COMPUTATION OF THE GREEKS IN BLACK-SCHOLES-MERTON AND STOCHASTIC VOLATILITY MODELS USING MALLIAVIN CALCULUS

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ABSTRACT

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The computation of the Greeks of options is an essential aspect of financial mathematics. The investors use the information gained from this aspect for hedging purposes or to decide whether to invest in an option or not. However, computation of the Greeks is not straightforward in some cases due to technical difficulties. For instance, the value function of some options are complicated or moreover in some cases they might not have a closed form solution which makes the computation of their Greeks cumbersome. If this is the case, the Greeks have to be computed numerically. In this thesis, the Greeks of European call options are computed under Black-Scholes-Merton and stochastic volatility models assumptions with Malliavin calculus in particular "infinite dimensional integration by parts formula". Moreover, the results for Black-Scholes-Merton assumptions Greeks are compared with finite difference and pathwise methods.

This thesis provides a contribution to computation of Greeks literature by means of the Malliavin calculus. The advantage of the methodology followed in this thesis is that, once the Greeks formula is obtained, it can be applied to any options with continuous and discontinuous payoffs.

Keywords: Malliavin calculus, integration by parts formula, options, computation of Greeks, stochastic volatility models.

ÖZ

BLACK-SCHOLES-MERTON VE STOKASTİK VOLATİLİTE MODELLERDE MALLİAVİN ANALİZİ KULLANILARAK GREEK'LERİN HESAPLANMASI

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Opsiyonların parametrelerine göre hassasiyet analizi (Greeks) finansal matematiğin temel konularından birisidir. Yatırımcılar, bu hesaplamalardan elde ettikleri bilgileri, hedging ya da alım satım kararlarında kullanmaktadırlar. Ancak Greek'lerin hesaplanması teknik zorluklardan dolayı bazı durumlarda kolay değildir. Bunun yanı sıra, bazı opsiyonların fiyat fonksiyonları karmaşık olduğundan ya da kapalı çözümlerinin bulunmaması Greek'lerinin hesaplanmasını karmaşık hale getirmektedir. Bu tezde, Black-Scholes-Merton ve stokastik volatiliti modelleri varsayımları altında Avrupa alım opsiyonlarının Greek'leri Malliavin analiz özellikle, "sonsuz boyutlu uzayda kısmi integrasyon" kullanılarak hesaplanmıştır. Buna ek olarak, Black-Scholes-Merton varsayımı altında hesaplanan Greekler sonlu farklar ve pathwise yötemi ile karşılaştırılmıştır.

Bu tez, son zamanlarda Malliavin analiz alanındaki gelişmeler vasıtasıyla Greek hesaplama literatürüne katkı sağlamaktadır. Bu yöntemin avantajı, elde edilen Greek formüllerinin sürekli ve süreksiz ödeme fonksiyonlarının her ikisine de uygulanabilmesidir.

Anahtar Kelimeler: Malliavin analiz, kısmi integrasyon formülü, Greeklerin hesabı, stokastik volatilite modeller.

To My Family

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CHAPTER 1

MOTIVATION

1.1 Introduction

The fundamental key of option pricing is to calculate the cost of replication of the sold option. For some cases, investors can construct a static replication strategy. The price that they need to charge for these cases is then given by the cost of setting up the initial hedging. After it is done, since investors are hedged their positions, they may forget their position, hence some people call this situation, hedge and forget [35]. But, for many options there is no static replication and the dynamic hedging strategy has to be used in order to protect from the market risk posed by the position. The choice of a model for underlying asset process and setting the risk neutral measure and solving mathematically gives the value of the derivative. In some sense, for the quantitative analyst, the job finishes there, but for the trader who has to manage the position, the real work starts after this point. The trader has to re-hedge the position dynamically which requires knowledge of the various hedge parameters known as sensitivities, called Greeks. This means, the value of the option is not enough itself, also need how its value depends on changes of the model parameters and the traded price of the underlying asset.

The growing significance of risk management issues and the development of more complicated financial products in the market have impose the researchers to develop efficient techniques for the computation of price sensitivities with respect to its model parameters. The Greeks in finance are the partial derivatives of the price value with respect to any parameter of the value function. These derivatives could serve to measure the stability of the financial derivative under study (e.g. delta is the derivative of an option price with respect to its initial price). They may also be used to hedge a certain payoff [39]. The Greeks are useful tools in finance which help to understand how the contingent claim reacts to a change in the parameters of the model. The information gained from computed Greeks of financial derivatives guide the investors in their portfolio management decisions. However, the computation of Greeks are not always straightforward, because of some technical difficulties. For example, the structure of the payoff function of some contingent claims might be very complex. For this reason computing the Greeks could be cumbersome. On the other hand, sometimes the payoff function does not have a closed form solution and one may have to estimate the Greeks numerically, which could be a time consuming and disturbing workout. There are essentially four methods used in computation of the Greeks: the finite difference, pathwise derivative estimates, the likelihood and the Malliavin calculus method [62].

The most widely used method to compute the Greeks is the finite difference method. This method requires to compute the contingent claim of interest at two nearby points and approximate the differential of the payoff function at that point. The problem of this method is, the identification of "two nearby points" is not clear. On the other hand, pathwise derivative estimates technique and the likelihood method also contain some drawbacks in practical issues. For instance, the drawback of the likelihood method is the one that has to know the probability density function of the underlying financial asset in order to compute its Greeks. However, in many cases it is not possible to figure out the probability density function of underlying asset at the maturity.

All methods discussed above use two approach to obtain a result for the Greeks: computation explicitly or estimation numerically by Monte Carlo simulations. Both approach have some difficulties. In the first place, the payoff function of options can be complex and carrying out the differentiation can be unfavorable. Moreover, if the option has no analytic solution and the Greeks are estimated by numerical methods, such as finite difference methods, the estimation may be computationally expensive and the result will be inaccurate due to estimation of expectation and the derivative of the payoff function.

The Malliavin calculus, also known as stochastic calculus of variations or calculus in infinite dimensions introduced by Paul Malliavin in 1976 [48], is used recently in computation of the Greeks to avoid the drawbacks outlined above. It is first used as a tool to prove results in calculus through the use of probabilistic theory and it is an area of research which has been considered as highly theoretical and technical from the mathematical point of view for many years. In recent years, it plays a significant role in applications of mathematical and computational finance. Due to a famous result in Malliavin calculus "infinite dimensional integration by parts formula", one can skip having to evaluate the derivative of the payoff function. Instead of evaluating the derivative, just computing the expected value of the option's payoff multiplied with a weight, called "Malliavin weight" will be enough to compute the Greeks. This expectation can either be computed explicitly or estimated by using Monte Carlo simulations. Since there is no need to estimate the derivative, each way will be less expensive than evaluating the derivative of the payoff function and using it in computations. Moreover, all Greeks can be expressed as the expected value of the payoff times a Malliavin weight and these weights are independent from the payoff functions which is a great advantage in computations. Using this independence feature, one can construct a Monte Carlo algorithm for general options and not specifically for each option. Therefore, the efficiency of this method increases for options which have discontinuous payoff function. Much work concerning numerical applications of Malliavin calculus has been carried out in the pioneering papers of Fournié et al. ([22], [23]).

This thesis focus on the computation of Greeks using Malliavin calculus. The motivation to use Malliavin calculus is that, this method is applicable to a wide class of option prices. Moreover, it allows us to obtain tractable formulas for Greeks which can be simulated using Monte Carlo methods as Fournié et al. [23] showed.

The aim of this chapter is first to discuss the Monte Carlo method and its application in option pricing. Then, the biased and unbiased estimators to determine the sensitivity of a contingent claim to its parameters are discussed. A brief discussion on Monte Carlo method in finance is explained. Then, the details of the three methods, the finite difference method, pathwise derivative estimates technique and the likelihood method are examined in a preliminary concept.

In second chapter, fundamental tools of Malliavin calculus used in computation of the Greeks are provided. This is a powerful alternative method for several reasons. In particular, since it is important in computation of Greeks, the Malliavin derivative, Skorohod integral and integration by parts formula are presented.

In chapter three, the definitions of options and the Greeks are presented. For a smoother reading, basic definitions of option types used in financial markets are provided rather than giving all of the options and Greeks in detail.

In chapter four and five, the Malliavin calculus (famous integration by parts formula) is used to compute the most commonly used Greeks in finance. To be specifies in chapter four the Greeks of Black-Scholes-Merton model and in chapter five the delta of stochastic volatility models is computed.

1.2 Monte Carlo Simulation

The logic of Monte Carlo simulation is to approximate an expected value $\mathbb{E}[X]$ with an average of independent experiments each of them have the same distribution as X. Which is the result of strong law of large numbers theory [42]. Since, in financial markets, one can express the price of financial derivatives or financial quantities as expected values of the associated payoff which is usually defined as a function of the underlying asset, this simulation can be used in computations.

In this section, the tools which are used to obtain numerical approximations of quantities in stochastic systems are introduced. Monte Carlo method for the expectation of a given random variable X is defined as follows,

$$\hat{\mathbb{E}}[X] = \frac{1}{N}\sum_{i=1}^{N} X_i,$$

where N is the the number of simulation and X_i refers to the value that X takes in the ith simulation. The strong law of large numbers (see [20], page: 73) is a useful tool to obtain the result, $\hat{\mathbb{E}}[X] \longrightarrow \mathbb{E}[X]$, as the number of simulation N increases.

Consider a random variable Y such that $\mathbb{E}[Y] = \mathbb{E}[X]$. The simulator can generate average values of Y to construct $\hat{\mathbb{E}}[Y]$. Therefore, the law of large numbers implies that $\hat{\mathbb{E}}[Y] = \mathbb{E}[X]$ for the large N. In this case, Y is called an unbiased estimator of X if it is possible to find an unbiased estimator Y such that $var[Y] < var[X]$. Since the variance of $\hat{\mathbb{E}}[Y]$ will be smaller than the variance of $\hat{\mathbb{E}}[X]$, Y is preferable to X and it is expected to give more accurate results after a number of simulations.

In some cases, it may not be possible to find a better unbiased estimator Y for the given problem. Thus, a random variable Y is thought whose expectation is close to the expectation of X which is $\mathbb{E}[X]$. In this situation, Y is called a biased estimator and it is given with the formula [28];

$$Bias(Y) = \mathbb{E}[Y] - \mathbb{E}[X]. \tag{1.1}$$

1.3 Biased Sensitivity Estimation Methods in Finance

Consider an option's discounted payoff function φ depends on a parameter θ which ranges over an interval of real line. Then, the price of the option as follows;

$$V(\theta) = \mathbb{E}[\varphi(\theta)]. \tag{1.2}$$

The sensitivity estimation of the price, that is obtained as in Equation (1.2), with respect to the parameter θ is only the derivative of the price $V(\theta)$ in terms of θ. The parameter θ could be any of the parameter that effect the price. For example, if θ is the initial price S_0 of the underlying asset, then $V'(\theta)$ corresponds to the Delta of the option.

Let us consider the following example taken from [62] to understand the Monte Carlo method clearly.

Example: Suppose that we have a bag with red and blue balls in it. Assume there is a random variable X defined as,

$$X = \begin{cases} 1, & \text{if red ball is drawn,} \\ 0, & \text{if red ball is drawn,} \end{cases}$$

and $\lambda \in \mathbb{R}^+$is defined as the ratio of red to blue balls in the bag. How the change in $\mathbb{E}[X]$ is determined with the change in the parameter λ ?
The expectation of getting red is

$$\mathbb{E}\left[X\right] = \frac{\lambda}{\lambda + 1};$$

in this case, the effect of change in λ into the $\mathbb{E}\left[X\right]$ from the partial derivative of $[X]$ with respect to λ can be found as

$$\frac{d}{d\lambda}\mathbb{E}\left[X\right] = \frac{(\lambda + 1) - \lambda}{(\lambda + 1)^2} = \frac{1}{(\lambda + 1)^2}.$$

To obtain a more general formula, define $\Phi\left(\lambda\right) = \mathbb{E}\left[X|\lambda\right]$ for some random variable X that its value depends on λ. Hence the aim is to determine the value of the partial derivative at the point $\lambda = \lambda_0$,

$$\Phi'\left(\lambda_0\right) = \frac{d}{d\lambda}\mathbb{E}\left[X|\lambda_0\right].$$

1.3.1 Finite-Difference Methods

Finite difference methods are commonly used in sensitivity analysis. In these methods; for a given function φ depending on a parameter θ ranging over an interval on the real line assumed to be differentiable. For certain kinds of options there exist exact formula but in general one must resort to numerical techniques to approximate them. Hence, a bias occurred and the bias is the key point of this methodology (see Equation (1.1)).

There are essentially three different finite difference methods used in computation of Greeks and all of the finite difference methods can be derived from Taylor expansion neighborhood of a given point. Although these approximation methods are not very efficient, they are very popular among the practitioners because of the fact that they are easy to implement. The details of these three methods are given in the following subsections.

1.3.1.1 Forward-Difference Method

In the use of forward difference method, first independent replications $\varphi_1\left(\theta\right)$, $\varphi_2\left(\theta\right)$,..., $\varphi_n\left(\theta\right)$ of the model at parameter θ and n additional replications $\varphi_1\left(\theta + h\right), \varphi_2\left(\theta\right), \ldots, \varphi_n\left(\theta + h\right)$ at $\theta + h$ where h is a positive constant $(h > 0)$ are simulated. Here, the average of each set is represented by $\tilde{\varphi}_n\left(\theta\right)$ and $\tilde{\varphi}_n\left(\theta + h\right)$ respectively. Then, the forward difference estimator is given as;

$$\hat{\Delta}_F \equiv \hat{\Delta}_F\left(n, h\right) = \frac{\tilde{\varphi}_n\left(\theta + h\right) - \tilde{\varphi}_n\left(\theta\right)}{h}. \tag{1.3}$$

Taking expectation of both side and using the linearity of expectation

$$\begin{aligned}
\mathbb{E}\left[\hat{\Delta}_F\right] &= \frac{1}{h}\left(\mathbb{E}\left[\tilde{\varphi}_n\left(\theta+h\right)\right] - \mathbb{E}\left[\tilde{\varphi}_n\left(\theta\right)\right]\right) \\
&= \frac{1}{h}\left(V\left(\theta+h\right) - V\left(\theta\right)\right) \qquad (1.4)
\end{aligned}$$

is obtained. Assuming that the price function V is twice differentiable with respect to the parameter θ, the following equation can be obtained by Taylor expansion,

$$V\left(\theta+h\right) = V\left(\theta\right) + V'\left(\theta\right)h + \frac{1}{2}V''\left(\theta\right)h^2 + o\left(h^2\right). \qquad (1.5)$$

In this case, from Equation (1.4) the bias in the forward difference estimator is obtained as follows;

$$\begin{aligned}
Bias\left(\hat{\Delta}_F\right) &= \mathbb{E}\left[\hat{\Delta}_F - V'\left(\theta\right)\right] \\
&= \frac{1}{h}\left(V\left(\theta+h\right) - V\left(\theta\right)\right) \\
&- \frac{1}{h}\left(V\left(\theta+h\right) - V\left(\theta\right) - \frac{1}{2}V''\left(\theta\right)h^2 - o\left(h^2\right)\right) \\
&= \frac{1}{2}V''\left(\theta\right)h + o\left(h\right). \qquad (1.6)
\end{aligned}$$

The advantage of this method is very simple to implement. But, there are also several drawbacks of this method. The first one is, if h is too large, the bias (1.6) will give significant error. On the other hand if h is too small, a rounding error will occur. The second drawback is, discontinuous function and small h will increase the variance.

1.3.1.2 Backward-Difference Method

In the application of this method, $\varphi_1\left(\theta\right), \varphi_2\left(\theta\right), \ldots, \varphi_n\left(\theta\right)$ of the model at parameter θ and independent replications $\varphi_1\left(\theta-h\right)$, $\varphi_2\left(\theta-h\right)$, $\ldots$, $\varphi_n\left(\theta-h\right)$ for $\theta-h$ where h is a positive constant ($h>0$) are simulated. The average of these sets are $\tilde{\varphi}_n\left(\theta\right)$ and $\tilde{\varphi}_n\left(\theta-h\right)$ respectively. Like the forward difference method, the estimator is obtained in following equation,

$$\hat{\Delta}_B = \hat{\Delta}_B\left(n,h\right) = \frac{\tilde{\varphi}_n\left(\theta\right) - \tilde{\varphi}_n\left(\theta-h\right)}{h}. \qquad (1.7)$$

Taking expectation of both sides,

$$\mathbb{E}\left[\hat{\Delta}_B(n,h)\right] = \frac{V(\theta) - V(\theta - h)}{h}, \tag{1.8}$$

is obtained. The bias of this method is

$$Bias\left(\hat{\Delta}_B\right) = Bias\left(\hat{\Delta}_F\right). \tag{1.9}$$

The bias of backward finite difference method (1.9) is equal to the bias of forward difference method (1.6). Therefore, the advantage and drawbacks of forward finite method are also valid for this method.

1.3.1.3 Central-Difference Method

The central difference method might be considered as a combination of the forward and backward difference methods. In this method, estimation of the price function $V(\theta)$ and its derivative with respect to θ are emphasized. In the forward and backward methods to estimate $V(\theta)$ the simulation is applied at parameter θ. Therefore, these methods requires simulation at an additional points which are given above in terms of $\theta+h$ and $\theta-h$ where $h > 0$. However, as Glasserman [28] pointed out in the central difference method estimator requires simulation at two additional points which are $\theta + h$ and $\theta - h$. But, this additional computational effort is causing an increase in the convergence of the bias.

Assume that the payoff function is replicated n times for the points $\theta+h$ and $\theta-h$ where $h > 0$ and the average of them are $\tilde{\varphi}_n(\theta + h)$ and $\tilde{\varphi}_n(\theta - h)$ respectively. Then, the estimator of central difference method is obtained as follows;

$$\hat{\Delta}_C = \hat{\Delta}_C(n,h) = \frac{\tilde{\varphi}_n(\theta + h) - \tilde{\varphi}_n(\theta - h)}{2h}. \tag{1.10}$$

If the price function V is at least three times differentiable at the neighborhood of the point θ, the followings are satisfied.

$$V(\theta + h) = V(\theta) + V'(\theta)h + V''(\theta)\frac{h^2}{2} + V'''(\theta)\frac{h^3}{6} + o\left(h^3\right). \tag{1.11}$$

$$V(\theta - h) = V(\theta) - V'(\theta)h + V''(\theta)\frac{h^2}{2} - V'''(\theta)\frac{h^3}{6} + o\left(h^3\right). \tag{1.12}$$

Subtracting Equation (1.12) from (1.11) the following equation is reached,

$$\frac{V(\theta+h)-V(\theta-h)}{2h}=V'(\theta)+V'''(\theta)\frac{h^2}{6}+o\left(h^2\right). \tag{1.13}$$

By rearranging Equation (1.13) the bias obtained as,

$$Bias\left(\hat{\Delta}_C\right)=\frac{1}{6}V'''(\theta)\,h^2+o\left(h^2\right). \tag{1.14}$$

It is important to emphasize that the bias obtained from central-difference method of Equation (1.14) is smaller than the bias acquired from forward Equation (1.6) and backward-difference methods from Equation (1.9).

The advantage of this method is reaching smaller bias in addition to easy implementation as in the other finite difference methods. On the other hand, the drawbacks of other finite difference methods are still remain for this method.

1.3.2 Finite Difference Methods in Practice

On the Choice of h: In the finite difference methods, the choice of perturbation h ($h>0$) is very important. The smaller value of perturbation leads to improve the accuracy of the estimation. On the other hand, the effect of h on bias should be weighted against its variance. The variance of the forward difference estimator (1.3) is as follows,

$$Var\left[\hat{\Delta}_F(n,h)\right]=\frac{1}{h^2}Var\left[\tilde{\varphi}_n(\theta+h)-\tilde{\varphi}_n(\theta)\right]. \tag{1.15}$$

Hence, $\frac{1}{h^2}$ given in the equation plays an important role because it is possible to have chaotic consequences of taking h very small. From Equation (1.15) we can say the dependence between values simulated at different values of the parameter θ affects the variance of a finite difference estimator.

Consider the pairs of $(\varphi(\theta),\varphi(\theta+h))$ and $(\varphi_i(\theta),\varphi_i(\theta+h))$ where $i=1,2,\cdots$ are i.i.d. Therefore,

$$Var\left[\hat{\varphi}_n(\theta+h)-\hat{\varphi}_n(\theta)\right]=\frac{1}{n}Var\left[\varphi(\theta+h)-\varphi(\theta)\right].$$

The change of the variance in Equation (1.15) with respect to h is determined by the dependence of $Var\left[\varphi(\theta+h)-\varphi(\theta)\right]$. There are three primary cases occur in practice, which are

$$Var\left[\varphi(\theta+h)-\varphi(\theta)\right]=\begin{cases} O(1) & \textit{Case (i)}\ , \\ O(h) & \textit{Case (ii)}\ , \\ O\left(h^2\right) & \textit{Case (iii)}. \end{cases}$$

Case (i) applies if $\varphi(\theta)$ and $\varphi(\theta+h)$ are simulated independently. As h goes to 0 the variance is

$$\begin{aligned} Var[\varphi(\theta+h)-\varphi(\theta)] &= Var[\varphi(\theta+h)]+Var[\varphi(\theta)] \\ &\longrightarrow 2Var[\varphi(\theta)] \end{aligned}$$

under the assumptions $Var[\varphi(\theta)]$ is continuous for the parameter θ.

Case (ii) is a consequence of simulating $\varphi(\theta+h)$ and $\varphi(\theta)$ using the common random numbers.

In *Case (iii)* $\varphi(\theta)$ and $\varphi(\theta+h)$ are continuous for not only the same numbers, but also for all values of random numbers. Therefore, the output $\varphi(\cdot)$ is continuous for the parameter θ.

1.3.3 Optimal Relation between Increment and the Replication

An increase in h rises the variance, on the other hand decreases the bias. Thus, there should be a balance between variance and bias. Minimizing mean square error (MSE) can be used to find the balance between them. On the other hand, a decrease in the number of replication n, lowers the variance and it has no impact on the bias. Therefore, it is enough to find the optimal relation between the positive constant h and the replication number n [28].

Now consider the forward difference estimator in terms of independent simulation at θ and $\theta+h$. Assume that *Case(i)* holds and the estimator is denoted by $\hat{\Delta}_{F,i}=\hat{\Delta}_{F,i}(n,h)$. Through taking square of the bias in Equation (1.6) and adding it to Equation (1.15),

$$MSE\left(\hat{\Delta}_{F,i}(n,h)\right)=o\left(h^2\right)+o\left(n^{-1}h^{-2}\right)$$

is obtained. The minimal conditions for convergence are $h\longrightarrow 0$ and $nh\longrightarrow\infty$.

Glasserman [28] offers strengthening the *Cases (i)* and *(ii)* to derive a more exact result. He suggests four estimator to consider forward difference and central difference using independent sampling for different values of θ. Then, a generic estimator $\hat{\Delta}=\hat{\Delta}(n,h)$ for these cases can be obtained;

$$\mathbb{E}\left[\hat{\Delta}-\vartheta(\theta)\right]=bh^{\beta}+o\left(h^{\beta}\right),\quad Var\left[\hat{\Delta}\right]=\frac{\sigma^2}{nh^{\eta}}+o\left(h^{-\eta}\right),\tag{1.16}$$

for some $\beta>0$, $\eta>0$, $\sigma>0$ and some non-zero b. Forward and central difference estimators have; $\beta=1$, and $\beta=2$; choosing $\eta=2$ sharpens *Case (i)* and choosing $\eta=1$ sharpens *Case (ii)* in Subsection 1.3.2.

Consider a sequence of estimators, represented by $\hat{\Delta}(n, h_n)$, with the assumptions of bias and variance

$$h_n = h_* n^{-\gamma} \tag{1.17}$$

for some $h_* > 0$ and $\gamma > 0$.

$$MSE\left(\hat{\Delta}\right) = b^2 h_n^{2\beta} + \frac{\sigma^2}{n h_n^{\eta}}, \tag{1.18}$$

is obtained up to terms that are higher order in h_n. The value of γ which maximizes the MSE is $\gamma = \frac{1}{2\beta+\eta}$. Through substituting this in to (1.18) and taking the square root the following result is achieved:

$$RMSE\left(\hat{\Delta}\right) = O\left(n^{-\frac{\beta}{2\beta+\eta}}\right).$$

From this result,

$$n^{\frac{2\beta}{2\beta+\eta}} MSE\left(\hat{\Delta}\right) = b^2 h_*^{2\beta} + \sigma^2 h_*^{-\eta},$$

is reached and minimizing this for h_* it yields an optimal value of

$$h_* = \left(\frac{\eta\sigma^2}{2\beta b^2}\right)^{\frac{1}{2\beta+\eta}}.$$

Glasserman [28] claims that in the situation of decrease h lowers the error, the standard call option is valid for *Case (iii).* On the other hand, in the situation of increase in error as h approaches to zero, the digital option fits *Case (iii).*

1.4 Unbiased Sensitivity Estimation Methods in Finance

In this section, unbiased estimators θ for the sensitivity of $\mathbb{E}[X]$ with respect to the parameter θ is emphasized. When the unbiased estimators θ are used, a compromise between the bias and the computation time are not encountered. In essence, this does not mean that the number of simulations can be decreased, but the obtained values are correct for sure.

Two approaches of unbiased Monte Carlo estimation of sensitivities, the "pathwise derivative" and "likelihood" are discussed in this section. Since each of these approaches are limited by the narrower scope of problems and are powerful methods in the case of their implementations, they can be successfully addressed.

1.4.1 Pathwise Derivative Estimates Method

Consider a collection of random variables $\{\varphi(\theta), \theta \in \Theta\}$ which are defined on a probability space $(\Omega, \mathcal{F}, P)$. Assume that $\Theta \subseteq \mathbb{R}$ is an interval. For a fix $w \in \Omega$ consider a mapping $\theta \longmapsto \varphi(\theta, w)$ as a random function on the interval Θ. Then, $\varphi'(\theta) = \varphi'(\theta, w)$ is the derivative of the random function with respect to the parameter θ. In this case, it is assumed that the derivative exists on this interval. If this holds, $\varphi'(\theta)$ is called the pathwise derivative of φ with respect to θ [28].

In the discussion of the finite difference methodologies in practice, in *Case (iii)* in Subsection 1.3.2 the mean square error (MSE) decreases as h increases. Therefore, the parameter h to zero should be decreased and tried to estimate the derivative of the option price $V(\theta) = \mathbb{E}[\varphi(\theta)]$ (where $\varphi(\theta) = \varphi(\theta, w)$) using,

$$\varphi^{'}(\theta) = \lim_{h \to 0} \frac{\varphi(\theta + h) - \varphi(\theta)}{h}. \tag{1.19}$$

This is an estimator with mean $\mathbb{E}[\varphi'(\Theta)]$. It is an unbiased estimator of the derivative of the option price $V'(\theta)$. Under the assumption of Leibniz rule, it can be written as follows,

$$\mathbb{E}\left[\frac{d}{d\theta}\varphi(\theta)\right] = \frac{d}{d\theta}\mathbb{E}[\varphi(\theta)]. \tag{1.20}$$

There are two advantages of pathwise derivative estimates method. The first advantage of this method is is that the method does not have a problem of variance “blow up” as the number of time steps increases, which makes it more suitable for dealing with path-dependent options. The second advantage of this method is the computational cost is independent of the number of first derivatives to be calculated. It is especially useful in computation of multiple Greeks simultaneously.

On the other hand, there are also some drawbacks of this method. The most crucial feature of pathwise derivative estimate method is; the payoff function has to be differentiable. Since the payoff function is not differentiable in general, derivative may not exist. However, there are methods used to overcome this problem. For instance, using a continuous piecewise linear function to approximate the payoff function, called “payoff smoothing”, is one of the ways to handle (see [37] for further details). For some cases, even the option payoff function is differentiable, it can be difficult to calculate the derivatives of the function in practice [37]. Another drawback since Monte Carlo estimator requires computations for each Greek, this method is computationally expensive. In addition to these drawbacks, this method is not applicable for the barrier options and on estimation of second order derivatives.

Despite the fact that this method is limited by the requirement of continuity condition in the discounted payoff as a function of the parameter for differentiation [28], it gives efficient results in some cases.

1.4.2 The Likelihood Method

As Broadie and Glasserman [6] point it, the likelihood method is one of the ways to compute the Greeks in such cases where the joint density of the random variables are used in the problem which are explicitly known or can be approximated. This method is proven to be highly effective to estimate the Greeks when it is applicable. However, it is not possible to know the density in all cases. In some cases, such as the density is not known explicitly, Kernel type approximation of the joint density can be used in computations of sensitivity analysis.

Consider the discounted payoff $\varphi = f(X, \theta)$ where $X = (X_1, \ldots, X_m)$. Here, X_i $(i = 1, 2, \ldots, m)$ can be different underlying asset or values of only one underlying asset at multiple dates. Suppose that the density of X is known and represented by g and θ which are parameters of this density. Now suppose that the density represented with g_θ. Then, the expected discounted payoff is given by,

$$\mathbb{E}[\varphi] = \mathbb{E}[f(X_1, \ldots, X_m)] = \int_{\mathbb{R}^m} f(x)\, g_\theta(x) dx.$$

The payoff function f and the density g depends on model parameters. Fixing the parameter θ and, if it is applicable, applying Leibniz rule one can have the Greek as follows,

$$\frac{\partial}{\partial\theta}\mathbb{E}[\varphi] = \int_{\mathbb{R}^m} f(x)\, \frac{\partial}{\partial\theta} g_\theta(x)\, dx. \tag{1.21}$$

One may further, can rewrite Equation (1.21) as,

$$\begin{aligned}\frac{\partial}{\partial\theta}\mathbb{E}[\varphi] &= \int_{\mathbb{R}^m} f(x)\, \frac{g'_\theta(x)}{g_\theta(x)} g_\theta(x)\, dx \\ &= \mathbb{E}\left[f(X)\, \frac{g'_\theta(X)}{g_\theta(X)}\right],\end{aligned} \tag{1.22}$$

under the assumption of interchangeability of differentiation and integration. Note that $\mathbb{E}\left[f(X)\, \frac{g'_\theta(X)}{g_\theta(X)}\right]$ is an unbiased estimator of $\frac{\partial}{\partial\theta}\mathbb{E}[\varphi]$.

The computation in Equation (1.22) can be seen as an option pricing problem with a payoff function $f(x)\, \frac{g'_\theta(x)}{g_\theta(x)}$, which is the initial payoff multiplied by a weight $\omega(\theta) = \frac{g'_\theta(x)}{g_\theta(x)}$.

Having carried out the computation of the weights the Greeks will become,

$$Greek_{\theta} = \frac{1}{N}\sum_{i}^{N} f(x)\,\omega(\theta).$$

The advantage of likelihood method is, contrary to pathwise method, that it does not require any regularity assumption on the payoff function, which means that it can be applied to non-differentiable payoff functions, too because the derivative of the payoff function is not needed [7].

On the other hand, there are two main drawbacks of this method: first, for some cases the density function of the underlying asset is not known. Second, this method leads to a Monte-Carlo estimator with variance of order N^{-1}. Indeed, it becomes infinity as N approaches to zero and is unsuitable for calculations involving path-dependent options [37].

This method requires probability density function of the underlying asset which the option is written on and interchangeability of integration and differentiation. Fournié et al. [23] generalizes this method to path space using Malliavin calculus.

1.5 Summary

In this chapter, the finite difference, pathwise and likelihood methods are discussed in a preliminary level. It is seen that these methods cannot be used for all type of options because of complexity of the some underlying asset pricing model or option's payoff functions, like stochastic volatility models and Asian options. For example; if the option's payoff function is differentiable, the finite difference and pathwise method can be used in computation of Greeks. But, for many option, the payoff functions are not differentiable. Sometimes even if it is differentiable, differentiation of the function can be confusing. Also, the finite difference method is highly biased and the pathwise method can not be used in computation of second and higher order Greeks. On the other hand; if the density of the underlying asset is known, the likelihood method can be used to estimate both continuous and discontinuous payoff functions. However, the density function can not be determined in many cases. Further, it cannot be applied to complex payoffs such as Asian options. For further details see [18] and [28].

Finite difference, pathwise and likelihood methods used in computation of the Greeks are summarized in the following Table 1.1.

Table 1.1: Summary of the Methods

FDM	Enough information. No need any additional information from the model.
	Problematic for discontinuous payoffs.
	This method is very simple to implement.
	Always applicable in the existence of corresponding derivative.
	The choice of h is confusing.
	Highly biased result.
	Three source of error; the Monte Carlo, the possible discretization, Finite difference.
	Computationally expensive.
PDEM	Need for additional information from the model that used.
	Not applicable for discontinuous payoffs due to need of partial derivative of the payoff function calculation.
	Computationally expensive.
	No problem of variance blow up as the number of time steps increases.
	Independent computational cost of the number of first derivatives.
	Leads to an unbiased of the Greeks.
	Not applicable for second order Greeks.
LM	Need of determination of the probability density function of the model under consideration.
	Applicable to discontinuous payoffs.
	Leads to an unbiased estimator of the Greeks.
	Computationally effective.

CHAPTER 2

PRELIMINARIES

2.1 Introduction

The calculus of variations for stochastic processes, called Malliavin calculus, is introduced by Paul Malliavin in 1976 [48]. This calculus is defined on the Wiener space. The theory of Malliavin calculus has been established to prove results on the regularity of the density of solutions of stochastic differential equations driven by a Brownian motion. It is a research area that has been considered highly theoretical and practical from the mathematical perspective for many years. However, in recent years it has playes a major role in applications of mathematical and computational finance. Since it can be easily applied to Monte- Carlo methodology, it is also a useful tool in computation of the Greeks in finance.

While dealing with sophisticated models in option pricing with non standard and discontinuous payoff, the classical methods like finite differences, pathwise derivative estimates, the likelihood and other numerical methods for partial differential equations could be inefficient in solutions. Researchers use the Monte-Carlo and quasi Monte-Carlo methods to overcome this insufficiency. The main drawback of these two method is, in the case of a discontinuous payoff function, they have weak convergence to the exact solution [4]. Because of this reason recently Malliavin calculus has been used in computation of the Greeks.

In financial mathematics, the computation of Greeks with Malliavin calculus for the Black-Scholes-Merton asset price dynamics model and application to Monte Carlo simulation was first introduced by Fornié et. al (1999, 2001) (see [22] and [23] for further readings). Malliavin method does not require differentiable payoff function and no need to know the probability density function of the underlying asset. The main tool in computation of the Greeks with this method is the integration by parts formula. Through Malliavin method all Greeks can be expressed as the expected value of the payoff function multiplied by a weight function, so called Malliavin weight. The main advantage of this method is; the independence of the Malliavin weight and payoff function. Therefore, it is applicable for both discontinuous and continuous payoffs. Using this feature, a Monte Carlo algorithm for general options can be constructed at first and then, this algorithm can be used for specific payoffs.

As for prerequisites, the reader is expected to be familiar with some findings in Malliavin calculus to understand this method. In this chapter, without going into technical details, concepts and theorems used in computation of Greeks are discussed briefly. Therefore, a basic introduction to Malliavin calculus and its usage within the area of Monte Carlo simulations in finance are provided (see [16], [28] and [54] for further details). Instead of giving an overview of the theory on Malliavin calculus, the theory (without proofs) of the computation of Greeks is emphasized. This chapter follows up on some ideas from David Nualart (2006) [54], which contains topics about Malliavin calculus application to finance.

2.2 Wiener Space

The Malliavin calculus has been developed on the Wiener space framework. Indeed it is an infinite dimensional differential calculus. To define this infinite dimensional space, first consider the space of real valued continuous functions defined on $[0,T]$ with value 0 at $t=0$, i.e.

$$\Omega = C_0([0,T]) = \{w_t : [0,T] \longrightarrow \mathbb{R} |\ w_t \text{ is continuous}, w_0 = 0\}.$$

Second, consider a probability space $(\Psi, \mathcal{G}, \nu)$ and then, define $(B_t)_{t\in[0,T]}$ which is assumed to be a Brownian motion with respect to the probability measure ν. Here, $\mathcal{G}$ is the σ-algebra generated by the Brownian motion $(B_t)_{t\in[0,T]}$. Because of continuity of Brownian motion, it is reasonable to construct a mapping from Ψ to Ω namely, an element X of Ψ is mapped to an element $B_t(X)$ of Ω. This can be viewed as

$$\begin{aligned} B_t \ : \ & \Psi \longrightarrow \Omega, \\ & X \longmapsto B_t(X), \end{aligned}$$

where Ω is equipped with the σ-algebra $\mathcal{F}$ generated by the sets

$$\{w | w_{t_1} \in A_1, \ldots, w_{t_n} \in A_n, \},$$

where, $0 \leq t_1 \leq t_2 \leq \ldots \leq t_n \leq T$ and $A_1, A_2, \ldots, A_n \in \mathcal{B}$ are Borel sets in $\mathbb{R}$. Here, the Brownian motion $X \longmapsto B_t(X)$ is a measurable mapping from the space $(\Psi, \mathcal{G}, \nu)$ to the space $(\Omega, \mathcal{F})$. Hence, it can be induced to a probability measure on $(\Omega, \mathcal{F})$ given by,

$$P(w | w_{t_1} \in A_1, \ldots, w_{t_n} \in A_n) = \nu(B_{t_1} \in A_1, \ldots, B_{t_n} \in A_n). \tag{2.1}$$

The probability measure (2.1) is called Wiener measure [58].

After setting this define the coordinate mapping process, which is a connection between Ω and the real line $\mathbb{R}$,

$$\begin{aligned} W_t \ : \ & \Omega \longrightarrow \mathbb{R} \\ & W_t(w) = w_t. \end{aligned}$$

Here, the process $W = (W_t)_{t\in[0,T]}$ has the same distribution under the measure P like $(B_t)_{t\in[0,T]}$ has under the measure ν [54]. Therefore, the process $W = (W_t)_{t\in[0,T]}$ is a Brownian motion in the probability space $(\Omega, \mathcal{F}, P)$ and this space is called Wiener space. Note that, $\mathcal{F}$ is the σ-algebra generated by the Brownian motion $W = (W_t)_{t\in[0,T]}$.

With this expression the coordinate mapping on the Wiener space becomes a Brownian motion under the Wiener measure defined in Equation (2.1). By this fact, when it is need to study on Brownian motion, the coordinate mapping on Wiener space can be used instead of the probability space $(\Psi, \mathcal{G}, \nu)$ (See [30] and [54] for further details).

2.3 The Malliavin Derivative on Wiener Space

Let us first introduce the Gaussian isonormal processes, defined by Dudeley in [17], before beginning to explain the Malliavin derivative.

Definition 2.1. A centered Gaussian family $W = (W(h), h \in H)$ is an isonormal Gaussian process on the Hilbert space H if it is parametrized by the elements of this Hilbert space, such as

$$\mathbb{E}[W(g)W(h)] = \langle g, h\rangle_H,$$

where $g, h \in H$ and $\langle ., .\rangle$ is the inner product of the Hilbert space H.

Assume that $W = (W(h), h \in H)$ be an isonormal Gaussian process with respect to a separable Hilbert space H. The process W is defined on the given complete probability space $(\Omega, \mathcal{F}, P)$.

Now, the derivative DF for a square integrable random variable F

$$F : \Omega \longrightarrow \mathbb{R}$$

can be discussed. Actually, in this part the derivative of the random variable F means derivative with respect to $\omega \in \Omega$. The interesting point in this scheme is, although F does not possess a continuous version, the derivative of F with respect to chance parameter ω exist. Hence, a notion of derivative does not have a continuous version which is defined in a weak sense [54].

The space of all infinitely differentiable functions $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is denoted by $C_p^\infty(\mathbb{R}^n)$, with partial derivatives are bounded, that is, all of its partial derivatives have polynomial growth.

Definition 2.2. 1. The family of smooth random variables is the set $\mathcal{S}$ of random variables $F : \Omega \longrightarrow \mathbb{R}$ such that there exist a function f in $C_p^\infty(\mathbb{R}^n)$ and $h_1, h_2, \ldots, h_n \in H$ such that

$$F = f\left(W\left(h_1\right), W\left(h_2\right), \cdots, W\left(h_n\right)\right). \tag{2.2}$$

2. The family $\mathcal{P}$ represents the set of random variables of the form (2.2) with a polynomial function f.

3. The family $\mathcal{S}_b$ denotes the space of random variables of the form (2.2) where f and all of its partial derivatives are bounded ($f \in C_b^\infty(\mathbb{R}^n)$).

4. The family $\mathcal{S}_0$ denotes the space of random variables of the form (2.2) where f has a compact support ($f \in C_0^\infty(\mathbb{R}^n)$).

The families given in Definition 2.2 can be associated with the following remark (see [54], [62], and [54] for further details).

Remark 2.1. The families given in Definition 2.2 are ordered like $\mathcal{S}_0 \subset \mathcal{S}_b \subset \mathcal{S}$ and $\mathcal{P} \subset \mathcal{S}$. Also, the families $\mathcal{S}_0$ and $\mathcal{P}$ are dense in the space $L^2(\Omega, \mathcal{F}, P)$.

Brownian motion W has continuous paths, the sensitivity of F's with respect to change in the paths ω_t for any t can be measured. Therefore, the Malliavin derivative DF of the random variable $F \in \mathcal{S}$ is a process and it measures the sensitivity of F with respect to changes in ω_t. Generally, it is represented as $DF = (D_t F)_{t\in[0,T]}$ and the operator D_t is defined on the smooth random variable space. Definition of the Malliavin derivative on the Wiener space $(\Omega, \mathcal{F}, P)$ is characterized with following proposition without proof (see the proof in [54]).

Proposition 2.1. *The derivative of a smooth random variable $F \in \mathcal{S}$ which has a form as (2.2) is a H-valued random variable given by*

$$DF = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\left(W\left(h_1\right), \ldots, W\left(h_n\right)\right) h_i. \tag{2.3}$$

Here, $\frac{\partial f}{\partial x_i}$ means the partial derivative with respect to ith component.

The following basic example provides an intuition to apply Proposition 2.1.

Example 2.1. Consider the simple case, $f(x) = x^2$. In this case, $F = W(h)^2$. Then, the derivative of $W(h)^2$ is,

$$D\,W\left(h\right)^2 = D\,f\left(W\left(h\right)\right) = 2W\left(h\right)h.$$

The Malliavin derivative DF can be interpreted as a directional derivative for any $h \in H$.

Remark 2.2. Suppose that F is a smooth random variable defined by (2.2), and DF is the Malliavin derivative, then,

$$\begin{aligned}\langle DF, h\rangle_H &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\Bigg[f\left(W(t_1)+\epsilon\left\langle t_1,h\right\rangle_H,\ldots,W(t_n)+\epsilon\left\langle t_n,h\right\rangle_H\right)\\ &- f\left(W(t_1),\ldots,W(t_n)\right)\Bigg].\end{aligned}$$

where $\langle DF, h\rangle_H$ is the inner product of the Hilbert space H.

By Remark 2.2 it can be said that the scalar product $\langle DF, h\rangle_H$ is the Malliavin derivative of F at $\epsilon = 0$.

The Malliavin derivative of a random variable $F \in \mathcal{S}$ given in Proposition 2.1 satisfies two important properties; product rule and linearity just like the derivative in ordinary differential calculus. The properties are given with the following propositions.

Proposition 2.2. *Suppose that F and G are any smooth random variables. Then, $D\left(FG\right)$ is*

$$D\left(FG\right) = FD\left(G\right) + GD\left(F\right). \tag{2.4}$$

Proposition 2.3. *Suppose that F and G are any smooth random variables and $a, b \in \mathbb{R}$. Then,*

$$D\left(aF + bG\right) = aD\left(F\right) + bD\left(G\right).$$

The following proposition is called integration by parts formula. It is a major tool in Malliavin calculus both application and for theoretical sides. Particularly, it can be used to get a great advantage in computation of the Greeks.

Proposition 2.4. *Suppose that F is a random variable of the form (2.2) and h is the element of the Hilbert space H. Then, the following formula holds.*

$$\mathbb{E}\left[\langle DF, h\rangle_H\right] = \mathbb{E}\left[FW(h)\right]. \tag{2.5}$$

Proof. The random variable F is denoted by $F = f\left(W\left(h_1\right), W\left(h_2\right), \cdots, W\left(h_n\right)\right)$ in Equation 2.2 where $f \in C_p^\infty\left(\mathbb{R}^n\right)$. Assume that the n-fold Wiener measure denoted by μ_n, then,

$$\begin{aligned}
\mathbb{E}\left[\langle DF, h\rangle_H\right] &= \int_{\mathbb{R}^n} \frac{\partial f}{\partial x_1}\left(x\right) d\mu_n\left(x\right) \\
&= \int_{\mathbb{R}^n} f\left(x\right) x_1 d\mu_n\left(x\right) \\
&= \mathbb{E}\left[FW(h)\right].
\end{aligned}$$

The detailed proof can be found in [54] on page 26. □

If the integration by parts formula is applied to the product FG where $F, G \in \mathcal{S}$ then, the following consequence is observed.

Lemma 2.5. *Suppose F and G are two smooth random variables and $h \in H$. Then,*

$$\mathbb{E}\left[G\left\langle DF, h\right\rangle_H\right] = \mathbb{E}\left[-F\left\langle DG, h\right\rangle_H\right] + FGW\left(h\right). \tag{2.6}$$

As a consequence of Lemma 2.5 the following proposition is obtained.

Corollary 2.6. *The Malliavin derivative operator D is closeable from the space $L^P\left(\Omega\right)$ to the space $L^P\left(\Omega; H\right)$ for any $p \geq 1$.*

Proof. The detailed proof is given by Nualart in [54]. □

In Proposition 2.1, the Malliavin derivative is defined for the elements of $\mathcal{S}$ and then extended to nonsmooth random variables which encountered in finance. For this purpose, define the domain of k^{th} order Malliavin derivative denoted by $\mathbb{D}^{k,p}$ for any $p \geq 1$ and $k \geq 1$:

The iteration of the derivative operator D k times for a random variable $F \in \mathcal{S}$ can be defined as a random variable denoted by $D^k F \in H^{\otimes k}$. The following seminorm can be defined on the family $\mathcal{S}$:

$$\|F\|_{k,p} = \left[\mathbb{E}\left[|F|^p\right] + \sum_{j=1}^{k} \mathbb{E}\left[\left\|D^j F\right\|_{H^{\otimes j}}^p\right]\right]^{\frac{1}{p}}. \tag{2.7}$$

Then, the domain of k^{th} order Malliavin derivative, denoted by $\mathbb{D}^{k,p}$, is a completion of $\mathcal{S}$ with respect to the semi-norm defined in Equation (2.7).

Choosing $k = 1$, for any $p \geq 1$, the domain of the first order Malliavin derivative, denoted by $\mathbb{D}^{1,p}$, is in the space $L^P(\Omega)$. In this case, $\mathbb{D}^{1,p}$ is the closure of smooth random variables family $\mathcal{S}$ with respect to the following semi-norm,

$$\|F\|_{1,p} = \|F\|_{L^p(\Omega)} + \|D\,F\|_{L^p([0,T]\times\Omega)}\,, F \in L^p(\Omega)\,.$$

or,

$$\|F\|_{1,p} = \left[\mathbb{E}\left[|F|^p\right] + \mathbb{E}\left[\|DF\|_H^p\right]\right]^{\frac{1}{p}}$$

If $p = 2$ the space $\mathbb{D}^{1,2}$ becomes a Hilbert space with the scalar product,

$$\langle F, G\rangle_{1,2} = \mathbb{E}\left[F, G\right] + \mathbb{E}\left[\langle DF, DG\rangle_H\right]. \tag{2.8}$$

Now on, the completion of the family of smooth random variables $\mathcal{S}$ by $\mathbb{D}^{k,p}$ will denoted with the norm $\|.\|_{k,p}$. Note that for a fix $h \in H$, the Malliavin derivative of a smooth random variable F can be defined as in the following remark.

Remark 2.3. Let us take a fix element h of the Hilbert space H. Then, the derivative operator is denoted with D^h and it is defined on the smooth random variables in $\mathcal{S}$ by

$$D^h F = \langle DF, h\rangle_H = \int_0^T (D_t F)\, h_t dt. \tag{2.9}$$

As a matter of fact, the Malliavin derivative operator D satisfies the chain rule property which is satisfied by the derivative operator of ordinary differential calculus [54].

Proposition 2.7. *Let $\varphi : \mathbb{R}^m \longmapsto \mathbb{R}$ be a continuously differentiable function with polynomial growth partial derivatives. Then, for a given random vector $F = (F^1, F^2, \cdots, F^m)$ such that each $F^i \in \mathbb{D}^{1,p}$ for all $i = 1, 2, \cdots, m$ for a given $p \geq 1$. Then, $\varphi(F) \in \mathbb{D}^{1,p}$ and,*

$$D(\varphi(F)) = \sum_{i=1}^{m} \frac{\partial}{\partial x_i}\varphi(F)\, DF^i. \tag{2.10}$$

Proof. Proof can be found in [16]. □

It is important to emphasize that, Proposition 2.7 can be extended to the functions satisfying Lipschitz continuity which will be helpful to compute the Greeks of many options with complex payoffs (see [16] and [54]). The extension is given with following proposition.

Proposition 2.8. *Let $\varphi : \mathbb{R}^m \longrightarrow \mathbb{R}$ be a such that it satisfies the condition*

$$|\varphi(x) - \varphi(y)| \leq K \|x - y\|$$

for any $x, y \in \mathbb{R}^m$. Suppose that $F = (F^1, F^2, \ldots, F^m)$ is a random vector whose components F^i's are in the space $\mathbb{D}^{1,p}$. Then, $\varphi(F) \in \mathbb{D}^{1,p}$ and there exist bounded random variables G^i for $i = 1, 2, \ldots, m$ such that

$$D(\varphi(F)) = \sum_{i=1}^{m} G^i DF^i. \tag{2.11}$$

Proof. For the proof of this proposition [54] page: 29 is referred to the readers. □

Remark 2.4. In Proposition 2.8 if the random variable is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^m$ then, $G^i = \frac{\partial}{\partial x_i}\varphi(F)$. Moreover, since the function φ is Lipschitz, the derivative $\frac{\partial}{\partial x_i}\varphi(x)$ exists for almost all $x \in \mathbb{R}^m$.

2.4 The Divergence Operator

The divergence operator is defined as the adjoint of the derivative operator. In this section, the separable Hilbert space H is taken as L^2 space and it is represented by $H = L^2([0,T], \mathcal{B}, \mu)$, where μ is a σ-finite atomless measure on the measurable space $([0,T], \mathcal{B})$. The divergence operator is interpreted as a stochastic integral which is called the Skorohod integral because it coincides to the generalization of the Itô stochastic integral introduced for the first time by Skorohod 1976 [66]. Throughout this study, the divergence operator is denoted by $\delta(.)$ and operates on $u = u_t \in L^2([0,T] \times \Omega)$ for $t \in [0,T]$. The elements of $Dom(\delta)$, the domain of Skorohod integrable processes, are the subsets of the space $L^2([0,T] \times \Omega)$. The Skorohod integral of $\{u_t\}$ is defined as follows:

$$\delta(u) = \int_0^T u_t \, dW_t.$$

The divergence operator is introduced in the framework of a Gaussian isonormal process $W = (W(h), h \in H)$. Here, it is assumed that the process W is defined on a complete Wiener space $(\Omega, \mathcal{F}, P)$, where $\mathcal{F}$ is the σ-algebra generated by W.

Remember that, the derivative operator D defined in Proposition 2.1 is a closed and unbounded operator with values in the space $L^2(\Omega; H)$ and defined on the set $\mathbb{D}^{1,2} \subset L^2(\Omega)$.

Definition 2.3. The adjoint of the derivative operator D is denoted by δ. It is an unbounded operator on $L^2(\Omega; H)$ with values in $L^2(\Omega)$ and it satisfies:

1. The domain of δ ($Dom(\delta)$) is denoted by the set of H valued random variables $u \in L^2(\Omega; H)$ which are square integrable such that,

$$|\mathbb{E}[\langle DF, u\rangle_H]| \leq c \|F\|_2, \tag{2.12}$$

for all $F \in \mathbb{D}^{1,2}$ and some constant c which depends on u.

2. If $u \in Dom\delta$ then $\delta(u) \in L^2(\Omega)$ and characterized by

$$\mathbb{E}[F\delta(u)] = \mathbb{E}[\langle DF, u\rangle_H], \tag{2.13}$$

for any $F \in \mathbb{D}^{1,2}$.

Note that as in the derivative operator D, the divergence operator δ is closed and a densely defined operator.

Proposition 2.9. *Let $u \in \mathcal{S}$, $F \in \mathcal{S}$ and $h \in H$. Then the following property is satisfied.*

$$D^h(\delta(u)) = \langle u, h\rangle_H + \delta\left(D^h u\right). \tag{2.14}$$

Proof. Detailed proof can be found in [54], page:38. □

Remark 2.5. If $u \in \mathbb{D}^{1,2}$ then the derivative Du is a square integrable random variable.

The following proposition provides a large class of H-valued random variables in the domain of the δ for computation purposes. Proposition 2.9 can be extended to more general sets of random variables by the help of following lemma.

Lemma 2.10. *Let the random variable G is square integrable and suppose there exist a random variable $Y \in L^2(\Omega)$ such that,*

$$\mathbb{E}[G\delta(hF)] = \mathbb{E}[YF], \tag{2.15}$$

for all $F \in \mathbb{D}^{1,2}$. Then G is an element of $\mathbb{D}^{h,2}$ and $D^h G = Y$.

Proof. For the proof [54], page:39 is referred to the reader. □

Using following proposition, a scalar random variable can be factored out from the divergence operator.

Proposition 2.11. *Let the random variable F be an element of $\mathbb{D}^{1,2}$ and $u \in Dom(\delta)$ such that $Fu \in L^2(\Omega; F)$. Then, $Fu \in Dom(\delta)$ and the following equality holds:*

$$\delta(Fu) = F\delta(u) - \langle DF, u\rangle_H. \tag{2.16}$$

Proposition 2.11 can be modified by replacing u with a deterministic function h of Hilbert space H. In this case, it suffices to note that the random variable F is differentiable in the direction of h.

Proposition 2.12. *Let the random variable F is an element of $\mathbb{D}^{1,2}$ and $h \in H$. Then, the product Fu in the domain of δ and the following equality holds:*

$$\delta(Fu) = FW(h) - D^h F. \tag{2.17}$$

Proposition 2.11 can be extended by the following proposition, which will be very useful in computations.

Proposition 2.13. *Suppose that the Hilbert space H is $H = L^2([0,T], \mathcal{B}, \mu)$. Consider $A \in \mathcal{B}$ and random variable $F \in \mathcal{D}^{A,2}$. Let u be an element of the space $L^2(\Omega; H)$ such that $u\mathbb{1}_A \in Dom(\delta)$ and such that $Fu\mathbb{1}_A \in L^2(\Omega; H)$ for indicator function $\mathbb{1}_A$. Then, $Fu\mathbb{1}_A$ in $Dom(\delta)$ and the following holds:*

$$\delta(Fu\mathbb{1}_A) = F\delta(u\mathbb{1}_A) - \int_A D_t F \, u_t \, \mu(dt). \tag{2.18}$$

The following proposition provides a useful criterion to for the existence of the divergence operator [54].

Proposition 2.14. *Consider an element u of the space $L^2(\Omega; H)$ such that there exists a sequence u^n in $Dom\delta$ and the sequence converges to u in $L^2(\Omega; H)$. Suppose that there exists a random variable $G \in L^2(\Omega)$ such that*

$$\lim_{n \mapsto \infty} \mathbb{E}[\delta(u^n) F] = \mathbb{E}[GF],$$

for all smooth random variables F. Then, $u \in Dom(\delta)$ and $\delta(u) = G$.

2.4.1 The Skorohod Integral

In this part, consider the separable Hilbert space H is $H = L^2([0,T], \mathcal{B}, \mu)$, where μ is a σ-finite atmoless measure on the measurable space $([0,T], \mathcal{B})$.

The elements of $Dom(\delta)$ are the subsets of the space $L^2([0,T] \times \Omega)$ which are square integrable processes. Through out this study, the divergence operator is denoted by $\delta(.)$ and operates on the processes $u = u_t$ for $t \in [0,T]$. This operator is called Skorohod stochastic integral of the given process u. The integral notation is,

$$\delta(u) = \int_0^T u_t \, \delta(W_t).$$

Remark 2.6. Since the Skorohod integral δ is closed, it can be deduced that $u \in Dom\delta$ and $\delta(u)$ is equal to the Itô integral of the process u. More generally, any type of adapted stochastic integral with a multiparameter Gaussian white noise W integrator can be thought as a Skorohod integral [54].

The following theorem is explaining this remark.

Theorem 2.15. *Let the pair $(t, w) \in [0, T] \times \Omega$ and $H \in L^2([0, T] \times \Omega)$ be a process on the probability space $(\Omega, \mathcal{F}_t, P)$. In this setting, $\mathcal{F}_t$ is a σ-algebra generated by the Brownian motion at time $s \leq t$. Under these circumstances the Skorohod integral coincides with Itô integral, that is,*

$$\delta(H) = \int_0^T H_t \delta(W_t) = \int_0^T H_t dW_t, \tag{2.19}$$

for all $(t, w) \in [0, T] \times \Omega$.

Proof. See [54] for further details. □

The following example is mainly used in computation of Greeks.

Example 2.2. Suppose that the process $u_t = 1$. Since any constant process is adapted to W_t for all t,

$$\delta(1) = \int_0^T 1d\, W_t = W_T - W_0 = W_T. \tag{2.20}$$

In this example the integral is in the Itô sense.

Let the space $\mathbb{D}^{1,2}(L^2([0, T]))$, denoted by $\mathbb{L}^{1,2}$, coincides with the set of processes $u \in L^2([0, T] \times \Omega)$ such that $u(t) \in \mathbb{D}^{1,2}$ for almost all $t \in [0, T]$.

Proposition 2.16. *Consider a process u which is an element of $\mathbb{L}^{1,2}$ and for almost all t the process $\{D_t u_s, 0 \leq s \leq T\}$ is Skorohod integrable. In addition, assume there is a version of the process $\left\{\int_0^T D_t u_s dW_s, 0 \leq t \leq T\right\}$ which is in $L^2(T \times \Omega)$. Then, $\delta(u)$ is in $\mathbb{D}^{1,2}$ and,*

$$D_t(\delta(u)) = u_t + \int_0^T D_t u_s\, dW_s. \tag{2.21}$$

Processes u, which are Skorohod integrable, can be constructed and they do not belong to the space $\mathbb{L}^{1,2}$. In this setting one can always find a measurable version of the process $D_s u_t$ verifies

$$\mathbb{E}\left[\int_0^T \int_0^T (D_s u_t)^2 \mu(ds)\mu(dt)\right] < \infty, \tag{2.22}$$

and the $\mathbb{L}^{1,2}$ is a Hilbert space with a norm as follows,

$$\|u\|^2_{1,2,L^2([0,T])} = \|u\|^2_{L^2([0,T]\times\Omega)} + \|Du\|^2_{L^2\left([0,T]^2\times\Omega\right)}.$$

Note that the space $\mathbb{L}^{1,2}$ is isomorphic to the space $L^2\left([0,T];\mathbb{D}^{1,2}\right)$ [54].

One can construct processes a process u that is Skorohod integrable and do not in the space $\mathbb{L}^{1,2}$. The next lemma provides a major information about how to construct the processes of this type.

Lemma 2.17. *Let $A \in \mathcal{B}_0 = \{A \in \mathcal{B} : \mu(A) < \infty\}$ and let the random variable F be square integrable and it is measurable with respect to the σ-algebra $\mathcal{F}_{A^c}$. In this case, the process $F\mathbb{1}_A$ is Skorohod integrable and the following equation is hold*

$$\delta(F\mathbb{1}_A) = FW(A). \tag{2.23}$$

Proposition 2.11 can be written as in the following notation by the inner product in Hilbert space H.

Proposition 2.18. *Suppose that there exists a Malliavin differentiable random variable F and a Skorohod integrable process u_t. Then,*

$$\delta(Fu) = F\delta(u) - \int_0^T (D_t F)\, u_t dt. \tag{2.24}$$

Proof. For the proof see [62] page:19 □

Example 2.3. Using the previous proposition we can compute the Skorohod integral of $\delta(W_T)$ can be computed by letting $u = 1$ and $F = W_T$. Then,

$$\begin{aligned}\delta(W_T) &= W_T\delta(1) - \int_0^T 1dt \\ &= W_T^2 - T.\end{aligned}$$

CHAPTER 3

OPTIONS AND THE GREEKS IN FINANCE

3.1 Introduction

Financial market instruments, which are used for investment, require a careful risk control in order to avoid undesirable results based on the unexpected large movements in their price and volatility. One of the most important components such kind of issues is the risk control problem of an option contract. The quantity that interpret risk is represented by the derivative of the option price with respect to its parameters. In finance theory, this concept is called the sensitivity analysis of option prices.

Estimating the price sensitivity of an underlying asset against to the change in the parameters is an important part of the validity of an investment decision. Particularly, in finance, certain derivatives of a contingent claim or portfolio value with respect to underlying model parameters. Since, many of the derivatives are denoted by Greek letters, they are called "Greeks". These derivatives are useful tools for investors to measure the stability of the quantities under variations of the parameters.

This chapter consists of two main sections. In the first section, the options used in financial markets are discussed. The main focus on the definition of them, does not rely on the pricing methods. There are several methods for pricing such kind of instruments in literature (for details see [14], [41], [44] and [56]). In the second section, certain Greeks, first order and higher order Greeks are discussed.

3.2 Options

Options are the right to buy or sell risky assets at a predetermined fixed price within a specified period. Indeed they are financial instruments which are allowed to bet on rising or falling values of a contingent claim [32]. Options are contracts between two parties interms of buying and selling the contingent claim at a certain time in the future. One of the parties is the writer who fixes the indispensable terms of an option contract and then sells the option to another party. The other party is the holder of the option, who purchases the option from the writer [65].

In recent years, the options have become very popular among the investors and are traded actively on many exchange throughout the world. They are attractive to investors because options are relatively cheaper to buy and also offer higher net returns than the contingent claims on which they are written. They can be used by investors to hedge against to the risks associated with sharp movements of underlying asset price. Investors can use them to generate higher returns under the assumptions of future market behavior and their prediction [10].

In accordance with the right, there are two types of option in financial markets which are called “call option” and “put option” [32]. A call option gives the holder the right to buy an underlying asset, which the option write on, by a predetermined date for a predetermined price, and the put option contrary to call option, gives the holder the right to sell an underlying asset by a predetermined date for a predetermined price. The price is known and called strike or exercise price and the date in the contract is called maturity or expiration date. For the sake of both party to make it clear on details, the writers have to specify the following instructions in an option contract:

- Underlying asset price,
- Exercise price,
- Time to maturity,
- Payoff function.

3.2.1 Vanilla Options

The value of a vanilla option at any particular time depends only on the current price of a fixed underlying asset, a predetermined fixed strike price, time to maturity and a dividend rate on the underlying asset and addition to these a risk-free rate, the volatility of the underlying asset. Furthermore, there are no special conditions on any of these parameters [40].

European Option

A European option written on an underlying asset is a financial security that gives its holder the right (but it is important to emphasize that not the obligation) to buy or sell the underlying asset at some given date and for a predetermined price. If the holder makes this transaction, it is referred to exercising the option. Since there is no obligation, in some conditions the holder does not exercise the option. If the holder does not exercise the option, it is abandoned [52].

Consider a European call option with exercise price K and maturity T. Suppose that the underlying asset price at maturity denoted with S_T. Since the price S_T is unknown at time $t = 0$, this price S_T gives an uncertainty to the model. The exercise price K is known and there are two possibilities for S_T which are, $S_T > K$ or $S_T < K$.

If $S_T > K$, the payoff of the European call option is equal to $(S_T - K)$. Otherwise, exercising the option is not profitable. In this case there will be no transaction thus the payoff is zero. In mathematical representation the payoff is,

$$payoff = \max (S_T - K, 0) = (S_T - K)^+ .$$

In Figure 3.1 the results for the payoff of a European call option is represented with parameters $K = 50$, $T = 2$. S_T changes from 30 to 70, the interest rate $r = 0.012$ and the volatility is $\sigma = 0.4$.

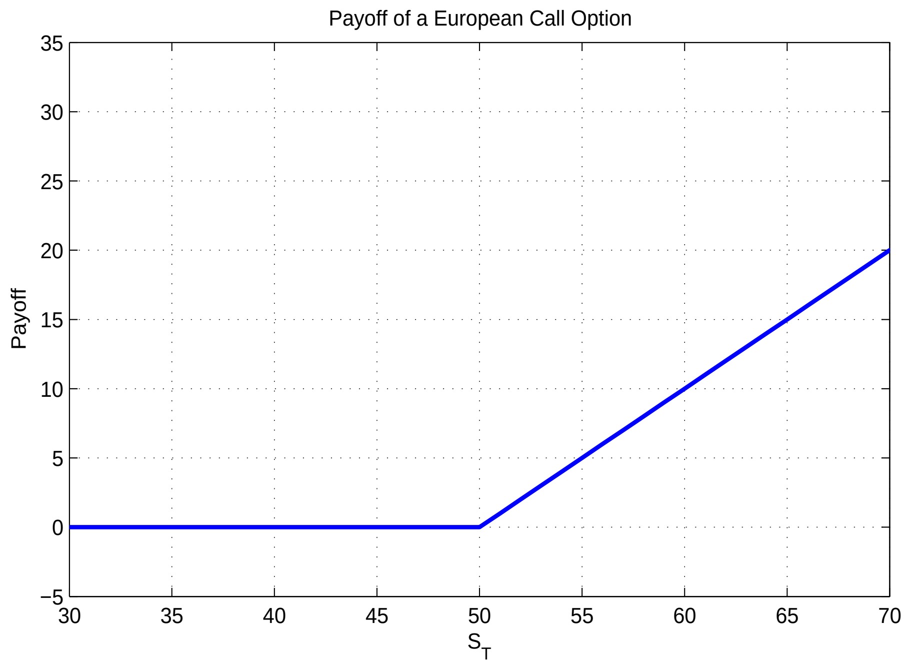

Figure 3.1: Payoff of a European Call Option

The payoff of this option will be zero in the interval $(30, 50)$ and it is grater than zero in the interval $(50, 70)$.

American Option

An American option is characterized by the opportunity of early exercise at any time during the life span of the contract. In contrast to European options which can only be exercised at expiration, an American option can be exercised at any time of its horizon. All other things are similar to the European options. In order to describe an American option it is crucial to specify the premium that has to be paid to the owner in case of early exercise.

It is important to emphasis that American vanilla options have not closed form solutions. Due to early exerciser opportunity, the price of an American call option is always greater than the corresponding European type of call option as long as it pays dividends, as it is seen in the Figure 3.2 [52]. As a matter of fact, if the underlying asset pays no dividend an American option price will be equal to the corresponding European call option price. Thus, in this case exercising an American option is not preferable.

Consider an American put option with exercise price K and maturity T. Let's denote the underlying asset price at time $0 \leq t \leq T$ with S_t. Since, at time t, the price S_t is not known, hence this price gives an uncertainty to the model. The exercise price K is known so there are two possibilities for S_t which are, $S_t > K$ or $S_t < K$. In mathematical representation,

$$payoff = \max\left(K - S_t, 0\right) = \left(K - S_t\right)^+ .$$

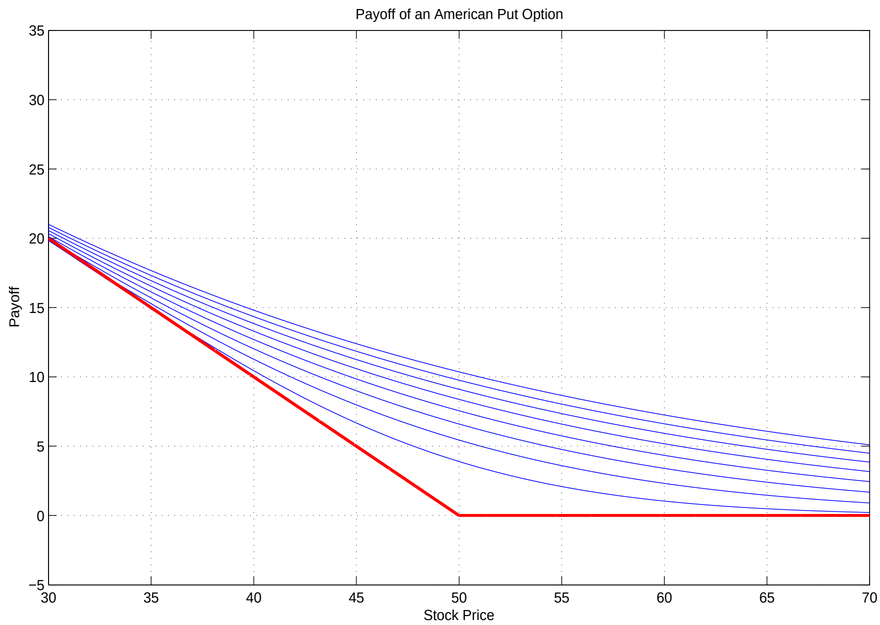

Figure 3.2: Payoff of an American Put Option

In Figure 3.2 the results for a payoff of a American put option is represented parameters with $K = 50$, $T = 2$. S_T changes from 30 to 70, the interest rate $r = 0.013$ and the volatility is $\sigma = 0.4$.

3.2.2 Russian Option

Russian option is a special type of American option which has lookback payoff and undetermined maturity. It guarantees that the holder of the option receives the historical maximum value of the underlying asset price on the path of exercising the option. The holder can exercise it at any time. Suppose that the historical realized maximum of the asset price denoted with M and the asset price is S_t, and these prices are taken at the same time. The option value is independent of time because it is a perpetual option. Let $V = (M, S_t)$ be the option price and S_t^* be the optimal exercise price at which the Russian option should be exercised. At a sufficiently low asset price, the Russian option becomes more attractive to the holder to exercise and receive the amount M instead of hold and wait. Therefore, the holder will keep it when $S_t^* < S_t \leq M$ and exercise when $S_t \leq S_t^*$. The payoff function of the Russian option upon exercising is [43]

$$V\left(S_t^*, M\right) = M \ .$$

As in any American option, a Russian option value is higher than its exercise payoff in the case of the option is alive.

3.2.3 Exotic Options

In recent years, a variety of complex options developed by financial engineers which are jointly known as exotic options, which can be contrasted with vanilla options. A vanilla option does not consider the past values of the underlying asset. Therefore, the price of it depends only on the current price of the underlying asset. On the other hand, many exotic options consider the historical prices of the underlying asset and the price of the option today, depend on the previous or future price path followed by the underlying asset [40].

Asian Options: Asian options are the options where the payoff depends on the average of the underlying asset during the life of the option. In an Asian option, the average price of the underlying asset is used as the terminal price of the underlying asset in determining the payoff. The average can be computed in several ways and each of them is a different type of Asian option. These types of options are suited to hedge risk at foreign exchange markets. Kolb and Overdahl [40] stressed that, because of the averaging effect, they are extremely useful against price manipulations and cheaper than vanilla options.

For simple constructions, the average can be computed arithmetically over a finite set of times. Such an example, the payoff function might be $\phi\left(S_{T/4}, S_{T/2}, S_{3T/4}, S_T\right)$ [62]. In this thesis the case of arithmetic Asian options is focused on. In particular we focus on the most important and most popular case of a continuous arithmetic average price call is,

$$\left(\frac{1}{T}\int_0^T S_t dt - K\right)^+ .$$

where K is the strike price [71].

Lookback Options: Lookback options are the options whose payoff depends on the maximum or minimum of the underlying asset price reached during the life of the option. The purpose of this option is to give the opportunity to the investor for gaining the maximum payment. These kinds of options become very popular on derivative markets, particularly in the currency options.

Mainly there are two different form of lookback options. The first form is that the settlement price of the option chosen with perfect hindsight of the stock's price path during the life of the option and the fixed strike price. The second form is the one that the strike price is chosen with perfect hindsight and the settlement price is the price of the option at maturity [14]. By these forms one can say there are four different lookback options:

1. Fixed strike call lookback or Max lookback: This type of lookback option pays the difference between the strike price and the highest stock price during the life of the option.

2. Fixed strike put lookback or Min lookback: This type of lookback option pays the difference between the strike and the lowest stock price during the life of the option.

3. Floating strike call lookback: This type of lookback option pays the difference between the stock price at expiration date and the lowest stock price during the life of the option.

4. Floating strike put lookback: This type of lookback option pays the difference between the stock price at expiration date and the highest stock price during the life of the option.

Consider a lookback option which is decided to exercised at time t with time to maturity T. The stock price at time t is denoted by S_t. Then, the payoffs on the lookback call is,

$$payoff = \max\left\{0, S_T - min\left\{S_t, S_{t+1}, \ldots, S_T\right\}\right\}.$$

As Kolb and Overdahl emphasized [40], a lookback call option allows to the buyer to receive the underlying asset at its minimum price over the life of the option.

Ladder Options: Ladder options are modeled for the investors, who want to get baring the upside of a stock price while at the same time locking in the performance of the stock if it ever goes above certain levels. These type of options are particularly popular among individual investors. They are typically structured as a capital guaranteed note with unlimited upside participation and the added advantage that a certain performance is guaranteed once the stock goes above a certain level [14].

A European type of ladder call option has the following payoff at maturity:

$$C_T = \max\left(0, S_T - K, \max\left(0, L_k - K\right)\right),$$

where L_k is the specified k^{th} rung in the ladder of strike prices, K is the strike price and S_T is the stock price at maturity [12].

Bermudan Option: Bermudan option is a combination of European option and American option. These type of options can be exercised not only at the expiration date but also exercised on certain specified dates which are occurred between the purchase date and the expiration date.

Mathematically, the Bermudan option is a pair (U, R) where $R \subseteq [0, T]$ is the region of permitted exercise dates and $U = (U_t)$ $(0 \leq t \leq T)$ is a non-negative adapted right continuous process with left limit (RCLL) called the payoff process. The holder of a Bermudan option can choose a stopping time τ with values in R; then obtain the payoff U_τ at time τ from the writer of the option [61].

Barrier Option: In recent years, options with payoff which depend on the complete path taken by the underlying price to reach its exercise value are becoming increasingly popular among the investors. The most popular of these path dependent options are barrier options. There are two types of barrier options [11]:

- **Knock-out option:** This type of option cancels immediately when the underlying price hits or crosses a predetermined level of price.
- **Knock-in option:** If the underlying price does not hit or cross the barrier, the option does not come into existence and therefore it becomes worthless.

Since barrier option provides the investor with additional protection or leverage, it is very popular among the individual investors. As Weert ([14]) emphasized; from the risk management point of view the associated risks are discontinuous around the barrier. Thus, the Greeks become less predictable and very often change sign around the barrier. But, it is possible to capture all these risks in the price and also able to manage the barrier risk properly because the risks associated with the barrier are typically of such a nature.

Digital Option: Digital option, also known binary options, is an option which gives a fixed payout if it is below or above a certain point and does not give a payout at all in all other cases. This option have an easy payoff comparing to the other exotic options but still it is assumed to be in this class by some researchers [14].

These options are an ideal trading instrument for beginners to test their skills. At a core level, binary option trading starts with anticipating on right direction is one of the most important skills to trading at any market.

Just like the put or call options, digital has an underlying asset, that is stock indexes, foreign currency, and futures, strike price and an expiration date. Digital options cannot be exercised before the expiration date as European options. But, they have some differences such as [53],

- They have fixed risk and fixed payoff. Thus, investors know maximum risk and payoff before trading. This is the main advantage of digital options.

- These options have no put or call options. They have only price conditions on the underlying asset

- They are fully collateralized, thus investor can never lose more than they put into a trade.

- With these options investor can take long and short position with minimal collateral. Thus, they can customize their trading strategies without having to put up large margin requirements.

There are some drawback of trading in digital options. The obvious one is the limited gain. Another disadvantage is, the market is not as big as European options market. Therefore, sometimes it is difficult to find the strike prices, expiration dates or contract.

In Figure 3.3, a digital option payoff of 0 or 1 is graphed. For further reading see [43], [70].

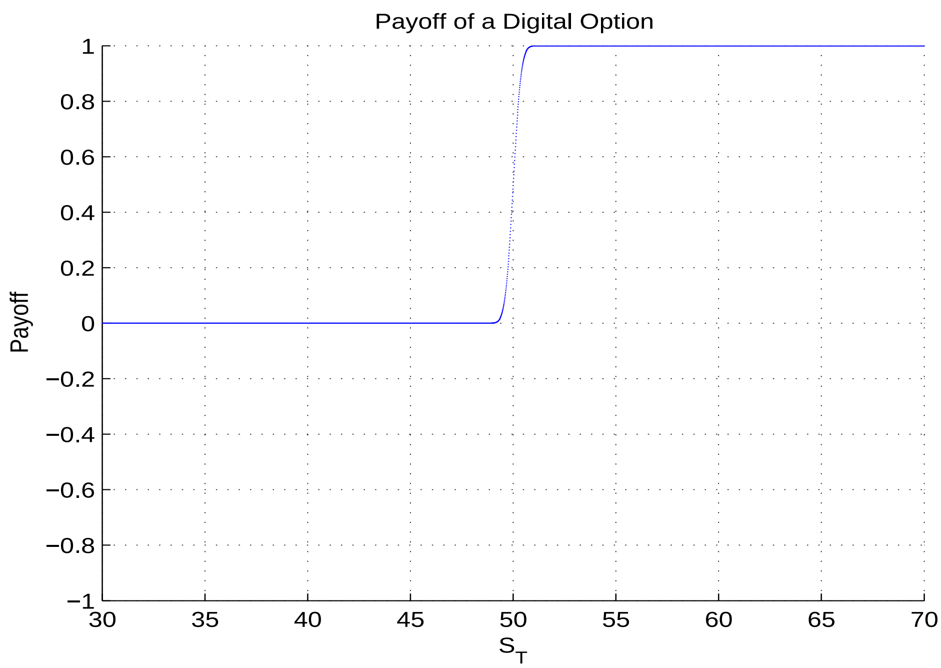

Figure 3.3: Payoff of a Digital Option

3.3 The Greeks

Greek letters delta, gamma, theta, rho and vega (actually vega is not a Greek letter) are used commonly by option traders. Each of them measures a unique risk associated with the model parameters. A Greek is essentially the derivative of a financial quantity's discounted payoff function with respect to any parameters associated with the problem [10]. Suppose V denotes the value of a portfolio based on an asset S_t, whose volatility is represented by σ and the current spot interest rate is r. If $V = V(S_t, K, t, \sigma, r)$, the most commonly used Greeks in finance are given in Table 3.1. One of the most frequently used Greek in hedging strategy is "Delta" because it measures the sensitivity of the option value within changes in the price of the underlying asset [1]. These derivatives are important because they are relevant to hedging strategy. Moreover, they also give an idea, about how rapidly the value of our portfolio is effected when there is a change in one of the parameters.

Suppose an option with payoff H which is square integrable ($E_Q[H^2] < \infty$). The price of this option at time $t = 0$ will be

$$V_0 = E_Q[e^{-\int_0^T r_s \,\mathrm{d}s} H]. \tag{3.1}$$

Since the value of the option at time $t = 0$ is equal to Equation (3.1), the derivative of this expectation should be taken, with respect to a parameter λ which is one of the parameter of H (S_0, K, σ, t or r). Assuming that the payoff function H can be written as a function of λ, it can be represented as $H = f(\lambda)$. Then, the derivative of H with respect to λ as follows,

$$\frac{\partial V_0}{\partial \lambda} = e^{-rT} E_Q \left[f'(F_\lambda) \frac{\partial F_\lambda}{\partial \lambda} \right].$$

As the parameter λ changes different Greeks are obtained. The name of the Greeks are given in Table 3.1. These are the most common used Greeks but there also some other Greeks that used rarely, called second and higher order Greeks. In the following subsections fundamental definitions of the first order and second order Greeks are provided.

Table 3.1: Common Used Greeks in Finance

Name	Symbol	Derivative
Delta	Δ	$\frac{\partial V}{\partial S_0}$
Gamma	Γ	$\frac{\partial^2 V}{\partial S_0^2}$
Rho	ρ	$\frac{\partial V}{\partial r}$
Theta	Θ	$\frac{\partial V}{\partial t}$
Vega	ϑ	$\frac{\partial V}{\partial \sigma}$

3.3.1 First Order Greeks

Delta: Delta measures the speed of option price changes with in the change of the underlying asset price [14]. This explanation is not unique, there are other explanations to option's Delta such as: mathematicians perceive an options Delta as the first order partial derivative of the option price with respect to underlying asset price, and economists perceive it as the sensitivity of the option price to a change in the price of its underlying asset:

$$\Delta = \frac{\partial V}{\partial S_0} \, .$$

Call options have positive Delta in the interval $[0, 1]$. This is clearly mean that if the stock price goes up, assuming other parameters fixed, the price of the call option will increase.

The investors purchase the option in case of making profit by an increase in the underlying asset price S_t. However, there is also a risk that the owner of the option can face if the price of underlying asset goes down. Using the positive correlation between the call option and the underlying asset S_t, it is possible to hedge against the risk by shorting the underlying asset [26].

Vega: Even if there is no change in futures or asset price risk or in time risk, an option price can be affected by changes in implied volatility. This change is formally referred to as Kappa or Vega. An option's Vega measures the speed of option price changes with in the change of its underlying asset volatility. In other words, mathematically, it is the first order derivative of option price with respect to volatility of its underlying asset:

$$\nu = \frac{\partial V}{\partial \sigma}.$$

Since the underlying asset volatility is essential in option trading strategies, the option's Vega is very crucial. The options can not exist without volatility, on the other hand they cannot negotiate smoothly in the market if there is too much volatility. If enough volatility does not exist in the market, the price of the underlying assets can remain relatively stable, therefore there is no need for options to be written on these assets.

The Vega is completely different from the other Greeks of the option because it is a partial derivative with respect to a parameter, rather than a variable. When we come to find numerical solutions, this makes it harder. In real world, volatility of an underlying asset is not known with certainty and it is very difficult to measure at any time and harder to predict it's future value [69].

If Vega is high in absolute term, the options value is very sensitive against to a small change in volatility. On contrary, if it is low in absolute term, a change in volatility have little impact on the value of the option.

Theta: Theta of an option measures the sensitivity of an option price with respect to change in time to maturity which is called the time decay of an option. It is known that the value of a European type option at exercising date depends on the relative price level of the underlying asset and the exercise price of the option. At the expiration date, the value of an option is called the intrinsic value. At exercise date, time to maturity will be zero or another word there will be no time value of money. Hence, the intrinsic value is only one part of an option's value. Otherwise, an option with positive time to maturity will have a value which is changing with time, and this part of value is called the time value of an option. Since, there is always a possibility of the underlying assets price to change when there is still remaining time to the expiration, options have time values. Mathematically this Greek is

$$\Theta = \frac{\partial V}{\partial t} \, .$$

The Theta is always positive because of higher possibility for the prices of the underlying assets to change as long as there is still time before expiration.

This risk is obvious since, all else being equal, an option contract with fewer days remaining worth less than an equivalent one with more days to expiration. The option will add value for the extra days. Thus, option values tend to decline as expiration date approaches, and decline more rapidly the closer the expiration date is [2].

Rho: Rho of an option measures option value sensitivity against to changes in interest rates. The interest rate level reflects the opportunity cost of holding options. The Rho of a call option written on an ordinary stock should be positive because higher interest rate cause to decline in the present value of the strike price, which in turn increases the value of the call option [46].

$$\rho = \frac{\partial V}{\partial r} \ .$$

Commonly in practice, time dependent interest rate $r(t)$ is used. Hence, the Rho is the sensitivity against the level of the rates under the assumption of a parallel shift in rates at all times [69].

It also worth to not that, since interest rates rarely move so dramatically over six month period, a change in the risk free rate is not affect the option prices as much as other risks [2].

3.3.2 Second and Higher Order Greeks

The Greeks discussed so far are not the only sensitivities in finance. Investors can imagine many other sensitivities for their investment analysis. In this section, the second and higher order Greeks are presented.

Gamma: Delta of an option is not constant, but changes as the underlying asset price changes, and makes the option in, at or out of the money. The change in an option's Delta as the underlying asset price moves up or down is measured with Gamma. Since it measures the option's Delta sensitivity it is a second order sensitivity. This Greek is the second partial derivative of the asset with respect to underlying asset price:

$$\Gamma = \frac{\partial^2 V}{\partial S_0^2} \ .$$

Since the Gamma is the sensitivity of the Delta against a change in the underlying asset price, it is a measure of how much and how often a position has to be rehedged in order to continue a Delta-neutral position [69]. Gamma of an option can be negative as well as positive. Such as, a long call and long put both have a positive Gamma. On the other hand, a short call and short put have a negative Gamma. If Gamma is small, the change in the Delta will be slow. However, if Gamma is large in absolute term, the sensitivity of the Delta with respect to underlying asset is high.

The Gamma also plays an important role when there is a gap between the market's view of volatility and the actual volatility of the underlying asset. In the case of high cost the investor wants to reduce the disclosure of the model error and try to minimize the need of rebalance of the portfolio. Since the Gamma is a measure of sensitivity of the hedge ratio Delta to the change in the underlying asset, the hedging requirement can be decreased by a Gamma-neutral strategy.

Speed: Speed measures how fast the Gamma of an option changes against in the change of underlying asset price. Hence, it is sometimes called as "Gamma of

Gamma" In the economic environment, it is the sensitivity of the Gamma with respect to underlying asset price:

$$speed = \frac{\partial^3 V}{\partial S_0^3}\ .$$

The Gamma is used to estimate how much investors will have to rehedge if the stock price moves. If the stock price changes by 1 unit, the Delta changes in accordance to the Gamma. But this change is only an approximation. The Delta may change more or less than the change, especially if the stock price moves by a larger amount, or the option price is close to the strike at expiration [69].

Charm: Charm or Delta decay is the second derivative of the value of an option with respect to initial price of the underlying asset and time. It measures the instantaneous rate of change at which the Delta of a derivative asset changes with its time to maturity. Hence, it is the derivative of Theta with respect to underlying asset price:

$$Charm = \frac{\partial \Delta}{\partial t} = \frac{\partial^2 V}{\partial S_0 \partial t}\ .$$

Charm can be one of the important Greeks when an investor want to make a Delta hedging position over a week.

Color: Color is called Gamma decay and it measures the speed of change of Gamma of an option over time to maturity. Color is used by the investors who use Gamma hedging strategy. It helps the investors to maintain their Gamma hedging positions and to see their hedging effectiveness:

$$color = \frac{\partial \Gamma}{\partial t} = \frac{\partial^3 V}{\partial S_0^2 \partial t}\ .$$

This Greek provides information on the Gamma of an option as time passed. As the time approaches to expiration, Color becomes more volatile.

Vomma (Volga): Vomma is the second partial derivative of the option value with respect to volatility and measures the change in the Vega as volatility change. It is used to determine how closely an option will track the market:

$$Vomma = \frac{\partial \nu}{\partial \sigma} = \frac{\partial^2 V}{\partial \sigma^2}\ .$$

Vanna: Vega of an option is not change only within the change of volatility, it also changes with the underlying asset price. Vanna measures the changes in Vega of an option as the underlying asset price changes. This Greek is the second derivative of the option value with respect to volatility and initial price of the underlying asset:

$$Vanna = \frac{\partial^2 V}{\partial \sigma \partial S_0} = \frac{\partial \nu}{\partial S_0} \ .$$

Veta: The change in Vega of an option with the change in time is measured with Veta. This Greek is the second derivative of the option value with respect to volatility and time:

$$Veta = \frac{\partial^2 V}{\partial \sigma \partial t} = \frac{\partial \nu}{\partial t} \ .$$

Vera: The change in the Vega with respect to interest rate is measured with Vera and this Greek is the first derivative of Vega of an option with respect to r:

$$Vera = \frac{\partial V}{\partial \sigma \partial r} = \frac{\partial \nu}{\partial r} \ .$$

CHAPTER 4

COMPUTATION OF THE GREEKS IN BLACK-SCHOLES-MERTON MODEL USING MALLIAVIN CALCULUS

4.1 Introduction

The option pricing model first presented to literature by Black and Scholes (1973) and further extended by Merton (1973) is a landmark in financial application and theory. Despite the further development of the theory of option pricing, for a European option pricing the original Black-Scholes formula remains the most successful and widely used application [31]. This model is particularly useful as it relates to the distribution of returns for cash transverse of option prices and successful in explaining option prices.

Modeling the underlying asset price with a geometric Brownian motion provides a useful approximation to stock prices accepted by practitioners for short and medium time to maturity. The Black-Scholes-Merton approach is still popular among practitioners to approximate option prices and its basic idea to derive option prices can be applied to more general option price models [24].

Although the Black-Scholes-Merton's restrictive assumptions and the improvements to the model available today, it remains an important reference to option pricing and the cornerstone of the financial modeling [49].

In this chapter first, the Black-Sholes-Merton model assumptions and its basics are introduced. Then, the Greeks are computed using Malliavin calculus (Integration by parts formula) in Black-Scholes-Merton environment. Moreover, the results are compared with the numerical Greeks and the estimation results of finite difference and pathwise derivative method.

Black-Scholes-Merton Assumptions

1. *Markovian property:* The dynamics of the underlying asset is characterized by a random component whose increments are independent and identically distributed. Indeed, this mean that the increase in the relative returns are not effected by the previous return values.

2. *Frictionless Market:* There are no transaction costs, no cost of adjustment, no stamp tax, or exchange controls. This assumption implies that the investor can buy and sell in large quantities to adjust the Delta. The existence of the transaction costs would necessarily change the argument for a hedging policy of an isolated operator, but it would not effect the fair value of the underlying asset.

3. *Constant Volatility:* According to the model, the daily variations are drawn from the same distribution and that the variance is known. It leads to a constant correlation between different assets.

4. *Geometric Brownian Motion:* To derive an exact option pricing model, it is assumed that stock prices follow a diffusion process or a geometric random walk. It implies that the dynamics of underlying asset is geometric that the expected variance of the logarithms of the returns remains constant [55].

5. *Constant Drift:* In trader's side, the structure of the forwards slope is constant.

6. *The underlying asset price follows the log-normal random walk.* This assumption implies that a smaller probability of significant deviations from the mean than is generally the case in practice. This is reflected in how fat or thin the "tails" of the bell-shaped probability curve are and affects the pricing of deep in-the-money and deep out-of-the-money options [68].

7. *The risk-free interest rate r and the underlying asset volatility σ assumed to be constant over the life of the option.* However, in practice, the volatility of the underlying asset and interest rates are not constant throughout the life of the option.

8. *The underlying asset pays no dividends during the life of the option.*

9. *The short selling is allowed and the assets are divisible.*

10. *There are no riskless arbitrage opportunities in the market.* Arbitrage means; making money out of nothing with no risk. There might be short time periods which includes arbitrage opportunities in financial markets. But, these opportunities tend to disappear quickly. It is because market participants observes the mismatch asset price, then the demand for the cheaper asset increase and the supply of the expensive asset decrease. This process drives the price to the no-arbitrage level [13].

4.2 Description of the Black-Scholes-Merton Model

The Chicago Board of Options Exchange started the trading of options in exchanges, in early 1973. In the same year, Fischer Black and Myron Scholes (1973) and Robert C. Merton (1973) derived the most widely used pricing model for underlying asset. Therefore, it is known as the Black-Scholes-Merton or Black-Scholes formula in finance.

The Behavior of the Prices

Consider a continuous time economy with a trading range $[0, T]$, where T is the time to maturity and $T > 0$. Assume that trading can be take place continuously in $[0, T]$.

Let $(\Omega, \mathcal{F}, P)$ be a probability space, and $(W_t)_{0 \le t \le T}$ be a a Brownian motion defined on this space. Here, P is an objective probability measure. The information in the economy is represented by a filtration $(\mathcal{F}_t)_{0 \le t \le T} = \sigma(W_s, 0 \le s \le t)$ which is an icreasing family of σ-algebras such that $\mathcal{F}_T = \mathcal{F}$. The filtration is assumed that right continuous and $\mathcal{F}_0$ contains all the null sets [49].

Lamberton [45] emphasis that the model proposed by Black and Scholes to describe the price behavior is a continuous time model with a risky asset and a riskless asset. Assume that the behavior of riskless asset S_t^0 is given by,

$$dS_t^0 = rS_t^0 dt, \tag{4.1}$$

where r represent the risk-free rate and a non-negative constant number. Suppose that $S_0^0 = 1$. Hence, $S_t^0 = e^{rt}$ for all $t \geq 0$. A model for the stock price, the basis of the classic Black-Scholes-Merton approach called geometric Brownian motion. In this model, suppose that the underlying asset, is a stock and it pays no dividends, is modeled with a stochastic process S_t, depending on time is a solution of the following stochastic differential Equation 4.2,

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t, \tag{4.2}$$

with an initial underlying asset price $S_0 \in \mathbb{R}$. Here, it is assumed that the drift term μ is the expected return of the underlying asset, σ is the volatility of the returns on this asset and W is a standard Brownian motion. In the investment decision both parameters μ and σ are important factors and they are dependent on each other. As the the expected return μ of the underlying asset increases, the volatility σ is increases too. The model is satisfied on the interval $[0, T]$ where T stands for the time to maturity of the option.

Proposition 4.1. *The stochastic differential Equation (4.2) has a closed form solution,*

$$S_t = S_0 \exp\left\{\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t\right\}, \tag{4.3}$$

where S_0 is the spot price observed at time $t = 0$ and the process S_t satisfies the properties; continuity, independent increments and stationary increments [45].

Proof. Applying Itô lemma for the function $f(x) = \log(x)$ to Equation (4.2). The first partial derivative of this function is $f'(x) = \frac{1}{x}$ and the second partial derivative of this function is $f''(x) = -\frac{1}{x^2}$. Substituting the function and the derivatives in to the Itô lemma the following is obtained,

$$\log(S_t) = \log(S_0) + \int_0^t \frac{1}{S_t} dS_t - \frac{1}{2}\int_0^t \frac{2}{S_t^2} d\langle S, S\rangle_t . \tag{4.4}$$

Substituting Equation (4.2) and $d\langle S, S\rangle_t = S_t^2\sigma^2 dt$ into Equation (4.4),

$$\begin{aligned} \log\left(\frac{S_t}{S_0}\right) &= \int_0^t \frac{1}{S_s}\left[\mu\, S_s ds + \sigma\, S_s\, dW_s\right] ds - \frac{1}{2}\int_0^t \frac{1}{S_s^2}\, S_s^2\, \sigma^2\, ds \\ &= \int_0^t \sigma dW_s + \int_0^t \left(\mu - \frac{1}{2}\sigma^2\right) ds \\ &= \left\{\sigma W_t + \left(\mu - \frac{1}{2}\sigma^2\right) t\right\}. \end{aligned} \tag{4.5}$$

is obtained. By rearranging Equation (4.5) the stock price is obtained as follows,

$$S_t = S_0 \exp\left\{\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t\right\}. \tag{4.6}$$

□

4.3 Pricing European Type Options under Black-Scholes-Merton Model

In the Black-Scholes-Merton model it is assumed that the market is arbitrage free. Hence, the pricing of an option has to be done under a risk neutral probability measure. Before giving a risk neutral probability measure, let us define the equivalent probability measure.

Definition 4.1. Let $(\Omega, \mathcal{A}, P)$ be a probability space. Consider a probability measure on $\Omega, \mathcal{A}$ which is continuous with respect to the measure P if

$$\forall A \in \mathcal{A} \qquad , \qquad P(A) = 0 \Rightarrow Q(A) = 0.$$

The probability measures P and Q are equivalent if each of them are continuous with respect to each other.

The following theorem is useful to show the measure Q is continuous or not with respect to the measure P.

Theorem 4.2. *A probability measure Q is continuous with respect to the measure P if and only if there exist a random variable $Z \in (\Omega, \mathcal{A})$ which is non-negative and satisfy,*

$$\forall A \in \mathcal{A}, \qquad Q(A) = \int_A Z(\omega)\, dP(\omega),$$

where the random variable Z is denoted by dQ/dP.

The Brownian motion under the risk neutral probability Q can be found with the Girsanov theorem. This theorem is as follows,

Theorem 4.3. *Consider a filtered probability space such as $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, P\right)$. $(B_t)_{0\leq t\leq T}$ is a $\mathcal{F}_t$-standard Brownian motion and $(\theta_t)_{0\leq t\leq T}$ is a square integrable adapted process. The process,*

$$L_t = \exp\left(-\int_0^t \theta_s dB_s - \frac{1}{2}\int_0^t \theta_s^2 ds\right),$$

is a martingale. Then, the process $W_t = B_t + \int_0^t \theta_s\, ds$ is a Brownian motion under the measure Q.

Using the new Brownian motion under the risk neutral probability Q obtained from Theorem 4.3, the stochastic differential equation of the stock price can be rearrange. By doing this, a new stochastic differential equation is obtained as follows with initial price S_0,

$$dS_t = rS_t dt + \sigma S_t dW_t.$$

In this case, the solution of this equation is under risk neutral probability measure and it corresponds to,

$$S_t = S_0 \exp\left\{\left(r - \frac{\sigma^2}{2}\right)t + \sigma W_t\right\}. \tag{4.7}$$

At time t the stock price at maturity (S_T) is not known. Therefore, the option price is not known as well. Then, the price of the option V_t is nothing but the discounted expected value of the option's payoff function. By this feature, the arbitrage free price of the option at any time $t \in [0, T]$ is defined as follows with the following theorem.

Theorem 4.4. *In the famous Black-Scholes-Merton model, any option defined by a non-negative, $\mathcal{F}_T$ measurable random variable H, where H is square integrable ($E_Q\left[H^2\right] < \infty$) under the probability measure Q, which is equivalent to probability measure P, price of the option at time t is given by,*

$$V_t = \mathbb{E}_Q\left[e^{-r(T-t)}H|\mathcal{F}_t\right]. \tag{4.8}$$

4.4 Computation of the Greeks under Black-Scholes-Merton Model

In the Theorem 4.4, suppose that $t = 0$. Then, the price of this option at time zero ($t = 0$) becomes [54],

$$V_0 = \mathbb{E}_Q \left[e^{-rT} H \right]. \tag{4.9}$$

Here, we want to compute the derivative of the expectation at (4.9), with respect to a parameter λ which is one of the parameter of the option payoff formula H that is S_0, r, σ or T. Assume that the payoff function H is written as a given differentiable function of λ, which is represented by $H = f(F_\lambda)$. Then the derivative of H with respect to λ will be as below,

$$\frac{\partial V_0}{\partial \lambda} = e^{-rT} \mathbb{E}_Q \left[f'(F_\lambda) \frac{\partial F_\lambda}{\partial \lambda} \right]. \tag{4.10}$$

In particular for a European call option the function f becomes $f(S_T) = (S_T - 0)^+$, where $F = S_T$ and λ is one of the parameters of the stock price which are, S_0, r, σ and T.

The methods, finite difference, pathwise derivative estimate and likelihood that used in computation of Greeks are discussed in Chapter 1. In this chapter, the Greeks are computed with using the Malliavin calculus. The necessary properties used in computations are given in Chapter 2.

Proposition 4.5. *Suppose that F, G are two random variables and $F \in \mathbb{D}^{1,2}$. Consider an H valued random variable u and $D^u F = \langle DF, u \rangle \neq 0$ a.s. and also $Gu (D^u F)^{-1} \in Dom(\delta)$. Then, one can said that any continuously differentiable function f with bounded derivative have the following;*

$$\mathbb{E}_Q [f'(F) G]) = \mathbb{E}_Q [f(F) H(F, G)], \tag{4.11}$$

where

$$H(F, G) = \delta \left(Gu (D^u F)^{-1} \right).$$

Proof. By Remark 2.3 and Proposition 2.7,

$$\begin{aligned} D^u f(F) &= \langle Df(F), u \rangle_H \\ &= \langle f'(F) DF, u \rangle_H \\ &= f'(F) \langle DF, u \rangle_H. \end{aligned}$$

Since it is assumed that $\langle DF, u\rangle \neq 0$, one can write;

$$f'(F) = \langle Df(F), u\rangle_H \left(\langle DF, u\rangle_H\right)^{-1}. \tag{4.12}$$

Now multiplying both side of Equation (4.12) with G,

$$f'(F)\,G = \langle Df(F), u\rangle_H\, G \left(\langle DF, u\rangle_H\right)^{-1} \tag{4.13}$$

is reached. Taking the expectation of both sides under the risk neutral probability measure,

$$\begin{aligned}
\mathbb{E}_Q\left[f'(F)\,G\right] &= \mathbb{E}_Q\left[\langle Df(F), u\rangle_H\, G\left(\langle DF, u\rangle_H\right)^{-1}\right] \\
&= \mathbb{E}_Q\left[\left\langle Df(F), Gu\left(\langle DF, u\rangle_H\right)^{-1}\right\rangle_H\right] \\
&= \mathbb{E}_Q\left[\left\langle Df(F), Gu\left(D^u F\right)^{-1}\right\rangle_H\right] \\
&= \mathbb{E}_Q\left[f(F)\,\delta\left(Gu\left(D^u F\right)^{-1}\right)\right].
\end{aligned}$$

□

Proposition 4.5 given above is called integration by parts formula. As it is seen in this proposition the aim of integration by parts formula is to convert the derivative of f' into its antiderivative f. Moreover, one can extend this proposition to Lipschitz functions.

Corollary 4.6. *If $F, G \in \mathbb{D}^{1,2}$, f is smooth function, $D_v F$ is differentiable with respect to v for all $v \in [0,T]$ and $\int_0^T \left(\frac{\partial}{\partial v} D_v^W F\right) dv \neq 0$. Then,*

$$\mathbb{E}_Q\left[f'(F)\,G\right] = \mathbb{E}_Q\left[f(F)\,\delta\left(\frac{2G\frac{\partial}{\partial s}D_s F}{\left(D_T F\right)^2 - \left(D_0 F\right)^2}\right)\right]. \tag{4.14}$$

Proof. See the proof in [62] $page: 22$. □

Lemma 4.7. *Let S_t be given as in Equation (4.7). Then,*

$$D_t S_\tau = \sigma S_\tau D_t W_\tau = \sigma S_\tau \mathbb{1}_{t<\tau}. \tag{4.15}$$

Moreover for $\tau = T$,

$$D^u S_T = \int_0^T D_t S_T dt = \sigma T S_T. \tag{4.16}$$

Proof.

$$\begin{aligned}
D_t S_\tau &= D_t \left(S_0 \exp\left\{ \left(\mu - \frac{\sigma^2}{2}\right)\tau + \sigma W_\tau \right\} \right) \\
&= S_0 \exp\left\{ \left(r - \frac{\sigma^2}{2}\right)\tau \right\} D_t \left(\exp\left\{\sigma W_\tau\right\}\right) \\
&= S_0 \exp\left\{ \left(r - \frac{\sigma^2}{2}\right)\tau \right\} \exp\left\{\sigma W_\tau\right\} D_t \int_0^\tau \sigma dW_u \\
&= \sigma S_\tau \mathbb{1}_{t<\tau}. \qquad (4.17)
\end{aligned}$$

To prove Equation (4.16) use Equation (4.15) with choosing $\tau = T$. Then,

$$D_t S_T = \sigma S_T \mathbb{1}_{t<T}. \qquad (4.18)$$

Integrating both side of Equation (4.18) following is obtain,

$$\begin{aligned}
\int_0^T D_t S_T dt &= \int_0^T \sigma S_T \mathbb{1}_{t<T} dt \\
&= \sigma S_T T. \qquad (4.19)
\end{aligned}$$

□

Using Equation (4.19),

$$S_T = \frac{1}{\sigma T} \int_0^T D_t S_T dt, \qquad (4.20)$$

is obtained. Note that, Equality (4.20) above will be used in the computations of the Greeks.

4.4.1 Computation of the Greeks of European Type Options

Computation of Delta:

Proposition 4.8. *Consider a European type option with payoff function f and its underlying asset is following a geometric Brownian motion $(W_t)_{t\in[0,T]}$ with constant risk-free rate r, time to maturity T, volatility σ and initial price S_0. Let the payoff function f of the option is continuously differentiable and given as $f : \mathbb{R} \longrightarrow \mathbb{R}$. Then the Delta is given by,*

$$\Delta = \frac{e^{-rT}}{\sigma S_0 T} E_Q\left[f\left(S_T\right) W_T\right]. \qquad (4.21)$$

Proof. In Equation (4.7) if $t = T$ the price of underlying asset at maturity S_T is obtained as follows,

$$S_T = S_0 \exp\left\{\left(r - \frac{\sigma^2}{2}\right)T + \sigma W_T\right\}.$$

Then the partial derivative of S_T with respect to S_0 is,

$$\begin{aligned}
\frac{\partial S_T}{\partial S_0} &= \exp\left\{\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma W_T\right\} \\
&= \frac{1}{S_0} S_0 \exp\left\{\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma W_T\right\} \\
&= \frac{1}{S_0} S_T. \qquad (4.22)
\end{aligned}$$

The Delta of the option is,

$$\begin{aligned}
\Delta = \frac{\partial V_0}{\partial S_0} &= E_Q\left[e^{-rT} f'(S_T) \frac{\partial S_T}{\partial S_0}\right] \\
&= E_Q\left[e^{-rT} f(S_T) \frac{1}{S_0} S_T\right] \\
&= \frac{e^{-rT}}{S_0} E\left[f'(S_T) S_T\right]. \qquad (4.23)
\end{aligned}$$

Apply Proposition 4.5 to Equation (4.23) for $u = 1$, $F = S_T$ and $G = S_T$, then

$$\begin{aligned}
\Delta &= \frac{e^{-rT}}{S_0} \mathbb{E}_Q\left[f'(S_T) \frac{1}{\sigma T} \int_0^T D_t S_T dt\right] \\
&= \frac{e^{-rT}}{\sigma S_0 T} \mathbb{E}_Q\left[\int_0^T D_t\left(f(S_T)\right) dt\right] \\
&= \frac{e^{-rT}}{\sigma S_0 T} \mathbb{E}_Q\left[f(S_T) \delta(1)\right] \\
&= \frac{e^{-rT}}{\sigma S_0 T} \mathbb{E}_Q\left[f(S_T) W_T\right],
\end{aligned}$$

is reached. Here,

$$
\begin{aligned}
H(F,G) &= \delta\left(S_T\left(\int_0^T D_t S_T dt\right)^{-1}\right) \\
&= \delta\left(\frac{1}{\sigma T}\right) \\
&= \frac{1}{\sigma T}\delta(1).
\end{aligned}
$$

Using the feature,

$$
\delta(1) = \int_0^T dW_t = W_T,
$$

the following result is obtained,

$$
H(F,G) = \frac{W_T}{\sigma T}.
$$

□

Computation of Gamma:

Proposition 4.9. *Consider a European type option with payoff function f and its underlying asset is following a geometric Brownian motion $(W_t)_{t\in[0,T]}$ with constant risk-free rate r, time to maturity T, volatility σ and initial price S_0. Let the payoff function f of the option is continuously differentiable and given as $f:\mathbb{R}\longrightarrow\mathbb{R}$. Then the Gamma of this option is,*

$$
\Gamma = \frac{e^{-rT}}{\sigma S_0^2 T} E_Q\left[f(S_T)\left(\frac{W_T^2}{\sigma T} - \frac{1}{\sigma} - W_T\right)\right]. \tag{4.24}
$$

Proof. The Gamma is the second derivative of the option's value with respect to the underlying asset price. Hence, the calculations are slightly different from the calculations for delta. Indeed, it requires to perform integration by parts twice. Since f is continuously differentiable, the second partial derivative of the payoff function with respect to initial price S_0 is,

$$
\begin{aligned}
\Gamma &= \frac{\partial^2}{\partial S_0^2} V_0 = \frac{\partial}{\partial S_0} \Delta \\
&= \frac{\partial}{\partial S_0} \left(\frac{e^{-rT}}{\sigma S_0 T} \mathbb{E}_Q \left[f\left(S_T\right) W_T \right] \right) \\
&= -\frac{e^{-rT}}{\sigma S_0^2 T} \mathbb{E}_Q \left[f\left(S_T\right) W_T \right] + \frac{e^{-rT}}{\sigma S_0^2 T} \mathbb{E}_Q \left[f'\left(S_T\right) S_T W_T \right].
\end{aligned}
$$

Then from Equation (4.20) the following is obtained,

$$
\begin{aligned}
\Gamma &= -\frac{e^{-rT}}{\sigma S_0^2 T} \mathbb{E}_Q \left[f\left(S_T\right) W_T \right] + \frac{e^{-rT}}{\sigma S_0^2 T} \mathbb{E}_Q \left[f'\left(S_T\right) \frac{1}{\sigma T} \int_0^T D_t S_T dt W_T \right] \\
&= -\frac{e^{-rT}}{\sigma S_0^2 T} \mathbb{E}_Q \left[f\left(S_T\right) W_T \right] + \frac{e^{-rT}}{\sigma S_0^2 T} \mathbb{E}_Q \left[\int_0^T D_t \left(f\left(S_T\right) \right) \frac{1}{\sigma T} W_T dt \right] \\
&= -\frac{e^{-rT}}{\sigma S_0^2 T} \mathbb{E}_Q \left[f\left(S_T\right) W_T \right] + \frac{e^{-rT}}{\sigma S_0^2 T} \mathbb{E}_Q \left[f\left(S_T\right) \frac{1}{\sigma T} \delta\left(W_T\right) \right]. \qquad (4.25)
\end{aligned}
$$

First compute the Skorohod integral of W_T. Using Proposition 2.18 it is computed as,

$$
\begin{aligned}
\delta\left(W_T\right) &= \int_0^T W_T dW_t - \int_0^T D_t W_T dt \\
&= W_T \int_0^T dW_t - \int_0^T dt \\
&= W_T^2 - T. \qquad (4.26)
\end{aligned}
$$

Substituting Equation (4.26) into Equation (4.25) and using the linearity of the expectation,

$$
\Gamma = \frac{e^{-rT}}{\sigma S_0^2 T} E_Q \left[f\left(S_T\right) \left(\frac{W_T^2}{\sigma T} - \frac{1}{\sigma} - W_T \right) \right]
$$

is reached. □

Computation of Vega:

Proposition 4.10. *Consider a European type option with payoff function f and its underlying asset is following a geometric Brownian motion $(W_t)_{t\in[0,T]}$ with constant risk-free rate r, time to maturity T, volatility σ and initial price S_0.*

Let the payoff function f of the option is continuously differentiable and given as $f : \mathbb{R} \longrightarrow \mathbb{R}$. Then the Vega of this option's value is,

$$\vartheta = e^{-rT}\mathbb{E}_Q\left[f\left(S_T\right)\left(\frac{W_T^2}{\sigma T} - \frac{1}{\sigma} - W_T\right)\right]. \tag{4.27}$$

Proof. From Equation (4.20) S_T is,

$$S_T = S_0 exp\left\{\left(r - \frac{\sigma^2}{2}\right)T + \sigma W_T\right\}$$

Then the partial derivative of S_T with respect to volatility σ is

$$\begin{aligned}\frac{\partial S_T}{\partial \sigma} &= S_0\left(W_T - \sigma T\right)\exp\left\{\left(r - \frac{\sigma^2}{2}\right)T + \sigma W_T\right\}\\ &= S_T\left(W_T - \sigma T\right).\end{aligned} \tag{4.28}$$

Vega of an option is the first order partial derivative of the option with respect to volatility σ. Using the definition of Vega and Equation 4.28 Vega is obtained as;

$$\begin{aligned}\vartheta &= \frac{\partial}{\partial \sigma}\mathbb{E}_Q\left[e^{-rT}f\left(S_T\right)\right]\\ &= \mathbb{E}_Q\left[e^{-rT}f'\left(S_T\right)\frac{\partial S_T}{\partial \sigma}\right]\\ &= e^{-rT}\mathbb{E}_Q\left[f' S_T S_T\left(W_T - \sigma T\right)\right].\end{aligned} \tag{4.29}$$

Substituting Equation (4.20) into Equation 4.29 the Vega equation becomes,

$$\begin{aligned}\vartheta &= e^{-rt}\mathbb{E}_Q\left[f'\left(S_T\right)\frac{1}{\sigma T}\int_0^T D_t S_T dt\left(W_T - \sigma T\right)\right]\\ &= \frac{e^{-rT}}{\sigma T}\mathbb{E}_Q\left[\int_0^T f'\left(S_T\right)D_t S_T\left(W_T - \sigma T\right)dt\right]\\ &= \frac{e^{-rT}}{\sigma T}\mathbb{E}_Q\left[\int_0^T D_t\left(f\left(S_T\right)\right)\left(W_T - \sigma T\right)dt\right]\\ &= \frac{e^{-rT}}{\sigma T}\mathbb{E}_Q\left[f\left(S_T\right)\delta\left(W_T - \sigma T\right)\right].\end{aligned} \tag{4.30}$$

By Equation (2.13) in Definition 2.3, Equation (4.30) is obtained for $\forall F \in \mathbb{D}^{1,2}$. The remaining part is to calculate the Skorohod integral of $(W_T - \sigma T)$. Using the linearity property of Skorohod integral one can obtain;

$$\delta\left(W_T - \sigma T\right) = \delta\left(W_T\right) - \sigma T \delta\left(1\right), \tag{4.31}$$

where

$$\delta\left(W_T\right) = W_T \int_0^T dW_t - \int_0^T dt = W_T^2 - T, \tag{4.32}$$

and

$$\delta\left(1\right) = \int_0^T dW_t = W_T. \tag{4.33}$$

Substituting Equation (4.31) into Equation (4.30) the following is obtained,

$$\vartheta = \frac{e^{-rT}}{\sigma T} \mathbb{E}_Q \left[f\left(S_T\right) \left(W_T^2 - T - \sigma T W T\right) \right] \tag{4.34}$$

Rearranging Equation (4.34) the result;

$$\vartheta = e^{-rT} \mathbb{E}_Q \left[\phi\left(S_T\right) \left(\frac{W_T^2}{\sigma T} - \frac{1}{\sigma} - W_T \right) \right],$$

is obtained. Moreover, one can write the Vega of an option as a function of Γ, initial stock price S_0, volatility σ and maturity T as,

$$\vartheta = \Gamma \sigma S_0^2 T.$$

□

Computation of Rho:

Proposition 4.11. *Consider a European type option with payoff function f and its underlying asset is following a geometric Brownian motion $(W_t)_{t\in[0,T]}$ with a constant risk-free rate r, time to maturity T, volatility σ and initial price S_0. Let the payoff function f of the option is continuously differentiable and given as $f : \mathbb{R} \longrightarrow \mathbb{R}$. Then the Rho of this option is,*

$$\rho = T e^{-rT} \mathbb{E}_Q \left[f\left(S_T\right) \left(\frac{W_T}{\sigma T} - 1 \right) \right]. \tag{4.35}$$

Proof. The value of option is given as,

$$V_0 = e^{-rT}\mathbb{E}_Q\left[f\left(S_T\right)\right], \tag{4.36}$$

where the stock price S_T is given as,

$$S_T = S_0 exp\left\{\left(r - \frac{\sigma^2}{2}\right)T + \sigma W_T\right\} \tag{4.37}$$

The first order partial derivative of S_T with respect to r is,

$$\begin{aligned} \frac{\partial S_T}{\partial r} &= TS_0 \exp\left\{\left(r - \frac{\sigma^2}{2}\right)T + \sigma W_T\right\} \\ &= TS_T. \end{aligned} \tag{4.38}$$

Rho is the partial derivative of the value of the option at the given time with respect to interest rate r. Thus,

$$\begin{aligned} \rho &= \frac{\partial V_0}{\partial r} \\ &= -Te^{-rT}\mathbb{E}_Q\left[f(S_T\right] + e^{-rT}\mathbb{E}_Q\left[\frac{\partial}{\partial r}f\left(S_T\right)\right] \\ &= -Te^{-rT}\mathbb{E}_Q\left[f(S_T\right] + e^{-rT}\mathbb{E}_Q\left[f'\left(S_T\right)\frac{\partial}{\partial r}S_T\right]. \end{aligned} \tag{4.39}$$

Substituting Equation (4.38) in to Equation (4.39)

$$\begin{aligned} \rho &= -TV_0 + e^{-rT}\mathbb{E}_Q\left[f'\left(S_T\right)\left(TS_T\right)\right] \\ &= -TV_0 + Te^{-rT}\mathbb{E}_Q\left[f'\left(S_T\right)S_T\right] \\ &= -Te^{-rT}\mathbb{E}_Q\left[f(S_T\right] + \frac{T}{\sigma T}e^{-rT}\mathbb{E}_Q\left[f\left(S_T\right)W_T\right] \\ &= Te^{-rT}\mathbb{E}_Q\left[f\left(S_T\right)\left(\frac{W_T}{\sigma T} - 1\right)\right] \end{aligned} \tag{4.40}$$

is obtained. Moreover, by $e^{-rT}\mathbb{E}_Q\left[f'\left(S_T\right)S_T\right]$ which is computed in Delta computation,

$$\Delta = \frac{e^{-rT}}{S_0}\mathbb{E}_Q\left[f'\left(S_T\right)S_T\right] \tag{4.41}$$

is obtained. From Equation (4.41),

$$e^{-rT}E_Q\left[f'\left(S_T\right)S_T\right]=\Delta S_0 \tag{4.42}$$

can be written. Substituting Equation (4.42) into Equation (4.40)

$$\rho=-TV_0+TS_0\Delta,$$

is reached. □

4.4.2 Computation of the Greeks of Asian Options

In this subsection, the Greeks of Asian options with fixed strike price are computed. The payoff function of these options is given with a general formula as follows,

$$Payoff=f\left(\frac{1}{T}\int_0^T S_t dt\right), \tag{4.43}$$

where the function f is a deterministic function and the underlying asset $\{S_t, 0\leq t\leq T\}$ is given with Equation (4.7). For example, the European call option payoff function with a strike price K is denoted by,

$$Payoff=f\left(\frac{1}{T}\int_0^T S_t dt\right)=\max\left(\frac{1}{T}\int_0^T S_t dt-K,0\right).$$

Using the same feature in previous subsection 4.4 by Equation (4.9) the Asian options value at time $t=0$ can be defined as follows,

$$V_0=E_Q\left[e^{-rT}f\left(\frac{1}{T}\int_0^T S_t dt\right)\right]. \tag{4.44}$$

The Greeks of these type options can be computed, as Vanilla option's Greeks, from Equation (4.44).

Computation of Delta:

Proposition 4.12. *Consider an Asian type option with payoff function f and its underlying asset is following a geometric Brownian motion $(W_t)_{t\in[0,T]}$ with a constant risk-free rate r, time to maturity T, volatility σ and initial price S_0. Let the payoff function f of the option is continuously differentiable and given as $f:\mathbb{R}\longrightarrow\mathbb{R}$. Then the Delta of Asian option is given by,*

$$\Delta = \frac{2e^{-rT}}{\sigma^2 S_0}\mathbb{E}_Q\left[f\left(\frac{1}{T}\int_0^T S_t dt\right)\left(\frac{S_T - S_0}{\int_0^T S_t dt} - \left(r - \frac{\sigma^2}{2}\right)\right)\right]. \tag{4.45}$$

Proof. The Delta is the first partial derivative of the option price with respect to initial price S_0 at time $t = 0$. Therefore, the partial derivative, of an Asian option value, Equation 4.44 with respect to S_0 is

$$\begin{aligned}
\Delta &= \frac{\partial}{\partial S_0}\mathbb{E}_Q\left[e^{-rT} f\left(\frac{1}{T}\int_0^T S_t dt\right)\right] \\
&= e^{-rT}\mathbb{E}_Q\left[f'\left(\frac{1}{T}\int_0^T S_t dt\right)\frac{1}{T}\int_0^T \frac{\partial S_t}{\partial S_0} dt\right].
\end{aligned} \tag{4.46}$$

In the right hand side of the previous Equation (4.46) the derivative of S_t is occurred. Therefore, let us compute the partial derivative of S_t with respect to S_0 independently. Doing this, we have

$$\begin{aligned}
\frac{\partial S_t}{\partial S_0} &= \frac{\partial}{\partial S_0} S_0 \exp\left(\left(r - \frac{\sigma^2}{2}\right)t + \sigma W_t\right) \\
&= \frac{1}{S_0} S_0 \exp\left(\left(r - \frac{\sigma^2}{2}\right)t + \sigma W_t\right) \\
&= \frac{1}{S_0} S_t.
\end{aligned} \tag{4.47}$$

Substituting Equation (4.47) into Equation 4.45 the following is obtained,

$$\begin{aligned}
\Delta &= e^{-rT}\mathbb{E}_Q\left[f'\left(\frac{1}{T}\int_0^T S_t dt\right)\frac{1}{T}\int_0^T \frac{S_t}{S_0} dt\right] \\
&= \frac{e^{-rT}}{S_0}\mathbb{E}_Q\left[f'\left(\frac{1}{T}\int_0^T S_t dt\right)\frac{1}{T}\int_0^T S_t dt\right].
\end{aligned} \tag{4.48}$$

Applying Proposition 4.5 with $F = \frac{1}{T}\int_0^T S_u du$, $G = \frac{1}{T}\int_0^T S_u du$ and $u = S_t$ the following is obtained,

$$
\begin{aligned}
\Delta &= \frac{e^{-rt}}{S_0}\mathbb{E}_Q\left[f\left(\frac{1}{T}\int_0^T S_t dt\right)\delta\left(\frac{\frac{1}{T}\int_0^T S_\tau d\tau S_t}{\int_0^T D^u\left(\frac{1}{T}\int_0^T S_\tau d\tau\right)}\right)\right] \\
&= \frac{e^{-rT}}{S_0}\mathbb{E}_Q\left[f\left(\frac{1}{T}\int_0^T S_t dt\right)\delta\left(\frac{\frac{1}{T}\int_0^T S_\tau d\tau S_t}{\int_0^T D_t\left(\frac{1}{T}\int_0^T S_\tau d\tau\right) S_t dt}\right)\right], \quad (4.49)
\end{aligned}
$$

where,

$$
D_t\left(\frac{1}{T}\int_0^T S_\tau d\tau\right) = \frac{\sigma}{T}\int_t^T S_\tau d\tau. \quad (4.50)
$$

Then using Equation (4.50) and fundamental theorem of calculus it becomes

$$
\begin{aligned}
\int_0^T D_t\left(\frac{1}{T}\int_0^T S_\tau d\tau\right) S_t dt &= \frac{\sigma}{T}\int_0^T\left(\int_t^T S_\tau d\tau\right) S_t dt \\
&= \frac{\sigma}{T}\int_0^T S_\tau\left(\int_0^\tau S_t dt\right) d\tau \\
&= \frac{\sigma}{T}\int_0^T \frac{1}{2} d\left(\int_0^\tau S_t dt\right)^2 \\
&= \frac{\sigma}{2T}\left(\int_0^T S_t dt\right)^2. \quad (4.51)
\end{aligned}
$$

Substituting Equation 4.51 into Equation (4.49) the following is obtained,

$$
\begin{aligned}
\Delta &= \frac{e^{-rT}}{S_0}\mathbb{E}_Q\left[f\left(\frac{1}{T}\int_0^T S_t dt\right)\delta\left(\frac{\int_0^T S_\tau d\tau S_t}{\frac{\sigma}{2}\left(\int_0^T S_\tau d\tau\right)^2}\right)\right] \\
&= \frac{e^{-rT}}{S_0}\mathbb{E}_Q\left[f\left(\frac{1}{T}\int_0^T S_t dt\right)\frac{2}{\sigma}\delta\left(\frac{S_t}{\int_0^T S_\tau d\tau}\right)\right]. \quad (4.52)
\end{aligned}
$$

In Equation (4.52) a Skorohod integral is show up. First compute the Skrohod integral on the right hand side. By using Propositions 2.18 and 2.7, the computation is obtained as follows;

$$
\begin{aligned}
\delta\left(\frac{1}{\int_0^T S_\tau d\tau} S_t\right) &= \frac{1}{\int_0^T S_\tau d\tau}\int_0^T S_t dW_t - \int_0^T D_t\left(\frac{1}{\int_0^T S_\tau d\tau}\right) S_t dt \\
&= \frac{\int_0^T S_t dW_t}{\int_0^T S_\tau d\tau} - \int_0^T \frac{-\sigma\int_t^T S_\tau d\tau}{\left(\int_0^T S_\tau d\tau\right)^2} S_t dt \\
&= \frac{\int_0^T S_t dW_t}{\int_0^T S_\tau d\tau} + \sigma\int_0^T \frac{\int_t^T S_\tau d\tau}{\left(\int_0^T S_\tau d\tau\right)^2} S_t dt. \qquad (4.53)
\end{aligned}
$$

In Equation (4.53) the second part of right hand side can be computed independently. The computation is as follows;

$$
\begin{aligned}
\int_0^T \frac{\int_t^T S_\tau d\tau}{\left(\int_0^T S_\tau d\tau\right)^2} S_t dt &= \frac{1}{\int_0^T S_\tau d\tau}\int_0^T\left(\int_t^T S_\tau d\tau\right) S_t dt \\
&= \left(\frac{1}{\left(\int_0^T S_\tau d\tau\right)^2}\right)\frac{1}{2}\left(\int_0^T S_t dt\right)^2 \\
&= \frac{1}{2}. \qquad (4.54)
\end{aligned}
$$

Then substituting Equation (4.54) into Equation (4.53) it becomes,

$$
\delta\left(\frac{1}{\int_0^T S_\tau d\tau} S_t\right) = \frac{\int_0^T S_t dW_t}{\int_0^T S_\tau d\tau} + \frac{\sigma}{2}. \qquad (4.55)
$$

Substituting Equation (4.55) into Equation (4.52) the following result is obtained for the Delta,

$$
\Delta = \frac{e^{-rT}}{S_0} E_Q\left[f\left(\frac{1}{T}\int_0^T S_t dt\right)\left(\frac{2\int_0^T S_t dW_t}{\sigma\int_0^T S_t dt} + 1\right)\right]. \qquad (4.56)
$$

On the other hand, by Itô lemma one can write the following,

$$
S_t - S_0 = \int_0^t r S_\tau d\tau + \int_0^t \sigma S_\tau dW_\tau. \qquad (4.57)
$$

If both sides of Equation (4.57) divided by $\sigma \int_0^T S_\tau d\tau$, the following is obtained,

$$\frac{S_t - S_0}{\sigma \int_0^T S_\tau d\tau} = \frac{\int_0^T S_\tau dW_\tau}{\int_0^T S_\tau d\tau} + \frac{r}{\sigma}. \tag{4.58}$$

By rearranging Equation (4.58),

$$\frac{\int_0^T S_\tau dW_\tau}{\int_0^T S_\tau d\tau} = \frac{S_T - S_0}{\sigma \int_0^T S_\tau d\tau} - \frac{r}{\sigma}. \tag{4.59}$$

is obtained. Now it is clearly seen that by substituting Equation (4.59) into Equation (4.56), it becomes as follows,

$$\begin{aligned}\Delta &= \frac{e^{-rT}}{S_0}\mathbb{E}_Q\left[f\left(\frac{1}{T}\int_0^T S_t dt\right)\left(\frac{2\left(S_T - S_0\right)}{\sigma^2 \int_0^T S_t dt} - \frac{2r}{\sigma^2} + 1\right)\right] \\ &= \frac{2e^{-rT}}{\sigma^2 S_0}\mathbb{E}_Q\left[f\left(\frac{1}{T}\int_0^T S_t dt\right)\left(\frac{S_T - S_0}{\int_0^T S_t dt} - \left(r - \frac{\sigma^2}{2}\right)\right)\right].\end{aligned}$$

□

Computation of Gamma:

Proposition 4.13. *Consider an Asian type option with payoff function f and its underlying asset is following a geometric Brownian motion $(W_t)_{t\in[0,T]}$ with a constant risk-free rate r, time to maturity T, volatility σ and initial price S_0. Let the payoff function f of the option is continuously differentiable and given as $f : \mathbb{R} \longrightarrow \mathbb{R}$. For the sake of simplicity let us define $\bar{S_T} = \frac{1}{T}\int_0^T S_t dt$. Then, the Gamma of Asian option is given by,*

$$\Gamma = \frac{4e^{-rT}}{\sigma^3 S_0^2}\mathbb{E}_Q\left[f\left(\bar{S_T}\right)\left(\frac{\left(S_T - S_0\right)^2 - \left(S_T - S_0\right) r\bar{S_T}}{\sigma \bar{S_T}^2} - \frac{\sigma S_0}{\bar{S_T}}\right)\right] - \frac{2r}{\sigma^2 S_0}\Delta. \tag{4.60}$$

Proof. Since the function f is continuously differentiable, the Gamma can be computed. Using the definition of Gamma the following, namely by differentiating the value of the option two times,

$$\begin{aligned}
\Gamma &= \frac{\partial^2}{\partial S_0^2}\mathbb{E}_Q\left[e^{-rT}f\left(\bar{S_T}\right)\right] \\
&= e^{-rT}\mathbb{E}_Q\left[\frac{\partial^2}{\partial S_0^2}f\left(\bar{S_T}\right)\right] \\
&= e^{-rT}\mathbb{E}_Q\left[f''\left(\bar{S_T}\right)\frac{\partial^2}{\partial S_0^2}\bar{S_T}\right] \\
&= \frac{e^{-rT}}{S_0^2}\mathbb{E}_Q\left[f''\left(\bar{S_T}\right)\bar{S_T}^2\right] \\
&= \frac{e^{-rT}}{S_0^2}\mathbb{E}_Q\left[f'\left(\bar{S_T}\right)\delta\left(u\right)\right], \qquad (4.61)
\end{aligned}$$

where u_s is specified by Proposition 4.5 with $F = \bar{S_T}$ and $G = \bar{S_T}^2$as,

$$u_s = \frac{2\bar{S_T}\frac{\partial}{\partial s}D_s\bar{S_T}}{\left(D_T\bar{S_T}\right)^2 - \left(D_0\bar{S_T}\right)^2}.$$

First compute $\frac{\partial}{\partial s}D_s\bar{S_T}$, $D_T\bar{S_T}$ and $D_0^W\bar{S_T}$.

$$\frac{\partial}{\partial s}D_s\bar{S_T} = \frac{\partial}{\partial s}\sigma\int_s^T S\tau d\tau = -\sigma\frac{\partial}{\partial s}\int_T^s S_\tau d\tau = -\sigma S_s. \qquad (4.62)$$

Using the fact that $D_s\bar{S_T} = \sigma\int_s^T S_t dt$,

$$D_T\bar{S_T} = \sigma\int_T^T S_t dt = 0, \qquad (4.63)$$

$$D_0\bar{S_T} = \sigma\int_0^T S_t dt = \sigma\bar{S_T}, \qquad (4.64)$$

are obtained. Then, substituting Equations (4.63) and (4.64) into Equation 4.61 the following is obtained,

$$\begin{aligned}
\Gamma &= \frac{e^{-rT}}{S_0^2}\mathbb{E}\left[f'\left(\bar{S_T}\right)\delta\left(\frac{-2\bar{S_T^2}\sigma S_.}{-\left(\sigma\bar{S_T}\right)^2}\right)\right] \\
&= \frac{2e^{-rT}}{\sigma S_0^2}\mathbb{E}\left[f'\left(\bar{S_T}\right)\delta\left(S_.\right)\right]. \qquad (4.65)
\end{aligned}$$

Note that the solution of S_T is obtained from the Itô lemma. Using this lemma one can compute $\delta\left(S_.\right)$ as follows,

$$\begin{aligned} dS_t &= \left(r - \frac{1}{2}\sigma^2\right) S_t dt + \sigma S_t dW_t + \frac{1}{2}\sigma^2 S_t dt \\ &= r S_t dt + \sigma S_t dW_t, \end{aligned} \tag{4.66}$$

writing Equation (4.66) in integration form,

$$\begin{aligned} \int_0^T dS_t = \int_0^T r S_t dt + \int_0^T \sigma S_t dW_t, \\ S_T - S_0 = r \int_0^T S_t dt + \sigma \int_0^T S_t dW_t \end{aligned} \tag{4.67}$$

is obtained. By rearranging Equation (4.67),

$$\int_0^T S_t dW_t = \frac{1}{\sigma}\left(S_T - S_0 - r\int_0^T S_t dt\right) \tag{4.68}$$

is reached. On the other hand, by the definition of Skorohod integral it can be said,

$$\delta\left(S_.\right) = \int_0^T S_t dW_t = \frac{1}{\sigma}\left(S_T - S_0 - r\int_0^T S_t dt\right). \tag{4.69}$$

Substituting Equation (4.69) into Equation (4.65) and using the linearity of expectation the following is obtained,

$$\Gamma = \frac{2e^{-rT}}{\sigma^2 S_0^2}\left(\mathbb{E}_Q\left[f'\left(\bar{S_T}\right)\left(S_T - S_0\right)\right] - \mathbb{E}_Q\left[f'\left(S_T\right)\left(r\bar{S_T}\right)\right]\right). \tag{4.70}$$

Using the $\Delta = \frac{e^{rT}}{S_0}\mathbb{E}_Q\left[f'\left(\bar{S_T}\right)\left(r\bar{S_T}\right)\right]$ of Asian option one can say,

$$\mathbb{E}_Q\left[f'\left(\bar{S_T}\right)\left(r\bar{S_T}\right)\right] = \frac{S_0 r}{e^{-rT}}\Delta. \tag{4.71}$$

If the integration by parts formula applied to the first term of the right hand side of Equation (4.70),

$$\begin{aligned}\mathbb{E}_Q\left[f'\left(\bar{S_T}\right)\left(S_T-S_0\right)\right] &= \mathbb{E}_Q\left[f\left(\bar{S_T}\right)\delta\left(\frac{2\left(S_T-S_0\right)\left(-\sigma S_.\right)}{-\left(\sigma\bar{S_T}\right)^2}\right)\right] \\ &= \frac{2}{\sigma}\mathbb{E}\left[\Phi\left(\bar{S_T}\right)\delta\left(\frac{\left(S_T-S_0\right)S_.}{\bar{S_T}^2}\right)\right] \qquad (4.72)\end{aligned}$$

is obtained. The Skorohod integral in this final expression is computed as follows;

$$\delta\left(\frac{\left(S_T-S_0\right)S_.}{\bar{S_T}^2}\right) = \frac{S_T-S_0}{\bar{S}_T^2}\delta\left(S_.\right) - \int_0^T S_t D_t\left(\frac{S_T-S_0}{\bar{S_T}^2}\right)dt. \qquad (4.73)$$

Now computing the Malliavin derivative in the integral,

$$D_t\left(\frac{\left(S_T-S_0\right)S_.}{\bar{S_T}^2}\right) = \frac{\bar{S_T}^2 D_t\left(S_T-S_0\right) - \left(S_T-S_0\right)\left(2\bar{S_T}\right)D_t\bar{S_T}}{\bar{S_T}^4}.$$

Note that The Malliavin operator is linear. Using this fact,

$$D_t^W\left(S_T-S_0\right) = D_tS_T - D_tS_0 = \sigma S_T\mathbf{1}_{t<T} - 0$$

and

$$\begin{aligned}D_t\left(\frac{S_T-S_0}{\bar{S_t}^2}\right) &= \frac{\bar{S_T}^2\left(\sigma S_T\mathbf{1}_{t<T}-0\right) - 2\left(S_T-S_0\right)\bar{S_T}\sigma\int_t^T S_\tau d\tau}{\bar{S_T}^2} \\ &= \sigma\frac{S_T}{\bar{S_T}^2}\mathbf{1}_{t<T} - 2\sigma\frac{\left(S_T-S_0\right)}{\bar{S_T}^3}\int_t^T S_\tau d\tau\end{aligned}$$

is obtained. Plugging this result into the integral in Equation (4.73),

$$\begin{aligned}\int_0^T S_tD_t\left(\frac{S_{T-S_0}}{\bar{S_T}^2}\right)st &= \sigma\int_0^T S_t\left(\frac{S_T}{\bar{S_T}^2}\mathbf{1}_{t<T}\right)dt - 2\sigma\int_0^T S_t\left(\frac{S_T-S_0}{\bar{S}_t^3}\int_t^T S_\tau d\tau\right)dt \\ &= \frac{\sigma S_T}{\bar{S_T}} - \frac{\sigma\left(S_T-S_0\right)}{\bar{S_T}} \\ &= \frac{\sigma S_0}{\bar{S_T}} \qquad (4.74)\end{aligned}$$

is obtained. Note that $\int_0^T S_t \left(\int_t^T S_\tau d\tau\right) dt$ is computed in the previous section while computing the Δ of Asian option. Now one can conclude that,

$$\mathbb{E}_Q\left[f'\left(\bar{S_T}\right)\left(S_T - S_0\right)\right] = \frac{2}{\sigma}\mathbb{E}_Q\left[f\left(\bar{S_T}\right)\left(\frac{(S_T - S_0)}{\sigma \bar{S_T}^2}\left(S_T - S_0 - r\bar{S_T}\right) - \frac{\sigma S_0}{\bar{S_T}}\right)\right].$$

By substituting Equations (4.71) and (4.4.2) into Equation (4.70),

$$\begin{aligned}\Gamma &= \frac{2e^{-rT}}{\sigma^2 S_0^2}\left(\frac{2}{\sigma}\mathbb{E}_Q\left[f\left(\bar{S_T}\right)\left(\frac{(S_T - S_0)}{\bar{S_T}^2}\left(S_T - S_0 - r\bar{S_T}\right) - \frac{\sigma S_0}{\bar{S_T}}\right)\right] - \frac{rS_0}{e^{-rT}}\Delta\right)\\ &= \frac{4e^{-rT}}{\sigma^3 S_0^2}\mathbb{E}_Q\left[f\left(\bar{S_T}\right)\left(\frac{(S_T - S_0)^2 - (S_T - S_0)\, r\bar{S_T}}{\sigma \bar{S}_T^2} - \frac{\sigma S_0}{\bar{S_T}}\right)\right] - \frac{2r}{\sigma^2 S_0}\Delta\end{aligned}$$

□

is obtained.

Computation of Vega:

Proposition 4.14. *Consider an Asian type option with payoff function f and its underlying asset is following a geometric Brownian motion $(W_t)_{t\in[0,T]}$ with a constant risk-free rate r, time to maturity T, volatility σ and initial price S_0. Let the payoff function f of the option is continuously differentiable and given as $f : \mathbb{R} \longrightarrow \mathbb{R}$. Then, the Vega of Asian option is given by,*

$$\begin{aligned}\vartheta = e^{-rT}\mathbb{E}_Q\Bigg[\quad & f\left(\frac{1}{T}\int_0^T S_t dt\right) \times \\ & \left(\frac{\int_0^T \int_0^T S_t W_t dt dW_t}{\sigma \int_0^T t S_t dt} + \frac{\int_0^T t^2 S_t dt \int_0^T S_t W_t dt}{\left(\int_0^T t S_t dt\right)^2} - W_T\right)\Bigg]\end{aligned} \tag{4.75}$$

Proof. Using the definition of Vega;

$$\begin{aligned}\vartheta &= \frac{\partial}{\partial\sigma}\mathbb{E}_Q\left[e^{-rT} f\left(\frac{1}{T}\int_0^T S_t dt\right)\right]\\ &= e^{-rT}\mathbb{E}_Q\left[f'\left(\frac{1}{T}\int_0^T S_t dt\right)\frac{1}{T}\int_0^T \frac{\partial}{\partial\sigma} S_t dt\right]\end{aligned} \tag{4.76}$$

can be found. Note that S_t is given above with Equation (4.3). Using that equation the partial derivative of the stock price with respect to the volatility σ at time t, can be computed as follows,

$$\begin{aligned}\frac{\partial}{\partial\sigma}S_t &= (-\sigma t+W_t)\,S_0\exp\left(\left(r-\frac{\sigma^2}{2}\right)t+\sigma W_t\right)\\ &= (W_t-\sigma t)\,S_t. \end{aligned}\tag{4.77}$$

Substituting Equation (4.77) into Equation (4.76),

$$\vartheta = e^{-rT}\mathbb{E}_Q\left[f'\left(\frac{1}{T}\int_0^T S_t dt\right)\left(\frac{1}{T}\int_0^T (W_t-\sigma t)\,S_t dt\right)\right]\tag{4.78}$$

is obtained. Applying Proposition 4.5 to Equation (4.78) with $F=\frac{1}{T}\int_0^T S_t dt$, $G=\frac{1}{T}\int_0^T (W_t-\sigma t)\,S_t dt$ and $u=1$, the following is obtained,

$$\vartheta = e^{-rT}\mathbb{E}_Q\left[f\left(\frac{1}{T}\int_0^T S_t dt\right)H\left(\frac{1}{T}\int_0^T S_t dt,\frac{1}{T}\int_0^T (W_t-\sigma t)\,S_t dt\right)\right],\tag{4.79}$$

where,

$$H\left(\frac{1}{T}\int_0^T S_t dt,\frac{1}{T}\int_0^T (W_t-\sigma t)\,S_t dt\right)=\delta\left(\frac{\frac{1}{T}\int_0^T (W_t-\sigma t)\,S_t dt}{\int_0^T D_t\left(\frac{1}{T}\int_0^T S_\tau d\tau\right)dt}\right).\tag{4.80}$$

Now compute the Malliavin derivative of $\frac{1}{T}\int_0^T S_\tau dt\tau$. It can be compute with the help of Lemma 4.7 (using Equation (4.15)) as follows,

$$D_t\left(\frac{1}{T}\int_0^T S_\tau dt\tau\right)=\frac{\sigma}{T}\int_t^T S_\tau d\tau.\tag{4.81}$$

Then, using Equation (4.81) the following can be written as;

$$\begin{aligned}\int_0^T D_t\left(\frac{1}{T}\int_0^T S_\tau dt\tau\right)dt &= \int_0^T\left(\frac{\sigma}{T}\int_t^T S_\tau d\tau\right)dt\\ &= \frac{\sigma}{T}\int_0^T S_\tau\left(\int_0^\tau dt\right)d\tau\\ &= \frac{\sigma}{T}\int_0^T \tau S_\tau d\tau.\end{aligned}\tag{4.82}$$

By substituting Equation 4.82 into Equation 4.80 and using the linearity of Skorohod integral,

$$H\left(\frac{1}{T}\int_0^T S_t dt, \frac{1}{T}\int_0^T (W_t - \sigma t) S_t dt\right) = \delta\left(\frac{\frac{1}{T}\left(\int_0^T S_t W_t dt - \sigma \int_0^T t S_t dt\right)}{\frac{\sigma}{T}\int_0^T t S_t dt}\right)$$
$$= \delta\left(\frac{\int_0^T S_t W_t dt}{\sigma \int_0^T t S_t dt} - 1\right) \tag{4.83}$$

is obtained. Now plugging the function H, which is computed as in Equation (4.83), into Equation (4.79)

$$\vartheta = e^{-rT}\mathbb{E}_Q\left[f\left(\frac{1}{T}\int_0^T S_t dt\right)\delta\left(\frac{\int_0^T S_t W_t dt}{\sigma\int_0^T t S_t dt} - 1\right)\right] \tag{4.84}$$

is obtained. By the linearity of Skorohod integral,

$$\delta\left(\frac{\int_0^T S_t W_t dt}{\sigma\int_0^T t S_t dt} - 1\right) = \delta\left(\frac{\int_0^T S_t W_t dt}{\sigma\int_0^T t S_t dt}\right) - \delta(1) \tag{4.85}$$

can be written. Then, applying Proposition 2.18 to Equation (4.85) with $F = \frac{1}{\sigma\int_0^T t S_t dt}$ and $u = \int_0^T S_t W_t dt$ and also using Equation (4.33) one can obtain the following,

$$\delta\left(\frac{\int_0^T S_t W_t dt}{\sigma\int_0^T t S_t dt}\right) - \delta(1) = \frac{\int_0^T\left(\int_0^T S_t W_t dt\right) dW_\tau}{\sigma\int_0^T t S_t dt}$$
$$- \int_0^T\left(D_t^W\left(\frac{1}{\sigma\int_0^T \tau S_\tau d\tau}\right)\int_0^T S_t W_t dt\right) dt$$
$$- W_T. \tag{4.86}$$

Then, by the help of Proposition 2.7 with choosing $\Phi(x) = \frac{1}{x}$ and $F = \sigma\int_0^T \tau S_\tau d\tau$,

$$D_t\left(\frac{1}{\sigma\int_0^T \tau S_\tau d\tau}\right) = -\frac{1}{\left(\sigma\int_0^T \tau S_\tau d\tau\right)^2} D_t\left(\sigma\int_0^T \tau S_\tau d\tau\right) \tag{4.87}$$

is reached. By substituting Equation (4.87) into previous Equation (4.86),

$$
\begin{aligned}
\delta\left(\frac{\int_0^T S_t W_t dt}{\sigma \int_0^T t S_t dt}\right) - \delta(1) &= \frac{\int_0^T \left(\int_0^T S_t W_t dt\right) dW_\tau}{\sigma \int_0^T t S_t dt} \\
&+ \frac{\int_0^T D_t^W \left(\sigma \int_0^T \tau S_\tau d\tau\right) \left(\int_0^T S_t W_t dt\right) dt}{\left(\sigma \int_0^T t S_t dt\right)^2} - W_T \\
&= \frac{\int_0^T \left(\int_0^T S_t W_t dt\right) dW_\tau}{\sigma \int_0^T t S_t dt} + \frac{\sigma^2 \int_0^T t^2 S_t dt \int_0^T S_t W_t dt}{\sigma^2 \left(\int_0^T t S_t dt\right)^2} \\
&- W_T \\
&= \frac{\int_0^T \left(\int_0^T S_t W_t dt\right) dW_\tau}{\sigma \int_0^T t S_t dt} + \frac{\int_0^T t^2 S_t dt \int_0^T S_t W_t dt}{\left(\int_0^T t S_t dt\right)^2} \\
&- W_T \qquad (4.88)
\end{aligned}
$$

is obtained. Here, it is better to remember the following equation,

$$
D_t \left(\sigma \int_0^T \tau S_\tau d\tau\right) = \sigma^2 \int_0^T \tau S_\tau \int_0^\tau dt d\tau = \sigma^2 \int_0^T \tau^2 S_\tau d\tau.
$$

Therefore, the Vega of an Asian option is obtained as following,

$$
\begin{aligned}
\vartheta = e^{-rT} \mathbb{E}_Q \Bigg[\quad & f\left(\frac{1}{T} \int_0^T S_t dt\right) \times \\
& \left(\frac{\int_0^T \int_0^T S_t W_t dt dW_t}{\sigma \int_0^T t S_t dt} + \frac{\int_0^T t^2 S_t dt \int_0^T S_t W_t dt}{\left(\int_0^T t S_t dt\right)^2} - W_T\right)\Bigg].
\end{aligned}
$$

□

4.5 Numerical Investigation and Efficiency

In Section 4.4, the Greeks for European type options are computed with Malliavin calculus method, in particular integration by parts formula. In this section, the Greeks obtained in the previous section are estimated with Monte Carlo Simulation. Moreover, the estimation results of Malliavin Greeks and the analytical value of the Greeks of a European call option are compared. Then, the estimation results are compared with the other techniques, finite difference and pathwise methods, we described in Chapter 1. Therefore, in this section, first the analytical values of the Greeks of a European call option are computed from close from solution. Then, without no computation details, for finite difference method and pathwise method Monte Carlo method applied to estimate the Greeks. And finally, the estimation results of the Greeks are compared on figures.

For the estimation of the Greeks [62] (a case study together with codes) is referred to reader.

4.5.1 Analytical Value of the Greeks

Consider a European call option which has a payoff function defined as

$$\phi(S_T) = \max(S_T - K, 0) = (S_T - K)^+ .$$

The analytic solution for the Value V_0 of Black-Sholes-Merton equation is given as as a function of maturity T, underlying asset price S_0, strike price K, risk free interest rate r, and volatility σ [62] as follows,

$$V_0 = S_0 \mathcal{N}(d_1) - e^{-rT} K \mathcal{N}(d_2), \tag{4.89}$$

where $\mathcal{N}(x) = \int_{-\infty}^{x} exp(-z^2/2)\sqrt{2\pi} dz$ stands for the cumulative density function of a normalized Gaussian distribution and ;

$$d_1 = \frac{\log(S_0/K) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}},$$
$$d_2 = \frac{\log(S_0/K) + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

Analytical Delta:

Proposition 4.15. *Suppose that the the price is given as in Equation (4.89). Then analytical delta of a European call option is,*

$$\Delta = \mathcal{N}(d_1). \tag{4.90}$$

Proof. Delta is the first derivative of discounted asset price with respect to S_0. Now differentiating the price function with respect to initial underlying asset price S_0 Delta found as follows;

$$
\begin{aligned}
\Delta &= \frac{\partial}{\partial S_0} V_0 \\
&= \frac{\partial}{\partial S_0}\left(S_0 \mathcal{N}\left(d_1\right) - e^{-rT} K \mathcal{N}\left(d_2\right)\right) \\
&= \frac{\partial}{\partial S_0}\left(S_0 \mathcal{N}\left(d_1\right) - e^{-rT} K \mathcal{N}\left(d_1 - \sigma\sqrt{T}\right)\right) \\
&= \mathcal{N}\left(d_1\right) + S_0 \frac{\partial}{\partial S_0} \mathcal{N}\left(d_1\right) - \frac{\partial}{\partial S_0}\left(e^{-rT} K \mathcal{N}\left(d_1 - \sigma\sqrt{T}\right)\right) \\
&= \mathcal{N}\left(d_1\right) + S_0 \mathcal{N}'\left(d_1\right) \frac{\partial}{\partial S_0} d_1 - e^{-rT} K \mathcal{N}'\left(d_1 - \sigma\sqrt{T}\right) \frac{\partial}{\partial S_0}\left(d_1 - \sigma\sqrt{T}\right).
\end{aligned}
$$

It is sure that $\mathcal{N}'$ is a density of a normalized Gaussian processes. Therefore,

$$
\begin{aligned}
\mathcal{N}'\left(d_1 - \sigma\sqrt{T}\right) &= \frac{1}{2\pi} exp\left(\frac{-\left(d_1^2 - 2 d_1 \sigma\sqrt{T} + \sigma^2 T\right)}{2}\right) \\
&= \frac{1}{2\pi} exp\left(\frac{-d_1^2}{2}\right) exp\left(d_1 \sigma\sqrt{T} - \frac{\sigma^2 T}{2}\right) \\
&= \mathcal{N}'\left(d_1\right) exp\left(\frac{\log\left(S_0/K\right) + \left(r + \frac{\sigma^2}{2}\right) T}{\sigma\sqrt{T}} \sigma\sqrt{T} - \frac{\sigma^2 T}{2}\right) \\
&= \frac{e^{rT} S}{K} \mathcal{N}'\left(d_1\right).
\end{aligned}
$$

Thus, the analytical delta becomes,

$$
\begin{aligned}
\Delta &= \mathcal{N}\left(d_1\right) + S \mathcal{N}'\left(d_1\right) - e^{-rT} K \left(\frac{e^{rT} S}{K} \mathcal{N}'\left(d_1\right)\right) \frac{\partial}{\partial S_0}\left(d_1 - \sigma\sqrt{T}\right) \\
&= \mathcal{N}\left(d_1\right) + S \mathcal{N}' \frac{\partial}{\partial S_0}\left(d_1 - d_1 + \sigma\sqrt{T}\right) \\
&= \mathcal{N}\left(d_1\right).
\end{aligned}
$$

$\square$

Analytical Gamma:

Proposition 4.16. *Suppose that the the price is given as in Equation (4.89). Then analytical gamma of a European call option is,*

$$
\Gamma = \frac{1}{S_0 \sigma\sqrt{T}} \mathcal{N}'\left(d_1\right). \tag{4.91}
$$

Proof. Using the definition of Gamma we and Delta one can find,

$$\begin{aligned}
\Gamma &= \frac{\partial^2}{\partial S_0^2} V_0 \\
&= \frac{\partial}{\partial S_0} \Delta \\
&= \frac{\partial}{\partial S_0} \mathcal{N}(d_1) \\
&= \mathcal{N}'(d_1) \frac{\partial}{\partial S_0}(d_1) \\
&= \mathcal{N}'(d_1) \frac{\partial}{\partial S_0} \left(\frac{\log(S_0/K) + \left(r + \frac{\sigma^2}{2}\right) T}{\sigma\sqrt{T}} \right) \\
&= \frac{1}{S_0 \sigma \sqrt{T}} \mathcal{N}'(d_1).
\end{aligned}$$

□

Analytical Vega:

Proposition 4.17. *Suppose that the the price is given as in Equation (4.89). Then analytical Vega of a European call option is,*

$$\vartheta = S_0 \mathcal{N}'(d_1) \sqrt{T}. \tag{4.92}$$

Proof. By the definition of Vega, one can find it as follows;

$$\begin{aligned}
\vartheta &= \frac{\partial}{\partial \sigma} V_0 \\
&= \frac{\partial}{\partial \sigma} \left(S_0 \mathcal{N}(d_1) - e^{-rT} K \mathcal{N}(d_2) \right) \\
&= S_0 \mathcal{N}'(d_1) \frac{\partial}{\partial \sigma}(d_1) - \frac{\partial}{\partial \sigma} \left(e^{-rT} K \mathcal{N}\left(d_1 - \sigma\sqrt{T}\right)\right) \\
&= S_0 \mathcal{N}'(d_1) \frac{\partial}{\partial \sigma}(d_1) - e^{-rT} \mathcal{N}'\left(d_1 - \sigma\sqrt{T}\right) \frac{\partial}{\partial \sigma}\left(d_1 - \sigma\sqrt{T}\right) \\
&= S_0 \mathcal{N}'(d_1) \frac{\partial}{\partial \sigma}(d_1) - e^{-rT} \frac{S_0 e^{rT}}{K} \mathcal{N}'(d_1) \left(\frac{\partial}{\partial \sigma} d_1 - \frac{\partial}{\partial \sigma} \sigma\sqrt{T} \right) \\
&= S_0 \mathcal{N}'(d_1) \sqrt{T}.
\end{aligned}$$

□

Analytical Rho:

Proposition 4.18. *Suppose that the the price is given as in Equation (4.89). Then analytical Rho of a European call option is,*

$$\rho = KT\mathcal{N}(d_2). \tag{4.93}$$

Proof. By the definition of Rho, one can find it as follows;

$$\begin{aligned}
\rho &= \frac{\partial}{\partial r} V_0 \\
&= \frac{\partial}{\partial r}\partial\sigma\left(S_0\mathcal{N}(d_1) - e^{-rT}K\mathcal{N}(d_2)\right) \\
&= S_0\mathcal{N}'(d_1)\frac{\partial}{\partial r}d_1 - \frac{\partial}{\partial r}\left(e^{-rT}K\mathcal{N}\left(d_1 - \sigma\sqrt{T}\right)\right) \\
&= S_0\mathcal{N}'(d_1)\frac{\partial}{\partial r}d_1 + KTe^{-rT}\mathcal{N}(d_2) - e^{-rT}\mathcal{N}'\left(d_1 - \sigma\sqrt{T}\right)\frac{\partial}{\partial r}\left(d_1 - \sigma\sqrt{T}\right) \\
&= KTe^{-rT}\mathcal{N}(d_2).
\end{aligned}$$

□

4.5.2 Comparison of Malliavin and Analytical Greeks

In computations the parameters are chosen as; $S0 = 100$, $r = 0.05$, $\sigma = 0.1$, $T = 1$, $K = 105$ and NumSimulations = 20000 to simulate the Greeks and plot to show how quickly the Malliavin estimators converges to the analytical values as the number of simulations increases. The figures that obtained for the Greeks are given as follows.

Figure 4.1 represents the Malliavin and numerical Delta as a function of simulation number. The purpose of this figure is to demonstrate the convergence behavior of the Delta of a European call option computed with Malliavin calculus to the numerical Delta of this option. In this figure, the result indicates the Delta estimated with Malliavin calculus method converge to the numerical Delta. Also, it is worth to emphasized that as the number of simulation increases, the estimated delta with Mallavin calculus gives better result. According to this result, the Malliavin calculus provides an efficient method to estimate the Delta of a European type option.

One of the key advantages of Malliavin calculus method in estimation of Delta is the accuricy of the estimation is increasing as the number of simulation increase. Thus, Figure 4.1 is helpful because it shows, how many times the simulation has to be done to obtain a close estimation to real Delta.

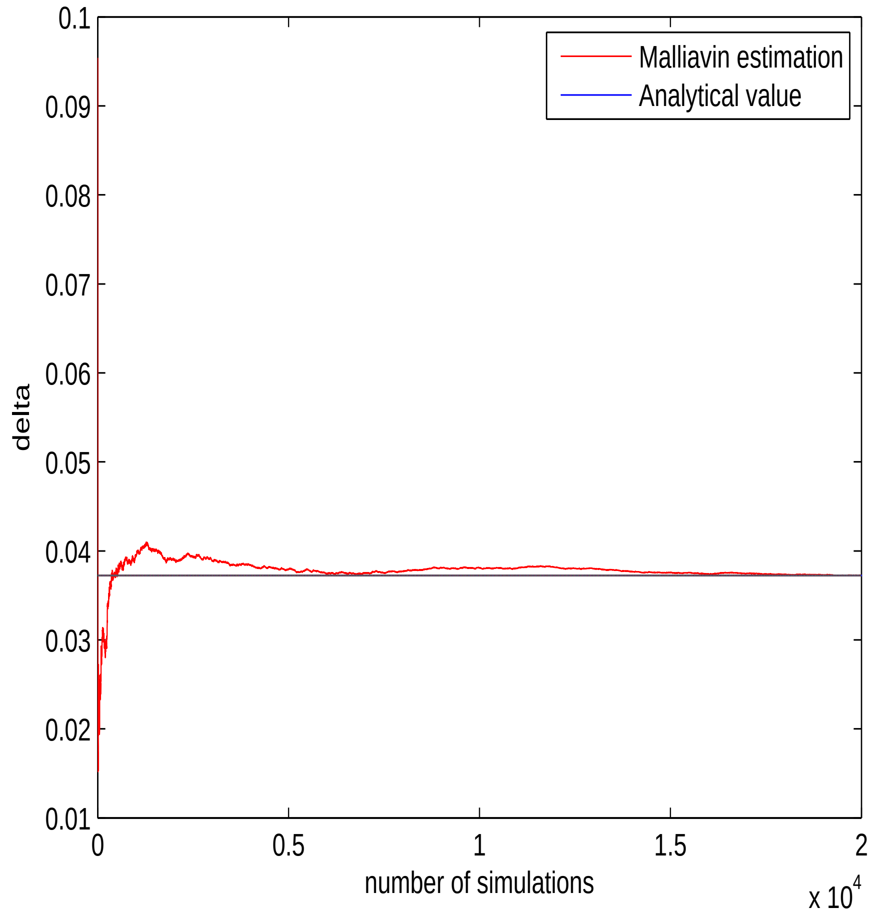

Figure 4.1: Monte-Carlo Estimation of Delta for European Call Option Using Malliavin Method

Figure 4.2 represents the Malliavin and numerical Gamma as a function of simulation number. The purpose of this figure is to demonstrate the convergence behavior of the Gamma of a European call option computed with Malliavin calculus to the numerical Gamma of this option. The result indicates the Malliavin Gamma is converge to the numerical Gamma. Also, it is worth to emphasized that as the number of simulation increases Malliavin Gamma gives better result. According to this result, the Malliavin calculus provide us an efficient method to estimate the Delta of a European option.

As in the Delta of the option, the key feature in estimation of Gamma with Malliavin calculus method is the accuricy of the estimation is increasing as the number of simulation increase. Thus, Figure 4.2 is helpful to analyze the number of necessary simulation because it shows, how many simulation has to be done to obtain a close estimation to real Gamma.

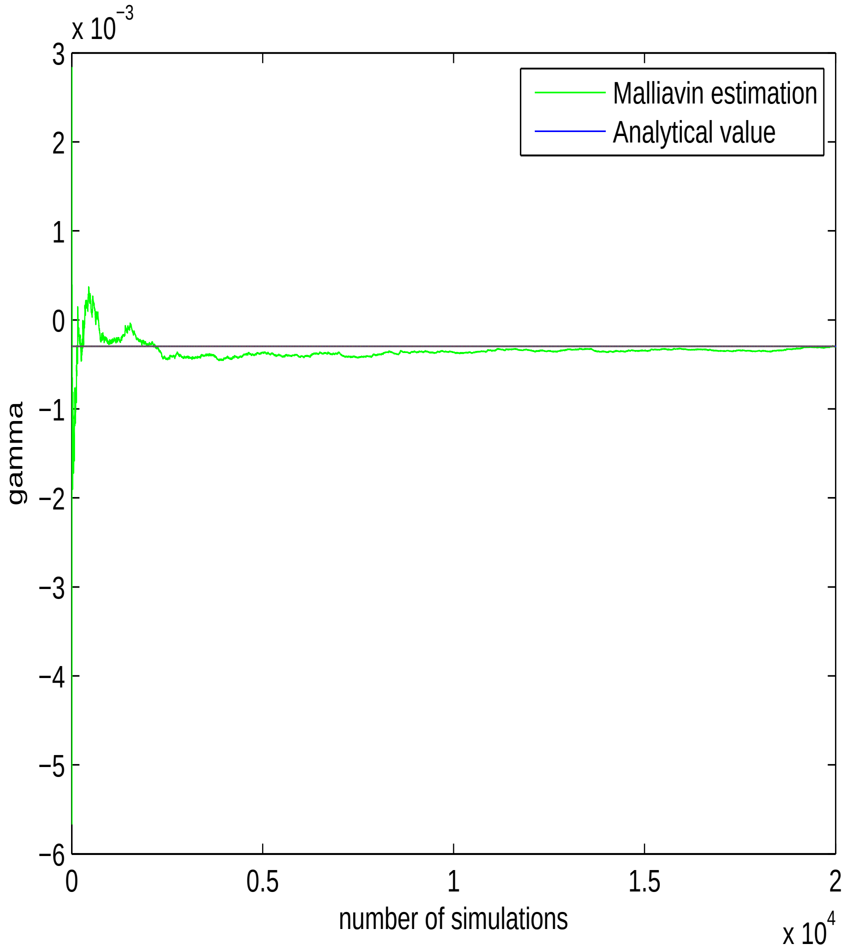

Figure 4.2: Monte-Carlo estimation of Gamma of European Call Option Using Malliavin Method

Figure 4.3 represent the Malliavin and numerical Vega as a function of simulation number. The purpose of this figure is to demonstrate the convergence behavior of the Vega of a European call option computed with Malliavin calculus to the numerical Vega of this option. In this figure, one can see from the result that the Vega estimated with Malliavin calculus method converge to the numerical Vega.

The benefit of Figure 4.3 is one can observe the necessary simulation number to have an accurate estimation result with Malliavin Calculus method. This is because as the number of simulation increases Malliavin Vega gives better result. According to this result, the Malliavin calculus provide us an efficient method to estimate the Vega of a European call option. Hence, Figure 4.3 is helpful to researchers because it shows, how many times the simulation has to be done to obtain a good estimation to real Vega.

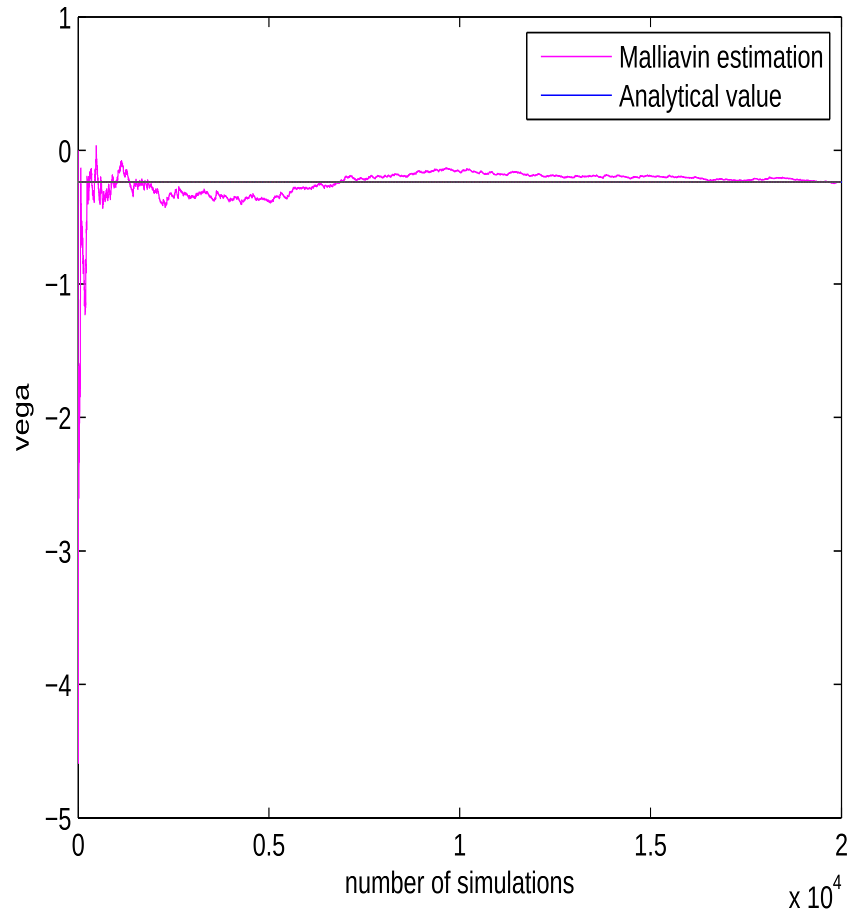

Figure 4.3: Monte-Carlo Estimation of Vega of European Call Option Using Malliavin Method

Figure 4.4 represent the Malliavin and numerical Rho as a function of simulation number. The purpose of this figure is to demonstrate the convergence behavior of the Rho of a European call option computed with Malliavin calculus to the numerical Rho of this option. The result indicates the Malliavin Rho is converge to the numerical Rho. Also, it is worth to emphasized that as the number of simulation increases Malliavin Rho gives better result. According to this result, the Malliavin calculus provide us an efficient method to estimate the Rho of a European option.

As in previous figures, Figure 4.4 shows the convergence speed of the Malliavin Rho to real Rho of the option. From this figure, one can observe the necessary simulation number to have an accurate estimation result with Malliavin Calculus method to have a nice estimation result for Rho of a European call option.

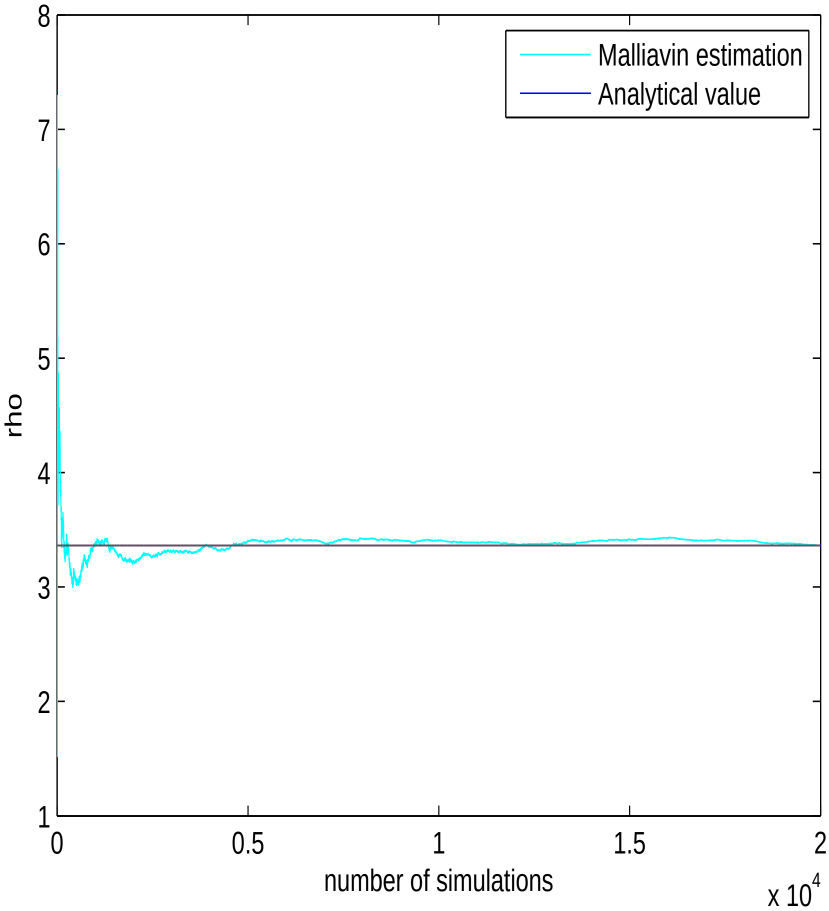

Figure 4.4: Monte-Carlo Estimation of Rho of European Call Option Using Malliavin Method

4.5.3 Comparison of the Estimation Methods

The convergence behavior of Malliavin calculus is examined in Subsection 4.5.2. From those figures, about convergent property, one can see this method is a good estimator. But, there is no clue yet which method is better. Thus, in this section, a numerical experiment presented in order to compare the Malliavin approach to the finite difference and pathwise method. The Likelihood method is avoided because it coincides to Malliavin method for the Greeks of a European call option. The comparisons are illustrated by the following figures for each Greek independently.

Comparison of Delta Estimators:

Now here, the formulas obtained for the Delta in this chapter and given in motivation for a European call option discussed. The fact that there are three different formula for Delta may seem confusing at first. However, these three formulas of Delta are different and thus the simulation results of them will lead different estimations. These features can be observed in Figure 4.5 where the outcome showed by the Monte-Carlo simulations using these estimators. In this figure the Delta estimations results with analytical Delta compared. For this figure; stock price $S_0 = 100$, strike price $K = 100$, interest rate $r = 0.05$, volatility $\sigma = 0.1$, and number of simulations 1000 are chosen.

In Figure 4.5 the colors red, green and blue represents finite difference estimation, pathwise estimation and Malliavin estimation methods results respectively. As it is seen in this figure, the finite difference method is the worst estimator of these three method because it is not converging to analytical value of Delta as good as others. The other two methods, pathwise and Malliavin estimation methods are converge to analytical value almost with same speed.

A crucial observation in this case is that, the pathwise method is computationally less expensive then other methods if it is possible to break it down to evaluating a payoff function. However, to this end an exact solution of the underlying asset price processes is needed. Otherwise, the dynamics of the sensitivity under consideration have to simulated and this will make the method computationally expensive. Finally, the Malliavin calculus method can be applied to any kind of payoff functions and no need to know the probability density as in the likelihood method.

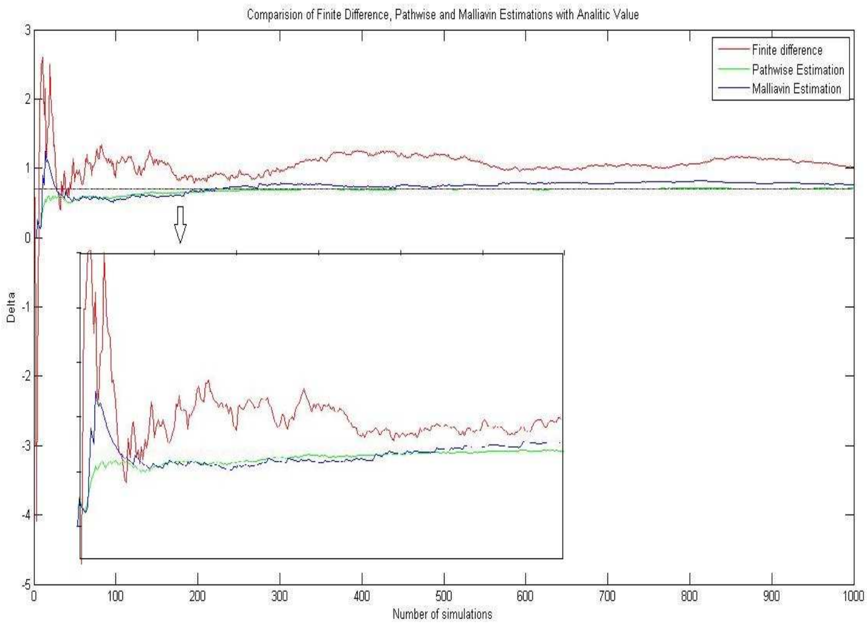

Figure 4.5: Comparison of Finite Difference, Pathwise and Malliavin Methods with Analytic Delta

In Figure 4.6 the values of Deltas obtained by Monte Carlo simulation with the choice of strike price $K = 55$, initial stock price $S0 = 57$, risk free interest rate $r = 0.05$, volatility $\sigma = 0.1$ and time to maturity $T = 1$. The corresponding 95%confidence intervals to the number of replications used by the Monte Carlo simulations are shown on the vertical line segments. Such a graph illustrates the convergence of the estimated Deltas to the real Delta of a European call option represented by the horizontal line. This figure supported the comments on the Figure 4.5.

As it is seen by the Figure 4.6 the confidence interval of Malliavin Delta is less then the other and the estimated values are more consistent. Hence, one can say this method gives more accurate result then other two methods.

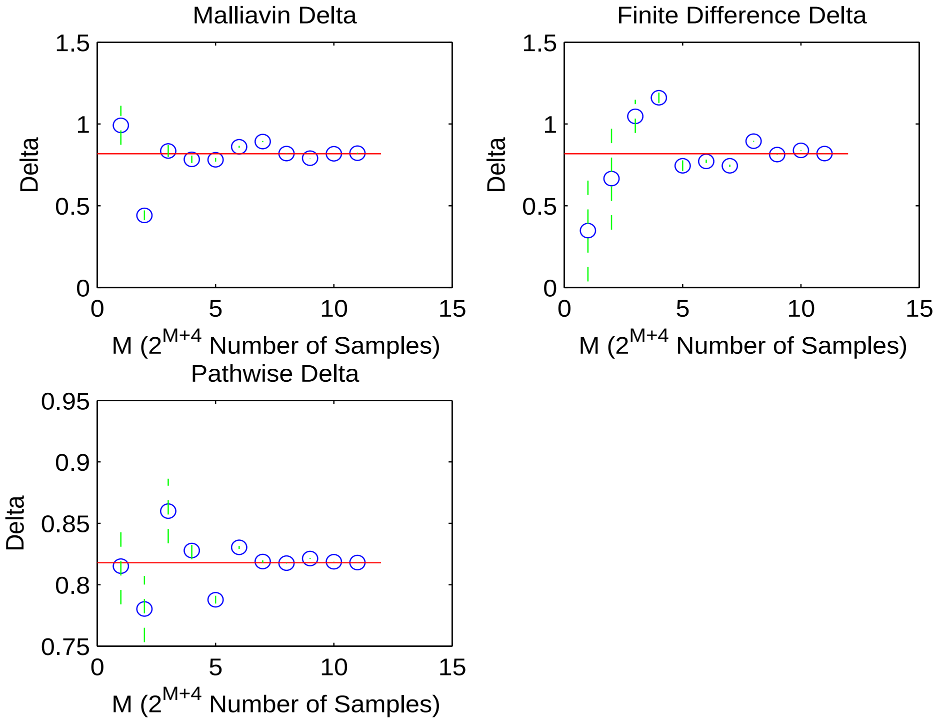

Figure 4.6: Confidence Intervals of Delta Estimators

Comparison of Gamma Estimators:

In Figure 4.7 the Gamma estimations results with analytical Gamma of European call option compared. This figure shows three simulated European call option Gamma. Since there are three different formula for the Gamma of a European option in hand, three different estimation result is obtained from the simulation. To obtain this figure; stock price $S_0 = 100$, strike price $K = 100$, interest rate $r = 0.05$, volatility $\sigma = 0.1$, and number of simulations 1000 are chosen.

In Figure 4.7 the colors red, green and blue represents finite difference estimation, pathwise estimation and Malliavin estimation methods results respectively. The estimation results shows that, since pathwise method is not applicable to second order Greeks, Malliavin calculus method versus finite different method. Therefore, Malliavin calculus method is the best among the three method in computation of the Gamma of a European call option.

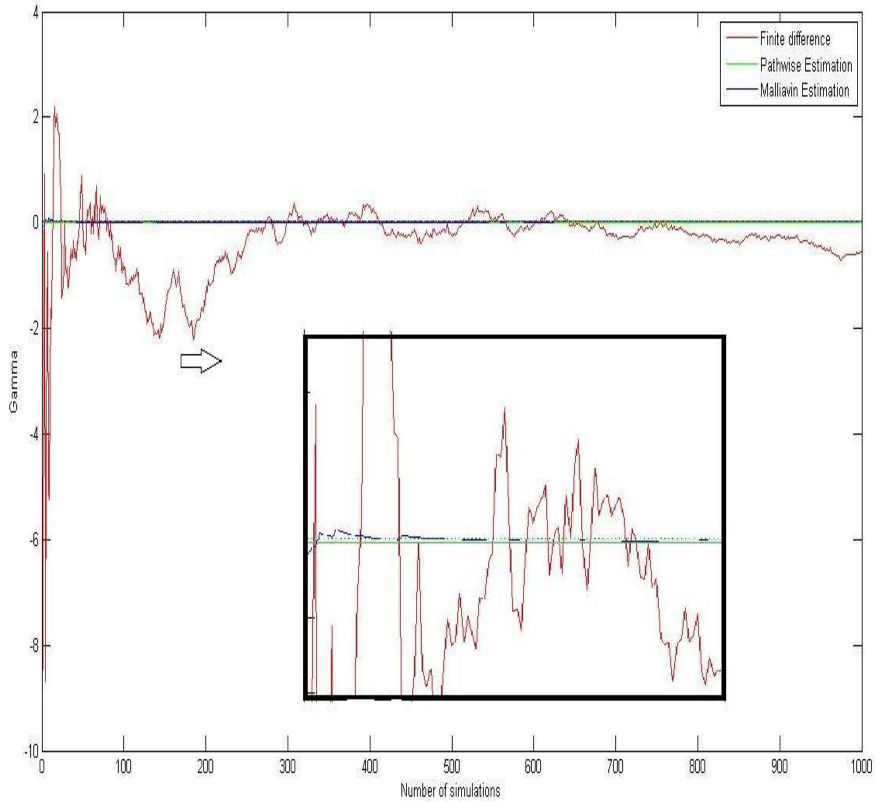

Figure 4.7: Comparison of Finite Difference, Pathwise and Malliavin Methods with Analytic Gamma

In Figure 4.8 the values of Gammas obtained by Monte Carlo simulation with the choice of strike price $K = 55$, initial stock price $S0 = 57$, risk free interest rate $r = 0.05$, volatility $\sigma = 0.1$ and time to maturity $T = 1$. The corresponding 95%confidence intervals to the number of replications used by the Monte Carlo simulations are shown on the vertical line segments. Such a graph illustrates the convergence of the estimated Gammas to the real Gamma of a European call option represented by the horizontal line. Since the pathwise metthod gives no result the confidence interval of this method is skipped. The Figure supported the comments on the Figure 4.7.

By looking Figure 4.8 one can say, Malliavin method is more consistent than finite difference method because it has a less confidence interval.

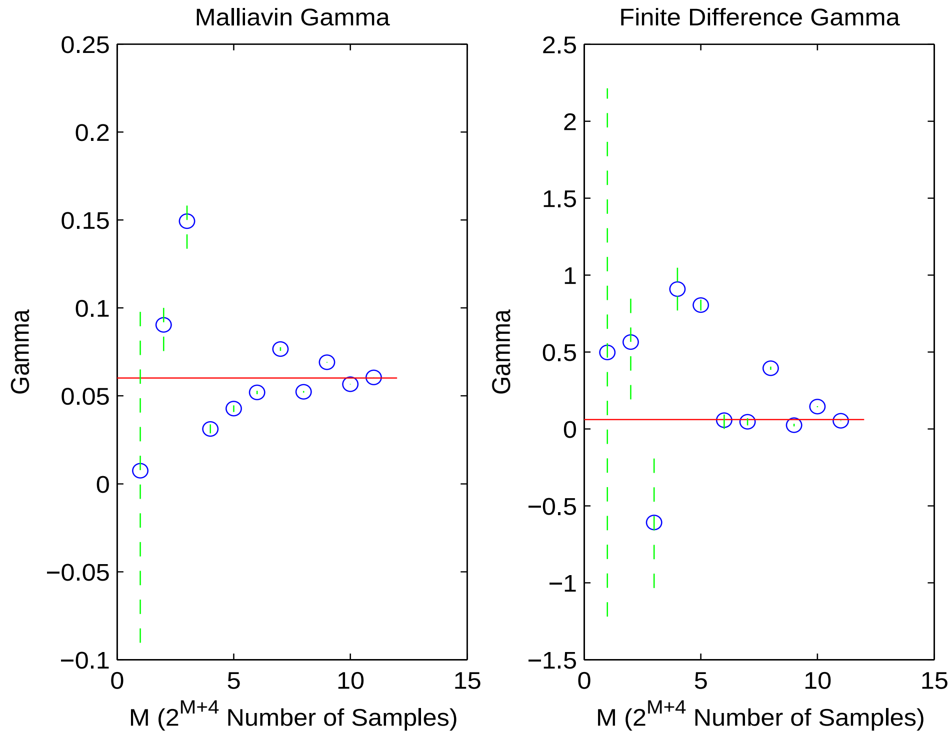

Figure 4.8: Confidence Intervals of Gamma Estimators

Comparison of Vega Estimators:

In Figure 4.9 the Vega estimations results with analytical Vega of a European call option compared. In simulations the paramaters are chosen as, stock price $S_0 = 100$, strike price $K = 100$, interest rate $r = 0.05$, volatility $\sigma = 0.1$, and number of simulations 1000.

In Figure 4.9, the colors red, green and blue represents finite difference estimation, pathwise estimation and Malliavin estimation methods results respectively. In this figure the pathwise method gives better result than both finite difference and Malliavin method. Thus, one can come up with the idea that for the Vega of a European call option finite difference estimation method is the worst of these three method. It is also clearly seen the estimation of Vega with Malliavin calculus method is getting better as the number of simulation increasing. Since Malliavin calculus method can be used for both both continuous and discontinuous payoffs, it can be claimed that Mallavin calculus method is a good method in computation of Vega of a European call option.

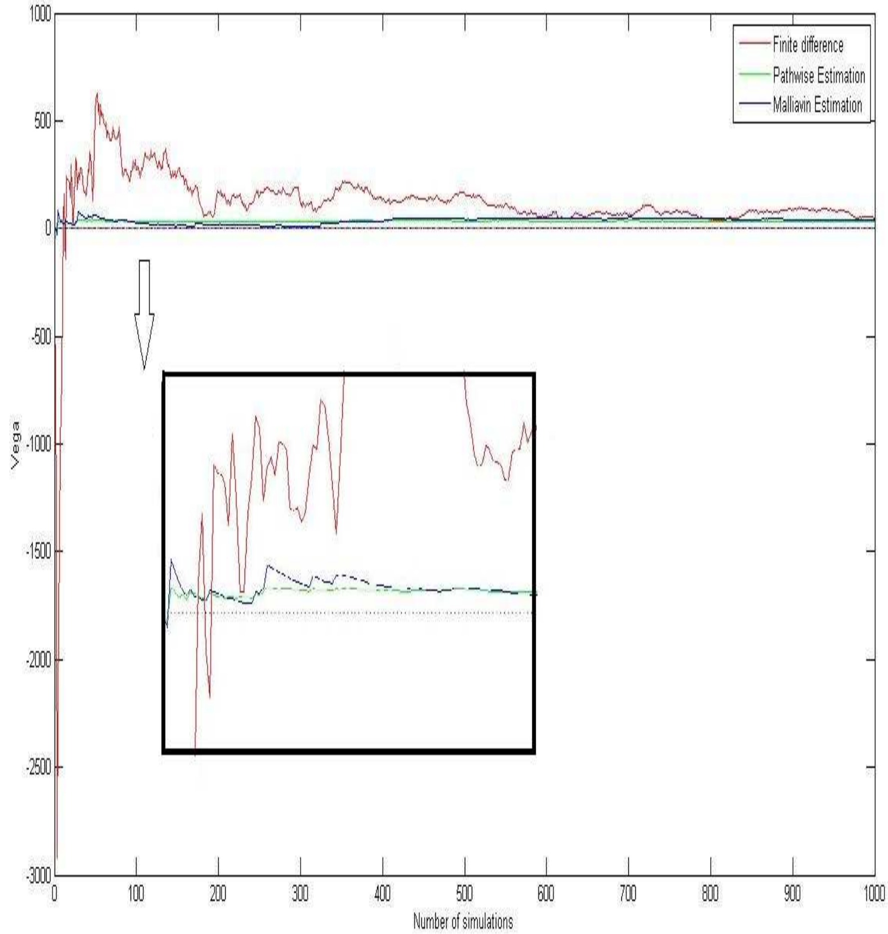

Figure 4.9: Comparison of Finite Difference, Pathwise and Malliavin Methods with Analytic Vega

In Figure 4.10 the values of Vegas obtained by Monte Carlo simulation with the choice of strike price $K = 55$, initial stock price $S0 = 57$, risk free interest rate $r = 0.05$, volatility $\sigma = 0.1$ and time to maturity $T = 1$. The corresponding 95% confidence intervals to the number of replications used by the Monte Carlo simulations are shown on the vertical line segments. Such a graph illustrates the convergence of the estimated Vegas to the real Vega of a European call option represented by the horizontal line. The Figure supported the comments on the Figure 4.9. The Estimation result of Malliavin Vega's confidence intervals are smaller than other two method. Hence the result of this method is more consistent than other methods.

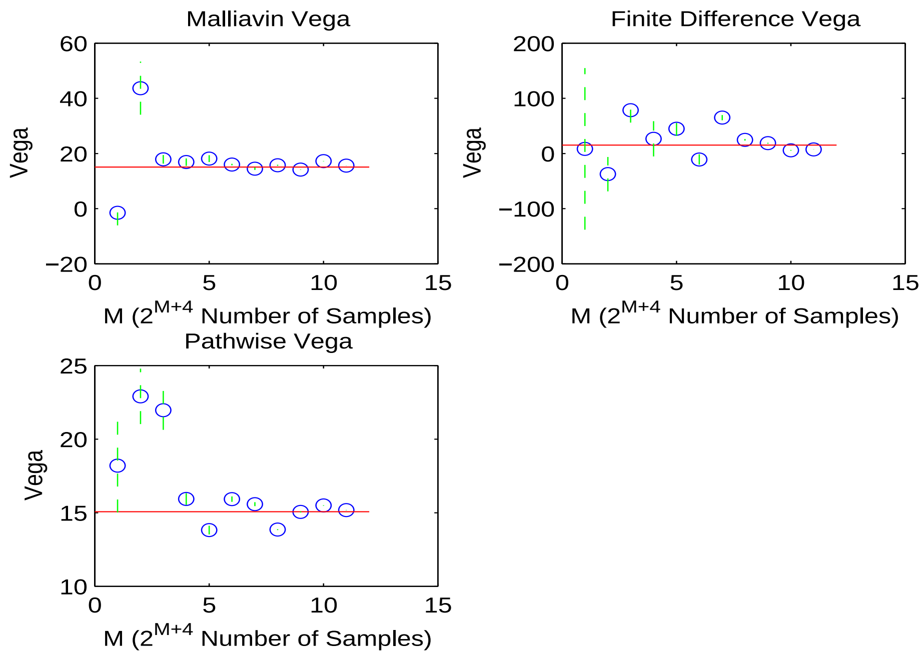

Figure 4.10: Confidence Intervals of Vega Estimators

4.6 Summary

In this section first the Black-Scholes-Merton model [5] introduced to readers. Then, the Greeks of European call options are computed with Malliavin calculus method. In addition to the computation, the results are compared with other methods—finite difference, pathwise, likelihood methods—given in Chapter 1. These four method of the estimation of Greeks have their own advantages and disadvantages in practical applications. Although it seems plausible to decide on the choice of pathwise method for European options and estimate the Greeks with it, however, the key point on the choice lies in the independence property of Malliavin weight. Since, the Malliavin weight and the payoff function of option are independent, Malliavin calculus method can be applied to both continuous and discontinuous type of payoff. Further details on Malliavin calculus as well as computation of Greeks, see [16], [51], [54], [62].

CHAPTER 5

COMPUTATION OF THE DELTA IN STOCHASTIC VOLATILITY MODELS USING MALLIAVIN CALCULUS

5.1 Introduction

The most widely used model, for stock prices in financial markets is Black-Scholes-Merton model. This model based on the assumption of geometric Brownian motion dynamics and constant volatility. However, while computing the implied volatilities for the given model, it is observed that the implied volatilities are not constant for different strikes and maturities of the options and indeed they tend to be shaped like a smile. Over the past three decades, researchers seek to find some extensions of the Black-Scholes-Merton model which will explain this empirical feature [57].

Different approaches applied to obtain new models for stock returns. One of the most popular approach is to consider the volatility as a stochastic process. These new models are called stochastic volatility models and they have fit the implied skew in the market [8]. However, for some of the models the closed form solution for the price process can not be obtained. Hence, for these cases the numerical methods are needed to handle this problem.

Merton [50], suggested that the volatility can be explained by a deterministic function of time. This approach explains the different implied volatility levels for different times of maturity, but it still does not explain the smile shape for different strike prices. Derman and Kani [15], Dupire [19] and Rubinstein [60] offer the idea that time and volatility coefficients are both dependent. This deterministic volatility approach on the model yields a complete market model. This model fits the local volatiliy surface, but it is not enough to explain the volatility smile which does not vanish as time passes and also it can not be used to price exotic options.

Then researchers came up the idea that a stochastic process can be used to model the volatility structure. The study of Heston [31], Hull and White [34] and Stein & Stein [67] lead to the development of stochastic volatility models. The models they have introduced are multi-factor models.

Gatheral [27] implies that the stochastic volatility models are useful because they explain the reason of different strike prices and maturities have different Black-Scholes-Merton implied volatilities, called, “the volatility smile”, in a self-consistent way. Moreover, unlike the other models, stochastic volatility models can fit the smile. Stochastic volatility models assume realistic dynamics for the underlying asset.

Once a closed form solution is obtained, it is easy to compute the Greeks. However, because of complex payoffs it might not be possible to obtain a closed form solution for all type of options (see [59]). Therefore, most researchers have demonstrated that the model they proposed are efficient for pricing purposes, but only a few paper, has interested in the subject of computing sensitivities. The studies Benhamou (2000) [4] and Mhlanga (2011) [51] are pioneering studies on computation of Greeks under the stochastic volatility models.

This chapter consists of two main parts. In the first part we reviewed the general ideas about stochastic volatility models will be reviewed and some of the special stochastic volatility models will be introduced. In the second part firstly the Delta for a general stochastic volatility model is computed and then the Delta of Stein & Stein and Heston models are obtained by using the general Delta formula.

5.2 Stochastic Volatility Models

The volatility is not traded in the market. Thus, it can not be observed directly from the market. However, from empirical studies of the stock price, the stock price return dS/S can be derived, and from this return the volatility can be estimated. This observation shows that, the volatility seems to be low for several days, then high for a period and so on. Therefore, one can say the process behaves like a mean-reverting process.

The pioneering studies about the stochastic volatility models are the works of Hull and White and Scott (see [33], [64]). These studies are too complicated and at that time there was no analytical solution for them. Later on as studies goes on in this research area, a variety of studies have been done and different models are developed. The first study that provides semi analytical solution was first done by Stein & Stein [67]. However, there are two crucial drawbacks of this model; it is not flexible enough to represent observable market prices and allows volatility become negative which is an undesirable matter. In 1993, Heston [31] proposed the first stochastic volatility model that allows practitioners to have reasonable amount of calibration freedom. Moreover, this well-known model permits semi analytical solutions (see [59]). Later on, various other stochastic volatility models have been developed by researchers [36]. In this section some of the stochastic volatility models are introduced to the readers.

5.2.1 Generalized Stochastic Volatility Model

There are several stochastic volatility models which are widely used in financial market modeling. In this thesis, a general stochastic volatility model is considered for computation purposes as in the following definition.

Definition 5.1. Suppose that $\left(\Omega, \mathcal{F}, \mathcal{F}_{t\in[0,T]}, Q\right)$ is the filtered probability space where $(\mathcal{F}_t)_{t\in[0,T]}$ is the filtration generated by two independent Brownian motions $(W_t)_{t\in[0,T]}$ and $(W_t^1)_{t\in[0,T]}$. The stock prices are denoted by $(S_t)_{t\in[0,T]}$ as in the celebrated Black-Scholes-Merton model. Then, the dynamics of the general stochastic volatility model under the risk-neutral measure is,

$$\frac{dS_t}{S_t} = r_t dt + \sigma(t, V_t)\, dW_t, \tag{5.1}$$

$$dV_t = u(t, V_t)\, dt + v(t, V_t)\, dZ_t, \tag{5.2}$$

where

$$\begin{aligned} dZ_t &= \left[\rho\, dW_t + \sqrt{1-\rho^2}\, dW_t^1\right], \\ \langle dW_t, dZ_t \rangle &= \rho\, dt, \end{aligned}$$

for $t \in [0,T]$ and initial values S_0 and V_0. In this stochastic differential equation system, $u(t, V_t)$ and $v(t, V_t)$ are deterministic functions in $C^2([0,T] \times \mathbb{R})$, which satisfy certain conditions to have unique solution. Moreover,

- ρ stands for the correlation between standard Brownian motions $(W_t)_{t\in[0,T]}$ and $(Z_t)_{t\in[0,T]}$.
- r_t is the risk free interest rate,
- $\sigma(t, V_t)$ is the volatility.

Since $u(t, V_t)$ and $v(t, V_t)$ are not specified in the stochastic process system given in the Definition 5.1, the system is a general model and the specific stochastic models can be drawn from this model by determining the functions σ, u and v.

The two Brownian motions $(W_t)_{t\in[0,T]}$ and $(Z_t)_{t\in[0,T]}$ are independent if $\rho = 0$. If $\rho = 1$ the Brownian motions are perfectly correlated. But these are the particular cases. In many cases, ρ is a positive or negative number within range $-1 \leq \rho \leq 1$.

In the Black-Scholes-Merton model, there is only one source of randomness, which is the change in stock price but in stochastic volatility models, one more source of randomness is presented, which is the change in volatility.

5.2.2 Hull and White Model

Hull and White 1987 [33] considered a stock with a price S_t which has an instantaneous variance $V_t = \sigma^2$, which assumed to have the following stochastic processes for a security price and its volatility of return,

$$\frac{dS_t}{S_t} = \phi\left(S_t, \sigma, t\right)\ dt + \sqrt{V_t} dW_t, \tag{5.3}$$

$$dV_t = \mu\left(\sigma, t\right) V_t dt + \xi\left(\sigma, t\right) V_t dZ_t, \tag{5.4}$$

$$\tag{5.5}$$

where

$$dZ_t = \left[\rho dW_t + \sqrt{1-\rho^2} dW_t^1\right],$$

$$\langle dW_t, dZ_t \rangle = \rho dt,$$

for $t \in [0, T]$. The Brownian motion processes W and Z have a correlation and this correlation is denoted by ρ. In terms of the general SV model given by Equations (5.1) and (5.2) we have

$$\sigma\left(t, V_t\right) = \sqrt{V_t}, \tag{5.6}$$

$$u\left(t, V_t\right) = \mu\left(\sigma, t\right) V_t, \tag{5.7}$$

$$v\left(t, V_t\right) = \xi\left(\sigma, t\right) V_t. \tag{5.8}$$

5.2.3 Stein and Stein Model

Stein & Stein's (1991) [67] study is one of the first article which provides semi analytical solutions. This model is capable of handling a nonzero mean reversion parameter δ, given the empirical evidence that volatility is strongly mean reverting. However, their model does not provide enough flexibility to represent the available market prices. The other important fact about this model is it allows the volatility to become negative which is an undesirable feature. The stochastic differential equation system of this model is as follows,

$$\frac{dS_t}{S_t} = r dt + V_t\ dW_t, \tag{5.9}$$

$$dV_t = \gamma\left(\Theta - V_t\right)\ dt + \kappa\ dZ_t, \tag{5.10}$$

where

$$dZ_t = \left[\rho\ dW_t + \sqrt{1-\rho^2}\ dW_t^1\right],$$

$$\langle dW_t, dZ_t \rangle = \rho\ dt,$$

for $t \in [0, T]$.

In this system, S and V denote the stock price and volatility process with initial values S_0 and V_0 respectively. Further, r, γ, Θ and κ are fixed constants and W_t and W_t^1 are two independent Brownian motion processes. This model can be obtained from the general stochastic volatility model given by Equations (5.1) and (5.2) with the choices,

$$\sigma(t, V_t) = V_t, \tag{5.11}$$
$$u(t, V_t) = \kappa(\Theta - V_t), \tag{5.12}$$
$$v(t, V_t) = \kappa. \tag{5.13}$$

This model is an arithmetic Ornstein-Uhlenbeck process, with a tendency to revert back to a long run average of Θ [67].

5.2.4 Heston Model

The Black-Scholes Merton model (1973) [5] shows that the average return for cash does not affect the prices of options at all, while the variance has a significant effect [31]. The Heston model [31] yield a closed form solution and this model is better to explain the correlation between the asset price and the asset volatility. Heston [31] suggested that a diffusion process for the stock price is same as the Black-Scholes-Merton, except that the volatility is allowed to be change in time. Thus, this model is a generalization of the Black-Scholes-Merton model for time-varying volatility. Moreover, the Heston model is a superior choice to the Black-Scholes-Merton model due to the fact that it has a stochastic volatility dynamics.

The Heston model is popular among academicians and practitioners today. It is attractive, because it provides a natural extension beyond geometric Brownian motion as a description of asset price dynamics, modeling volatility of underlying asset return as a positive, mean reverting, stochastic process and the powerful duality of its tractability and robustness relative to other stochastic volatility models.

This model is proposed as a pair of stochastic differential equations as follows;

$$\frac{dS_t}{S_t} = rdt + \sqrt{V_t}dW_t, \tag{5.14}$$
$$dV_t = \kappa(\Theta - V_t)\,dt + \epsilon\sqrt{V_t}dZ_t, \tag{5.15}$$

where

$$dZ_t = \left[\rho dW_t + \sqrt{1-\rho^2}dW_t^1\right],$$
$$\langle dW_t, dZ_t\rangle = \rho dt.$$

In this system, r denotes the risk free interest rate, κ is the mean-reversion rate, Θ reflects the long-term average variance and ϵ is the volatility of variance.

Further more, this model can be obtained from the general stochastic volatility model given in Definition 5.1 under the following conditions.

$$\sigma(t, V_t) = \sqrt{V_t}, \tag{5.16}$$
$$u(t, V_t) = \kappa(\Theta - V_t), \tag{5.17}$$
$$v(t, V_t) = \epsilon\sqrt{V_t}. \tag{5.18}$$

5.2.5 Schöbel and Zhu Model

Schöbel and Zhu [63] considered that the volatility follows an Ornstein-Uhlenbeck processes. They described the model as Stein and Stein [67] and it is given as following,

$$dS_t = \left(r - \frac{1}{2}V(t)^2\right)dt + V(t)\,dW_t, \tag{5.19}$$
$$dV(t) = \kappa(\Theta - V(t))\,dt + \sigma dZ_t, \tag{5.20}$$

where

$$dZ_t = \left[\rho dW_t + \sqrt{1-\rho^2}dW_t^1\right],$$
$$\langle dW_t, dZ_t\rangle = \rho dt.$$

In this model the Brownian motion processes W_t and Z_t are correlated and the correlation between these processes is represented by ρ.

This model can be obtained from the general stochastic volatility model given in Definition 5.1 under the choice of

$$\sigma(t, V_t) = V_t, \tag{5.21}$$
$$u(t, V_t) = \kappa(\Theta - V(t)), \tag{5.22}$$
$$v(t, V_t) = \sigma. \tag{5.23}$$

5.2.6 Bates Model

Bates model [3] is a combination of two model which are Heston model and Merton model and the SDE system of this model is given as following,

$$\frac{dS_t}{S_t} = (r - \delta)\,dt + \sqrt{V_t}dW_t + dN_t, \tag{5.24}$$
$$dV_t = -\kappa(V_t - \Theta)\,dt + \sigma\sqrt{V_t}dZ_t, \tag{5.25}$$

where r is the interest rate, δ is the dividend paid by the underlying asset, V is the value of value of spot volatility, Θ is the long run volatility, σ is the volatility of volatility and W_t, Z_t are two correlated processes and the correlation between them is ρ. In this model N_t is a compound Poisson process.

It is important to emphasize that in this model the stock prices are modeled with an additional jump process, that makes the model as a flexible and robust alternative model.

Bates model can be derived from the general stochastic volatility model given in Definition 5.1 with the choice of

$$\sigma(t, V_t) = \sqrt{V_t}, \tag{5.26}$$

$$u(t, V_t) = -\kappa (V_t - \Theta), \tag{5.27}$$

$$v(t, V_t) = \sigma \sqrt{V_t} \tag{5.28}$$

and a compound Poisson process N_t .

5.3 Computation of the Delta in Stochastic Volatility Models

Most researchers have demonstrated that the model they proposed are efficient for pricing purposes, but only a few paper, has interested in the subject of computing sensitivities. Thus, there is not much study has been worked on computing sensitivities with respect to the parameters of the volatility models. This is very surprising given the importance of Greeks for hedging and risk-management.

In general, while computing the sensitivities of an option price, one differentiates the underlying asset and, by doing so, one is differentiating the approximation. This provides to researchers the advantage that differentiating the approximation price is usually much easier, particularly for the Heston model. If one has a different approximation to the process, the derivative of this approximation will also be different. Therefore, numerical computations can give different results for the Greeks even if they have similar effects on pricing [9].

The abstract framework of this study is quite similar to the one developed in [21], but in this thesis some modifications introduced in order to take into account stochastic volatility models such as Heston stochastic volatility and Stein & Stein models. In this thesis a general stochastic volatility model is considered and the Delta of this general formula is obtained. Then as an example, by the generalized Delta, derived by using Malliavin calculus to the general formula, the Heston and Stein & Stein models Delta is presented.

The stock price process S_t given in Equation (5.1) can be solved and the solution is given with the following proposition.

Proposition 5.1. *Let the stock price process be given as in Definition 5.1. Then, the stock price process S_t has a solution as,*

$$S_T = S_0 \exp\left\{ \int_0^T \sigma(t, V_t)\, dW_t + \int_0^T \left(r_t - \frac{1}{2}\sigma^2(t, V_t) \right) dt \right\}. \tag{5.29}$$

Proof. The proof of this proposition can be done by applying Itô lemma with $f(x) = \ln x$. The first and second order partial derivatives of function $f(x) = lnx$ are $f^{'} = \frac{1}{x}$ and $f^{''} = \frac{-1}{x^2}$ respectively. Using these equalities the following is obtained,

$$\ln S_T = \ln S_0 + \int_0^T \frac{1}{S_t} dS_t - \frac{1}{2}\int_0^T \frac{1}{S_t^2} d\langle S, S\rangle_t . \tag{5.30}$$

It is known that $dS_t = S_t r_t dt + S_t \sigma(t, V_t) dW_t$ and the quadratic variation of the process is $d\langle S, S\rangle_t = S_t^2 \sigma^2(t, V_t) dt$. Substituting them into Equation (5.30),

$$\begin{aligned}
\ln\left(\frac{S_T}{S_0}\right) &= \int_0^T \frac{1}{S_t}\left[r_t dt + \sigma(t, V_t) dW_t\right] - \frac{1}{2}\int_0^T \frac{1}{S_t^2} S_t^2 \sigma^2(t, V_t) dt \\
&= \int_0^T \sigma(t, V_t) dW_t + \int_0^T \left(r_t - \frac{1}{2}\sigma^2(t, V_t)\right) dt
\end{aligned}$$

is found. Then using exponential function,

$$S_T = S_0 \exp\left\{\int_0^T \sigma(t, V_t) dW_t + \int_0^T \left(r_t - \frac{1}{2}\sigma^2(t, V_t)\right) dt\right\}.$$

□

The integration by parts formula given in Proposition 2.4 is the most important formula that is used to compute the Greeks with Malliavin calculus. In this chapter, to make computations more clear, a generalized version of the integration by parts formula, given in the following Proposition 5.2.

Proposition 5.2. *Suppose I is an open interval of $\mathbb{R}$. Consider the families of random variables $\left(F^{\zeta}\right)_{\zeta \in I}$ and $\left(H^{\zeta}\right)_{\zeta \in I}$. These families are continuously differentiable in $Dom(\delta)$ with respect to the parameter $\zeta \in I$. Assume that $(u_t)_{t\in[0,T]} \in \mathbb{D}^{1,2}$ satisfying*

$$D^u F^{\zeta} \neq 0 \ a.s. \ on \ \left\{\partial_{\zeta} F^{\zeta} \neq 0\right\}, \zeta \in I.$$

Furthermore assume that $\frac{uH^{\zeta}\partial_{\zeta}F^{\zeta}}{D^u F^{\zeta}}$ is continuous with respect to ζ in $Dom(\delta)$. Then we have

$$\frac{\partial}{\partial \zeta}\mathbb{E}_Q\left[H^{\zeta} f\left(F^{\zeta}\right)\right] = \mathbb{E}_Q\left[f\left(F^{\zeta}\right)\left(\frac{H^{\zeta}\partial_{\zeta}F^{\zeta}}{D^u F^{\zeta}}\delta(u) - D^u\left(\frac{H^{\zeta}\partial_{\zeta}F^{\zeta}}{D^u F^{\zeta}}\right) + \partial_{\zeta} H^{\zeta}\right)\right]$$

for any function f such that $f\left(F^{\zeta}\right) \in L^2(\Omega)$, $\zeta \in I$.

Proof. Using the chain rule and the linearity property of expectation to $\mathbb{E}\left[f\left(F^{\zeta}\right)H^{\zeta}\right]$,

$$\begin{aligned}
\frac{\partial}{\partial\zeta}\mathbb{E}_Q\left[f\left(F^{\zeta}\right)H^{\zeta}\right] &= \mathbb{E}_Q\left[\left(\left(\frac{\partial}{\partial\zeta}f\left(F^{\zeta}\right)\right)H^{\zeta}\right)+f\left(F^{\zeta}\right)\frac{\partial}{\partial\zeta}H^{\zeta}\right] \\
&= \mathbb{E}_Q\left[f'\left(F^{\zeta}\right)H^{\zeta}\frac{\partial}{\partial\zeta}F^{\zeta}+f\left(F^{\zeta}\right)\frac{\partial}{\partial\zeta}H^{\zeta}\right] \\
&= \mathbb{E}_Q\left[f'\left(F^{\zeta}\right)H^{\zeta}\frac{\partial}{\partial\zeta}F^{\zeta}\right]+\mathbb{E}_Q\left[f\left(F^{\zeta}\right)\frac{\partial}{\partial\zeta}H^{\zeta}\right]. \qquad (5.31)
\end{aligned}$$

is obtained. For the sake of simplicity, first compute the right hand side. Applying the integration by parts formula Proposition 2.4 to achieve what is desired, it is also necessary to use Remark 2.3 and Proposition 2.11. Now using them,

$$\begin{aligned}
\mathbb{E}_Q\left[f'\left(F^{\zeta}\right)H^{\zeta}\frac{\partial}{\partial\zeta}F^{\zeta}\right] &= \mathbb{E}_Q\left[f\left(F^{\zeta}\right)\delta\left(\frac{H^{\zeta}\frac{\partial}{\partial\zeta}F^{\zeta}u_{.}}{\int_0^T u_v D_v F^{\zeta}dv}\right)\right] \\
&= \mathbb{E}_Q\left[f\left(F^{\zeta}\right)\delta\left(\frac{H^{\zeta}\frac{\partial}{\partial\zeta}F^{\zeta}u_{.}}{D_u F^{\zeta}}\right)\right] \\
&= \mathbb{E}_Q\left[f\left(F^{\zeta}\right)\left(\frac{H^{\zeta}\frac{\partial}{\partial\zeta}F^{\zeta}}{D_u F^{\zeta}}\delta\left(u_{.}\right)-\int_0^T\left(D_v\frac{H^{\zeta}\frac{\partial}{\partial\zeta}F^{\zeta}}{D_u F^{\zeta}}\right)u_v dv\right)\right] \\
&= \mathbb{E}_Q\left[f\left(F^{\zeta}\right)\left(\frac{H^{\zeta}\frac{\partial}{\partial\zeta}F^{\zeta}}{D_u F^{\zeta}}\delta\left(u_{.}\right)-D_u\left(\frac{H^{\zeta}\frac{\partial}{\partial\zeta}F^{\zeta}}{D_u F^{\zeta}}\right)\right)\right] \qquad (5.32)
\end{aligned}$$

is obtained. Then, substituting Equation (5.32) into Equation (5.31), get

$$\frac{\partial}{\partial\zeta}E_Q\left[H^{\zeta}f\left(F^{\zeta}\right)\right]=\mathbb{E}_Q\left[f\left(F^{\zeta}\right)\left(\frac{H^{\zeta}\partial_{\zeta}F^{\zeta}}{D_u^W F^{\zeta}}\delta^W(u)-D_u^W\left(\frac{H^{\zeta}\partial_{\zeta}F^{\zeta}}{D_u^W F^{\zeta}}\right)+\partial_{\zeta}H^{\zeta}\right)\right].$$

□

In this section, a general formula for the delta of a European type options whose underlying asset follows the stochastic volatility dynamics given by Equations (5.1) and (5.2) is obtained. But for computation purposes it is assumed that that $S_t \in \mathbb{D}^{2,2}$ and $V_t \in \mathbb{D}^{2,2}$ for $t \in [0,T]$.

Before beginning the computations, define the following function $G(t,T)$ for $t \in [0,T]$, which plays a key role in further computations of Delta.

Definition 5.2. For the general stochastic volatility model, define

$$\begin{aligned}
G(t,T) = \sigma(t,V_t) &+ \int_t^T \frac{\partial\sigma}{\partial y}(s,V_s)\,D_tV_s dW_s \\
&- \int_t^T \frac{\partial\sigma}{\partial y}(s,V_s)\,D_tV_s\sigma(s,V_s)\,ds, \qquad (5.33)
\end{aligned}$$

where $\frac{\partial}{\partial y}$ denotes the first order partial derivative of the function $\sigma(s,V_s)$ with respect to second component throughout this thesis and V_t, $\sigma(t,V_t)$, $t \in [0,T]$ are defined in Definition 5.1.

Remark 5.1. Note that the general stochastic volatility model given by Equations (5.1) and (5.2) is represented by two independent Brownian motions, namely W and W^1. Since, in this study it is interested in the derivative of the stock price dynamics with respect to chance parameter ω and its dynamic is generated by the Brownian motion W, in the computations, focused on the Malliavin derivative of random variables in the Wiener space generated by the paths of $W(t,\omega)$, $t \in [0,T]$ and $\omega \in \Omega$.

The function $G(t,T), t \in [0,T]$ given in Definition 5.2 has several useful properties which are summarized as in the following proposition.

Proposition 5.3. *Let the dynamics of $\{S_t\}_{0\le t\le T}$ be given by Equations (5.1) and (5.2). Then, the following equalities hold for $s \le t \le v \le T$.*

$$D_t S_T = S_T G\left(t,T\right) \tag{5.34}$$

$$D^u S_T := \langle DS_T, u\rangle_{\mathbb{L}^2([0,T])} = S_T \int_0^T u_t G\left(t,T\right) dt \tag{5.35}$$

$$\begin{aligned} D^u D^u S_T &= S_T \left(\int_0^T u_t G\left(t,T\right) dt \right)^2 \\ &+ S_T \int_0^T \int_s^T u_s u_t D_s G\left(t,T\right) dt ds, \end{aligned} \tag{5.36}$$

$$\begin{aligned} D_s G\left(t,T\right) &= \frac{\partial \sigma}{\partial y}\left(t,V_t\right) D_s V_t + \int_t^T \left(\frac{\partial \sigma}{\partial y}\left(v,V_v\right) \right)^2 D_t V_v D_s V_v dv \\ &+ \int_t^T D_s V_v D_t V_v \left(\frac{\partial^2 \sigma}{\partial y^2}\left(v,V_v\right) \left(dW_v - \sigma\left(v,V_v\right) dv\right) \right) \\ &+ \int_t^T \frac{\partial \sigma}{\partial y}\left(v,V_v\right) D_s D_t V_v \left(dW_v - \sigma\left(v,V_v\right) dv\right), \end{aligned} \tag{5.37}$$

Where the function $G\left(t,T\right)$ is defined in Definition 5.2 and $\frac{\partial^2}{\partial y^2}$ is the second order partial derivative of the function $\sigma\left(v,V_v\right)$ with respect to the second component.

Proof. Using the dynamics of the stock price Equation (5.29) given in Remark 5.1, first compute the Malliavin derivative of stock price at maturity,

$$
\begin{aligned}
D_t S_T &= D_t \left(S_0 \exp \left(\int_0^T \sigma(s, V_s)\, dW_s + \int_0^T \left(r_s - \frac{\sigma^2(s, V_s)}{2} \right) ds \right) \right) \\
&= S_0 \exp \left\{ \int_0^T \sigma(s, V_s)\, dW_s + \int_0^T \left(r_s - \frac{\sigma^2(s, V_s)}{2} \right) ds \right\} \\
&\times D_t \left(\int_0^T \sigma(s, V_s)\, dW_s + \int_0^T \left(r_s - \frac{\sigma^2(s, V_s)}{2} \right) ds \right) \\
&= S_T \left(\int_t^T \frac{\partial \sigma}{\partial y}(s, V_s)\ D_t V_s dW_s + \sigma(t, V_t) - \int_t^T \sigma(s, V_s) \frac{\partial \sigma}{\partial y}(s, V_s)\ ds \right) \\
&= S_T \left(\sigma(t, V_t) + \int_t^T \frac{\partial \sigma}{\partial y}(s, V_s)\, D_t V_s \left[dW_s - \sigma(s, V_s)\, ds \right] \right) \\
&= S_T G(t, T).
\end{aligned}
$$

The directional derivative of S_T can be computed by using Remark 2.3 as follows,

$$
D^u S_T = \langle D.S_T, u \rangle_{\mathbb{L}^2([0,T])} = \int_0^T u_t\, D_t\, S_T\, dt = S_T \int_0^T u_t\, G(t, T)\, dt.
$$

Applying Proposition 2.2 to Equation (5.35), the second Malliavin derivative is obtained as,

$$
\begin{aligned}
D^u D^u S_T &= D^u \left(S_T \int_0^T u_t\, G(t, T)\ dt \right) \\
&= (D^u S_T) \left(\int_0^T u_t G(t, T)\, dt \right) + S_T D^u \left(\int_0^T u_t\, G(t, T)\, dt \right) \\
&= S_T \left(\int_0^T u_t G(t, T)\, dt \right) \left(\int_0^T u_t G(t, T)\, dt \right) \\
&+ S_T \int_0^T u_s\, D_s \left(\int_0^T u_t\, G(t, T)\ dt \right) ds \\
&= S_T \left[\left(\int_0^T u_t\, G(t, T)\, dt \right)^2 + \int_0^T \int_s^T u_s\, u_t\, D_s G(t, T)\ dt\, ds \right].
\end{aligned}
$$

Now $D_sG(t,T)$ can be computed for $0 \leq s \leq t \leq v \leq T$ as follows

$$
\begin{aligned}
D_sG(t,T) &= D_s\left[\sigma(t,V_t) + \int_t^T \frac{\partial\sigma}{\partial y}(v,V_v)\ D_tV_v\left[dW_v - \sigma(v,V_v)\,dv\right]\right] \\
&= \frac{\partial\sigma}{\partial y}(t,V_t)\ D_sV_t + \int_t^T D_s\left(\frac{\partial\sigma}{\partial y}(v,V_v)\ D_tV_v\right)dW_v \\
&- \int_t^T D_s\left(\frac{\partial\sigma}{\partial y}(v,V_v)\ D_tV_v\sigma(v,V_v)\right)dv \\
&= \frac{\partial\sigma}{\partial y}(t,V_t)\ D_sV_t \\
&+ \int_t^T D_sV_vD_tV_v\left[\frac{\partial^2\sigma}{\partial y^2}(v,V_v)\left(dW_v - \sigma(v,V_v)\,dv\right)\right] \\
&+ \int_t^T \frac{\partial\sigma}{\partial y}(v,V_v)\ D_sD_tV_v\left(dW_v - \sigma(v,V_v)\,dv\right) \\
&+ \int_t^T \frac{\partial\sigma}{\partial y}(v,V_v)\ D_tV_v\frac{\partial\sigma}{\partial y}(v,V_v)\ D_sV_vdv,
\end{aligned}
$$

where D_tV_v can be computed explicitly for $0 \leq t \leq v \leq T$ as

$$
\begin{aligned}
D_tV_v &= D_t\left(V_0 + \int_0^v u(s,V_s)\,ds + \int_0^v \rho v(s,V_s)\,dW_s + \int_0^v \sqrt{1-\rho^2}v(s,V_s)\,dW_s^1\right) \\
&= \int_t^v D_tu(s,V_s)\,ds + D_t\int_0^v \rho v(s,V_s)\,dW_s + D_t\int_0^v \sqrt{1-\rho^2}v(s,V_s)\,dW_s^1 \\
&= \int_t^v D_tu(s,V_s)\,ds + \int_t^v \rho D_tv(s,V_s)\,dW_s + \rho v(t,V_t) \\
&= \int_t^v \frac{\partial u}{\partial y}(s,V_s)\ D_tV_sds + \int_t^v \rho\frac{\partial v}{\partial y}(s,V_s)\ D_tV_sdW_s + \rho v(y,V_t) \\
&= \int_t^v D_tV_s\left[\frac{\partial u}{\partial y}(s,V_s)\ ds + \rho\frac{\partial v}{\partial y}(s,V_s)\ dW_s\right] + \rho v(t,V_t).
\end{aligned}
$$

□

The following Lemma 5.4 and Proposition 5.5 give the general formula of Delta for stochastic volatility models.

Lemma 5.4. *Let the dynamics of the stock price process $\{S_t\}_{t\in[0,T]}$ be given by Equations (5.1) and (5.2). The first variation $Y_t := \frac{\partial}{\partial V_0}V_t$ of V_t is given as*

$$
\begin{aligned}
Y_t = &\exp\left(\int_0^t u'(s,V_s) - \frac{\left(v'(s,V_s)\right)^2}{2}ds\right) \\
&\times\exp\left(\int_0^t v'(s,V_s)\left[\rho\,dW_s + \sqrt{1-\rho^2}\,dW_s^1\right]\right),
\end{aligned}
$$

for all $t \in [0,T]$.

For $s \le t$, the Malliavin derivative of the stochastic volatility process V_t at time s is given by

$$D_s V_t = \frac{Y_t \, \rho \, v(s, V_s) \exp\left(\frac{1}{2} \int_s^t (1-\rho^2) \left(v'(r, V_r)\right)^2 dr\right)}{Y_s \exp\left(\int_s^t \sqrt{1-\rho^2}\, v'(r, V_r)\, dW_r^1\right)}.$$

Proof. The first variation process of V_t is defined by the following equations system:

$$\begin{aligned} dY_t &= u'(t, V_t) Y_t dt + v'(t, V_t) Y_t \rho dW_t + v'(t, V_t) \sqrt{1-\rho^2} Y_t dW_t^1, \\ Y_0 &= 1. \end{aligned}$$

By Itô's formula,

$$Y_t = \exp\left(\int_0^t u'(s, V_s) - \frac{\left(v'(s, V_s)\right)^2}{2} ds + \int_0^t v'(s, V_s) \left(\rho dW_s + \sqrt{1-\rho^2} dW_s^1\right)\right),$$

is obtained for all $t \in [0, T]$. Now, the following process can be defined

$$Z_t := D_s V_t = \int_s^t u'(r, V_r) D_s V_r \, dr + \rho \int_s^t v'(r, V_r) D_s V_r \, dW_r + \rho \, v(s, V_s),$$

which can be expressed more formally by the following system of equations

$$\begin{aligned} dZ_t &= u'(t, V_t) Z_t dt + \rho \, v'(t, V_t) Z_t dW_t, \\ Z_s &= \rho \, v(s, V_s), \end{aligned}$$

for $t \ge s$. The solution of the above system is given as

$$Z_t = \rho \, v(s, V_s) \exp\left(\int_s^t \left\{u'(r, V_r) - \frac{\left(\rho \, v'(r, V_r)\right)^2}{2}\right\} dr + \int_s^t v'(r, V_r) \rho \, dW_r\right)$$

and finally the following equality obtained for the Malliavin derivative of V_t assuming the general stochastic volatility model

$$D_s V_t = \rho \, v(s, V_s) \frac{Y_t}{Y_s} \exp\left(\int_s^t \frac{1-\rho^2}{2} \left(v'(r, V_r)\right)^2 dr - \int_s^t v'(r, V_r) \sqrt{1-\rho^2} dW_r^1\right).$$

$\square$

Proposition 5.5. *Let $\zeta = S_0$ in Proposition 5.2. Then, $F^{S_0} := S_T \in Dom(\delta)$, $H^{S_0} := \exp\left(-\int_0^T r_t \, dt\right) \in Dom(\delta)$, $(u_t)_{t \in [0,T]} \in \mathbb{D}^{1,2}$ and*

$$D^u S_T \neq 0 \quad a.s. \ on \quad \{\partial_{S_0} S_T \neq 0\}.$$

Further, assume that $\partial_{S_0} = \frac{\partial}{\partial S_0}$ and $\frac{uH^{S_0}\partial_{S_0}S_T}{D^u S_T}$ be continuous with respect to S_0 in $Dom(\delta)$. Then, the Delta within the general stochastic volatility model defined in Equations (5.1) and (5.2) is given as follows

$$\Delta = \frac{e^{-\int_0^T r_t dt}}{S_0} \times \mathbb{E}\left[f\left(S_T\right) \left(\frac{\delta\left(u_.\right)}{\int_0^T G\left(t,T\right)dt} - \frac{\int_0^T \int_s^T u_s u_t D_s G\left(t,T\right) dtds}{\left(\int_0^T u_t G\left(t,T\right)dt\right)^2} \right) \right], \tag{5.38}$$

where $G(t,T)$ is defined in Definition 5.2.

Proof. In order to compute the delta under a specified stochastic volatility model, the Malliavin derivative of $G(t,T)$ is necessary, and choose a Skorohod integrable process u_t in an appropriate way to obtain useful results. In fact, the choice $u_t = 1$ for $t \in [0,T]$ allows to prove the desired result. Start the proof by applying the result of Proposition 5.2 with $\zeta = S_0$ and $F^\zeta = S_T$.

In the Proposition 5.2 H^{S_0} can be thought as a discount factor. Since we define it as an independent process of S_0, the partial derivative

$$\frac{\partial H^{S_0}}{\partial S_0} = 0.$$

The partial derivative of S_T with respect to S_0 is as follows;

$$\frac{\partial S_T}{\partial S_0} = \exp\left(\int_0^T \sigma\left(t,V_t\right)dW_t + \int_0^T \left(r_t - \frac{1}{2}\sigma^2\left(t,V_t\right)\right)dt\right) = \frac{S_T}{S_0}.$$

By these features, the following is obtained,

$$\Delta = \mathbb{E}_Q\left[f\left(S_T\right)\left(\frac{\partial_{S_0}S_T}{D^u S_T}\delta\left(u_.\right) - D^u\left(\frac{\partial_{S_0}S_T}{D^u S_T}\right)\right)\right], \tag{5.39}$$

where $\partial_{S_0} = \frac{\partial}{\partial S_0}$. A direct computation yields

$$\begin{aligned}
\Delta &= \mathbb{E}\left[f\left(S_T\right)\left(\frac{\frac{1}{S_0}S_T}{D^u S_T}\delta\left(u_.\right) - D^u\left(\frac{\frac{1}{S_0}S_T}{D^u S_T}\right)\right)\right] \\
&= \frac{e^{-\int_0^T r_t dt}}{S_0}\mathbb{E}_Q\left[f\left(S_T\right)\left(\frac{S_T}{D^u S_T}\delta\left(u_.\right) - D^u\left(\frac{S_T}{D^u S_T}\right)\right)\right] \\
&= \frac{e^{-\int_0^T r_t dt}}{S_0}\mathbb{E}_Q\left[f\left(S_T\right)\left(\frac{S_T}{D^u S_T}\delta\left(u_.\right) - \left(\frac{1}{D^u S_T}D^u S_T - S_T\frac{D^u D^u S_T}{\left(D^u S_T\right)^2}\right)\right)\right].
\end{aligned} \tag{5.40}$$

Now substituting Equations (5.35) and (5.36) into Equation (5.40),

$$\begin{aligned}
\Delta &= \frac{e^{-\int_0^T r_t dt}}{S_0}\mathbb{E}_Q\left[f(S_T)\left(\frac{1}{\int_0^T u_t\, G(t,T)\,dt}\delta(u_.)\right)\right] \\
&- \frac{e^{-\int_0^T r_t dt}}{S_0}\mathbb{E}_Q\left[f(S_T)\left(1-\frac{S_T\left(\int_0^T u_t\, G(t,T)\,dt\right)^2}{S_T\left(\int_0^T u_t\, G(t,T)\,dt\right)^2}\right)\right] \\
&- \frac{e^{-\int_0^T r_t dt}}{S_0}\mathbb{E}_Q\left[f(S_T)\,()\,\frac{S_T\int_0^T\int_s^T u_s\, u_t\, D_sG(t,T)\,dtds}{S_T\left(\int_0^T u_t\, G(t,T)\,dt\right)^2}\right]
\end{aligned}$$

is obtained which leads to the required result for the delta under the general stochastic volatility model:

$$\Delta = \frac{e^{-\int_0^T r_t dt}}{S_0}\mathbb{E}_Q\left[f(S_T)\left(\frac{1}{\int_0^T u_t\, G(t,T)\,dt}\delta(u_.) - \frac{\int_0^T\int_s^T u_s u_t D_sG(t,T)\,dtds}{\left(\int_0^T u_t\, G(t,T)\,dt\right)^2}\right)\right].$$

□

Remark 5.2. According to Proposition 5.5, the computation of the Delta results in the computation of the expectation of the payoff function multiplied by the term

$$\Delta_{MW} := \left(\frac{\delta(u_.)}{\int_0^T G(t,T)\,dt} - \frac{\int_0^T\int_s^T u_s\, u_t\, D_sG(t,T)dtds}{\left(\int_0^T u_t\, G(t,T)\,dt\right)^2}\right), \tag{5.41}$$

which is the Malliavin weight of the Delta of a European option under the general stochastic volatility model.

5.4 Applications

Having the general formula for the Delta at hand, one can drive the formula of Delta for particular stochastic volatility models. In this section, the Proposition 5.5 is applied to particular cases, Stein & Stein and Heston models and obtained the Deltas of these two models.

5.4.1 Stein and Stein Model

In this section the main result applied to compute the delta of a European option under the Stein & Stein model. The model assumes an Ornstein-Uhlenbeck process for the volatility dynamics and is defined by the following system of equations

$$\frac{dS_t}{S_t} = rdt + V_t dW_t, \tag{5.42}$$

$$dV_t = \gamma\left(\Theta - V_t\right)dt + \kappa dZ_t, \tag{5.43}$$

where

$$dZ_t = \left[\rho dW_t + \sqrt{1-\rho^2}dW_t^1\right],$$

$$\langle dW_t, dZ_t\rangle = \rho dt,$$

for $t \in [0, T]$. Here, S_t and V_t denote the stock price and volatility process with initial values S_0 and V_0. Further, r, γ, Θ and κ are fixed constants and W_t and W_t^1 are two independent Wiener processes. In terms of the general SV model given by Equations (5.1) and (5.2) we have

$$\sigma\left(t, V_t\right) = V_t, \tag{5.44}$$

$$u\left(t, V_t\right) = \kappa\left(\Theta - V_t\right), \tag{5.45}$$

$$v\left(t, V_t\right) = \kappa. \tag{5.46}$$

Proposition 5.6. *Let $u = 1$ and the interest rate r_t is constant in Proposition 5.5. Then, the Delta of Stein & Stein model is,*

$$\Delta = \frac{e^{-rT}}{S_0}\mathbb{E}_Q\left[f\left(S_T\right)\left(\frac{W_T}{\int_0^T G\left(t,T\right)dt} - \frac{\int_0^T\int_s^T D_s G(t,T)dtds}{\left(\int_0^T G\left(t,T\right)dt\right)^2}\right)\right].$$

Proof. By Lemma 5.4 the first variation Y_t of V_t can be compute as follow,

$$Y_t = \exp\left(-\int_0^t \gamma ds\right) = \exp\left(-\gamma t\right), \forall t \in [0, T],$$

and by using the relation between the first variation and the Malliavin derivative of V_t,

$$D_s V_t = \kappa\rho\exp\left(-\gamma\left(t-s\right)\right)$$

is computed for $s \leq t \leq T$. Next, the function $G(t, T)$ is computed as;

$$\begin{aligned} G\left(t,T\right) &= V_t + \int_t^T D_t V_u dW_u - \int_t^T \left(D_t V_u\right) V_u du \\ &= V_t + \int_t^T \kappa\rho\exp\left(-\gamma\left(u-t\right)\right)dW_u - \int_t^T \kappa\rho\exp\left(-\gamma\left(u-t\right)\right)V_u du \\ &= V_t + \kappa\rho e^{\gamma t}\left(\int_t^T \exp\left(-\gamma u\right)dW_u - \int_t^T \exp\left(-\gamma u\right)V_u du\right), \end{aligned} \tag{5.47}$$

and compute the Malliavin derivative of $G(t,T)$ using Equation (5.37) as

$$\begin{aligned} D_sG(t,T) &= D_sV_t - \int_t^T \left(\frac{\partial\sigma\left(u,V_u\right)}{\partial y}\right)^2 D_tV_uD_sV_udu \\ &= \kappa\rho e^{-\gamma(t-s)}\left(1+\frac{\kappa\rho\left(1+e^{-2T\gamma}\right)}{2\gamma}\right). \end{aligned} \tag{5.48}$$

□

As a consequence of Proposition 5.6, the Malliavin weight of Delta of a European type option under the Stein & Stein model can be given as in the following remark.

Remark 5.3. Let the Delta of a European type of option has a Delta has an expression as in the Proposition 5.6. Then The Malliavin weight is

$$\Delta_{MW} = \left(\frac{W_T}{\int_0^T G\left(t,T\right)dt} - \frac{\int_0^T\int_s^T D_sG(t,T)dtds}{\left(\int_0^T G\left(t,T\right)dt\right)^2}\right),$$

or more explicitly,

$$\begin{aligned} \Delta_{MW}^H = & \left(\frac{W_T}{\int_0^T\left(V_t+\kappa\rho e^{\gamma t}\left(\int_t^T e^{-\gamma u}\left(dW_u-V_udu\right)\right)\right)dt}\right. \\ & \left. -\frac{\int_0^T\int_s^T D_sG(t,T)dtds}{\left(\int_0^T\left(V_t+\kappa\rho e^{\gamma t}\left(\int_t^T e^{-\gamma u}\left(dW_u-V_udu\right)\right)\right)dt\right)^2}\right). \end{aligned} \tag{5.49}$$

In [67], it is assumed that $\rho = 0$. This assumption simplifies the calculations and the result is given by the following corollary.

Corollary 5.7. *Under the assumption that W_t and Z_t are two independent Brownian motion processes, the delta of a European option under the Stein & Stein model is given by*

$$\Delta = \frac{e^{-rT}}{S_0}\mathbb{E}\left[f\left(S_T\right)\left(\frac{W_T}{\int_0^T V_tdt}\right)\right]. \tag{5.50}$$

5.4.2 Heston Model

In this section, the delta of a European option under the Heston model computed by using Proposition 5.5. Heston stochastic volatility model is a special case of the general stochastic volatility model given in Equations (5.1) and (5.2) with the choice of functions as:

$$\sigma\left(t,V_t\right) = \sqrt{V_t}, \tag{5.51}$$

$$u\left(t,V_t\right) = \kappa\left(\Theta - V_t\right), \tag{5.52}$$

$$v\left(t,V_t\right) = \epsilon\sqrt{V_t}. \tag{5.53}$$

Formally, Heston model is given as follows:

$$\frac{dS_t}{S_t} = rdt + \sqrt{V_t} dW_t, \tag{5.54}$$
$$dV_t = \kappa (\Theta - V_t) dt + \epsilon \sqrt{V_t} dZ_t, \tag{5.55}$$

where

$$dZ_t = \left[\rho dW_t + \sqrt{1-\rho^2} dW_t^1\right],$$
$$\langle dW_t, dZ_t \rangle = \rho dt.$$

Further, r denotes the risk free rate, κ is the mean-reversion rate, Θ reflects the long-term average variance and ϵ is the volatility of variance. As we know the explicit expressions of the functions $v(t, Y_t)$, $u(t, Y_t)$ and $\sigma(t, Y_t)$, we can compute the function $G(t, T)$ defined in Definition 5.2 with Equation (5.33) and then apply Proposition 5.3 to compute the Delta within Heston model. Substituting Equations (5.51), (5.52) and (5.53) into Equation (5.33),

$$G(t,T) = \sqrt{V_t} + \int_t^T \frac{1}{2\sqrt{V_v}} D_t V_v dW_v - \frac{1}{2} \int_t^T D_t V_v dv. \tag{5.56}$$

is obtained. By employing Itô's Lemma, a solution to Equation (5.54) is obtained as

$$S_T = S_0 \exp\left\{\int_0^T \sqrt{V_t}\, dW_t + \int_0^T \left(r - \frac{1}{2} V_t\right) dt\right\}. \tag{5.57}$$

Having $G(t, T)$ and S_T at hand, one can apply Proposition 5.3 and state the result as in the following proposition.

Proposition 5.8. *Let the dynamics of $\{S_t\}_{0 \le t \le T}$ be given by Equations (5.54) and (5.55). Let $G(t, T)$ be defined in Equation (5.56) and let $u_t = 1$ for all $t \in [0, T]$. Moreover, suppose that the riskless interest rate is constant. Then, for $s \le t \le v \le T$*

$$\Delta = \frac{e^{-rT}}{S_0} \mathbb{E}\left[f(S_T)\left(\frac{W_T}{\int_0^T G(t,T)\,dt} - \frac{\int_0^T \int_s^T D_s G(t,T)\,dtds}{\left(\int_0^T G(t,T)\,dt\right)^2}\right)\right],$$

where

$$
\begin{aligned}
D_sG(t,T) &= \frac{1}{2\sqrt{V_t}}D_sV_t - \frac{1}{4}\int_t^T \frac{1}{V_v}D_tV_vD_sV_v\,dv \\
&+ \frac{1}{2}\int_t^T \left(\frac{1}{\sqrt{V_v}}D_s(D_tV_v) - \frac{1}{2V_v\sqrt{V_v}}D_sV_vD_tV_v\right)dW_v \\
&- \frac{1}{2}\int_t^T \left(D_s(D_tV_v) - \frac{1}{2V_v}D_sV_vD_tV_v\right)dv, \\
&= \frac{1}{2\sqrt{V_t}}D_sV_t + \int_t^T \frac{1}{2\sqrt{V_t}}\left[D_s(D_tV_v) - \frac{1}{2V_t}D_sV_vD_tV_v\right]dW_v \\
&- \frac{1}{2}\int_t^T D_s(D_tV_v)\,dv, \\
D_tV_v &= \rho\,\epsilon\sqrt{V_t}\exp\left(\frac{\rho\,\epsilon}{2}\left(\int_t^v \frac{1}{\sqrt{V_s}}dW_s - \frac{\rho\,\epsilon}{4}\int_t^v \frac{1}{V_s}\,ds\right) - \kappa(v-t)\right), \\
D_s(D_tV_v) &= \frac{D_tV_vD_sV_t}{2V_t} \\
&+ \frac{\rho\,\epsilon}{2}D_tV_v\left(\frac{\rho\,\epsilon}{4}\int_t^v \frac{1}{V_u^2}D_sV_u\,du + \frac{1}{\sqrt{V_s}} - \frac{1}{2}\int_t^v \frac{1}{V_u\sqrt{V_u}}\,dW_u\right).
\end{aligned}
$$

As an immediate result of the Proposition 5.8 the following remark is obtained for the Malliavin weight of a European option under the Heston stochastic volatility model.

Remark 5.4. The Malliavin weight of a European option under the assumption of the stock price is a Heston process is

$$
\Delta_{MW} = \left(\frac{W_T}{\int_0^T G(t,T)\,dt} - \frac{\int_0^T\int_s^T D_sG(t,T)\,dtds}{\left(\int_0^T G(t,T)\,dt\right)^2}\right) \tag{5.58}
$$

or a more appropriate way,

$$
\Delta_{MW}^H = \Delta_1^H - \Delta_2^H,
$$

where

$$
\begin{aligned}
\Delta_1^H &= \frac{W_T}{\int_0^T\left(\sqrt{V_t} + \int_t^T \frac{1}{2\sqrt{V_v}}D_tV_vdW_v - \frac{1}{2}\int_t^T D_tV_vdv\right)dt}, \\
\Delta_2^H &= \frac{\int_0^T\int_s^T\left(\frac{D_sV_t}{2\sqrt{V_t}} + \int_t^T \frac{1}{2\sqrt{V_t}}\left[D_s(D_tV_v) - \frac{D_sV_vD_tV_v}{2V_t}\right]dW_v - \frac{\int_t^T D_s(D_tV_v)\,dv}{2}\right)dtds}{\left(\int_0^T\left(\sqrt{V_t} + \int_t^T \frac{1}{2\sqrt{V_v}}D_tV_vdW_v - \frac{1}{2}\int_t^T D_tV_vdv\right)dt\right)^2},
\end{aligned}
$$

CHAPTER 6

CONCLUSION AND OUTLOOK

It is very important for the traders to understand how option prices behave within the change in the parameters of the model. If the option has a closed form solution for pricing, the Greeks can be computed analytically. However, driving a close form pricing formula is not possible for all cases, but still the traders need a thorough knowledge of how option values react in response to changes in model parameters during the life of the option. Because, the Greeks provide necessary information to manage the risks of options and portfolios.

There are two approaches in computation of the Greeks: if an analytic solution is available for option pricing, they can be computed explicitly by differentiating the solution directly. However, due to complicated rules of some dynamic systems, analytical solutions can be found rarely. In this case, a numerical approximation can be performed by simulating the evolution of the system in a computer programmed with the governing laws. But, these two approaches are not perfect indeed. In the first place, the payoff function can be complex and carrying out the differentiation can be unfavorable. Moreover, if the option has no analytic solution and the Greeks are estimated by numerical methods, such as "finite difference, pathwise derivative estimation and likelihood methods", the estimation may be computationally expensive and the results will be inaccurate due to estimation error of expectation and the derivative of the payoff function.

This is why the Malliavin calculus become popular in computation of Greeks in recent years. In this method, due to a famous result known as "infinite dimensional integration by parts formula", one can skip having to evaluate the derivative of the payoff function. Instead of evaluating the derivative, computing the expected value of the option's payoff multiplied with a weight, called "Malliavin weight" is enough. This expectation can either be computed explicitly or estimated by using Monte Carlo simulations. Since there is no need to estimate the derivative, each way will be less expensive than evaluating the payoff function and assign this for computing the partial derivatives. Further more, with this method, all Greeks can be written as an expected value of the payoff times a unique weight and these weights are independent from the payoff function which is a great advantage in computations. Due to this independence feature of payoff function and Malliavin weight, once a Monte Carlo algorithm constructed for general options, it can be used for all type of payoff functions. Hence, the method efficiency is increased for options that have a discontinuous payoff function.

If the option has smooth payoff functions such as European type options, the Malliavin calculus give comparable results to finite difference, pathwise derivative and the likelihood method (the comparisons are presented in Chapter 4). For example, if the density of the underlying asset is known, both the Malliavin calculus and the likelihood method give the same weight function [39]. Thus, the Malliavin calculus method can be thought as an extended version of the likelihood method. The most valuable advantage of the Malliavin calculus method is that, it is applicable to both complicated continuous and discontinuous payoff functions. This method also can be used when dealing with underlying assets whose density is not known explicitly, such as Asian type options [23] and the results are presented in Chapter 4.

The numerical tests in Chapter 4 indicate promising convergence results for European type of options when the number of simulations increases, so the algorithm of the Malliavin calculus seems to be applicable in computation of Greek problems. Moreover, by the comparison results it is seen that, this method has a faster convergence Monte-Carlo simulations for the Greeks.

The Malliavin calculus can also be applied to the options which are written on assets that follows stochastic volatility processes and the results are presented in Chapter 5. Theoretically, the results are promising but they did not prove to yield any satisfactory results to practitioners because practitioners concern with numerical aspects. The implementation to stochastic volatility models underline the fact that the computation algorithm is hard to deal because of the double integrals inherent within the Greek (Delta) formula. However, it is believed that once the algorithm is finished, it most certainly would have enabled the practitioners to obtain more stable and more accurate Delta results. Indeed, the results presented for Black-Scholes- Merton assumption seem to be reasonably in line with actual results when the number of simulation is relatively high, but this will mechanically increase the computation time because of the double integrals.

Note that in this thesis, the interest rate assumed to be constant but it might not be constant. Hence, the famous Black-Scholes-Merton model and stochastic volatility models are extended by assuming the interest rate is a deterministic function or a stochastic process. An extended version of the Black-Scholes-Merton and stochastic volatility models which allows for the rate being a stochastic process can be found in [25], [29], [38], [47].

If the interest rate assumed to be a deterministic function, the only change on computations of the Greeks will be the discount factor. If the function is numerically integrable, the discount factor can be computed easily. However, every function appears in practice are not necessarily has to be integrable analytically. In this case, some approximation methods are used in computations. Basic numerical methods for these kind of integrals are midpoint and triangle rules. Both of them use the definition of Riemann integral. For the given deterministic interest rate r_t on the interval $[0, T]$,

$$\int_0^T r_t d\,s = \sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} r\left(s\right) d\,s. \tag{6.1}$$

Applying midpoint and triangle rule methods to Equation (6.1), the integral can be computed approximately as follows,

$$\begin{aligned} \int_0^T r_t ds &\approx \sum_{i=0}^{N-1} r_{t_i}\left(t_{i+1} - t_i\right) \quad \text{(triangle rule)} \\ &\approx \sum_{i=0}^{N-1} r_{\left(\frac{t_i + t_{i+1}}{2}\right)}\left(t_{i+1} - t_i\right) \quad \text{(midpoint rule).} \end{aligned}$$

On the other hand, if the interest rate assumed to be a stochastic process, the computation of the Greeks become more technical. Consider Heston-Hull-White model [38] which replace the constant interest rate in Heston model by Hull-White short term stochastic interest rate model. The model represented by the following stochastic differential equations:

$$\begin{aligned} \frac{dS_t}{S_t} &= \left(r_t\right) dt + \sqrt{V_t} dW_t^1 && (6.2) \\ dV_t &= \kappa\left(\Theta - V_t\right) dt + \epsilon\sqrt{V_t} dW_t^2 && (6.3) \\ dr_t &= \lambda\left(\Theta_r\left(t\right) - r_t\right) dt + \zeta dW_t^3 && (6.4) \\ \left\langle dW^i, dW^j \right\rangle &= \rho_{ij} dt, \quad i, j = 1, 2, 3, && (6.5) \end{aligned}$$

with initial prices S_0, V_0 and r_0. Note that in this system; κ is the mean-reversion rate, θ reflects the long-term average variance, ϵ is the volatility of variance, λ is the mean reversion speed and determines the speed of diverging from θ, the function $\theta_r\left(t\right)$ is used to recover the initial term structure of interest rates at time and $\zeta > 0$ is the volatility parameter. In this stochastic differential equation system; Equation (6.2) describes the evolution of the price of an underlying asset, the volatility is given with $\sqrt{V_t}$ where V_t evolves as a Cox-Ingersoll-Ross mean reverting process determined by Equation (6.3), which is the dynamic from the Heston stochastic volatility model, and the short rate $r\left(t\right)$ is given by a mean reverting Ornstein Uhlenbeck process given by Equation (6.4) with time dependent but deterministic mean reversion level $\Theta_r\left(t\right)$ which is known as the Hull-White model. The correlation between Brownian motions W^1, W^2 and W^3 are given with $\rho_{i,j}, i, j = 1, 2, 3$. Note that the computation of Greeks under this model assumptions is a difficult issue. Even a formula can be found for a Greek of this model, the simulation will require a significant additional amount of time for a result because of the simulating effort to evaluate the volatility and interest rate at each time step.

There are many possible extensions and applications of this thesis, especially to more complicated models than the Black-Scholes-Merton. One extension of research is to extend the results in Chapter 5 to other Geeks and build efficient algorithms for these Greeks. Indeed, deriving formulas for the Greeks of stochastic volatility models and writing algorithms will be an interesting enlargement. Another interesting future work is to compute the Greeks under the assumption of deterministic and/or stochastic interest rate rather than being constant, which will increase the computational complexity of the Greeks.

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