transactions*, 29, 703-709.

[11] Gürler, U., Parlar, M. (1997). An inventory problem with two randomly available suppliers. *Operations Research*, 45(6), 904-918.

[12] Heath, D. C., Jackson, P. L. (1994). Modeling the evoluation of demand forecasts with application to safety stock analysis in production/distribution systems. *IIE Transactions*, 26, 28-35.

[13] Heimann, D., Waage, F. (2007). A closed-form approximation solution for an inventory model with supply disruptions and non-ZIO reorder policy. *Journal of Systemics, Cybernetics, and Informatics*, 5(4), 1-12.

[14] Hendricks, K. B., Singhal, V. R. (2003). The effect of supply chain glitches on shareholder wealth. *Journal of Operations Management*, 21(5), 501-522.

[15] Hendricks, K. B., Singhal, V. R. (2005). Association between supply chain glitches and operating performance. *Management Science*, 51(5), 695-711.

[16] Jung, J. Y., Blau, G., Pekny, J. F., Reklaitis, G. V., Eversdyk, D. (2004). A simulation based optimization approach to supply chain management under demand uncertainty. *Computers and Chemical Engineering*, 28, 2087–2106.

[17] Kalpakam, S., Sapna, K. P. (1997). A lost sales inventory system with supply uncertainty. *Computers & Mathematics with Applications*, 33(3), 81-93.

[18] Li, Z., Xu, S. H., Hayya, J. (2004). A periodic-review inventory system with supply interruptions. *Probability in the Engineering and Informational Sciences*, 18, 33-53.

[19] Liu, B., Cao, J. (1999). Analysis of a production-inventory system with machine breakdowns and shutdowns. *Computers and Operations Research*, 26, 73-91.

[20] Mohebbi, E. (2003). Supply interruptions in a lost-sales inventory system with random lead time. *Computers and Operations Research*, 30, 411-426.

[21] Mohebbi, E. (2004). A replenishment model for the supply-uncertainty problem. *International Journal of Production Economics*, 87(1), 25-37.

[22] Mohebbi, E., Hao, D. (2006). When supplier's availability affects the replenishment lead time: An extension of the supply-interruption problem. *European Journal of Operational Research*, 175, 992-1008.

[23] Mohebbi, E., Hao, D. (2008). An inventory model with non-resuming randomly interruptible lead time. *International Journal of Production Economics*, 114, 755-768.

[24] Moinzadeh, K., Aggarwal, P. (1997). Analysis of a production/inventory system subject to random disruptions. *Management Science*, 43(11), 1577-1588.

[25] Özekici, S., Parlar, M. (1999). Inventory models with unreliable suppliers in a random environment. *Annals of Operations Research*, 91, 123-136.

[26] Parlar, M. (1997). Continuous-review inventory problem with random supply interruptions. *European Journal of Operational Research*, 99, 366-385.

[27] Parlar, M., Berkin, D. (1991). Future supply uncertainty in EOQ models. *Naval Research Logistics*, 38, 107-121.

[28] Parlar, M., Perry, D. (1995). Analysis of a (q, r, t) inventory policy with deterministic and random yields when future supply is uncertain. *European Journal of Operational Research*, 84, 431-443.

[29] Parlar, M., Perry, D. (1996). Inventory models of future supply uncertainty with single and multiple suppliers. *Naval Research Logistics*, 43, 191-210.

[30] Qi, L., Shen, Z.-J. M., Snyder, L. V. (2009). A continuous-review inventory model with disruptions at both supplier and retailer. *Production and Operations Management*, 18(5), 516-532.

[31] Saghafian, S., Van Oyen, M. P. (2008). On the value of flexible suppliers and disruption risk level information in supply chains. Working Paper.

[32] Snyder, L. V. (2008). A tight approximation for a continuous-review inventory model with supplier disruptions. Working Paper.

[33] Snyder, L. V., Atan, Z., Peng, P., Rong, Y., Schmitt. A. J., Sinsoysal, B. (2012). OR/MS models for supply chain disruptions: A review. Submitted for publication.

[34] Song, J.-S., Zipkin, P. H. (1996). Inventory control with information about supply conditions. *Management Science*, 42(10), 1409-1419.

[35] Tomlin, B. T. (2006). On the value of mitigation and contingency strategies for managing supply chain disruption risks. *Management Science*, 52(5), 639–657.

[36] Xia, Y., Yang, M.-H., Golany, B., Gilbert, S. M., Yu, G. (2004). Real-time disruption management in a two-stage production and inventory system. *IIE Transactions*, 36(2), 111-125.

[37] Yang, Z., Aydin, G., Babich, V., Beil, D. R. (2008). Supply disruptions, asymmetric information and a backup production option. Technical report, University of Michigan.

APPENDIX A

DETAILED DERIVATIONS

A.1 Derivation of Equation 4.10

The CDF of $X + Y|X + Y < S/D$ is calcuated as follows:

$$F_{(X+Y)}(z|X + Y \leq \frac{S}{D}) = P\left\{X + Y \leq z \middle| X + Y \leq \frac{S}{D}\right\} = \frac{P\left\{X + Y \leq z, X + Y \leq \frac{S}{D}\right\}}{P\left\{X + Y \leq \frac{S}{D}\right\}}$$

$$P\left\{X + Y \leq z \middle| X + Y \leq \frac{S}{D}\right\} = \begin{cases} \dfrac{P\{X + Y \leq z\}}{P\left\{X + Y \leq \frac{S}{D}\right\}} = \dfrac{1 - \frac{\mu e^{-\lambda z} - \lambda e^{-\mu z}}{\mu - \lambda}}{1 - \frac{\mu e^{-\lambda \frac{S}{D}} - \lambda e^{-\mu \frac{S}{D}}}{\mu - \lambda}} & \\ = \dfrac{\mu - \lambda - \mu e^{-\lambda z} + \lambda e^{-\mu z}}{\mu - \lambda - \mu e^{-\lambda \frac{S}{D}} + \lambda e^{-\mu \frac{S}{D}}} & z \leq \frac{S}{D} \\ 1 & z > \frac{S}{D} \end{cases}$$

A.2 Derivation of Equation 4.12

$$E\left[H \middle| Y \le \frac{S}{D}, X+Y > \frac{S}{D}\right]$$
$$= h\frac{S^2}{2D} + \sum_{k=1}^{\infty} E\left[H \middle| Y \le \frac{S}{D}, X+Y > \frac{S}{D}, N=k\right] P\left\{N=k \middle| Y \le \frac{S}{D}, X+Y > \frac{S}{D}\right\}$$
$$= h\left(\frac{S^2}{2D} + \sum_{k=1}^{\infty}\left(\frac{(k-1)Q^2}{2D} + E\left[\frac{((k+1)\,Q - XD - YD + S)\left(X+Y-\frac{S}{D}-\frac{(k-1)Q}{D}\right)}{2}\right.\right.\right.$$
$$\left.\left.\left.\middle| Y \le \frac{S}{D}, \frac{S+(k-1)Q}{D} < X+Y < \frac{S+kQ}{D}\right]\right) P\left\{N=k \middle| Y \le \frac{S}{D}, X+Y > \frac{S}{D}\right\}\right)$$
$$= h\left(\frac{S^2}{2D} + \sum_{k=1}^{\infty}\frac{(k-1)Q^2}{2D} P\left\{N=k \middle| Y \le \frac{S}{D}, X+Y > \frac{S}{D}\right\}\right.$$
$$+ \sum_{k=1}^{\infty}\left(E\left[(kQ+S)X + (kQ+S)Y - \frac{D}{2}X^2 - \frac{D}{2}Y^2 - DXY - \frac{kQ}{D}S - \frac{S^2}{2D} - \frac{(k^2-1)Q^2}{2D}\right.\right.$$
$$\left.\left.\left.\middle| Y \le \frac{S}{D}, \frac{S+(k-1)Q}{D} < X+Y < \frac{S+kQ}{D}\right]\right) P\left\{N=k \middle| Y \le \frac{S}{D}, X+Y > \frac{S}{D}\right\}\right)$$
$$= h\left(\frac{S^2}{2D} + \frac{Q^2}{2D}\left(\sum_{k=1}^{\infty}(k-1)P\left\{N=k \middle| Y \le \frac{S}{D}, X+Y > \frac{S}{D}\right\}\right)\right.$$
$$+ \sum_{k=1}^{\infty}\left((kQ+S)\,E\left[X+Y \middle| Y \le \frac{S}{D}, \frac{S+(k-1)}{D} < X+Y < \frac{S+kQ}{D}\right]\right.$$
$$\left. - \frac{D}{2}E\left[(X+Y)^2 \middle| Y \le \frac{S}{D}, \frac{S+(k-1)Q}{D} < X+Y < \frac{S+kQ}{D}\right] - \frac{kQ}{D}S - \frac{S^2}{2D} - \frac{(k^2-1)Q^2}{2D}\right)$$
$$\left.P\left\{N=k \middle| Y \le \frac{S}{D}, X+Y > \frac{S}{D}\right\}\right).$$

So in order to derive $E\left[X+Y \middle| Y \le \frac{S}{D}, \frac{S+(k-1)Q}{D} < X+Y \le \frac{S+kQ}{D}\right]$ and $E\left[(X+Y)^2 \middle| Y \le \frac{S}{D}, \frac{S+(k-1)Q}{D} < X+Y \le \frac{S+kQ}{D}\right]$ we need the CDF of $X+Y \middle| \frac{(S+(k-1)Q)}{D} < X+Y \le \frac{(S+kQ)}{D}$

$$P\left\{X+Y \le z \middle| Y \le \frac{S}{D}, \frac{S+(k-1)Q}{D} < X+Y < \frac{S+kQ}{D}\right\}$$
$$= \frac{P\left\{X+Y \le z, Y \le \frac{S}{D}, \frac{S+(k-1)Q}{D} < X+Y < \frac{S+kQ}{D}\right\}}{P\left\{Y \le \frac{S}{D}, \frac{S+(k-1)Q}{D} < X+Y < \frac{S+kQ}{D}\right\}}$$

If $z \leq \frac{S+(k-1)Q}{D}$ then

$$P\left\{X+Y \leq z \middle| Y \leq \frac{S}{D}, \frac{S+(k-1)Q}{D} < X+Y \leq \frac{S+kQ}{D}\right\} = 0$$

If $\frac{S+(k-1)Q}{D} < z \leq \frac{S+kQ}{D}$ then

$$P\left\{X+Y \leq z \middle| Y \leq \frac{S}{D}, \frac{S+(k-1)Q}{D} < X+Y \leq \frac{S+kQ}{D}\right\}$$
$$= \frac{P\left\{Y \leq \frac{S}{D}, \frac{S+(k-1)Q}{D} < X+Y \leq z\right\}}{P\left\{Y \leq \frac{S}{D}, \frac{S+(k-1)Q}{D} < X+Y < \frac{S+kQ}{D}\right\}} = \frac{\left(e^{-\lambda\frac{S+(k-1)Q}{D}} - e^{-\lambda z}\right)}{\left(e^{-\lambda\frac{S+(k-1)Q}{D}} - e^{-\lambda\frac{S+kQ}{D}}\right)}$$

If $z > \frac{S+kQ}{D}$ then

$$P\left\{X+Y \leq z \middle| Y \leq \frac{S}{D}, \frac{S+(k-1)Q}{D} < X+Y < \frac{S+kQ}{D}\right\} = 1$$

Therefore;

$$E\left[X+Y \middle| Y \leq \frac{S}{D}, \frac{S+(k-1)Q}{D} < X+Y \leq \frac{S+kQ}{D}\right]$$
$$= \int_0^{\frac{S+(k-1)Q}{D}} (1-0)\ \mathrm{d}z + \int_{\frac{S+(k-1)Q}{D}}^{\frac{S+kQ}{D}} \left(1 - \frac{\left(e^{-\lambda\frac{S+(k-1)Q}{D}} - e^{-\lambda z}\right)}{\left(e^{-\lambda\frac{S+(k-1)Q}{D}} - e^{-\lambda\frac{S+kQ}{D}}\right)}\right) \mathrm{d}z + \int_{\frac{S+kQ}{D}}^{\infty} (1-1)\ \mathrm{d}z$$
$$= \frac{S+(k-1)Q}{D} + \left(\frac{S+kQ}{D} - \frac{S+(k-1)Q}{D}\right) - \frac{1}{e^{-\lambda\frac{S+kQ}{D}}\left(e^{\lambda\frac{Q}{D}}-1\right)} \int_{\frac{S+(k-1)Q}{D}}^{\frac{S+kQ}{D}} \left(e^{-\lambda\frac{S+(k-1)Q}{D}} - e^{-\lambda z}\right) \mathrm{d}z$$
$$= \frac{S+kQ}{D} - \frac{1}{e^{-\lambda\frac{S+kQ}{D}}\left(e^{\lambda\frac{Q}{D}}-1\right)}\left(e^{-\lambda\frac{S+(k-1)Q}{D}}\left(\frac{S+kQ}{D} - \frac{S+(k-1)Q}{D}\right) + \frac{\left(e^{-\lambda\frac{S+kQ}{D}} - e^{-\lambda\frac{S+(k-1)Q}{D}}\right)}{\lambda}\right)$$
$$= \frac{S+kQ}{D} - \frac{e^{\lambda\frac{Q}{D}}Q}{\left(e^{\lambda\frac{Q}{D}}-1\right)} - \frac{e^{-\lambda\frac{S+kQ}{D}}\left(1-e^{\lambda\frac{Q}{D}}\right)}{e^{-\lambda\frac{S+kQ}{D}}\left(e^{\lambda\frac{Q}{D}}-1\right)\lambda}$$
$$= \frac{S+kQ}{D} - \frac{Q}{D\left(1-e^{-\lambda\frac{Q}{D}}\right)} + \frac{1}{\lambda}$$

And;

$$
\begin{aligned}
&E\left[(X+Y)^2\middle|Y\le\frac{S}{D},\frac{S+(k-1)Q}{D}<X+Y\le\frac{S+kQ}{D}\right]\\
&=2\left(\int_0^{\frac{S+(k-1)Q}{D}} z(1-0)\,\mathrm{d}z+\int_{\frac{S+(k-1)Q}{D}}^{\frac{S+kQ}{D}}\left(z-z\left(\frac{\left(\mathrm{e}^{-\lambda\frac{S+(k-1)Q}{D}}-\mathrm{e}^{-\lambda z}\right)}{\left(\mathrm{e}^{-\lambda\frac{S+(k-1)Q}{D}}-\mathrm{e}^{-\lambda\frac{S+kQ}{D}}\right)}\right)\right)\mathrm{d}z+\int_{\frac{S+kQ}{D}}^{\infty} z(1-1)\,\mathrm{d}z\right)\\
&=2\left(\frac{(S+(k-1)Q)^2}{2D^2}+\left(\frac{(S+kQ)^2}{2D^2}-\frac{(S+(k-1)Q)^2}{2D^2}\right)\right.\\
&\left.-\frac{1}{\mathrm{e}^{-\lambda\frac{S+kQ}{D}}\left(\mathrm{e}^{\lambda\frac{Q}{D}}-1\right)}\left(\mathrm{e}^{-\lambda\frac{S+(k-1)Q}{D}}\left(\frac{(S+kQ)^2}{2D^2}-\frac{(S+(k-1)Q)^2}{2D^2}\right)\int_{\frac{S+(k-1)Q}{D}}^{\frac{S+kQ}{D}} z\mathrm{e}^{-\lambda z}\,\mathrm{d}z\right)\right)\\
&=2\left(\frac{(S+kQ)^2}{2D^2}-\frac{\left(2SQ+2kQ^2-Q^2\right)}{2D^2\left(1-\mathrm{e}^{-\lambda\frac{Q}{D}}\right)}-\frac{S+kQ}{D\lambda\mathrm{e}^{\lambda\frac{Q}{D}}}+\frac{S+(k-1)Q}{D\lambda\left(1-\mathrm{e}^{-\lambda\frac{Q}{D}}\right)}+\frac{1}{\lambda^2}\right)
\end{aligned}
$$

Then by using the expectations, we calculate the first summation;

$$
\begin{aligned}
&\frac{Q^2}{2D}\left(\sum_{k=1}^{\infty}(k-1)P\left\{N=k\middle|Y\le\frac{S}{D},X+Y>\frac{S}{D}\right\}\right)\\
&=\frac{Q^2}{2D}\left(\left(\sum_{k=1}^{\infty}kP\left\{N=k\middle|Y\le\frac{S}{D},X+Y>\frac{S}{D}\right\}\right)-\left(\sum_{k=1}^{\infty}P\left\{N=k\middle|Y\le\frac{S}{D},X+Y>\frac{S}{D}\right\}\right)\right)\\
&=\frac{Q^2}{2D}\left(E\left[N\middle|Y\le\frac{S}{D},X+Y>\frac{S}{D}\right]-1\right)=\frac{Q^2}{2D}\left(\frac{1}{(1-\mathrm{e}^{-\lambda\frac{Q}{D}})}-1\right)=\frac{Q^2}{2D(\mathrm{e}^{\lambda\frac{Q}{D}}-1)}.
\end{aligned}
$$

The first part of second summation is calculated as follows;

$$
\begin{aligned}
&\sum_{k=1}^{\infty} P\left\{N=k \middle| Y \leq \frac{S}{D}, X+Y>\frac{S}{D}\right\}(S+kQ)\left(\frac{S+kQ}{D}-\frac{Q}{D\left(1-\mathrm{e}^{-\lambda\frac{Q}{D}}\right)}+\frac{1}{\lambda}\right)\\
&=S\sum_{k=1}^{\infty}\mathrm{e}^{-\lambda\frac{(k-1)Q}{D}}\left(1-\mathrm{e}^{-\lambda\frac{Q}{D}}\right)\left(\frac{S+kQ}{D}-\frac{Q}{D\left(1-\mathrm{e}^{-\lambda\frac{Q}{D}}\right)}+\frac{1}{\lambda}\right)\\
&+Q\sum_{k=1}^{\infty}\mathrm{e}^{-\lambda\frac{kQ}{D}}\left(\mathrm{e}^{\lambda\frac{Q}{D}}-1\right)(k)\left(\frac{S+kQ}{D}-\frac{Q}{D\left(1-\mathrm{e}^{-\lambda\frac{Q}{D}}\right)}+\frac{1}{\lambda}\right)\\
&=S\left(\frac{S}{D}+\frac{Q}{D\left(1-\mathrm{e}^{-\lambda\frac{Q}{D}}\right)}-\frac{Q}{D\left(1-\mathrm{e}^{-\lambda\frac{Q}{D}}\right)}+\frac{1}{\lambda}\right)+Q\left(\frac{S\left(\mathrm{e}^{\lambda\frac{Q}{D}}-1\right)\mathrm{e}^{-\lambda\frac{Q}{D}}}{D\left(1-\mathrm{e}^{-\lambda\frac{Q}{D}}\right)^2}\right.\\
&\left.+\frac{Q\left(\mathrm{e}^{\lambda\frac{Q}{D}}-1\right)\mathrm{e}^{-\lambda\frac{Q}{D}}\left(1+\mathrm{e}^{-\lambda\frac{Q}{D}}\right)}{D\left(1-\mathrm{e}^{-\lambda\frac{Q}{D}}\right)^3}-\frac{Q\left(\mathrm{e}^{\lambda\frac{Q}{D}}-1\right)\mathrm{e}^{-\lambda\frac{Q}{D}}}{D\left(1-\mathrm{e}^{-\lambda\frac{Q}{D}}\right)\left(1-\mathrm{e}^{-\lambda\frac{Q}{D}}\right)^2}+\frac{\left(\mathrm{e}^{\lambda\frac{Q}{D}}-1\right)\mathrm{e}^{-\lambda\frac{Q}{D}}}{\lambda\left(1-\mathrm{e}^{-\lambda\frac{Q}{D}}\right)^2}\right)\\
&=\frac{S^2}{D}+\frac{SQ}{D\left(1-\mathrm{e}^{-\lambda\frac{Q}{D}}\right)}-\frac{SQ}{D\left(1-\mathrm{e}^{-\lambda\frac{Q}{D}}\right)}+\frac{S}{\lambda}\\
&+\frac{SQ}{D\left(1-\mathrm{e}^{-\lambda\frac{Q}{D}}\right)}+\frac{Q^2\left(1+\mathrm{e}^{-\lambda\frac{Q}{D}}\right)}{D\left(1-\mathrm{e}^{-\lambda\frac{Q}{D}}\right)^2}-\frac{Q^2}{D\left(1-\mathrm{e}^{-\lambda\frac{Q}{D}}\right)^2}+\frac{Q}{\lambda\left(1-\mathrm{e}^{-\lambda\frac{Q}{D}}\right)}\\
&=\frac{S^2}{D}+\frac{S}{\lambda}+\frac{SQ}{D\left(1-\mathrm{e}^{-\lambda\frac{Q}{D}}\right)}+\frac{Q^2\mathrm{e}^{-\lambda\frac{Q}{D}}}{D\left(1-\mathrm{e}^{-\lambda\frac{Q}{D}}\right)^2}+\frac{Q}{\lambda\left(1-\mathrm{e}^{-\lambda\frac{Q}{D}}\right)}.
\end{aligned}
$$

Now the second part of the second summation is;

$$
\begin{aligned}
&\sum_{k=1}^{\infty} P\left\{N=k \middle| Y \le \frac{S}{D}, X+Y>\frac{S}{D}\right\}\left(-\frac{D}{2}\right)2 \\
&\left(\frac{S^2+2SQk+k^2Q^2}{2D^2}-\frac{\left(2SQ+2kQ^2-Q^2\right)}{2D^2\left(1-e^{-\lambda\frac{Q}{D}}\right)}-\frac{S+kQ}{D\lambda\left(e^{\lambda\frac{Q}{D}}-1\right)}+\frac{S+(k-1)Q}{D\lambda\left(1-e^{-\lambda\frac{Q}{D}}\right)}+\frac{1}{\lambda^2}\right) \\
&=-D\left(\sum_{k=1}^{\infty} e^{-\lambda\frac{kQ}{D}}\left(e^{\lambda\frac{Q}{D}}-1\right)\left(\frac{S^2}{2D^2}+\frac{2SQk}{2D^2}+\frac{k^2Q^2}{2D^2}-\frac{2SQ}{2D^2\left(1-e^{-\lambda\frac{Q}{D}}\right)}-\frac{2kQ^2}{2D^2\left(1-e^{-\lambda\frac{Q}{D}}\right)}\right.\right. \\
&+\frac{Q^2}{2D^2\left(1-e^{-\lambda\frac{Q}{D}}\right)}-\frac{S}{D\lambda\left(e^{\lambda\frac{Q}{D}}-1\right)}-\frac{kQ}{D\lambda\left(e^{\lambda\frac{Q}{D}}-1\right)}+\frac{S}{D\lambda\left(1-e^{-\lambda\frac{Q}{D}}\right)}+\frac{kQ}{D\lambda\left(1-e^{-\lambda\frac{Q}{D}}\right)} \\
&\left.\left.-\frac{Q}{D\lambda\left(1-e^{-\lambda\frac{Q}{D}}\right)}+\frac{1}{\lambda^2}\right)\right) \\
&=-D\left(\frac{S^2}{2D^2}+\frac{SQ}{D^2\left(1-e^{-\lambda\frac{Q}{D}}\right)}+\frac{Q^2\left(1+e^{-\lambda\frac{Q}{D}}\right)}{2D^2\left(1-e^{-\lambda\frac{Q}{D}}\right)^2}-\frac{SQ}{D^2\left(1-e^{-\lambda\frac{Q}{D}}\right)}-\frac{Q^2}{D^2\left(1-e^{-\lambda\frac{Q}{D}}\right)^2}\right. \\
&+\frac{Q^2}{2D^2\left(1-e^{-\lambda\frac{Q}{D}}\right)}-\frac{S}{D\lambda\left(e^{\lambda\frac{Q}{D}}-1\right)}-\frac{Qe^{-\lambda\frac{Q}{D}}}{D\lambda\left(1-e^{-\lambda\frac{Q}{D}}\right)^2}+\frac{S}{D\lambda\left(1-e^{-\lambda\frac{Q}{D}}\right)}+\frac{Q}{D\lambda\left(1-e^{-\lambda\frac{Q}{D}}\right)^2} \\
&\left.-\frac{Q}{D\lambda\left(1-e^{-\lambda\frac{Q}{D}}\right)}+\frac{1}{\lambda^2}\right) \\
&=-\frac{S^2}{2D}-\frac{S}{\lambda}-\frac{D}{\lambda^2}.
\end{aligned}
$$

Finally the last part of the second summation;

$$
\begin{aligned}
&\sum_{k=1}^{\infty} P\left\{N=k \middle| Y \le \frac{S}{D}, X+Y>\frac{S}{D}\right\}\left(-\frac{kQ}{D}S-\frac{S^2}{2D}-\frac{\left(k^2-1\right)Q^2}{2D}\right) \\
&=-\frac{Q^2}{2D}\sum_{k=1}^{\infty} k^2 e^{-\lambda\frac{kQ}{D}}\left(e^{\lambda\frac{Q}{D}}-1\right)-\frac{SQ}{D}\sum_{k=1}^{\infty} k e^{-\lambda\frac{kQ}{D}}\left(e^{\lambda\frac{Q}{D}}-1\right)-\frac{S^2-Q^2}{'D}\sum_{k=1}^{\infty} e^{-\lambda\frac{kQ}{D}}\left(e^{\lambda\frac{Q}{D}}-1\right) \\
&=-\frac{Q^2\left(1+e^{-\lambda\frac{Q}{D}}\right)}{2D\left(1-e^{-\lambda\frac{Q}{D}}\right)^2}-\frac{SQ}{D\left(1-e^{-\lambda\frac{Q}{D}}\right)}-\frac{S^2-Q^2}{2D}.
\end{aligned}
$$

A.3 Derivation of Equation 4.15

$$
\begin{aligned}
E\left[H \middle| Y > \frac{S}{D}\right] &= h\frac{S^2}{2D} + \sum_{k=1}^{\infty} E\left[H \middle| Y > \frac{S}{D}, N = k\right] P\left\{N = k \middle| Y > \frac{S}{D}\right\} \\
&= h\left(\frac{S^2}{2D} + \sum_{k=1}^{\infty}\left(\frac{(k-1)Q^2}{2D} + E\left[\frac{((k+1)Q - XD)(X - \frac{(k-1)Q}{D})}{2} \middle| \frac{(k-1)Q}{D} < X < \frac{kQ}{D}\right]\right)\right. \\
&\quad \left. P\left\{N = k \middle| Y > \frac{S}{D}\right\}\right) \\
&= h\left(\frac{S^2}{2D} + \sum_{k=1}^{\infty} \frac{(k-1)Q^2}{2D} P\left\{N = k \middle| Y > \frac{S}{D}\right\}\right. \\
&\quad \left. + \sum_{k=1}^{\infty}\left(E\left[kQX - \frac{D}{2}X^2 - \frac{\left(k^2-1\right)Q^2}{2D} \middle| \frac{(k-1)Q}{D} < X < \frac{kQ}{D}\right]\right) P\left\{N = k \middle| Y > \frac{S}{D}\right\}\right) \\
&= h\left(\frac{S^2}{2D} + \frac{Q^2}{2D}\left(\sum_{k=1}^{\infty}(k-1)P\left\{N = k \middle| Y > \frac{S}{D}\right\}\right)\right. \\
&\quad + \sum_{k=1}^{\infty}\left(kQE\left[X \middle| \frac{(k-1)Q}{D} < X < \frac{kQ}{D}\right] - \frac{D}{2}E\left[X^2 \middle| \frac{(k-1)Q}{D} < X < \frac{kQ}{D}\right] - \frac{(k^2-1)Q^2}{2D}\right) \\
&\quad \left. P\left\{N = k \middle| Y > \frac{S}{D}\right\}\right).
\end{aligned}
\tag{A.1}
$$

In order to derive $E\left[X \middle| \frac{(k-1)Q}{D} < X < \frac{kQ}{D}\right]$ and $E\left[X^2 \middle| \frac{(k-1)Q}{D} < X < \frac{kQ}{D}\right]$ we need the CDF of $X \middle| \frac{(k-1)Q}{D} < X < \frac{kQ}{D}$

$$
P\left\{X \le z \middle| \frac{(k-1)Q}{D} < X < \frac{kQ}{D}\right\} = \frac{P\left\{X \le z, \frac{(k-1)Q}{D} < X < \frac{kQ}{D}\right\}}{P\left\{\frac{(k-1)Q}{D} < X < \frac{kQ}{D}\right\}}
$$

If $z \le \frac{(k-1)Q}{D}$ then

$$
P\left\{X \le z \middle| \frac{(k-1)Q}{D} < X < \frac{kQ}{D}\right\} = 0
$$

If $\frac{(k-1)Q}{D} < z < \frac{kQ}{D}$ then

$$P\left\{X \le z \middle| \frac{(k-1)Q}{D} < X < \frac{kQ}{D}\right\} = \frac{\int_{\frac{(k-1)Q}{D}}^{z} \lambda e^{-\lambda z}\, dz}{1 - e^{-\lambda\frac{kQ}{D}} - 1 + e^{-\lambda\frac{(k-1)Q}{D}}} = \frac{e^{-\lambda\frac{(k-1)Q}{D}} - e^{-\lambda z}}{e^{-\lambda\frac{kQ}{D}}\left(e^{\lambda\frac{Q}{D}} - 1\right)}$$

If $z > \frac{kQ}{D}$ then

$$P\left\{X \le z \middle| \frac{(k-1)Q}{D} < X < \frac{kQ}{D}\right\} = 1$$

Therefore;

$$E\left[X \middle| \frac{(k-1)Q}{D} < X < \frac{kQ}{D}\right] = \int_{0}^{\frac{(k-1)Q}{D}} (1-0)\, dz + \int_{\frac{(k-1)Q}{D}}^{\frac{kQ}{D}} \left(1 - \frac{e^{-\lambda\frac{(k-1)Q}{D}} - e^{-\lambda z}}{e^{-\lambda\frac{kQ}{D}}\left(e^{\lambda\frac{Q}{D}} - 1\right)}\right) dz + \int_{\frac{kQ}{D}}^{\infty} (1-1)\, dz$$

$$= \frac{(k-1)Q}{D} + \left(\frac{kQ}{D} - \frac{(k-1)Q}{D}\right) - \frac{1}{e^{-\lambda\frac{kQ}{D}}\left(e^{\lambda\frac{Q}{D}} - 1\right)} \int_{\frac{(k-1)Q}{D}}^{\frac{S+kQ}{D}} \left(e^{-\lambda\frac{(k-1)Q}{D}} - e^{-\lambda z}\right) dz$$

$$= \frac{kQ}{D} - \frac{1}{e^{-\lambda\frac{kQ}{D}}\left(e^{\lambda\frac{Q}{D}} - 1\right)} \left(e^{-\lambda\frac{(k-1)Q}{D}}\frac{Q}{D} + \frac{\left(e^{-\lambda\frac{kQ}{D}} - e^{-\lambda\frac{(k-1)Q}{D}}\right)}{\lambda}\right)$$

$$= \frac{kQ}{D} - \frac{Q}{D\left(1 - e^{-\lambda\frac{Q}{D}}\right)} + \frac{1}{\lambda}$$

And;

$$
\begin{aligned}
&E\left[X^2\middle|\frac{(k-1)Q}{D}<X<\frac{kQ}{D}\right]\\
&=2\left(\int\limits_0^{\frac{(k-1)Q}{D}} z\,(1-0)\;\mathrm{d}z+\int\limits_{\frac{(k-1)Q}{D}}^{\frac{kQ}{D}}\left(z-z\frac{\mathrm{e}^{-\lambda\frac{(k-1)Q}{D}}-\mathrm{e}^{-\lambda z}}{\mathrm{e}^{-\lambda\frac{kQ}{D}}\left(\mathrm{e}^{\lambda\frac{Q}{D}}-1\right)}\right)\mathrm{d}z+\int\limits_{\frac{kQ}{D}}^{\infty} z\,(1-1)\;\mathrm{d}z\right)\\
&=2\left(\frac{k^2Q^2-2kQ^2+Q^2}{2D^2}+\left(\frac{2kQ^2-Q^2}{2D^2}-\frac{1}{\mathrm{e}^{-\lambda\frac{kQ}{D}}\left(\mathrm{e}^{\lambda\frac{Q}{D}}-1\right)}\right.\right.\\
&\left(\mathrm{e}^{-\lambda\frac{(k-1)Q}{D}}\left(\frac{k^2Q^2-2kQ^2+Q^2}{2D^2}-\frac{2kQ^2-Q^2}{2D^2}\right)\int\limits_{\frac{(k-1)Q}{D}}^{\frac{kQ}{D}} z\mathrm{e}^{-\lambda z}\,\mathrm{d}z\right)\\
&=2\left(\frac{k^2Q^2}{2D^2}-\frac{\left(2kQ^2-Q^2\right)}{2D^2\left(1-\mathrm{e}^{-\lambda\frac{Q}{D}}\right)}-\frac{kQ}{D\lambda\mathrm{e}^{\lambda\frac{Q}{D}}}+\frac{(k-1)Q}{D\lambda\left(1-\mathrm{e}^{-\lambda\frac{Q}{D}}\right)}+\frac{1}{\lambda^2}\right)
\end{aligned}
$$

Then, the first summation is calculated as follows;

$$
\begin{aligned}
&\frac{Q^2}{2D}\left(\sum_{k=1}^{\infty}(k-1)P\left\{N=k\middle|Y>\frac{S}{D}\right\}\right)\\
&=\frac{Q^2}{2D}\left(\left(\sum_{k=1}^{\infty}kP\left\{N=k\middle|Y>\frac{S}{D}\right\}\right)-\left(\sum_{k=1}^{\infty}P\left\{N=k\middle|Y>\frac{S}{D}\right\}\right)\right)\\
&=\frac{Q^2}{2D}\left(E\left[N\middle|Y>\frac{S}{D}\right]-1\right)=\frac{Q^2}{2D}\left(\frac{1}{(1-\mathrm{e}^{-\lambda\frac{Q}{D}})}-1\right)=\frac{Q^2}{2D(\mathrm{e}^{\lambda\frac{Q}{D}}-1)}.
\end{aligned}
$$

Now, the first part in the second summation;

$$
\begin{aligned}
&\sum_{k=1}^{\infty} P\left\{N=k \middle| Y>\frac{S}{D}\right\}(kQ)\left(\frac{kQ}{D}-\frac{Q}{D\left(1-e^{-\lambda\frac{Q}{D}}\right)}+\frac{1}{\lambda}\right)\\
&=\sum_{k=1}^{\infty} e^{-\lambda\frac{kQ}{D}}\left(e^{\lambda\frac{Q}{D}}-1\right)(kQ)\left(\frac{kQ}{D}-\frac{Q}{D\left(1-e^{-\lambda\frac{Q}{D}}\right)}+\frac{1}{\lambda}\right)\\
&=\frac{Q^2}{D}\sum_{k=1}^{\infty} k^2 e^{-\lambda\frac{kQ}{D}}\left(e^{\lambda\frac{Q}{D}}-1\right)+Q\left(-\frac{Qe^{\lambda\frac{Q}{D}}}{D}-\frac{1}{\lambda}+\frac{e^{\lambda\frac{Q}{D}}}{\lambda}\right)\sum_{k=1}^{\infty} k e^{-\lambda\frac{kQ}{D}}\\
&=\frac{Q^2}{D}\left(\frac{e^{-\lambda\frac{Q}{D}}\left(1+e^{-\lambda\frac{Q}{D}}\right)\left(e^{\lambda\frac{Q}{D}}-1\right)}{\left(1-e^{-\lambda\frac{Q}{D}}\right)^3}\right)+Q\left(-\frac{Qe^{\lambda\frac{Q}{D}}}{D}-\frac{1}{\lambda}+\frac{e^{\lambda\frac{Q}{D}}}{\lambda}\right)\left(\frac{e^{-\lambda\frac{Q}{D}}}{\left(1-e^{-\lambda\frac{Q}{D}}\right)^2}\right)\\
&=\frac{Q^2\left(1+e^{-\lambda\frac{Q}{D}}\right)}{D\left(1-e^{-\lambda\frac{Q}{D}}\right)^2}-\frac{Q^2}{D\left(1-e^{-\lambda\frac{Q}{D}}\right)^2}-\frac{Qe^{-\lambda\frac{Q}{D}}}{\lambda\left(1-e^{-\lambda\frac{Q}{D}}\right)^2}+\frac{Q}{\lambda\left(1-e^{-\lambda\frac{Q}{D}}\right)^2}\\
&=\frac{Q^2 e^{-\lambda\frac{Q}{D}}}{D\left(1-e^{-\lambda\frac{Q}{D}}\right)^2}+\frac{Q}{\lambda\left(1-e^{-\lambda\frac{Q}{D}}\right)}.
\end{aligned}
$$

Also, the last part of second summation is as follows;

$$
\begin{aligned}
&\sum_{k=1}^{\infty} P\left\{N=k \middle| Y>\frac{S}{D}\right\}\left(-\frac{D}{2}\right)2 \\
&\left(\frac{k^2Q^2}{2D^2}-\frac{\left(2kQ^2-Q^2\right)}{2D^2\left(1-\mathrm{e}^{-\lambda\frac{Q}{D}}\right)}-\frac{kQ}{D\lambda\left(\mathrm{e}^{\lambda\frac{Q}{D}}-1\right)}+\frac{(k-1)Q}{D\lambda\left(1-\mathrm{e}^{-\lambda\frac{Q}{D}}\right)}+\frac{1}{\lambda^2}\right) \\
&=-D\left(\frac{Q^2}{D}\sum_{k=1}^{\infty}k^2\mathrm{e}^{-\lambda\frac{kQ}{D}}\left(\mathrm{e}^{\lambda\frac{Q}{D}}-1\right)+\left(-\frac{2Q^2\mathrm{e}^{\lambda\frac{Q}{D}}}{2D^2}-\frac{Q}{D\lambda}+\frac{Q\mathrm{e}^{\lambda\frac{Q}{D}}}{D\lambda}\right)\sum_{k=1}^{\infty}k\mathrm{e}^{-\lambda\frac{kQ}{D}}\right. \\
&\left.+\left(\frac{Q^2\mathrm{e}^{\lambda\frac{Q}{D}}}{2D^2}-\frac{Q\mathrm{e}^{\lambda\frac{Q}{D}}}{D\lambda}-\frac{1}{\lambda^2}+\frac{\mathrm{e}^{\lambda\frac{Q}{D}}}{\lambda^2}\right)\sum_{k=1}^{\infty}\mathrm{e}^{-\lambda\frac{kQ}{D}}\right) \\
&=-D\left(\frac{Q^2}{D}\frac{\left(1+\mathrm{e}^{-\lambda\frac{Q}{D}}\right)}{\left(1-\mathrm{e}^{-\lambda\frac{Q}{D}}\right)^2}\left(\mathrm{e}^{\lambda\frac{Q}{D}}-1\right)+\left(-\frac{2Q^2\mathrm{e}^{\lambda\frac{Q}{D}}}{2D^2}-\frac{Q}{D\lambda}+\frac{Q\mathrm{e}^{\lambda\frac{Q}{D}}}{D\lambda}\right)\frac{\mathrm{e}^{-\lambda\frac{Q}{D}}}{\left(1-\mathrm{e}^{-\lambda\frac{Q}{D}}\right)^2}\right. \\
&\left.+\left(\frac{Q^2\mathrm{e}^{\lambda\frac{Q}{D}}}{2D^2}-\frac{Q\mathrm{e}^{\lambda\frac{Q}{D}}}{D\lambda}-\frac{1}{\lambda^2}+\frac{\mathrm{e}^{\lambda\frac{Q}{D}}}{\lambda^2}\right)\frac{\mathrm{e}^{-\lambda\frac{Q}{D}}}{\left(1-\mathrm{e}^{-\lambda\frac{Q}{D}}\right)}\right) \\
&=-D\left(\frac{Q^2\left(1+\mathrm{e}^{-\lambda\frac{Q}{D}}\right)}{2D^2\left(1-\mathrm{e}^{-\lambda\frac{Q}{D}}\right)^2}-\frac{Q^2}{D^2\left(1-\mathrm{e}^{-\lambda\frac{Q}{D}}\right)^2}-\frac{Q\mathrm{e}^{-\lambda\frac{Q}{D}}}{D\lambda\left(1-\mathrm{e}^{-\lambda\frac{Q}{D}}\right)^2}+\frac{Q}{D\lambda\left(1-\mathrm{e}^{-\lambda\frac{Q}{D}}\right)^2}\right. \\
&\left.+\frac{Q^2}{2D^2\left(1-\mathrm{e}^{-\lambda\frac{Q}{D}}\right)}-\frac{Q}{D\lambda\left(1-\mathrm{e}^{-\lambda\frac{Q}{D}}\right)}-\frac{\mathrm{e}^{-\lambda\frac{Q}{D}}}{\lambda^2\left(1-\mathrm{e}^{-\lambda\frac{Q}{D}}\right)}+\frac{1}{\lambda^2\left(1-\mathrm{e}^{-\lambda\frac{Q}{D}}\right)}\right) \\
&=-\frac{D}{\lambda^2}.
\end{aligned}
$$

Finally, the last part of the holding cost is derived;

$$
\begin{aligned}
\sum_{k=1}^{\infty} P\left\{N=k \middle| Y>\frac{S}{D}\right\}\left(-\frac{\left(k^2-1\right)Q^2}{2D}\right) &= -\frac{Q^2}{D}\sum_{k=1}^{\infty}k^2\mathrm{e}^{-\lambda\frac{kQ}{D}}\left(\mathrm{e}^{\lambda\frac{Q}{D}}-1\right)+\frac{Q^2}{D}\sum_{k=1}^{\infty}\mathrm{e}^{-\lambda\frac{kQ}{D}}\left(\mathrm{e}^{\lambda\frac{Q}{D}}-1\right) \\
&= \frac{Q^2}{2D}-\frac{Q^2\left(1+\mathrm{e}^{-\lambda\frac{Q}{D}}\right)}{2D\left(-\mathrm{e}^{-\lambda\frac{Q}{D}}\right)^2}.
\end{aligned}
$$

A.4 Derivation of Equation 4.19 and 4.23

Since type (3) cycles in Figure 4.11 (Figure 4.13) are independent of whether the supplier's state ON (OFF) at the first time the inventory level is S,

$$E\left[S\,(X+Y)-\frac{D}{2}\,(X+Y)^2\,\middle|\,X+Y\leq\frac{S}{D},Z\left(\frac{Q-S}{D}\right)=1\right]$$
$$=E\left[S\,(X+Y)-\frac{D}{2}\,(X+Y)^2\,\middle|\,X+Y\leq\frac{S}{D}\right]$$

and,

$$E\left[(Q-S)\,X-\frac{D}{2}X^2\middle|X\leq\frac{S}{D},Z\left(\frac{Q-S}{D}\right)=1\right]=E\left[(Q-S)\,X-\frac{D}{2}X^2\middle|X\leq\frac{S}{D}\right]$$

By the derivation in Section A.1;

$$E\left[X+Y\middle|X+Y\leq\frac{S}{D}\right]$$
$$=\int_0^{\frac{S}{D}}\left(1-\frac{\mu-\lambda-\mu e^{-\lambda z}+\lambda e^{-\mu z}}{\mu-\lambda-\mu e^{-\lambda\frac{S}{D}}+\lambda e^{-\mu\frac{S}{D}}}\right)\,dz+\int_{\frac{S}{D}}^{\infty}(1-1)\;dz$$
$$=\frac{S}{D}-\left(\frac{1}{\mu-\lambda-\mu e^{-\lambda\frac{S}{D}}+\lambda e^{-\mu\frac{S}{D}}}\right)\left(\frac{S\,(\mu-\lambda)}{D}+\frac{\mu}{\lambda}\left(e^{-\lambda\frac{S}{D}}-1\right)-\frac{\lambda}{\mu}\left(e^{-\mu\frac{S}{D}}-1\right)\right)$$

And;

$$E\left[(X+Y)^2\,\middle|X+Y\leq\frac{S}{D}\right]$$
$$=2\left(\int_0^{\frac{S}{D}}\left(z-\frac{z(\mu-\lambda-\mu e^{-\lambda z}+\lambda e^{-\mu z})}{\mu-\lambda-\mu e^{-\lambda\frac{S}{D}}+\lambda e^{-\mu\frac{S}{D}}}\right)\,dz+\int_{\frac{S}{D}}^{\infty}z(1-1)\;dz\right)$$
$$=2\left(\frac{S^2}{2D^2}-\left(\frac{1}{\mu-\lambda-\mu e^{-\lambda\frac{S}{D}}+\lambda e^{-\mu\frac{S}{D}}}\right)\left(\frac{S^2\,(\mu-\lambda)}{2D^2}-\mu\left(-\frac{S}{D\lambda}e^{-\lambda\frac{S}{D}}-\frac{\left(e^{-\lambda\frac{S}{D}}-1\right)}{\lambda^2}\right)\right.\right.$$
$$\left.\left.+\lambda\left(-\frac{S}{D\mu}e^{-\mu\frac{S}{D}}-\frac{\left(e^{-\mu\frac{S}{D}}-1\right)}{\mu^2}\right)\right)\right)$$

In order to derive $E\left[X\middle|X\leq\frac{S}{D}\right]$ and $E\left[(X)^2\,\middle|X+Y\leq\frac{S}{D}\right]$ we need the CDF of $X\middle|X\leq\frac{S}{D}$.

$$P\left\{X \le z \middle| X < \frac{S}{D}\right\} = \frac{P\left\{X \le z, X + Y < \frac{S}{D}\right\}}{P\left\{X < \frac{S}{D}\right\}}$$

$$P\left\{X \le z \middle| X \le \frac{S}{D}\right\} = \begin{cases} \dfrac{P\{X \le z\}}{P\left\{X \le \frac{S}{D}\right\}} = \dfrac{\left(1-\mathrm{e}^{-\lambda z}\right)}{\left(1-\mathrm{e}^{-\lambda \frac{S}{D}}\right)} & z \le \frac{S}{D} \\ 1 & z > \frac{S}{D} \end{cases}$$

So;

$$E\left[X \middle| X \le \frac{S}{D}\right] = \int_0^{\frac{S}{D}} \left(1 - \frac{\left(1 - \mathrm{e}^{-\lambda z}\right)}{\left(1 - \mathrm{e}^{-\lambda \frac{S}{D}}\right)}\right) \mathrm{d}z = \frac{1}{\lambda} - \frac{S\,\mathrm{e}^{-\lambda \frac{S}{D}}}{D\left(1 - \mathrm{e}^{-\lambda \frac{S}{D}}\right)}$$

And;

$$\begin{aligned} E\left[X^2 \middle| X \le \frac{S}{D}\right] &= 2\left(\int_0^{\frac{S}{D}} z\left(1 - \frac{\left(1 - \mathrm{e}^{-\lambda z}\right)}{\left(1 - \mathrm{e}^{-\lambda \frac{S}{D}}\right)}\right) \mathrm{d}z\right) \\ &= 2\left(-\frac{S^2 \mathrm{e}^{-\lambda \frac{S}{D}}}{2D^2\left(1 - \mathrm{e}^{-\lambda \frac{S}{D}}\right)} - \frac{S\,\mathrm{e}^{-\lambda \frac{S}{D}}}{D\lambda\left(1 - \mathrm{e}^{-\lambda \frac{S}{D}}\right)} + \frac{1}{\lambda^2}\right) \end{aligned}$$

We plug these equations into Equations 4.19 and 4.23.

APPENDIX B

BASE SETTING RESULTS

Table B.1: Sensitivity Analysis for $1/\lambda$

$K = 10, h = 1, b = 10, D = 100, 1/\mu = 1, EOQ = 44.72, EOQw/b = 46.9$						
$1/\lambda$	**No-order Opp (Q)**	**No-order Opp Cost**	**Order Opp (Q)**	**Order Opp (S)**	**Order Opp Cost**	**%Δ**
1	269.35	272.77	41.59	201.88	166.97	38.79
2	210.78	227.17	43.09	195.71	127.90	43.70
4	137.56	174.56	43.89	192.38	95.17	45.48
10	70.20	111.26	44.39	190.34	67.81	39.06
20	54.72	80.60	44.55	189.67	56.84	29.48
30	50.79	68.99	44.60	189.51	52.86	23.38
40	49.17	63.32	44.63	189.30	50.94	19.55
50	48.20	59.70	44.64	189.23	49.72	16.72
60	47.63	57.51	44.65	189.14	48.98	14.83
70	47.09	55.30	44.67	189.11	48.24	12.76
80	46.82	54.18	44.67	189.15	47.87	11.66
90	46.57	53.14	44.67	189.11	47.52	10.58
100	46.37	52.32	44.68	189.00	47.25	9.69
1000	44.87	45.49	44.71	189.01	44.98	1.13

Table B.2: Sensitivity Analysis for K

$h = 1, b = 10, D = 100, 1/\lambda = 4, 1/\mu = 1$						
K	**No-order Opp (Q)**	**No-order Opp Cost**	**Order Opp (Q)**	**Order Opp (S)**	**Order Opp Cost**	$\%\Delta$
5	130.93	171.24	31.22	188.59	85.49	50.08
10	137.56	174.56	43.89	192.38	95.17	45.48
15	143.50	177.74	53.52	195.49	102.91	42.10
20	148.96	180.81	61.63	198.15	109.64	39.36
25	153.96	183.77	68.68	200.65	115.73	37.03
30	158.65	186.66	75.02	202.87	121.35	34.98
35	163.01	189.46	80.85	205.02	126.63	33.16
40	167.23	192.20	86.18	207.06	131.63	31.51
45	171.12	194.88	91.23	208.99	136.41	30.00
50	174.97	197.50	95.98	210.83	140.99	28.61
55	178.57	200.07	100.45	212.65	145.41	27.32
60	182.04	202.59	104.71	214.41	149.69	26.11
65	185.51	205.06	108.82	216.07	153.85	24.98
70	188.78	207.50	112.71	217.77	157.90	23.91
75	191.91	209.90	116.52	219.36	161.84	22.89
80	195.00	212.26	120.14	221.00	165.70	21.93
85	198.10	214.59	123.62	222.54	169.48	21.02
90	200.95	216.88	127.06	224.06	173.18	20.15
95	203.89	219.15	130.34	225.58	176.81	19.32
100	206.62	221.38	133.53	227.15	180.39	18.52

Table B.3: Sensitivity Analysis for h

$K = 10, b = 10, D = 100, 1/\lambda = 4, 1/\mu = 1$						
h	**No-order Opp (Q)**	**No-order Opp Cost**	**Order Opp (Q)**	**Order Opp (S)**	**Order Opp Cost**	**%Δ**
0.1	627.16	62.83	133.52	382.12	25.41	59.56
0.5	245.42	132.81	61.61	244.65	65.59	50.62
1	137.56	174.56	43.89	192.38	95.17	45.48
1.5	88.20	198.84	35.96	164.17	116.63	41.34
2	62.31	214.74	31.20	145.28	133.78	37.70
2.5	48.10	226.29	27.94	131.35	148.18	34.52
3	39.66	235.42	25.54	120.47	160.64	31.76
3.5	34.17	243.07	23.66	111.64	171.65	29.38
4	30.34	249.74	22.15	104.25	181.54	27.31
4.5	27.49	255.70	20.89	97.98	190.54	25.49
5	25.29	261.13	19.83	92.52	198.79	23.88

Table B.4: Sensitivity Analysis for b

$K = 10, h = 1, D = 100, 1/\lambda = 4, 1/\mu = 1$						
b	**No-order Opp (Q)**	**No-order Opp Cost**	**Order Opp (Q)**	**Order Opp (S)**	**Order Opp Cost**	$\%\Delta$
5	76.27	113.53	43.89	149.69	59.56	31.89
10	137.56	174.56	43.89	192.38	50.62	45.48
15	193.88	222.23	43.88	219.59	45.48	51.47
20	240.15	261.80	43.90	239.76	41.34	54.97
25	279.41	296.23	43.91	255.88	37.70	57.38
30	313.75	327.07	43.89	269.31	34.52	59.19
35	344.62	355.26	43.86	280.91	31.76	60.63
40	372.74	381.38	43.87	291.05	29.38	61.83
45	398.72	405.83	43.88	300.13	27.31	62.85
50	422.98	428.90	43.88	308.31	25.49	63.73
55	445.95	450.80	43.90	315.81	23.88	64.52
60	467.61	471.71	43.87	322.63	322.63	65.22
65	488.23	491.74	43.88	328.96	328.96	65.85
70	508.17	510.99	43.87	334.91	334.91	66.43
75	526.98	529.57	43.90	340.51	340.51	66.96
80	545.34	547.52	43.87	345.65	345.65	67.46
85	563.07	564.91	43.91	350.57	350.57	67.91
90	580.36	581.80	43.89	355.21	355.21	68.34
95	596.83	598.22	43.89	359.63	359.63	68.74
100	613.06	614.21	43.91	363.88	363.88	69.12

Table B.5: Sensitivity Analysis for D

$K = 10, h = 1, b = 10, 1/\lambda = 4, 1/\mu = 1$						
D	**No-order Opportunity (Q)**	**No-order Opportunity Cost**	**Order Opportunity (Q)**	**Order Opportunity (S)**	**Order Opportunity Cost**	**%Δ**
10	20.67	22.14	13.35	22.71	18.04	18.52
50	74.50	90.40	30.80	99.10	54.82	39.36
100	137.56	174.56	43.89	192.38	95.17	45.48
200	261.60	342.48	62.39	377.02	170.98	50.08
300	385.15	510.29	76.62	560.73	244.03	52.18
400	508.24	678.06	88.55	743.53	315.67	53.45
500	631.65	845.83	99.14	926.17	386.40	54.32
600	754.90	1013.59	108.69	1108.62	456.49	54.96
700	877.94	1181.34	117.43	1290.79	526.08	55.47
800	1000.76	1349.09	125.63	1472.64	595.29	55.87
900	1124.00	1516.84	133.22	1654.77	664.19	56.21
1000	1247.38	1684.58	140.52	1836.40	732.82	56.50