LYAPUNOV TYPE INEQUALITIES AND THEIR APPLICATIONS FOR LINEAR AND NONLINEAR SYSTEMS UNDER IMPULSE EFFECT

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submitted by **ZEYNEP KAYAR** in partial fulfillment of the requirements for the degree of **Doctor of Philosophy in Mathematics Department, Middle East Technical University** by,

Prof. Dr. Canan Özgen
Dean, Graduate School of **Natural and Applied Sciences**

Prof. Dr. Mustafa Korkmaz
Head of Department, **Mathematics**

Prof. Dr. Ağacık Zafer
Supervisor, **Mathematics Department, METU**

Examining Committee Members:

Prof. Dr. Hüseyin Şirin Hüseyin
Mathematics Department, Atılım University

Prof. Dr. Ağacık Zafer
Mathematics Department, METU

Prof. Dr. Hasan Taşeli
Mathematics Department, METU

Prof. Dr. Hüseyin Bereketoğlu
Mathematics Department, Ankara University

Prof. Dr. Billur Kaymakçalan
Mathematics and Comp. Sci. Department, Çankaya University

Date:

I hereby declare that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.

Name, Last Name: ZEYNEP KAYAR

Signature :

ABSTRACT

LYAPUNOV TYPE INEQUALITIES AND THEIR APPLICATIONS FOR LINEAR AND NONLINEAR SYSTEMS UNDER IMPULSE EFFECT

Kayar, Zeynep
Ph.D., Department of Mathematics
Supervisor : Prof. Dr. Ağacık Zafer

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In this thesis, Lyapunov type inequalities and their applications for impulsive systems of linear/nonlinear differential equations are studied. Since systems under impulse effect are one of the fundamental problems in most branches of applied mathematics, science and technology, investigation of their theory has developed rapidly in the last three decades. In addition, Lyapunov type inequalities have become a popular research area in recent years due to the fact that they provide not only better understanding of the qualitative nature of the solutions of ordinary and impulsive systems, for instance oscillation, disconjugacy, stability and asymptotic behavior of solutions, but also deeper analysis for boundary and eigenvalue problems.

This thesis consists of 7 chapters. Chapter 1 is introductory and contains detailed literature review, and brief information about the linear systems of impulsive differential equations and Hamiltonian systems. The main contributions of the thesis, which are presented in the second and third chapters, are to derive Lyapunov type inequalities for the linear $2n \times 2n$ Hamiltonian system with impulsive perturbations and to prove the existence and uniqueness criteria for the solutions of inhomogenous boundary value problems to such systems, respectively. Since changing the impulsive perturbation or assuming different conditions on the impulses leads to different inequalities, presence of the impulse effect provides various Lyapunov type inequalities. This shows that the systems of impulsive equations is richer and more fruitful

in the applications than the systems of ordinary differential equations and that is why we are interested in these systems. Besides, the obtained inequalities are new even in the nonimpulsive case and therefore they improve and generalize the previous ones existing in the literature. In Chapter 3, the connection, which has not been noticed even for the nonimpulsive case, between Lyapunov type inequalities and boundary value problems has been revealed for the first time and two existence and uniqueness criteria for the solutions of inhomogenous BVPs are proved by using the Lyapunov type inequalities obtained in the previous chapter. Furthermore, the unique solution of inhomogenous BVPs is expressed in terms of Green's function (pair) and the properties of Green's function (pair) are listed. Chapter 4 is devoted to the stability theory, which is the application of Lyapunov type inequalities, for the linear planar Hamiltonian systems with impulsive perturbations. Two pairs of stability criteria are obtained, one of which is the generalization of the results obtained for systems of ordinary differential equations to the impulsive case and the latter is new and alternative to the former. In Chapter 5 and 6, we establish several Lyapunov type inequalites, some of which are generalizations of the nonimpulsive case while the others are new for nonlinear and quasilinear impulsive systems, respectively. As an application of Lyapunov type inequalities, we investigate disconjugacy intervals and study the asymptotic behaviour of oscillatory solutions for the systems under considerations and find a lower bound for the eigenvalues of the associated eigenvalue problems. The last chapter serves as a conclusion and is a summary of our findings.

Keywords: Lyapunov Type Inequalities, Impulsive Differential Equations, Boundary Value Problems, Stability

ÖZ

İMPALS ETKİSİ ALTINDAKİ LİNEER VE LİNEER OLMAYAN SİSTEMLER İÇİN LYAPUNOV TİPİ EŞİTSİZLİKLER VE UYGULAMALARI

Kayar, Zeynep
Doktora, Matematik Bölümü
Tez Yöneticisi : Prof. Dr. Ağacık Zafer

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Bu tezde impalsif lineer/lineer olmayan diferansiyel denklem sistemleri için Lyapunov tipi eşitsizlikler ve uygulamaları çalışılmıştır. İmpals etkisi altındaki sistemler uygulamalı matematik, bilim ve teknolojinin çoğu dalının temel problemlerinden oldukları için, son otuz yılda bu sistemlerin teorisinin incelenmesi hızlı bir şekilde gelişmiştir. Lyapunov tipi eşitsizlikler ise sadece adi ve impalsif denklem sistemlerinin çözümlerinin salınım, konjuge olmama (diskonjuge), kararlılık, asimptotik davranış gibi niteliksel yapılarının daha iyi anlaşılmasını değil aynı zamanda da sınır ve özdeğer problemlerinin daha derin analiz edilmesini sağladıkları için son yıllarda popüler araştırma alanı haline gelmişlerdir.

Bu tez 7 bölümden oluşmaktadır. Birinci bölüm giriş niteliğinde olup detaylı literatür taraması ve impalsif lineer diferansiyel denklem sistemleri ve Hamiltonian sistemler hakkında kısa bilgiler içermektedir. Bu tezin temel katkıları, ikinci ve üçüncü bölümde sunulan, sırasıyla, impalsif perturbasyonlu lineer $2n \times 2n$ Hamiltonian sistemler için Lyapunov tipi eşitsizlikler elde etmek ve bu sistemlere karşılık gelen homojen olmayan sınır değer problemlerinin çözümlerinin varlık teklik kriterlerini ispatlamaktır. İmpalsif perturbasyonun değiştirilmesi ya da impals üzerinde farklı koşulların kabul edilmesi muhtelif eşitsizliklere sebep olduğu için impals etkisinin varlığı çeşitli Lyapunov tipi eşitsizlikler vermektedir. Bu ise impalsif diferansiyel denklem sistemlerinin uygulamalarda adi diferansiyel denklem sistemlerinden daha zengin ve daha verimli olduğunu ve neden bu sistemlerle ilgilendiğimizi göstermektedir. Üstelik elde edilen bu eşitsizlikler impals olmayan durumda bile yenidirler ve bu yüzden literatürde var olan eski eşitsizlikleri geliştirmiş ve genelleştirmişlerdir. 3. bölümde Lyapunov tipi eşitsizlikler ve sınır değer problemleri arasındaki impals olmayan durumda bile fark edilmeyen bağlantı ilk kez ortaya çıkarılmış ve homojen olmayan sınır değer problemleri için önceki bölümde elde edilen Lyapunov tipi eşitsizlikler kullanılarak iki tane varlık teklik kriteri ispat edilmiştir. Ayrıca homojen olmayan sınır değer probleminin tek çözümü Green's fonksiyonu (çifti) cinsinden ifade edilmiş ve Green's fonksiyonunun (çiftinin) özellikleri listelenmiştir. 4. bölüm Lyapunov tipi eşitsizliklerin uygulaması olan kararlılık teorisine ayrılmıştır. Birinci çifti adi diferansiyel denklem sistemleri için elde edilen sonuçların impals içeren duruma iki farklı şekilde genelleştirilmesi ve ikinci çifti yeni ve birinciye alternatif olan iki çift kararlılık kriteri elde edilmiştir. 5. ve 6. bölümde, sırasıyla, lineer olmayan ve yarı lineer (quasilineer) impalsif sistemler için bazısı impals olmayan durumların genelleştirilmesi iken diğerleri yeni olan çeşitli Lyapunov tipi eşitsizlikler oluşturulmuştur. Lyapunov tipi eşitsizliklerin uygulaması olarak, ele alınan sistemlerin konjuge olmama aralıkları incelenmiş, salınımlı çözümlerin asimptotik davranışı çalışılmış ve ilgili özdeğer probleminin özdeğerleri için bir alt sınır bulunmuştur. Son bölüm sonuç niteliğinde olup bu tezde yaptıklarımızın özeti şeklindedir.

Anahtar Kelimeler: Lyapunov Tipi Eşitsizlikler, İmpalsif Diferansiyel Denklemler, Sınır Değer Problemleri, Kararlılık

To the memory of my grandmother and my aunt,

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to my parents Hamiyet and Hasan and my brother Ali Rıza,

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CHAPTER 1

INTRODUCTION

Describing the evolution processes of several real world problems, which have a sudden change in their states, by using ordinary differential equations is not adequate due to the fact that they are subject to short time perturbation (harvesting, diseases, wars, etc.) whose duration is negligible in comparison with the duration of the whole process. Therefore, in the mathematical simulation of such processes it is convenient to assume that this change takes place momentarily or the perturbations occur immediately as impulses and the processes change their states by jump. For instance, when a hammer hits a string which is already oscillating, it experiences a rapid change of velocity [107]; a pendulum of a clock undergoes a sudden change of momentum when it crosses its equilibruim position [107]; in a real evolutionary process of the population of a given species, since the perturbation or the influence from outside occurs at certain moments as impulses, not continuosly, the population has some jumps at these moments and these jumps follow a specific pattern [70]; when the configuration space of the system collapses instantaneously because of an inelastic collision, the system suffers a sudden change of kinetic energy [28]; when passing from one optical media to another, a ray of light splits into reflected and refracted rays [28].

Since discontinuity is defined as an instantenous interruption at anytime of a continuous process, in order to analyse dynamical systems with discontinous trajectories, or with impulse effect, it becomes necessary to introduce *impulsive differential equations*, or sometimes *differential equations under impulse effect*, arising from the real world phenomena and describing the dynamics of processes in which sudden, discontinuous jumps occur at the points of impulses or jumps.

Impulsive differential equations have attracted a great deal of attention and the theory of it has developed rapidly in the last three decades because they are appropriate description of simulation processes and various phenomena encountered in mechanical systems with impact [90, 37], biological systems such as heart beats, blood flows [3, 5], population dynamics [32, 122, 31], theoretical physics [65, 20, 92], mathematical economy [121, 94], electrical technology [62, 61], ecology [72, 123], biology [73, 53], epidemiology [97, 111], chemistry [75], engineering [30, 17], control theory [96, 52], medicine [48, 74], networks (such as food webs, communication networks, social networks, power grids, cellular networks, World Wide Web, metabolic systems, disease transmission networks, neural networks) [71] and chaos synchronization (for example secure communication, parallel image processing) [66, 23]. Moreover it has been recognized that impulsive differential equations not only generalize the corresponding theory of ordinary differential equations [25, 45, 44, 76] but also provide more mathematical description for many real world phenomena. Therefore, impulsive differential equations are richer and more fruitful in applications compared to the corresponding theory of ordinary differential equations. However, the dynamic behaviour of systems under impulse effects is more complex than the behaviour of dynamical systems without impulsive effects. Although there is a large body of literature on impulsive differential equations that we can not cover completely, we want to mention the seminal books of Lakshmikantham, Baĭnov and Simeonov [60], Baĭnov and Simeonov [10, 9], and Samoĭlenko and Perestyuk [93] in which the qualitative theory such as existence and uniqueness theorems for solutions, comparison theory, stability, periodicity are investigated. In the book of Akhmet [4] in addition to the fundamental theory, the new concepts B-equivalence and chaos teory of impulsive differential equations are introduced as well.

1.1 Structure of the Thesis

The thesis is organized as follows:

In the present chapter, we provide related literature review for Lyapunov type inequalities, boundary value problems and stability and summarize the previous results obtained for the special cases of systems that we will study. Then we outline some facts about systems of impulsive differential equations and introduce Hamiltonian systems. The main result of the thesis presented in Chapter 2 is to obtain new Lyapunov type inequalities for linear $2n \times 2n$ Hamiltonian systems under impulse effect. As applications, these inequalities are used to derive disconjugacy criteria and to find lower bounds for associated eigenvalue problems. In Chapter 3, we consider inhomogenous boundary value problems (BVPs) to the linear $2n \times 2n$ Hamiltonian systems under impulse effect and establish the existence and uniqness criteria for such BVPs. Moreover we express the unique solution of the considered BVPs in terms of Green's function (pair) and properties of Green's function (pair) is listed. Chapter 4 is devoted to derive the stability criteria for the linear planar Hamiltonian systems under impulse effect by using the connection between stability theory and Lyapunov type inequalities. In Chapter 5 and Chapter 6, we consider nonlinear and quasilinear systems with impulsive perturbations, respectively. For such systems Lyapunov type inequalities are obtained and their applications in studying qualitative nature of the solutions such as disconjugacy criteria, lower bounds for associated eigenvalue problems, boundedness and asymptotic behaviour of oscillatory solutions are demonstrated. Finally, in Chapter 7, we summarize our findings in this thesis.

1.2 Literature Review

Since we are interested in different subjects or different systems in each chapter, the detailed literature review for all chapters is given in this section. In the sequel we assume $m^+(t) = \max\{m(t), 0\},\ \ m_i^+ = \max\{m_i, 0\},\ \ i \in \mathbb{N}$.

1.2.1 Lyapunov Type Inequalities For Linear Hamiltonian Systems Under Impulse Effect

Now we want to give a related literature review for Lyapunov type inequalities obtained for linear equations and systems with or without impulses.

Let us consider the following second order ordinary differential equation

$$x'' + q(t)x = 0. \tag{1.1}$$

In a celebrated paper of 1893, Lyapunov [69] prove the following result for (1.1) in attemp to find sufficient conditions for the stability of the related periodic equation to (1.1).

Theorem 1.2.1 ([69]) *Let*

$$q(t) \geq 0,\ q(t) \not\equiv 0.$$

If $x(t)$ is a nontrivial solution of (1.1) *with $x(t_1) = 0 = x(t_2)$, where $t_1, t_2 \in \mathbb{R}$ with $t_1 < t_2$ are consecutive zeros, i.e $x(t) \neq 0$ for $t \in (t_1, t_2)$, then the so-called Lyapunov inequality*

$$(t_2 - t_1) \int_{t_1}^{t_2} q(t)dt > 4, \tag{1.2}$$

holds.

Inequality (1.2) is the best possible in the sense that if the constant 4 in (1.2) is replaced by any larger constant, then there exists an example of (1.1) for which (1.2) no longer holds, see [54, 13]. After the initiated work of Lyapunov [69], many authors have paid a considerable attention to Lyapunov type inequalities and various proofs and generalizations or improvements have appeared in the literature such as [13, 114, 46, 79, 57, 80, 45, 47, 36, 64, 26, 34, 59, 14]. For a comprehensive exibition of these results we refer two surveys [24, 104] and references therein. We should also mention the following theorems to clarify the main generalizations or improvements of Lyapunov type inequalities.

When analyzing stability of the related periodic equation to (1.1), Borg [13] changed the nonnegativity condition of $q(t)$ by nonnegative integral of $q(t)$ and improved inequality (1.2).

Theorem 1.2.2 ([13]) *Let*

$$\int_{t_1}^{t_2} q(t)dt \geq 0,\ q(t) \not\equiv 0.$$

If $x(t)$ is a nontrivial solution of (1.1) *with $x(t_1) = 0 = x(t_2)$, where $t_1, t_2 \in \mathbb{R}$ with $t_1 < t_2$ and $x(t) \neq 0$ for $t \in (t_1, t_2)$, then we have the Lyapunov type inequality*

$$\int_{t_1}^{t_2} |q(t)|dt > \frac{4}{t_2 - t_1}. \tag{1.3}$$

Wintner [114] used the same conditions of Theorem 1.2.2 and obtained the following better inequality by replacing $|q(t)|$ by $q^+(t) = \max\{q(t), 0\}$ whereas Krein [57] established the same result as in [114] while studying the stability of the related periodic equation to (1.1).

Theorem 1.2.3 ([114, 57]) *Let*

$$\int_{t_1}^{t_2} q(t)dt \geq 0, \ q(t) \not\equiv 0.$$

If $x(t)$ is a nontrivial solution of (1.1) *with $x(t_1) = 0 = x(t_2)$, where $t_1, t_2 \in \mathbb{R}$ with $t_1 < t_2$ and $x(t) \neq 0$ for $t \in (t_1, t_2)$, then we have the Lyapunov type inequality*

$$\int_{t_1}^{t_2} q^+(t)dt > \frac{4}{t_2 - t_1}.$$

Hartman [45] has generalized the classical Lyapunov inequality for the linear differential equation

$$(p(t)x')' + q(t)x = 0. \tag{1.4}$$

as follows.

Theorem 1.2.4 ([45]) *Let $p(t) > 0$. If $x(t)$ is a nontrivial solution of* (1.4) *with $x(t_1) = 0 = x(t_2)$, where $t_1, t_2 \in \mathbb{R}$ with $t_1 < t_2$ and $x(t) \neq 0$ for $t \in (t_1, t_2)$, then we have the Lyapunov type inequality*

$$\int_{t_1}^{t_2} q^+(t)dt > \frac{4}{\displaystyle\int_{t_1}^{t_2} p^{-1}(t)dt}.$$

The results for (1.1) in [59, 14] are worth mentioning due to their contribution to these subject. In [59] it is shown that

$$\int_{t_1}^{t_0} q^+(t)dt > \frac{1}{t_0 - t_1}, \qquad \int_{t_0}^{t_2} q^+(t)dt > \frac{1}{t_2 - t_0},$$

where $t_0 \in (t_1, t_2)$ such that $y'(t_0) = 0$. Hence

$$\int_{t_1}^{t_2} q^+(t)dt > \frac{1}{t_0 - t_1} + \frac{1}{t_2 - t_0} = \frac{t_2 - t_1}{(t_0 - t_1)(t_2 - t_0)} \geq \frac{4}{t_2 - t_1}.$$

In [14] the authors obtained

$$\left|\int_{t_1}^{t_2} q(t)dt\right| > \frac{4}{t_2 - t_1}$$

which implies (1.3).

Although Lyapunov-type inequalities are well developed for ordinary differential equations (ODEs) after the appearance of Lyapunov's well-known inequality, the impulsive version of it has not been studied until 2008. The second-order differential equations under impulse effect

$$\begin{aligned} &(p(t)x')' + q(t)x = 0, && t \neq \tau_i \\ &x(\tau_i^+) = k_i x(\tau_i^-), \quad (px')(\tau_i^+) = -l_i x(\tau_i^-) + k_i (px')(\tau_i^-), && i \in \mathbb{N}. \end{aligned} \tag{1.5}$$

was considered first in [43] and the extended Lyapunov-type inequality is given therein.

Theorem 1.2.5 ([43]) *Let* $p(t) > 0$ *and* $k_i \neq 0$ *for* $i \in \mathbb{N}$. *If* $x(t)$ *is a nontrivial solution of* (1.5) *with* $x(t_1^+) = 0 = x(t_2^-)$, *where* $t_1, t_2 \in \mathbb{R}$ *with* $t_1 < t_2$ *and* $x(t) \neq 0$ *for* $t \in (t_1, t_2)$, *then we have the Lyapunov type inequality*

$$\left[\int_{t_1}^{t_2} \frac{1}{p(t)} dt\right] \left[\int_{t_1}^{t_2} q^+(t) dt + \sum_{\tau_i \in [t_1, t_2)} \left(\frac{l_i}{k_i}\right)^+\right] > 4.$$

To the best of our knowledge, the first result concerning Hamiltonian system

$$x' = A(t)x + B(t)u, \quad u' = -C(t)x - A^T(t)u, \tag{1.6}$$

is due to Krein [58]. While investigating the stability criterion for the related periodic system to the system (1.6), Krein proved a Lyapunov type inequality. When $n = 1$, i.e., for

$$x' = a(t)x + b(t)u, \quad u' = -c(t)x - a(t)u \tag{1.7}$$

this inequality is reduced to the following one.

Theorem 1.2.6 ([58]) *Let* $b(t) \geq 0$ *and* $c(t) \geq 0$. *If system* (1.7) *has a solution* $(x(t), u(t))$ *with* $x(t_1) = x(t_2) = 0,\ x(t) \not\equiv 0$ *on* (t_1, t_2), *then*

$$\int_{t_1}^{t_2} |a(t)| dt + \left(\int_{t_1}^{t_2} b(t) dt\right)^{1/2} \left(\int_{t_1}^{t_2} c(t) dt\right)^{1/2} \geq 2.$$

Since the conditions are weakened, the improved version of Theorem 1.2.6 is as follows.

Theorem 1.2.7 ([41]) *Let $b(t) \geq 0$. If system (1.7) has a solution $y(t) = (x(t), u(t))$ with $x(t_1) = x(t_2) = 0,\ x(t) \not\equiv 0$, then*

$$\int_{t_1}^{t_2} |a(t)|dt + \left(\int_{t_1}^{t_2} b(t)dt\right)^{1/2} \left(\int_{t_1}^{t_2} c^+(t)dt\right)^{1/2} \geq 2. \tag{1.8}$$

While studying the stability for the periodic case, Wang derived the following Lyapunov type inequality as an alternative to (1.8).

Theorem 1.2.8 ([112]) *Let $b(t) \geq 0$. If system (1.7) has a solution $(x(t), u(t))$ with $x(t) \not\equiv 0$ on (t_1, t_2), then for some $t_0 \in (t_1, t_2)$,*

$$\left[\int_{t_1}^{t_2} b(t) \exp\left(-2\int_{t_0}^{t} a(s)ds\right) dt\right] \left(\int_{t_1}^{t_2} c^+(t)dt\right) \geq 4. \tag{1.9}$$

Theorem 1.2.7 and Theorem 1.2.8 have been extended to impulsive system

$$\begin{aligned} &x' = a(t)x + b(t)u, \quad u' = -c(t)x - a(t)u, && t \neq \tau_i \\ &x(\tau_i^+) = k_i x(\tau_i^-), \quad u(\tau_i^+) = -l_i x(\tau_i^-) + k_i u(\tau_i^-), && i \in \mathbb{N} \end{aligned} \tag{1.10}$$

in the next two theorems, respectively.

Theorem 1.2.9 ([42]) *Let $b(t) > 0$ and $k_i \neq 0$ for $i \in \mathbb{Z}$. If the impulsive system* (1.10) *has a solution $(x(t), u(t))$ with $x(t_1^+) = x(t_2^-) = 0,\ x(t) \neq 0$ on (t_1, t_2), then*

$$\int_{t_1}^{t_2} |a(t)|dt + \left(\int_{t_1}^{t_2} b(t)dt\right)^{1/2} \left(\int_{t_1}^{t_2} c^+(t)dt + \sum_{t_1 \leq \tau_i < t_2} \left(\frac{l_i}{k_i}\right)^+\right)^{1/2} \geq 2.$$

Theorem 1.2.10 ([55]) *Let $b(t) > 0$ and $k_i \neq 0$ for $i \in \mathbb{Z}$. If the impulsive system* (1.10) *has a solution $(x(t), u(t))$ with $x(t_1^+) = x(t_2^-) = 0,\ x(t) \neq 0$ on (t_1, t_2), then for some $t_0 \in (t_1, t_2)$,*

$$\left[\int_{t_1}^{t_2} b(t) \exp\left(-2\int_{t_0}^{t} a(s)ds\right) dt\right] \left[\int_{t_1}^{t_2} c^+(t)dt + \sum_{t_1 \leq \tau_i < t_2} \left(\frac{l_i}{k_i}\right)^+\right] \geq 4.$$

More recently, Tang and Zhang [99] improved and generalized the Lyapunov type inequalities (1.8) and (1.9) to the general $2n \times 2n$ system (1.6). To state their main result, the following conventions should be made.

For any $x \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$ (the space of real $n\times n$ matrices), denote the Euclidean norm of vectors and the matrix norm of matrices as

$$|x| = x^T x, \quad |A| = \max_{|x|=1} |Ax|,$$

respectively.

Definition 1.2.1 *Let $\mathbb{R}_s^{n\times n}$ be the space of all real $n \times n$ symmetric matrices. By $B \geq 0$, we mean $x^T B(t)x \geq 0$ for all $x \in \mathbb{R}^n$ and say that B is semi positive definite. More generally, by $B_1 \geq B_2$ it is meant that $B_1 - B_2 \geq 0$.*

Theorem 1.2.11 ([99]) *Let B and C be symmetric matrices and $B(t) \geq 0$. If system (1.6) has a solution $y(t) = (x(t), u(t))$ with $x(t_1) = x(t_2) = 0,\ x(t) \neq 0$ on (t_1, t_2), then*

$$\exp\left(\int_{t_2}^{t_1} |A(s)|ds\right)\left(\int_{t_1}^{t_2} |B(t)|dt\right)\left(\int_{t_1}^{t_2} |C^+(t)|dt\right) \geq 4, \tag{1.11}$$

where $C^+(t)$ is the matrix obtained from C by replacing the diagonal elements c_{ii} by $\max\{0, c_{ii}\}$ for $i = 1, 2, \ldots, 2n$.

Since $2\exp(-x/2) \geq 2 - x$ for all $x \in \mathbb{R}$, inequality (1.11) implies

$$\int_{t_2}^{t_1} |A(s)|ds + \left(\int_{t_1}^{t_2} |B(t)|dt\right)^{1/2}\left(\int_{t_1}^{t_2} |C^+(t)|dt\right)^{1/2} \geq 2,$$

and so all the previous results are also recovered by Theorem 1.2.11. As a special case of Theorem 1.2.11 one also has the following result, which gives improvements of Theorem 1.2.7 and Theorem 1.2.8.

Theorem 1.2.12 ([99], $n = 1$) *Let $b(t) \geq 0$. If system (1.7) has a solution $(x(t), u(t))$ with $x(t_1) = x(t_2) = 0,\ x(t) \neq 0$ on (t_1, t_2), then*

$$\exp\left(\int_{t_2}^{t_1} |a(s)|ds\right)\left(\int_{t_1}^{t_2} b(t)dt\right)\left(\int_{t_1}^{t_2} c^+(t)dt\right) \geq 4. \tag{1.12}$$

1.2.2 Boundary Value Problems For Linear Hamiltonian Systems Under Impulse Effect

To the best of our knowledge although many results have been obtained for linear impulsive boundary value problems by using different techiques, there is little known for the linear $2n \times 2n$ Hamiltonian system under impulse effect. The first order impulsive differential equations are considered in [81] and [67] and the existence and uniqueness criteria for linear constant coefficient impulsive boundary value problem are shown by using Green's function of the equation. Moreover the authors presented the expression of Green's functions of the related linear operator in the space of piecewise continuous functions and obtained the existence and uniqueness criteria for the nonlinear equations. The variable coefficient case is considered in [83] and [82] and Green's function of linear equations is obtained. By using the operator theory and Schaefer's fixed-point theorem, the solvability and the existence of solutions of nonlinear problem are given. After defining Green's function, some comparison results and presentation of the upper and lower solution method and the monotone iterative scheme are given in [39, 38, 2]. In [84], the authors obtained the explicit representation of the solution by providing the expression of the corresponding Green's function and by using this expression, they deduce sufficient conditions for the existence of solutions with constant sign for the boundary value problem.

Green's function of second order linear differential equations subject to linear impulse conditions at the one impulse point and periodic boundary conditions is obtained and its sign properties are investigated in [49]. Since the study of the existence of a solution of linear differential equations has an important role in the analysis of nonlinear problems, the integral representation of the general solution of second order linear impulsive boundary value problems is obtained by employing Green's function and by using this representation and monotone iterative method, which is based on finding upper and lower solutions of the equations, the uniqueness and the existence of solutions of nonlinear problem are obtained in [16, 29, 22, 110]. By defining integral representation of solution of nonlinear second order impulsive boundary value problems, which is obtained by using Green's function for linear problem, as an operator, and by using the operator theory and some fixed point theorems such as Contraction Mapping Theorem (or Banach Fixed Point Theorem), Schauder Fixed Point Thereom, Schaefer's Fixed Point Thereom and Krasnolesskii's Fixed Point Thereom, the existence and uniqness of solutions of nonlinear second order impulsive boundary value problems are investigated in [68, 8, 103, 102, 120, 50, 51].

The higher order linear impulsive boundary value problems are considered in [15, 108]. In [15], the solvability of linear impulsive equations with constant coefficient are investigated by making use of Green's function and the integral representation of the general solution. Then the nonlinear problem is considered and the existence criteria for such equations is derived by employing the method of upper and lower solution coupled with the monotone iterative technique. The theory of higher order linear impulsive boundary value problems, which is the generalization of nonimpulsive case given in [25, 77, 76], can be found in [108] or [107] in detail. In this work, Green's formula is defined and properties of Green's function are introduced. Since higher order linear differential equations can be written as a system of first order equations as long as the leading coefficient of the equation is different from zero, these results can be generealized to system of n first order equations.

The boundary value problem of system of ordinary differential equations are considered in [11] and references cited therein. In this paper adjoint form of the system of boundary value problem is introduced and the relation of solutions of the original system and its adjoint is proved. Similar to the ordinary diferential equations, it is obtained that the nonexistence of nontrivial solution of the corresponding homogenous system implies the uniqueness of the solution of nonhomogenous boundary value problem.

A boundary value problem for impulsive system is studied in [35, 12, 21, 85]. In [35], the method of upper and lower solutions is employed to obtain the existence of solutions of nonlinear impulsive boundary value problem. The necessary and sufficient conditions of the solvability of the linear nonhomogenous impulsive system is given in [12]. In [21], new results are obtained for the existence of solutions to an impulsive first-order nonlinear ordinary differential system with periodic boundary conditions by defining a suitable integral operator whose fixed-points are the solutions of the considered system. In [85], the existence and uniqueness of solutions of the nonlinear first–order impulsive differential system are considered and new results are obtained for different right hand side of the equation which may grow linearly, or sub– or super–linearly in its second argument.

1.2.3 Stability of Linear Planar Hamiltonian Systems Under Impulse Effect

As is well known, stability is not only one of the major problems encountered in theory of differential equations, but also it has attracted considerable attention due to its important role in applications. Although there is an extensive literature on this topic, we restrict ourselves on obtaining sufficient conditions for the boundedness of solutions on $\mathbb{R}$ of periodic linear equations and systems with or without impulses.

As far as we know, stability analysis of linear Hamiltonian systems with periodic coefficients goes back to Lyapunov [69]. The first stability criterion for the following second order $T-periodic$ ordinary differential equation

$$y'' + q(t)y = 0. \tag{1.13}$$

is obtained by Lyapunov [69].

Theorem 1.2.13 ([69]) *Let* $q(t+T) = q(t)$. *If*

$$\begin{aligned} &q(t) \geq 0,\ q(t) \not\equiv 0; \\ &\int_0^T q(t)dt \leq \frac{4}{T}, \end{aligned} \tag{1.14}$$

then equation (1.13) is stable.

The alternative proof of Theorem 1.2.13 can be found in the monograph [1].

Remark 1.2.1 *The condition* (1.14) *is the best possible in the sense that if it is replaced by* $\int_0^T q(t)dt < \frac{4}{T} + \epsilon$, *then the conclusion of Theorem 1.2.13 is no longer true, see [54, 13].*

Then Borg [13] changed the conditions of the Theorem 1.2.13 and obtained improved result for equation (1.13) by using different technique in the proof. In his result the nonnegativity condition of $q(t)$ is replaced by the nonnegativity integral of $q(t)$.

Theorem 1.2.14 ([13]) *Let* $q(t+T) = q(t)$. *If*

$$\int_0^T q(t)dt \geq 0,\ q(t) \not\equiv 0;$$

$$\int_0^T |q(t)|dt \le \frac{4}{T},$$

then equation (1.13) is stable.

Krein [57] make an improvement on the above results for the equation (1.13) by replacing the condition of Borg's theorem by a weaker condition, i.e $|q(t)|$ by $q^+(t) = \max\{q(t), 0\}$.

Theorem 1.2.15 ([57]) *Let* $q(t+T) = q(t)$. *If*

$$\int_0^T q(t)dt \ge 0,\ q(t) \not\equiv 0;$$

$$\int_0^T q^+(t)dt \le \frac{4}{T},$$

then equation (1.13) is stable.

The impulsive version of Theorem 1.2.15 is proven in the next theorem for the impulsive equation (1.5).

Theorem 1.2.16 ([43]) *Let equation (1.5) be* $(T, r) - periodic$. *If*

$$p(t) > 0,\ k_i \neq 0\ \textit{for}\ \ i \in \mathbb{Z};$$

$$\prod_{i=1}^{r} k_i^2 = 1;$$

$$\int_0^T q(t)dt + \sum_{i=1}^{r} \frac{l_i}{k_i} \ge 0,\ \ \textit{either}\ q(t) \not\equiv 0\ \textit{on}\ [0,T]\backslash\ \{\tau_1, \ldots, \tau_r\}$$

$$\textit{or}\ \ l_i \neq 0\ \ \textit{some}\ i \in \{1, \ldots, r\}\,;$$

$$\left[\int_0^T \frac{1}{p(t)}dt\right]\left[\int_0^T q^+(t)dt + \sum_{\tau_i \in [0,T)} \left(\frac{l_i}{k_i}\right)^+\right] \le 4,$$

then equation (1.5) is stable.

To the best of our knowledge, the first result carried over for $2n-$dimensional Hamiltonian system (1.6) is due to Krein whose main objective is to generalize Lyapunov's theorem 1.2.13 to the general case, see [56]. When $n = 1$, i.e., for system (1.7), Krein's second result is reduced to the following theorem.

Theorem 1.2.17 ([58]) *Let system (1.7) be $T-periodic$. If*

$$b(t) \geq 0, \quad c(t) \geq 0, \quad b(t)c(t) - a^2(t) \geq 0;$$

$$\int_0^T b(t)dt \int_0^T c(t)dt - \left[\int_0^T a(t)\right]^2 > 0;$$

$$\int_0^T |a(t)|dt + \left[\int_0^T b(t)dt \int_0^T c(t)dt\right]^{1/2} < 2,$$

then system (1.7) is stable.

After the inspired work of Krein, many works have been devoted to stability of Hamiltonian systems [95, 27, 40, 118, 116, 117].

An improved version of Theorem 1.2.17 is obtained by assuming weaker conditions on the coefficient functions, a, b, c, and by using Floquet Theory in the proof of the stability theorem for the first time.

Theorem 1.2.18 ([41]) *Let system (1.7) be $T-periodic$. If*

$$b(t) > 0, \quad c(t) \geq 0, \quad b(t)c(t) - a^2(t) \geq 0;$$

$$b(t)c(t) - a^2(t) \not\equiv 0;$$

$$\int_0^T |a(t)|dt + \left[\int_0^T b(t)\,dt \int_0^T c(t)dt\right]^{1/2} < 2,$$

then system (1.7) is stable.

Since Theorem 1.2.17 and Theorem 1.2.18 have limitations, in other words they are not applicable in the case $\int_0^T |a(t)|\,dt > 2$ or $\left[\int_0^T b(t)\,dt \int_0^T c(t)\,dt\right]^{1/2} > 2$, an alternative stability criterion to these theorems, which can be used in either of such cases, is obtained by Wang [112].

Theorem 1.2.19 ([112]) *Let system (1.7) be $T-periodic$. If*

$$\int_0^T \left[c(t) - \frac{a^2(t)}{b(t)}\right] dt > 0;$$

$$\exp\left(\int_0^T |a(u)|\,du\right) \left[\int_0^T b(t)dt\right]^{\frac{1}{2}} \left[\int_0^T c^+(t)dt\right]^{\frac{1}{2}} \leq 2,$$

then system (1.7) is stable.

In [42] and [55], the extended versions of Theorem 1.2.18 and Theorem 1.2.19 to system (1.10) are obtained by the following two theorems, respectively.

Theorem 1.2.20 ([42]) *Let system (1.10) be* $(T, r) - periodic$*. If*

$$\prod_{i=1}^{r} k_i^2 = 1;$$

$$\int_0^T \left[c(t) - \frac{a^2(t)}{b(t)}\right] dt + \sum_{i=1}^{r} \frac{l_i}{k_i} > 0;$$

$$\int_0^T |a(t)|dt + \left[\int_0^T b(t)dt\right]^{1/2} \left[\int_0^T c^+(t)dt + \sum_{i=1}^{r} \left(\frac{l_i}{k_i}\right)^+\right]^{1/2} \leq 2,$$

then impulsive system (1.10) is stable.

Theorem 1.2.21 ([55]) *Let system (4.3) be* $(T, r) - periodic$*. If*

$$\prod_{i=1}^{r} k_i^2 = 1;$$

$$\int_0^T \left[c(t) - \frac{a^2(t)}{b(t)}\right] dt + \sum_{i=1}^{r} \frac{l_i}{k_i} > 0;$$

$$\exp\left(\int_0^T |a(t)|\, dt\right) \left[\int_0^T b(t)dt\right]^{\frac{1}{2}} \left[\int_0^T c^+(t)dt + \sum_{i=1}^{r} \left(\frac{l_i}{k_i}\right)^+\right]^{\frac{1}{2}} \leq 2,$$

then impulsive system (1.10) is stable.

Remark 1.2.2 *With or without impulse effect, Theorem 1.2.20 and Theorem 1.2.21 are alternative to each other. Let* $x = \int_0^T |a(t)|\, dt$*. If we compare the functions* $f(x) = 2 - x$ *and* $g(x) = 2\exp(-x)$*, it can be seen that* $g(x) < f(x)$ *if* $0 < x < 1.594$*, therefore Theorem 1.2.21 is better than Theorem 1.2.20. When* $1.594 < x < 2$*, then* $f(x) < g(x)$ *and so, Theorem 1.2.20 is better than Theorem 1.2.21. For* $x > 2$*, Theorem 1.2.20 can not be used whereas Theorem 1.2.21 can.*

More recently, Tang and Zhang [99] improved the stability criterion for the system (1.7) in the sense that $|a(t)|$ is replaced by $\frac{|a(t)|}{2}$.

Theorem 1.2.22 ([99]) *Let system (1.7) be $T-periodic$. If*

$$\int_0^T \left[c(t) - \frac{a^2(t)}{b(t)}\right] dt > 0;$$

$$\exp\left(\frac{1}{2}\int_0^T |a(t)|\,dt\right)\left[\int_0^T b(t)dt\right]^{\frac{1}{2}}\left[\int_0^T c^+(t)dt\right]^{\frac{1}{2}} \leq 2,$$

then system (1.7) is stable.

Remark 1.2.3 *Let* $\int_0^T |a(t)|\,dt = x$. *Since* $2\exp(-x/2) \geq 2-x$ *for all* $x \in \mathbb{R}$, *all the previous results are also recovered by Theorem 1.2.22.*

1.2.4 Lyapunov Type Inequalities For Nonlinear Impulsive Systems

Since Lyapunov type inequalities are important tools in many applications such as oscillation theory, stability criteria for periodic differential equations, estimates for intervals of disconjugacy, asymptotic behaviour of solutions, boundary and eigenvalue problems, it is necessary to generalize Lyapunov's inequality (1.2), which is obtained for linear equation (1.1), to the nonlinear equations and systems with or without impulses.

In 1974, Eliason [34] has generalized the Lyapunov inequality for differential equations of the form

$$(r(t)y')' + p(t)f(y(t)) = 0 \tag{1.15}$$

$$y'' + m(t)y + n(t)f(y(t)) = 0 \tag{1.16}$$

as follows.

Theorem 1.2.23 ([34]) *If the equation (1.15) has a real nontrivial solution y such that $y(a) = y(b) = 0$, $a, b \in \mathbb{R}$ with $a < b$ are consecutive zeros, then we have the following Lyapunov type inequality*

$$4^2 < f_1^2(y(c))S^2(a,b;p)\left(\int_a^b \frac{1}{r(t)}dt\right)^2,$$

where $f_1(y) = \frac{f(y)}{y}$, $y'(c) = 0$ *with* $c \in (a,b)$ *and* $S(a,b;p) = \sup_{a\leq u\leq v\leq b}\int_u^v p(s)ds$.

Theorem 1.2.24 ([34]) *If the equation (1.16) has a real nontrivial solution y such that $y(a) = y(b) = 0$, $a, b \in \mathbb{R}$ with $a < b$ are consecutive zeros, then we have the following Lyapunov type inequality*

$$4^2 < (b-a)^2 Q^2 \exp\left([S(a,b;m) - I(a,b;m)]\right) \left(\int_a^b \eta^+(t)dt\right)^2,$$

where $Q = \max\{f_1(y(c_1)), f_1(y(c_2))\}$ with $a < c_1 \leq c_2 < b$ such that $y'(c_1) = y'(c_2) = 0$, $I(a,b;p) = \inf_{a \leq u \leq v \leq b} \int_u^v p(s)ds$ and f_1, S are defined as in previous theorem.

Besides the works [86, 87] on higher order differential equations, Pachpatte consider the second order nonlinear differential equations and obtained generalized Lyapunov type inequality for the following equations.

$$(r(t)|y'|^{\alpha-1}y')' + p(t)y' + q(t)y + f(t,y) = 0 \tag{1.17}$$

$$(r(t)|y|^{\beta}|y'|^{\gamma-2}y')' + p(t)y' + q(t)y + f(t,y) = 0 \tag{1.18}$$

$$(r(t)|y'|^{\alpha-1}y')' + q(t)|y|^{\beta-1}y = 0 \tag{1.19}$$

$$(r(t)|y|^{p}|y'|^{k-2}y')' + q(t)|y|^{p+k-2}y = 0 \tag{1.20}$$

Theorem 1.2.25 ([88]) *Let $\alpha \geq 1$. If the equation (1.17) has a real nontrivial solution y such that $y(a) = y(b) = 0$, $a, b \in \mathbb{R}$ with $a < b$ are consecutive zeros, then we have the following Lyapunov type inequality*

$$2^{\alpha+1} \leq \left(\int_a^b r^{\frac{-1}{\alpha}}(t)dt\right)^{\alpha} \left(M^{1-\alpha}\int_a^b \left|q(t) - \frac{p'(t)}{2}\right| dt + M^{-\alpha}\int_a^b w(t,M)dt\right),$$

where $M = \max\{|y(t)| : a < t < b\}$ and $f(t,y) \leq w(t,|y|)$.

Theorem 1.2.26 ([88]) *Let $\beta \geq 0,\ \gamma \geq 2$. If the equation (1.18) has a real nontrivial solution y such that $y(a) = y(b) = 0$, $a, b \in \mathbb{R}$ with $a < b$ are consecutive zeros, then we have the following Lyapunov type inequality*

$$1 \leq \left(\int_a^b r^{\frac{-1}{\gamma-1}}(t)dt\right)^{\gamma-1} \left(M^{2-\beta-\gamma}\int_a^b \left|q(t) - \frac{p'(t)}{2}\right| dt + M^{1-\beta-\gamma}\int_a^b w(t,M)dt\right),$$

where M and w are defined as in previous theorem.

Theorem 1.2.27 ([89]) *Let $\alpha \geq 1$ and $\beta \geq 1$. If the equation (1.19) has a real nontrivial solution y such that $y(a) = y(b) = 0$, $a, b \in \mathbb{R}$ with $a < b$ are consecutive zeros, then we have the following Lyapunov type inequalities*

$$1 \leq M^{\beta-\alpha} \left(\int_a^b r^{\frac{-1}{\alpha}}(t)dt \right)^{\alpha} \int_a^b |q(t)|dt,$$

$$1 \leq M^{\beta-\alpha} 2^{\alpha+1} \left(\int_a^c r^{\frac{-1}{\alpha}}(t)dt \right)^{\alpha} \int_a^c |q(t)|dt,$$

$$1 \leq M^{\beta-\alpha} 2^{\alpha+1} \left(\int_c^b r^{\frac{-1}{\alpha}}(t)dt \right)^{\alpha} \int_c^b |q(t)|dt,$$

where $M = \max\{|y(t)| : a < t < b\} = |y(c)|$ with $c \in (a, b)$.

Theorem 1.2.28 ([89]) *Let $p \geq 0$ and $k \geq 2$. If the equation (1.20) has a real nontrivial solution y such that $y(a) = y(b) = 0$, $a, b \in \mathbb{R}$ with $a < b$ are consecutive zeros, then we have the following Lyapunov type inequalities*

$$1 \leq \left(\int_a^b r^{\frac{-1}{k}}(t)dt \right)^{k} \int_a^b |q(t)|dt,$$

$$1 \leq 2^k \left(\int_a^c r^{\frac{-1}{k-1}}(t)dt \right)^{k-1} \int_a^c |q(t)|dt,$$

$$1 \leq 2^k \left(\int_c^b r^{\frac{-1}{k-1}}(t)dt \right)^{k-1} \int_c^b |q(t)|dt.$$

Although there is extensive literature on linear and nonlinear equations, there is not much done for the following nonlinear system.

$$x' = \alpha_1(t)x + \beta_1(t)|u|^{\gamma-2}u, \quad u' = -\alpha_1(t)u - \beta_2(t)|x|^{\beta-2}x, \tag{1.21}$$

Note that if $\gamma = \beta = 2$, the system (1.21) is reduced to system of 2-linear first order differential equations (1.7). The first result for system (1.21) is obtained in [106].

Theorem 1.2.29 ([106]) *Let $\gamma \geq 2$ and $\beta \geq 2$. If the system (1.21) has a real nontrivial solution $y = (x, u)$ such that $x(a) = x(b) = 0$, and x is not identically zero on $[a, b]$, where $a, b \in \mathbb{R}$ with $a < b$ are consecutive zeros, then we have the following Lyapunov type inequalities*

$$2 \leq \int_a^b |\alpha_1(t)|dt + M^{\frac{\beta}{\alpha}-1} \left(\int_a^b \beta_1(t)dt \right)^{\frac{1}{\gamma}} \left(\int_a^b \beta_2^+(t)dt \right)^{\frac{1}{\alpha}},$$

$$1 \leq M^{\beta-\alpha} \left(\int_a^\tau \beta_1(t)dt \right)^{\alpha-1} \int_a^\tau \beta_2^+(t)dt,$$

$$1 \leq M^{\beta-\alpha} \left(\int_\tau^b \beta_1(t)dt \right)^{\alpha-1} \int_\tau^b \beta_2^+(t)dt,$$

$$2^\alpha \leq M^{\beta-\alpha} \left(\int_a^b \beta_1(t)dt \right)^{\alpha-1} \int_a^b \beta_2^+(t)dt,$$

where $\frac{1}{\alpha} + \frac{1}{\gamma} = 1$ *and* $M = \max\{|x(t)| : a < t < b\} = |x(\tau)|$ *with* $\tau \in (a, b)$.

The better and alternative results to previous theorem are derived in [100, 105], respectively for system (1.21). Like the above results, these works also include Lyapunov type inequalities which relate the points where the first component of the solution $(x(t), u(t))$ of system (1.21) has consecutive zeros but also the point where the first component of the solution $(x(t), u(t))$ of system (1.21) is maximized.

For convenience the following definitions are made in [100, 105].

$$\begin{aligned} h_a(t) &= \int_a^t \beta_1(w) \exp\left(\gamma \int_w^t \alpha_1(s)ds\right) dw \\ h_b(t) &= \int_t^b \beta_1(w) \exp\left(\gamma \int_w^t \alpha_1(s)ds\right) dw. \end{aligned} \tag{1.22}$$

Theorem 1.2.30 ([100]) *Let* $\gamma \geq 2$ *and* $\beta \geq 2$. *If the system (1.21) has a real nontrivial solution* $y = (x, u)$ *such that* $x(a) = x(b) = 0$, *and* x *is not identically zero on* $[a, b]$, *where* $a, b \in \mathbb{R}$ *with* $a < b$ *are consecutive zeros, then we have the following Lyapunov type inequalities*

$$1 \leq \int_a^b \frac{h_a^{\beta-1}(t)h_b^{\beta-1}(t)}{h_a^{\beta-1}(t) + h_b^{\beta-1}(t)} \beta_2^+(t)dt,$$

$$2 \leq \exp\left(\frac{1}{2}\int_a^b \alpha_1(t)dt\right) \left(\int_a^b \beta_1(t)dt\right)^{\frac{1}{\gamma}} \left(\int_a^b \beta_2^+(t)dt\right)^{\frac{1}{\beta}}$$

where $\frac{1}{\beta} + \frac{1}{\gamma} = 1$.

Theorem 1.2.31 ([105]) *Let* $\gamma \geq 2$ *and* $\beta \geq 2$. *If the system (1.21) has a real nontrivial solution* $y = (x, u)$ *such that* $x(a) = x(b) = 0$, *and* x *is not identically zero on*

$[a,b]$*, where* $a,b \in \mathbb{R}$ *with* $a<b$ *are consecutive zeros, then we have the following Lyapunov type inequalities*

$$1 \leq \int_a^b \frac{\beta_2^+(t)}{h_a^{1-\alpha}(t)+h_b^{1-\alpha}(t)}dt,$$

$$2^{2-\alpha} \leq \int_a^b \left(\frac{1}{h_a(t)}+\frac{1}{h_b(t)}\right)^{1-\alpha} \beta_2^+(t)dt$$

$$h_a^{1-\alpha}(\tau)+h_b^{1-\alpha}(\tau) \leq M^{\beta-\alpha}\int_a^b \beta_2^+(t)dt$$

$$2^{2-\alpha}\left(\frac{1}{h_a(t)}+\frac{1}{h_b(t)}\right)^{\alpha-1} \leq M^{\beta-\alpha}\int_a^b \beta_2^+(t)dt$$

$$2^{\alpha} \leq M^{\beta-\alpha}\left(\int_a^b \beta_1(t)\exp\left(\int_t^{\tau}\alpha_1(t)dt\right)dt\right)^{\alpha-1}\left(\int_a^b \beta_2^+(t)dt\right)$$

where $\frac{1}{\alpha}+\frac{1}{\gamma}=1$ *and* $M=\max\{|x(t)|: a<t<b\}=|x(\tau)|$ *with* $\tau \in (a,b)$.

1.2.5 Lyapunov Type Inequalities For Quasilinear Impulsive Systems

Due to the fact that Lyapunov type inequalities are useful tools in oscillation theory, disconjugacy, stability and boundary and eigenvalue problems, after the pioneering work of Lyapunov in [69], many papers have followed to extend Lyapunov inequality to half linear equations and in general to quasilinear systems. The second order half linear equations (sometimes it is called differential equations with the one dimensional α-Laplacian) can be defined as

$$(r(t)|y'|^{\alpha-2}y')'+p(t)|y'|^{\alpha-2}y'+q(t)|y|^{\alpha-2}y=0, \tag{1.23}$$

where $r(t)>0,\ \alpha>1$ and $\varphi(u)=|u|^{\alpha-2}u$ is α-Laplacian operator. Recall that solution space of equation (1.23) is homogenous but not additive. Since half linear equations describe various physical, biological and chemical phenomena, qualitative nature of solutions of these equations have been investigated and half linear counterparts of Lyapunov inequlity have been established by many authors.

Theorem 1.2.32 ([91]) *Let* $r(t)=1,\ p(t)=0,\ q(t)>0$. *If the equation (1.23) has a real nontrivial solution* y *such that* $y(a)=y(b)=0$, $a,b \in \mathbb{R}$ *with* $a<b$ *are consecutive zeros, then we have the following Lyapunov type inequalities*

$$\left(\frac{1}{c-a}\right)^{\frac{\alpha}{\beta}} \leq \int_a^c q(t)dt, \qquad \left(\frac{1}{b-c}\right)^{\frac{\alpha}{\beta}} \leq \int_c^b q(t)dt$$

and

$$2^{\alpha} \leq (b-a)^{\alpha-1} \int_a^b q(t)dt,$$

where $c \in (a,b)$ *such that* $y(c) = \max\{|y(t)| : a < t < b\}$ *and* $\dfrac{1}{\alpha} + \dfrac{1}{\beta} = 1.$

Theorem 1.2.33 ([63]) *Let* $r(t) = 1,\ p(t) = 0$. *If the equation (1.23) has a real nontrivial solution* y *such that* $y(a) = y(b) = 0,\ a, b \in \mathbb{R}$ *with* $a < b$ *are consecutive zeros, then we have the following Lyapunov type inequalities*

$$\left(\frac{1}{c-a}\right)^{\frac{\alpha}{\beta}} \leq \int_a^c q^+(t)dt, \qquad \left(\frac{1}{b-c}\right)^{\frac{\alpha}{\beta}} \leq \int_c^b q^+(t)dt$$

where $c \in (a,b)$ *such that* $y(c) = \max\{|y(t)| : a < t < b\}$ *and* $\dfrac{1}{\alpha} + \dfrac{1}{\beta} = 1.$

Theorem 1.2.34 ([119, 33, 113]) *Let* $p(t) = 0$. *If the equation (1.23) has a real nontrivial solution* y *such that* $y(a) = y(b) = 0,\ a, b \in \mathbb{R}$ *with* $a < b$ *are consecutive zeros, then we have the following Lyapunov type inequalities*

$$2^{\alpha} \leq \left(\int_a^b r^{1-\beta}(t)dt\right)^{\alpha-1} \int_a^b q^+(t)dt,$$

where $\dfrac{1}{\alpha} + \dfrac{1}{\beta} = 1.$

Theorem 1.2.35 ([63]) *Let* $r(t) = 1$. *If the equation (1.23) has a real nontrivial solution* y *such that* $y(a) = y(b) = 0,\ a, b \in \mathbb{R}$ *with* $a < b$ *are consecutive zeros, then we have the following Lyapunov type inequalities.*

(i) If $\alpha \geq 2$, *then*

$$4 < \exp\left(\int_a^b |p(t)|dt\right)(b-a)^{\alpha-1} \int_a^b q^+(t)dt,$$

$$4 < 4\int_a^b |p(t)|dt + (b-a)^{\alpha-1} \int_a^b q^+(t)dt.$$

(ii) If $1 < \alpha \leq 2$, *then*

$$2^{\alpha} < \exp\left(\int_a^b |p(t)|dt\right)(b-a)^{\alpha-1} \int_a^b q^+(t)dt,$$

$$2^{\alpha} < 2^{\alpha}\int_a^b |p(t)|dt + (b-a)^{\alpha-1} \int_a^b q^+(t)dt.$$

Theorem 1.2.36 ([113]) *Let $r(t) = 1$. If the equation (1.23) has a real nontrivial solution y such that $y(a) = y(b) = 0$, $a, b \in \mathbb{R}$ with $a < b$ are consecutive zeros, then we have the following Lyapunov type inequality*

$$2^{\alpha} < \exp\left(\frac{1}{2}\int_a^b |p(t)|dt\right)(b-a)^{\alpha-1}\int_a^b q^{+}(t)dt.$$

The quasilinear elliptic system of partial differential equations can be reduced to the following one dimensional system of two ordinary differential equations (it is called (p, q)-Laplacian operators)

$$\begin{aligned} -\left(h(t)\left|u'\right|^{p-2}u'\right)' &= f(t)\left|u\right|^{\alpha-2}u\left|v\right|^{\beta}, \\ -\left(m(t)\left|v'\right|^{q-2}v'\right)' &= g(t)\left|u\right|^{\theta}\left|v\right|^{\gamma-2}v, \end{aligned} \tag{1.24}$$

and to its generalization $(p_1, p_2, \ldots, p_n)$-Laplacian operators

$$-\left(r_k(t)\phi_{p_k}(u_k')\right)' = f_k(t)\phi_{q_{kk}}(u_k)\prod_{j=1(j\neq k)}^{n}\psi_{q_{kj}}(u_j), \tag{1.25}$$

where $h(t), m(t), r_k(t) > 0$ and $p, q, p_k > 1$ for $k = 1, \ldots, n$, and $\alpha, \beta, \theta, \gamma, q_{kj} > 0$ for $k, j = 1, 2, \ldots, n$ and $\phi_p(z) = |z|^{p-2}z,\ \psi_q(z) = |z|^q$.

Because of the usefulness of Lyapunov type inequalities in investigating the qualitative behaviour of solutions of differential equations, such as oscillation, disconjugacy and stability and utility of such inequalities in studying boundary and eigenvalue problems, many authors generalize the pioneering work of Lyapunov in [69] to quasilinear systems (1.24) and (1.25) in order to analysis the properties of solutions of such systems.

Theorem 1.2.37 ([78]) *Let $h(t) = m(t) = 1$, $f(t), g(t) > 0$, $\alpha = \theta$, $\beta = \gamma$, $\dfrac{\alpha}{p} + \dfrac{\beta}{q} = 1$, and p' and q' be conjugate numbers for p and q, respectively. If the system (1.24) has a real nontrivial solution $(u(t), v(t))$ such that $u(a) = u(b) = v(a) = v(b) = 0$, $a, b \in \mathbb{R}$ with $a < b$ are consecutive zeros, and u, v are not identically zero on $[a, b]$, then we have the following Lyapunov type inequality*

$$2^{\alpha+\beta} \leq (b-a)^{\frac{\alpha}{p'}+\frac{\beta}{q'}}\left(\int_a^b f(t)dt\right)^{\frac{\alpha}{p}}\left(\int_a^b g(t)dt\right)^{\frac{\beta}{q}}.$$

Theorem 1.2.38 ([19, 98, 7]) *Let* $\dfrac{\alpha}{p}+\dfrac{\beta}{q}=1$, $\dfrac{\theta}{p}+\dfrac{\gamma}{q}=1$, *and* p' *and* q' *be conjugate numbers for* p *and* q, *respectively. If the system (1.24) has a real solution* $(u(t),v(t))$ *such that* $u(a)=u(b)=v(a)=v(b)=0$, $a,b\in\mathbb{R}$ *with* $a<b$ *are consecutive zeros, and* u,v *are not identically zero on* $[a,b]$, *then we have the following Lyapunov type inequality*

$$2^{\theta+\beta}\le\left(\int_a^b h^{1-p'}(t)dt\right)^{\frac{\theta}{p'}}\left(\int_a^b m^{1-q'}(t)\right)^{\frac{\beta}{q'}}\left(\int_a^b f^{+}(t)dt\right)^{\frac{\theta}{p}}\left(\int_a^b g^{+}(t)dt\right)^{\frac{\beta}{q}}.$$

The generalization of Lyapunov inequality to system of $(p_1,p_2,\ldots,p_n)$-Laplacian operators is made as in the following theorems.

Theorem 1.2.39 ([18, 98]) *Let* $r_k(t)=1$, $q_{kj}=q_{jj}$ *for* $k,j=1,2,\ldots,n$ *and* $\sum_{j=1}^{n}\dfrac{q_{jj}}{p_j}=1$ *and* p'_j *be the conjugate number for* p_j *for* $j=1,2,\ldots,n$. *If the system (1.25) has a real solution* $(u_1(t),u_2(t),\ldots,u_n(t))$ *such that* $u_k(a)=u_k(b)=0$ *for* $k=1,2,\ldots,n$ *and* $a,b\in\mathbb{R}$ *with* $a<b$ *are consecutive zeros and* $u_1,u_2,\ldots,u_n$ *are not identically zero on* $[a,b]$, *then we have the following Lyapunov type inequality*

$$2^{\sum_{j=1}^{n}q_{jj}}\le(b-a)^{-1+\sum_{j=1}^{n}q_{jj}}\prod_{j=1}^{n}\left(\int_a^b f_j^{+}(t)dt\right)^{\frac{q_{jj}}{p_j}}. \tag{1.26}$$

Theorem 1.2.40 ([7]) *Let* $q_{kj}=q_{jj}$ *for* $k,j=1,2,\ldots,n$ *and* $\sum_{j=1}^{n}\dfrac{q_{jj}}{p_j}=1$ *and* p'_j *be the conjugate number for* p_j *for* $j=1,2,\ldots,n$. *If the system (1.25) has a real solution* $(u_1(t),u_2(t),\ldots,u_n(t))$ *such that* $u_k(a)=u_k(b)=0$ *for* $k=1,2,\ldots,n$ *and* $a,b\in\mathbb{R}$ *with* $a<b$ *are consecutive zeros and* $u_1,u_2,\ldots,u_n$ *are not identically zero on* $[a,b]$, *then we have the following Lyapunov type inequality*

$$2^{\sum_{j=1}^{n}q_{jj}}\le\prod_{j=1}^{n}\left(\int_a^b r_j^{1-p'_j}(t)dt\right)^{\frac{q_{jj}}{p'_j}}\left(\int_a^b f_j^{+}(t)dt\right)^{\frac{q_{jj}}{p_j}}. \tag{1.27}$$

Theorem 1.2.41 ([6]) *Let* $q_{kj}=q_{jk}$ *for* $k,j=1,2,\ldots,n$. *If the system (1.25) has a real solution* $(u_1(t),u_2(t),\ldots,u_n(t))$ *such that* $u_k(a)=u_k(b)=0$ *for* $k=$

$1, 2, \ldots, n$ *and* $a, b \in \mathbb{R}$ *with* $a < b$ *are consecutive zeros and* $u_1, u_2, \ldots, u_n$ *are not identically zero on* $[a, b]$*, then the following Lyapunov type inequality*

$$2^{\sum_{j=1}^{n} p_j e_j} \leq \prod_{k=1}^{n} \left(\int_a^b r_k^{\frac{1}{1-p_k}}(t)dt \right)^{e_k(p_k-1)} \left(\int_a^b f_k^+(t)dt \right)^{e_k}$$

holds for $k = 1, 2, \ldots, n$ *where* $(e_1, e_2, \ldots, e_n)$ *is nontrivial solution of the homogenous system*

$$e_k \left(1 - \frac{q_{kk}}{p_k} \right) - \sum_{j=1 (j \neq k)}^{n} \frac{q_{jk}}{p_k} e_j = 0 \tag{1.28}$$

where $e_k \geq 0$ *for* $k = 1, 2, \ldots, n$ *and* $\sum_{j=1}^{n} e_j^2 > 0$.

Theorem 1.2.42 ([115]) *If the quasilinear system (1.25) has a real solution* $(u_1(t), u_2(t), \ldots, u_n(t))$ *such that* $u_k(a) = u_k(b) = 0$ *for* $k = 1, 2, \ldots, n$ *and* $a, b \in \mathbb{R}$ *with* $a < b$ *are consecutive zeros and* $u_1, u_2, \ldots, u_n$ *are not identically zero on* $[a, b]$*, then the following Lyapunov type inequality*

$$2^{\sum_{j=1}^{n} p_j e_j} \leq \prod_{k=1}^{n} \left(\int_a^b r_k^{\frac{1}{1-p_k}}(t)dt \right)^{e_k(p_k-1)} \left(\int_a^b f_k^+(t)dt \right)^{e_k}$$

holds for $k = 1, 2, \ldots, n$ *where* $(e_1, e_2, \ldots, e_n)$ *is nontrivial solution of the homogenous system* (1.28).

1.3 Linear System of Impulsive Differential Equations

The theory of impulsive differential equations has become an important object of investigation because of its wide applicability in biology, medicine, mechanics, control and in more fields mentioned at the begining of Chapter 1. The impulse condition is the appropriate model for describing physical phenomena if the system changes its state rapidly at certain moments. In this case system can not be modeled by traditional ways, i.e by ordinary differential equations.

In this section we outline some basic facts about linear system of impulsive differential equations with fixed moments, for details and for systems with variable moments of impulses, see [60, 10, 9, 93, 4] and the references therein.

For any interval J of $\mathbb{R}$, let τ_i be the given strictly increasing sequence of impulse points in J, i.e $\tau_i < \tau_{i+1}$. If J is a bounded interval of $\mathbb{R}$, then τ_i is a finite sequence, otherwise, that is if J is an infinite interval, then the sequence τ_i may be infinite and $\lim_{i\to\infty} \tau_i = \infty$ because this sequence have no finite accumulation points.

Let $J \subset \mathbb{R}$ and the sequence τ_i be fixed in J. We denote by $PLC(J)$ the space of all piecewise left continuous functions $\omega : J \to \mathbb{R}$ having discontinuities of the first kind at $\tau_i \in J,\ i \in \mathbb{Z}$. As usual, by $PLC^1(J)$ we mean the set of functions $\omega : J \to \mathbb{R}$ such that $\omega, \omega' \in PLC(J)$.

Linear system of impulsive equations can be described by three components: a continuous time ordinary differential equation, which governs the state of the system between impulses; an impulse equation, which models an impulsive jump defined by a jump function at the instant an impulse occurs; and a jump criterion, which defines a set of jump points τ_i at which the impulse equation is active. Mathematically a *linear system of impulsive differential equation* takes the form

$$\begin{aligned} \omega' &= A(t)\omega, \quad t \neq \tau_i \\ \Delta\omega|_{t=\tau_i} &= B_i\omega, \qquad i \in \mathbb{Z}. \end{aligned} \tag{1.29}$$

where $A(t)$ is an $n \times n$ matrix with entries $a_{ij} \in PLC(J)$, B_i is an $n \times n$ constant matrix, i.e $B_i \in \mathbb{R}^{n\times n}$ for all $i \in \mathbb{Z}$ and $\Delta\omega$ denotes the jump operator at $t = \tau_i$ defined as

$$\Delta\omega|_{t=\tau_i} = \omega(\tau_i^+) - \omega(\tau_i^-)$$

such that $\omega(\tau_i^\pm) = \lim_{h\to 0^+} \omega(\tau_i \pm h)$.

By a solution of system (1.29), we mean a vector valued function ω defined for $t \in \mathbb{R}$ such that $\omega \in PLC(J)$ and system (1.29) is fulfilled for all $t \in \mathbb{R}$.

The main result for the existence and uniquness of the solutions of homogenous system (1.29) is the following.

Theorem 1.3.1 ([93]) *Let $A(t) \in PLC(J)$ and $B_i \in \mathbb{R}^{n\times n}$ for all $i \in \mathbb{Z}$. Then for any $t_0 \in J$ and $\delta = [\delta_1, \delta_2, \ldots, \delta_n]$, there exists a unique solution $\omega(t) = \omega(t, t_0, \delta)$ of system (1.29) on J satisfying the initial condition $\omega(t_0) = \delta$ provided $\det(I + B_i) \neq 0$ for all $i \in \mathbb{Z}$.*

Remark 1.3.1 *The condition* $\det(I + B_i) \neq 0,\ i \in \mathbb{Z}$ *provides the existence, uniqueness and continuability of solutions of system (1.29) throughout* J.

Clearly, $\omega(t) = 0$ for all J is a solution of system (1.29). Therefore the solution $\omega(t) \equiv 0$ is called trivial solution of the system (1.29).

Theorem 1.3.2 ([93, 107]) *The set* Ω_n *of all the solutions of the* $n-$*dimensional homogenous system (1.29) defined on* J *is a* $n-$*dimensional vector space.*

Theorem 1.3.3 ([93, 107]) *If* $\phi(t) = \{\phi_1(t), \phi_2(t), \ldots, \phi_n(t)\}$ *is any set of* n *linearly independent solutions of system (1.29) on* J*, then the set* $\phi(t)$*, which is a basis in the space* Ω_n*, is called a fundamental set of solutions of system (1.29) and the* $n \times n$ *matrix* $\Phi = [\phi_1\ \phi_2\ \ldots\ \phi_n]$ *is called fundamental matrix of system (1.29). Every solution of system (1.29) is a linear combination of solutions of the fundamental set, i.e any solution* $\omega(t)$ *of system (1.29) can be written in this form:*

$$\omega(t) = \Phi(t)c = c_1\phi_1(t) + c_2\phi_2(t) + \ldots + c_n\phi_n(t)$$

where $c = [c_1, c_2, \ldots c_n]^T$ *is any column vector.*

In the sequel, it is convenient to use the notation

$$\Phi = [\phi_1\,\phi_2 \ldots \phi_n] = \begin{bmatrix} \phi_{11} & \phi_{12} & \ldots & \phi_{1\,n} \\ \phi_{21} & \phi_{22} & \ldots & \phi_{2\,n} \\ \vdots & \vdots & \vdots & \vdots \\ \phi_{n1} & \phi_{n2} & \ldots & \phi_{n\,n} \end{bmatrix} \tag{1.30}$$

Theorem 1.3.4 ([93]) *The determinant of* $\Phi(t)$ *is called Wronskian* $W(t)$ *of solutions of system (1.29) and it is computed as*

$$W(t) = \det\Phi(t) = \det\Phi(t_0)\exp\left(\int_{t_0}^{t} trace(A(s))ds\right)\prod_{\upsilon=1}^{k+1}\det(B_{j-\upsilon-1}) \tag{1.31}$$

for $\tau_{j-1} \leq t_0 \leq \tau_j < \tau_{j+k} < t \leq \tau_{j+k+1}$.

1.4 Hamiltonian Systems

In this section, we introduce conservative dynamical systems which were first studied in mechanics and contain no energy dissipating elements, namely Hamiltonian systems, see [76, 101]. Hamiltonian mechanics arising from Lagrangian mechanics, a previous re-formulation of classical mechanics, is a re-formulation of classical mechanics that was introduced in 1833 by Irish mathematician William Rowan Hamilton. The Hamiltonian method differs from the Lagrangian method in that instead of expressing second-order differential constraints on an n-dimensional coordinate space, it expresses first-order constraints on a 2n-dimensional phase space. Therefore the number of degrees of freedom of a Hamiltonian system is n but the dimension of the phase space is $2n$. Let $H(p,q)$ be Hamiltonian function with n degrees of freedom where $q = [q_1, q_2, \dots, q_n]^T$ and $p = [p_1, p_2, \dots, p_n]^T$ denote n generalized position coordinates and n generalized momentum coordinates, respectively. $H(p,q)$ has the form that $H(p,q) = T(q,q') + W(q)$, where T denotes the kinetic energy and W denotes the potential energy of the system. These energy terms are obtained from the path independent line integrals

$$T(q,q') = \int_0^{q'} p(q,\xi)^T d\xi = \int_0^{q'} \sum_{i=1}^{n} p_i(q,\xi) d\xi_i, \tag{1.32}$$

$$W(q) = \int_0^{q} f^T(\eta) d\eta = \int_0^{q} \sum_{i=1}^{n} f_i^T(\eta) d\eta_i, \tag{1.33}$$

where $f_i,\ i = 1, 2, \dots, n$ denote generalized potential forces. Integrals (1.32) and (1.33) are path independent if and only if

$$\frac{\partial p_i(q,q')}{\partial q'_j} = \frac{\partial p_j(q,q')}{\partial q'_i},\ i,j = 1,2,\dots,n.$$

Conservative dynamical systems are described by the system of $2n$ ordinary differential equations

$$q'_i = \frac{\partial H}{\partial p_i}(p,q), \quad p'_i = -\frac{\partial H}{\partial q_i}(p,q), \qquad i = 1, \dots, n. \tag{1.34}$$

Note that along the solutions $q_i(t), p_i(t), i = 1, \ldots, n$ the derivative of $H(p, q)$ can be computed by employing chain rule as

$$\begin{aligned}
\frac{dH}{dt}(p(t), q(t)) &= \sum_{i=1}^{n} \frac{\partial H}{\partial p_i}(p, q)p_i' + \sum_{i=1}^{n} \frac{\partial H}{\partial q_i}(p, q)q_i' \\
&= \sum_{i=1}^{n} \frac{-\partial H}{\partial p_i}(p, q)\frac{\partial H}{\partial q_i}(p, q) + \sum_{i=1}^{n} \frac{\partial H}{\partial q_i}(p, q)\frac{\partial H}{\partial p_i}(p, q) \\
&= -\sum_{i=1}^{n} \frac{\partial H}{\partial p_i}(p, q)\frac{\partial H}{\partial q_i}(p, q) + \sum_{i=1}^{n} \frac{\partial H}{\partial q_i}(p, q)\frac{\partial H}{\partial p_i}(p, q) = 0.
\end{aligned}$$

In other words, in a conservative system (1.34) the Hamiltonian, i.e the total energy will be constant along the solutions of system (1.34). This constant is determined by the initial data $p(0), q(0)$.

If the Hamiltoian function H is of quadratic form

$$H(p, q) = \frac{1}{2}(p^T \ q^T)H(t)\begin{bmatrix} p(t) \\ q(t) \end{bmatrix}$$

where $H(t)$ is symmetric matrix, then by replacing $p(t)$ and $q(t)$ by $x(t)$ and $u(t)$, respectively, and setting the function $y(t)$ as $y = (x, u)$, one can rewrite system (1.34) as a single vector differential equation

$$\frac{dy}{dt} = JH(t)y \tag{1.35}$$

which is a standart form of the Hamiltonian systems, where J is a symplectic identity defined as $J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$.

In general the Hamiltonian system of $2n$-linear first-order equations has the form of (1.35), where $y \in \mathbb{R}^{2n\times1}$, H is a $2n\times2n$ symmetric matrix with piece-wise continuous real-valued entries, and J is defined as above. Letting $y = (x, u)^T$ and

$$H(t) = \begin{bmatrix} C(t) & A^T(t) \\ A(t) & B(t) \end{bmatrix}$$

we may rewrite the Hamiltonian system in an alternative way

$$x' = A(t)x + B(t)u, \quad u' = -C(t)x - A^T(t)u. \tag{1.36}$$

Definition 1.4.1 *The matrix M is Hamiltonian if it satisfies*

$$M^T(t)J + JM(t) = 0, \textit{ for all } t.$$

In our case $M = JH(t)$.

Definition 1.4.2 *The matrix $H(t)$ is symplectic matrix if it satisfies*

$$H^T(t)JH(t) = J.$$

In this case the Hamiltonian system is of symplectic structure.

CHAPTER 2

LYAPUNOV TYPE INEQUALITIES FOR $2N \times 2N$ LINEAR HAMILTONIAN SYSTEMS WITH IMPULSIVE PERTURBATIONS

2.1 Introduction

As is defined in Chapter 1 by (1.35) or (1.36), the Hamiltonian system of $2n$-linear first-order equations has the form

$$y' = JH(t)y, \quad t \in \mathbb{R},$$

or

$$x' = A(t)x + B(t)u, \quad u' = -C(t)x - A^T(t)u. \tag{2.1}$$

With regard to Definition 1.4.2, we want to remark that if

$$\begin{aligned} A^T(t)C(t) = C(t)A(t), \quad -B(t)C(t) + A(t)A(t) = -I_n, \\ B(t)A^T(t) = A(t)B(t), \quad -A^T(t)A^T(t) + C(t)B(t) = I_n, \end{aligned}$$

then system (2.1) is of symplectic structure and therefore our results are also valid for symplectic systems under impulse effect.

In the present chapter we consider (2.1) under impulse effect, that is,

$$\begin{aligned} &x' = A(t)x + B(t)u, \quad u' = -C(t)x - A^T(t)u, && t \ge t_0, \quad t \neq \tau_i \\ &x(\tau_i^+) = K_i x(\tau_i^-), \quad u(\tau_i^+) = -L_i x(\tau_i^-) + K_i u(\tau_i^-), && i \in \mathbb{N} = \{1, 2, \ldots\}. \end{aligned} \tag{2.2}$$

where

(i) $\{\tau_i\}$ is a strictly increasing sequence of real numbers,

(ii) A, B, and C are $n \times n$ matrices with entries left continuous on each interval (τ_i, τ_{i+1}) and two-sided limits at the point $\tau_i,\ i \in \mathbb{N}$ exist, and $B = B^T$ and $C = C^T$,

(iii) $\{K_i\}$ and $\{L_i\}$ are sequences of $n \times n$ matrices such that each K_i has an inverse for $i \in \mathbb{N}$.

By a solution of system (2.2), we mean a vector valued function $y(t) = (x(t), u(t))$ defined for $t \geq t_0$ such that $y \in PLC(\Gamma)$ and system (2.2) is fulfilled for all $t \geq t_0$, where the set $PLC(\Gamma)$ is defined by

$PLC(\Gamma) = \{\omega : \Gamma = [t_0, \infty) \to \mathbb{R}$ is continuous on each interval (τ_i, τ_{i+1}), the limits $w(\tau_i^{\pm})$ exist and $w(\tau_i^-) = w(\tau_i)$ for $i \in \mathbb{N}\}$.

Note that the second-order impulsive system

$$\begin{aligned} &x'' + C(t)x = 0, && t \geq t_0, \quad t \neq \tau_i \\ &x(\tau_i^+) = K_i x(\tau_i^-), \quad x'(\tau_i^+) = -L_i x(\tau_i^-) + K_i x'(\tau_i^-), && i \in \mathbb{N} \end{aligned} \tag{2.3}$$

is equivalent to (2.2) with $A = 0,\ B = I$ and $u(t) = x'(t)$. In particular, choosing $n = 1$ in system (2.2) yields the following planar Hamiltonian system under impulse effect

$$\begin{aligned} &x' = a(t)x + b(t)u, \quad u' = -c(t)x - a(t)u, && t \geq t_0 \quad t \neq \tau_i \\ &x(\tau_i^+) = k_i x(\tau_i^-), \quad u(\tau_i^+) = -l_i x(\tau_i^-) + k_i u(\tau_i^-), && i \in \mathbb{N} \end{aligned} \tag{2.4}$$

where $a, b, c \in PLC(\Gamma)$ and k_i, l_i are real sequences for $i \in \mathbb{N}$. It is worth mentioning that system (2.4) is of symplectic structure if $b(t)c(t) - a^2(t) = 1$. Note that when $b(t) > 0$ if we take $a(t) \equiv 0$, $b(t) = 1/p(t)$, $c(t) = q(t)$ and $u(t) = p(t)x'(t)$, then we obtain the special case of (2.2), the impulsive second-order differential equations of the form

$$\begin{aligned} &(p(t)x')' + q(t)x = 0, && t \geq t_0 \quad t \neq \tau_i \\ &x(\tau_i^+) = k_i x(\tau_i^-), \quad (px')(\tau_i^+) = -l_i x(\tau_i^-) + k_i (px')(\tau_i^-), && i \in \mathbb{N}. \end{aligned} \tag{2.5}$$

The next definition is adapted from [76].

Definition 2.1.1 ***(Dini derivative for piece-wise continuos functions)*** *Let $f \in PLC(\Gamma)$. Then the upper right Dini derivative D^+f is defined by*

$$D^+f(t) = \limsup_{h \to 0^+} \frac{x(t+h) - x(t)}{h}, \quad t \neq \tau_i.$$

Similarly, the upper left Dini derivative, lower right Dini derivative and lower left Dini derivative are defined as follows, respectively.

$$D^- f(t) = \limsup_{h \to 0^-} \frac{x(t+h) - x(t)}{h}, \quad t \neq \tau_i$$

$$D_+ f(t) = \limsup_{h \to 0^+} \frac{x(t+h) - x(t)}{h}, \quad t \neq \tau_i$$

and

$$D_- f(t) = \limsup_{h \to 0^-} \frac{x(t+h) - x(t)}{h}, \quad t \neq \tau_i.$$

For impulsive differential equations or systems, in general for piece-wise continuos functions, the concept of a zero of a function is replaced by a so-called generalized zero.

Definition 2.1.2 ([45, 43, 42]) *A real number c is called a zero (generalized zero) of a function f if and only if $f(c^-) = 0$ or $f(c^+) = 0$. If f is continuous function at c, then c becomes a real zero.*

Now we give the definition of disconjugacy which is about the zeros of the solutions of differential equations or systems.

Definition 2.1.3 ([45, 43]) *Equation* (2.5) *is called disconjugate on an interval $[t_1, t_2]$ if and only if all solutions of equation (2.5) have at most one zero (generalized zero) on an interval $[t_1, t_2]$.*

We generalize the definition of disconjugacy given in [45, 43].

Definition 2.1.4 ([45, 42]) *System* (2.2) *(or* (2.3)*,* (2.4)*) is called disconjugate (relatively disconjugate with respect to x) on an interval $[t_1, t_2]$ if and only if there is no real solution $(x(t), u(t))$ of system* (2.2) *(or* (2.3)*,* (2.4)*) with a nontrivial x having two or more zeros (generalized zeros) on $[t_1, t_2]$.*

Our aim in this chapter, which constitutes for the main part of the thesis, is to improve and extend Theorem 1.2.11 to the more general impulsive system (2.2), that is to obtain Lyapunov type inequalities sharper than all the results existing in the literature.

It is of great importance to obtain Lyapunov type inequalities since they are useful tools not only in boundary value problems but also in oscillation theory, asymptotic behaviour of solutions, disconjugacy, stability theory and eigenvalue problems. It turns out that there is more than one way to approach the problem due to impulses. Note that since changing the impulsive perturbation or assuming different condition on the impulses leads to variety of inequalities, presence of impulse effect yields different and new inequalities. This shows that systems with impulses are richer and more fruitful than systems without impulses. Besides, we are able to improve Theorem 1.2.11 in the special cases when the impulses are absent.

This chapter of the thesis is organized as follows. Similar to [99], the proof of the theorems are based on estimating the involved fundamental matrices by using matrix measure. Therefore, in the next section, we mention some properties of matrices and prove an auxiliary lemma providing an estimation for fundamental matrix of homogenous impulsive system. By the help of the lemmas presented in Section 2.2, we derive new Lyapunov type inequalities in Section 2.3. As applications of Lyapunov type inequalities, we present disconjugacy criteria and find lower bounds for the eigenvalues of the related eigenvalue problems in the last section.

2.2 Matrix measure and fundamental matrices

For $x \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$, $|x| = x^T x$ and $|A| = \sqrt{\lambda_{\max}(A^T A)}$ denote the Euclidean norm and induced matrix norm, respectively. Let $B_1, B_2 \in \mathbb{R}_s^{n\times n}$. Then the property $B_1 \geq B_2$ is defined as in Definition 1.2.1 and B_i has a unique square root $B_i^{1/2} \in \mathbb{R}_s^{n\times n}$ such that $B_i^{1/2} \geq 0$, and $(B_i^{1/2})^2 = B_i,\ i = 1, 2$.

Now, we give some elementary inequalities for norms.

Lemma 2.2.1 ([99]) *(a) Let $C \in \mathbb{R}_s^{n\times n}$. Then for any $C^* \in \mathbb{R}_s^{n\times n}$ with $C^* \geq C$, one has*

$$x^T C x \leq |C^*||x|^2, \quad x \in \mathbb{R}^n \tag{2.6}$$

(b) Let $P \in \mathbb{R}^{n\times n}$ and $Q \in \mathbb{R}_s^{n\times n}$ with $Q \geq 0$. One has

$$|PQx| \leq |Q^{1/2} P^T P Q^{1/2}|^{1/2} (x^T Q x)^{1/2}, \quad x \in \mathbb{R}^n \tag{2.7}$$

Proof.

(a) Let $C \in \mathbb{R}_s^{n\times n}$. By definition we have

$$x^T C x \le x^T C^* x \le |x||C^* x| \le |x||C^*||x| = |C^*||x|^2.$$

(b) Let $P \in \mathbb{R}^{n\times n}$ and $Q \in \mathbb{R}_s^{n\times n}$ with $Q \ge 0$. Then

$$\begin{aligned}
|PQx|^2 &= x^T Q P^T P Q x = (Q^{1/2}x)^T Q^{1/2} P^T P Q^{1/2} (Q^{1/2}x) \\
&\le \left|Q^{1/2}x\right| \left|Q^{1/2} P^T P Q^{1/2}\right| \left|Q^{1/2}x\right| \\
&= \left|Q^{1/2} P^T P Q^{1/2}\right| (Q^{1/2}x)^T (Q^{1/2}x) = \left|Q^{1/2} P^T P Q^{1/2}\right| x^T Q x.
\end{aligned}$$

□

First we want to define the concept of matrix measure and show the relationship of it with fundamental matrices of system of ordinary differential equations. Then this relationship will be obtained for fundamental matrices of the impulsive systems.

Lemma 2.2.2 ([109]) *For a matrix $A \in \mathbb{R}^{n\times n}$, the matrix measure $\mu(A) \in \mathbb{R}$ is defined by*

$$\mu(A) = \lim_{h\to 0} \frac{|I + hA| - 1}{h}.$$

Trivially, for any matrix norm one has

$$-|A| \le -\mu(-A) \le \Re(\lambda_i(A)) \le \mu(A) \le |A| \tag{2.8}$$

where $\Re(\lambda_i(A))$ denotes the real part of eigenvalue $\lambda_i(A)$ of matrix A for $i = 1, \ldots, n$.

Remark 2.2.1 *The matrix measure of a matrix A can also be written as follows:*

$$\mu(A) = \frac{\lambda_{\max}(A^T + A)}{2}. \tag{2.9}$$

The importance of the matrix measure in making estimations for fundamental matrix solutions of ordinary differential equations is presented in the next lemma.

Lemma 2.2.3 ([99]) *If $Z(t,s)$ is a fundamental matrix (state transition matrix) for*

$$x' = A(t)x \tag{2.10}$$

satisfying $Z(s,s) = I$, *then*

$$|Z(t,s)| \leq \exp\left(\int_s^t \mu(A(r))dr\right), \quad t \geq s \tag{2.11}$$

and

$$|Z(s,t)| \leq \exp\left(\int_s^t \mu(-A(r))dr\right), \quad t \geq s. \tag{2.12}$$

The main contribution of this section, which is to derive estimations similarly to (2.11) and (2.12) for the following impulsive systems,

$$\begin{aligned} x' &= A(t)x, \quad t \neq \tau_i, \\ x(\tau_i^+) &= K_i x(\tau_i^-), \quad i \in \mathbb{N}. \end{aligned} \tag{2.13}$$

can be given as in the next lemma.

Lemma 2.2.4 *Let (i)-(iii) hold and denote by* $X(t,s)$*,* $X(s,s) = I$*, the fundamental matrix of* (2.13)*. Then we have the estimates:*

$$|X(t,s)| \leq \exp\left(\int_s^t \mu(A(r))dr\right) \prod_{s \leq \tau_i < t} |K_i|, \quad t \geq s. \tag{2.14}$$

and

$$|X(s,t)| \leq \exp\left(\int_s^t \mu(-A(r))dr\right) \prod_{s \leq \tau_i < t} |K_i|, \quad t \geq s. \tag{2.15}$$

Proof. Consider the initial value problem

$$\begin{aligned} x' &= A(t)x, \quad t \neq \tau_i, \\ x(\tau_i^+) &= K_i x(\tau_i^-), \\ x(s) &= x_0. \end{aligned} \tag{2.16}$$

For $t \neq \tau_i$, we may write that

$$A(t)x = \frac{x(t+h) - x(t)}{h} + \eta(h), \quad \lim_{h \to 0^+} \eta(h) = 0.$$

After some manipulations and taking the norms of the both sides, we get

$$|x(t+h)| \leq |hA(t) + I|\,|x(t)| + h\,|\eta(h)|, \quad t \neq \tau_i,$$

or

$$\frac{|x(t+h)| - |x(t)|}{h} \leq \frac{\big(|hA(t) + I| - 1\big)|x(t)|}{h} + |\eta(h)|, \quad t \neq \tau_i. \tag{2.17}$$

By taking limsup of both sides of (2.17) as $t \to 0^+$, we have

$$D^+|x(t)| \le \mu(A(t))|x(t)|, \quad t \ne \tau_i,$$

where $D^+|x(t)|$ denotes the upper right Dini derivative of $|x(t)|$. Setting $u(t) = |x(t)|$ and using (2.16), we obtain

$$\begin{aligned} &D^+u \le \mu(A(t))u, \quad t \ne \tau_i,\\ &u(\tau_i^+) \le |K_i|u(\tau_i^-),\\ &u(s) = |x_0|. \end{aligned} \tag{2.18}$$

From the classical comparison theory [76, 60], we know that any solution $u(t)$ of equation (2.18) for $t \ge s$ is not greater than the maximal solution $v_M(t)$ of

$$\begin{aligned} &v' = \mu(A(t))v, \quad t \ne \tau_i,\\ &v(\tau_i^+) = |K_i|v(\tau_i^-),\\ &v(s) = |x_0|. \end{aligned} \tag{2.19}$$

Since (2.19) has the unique solution as

$$v(t) = |x_0| \exp\left(\int_s^t \mu(A(r))dr\right) \prod_{s \le \tau_i < t} |K_i|, \quad t \ge s,$$

for the solution of (2.18) we have

$$u(t) \le |x_0| \exp\left(\int_s^t \mu(A(r))dr\right) \prod_{s \le \tau_i < t} |K_i|, \quad t \ge s. \tag{2.20}$$

It follows that the counterpart of (2.11) is

$$|X(t,s)| \le \exp\left(\int_s^t \mu(A(r))dr\right) \prod_{s \le \tau_i < t} |K_i|, \quad t \ge s,$$

as desired.

To show that the estimate (2.15) holds as well, we start with the adjoint system of the impulsive system (2.13), which reads, see [93], as

$$\begin{aligned} &y' = -A^T(t)y, \quad t \ne \tau_i,\\ &y(\tau_i^+) - y(\tau_i^-) = -\left[I + (K_i - I)^T\right]^{-1} (K_i - I)^T y(\tau_i^-) = (K_i - I)\, y(\tau_i^-). \end{aligned} \tag{2.21}$$

In this case, we have

$$
\begin{aligned}
&D^+|y(t)| \le \mu(-A(t))|y(t)|, \quad t \ne \tau_i, \\
&|y(\tau_i^+)| \le |K_i||y(\tau_i^-)|.
\end{aligned}
\tag{2.22}
$$

Hence, if $Y(t,s)$ is the fundamental matrix of the adjoint system (2.21) with $Y(s,s) = I$, then in a similar manner of (2.20), from (2.22) one has

$$
|y(t)| \le |y(s)| \exp\left(\int_s^t \mu(-A(r))dr\right) \prod_{s \le \tau_i < t} |K_i|, \quad t \ge s,
$$

and hence

$$
|Y(t,s)| \le \exp\left(\int_s^t \mu(-A(r))dr\right) \prod_{s \le \tau_i < t} |K_i|, \quad t \ge s. \tag{2.23}
$$

Using $Y^T(t,s)X(t,s) = I$ gives $Y(t,s) = X^{-T}(t,s) = X^T(s,t)$ and so from (2.23), the estimate (2.15) is obtained. $\square$

2.3 Lyapunov Type Inequalities

In this section we focus on obtaining different Lyapunov type inequalities for system (2.2) and for its particular cases, (2.3), (2.4) and (2.5). These inequalities are so important due to the fact that they are used to prove disconjugacy criterion for the solutions of systems, to show the uniqueness of the solutions of associated inhomogeneous BVP, to study the stability of the solutions of planar periodic systems, to find lower bounds for the eigenvalues of the associated eigenvalue problems and to analyse the asymptotic behaviour of solutions of systems. Variety of the conditions on K_i yields applying different techniques in the proofs and establishing new Lyapunov type inequalities.

In what follows, let $\alpha(t) = \max\left\{\mu^+(A(t)), \mu^+(-A(t))\right\}$, where $\mu(A(t))$ and $\mu(-A(t))$ are matrix measures of the matrices $A(t)$ and $-A(t)$, respectively, and $m^+(t) := \max\{m(t), 0\}$. Note that in view of (2.8) we have

$$
\alpha(t) \le |A(t)|. \tag{2.24}
$$

It is easy to see that if A is a diagonal, then $\alpha(t) = |A(t)|$, but the inequality (2.24) is in general strict, which can be verified through (2.9) by examples.

Theorem 2.3.1 *Suppose that the matrices A, A^T, B, and C all commute with K_i for all $i \in \mathbb{N}$ such that*

$$B(t) \geq 0, \qquad \int_{t_1}^{t_2} |B(t)|\, dt > 0. \tag{2.25}$$

If system (2.2) has a solution $y(t) = (x(t), u(t))$ with nontrivial x such that $x(t_1^+) = x(t_2^-) = 0$, $x(t) \neq 0$ on (t_1, t_2), then for any $C^*(t) \geq C(t)$ we have the Lyapunov type inequality

$$\exp\left(\int_{t_1}^{t_2} \alpha(t)dt\right)\left(\int_{t_1}^{t_2} |B(t)|dt\right)\left(\int_{t_1}^{t_2} |C^*(t)|dt + \sum_{t_1 \leq \tau_i < t_2} |S_i|\right) \geq 4, \tag{2.26}$$

where $S_i = K_1^{-1}K_2^{-1}\ldots K_i^{-1}L_iK_{i-1}\ldots K_1,\ i = 2, 3\ldots.$

Proof. Define

$$M_0 = I, \text{ and } M_i = K_iK_{i-1}\ldots K_1 \text{ for } i = 1, 2, \ldots, m.$$

Let for each $i = 1, 2, \ldots, m$,

$$M_iz(t) = x(t), \quad M_iv(t) = u(t), \quad t \in (\tau_i, \tau_{i+1}).$$

where we put $t_1 = \tau_0$ and $t_2 = \tau_{m+1}$.

It is easy to see that with the above transformation system (2.2) becomes the following system.

$$\begin{aligned} &z' = A(t)z + B(t)v, \quad v' = -C(t)y - A^T(t)v, \quad t \neq \tau_i \\ &z(\tau_i^+) = z(\tau_i^-), \quad v(\tau_i^+) = v(\tau_i^-) - S_iz(\tau_i^-), \quad i = 0, 1, 2\ldots, m. \end{aligned} \tag{2.27}$$

where $S_i = M_i^{-1}L_iM_{i-1}$. Since $z(\tau_i) = z(\tau_i^-) = z(\tau_i^+)$, z is continuous on $[t_1, t_2]$ and $z(t_1) = z(t_2) = 0$, and $z(t) \neq 0$ on (t_1, t_2). Let $t_0 \in (t_1, t_2)$ be such that

$$|z(t_0)| = \max_{t \in [t_1, t_2]} |z(t)|\,.$$

Let $Z(t, s)$ be the fundamental matrix of (2.10). Then from the first equation of system (2.27) we may write that

$$z(t) = Z(t, \xi)z(\xi) + \int_{\xi}^{t} Z(t, s)B(s)v(s)ds. \tag{2.28}$$

Putting $t = t_0$ and replacing ξ by t_1 and t_2 in (2.28), we have

$$z(t_0) = \int_{t_1}^{t_0} Z(t_0, s)B(s)v(s)ds \tag{2.29}$$

and

$$z(t_0) = -\int_{t_0}^{t_2} Z(t_0, s)B(s)v(s)ds. \tag{2.30}$$

From (2.7), (2.11) and (2.29), we obtain

$$|z(t_0)| \le \int_{t_1}^{t_0} \exp\left(\int_s^{t_0} \mu(A(r))dr\right) |B(s)|^{1/2} \left[v^T(s)B(s)v(s)\right]^{1/2} ds. \tag{2.31}$$

Applying the Cauchy-Schwarz inequality to the square of inequality (2.31), we get

$$\begin{aligned}
|z(t_0)|^2 &\le \left[\int_{t_1}^{t_0} \exp\left(2\int_s^{t_0} \mu(A(r))dr\right) |B(s)|ds\right] \left[\int_{t_1}^{t_0} v^T(s)B(s)v(s)ds\right] \\
&\le \exp\left(2\int_{t_1}^{t_0} \mu^+(A(r))dr\right) \int_{t_1}^{t_0} |B(s)|ds \int_{t_1}^{t_0} v^T(s)B(s)v(s)ds \\
&\le \exp\left(2\int_{t_1}^{t_0} \alpha(r)dr\right) \int_{t_1}^{t_0} |B(s)|ds \int_{t_1}^{t_0} v^T(s)B(s)v(s)ds,
\end{aligned}$$

which yields

$$\Lambda_1^2 = \frac{|z(t_0)|^2}{\exp\left(2\int_{t_1}^{t_0} \alpha(r)dr\right)} \le \int_{t_1}^{t_0} |B(s)|ds \int_{t_1}^{t_0} v^T(s)B(s)v(s)ds. \tag{2.32}$$

Similarly from (2.7),(2.12) and (2.30), we obtain

$$\begin{aligned}
|z(t_0)|^2 &\le \exp\left(2\int_{t_0}^{t_2} \mu^+(-A(r))dr\right) \int_{t_0}^{t_2} |B(s)|ds \int_{t_0}^{t_2} v^T(s)B(s)v(s)ds \\
&\le \exp\left(2\int_{t_0}^{t_2} \alpha(r)dr\right) \int_{t_0}^{t_2} |B(s)|ds \int_{t_0}^{t_2} v^T(s)B(s)v(s)ds
\end{aligned}$$

or

$$\Lambda_2^2 = \frac{|z(t_0)|^2}{\exp\left(2\int_{t_0}^{t_2} \alpha(r)dr\right)} \le \int_{t_0}^{t_2} |B(s)|ds \int_{t_0}^{t_2} v^T(s)B(s)v(s)ds. \tag{2.33}$$

On the other hand, it is easy to see that

$$(z^T(t)v(t))' = -z^T(t)C(t)z(t) + v^T(t)B(t)v(t), \quad t \ne \tau_i \tag{2.34}$$

$$\Delta(z^T(\tau_i)v(\tau_i)) = z^T(\tau_i)v(\tau_i^+) - z^T(\tau_i)v(\tau_i^-) = -z^T(\tau_i)S_i z(\tau_i). \tag{2.35}$$

Integrating (2.34) from t_1 to t_2 and using (2.35), we have

$$\int_{t_1}^{t_2} v^T(t)B(t)v(t)dt = \int_{t_1}^{t_2} z^T(t)C(t)z(t)dt + \sum_{t_1 \le \tau_i < t_2} z^T(\tau_i)S_i z(\tau_i).$$

Since $C^*(t) \geq C(t)$ and $|z(t)| \leq |z(t_0)|$ for $t \in [t_1, t_2]$ and by employing (2.6), we can estimate the right-hand side of the previous inequality as

$$\begin{aligned} \int_{t_1}^{t_2} v^T(t)B(t)v(t)dt &\leq \int_{t_1}^{t_2} z^T(t)C^*(t)z(t)dt + \sum_{t_1 \leq \tau_i < t_2} z^T(\tau_i)S_i z(\tau_i) \\ &\leq |z(t_0)|^2 \left(\int_{t_1}^{t_2} |C^*(t)|dt + \sum_{t_1 \leq \tau_i < t_2} |S_i| \right). \end{aligned} \tag{2.36}$$

By using inequalities (2.32) and (2.33), we see from (2.36) that

$$\begin{aligned} & |z(t_0)|^2 \left(\int_{t_1}^{t_2} |B(t)|dt \right) \left(\int_{t_1}^{t_2} |C^*(t)|dt + \sum_{t_1 \leq \tau_i < t_2} |S_i| \right) \\ & \geq \frac{|z(t_0)|^2 \int_{t_1}^{t_2} |B(t)|dt}{\exp\left(2\int_{t_1}^{t_0} \alpha(s)ds\right) \int_{t_1}^{t_0} |B(t)|dt} + \frac{|z(t_0)|^2 \int_{t_1}^{t_2} |B(t)|dt}{\exp\left(2\int_{t_0}^{t_2} \alpha(s)ds\right) \int_{t_0}^{t_2} |B(t)|dt} \\ & \geq \frac{\Lambda_1^2}{q_1} + \frac{\Lambda_2^2}{q_2}, \end{aligned} \tag{2.37}$$

where

$$q_1 = \frac{\int_{t_1}^{t_0} |B(t)|dt}{\int_{t_1}^{t_2} |B(t)|dt}, \qquad q_2 = \frac{\int_{t_0}^{t_2} |B(t)|dt}{\int_{t_1}^{t_2} |B(t)|dt}.$$

As $q_1 + q_2 = 1$, we have

$$\frac{\Lambda_1^2}{q_1} + \frac{\Lambda_2^2}{q_2} \geq 4\Lambda_1\Lambda_2. \tag{2.38}$$

Therefore, from (2.37) we obtain

$$\left(\int_{t_1}^{t_2} |B(t)|dt \right) \left(\int_{t_1}^{t_2} |C^*(t)|dt + \sum_{t_1 \leq \tau_i < t_2} |S_i| \right) \geq 4 \exp\left(-\int_{t_1}^{t_2} \alpha(t)dt \right).$$

□

An alternative Lyapunov inequality is possible by using (2.14) and (2.15) instead of (2.11) and (2.12), respectively.

Theorem 2.3.2 *Suppose that* $K_i^T K_i = I$ *for all* $i \in \mathbb{N}$ *and* (2.25) *holds. If system (2.2) has a solution* $y(t) = (x(t), u(t))$ *with nontrivial* x *such that* $x(t_1^+) = x(t_2^-) =$

$0, x(t) \neq 0$ *on* (t_1, t_2)*, then for any* $C^*(t) \geq C(t)$ *we have the Lyapunov type inequality*

$$\exp\left(\int_{t_1}^{t_2} \alpha(t)dt\right)\left(\prod_{t_1 \leq \tau_i < t_2} |K_i|^2\right)\left(\int_{t_1}^{t_2} |B(t)|dt\right) \times \left[\int_{t_1}^{t_2} |C^*(t)|dt + \sum_{t_1 \leq \tau_i < t_2} |K_i^T L_i|\right] \geq 4. \tag{2.39}$$

Proof. Let $X(t,s)$ be the fundamental matrix of (2.13) given in Lemma 2.2.4. We can write from system (2.2) that

$$x(t) = X(t,\xi)x(\xi) + \int_{\xi}^{t} X(t,s)B(s)u(s)ds.$$

Letting $\xi \to t_1^+$ and $\xi \to t_2^-$, and if $t = t_0$, we get

$$x(t_0) = \int_{t_1}^{t_0} X(t_0,s)B(s)u(s)ds \tag{2.40}$$

and

$$x(t_0) = -\int_{t_0}^{t_2} X(t_0,s)B(s)u(s)dr, \tag{2.41}$$

where $t_0 \in (t_1, t_2)$ is a point such that

$$|x(t_0)| = \sup_{t \in (t_1,t_2)} |x(t)|.$$

In view of (2.14) and (2.40), (2.15) and (2.41), proceeding along the similar lines as in the proof of Theorem 2.3.1, we have

$$|x(t_0)| \leq \int_{t_1}^{t_0} \exp\left(\int_{s}^{t_0} \mu(A(w))dw\right) \prod_{s \leq \tau_i < t_0} |K_i||B(s)|^{1/2}\left[u^T(s)B(s)u(s)\right]^{1/2} ds$$

and

$$|x(t_0)| \leq \int_{t_0}^{t_2} \exp\left(\int_{t_0}^{s} \mu(-A(w))dw\right) \prod_{t_0 \leq \tau_i < s} |K_i||B(s)|^{1/2}\left[u^T(s)B(s)u(s)\right]^{1/2} ds.$$

and hence by applying Cauchy-Schwarz inequality to the square of the above inequalities and similarly to the proof of Theorem 2.3.1, we obtain

$$|x(t_0)|^2 \leq \exp\left(2\int_{t_1}^{t_0} \alpha(t)dt\right) \prod_{t_1 \leq \tau_i < t_0} |K_i|^2 \left[\int_{t_1}^{t_0} |B(s)|ds\right]\left[\int_{t_1}^{t_0} u^T(s)B(s)u(s)ds\right]$$

and

$$|x(t_0)|^2 \leq \exp\left(2\int_{t_0}^{t_2} \alpha(t)dt\right) \prod_{t_0 \leq \tau_i < t_2} |K_i|^2 \left[\int_{t_0}^{t_2} |B(s)|ds\right]\left[\int_{t_0}^{t_2} u^T(s)B(s)u(s)ds\right].$$

Thus, we have

$$Q_1 \le \left[\int_{t_1}^{t_0} |B(s)|ds\right]\left[\int_{t_1}^{t_0} u^T(s)B(s)u(s)ds\right] \tag{2.42}$$

and

$$Q_2 \le \left[\int_{t_0}^{t_2} |B(s)|ds\right]\left[\int_{t_0}^{t_2} u^T(s)B(s)u(s)ds\right], \tag{2.43}$$

where $Q_1 = \dfrac{|x(t_0)|}{\exp\left(\int_{t_1}^{t_0}\alpha(s)ds\right)\prod_{t_1\le\tau_i<t_0}|K_i|}$, $\quad Q_2 = \dfrac{|x(t_0)|}{\exp\left(\int_{t_0}^{t_2}\alpha(s)ds\right)\prod_{t_0\le\tau_i<t_2}|K_i|}$.

On the other hand, in view of

$$\Delta(x^T(\tau_i)u(\tau_i)) = -x^T(\tau_i)K_i^T L_i x(\tau_i),$$

by integrating

$$(x^T(t)u(t))' = -x^T(t)C(t)x(t) + u^T(t)B(t)u(t), \quad t \ne \tau_i$$

from s_1 to s_2, and then letting $s_1 \to t_1^+$ and $s_2 \to t_2^-$, we get

$$\int_{t_1}^{t_2} u^T(t)B(t)u(t)dt \;=\; \int_{t_1}^{t_2} x^T(t)C(t)x(t)dt + \sum_{t_1\le\tau_i<t_2} x^T(\tau_i)K_i^T L_i x(\tau_i).$$

Since $C^*(t) \ge C(t)$ and $|x(t)| \le |x(t_0)|$ for $t \in [t_1, t_2]$ and from (2.6), we have

$$\int_{t_1}^{t_2} u^T(t)B(t)u(t)dt \le |x(t_0)|^2\left[\int_{t_1}^{t_2}|C^*(s)|ds + \sum_{t_1\le\tau_i<t_2}|K_i^T L_i|\right]. \tag{2.44}$$

By using inequalities (2.42), (2.43) and (2.44), we obtain

$$\begin{aligned}
&\left[\int_{t_1}^{t_2}|B(t)|dt\right]|x(t_0)|^2\left[\int_{t_1}^{t_2}|C^*(s)|ds + \sum_{t_1\le\tau_i<t_2}|K_i^T L_i|\right]\\
&\qquad \ge \left[\int_{t_1}^{t_2}|B(t)|dt\right]\left[\int_{t_1}^{t_2} u^T(t)B(t)u(t)dt\right]\\
&\qquad = \left[\int_{t_1}^{t_2}|B(t)|dt\right]\left[\int_{t_1}^{t_0} u^T(t)B(t)u(t)dt + \int_{t_0}^{t_2} u^T(t)B(t)u(t)dt\right].
\end{aligned} \tag{2.45}$$

By employing (2.38) in (2.45) with Q_1, Q_2 which are defined as above and with the same q_1, q_2 defined as in Theorem 2.3.1, we arrive at the desired Lyapunov inequality (2.39). □

Remark 2.3.1 *Since $\alpha(t) \le |A(t)|$, Theorem 2.3.1 and Theorem 2.3.2 improve and generalize all the results existing in the literature, in particular Theorem 1.2.11, [99,* *Theorem 2.4]. Therefore, our results are new and alternative to each other. If there is no impulse, i.e $K_i = I,\ L_i = 0$ for all $i \in \mathbb{N}$, then Theorem 2.3.1 and Theorem 2.3.2 coincide but still improve [99, Theorem 2.4], which implies that Theorem 2.3.1 and Theorem 2.3.2 are new even for the nonimpulsive case.*

Remark 2.3.2 *Let us consider the special cases of the matrix $C^*(t)$. If $C^*(t)$ is taken as $C^+(t)$ or $C_+(t)$ in* (2.26) *and* (2.39)*, where $C^+(t)$ is defined as in Theorem 1.2.11 and $C_+(t)$ is defined by $C_+(t) = \frac{1}{2}\{C(t) + [C(t)C^T(t)]^{1/2}\}$, then the condition $C^*(t) \geq C(t)$ is satisfied. Thus $C^*(t)$ can be replaced by $C^+(t)$ or $C_+(t)$.*

Remark 2.3.3 *In view of* (2.24)*, we may replace the Lyapunov type inequalities* (2.26) *and* (2.39)*, respectively, by*

$$\exp\left(\int_{t_1}^{t_2} |A(t)|dt\right)\left(\int_{t_1}^{t_2} |B(t)|dt\right)\left(\int_{t_1}^{t_2} |C^*(t)|dt + \sum_{t_1\leq\tau_i<t_2} |S_i|\right) \geq 4, \quad (2.46)$$

and

$$\begin{aligned}\exp\left(\int_{t_1}^{t_2} |A(t)|dt\right)&\left(\prod_{t_1\leq\tau_i<t_2} |K_i|^2\right)\left[\int_{t_1}^{t_2} |B(s)|ds\right] \\ &\times\left[\int_{t_1}^{t_2} |C^*(s)|ds + \sum_{t_1\leq\tau_i<t_2} |K_i^T L_i|\right] \geq 4.\end{aligned} \quad (2.47)$$

Since inequalities (2.26), (2.39) *and* (2.46), (2.47) *are obtained due to assuming different conditions on the impulses, changing the conditions of the coefficient functions appearing on the impulse effect or choosing different impulsive perturbation yields more various inequalities than we present. Therefore existence of impulse effect provides new Lyapunov type inequalities. That is why we are interested in system of impulsive differential equations than system of ordinary differential equations. In the absence of impulse, inequalities* (2.46), (2.47) *and inequality (2.22) in [99] coincide.*

The following results are obtained from Theorem 2.3.1 and Theorem 2.3.2 for the second-order impulsive system (2.3).

Corollary 2.3.1 *Suppose that C commutes with K_i for all $i \in \mathbb{N}$. If system* (2.3) *has a nontrivial solution $x(t)$ such that $x(t_1^+) = x(t_2^-) = 0, x(t) \neq 0$ on (t_1, t_2), then for* *any $C^*(t) \geq C(t)$ we have the Lyapunov type inequality*

$$\int_{t_1}^{t_2} |C^*(t)|dt + \sum_{t_1 \leq \tau_i < t_2} |S_i| \geq \frac{4}{t_2 - t_1},$$

where $S_i = K_1^{-1} K_2^{-1} \ldots K_i^{-1} L_i K_{i-1} \ldots K_1,\ i = 1, 2, \ldots$.

Corollary 2.3.2 *Suppose that $K_i^T K_i = I$ for all $i \in \mathbb{N}$. If system (2.3) has a nontrivial solution $x(t)$ such that $x(t_1^+) = x(t_2^-) = 0, x(t) \neq 0$ on (t_1, t_2), then for any $C^*(t) \geq C(t)$ we have the Lyapunov type inequality*

$$\left(\prod_{t_1 \leq \tau_i < t_2} |K_i|^2 \right) \left[\int_{t_1}^{t_2} |C^*(s)|ds + \sum_{t_1 \leq \tau_i < t_2} |K_i^T L_i| \right] \geq \frac{4}{t_2 - t_1}.$$

In the case $n = 1$, or for system (2.4), the commutativity of the coefficient functions a, b, c with k_i for all $i \in \mathbb{N}$ is satisfied automatically, $\alpha(t) = |a(t)|$ and $S_i = l_i/k_i$. Besides, if $c^+(t)$ is the function which satisfies the condition $c^+(t) \geq c(t)$, then Theorem 2.3.1 is reduced to the following corollary under the following version of condition (2.25)

$$b(t) \geq 0, \qquad \int_{t_1}^{t_2} b(t)dt > 0. \tag{2.48}$$

Corollary 2.3.3 *Assume* (2.48) *holds. If system (2.4) has a solution $y(t) = (x(t), u(t))$ with nontrivial x such that $x(t_1^+) = x(t_2^-) = 0, x(t) \neq 0$ on (t_1, t_2), then we have the Lyapunov type inequality*

$$\exp\left(\int_{t_1}^{t_2} |a(t)|dt \right) \left[\int_{t_1}^{t_2} b(t)dt \right] \left[\int_{t_1}^{t_2} c^+(t)dt + \sum_{t_1 \leq \tau_i < t_2} \left(\frac{l_i}{k_i} \right)^+ \right] \geq 4. \tag{2.49}$$

Theorem 2.3.2 is adapted to system (2.4) as follows: The condition $K_i^T K_i = I$ for all $i \in \mathbb{N}$ turns out to be $k_i^2 = 1$ for all $i \in \mathbb{N}$. Therefore we have two cases due to the given k_i. Let us consider the case that $k_i = 1$ for all $i \in \mathbb{N}$ which implies the continuity of $x(t)$ for all $t \geq t_0$. Hence there is impulse condition only on $u(t)$ and Theorem 2.3.1 and Theorem 2.3.2 coincide. In the latter case, i.e there exists an $i_0 \in \mathbb{N}$ such that $k_{i_0} = -1$, system (2.4) has an impulse effect on both $x(t)$ and $u(t)$ and Theorem 2.3.2 is reduced to following corollary.

Corollary 2.3.4 *Let $k_i^2 = 1$ for all $i \in \mathbb{N}$. Suppose that* (2.48) *holds. If system (2.4) has a solution $y(t) = (x(t), u(t))$ with nontrivial x such that $x(t_1^+) = x(t_2^-) = 0, x(t) \neq 0$ on (t_1, t_2), then we have the Lyapunov type inequality*

$$\exp\left(\int_{t_1}^{t_2} |a(t)|dt\right)\left[\int_{t_1}^{t_2} b(t)dt\right]\left[\int_{t_1}^{t_2} c^+(t)dt + \sum_{t_1 \leq \tau_i < t_2} (l_i k_i)^+\right] \geq 4. \quad (2.50)$$

Remark 2.3.4 *Inequalities* (2.49) *and* (2.50) *generalize all the results obtained for planar Hamiltonian systems without impulse effect. In particular, Corollary 2.3.3 is an extension of [99, Theorem 2.4] while Corollary 2.3.4 is new and alternative to Corollary 2.3.3.*

The following corollaries are obtained directly from Theorem 2.3.1 and Theorem 2.3.2 for equation (2.5).

Corollary 2.3.5 *If equation* (2.5) *has a nontrivial solution $x(t)$ such that $x(t_1^+) = x(t_2^-) = 0, x(t) \neq 0$ on (t_1, t_2), then we have the Lyapunov type inequality*

$$\left[\int_{t_1}^{t_2} \frac{1}{p(t)}dt\right]\left[\int_{t_1}^{t_2} q^+(t)dt + \sum_{t_1 \leq \tau_i < t_2} \left(\frac{l_i}{k_i}\right)^+\right] \geq 4. \quad (2.51)$$

Corollary 2.3.6 *Let $k_i^2 = 1$ for all $i \in \mathbb{N}$. If equation* (2.5) *has a nontrivial solution $x(t)$ such that $x(t_1^+) = x(t_2^-) = 0, x(t) \neq 0$ on (t_1, t_2), then we have the Lyapunov type inequality*

$$\left[\int_{t_1}^{t_2} \frac{1}{p(t)}dt\right]\left[\int_{t_1}^{t_2} q^+(t)dt + \sum_{t_1 \leq \tau_i < t_2} (l_i k_i)^+\right] \geq 4.$$

Remark 2.3.5 *Corollary 2.3.5 and Corollary 2.3.6 provide the same result as and alternative result to [43, Theorem 4.5], respectively, for the case of the second order impulsive differential equations* (2.5). *Moreover if there is no impulse effect, they give the same result of Wintner, Hartman and Krein [114, 45, 57].*

2.4 Applications

In this section we give some applications of Lyapunov type inequalities which are used as a handy tool in studying of the qualitative nature of solutions. By making use of Lyapunov type inequalities, we prove disconjugacy criteria and find lower bounds for the eigenvalues of the related eigenvalue problems to systems (2.2), (2.3), (2.4) and to equation (2.5).

2.4.1 Disconjugacy

Since Lyapunov type inequality leads immediately to disconjugacy criteria, in this section we prove disconjugacy criteria for systems (2.2), (2.3), (2.4) and equation (2.5).

Note that if $B \equiv 0$, system (2.2) is already disconjugate.

Theorem 2.4.1 *Suppose that the matrices A, A^T, B, and C all commute with K_i for all $i \in \mathbb{N}$ such that* (2.25) *holds. If for some $C^*(t) \geq C(t)$*

$$\exp\left(\int_{t_1}^{t_2} \alpha(t)dt\right)\left(\int_{t_1}^{t_2} |B(t)|dt\right)\left(\int_{t_1}^{t_2} |C^*(t)|dt + \sum_{t_1 \leq \tau_i < t_2} |S_i|\right) < 4, \quad (2.52)$$

then system (2.2) is disconjugate on $[t_1, t_2]$, where $\alpha(t)$ and S_i are defined as in Theorem 2.3.1.

Proof. Suppose on the contrary that there is a real solution $y(t) = (x(t), u(t))$ with nontrivial $x(t)$ having two zeros $s_1, s_2 \in [t_1, t_2]$ $(s_1 < s_2)$ such that $x(t) \neq 0$ for all $t \in (s_1, s_2)$. Applying Theorem 2.3.1 we see that

$$\begin{aligned} 4 &\leq \exp\left(\int_{s_1}^{s_2} \alpha(t)dt\right)\left(\int_{s_1}^{s_2} |B(t)|dt\right)\left(\int_{s_1}^{s_2} |C^*(t)|dt + \sum_{s_1 \leq \tau_i < s_2} |S_i|\right) \\ &\leq \exp\left(\int_{t_1}^{t_2} \alpha(t)dt\right)\left(\int_{t_1}^{t_2} |B(t)|dt\right)\left(\int_{t_1}^{t_2} |C^*(t)|dt + \sum_{t_1 \leq \tau_i < t_2} |S_i|\right). \end{aligned}$$

Clearly, the last inequality contradicts (2.52). The proof is complete. □

Since the proof of the following theorem is exactly same as the proof of Theorem 2.4.1, it is omitted.

Theorem 2.4.2 *Suppose that $K_i^T K_i = I$ for all $i \in \mathbb{N}$ and* (2.25) *holds. If for some*

$C^*(t) \geq C(t)$

$$\exp\left(\int_{t_1}^{t_2} \alpha(t)dt\right)\left(\prod_{t_1\leq\tau_i<t_2} |K_i|^2\right)\left[\int_{t_1}^{t_2} |B(s)|ds\right] \times \left[\int_{t_1}^{t_2} |C^*(s)|ds + \sum_{t_1\leq\tau_i<t_2} |K_i^T L_i|\right] < 4,$$

then system (2.2) is disconjugate on $[t_1, t_2]$, where $\alpha(t)$ is defined as in Theorem 2.3.1.

Remark 2.4.1 *Disconjugacy on $[t_1, t_2]$ is equivalent to nonexistence of a nontrivial solution of system (2.2) satisfying $x(t_1) = x(t_2) = 0$. This gives a sufficient condition for the uniqueness of solutions of the corresponding nonhomogeneous boundary problem which is studied in the next chapter.*

We have the following corollaries, which are obtained direcly from Theorem 2.4.1 and 2.4.2 for system (2.3) and system (2.4), whose proofs are exactly the same as the proof of Theorem 2.4.1, and so, omitted.

Corollary 2.4.1 *Suppose that the matrices C commutes with K_i for all $i \in \mathbb{N}$. If for some $C^*(t) \geq C(t)$*

$$\left(\int_{t_1}^{t_2} |C^*(t)|dt + \sum_{t_1\leq\tau_i<t_2} |S_i|\right) < \frac{4}{t_2 - t_1}, \tag{2.53}$$

then system (2.3) is disconjugate on $[t_1, t_2]$, where S_i is defined as in Theorem 2.3.1.

Corollary 2.4.2 *Suppose that $K_i^T K_i = I$ for all $i \in \mathbb{N}$. If for some $C^*(t) \geq C(t)$*

$$\left(\prod_{t_1\leq\tau_i<t_2} |K_i|^2\right)\left[\int_{t_1}^{t_2} |C^*(s)|ds + \sum_{t_1\leq\tau_i<t_2} |K_i^T L_i|\right] < \frac{4}{t_2 - t_1},$$

then system (2.3) is disconjugate on $[t_1, t_2]$.

Corollary 2.4.3 *Assume* (2.48) *holds. If*

$$\exp\left(\int_{t_1}^{t_2} |a(t)|dt\right)\left[\int_{t_1}^{t_2} b(t)dt\right]\left[\int_{t_1}^{t_2} c^+(t)dt + \sum_{t_1\leq\tau_i<t_2} \left(\frac{l_i}{k_i}\right)^+\right] < 4,$$

then system (2.4) is disconjugate on $[t_1, t_2]$.

Corollary 2.4.4 *Let $k_i^2 = 1$ for all $i \in \mathbb{N}$. Assume* (2.48) *holds. If*

$$\exp\left(\int_{t_1}^{t_2} |a(t)|dt\right)\left[\int_{t_1}^{t_2} b(t)dt\right]\left[\int_{t_1}^{t_2} c^+(t)dt + \sum_{t_1 \le \tau_i < t_2} (l_i k_i)^+\right] < 4,$$

then system (2.4) is disconjugate on $[t_1, t_2]$.

The next two corollaries are direct consequences of Corollary 2.4.3 and Corollary 2.4.4, respectively, in the case $b(t) > 0$.

Corollary 2.4.5 *If*

$$\left[\int_{t_1}^{t_2} \frac{1}{p(t)}dt\right]\left[\int_{t_1}^{t_2} q^+(t)dt + \sum_{t_1 \le \tau_i < t_2} \left(\frac{l_i}{k_i}\right)^+\right] < 4, \tag{2.54}$$

then equation (2.5) *is disconjugate on $[t_1, t_2]$.*

Proof. Suppose on the contrary that there is a real solution $x(t) \not\equiv 0$ having two zeros $s_1, s_2 \in [t_1, t_2]$ $(s_1 < s_2)$ such that $x(t) \neq 0$ for all $t \in (s_1, s_2)$. Applying Corollary 2.3.5 we see that

$$\begin{aligned} 4 &\le \left[\int_{s_1}^{s_2} \frac{1}{p(t)}dt\right]\left[\int_{s_1}^{s_2} q^+(t)dt + \sum_{s_1 \le \tau_i < s_2} \left(\frac{l_i}{k_i}\right)^+\right] \\ &\le \left[\int_{t_1}^{t_2} \frac{1}{p(t)}dt\right]\left[\int_{t_1}^{t_2} q^+(t)dt + \sum_{t_1 \le \tau_i < t_2} \left(\frac{l_i}{k_i}\right)^+\right]. \end{aligned}$$

Clearly, the last inequality contradicts (2.54). The proof is complete. $\square$

Since the proof of the last corollary is the same as the proof of Corollary 2.4.5, it is omitted.

Corollary 2.4.6 *Let $k_i^2 = 1$ for all $i \in \mathbb{N}$. If*

$$\left[\int_{t_1}^{t_2} \frac{1}{p(t)}dt\right]\left[\int_{t_1}^{t_2} q^+(t)dt + \sum_{t_1 \le \tau_i < t_2} (l_i k_i)^+\right] < 4,$$

then equation (2.5) *is disconjugate on $[t_1, t_2]$.*

2.4.2 Lower Bounds on Eigenvalues

Another application of Lyapunov type inequalities is that it can be used to provide lower bounds for the eigenvalues of the associated eigenvalue problems. The proofs of the following theorems and corollaries are based on the Lyapunov type inequalities derived in Section 2.3.

Consider the following impulsive eigenvalue problems

$$\begin{aligned}
&x' = A(t)x + B(t)u, \quad u' = -\lambda\, C(t)x - A^T(t)u, \quad t \in (t_1, t_2)\backslash\{\tau_1, \tau_1, \dots, \tau_m\},\\
&x(\tau_i^+) = K_i x(\tau_i^-), \quad u(\tau_i^+) = -\mu\, L_i x(\tau_i^-) + K_i u(\tau_i^-), \quad i = 1, 2, \dots, m,\\
&x(t_1) = x(t_2) = 0
\end{aligned} \tag{2.55}$$

and

$$\begin{aligned}
&x'' + \lambda C(t)x = 0, \quad t \in (t_1, t_2)\backslash\{\tau_1, \tau_1, \dots, \tau_m\},\\
&x(\tau_i^+) = K_i x(\tau_i^-),\\
&x'(\tau_i^+) = -\mu\, L_i x(\tau_i^-) + K_i x'(\tau_i^-), \quad i = 1, 2, \dots, m,\\
&x(t_1) = x(t_2) = 0
\end{aligned} \tag{2.56}$$

where $\lambda, \mu \in \mathbb{R}$.

Definition 2.4.1 *A pair* (λ, μ) *is called an eigenvalue of* (2.55) *if there is a corresponding solution* (x, u) *such that* $x(t) \not\equiv 0$ *on* (t_1, t_2).

Definition 2.4.2 *A pair* (λ, μ) *is called an eigenvalue of* (2.56) *if there is a corresponding nontrivial solution* x *on* (t_1, t_2).

Theorem 2.4.3 *Suppose that the matrices* A, A^T, B, *and* C *all commute with* K_i *for all* $i \in \mathbb{N}$ *such that* (2.25) *holds. If* (λ, μ) *is a positive eigenvalue pair of* (2.55)*, then*

$$\lambda \geq \frac{4}{\exp\left(\int_{t_1}^{t_2} \alpha(t)dt\right)\left(\int_{t_1}^{t_2} |B(t)|dt\right)\left(\int_{t_1}^{t_2} |C^*(t)|dt + \eta \sum_{i=1}^{m} |S_i|\right)},$$

for any $C^*(t) \geq C(t)$*, where* $\eta = \mu/\lambda$ *and* $\alpha(t),\ S_i$ *are defined as in Theorem 2.3.1.*

Proof. Let (λ, μ) be a positive eigenvalue and (x, u) be the corresponding eigenfunctions of the system (2.55). If we apply Lyapunov type inequality obtained in Theorem

2.3.1 for system (2.55), we get

$$4 \leq \exp\left(\int_{t_1}^{t_2} \alpha(t)dt\right)\left(\int_{t_1}^{t_2} |B(t)|dt\right)\left(\lambda \int_{t_1}^{t_2} |C^*(t)|dt + \mu \sum_{t_1 \leq \tau_i < t_2} |S_i|\right)$$

Then for the eigenvalue λ we can find the desired lower bound. $\square$

Since the proofs of the following theorem and corollaries are same as the proof of Theorem 2.4.3, we skip them.

Theorem 2.4.4 *Suppose that $K_i^T K_i = I$ for all $i \in \mathbb{N}$ and* (2.25) *holds. If (λ, μ) is a positive eigenvalue pair of* (2.55)*, then*

$$\lambda \geq \frac{4}{\exp\left(\int_{t_1}^{t_2} \alpha(t)dt\right)\left(\prod_{i=1}^{m} |K_i|^2\right)\left[\int_{t_1}^{t_2} |B(t)|dt\right]\left[\int_{t_1}^{t_2} |C^*(t)|dt + \eta \sum_{i=1}^{m} |K_i^T L_i|\right]},$$

for any $C^*(t) \geq C(t)$, where $\eta = \mu/\lambda$ and $\alpha(t)$ is defined as in Theorem 2.3.1.

For the eigenvalue problem (2.56), the above theorems take the following simpler forms.

Corollary 2.4.7 *Suppose that C commute with K_i for all $i \in \mathbb{N}$. If (λ, μ) is a positive eigenvalue pair of* (2.56)*, then*

$$\lambda \geq \frac{4}{\left[\int_{t_1}^{t_2} |C^*(t)|dt + \eta \sum_{i=1}^{m} |S_i|\right](t_2 - t_1)},$$

for any $C^*(t) \geq C(t)$, where $\eta = \mu/\lambda$ and S_i is defined as in Theorem 2.3.1.

Corollary 2.4.8 *Suppose that $K_i^T K_i = I$ for all $i \in \mathbb{N}$. If (λ, μ) is a positive eigenvalue pair of* (2.56)*, then*

$$\lambda \geq \frac{4}{\left(\prod_{i=1}^{m} |K_i|^2\right)\left[\int_{t_1}^{t_2} |C^*(t)|dt + \eta \sum_{i=1}^{m} |K_i^T L_i|\right](t_2 - t_1)},$$

for any $C^*(t) \geq C(t)$, where $\eta = \mu/\lambda$.

In system (2.55) if $n = 1$, then the planar eigenvalue problem can be obtained as follows.

$$
\begin{aligned}
& x' = a(t)x + b(t)u, \quad u' = -\lambda\, c(t)x - a(t)u, \quad t \in (t_1, t_2)\backslash\{\tau_1, \tau_1, \dots, \tau_m\}, \\
& x(\tau_i^+) = k_i x(\tau_i^-), \quad u(\tau_i^+) = -\mu\, l_i x(\tau_i^-) + k_i u(\tau_i^-), \quad i = 1, 2, \dots, m, \\
& x(t_1) = x(t_2) = 0.
\end{aligned} \tag{2.57}
$$

where $\lambda, \mu \in \mathbb{R}$.

Definition 2.4.3 *A pair* (λ, μ) *is called an eigenvalue of* (2.57) *if there is a corresponding solution* (x, u) *such that* $x(t) \not\equiv 0$ *on* (t_1, t_2).

Then Theorem 2.4.3 and Theorem 2.4.4 are reduced to the following corollaries.

Corollary 2.4.9 *Assume* (2.48) *holds. If* (λ, μ) *is a positive eigenvalue pair of* (2.57)*, then*

$$
\lambda \geq \frac{4}{\exp\left(\int_{t_1}^{t_2} |a(t)| dt\right) \left[\int_{t_1}^{t_2} b(t) dt\right] \left[\int_{t_1}^{t_2} c^+(t) dt + \eta \sum_{i=1}^{m} \left(\frac{l_i}{k_i}\right)^+\right]},
$$

where $\eta = \mu/\lambda$.

Corollary 2.4.10 *Let* $k_i^2 = 1$ *for all* $i \in \mathbb{N}$. *Assume* (2.48) *holds. If* (λ, μ) *is a positive eigenvalue pair of* (2.57)*, then*

$$
\lambda \geq \frac{4}{\exp\left(\int_{t_1}^{t_2} |a(t)| dt\right) \left[\int_{t_1}^{t_2} b(t) dt\right] \left[\int_{t_1}^{t_2} c^+(t) dt + \eta \sum_{i=1}^{m} (k_i l_i)^+\right]},
$$

where $\eta = \mu/\lambda$.

By taking $a(t) \equiv 0$, $b(t) = 1/p(t)$, $c(t) = q(t)$ and $u(t) = p(t)x'(t)$, we obtain the special case of (2.57), the impulsive second-order eigenvalue problem, which has the form

$$
\begin{aligned}
& (p(t)x')' + \lambda q(t)x = 0, \qquad\qquad t \neq \tau_i \\
& x(\tau_i^+) = k_i x(\tau_i^-), \quad (px')(\tau_i^+) = -\mu l_i x(\tau_i^-) + k_i (px')(\tau_i^-), \quad i = 1, 2, \dots, m, \\
& x(t_1) = x(t_2) = 0
\end{aligned} \tag{2.58}
$$

where $\lambda, \mu \in \mathbb{R}$.

Definition 2.4.4 *A pair* (λ, μ) *is called an eigenvalue of* (2.58) *if there is a corresponding nontrivial solution* x *on* (t_1, t_2).

Theorem 2.4.3 and Theorem 2.4.4 lead to the following corollaries which are the final results of the present chapter.

Corollary 2.4.11 *If* (λ, μ) *is a positive eigenvalue pair of* (2.58)*, then*

$$\lambda \geq \frac{4}{\left[\int_{t_1}^{t_2} \frac{1}{p(t)} dt\right] \left[\int_{t_1}^{t_2} q^+(t) dt + \eta \sum_{i=1}^{m} \left(\frac{l_i}{k_i}\right)^+\right]},$$

where $\eta = \mu/\lambda$.

Corollary 2.4.12 *Let* $k_i^2 = 1$ *for all* $i \in \mathbb{N}$. *If* (λ, μ) *is a positive eigenvalue pair of* (2.58)*, then*

$$\lambda \geq \frac{4}{\left[\int_{t_1}^{t_2} \frac{1}{p(t)} dt\right] \left[\int_{t_1}^{t_2} q^+(t) dt + \eta \sum_{i=1}^{m} (k_i l_i)^+\right]},$$

where $\eta = \mu/\lambda$.

CHAPTER 3

BOUNDARY VALUE PROBLEMS FOR $2N$-DIMENSIONAL LINEAR HAMILTONIAN SYSTEMS WITH IMPULSIVE PERTURBATIONS

3.1 Introduction

In the present chapter our main aim is to prove an existence and uniqueness theorem for solutions of the related BVP of (2.2), which is called as inhomogeneous Hamiltonian system under impulse effect, of the form

$$x' = A(t)x + B(t)u + f(t), \quad u' = -C(t)x - A^T(t)u + g(t), \qquad t \neq \tau_i \tag{3.1a}$$

$$x(\tau_i^+) = K_i x(\tau_i^-) + a_i, \quad u(\tau_i^+) = -L_i x(\tau_i^-) + K_i u(\tau_i^-) + b_i, \tag{3.1b}$$

$$x(t_1) = \xi, \quad x(t_2) = \zeta, \tag{3.1c}$$

where

(i) The entries of the given $n \times n$ matrices A and symmetric matrices B, C and of the given $n \times 1$ vectors f, g are real valued and left continuous functions on each interval (τ_i, τ_{i+1}) having finite limit from both sides at τ_i;

(ii) $\{K_i\}, \{L_i\}$ are given sequence of $n \times n$ matrices $\{a_i\}, \{b_i\}$ are given sequence of $n \times 1$ vectors, $\{\tau_i\}$ is a real sequence of numbers for $i = 1, 2, \ldots, p$ with $t_1 = \tau_0 < \tau_1 < \tau_2 < \ldots < \tau_p < \tau_{p+1} = t_2$;

(iii) $B(t) \geq 0$ for $t \in (t_1, t_2)$ in the sense that $x^T B(t) x \geq 0$ for all $x \in \mathbb{R}^n$ and K_i^{-1} exists for all $i = 1, 2, \ldots, p$; ξ and ζ are given $n \times 1$ vectors.

By a solution of system (3.1), we mean a vector valued function $y(t) = (x(t), u(t))$ defined on $J = [t_1, t_2]$ such that $y \in PLC(J)$ and system (3.1) is fulfilled for all $t \in J$, where $PLC(J) = \{\omega : J \to \mathbb{R}$ is continuous on each interval (τ_i, τ_{i+1}), the limits $w(\tau_i^{\pm})$ exist and $w(\tau_i^-) = w(\tau_i)$ for $i \in \mathbb{N}\}$.

The corresponding homogeneous BVP of system (3.1) takes the form

$$x' = A(t)x + B(t)u, \quad u' = -C(t)x - A^T u, \qquad t \in (t_1, t_2)\backslash\{\tau_i\} \tag{3.2a}$$

$$x(\tau_i^+) = K_i x(\tau_i^-), \quad u(\tau_i^+) = -L_i x(\tau_i^-) + K_i u(\tau_i^-), \tag{3.2b}$$

$$x(t_1) = 0, \quad x(t_2) = 0, \tag{3.2c}$$

If $y = (x, u)$, system (3.2a)-(3.2a) can be written in the form of

$$y' = JH(t)y \qquad t \neq \tau_i \tag{3.3a}$$

$$y(\tau_i^+) = B_i y(\tau_i^-), \quad i = 1, 2, \ldots, p \tag{3.3b}$$

where

$$J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}, \; H(t) = \begin{bmatrix} C(t) & A^T(t) \\ A(t) & B(t) \end{bmatrix}, \; B_i = \begin{bmatrix} K_i & 0 \\ -L_i & K_i \end{bmatrix}.$$

The impulse condition (3.3b) (or (3.2b)) can be written in another form by using delta operator as

$$\Delta y|_{t=\tau_i} = y(\tau_i^+) - y(\tau_i^-) = (B_i - I)y(\tau_i^-) = C_i y(\tau_i^-) \tag{3.4}$$

Let us recall the following definition. With regard to Definition 1.4.2, we want to remark that if

$$\begin{aligned} A^T(t)C(t) = C(t)A(t), \quad -B(t)C(t) + A(t)A(t) = -I_n, \\ B(t)A^T(t) = A(t)B(t), \quad -A^T(t)A^T(t) + C(t)B(t) = I_n, \end{aligned}$$

then system (3.2a) is of symplectic structure and therefore our results are also valid for symplectic systems under impulse effect.

In particular, choosing $n = 1$ in system (3.1) yields the following inhomogenous BVP for planar Hamiltonian system under impulse effect

$$x' = a(t)x + b(t)u + f(t), \quad u' = -c(t)x - a(t)u + g(t), \qquad t \neq \tau_i \tag{3.5a}$$

$$x(\tau_i^+) = k_i x(\tau_i^-) + a_i, \quad u(\tau_i^+) = -l_i x(\tau_i^-) + k_i u(\tau_i^-) + b_i, \tag{3.5b}$$

$$x(t_1) = \xi, \quad x(t_2) = \zeta, \tag{3.5c}$$

where a, b, c, f, g are real valued piece-wise left continous functions having discontinuities at the points τ_i and k_i, l_i, a_i, b_i are real sequences for $i = 1, 2, \dots, p$ and ξ and ζ are given real numbers. The associated homogenous BVP is obtained for system (3.5) if $f(t) = g(t) = 0,\ t \in J,\ a_i = b_i = 0,\ i = 1, 2, \dots, p$ and $\xi = \zeta = 0$, i.e

$$x' = a(t)x + b(t)u, \quad u' = -c(t)x - a(t)u, \qquad t \neq \tau_i \tag{3.6a}$$

$$x(\tau_i^+) = k_i x(\tau_i^-), \quad u(\tau_i^+) = -l_i x(\tau_i^-) + k_i u(\tau_i^-), \tag{3.6b}$$

$$x(t_1) = 0, \quad x(t_2) = 0, \tag{3.6c}$$

Note that when $b(t) > 0$, if we take $a(t) \equiv 0$, $b(t) = 1/p(t)$, $c(t) = q(t)$, $f(t) \equiv 0$ and $u(t) = p(t)x'(t)$, then we obtain the following impulsive BVP for second-order differential equations, as a special case of (3.5), that is,

$$(p(t)x')' + q(t)x = g(t), \qquad t \in (t_1, t_2)\backslash\{\tau_i\} \tag{3.7a}$$

$$x(\tau_i^+) = k_i x(\tau_i^-) + a_i, \quad (px')(\tau_i^+) = -l_i x(\tau_i^-) + k_i (px')(\tau_i^-) + b_i, \tag{3.7b}$$

$$x(t_1) = \xi, \quad x(t_2) = \zeta. \tag{3.7c}$$

The following BVP represents the associated homogenous BVP of (3.7).

$$(p(t)x')' + q(t)x = 0, \qquad t \in (t_1, t_2)\backslash\{\tau_i\} \tag{3.8a}$$

$$x(\tau_i^+) = k_i x(\tau_i^-), \quad (px')(\tau_i^+) = -l_i x(\tau_i^-) + k_i (px')(\tau_i^-), \tag{3.8b}$$

$$x(t_1) = 0, \quad x(t_2) = 0. \tag{3.8c}$$

This chapter of the thesis is organized as follows. The proof of our result is based on establishing Lyapunov type inequalities for linear Hamiltonian system under impulse effect. Therefore after defining inhomogenous impulsive BVPs of $2n-$dimensional Hamiltonian systems, planar systems and second order equations as well as their homogenous counterparts, in Section 3.2 we restate Lyapunov type inequalities obtained in Section 2.3 to show nonexistence of nontrivial solutions of systems (3.2), (3.6) and equation (3.8). Then in Section 3.3 we present fundamental theorems about homogenous and nonhomogenous system of impulsive differential equations and give the relationship between solutions of them. Section 3.4 is divided into subsections to mention the properties of impulsive BVPs in detail. After introducing inhomogenous BVP and defining Green's function, the derivation of Green's function is shown and integral representation of unique solution of system (3.1) is expressed by Green's function. Then the properties of Green's function is given. After that system (3.5) and equation (3.7) is considered as a special case of system (3.1) and their Green's functions are written in terms of the solutions of corresponding homogenous system and equation. Section 3.5 contains the main results of the present chapter and is devoted to the existence and uniqueness criteria of solutions to (3.1), (3.5) and equation (3.7). Since the proofs of the theorems are based on Lyapunov type inequalities, two different inequalities for each system (3.2), (3.6) and for equation (3.8) yield two alternative criteria for the uniqness of the solutions of systems (3.1), (3.5) and equation (3.7). To the best of our knowledge, our approach is quite new and our result is the first work which gives the relation between existence and uniqueness theory of boundary value problems and Lyapunov type inequalities. This relation has not been noted even for the ordinary differential equations case.

3.2 Lyapunov type inequality for homogeneous problems

The following theorems are obtained in Section 2.3 to derive Lyapunov type inequalities for systems (3.2), (3.6) and equation (3.8). The importance of these inequalities in showing the uniqueness of the solutions of inhomogeneous BVP (3.1), (3.5) and (3.7) is the main result of this chapter and presented in the last section. Before concerning the connection between Lyapunov type inequalities and inhomogeneous BVP, we want to remind these inequalities obtained for system (3.2), (3.6) and equation (3.8). In the sequel, $|A| = \sqrt{\lambda_{\max}(A^T A)}$ and $\mu(A) = \lambda_{\max}(A^T + A)/2$, which are defined as in Chapter 2, denote the induced matrix norm and matrix measure of a matrix A, respectively, $m^+(t) = \max\{m(t), 0\}$ and $m_i^+ = \max\{m_i, 0\}$. The first two theorems are the main results which yield Lyapunov type inequalities for system (3.2).

Theorem 3.2.1 *Suppose that the matrices A, A^T, B, and C all commute with K_i for all $i \in \mathbb{N}$ such that*

$$B(t) \geq 0, \quad \int_{t_1}^{t_2} |B(t)|\, dt > 0. \tag{3.9}$$

If the homomogeneous BVP (3.2) *has a real solution $(x(t), u(t))$ such that $x(t) \not\equiv 0$*

on (t_1, t_2)*, then for any* $C^*(t) \geq C(t)$*, we have the Lyapunov type inequality*

$$\exp\left(\int_{t_1}^{t_2} \alpha(t)dt\right)\left[\int_{t_1}^{t_2} |B(t)|dt\right]\left[\int_{t_1}^{t_2} |C^*(t)|dt + \sum_{t_1 \leq \tau_i < t_2} |S_i|\right] \geq 4 \qquad (3.10)$$

where $S_i = K_1^{-1}K_2^{-1}\dots K_i^{-1}L_iK_{i-1}\dots K_1,\ i = 2, 3\dots, p$*, and*

$\alpha(t) = \max\left\{\mu^+(A(t)), \mu^+(-A(t))\right\}$.

Theorem 3.2.2 *Suppose that* $K_i^T K_i = I$ *for all* $i \in \mathbb{N}$ *and* (3.9) *holds. If the homomogeneous BVP* (3.2) *has a real solution* $(x(t), u(t))$ *such that* $x(t) \not\equiv 0$ *on* (t_1, t_2)*, then for any* $C^*(t) \geq C(t)$ *we have the Lyapunov type inequality*

$$\begin{aligned}\exp\left(\int_{t_1}^{t_2} \alpha(t)dt\right)&\left[\prod_{t_1 \leq \tau_i < t_2} |K_i|^2\right]\left[\int_{t_1}^{t_2} |B(s)|ds\right] \\ &\times\left[\int_{t_1}^{t_2} |C^*(s)|ds + \sum_{t_1 \leq \tau_i < t_2} |K_i^T L_i|\right] \geq 4,\end{aligned} \qquad (3.11)$$

where $\alpha(t)$ *is defined as in Theorem 3.2.1.*

The next two theorems, which are corollaries of the above theorems, provide different Lyapunov type inequalities for system (3.6).

Theorem 3.2.3 *Suppose that*

$$b(t) \geq 0, \qquad \int_{t_1}^{t_2} b(t)dt > 0. \qquad (3.12)$$

If the homomogeneous BVP of (3.6) *has a real solution* $(x(t), u(t))$ *such that* $x(t) \not\equiv 0$ *on* (t_1, t_2)*, then we have the Lyapunov type inequality*

$$\exp\left(\int_{t_1}^{t_2} |a(t)|dt\right)\left[\int_{t_1}^{t_2} b(t)dt\right]\left[\int_{t_1}^{t_2} c^+(t)dt + \sum_{t_1 \leq \tau_i < t_2} \left(\frac{l_i}{k_i}\right)^+\right] \geq 4. \qquad (3.13)$$

Theorem 3.2.4 *Suppose* (3.12) *holds. If the homomogeneous BVP of* (3.6) *has a real solution* $(x(t), u(t))$ *such that* $x(t) \not\equiv 0$ *on* (t_1, t_2)*, then we have the Lyapunov type inequality*

$$\exp\left(\int_{t_1}^{t_2} |a(t)|dt\right)\left[\int_{t_1}^{t_2} b(t)dt\right]\left[\int_{t_1}^{t_2} c^+(t)dt + \sum_{t_1 \leq \tau_i < t_2} (l_ik_i)^+\right] \geq 4. \qquad (3.14)$$

Finally, we have the next two theorems giving Lyapunov type inequalities for equation (3.8).

Theorem 3.2.5 *If the homomogeneous BVP of* (3.8) *has a real solution* $x(t)$ *such that* $x(t) \not\equiv 0$ *on* (t_1, t_2)*, then we have the Lyapunov type inequality*

$$\left[\int_{t_1}^{t_2} \frac{1}{p(t)} dt\right] \left[\int_{t_1}^{t_2} q^+(t) dt + \sum_{t_1 \le \tau_i < t_2} \left(\frac{l_i}{k_i}\right)^+\right] \ge 4. \tag{3.15}$$

Theorem 3.2.6 *If the homomogeneous BVP of* (3.8) *has a real solution* $x(t)$ *such that* $x(t) \not\equiv 0$ *on* (t_1, t_2)*, then we have the Lyapunov type inequality*

$$\left[\int_{t_1}^{t_2} \frac{1}{p(t)} dt\right] \left[\int_{t_1}^{t_2} q^+(t) dt + \sum_{t_1 \le \tau_i < t_2} (l_i k_i)^+\right] \ge 4. \tag{3.16}$$

3.3 System of Linear Homogenous and Nonhomogenous Impulsive Differential Equations

Before considering impulsive inhomogeneous BVP (3.1), we will give fundamental properties of system of linear impulsive equations (IDEs) of homogenous type (3.2a)-(3.2b) (or equivalently (3.3)). These properties are similar to that of in system of ordinary differential equations, see [25, 45, 76] and can be found in detail in [60, 9, 93, 107]. A nonhomogenous system of impulsive differential equations (3.1a)-(3.1b) can be defined similar to the theory of nonhomogenous system of ordinary differential equations and can also be written in the form

$$y' = JH(t)y + h(t) \qquad t \neq \tau_i \tag{3.17a}$$

$$y(\tau_i^+) = B_i y(\tau_i^-) + c_i, \quad i = 1, 2, \ldots, p, \tag{3.17b}$$

where $J, H(t), B_i$ are defined as in system (3.3) and

$$h(t) = \begin{bmatrix} f(t) \\ g(t) \end{bmatrix}, \quad c_i = \begin{bmatrix} a_i \\ b_i \end{bmatrix}.$$

The fundamental matrix of homogenous system (3.3) (or (3.2a)-(3.2b)) has the following property due to the Theorem 1.3.4.

Remark 3.3.1 ([42]) *Let $\Phi(t)$ be the fundamental matrix of system (3.3) satisfying $\det \Phi(0) = I$. Since $trace(JH(s)) = 0$, then Wronskian $W(t)$ of solutions of system (3.3) is*

$$W(t) = \det \Phi(t) = \prod_{i=1}^{p} \det B_i = \prod_{i=1}^{p} K_i^2$$

for $t_1 = \tau_0 < \tau_1 < \ldots < \tau_p < \tau_{p+1} = t_2$.

After defining the solution of homogenous system (3.3), now we are ready to give the particular solution, general solution and unique solution of nonhomogenous system (3.17). The relationship between the solutions of nonhomogenous system (3.17) and the associated homogenous system (3.3) can be given in the next theorem.

Theorem 3.3.1 ([93, 107]) *If $\varphi(t)$ and $\psi(t)$ are the solutions of homogenous system (3.3) and nonhomogenous system (3.17), respectively, then $\varphi(t) + \psi(t)$ is again a solution of system (3.17). Conversely, if $\psi_1(t), \psi_2(t)$ are solutions of nonhomogenous system (3.17), then the difference $\psi_1(t) - \psi_2(t)$ is a solution of homogenous system (3.3).*

We find the general solution of nonhomogenous system (3.17) in terms of the solutions of homogenous system (3.3) by using Variation of Parameters Formula.

Theorem 3.3.2 ([93, 107]) *(Variation of Parameters Formula) Let $\Phi(t)$ be a fundamental matrix solution of system (3.3). Then the general solution y of nonhomogenous system (3.17) is of the form*

$$y(t) = \Phi(t)c + \int_{t_0}^{t} \Phi(t,s)h(s)ds + \sum_{t_0 \le \tau_i < t} \Phi(t, \tau_i^+)c_i, \;\; t \ge t_0, \tag{3.18}$$

$$y(t) = \Phi(t)c + \int_{t_0}^{t} \Phi(t,s)h(s)ds + \sum_{t_0 \le \tau_i < t} \Phi(t, \tau_i^+)c_i, \;\; t \le t_0,$$

where c is the column vector defined as $c = [c_1, c_2, \ldots, c_n]^T$. The unique solution $y(t, t_0)$ of nonhomogenous system (3.17) satisfying the initial condition $y(t_0) = \delta$ is of the following form

$$y(t, t_0) = \Phi(t, t_0)\delta + \int_{t_0}^{t} \Phi(t,s)h(s)ds + \sum_{t_0 \le \tau_i < t} \Phi(t, \tau_i^+)c_i, \;\; t \ge t_0,$$

and

$$y(t,t_0) = \Phi(t,t_0)\delta + \int_{t_0}^{t} \Phi(t,s)h(s)ds - \sum_{t \le \tau_i < t_0} \Phi(t,\tau_i^+)c_i, \;\; t \le t_0,$$

where $\Phi(t,s) = \Phi(t)\Phi^{-1}(s)$.

The previous theorem states that like linear system of ordinary differential equations, the general solution y of linear nonhomogenous system (3.17) can be written as a sum of complementary solution $y_h(t) = \Phi(t)c$ of homogenous system (3.3) and particular solution $y_p(t) = \displaystyle\int_{t_0}^{t} \Phi(t,s)h(t) + \sum_{t_0 \le \tau_i < t} \Phi(t,\tau_i^+)c_i$ of nonhomogenous system (3.17), i.e $y(t) = y_h(t) + y_p(t)$, see [93, 107].

3.4 Boundary Value Problems For $2n$-dimensional Impulsive Systems

Throughout this section, we consider impulsive BVP (3.1) and present the unique solution of this system. By using the connection between the solutions of homogenous systems (3.2) and inhomogenous BVP's (3.1), we introduce Green's function as well as its properties. In contrast to system of ODEs, there is a pair of Green's function which has discontinuities at the jump points due to the impulses. We also remark the importance of Green's function in obtaining the representation of unique solution of inhomogenous BVP (3.1). Moreover for the special cases of impulsive BVP (3.1), system (3.5) and equation (3.7), Green's function (pair) and representation of unique solution are obtained as an application.

3.4.1 Inhomogenous Boundary Value Problems

In this subsection, we are interested in inhomogenous BVP, system (3.1). If $y(t) = (x(t), u(t))$, then system (3.1) can be rewritten in the form of $2n$-dimensional Hamiltonian system with impulsive perturbation as

$$y' = JH(t)y + h(t) \qquad t \neq \tau_i \tag{3.19a}$$

$$y(\tau_i^+) = B_i y(\tau_i^-) + c_i, \quad i = 1, 2, \ldots, p \tag{3.19b}$$

$$U(y) = My(t_1) + Ny(t_2) = \eta, \tag{3.19c}$$

where $J, H(t), B_i$ are defined as in Section 3.1 and $h(t), c_i$ is defined as in Section 3.3 and $M = \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix}$, $N = \begin{bmatrix} 0 & 0 \\ I_n & 0 \end{bmatrix}$, and $\eta = \begin{bmatrix} \xi \\ \zeta \end{bmatrix}$.

Here, $U(y)$ is called a boundary form and in general it is defined as follows

$$U_\nu(y) = \sum_{j=1}^{2n} M_{\nu j} y_j(t_1) + N_{\nu j} y_j(t_2), \quad \nu = 1, 2, \ldots, 2n.$$

Remark 3.4.1 ([107]) *The vector boundary form U has rank m if $U_\upsilon,\ \upsilon = 1, 2, \ldots, m$, are linearly independent boundary forms. In other words, the vector boundary form U has rank m if and only if $rank(M : N) = m$, where the matrix $(M : N)$ is defined by*

$$(M : N) = \begin{bmatrix} M_{11} & \ldots & M_{12n} & N_{11} & \ldots & N_{12n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ M_{2n1} & \ldots & M_{2n2n} & N_{2n1} & \ldots & N_{2n2n} \end{bmatrix}.$$

Remark 3.4.2 *In our case, by definition of M and N, it is easy to see that $rank(M : N) = 2n$. Therefore $U_\upsilon,\ \upsilon = 1, 2, \ldots, 2n$, is linearly independent boundary form.*

As in the theory of system of ordinary differential equations, the uniqueness of solution of inhomogenous system (3.1) (or (3.19)) depends on the nonexistence of nontrivial solution of homogenous system (3.2). The next theorem stating this fact is the main argument to show the uniqueness of solution of inhomogenous system (3.1).

Theorem 3.4.1 ([107]) *Since the rank of the vector boundary form U is equal to the dimension of system (3.1), that is $rank(M : N) = 2n$, if the homogenous system (3.2) has only trivial solution, then the inhomogenous system (3.1) has a unique solution.*

3.4.2 Derivation of Green's Function

In the next theorem with regard of Theorem 3.4.1, by assuming that the homogenous system (3.2) has only trivial solution, we show that the unique solution of inhomogenous system (3.1) can be given in terms of *Green's Function.* To find Green's function, firstly we need to write the general solution of inhomogenous system (3.1) by using Variation Parameters Formula on the interval $J = [t_1, t_2]$. The next theorem provides the detailed derivation and explicit formula for Green's function.

Theorem 3.4.2 *Let (i)–(iii) hold. Assume that the homogenous system (3.2) has only trivial solution. Then the unique solution $y = (x(t), u(t))$ of the inhomogenous system (3.1) is expressible as*

$$y(t) = w(t) + \int_{t_1}^{t_2} G(t,s)h(s)ds + \sum_{t_1 \le \tau_i < t_2} \tilde{G}(t, \tau_i^+)c_i \tag{3.20}$$

where

$$w(t) = \Phi(t)\left[M\Phi(t_1) + N\Phi(t_2)\right]^{-1}\eta,$$

and the Green's function pair is given by

$$G(t,s) = \begin{cases} \Phi(t)(I+K)\Phi^{-1}(s), & s < t \\ \Phi(t)K\Phi^{-1}(s), & s \ge t \end{cases} \tag{3.21}$$

and

$$\tilde{G}(t, \tau_i^+) = \begin{cases} \Phi(t)(I+K)\Phi^{-1}(\tau_i^+), & \tau_i < t \\ \Phi(t)K\Phi^{-1}(\tau_i^+), & \tau_i \ge t. \end{cases} \tag{3.22}$$

Proof. We start with the variation of parameters formula (3.18) and write the general solution of system (3.1a), (3.1b) as

$$y(t) = \Phi(t)c + \Phi(t)\left[\int_{t_1}^{t} \Phi^{-1}(s)h(s) + \sum_{t_1 \le \tau_i < t} \Phi^{-1}(\tau_i^+)c_i\right], \tag{3.23}$$

where c is a constant column vector which will be determined from the boundary conditions (3.1c). Now, imposing the vector boundary conditions (3.1c) on (3.23) yields

$$\left[M\Phi(t_1) + N\Phi(t_2)\right]c = \eta - N\Phi(t_2)\left[\int_{t_1}^{t_2} \Phi^{-1}(s)h(s) + \sum_{t_1 \le \tau_i < t_2} \Phi^{-1}(\tau_i^+)c_i\right]. \tag{3.24}$$

Since the homogenous system (3.2) has only trivial solution, the matrix $M\Phi(t_1) + N\Phi(t_2)$ has an inverse. Observe that $y(t) = \Phi(t)d$ is a general solution of homogenous system (3.2) where d is a constant column vector. The homogenous boundary conditions satisfy if $\left[M\Phi(t_1) + N\Phi(t_2)\right]d = 0$. If the matrix $M\Phi(t_1) + N\Phi(t_2)$ did not have an inverse, then d would be different than zero and the homogenous system

(3.2) would have other solutions than the trivial one. This contradiction leads us that the matrix $M\Phi(t_1) + N\Phi(t_2)$ has an inverse and we have the uniqueness of solutions of inhomogenous system (3.1). Setting

$$K = -\left[M\Phi(t_1) + N\Phi(t_2)\right]^{-1} N\Phi(t_2),$$

we may solve c as

$$c = \left[M\Phi(t_1) + N\Phi(t_2)\right]^{-1}\eta + K\left[\int_{t_1}^{t_2} \Phi^{-1}(s)h(s)ds + \sum_{t_1 \le \tau_i < t_2} \Phi^{-1}(\tau_i^+)c_i\right].$$

Hence,

$$\begin{aligned} y(t) &= \Phi(t)\left[M\Phi(t_1) + N\Phi(t_2)\right]^{-1}\eta \\ &\quad + \Phi(t)(I+K)\left[\int_{t_1}^{t} \Phi^{-1}(s)h(s)ds + \sum_{t_1 \le \tau_i < t} \Phi^{-1}(\tau_i^+)c_i\right] \\ &\quad + \Phi(t)K\left[\int_{t}^{t_2} \Phi^{-1}(s)h(s)ds + \sum_{t \le \tau_i < t_2} \Phi^{-1}(\tau_i^+)c_i\right]. \end{aligned}$$

Therefore the unique solution of the BVP (3.1a)–(3.1c) can be expressed as

$$y(t) = w(t) + \int_{t_1}^{t_2} G(t,s)h(s)ds + \sum_{t_1 \le \tau_i < t_2} \tilde{G}(t,\tau_i^+)c_i.$$

where the pair of functions (3.21) and (3.22) constitutes the Green's function for (3.1). □

3.4.3 Properties of Green's Function

Similar to the theory of system of ordinary differential equations, Green's fuction (pair) have some continuity and differenbility properties. Compared with the nonimpulsive case, Green's function (pair) $G(t,s)$ and $\tilde{G}(t,\tau_i^+),\ i = 1,2,\ldots,p$ of linear impulsive system (3.1) are left continuous functions having discontinuities of the first kind at the jump points $\tau_i,\ i = 1,2,\ldots,p$. To obtain more properties, we need to set up the following rectangles, see [107],

$$\begin{aligned} R_{11} &= [t_1,\tau_1] \times [t_1,\tau_1], \\ R_{i1} &= (\tau_{i-1},\tau_i] \times [t_1,\tau_1], \quad i = 2,3,\ldots,p+1 \\ R_{1j} &= [t_1,\tau_1] \times (\tau_{j-1},\tau_j], \quad j = 2,3,\ldots,p+1 \\ R_{ij} &= (\tau_{i-1},\tau_i] \times (\tau_{j-1},\tau_j], \quad i,j = 2,3,\ldots,p+1 \end{aligned}$$

and triangles

$$
\begin{aligned}
T^u &= \{(t,s) \in [t_1,t_2] \times [t_1,t_2] : s > t\} \\
T^l &= \{(t,s) \in [t_1,t_2] \times [t_1,t_2] : s < t\} \\
T^u_{ii} &= \{(t,s) \in R_{ii} : s > t\}, \quad i = 1,2,3,\ldots,p+1 \\
T^l_{ii} &= \{(t,s) \in R_{ii} : s < t\}, \quad i = 1,2,3,\ldots,p+1.
\end{aligned}
$$

as we construct for system of ordinary differential equations, see [25, 76]. In the impulsive case, since instead of one interval $[a,b]$, there are p subintervals which are in the form of $(\tau_i, \tau_{i+1}),\ i = 0,1,\ldots,p$, the only difference between continous and discontinous case occurs at τ_i, the points of discontinuities. We can summarized the mentioned properties of Green's function (pair) and give more of them in the next theorem.

Theorem 3.4.3 ([107]) *Let Green's function (pair) be defined as (3.21) and (3.22) for system* (3.1)*. Then we have the following properties.*

(G1) $G(t,s)$ *is continuous and bounded for* (t,s) *on the rectangles* R_{ij}, $i,j = 1,2\ldots,p+1$.

(G2) $\dfrac{\partial G(t,s)}{\partial t}$ *is continuous and bounded on the rectangles* R_{ij} *with* $i \neq j$ *and on the triangles* T^u_{ii} *and* T^l_{ii}*, i.e at the points* $t = s$ *and* $t = \tau_i,\ i = 1,2,3,\ldots,p$, $G(t,s)$ *fails to be continous and bounded.*

(G3) At the points $t = s$ *and* $t = \tau_i,\ i = 1,2,3,\ldots,p,\ G(t,s)$ *satisfies the following jump conditions;*

(a) $G(s^+,s) - G(s^-,s) = I,\ s \neq \tau_i$

(b) $G(\tau_i^+,\tau_i) - B_i G(\tau_i^-,\tau_i) = B_i$

(c) $\dfrac{\partial G(s^+,s)}{\partial t} - \dfrac{\partial G(s^-,s)}{\partial t} = JH(s),\ s \neq \tau_i$

(G4) $G(t,s)$*, considered as a function of t, is left continuous and satisfies*

$$
\begin{aligned}
& y' = JH(t)y, && t \in J_s \backslash \{\tau_i\} \\
& y(\tau_i+) = B_i y(\tau_i^-), && i \in \{i : \tau_i \in J_s\} \\
& My(t_1) + Ny(t_2) = 0
\end{aligned} \tag{3.25}
$$

where J_s *is any of the intervals* $[t_1,s)$ *or* $(s,t_2]$

(G5) $\Delta|_{t=\tau_i}\tilde{G}(t,\tau_i^+) = \tilde{G}(\tau_i^+,\tau_i^+) - \tilde{G}(\tau_i^-,\tau_i^+) = (B_i - I)\tilde{G}(\tau_i^-,\tau_i^+)$

(G6) $\tilde{G}(t,s)$, considered as a function of t, is left continuous and satisfies (3.25).

Proof. The proofs of (G1) and (G2) are similar to the that of in ordinary differential equations. Let us consider (G3).

(a) To see (a), we write for $s \neq \tau_i$,

$$\begin{aligned} G(s^+,s) - G(s^-,s) &= \Phi(s^+)(I+K)\Phi^{-1}(s) - \Phi(s^-)K\Phi^{-1}(s) \\ &= \Phi(s^-)(I+K)\Phi^{-1}(s) - \Phi(s^-)K\Phi^{-1}(s) = I. \end{aligned}$$

(b) (b) follows from

$$\begin{aligned} G(\tau_i^+,\tau_i) - B_iG(\tau_i^-,\tau_i) &= \left[\Phi(\tau_i^+)(I+K) - B_i\Phi(\tau_i^-)K\right]\Phi^{-1}(\tau_i) \\ &= \left[B_i\Phi(\tau_i^-)(I+K) - B_i\Phi(\tau_i^-)K\right]\Phi^{-1}(\tau_i) = B_i. \end{aligned}$$

(c) For (c), let $t \neq \tau_i$, then

$$\frac{\partial G(t,s)}{\partial t} = \begin{cases} \Phi'(t)(I+K)\Phi^{-1}(s) = JH(t)\Phi(t)(I+K)\Phi^{-1}(s), & s < t \\ \Phi'(t)K\Phi^{-1}(s) = JH(t)\Phi(t)K\Phi^{-1}(s), & s \geq t \end{cases}$$

and since $s \neq \tau_i$,

$$\begin{aligned} \frac{\partial G(s^+,s)}{\partial t} - \frac{\partial G(s^-,s)}{\partial t} &= J\left[H(s^+)\Phi(s^+)(I+K) - H(s^-)\Phi(s^-)K\right]\Phi^{-1}(s) \\ &= JH(s). \end{aligned}$$

Next, we consider (G4). By definition, it is easy to see that $G(t,s)$ is left continuous function at $t = \tau_i$. Let us consider the interval $[t_1, s)$. The latter is similar. The first equation in (3.25) is a direct consequences of (c) and the definition of $G(t,s)$. Clearly,

$$G(\tau_i^+,s) = \Phi(\tau_i^+)K\Phi^{-1}(s) = B_i\Phi(\tau_i^-)K\Phi^{-1}(s) = B_iG(\tau_i^-,s)$$

and

$$\begin{aligned} MG(t_1,s) + NG(t_2,s) &= M\Phi(t_1)K\Phi^{-1}(s) + N\Phi(t_2)(I+K)\Phi^{-1}(s) \\ &= \left\{\left[M\Phi(t_1) + N\Phi(t_2)\right]K + N\Phi(t_2)(I+K)\right\}\Phi^{-1}(s) = 0. \end{aligned}$$

The proofs of (G5) and (G6) are similar to (b) and (G4), respectively. □

The following theorem is adapted from [107] and is presented to introduce the method of finding Green's function $G(t,s)$.

Theorem 3.4.4 ([107]) *If the homogenous system (3.2) has only trivial solution, then the properties (G1)-(G3) uniquely determine the Green's function $G(t,s)$.*

Proof. Since $G(t,s)$ satisfies the first two homogenous equation of (3.25), it can be written as

$$G(t,s) = \begin{cases} \Phi(t)c(s), & s < t \\ \Phi(t)d(s), & s > t \end{cases}$$

for $t \in [t_1, t_2]$, where $\Phi(t)$ is the fundamental matrix solution of

$$\begin{aligned} y' &= JH(t)y, \qquad & t &\in J_s \backslash \{\tau_i\} \\ y(\tau_i +) &= B_i y(\tau_i^-), & i &\in \{i : \tau_i \in J_s\}. \end{aligned}$$

In view of (a) of (G2), if $s \neq \tau_i,\ i = 1, 2, \ldots, p$, then

$$I = G(s^+, s) - G(s^-, s) = \Phi(s^+)c(s) - \Phi(s^-)d(s) = \Phi(s^-)[c(s) - d(s)].$$

Therefore we have

$$c(s) - d(s) = \Phi^{-1}(s),\ s \neq \tau_i.$$

In the case $s = \tau_i,\ i \in \{1, 2, \ldots, p\}$, from (b) one has that

$$\begin{aligned} B_i &= G(\tau_i^+, \tau_i) - B_i G(\tau_i^-, \tau_i) = \Phi(\tau_i^+)c(\tau_i) - B_i \Phi(\tau_i^-)d(\tau_i) \\ &= B_i \Phi(\tau_i^-)c(\tau_i) - B_i \Phi(\tau_i^-)d(\tau_i) \end{aligned}$$

which implies

$$c(\tau_i) - d(\tau_i) = \Phi^{-1}(\tau_i),\ i \in \{1, 2, \ldots, p\}.$$

Therefore, for all $s \in [t_1, t_2]$,

$$c(s) - d(s) = \Phi^{-1}(s). \tag{3.26}$$

Due to the fact that the boundary condition $U(y) = 0$ must be satisfied by $G(t,s)$, which is considered as a function of t, one can obtain

$$0 = MG(t_1, s) + NG(t_2, s) = M\Phi(t_1)d(s) + N\Phi(t_2)c(s).$$

By using the relation in (3.26), we have

$$0 = M\Phi(t_1)d(s) + N\Phi(t_2)(d(s) + \Phi^{-1}(s)) = [M\Phi(t_1) + N\Phi(t_2)]d(s) + N\Phi(t_2)\Phi^{-1}(s).$$

Because of the argument used in Theorem 3.4.2, the matrix $M\Phi(t_1)+N\Phi(t_2)$ has as inverse, which yields

$$d(s)=K\Phi^{-1}(s),\ c(s)=(I+K)\Phi^{-1}(s),\ s\in[t_1,t_2].$$

Hence

$$G(t,s)=\begin{cases}\Phi(t)(I+K)\Phi^{-1}(s), & s<t\\ \Phi(t)K\Phi^{-1}(s), & s\geq t\end{cases}$$

exists and uniquely determined as a result of the left continuity of $G(t,s)$. □

3.4.4 Green's Function For Planar System

Since system (3.5) is 2-dimensional, we can present its Green's function explicitly.

Let

$$\Phi(t)=\begin{bmatrix}x_1(t) & x_2(t)\\ u_1(t) & u_2(t)\end{bmatrix},\ \Phi(0)=I$$

be a fundamental matrix for (3.6a)-(3.6b) (or equivalently (3.3), $n=1$). Define

$$M=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\text{ and }N=\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}.$$

For impulsive differential system (3.6a)-(3.6b) (or equivalently (3.3), $n=1$), it is known from Remark 3.3.1 that

$$\det\Phi(t)=\prod_{i=1}^{p}\det B_i=\prod_{i=1}^{p}k_i^2.$$

Therefore

$$\Phi^{-1}(t)=\frac{1}{\det\Phi(t)}\begin{bmatrix}u_2(t) & -x_2(t)\\ -u_1(t) & x_1(t)\end{bmatrix}=\prod_{i=1}^{p}k_i^{-2}\begin{bmatrix}u_2(t) & -x_2(t)\\ -u_1(t) & x_1(t)\end{bmatrix}.$$

The matrices K and $I+K$ can be computed as the following, respectively.

$$\begin{aligned}K &= -\left[M\Phi(t_1)+N\Phi(t_2)\right]^{-1}N\Phi(t_2)\\ &=\frac{1}{x_1(t_1)x_2(t_2)-x_1(t_2)x_2(t_1)}\begin{bmatrix}x_1(t_2)x_2(t_1) & x_2(t_1)x_2(t_2)\\ -x_1(t_1)x_1(t_2) & -x_1(t_1)x_2(t_2)\end{bmatrix}\\ &=\frac{1}{x_1(t_1)x_2(t_2)-x_1(t_2)x_2(t_1)}\begin{bmatrix}x_2(t_1) & 0\\ 0 & x_1(t_1)\end{bmatrix}\begin{bmatrix}x_1(t_2) & x_2(t_2)\\ -x_1(t_2) & -x_2(t_2)\end{bmatrix},\end{aligned}$$

and

$$
\begin{aligned}
I+K &= \frac{1}{x_1(t_1)x_2(t_2)-x_1(t_2)x_2(t_1)}\begin{bmatrix} x_1(t_1)x_2(t_2) & x_2(t_1)x_2(t_2) \\ -x_1(t_1)x_1(t_2) & -x_1(t_2)x_2(t_1) \end{bmatrix} \\
&= \frac{1}{x_1(t_1)x_2(t_2)-x_1(t_2)x_2(t_1)}\begin{bmatrix} x_2(t_2) & 0 \\ 0 & x_1(t_2) \end{bmatrix}\begin{bmatrix} x_1(t_1) & x_2(t_1) \\ -x_1(t_1) & -x_2(t_1) \end{bmatrix}.
\end{aligned}
$$

Now, we are ready to rewrite the Green's function (pair) in terms of the solutions of system (3.6a)-(3.6b) (or equivalently (3.3), $n=1$) as follows

$$
G(t,s) = S\begin{cases} M_1(t)N_1(s), & s<t \\ M_2(t)N_1(s), & s\geq t \end{cases} \tag{3.27}
$$

$$
\tilde{G}(t,\tau_i^+) = S\begin{cases} M_1(t)N_1(\tau_i^+), & \tau_i<t \\ M_2(t)N_1(\tau_i^+), & \tau_i\geq t \end{cases} \tag{3.28}
$$

where $S = \dfrac{\prod_{i=1}^{p} k_i^{-2}}{x_1(t_2)x_2(t_1)-x_1(t_1)x_2(t_2)}$

$$
M_1(t) = \begin{bmatrix} x_1(t_1)\left[x_1(t)x_2(t_2)-x_2(t)x_1(t_2)\right] & x_2(t_1)\left[x_1(t)x_2(t_2)-x_2(t)x_1(t_2)\right] \\ \\ x_1(t_1)\left[u_1(t)x_2(t_2)-u_2(t)x_1(t_2)\right] & x_2(t_1)\left[u_1(t)x_2(t_2)-u_2(t)x_1(t_2)\right] \end{bmatrix}
$$

and

$$
M_2(t) = \begin{bmatrix} x_1(t_2)\left[x_1(t)x_2(t_1)-x_2(t)x_1(t_1)\right] & x_2(t_2)\left[x_1(t)x_2(t_1)-x_2(t)x_1(t_1)\right] \\ \\ x_1(t_2)\left[u_1(t)x_2(t_1)-u_2(t)x_1(t_1)\right] & x_2(t_2)\left[u_1(t)x_2(t_1)-u_2(t)x_1(t_1)\right] \end{bmatrix}
$$

and

$$
N_1(s) = \begin{bmatrix} u_2(s) & -x_2(s) \\ -u_1(s) & x_1(s) \end{bmatrix}
$$

3.4.5 Green's Function For Second Order Equation

Since the case of impulsive differential equations are slightly different than impulsive differential systems, before introducing Green's function of equation (3.7), we need to present Variation of Parameters Formula and state and prove a theorem about the uniqueness of solution of equation (3.7).

Let $\{\psi_1, \psi_2\}$ be the fundamental set of solutions of corresponding homogenous equation (3.8a)-(3.8b). Then $\psi(t) = [\psi_1, \psi_2]$ is the first row of the (Wronskian) matrix

$$\Psi(t) = \begin{bmatrix} \psi_1(t) & \psi_2(t) \\ \psi_1'(t) & \psi_2'(t) \end{bmatrix}.$$

Theorem 3.4.5 ([107]) *(Variation of Parameters Formula) Let $\psi(t) = [\psi_1, \psi_2]$ be row vector of fundamental solutions of* (3.8a)-(3.8b)*, then any solution of* (3.7a)-(3.7b) *is of the form*

$$x(t) = \psi(t)\left[c + \int_{t_0}^{t} \Psi^{-1}(s)\frac{g(s)}{p(s)}e_2 ds + \sum_{t_0 \le \tau_i < t} \Psi^{-1}(\tau_i^+)c_i\right], \ t \ge t_0$$

and

$$x(t) = \psi(t)\left[c + \int_{t_0}^{t} \Psi^{-1}(s)\frac{g(s)}{p(s)}e_2 ds - \sum_{t \le \tau_i < t_0} \Psi^{-1}(\tau_i^+)c_i\right], \ t \le t_0,$$

where $c_i = [a_i, b_i]^T$. In particular

$$x(t) = \psi(t)\left[\Psi^{-1}(t_0)\delta + \int_{t_0}^{t} \Psi^{-1}(s)\frac{g(s)}{p(s)}e_2 ds + \sum_{t_0 \le \tau_i < t} \Psi^{-1}(\tau_i^+)c_i\right], \ t \ge t_0$$

satisfies the initial condition

$$x(t_0) = \delta_1, \ x'(t_0) = \delta_2,$$

where the column vector δ is $\delta = [\delta_1, \delta_2]^T$. A similar result holds for $t_0 \ge t$.

Our proof is again based on the fact that if the homogenous BVP (3.8) has only trivial solution then associated inhomogenous BVP (3.7) has a unique solution.

Corollary 3.4.1 *Suppose that $p, c \in PLC(J)$, $p(t) > 0$, and $\alpha_i \neq 0$ for $i = 1, 2, \ldots, p$. Assume that the homogenous system (3.8) has only trivial solution. Then the unique solution $x(t)$ of the inhomogenous system (3.7) is expressible as*

$$x(t) = w(t) + \int_{t_1}^{t_2} G(t,s)g(s)ds + \sum_{t_1 \le \tau_i < t_2} \tilde{G}(t, \tau_i^+)c_i \tag{3.29}$$

where

$$w(t) = \psi(t) \left[M\Psi(t_1) + N\Psi(t_2)\right]^{-1} \eta,$$

and the Green's function pair is given by

$$G(t,s) = \begin{cases} \psi(t)(I+K)\Psi^{-1}(s)\frac{1}{p(s)}e_2, & s < t \\ \psi(t)K\Psi^{-1}(s)\frac{1}{p(s)}e_2, & s \geq t \end{cases} \tag{3.30}$$

and

$$\tilde{G}(t,\tau_i^+) = \begin{cases} \psi(t)(I+K)\Psi^{-1}(\tau_i^+), & \tau_i < t \\ \psi(t)K\Psi^{-1}(\tau_i^+), & \tau_i \geq t, \end{cases} \tag{3.31}$$

where $K = -\left[M\Psi(t_1) + N\Psi(t_2)\right]^{-1} N\Psi(t_2)$, *and* $e_2 = [0,1]^T$.

Proof. We start with the variation of parameters formula (3.4.5) and write the general solution of equation (3.7a)-(3.7b) as

$$x(t) = \psi(t) \left[c + \int_{t_1}^{t} \Psi^{-1}(s)\frac{g(s)}{p(s)}e_2 ds + \sum_{t_1 \leq \tau_i < t} \Psi^{-1}(\tau_i^+)c_i \right], \quad t \geq t_1, \tag{3.32}$$

where c is a constant column vector which will be determined from the boundary conditions (3.7c). Now, imposing the boundary conditions (3.7c) on (3.32) yields

$$\begin{aligned} x(t_1) &= \xi = \psi(t_1)c \\ x(t_2) &= \zeta = \psi(t_2)c + \psi(t_2) \left[\int_{t_1}^{t_2} \Psi^{-1}(s)\frac{g(s)}{p(s)}e_2 ds + \sum_{t_1 \leq \tau_i < t_2} \Psi^{-1}(\tau_i^+)c_i \right]. \end{aligned}$$

Therefore we obtain linear system of algebraic equations

$$\begin{bmatrix} \psi_1(t_1) & \psi_2(t_1) \\ \psi_1(t_2) & \psi_2(t_2) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \xi \\ \zeta - \psi(t_2)\left(\int_{t_1}^{t_2} \Psi^{-1}(s)\frac{g(s)}{p(s)}e_2 ds + \sum_{t_1 \leq \tau_i < t_2} \Psi^{-1}(\tau_i^+)c_i \right) \end{bmatrix}$$

which has a unique solution, c_1, c_2 if $\det P \neq 0$ where $P = \begin{bmatrix} \psi_1(t_1) & \psi_2(t_1) \\ \psi_1(t_2) & \psi_2(t_2) \end{bmatrix}$.

Since the homogenous equation (3.8) has only trivial solution, the matrix P has an inverse. Observe that $x(t) = \psi(t)d = d_1\psi_1(t) + d_2\psi_2(t)$ is a general solution of homogenous system (3.8) where $d = [d_1 \ d_2]^T$ is a constant column vector. The homogenous boundary conditions satisfy if

$$\begin{aligned} x(t_1) = \psi(t_1)d &= d_1\psi_1(t_1) + d_2\psi_2(t_1) = 0 \\ x(t_2) = \psi(t_2)d &= d_1\psi_1(t_2) + d_2\psi_2(t_2) = 0 \end{aligned}$$

If the matrix P did not have an inverse, then d would be different than zero and the homogenous system (3.8) would have other solutions than the trivial one. This contradiction leads us that the matrix P has an inverse and we have the uniqueness of solutions of inhomogenous system (3.7).

After applying the same steps of proof of Theorem 3.4.2, it is not difficult see that the corresponding Green's function (pair) becomes as (3.30) and (3.31). $\square$

Let $W(\psi_1, \psi_2)(t)$ be Wronskian of the fundamental solutions of the corresponding homogenous equation (3.8a)-(3.8b) and

$$\begin{gathered}
(p(t)W(t))' = (p(t)\psi_1'\psi_2 - p(t)\psi_2'\psi_1)' = 0, \quad t \neq \tau_i \\
p(\tau_i^+)W(\tau_i^+) = k_i p(\tau_i^-)W(\tau_i^-)
\end{gathered}$$

implies that $p(t)W(\psi_1, \psi_2)(t) = C_i,\ t \in (\tau_i, \tau_{i+1}],\ C_i \in \mathbb{R}$, is piecewise constant function for $t \in J$. Therefore one can obtain inverse of the matrix $\Psi(t)$ as

$$\Psi^{-1}(t) = \frac{1}{W(\psi_1, \psi_2)(t)} \begin{bmatrix} \psi_2'(t) & -\psi_2(t) \\ -\psi_1'(t) & \psi_1(t) \end{bmatrix} = \frac{1}{W(\psi_1, \psi_2)(t)} \begin{bmatrix} \psi_2'(t) & -\psi_2(t) \\ -\psi_1'(t) & \psi_1(t) \end{bmatrix}.$$

Note that the matrices K and $I + K$ can be computed as the following, respectively.

$$\begin{aligned}
K &= -\left[M\Psi(t_1) + N\Psi(t_2)\right]^{-1} N\Psi(t_2) \\
&= \frac{1}{\psi_1(t_1)\psi_2(t_2) - \psi_1(t_2)\psi_2(t_1)} \begin{bmatrix} \psi_1(t_2)\psi_2(t_1) & \psi_2(t_1)\psi_2(t_2) \\ -\psi_1(t_1)\psi_1(t_2) & -\psi_1(t_1)\psi_2(t_2) \end{bmatrix} \\
&= \frac{1}{\psi_1(t_1)\psi_2(t_2) - \psi_1(t_2)\psi_2(t_1)} \begin{bmatrix} \psi_2(t_1) & 0 \\ 0 & \psi_1(t_1) \end{bmatrix} \begin{bmatrix} \psi_1(t_2) & \psi_2(t_2) \\ -\psi_1(t_2) & -\psi_2(t_2) \end{bmatrix},
\end{aligned}$$

and

$$\begin{aligned}
I + K &= \frac{1}{\psi_1(t_1)\psi_2(t_2) - \psi_1(t_2)\psi_2(t_1)} \begin{bmatrix} \psi_1(t_1)\psi_2(t_2) & \psi_2(t_1)\psi_2(t_2) \\ -\psi_1(t_1)x_1(t_2) & -\psi_1(t_2)x_2(t_1) \end{bmatrix} \\
&= \frac{1}{\psi_1(t_1)\psi_2(t_2) - \psi_1(t_2)\psi_2(t_1)} \begin{bmatrix} \psi_2(t_2) & 0 \\ 0 & \psi_1(t_2) \end{bmatrix} \begin{bmatrix} \psi_1(t_1) & \psi_2(t_1) \\ -\psi_1(t_1) & -\psi_2(t_1) \end{bmatrix}.
\end{aligned}$$

Now, we are ready to rewrite the Green's function (pair) in terms of the solutions of system (3.8a)-(3.8b) as follows

$$G(t,s) = S_1 \begin{cases} \left[\psi_1(t)\psi_2(t_2) - \psi_2(t)\psi_1(t_2)\right]\left[-\psi_2(s)\psi_1(t_1) + \psi_1(s)\psi_2(t_1)\right], & s < t \\ \left[\psi_1(t)\psi_2(t_1) - \psi_2(t)\psi_1(t_1)\right]\left[-\psi_2(s)\psi_1(t_2) + \psi_1(s)\psi_2(t_2)\right], & s \geq t \end{cases}$$

and

$$\tilde{G}(t,\tau_i^+) = S_2 \begin{cases} [\psi_1(t)\psi_2(t_2)\ \psi_2(t)\psi_1(t_2)]\, M_3, & s < t \\ [\psi_1(t)\psi_2(t_1)\ \psi_2(t)\psi_1(t_1)]\, M_4, & s \geq t \end{cases}$$

where

$$S_1 = \frac{1}{\psi_2(t_1)\psi_1(t_2) - \psi_1(t_1)\psi_2(t_2)} \frac{1}{p(s)W(\psi_1(s),\psi_2(s))},$$

$$M_3 = \begin{bmatrix} \psi_1(t_1)\psi_2'(\tau_i^+) - \psi_2(t_1)\psi_1'(\tau_i^+) & -\psi_1(t_1)\psi_2(\tau_i^+) + \psi_2(t_1)\psi_1(\tau_i^+) \\ -\psi_1(t_1)\psi_2'(\tau_i^+) + \psi_2(t_1)\psi_1'(\tau_i^+) & \psi_1(t_1)\psi_2(\tau_i^+) - \psi_2(t_1)\psi_1(\tau_i^+) \end{bmatrix},$$

$$M_4 = \begin{bmatrix} \psi_1(t_2)\psi_2'(\tau_i^+) - \psi_2(t_2)\psi_1'(\tau_i^+) & -\psi_1(t_2)\psi_2(\tau_i^+) + \psi_2(t_2)\psi_1(\tau_i^+) \\ -\psi_1(t_2)\psi_2'(\tau_i^+) + \psi_2(t_2)\psi_1'(\tau_i^+) & \psi_1(t_2)\psi_2(\tau_i^+) - \psi_2(t_2)\psi_1(\tau_i^+) \end{bmatrix},$$

and

$$S_2 = \frac{1}{\psi_2(t_1)\psi_1(t_2) - \psi_1(t_1)\psi_2(t_2)} \frac{1}{W(\psi_1(\tau_i^+),\psi_2(\tau_i^+))}.$$

We should remark that $\tilde{G}(t,\tau_i^+)$ is a row vector whereas $G(t,s)$ is a scalar function.

3.5 The Main Result

The main result of the present chapter is the following two theorems and corollaries. By employing Lyapunov type inequalities given in Section 3.2, we can prove the uniqueness of solutions of inhomogenous system (3.1), (3.5) and equation (3.7). We should remark that Lyapunov type inequalities are obtained for homogenous system (3.2), (3.6) and equation (3.8). Since for each system (3.2), (3.6) and equation (3.8), two different Lyapunov type inequalities are derived and corresponding to each Lyapunov type inequality there is one uniqueness criterion, we obtain two uniqueness criteria which are alternative to each other for each system (3.1), (3.5) and for equation (3.7).

Theorem 3.5.1 *Let (i)–(iii) hold. Suppose that the matrices A, A^T, B, and C all commute with K_i for all $i \in \mathbb{N}$ such that (3.9) holds. If for any $C^*(t) \geq C(t)$*

$$\exp\left(\int_{t_1}^{t_2} \alpha(t)dt\right) \left[\int_{t_1}^{t_2} |B(t)|dt\right] \left[\int_{t_1}^{t_2} |C^*(t)|dt + \sum_{t_1 \leq \tau_i < t_2} |S_i|\right] < 4, \qquad (3.33)$$

then there exist a unique solution $y(t) = (x(t),u(t))$ of BVP (3.1) *which can be expressed as (3.20) where $\alpha(t)$ and S_i are defined as in Theorem 3.2.1.*

Proof. In order to prove the uniqueness of solutions of BVP (3.1), it suffices to show that the homogeneous BVP (3.2a)–(3.2c) has only the trivial solution. Suppose to the contrary that $x(t) \not\equiv 0$ on (t_1, t_2). By Theorem 3.2.1, we see that Lyapunov type inequality (3.10) holds contradicting the inequality (3.33). Thus $x(t) = 0$ for all $t \in [t_1, t_2]$. Moreover, by (3.1a) we have

$$b(t)u = 0, \quad t \neq \tau_i,$$

which results in $u(t) = 0$ for $t \neq \tau_i$. Taking limit we see that $u(\tau_i^{\pm}) = 0$. As a result we obtain $(x(t), u(t)) = (0, 0)$ for all $t \in [t_1, t_2]$. This completes the uniqueness of the solutions. Since the form of unique solution of BVP (3.1) is given in Theorem 3.4.2 as (3.20), the proof is completed. $\square$

The next theorem can be used when Theorem 3.5.1 is not applicable, i.e in the case (3.33) fails. Since the proofs of following theorem and corollaries are exactly same as the proof of Theorem 3.5.1, they are omitted.

Theorem 3.5.2 *Let (i)–(iii) hold. Suppose that $K_i^T K_i = I$ for all $i \in \mathbb{N}$ and* (3.9) *holds. If for any $C^*(t) \geq C(t)$*

$$\begin{aligned}\exp\left(\int_{t_1}^{t_2} \alpha(t)dt\right) &\left[\prod_{t_1 \leq \tau_i < t_2} |K_i|^2\right] \left[\int_{t_1}^{t_2} |B(s)|ds\right] \\ &\times \left[\int_{t_1}^{t_2} |C^*(s)|ds + \sum_{t_1 \leq \tau_i < t_2} |K_i^T L_i|\right] < 4,\end{aligned} \tag{3.34}$$

then there exist a unique solution $y(t) = (x(t), u(t))$ of BVP (3.1) *which can be expressed as (3.20) where $\alpha(t)$ is defined as in Theorem 3.2.2.*

Corollary 3.5.1 *Suppose* (3.12) *holds. If*

$$\exp\left(\int_{t_1}^{t_2} |a(t)|dt\right) \left[\int_{t_1}^{t_2} b(t)dt\right] \left[\int_{t_1}^{t_2} c^+(t)dt + \sum_{t_1 \leq \tau_i < t_2} \left(\frac{l_i}{k_i}\right)^+\right] < 4, \tag{3.35}$$

then there exist a unique solution $y(t) = (x(t), u(t))$ of BVP (3.5) *which can be expressed as (3.20).*

In case (3.35) does not hold, we have the following alternative for Corollary 4.30.

Corollary 3.5.2 *Suppose* (3.12) *holds. If*

$$\exp\left(\int_{t_1}^{t_2} |a(t)|dt\right)\left[\int_{t_1}^{t_2} b(t)dt\right]\left[\int_{t_1}^{t_2} c^+(t)dt + \sum_{t_1\le\tau_i<t_2} (l_i k_i)^+\right] < 4, \qquad (3.36)$$

then there exist a unique solution $y(t) = (x(t), u(t))$ *of BVP* (3.5) *which can be expressed as (3.20).*

Corollary 3.5.3 *If*

$$\left[\int_{t_1}^{t_2} \frac{1}{p(t)}dt\right]\left[\int_{t_1}^{t_2} q^+(t)dt + \sum_{t_1\le\tau_i<t_2} \left(\frac{l_i}{k_i}\right)^+\right] < 4, \qquad (3.37)$$

then there exist a unique solution $x(t)$ *of BVP* (3.7) *which can be expressed as (3.29).*

When (3.37) is not satisfed, one can use the following alternative criterion for Corollary 3.5.3.

Corollary 3.5.4 *If*

$$\left[\int_{t_1}^{t_2} \frac{1}{p(t)}dt\right]\left[\int_{t_1}^{t_2} q^+(t)dt + \sum_{t_1\le\tau_i<t_2} (l_i k_i)^+\right] < 4, \qquad (3.38)$$

then there exist a unique solution $x(t)$ *of BVP* (3.7) *which can be expressed as (3.29).*

Remark 3.5.1 *Note that there are two criteria providing the unique solution of systems* (3.1)*,* (3.5) *and equation* (3.7)*. These criteria are not only new but also alternative to each other. Since changing the impulsive perturbation or assuming different condition on the impulses leads to variety of inequalities, presence of impulse effect yields different uniqueness criteria.*

CHAPTER 4

STABILITY OF LINEAR PERIODIC PLANAR HAMILTONIAN SYSTEMS UNDER IMPULSE EFFECT

4.1 Introduction

The planar Hamiltonian system has the form

$$y' = JH(t)y, \quad t \in \mathbb{R}, \tag{4.1}$$

where

$$H(t) = \begin{bmatrix} c(t) & a(t) \\ a(t) & b(t) \end{bmatrix}$$

is a symmetric matrix with piece-wice continuous real-valued entries, and

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

is the so called symplectic identity. Setting $y_1(t) = x(t)$ and $y_2(t) = u(t)$, we may rewrite system (4.1) in a more convenient way, i.e as a system of 2-linear first-order equations

$$x' = a(t)x + b(t)u, \quad u' = -c(t)x - a(t)u. \tag{4.2}$$

With regard to Definition 1.4.2, we want to remark that if $b(t)c(t) - a^2(t) = 1, t \in \mathbb{R}$, then system (4.2) is of symplectic structure and therefore our results are also valid for symplectic systems under impulse effect.

In this chapter we establish stability criteria for a special case of system (2.2), which is the following 2×2 linear periodic Hamiltonian systems under impulse effect

$$\begin{aligned} &x' = a(t)x + b(t)u, \quad u' = -c(t)x - a(t)u, && t \neq \tau_i \\ &x(\tau_i^+) = k_i x(\tau_i^-), \quad u(\tau_i^+) = l_i x(\tau_i^-) + k_i u(\tau_i^-), && i \in \mathbb{Z}. \end{aligned} \tag{4.3}$$

Unless otherwise stated, we assume that

(i) $\{\tau_i\}$ is a strictly increasing sequence of real numbers,

(ii) $a, b, c \in PLC(-\infty, \infty) = \{\omega : (-\infty, \infty) \to \mathbb{R}$ is continuous on each interval (τ_i, τ_{i+1}), the limits $w(\tau_i^\pm)$ exist and $w(\tau_i^-) = w(\tau_i)$ for $i \in \mathbb{Z}\}$, and

$$b(t) \geq 0, \qquad \int_{t_1}^{t_2} b(t)dt > 0, \tag{4.4}$$

(iii) k_i, l_i are sequence of real numbers such that $k_i \neq 0$ for $i \in \mathbb{Z}$.

By a solution of system (4.3), we mean a vector valued function $y(t) = (x(t), u(t))$ defined for $t \in \mathbb{R}$ such that $y \in PLC(-\infty, \infty)$ and system (4.3) is fulfilled for all $t \in \mathbb{R}$.

Note that if $b(t) > 0$, then the second-order impulsive differential equation

$$\begin{aligned} &(p(t)x')' + q(t)x = 0, && t \neq \tau_i \\ &x(\tau_i^+) = k_i x(\tau_i^-), \quad x'(\tau_i^+) = -l_i x(\tau_i^-) + k_i (px')(\tau_i^-), && i \in \mathbb{Z} \end{aligned} \tag{4.5}$$

is equivalent to system (4.3) with $a(t) = 0,\ b(t) = 1/p(t)$ and $c(t) = q(t)$.

Periodicity of impulsive system (4.3) is defined as in the next theorem. It can be seen that periodicity conditions depend not only on the periodicity of coefficient functions, a, b, c but also on the periodicity of the constants appearing as an impulse conditions, k_i, l_i and on the periodicity of the impulse points τ_i, as expected.

Definition 4.1.1 ([93]) *A linear impulsive system (4.3) is* $(T, r) - periodic$ *if*

$$\begin{aligned} &a(t+T) = a(t),\ b(t+T) = b(t),\ c(t+T) = c(t) \\ &k_{i+r} = k_i,\ l_{i+r} = l_i \\ &\tau_{i+r} = \tau_i + T \end{aligned} \tag{4.6}$$

Since system (4.3) is linear, stability of the system is equivalent to boundedness of the all solutions of the system. Moreover periodicity of the system suggest that stability of system on $\mathbb{R}^+ = (0, \infty)$ implies stability on $\mathbb{R}$. Therefore we have the following definition.

Definition 4.1.2 ([58, 42]) *System (4.3) is said to be stable if all solutions are bounded on $\mathbb{R}$, unstable if all nontrivial solutions are unbounded on $\mathbb{R}$, and conditionally stable if there exits a nontrivial solution bounded on $\mathbb{R}$.*

For impulsive differential equations or systems, in general for piece-wise continuos functions, the concept of a zero of a function is replaced by a so-called generalized zero.

Definition 4.1.3 ([45, 43, 42]) *A real number c is called a zero (generalized zero) of a function f if and only if $f(c^-) = 0$ or $f(c^+) = 0$. If f is continuous function at c, then c becomes a real zero.*

In this chapter our aim is to establish sufficient conditions for the stability of system (4.3) by extending some continuous results from system of ordinary differential equations to system of impulsive differential equations and by deriving new results. The proofs of the obtained stability theorems are based on both Floquet Theory due to the periodicity and Lyapunov type inequalities which are given in Section 2.3. Therefore the present chapter of the thesis is organized as follows. In the next section Lyapunov type inequalities, which are derived in Section 2.3 and are the main arguments of the proofs of stability theorems, are reminded. In Section 4.3 we outline the basic facts about Floquet Theory whose detailed information can be found in [9, 93] in the presence of impulse and in [25, 76] for equations without impulse. Section 4.4 is devoted to two auxiliary lemmas which are essential for the proofs of stability theorems. The main results of the paper, four stability criteria and their proofs and corollaries and some remarks are given in the last section.

4.2 Lyapunov Type Inequality

As far as Lyapunov type inequality is concerned, system (4.3) and equation (4.5) need not to be periodic. So the periodicity conditions are not necessary. In the sequel, we assume that $m^+(t) = \max\{m(t), 0\}$ and $m_i^+ = \max\{m_i, 0\}$.

For system (4.3), let us recall Lyapunov type inequalities obtained in Corollary 2.3.3 and Corollary 2.3.4.

Theorem 4.2.1 *If system (4.3) has a solution $y(t) = (x(t), u(t))$ with nontrivial x such that $x(t_1^+) = x(t_2^-) = 0, x(t) \neq 0$ on (t_1, t_2), then we have the Lyapunov type inequality*

$$4 \leq \exp\left(\int_{t_1}^{t_2} |a(t)|dt\right)\left[\int_{t_1}^{t_2} b(t)dt\right]\left[\int_{t_1}^{t_2} c^+(t)dt + \sum_{t_1 \leq \tau_i < t_2}\left(\frac{l_i}{k_i}\right)^+\right]. \tag{4.7}$$

Theorem 4.2.2 *If system (4.3) has a solution $y(t) = (x(t), u(t))$ with nontrivial x such that $x(t_1^+) = x(t_2^-) = 0, x(t) \neq 0$ on (t_1, t_2), then we have the Lyapunov type inequality*

$$4 \leq \exp\left(\int_{t_1}^{t_2} |a(t)|dt\right)\left[\int_{t_1}^{t_2} b(t)dt\right]\left[\int_{t_1}^{t_2} c^+(t)dt + \sum_{t_1 \leq \tau_i < t_2}(l_i k_i)^+\right]. \tag{4.8}$$

Lyapunov type inequalities for equation (4.5) are obtained in Corollary 2.3.5 and Corollary 2.3.6 as follows.

Theorem 4.2.3 *If equation* (4.5) *has a nontrivial solution x such that $x(t_1^+) = x(t_2^-) = 0, x(t) \neq 0$ on (t_1, t_2), then we have the Lyapunov type inequality*

$$4 \leq \left[\int_{t_1}^{t_2} \frac{1}{p(t)}dt\right]\left[\int_{t_1}^{t_2} q^+(t)dt + \sum_{t_1 \leq \tau_i < t_2}\left(\frac{l_i}{k_i}\right)^+\right]. \tag{4.9}$$

Theorem 4.2.4 *If equation* (4.5) *has a nontrivial solution x such that $x(t_1^+) = x(t_2^-) = 0, x(t) \neq 0$ on (t_1, t_2), then we have the Lyapunov type inequality*

$$4 \leq \left[\int_{t_1}^{t_2} \frac{1}{p(t)}dt\right]\left[\int_{t_1}^{t_2} q^+(t)dt + \sum_{t_1 \leq \tau_i < t_2}(l_i k_i)^+\right]. \tag{4.10}$$

4.3 Floquet Theory

Floquet theory for periodic impulsive system (4.3) can be summarized as follows. Let

$$X(t) = \begin{bmatrix} x_1(t) & x_2(t) \\ u_1(t) & u_2(t) \end{bmatrix}, \quad X(0) = I_2$$

be a fundamental matrix solution of system (4.3). Since the coefficients of linear system (4.3) are real, the components of solutions $(x_1(t), u_1(t))$ and $(x_2(t), u_2(t))$ can be taken real-valued. Since system (4.3) is periodic, it is easy to see that $X(t+T) = X(t)X(T)$ is also fundamental matrix and the matrix $X(T)$ is called monodromy matrix of the system (4.3). The Floquet multipliers (real or complex) of system (4.3) are the eigenvalues of the monodromy matrix and they are the roots of

$$\det(\rho I_2 - X(T)) = 0,$$

which is equivalent to

$$\rho^2 - A\rho + B = 0,$$

where

$$A = x_1(T) + u_2(T) = \rho_1 + \rho_2, \;\; B = \prod_{i=1}^{r} k_i^2 = \rho_1\rho_2.$$

Theorem 4.3.1 ([93]) *For any multiplier ρ there exists a nontrivial solution $y(t) = (x(t), u(t))$, which satisfies $y(t+T) = \rho y(t)$ of periodic impulsive system (4.3). Conversely, if for a nontrivial solution $y(t)$ and some number ρ relation $y(t+T) = \rho y(t)$ holds, then ρ is a multiplier of this system.*

In view of Theorem 4.3.1 we easily obtain that $y(t+mT) = \rho^m y(t), \;\; m \in \mathbb{Z}$. Since $|\rho| \neq 1$ implies that $y(t)$ is an unbounded solution of system (4.3), $|\rho| = 1$ is necessary condition in order to have stability. It follows that if $\prod_{i=1}^{r} k_i^2 \neq 1$ then $B = \rho_1\rho_2 \neq 1$ and so at least one of the multipliers will have modulus different from 1. Therefore system (4.3) cannot be stable unless $B = 1$. Clearly, if $B = 1$ then (4.3) becomes

$$\rho^2 - A\,\rho + 1 = 0$$

and $|A|$ determines the stability criteria.

Lemma 4.3.1 *Assume that $B = 1$. Then system (4.3) is unstable if $|A| > 2$, and stable if $|A| < 2$.*

Proof. The roots of the quadratic equation (4.3) is given as $\rho_{1,2} = \dfrac{A + \sqrt{A^2 - 4}}{2}$. Therefore the two roots ρ_1 and ρ_2 of (4.3) are distinct if $A^2 - 4 \neq 0$. We may assume without any loss of generality that $|\rho_1| \geq |\rho_2|$. It follows that system (4.3) has two linearly independent solutions $\psi_1(t)$ and $\psi_2(t)$ such that for all $t \in \mathbb{R}\backslash\{\tau_i : \ i \in \mathbb{Z}\}$,

$$\psi_1(t + T) = \rho_1 \, \psi_1(t), \quad \psi_2(t + T) = \rho_2 \, \psi_2(t)$$

and the general solution of system (4.3) can be written as $y(t) = c_1\psi_1(t) + c_2\psi_2(t)$. Hence for all $m \in \mathbb{Z}$,

$$y(t + mT) = c_1\psi_1(t + mT) + c_2\psi_2(t + mT) = c_1\rho_1^m\psi_1(t) + c_2\rho_2^m\psi_2(t) \tag{4.11}$$

Now let us consider the case $|A| > 2$. Then the numbers ρ_1 and ρ_2 are real. Since $\rho_1\rho_2 = 1$, we see that $|\rho_1| > 1$ and $|\rho_2| < 1$. Fixing t and letting $m \to \infty$ in (4.11) yield $\rho_1 \to \infty$ and $\rho_2 \to 0$ but letting $m \to -\infty$ implies $\rho_1 \to 0$ and $\rho_2 \to \infty$. Therefore we see that one solution of system (4.3) becomes unbounded on $\mathbb{R}$, implying that system (4.3) is unstable.

If $|A| < 2$ then ρ_1 and ρ_2 are complex conjugate with $|\rho_1| = |\rho_2| = 1$. In this case we have

$$|\psi_1(t + T)| = |\psi_1(t)|, \quad |\psi_2(t + T)| = |\psi_2(t)|, \qquad t \in \mathbb{R}\backslash\{\tau_i : \ i \in \mathbb{Z}\},$$

which implies that both $\psi_1(t)$ and $\psi_2(t)$ are bounded on $\mathbb{R}$, and hence system (4.3) is stable. □

Remark 4.3.1 *If $|A| = 2$ then system (4.3) is stable when $u_1(T) = x_2(T) = 0$; but conditionally stable and not stable otherwise.*

4.4 Preparatory Lemmas

In this section we give the following lemmas which are used to prove stability criteria for $(T, r)-$periodic impulsive system (4.3) and in particular for $(T, r)-$periodic impulsive equation (4.5). Although these lemmas are stated and proved in [42], we give them and their proofs for completeness of the thesis.

Lemma 4.4.1 ([42]) *Let system (4.3) be* (T, r) *periodic. Suppose that*

$$\prod_{i=1}^{r} k_i^2 = 1 \tag{4.12}$$

and

$$\int_0^T \left[c(t) - \frac{a^2(t)}{b(t)}\right] dt + \sum_{i=1}^{r} \frac{l_i}{k_i} > 0 \tag{4.13}$$

hold. If

$$A^2 \geq 4, \tag{4.14}$$

then system (4.3) has a nontrivial solution $y(t) = (x(t), u(t))$ *with* $x(t_1) = x(t_2) = 0$ $x(t) \neq 0$ *for all* $t_1 < t < t_2$ *such that* $0 \leq t_1 \leq T,\ t_2 > t_1,\ t_2 - t_1 \leq T.$

Proof. Since $A^2 \geq 4$ by (4.14), it follows from Theorem 4.3.1 that system (4.3) has a nontrivial solution $y(t) = (x(t), u(t))$ having the property that

$$y(t + T) = \rho y(t) \quad \text{for all } t \in \mathbb{R} \setminus \{\tau_i : i \in \mathbb{Z}\}, \tag{4.15}$$

where ρ is a nonzero real number. Since system (4.3) is linear and its coefficients are real, we may assume without loss of generality that the components $x(t)$ and $u(t)$ of $y(t)$ are real.

Firstly, our aim is to show the existence of a zero of $x(t)$ on $[0, T]$. Suppose not, then $x(t) \neq 0$ for all $t \in \mathbb{R}$. Define

$$w(t) = \frac{u(t)}{x(t)}, \quad t \in [0, T].$$

and by (4.15) we have

$$w(0) = w(T). \tag{4.16}$$

From system (4.3) it is easy to see that

$$w' = -c(t) + \frac{a^2(t)}{b(t)} - \left[\sqrt{b(t)}w + \frac{a(t)}{\sqrt{b(t)}}\right]^2, \quad t \neq \tau_i, \tag{4.17}$$

$$w(\tau_i+) - w(\tau_i-) = -\frac{\beta_i}{\alpha_i}. \tag{4.18}$$

In view of (4.18) and (4.16), integrating (4.17) over $[0, T]$ we get

$$\int_0^T \left[c(t) - \frac{a^2(t)}{b(t)}\right] dt + \sum_{i=1}^{r} \frac{l_i}{k_i} = -\int_0^T \left[\sqrt{b(t)}w + \frac{a(t)}{\sqrt{b(t)}}\right]^2 dt \leq 0, \tag{4.19}$$

which contradicts (4.13). Thus $x(t)$ must have a zero at a point $t_1 \in [0, T]$.

From (4.15) we see that $x(t)$ has also a zero at $t_1 + T$. It is easy to show that on the segment $[t_1, t_1 + T]$, $x(t)$ may have only finitely many zeros. Denote by t_2 the smallest zero of $x(t)$ lying to right of t_1 and different from t_1. Clearly $t_2 \leq t_1 + T$, $t_2 > t_1$, and $x(t) \neq 0$ for all $t \in (t_1, t_2)$. Thus the lemma is proved. $\square$

Lemma 4.4.2 ([42]) *Suppose that (4.12), (4.14) hold, $a/b \in \mathrm{C}[0, T]$, and*

$$\int_0^T \left[c(t) - \frac{a^2(t)}{b(t)} \right] dt + \sum_{i=1}^{r} \frac{l_i}{k_i} = 0. \tag{4.20}$$

If either

$$\textit{there exist } i \in N_1^r = \{1, 2, \ldots, r\} \textit{ such that } l_i \neq 0, \tag{4.21}$$

or

$$a/b \notin PLC^1[0, T] \textit{ and } l_i = 0 \textit{ for all } i \in N_1^r, \tag{4.22}$$

or

$$a/b \in PLC^1[0, T],\ \left(\frac{a}{b}\right)' - c(t) + \frac{a^2(t)}{b(t)} \not\equiv 0 \textit{ and } l_i = 0 \textit{ for all } i \in N_1^r, \tag{4.23}$$

holds true then the conclusion of Lemma 4.4.1 remains valid.

Proof. We proceed as in the proof of the previous lemma until (4.19). Using (4.20) in (4.19) we have

$$\sqrt{b(t)}w + \frac{a(t)}{\sqrt{b(t)}} = 0, \quad t \in [0, T],$$

which is equivalent to

$$b(t)u + a(t)x = 0, \quad t \in [0, T]. \tag{4.24}$$

In view of the first equation in (4.3) and (4.24) we see that

$$x(t) = c_i, \quad t \in (\tau_i, \tau_{i+1}), \quad i \in N_0^r \equiv \{0, 1, \ldots, r\}, \tag{4.25}$$

where c_i is a constant and we put $\tau_0 = 0$ and $\tau_{r+1} = T$. Then by (4.24) we obtain

$$u(t) = -\frac{a(t)}{b(t)} c_i, \quad t \in (\tau_i, \tau_{i+1}), \quad i \in N_0^r. \tag{4.26}$$

Since $x(t)$ has no zero by our assumption, we have that $c_i \neq 0$ for any $i \in N_1^r$. From the impulse conditions in (4.3) we also have, taking into account that a/b is continuous on $[0, T]$,

$$c_i = k_i c_{i-1}, \quad i \in N_1^r;$$
$$\frac{a(t)}{b(t)}(-c_i + k_i c_{i-1}) = -l_i c_{i-1}, \quad i \in N_1^r$$

which implies

$$l_i c_{i-1} = 0, \quad i \in N_1^r. \tag{4.27}$$

Now under the condition (4.21), (4.27) gives a contradiction because $c_i \neq 0$ for any $i \in N_1^r$. (4.26) contradicts condition (4.22) since $u(t)$ belongs to $\mathrm{PLC}^1[0, T]$ as solution of system (4.3). Finally, under the condition (4.23) substituting (4.25) and (4.26) into the second equation in (4.3), we get

$$\left[\left(\frac{a(t)}{b(t)} \right)' - c(t) + \frac{a^2(t)}{b(t)} \right] c_i = 0, \quad t \in (\tau_i, \tau_{i+1}), \quad i \in N_0^r.$$

But this contradicts (4.23). The remainder of the proof is exactly the same as that of the previous lemma after (4.19) therein. □

4.5 Stability Criteria

The following theorems and corollaries are the main results of the present chapter providing the sufficient conditions for the stability of system (4.3) and equation (4.5). Since there are two different Lyapunov type inequalities obtained for system (4.3), we derived two different and new stability criteria for such systems. Moreover we prove the alternative results to those criteria in the case they are not applicable. Therefore we have four new stability criteria which can be used in place of each other. Besides our results generalize the previous ones to the impulsive case.

Theorem 4.5.1 *Assume that (4.12), (4.13) and*

$$\exp\left(\int_0^T |a(t)| dt \right) \left[\int_0^T b(t) dt \right] \left[\int_0^T c^+(t) dt + \sum_{0 \leq \tau_i < T} \left(\frac{l_i}{k_i} \right)^+ \right] < 4. \tag{4.28}$$

Then impulsive system (4.3) is stable.

Proof. In virtue of Lemma 4.3.1 in order to prove the stability, it is sufficient to show that $A^2 < 4$. Assume on the contrary that $A^2 \geq 4$. From Lemma 4.4.1, the conditions (4.12), (4.13) and $A^2 \geq 4$ imply that $x(t)$, the first component of the solution $y(t) = (x(t), u(t))$, has two zeros at some points s_1, s_2 with $s_1 \in [0,T],\ s_1 < s_2 \leq s_1 + T$. Then applying Theorem 4.2.1 by using these zeros of $x(t)$, i.e employing Lyapunov type inequality, we see that

$$\begin{aligned} 4 &\leq \exp\left(\int_{s_1}^{s_2} |a(t)|dt\right)\left[\int_{s_1}^{s_2} b(t)dt\right]\left[\int_{s_1}^{s_2} c^+(t)dt + \sum_{s_1\leq\tau_i<s_2}\left(\frac{l_i}{k_i}\right)^+\right] \\ &\leq \exp\left(\int_{s_1}^{s_1+T} |a(t)|dt\right)\left[\int_{s_1}^{s_1+T} b(t)dt\right]\left[\int_{s_1}^{s_1+T} c^+(t)dt + \sum_{s_1\leq\tau_i<s_1+T}\left(\frac{l_i}{k_i}\right)^+\right]. \end{aligned}$$

For $T-periodic$ function f, it is known that $\displaystyle\int_{s_1}^{s_1+T} f(t)dt = \int_0^T f(t)dt$. Similarly by using periodicity conditions given in Definition 4.1.1 it can be seen that

$$\begin{aligned} \sum_{s_1\leq\tau_i<s_1+T} a_i &= \sum_{s_1\leq\tau_i<s_1+T} a_i + \sum_{0\leq\tau_i<s_1} a_i - \sum_{0\leq\tau_i<s_1} a_i \\ &= \sum_{0\leq\tau_i<s_1+T} a_i - \sum_{T\leq\tau_i+T<s_1+T} a_i = \sum_{0\leq\tau_i<s_1+T} a_i - \sum_{T\leq\tau_{i+r}<s_1+T} a_i \\ &= \sum_{0\leq\tau_i<s_1+T} a_i - \sum_{T\leq\tau_i<s_1+T} a_{i+r} = \sum_{0\leq\tau_i<s_1+T} a_i - \sum_{T\leq\tau_i<s_1+T} a_i \\ &= \sum_{0\leq\tau_i<T} a_i \end{aligned}$$

Hence we obtain

$$4 \leq \exp\left(\int_0^T |a(t)|(t)dt\right)\left[\int_0^T b(t)dt\right]\left[\int_0^T c^+(t)dt + \sum_{0\leq\tau_i<T}\left(\frac{l_i}{k_i}\right)^+\right],$$

which contradicts condition (4.28). Thus $A^2 < 4$ and hence system (4.3) is stable. This completes the proof. $\square$

The next theorem can be used when Theorem 4.5.1 is not applicable, i.e in the case (4.28) fails. Since the proofs of the following theorems are exactly the same as the proof of Theorem 4.5.1, they are skipped.

Theorem 4.5.2 *Assume that (4.12), (4.13) and*

$$\exp\left(\int_0^T |a(t)|(t)dt\right)\left[\int_0^T b(t)dt\right]\left[\int_0^T c^+(t)dt + \sum_{0\leq\tau_i<T}(l_ik_i)^+\right] < 4. \quad (4.29)$$

Then impulsive system (4.3) is stable.

In case (4.13) fails we have the following alternative for Theorem 4.5.1.

Theorem 4.5.3 *Assume that (4.12), (4.28) hold and $a/b \in C(0,T)$. If (4.20) and either (4.21) or (4.22) or (4.23), then impulsive system (4.3) is stable.*

If (4.13) and (4.28) do not hold, the following alternative stability criteria can be used.

Theorem 4.5.4 *Assume that (4.12), (4.29) hold and $a/b \in C(0,T)$. If (4.20) and either (4.21) or (4.22) or (4.23), then impulsive system (4.3) is stable.*

Remark 4.5.1 *As it is seen above, we have possibility to use four different stability criteria two of which are new and alternative to the other two. Since in the absence of impulse Theorem 4.5.1 and Theorem 4.5.2 coincide, we can conclude that existence of impulse effect provides new inequalities, such as* (4.28) *and* (4.29)*. This shows that systems with impulses are richer and more fruitful than systems without impulses.*

The next two corollaries can be given in the absence of impulse effect, i.e $k_i = 1$ and $l_i = 0$ for $i \in \mathbb{Z}$. Since they are immediate consequences of Theorem 4.5.1 and Theorem 4.5.3, their proofs are omitted.

Corollary 4.5.1 *Assume that*

$$\int_0^T \left[c(t) - \frac{a^2(t)}{b(t)}\right] dt > 0, \tag{4.30}$$

$$\exp\left(\int_0^T |a(t)|dt\right)\left[\int_0^T b(t)dt\right]\left[\int_0^T c^+(t)dt\right] < 4. \tag{4.31}$$

Then system (4.3) is stable.

Corollary 4.5.2 *Assume that (4.31) holds and $a/b \in C(0,T)$. If either*

$$\int_0^T \left[c(t) - \frac{a^2(t)}{b(t)}\right] dt = 0$$

or

$$a/b \notin PLC^1[0,T],$$

or

$$a/b \in PLC^1[0,T],\ \left(\frac{a}{b}\right)' - c(t) + \frac{a^2(t)}{b(t)} \not\equiv 0,$$

then system (4.3) is stable.

Remark 4.5.2 *For the nonimpulsive case Theorem 4.5.1 and Theorem 4.5.2 are reduced to Corollary 4.5.1, which is the same as Theorem 1.2.22. Besides Corollary 4.5.2 can be used in the place of Corollary 4.5.1 when (4.30) does not hold.*

When $b(t) > 0$, we can set $a(t) \equiv 0$, $b(t) = 1/p(t)$, and $c(t) = q(t)$ in Theorem 4.5.1 and Theorem 4.5.2 and obtained the following corollaries whose proofs are exactly the same as the proof of Theorem 4.5.1 and so, they are skipped.

Corollary 4.5.3 *Assume that (4.12) holds. If*

$$\int_0^T q(t)dt + \sum_{i=1}^{r} \frac{l_i}{k_i} \geq 0,\ \textit{either}\ q(t) \not\equiv 0\ \textit{on}\ [0,T]\backslash\ \{\tau_1, \dots, \tau_r\}$$

or $l_i \neq 0$ *some* $i \in \{1, \dots, r\}$;

$$\left[\int_0^T \frac{dt}{p(t)}\right]\left[\int_0^T q^+(t)dt + \sum_{i=1}^{r} \left(\frac{l_i}{k_i}\right)^+\right] \leq 4.$$

Then equation (4.5) is stable.

The alternative result for Corollary 4.5.3 is obtained as follows.

Corollary 4.5.4 *Assume that (4.12) holds. If*

$$\int_0^T q(t)dt + \sum_{i=1}^{r} \frac{l_i}{k_i} \geq 0,\ \textit{either}\ q(t) \not\equiv 0\ \textit{on}\ [0,T]\backslash\ \{\tau_1, \dots, \tau_r\}$$

or $l_i \neq 0$ *some* $i \in \{1, \dots, r\}$;

$$\left[\int_0^T \frac{dt}{p(t)}\right]\left[\int_0^T q^+(t)dt + \sum_{i=1}^{r} (l_i k_i)^+\right] \leq 4.$$

Then equation (4.5) is stable.

Remark 4.5.3 *Corollary 4.5.3 coincides with Theorem 1.2.16 while Corollary 4.5.4 yields an alternative result for them.*

CHAPTER 5

LYAPUNOV TYPE INEQUALITIES AND APPLICATIONS FOR NONLINEAR IMPULSIVE SYSTEMS

5.1 Lyapunov Type Inequalities For The First Order Nonlinear Impulsive Systems

In this chapter we are interested in obtaining Lyapunov type inequalities for the nonlinear impulsive systems of the form

$$\begin{aligned}
&x' = \alpha_1(t)x + \beta_1(t)|u|^{\gamma-2}u, \quad u' = -\alpha_1(t)u - \beta_2(t)|x|^{\beta-2}x, \quad t \neq \tau_i \\
&x(\tau_i^+) = \xi_i x(\tau_i^-), \quad u(\tau_i^+) = \xi_i u(\tau_i^-) - \eta_i |x(\tau_i^-)|^{\beta-2}x(\tau_i^-), \\
&t \geq t_0, \quad i \in \mathbb{N} := \{1, 2, \ldots\}
\end{aligned} \tag{5.1}$$

where $\gamma > 1, \beta > 1$ are real constants and without further mention we assume that

(i) $\alpha_1, \beta_1, \beta_2 \in \mathrm{PLC}[t_0, \infty) = \{\omega : [t_0, \infty) \to \mathbb{R}$ is continuous on each interval (τ_i, τ_{i+1}), the limits $w(\tau_i^{\pm})$ exist and $w(\tau_i^-) = w(\tau_i)$ for $i \in \mathbb{N}\}$, $\beta_1(t) > 0$,

(ii) $\{\tau_i\}$ is a strictly increasing sequence of real numbers,

(iii) ξ_i, η_i are sequence of real numbers such that $\xi_i \neq 0$ for $i = 1, 2, \ldots$.

By a solution of system (5.1), we mean $x, u \in \mathrm{PLC}[t_0, \infty)$ satisfying system (5.1) for $t \geq t_0$. Such a solution is said to be proper if $\sup\{|x(s)| + |u(s)| : t \leq s < \infty\} > 0$ for any $t \geq t_0$.

In the special case, $\alpha_1(t) = 0$, system (5.1) reduces to

$$\begin{aligned}
&x' = \beta_1(t)|u|^{\gamma-2}u, \quad u' = -\beta_2(t)|x|^{\beta-2}x, \quad t \neq \tau_i \\
&x(\tau_i^+) = \xi_i x(\tau_i^-), \quad u(\tau_i^+) = \xi_i u(\tau_i^-) - \eta_i |x(\tau_i^-)|^{\beta-2}x(\tau_i^-) \\
&t \geq t_0, \quad i \in \mathbb{N}
\end{aligned} \tag{5.2}$$

which can be written in the form of the following impulsive Emden-Fowler type differential equations by using the transformation $u(t) = p(t)|x'|^{\alpha-2}x'$,

$$\begin{aligned}
&(p(t)|x'|^{\alpha-2}x')' + q(t)|x|^{\beta-2}x = 0, \quad t \neq \tau_i, \\
&x(\tau_i^+) = \xi_i\, x(\tau_i^-), \\
&p(\tau_i^+)|x'(\tau_i^+)|^{\alpha-2}x'(\tau_i^+) = \xi_i\, p(\tau_i^-)|x'(\tau_i^-)|^{\alpha-2}x'(\tau_i^-) - \eta_i|x(\tau_i^-)|^{\beta-2}x(\tau_i^-) \\
&t \geq t_0, \quad i \in \mathbb{N}
\end{aligned} \tag{5.3}$$

with

$$\frac{1}{\gamma} + \frac{1}{\alpha} = 1, \quad \beta_1(t) = p^{1-\gamma}(t), \quad \beta_2(t) = q(t). \tag{5.4}$$

Equation (5.3) becomes half-linear if $\alpha = \beta$, i.e.,

$$\begin{aligned}
&(p(t)|x'|^{\beta-2}x')' + q(t)|x|^{\beta-2}x = 0, \quad t \neq \tau_i, \\
&x(\tau_i^+) = \xi_i\, x(\tau_i^-), \\
&p(\tau_i^+)|x'(\tau_i^+)|^{\beta-2}x'(\tau_i^+) = \xi_i\, p(\tau_i^-)|x'(\tau_i^-)|^{\beta-2}x'(\tau_i^-) - \eta_i|x(\tau_i^-)|^{\beta-2}x(\tau_i^-) \\
&t \geq t_0, \quad i \in \mathbb{N},
\end{aligned} \tag{5.5}$$

which is equivalent to (5.2) with $u(t) = p(t)|x'|^{\beta-2}x'$ and α replaced by β in (5.4).

Impulsive Emden-Fowler equation (5.3) is called super-half linear if $\beta > \alpha$ and sub-half-linear if $\beta < \alpha$, see [105].

For convenience let us define a piece-wise constant function k as

$$k(t) = \begin{cases} \xi_1\xi_2\ldots\xi_j, & t \in (\tau_j, \tau_{j+1}], \quad j \in \mathbb{N}, \\ 1, & t \in [t_0, \tau_1] \end{cases} \tag{5.6}$$

and the sequence $\{k_j\}$ as

$$k_j = \begin{cases} \xi_1\xi_2\ldots\xi_j, & j \geq 1 \\ 1, & j \leq 0. \end{cases} \tag{5.7}$$

For impulsive differential equations or systems, in general for piece-wise continuos functions, the concept of a zero of a function is replaced by a so-called generalized zero.

Definition 5.1.1 ([45, 42]) *A real number c is called a zero (generalized zero) of a function f if and only if $f(c^-) = 0$ or $f(c^+) = 0$. If f is continuous function at c, then c becomes a real zero.*

Now we give the definition of disconjugacy which is about the zeros of the solutions of differential equations or systems.

Definition 5.1.2 ([45, 43]) *Equation* (5.3) *(equation* (5.5)*) is called disconjugate on an interval $[t_1, t_2]$ if and only if all solutions of equation* (5.3) *(equation* (5.5)*) have at most one (generalized) zero on an interval $[t_1, t_2]$.*

We generalize the definition of disconjugacy given in [45, 42].

Definition 5.1.3 ([45, 42]) *System* (5.1) *is called disconjugate (relatively disconjugate with respect to x) on an interval $[t_1, t_2]$ if and only if there is no real solution $(x(t), u(t))$ of system* (5.1) *with a nontrivial x having two or more zeros (generalized zeros) on $[t_1, t_2]$.*

We will make use of the following definitions.

Definition 5.1.4 *A proper solution $y(t) = (x(t), u(t))$ of system (5.1) is said to be weakly oscillatory if $x(t)$ has arbitrarily large (generalized) zeros. This solution is said to be oscillatory if both components of y have arbitrarily large (generalized) zeros. If both components (at least one component) of y are different from zero for large t, then the solution y of system (5.1) is called nonoscillatory (weakly nonoscillatory). System (5.1) is said to be oscillatory if all the solutions are oscillatory.*

Definition 5.1.5 *A proper solution $y(t) = (x(t), u(t))$ of system (5.1) is said to be weakly bounded if $x(t)/k(t)$ is bounded on $[t_0, \infty)$. The solution y is said to be bounded if both $x(t)/k(t)$ and $u(t)/k(t)$ are bounded on $[t_0, \infty)$. If both $x(t)/k(t)$ and $u(t)/k(t)$ (at least one of them) are not bounded on $[t_0, \infty)$, then the solution y of system (5.1) is called unbounded on $[t_0, \infty)$.*

Since we restrict ourselves to establish Lyapunov type inequalities for system (5.1), we tacitly assume that system (5.1) has proper solutions. Although there is an extensive literature on the Lyapunov type inequalities mentioned in Chapter 1, there is not much done for nonlinear systems with or without impulse [106, 100, 105]. The present chapter which stems from [106, 105] is about nonlinear impulsive systems (5.1) whose special cases are linear Hamiltonian systems under impulse effect (4.3), impulsive Emden-Fowler type equations (5.3), impulsive half linear equations (5.5) and impulsive linear equations (1.5). The main objective of this chapter is to establish several Lyapunov type inequalities, which are generalization of the existing ones in the literature, for system (5.1) and its particular cases. Our results relate not only points where the first component of the solution $(x(t), u(t))$ of system (5.1) has consecutive zeros but also the point at which $\left|\frac{x(t)}{k(t)}\right|$ has supremum where $k(t)$ is defined as in equation (5.6).

5.1.1 Lyapunov Type Inequality

In this chapter, although the proofs of the theorems are based on the same argument, by using different well-known inequalities we obtain several Lyapunov type inequalities. Throughout this section, we define

$$\begin{aligned} h_{t_1}(t) &= \int_{t_1}^{t} \beta_1(w)|k(w)|^{\gamma-2} \exp\left(\gamma \int_{w}^{t} \alpha_1(s)ds\right) dw \\ h_{t_2}(t) &= \int_{t}^{t_2} \beta_1(w)|k(w)|^{\gamma-2} \exp\left(\gamma \int_{w}^{t} \alpha_1(s)ds\right) dw. \end{aligned} \tag{5.8}$$

Recall that the numbers $p_1, p_2 > 1$ are said to be conjugate if

$$\frac{1}{p_1} + \frac{1}{p_2} = 1.$$

In the sequel, we denote $m^+(t) = \max\{m(t), 0\}$ and $m_i^+ = \max\{m_i, 0\}$.

Theorem 5.1.1 *Let α be the conjugate of γ. If system (5.1) has a solution $y(t) = (x(t), u(t))$ with nontrivial x such that $x(t_1^+) = x(t_2^-) = 0, x(t) \neq 0$ on (t_1, t_2), then we have the following Lyapunov type inequality*

$$h_{t_1}^{1-\alpha}(\tau) + h_{t_2}^{1-\alpha}(\tau) \leq M^{\beta-\alpha} \left[\int_{t_1}^{t_2} \beta_2^+(t)|k(t)|^{\beta-2}dt + \sum_{t_1 \leq \tau_i < t_2} (\eta_i/\xi_i)^+ |k_{i-1}|^{\beta-2}\right], \tag{5.9}$$

where

$$M = \sup_{t\in(t_1,t_2)} \left|\frac{x(t)}{k(t)}\right| = \left|\frac{x(\tau)}{k(\tau)}\right| \tag{5.10}$$

and $h_{t_1}(\tau), h_{t_2}(\tau), k(t)$ and k_i are defined as in equation (5.8), (5.6) and (5.7), respectively.

Proof. Let us define

$$z(t) = \frac{x(t)}{k(t)}, \quad v(t) = \frac{u(t)}{k(t)}, \qquad t \geq t_0.$$

It is easy to see that with the above transformation system (5.1) becomes as

$$\begin{aligned} & z' = \alpha_1(t)z + \beta_1(t)|k(t)|^{\gamma-2}|v|^{\gamma-2}v, \quad v' = -\alpha_1(t)v - \beta_2(t)|k(t)|^{\beta-2}|z|^{\beta-2}z, \quad t \neq \tau_i \\ & z(\tau_i^+) = z(\tau_i^-), \quad v(\tau_i^+) = v(\tau_i^-) - (\eta_i/\xi_i)|k_{i-1}|^{\beta-2}|z(\tau_i^-)|^{\beta-2}z(\tau_i^-), \\ & t \geq t_0, \quad i \in \mathbb{N}. \end{aligned} \tag{5.11}$$

Because of the definition of z, it is obvious that $z \in PLC[t_0, \infty)$. Therefore $z(t)$ is continuous on $[t_1, t_2]$. Thus, $z(t_1) = z(t_2) = 0$ and $z(t) \neq 0$ for all $t \in (t_1, t_2)$. Since $z(t)$ is continuous, we can choose $\tau \in (t_1, t_2)$ such that

$$|z(\tau)| = \max_{t \in (t_1, t_2)} |z(t)| = M > 0.$$

From (5.11), one can obtain

$$\begin{aligned} & (vz)' = \beta_1(t)|k(t)|^{\gamma-2}|v(t)|^{\gamma} - \beta_2(t)|k(t)|^{\beta-2}|z(t)|^{\beta}, \quad t \neq \tau_i \\ & (vz)(\tau_i^+) - (vz)(\tau_i^-) = -(\eta_i/\xi_i)|k_{i-1}|^{\beta-2}|z(\tau_i)|^{\beta}. \end{aligned} \tag{5.12}$$

Integrating the first equation of (5.12) from t_1 to t_2 and using $\beta_2^+(t) = \max\{\beta_2(t), 0\}$ and $(\eta_i/\xi_i)^+ = \max\{\eta_i/\xi_i, 0\}$ yields,

$$\begin{aligned} \int_{t_1}^{t_2} \beta_1(t)|k(t)|^{\gamma-2}|v(t)|^{\gamma} dt \leq & \int_{t_1}^{t_2} \beta_2^+(t)|k(t)|^{\beta-2}|z(t)|^{\beta} dt \\ & + \sum_{t_1 \leq \tau_i < t_2} (\eta_i/\xi_i)^+ |k_{i-1}|^{\beta-2}|z(\tau_i)|^{\beta}. \end{aligned} \tag{5.13}$$

From the first equation in (5.11), we have

$$\left[z(t)\exp\left(-\int_{t_1}^{t} \alpha_1(s)ds\right)\right]' = \beta_1(t)|k(t)|^{\gamma-2}|v(t)|^{\gamma-2}v(t)\exp\left(-\int_{t_1}^{t} \alpha_1(s)ds\right) \tag{5.14}$$

and

$$\left[z(t)\exp\left(\int_{t}^{t_2} \alpha_1(s)ds\right)\right]' = \beta_1(t)|k(t)|^{\gamma-2}|v(t)|^{\gamma-2}v(t)\exp\left(\int_{t}^{t_2} \alpha_1(s)ds\right). \tag{5.15}$$

Integrating (5.14) from t_1 to t implies

$$z(t) = \int_{t_1}^{t} \beta_1(w)|k(w)|^{\gamma-2}|v(w)|^{\gamma-2}v(w)\exp\left(\int_{w}^{t}\alpha_1(s)ds\right)dw$$

Taking absolute values of both sides and applying Hölder inequality with indices γ and α, we get

$$\begin{aligned}
|z(t)| &\leq \int_{t_1}^{t} \beta_1(w)|k(w)|^{\gamma-2}|v(w)|^{\gamma-1}\exp\left(\int_{w}^{t}\alpha_1(s)ds\right)dw \\
&\leq \left[\int_{t_1}^{t} \beta_1(w)|k(w)|^{\gamma-2}\exp\left(\gamma\int_{w}^{t}\alpha_1(s)ds\right)dw\right]^{\frac{1}{\gamma}} \\
&\times \left[\int_{t_1}^{t} \beta_1(w)|k(w)|^{\gamma-2}|v(w)|^{\gamma}dw\right]^{\frac{1}{\alpha}} \\
&= h_{t_1}^{\frac{1}{\gamma}}(t)\left[\int_{t_1}^{t} \beta_1(w)|k(w)|^{\gamma-2}|v(w)|^{\gamma}dw\right]^{\frac{1}{\alpha}}.
\end{aligned}$$

Taking α-th power of both sides yields

$$|z(t)|^{\alpha}h_{t_1}^{1-\alpha}(t) \leq \int_{t_1}^{t} \beta_1(w)|k(w)|^{\gamma-2}|v(w)|^{\gamma}dw. \tag{5.16}$$

Similarly, if the above procedure is followed for equation (5.15) on the interval $[t, t_2]$ then one can obtain

$$|z(t)|^{\alpha}h_{t_2}^{1-\alpha}(t) \leq \int_{t}^{t_2} \beta_1(w)|k(w)|^{\gamma-2}|v(w)|^{\gamma}dw. \tag{5.17}$$

Adding (5.16) and (5.17) and replacing t by τ in the resulting inequality and using inequality (5.13) yield

$$\begin{aligned}
|z(\tau)|^{\alpha}\left[h_{t_1}^{1-\alpha}(\tau) + h_{t_2}^{1-\alpha}(\tau)\right] &\leq \int_{t_1}^{t_2} \beta_1(t)|k(t)|^{\gamma-2}|v(t)|^{\gamma}dt \\
&\leq \int_{t_1}^{t_2} \beta_2^{+}(t)|k(t)|^{\beta-2}|z(t)|^{\beta}dt \\
&+ \sum_{t_1\leq\tau_i<t_2}\left(\frac{\eta_i}{\xi_i}\right)^{+}|k_{i-1}|^{\beta-2}|z(\tau_i)|^{\beta}.
\end{aligned} \tag{5.18}$$

Since $|z(\tau)| \geq |z(t)|$ for $t \in [t_1, t_2]$, we obtain from inequality (5.18) that

$$\begin{aligned}
|z(\tau)|^{\alpha}\left[h_{t_1}^{1-\alpha}(\tau) + h_{t_2}^{1-\alpha}(\tau)\right] &\leq |z(\tau)|^{\beta}\left[\int_{t_1}^{t_2} \beta_2^{+}(t)|k(t)|^{\beta-2}dt\right. \\
&\left.+ \sum_{t_1\leq\tau_i<t_2}(\eta_i/\xi_i)^{+}|k_{i-1}|^{\beta-2}\right].
\end{aligned}$$

Finally, if we use (5.10) in the last inequality the desired inequality (5.9) can be obtained. $\square$

Theorem 5.1.2 *Let α be the conjugate of γ. If system (5.1) has a solution $y(t) = (x(t), u(t))$ with nontrivial x such that $x(t_1^+) = x(t_2^-) = 0, x(t) \neq 0$ on (t_1, t_2), then we have the following Lyapunov type inequality*

$$\begin{aligned} 2^{2-\alpha}\left[\frac{1}{h_{t_1}(\tau)}+\frac{1}{h_{t_2}(\tau)}\right]^{\alpha-1} \quad &\leq M^{\beta-\alpha}\left[\int_{t_1}^{t_2}\beta_2^+(t)|k(t)|^{\beta-2}dt\right. \\ &\left.\qquad + \sum_{t_1\leq\tau_i<t_2}(\eta_i/\xi_i)^+|k_{i-1}|^{\beta-2}\right], \end{aligned} \tag{5.19}$$

where $h_{t_1}(\tau), h_{t_2}(\tau), M, k(t)$ and k_i are defined as in equation (5.8), (5.10), (5.6) and (5.7), respectively.

Proof. After following the same steps of the proof of Theorem 5.1.1 and arriving inequalities (5.16) and (5.17), one can observe that $h_{t_1}(t_1) = h_{t_2}(t_2) = 0$ and $h_{t_1}(t_2) > 0, h_{t_2}(t_1) > 0$. Since $h_{t_1}(t)$ and $h_{t_2}(t)$ are continous functions, there exist $c \in (t_1, t_2)$ such that $h_{t_1}(c) = h_{t_2}(c) > 0$. Hence, for $t \in [t_1, c]$, $h_{t_1}(t) \leq h_{t_2}(t)$ and for $t \in [c, t_2]$, $h_{t_2}(t) \leq h_{t_1}(t)$. Moreover, it is obvious that

$$h_{t_1}(t) \leq \frac{2h_{t_1}(t)h_{t_2}(t)}{h_{t_1}(t)+h_{t_2}(t)}, \quad t \in [t_1, c] \tag{5.20}$$

and

$$h_{t_2}(t) \leq \frac{2h_{t_1}(t)h_{t_2}(t)}{h_{t_1}(t)+h_{t_2}(t)}, \quad t \in [c, t_2] \tag{5.21}$$

Adding (5.16) and (5.17) and using (5.20) and (5.21) lead to

$$2|z(t)|^\alpha \leq \left[\frac{2h_{t_1}(t)h_{t_2}(t)}{h_{t_1}(t)+h_{t_2}(t)}\right]^{\alpha-1}\left[\int_{t_1}^{t_2}\beta_1(w)|k(w)|^{\gamma-2}|v(w)|^\gamma dw\right]. \tag{5.22}$$

Replacing t by τ in inequality (5.22), using (5.13) and employing $z(\tau) \geq |z(t)|$ for $t \in [t_1, t_2]$ result in

$$\begin{aligned} 2^{2-\alpha}\left[\frac{1}{h_{t_1}(\tau)}+\frac{1}{h_{t_2}(\tau)}\right]^{\alpha-1}|z(\tau)|^\alpha \quad &\leq |z(\tau)|^\beta\left[\int_{t_1}^{t_2}\beta_2^+(t)|k(t)|^{\beta-2}dt\right. \\ &\left.\qquad + \sum_{t_1\leq\tau_i<t_2}(\eta_i/\xi_i)^+|k_{i-1}|^{\beta-2}\right]. \end{aligned}$$

Finally, if we use (5.10) in the last inequality the desired inequality (5.19) holds. $\square$

Theorem 5.1.3 *Let α be the conjugate of γ. If system (5.1) has a solution*

$y(t) = (x(t), u(t))$ with nontrivial x such that $x(t_1^+) = x(t_2^-) = 0, x(t) \neq 0$ on (t_1, t_2), then for some $\tau \in (t_1, t_2)$ we have the following Lyapunov type inequality

$$\begin{aligned} 2^\alpha &\leq M^{\beta-\alpha} \left[\int_{t_1}^{t_2} \beta_1(t)|k(t)|^{\gamma-2} \exp\left(\gamma \int_t^\tau \alpha_1(s)ds \right) dt \right]^{\alpha-1} \\ &\quad \times \left[\int_{t_1}^{t_2} \beta_2^+(t)|k(t)|^{\beta-2}dt + \sum_{t_1 \leq \tau_i < t_2} (\eta_i/\xi_i)^+ |k_{i-1}|^{\beta-2} \right], \end{aligned} \tag{5.23}$$

where $M, k(t)$ and k_i are defined as in equation (5.10), (5.6) and (5.7), respectively.

Proof. Since $h(t) = t^{1-\alpha}$ is a convex function for $t > 0$ and $\alpha > 1$, Jensen inequality $h\left(\frac{\mu+\xi}{2}\right) \leq \frac{1}{2}[h(\mu) + h(\xi)]$ with $\mu = h_{t_1}(\tau)$ and $\xi = h_{t_2}(\tau)$ and inequality (5.9) yield

$$\begin{aligned} \left[\frac{h_{t_1}(\tau) + h_{t_2}(\tau)}{2} \right]^{1-\alpha} &\leq \frac{1}{2}\left[h_{t_1}^{1-\alpha}(\tau) + h_{t_2}^{1-\alpha}(\tau) \right] \\ &\leq \frac{1}{2} M^{\beta-\alpha} \left[\int_{t_1}^{t_2} \beta_2^+(t)|k(t)|^{\beta-2}dt \right. \\ &\qquad \left. + \sum_{t_1 \leq \tau_i < t_2} (\eta_i/\xi_i)^+ |k_{i-1}|^{\beta-2} \right]. \end{aligned}$$

By using the definition of $h_{t_1}(\tau)$ and $h_{t_2}(\tau)$, one can obtain the desired result. □

The next theorem differs from the previous ones for the reason that the point τ where $|x(t)/k(t)|$ has supremum disappears in the Lyapunov type inequality. It is the main result of the present chapter and is the generalization of all the existing results mentioned in the literature review in Section 1.2.4.

Theorem 5.1.4 *Let α be the conjugate of γ. If system (5.1) has a solution $y(t) = (x(t), u(t))$ with nontrivial x such that $x(t_1^+) = x(t_2^-) = 0, x(t) \neq 0$ on (t_1, t_2), then we have the following Lyapunov type inequality*

$$\begin{aligned} 2^\alpha &\leq M^{\beta-\alpha} \exp\left(\frac{\alpha}{2} \int_{t_1}^{t_2} |\alpha_1(t)|dt \right) \left[\int_{t_1}^{t_2} \beta_1(t)|k(t)|^{\gamma-2}dt \right]^{\alpha-1} \\ &\quad \times \left[\int_{t_1}^{t_2} \beta_2^+(t)|k(t)|^{\beta-2}dt + \sum_{t_1 \leq \tau_i < t_2} (\eta_i/\xi_i)^+ |k_{i-1}|^{\beta-2} \right] \end{aligned} \tag{5.24}$$

where $M, k(t)$ and k_i are defined as in equation (5.10), (5.6) and (5.7), respectively.

Proof. Proceeding exactly as in the proof of Theorem 5.1.1 we arrive at (5.15). Then integrating (5.14) from t_1 to τ, we get

$$z(\tau) = \int_{t_1}^{\tau} \beta_1(t)|k(t)|^{\gamma-2}|v(t)|^{\gamma-2}v(t) \exp\left(\int_t^\tau \alpha_1(w)dw \right) dt,$$

which implies

$$|z(\tau)| \leq \exp\left(\int_{t_1}^{\tau} |\alpha_1(t)|dt\right) \int_{t_1}^{\tau} \beta_1(t)|k(t)|^{\gamma-2}|v(t)|^{\gamma-1}dt. \tag{5.25}$$

Similarly, by integrating (5.15) from τ to t_2, we have

$$|z(\tau)| \leq \exp\left(\int_{\tau}^{t_2} |\alpha_1(t)|dt\right) \int_{\tau}^{t_2} \beta_1(t)|k(t)|^{\gamma-2}|v(t)|^{\gamma-1}dt. \tag{5.26}$$

Let us define $Q_1 = \dfrac{|z(\tau)|}{\exp\left(\int_{t_1}^{\tau} |\alpha_1(t)|dt\right)}$ and $Q_2 = \dfrac{|z(\tau)|}{\exp\left(\int_{\tau}^{t_2} |\alpha_1(t)|dt\right)}$.

Observe that

$$\begin{aligned}\frac{|z(\tau)|^{\alpha/2}}{\exp\left(\frac{\alpha}{4}\int_{t_1}^{t_2} |\alpha_1(t)|dt\right)} &= \frac{|z(\tau)|^{\alpha/4}|z(\tau)|^{\alpha/4}}{\exp\left(\frac{\alpha}{4}\int_{t_1}^{\tau} |\alpha_1(t)|dt\right)\exp\left(\frac{\alpha}{4}\int_{\tau}^{t_2} |\alpha_1(t)|dt\right)} \\ &= (Q_1Q_2)^{\alpha/4} \leq \left(\frac{Q_1+Q_2}{2}\right)^{\alpha/2},\end{aligned}$$

where we have used the well known inequality $ab \leq \left(\dfrac{a+b}{2}\right)^2$. Therefore, from (5.25) and (5.26) we have

$$\begin{aligned}\frac{|z(\tau)|^{\alpha/2}}{\exp\left(\dfrac{\alpha}{4}\int_{t_1}^{t_2} |\alpha_1(t)|dt\right)} &\leq \left[\frac{|z(\tau)|}{2\exp\left(\int_{t_1}^{\tau} |\alpha_1(t)|dt\right)} + \frac{|z(\tau)|}{2\exp\left(\int_{\tau}^{t_2} |\alpha_1(t)|dt\right)}\right]^{\frac{\alpha}{2}} \\ &\leq 2^{\alpha/2}\left[\int_{t_1}^{t_2} \beta_1(t)|k(t)|^{\gamma-2}|v(t)|^{\gamma-1}dt\right]^{\frac{\alpha}{2}},\end{aligned}$$

which implies

$$\frac{2^{\alpha/2}|z(\tau)|^{\alpha/2}}{\exp\left(\dfrac{\alpha}{4}\int_{t_1}^{t_2} |\alpha_1(t)|dt\right)} \leq \left[\int_{t_1}^{t_2} \beta_1(t)|k(t)|^{\gamma-2}|v(t)|^{\gamma-1}dt\right]^{\frac{\alpha}{2}}.$$

Taking $\dfrac{2}{\alpha}$-th power, applying Hölder inequality with indices γ and α and using (5.13), we obtain

$$\begin{aligned}
\frac{2|z(\tau)|}{\exp\left(\dfrac{1}{2}\displaystyle\int_{t_1}^{t_2}|\alpha_1(t)|dt\right)} &\leq \int_{t_1}^{t_2}\beta_1(t)|k(t)|^{\gamma-2}|v(t)|^{\gamma-1}dt \\
&\leq \left[\int_{t_1}^{t_2}\beta_1(t)|k(t)|^{\gamma-2}dt\right]^{\frac{1}{\gamma}}\left[\int_{t_1}^{t_2}\beta_1(t)|k(t)|^{\gamma-2}|v(t)|^{\gamma}dt\right]^{\frac{1}{\alpha}} \\
&\leq \left[\int_{t_1}^{t_2}\beta_1(t)|k(t)|^{\gamma-2}dt\right]^{\frac{1}{\gamma}} \\
&\times\left[\int_{t_1}^{t_2}\beta_2^{+}(t)|k(t)|^{\beta-2}|z(t)|^{\beta}dt\right. \\
&\qquad\left.+\sum_{t_1\leq\tau_i<t_2}(\eta_i/\xi_i)^{+}|k_{i-1}|^{\beta-2}|z(\tau_i)|^{\beta}\right]^{\frac{1}{\alpha}}.
\end{aligned}$$

Since $|z(\tau)|\geq|z(t)|$ for all $t\in[t_1,t_2]$, we obtain from the above inequality that

$$\begin{aligned}
\frac{2|z(\tau)|}{\exp\left(\dfrac{1}{2}\displaystyle\int_{t_1}^{t_2}|\alpha_1(t)|dt\right)} &\leq |z(\tau)|^{\frac{\beta}{\alpha}}\left[\int_{t_1}^{t_2}\beta_1(t)|k(t)|^{\gamma-2}dt\right]^{\frac{1}{\gamma}} \\
&\times\left[\int_{t_1}^{t_2}\beta_2^{+}(t)|k(t)|^{\beta-2}dt+\sum_{t_1\leq\tau_i<t_2}(\eta_i/\xi_i)^{+}|k_{i-1}|^{\beta-2}\right]^{\frac{1}{\alpha}}.
\end{aligned}$$

Finally, we use (5.10) in the last inequality and take α-th power of both sides to see that (5.24) holds. □

Remark 5.1.1 *When $\alpha_1(t)=0$, Theorem 5.1.3 and Theorem 5.1.4 coincide.*

In the special case when β and γ are conjugates, ($\alpha=\beta$), then M disappears and we have the following theorems.

Theorem 5.1.5 *Let β be the conjugate of γ. If system (5.1) has a solution $y(t)=(x(t),u(t))$ with nontrivial x such that $x(t_1^{+})=x(t_2^{-})=0, x(t)\neq 0$ on (t_1,t_2), then we have the following Lyapunov type inequality*

$$h_{t_1}^{1-\beta}(\tau)+h_{t_2}^{1-\beta}(\tau)\leq\left[\int_{t_1}^{t_2}\beta_2^{+}(t)|k(t)|^{\beta-2}dt+\sum_{t_1\leq\tau_i<t_2}(\eta_i/\xi_i)^{+}|k_{i-1}|^{\beta-2}\right],$$

where $h_{t_1}(\tau), h_{t_2}(\tau), k(t)$ and k_i are defined as in equation (5.8), (5.6) and (5.7), respectively.

Theorem 5.1.6 *Let β be the conjugate of γ. If system (5.1) has a solution $y(t) = (x(t), u(t))$ with nontrivial x such that $x(t_1^+) = x(t_2^-) = 0, x(t) \neq 0$ on (t_1, t_2), then we have the following Lyapunov type inequality*

$$2^{2-\beta}\left[\frac{1}{h_{t_1}(\tau)}+\frac{1}{h_{t_2}(\tau)}\right]^{\beta-1} \leq \left[\int_{t_1}^{t_2}\beta_2^+(t)|k(t)|^{\beta-2}dt + \sum_{t_1\leq\tau_i<t_2}(\eta_i/\xi_i)^+|k_{i-1}|^{\beta-2}\right],$$

where $h_{t_1}(\tau), h_{t_2}(\tau), k(t)$ and k_i are defined as in equation (5.8), (5.6) and (5.7), respectively.

Theorem 5.1.7 *Let β be the conjugate of γ. If system (5.1) has a solution $y(t) = (x(t), u(t))$ with nontrivial x such that $x(t_1^+) = x(t_2^-) = 0, x(t) \neq 0$ on (t_1, t_2), then we have the following Lyapunov type inequality*

$$\begin{aligned} 2^{\beta} \quad &\leq \left[\int_{t_1}^{t_2}\beta_1(t)|k(t)|^{\gamma-2}\exp\left(\gamma\int_t^{\tau}\alpha_1(s)ds\right)dt\right]^{\beta-1} \\ &\times\left[\int_{t_1}^{t_2}\beta_2^+(t)|k(t)|^{\beta-2}dt + \sum_{t_1\leq\tau_i<t_2}(\eta_i/\xi_i)^+|k_{i-1}|^{\beta-2}\right], \end{aligned} \tag{5.27}$$

where $k(t)$ and k_i are defined as in equation (5.6) and (5.7), respectively.

Theorem 5.1.8 *Let β be the conjugate of γ. If system (5.1) has a solution $y(t) = (x(t), u(t))$ with nontrivial x such that $x(t_1^+) = x(t_2^-) = 0, x(t) \neq 0$ on (t_1, t_2), then we have the following Lyapunov type inequality*

$$\begin{aligned} 2^{\beta} \leq \exp\left(\frac{\beta}{2}\int_{t_1}^{t_2}|\alpha_1(t)|dt\right)&\left[\int_{t_1}^{t_2}\beta_1(t)|k(t)|^{\gamma-2}dt\right]^{\beta-1} \\ \times&\left[\int_{t_1}^{t_2}\beta_2^+(t)|k(t)|^{\beta-2}dt + \sum_{t_1\leq\tau_i<t_2}(\eta_i/\xi_i)^+|k_{i-1}|^{\beta-2}\right], \end{aligned}$$

where $k(t)$ and k_i are defined as in equation (5.6) and (5.7), respectively.

The following corollaries obtained for second-order impulsive Emden-Fowler differential equations (5.3) and impulsive half linear equations (5.5) are immediate from the previous theorems. In this case since $\alpha_1(t) = 0$, the results of Theorem 5.1.3 and Theorem 5.1.4 coincide, hence they provide the same corollary.

Corollary 5.1.1 *Let α be the conjugate of γ. If the impulsive Emden-Fowler equation* (5.3) *has a real nontrivial solution x such that $x(t_1^+) = x(t_2^-) = 0, x(t) \neq 0$ on (t_1, t_2), then we have the following Lyapunov type inequality*

$$h_{t_1}^{1-\alpha}(\tau) + h_{t_2}^{1-\alpha}(\tau) \leq M^{\beta-\alpha} \left[\int_{t_1}^{t_2} q^+(t)|k(t)|^{\beta-2}dt + \sum_{t_1 \leq \tau_i < t_2} (\eta_i/\xi_i)^+ |k_{i-1}|^{\beta-2} \right], \tag{5.28}$$

where

$$M = \sup_{t \in (t_1,t_2)} \left| \frac{x(t)}{k(t)} \right| = \left| \frac{x(\tau)}{k(\tau)} \right|, \tag{5.29}$$

$$\begin{aligned} h_{t_1}(t) &= \int_{t_1}^{t} p^{1-\gamma}(w)|k(w)|^{\gamma-2}dw, \\ h_{t_2}(t) &= \int_{t}^{t_2} p^{1-\gamma}(w)|k(w)|^{\gamma-2}dw, \end{aligned} \tag{5.30}$$

and $k(t)$ and k_i are defined as in equation (5.6) and in equation (5.7), respectively.

Corollary 5.1.2 *Let α be the conjugate of γ. If the impulsive Emden-Fowler equation* (5.3) *has a real nontrivial solution x such that $x(t_1^+) = x(t_2^-) = 0, x(t) \neq 0$ on (t_1, t_2), then we have the following Lyapunov type inequality*

$$\begin{aligned} 2^{2-\alpha} \left[\frac{1}{h_{t_1}(\tau)} + \frac{1}{h_{t_2}(\tau)} \right]^{\alpha-1} \leq M^{\beta-\alpha} \Bigg[& \int_{t_1}^{t_2} q^+(t)|k(t)|^{\beta-2}dt \\ & + \sum_{t_1 \leq \tau_i < t_2} (\eta_i/\xi_i)^+ |k_{i-1}|^{\beta-2} \Bigg], \end{aligned} \tag{5.31}$$

where $h_{t_1}(t), h_{t_1}(t), M, k(t)$ and k_i are defined as in equation (5.30), (5.29), (5.6) and in equation (5.7), respectively.

Corollary 5.1.3 *Let α be the conjugate of γ. If the impulsive Emden-Fowler equation* (5.3) *has a real nontrivial solution x such that $x(t_1^+) = x(t_2^-) = 0, x(t) \neq 0$ on (t_1, t_2), then we have the following Lyapunov type inequality*

$$\begin{aligned} 2^{\alpha} \leq M^{\beta-\alpha} & \left[\int_{t_1}^{t_2} p^{1-\gamma}(t)|k(t)|^{\gamma-2}dt \right]^{\alpha-1} \\ & \times \left[\int_{t_1}^{t_2} q^+(t)|k(t)|^{\beta-2}dt + \sum_{t_1 \leq \tau_i < t_2} (\eta_i/\xi_i)^+ |k_{i-1}|^{\beta-2} \right], \end{aligned} \tag{5.32}$$

where $M, k(t)$ and k_i are defined as in equation (5.29), (5.6) and in equation (5.7), respectively.

Corollary 5.1.4 *Let β be the conjugate of γ. If the impulsive Emden-Fowler equation* (5.5) *has a real nontrivial solution x such that $x(t_1^+) = x(t_2^-) = 0, x(t) \neq 0$ on (t_1, t_2), then we have the following Lyapunov type inequality*

$$h_{t_1}^{1-\beta}(\tau) + h_{t_2}^{1-\beta}(\tau) \leq \left[\int_{t_1}^{t_2} q^+(t)|k(t)|^{\beta-2}dt + \sum_{t_1 \leq \tau_i < t_2} (\eta_i/\xi_i)^+|k_{i-1}|^{\beta-2}\right], \tag{5.33}$$

where $h_{t_1}(t), h_{t_2}(t), k(t)$ and k_i are defined as in equation (5.30), (5.6) and in equation (5.7), respectively.

Corollary 5.1.5 *Let β be the conjugate of γ. If the impulsive Emden-Fowler equation* (5.5) *has a real nontrivial solution x such that $x(t_1^+) = x(t_2^-) = 0, x(t) \neq 0$ on (t_1, t_2), then we have the following Lyapunov type inequality*

$$2^{2-\beta}\left[\frac{1}{h_{t_1}(\tau)} + \frac{1}{h_{t_2}(\tau)}\right]^{\beta-1} \leq \left[\int_{t_1}^{t_2} q^+(t)|k(t)|^{\beta-2}dt + \sum_{t_1 \leq \tau_i < t_2} (\eta_i/\xi_i)^+|k_{i-1}|^{\beta-2}\right]$$

where $h_{t_1}(t), h_{t_2}(t), k(t)$ and k_i are defined as in equation (5.30), (5.6) and in equation (5.7), respectively.

Corollary 5.1.6 *Let β be the conjugate of γ. If the impulsive half-linear equation* (5.5) *has a real nontrivial solution x such that $x(t_1^+) = x(t_2^-) = 0, x(t) \neq 0$ on (t_1, t_2), then we have the following Lyapunov type inequality*

$$\begin{aligned} 2^\beta \leq & \left[\int_{t_1}^{t_2} p^{1-\gamma}(t)|k(t)|^{\gamma-2}dt\right]^{\beta-1} \\ & \times \left[\int_{t_1}^{t_2} q^+(t)|k(t)|^{\beta-2}dt + \sum_{t_1 \leq \tau_i < t_2} (\eta_i/\xi_i)^+|k_{i-1}(t)|^{\beta-2}\right]. \end{aligned} \tag{5.34}$$

where $k(t)$ and k_i are defined as in equation (5.30), (5.6) and in equation (5.7), respectively.

The next theorems contain new Lyapunov type inequalities given in the subintervals $[a, \tau]$ and $[\tau, b]$ where τ is the point at which $|x(t)/k(t)|$ has supremum. When τ is an impulsive point, additional term occurs in Lyapunov type inequality. Therefore, in the next results the location of τ is important.

Theorem 5.1.9 *Let α be the conjugate of γ and M be given by* (5.29). *If the impulsive Emden-Fowler equation (5.3) has a real nontrivial solution x such that $x(t_1^+) = x(t_2^-) = 0, x(t) \neq 0$ on (t_1, t_2), then there exists $\tau \in (t_1, t_2)$ such that the following inequalities hold:*

(i) If $\tau \in (\tau_{n-1}, \tau_n)$ for some n, then

$$1 \leq M^{\beta-\alpha} \left[\int_{t_1}^{\tau} p^{1-\gamma}(t)|k(t)|^{\gamma-2} dt \right]^{\alpha-1} \times \left[\int_{t_1}^{\tau} q^+(t)|k(t)|^{\beta-2} dt + \sum_{t_1 \leq \tau_i < \tau} (\eta_i/\xi_i)^+ |k_{i-1}|^{\beta-2} \right]$$

and

$$1 \leq M^{\beta-\alpha} \left[\int_{\tau}^{t_2} p^{1-\gamma}(t)|k(t)|^{\gamma-2} dt \right]^{\alpha-1} \times \left[\int_{\tau}^{t_2} q^+(t)|k(t)|^{\beta-2} dt + \sum_{\tau \leq \tau_i < t_2} (\eta_i/\xi_i)^+ |k_{i-1}|^{\beta-2} \right].$$

(ii) If $\tau = \tau_n$, then

$$1 \leq M^{\beta-\alpha} \left[\int_{t_1}^{\tau} p^{1-\gamma}(t)|k(t)|^{\gamma-2} dt \right]^{\alpha-1} \times \left[\int_{t_1}^{\tau} q^+(t)|k(t)|^{\beta-2} dt + \sum_{t_1 \leq \tau_i < \tau} (\eta_i/\xi_i)^+ |k_{i-1}|^{\beta-2} + \max_{i=1,2,\ldots,m} (\eta_i/\xi_i)^+ |k_{i-1}|^{\beta-2} \right]$$

and

$$1 \leq M^{\beta-\alpha} \left[\int_{\tau}^{t_2} p^{1-\gamma}(t)|k(t)|^{\gamma-2} dt \right]^{\alpha-1} \times \left[\int_{\tau}^{t_2} q^+(t)|k(t)|^{\beta-2} dt + \sum_{\tau \leq \tau_i < t_2} (\eta_i/\xi_i)^+ |k_{i-1}|^{\beta-2} \right].$$

Proof. (i) The proof is obtained by applying the proof of the Theorem 5.1.4 step by step for the intervals (t_1, τ) and (τ, t_2) separately and using $z'(\tau) = 0$ which implies $v(\tau) = 0$.

(ii) Let $\tau = \tau_n$ and $\tau_n < s < \tau_{n+1}$. We set

$$\beta_1 = p^{1-\gamma}(t), \quad \beta_2(t) = q(t).$$

If we repeat the same procedure of the proof of Theorem 2.1, for the interval (t_1, s), we get

$$|z(s)| \leq \left[\int_{t_1}^{s} \beta_1(t)|k(t)|^{\gamma-2}dt\right]^{\frac{1}{\gamma}} \left[\int_{t_1}^{s} \beta_1(t)|k(t)|^{\gamma-2}|v(t)|^{\gamma}dt\right]^{\frac{1}{\alpha}}. \tag{5.35}$$

On the other hand, one can show that

$$\begin{aligned}\int_{t_1}^{s}(vz)'dt &= z(s)v(s^-) + \sum_{t_1 \leq \tau_i < s} (\eta_i/\xi_i)|k_{i-1}|^{\beta-2}|z(\tau_i)|^{\beta} \\ &= \int_{t_1}^{s} \beta_1(t)|k(t)|^{\gamma-2}|v(t)|^{\gamma}dt - \int_{t_1}^{s} \beta_2(t)|k(t)|^{\beta-2}|z(t)|^{\beta}dt.\end{aligned} \tag{5.36}$$

Substituting (5.36) into (5.35), we have

$$\begin{aligned}|z(s)| \leq &\left[\int_{t_1}^{s} \beta_1(t)|k(t)|^{\gamma-2}dt\right]^{\frac{1}{\gamma}} \\ &\times \left[\int_{t_1}^{s} \beta_2(t)|k(t)|^{\beta-2}|z(t)|^{\beta}dt + \sum_{t_1 \leq \tau_i < s} (\eta_i/\xi_i)|k_{i-1}|^{\beta-2}|z(\tau_i)|^{\beta} + z(s)v(s^-)\right]^{\frac{1}{\alpha}}.\end{aligned}$$

Now, letting $s \to \tau^+$,

$$\begin{aligned}|z(\tau)| \leq &\left[\int_{t_1}^{\tau} \beta_1(t)|k(t)|^{\gamma-2}dt\right]^{\frac{1}{\gamma}} \\ &\times \left[\int_{t_1}^{\tau} \beta_2(t)|k(t)|^{\beta-2}|z(t)|^{\beta}dt + \sum_{t_1 \leq \tau_i < \tau^+} (\eta_i/\xi_i)|k_{i-1}|^{\beta-2}|z(\tau_i)|^{\beta} + z(\tau)v(\tau^+)\right]^{\frac{1}{\alpha}}.\end{aligned}$$

Note that $z(\tau)v(\tau^+) \leq 0$ and $z(\tau)v(\tau^-) \geq 0$. Therefore,

$$\begin{aligned}|z(\tau)| \leq |z(\tau)|^{\frac{\beta}{\alpha}} &\left[\int_{t_1}^{\tau} \beta_1(t)|k(t)|^{\gamma-2}dt\right]^{\frac{1}{\gamma}} \\ &\times \left[\int_{t_1}^{\tau} \beta_2(t)|k(t)|^{\beta-2}dt + \sum_{t_1 \leq \tau_i < \tau} (\eta_i/\xi_i)|k_{i-1}|^{\beta-2} + \frac{\eta_n}{\xi_n}|k_{n-1}|^{\beta-2}\right]^{\frac{1}{\alpha}}.\end{aligned}$$

Finally, by using $|z(t)| \leq |z(\tau)|$ for all $t \in [t_1, t_2]$ and taking α-th power of both sides, we obtain the desired inequality

$$\begin{aligned}1 \leq M^{\beta-\alpha} &\left[\int_{t_1}^{\tau} \beta_1(t)|k(t)|^{\gamma-2}dt\right]^{\alpha-1} \\ &\times \left[\int_{t_1}^{\tau} \beta_2^+(t)|k(t)|^{\beta-2}dt + \sum_{t_1 \leq \tau_i < \tau} (\eta_i/\xi_i)^+|k_{i-1}|^{\beta-2} + \max_{i=1,2,\ldots,m} (\eta_i/\xi_i)^+|k_{i-1}|^{\beta-2}\right].\end{aligned}$$

Now, let $\tau = \tau_n$ and $s < \tau_n < \tau_{n+1}$. By the same procedure applied on (s, t_2), we get

$$|z(s)| \leq \left[\int_s^{t_2} \beta_1(t)|k(t)|^{\gamma-2}dt\right]^{\frac{1}{\gamma}} \left[\int_s^{t_2} \beta_1(t)|k(t)|^{\gamma-2}|v(t)|^{\gamma}dt\right]^{\frac{1}{\alpha}},$$

which in a similar manner above leads to

$$\begin{aligned}|z(s)| \leq & \left[\int_s^{t_2} \beta_1(t)|k(t)|^{\gamma-2}dt\right]^{\frac{1}{\gamma}} \\ & \times \left[\int_s^{t_2} \beta_2(t)|k(t)|^{\beta-2}|z(t)|^{\beta}dt + \sum_{s \leq \tau_i < t_2} (\eta_i/\xi_i)|k_{i-1}|^{\beta-2}|z(\tau_i)|^{\beta} - z(s)v(s^+)\right]^{\frac{1}{\alpha}}\end{aligned}$$

and so, as $s \to \tau^-$, we obtain

$$\begin{aligned}|z(\tau)| \leq & \left[\int_\tau^{t_2} \beta_1(t)|k(t)|^{\gamma-2}dt\right]^{\frac{1}{\gamma}} \\ & \times \left[\int_\tau^{t_2} \beta_2(t)|k(t)|^{\beta-2}|z(t)|^{\beta}dt + \sum_{\tau \leq \tau_i < t_2} (\eta_i/\xi_i)|k_{i-1}|^{\beta-2}|z(\tau_i)|^{\beta} - z(\tau)v(\tau^-)\right]^{\frac{1}{\alpha}},\end{aligned}$$

which yields

$$\begin{aligned}1 \leq M^{\beta-\alpha} & \left[\int_\tau^{t_2} \beta_1(t)|k(t)|^{\gamma-2}dt\right]^{\alpha-1} \\ & \times \left[\int_\tau^{t_2} \beta_2^+(t)|k(t)|^{\beta-2}dt + \sum_{\tau \leq \tau_i < t_2} (\eta_i/\xi_i)^+|k_{i-1}|^{\beta-2}\right].\end{aligned}$$

□

Theorem 5.1.10 *Let β be the conjugate of γ. If the impulsive half-linear equation (5.5) has a real nontrivial solution x such that $x(t_1^+) = x(t_2^-) = 0, x(t) \neq 0$ on (t_1, t_2), then there exists $\tau \in (t_1, t_2)$ such that the following inequalities hold:*

(i) If $\tau \in (\tau_{n-1}, \tau_n)$, for some $n = 1, 2, \ldots, m$, then

$$1 \leq \left[\int_{t_1}^{\tau} p^{1-\gamma}(t)|k(t)|^{\gamma-2}dt\right]^{\beta-1} \left[\int_{t_1}^{\tau} q^+(t)|k(t)|^{\beta-2}dt + \sum_{t_1 \leq \tau_i < \tau} (\eta_i/\xi_i)^+|k_{i-1}|^{\beta-2}\right]$$

and

$$1 \leq \left[\int_{\tau}^{t_2} p^{1-\gamma}(t)|k(t)|^{\gamma-2}dt\right]^{\beta-1} \left[\int_{\tau}^{t_2} q^+(t)|k(t)|^{\beta-2}dt + \sum_{\tau \leq \tau_i < t_2} (\eta_i/\xi_i)^+|k_{i-1}|^{\beta-2}\right].$$

(ii) If $\tau = \tau_n$, then

$$1 \leq \left[\int_{t_1}^{\tau} p^{1-\gamma}(t)|k(t)|^{\gamma-2}dt\right]^{\beta-1}$$
$$\times \left[\int_{t_1}^{\tau} q^{+}(t)|k(t)|^{\beta-2}dt + \sum_{t_1\leq\tau_i<\tau}(\eta_i/\xi_i)^{+}|k_{i-1}|^{\beta-2} + \max_{i=1,2,\ldots,m}(\eta_i/\xi_i)^{+}|k_{i-1}|^{\beta-2}\right]$$

and

$$1 \leq \left[\int_{\tau}^{t_2} p^{1-\gamma}(t)|k(t)|^{\gamma-2}dt\right]^{\beta-1}\left[\int_{\tau}^{t_2} q^{+}(t)|k(t)|^{\beta-2}dt + \sum_{\tau\leq\tau_i<t_2}(\eta_i/\xi_i)^{+}|k_{i-1}|^{\beta-2}\right].$$

Remark 5.1.2 *Our results generalize all the previous results obtained for nonlinear systems, nonlinear equations, linear systems and linear equations without impulse effect.*

Remark 5.1.3 *If there is no impulse effect, i.e $\xi_i = 1$ and $\eta_i = 0$ for all $i \in \mathbb{N}$, then inequality (5.9) and (5.19) coincide with inequalities (2.27) and (2.34) of [105], respectively. Theorem 5.1.3 yields the same result as [105, Corollary 2.3]. Moreover, it is the generalization of [112] from the case $\alpha = \beta = \gamma = 2$ to the nonlinear system (5.1). Besides, inequality (5.24) gives the same inequality as in Tiryaki and et al. [106, Theorem 1]. Furtermore Theorem 5.1.9 is reduced to Theorem 3 of the same work.*

Remark 5.1.4 *Let $\alpha = \beta = \gamma = 2$. Then system (5.1) is reduced to linear system of two first order impulsive differential equations which is considered in [55] and [42]. It can be observed that Theorem 5.1.3 gives the same result as [55, Theorem 5.1]. Let $x = \int_{t_1}^{t_2} \alpha_1(t)dt$. Since $2\exp(-x/2) \geq 2 - x$, Theorem 5.1.4 is better than [42, Theorem 5.1]. In the absence of impulse effect, inequality (5.24) in Theorem 5.1.4 is sharper than the results obtained in [58], [41] while it coincides with [99, Theorem 2.4] for the case $n = 1$.*

Remark 5.1.5 *Let $\alpha_1(t) = 0$. Then system (5.1) can be written in the form of equation (5.3). Therefore Corollary 5.1.1, Corollary 5.1.2 and Corollary 5.1.4, Corollary 5.1.5 provide new inequalities for impulsive Emden-Fowler equations (5.3) and impulsive half linear equations (5.5) for $\alpha = \beta$, respectively. Furthermore, Corollary*

5.1.3, Theorem 5.1.9 and Corollary 5.1.6, Theorem 5.1.10 generalize [106, Theorem 3] and [106, Corollary 4], respectively. If there is no impulse effect and $\alpha = \beta$, then Corollary 5.1.6 reduces to [33, Theorem 5.1.1] for half-linear differential equation. Moreover, since the restricted condition, i.e. a bounded positive function, on the function r (in our case it is $q(t) = \beta_2(t)$) in [91, Theorem 2.3] is dropped, Corollary 5.1.6 improves [91, Theorem 2.3]. In this case Theorem 5.1.10 coincides with [63, Lemma 1].

Remark 5.1.6 *When $\alpha_1(t) = 0$ and $\alpha = \beta = \gamma = 2$, Corollary 5.1.6 provides the same result as [43, Theorem 4.5] for the case of the second order impulsive differential equations. Moreover if there is no impulse effect, then the same result of Krein [57] is obtained.*

5.2 Applications

This section is devoted to show the applicability of Lyapunov type inequalities, obtained in Section 5.1.1, to investigate the asymptotic behaviour of solutions of system (5.1). We are concerned with disconjugacy, finding lower bounds for the eigenvalues of the associated eigenvalue problems and boundedness of weakly oscillatory as well as weakly bounded solutions.

5.2.1 Disconjugacy

In this section by using the inequalities derived in Theorem 5.1.3, Theorem 5.1.4, Theorem 5.1.7, Theorem 5.1.8, Corollary 5.1.3 and Corollary 5.1.6, we establish some disconjugacy results.

Theorem 5.2.1 *Let α be the conjugate number of γ and M, $k(t)$ and k_i be defined as in equations* (5.10)*, (5.6) and (5.7), respectively. If for every $\tau \in (t_1, t_2)$*

$$\begin{aligned} 2^{\alpha} &> M^{\beta-\alpha} \left[\int_{t_1}^{t_2} \beta_1(t)|k(t)|^{\gamma-2} \exp\left(\gamma \int_t^{\tau} \alpha_1(s)ds \right) dt \right]^{\alpha-1} \\ &\times \left[\int_{t_1}^{t_2} \beta_2^+(t)|k(t)|^{\beta-2} dt + \sum_{t_1 \le \tau_i < t_2} (\eta_i/\xi_i)^+ |k_{i-1}|^{\beta-2} \right] \end{aligned} \tag{5.37}$$

holds, then system (5.1) *is disconjugate on* $[t_1, t_2]$.

Proof. Suppose on the contrary that there is a real solution $y(t) = (x(t), u(t))$ with nontrivial x having two zeros $s_1, s_2 \in [t_1, t_2]$ $(s_1 < s_2)$ such that $x(t) \neq 0$ for all $t \in (s_1, s_2)$. Applying Theorem 5.1.3 we see that

$$\begin{aligned} 2^\alpha &\leq M^{\beta-\alpha} \left[\int_{s_1}^{s_2} \beta_1(t)|k(t)|^{\gamma-2} \exp\left(\gamma \int_t^\tau \alpha_1(s)ds \right) dt \right]^{\alpha-1} \\ &\times \left[\int_{s_1}^{s_2} \beta_2^+(t)|k(t)|^{\beta-2}dt + \sum_{s_1 \leq \tau_i < s_2} (\eta_i/\xi_i)^+ |k_{i-1}|^{\beta-2} \right] \\ &\leq M^{\beta-\alpha} \left[\int_{t_1}^{t_2} \beta_1(t)|k(t)|^{\gamma-2} \exp\left(\gamma \int_t^\tau \alpha_1(s)ds \right) dt \right]^{\alpha-1} \\ &\times \left[\int_{t_1}^{t_2} \beta_2^+(t)|k(t)|^{\beta-2}dt + \sum_{t_1 \leq \tau_i < t_2} (\eta_i/\xi_i)^+ |k_{i-1}|^{\beta-2} \right]. \end{aligned}$$

Clearly, the last inequality contradicts (5.37). The proof is complete. $\square$

Since the proofs of the following theorems are exactly the same as proof of the Theorem 5.2.1, we omit them.

Theorem 5.2.2 *Let* β *be the conjugate number of* γ *and* M*,* $k(t)$ *and* k_i *be defined as in equations* (5.10)*, (5.6) and (5.7), respectively. If for every* $\tau \in (t_1, t_2)$

$$\begin{aligned} 2^\beta \quad &> \left[\int_{t_1}^{t_2} \beta_1(t)|k(t)|^{\gamma-2} \exp\left(\gamma \int_t^\tau \alpha_1(s)ds \right) dt \right]^{\beta-1} \\ &\times \left[\int_{t_1}^{t_2} \beta_2^+(t)|k(t)|^{\beta-2}dt + \sum_{t_1 \leq \tau_i < t_2} (\eta_i/\xi_i)^+ |k_{i-1}|^{\beta-2} \right] \end{aligned}$$

holds, then system (5.1) *is disconjugate on* $[t_1, t_2]$.

Theorem 5.2.3 *Let* α *be the conjugate number of* γ *and* M*,* $k(t)$ *and* k_i *be defined as in equations* (5.10)*, (5.6) and (5.7), respectively. If*

$$\begin{aligned} 2^\alpha > M^{\beta-\alpha} &\exp\left(\frac{\alpha}{2} \int_{t_1}^{t_2} |\alpha_1(t)|dt \right) \left[\int_{t_1}^{t_2} \beta_1(t)|k(t)|^{\gamma-2}dt \right]^{\alpha-1} \\ &\times \left[\int_{t_1}^{t_2} \beta_2^+(t)|k(t)|^{\beta-2}dt + \sum_{t_1 \leq \tau_i < t_2} (\eta_i/\xi_i)^+ |k_{i-1}|^{\beta-2} \right] \end{aligned}$$

holds, then system (5.1) *is disconjugate on* $[t_1, t_2]$.

Theorem 5.2.4 *Let β be the conjugate of γ and $k(t)$ and k_i be defined as in equations (5.6) and (5.7), respectively. If*

$$\begin{aligned}2^{\beta} > \exp\left(\frac{\beta}{2}\int_{t_1}^{t_2}|\alpha_1(t)|dt\right)\left[\int_{t_1}^{t_2}\beta_1(t)|k(t)|^{\gamma-2}dt\right]^{\beta-1}\\ \times\left[\int_{t_1}^{t_2}\beta_2^{+}(t)|k(t)|^{\beta-2}dt+\sum_{t_1\le\tau_i<t_2}(\eta_i/\xi_i)^{+}|k_{i-1}|^{\beta-2}\right]\end{aligned}$$

holds, then system (5.1) *is disconjugate on* $[t_1, t_2]$.

Again, we have the corresponding corollaries.

Corollary 5.2.1 *Let α be the conjugate of γ and M, $k(t)$ and k_i be defined as in equations* (5.29)*, (5.6) and (5.7), respectively. If*

$$\begin{aligned}2^{\alpha} > M^{\beta-\alpha}\left[\int_{t_1}^{t_2}p^{1-\gamma}(t)|k(t)|^{\gamma-2}dt\right]^{\alpha-1}\\ \times\left[\int_{t_1}^{t_2}q^{+}(t)|k(t)|^{\beta-2}dt+\sum_{t_1\le\tau_i<t_2}(\eta_i/\xi_i)^{+}|k_{i-1}|^{\beta-2}\right]\end{aligned}\tag{5.38}$$

holds, then equation (5.3) *is disconjugate on* $[t_1, t_2]$.

Proof. Suppose on the contrary that there is a real nontrivial solution x of eqution (5.3) having two zeros $s_1, s_2 \in [t_1, t_2]$ $(s_1 < s_2)$ such that $x(t) \neq 0$ for all $t \in (s_1, s_2)$. Applying Corollary 5.1.3 we see that

$$\begin{aligned}2^{\alpha} &\le M^{\beta-\alpha}\left[\int_{s_1}^{s_2}p^{1-\gamma}(t)|k(t)|^{\gamma-2}dt\right]^{\alpha-1}\\ &\quad\times\left[\int_{s_1}^{s_2}q^{+}(t)|k(t)|^{\beta-2}dt+\sum_{s_1\le\tau_i<s_2}(\eta_i/\xi_i)^{+}|k_{i-1}|^{\beta-2}\right]\\ &\le M^{\beta-\alpha}\left[\int_{t_1}^{t_2}p^{1-\gamma}(t)|k(t)|^{\gamma-2}dt\right]^{\alpha-1}\\ &\quad\times\left[\int_{t_1}^{t_2}q^{+}(t)|k(t)|^{\beta-2}dt+\sum_{t_1\le\tau_i<t_2}(\eta_i/\xi_i)^{+}|k_{i-1}|^{\beta-2}\right].\end{aligned}$$

Clearly, the last inequality contradicts (5.38). The proof is complete. □

Since the proof of the following corollary is exactly the same as proof of the Corollary 5.2.1, we omit it.

Corollary 5.2.2 *Let β be the conjugate of γ and $k(t)$ and k_i be defined as in equations (5.6) and (5.7), respectively. If*

$$2^{\beta} > \left[\int_{t_1}^{t_2} p^{1-\gamma}(t)|k(t)|^{\gamma-2}dt\right]^{\beta-1}\left[\int_{t_1}^{t_2} q^{+}(t)|k(t)|^{\beta-2}dt + \sum_{t_1\le\tau_i<t_2}(\eta_i/\xi_i)^{+}|k_{i-1}|^{\beta-2}\right]$$

holds, then equation (5.5) *is disconjugate on* $[t_1, t_2]$.

5.2.2 Eigenvalue Problems

Now, we present an application of the obtained Lyapunov-type inequalities for system (5.1) and equations (5.3) and (5.5). The proofs of the following theorems are based on the Lyapunov type inequalities derived in Theorem 5.1.3 and Theorem 5.1.4 and Corollary 5.1.3 and Corollary 5.1.6.

Consider the system of impulsive eigenvalue problem

$$\begin{aligned}
&x' = \alpha_1(t)x + \beta_1(t)|u|^{\gamma-2}u, \quad u' = -\alpha_1(t)u - \lambda\beta_2(t)|x|^{\beta-2}x, \quad t \neq \tau_i \\
&x(\tau_i^+) = \xi_i x(\tau_i^-), \quad u(\tau_i^+) = \xi_i u(\tau_i^-) - \mu\eta_i|x(\tau_i^-)|^{\beta-2}x(\tau_i^-), \\
&x(t_1) = x(t_2) = 0
\end{aligned} \tag{5.39}$$

and the impulsive Emden-Fowler type eigenvalue problem

$$\begin{aligned}
&(p(t)|x'|^{\alpha-2}x')' + \lambda q(t)|x|^{\beta-2}x = 0, \quad t \neq \tau_i, \\
&x(\tau_i^+) = \xi_i\, x(\tau_i^-), \\
&p(\tau_i^+)|x'(\tau_i^+)|^{\alpha-2}x'(\tau_i^+) = \xi_i\, p(\tau_i^-)|x'(\tau_i^-)|^{\alpha-2}x'(\tau_i^-) - \mu\eta_i|x(\tau_i^-)|^{\beta-2}x(\tau_i^-) \\
&x(t_1) = x(t_2) = 0
\end{aligned} \tag{5.40}$$

where $\lambda, \mu \in \mathbb{R}$.

If $\alpha = \beta$ in (5.40), then impulsive Emden-Fowler type eigenvalue problem (5.40) becomes impulsive half linear eigenvalue problem.

Definition 5.2.1 *A pair (λ, μ) is called an eigenvalue of* (5.39) *if there is a corresponding solution (x, u) such that $x(t) \not\equiv 0$ on (t_1, t_2).*

Definition 5.2.2 *A pair (λ, μ) is called an eigenvalue of* (5.40) *if there is a corresponding nontrivial solution x on (t_1, t_2).*

Theorem 5.2.5 *Let α be the conjugate of γ. If (λ, μ) is a positive eigenvalue pair of* (5.39)*, then*

$$\lambda \geq \frac{2^\alpha}{A}$$

where

$$\begin{aligned} A &= M^{\beta-\alpha}\left[\int_{t_1}^{t_2} \beta_1(t)|k(t)|^{\gamma-2}\exp\left(\gamma\int_t^\tau \alpha_1(s)ds\right)dt\right]^{\alpha-1} \\ &\times\left[\int_{t_1}^{t_2} \beta_2^+(w)|k(t)|^{\beta-2}dt + \frac{\mu}{\lambda}\sum_{t_1\leq\tau_i<t_2}(\eta_i/\xi_i)^+|k_{i-1}|^{\beta-2}\right] \end{aligned}$$

and $M, k(t)$ and k_i are defined as in equation (5.10), (5.6) and (5.7), respectively.

Proof. Let (λ, μ) be a positive eigenvalue and (x, u) be the corresponding eigenfunctions of the system (5.39). If for some $\tau \in (t_1, t_2)$ we apply Lyapunov inequality obtained in Theorem 5.1.3 for system (5.39), we get

$$\begin{aligned} 2^\alpha &\leq M^{\beta-\alpha}\left[\int_{t_1}^{t_2} \beta_1(t)|k(t)|^{\gamma-2}\exp\left(\gamma\int_t^\tau \alpha_1(s)ds\right)dt\right]^{\alpha-1} \\ &\times\left[\lambda\int_{t_1}^{t_2} \beta_2^+(t)|k(t)|^{\beta-2}dt + \mu\sum_{t_1\leq\tau_i<t_2}(\eta_i/\xi_i)^+|k_{i-1}|^{\beta-2}\right]. \end{aligned}$$

Then for the eigenvalue λ we can find the desired lower bound. □

Since the proofs of the following theorem and corollaries are same as the proof of Theorem 5.2.5, we skip them.

Theorem 5.2.6 *Let α be the conjugate of γ. If (λ, μ) is a positive eigenvalue pair of* (5.39)*, then*

$$\lambda \geq \frac{2^\alpha}{B}$$

where

$$\begin{aligned} B = M^{\beta-\alpha}\exp\left(\frac{\alpha}{2}\int_{t_1}^{t_2}|\alpha_1(t)|dt\right)\left[\int_{t_1}^{t_2}\beta_1(t)|k(t)|^{\gamma-2}dt\right]^{\alpha-1} \\ \times\left[\int_{t_1}^{t_2} \beta_2^+(t)|k(t)|^{\beta-2}dt + \frac{\mu}{\lambda}\sum_{t_1\leq\tau_i<t_2}(\eta_i/\xi_i)^+|k_{i-1}|^{\beta-2}\right] \end{aligned}$$

and $M, k(t)$ and k_i are defined as in equation (5.10), (5.6) and (5.7), respectively.

Corollary 5.2.3 *Let α be the conjugate of γ. If (λ, μ) is a positive eigenvalue pair of* (5.40) *then*

$$\lambda \geq \frac{2^{\alpha}}{C}$$

where

$$C = M^{\beta-\alpha}\left[\int_{t_1}^{t_2} p^{1-\gamma}(t)|k(t)|^{\gamma-2}dt\right]^{\alpha-1} \\ \times\left[\int_{t_1}^{t_2} q^{+}(t)|k(t)|^{\beta-2}dt + \frac{\mu}{\lambda}\sum_{t_1\leq\tau_i<t_2}(\eta_i/\xi_i)^{+}|k_{i-1}|^{\beta-2}\right],$$

and $M, k(t)$ and k_i are defined as in equation (5.29), (5.6) and in equation (5.7), respectively.

Corollary 5.2.4 *Let β be the conjugate of γ. If (λ, μ) is a positive eigenvalue pair of impulsive half linear eigenvalue problem then*

$$\lambda \geq \frac{2^{\beta}}{D}$$

where

$$D = \left[\int_{t_1}^{t_2} p^{1-\gamma}(t)|k(t)|^{\gamma-2}dt\right]^{\beta-1} \\ \times\left[\int_{t_1}^{t_2} q^{+}(t)|k(t)|^{\beta-2}dt + \frac{\mu}{\lambda}\sum_{t_1\leq\tau_i<t_2}(\eta_i/\xi_i)^{+}|k_{i-1}|^{\beta-2}\right],$$

and $k(t)$ and k_i are defined as in equation (5.6) and in equation (5.7), respectively.

5.2.3 Boundedness

In this section, as an application of Lyapunov type inequality given in Section 5.1.1, we obtain a sufficient condition for the boundedness of weakly oscillatory and weakly bounded solutions of system (5.1).

Theorem 5.2.7 *Suppose that for some $\tau \in (t_1, t_2)$*

$$\left[\int^{\infty} \beta_1(t)|k(t)|^{\gamma-2}\exp\left(\gamma\int_t^{\tau}\alpha_1(s)ds\right)dt\right]^{\alpha-1} < \infty \\ \int^{\infty}\beta_2^{+}(t)|k(t)|^{\beta-2}dt < \infty, \\ \sum_{\tau_i<\infty}(\eta_i/\xi_i)^{+}|k_{i-1}|^{\beta-2} < \infty. \tag{5.41}$$

Then the following hold:

(a) Every weakly oscillatory proper solution $(x(t), u(t))$ of (5.1) is weakly bounded.

(b) For each weakly oscillatory proper solution $(x(t), u(t))$ of (5.1), we have

$$\lim_{t\to\infty} \frac{x(t)}{k(t)} = 0.$$

Proof. (a) Let $(x(t), u(t))$ be a weakly oscillatory proper solution of (5.1). Let $z(t) = x(t)/k(t)$. Suppose on the contrary that $z(t)$ is unbounded. Then given any positive number M_1, we can find a positive number $T = T(M_1)$ such that $|z(t)| > M_1$ for all $t > T$. Since z is also oscillatory, there exists an interval (t_1, t_2) with $t_1 \geq T$ such that $z(t_1) = z(t_2) = 0$. Choose $\tau \in (t_1, t_2)$ such that

$$M = |z(\tau)| = \max\{|z(t)| : t_1 < t < t_2\} > M_1.$$

Because of (5.41), one can choose $T \geq t_0$ large enough so that for every $t_1 \geq T$,

$$\left[\int_{t_1}^{t_2} \beta_1(t)|k(t)|^{\gamma-2} \exp\left(\gamma \int_t^\tau \alpha_1(s)ds\right) dt\right]^{\alpha-1} < M^{\frac{\alpha-\beta}{\alpha-1}} \tag{5.42}$$

and

$$\int_{t_1}^{\infty} \beta_2^+(t)|k(t)|^{\beta-2}dt + \sum_{t_1\leq\tau_i<\infty} (\eta_i/\xi_i)^+|k_{i-1}|^{\beta-2} < 1. \tag{5.43}$$

In view of (5.42) and (5.43), we see from (5.23) that

$$2 \leq M^{\frac{\beta-\alpha}{\alpha}} M^{\frac{\alpha-\beta}{\alpha}} = 1,$$

which implies a contradiction.

(b) From (a) we know that every weakly oscillatory solution is weakly bounded. Suppose on the contrary that $z(t)$ does not tend to zero as $t \to \infty$. Then

$$\limsup_{t\to\infty} |z(t)| = L > 0.$$

Since z has arbitrarily large zeros, there exists an interval (t_1, t_2) with $t_1 \geq T$, where T is sufficiently large, such that $z(t_1) = z(t_2) = 0$. Choose τ in (t_1, t_2),

$$M = |z(\tau)| = \max\{|z(t)| : t \in (t_1, t_2)\} > L/2.$$

The remainder of the proof is similar to that of part (a), hence it is omitted. $\square$

Theorem 5.2.8 *Suppose that*

$$\exp\left(\frac{\alpha}{2}\int^{\infty}|\alpha_1(t)|dt\right)\left[\int^{\infty}\beta_1(t)|k(t)|^{\gamma-2}dt\right]^{\alpha-1}<\infty,$$
$$\int^{\infty}\beta_2^{+}(t)|k(t)|^{\beta-2}dt<\infty, \tag{5.44}$$
$$\sum_{\tau_i<\infty}(\eta_i/\xi_i)^{+}|k_{i-1}|^{\beta-2}<\infty.$$

Then the following hold:

(a) Every weakly oscillatory proper solution $(x(t),u(t))$ of (5.1) is weakly bounded.

(b) For each weakly oscillatory proper solution $(x(t),u(t))$ of (5.1), we have

$$\lim_{t\to\infty}\frac{x(t)}{k(t)}=0.$$

Proof. (a) Let $(x(t),u(t))$ be a weakly oscillatory proper solution of (5.1). Let $z(t)=x(t)/k(t)$. Suppose on the contrary that $z(t)$ is unbounded. Then given any positive number M_1 we can find a positive number $T=T(M_1)$ such that $|z(t)|>M_1$ for all $t>T$. Since z is also oscillatory, there exist an interval (t_1,t_2) with $t_1\geq T$ such that $z(t_1)=z(t_2)=0$. Choose $\tau\in(t_1,t_2)$ such that

$$M=|z(\tau)|=\max\{|z(t)|:t_1<t<t_2\}>M_1.$$

Because of (5.44), one can choose $T\geq t_0$ large enough so that for every $t_1\geq T$,

$$\exp\left(\frac{\alpha}{2}\int_{t_1}^{t_2}|\alpha_1(t)|dt\right)\left[\int_{t_1}^{t_2}\beta_1(t)|k(t)|^{\gamma-2}dt\right]^{\alpha-1}<M^{\frac{\alpha-\beta}{\alpha-1}} \tag{5.45}$$

and

$$\int_{t_1}^{t_2}\beta_2^{+}(t)|k(t)|^{\beta-2}dt+\sum_{t_1\leq\tau_i<t_2}(\eta_i/\xi_i)^{+}|k_{i-1}|^{\beta-2}<1. \tag{5.46}$$

In view of (5.45) and (5.46), we see from (5.9) that

$$2\leq M^{\frac{\beta-\alpha}{\alpha}}M^{\frac{\alpha-\beta}{\alpha}}=1,$$

which implies a contradiction.

(b) From (a) we know that every weakly oscillatory solution is weakly bounded. Suppose on the contrary that $z(t)$ does not tend to zero as $t\to\infty$. Then

$$\limsup_{t\to\infty}|z(t)|=L>0.$$

Since z has arbitrarily large zeros, there exist interval (t_1, t_2) with $t_1 \geq T$, where T is sufficiently large, such that $z(t_1) = z(t_2) = 0$. Choose τ in (t_1, t_2),

$$M = |z(\tau)| = \max\{|z(t)| : t \in (t_1, t_2)\} > L/2.$$

The remainder of the proof is similar to that of part (a), hence it is omitted. $\square$

Corollary 5.2.5 *Let α be the conjugate of γ. Suppose that*

$$\int^{\infty} p^{1-\gamma}(t)|k(t)|^{\gamma-2} < \infty,$$
$$\int^{\infty} q^{+}(t)|k(t)|^{\beta-2}dt < \infty,$$
$$\sum_{\tau_i<\infty} (\eta_i/\xi_i)^{+}|k_{i-1}|^{\beta-2} < \infty.$$

Then every oscillatory solution $x(t)$ of impulsive Emden-Fowler equation (5.3) *satisfies*

$$\lim_{t\to\infty} \frac{x(t)}{k(t)} = 0.$$

Remark 5.2.1 *Corollary 5.2.5 is valid for solutions of impulsive half-linear equation* (5.5) *by taking $\alpha = \beta$.*

We conclude the paper with a theorem on boundedness of the weakly bounded solutions of (5.1).

Theorem 5.2.9 *Suppose that*

$$\int^{\infty} \alpha_1(t)dt > -\infty,$$
$$\int^{\infty} |\beta_2(t)||k(t)|^{\beta-2} \exp\left(-\int_t^{\infty} \alpha_1(s)ds\right) dt < \infty,$$
$$\sum_{\tau_i<\infty} |\eta_i/\xi_i||k_{i-1}|^{\beta-2} \exp\left(-\int_{\tau_i}^{\infty} \alpha_1(t)dt\right) < \infty.$$

Then every weakly bounded solution of (5.1) is bounded.

Proof. Given $z(t) = x(t)/k(t)$ is bounded, we only need to show that $v(t) = u(t)/k(t)$ is bounded as well. We know that

$$v' + \alpha_1(t)v = -\beta_2(t)|k(t)|^{\beta-2}|z|^{\beta-2}z, \quad t \neq \tau_i,$$
$$v(\tau_i^{+}) = v(\tau_i^{-}) - (\eta_i/\xi_i)|k_{i-1}|^{\beta-2}|z(\tau_i^{-})|^{\beta-2}z(\tau_i^{-}).$$

and hence

$$\left[v(t)\exp\left(\int_\tau^t \alpha_1(s)ds\right)\right]' = -\exp\left(\int_\tau^t \alpha_1(s)ds\right)\beta_2(t)|k(t)|^{\beta-2}|z|^{\beta-2}z, \quad t \neq \tau_i,$$
$$v(\tau_i^+) = v(\tau_i^-) - (\eta_i/\xi_i)|k_{i-1}|^{\beta-2}|z(\tau_i^-)|^{\beta-2}z(\tau_i^-).$$

Integrating from τ to $t, \tau \leq t \leq t_2$, we get

$$\begin{aligned} v(t) &= v(\tau)\exp\left(-\int_\tau^t \alpha_1(s)ds\right) \\ &\quad - \int_\tau^t \exp\left(-\int_w^t \alpha_1(s)ds\right)\beta_2(w)|k(w)|^{\beta-2}|z(w)|^{\beta-2}z(w)dw \\ &\quad - \sum_{\tau\leq\tau_i\leq t}(\eta_i/\xi_i)|k_{i-1}|^{\beta-2}|z(\tau_i^-)|^{\beta-2}z(\tau_i^-)\exp\left(\int_{\tau_i}^t -\alpha_1(s)ds\right), \end{aligned}$$

from which we easily obtain that $v(t)$ is bounded. $\square$

CHAPTER 6

LYAPUNOV TYPE INEQUALITIES AND APPLICATIONS FOR QUASILINEAR IMPULSIVE SYSTEMS

6.1 Quasilinear Impulsive Systems For (p, q)-Laplacian

In this section we obtain Lyapunov-type inequality for Dirichlet problem associated with the quasilinear impulsive system involving the (p, q)-Laplacian operator

$$\begin{aligned} -\left(h(t)\left|u'\right|^{p-2} u'\right)' &= f(t)\left|u\right|^{\alpha-2} u\left|v\right|^{\beta}, \quad t \neq \tau_i, \\ -\left(m(t)\left|v'\right|^{q-2} v'\right)' &= g(t)\left|u\right|^{\theta}\left|v\right|^{\gamma-2} v, \quad t \neq \tau_i, \\ -\Delta\left(h(t)\left|u'\right|^{p-2} u'\right) &= a_i\left|u\right|^{\alpha-2} u\left|v\right|^{\beta}, \quad t = \tau_i \\ -\Delta\left(m(t)\left|v'\right|^{q-2} v'\right) &= b_i\left|u\right|^{\theta}\left|v\right|^{\gamma-2} v, \quad t = \tau_i \end{aligned} \tag{6.1}$$

and for its more general form where the solution is not continuous, i.e, the case where $\Delta u|_{t=\tau_i} \neq 0$ and $\Delta v|_{t=\tau_i} \neq 0$ with $\alpha = \theta$ and $\beta = \gamma$,

$$\begin{aligned} & -\left(h(t)\left|u'\right|^{p-2} u'\right)' = f(t)\left|u\right|^{\alpha-2} u\left|v\right|^{\beta}, && t \neq \tau_i, \\ & -\left(m(t)\left|v'\right|^{q-2} v'\right)' = g(t)\left|u\right|^{\alpha}\left|v\right|^{\beta-2} v, && t \neq \tau_i, \\ & \Delta u = \alpha_i u, \quad \Delta v = \hat{\alpha}_i v, && t = \tau_i, \\ & \Delta\left(h(t)\left|u'\right|^{p-2} u'\right) = -\beta_i\left(h(t)\left|u'\right|^{p-2} u'\right) + \gamma_i\left|u\right|^{\alpha-2} u\left|v\right|^{\beta}, && t = \tau_i, \\ & \Delta\left(m(t)\left|v'\right|^{q-2} v'\right) = -\delta_i\left(m(t)\left|v'\right|^{q-2} v'\right) + \mu_i\left|u\right|^{\alpha}\left|v\right|^{\beta-2} v, && t = \tau_i. \end{aligned} \tag{6.2}$$

Throughout this section, we assume that

(ii) $p, q > 1$ and $\alpha, \beta, \gamma, \theta > 0$ are real numbers,

(iii) $\{\tau_i\}$ is a strictly increasing sequence of real numbers,

(i) $h, m, f, g \in PLC[t_0, \infty) = \{\omega : [t_0, \infty) \to \mathbb{R}$ is continuous on each interval (τ_i, τ_{i+1}), the limits $w(\tau_i^{\pm})$ exist and $w(\tau_i^{-}) = w(\tau_i)$ for $i \in \mathbb{N}\}$, $h, m > 0$,

(iv) $a_i, b_i, \alpha_i, \hat{\alpha}_i, \beta_i, \gamma_i, \delta_i, \mu_i$ are sequence of real numbers and $\alpha_i \neq -1$, $\hat{\alpha}_i \neq -1$ for $i \in \mathbb{N}$.

Definition 6.1.1 *By a solution $w(t) = (u(t), v(t))$ of system (6.1) on the interval $[t_0, \infty)$, we mean a nontrivial pair of continuous functions $(u(t), v(t))$ defined on $[t_0, \infty)$ such that $(h\,|u'|^{p-2}\,u'), (m\,|v'|^{q-2}\,v') \in PLC[t_0, \infty)$ satisfying (6.1) for $t \geq t_0$.*

Definition 6.1.2 *By a solution $w(t) = (u(t), v(t))$ of system (6.2) on the interval $[t_0, \infty)$, we mean a nontrivial pair of functions $(u(t), v(t))$ defined on $[t_0, \infty)$ such that u, v, $(h\,|u'|^{p-2}\,u'), (m\,|v'|^{q-2}\,v') \in PLC[t_0, \infty)$ satisfying (6.2) for $t \geq t_0$.*

For the sake of brevity, let us define

$$(\alpha_1 + 1)(\alpha_2 + 1) \ldots (\alpha_i + 1) = M_i, \qquad i \in \mathbb{N} \tag{6.3}$$

and make convention that $(\alpha_1 + 1)(\alpha_2 + 1) \ldots (\alpha_i + 1) = 1$ if $i = 0$.

For impulsive differential equations or systems, in general for piece-wise continuos functions, the concept of a zero of a function is replaced by a so-called generalized zero.

Definition 6.1.3 ([45, 42]) *A real number c is called a zero (generalized zero) of a function f if and only if $f(c^-) = 0$ or $f(c^+) = 0$. If f is continuous function at c, then c becomes a real zero.*

Definition 6.1.4 *The solution $w(t) = (u(t), v(t))$ of system (6.1) (or system (6.2)) has a zero (or generalized zero) at the point c if both components of the solution w have a zero (or generalized zero) at this point.*

We also need the following definitions.

Definition 6.1.5 *System (6.1) (or system (6.2)) is called disconjugate on an interval* $[a, b]$ *if and only if there is no real nontrivial solution* $w(t) = (u(t), v(t))$ *of system (6.1) (or system (6.2)) having two or more zeros (or generalized zeros) on* $[a, b]$.

Definition 6.1.6 *A nontrivial solution* $w(t) = (u(t), v(t))$ *of system (6.1) is bounded on* $[t_0, \infty)$ *if both components of* w *are bounded on* $[t_0, \infty)$. *If at least one component of* w *is not bounded on* $[t_0, \infty)$*, then this solution is called unbounded.*

Definition 6.1.7 *A nontrivial solution* $w(t) = (u(t), v(t))$ *of system (6.1) is said to be oscillatory if both components of* w *are oscillatory on* $[T_0, \infty)$*, i.e if for each* $T > T_0$ *there is a point* $T_1 \in (T, \infty)$ *such that* $u(T_1) = v(T_1) = 0$*. If either at least one component of* w *is not oscillatory or they are oscillatory but they become zero at different points, this solution is called nonoscillatory.*

Definition 6.1.8 *A nontrivial solution* $w(t) = (u(t), v(t))$ *of system (6.1) tends to zero as* $t \to \infty$ *if both components of* w *tend to zero as* $t \to \infty$*. If at least one component of* w *does not approach zero as* $t \to \infty$*, then this solution does not approach zero as* $t \to \infty$*.*

Definition 6.1.9 *A nontrivial solution* $w(t) = (u(t), v(t))$ *of system (6.2) is said to be bounded on* $[t_0, \infty)$ *if both components of* $\hat{w}(t) = \left(\dfrac{u(t)}{M_i}, \dfrac{v(t)}{|M_i|^{p/q}}\right)$ *are bounded on* $[t_0, \infty)$*, where* M_i *is defined as in (6.3). If at least one component of* $\hat{w}$ *is not bounded on* $[t_0, \infty)$*, then* w *is called unbounded.*

Definition 6.1.10 *A nontrivial solution* $w(t) = (u(t), v(t))$ *of system (6.2) tends to zero as* $t \to \infty$ *if both components of* $\hat{w}(t) = \left(\dfrac{u(t)}{M_i}, \dfrac{v(t)}{|M_i|^{p/q}}\right)$ *tend to zero as* $t \to \infty$*, where* M_i *is defined as in (6.3). If at least one component of* $\hat{w}$ *does not approach zero as* $t \to \infty$*, then* w *does not approach zero as* $t \to \infty$*.*

Since our main interest is Lyapunov type inequalities for system (6.1) and system (6.2), we assume the existence of nontrivial solution of these systems. Our main purpose is to establish Lyapunov type inequalities for the impulsive system of differential equations (6.1) satisfying Dirichlet boundary conditions. We will also consider a related problem (6.2) where the solutions are discontinuous. Although our motivation comes from the papers of [78, 115], our results not only extend the results of such papers and that of [19, 98, 7] but also generalize them to the impulsive case.

6.1.1 Lyapunov Type Inequality

Recall that the numbers $p_1, p_2 > 1$ are said to be conjugate if $\dfrac{1}{p_1} + \dfrac{1}{p_2} = 1$.

In the sequel we denote $m^+(t) = \max\{m(t), 0\}$ and $m_i^+ = \max\{m_i, 0\}$.

The main result of this section is the following theorem.

Theorem 6.1.1 *Let p' and q' be conjugate numbers for p and q, respectively and (e_1, e_2) be a nontrivial solution of the homogenous system*

$$\begin{aligned} e_1(\alpha - p) + e_2\theta &= 0 \\ e_1\beta + e_2(\gamma - q) &= 0 \end{aligned} \tag{6.4}$$

where $e_k \geq 0$ for $k = 1, 2$ and $e_1^2 + e_2^2 > 0$. If the system (6.1) has a real nontrivial solution $(u(t), v(t))$ such that $u(a) = u(b) = v(a) = v(b) = 0$, $a, b \in \mathbb{R}$ with $a < b$ are consecutive zeros, and u, v are not identically zero on $[a, b]$, then we have the following Lyapunov type inequality

$$\begin{aligned} 2^{e_1 p + e_2 q} \leq & \left(\int_a^b h^{-p'/p}(t) dt \right)^{\frac{e_1 p}{p'}} \left(\int_a^b m^{-q'/q}(t) dt \right)^{\frac{e_2 q}{q'}} \left(\int_a^b f^+(t) dt + \sum_{a \leq \tau_i < b} a_i^+ \right)^{e_1} \\ & \times \left(\int_a^b g^+(t) dt + \sum_{a \leq \tau_i < b} b_i^+ \right)^{e_2}. \end{aligned} \tag{6.5}$$

Proof. Multiplying the first equation of system (6.1) by u and integrating from a to b, we have

$$\begin{aligned} \int_a^b - \left(h(t) |u'(t)|^{p-2} u'(t) \right)' u(t) dt &= \int_a^b f(t) |u(t)|^\alpha |v(t)|^\beta dt \\ &= -u(t) h(t) |u'(t)|^{p-2} u'(t) \mid_a^b + \int_a^b h(t) |u'(t)|^p dt \\ &+ \sum_{a \leq \tau_i < b} u(\tau_i) \Delta \left(h(\tau_i) |u'(\tau_i)|^{p-2} u'(\tau_i) \right). \end{aligned}$$

Clearly, using $f^+(t) = \max\{f(t), 0\}$ and $a_i^+ = \max\{a_i, 0\}$ yield

$$\begin{aligned}\int_a^b h(t)|u'(t)|^p dt &= \int_a^b f(t)|u(t)|^\alpha|v(t)|^\beta dt + \sum_{a\le\tau_i<b} a_i|u(\tau_i)|^\alpha|v(\tau_i)|^\beta \\ &\le \int_a^b f^+(t)|u(t)|^\alpha|v(t)|^\beta dt + \sum_{a\le\tau_i<b} a_i^+|u(\tau_i)|^\alpha|v(\tau_i)|^\beta.\end{aligned} \tag{6.6}$$

Similarly from the second equation of system (6.1), we get

$$\int_a^b m(t)|v'(t)|^q dt \le \int_a^b g^+(t)|u(t)|^\theta|v(t)|^\gamma dt + \sum_{a\le\tau_i<b} b_i^+|u(\tau_i)|^\theta|v(\tau_i)|^\gamma. \tag{6.7}$$

On the other hand by employing Hölder inequality with indices p' and p, one can obtain

$$\begin{aligned}2\,|u(c)| &= \left|\int_a^c u'(t)dt\right| + \left|\int_c^b u'(t)dx\right| \le \int_a^b |u'(t)|\,dt = \int_a^b h^{\frac{-1}{p}}(t)h^{\frac{1}{p}}(t)\,|u'(t)|\,dt \\ &\le \left(\int_a^b h^{\frac{-p'}{p}}(t)dt\right)^{\frac{1}{p'}} \left(\int_a^b h(t)\,|u'(t)|^p\,dt\right)^{\frac{1}{p}}\end{aligned}$$

and so, combining with (6.6) implies,

$$\begin{aligned}2\,|u(c)| \le& \left(\int_a^b h^{\frac{-p'}{p}}(t)dt\right)^{\frac{1}{p'}} \\ &\times\left[\int_a^b f^+(t)\,|u(t)|^\alpha\,|v(t)|^\beta\,dt + \sum_{a\le\tau_i<b} a_i^+\,|u(\tau_i)|^\alpha\,|v(\tau_i)|^\beta\right]^{\frac{1}{p}}.\end{aligned} \tag{6.8}$$

Let $|u(c)| = \max_{a\le t\le b} u(t)$ and $|v(d)| = \max_{a\le t\le b} v(t)$, then from inequality (6.8) we have

$$2^p \le \left(\int_a^b h^{\frac{-p'}{p}}(t)dt\right)^{\frac{p}{p'}} |u(c)|^{\alpha-p}\,|v(d)|^\beta \left[\int_a^b f^+(t)dt + \sum_{a\le\tau_i<b} a_i^+\right]. \tag{6.9}$$

In view of (6.7) repeating the above procedure with

$$2\,|v(d)| = \left|\int_a^d v'(t)dt\right| + \left|\int_d^b v'(t)dt\right| \le \int_a^b |v'(t)|\,dt = \int_a^b m^{\frac{-1}{q}}(t)m^{\frac{1}{q}}(t)\,|v'(t)|\,dt$$

one can obtain the following inequality

$$2^q \le \left(\int_a^b m^{\frac{-q'}{q}}(t)dt\right)^{\frac{q}{q'}} |u(c)|^\theta\,|v(d)|^{\gamma-q} \left[\int_a^b g^+(t)dt + \sum_{a\le\tau_i<b} b_i^+\right]. \tag{6.10}$$

Raising inequalities (6.9) and (6.10) by e_1 and e_2, respectively, then multiplying the resulting inequalities yield

$$\begin{aligned}2^{pe_1+qe_2} \le& |u(c)|^{(\alpha-p)e_1+\theta e_2}\,|v(d)|^{\beta e_1+(\gamma-q)e_2} \left(\int_a^b h^{\frac{-p'}{p}}(t)dt\right)^{\frac{e_1 p}{p'}} \left(\int_a^b m^{\frac{-q'}{q}}(t)dt\right)^{\frac{qe_2}{q'}} \\ &\times\left[\int_a^b f^+(t)dt + \sum_{a\le\tau_i<b} a_i^+\right]^{e_1} \left[\int_a^b g^+(t)dt + \sum_{a\le\tau_i<b} b_i^+\right]^{e_2}.\end{aligned}$$

In view of the homogenous system (6.4) we finally arrive at (6.5). ☐

Remark 6.1.1 *Theorem 6.1.1 is an impulsive generalization of [115, Theorem 1] in the case $n = 2$. Since system (6.1) is more general than system (20) of [19] and system (1.16) of [98], Theorem 6.1.1 extends [19, Corollary 2] and [98, Corollary 2.6]. Moreover since no sign condition is assumed for $h(x)$ and $k(x)$, Theorem 6.1.1 improves and generalizes [78, Theorem 1.5].*

The following corollaries provide new Lyapunov type inequalities for the particular cases of system (6.1). Assuming different conditions on the relations between $\alpha, \beta, \theta, \gamma, p$ and q yields more inequalities than we will show.

Corollary 6.1.1 *Let p' and q' be conjugate numbers for p and q, respectively. Assume that*

$$\begin{aligned} \alpha + \gamma &= p \\ \beta + \theta &= q \end{aligned} \tag{6.11}$$

or

$$\begin{aligned} \alpha + \theta &= p \\ \beta + \gamma &= q \end{aligned} \tag{6.12}$$

If system (6.1) has a real solution $(u(t), v(t))$ such that $u(a) = u(b) = v(a) = v(b) = 0$, $a, b \in \mathbb{R}$ with $a < b$ are consecutive zeros and u, v are not identically zero on $[a, b]$, then we have the following Lyapunov type inequality

$$\begin{aligned} 2^{p+q} \leq &\left(\int_a^b h^{-p'/p}(t)dt \right)^{\frac{p}{p'}} \left(\int_a^b m^{-q'/q}(t)dt \right)^{\frac{q}{q'}} \left(\int_a^b f^+(t)dt + \sum_{a \leq \tau_i < b} a_i^+ \right) \\ &\times \left(\int_a^b g^+(t)dt + \sum_{a \leq \tau_i < b} b_i^+ \right). \end{aligned}$$

Proof. From the proof of Theorem 6.1.1, we see that condition (6.11) or (6.12) implies that $e_1 = e_2 = 1$ is a nonzero solution of (6.4). Now, Corollary 6.1.1 is a direct consequence of Theorem 6.1.1. ☐

Corollary 6.1.2 *Let p' and q' be conjugate numbers for p and q, respectively. Assume that*

$$\frac{\alpha}{p} + \frac{\beta}{q} = 1, \quad \frac{\theta}{p} + \frac{\gamma}{q} = 1 \tag{6.13}$$

If system (6.1) has a real solution $(u(t), v(t))$ such that $u(a) = u(b) = v(a) = v(b) = 0$, $a, b \in \mathbb{R}$ with $a < b$ are consecutive zeros and u, v are not identically zero on $[a, b]$, then we have the following Lyapunov type inequality

$$2^{\theta+\beta} \leq \left(\int_a^b h^{-p'/p}(t)dt\right)^{\frac{\theta}{p'}} \left(\int_a^b m^{-q'/q}(t)dt\right)^{\frac{\beta}{q'}} \left(\int_a^b f^+(t)dt + \sum_{a\leq\tau_i<b} a_i^+\right)^{\theta/p} \times \left(\int_a^b g^+(t)dt + \sum_{a\leq\tau_i<b} b_i^+\right)^{\beta/q}.$$

Proof. From the proof of Theorem 6.1.1, we see that condition (6.13) implies that $e_1 = \dfrac{\theta}{p}$ and $e_2 = \dfrac{\beta}{q}$ is a nonzero solution of (6.4). Now, Corollary 6.1.2 is a direct consequence of Theorem 6.1.1. □

Remark 6.1.2 *In the absence of impulse effect, corollary 6.1.2 gives the same result in [19, Corollary 2] but it still improves and generalizes [78, Theorem 1.5].*

Corollary 6.1.3 *Let p' and q' be conjugate numbers for p and q, respectively and $\alpha = \theta$ and $\beta = \gamma$. Assume that*

$$\frac{\alpha}{p} + \frac{\beta}{q} = 1 \tag{6.14}$$

If system (6.1) has a real solution $(u(t), v(t))$ such that $u(a) = u(b) = v(a) = v(b) = 0$, $a, b \in \mathbb{R}$ with $a < b$ are consecutive zeros and u, v are not identically zero on $[a, b]$, then we have the following Lyapunov type inequality

$$2^{\alpha+\beta} \leq \left(\int_a^b h^{-p'/p}(t)dt\right)^{\frac{\alpha}{p'}} \left(\int_a^b m^{-q'/q}(t)dt\right)^{\frac{\beta}{q'}} \left(\int_a^b f^+(t)dt + \sum_{a\leq\tau_i<b} a_i^+\right)^{\alpha/p} \times \left(\int_a^b g^+(t)dt + \sum_{a\leq\tau_i<b} b_i^+\right)^{\beta/q}. \tag{6.15}$$

Proof. From the proof of Theorem 6.1.1, we see that condition (6.14) implies that $e_1 = \dfrac{\alpha}{p}$ and $e_2 = \dfrac{\beta}{q}$ is a nonzero solution of (6.4). Now, Corollary 6.1.3 is a direct consequence of Theorem 6.1.1. □

Remark 6.1.3 *Corollary 6.1.3 recovers [78, Theorem 1.5].*

We next consider system (6.2) which has discontinuos solutions.

Theorem 6.1.2 *Let p' and q' be conjugate numbers for p and q, respectively, and* (6.14) *hold. Suppose that*

$$\hat{\alpha}_i = |\alpha_i+1|^{p/q}-1, \quad \beta_i = |\alpha_i+1|^{p-2}(\alpha_i+1)-1, \quad \delta_i = |\alpha_i+1|^{p/q'}-1, \quad i \in \mathbb{N}.$$

If system (6.2) has a real nontrivial solution $(u(t), v(t))$ such that $u(a^+) = u(b^-) = v(a^+) = v(b^-) = 0$, $a, b \in \mathbb{R}$ with $a < b$ are consecutive zeros and u, v are not identically zero on $[a, b]$, then we have the Lyapunov type inequality

$$2^{\alpha+\beta} \leq \left(\int_a^b h^{-p'/p}(t)dt\right)^{\frac{\alpha}{p'}} \left[\int_a^b f^+(t)dt + \sum_{a\leq\tau_i<b} |\alpha_i+1|^{2-p}\left(\frac{\gamma_i}{\alpha_i+1}\right)^+\right]^{\frac{\alpha}{p}}$$
$$\times \left(\int_a^b m^{-q'/q}(t)dt\right)^{\frac{\beta}{q'}} \left(\int_a^b g^+(t)dt + \sum_{a\leq\tau_i<b} |\alpha_i+1|^{-p/q'}\mu_i^+\right)^{\frac{\beta}{q}}.$$

Proof. Let a, b be generalized zeros of u, v, in other words $u(a^+) = u(b^-) = 0$ and $v(a^+) = v^+(b^-) = 0$ where $a = \tau_0 < \tau_1 < \ldots < \tau_m < b$ and M_i be given as in equation (6.3). Define

$$y(t) = \frac{1}{M_i}u(t) \quad z(t) = \frac{1}{|M_i|^{p/q}}v(t), \quad t \in (\tau_i, \tau_{i+1}), \quad i = 0, 1, \ldots, m.$$

where we put $a = \tau_0$ and $b = \tau_{m+1}$. It is easy to see that with the above transformation system (6.2) becomes the following system

$$\begin{aligned}
&-\left(h(t)\,|y'|^{p-2}\,y'\right)' = f(t)\,|y|^{\alpha-2}\,y\,|z|^{\beta}, && t \neq \tau_i\\
&-\left(m(t)\,|z'|^{q-2}\,z'\right)' = g(t)\,|y|^{\alpha}\,|z|^{\beta-2}\,z, && t \neq \tau_i\\
&\Delta y = 0, \quad \Delta z = 0, && t = \tau_i\\
&-\Delta\left(h(t)\,|y'|^{p-2}\,y'\right) = \frac{\gamma_i}{|\alpha_i+1|^{p-2}(\alpha_i+1)}\,|y|^{\alpha-2}\,y\,|z|^{\beta}, && t = \tau_i\\
&-\Delta\left(m(t)\,|z'|^{q-2}\,y'\right) = \frac{\mu_i}{|\alpha_i+1|^{p/q'}}\,|y|^{\alpha}\,|z|^{\beta-2}\,z, && t = \tau_i.
\end{aligned} \tag{6.16}$$

Applying Corollary 6.1.3 to system (6.16) with

$$a_i = \frac{\gamma_i}{|\alpha_i+1|^{p-2}(\alpha_i+1)}, \quad b_i = \frac{\mu_i}{|\alpha_i+1|^{p/q'}}$$

we easily obtain the desired result. □

Remark 6.1.4 *Since system (6.1), with $\theta = \alpha$ and $\gamma = \beta$, is obtained from system (6.2) by choosing $\alpha_i = 0$ for $i \in \mathbb{N}$, Corollary 6.1.3 is generalized by Theorem 6.1.2.*

6.2 Quasilinear Impulsive Systems For $(p_1, p_2, \ldots, p_n)$-Laplacian

Here we consider n-dimensional quasilinear impulsive systems

$$
\begin{aligned}
&-\left(h_k(t)\,|u_k'|^{p_k-2}\,u_k'\right)' = f_k(t)\,|u_k|^{q_{kk}-2}\,u_k \prod_{j=1(j\neq k)}^{n} |u_j|^{q_{kj}}\,, \quad t \neq \tau_i \\
&-\Delta\left(h_k(t)\,|u_k'|^{p_k-2}\,u_k'\right) = a_{ik}\,|u_k|^{q_{kk}-2}\,u_k \prod_{j=1(j\neq k)}^{n} |u_j|^{q_{kj}}\,, \quad t = \tau_i \\
&\quad (k = 1, 2, \ldots, n, \quad i \in \mathbb{N})
\end{aligned} \tag{6.17}
$$

and its general form in which solutions are not contiuous,

$$
\begin{aligned}
-\left(h_k(t)\,|u_k'|^{p_k-2}\,u_k'\right)' &= f_k(t)\,|u_k|^{q_k-2}\,u_k \prod_{j=1(j\neq k)}^{n} |u_j|^{q_j}, \quad t \neq \tau_i \\
\Delta u_k &= \alpha_{ik} u_k, \qquad\qquad\qquad\qquad t = \tau_i \\
-\Delta\left(h_k(t)\,|u_k'|^{p_k-2}\,u_k'\right) &= -\beta_{ik}\left(h_k(t)\,|u_k'|^{p_k-2}\,u_k'\right) \\
&\quad + \gamma_{ik}\,|u_k|^{q_k-2}\,u_k \prod_{j=1(j\neq k)}^{n} |u_j|^{q_j}, \quad t = \tau_i \\
&(k = 1, 2, \ldots, n, \quad i \in \mathbb{N}).
\end{aligned} \tag{6.18}
$$

Throughout this section, we assume that

(I) $f_k, h_k \in PLC[t_0, \infty) = \{\omega : [t_0, \infty) \to \mathbb{R}$ is continuous on each interval (τ_i, τ_{i+1}), the limits $w(\tau_i^{\pm})$ exist and $w(\tau_i^{-}) = w(\tau_i)$ for $i \in \mathbb{N}\}$, $h_k > 0$ for $k = 1, 2, \ldots, n$,

(II) $p_k > 1$ and $q_{kj}, q_k > 0$ are real numbers for $k, j = 1, 2, \ldots, n$,

(III) $\{\tau_i\}$ is a strictly increasing sequence of real numbers,

(IV) $a_{ik}, \alpha_{ik}, \beta_{ik}, \gamma_{ik}$ are sequence of real numbers and $\alpha_{ik} \neq -1$ for $k = 1, 2, \ldots, n$, $i \in \mathbb{N}$.

Definition 6.2.1 *By a solution $(u_1(t), u_2(t), \ldots, u_n(t))$ of system (6.17) on $[t_0, \infty)$, we mean an n-tuple of continuous functions $(u_1(t), u_2(t), \ldots, u_n(t))$ defined on $[t_0, \infty)$ such that $(h_k\,|u_k'|^{p_k-2}\,u_k') \in PLC[t_0, \infty)$ for $k = 1, 2, \ldots, n$ satisfying (6.17) for $t \geq t_0$.*

Definition 6.2.2 *By a solution* $(u_1(t), u_2(t), \ldots, u_n(t))$ *of system (6.18) on an interval* $[t_0, \infty)$*, we mean an* n*-tuple of functions* $(u_1(t), u_2(t), \ldots, u_n(t))$ *defined on* $[t_0, \infty)$ *such that* $u_k, (h_k |u'_k|^{p_k-2} u'_k) \in PLC[t_0, \infty)$ *for* $k = 1, 2, \ldots, n$ *satisfying (6.18) for* $t \geq t_0$*.*

Since our main interest is Lyapunov type inequalities for system (6.17) and system (6.18), we assume the existence of nontrivial solution of these systems. Our main purpose is to establish Lyapunov type inequalities for the impulsive system of differential equations (6.17) satisfying Dirichlet boundary conditions. We will also consider a related problem (6.18) where the solutions are discontinuous. Although our motivation comes from the papers of [18, 115], our results not only extend the results of such papers and that of [98, 7, 6] but also generalize them to the impulsive case.

6.2.1 Lyapunov Type Inequality

The main result of this section is the following theorem which is a generalization of Theorem 6.1.1 to the systems with n equations.

Theorem 6.2.1 *Let* p'_j *be the conjugate number for* p_j *for* $j = 1, 2, \ldots, n$*. If system (6.17) has a real solution* $(u_1(t), u_2(t), \ldots, u_n(t))$ *such that* $u_k(a) = u_k(b) = 0$ *for* $k = 1, 2, \ldots, n$ *and* $a, b \in \mathbb{R}$ *with* $a < b$ *are consecutive zeros and* $u_1, u_2, \ldots, u_n$ *are not identically zero on* $[a, b]$*, then the following Lyapunov type inequality*

$$2^{\sum_{j=1}^{n} p_j e_j} \leq \prod_{k=1}^{n} \left(\int_a^b h_k^{\frac{-p'_k}{p_k}}(t)dt \right)^{\frac{e_k p_k}{p'_k}} \left(\int_a^b f_k^+(t)dt + \sum_{a \leq \tau_i < b} a_{ik}^+ \right)^{e_k} \tag{6.19}$$

holds for $k = 1, 2, \ldots, n$ *where* $(e_1, e_2, \ldots, e_n)$ *is nontrivial solution of the homogenous system*

$$\begin{aligned} &e_1(p_1 - q_{11}) - e_2 q_{21} - e_3 q_{31} - \ldots - e_n q_{n1} = 0 \\ &- e_1 q_{12} + e_2(p_2 - q_{22}) - e_3 q_{32} - \ldots - e_n q_{n2} = 0 \\ &\qquad\vdots \\ &- e_1 q_{1n} - e_2 q_{2n} - e_3 q_{3n} - \ldots - e_n(p_n - q_{nn}) = 0 \end{aligned} \tag{6.20}$$

where $e_k \geq 0$ *for* $k = 1, 2, \ldots, n$ *and* $\displaystyle\sum_{j=1}^{n} e_j^2 > 0.$

Proof. Consider the k-th equation of system (6.17). Since for all k, u_k is continous on the interval $[a, b]$, there exist $c_k \in (a, b)$ such that $|u_k(c_k)| = \max_{a \leq t \leq b} |u_k(t)|$ for $k = 1, 2, \ldots, n$. By multiplying the k-th equation of system (6.17) by u_k and integrating from a to b, we get

$$\begin{aligned}
\int_a^b \left(h_k(t) \left|u_k'\right|^{p_k-2} u_k'\right)' u_k(t)dt &= \int_a^b f_k(t) \prod_{j=1}^{n} |u_j(t)|^{q_{kj}} dt \\
&= \int_a^b h_k(t)|u_k'(t)|^p dt - \sum_{a \leq \tau_i < b} a_{ik} \prod_{j=1}^{n} |u_j(\tau_i)|^{q_{kj}}
\end{aligned}$$

which implies

$$\begin{aligned}
\int_a^b h_k(t)|u_k'(t)|^p dt &= \int_a^b f_k(t) \prod_{j=1}^{n} |u_j(t)|^{q_{kj}} dt + \sum_{a \leq \tau_i < b} a_{ik} \prod_{j=1}^{n} |u_j(\tau_i)|^{q_{kj}} \\
&\leq \int_a^b f_k^+(t) \prod_{j=1}^{n} |u_j(t)|^{q_{kj}} dt + \sum_{a \leq \tau_i < b} a_{ik}^+ \prod_{j=1}^{n} |u_j(\tau_i)|^{q_{kj}}.
\end{aligned}$$

Now, it is easy to see that

$$\begin{aligned}
2\left|u_k(c_k)\right| &= \left|\int_a^{c_k} u_k'(t)dt\right| + \left|\int_{c_k}^b u_k'(t)dt\right| \leq \int_a^b \left|u_k'(t)\right| dt \\
&= \int_a^b h_k^{\frac{-1}{p_k}}(t) h_k^{\frac{1}{p_k}}(t) \left|u_k'(t)\right| dt \\
&\leq \left(\int_a^b h_k^{\frac{-p_k'}{p_k}}(t)dt\right)^{\frac{1}{p_k'}} \left(\int_a^b h_k(t) \left|u_k'(t)\right|^{p_k}\right)^{\frac{1}{p_k}} \\
&\leq \left(\int_a^b h_k^{\frac{-p_k'}{p_k}}(t)dt\right)^{\frac{1}{p_k'}} \\
&\quad \times \left[\int_a^b f_k^+(t) \prod_{j=1}^{n} |u_j(t)|^{q_{kj}} dt + \sum_{a \leq \tau_i < b} a_{ik}^+ \prod_{j=1}^{n} |u_j(\tau_i)|^{q_{kj}}\right]^{\frac{1}{p_k}}.
\end{aligned}$$

By taking $p_k - th$ power of the both sides of the above inequality and by using $|u_k(c_k)| = \max_{a \leq t \leq b} u_k(t)$, we have

$$2^{p_k} \left|u_k(c_k)\right|^{p_k} \leq \left(\int_a^b h_k^{\frac{-p_k'}{p_k}}(t)dt\right)^{\frac{p_k}{p_k'}} \prod_{j=1}^{n} |u_j(c_j)|^{q_{kj}} \left[\int_a^b f_k^+(t)dt + \sum_{a \leq \tau_i < b} a_{ik}^+\right]$$

and

$$2^{p_k}\,|u_k(c_k)|^{p_k-q_{kk}}\prod_{j=1(j\neq k)}^{n}|u_j(c_j)|^{-q_{kj}}\leq\left(\int_a^b h_k^{\frac{-p_k'}{p_k}}(t)dt\right)^{\frac{p_k}{p_k'}}$$
$$\times\left[\int_a^b f_k^{+}(t)dt+\sum_{a\leq\tau_i<b}a_{ik}^{+}\right].$$

Raising the both sides of the above inequality to the power e_k for each $k=1,2,\ldots,n$, respectively, and multiplying the resulting inequalities side by side, we obtain

$$2^{\sum_{k=1}^{n}p_k e_k}\prod_{k=1}^{n}|u_k(c_k)|^{\theta_k}\leq\prod_{k=1}^{n}\left(\int_a^b h_k^{\frac{-p_k'}{p_k}}(t)dt\right)^{\frac{e_k p_k}{p_k'}}\left[\int_a^b f_k^{+}(t)dt+\sum_{a\leq\tau_i<b}a_{ik}^{+}\right]^{e_k}$$

where $\theta_k=(p_k-q_{kk})e_k-\sum_{j=1(k\neq j)}^{n}q_{kj}e_j$ for $k=1,2,\ldots,n$. By assumption, equation (6.20) has nonzero solutions $(e_1,e_2,\ldots,e_n)$ such that $\theta_k=0$ for $k=1,2,\ldots,n$ where $e_k\geq 0$ for $k=1,2,\ldots,n$ and at least one $e_j>0$ for $j=1,2,\ldots,n$. Choosing one of the solutions $(e_1,e_2,\ldots,e_n)$, we obtain the desired inequality. This completes the proof of Theorem 6.2.1. □

Remark 6.2.1 *Theorem 6.2.1 is the generalization of [115, Theorem 1] to the impulsive case. Morever it is extension and improvement of [7, 6] due to the assuming weaker conditions on the exponents q_{kj}. Let there be no impulse effect, i.e $a_{ik}=0$ for $k=1,2,\ldots,n$ and $i\in\mathbb{N}$. Since system (6.17) is more general than system (7) of Çakmak and Tiryaki [18] and system (1.21) of Tang and He [98] in the sense that $q_{kj}\neq q_{pj}$ for $k\neq p$ and $k,j,p=1,2,\ldots,n$, inequality (6.19) extends inequality (27) of [18, Corollary 3] and inequality (3.22) of [98, Corollary 3.3].*

Remark 6.2.2 *Let $n=2$. Then inequality (6.19) is reduced to inequality (6.15) which is the generalization of inequality (42) of Çakmak and Tiryaki [19, Corollary 2] and inequality (2.32) of Tang and He [98, Corollary 2.6] to the impulsive case. If $\alpha_{kj}=\alpha_{jj}$ for $k,j=1,2,\ldots,n$, then Theorem 6.2.1 improves [78, Theorem 1.5].*

Remark 6.2.3 *Let $n=1$ and $a_{i1}=0$ for $i\in\mathbb{N}$. In this case, Theorem 6.2.1 reduces to [33, Theorem 5.1.1] and gives the result for half-linear differential equation. Moreover, since the restricted condition, i.e. a bounded positive function, on the function* *r (in our case it is $f_1(x)$) in [91, Theorem 2.3] is dropped, Theorem 6.2.1 improves [91, Theorem 2.3]. When $h_1(x) = 1$ and $p_1 = 2$, the same result of Krein [57] is obtained.*

Remark 6.2.4 *Let $n = 1$ and $p_1 = 2$. Then Theorem 6.2.1 coincides with [43, Theorem 4.5] which is obtained for the case of the second order impulsive differential equations.*

Corollary 6.2.1 *Let p'_j be the conjugate number for p_j for $j = 1, 2, \ldots, n$. Assume*

$$\sum_{j=1}^{n} q_{jk} = p_k, \; k = 1, 2, \ldots, n. \tag{6.21}$$

If system (6.17) has a real solution $(u_1(t), u_2(t), \ldots, u_n(t))$ such that $u_k(a) = u_k(b) = 0$ for $k = 1, 2, \ldots, n$ and $a, b \in \mathbb{R}$ with $a < b$ are consecutive zeros and $u_1, u_2, \ldots, u_n$ are not identically zero on $[a, b]$, then the following Lyapunov inequality

$$2^{\sum_{j=1}^{n} p_j} \leq \prod_{k=1}^{n} \left(\int_a^b h_k^{\frac{-p'_k}{p_k}}(t)dt \right)^{\frac{p_k}{p'_k}} \left(\int_a^b f_k^+(t)dt + \sum_{a \leq \tau_i < b} a_{ik}^+ \right)$$

holds for $k = 1, 2, \ldots, n$.

Proof. From the proof of Theorem 6.2.1, we see that condition (6.21) implies that $e_k = 1$ for $k = 1, 2, \ldots, n$ is a nonzero solution of (6.20). Now, Corollary 6.2.1 is a direct consequence of Theorem 6.2.1. □

Remark 6.2.5 *Corollary 6.2.1 shows that by choosing different conditions on q_{jk} and p_k for $k, j = 1, 2, \ldots, n$, Theorem 6.2.1 yields several new inequalities which were obtained in [115] for the nonimpulsive case. Therefore existence of impulse effect leads to more general result than the result of [115].*

Corollary 6.2.2 *Let p'_j be the conjugate number for p_j for $j = 1, 2, \ldots, n$. Assume $q_{kj} = q_{kk} = q_k$ for $k, j = 1, 2, \ldots, n$ in system (6.17) and*

$$\frac{q_1}{p_1} + \frac{q_2}{p_2} + \cdots + \frac{q_n}{p_n} = 1 \tag{6.22}$$

If system (6.17) has a real solution $(u_1(t), u_2(t), \ldots, u_n(t))$ such that $u_k(a) = u_k(b) = 0$ for $k = 1, 2, \ldots, n$ and $a, b \in \mathbb{R}$ with $a < b$ are consecutive zeros and $u_1, u_2, \ldots, u_n$ are not identically zero on $[a, b]$, then the following Lyapunov inequality

$$2^{\sum_{j=1}^{n} q_j} \leq \prod_{k=1}^{n} \left(\int_a^b h_k^{\frac{-p'_k}{p_k}}(t)dt \right)^{\frac{q_k}{p'_k}} \prod_{k=1}^{n} \left(\int_a^b f_k^+(t)dt + \sum_{a \leq \tau_i < b} a_{ik}^+ \right)^{\frac{q_k}{p_k}}$$

holds for $k = 1, 2, \ldots, n$.

Proof. From the proof of Theorem 6.2.1, we see that condition (6.22) implies that $e_k = \frac{q_k}{p_k}$ for $k = 1, 2, \ldots, n$ is a nonzero solution of (6.20). Now, Corollary 6.2.2 is a direct consequence of Theorem 6.2.1. □

Remark 6.2.6 *In the absence of impulse effect corollary 6.2.2 gives the same result of Çakmak and Tiryaki [18, Corollary 3].*

We now consider system (6.18), which has discontinuos solutions, similar to (p, q)-Laplacian case (6.2).

Theorem 6.2.2 *Let p'_k be conjugate numbers for p_k, $k = 1, 2, \ldots, n$ and* (6.22) *hold. Suppose that*

$$\alpha_{ik} = |\alpha_{i1}+1|^{p_1/p_k}-1, \ \beta_{i1} = |\alpha_{i1}+1|^{p_1-2}(\alpha_{i1}+1)-1, \ \beta_{ik} = |\alpha_{i1}+1|^{\frac{p_1}{p'_k}}-1, \ i \in \mathbb{N}$$

and $k = 2, 3, \ldots, n$. If system (6.18) has a real solution $(u_1(t), u_2(t), \ldots, u_n(t))$ such that $u_k(a^-) = u_k(b^+) = 0$ for $k = 1, 2, \ldots, n$ and $a, b \in \mathbb{R}$ with $a < b$ are consecutive zeros and $u_1, u_2, \ldots, u_n$ are not identically zero on $[a, b]$, then we have the following Lyapunov type inequality

$$2^{\sum_{j=1}^{n} q_j} \leq \left(\int_a^b h_1^{\frac{-p'_1}{p_1}}(t)dt \right)^{\frac{q_1}{p'_1}} \left(\int_a^b f_1^+(t)dt + \sum_{a \leq \tau_i < b} \left(\frac{\gamma_{i1}}{\alpha_{i1}+1} \right)^+ |\alpha_{i1}+1|^{2-p_1} \right)^{\frac{q_1}{p_1}}$$
$$\times \prod_{k=2}^{n} \left(\int_a^b h_k^{\frac{-p'_k}{p_k}}(t)dt \right)^{\frac{q_k}{p'_k}} \left(\int_a^b f_k^+(t)dt + \sum_{a \leq \tau_i < b} \gamma_{ik}^+ |\alpha_{i1}+1|^{-p_1/p'_k} \right)^{\frac{q_k}{p_k}}.$$

Proof. Let $u_k^+(a) = u_k(b^-) = 0$ and $a = \tau_0 < \tau_1 < \ldots < \tau_m < b$. Similar to the proof of Theorem 6.1.2, let us define $(\alpha_{11}+1)(\alpha_{21}+1)\ldots(\alpha_{i1}+1) = N_i$ and

$$y_1(t) = \frac{u_1(t)}{N_i} \quad y_k(t) = \frac{u_k(t)}{|N_i|^{p_1/p_k}}, \quad t \in (\tau_i, \tau_{i+1}), \quad i = 0, 1, \ldots, m, \ k = 2, \ldots, n.$$

where we put $a = \tau_0$ and $b = \tau_{m+1}$ and make convention that $(\alpha_{11}+1)(\alpha_{21}+1)\ldots(\alpha_{i1}+1) = 1$ if $i = 0$. It is easy to see that with the above transformation, system (6.18) turns into the following system

$$\begin{aligned}
&-\left(h_k(t)\,|y_k'|^{p_k-2}\,y_k'\right)' = f_k(t)\,|y_k|^{q_k-2}\,y_k \prod_{j=1(j\neq k)}^{n} |y_j|^{q_j}, && t \neq \tau_i \\
&\Delta y_k = 0, \ k = 1, 2, \ldots, n, && t = \tau_i \\
&-\Delta\left(h_1(t)\,|y_1'|^{p_1-2}\,y_1'\right) = \frac{\gamma_{i1}}{|\alpha_{i1}+1|^{p_1-2}\,(\alpha_{i1}+1)}\,|y_1|^{q_1-2}\,y_1 \prod_{j=2}^{n} |y_j|^{q_j}, && t = \tau_i \\
&-\Delta\left(h_k(t)\,|y_k'|^{p_k-2}\,y_k'\right) = \frac{\gamma_{ik}}{|\alpha_{i1}+1|^{p_1/p_k'}}\,|y_k|^{q_k-2}\,y_k \prod_{j=1(j\neq k)}^{n} |y_j|^{q_j}, && t = \tau_i.
\end{aligned} \tag{6.23}$$

Applying Corollary 6.2.2 to system (6.23) with

$$a_{i1} = \frac{\gamma_{i1}}{|\alpha_{i1}+1|^{p_1-2}\,(\alpha_{i1}+1)}, \quad a_{ik} = \frac{\gamma_{ik}}{|\alpha_{i1}+1|^{p_1/p_k'}}, \ k = 2, 3, \ldots, n$$

we easily obtain the desired result. □

6.3 Applications

In this section we give some applications of Lyapunov type inequalities which are used as a handy tool in studying of the qualitative nature of solutions. Here we only consider quasilinear systems with (p, q)-Laplacian but all the following results can be generalized to the quasilinear systems with $(p_1, \ldots, p_n)-$Laplacian (6.17) considered in Section 6.2.

6.3.1 Disconjugacy

In this part by using the inequalities obtained in Section 6.1.1, we establish disconjugacy criteria for system (6.1) and (6.2).

Theorem 6.3.1 *Let p' and q' be conjugate numbers for p and q, respectively and (e_1, e_2) be a nontrivial solution of the homogenous system (6.4). If*

$$2^{e_1 p+e_2 q} > \left(\int_a^b h^{-p'/p}(t)dt\right)^{\frac{e_1 p}{p'}} \left(\int_a^b m^{-q'/q}(t)dt\right)^{\frac{e_2 q}{q'}} \left(\int_a^b f^+(t)dt + \sum_{a\le\tau_i<b} a_i^+\right)^{e_1} \times \left(\int_a^b g^+(t)dt + \sum_{a\le\tau_i<b} b_i^+\right)^{e_2} \tag{6.24}$$

holds, then system (6.1) is disconjugate on $[a, b]$.

Proof. Suppose on the contrary that there is a real solution $w(t) = (u(t), v(t))$ with nontrivial $(u(t), v(t))$ having two zeros $s_1, s_2 \in [a, b]$ $(s_1 < s_2)$ such that $(u(t), v(t)) \neq 0$ for all $t \in (s_1, s_2)$. Applying Theorem 6.1.1 we see that

$$\begin{aligned}
2^{e_1 p+e_2 q} &\le \left(\int_{s_1}^{s_2} h^{\frac{-p'}{p}}(t)dt\right)^{\frac{e_1 p}{p'}} \left(\int_{s_1}^{s_2} m^{\frac{-q'}{q}}(t)dt\right)^{\frac{e_2 q}{q'}} \left(\int_{s_1}^{s_2} f^+(t)dt + \sum_{s_1\le\tau_i<s_2} a_i^+\right)^{e_1} \\
&\quad\times \left(\int_{s_1}^{s_2} g^+(t)dt + \sum_{s_1\le\tau_i<s_2} b_i^+\right)^{e_2} \\
&\le \left(\int_a^b h^{\frac{-p'}{p}}(t)dt\right)^{\frac{e_1 p}{p'}} \left(\int_a^b m^{\frac{-q'}{q}}(t)dt\right)^{\frac{e_2 q}{q'}} \left(\int_a^b f^+(t)dt + \sum_{a\le\tau_i<b} a_i^+\right)^{e_1} \\
&\quad\times \left(\int_a^b g^+(t)dt + \sum_{a\le\tau_i<b} b_i^+\right)^{e_2}.
\end{aligned}$$

Clearly, the last inequality contradicts (6.24). The proof is complete. $\square$

Again, we have the corresponding corollaries and theorem whose proofs are the same as proof of Theorem 6.3.1, hence omitted.

Corollary 6.3.1 *Let p' and q' be conjugate numbers for p and q, respectively. Assume that (6.11) or (6.12) holds. If*

$$2^{p+q} > \left(\int_a^b h^{-p'/p}(t)dt\right)^{\frac{p}{p'}} \left(\int_a^b m^{-q'/q}(t)dt\right)^{\frac{q}{q'}} \left(\int_a^b f^+(t)dt + \sum_{a\le\tau_i<b} a_i^+\right) \times \left(\int_a^b g^+(t)dt + \sum_{a\le\tau_i<b} b_i^+\right).$$

holds, then system (6.1) is disconjugate on $[a, b]$.

Corollary 6.3.2 *Let p' and q' be conjugate numbers for p and q, respectively. Assume (6.13) holds. If*

$$2^{\theta+\beta} > \left(\int_a^b h^{-p'/p}(t)dt\right)^{\frac{\theta}{p'}} \left(\int_a^b m^{-q'/q}(t)dt\right)^{\frac{\beta}{q'}} \left(\int_a^b f^+(t)dt + \sum_{a\le\tau_i<b} a_i^+\right)^{\theta/p} \times \left(\int_a^b g^+(t)dt + \sum_{a\le\tau_i<b} b_i^+\right)^{\beta/q}$$

holds, then system (6.1) is disconjugate on $[a,b]$.

Corollary 6.3.3 *Let p' and q' be conjugate numbers for p and q, respectively. Assume $\alpha=\theta$ and $\beta=\gamma$ and (6.14) hold. If*

$$2^{\alpha+\beta} > \left(\int_a^b h^{-p'/p}(t)dt\right)^{\frac{\alpha}{p'}} \left(\int_a^b m^{-q'/q}(t)dt\right)^{\frac{\beta}{q'}} \left(\int_a^b f^+(t)dt + \sum_{a\le\tau_i<b} a_i^+\right)^{\alpha/p} \times \left(\int_a^b g^+(t)dt + \sum_{a\le\tau_i<b} b_i^+\right)^{\beta/q}$$

holds, then system (6.1) is disconjugate on $[a,b]$.

We give the next theorem, which is a direct consequence of Theorem 6.3.1, for system (6.2).

Theorem 6.3.2 *Let p' and q' be conjugate numbers for p and q, respectively, and* (6.14) *hold. Suppose that*

$$\hat{\alpha}_i = |\alpha_i+1|^{p/q}-1, \quad \beta_i = |\alpha_i+1|^{p-2}(\alpha_i+1)-1, \quad \delta_i = |\alpha_i+1|^{p/q'}-1, \quad i\in\mathbb{N}.$$

If

$$2^{\alpha+\beta} > \left(\int_a^b h^{-p'/p}(t)dt\right)^{\frac{\alpha}{p'}} \left(\int_a^b f^+(t)dt + \sum_{a\le\tau_i<b} \left(\frac{\gamma_i}{\alpha_i+1}\right)^+ |\alpha_i+1|^{2-p}\right)^{\frac{\alpha}{p}} \times \left(\int_a^b m^{-q'/q}(t)dt\right)^{\frac{\beta}{q'}} \left(\int_a^b g^+(t)dt + \sum_{a\le\tau_i<b} \mu_i^+ |\alpha_i+1|^{-p/q'}\right)^{\frac{\beta}{q}}$$

holds, then system (6.2) is disconjugate on $[a,b]$.

6.3.2 Eigenvalue problems

Now, we present an application of the obtained Lyapunov-type inequalities for system (6.1) and (6.2). By using techniques similar to the technique in Napoli and Pinasco [78], we establish the following results which give lower bounds for eigenvalues of the associated eigenvalue problems of system (6.1) and (6.2). The proofs of the following theorems are based on the Lyapunov type inequalities derived in Theorem 6.1.1 and Theorem 6.1.2.

Let $f(t) = \lambda\alpha r_1(t)$, $g(t) = \mu\beta r_2(t)$, $a_i = \lambda\alpha c_{i1}$ and $b_i = \mu\beta c_{i2}$. Then system (6.1) reduces to the following eigenvalue problem

$$\begin{aligned}
&-\left(h(t)\left|u'\right|^{p-2}u'\right)' = \lambda\alpha r_1(t)\left|u\right|^{\alpha-2}u\left|v\right|^{\beta},\ t \neq \tau_i\\
&-\left(m(t)\left|v'\right|^{q-2}v'\right)' = \mu\beta r_2(t)\left|u\right|^{\theta}\left|v\right|^{\gamma-2}v,\ t \neq \tau_i\\
&-\Delta\left(h(t)\left|u'\right|^{p-2}u'\right)\Big|_{t=\tau_i^-} = \lambda\alpha c_{i1}\left|u\right|^{\alpha-2}u\left|v\right|^{\beta},\ i = 1, 2, \ldots, m\\
&-\Delta\left(m(t)\left|v'\right|^{q-2}u_2'\right)\Big|_{t=\tau_i^-} = \mu\beta c_{i2}\left|u\right|^{\theta}\left|v\right|^{\gamma-2}v,\ i = 1, 2, \ldots, m\\
&u(a) = u(b) = v(a) = v(b) = 0.
\end{aligned} \tag{6.25}$$

Definition 6.3.1 *A pair* (λ, μ) *is called an eigenvalue of* (6.25) *if there is a corresponding solution* (u, v) *such that* $u, v \not\equiv 0$ *on* (a, b).

Theorem 6.3.3 *Let* p' *and* q' *be conjugate numbers for* p *and* q*, respectively,* (e_1, e_2) *be a nontrivial solution of the homogenous system (6.4) and*

$$\int_a^b r_k(t)dt + \sum_{a\le\tau_i<b} c_{ik} > 0, \qquad k = 1, 2. \tag{6.26}$$

Then there exists a function $h(\lambda) = \dfrac{1}{\beta}\left(\dfrac{CD}{(\lambda\alpha)^{e_1}}\right)^{\frac{1}{e_2}}$ *such that* $\mu \ge h(\lambda)$ *for every positive eigenvalue pair* (λ, μ) *of the system (6.25) where the constants* C *and* D *are given as*

$$C = 2^{e_1p+e_2q}\left(\int_a^b h^{\frac{-p'}{p}}(t)dt\right)^{\frac{-e_1p}{p'}}\left(\int_a^b m^{\frac{-q'}{q}}(t)dt\right)^{\frac{-e_2q}{q'}},$$

$$D = \left(\int_a^b r_1(t)dt + \sum_{a\le\tau_i<b} c_{i1}\right)^{-e_1}\left(\int_a^b r_2(t)dt + \sum_{a\le\tau_i<b} c_{i2}\right)^{-e_2}.$$

Proof. Let (λ, μ) be a positive eigenvalue pair and (u, v) be the corresponding eigenfunctions of the system (6.25). If we apply Lyapunov inequality obtained in Theorem 6.1.1 for system (6.25), we get

$$\begin{aligned} 2^{e_1p+e_2q} &\leq \left(\int_a^b h^{-p'/p}(t)dt\right)^{\frac{e_1p}{p'}} \left(\int_a^b \lambda\alpha r_1(t)dt + \sum_{a\leq\tau_i<b} \lambda\alpha c_{i1}\right)^{e_1} \\ &\times \left(\int_a^b m^{-q'/q}(t)dt\right)^{\frac{e_2q}{q'}} \left(\int_a^b \mu\beta r_2(t)dt + \sum_{a\leq\tau_i<b} \mu\beta c_{i2}\right)^{e_2} \\ &= \left(\int_a^b h^{-p'/p}(t)dt\right)^{\frac{e_1p}{p'}} \left(\int_a^b m^{-q'/q}(t)dt\right)^{\frac{e_2q}{q'}} (\lambda\alpha)^{e_1}(\mu\beta)^{e_2} \\ &\times \left(\int_a^b r_1(t)dt + \sum_{a\leq\tau_i<b} c_{i1}\right)^{e_1} \left(\int_a^b r_2(t)dt + \sum_{a\leq\tau_i<b} c_{i2}\right)^{e_2}. \end{aligned} \tag{6.27}$$

For the eigenvalue μ we can find the following lower bound as

$$\begin{aligned} \mu\beta \geq 2^{\frac{e_1p+e_2q}{e_2}} (\lambda\alpha)^{\frac{-e_1}{e_2}} &\left(\int_a^b h^{-p'/p}(t)dt\right)^{\frac{-e_1p}{e_2p'}} \left(\int_a^b m^{-q'/q}(t)dt\right)^{\frac{-q}{q'}} \\ \times &\left(\int_a^b r_1(t)dt + \sum_{a\leq\tau_i<b} c_{i1}\right)^{\frac{-e_1}{e_2}} \left(\int_a^b r_2(t)dt + \sum_{a\leq\tau_i<b} c_{i2}\right)^{-1}. \end{aligned}$$

Also by rearranging terms in (6.27), we obtain

$$\lambda^{e_1}\mu^{e_2} \geq \frac{CD}{\alpha^{e_1}\beta^{e_2}}.$$

$\square$

Since the proofs of following corollaries are the same as that of Theorem 6.3.3, they are omitted.

Corollary 6.3.4 *Let p' and q' be conjugate numbers for p and q, respectively, (6.11) or (6.12) and (6.26) hold. Then there exists a function $h_1(\lambda) = \frac{1}{\beta}\left(\frac{C_1D_1}{\lambda\alpha}\right)$ such that $\mu \geq h_1(\lambda)$ for every positive eigenvalue pair (λ, μ) of the system (6.25) where the constants C_1 and D_1 are given as*

$$C_1 = 2^{p+q}\left(\int_a^b h^{\frac{-p'}{p}}(t)dt\right)^{\frac{-p}{p'}} \left(\int_a^b m^{\frac{-q'}{q}}(t)dt\right)^{\frac{-e_2q}{q'}},$$

$$D_1 = \left(\int_a^b r_1(t)dt + \sum_{a\leq\tau_i<b} c_{i1}\right)^{-1} \left(\int_a^b r_2(t)dt + \sum_{a\leq\tau_i<b} c_{i2}\right)^{-1}.$$

Corollary 6.3.5 *Let p' and q' be conjugate numbers for p and q, respectively, (6.13) and (6.26) hold. Then there exists a function $h_2(\lambda) = \frac{1}{\beta}\left(\frac{C_2 D_2}{(\lambda\alpha)^{\frac{\theta}{p}}}\right)^{\frac{q}{\beta}}$ such that $\mu \geq h_2(\lambda)$ for every positive eigenvalue pair (λ, μ) of the system (6.25) where the constants C_2 and D_2 are given as*

$$C_2 = 2^{\theta+\beta}\left(\int_a^b h^{\frac{-p'}{p}}(t)dt\right)^{\frac{-\theta}{p'}}\left(\int_a^b m^{\frac{-q'}{q}}(t)dt\right)^{\frac{-\beta}{q'}},$$

$$D_2 = \left(\int_a^b r_1(t)dt + \sum_{a\leq\tau_i<b} c_{i1}\right)^{\frac{-\theta}{p}}\left(\int_a^b r_2(t)dt + \sum_{a\leq\tau_i<b} c_{i2}\right)^{\frac{-\beta}{q}}.$$

Corollary 6.3.6 *Let p' and q' be conjugate numbers for p and q, respectively, and $\alpha = \theta$ and $\beta = \gamma$. Assume (6.14) and (6.26) hold. Then there exists a function $h_3(\lambda)$ such that $\mu \geq h_3(\lambda) = \frac{1}{\beta}\left(\frac{C_3 D_3}{(\lambda\alpha)^{\frac{\alpha}{p}}}\right)^{\frac{q}{\beta}}$ for every positive eigenvalue pair (λ, μ) of the system (6.25) where the constants C_3 and D_3 are given as*

$$C_3 = 2^{\alpha+\beta}\left(\int_a^b h^{\frac{-p'}{p}}(t)dt\right)^{\frac{-\alpha}{p'}}\left(\int_a^b m^{\frac{-q'}{q}}(t)dt\right)^{\frac{-\beta}{q'}},$$

$$D_3 = \left(\int_a^b r_1(t)dt + \sum_{a\leq\tau_i<b} c_{i1}\right)^{\frac{-\alpha}{p}}\left(\int_a^b r_2(t)dt + \sum_{a\leq\tau_i<b} c_{i2}\right)^{\frac{-\beta}{q}}.$$

Now we will consider system (6.2) and define an eigenvalue problem associated to this system. Let $f(t) = \lambda\alpha r_1(t)$, $g(t) = \mu\beta r_2(t)$, $\gamma_i = \lambda\alpha c_{i1}$ and $\mu_i = \mu\beta c_{i2}$. Then system (6.2) reduces to the following eigenvalue problem

$$\begin{aligned}
&-\left(h(t)\left|u'\right|^{p-2}u'\right)' = \lambda\alpha r_1(t)\left|u\right|^{\alpha-2}u\left|v\right|^{\beta},\ \ t\neq\tau_i\\
&-\left(m(t)\left|v'\right|^{q-2}v'\right)' = \mu\beta r_2(t)\left|u\right|^{\alpha}\left|v\right|^{\beta-2}v,\ \ t\neq\tau_i\\
&\Delta u\big|_{t=\tau_i} = \alpha_i u,\quad \Delta v\big|_{t=\tau_i} = \hat{\alpha}_i v,\\
&-\Delta\left(h(t)\left|u'\right|^{p-2}u'\right)\Big|_{t=\tau_i^-} = \beta_i\left(h(t)\left|u'\right|^{p-2}u'\right) - \lambda\alpha c_{i1}\left|u\right|^{\alpha-2}u\left|v\right|^{\beta},\\
&-\Delta\left(m(t)\left|v'\right|^{q-2}u_2'\right)\Big|_{t=\tau_i^-} = \delta_i\left(m(t)\left|v'\right|^{q-2}v'\right) - \mu\beta c_{i2}\left|u\right|^{\alpha}\left|v\right|^{\beta-2}v,\\
&u(a^+) = u(b^-) = v(a^+) = v(b^-) = 0
\end{aligned}\tag{6.28}$$

where $\hat{\alpha}_i = |\alpha_i+1|^{p/q} - 1,\quad \beta_i = |\alpha_i+1|^{p-2}(\alpha_i+1) - 1,\quad \delta_i = |\alpha_i+1|^{p/q'} - 1$ for $i \in \mathbb{N}$.

Definition 6.3.2 *A pair (λ,μ) is called an eigenvalue of* (6.28) *if there is a corresponding solution (u,v) such that $u,v\not\equiv 0$ on (a,b).*

Theorem 6.3.4 *Let p' and q' be conjugate numbers for p and q, respectively, and*

$$\int_a^b r_k(t)dt+\sum_{a\le\tau_i<b}\rho_{ik}>0,\qquad k=1,2$$

where $\rho_{i1}=c_{i1}|\alpha_i+1|^{2-p}/(\alpha_i+1)$ and $\rho_{i2}=c_{i2}|\alpha_i+1|^{-p/q'}$. Then there exists a function $h_4(\lambda)=\dfrac{1}{\beta}\left(\dfrac{EF}{(\lambda\alpha)^{\frac{\alpha}{p}}}\right)^{\frac{q}{\beta}}$ such that $\mu\ge h_4(\lambda)$ for every positive eigenvalue pair (λ,μ) of the system (6.28) where the constants E and F are given by

$$E=2^{\alpha+\beta}\left(\int_a^b h^{\frac{-p'}{p}}(t)dt\right)^{\frac{-\alpha}{p'}}\left(\int_a^b m^{\frac{-q'}{q}}(t)dt\right)^{\frac{-\beta}{q'}},$$

$$F=\left(\int_a^b r_1(t)dt+\sum_{a\le\tau_i<b}\rho_{i1}\right)^{\frac{-\alpha}{p}}\left(\int_a^b r_2(t)dt+\sum_{a\le\tau_i<b}\rho_{i2}\right)^{\frac{-\beta}{q}},$$

Proof. Let (λ,μ) be a positive eigenvalue pair and (u,v) be the corresponding eigenfunctions of the system (6.28). If we apply Lyapunov inequality obtained in Theorem 6.1.2 for system (6.28), we get

$$\begin{aligned}
2^{\alpha+\beta}\le&\left(\int_a^b h^{\frac{-p'}{p}}(t)dt\right)^{\frac{\alpha}{p'}}\left(\int_a^b\lambda\alpha r_1(t)dt+\sum_{a\le\tau_i<b}\lambda\alpha\frac{c_{i1}}{\alpha_i+1}|\alpha_i+1|^{2-p}\right)^{\frac{\alpha}{p}}\\
&\times\left(\int_a^b m^{\frac{-q'}{q}}(t)dt\right)^{\frac{\beta}{q'}}\left(\int_a^b\mu\beta r_2(t)dt+\sum_{a\le\tau_i<b}\mu\beta c_{i2}|\alpha_i+1|^{-p/q'}\right)^{\frac{\beta}{q}}\\
=&(\lambda\alpha)^{\frac{\alpha}{p}}\left(\int_a^b h^{\frac{-p'}{p}}(t)dt\right)^{\frac{\alpha}{p'}}\left(\int_a^b r_1(t)dt+\sum_{a\le\tau_i<b}\frac{c_{i1}}{\alpha_i+1}|\alpha_i+1|^{2-p}\right)^{\frac{\alpha}{p}}\\
&\times(\mu\beta)^{\frac{\beta}{q}}\left(\int_a^b m^{\frac{-q'}{q}}(t)dt\right)^{\frac{\beta}{q'}}\left(\int_a^b r_2(t)dt+\sum_{a\le\tau_i<b}c_{i2}|\alpha_i+1|^{-p/q'}\right)^{\frac{\beta}{q}}.
\end{aligned}$$

For the eigenvalue μ, we can obtain the following lower bound as

$$\begin{aligned}
\mu\beta\ge&\left(\int_a^b h^{\frac{-p'}{p}}(t)dt\right)^{\frac{-q\alpha}{\beta p'}}\left(\int_a^b r_1(t)dt+\sum_{a\le\tau_i<b}\frac{c_{i1}}{\alpha_i+1}|\alpha_i+1|^{2-p}\right)^{\frac{-q\alpha}{p\beta}}\\
&\times 2^{\frac{q}{\beta}(\alpha+\beta)}(\lambda\alpha)^{\frac{-q\alpha}{\beta p}}\left(\int_a^b m^{\frac{-q'}{q}}(t)dt\right)^{\frac{-q}{q'}}\left(\int_a^b r_2(t)dt+\sum_{a\le\tau_i<b}c_{i2}|\alpha_i+1|^{-p/q'}\right)^{-1}.
\end{aligned}$$

Also by rearranging terms in the above inequality, we obtain

$$\lambda^{\frac{\alpha}{p}}\mu^{\frac{\beta}{q}} \geq \frac{EF}{\alpha^{\frac{\alpha}{p}}\beta^{\frac{\beta}{q}}}.$$

$\square$

6.3.3 Asymptotic Behavior of Oscillatory Solutions

In this section as an application of Lyapunov type inequality given in Section 6.1.1, we establish the following results to study the asymptotic behavior of the oscillatory solutions of system (6.1) and (6.2).

Theorem 6.3.5 *Let p' and q' be conjugate numbers for p and q, respectively, and (e_1, e_2) be a nontrivial solution of the homogenous system (6.4). Let*

$$\left(\int^{\infty} h^{-p'/p}(t)dt\right)^{\frac{e_1 p}{p'}} \left(\int^{\infty} m^{-q'/q}(t)dt\right)^{\frac{e_2 q}{q'}} \left(\int^{\infty} f^{+}(t)dt + \sum_{\tau_i<\infty} a_i^{+}\right)^{e_1}$$
$$\times \left(\int^{\infty} g^{+}(t)dt + \sum_{\tau_i<\infty} b_i^{+}\right)^{e_2} < \infty.$$

Then every oscillatory solution $w(t) = (u(t), v(t))$ of system (6.1) is bounded and approaches zero as $t \to \infty$.

Proof. First we prove the boundedness of oscillatory solution $w(t) = (u(t), v(t))$. Let us suppose that $w(t)$ is oscillatory but not bounded. Then $\limsup_{t\to\infty} |w(t)| = \infty$. Then for every M_1, we can find $T = T(M_1)$ such that $|w(t)| > M_1$ for all $t > T$. Since w is oscillatory, there exists an interval (t_1, t_2) with $t_1 \geq T$ such that $w(t_1) = w(t_2) = 0$. By using Lyapunov inequality for $t_1 \geq T$, we get

$$\begin{aligned}
2^{e_1 p + e_2 q} \leq & \left(\int_{t_1}^{t_2} h^{-p'/p}(t)dt\right)^{\frac{e_1 p}{p'}} \left(\int_{t_1}^{t_2} m^{-q'/q}(t)dt\right)^{\frac{e_2 q}{q'}} \\
& \times \left(\int_{t_1}^{t_2} f^{+}(t)dt + \sum_{t_1\leq\tau_i<t_2} a_i^{+}\right)^{e_1} \left(\int_{t_1}^{t_2} g^{+}(t)dt + \sum_{t_1\leq\tau_i<t_2} b_i^{+}\right)^{e_2} \\
\leq & \left(\int_{t_1}^{\infty} h^{-p'/p}(t)dt\right)^{\frac{e_1 p}{p'}} \left(\int_{t_1}^{\infty} m^{-q'/q}(t)dt\right)^{\frac{e_2 q}{q'}} \\
& \times \left(\int_{t_1}^{\infty} f^{+}(t)dt + \sum_{t_1\leq\tau_i<\infty} a_i^{+}\right)^{e_1} \left(\int_{t_1}^{\infty} g^{+}(t)dt + \sum_{t_1\leq\tau_i<\infty} b_i^{+}\right)^{e_2} \leq 1
\end{aligned}$$

Then we get $e_1 p + e_2 q \leq 0$ which implies contradiction. Therefore w is bounded. Since w is bounded, $|w(t)| \leq N$ for $t > T$ for any T. If $w(t)$ does not approach zero as $t \to \infty$, then there exists a constant $d > 0$ such that $2d \leq \limsup_{t\to\infty} |w(t)| \leq N$. Since w has arbitrarily large zeros, there exists an interval (t_1, t_2) with $t_1 \geq T$, where T is sufficiently large, such that $w(t_1) = w(t_2) = 0$. The remainder of the proof is similar to above, hence it is omitted. $\square$

The following corollaries and their proofs follow easily from Theorem 6.3.5 and its proof, respectively.

Corollary 6.3.7 *Let p' and q' be conjugate numbers for p and q, respectively, and (6.11) or (6.12) hold. Let*

$$\left(\int^{\infty} h^{-p'/p}(t)dt\right)^{\frac{p}{p'}} \left(\int^{\infty} m^{-q'/q}(t)dt\right)^{\frac{q}{q'}} \left(\int^{\infty} f^{+}(t)dt + \sum_{\tau_i<\infty} a_i^{+}\right)$$
$$\times \left(\int^{\infty} g^{+}(t)dt + \sum_{\tau_i<\infty} b_i^{+}\right) < \infty.$$

Then every oscillatory solution $w(t) = (u(t), v(t))$ of system (6.1) is bounded and approaches zero as $t \to \infty$.

Corollary 6.3.8 *Let p' and q' be conjugate numbers for p and q, respectively, and (6.13) hold. Let*

$$\left(\int^{\infty} h^{-p'/p}(t)dt\right)^{\frac{\theta}{p'}} \left(\int^{\infty} m^{-q'/q}(t)dt\right)^{\frac{\beta}{q'}} \left(\int^{\infty} f^{+}(t)dt + \sum_{\tau_i<\infty} a_i^{+}\right)^{\frac{\theta}{p}}$$
$$\times \left(\int^{\infty} g^{+}(t)dt + \sum_{\tau_i<\infty} b_i^{+}\right)^{\frac{\beta}{q}} < \infty.$$

Then every oscillatory solution $w(t) = (u(t), v(t))$ of system (6.1) is bounded and approaches zero as $t \to \infty$.

Corollary 6.3.9 *Let p' and q' be conjugate numbers for p and q, respectively. Assume*

$\alpha = \theta$ and $\beta = \gamma$ and (6.14) hold. Let

$$\left(\int^{\infty} h^{-p'/p}(t)dt\right)^{\frac{\alpha}{p'}} \left(\int^{\infty} m^{-q'/q}(t)dt\right)^{\frac{\beta}{q'}} \left(\int^{\infty} f^{+}(t)dt + \sum_{\tau_i<\infty} a_i^{+}\right)^{\frac{\alpha}{p}}$$
$$\times \left(\int^{\infty} g^{+}(t)dt + \sum_{\tau_i<\infty} b_i^{+}\right)^{\frac{\beta}{q}} < \infty.$$

Then every oscillatory solution $w(t) = (u(t), v(t))$ of system (6.1) is bounded and approaches zero as $t \to \infty$.

Theorem 6.3.6 *Let p' and q' be conjugate numbers for p and q, respectively. Suppose that $\hat{\alpha}_i = |\alpha_i+1|^{p/q} - 1,\ \beta_i = |\alpha_i+1|^{p-2}(\alpha_i+1) - 1,\ \delta_i = |\alpha_i+1|^{p/q'} - 1,\ i \in \mathbb{N}$. Let*

$$\left(\int^{\infty} h^{-p'/p}(t)dt\right)^{\frac{\alpha}{p'}} \left(\int^{\infty} f^{+}(t)dt + \sum_{\tau_i<\infty} \left(\frac{\gamma_i}{\alpha_i+1}\right)^{+} |\alpha_i+1|^{2-p}\right)^{\frac{\alpha}{p}}$$
$$\times \left(\int^{\infty} m^{-q'/q}(t)dt\right)^{\frac{\beta}{q'}} \left(\int^{\infty} g^{+}(t)dt + \sum_{\tau_i<\infty} \mu_i^{+} |\alpha_i+1|^{-p/q'}\right)^{\frac{\beta}{q}} < \infty.$$

Then every oscillatory solution $w(t) = (u(t), v(t))$ of system (6.2) is bounded and $\hat{w}(t) = \left(\dfrac{u(t)}{M_i}, \dfrac{v(t)}{|M_i|^{p/q}}\right) \to 0$ as $t \to \infty$ where M_i is defined as in equation (6.3).

Proof. Let $w(t) = (u(t), v(t))$ be an oscillatory solution of (6.2). Let $y(t) = u(t)/M_i$ and $z(t) = v(t)/|M_i|^{p/q}$. Suppose on the contrary that $\hat{w}(t) = \left(\dfrac{u(t)}{M_i}, \dfrac{v(t)}{|M_i|^{p/q}}\right)$ is unbounded. Then given any positive number M_1, we can find a positive number $T = T(M_1)$ such that $|\hat{w}(t)| > M_1$ for all $t > T$. Since $\hat{w}$ is also oscillatory, there exists an interval (t_1, t_2) with $t_1 \geq T$ such that $\hat{w}(t_1) = \hat{w}(t_2) = 0$. Because of assumption, one can choose T large enough so that for every $t_1 \geq T$,

$$
\begin{aligned}
2^{\alpha+\beta} \leq & \left(\int_{t_1}^{t_2} h^{-p'/p}(t)dt\right)^{\frac{\alpha}{p'}} \left(\int_{t_1}^{t_2} f^{+}(t)dt + \sum_{t_1 \leq \tau_i < t_2} \left(\frac{\gamma_i}{\alpha_i+1}\right)^{+} |\alpha_i+1|^{2-p}\right)^{\frac{\alpha}{p}} \\
& \times \left(\int_{t_1}^{t_2} m^{-q'/q}(t)dt\right)^{\frac{\beta}{q'}} \left(\int_{t_1}^{t_2} g^{+}(t)dt + \sum_{t_1 \leq \tau_i < t_2} \mu_i^{+} |\alpha_i+1|^{-p/q'}\right)^{\frac{\beta}{q}} \\
\leq & \left(\int_{t_1}^{\infty} h^{-p'/p}(t)dt\right)^{\frac{\alpha}{p'}} \left(\int_{t_1}^{\infty} f^{+}(t)dt + \sum_{t_1 \leq \tau_i < \infty} \left(\frac{\gamma_i}{\alpha_i+1}\right)^{+} |\alpha_i+1|^{2-p}\right)^{\frac{\alpha}{p}} \\
& \times \left(\int_{t_1}^{\infty} m^{-q'/q}(t)dt\right)^{\frac{\beta}{q'}} \left(\int_{t_1}^{\infty} g^{+}(t)dt + \sum_{t_1 \leq \tau_i < \infty} \mu_i^{+} |\alpha_i+1|^{-p/q'}\right)^{\frac{\beta}{q}} \leq 1.
\end{aligned}
$$

Then we get $\alpha + \beta \leq 0$ which implies contradiction. Therefore w is bounded. Since w is bounded, $|\hat{w}(t)| \leq N$ for $t > T$ for any T. If $\hat{w}(t)$ does not approach zero as $t \to \infty$, then there exist a constant $d > 0$ such that $2d \leq \limsup_{t\to\infty} |\hat{w}(t)| \leq N$. Since $\hat{w}$ has arbitrarily large zeros, there exists an interval (t_1, t_2) with $t_1 \geq T$, where T is sufficiently large, such that $\hat{w}(t_1) = \hat{w}(t_2) = 0$. The remainder of the proof is similar to above, hence it is omitted. □

CHAPTER 7

CONCLUSION

This thesis is devoted to obtain Lyapunov type inequalities for linear and nonlinear systems under impulse effect. The importance of Lyapunov type inequalities in qualitative analysis of solutions of systems under considerations has been shown by means of applications, for instance by proving disconjugacy criterion, by showing the uniqueness of the solutions of associated inhomogeneous BVPs, by studying the stability of planar periodic systems, by finding lower bounds for the eigenvalues of the associated eigenvalue problems and by analysing asymptotic behaviour of oscillatory solutions of considered systems. Moreover it has been remarked that theory of system of impulsive differential equations is richer and more fruitful than the corresponding theory of system of ordinary differential equations due to the fact that existence of impulse effet yields various new inequalities.

In Chapter 2 we have established Lyapunov type inequalities, which are generalizations of the inspired work of Lyapunov [69] on second order linear ordinary differential equations, for linear $2n \times 2n$ Hamiltonian systems with impulsive perturbations. Since changing the impulsive perturbation or assuming different conditions on the impulses leads to variety of inequalities, presence of impulse effect provides different Lyapunov type inequalities. Moreover our result improve and generalize the previous ones, in particular Tang and Zhang [99], even in the special case when the impulses are dropped. As applications of Lyapunov type inequalities, we have found a disconjugacy interval and a lower bound for the associated eigenvalue problems for linear $2n-$dimensional Hamiltonian systems under impulse effect.

In Chapter 3 we have discussed the existence and uniqueness of solutions of inhomogenous BVPs to linear $2n \times 2n$ Hamiltonian systems with impulsive perturbations. The proof of our theorem has based on the fact that if corresponding homogenous BVP has only trivial solution, the sufficient condition of which is proved by Lyapunov type inequalities, then inhomogenous BVP has a unique solution. Moreover the unique solution of inhomogenous BVP has been expressed in terms of Green's function (pair) and properties of Green's function (pair) has been stated. To the best of our knowledge, our approach is quite new and our criteria are the first results which give the relation between existence and uniqueness theory of boundary value problems and Lyapunov type inequalities. This relation has not been noted even for the ordinary differential equations case.

In Chapter 4 we have dealt with stability problem for linear planar periodic Hamiltonian systems with impulsive perturbations. By combining Floquet theory and Lyapunov type inequalities, we have derived two pairs of stability criteria and each pair is alternative to the other one. The first pair of the criteria is generalization of Tang and Zhang [99] while the other one is new and can be used in place of the first one when it is not applicable. Therefore our results are new for the impulsive case.

In Chapter 5 we have derived several Lyapunov type inequalities for impulsive nonlinear systems and for their special cases, impulsive Emden-Fowler type equations and impulsive half linear equations. Our results have related not only points where the first component of the solution $(x(t), u(t))$ of considered system has consecutive zeros but also the point where the first component of the solution $(x(t), u(t))$ of such system is maximized. As an application we have derived disconjugacy criteria, found lower bounds for the associated eigenvalue problems and investigated asymptotic behavior of oscillatory solutions. Our results generalize the previous one existing in literature, in particular [105, 106].

In Chapter 6 we have obtained Lyapunov type inequalities for impulsive quasilinear systems with $(p, q)-$Laplacian and $(p_1, p_2, \ldots, p_n)$-Laplacian. First we have considered impulsive systems whose solutions are continous, i.e there is an impulse condition only on derivative of solutions. After establishing Lyapunov type inequalities for these systems, we have studied the systems with discontinous solutions which can be transformed to the continuous sytems. Moreover the applications of Lyapunov type inequalities such as obtaining disconjugacy criteria, finding lower bounds for the associated eigenvalue problems and investigating asymptotic behavior of oscillatory solutions are demonstrated. Our results generalize the result of [78, 19, 98, 7] and [18, 98, 7, 6, 115] for $(p, q)-$Laplacian and $(p_1, p_2, \ldots, p_n)$-Laplacian, respectively. Moreover we have derived new results by assuming different conditions on q_{jk} and p_k for $k, j = 1, 2, \ldots, n$.

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CURRICULUM VITAE

PERSONAL INFORMATION

Surname, Name: Kayar, Zeynep
Nationality: Turkish (TC)
Date and Place of Birth: 31 May 1982, Ankara
Marital Status: Single
Phone: +905052209928
E-mail: zkayar@metu.edu.tr, zykayar@gmail.com

EDUCATION

Degree	**Institution**	**Year**
Visiting Scholar	University of Santiago de Compostela, Department of Mathematical Analysis	2012
B.S.	Ankara University	2000-2004
High School	Yıldırım Beyazıt Anatolian High School	1997-2000

PROFESSIONAL EXPERIENCE

Year	**Place**	**Enrollment**
2005-present	METU, Department of Mathematics	Research Assistant

FOREIGN LANGUAGES

English (Advanced), Spanish (Intermediate)

RESEARCH INTEREST

- Ordinary Differential Equations
- Impulsive Differential Equations
- Functional (Delay) Differential Equations
- Fractional Differential Equations
- Difference Equations and Calculus on Time Scales
- Computational Methods

SCHOLARSHIPS

Ph.d. Scholarship Program, TUBITAK, 2006-2010

PUBLICATIONS

1) Kayar, Z. and Zafer, A., Stability Criteria for Linear Hamiltonian Systems Under Impulsive Perturbations, Applied Mathematics and Computation, 230 (2014) 680–686, http://dx.doi.org/10.1016/j.amc.2013.12.128.
2) Kayar, Z. and Zafer, A., Impulsive Boundary Value Problems for Planar Hamiltonian Systems, Abstract and Applied Analysis, Volume 2013 (2013), Article ID 892475, 6 pages, http://dx.doi.org/10.1155/2013/892475.
3) Priya, G. S., Prakash, P., Nieto, J. J. and Kayar, Z., Higher-Order Numerical Scheme for the Fractional Heat Equation with Dirichlet and Neumann Boundary Conditions, Numerical Heat Transfer Part B: Fundamentals, Volume 63, Number 6, 1 June 2013 , pp. 540-559.

INTERNATIONAL CONFERENCE PRESENTATIONS

1) Existence and Uniqueness of Solutions of Nonhomogenous Boundary Value Problem for System of First Order Linear Impulsive Differential Equations,

12th International Workshop on Dynamical Systems and Applications (IWDSA 2013), Atılım University, Ankara, Turkey, August 12-15, 2013.

2) Stability Criteria for Linear Hamiltonian Systems Under Impulsive Perturbations, Workshop on Qualitative Theory for Differential Equations, University of Santiago de Compostela, Santiago de Compostela, Spain, June 13, 2012.

3) Lyapunov Type Inequalities for Quasilinear Impulsive Differential Systems, 8th International ISAAC Congress, Peoples' Friendship University of Russia, Moscow, Russia, August 22-27, 2011.

4) Disconjugacy and Stability Criteria For Linear Hamiltonian Systems With Impulse Effect, International Conference on Differential & Difference Equations and Applications, Azores University, Ponta Delgada, Portugal, July 4 - 8, 2011.

5) Lyapunov Type Inequalities For Nonlinear Impulsive Differential Systems, 7th International ISAAC Congress, Imperial College, London, England, July 13-18, 2009.

NATIONAL CONFERENCE PRESENTATIONS

1) 2 Boyutlu Impulsive Hamiltonian Sistemler İçin Sınır Değer Problemleri ve Çözümler İçin Varlık ve Teklik Kriteri, Dinamik Sistemler Seminerleri, (Existence and Uniqueness Criteria For Solutions of Boundary Value Problems For Planar Impulsive Hamiltonian Systems, Dynamical Systems Seminars), Middle East Technical University, Ankara, Turkey, October 16, 2013.

2) Lineer Impulsive Hamiltonian Sistemler İçin Kararlılık Kriterleri, 10. Dinamik Sistemler Çalıştayı, (Stability Criteria For Linear Impulsive Hamiltonian Systems, 10th Workshop on Dynamical Systems, now International Workshop on Dynamical Systems and Applications), Tübitak Tüsside, Gebze, Turkey, October 6-8, 2011.

3) Hamiltonian Systems With Impulse: Lyapunov Inequalities, Disconjugacy, and Stability, Dinamik Sistemler Semineri (Dynamical Systems Seminars), Middle East Technical University, Ankara, Turkey, March 19, 2011.

4) Birinci Mertebeden Lineer Olmayan İmpulse İçeren Diferansiyel Denklem Sistemleri İçin Eşlenik Olmama Kriterleri, Matematik Lisansüstü Çalıştayı-1, (Disconjugacy Criteria For System of First Order Nonlinear Differential Equations with Impulsive Perturbations, Mathematics Graduate Workshop-1), Yeditepe University, İstanbul, Turkey, June 13-15, 2010.

5) Lineer Olmayan Süreksiz Diferansiyel Sistemler İçin Lyapunov Eşitsizlikleri, Dinamik Sistemler Semineri, (Lyapunov Inequalities For Nonlinear Discontinuous Differential Systems, Dynamical Systems Seminars), Middle East Technical University, Ankara, Turkey, October 19, 2009.

6) Lineer Olmayan Süreksiz Diferansiyel Sistemler İçin Lyapunov Eşitsizlikleri, 4. Ankara Matematik Günleri, (Lyapunov Inequalities For Nonlinear Discontinuous Differential Systems, 4th Ankara Mathematics Days), Middle East Technical University, Ankara, Turkey, May 4-5, 2009.

7) 2. Mertebeden Gecikmeli İmpalsif Diferansiyel Denklemlerin Aralık Salınımlılığı, 8. Ankara Diferansiyel Denklemler Çalıştayı (Interval Oscillation of Second Order Delay Differential Equations with Impulses, 8th Ankara Differential Equations Seminars, now International Workshop on Dynamical Systems and Applications), Yeditepe University, İstanbul, Turkey, June 19-21, 2008.

CONFERENCE PARTICIPATION

1) 11th International Workshop on Dynamical Systems and Applications (IWDSA 2012), Çankaya University, Ankara, Turkey, June 26-28, 2012.

2) 9th Ankara Differential Equations Seminars (now International Workshop on Dynamical Systems and Applications), İzmir University, İzmir, Turkey, June 18-19, 2009.

3) 14th International Conference on Difference Equations and Applications (ICDEA), Bahçeşehir University, İstanbul, Turkey, July 21-25, 2008.

4) 6th International Isaac Conference, Middle East Technical University, Ankara, Turkey, August 13-18, 2007.

ACADEMIC MEMBERSHIPS

- Turkish Mathematical Society (Ankara Branch)
- Association For Turkish Women in Maths

CONFERENCE ORGANIZATIONS

Member of the Organization Committee of 12th International Workshop on Dynamical Systems and Applications (IWDSA 2013), Atılım University, Ankara, Turkey, August 12-15, 2013.