AN INVESTIGATION OF TEACHERS' NOTICING OF STUDENTS' MATHEMATICAL THINKING IN THE CONTEXT OF A PROFESSIONAL DEVELOPMENT PROGRAM

A THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES
OF
MIDDLE EAST TECHNICAL UNIVERSITY

BY

SİNEM BAŞ

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY
IN
SECONDARY SCIENCE AND MATHEMATICS EDUCATION

DECEMBER 2013

Approval of the thesis:

AN INVESTIGATION OF TEACHERS' NOTICING OF STUDENTS' MATHEMATICAL THINKING IN THE CONTEXT OF A PROFESSIONAL DEVELOPMENT PROGRAM

submitted by **SİNEM BAŞ** in partial fulfillment of the requirements for the degree of **Doctor of Philosophy in Secondary Science and Mathematics Education Department, Middle East Technical University** by,

Prof. Dr. Canan Özgen ________
Dean, Graduate School of **Natural and Applied Sciences**

Prof. Dr. Ömer Geban ________
Head of Department, **Secondary Science and Mathematics Edu.**

Assoc. Prof. Dr. Ayhan Kürşat Erbaş ________
Supervisor, **Secondary Science and Mathematics Edu. Dept., METU**

Assoc. Prof. Dr. Bülent Çetinkaya ________
Co-Supervisor, **Secondary Science and Mathematics Edu. Dept., METU**

Examining Committee Members:

Prof. Dr. Cengiz Alacacı ________________
Primary Education Dept., Istanbul Medeniyet University

Assoc. Prof. Dr. Ayhan Kürşat Erbaş ________________
Secondary Science and Mathematics Education Dept., METU

Assoc. Prof. Dr. Selma Özçağ ________________
Mathematics Dept., Hacettepe University

Assoc. Prof. Dr. Erdinç Çakıroğlu ________________
Elementary Education Dept., METU

Assist. Prof. Dr. Ömer Faruk Özdemir ________________
Secondary Science and Mathematics Education Dept., METU

30. 12. 2013
Date:__________________

I hereby declare that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work

Name, Last name :

Signature :

ABSTRACT

AN INVESTIGATION OF TEACHERS' NOTICING OF STUDENTS' MATHEMATICAL THINKING IN THE CONTEXT OF A PROFESSIONAL DEVELOPMENT PROGRAM

Baş, Sinem
Ph.D., Department of Secondary Science and Mathematics Education
Supervisor: Assoc. Prof. Dr. Ayhan Kürşat Erbaş
Co-Supervisor: Assoc. Prof. Dr. Bülent Çetinkaya

December 2013, 223 pages

Teachers' noticing, i.e. attending to and understanding of, students' mathematical thinking is considered as a crucial component of teaching expertise. Research points out the pressing need for developing teachers' abilities to notice students' mathematical thinking through appropriate professional support, while also making sense of the dynamics of how such development occurs. The purpose of this study is to investigate evolution of teachers' noticing of students' mathematical thinking in the context of a professional development program based on principles of Models and Modeling Perspective. Case study approach was used as the strategy of inquiry. The study was conducted in 2011-2012 school year at two high schools, with the participation of four secondary mathematics teachers. The implementation of the program consisted of seven one-month periods, each of which included three one-week stages: an introductory meeting, implementation of the modeling task in teachers' classrooms, and a follow-up meeting, respectively. One-to-one interviews were also conducted with each teacher after each follow up meeting. An existing framework for learning to notice students' mathematical thinking was adopted and used for analyzing the data. Data analysis revealed that there was a gradual development in three of the four teachers' ability to notice students' mathematical thinking. Moreover, development of teachers' ability to notice occurred in different trajectories. These different trajectories provide clues about components of the professional development program that might have an influence on teachers' development. Implications of the findings for teachers, teacher educators and professional development program developers, and issues for further research are also discussed.

Keywords: Teacher Noticing, Modeling, Professional Development, Mathematics Education

ÖZ

BİR MESLEKİ GELİŞİM PROGRAMI ÇERÇEVESİNDE ÖĞRETMENLERİN ÖĞRENCİLERİN MATEMATİKSEL DÜŞÜNME BİÇİMLERİNİ FARK ETME BECERİLERİNİN İNCELENMESİ

Baş, Sinem
Doktora, Orta Öğretim Fen ve Matematik Alanları Eğitimi Bölümü
Tez Yöneticisi: Doç. Dr. Ayhan Kürşat Erbaş
Ortak Tez Yöneticisi: Doç. Dr. Bülent Çetinkaya

Aralık 2013, 223 sayfa

Öğretmenlikte mesleki uzmanlığın önemli bileşenlerinden biri öğretmenlerin öğrencilerin düşünme biçimlerine dikkat etmeleri ve anlamaları olarak da tanımlanabilecek fark etme becerileridir. Araştırmalar mesleki programlarla öğretmenlerin öğrencilerin düşünme becerilerini fark etme becerilerini geliştirmenin önemine vurgu yaparken, bir yandan da bu gelişimin nasıl gerçekleştiğini anlamanın gerekliliğine dikkat çekmektedir. Bu çalışmanın amacı Model ve Modelleme Perspektifinin ilkelerine göre hazırlanmış bir mesleki gelişim programı çerçevesinde öğretmenlerin öğrencilerin matematiksel düşünme biçimlerini fark etme becerilerinin değişimini incelemektir. Araştırmada durum çalışması yaklaşımı benimsenmiştir. Çalışma 2011-2012 öğretim yılında iki lisede, dört matematik öğretmeninin katılımıyla gerçekleştirilmiştir. Mesleki gelişim programının uygulaması yedi tane bir aylık dönemden oluşmuş ve her bir dönemde üç tane bir haftalık etaptan geçilmiştir. Bu etaplar sırasıyla başlangıç toplantısı, öğretmenlerin sınıflarında program çerçevesinde hazırlanan modelleme etkinliklerinin uygulanması ve takip toplantısıdır. Her bir takip toplantısından sonra öğretmenlerle birebir görüşmeler de yapılmıştır. Verilerin analizinde öğretmenlerin öğrencilerin matematiksel düşünme biçimlerini fark etme becerilerinin gelişimini incelemek için daha önce hazırlanmış bir çerçeve uyarlanmış ve kullanılmıştır. Analizler sonucunda çalışmada yer alan üç öğretmenin öğrencilerin matematiksel düşünme biçimlerini fark etme becerilerinde sürece yayılan bir gelişim olduğu ortaya çıkmıştır. Çalışmanın bir başka önemli bulgusu da fark etme becerilerindeki bu gelişimin farklı yörüngelerle gerçekleşmiş. olmasıdır. Bu farklı yörüngelerin mesleki gelişim programının öğretmenlerin gelişimi üzerinde etkisi olan bileşenleriyle ilgili ipuçları verdiği düşünülmektedir. Bulguların öğretmenler, öğretmen eğitimcileri ve mesleki gelişim program tasarımcıları açısından sonuçları ve bu alanda yapılabilecek çalışmalar tartışılmıştır.

Anahtar Kelimeler: Öğretmenlerin Fark Etme Becerileri, Modelleme, Mesleki Gelişim, Matematik Eğitimi

ACKNOWLEDGMENTS

First of all, I would like to thank my supervisor Assoc. Prof. Ayhan Kürşat Erbaş for all his support throughout this research study. I also thank my co-advisor, Assoc. Prof. Bülent Çetinkaya, for his support. I owe heartfelt thanks to all the members in the committee, Assoc. Prof. Erdinç Çakıroğlu, Prof. Cengiz Alacacı, Assist. Prof. Ömer Faruk Özdemir, Assoc. Prof. Selma Özçağ for their guidance and understanding.. I am also grateful to all the teachers and colleagues who took part in the project. This study would not run so smoothly without their cooperation. Biggest thanks go to my family who has always believed in me, given me their affection and supported me at good times and bad times.

This study was based on a research project supported by the Scientific and Technological Research Council of Turkey (TUBITAK) under grant number 110K250. Opinions expressed are those of the author and do not necessarily represent those of TUBITAK.

TABLE OF CONTENTS

LIST OF TABLES

TABLES

LIST OF FIGURES

FIGURES

CHAPTER 1

INTRODUCTION

In recent decades, the importance of teachers' eliciting, attending to, and understanding their students'thinking and building their instruction on this information in terms of students' achievement has been highlighted by many research studies in mathematics education (Ball & Cohen, 1999; Sowder, 2007). In the US, one of the standards for teaching mathematics listed by National Council of Teachers of Mathematics (NCTM) is that "the teacher of mathematics should create a learning environment that fosters development of each student's mathematical power by respecting and valuing students' ideas, ways of thinking, and mathematical dispositions" (NCTM, 1991, p.57). Similarly, according to Ball and Cohen (1999), mathematics teachers need to learn to attend to their students with insight in order to teach in a way endorsed by reform movements in mathematics education. Furthermore, many research findings revealed the link between instruction built on students' ways of thinking and increase in students' achievements. For example, literature on Cognitively Guided Instruction program (CGI) indicates that if teachers deeply understand students' reasoning and change their instructional decisions in a manner that is based on this understanding, the gains in students' learning are substantial (Carpenter, Fennema, & Franke, 1996).

However, many studies demonstrate teachers' lack of attending to and understanding what is significant in students' mathematical thinking (Kazemi & Franke, 2004; Koellner-Clark & Lesh, 2003). There are various reasons for teachers' difficulties in understanding students' thinking. Ball (1997) lists her explanations for why teachers find it difficult to pay attention to and make sense of students' thinking. She claims that one reason is that students tend to describe their thinking in ways different from how adults would do. Students also use multiple strategies for solving mathematics problems and explaining their ideas. Another issue Ball highlights is that teachers concentrate on implementing the curriculum in their classrooms and focusing on students' thinking at the same time can be difficult for them. Besides these factors, teachers' knowledge of mathematics, their disposition towards mathematics depending on their prior experiences with mathematics, and their personal beliefs about how mathematical knowledge can be expressed can have strong influence on teachers' difficulties in understanding mathematical ideas that students raise (Goldsmith & Seago, 2011).

In this context, attending to students' ideas with insight should be considered as a kind of expertise and like any other expertise, it should not be expected for teachers to possess inherently (Ball & Cohen, 1999; Sowder, 2007). Hence, teachers need professional development experiences to develop their ability to attend to and understand significant aspects of students' mathematical thinking (Jacobs, Lamb & Philipp, 2010). Research has shown that teachers can improve this ability with appropriate professional support (Goldsmith & Seago, 2011; Kazemi & Franke, 2004; Sherin & Han, 2004; Sherin & van Es, 2009; van Es, 2011; van Es & Sherin, 2008).

However, traditional professional development programs that are generally short term and fragmented do not have influence on changing teachers' practice in the directions mentioned above (Guskey, 1986, 2002; Hawley & Valli, 1999; NCTM, 2003). Although such programs encourage teachers to make specific changes in their teaching practice, these changes generally are not substantial and long lasting (Schorr & Koellner-Clark, 2003; Schorr & Lesh; 2003). Schorr and Koellner-Clark (2003) claim that the nature of the process is the reason why it is difficult for teachers to integrate what they learn in these programs into their actual teaching practice. By referring to a modelling perspective to teacher development, they describe this process as a painstaking one, being initiated by perturbations in teachers' models, followed by a mix of mental activities such as reflection, modification, and resulting in revisions in their models. As a result, such changes in teachers' model would yield certain shifts in their teaching practice. Similarly, Ball and Cohen (1999) highlight the importance of some kind of disequilibrium in teacher learning. Also, similar to Schorr and Koellner-Clark's modelling perspective, NCTM (2003) states that professional development experiences should provide teachers with *transformative* learning that promotes substantial changes in teachers' knowledge and beliefs, rather than *additive* learning in which teachers just attach isolated pieces of knowledge to existing knowledge without any revision in the existing one.

Practice-based professional development is a new professional development approach that has a potential for promoting the changes in teachers' practice in a way that is recommended by reform movements in mathematics education (Ball & Cohen, 1999; NCTM, 2003). Ball and Cohen (1999) point out this new approach by explaining the underpinning ideas as "learning in and from practice" (p. 10). They state that

> Recent reforms challenge this perspective on teaching and its improvement. A great deal of learning would be required for most teachers to be able to do the kind of teaching and produce the kind of student learning that reformers envision, for none of it is simple. This kind of teaching and learning would require that teachers become serious learners in and around their practice rather than amassing strategies and activities. (p. 4)

They also point out some crucial aspects of this professional development perspective based on principles of learning in practice. These features can be summarized as (a) an inquiry-oriented learning environment, (b) a collective endeavor to learning, and (c) situating discussion on concrete artifacts from the classroom. Similarly, NCTM (2003) highlights the importance of practice-based professional development experiences in terms of promoting changes in teachers' practice in a manner that is recommended by reform movements. These experiences provide teachers with a collaborative learning environment where they learn through critiquing, inquiring, and investigating authentic artifacts taken from real classrooms (NCTM, 2003).

Ball (1997) offers three approaches in order to equip teachers with the skills likely to be helpful in dealing with their difficulties in attending to and understanding students' mathematical thinking. One of them is engaging in investigation of artifacts from a real classroom environment such as video clips from a classroom environment, samples of students' written works, or teachers' observation notes on teaching. Findings of relevant research also confirm that such kinds of practice-based professional development experiences show great promise in terms of stimulating development of teachers' ability to attend to and understanding students' mathematical thinking (Ball, 1997; Ball & Cohen, 1999; Chamberlin, 2002, 2005; Goldsmith & Seago, 2011; Kazemi & Franke, 2004; van Es, 2011; van Es & Sherin, 2008).

However, researchers also caution that such kinds of artifacts are necessary but not sufficient for improving teachers' practices. They are raw materials and they should be artfully integrated into professional development programs (Ball & Cohen, 1999; Chamberlin, 2002; Goldsmith & Seago, 2011; NCTM, 2003). Some studies focus on how to design professional development environments for teachers that involve the artifacts that are previously mentioned. For instance, some studies offer various principles for designing learning environments where teachers are investigating together students' written works (Allen, 1998; Blythe, Allen &Powell, 1999). They also introduce two structures that can guide researchers or professional developers along designing such kind of learning environments for teachers. Similarly, Chamberlin (2005) proposes design principles of a professional development context where teachers investigate their students' responses to non-routine mathematical tasks, called Model Eliciting Activities (MEAs). To sum up, when practice-based materials are skillfully integrated into structured, well-organized professional development environments, such kinds of environments have the potential to equip teachers with the skills likely to be helpful in dealing with complex situations in teaching such as attending to and understanding key aspects of mathematical ideas that students raise.

An approach that can help researchers design well organized professional development programs for teachers' attending to student thinking is Multitier Professional Development Design. The underpinning theory is *Models and Modeling*

Perspective (Koellner-Clark & Lesh, 2003). This professional development approach can be considered as an example of a practice-based professional development program focusing on student thinking. Within models and modeling perspective (MMP), activities for all stakeholders in mathematics educators, i.e. students, teachers and researchers, can be designed, which are generally referred to as model eliciting activities (Lesh & Doerr, 2003). Teacher level model eliciting (MEAs) within this perspective can be tailored to address teachers' professional development focusing on student thinking.

1.1 Teacher Noticing

Teacher noticing is a new theoretical construct that many researchers in mathematics teacher education have focused on in order to understand and investigate teaching (Goldsmith & Seago, 2011; Jacobs, Lamb, & Philipp, 2010; Sherin & van Es, 2009; Star & Strickland, 2007; van Es, 2011). These researchers point out that noticing is a fundamental component of teaching expertise in mathematics. The driving idea behind this increasing focus on teacher noticing is to develop ways to equip teachers with the skills necessary to deal with the complex situations in teaching profession in alignment with reform movements. In the literature, there are various conceptualizations of noticing. However, there is a consensus that teacher noticing has two main components, that are (a) paying attention to significant interactions or events in an instructional environment and (b) interpreting these interactions or events by using existing knowledge (Sherin, Jacobs, & Philipp, 2011).

Many research findings point out that noticing of students' mathematical thinking is neither a skill that teachers possess inherently nor something that they acquire only through years of teaching experience and this claim is reinforced by many relevant studies on general aspects of teacher noticing (Blythe, Allen, & Powell, 1999; Rodgers, 2002; Sherin, Jacobs, & Philipp, 2011; Star & Strickland, 2007). Yet, research also revealed that through appropriate professional support, teachers' ability to attend to and understand significant aspects of students' mathematical thinking can develop (Goldsmith & Seago, 2011; Jacobs, Lamb, & Philipp, 2010; Schorr & Koellner-Clark, 2003; Sherin & Han, 2004; Sherin & van Es, 2009; Star &

Strickland, 2007; van Es, 2011; van Es & Sherin, 2008). Content and scope of professional support for teachers' noticing skills can change from one study to another according to which conceptualization of noticing is adopted and participant (Sherin, Jacobs, & Philipp, 2011). However, findings of the studies on teacher noticing reveal that professional development experiences where teachers collectively endeavor to investigate concrete artifacts from their classrooms in an inquiry-oriented way have potential to support teachers' noticing of students mathematical thinking (Chamberlin, 2002; Kazemi & Franke, 2004; van Es & Sherin, 2008). These classroom based artifacts can be video excerpts from the classroom, samples of students' works, or teachers' observational notes from their classrooms.

To sum up, teachers' noticing of, i.e., attending to and understanding of, their students' mathematical thinking and built their instruction on this understanding is considered as crucial in terms of their students' achivements. However, teachers' noticing of students' thinking is difficult for them and they need professional development experiences to improve this skill. Research reveals that teachers can learn to attend to and understand significant aspects of their students' thinking by collectively investigating classroom-based artifacts. For example, in the literature there are many examples of use of students' written works as a classroom based artifacts to improve teachers' learning to notice students' thinking (Chamberlin, 2002).

Yet, what still needs to be improved within research on teacher noticing is a through investigation of how teachers' noticing of students' thinking develops through their examination of students' written work (Sherin, Jacobs, & Philipp, 2011). While doing this, through which mechanicsm teachers examine students' written work is influential on making sense of teachers' professional development. Besides, using a framework that would underpin mathematics teaching and learning processes as well as the dynamics of teacher noticing and that would enable researchers to make sense of teachers' professional development is important. In the context of this study, MMP and multitier professional development program design provide opportunities for designing classroom implementations, research procedure and a framework for making sense teachers' noticing.

1.2 The Purpose of the Study

The main purpose of this study was to investigate nature and development of secondary mathematics teachers' ability to notice students' ways of mathematical thinking in the context of a professional development program, which is based on multitier professional development program design in two basic aspects. Firstly, student-level and teacher-level MEAs, which are the main tools of multitier design, were used to create classroom based artifacts that contain rich elements of students' ways of mathematical thinking and to reveal teachers' thoughts about students' ways of thinking. Secondly, design of teachers' investigation of students' written work was similar to multitier professional development program design (i.e. pre-implementation meeting, implementation of the activity, post-implementation meeting).

1.3 Research Questions

In light of what has been presented, this study is driven by the following research question:

How does secondary teachers' noticing of students' mathematical thinking evolve during a professional development program designed with models and modeling perspective?

While trying to find answers to this question through a detailed collection, analysis and interpretation of data, there will be two sub questions underpinning this overarching question:

1. How does secondary teachers' selective attention of students' mathematical thinking evolve during a professional development program designed with models and modeling perspective?

2. How does secondary teachers' knowledge-based reasoning of students' mathematical thinking evolve during a professional development program designed with models and modeling perspective?

1.4 Significance of the Study

The study of secondary mathematics teachers' ability to notice students' mathematical thinking in the context of a professional development environment with model and modeling perspective is important for several reasons. First of all, teacher noticing is a new research area which promises to address teachers' difficulties in making changes in their practice in accordance with reform documents (Sherin, Jacobs, & Philipp, 2011). The major tenet of this research is teachers' learning from their practice, which is considered as a way of learning the complex task of teaching (Ball & Cohen, 1999). More importantly, findings of many studies on teacher noticing indicated that focusing on improving teachers' noticing skills is important in terms of equipping teachers with the skills that are likely to be helpful in attending to and understanding their students' thinking and building their instruction on students' ways of thinking (Jacobs, Lamb, & Philipp, 2010). This study focused on only first part of these two important skills, i.e. attending to and understanding students' thinking, and the findings of the study will contribute to understanding the extent to which developing teachers' noticing through professional development experiences have influence on their understanding of students' ideas.

A significant feature of this study is its focus on secondary mathematics teachers and their noticing. Literature on teacher noticing is primarily on pre-service, elementary and middle school teachers. But, studies have highlighted a potential negative influence of high school teachers' advanced and structured mathematical knowledge on their abilities to attend to students' ideas in comparison with primary and middle school teachers (Hadjidemetriou & Williams, 2002; Nathan & Petrosino, 2003). Thus, a unique feature of this study is to work with secondary mathematics teachers, which add to the existing body of knowledge in the research area.

Another unique feature of this study is that the design of teachers' investigation of students work is adapted from the multitier professional development program design and model-eliciting activities for students were used. According to Ball and Cohen (1999), classroom-based artifacts are simply raw materials. For providing teachers with efficient professional development experiences, it is necessary to organize professional learning tasks into which these raw materials can be integrated. The findings of this study will offer insights on how teachers' noticing their students' thinking can be improved trough the design of teachers' investigation of students' work adopted from MMP. Similarly, use of MEAs in such design enabled the researcher to create classroom-based artifacts that contain rich elements of students' ways of mathematical thinking. These remarks related to the design of teachers' investigation of students' work have some implications for researchers who are interested in practice based teacher development.

In the literature, research also point out some components of professional development programs that have a potential to influence teachers' noticing of students' thinking such as facilitator's role (van Es & Sherin, 2008), the features of classroom-based artifacts (Goldsmith & Seago, 2011), or teachers' collective work (Kazemi & Franke, 2004). Although investigating contextual factors that have a potential influence on shifts in teachers' ability to notice students' mathematical thinking is not the main focus of this study, the result are expected to yield clues about the influence of such kind of factors on developing teachers' noticing, which contribute to the existing body of knowledge in the research area.

1.5. Definition of Terms

Definitions of the important terms involved in the study are listed below

Teacher Noticing

There are various conceptualizations of teacher noticing, which are presented in the literature review section. The conceptualization adopted in this study is given by Sherin, Jacobs, and Philipp (2011) as: "the processes through which teachers manage the blooming, buzzing confusion of sensory data with which they are faced, that is, the ongoing information with which they are presented during instruction" (p. 5).

Selective Attention

In this study, selective attention refers to what a teacher focuses her/his attention on while investigating students' written works (Sherin & van Es, 2009).

Knowledge-based Reasoning

In this study, selective attention refers to how a teacher reasons about what she/he attends to while investigating students' written works (Sherin & van Es, 2009).

Teacher-level and Student-level Model Eliciting Activities

These are particularly designed activities to "repeatedly challenge both teachers and students to reveal, test, refine, revise, and extend important aspects of their ways of thinking" (Schorr & Lesh, 2003, p. 149). These actitivies both trigger learning and at the same time document how this learning takes place.

CHAPTER 2

LITERATURE REVIEW

The main focus of this study is on nature and development of mathematics teachers' noticing of students' mathematical thinking throughout a professional development program. For this purpose, three relevant areas of literature were reviewed. The first research area is related to teachers' knowledge of student thinking. Research on teacher knowledge in general and teachers' knowledge of student thinking in specific will be discussed. The second research area is related to professional development of teachers. In this section, general features and components of effective professional development programs will be discussed and specifically literature on professional development projects focusing on student thinking to support teacher learning will be presented. In the last part of this chapter, literature on teacher noticing will be presented. Different conceptualizations of noticing will be considered. Then studies of mathematics teacher noticing will be discussed in terms of their differing methodological approaches, findings, and theoretical and practical implications for researchers and teacher educators.

2.1 Teacher Knowledge

What teachers need to know to teach mathematics effectively has been the focus of many studies. There are various conceptualizations of teacher knowledge and its components. One of the well-known and most cited conceptualizations of teacher knowledge is Shulman's (1986) characterization of content knowledge as (a) subject matter content knowledge, (b) pedagogical content knowledge, and (c) curricular knowledge. Subject matter content knowledge refers to "the amount and organization of knowledge per se in the mind of the teacher" (p. 9). Shulman defines this knowledge as not only knowing facts, concepts, and procedures of a domain but also understanding the underlying structures. Futhermore, this knowledge requires teachers to “not only understand that something is so, the teacher must further understand why it is so” (p. 9). He defines pedagogical content knowledge as “dimension of subject matter for teaching”. The dimensions of this knowledge is determined as (a) knowing the ways of representing subject matter in a way that makes it easy for students to understand (i.e., knowing most useful examples, illustrations, etc.), (b) knowing about students’ learning such as their difficulties, misconceptions, and preconceptions and strategies to help students overcome their difficulties and misconceptions. Curricular knowledge is defined as knowledge about a variety of curriculum materials to teach subject matter effectively.

Following Shulman’s conceptualization of three main dimensions of teacher knowledge, researchers have attempted to elaborate on these dimensions for different contexts and subject domains. For example, Grossman (1990) offers four components of pedagogical content knowledge in order to be used in research on professional development of mathematics teachers. These dimensions are (a) knowing the aim of teaching mathematics, (b) knowing students’ ways of mathematical thinking (e.g., students’ understanding, their misconceptions, and preconceptions), (c) knowing mathematics curriculum and alternative instructional materials, and (d) knowing instructional methods to teach some distinct mathematical topics. These characterizations closely resemble Shulman’s subdimensions of PCK.

Within efforts to characterize components of PCK for a variety of subject domains, a relevant example is a group of researchers’ attempt to specify dimensions of PCK for the domain of school algebra. In Artigue, Assude, Grugeon, and Lenfant’s (2001, as cited in Doerr, 2004) study, researchers introduced a framework to elaborate PCK for teachers of school algebra. This framework includes three dimensions of PCK. The first dimension is called the *epistemological dimension*. This dimension refers to knowledge about the content and structure of algebra, the relationship between algebra and the other areas of mathematics, etc. It is similar to Shulman’s subject matter content knowledge. The second dimension is called the *cognitive dimension.* This dimension includes knowledge about students’ algebraic thinking (e.g., knowing about how students interpret, understand some key concept and notations in algebra, what kind of misconceptions and/or difficulties students have in algebra, and how teachers can support students' motivations to learn algebra). This dimension can be considered as an extention of Shulman's dimensions of PCK to the domain of algebra. The third dimension, the *didactive dimension*, resembles Shulman's characterization of curricular knowledge and refers to knowledge of curriculum, effective use of variety of instructional materials.

Ball, Thames, and Phelps (2008) claimed that although Shulman's model of PCK is widely used in research on teacher knowledge, its definition is not clear enough for conducting empirical studies on this issue. Therefore, they extended Shulman's model of teacher knowledge and introduced the *Mathematical Knowledge for Teaching* (MKT). In this model, Shulman's subject matter knowledge is subdivided into *common content knowledge* (CCK) and *specialized content knowledge* (SCK). While CCK refers to mathematical knowledge that is not specific to teaching, SCK is defined as knowledge of mathematics that teachers need to teach mathematics. They also introduce *horizon content knowledge* as a subdimension of subject matter knowledge and define it as "awareness of how mathematical topics are related over the span of mathematics included in the curriculum" (p. 403). Ball and her colleagues (2008) extend PCK and divide it into three subcategories. The first one is called *knowledge of content and students* (KCS) and it is defined as knowledge which embodies elements from both content and students. According to this definition, Ball et al. (2008) describe KCS as a major driving force behind many teaching actions that teachers need to be carrying out.

> Teachers must anticipate what students are likely to think and what they will find confusing. When choosing an example, teachers need to predict what students will find interesting and motivating. When assigning a task, teachers need to anticipate what students are likely to do with it and whether they will find it easy or hard. Teachers must also be able to hear and interpret students' emerging and incomplete thinking as expressed in the ways that pupils use language. Each of these tasks requires an interaction between specific mathematical understanding and familiarity with students and their mathematical thinking. (p. 401)

All these actions point to the crucial role of having knowledge about students and their mathematical thinking together with an understanding of mathematical content. Furthermore, similar to one of the components of PCK, teachers' knowledge of conceptions and misconceptions about specific mathematical content students have commonly is considered as one of the subdimensions of KCS. The second subdimension of Ball et al.'s conceptualization of PCK is *knowledge of content and teaching* (KCT) which is at the intersection of knowledge of mathematics and knowledge of how to teach it. It refers to knowledge about the design of instruction (e.g., knowledge of the sequence in which a particular content is presented, which examples, representations, teaching strategies should be selected to teach a particular content). Lastly, they introduced *knowledge of curriculum* which is similar to Shulman's curricular knowledge.

To sum up, there are various conceptualizations of teacher knowledge and its dimensions. Particularly, Shulman's introduction of pedagogical content knowledge made valuable contribution to research on the nature and development of teacher knowledge (Petrou & Goulding, 2011). However, to conduct empirical research on teacher knowledge and to use it in different domains, more specific characterization of the components of PCK has been needed to. As previously discussed, researchers introduced a variety of different models of PCK (Ball, Thames, & Phelps; 2008; Fennema & Franke; 1992; Grossman, 1990). In all of these models, teachers' knowledge of student thinking is commonly considered as one of the main components of PCK. Shulman claimed that "it should be included at the heart of our definition of needed pedagogical knowledge" (p. 10). Similarly, Kieran (2007) highlighted the importance of teachers' knowledge of students' ways of thinking in terms of teaching expertise endorsed in reform documents.

In publications of NCTM, it has been highlighted that attending to and listening carefully to students' ideas and building the instruction based on this information are crucial components of an effective teaching (NCTM, 2003). In *Professional standards for teaching mathematics,* it is stated that "developing multiple perspectives on students as learners of mathematics enables teachers to build an environment in which students may learn mathematics with appropriate support and acceptance" (NCTM, 1991, p.144). Similarly, Schifter (2001) pointed to the crucial role of teachers' attending to mathematical ideas that students raise, in terms of acquiring necessary mathematical skills necessary for effective teaching as follows:

> Additional mathematical skills needed for teaching may evolve not from a focus on mathematical content but from attending to the mathematics in what one's students are saying and doing, assessing the mathematical validity of their ideas, listening for the sense in children's mathematical thinking even when something is amiss, and identifying the conceptual issues on which they are working. (p. 65)

Correspondingly, in Ball and Cohen's (1999) view, one of the things that teachers need to know for effective teaching is learning to attend to students ideas with insight. Consequently, besides being one of the main components of teacher knowledge in different conceptualizations, mathematics educators underscore the importance of teachers' knowing and anticipating students' ways of thinking and building their instructions on this information in terms of gains in student achievement (Carpenter, Fennema, & Franke, 1996; Jacobs, Lamb, & Philipp, 2010; Sowder, 2007). Chamberlin (2002) offered explanations of some benefits of teachers' attending to and understanding students' ways of thinking. Firstly, when teachers have knowledge about their students' mathematical thinking, they can be able to make more appropriate decisions on mathematical tasks. More importantly, teachers create a learning environment where students' conceptual understandings are elicited. Also, teachers can focus on problem solving activities where students take an active role.

However, research on teachers' knowledge of student thinking indicates that teachers have difficulties in attending to, anticipating, and understanding students' ways of mathematical thinking. In the next section, the findings of these studies and different perspectives on the reasons for these difficulties will be discussed.

2.1.1 Teacher Knowledge of Student Thinking

Findings of research on teachers' knowledge of students thinking revealed teachers' lack of ability to anticipate, attend to, and understand students' ways of thinking. As mentioned in the previous section, in Ball, Thames, and Phelps's (2008) conceptualization of KCS, teachers should be able to predict students' ways of thinking, their confusions, conceptions, misconceptions, and their difficulties. This is important because teachers' predictions and expectations of students' thinking influence their perceptions of students' thinking during the instruction and their instructional decisions (Nathan & Koedinger, 2000a).

Research on teachers' knowledge of student thinking revealed that there are considerable gaps between students' actual performance, difficulties, misconceptions and teachers' predictions about them (Kieran, 2007). For instance, in Nathan and Koedinger's (2000a) study with high school teachers, it was revealed teachers' predictions about story problems and word-equation problems being more difficult than symbol-equation problems contrasted with students' actual difficulties with symbolically presented problems. In addition, students' problem solving approaches were systematically different from those predicted by teachers. In a follow-up study, Nathan and Koedinger (2000b) added some points to these findings in terms of students' problem-solving strategies. This follow-up study indicated some specific ways that students' algebraic reasoning differed from that predicted by most teachers. For example, while the teachers anticipated that symbolic problem-solving developed prior to verbal reasoning for students, in reality, students' verbal competence preceded their symbol-manipulation skills for algebraic problems.

In a similar study, Nathan and Petrosino's (2003) found that high school teachers predicted students' problem-solving difficulties more inaccurately than middle school teachers. They contended that well-developed knowledge of subject matter could lead high school teachers to assume that learning should follow the structure of the subject-matter domain. Since high school teachers have more advanced mathematics education than middle school teachers, they view symbolic reasoning and mastery of equations as a necessary prerequisite for word equations and story problem solving.

Similarly, Hadjidemetriou and Williams's (2002) study on teachers' knowledge of students' algebraic thinking pointed out discrepancies between teachers' expectations of students' difficulties and students' actual difficulties. Like the previous study by Nathan and Petrosino, this study provided evidence to confirm that teachers' anticipations of students' difficulties were shaped by their mathematical knowledge and by the curriculum rather than their knowledge of students.

Another aspect of the gap in teachers' knowledge of students' thinking is related to teachers' views about students' level of algebraic thinking. In a case study, Bergqvist (2005) revealed that teachers underestimated students' reasoning levels in the conjecture tasks. Although students had a capability of abstract reasoning in mathematics, teachers were inclined to think that only a few students are able to use high level reasoning in mathematics (Bergqvist, 2005). Teachers' tendencies to underestimate students' mathematical reasoning are also confirmed by other studies. For example, in Goldsmith and Seago's (2011) study, it was revealed that teachers had a tendency to assess students' solution approaches to algebra problems in terms of whether students expressed their responses by using formal algebraic notations. They underestimated students' reasoning by comparing students' responses against what they had in their mind as a correct response. Similarly, in Kazemi and Franke's (2004) study, teachers had tendency to underestimate students' performances in solving mathematical problems. As they investigated a variety of students' solution strategies and learned to attend to details of the strategies, they expressed their surprise and appreciation in realizing the extent to which students' mathematical reasoning can be sophisticated.

Another subdimension of KCS is described as teachers' ability to hear and interpret students' mathematical ideas expressed in an incomplete and unstructured ways. However, findings of research highlighted teachers' difficulties and lack of capabilities in attending to and understanding students' thinking (Chamberlin, 2002; Jacobs, Lamb, & Philipp, 2010; Kazemi & Franke, 2004; Wallach & Even, 2005). In their study, Kazemi and Franke (2004) attempted to research teachers' collective investigation of their students' solution strategies to solve mathematical problems.

Findings of this research showed that teachers were unaware of details of the strategies and they had difficulties in understanding students' mathematical reasoning. Similarly, in Goldsmith and Seago's (2011) study, researchers investigated how teachers identify and interpret students' mathematical thinking revealed from classroom based artifacts such as samples of students' works and video excerpts, and focus on aspects of students' thinking in the classrooms. Findings of this research revealed teachers' lack of capability to identify mathematically significant details of students' works and interpret underlying mathematical ideas.

The findings of research on teachers' ability to notice students' mathematical thinking, also point out prospective and practicing teachers' lack of ability of attending to and understanding what is significant in students' mathematical thinking (Sherin & Han, 2004; Sherin & van Es, 2009; van Es, 2011; van Es & Sherin, 2008). For example, Sherin and Han (2004) investigated teachers' learning of attending to and understanding significant interactions in a complex classroom environment in a context of video-based professional development program. Findings revealed that, at the beginning of the program, teachers did not attend to students' mathematical thinking and they tended to focus on other issues such as the teachers in the video and what he/she was doing. Extensive works of Sherin and van Es on teacher noticing confirmed teachers' lack of ability to attend to significant aspects of students' mathematical thinking and make sense of underlying meaning. For example, in van Es and Sherin's (2008) study, teachers' learning to notice students' mathematical thinking in a classroom environment was investigated throughout a series of video-club meetings. Findings of the study showed that teachers were inclined to pay attention to range of issues (e.g., climate in the classroom, classroom management) other than mathematical ideas that students raised in the class. Also when they focused on students' mathematical thinking, their descriptions did not refer to mathematically significant aspects of students' thinking. Similar findings were found in van Es' (2011) study where she investigated developmental trajectories of teachers' learning to notice student thinking in the context of a video-based professional development program. The findings of the study confirmed that at the beginning of the program teachers focused on issues regarding classroom environment or instructional approach of the teacher and did not pay attention to mathematical ideas students shared and discussed. Even though they were prompted to look at specifics of students' solution strategies revealed from the video, they tended to make general comments that did not point to key aspects of students' thinking.

To sum up, all these studies point out teachers' difficulties in attending to and understanding students' mathematical thinking. Ball (2001) discussed several reasons why interpreting students' thinking might be challenging for teachers. The reasons were (a) students often use expressions different from those used commonly by adults in representing their thinking, (b) teachers can not easily find a balance between assuming too much to fill in the gaps of students' expressions and being too skeptical about what students are expressing. Thus they might be distorting students' expressions of their mathematical ideas by being positioning themselves towards one of the two approaches mentioned above, and (c) handling too many student ideas and strategies might be overwhelming for teachers. In the same way, Wallach and Even (2005) introduced the term *hearing through* to explain how it is difficult for teachers to hear what students really say by isolating their personal and social resources. Similarly, Confrey (1993) pointed to the importance of teachers' ability to isolate their own perspective and be able to think from students' perspectives. He also pointed out what kind of knowledge teachers need to have to be capable of attending to students' mathematical reasoning. In Confrey's view, content knowledge that teachers had learned in traditional ways is not enough to see and interpret students' sophisticated reasoning. He recommended that teachers should revise and refine their content knowledge by thinking from students' perspectives.

Rodgers (2002) claimed that teachers' difficulties in attending to students' thinking might ensue from their tendency to make quick evaluations and interpretations related to students' thinking. The findings from teachers' noticing of students' thinking confirmed this claim (Kazemi & Franke, 2004; Sherin & van Es, 2009; van Es, 2011; van Es & Sherin, 2008). In Rodgers's opinion, as opposed to what is generally assumed, making a specific, detailed and careful description is much more difficult than making an interpretation and/or evaluation. In the *reflective cycle framework*, which was created and used in teachers' professional development studies by her, she particularly focused on improving teachers' ability to carefully attend to and describe student learning. Through this reflective cycle, Rodgers introduced *the discipline of description* for teachers in order to help them slow down and see details and nuances before making quick judgments. According to Blythe, Allen, and Powell (1999), this is mainly related to pressures of the teaching profession. They claimed that since teachers feel the pressure from a curriculum that needs to be covered, they take on a responsibility to quickly identify any student outcome as good or bad, and make corrections whenever necessary.

Ball and Cohen (1999) stated that attending to students' ideas with insight should be considered as a kind of expertise and like any other expertise, it should not be expected for teachers to possess inherently. Teachers need professional supports to acquire this expertise. This claim is also reinforced by many relevant studies on teacher knowledge of student thinking (Jacobs, Lamb, & Philipp, 2010). These studies revealed that teachers can gain more insight into students' mathematical thinking through professional development programs organized with the principles of training teachers through investigating and interpreting student thinking (Carpenter, Fennema, & Franke, 1996; Koellner-Clark & Lesh, 2003; Sherin, Jacobs, & Philipp, 2011; Sowder, 2007).

In the next section, literature review on some examples of professional development projects focusing on student thinking will be presented. Their main purposes, how they were designed, the instructional tools they used, and their main findings on participating teachers' learning of attending to and understanding students' mathematical thinking will be discussed. However, before presenting these studies, literature on professional development programs will be presented. In general, essential components of an effective professional development program will be highlighted. Subsequently, different professional development approaches will be discussed. This section will be completed with a presentation of professional development programs that specifically focus on student thinking.

2.2 Teacher Education and Professional Development

Educational perspectives are changing and questions such as what mathematics is, how it should be taught and how mathematical knowledge can be evaluated are being discussed for plausible answers within varying approaches across the world. Standards published by NCTM (1989; 2003) are consequences of such efforts and they guide reform movements at both national and international levels. Knowledge and skills required for mathematics teachers have also changed within such new approaches to mathematics education. This necessitated a reconsideration and transformation of the current teacher education programs in order to train teachers equipped with the desired qualities. For example work has been initiated to replan the teacher education programs in Turkey in 1996 and radical changes have been implemented (Koca, Yaman, & Şen, 2005). Meanwhile, in-service teacher training programs that would enable the professional development of teachers to implement the reformed curricula, needed to be reconsidered.

Traditional models of professional development are known to be inadequate towards this aim (Guskey, 1986, 2002; Hawley & Valli, 1999; Sowder, 2007). Research in this area indicates various reasons why effects of in service training on teachers' professional development are far below the expected levels. According to Schorr and Koellner-Clark (2003), who approach this issue from a modelling perspective, why such programs are inadequate in making modifications in teaching practices is because of lack of provision of learning environments enabling them to display their conceptual models regarding such approaches, to test and develop them. Teachers' affective readiness is another factor influencing this process. Efforts to enroll teachers into professional development programs are claimed to be unsuccessful unless factors motivating teachers' participation or directing them to avoidance are taken into consideration (Guskey, 1996). For example teachers are known to show resistance against knowledge and skills that are expected to be acquired in short term in service training sessions, which take place in settings and conditions non-similar to theirs and which require allocation of long time beyond teachers' regular working hours (Hargreaves, 1995). To create changes in teachers' instructional practices following their engagement in professional development programs, these programs should have some characteristics different from traditional programs. Describing essential characteristics of an effective professional development program has been the focus of many teacher educators and researchers. In the following section, some of these characteristics will be reported.

2.2.1 Aims and Essential Properties of a Successful Professional Development Program

Even though enterprises to increase the quality of teacher training programs bring some positive outcomes, often pre-service teachers are unable to carry their knowledge, skills and attitudes from their training periods to their workplaces upon starting to teach. The ideas about teaching and being a teacher formed through observations during their years as a student are deep rooted and this issue prevents them from internalizing the desired knowledge, skills and attitudes targeted by their teacher training. Prejudices and beliefs from these early observations prevent them from developing their idiosyncratic teaching approaches and strategies, or in other words their personal teaching identities (Masingila & Doerr, 2002; Pajares, 1992).

This issue is also valid for in-service teacher training which an important part of professional development. For teachers, on top of experiences from their years as students, certain knowledge, attitude and beliefs acquired during their professional experiences might cause them to show resistance against change. In a world constantly changing by scientific and technological renovations, teachers educating future's workforce need to have an ongoing and continuous training process. Yet the quality of this training process is as important as its duration and continuity. According to this, the most important outcome and success criteria of a professional development program is the change in teachers' beliefs, knowledge and teaching methods (Guskey, 2002; Sowder, 2007). Such changes are reflected in issues related to teaching, such as selection of teaching and learning materials, lesson planning and classroom management for supporting and motivating all learners; hence directly influence students' learning processes. On a similar line of thought, Borko and Putnam (1995) claimed that professional development programs enabling teachers to develop knowledge and new ways of thinking about learning and students, and thus empowering their teaching skills can be considered successful.

For a professional development program to be successful, its aims need to be established very clearly (Haley & Valli, 1999). These aims need to be shaped according to the learning experiences of teachers who are expected to adopt a teaching approach parallel to the vision of reform movements in mathematics education (Sowder, 2007). Professional experiences that teachers need, have been explained by various researchers (Ball & Cohen, 1999). According to Sowder (2007), who posited 6 main aims of a professional development program in light of previous studies in the field, such programs need to support teachers in:

> a) developing a shared vision regarding mathematics teaching and learning, b) having profound mathematics knowledge at the level they are teaching, c) understanding how students learn mathematics, d) strengthening their pedagogical content knowledge, e) embracing the role of equity in education, and f) forming their own identities as mathematics teachers. (p. 161)

Results of studies investigating the features of programs that have given positive outcomes about the ultimate aim in education, improving student success and the factors that feed such success, display some criteria that could guide educators in designing professional development models. Haley and Valli (1999) and Elmore (2002), have analysed successful professional development programs and listed various features that stood out. According to this, the aim of the program needs to be decided well and the teaching style expected from teachers need to be modelled well. Some other outstanding features of such programs are getting related parties' approval and support while making changes in teaching approaches and strategies, providing teachers with different and rich learning environments in a cooperative setting, and giving teachers the feedback required for their development. According to Wilson and Berne (1999) teachers should be given the opportunities for diagnosing their own needs and their opinions need to be taken while planning professional development programs. Some other important features are about the learning environments provided for teachers. In successful professional development programs, teachers need to be in active learner positions just like students, communicate constantly with other teachers. Aslo, in a cooperative learning environment where risk taking and being critical are encouraged, they need to be constructing their content knowledge and pedagogical content knowledge (Knapp, 2003). On top of these, making the program more intensive and establishing its continuity makes the program more likely to bring success (Elmore, 2002a; Haley & Valli, 1999).

After considering what kind of features that a professional development program has to have in order to be successful in teacher learning, another important aspect of teacher professional development that is noteworthy for discussion is which context and strategies are used to provide teachers with an effective professional development experiences. In the literature, there are some professional development approaches different in terms of their context and strategies. In the following section, these different approaches will be mentioned.

2.2.2 Approaches to Professional Development

Ball and Cohen (1999) introduced a new perspective on professional development where the underpinning idea was "learning in and from practice" (p.10). Further, they stated that

> Recent reforms challenge this perspective on teaching and its improvement. A great deal of learning would be required for most teachers to be able to do the kind of teaching and produce the kind of student learning that reformers envision, for none of it is simple. This kind of teaching and learning would require that teachers become serious learners in and around their practice rather than amassing strategies and activities. (p.4)

Further in their work, they pointed out some crucial aspects of this new professional development perspective based on the principle of learning in practice. These can be summarized as (a) an inquiry-oriented learning environment, (b) collective endeavor to learning, and (c) situating discussion on concrete artifacts from the classroom (e.g., students written works, videotaped of classroom lessons, teachers' observation notes, etc.). Similarly, NCTM (2003) also advocated practice-based professional development experiences where teachers learn through critiquing, inquiring and investigating authentic artifacts taken from real classrooms.

Sowder (2007) categorized professional development approaches in the literature according to their focus, strategies, and tools. He offered four categories: a) approaches based on student thinking, b) approaches based on case studies, c) approaches based on curriculum, and d) approaches based on formal coursework (p. 173). Sowder stated that even though in these approaches different strategies and tools are used, an inquiry-oriented learning are commonly advocated, which is one of the three components of practice-based professional development perspective. In the following paragraphs approaches based on case studies, curriculum, and formal coursework will be mentioned briefly. Then, professional development approaches focusing on student thinking will be discussed with examples from the literature.

In professional development approach focusing on case studies, teachers' learning occurs through studying written cases or video cases from the classrooms. In this approach, teachers find opportunities to think on and discuss various teaching practices through case studies in a cooperative environment at the presence of an expert's guidance and ultimately analyse their teaching practices critically (Barnett, 1991, 1998). This professional development approach widely used in the fields of law and medicine, has recently been popular also in teacher education since it supports learning through practice. The underlying motive here is the recent consensus that social learning theories are more suited to making sense of teachers' learning processes, which have been done by adopting cognitive learning theories until recently (Lerman, 2001). On top of Piaget's individual and cognitive approach, Vygotsky's thoughts defining learning from a sociocultural perspective and placing social interaction and communication at the heart of learning, form the basis of professional development programs focusing on social learning. According to Lave and Wenger's (1991) situated learning definition developed from Vygotsky's thoughts on learning's social and cultural dimension, learning is context-bound and it can not be transferred from one context to the other. In other words, nature of knowledge is directly related with how it is constructed and how it is used. On a similar line of thought, Cooney (1999) posits that pre-service teachers need to observe the context in which teacher knowledge is developed and used in order to make sense of teaching practices. In the case study approach, which is the best practical manifestation of such theoretical underpinnings, teachers are provided with social learning environments through case studies based on written cases, video cases or multimedia cases. This teacher training strategy is often used for pre-service teachers who have limited opportunities for applying their theoretical knowledge in classroom environments and hence lacking the practical knowledge constructed within a context (Lampert & Ball, 1998; Masingila & Doerr, 2002). It is also an important professional development approach for teachers since it allows developing skills for analysing various teaching practices, analysing and evaluating alternative approaches, critiquing issues collaboratively with colleagues and questioning their own practices in such a collaborative environment.

Professional development approach focusing on the curriculum is based on the principle that teachers' learning of content they teach can be supported by using curriculum materials (Sowder, 2007). Even more than the quality of the developed learning materials, how teachers use these materials is crucial. Remillard (2000) claimed that textbooks prepared according to such renovated programs would not bring about change unless teachers are trained on how to use these books effectively. Curriculum-based approach aims at developing teachers' such skills and consequently assumes that by developing these skills teachers' content and pedagogic content knowledge will increase (Sowder, 2007). In this context, programs and instructional materials (textbooks, computer programs, etc) developed as a result of these reform movements in mathematics education can be considered as a learning environment that supports teachers' professional development. Yet, practical uses of these approaches might differ even though they are based on the same assumption. For example, some researchers use problems specifically selected for helping teachers make sense of the nature of mathematics and some important mathematical concepts, and develop their problem solving skills, as a learning environment (Borasi, Fonzi, Smith, & Rose, 1999). Some others try to contribute to teachers' professional development by enabling teachers to reconsider the mathematics program they are using in their classrooms and the mathematics they are teaching from a different perspective (Sowder, Armstrong, Lamon, Simon, Sowder & Thompson, 1998). Formal course work approach refers to supporting teachers' professional development through master's degree programs or certificate programs. Sowder (2007) recommended that when such kinds of programs are organized, main principles of professional development should be considered.

2.2.2.1 Professional development approaches focusing on student thinking

Sowder (2007) stated that one of the goals of professional development programs should be "developing an understanding of how students think about and learn mathematics" (p.163). Furthermore, he pointed out that besides being a goal of professional development programs, attending to and understanding student thinking were also used as a strategy in many professional development programs. Professional development approaches focusing on students' thinking are based on the principles of training teachers through investigating and interpreting students' ways of thinking gathered from research studies or from actual students' works. Furthermore, the main target of the programs based on this approach is to encourage teachers to make changes in their instructional practices in a way that supports students' learning. Studies have emphasised the importance of teachers having knowledge about students' different ways of thinking and more importantly, the need for shaping their knowledge, beliefs and teaching plans according to their knowledge on students' ways of thinking (Carpenter, Fennema, Peterson, Chiang, & Loef, 1989; Fennema, Franke, Carpenter, & Carey, 1993; Nathan & Koedinger, 2000a; 2000b). Moreover, Carpenter, Fennema and Franke (1996) claimed that if teachers understood students' thinking, this would help them develop their pedagogic knowledge and content knowledge.

Students' written works are commonly used artifacts in the professional development programs focusing on student thinking (Ball, 1997; Ball& Cohen, 1999; Blythe, Allen, & Powell, 1999; Chamberlin, 2002; Goldsmith & Seago, 2011; Koellner Clark & Lesh, 2003; Kazemi & Franke, 2004). Ball (1997) stated that "Investigating students' work hold promise for equipping teachers with the intellectual resources likely to be helpful in navigating the uncertainties of interpreting students thinking" (as cited in Chamberlin, 2004, p.52). According to NCTM (2003), students' written works give insight into students' mathematical thinking. Moreover, examining students' work can serve many purposes. For example, it expands teachers' view of what students can do when given the opportunity. Similarly, in Little's (2004) view, students' work is a good artifact for triggering teacher inquiries and subsequently teacher communities for inquiries can be built around students' work.

However, literature supporting the use of students' works as an artifact from real classroom highlighted some important points that should be considered. First of all, the task that is used to produce student work should be selected carefully (NCTM, 2003). In the context of an in-service professional development program targeting teacher training through working on students' thinking, it can be claimed that non-routine problems provide teachers with a rich environment in which they can focus on students' ways of thinking (Chamberlin, 2004). These problems should be based on significant mathematics and they should stimulate students' mathematical reasoning and problem solving abilities (NCTM, 1991).They should provide students with an environment where they think about a real life situation rather than following predetermined procedures (NCTM, 2003). Secondly, researchers in favor of organizing professional development experiences using students' works caution that, such kinds of materials are raw materials and they are not self-enacting (NCTM, 2003). They should be provided to teachers through well-organized "professional learning tasks" (Ball & Cohen, 1999; p.27). Similarly, Goldsmith and Seago (2011) refer to the importance of using such kinds of classroom-based artifacts artfully in order for professional development experiences around these materials to be effective. Researchers and teacher educators offer some principles of how such kind of professional learning tasks can be organized (Allen, 1998; Ball & Cohen, 1999; Blythe, Allen, & Powell, 1999). According to Allen (1998), although purposes of investigating students' works can vary, the investigation processes share some common principles:

- They involve collaboration, bringing multiple perspectives to bear in a focused inquiry.

- They move beyond grading and evaluation of the work to discussion that contributes to teachers' understandings of students' learning and their own instructional practice.
- Participants agree upon structure and norms for the inquiry (p.3)

These principles of investigating students' works correspond with Ball and Cohen's (1999) general principles of practice-based professional development that were mentioned previously. Similarly, Blythe, Allen, and Powell (1999) stated that such kinds of investigation process should be conducted by considering these principles. They also offered processes for designing professional development environments where teachers learn to attend to and understand student thinking through collaboratively investigating students' works. Figure 2.1 shows steps in the design of an investigation processes. Accordingly, in the first step, how teachers examine their students' works in their daily practice should be revealed to adapt it to the design. Then the purpose of investigating students' works should be clearly defined and relevant questions to guide the investigation process should be determined. Afterwards, a procedure should be selected to implement to achive the predetermined purposes. They introduced two examples of such kind of procedures called *The Tuning Protocol* and *The Colloborative Assessment Conference* (p. 11). After the implementation of the selected procedures, reflective discussions on the process should be held. As it can be seen in the figure, this process is cyclical. In the light of feedbacks received form the reflective discussions, the goals and the processes to achive these goals are revised and refined.

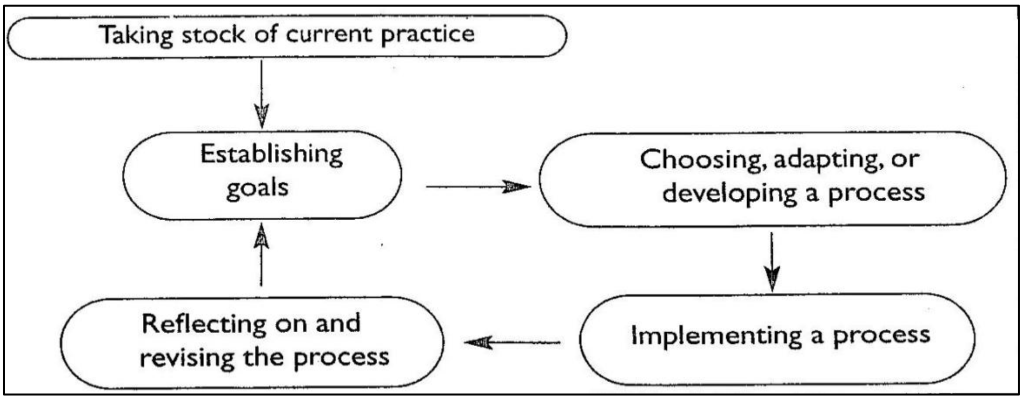


Figure 2.1. A process of designing teachers' investigation of students' works (Blythe, Allen, & Powell, 1999, p. 10)

Besides these general principles and steps in designing a professional development environment based on teachers' investigation of students' works, how the processes of investigation occur is another relevant issue for this study. In the literature, there are some suggestions of this investigation processes (Chamberlin, 2002, 2004; Koellner-Clark & Lesh, 2003; NCTM, 2003). In all these studies, the process of teachers' investigation of students' work are described in very similar ways. Accordingly, this process starts with teachers' work on a mathematical task. They solve the task and share their different solution approaches. Also they discuss underpinning mathematical ideas the task is based on. Then some teachers implement the task in their classrooms and collect students' works. After the implementation, teachers examine students' solution approaches and discuss underlying students' mathematical thinking and understandings. Teachers can use this information about student' ways of thinking in making plan for their instruction, designing mathematical tasks, or creating a tool to use in their teaching practices. As Allen (1998) and Blythe, Allen, and Powell (1999) recommend previously, teachers' investigation of students' work should have some purposes and how teachers may use this knowledge of their students' thinking should be determined at the beginning of the investigation.

In the literature, there are some examples of professional development programs based on the principle of teacher learning through investigating students' works. In the following paragraphs, these programs will be discussed by pointing out their design of investigations, the instructional tools that were used, and the main findings in terms of teacher learning.

Multitier Professional Development Design and Model-Eliciting Activities

Multitier professional development design is a professional development approach that is based on the principles of *Models and Modeling Perspective* (Koellner-Clark & Lesh, 2003). According to this perspective, Doerr and Lesh (2003) described conceptualization of teachers' learning and development as "creation and continued refinement of sophisticated models" that enable them to see and interpret teaching tasks such as "knowing how students learn important mathematical ideas about numbers, ratios, functions, selecting appropriate activities and curriculum materials that support further development of students' ideas" (p. 126). This professional development approach can be considered as an example of a practice-based professional development program focusing on student thinking. The main purpose of this program is to promote teachers' attending to and understanding students' modeling behaviors. The classroom-based artifacts around which the program is organized are students' written works that include their responses to non-routine mathematical tasks, called model-eliciting activities (MEAs). These special kinds of activities are created by using design principles of modelling perspectives and used for both research and education purposes (Lesh, Hoover, Hole, Kelly, & Post, 2000). One of the most prominent features of these activities is that they provide opportunities for students to create their own unique solutions in an open ended thinking environment, rather than using problems with a predetermined correct solution and having strict formulas and steps to be followed. Through this feature, they elicit students' different ways of thinking and provide opportunities for the teachers to analyse and make sense of their thinking (Lesh & Doerr, 2003).

In the multitier design, teachers first work on a model-eliciting activity in groups. Then, they implement the activity in their classrooms. Following the implementation, they investigate their students' works and based on this information they create a tool in order to use in instruction (e.g., concept map, student thinking sheet). Koellner-Clark and Lesh (2003) claim that this design creates a learning environment where student learning and teacher learning occur simultaneously and in a similar way; that is through developing "models" by engaging in student level and teacher level model-eliciting activities. In a context of in-service professional development program targeting teacher learning through working on students' thinking, MEAs provide teachers with a rich environment in which they can focus on students' ways of mathematical thinking (Chamberlin, 2002, 2004, 2005; Doerr & Lesh, 2003; Hjalmarson, 2004; Koellner-Clark & Lesh, 2003; Schorr & Koellner-Clark, 2003). Literature on this professional development program design indicate that it encourages teachers to engage in mini-inquiries, where teachers' inquiry into students' mathematical thinking are revealed by MEAs. Findings revealed that, these mini-inquiries enable teachers to overcome their challenges in attending to and understanding students' thinking (Chamberlin, 2005; Zawojewski, Chamberlin, Hjalmarson, & Lewis, 2008). This multitiered design also enables researchers or teacher educators to investigate how teachers' ways of interpretations of students' mathematical thinking are revealed, revised and refined throughout a series of model-eliciting activities. Therefore, in this study, for the purpose of investigating teachers' abilities to notice students' mathematical thinking, this professional development design was adopted. In studies of mathematics teacher noticing, various kinds of professional development approaches and tool are used to support teachers' noticing skills. This research area will be discussed in the next section.

Cognitively Guided Instruction (CGI)

Literature on Cognitively Guided Instruction program have shown that teachers' instructional practices can be changed by providing them with well-organized information about children's mathematical thinking revealed by arithmetic story problem solving processes (Carpenter, Fennema, Peterson, Chiang, & Loef, 1989). The driving idea behind CGI is that knowledge derived from research on students' mathematical thinking, to which CGI teachers are given access, can be used by teachers in instructional decisions that affect learning outcomes. In other words, if teachers understand deeply students' reasoning, and change their instructional decisions in a manner that is based on this understanding, the gains in students' learning are substantial.

The work of the CGI project focuses on helping teachers understand how children think about mathematical ideas and how those ideas develop in children. While the focus on pedagogical content knowledge generally seems to emphasize how the knowledge of pedagogy needs to take into account aspects of the content domain, this project focuses on how the knowledge of content needs to take into account aspects of the pedagogical domain in order to understand nature and development of teacher knowledge. The purpose of CGI studies is not to provide teachers with a body of knowledge that they would assimilate directly. Rather, the main purpose is to present teachers with a framework providing a detailed analysis of development of children's mathematical thinking to construct a well-organized knowledge base.

From this explicit purpose, it can be noticed that another assumption of CGI is that teachers construct their own models of students thinking (Carpenter, et al., 1996).
A number of experimental research studies have been conducted to test some hypotheses on possible changes in CGI teachers' beliefs about teaching and learning, and consequently their instructional decisions. A defining example is a study conducted by Carpenter, Fennema, Peterson, Chiang, and Loef (1989). The results of this study revealed that teachers given opportunities to access more specific and well-organized knowledge about their students' thinking differ significantly from other teachers in terms of their instruction and their students' achievement (Carpenter, et al., 1989).

Performance Assessment for Middle School Mathematics

Another teacher professional development program that used data and artifacts produced in teachers' own classrooms was carried out in Philadelphia, US (Katims & Tolbert, 1998). In their program, this group of researchers focused on teachers' integration of assessment and instruction through engaging the teachers on work with specifically designed activities and the implementation of these activities in their classrooms. This study had a similar methodological design in terms of the investigation process. Teachers were first involved in a workshop, where they solved and discussed an activity, which was followed by the actual implementation of the activity in their classrooms and further discussions of what took place in another workshop among teachers. Teachers used students' work from the implementation as the main source of discussion in the second workshop. This process of investigation with the teachers proceeded in cycles throughout the study. The main finding of this particular professional development program was that the teachers started to attend to students' mathematical ideas, as opposed to them focusing on covering the content and various classroom management issues at the beginning of the study. Teachers' self report data at the end of the study also indicated changes in their attitudes towards students' capabilities to produce sophisticated mathematical ideas.

Integrating Mathematics Assessment (IMA)

The idea of designing a school-based professional development program for teachers to assist their development for understanding students' mathematical ideas and learning how to use this understanding in instruction was also addressed in a project called Integrating Mathematics Assessment (Saxe, Gearhart & Nasir, 2001). Yet, this particular study focused on improvements in students' conceptual understanding of mathematics as influenced by developments in teachers' practices. The researchers, in a similar way to many studies discussed above, came up with a program design where teachers were supposed to go through cycles of stages. Throughout these cyles, one of the most important ideas was teachers to make sense of students' mathematical thinking and to use various ways to assess students' thinking during their implementation of the activities within the context of the study. The results indicated that students taught by teachers included in the program made more progress in both conceptual and procedural understanding in mathematics than students in comparison groups whose teachers did not get such support. The resarchers concluded that teachers supporting students through paying more attention to students' thinking was central for maintaining student progress in mathematics (Saxe, Gearhart & Nasir, 2001).

To sum up, all these professional development programs point out benefits of teachers' collective endeavor to investigate students' works in terms of the teachers' gaining more insights into their students' thinking and more importantly change in their instructional practices in a way that affects students' learning outcomes. Furthermore, the methodological designs of teachers' investigations are similar and compatible with the general design principles previously mentioned. Hence, they provide researchers or teacher educators with insights into the design of professional development environments to support teachers' ability to notice students' mathematical thinking. However investigation of how teachers' ability to attend to and understand students' mathematical reasoning can be supported through the context of a professional development program is not clear. Mason (2002) recommended focusing on teachers' noticing to increase their knowledge about their students' thinking and eventually make changes in their instructional practices. In the next section, first the construct of teacher noticing and its variety of conceptualizations will be presented. Then, literature on nature and development of teacher noticing will be discussed.

2.3 Teacher Noticing

In the following two sections, first the construct of teacher noticing and its varying conceptualizations will be presented. Then studies on mathematics teachers' noticing will be discussed in terms of their different methodological approaches, findings, and theoretical and practical implications for researchers and teacher educators

2.3.1 Various conceptualizations of teacher noticing

Teacher noticing is a new theoretical construct that many researchers in mathematics teacher education have focused on in order to equip teachers with skills necessary to manage complex classroom environment in a way that increases students' learning (Goldsmith & Seago, 2011; Jacobs, Lamb, & Philipp, 2010; Sherin & van Es, 2009; Star & Strickland, 2007; van Es, 2011). These researchers point out that noticing is a fundamental component of teaching expertise in mathematics. Although "noticing is part of everyday life", professional noticing or *intentional noticing* as Mason (2002) called is different from this kind of ordinary noticing. Professional noticing is characterized as a perceptual framework that enables members of a profession to view complex situations in particular ways (Jacobs, Lamb, & Philipp, 2010). Goodwin (1994) and Sherin and van Es (2009) use the term *professional vision* to refer to such kind of ability that "members of a professional group share for interpreting phenomena central to their work" (Sherin & van Es, 2009, p. 22). They conceptualize teachers' professional vision as expertise in seeing and interpreting significant features of complex classroom environment. This conceptualization is parallel to models and modeling perspective's conceptualization of teacher learning and development. Similar to previous definitions of professional noticing, Koellner Clark and Lesh (2003) consider teaching expertise as seeing and interpreting a complex and ill-structured domain of teaching. These perspectives on teaching expertise inspire many researchers to focus on developing mathematics teachers' noticing expertise (Sherin, Jacobs, & Philipp, 2011). The driving idea behind this increasing focus on teacher noticing is to equip teachers with the skills necessary to deal with the complex situations in teaching profession in a way that is endorsed in reform documents.

In the literature, there are various conceptualizations of noticing. For example, van Es and Sherin (2002) offered three aspects of teacher noticing in their learning-to-notice framework:

(a) Identifying what is important or noteworthy about a classroom situation.
(b) Making connections between the specifics of classroom interactions and the broader principles of teaching and learning they represent.
(c) Using what one knows about the context to reason about classroom interactions (p.573)

In this conceptualization, first aspect refers to teachers'ability to pay attention to something significant during the instruction. Second aspect refers to teachers' ability to consider what they attend to in terms of general principles of teaching and learning. Lastly, third aspect refers to teachers' ability to use their existing knowledge (e.g., subject matter knowledge, knowledge of student thinking, etc.) to make sense of significant interactions that they realize in the classroom. In this study, researchers specifically highlight the importance of the third aspects that is making sense of significant events. In another relevant study, Sherin and van Es (2009) conceptualized teacher noticing in terms of teacher professional vision and conceptualized professional vision as consisting of two subskills: (a) selective attention and (b) knowledge-based reasoning. In their conceptualization, selective attention refers to what a teacher focuses her/his attention on in a complex classroom environment. Knowledge-based reasoning refers to how a teacher reasons about what she/he attends to in a complex classroom environment (Sherin & van Es, 2009).

Jacobs, Lamb, and Philipp (2010) focused on a more specific aspect of teacher noticing, that is noticing of children's mathematical thinking. Rather than focusing on various things that teachers notice, they particularly investigated the extent to which and how teachers notice mathematical ideas that students raise. They also offered a conceptualization of teacher noticing. They defined noticing as consisting of three subskills: (a) "attending to children's strategies", (b) "interpreting children's mathematical understanding", and (c) "deciding how to respond on the basis of children's understanding" (p.173). The first two subskills are similar to van Es and Sherin's conceptualization with a difference that, in this conceptualization, the focus is particularly on children's mathematical thinking. Accordingly, the first subskill is characterized as the extent to which teachers pay attention to mathematically significant details of children's strategies. The second subskill is defined as how teachers make sense of students' ways of mathematical thinking revealed through their strategies. The third subskill is related to how teachers decide to respond to students by using what they learn about students's mathematical thinking from their strategies. As a result, there are various conceptualizations of noticing that researchers adopted in their studies on teacher noticing. However, there is a consensus that teacher noticing has two main components, which are (a) paying attention to significant interactions or events in an instructional environment and (b) interpreting these interactions or events by using existing knowledge (Sherin, Jacobs, & Philipp, 2011).

2.3.2 Nature and development of teacher noticing

In teacher education research, there is an increasing interest in nature and development of teacher noticing. The motivation behind this increasing focus on teacher noticing is an assumption that noticing is neither a skill that teachers possess inherently nor something that they acquire only through years of teaching experience. Therefore, teachers need professional development experiences to develop their ability to notice (Jacobs, Lamb, and Philipp, 2010). Research has confirmed that professional noticing of students' thinking is an expertise that can be learned through both teaching experience and professional support (Goldsmith & Seago, 2011; Kazemi & Franke, 2004; Sherin & Han, 2004; Sherin & van Es, 2009; van Es, 2011; van Es & Sherin, 2008).

Sherin and van Es conducted extensive studies on teacher noticing. They are particularly interested in how teachers learn to notice significant classroom interactions in a video-based professional development program. In one of their early studies, they investigated how a multimedia tool, i.e., video analysis support tool

(VAST), supported the development of intern teachers' noticing skills (van Es & Sherin, 2002). They created and used *learning to notice framework* to analyze the developmental pattern of this skill. This framework offers "a trajectory of the levels of development for learning to notice and interpret classroom interactions" (p.581). They formed this framework by using their conceptualization of noticing previously mentioned. To reveal the influence of the multimedia tool on development of teachers' ability to notice, they compared development of noticing skill of teachers who used this tool with those who did not use it during the study. Data were collected through teachers' written reflections on their analysis of video. The findings revealed that VAST contributed to the development of teachers' ability to notice significant interactions in a classroom environment. Accordingly, teachers who used VAST during their analysis of the videos began to attend to noteworthy aspects of classroom events. Also they supported their observations by referring to a specific event in the video, rather than giving references to events that were not in the video. Finally, they began taking an interpretive disposition in their analysis, which was considered as a robust evidence of development of their noticing.

In a later study, van Es & Sherin (2008) used this learning to notice framework to investigate the development of mathematics teachers' noticing skills in the context of a series of video-club meetings. The main purpose of this video-based professional development environment was to support teachers' learning to notice students' mathematical thinking raised in a complex classroom environment. In this program, elementary teachers attended ten video club meetings where they watched and discussed video excerpts from their own classrooms. One of the researchers had a role of facilitator and her main purpose was to support teachers in learning to attend to and interpret students' mathematical thinking as revealed in the videos. For this purpose, she prompted teachers to attend to and understand students' thinking by asking general and/or specific questions. Researchers supported their data from the meetings by using data from the interviews conducted with each teacher before and after the study. There were two main findings of this study. Firstly, it was revealed that what teachers attended to and how they reasoned about what they observed in the videos shifted over time. For example, teachers began to pay more attention to students and their mathematical ideas as revealed in the clips. Furthermore, they adopted an interpretive stance over time, as opposed to their evaluative stance at the beginning. They began to interpret what they attend to rather than literally describing or evaluating it. Secondly, researchers found out three different developmental trajectories along which teachers' learning to notice develop that is *direct path*, *cyclical path*, and *incremental path*. (p. 257). These learning trajectories gave the researchers insights into characteristics of video club meetings having the potential to contribute to the development of teachers' noticing skills. They highlighted two features of video club meetings that might have an influence on the development. Firstly, they pointed out the importance of use of video excerpts particularly containing rich elements of students' mathematical thinking. Secondly, they highlighted the crucial role of the facilitator in prompting teachers to attend to and interpret students' mathematical ideas (van Es & Sherin, 2008).

Similar findings were revealed in Sherin and Han's (2004) study. Like the previous study, the main purpose of this study was to investigate how teachers learn to pay attention to particular events in the classroom and to interpret them in particular way in a context of series of video club meetings. Design of the meetings and the role of the researcher as a facilitator in these meetings were similar to van Es and Sherin's study. The results of this study confirmed that video club meetings contributed to teachers' learning to notice. Throughout the study, what teachers attended to and how they reasoned about what they observed changed over time. While teachers mainly focused on pedagogical issues in the first meeting, they were primarily interested in issues related to students' mathematical thinking in the last meetings. Also, over time, teachers began to make efforts to make sense of students' mathematical reasoning. As opposed to being prompted by the facilitator in the early meetings, teachers took the initiative and led the discussions on students' thinking without the facilitator's prompt. Besides these main findings, researchers also speculated on features of video club meetings that might have influences on the shifts in teachers' learning to notice. They pointed out these features as (a) special kinds of video excerpts to use such kinds of professional development programs, (b) the crucial role of the researcher as facilitator, and (c) teachers' group working (Sherin & Han, 2004).

Besides the findings of previously discussed studies that video club meetings contributed to teachers' learning to notice significant features of complex classroom interactions, Sherin and van Es' (2009) study revealed that this development in teachers' noticing skills had influences on their instructional practices. For instance, while early in the school year teachers did not pay attention to mathematical ideas that students raised, over time teachers began to identify mathematically significant ideas and considered them as objects of inquiry. Similarly, there was a shift in ways that teachers reasoned about students' mathematical thinking in their classroom over time. They began to attempt to deeply analyse what students tried to do. All these findings pointed out that teachers' learning to notice in a video-based professional development program could extend to their teaching practice.

Unlike Sherin and van Es' characterization of noticing as both attending to and interpreting noteworthy aspects of classroom settings, Star and Strickland (2007) considered noticing as just attending to these aspects. While Sherin and van Es highlighted the importance of making sense of what is attended to, Star and Strickland claimed that, especially for pre-service teachers, attending to significant classroom events should be considered as the most important component of noticing. Like video-based professional development approach adopted by Sherin and van Es, Star and Strickland used videos (TIMSS videos from 8^{th} grade mathematics classrooms) to improve pre-service teachers' noticing skills. They focused on development of pre-service teachers' ability to attend to salient classroom features throughout a method course. In this course, some activities to improve teachers' noticing skills were integrated. Shifts in pre-service teachers' noticing were revealed through pre-and post assessments administered by researchers at the beginning and the end of the course. These assessments included questions that teachers were required to answer after waching the videos related to five observation categories (i.e., classroom environment, classroom management, tasks, mathematical content, and communication). Additionally, they were required to write a short paper to describe in detail what they noticed in the videos after viewing them for a second time. There were two main findings of this study. Accordingly, pre-assessments revealed pre-service teachers' lack of ability to notice details of classroom environments. However, it was also revealed that the method course contributed to improvement of their ability to notice classroom events particularly in four dimensions, i.e. classroom environment, tasks, mathematical content, and communication.

As discussed previously, in professional development programs focusing on student thinking as a means of teacher learning, students' works are effectively used as practice based-artifacts (Ball& Cohen, 1999; Chamberlin, 2002; Koellner-Clark & Lesh, 2003). Research on teachers' noticing also confirmed that students' works can be used as an effective tool in professional development environments supporting teachers' noticing of students' mathematical thinking. For example, Kazemi and Franke's (2004) study of teachers' learning to attend to and understand children's sophisticated mathematical reasoning was organized around students' written works generated through mathematical problems. In this study, researchers investigated teachers' collective inquiry into their students' mathematical thinking in a series of workgroup meetings throughout a year. In this professional development approach, teachers implemented mathematical problems in their classrooms and brought samples of students' works from their classroom to discuss them in the meetings. These discussions were moderated by the researcher as the facilitator. In these workgroup meetings, teachers were encouraged to pay their attention to a variety of solution strategies that students offered and understand underlying mathematical ideas. However, professional development experiences of teachers were not limited to these meetings. They were also encouraged to learn to notice their students' thinking in their classrooms. They had a chance to communicate with their students about their solution strategies to better understand underpinning mathematical thinking. Teachers were also prompted to generate strategies to reveal their students' thinking during the instruction.

Findings of this study revealed that at the beginning, teachers had difficulties in attending details of students' solution strategies. Also they had a tendency to point out students' incorrect solution strategies. At that time, the facilitator had a crucial role in forcing teachers to attend to and articulate details of the strategies. Over time, teachers began to spontaneously focus on mathematically significant details of the strategies and make sense of underlying reasoning processes. As opposed to their focus on how students fail to solve the problem in the first meeting, they began to realize and appreciate students' mathematical competencies. As a result, researchers highlighted the importance of the facilitator's role in prompting teachers to attend to and make sense of students' works particularly in the first meeting. Furthermore, they underscored that using students' works from teachers' own classrooms and teachers' opportunities to communicate with their students to clarify what they understood made valuable contributions to teachers' learning to notice their students' mathematical thinking.

In another similar professional development approach, Chamberlin (2005) investigated how a group of teachers' investigation of their students' written works generated through non-routine mathematical tasks, MEAs, improved their ability to attend to and understand students' mathematical thinking. In this study, the researcher specifically focused on how teachers collectively engaged in "mini inquiries", in which teachers endeavored to analyze students' solution approaches and underpinning mathematical ideas. In this design of teachers' investigations of students' works, teachers first worked on an MEA in an introductory meeting. Then they discussed underlying mathematical concepts, ideas and shared their predictions and expectations of how students would solve the MEA. Afterwards, teachers implemented the activity in their classrooms. They were required to individually create students' thinking sheets by examining students' works collected at the end of the implementation and by using their observation notes about students' solution processes they took during the implementation. In these sheets, teachers offered their observations on different solution strategies that students used, underlying mathematical concepts, ideas that solution strategies were based on. Then teachers attended follow-up meetings in order to share and discuss their ways of interpretations of students' mathematical thinking. Findings of the study pointed out that teachers' collective investigations of students' works promoted their mini inquiries where they carefully analyzed solution approach and reasoned about, speculated on underlying mathematical thinking. Furthermore, results revealed that these mini inquiries helped teachers overcome their difficulties in attending to and making sense of students' sophisticated reasoning revealed through MEAs. The researcher also underscored two contextual aspects of this professional development approach in terms of supporting teachers' learning to notice their students' mathematical thinking. Firstly, she referred to the importance of teachers' collaborative effort to examine and analyze students' works, in addressing teachers' challenges in attending to student thinking. In such kind of social learning environments, communications and/ or interactions among teachers supported their learning to notice students' thinking. Secondly, the facilitator's role was also crucial in teachers' learning to notice. She prompted teachers to carefully analyze and try to make sense of students' solutions even if they seemed to be complex and unreasonable. Additionally, she forced teachers to test, revise, refine their ways of interpretations of students' thinking by questioning their current interpretations, or by asking them to make clearer their current thoughts about students' thinking. This crucial role of the facilitator in teachers' learning to notice their students' thinking is similar to the impact of the facilitator role in the studies previously discussed (Kazemi & Franke, 2004; Sherin & Han, 2004; van Es & Sherin, 2008).

In another relevant study, Goldsmith and Seago (2011) attempted to understand how teachers' examining and discussions of different kinds of classroom-based artifacts brought about a shift in their noticing skills. For this purpose, they examined two professional development programs where different kinds of classroom-based artifacts were used to develop teachers'noticing and investigated what kinds of shifts occurred over time in teachers' noticing of students' mathematical thinking in these programs. Although these two programs had a common goal of improving teachers' ability to attend to students' mathematical thinking, they differed on kinds of artifacts around which professional development experiences were organized. While in one of them video excerpts from the teachers' classroom settings were used, in the other, samples of students' written works were used as classroom-based artifacts. Similar to the findings of the previously discussed studies, researchers found that in both programs, teachers' noticing skills developed over the course of these programs. For example, at the beginning, teachers focused on whether students reached the correct answer with an evaluative dispotion. They had a tendency to focus on students' shortcomings by assessing their solutions in terms of whether they were expressed in a formal, structured way. Over time, teachers began to attend to more details of the solutions and realized mathematically significant aspects of them. Also teachers attempted more to understand what students tried to do although students expressed what they did in a way that appeared illogical. The findings of this study also confirmed that the facilitator's prompts had strong influences on this shift in teachers' noticing. Additionally, researchers highlighted the impacts of using different kinds of classroom-based artifact on such kinds of professional development programs focusing on teacher noticing. For instance, in the two programs examined in this study, both video excerpts incuding students' ways of thinking and samples of students' works were used. Researchers concluded that different kinds of artifacts should be used together by paying attention to predetermining the purposes of using each of them and should be integrated into the programs artfully (Goldsmith & Seago, 2011).

CHAPTER 3

METHODOLOGY

In this chapter methodological issues and decisions underpinning the study will be presented. In general qualitative research design, and in particular, case study approach provides the guiding methodological framework of the study. The context of the study, participants, researcher's role in the study, data collection tools and methods, design considerations, data analysis procedures and trustworthiness of the study will be discussed.

3.1 Qualitative Methodology as the Underpinning Research Approach

Researchers in social sciences often adopt ways in which study of phenomena relating to human experiences can be conducted by obtaining as rich data as possible and as a result they can make sense of such experiences. Towards this aim, within qualitative methodology meanings of individual experiences are studied in detail with an assumption that meanings are socially, historically and contextually constructed as opposed to positivist knowledge claims in quantitative studies (Creswell, 2003). Such a qualitative research tradition has been growing rapidly in the field of education, where human experiences and the context in which such experiences take place are central (Cohen, Manion & Morrison, 2007).

Qualitative studies have important defining design considerations, some of which are listed below:

- The researcher positions herself in such a way that she can interact with the participants while trying to make sense of social issues,

- The participants are given opportunities to express their perspectives and experiences during data collection,

- The study is conducted in natural contexts of the participants,
- Data is often collected through a variety of methods to capture the richness of social phenomena (Creswell, 2003).

In the following sections such considerations will be discussed with references to specific considerations and decisions taken within this study. Next is an elaboration on why case study, a widely used qualitative approach, is adopted as the strategy of inquiry for this study.

3.2 Strategy of Inquiry: Case Study Approach

Case study is the study of a "specific instance that is frequently designed to illustrate a more general principle" (Cohen et al., 2007, p.253). In this study, a case study approach is seen as fit for purpose. The main focus of this study was to understand nature and development of teachers' ability to notice students' mathematical thinking. To investigate this issue, an appropriate case of this study was determined as a professional development program and four secondary mathematics teachers attending this program.

Cohen et al. (2007) claim that case studies can vary in the degree of structure in the natural setting and the degree of structure imposed by the observer. This study is exposing teachers to activities developed within a professional development program, hence structuring the natural environment the study is taking place in. Yet, teachers are working in their own schools, with their colleagues and students. Besides, the researcher is using a conceptual framework for making sense of the experiences teachers have within the professional development program. These considerations are in line with the case study approach and they are expected to allow the researcher to make sense of important phenomena relating to mathematics teachers' practice within a context of a teacher professional development program.

3.3 Researcher's Role

Since this study is a part of a three-year research project, necessary negotiations to gain entry to the setting were conducted by the director of the project and all formal approvals and permissions were received by him. So, the researcher had a smooth access to the schools research was conducted in and the teachers were already informed that they would take part in a professional development program.

According to Marshall and Rossman (2006), interpersonal skills that researchers have and good relations with the participants play a crucial role in success of a qualitative research. The researcher already had acquaintance with one of the participants in one of the schools since she had visited this school for research purposes in previous years. The prior interactions with the teacher and a positive previous research experience helped her to get to know the other teachers and eventually establish trust between the teachers and the researcher. Additionally, the researcher spent time with the teachers in their schools during break times and held formal and informal conversations with them in the teachers' rooms. This also helped her build up a good rapport with them.

The researcher conducting this study had various roles to play within this research. These roles can be classified as a facilitator, an assistant to teachers in the classrooms during the implementations of the activities, a researcher in collecting and interpreting data, and even a participant in the meetings and interviews. Each role might affect the data collection process directly or indirectly. In the meetings that were conducted before and after the implementations of the activities, researcher had a role of facilitator or moderator of the meetings. In the introductory meetings, she directed the teachers to share their own solutions with the others and moderate a discussion on various solution approaches that teachers offered. During the implementations of MEAs in the classrooms, she had a role of observer as participant and students were aware of her role of researcher. She circulated around classroom and observed the students while they were working. She avoided interacting with students except situations where students asked for clarification of statements in the problem. She also assisted teachers in various ways such as distributing and collecting materials.

3.4 Participants

This study was conducted at an Anatolian Teacher High School and an Anatolian High School in Ankara in the 2011-2012 school year. Participants of the study were selected from the participants of the larger research project previously mentioned. The participants of the research project were five secondary mathematics teachers from an Anatolian Teachers High School in Ankara and five secondary mathematics teachers from an Anatolian High School in Ankara. Their students in 9th-11th grades also participated in this study.

In the participant selection process of the project, purposeful and convenience sampling strategies were used. Since the purpose of the study was to provide mathematics teachers with professional development experiences on mathematical modeling, the decision was to study with teachers and students from Anatolian Teacher High Schools. Many of the students of this school would become teachers in the future, so this study can be considered as an early pre-service teacher training for them. Teachers from the Anatolian High School were selected by using convenience sampling strategy. Researcher of this study already knew teachers in this school because she had visited this school for the purpose of research in previous years. Practical issues such as ease of access were also taken into consideration. Short distance between the two schools influenced the selection decisions.

After the decision on the schools to conduct the study, these schools were visited to talk with the principles about the study, to get information about the teachers and students in the school, and eventually to receive permission to conduct the study. After getting permission, at the beginning of September, a meeting was conducted with the mathematics teachers in each school. In this meeting, teachers were briefly informed about the study and invited for the workshop that was planned to conduct at the beginning of the study. Fourteen teachers (six teachers from the Anatolian

Teacher High School and eight teachers from Anatolian High School) volunteered to attend the workshop.

The workshop was conducted with the participation of 14 secondary mathematics teachers from these schools on 13-16 September 2011. The purpose of this workshop was both to inform the teachers about the aims of the project and to prepare them for the activities in the professional development program. Many of the activities that were included in the program (e.g. investigation of students' written works on modeling problems, creating lesson plans that would include modeling problems) took place in the workshop. At the end of this workshop, the teachers who were willing to participate in the professional development program were determined. Five teachers from each school eagerly accepted being participant of the professional development program.

3.4.1 Participant selection for the study

To answer the research question in this study, the researcher worked more closely with four of the ten teachers (two teachers from Anatolian Teacher High School and two teachers from Anatolian High School) in the project. In the data collection process, it was not possible to isolate these four teachers from the other six participating teachers, because meetings were held with the participation of groups of teachers. In order to study with these four teachers more deeply, additional data were collected on these teachers, which will be described in detail in the data collection section.

Purposeful sampling strategy was used to select the four participants among the ten teachers. In the workshop and first month of the study, it was clear which teachers actively engaged in the investigation of students' work activities and had potential in terms of providing rich data. The researcher talked with these four teachers about her dissertation study and what additional requirements they were supposed to meet if they accepted to participate. Teachers agreed to participate in this dissertation study. A basic descriptive demographic account of these four teachers can be seen in Table 3.1.

3.5 Broader Context of the Study

This study was carried out as a part of a larger research project about mathematical modeling where the focus of research was on developing pre-service and in-service mathematics teachers' knowledge and skills about using modeling problems in teaching mathematics.

Table 3.1. Basic Demographic Information about the Teachers in the Study

Name*	Gender	Grade level	Years of teaching experience	Experience with modeling
Selda	Female	9^{th} -12^{th}	24	None
Kutay	Male	9^{th} -12^{th}	26	None
Ayfer	Female	10^{th} -11^{th}	19	None
Rana	Female	9^{th} -10^{th}	13	None

* All names are pseudonyms

There were three main purposes of this three year research project: (i) to develop mathematical modeling activities that can be used with both secondary school students and pre-service and in-service teacher education programs; (ii) to develop an in-service mathematics teacher professional development program about mathematical modeling and to investigate how the program would affect teachers' beliefs, knowledge and practices; (iii) to develop an academic course for pre-service mathematics teachers and investigate how the course would affect pre-service teachers' knowledge, competencies and attitudes in terms of mathematics, mathematical modeling and using mathematical modeling in mathematics education.

This study was related to the second purpose of the project. The general goal of the in-service teacher professional development program was to equip in-service teachers with the ability to use MEAs in their teaching practice. This ability was conceptualized as three interrelated sub skills that were (a) planning a lesson in which MEAs were integrated, (b) enacting the plan in the classroom, and (c) reflecting on students' thinking revealed through MEAs. This study focused on the last skill because it was considered that teachers' ability to reflect on significant mathematical ideas that students raise through MEAs was crucial in effective planning and implementing.

The professional development program was adapted from a multitier program development design introduced by Koellner Clark and Lesh (2003). They adopted a models and modeling perspective on teacher development. In the following sections, two main instructional tools used in this design, that is model-eliciting activities for students and model-eliciting activities for teachers will be described. Then, multitier program development design will be explained and the design of the professional development program will be presented.

3.5.1 Model eliciting activities for students

Model eliciting activities (MEAs) are non-routine problems that were developed in a multi-tiered teaching experiment by Lesh, Hoover, Hole, Kelly, and Post (2000). Researchers determined six design principles of MEAs with the help of teachers. These principles are (a) Model construction principle, (b) Reality principles, (c) Self-assessment principle, (d) Model documentation principle, (e) Construct shareability and re-usability principle, and (f) Effective prototype principle.These problems require students to mathematically interpret a real world situation in order to help a client in the situation (Lesh, et al., 2000). Rather than giving short answer as in traditional word problems, in MEAs, students are supposed to create a method, a tool, or procedure to solve a problem situation. Model-eliciting activities are also called thought-revealing activities because students' solutions to these problems reveal their mathematical thinking (Lesh & Doerr, 2003; Zawojewski, Chamberlin, Hjalmarson, & Lewis, 2008).

Seven modeling-eliciting activities that were used in the implementation of the professional development program were selected from a pool of MEAs, which were designed for the project by the project team. The development processes of MEAs are presented in Appendix A. A list of seven MEAs is presented in Appendix C. The titles of these activities and brief descriptions of them can be seen in Table 3.3. For determining the seven MEAs, firstly, a schedule including which teachers would implement the activities and appropriate dates of the implementations. The schedule of implementation can be seen in Table 3.2.

Table 3.2 A Schedule for the Implementations of the MEAs

MEAs	Dates of the implementations	Teachers who implemented the activity
Bank robbery	25 October-4 November 2011	Rana & Selda
Parking space on the street	22 November-2 December 2011	Ayfer & Kutay
Summer job	20-29 December 2011	Rana & Selda
Water tank	14-24 February 2012	Ayfer
How to store it?	6-16 March 2012	Rana & Selda
Magazine sales	3-13 April 2012	Kutay
Bouncing bal	8-11 May 2012	Ayfer & Kutay

Table 3.3. Description of the Modeling-Eliciting Activities

Title	Product	Mathematical Topic/Concept	Grade Level
Bank robbery	A method to decide on who is guilty	Logic	9^{th}
Parking space on the street	Parking space design,	Trigonometry Geometry	10^{th}-
Summer job	A method to decide on who to recruit	Ratio and proportionality,. Weighted average	9^{th} -12^{th}
Water tank	Coming up with directions for drawing a graph	Functions	9^{th} -12^{th}
How to store it?	A method to decide on the maximum profit	Geometry- Pythagorean Theorem	9^{th} -12^{th}
Magazine sales	A method to decide on the maximum profit	Quadratic equations	10^{th}
Bouncing ball	An interval of values for a rate	Exponential functions, Inequalities with exponential terms,	11^{th}

By considering the grade levels and the mathematics topics in the curriculum taught at the time of the implementations, appropriate MEAs were selected from the pool by the project team. When eventually deciding on the seven MEAs among the candidates, the activities implemented by teachers in the pilot study were preferred.

3.5.2 Model eliciting activities for teachers: Student thinking sheets

Student Thinking Sheet (STS) is a form designed to help teachers to think about and document students' mathematical thinking revealed through MEAs (Doerr & Lesh, 2003). Similar to the purpose of model-eliciting activities for students as revealing students' thinking, the main purpose of these forms to reveal teachers' ways of thinking. It consists of a two-page document formatted as a table. The first page is divided into rows in which teachers are required to report different solution strategies used by their students while working on modeling problems. The table in the second page includes sections in which teachers are asked to report mathematical concepts, skills and processes as well as students' errors and misconceptions for each different solution strategy. A sample STS can be seen in Appendix B.

3.5.3 Multitier program development design

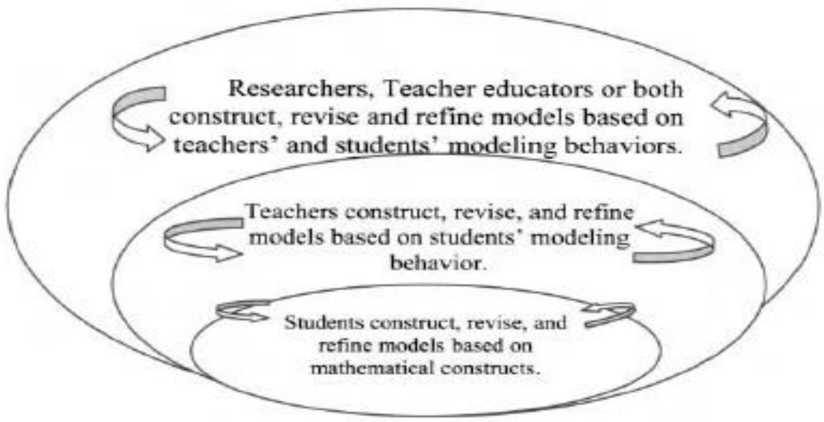


Figure 3.1. Multitier program development (Koellner-Clark & Lesh, 2003, p. 161)

Koellner-Clark and Lesh (2003) introduced a research design based on some theoretical assumptions of models and modeling perspective about learning and teaching. They called it multitier program development design and they explained its basic tenets in the following way: "Multitier program development and/or implementation designs that focus on the interacting development of students, teachers and researchers and/or teacher educators" (p. 160). The occurrence of such an interacting development is presented in Figure 3.1.

In this three-tiered research design, in the first tier, a group of three or four students engage in a series of model-eliciting activities that repeatedly challenge them to create, test, and refine mathematical models. These students' modeling processes reveal their ways of thinking about mathematical constructs embedded in the activity. In the second tier of the design, teachers are also involved in model-eliciting activities (e.g., creating student thinking sheets to document students' modeling behavior, or creating a concept map as an assessment tool) that challenge them to create, test, and refine their models of students' modeling behavior. In the third tier, researchers or teachers educators create, test, revise and refine their models to make sense of teachers' and students' modeling behaviors (Koellner-Clark & Lesh, 2003).

In this research, nature and development of teachers' noticing of students' mathematical thinking was investigated in the context of a professional development program. Main components of this program were adapted from multitier program development design and the tiers described in the previous paragraph are planned to take place during the course of the study. In the following sections, the implementation of this program and research procedure will be presented.

3.6 Research Design

Professional development program was carried out in the 2011-2012 school year and lasted approximately 9 months. In the first month, a workshop was organized with the participation of 15 secondary mathematics teachers from the two schools. The purposes of this workshop were (a) to inform the teachers about the aims/objectives/scope of the program, (b) to find out teachers' prior knowledge about mathematical modeling and their interest in it, (c) to enable teachers to gain deep

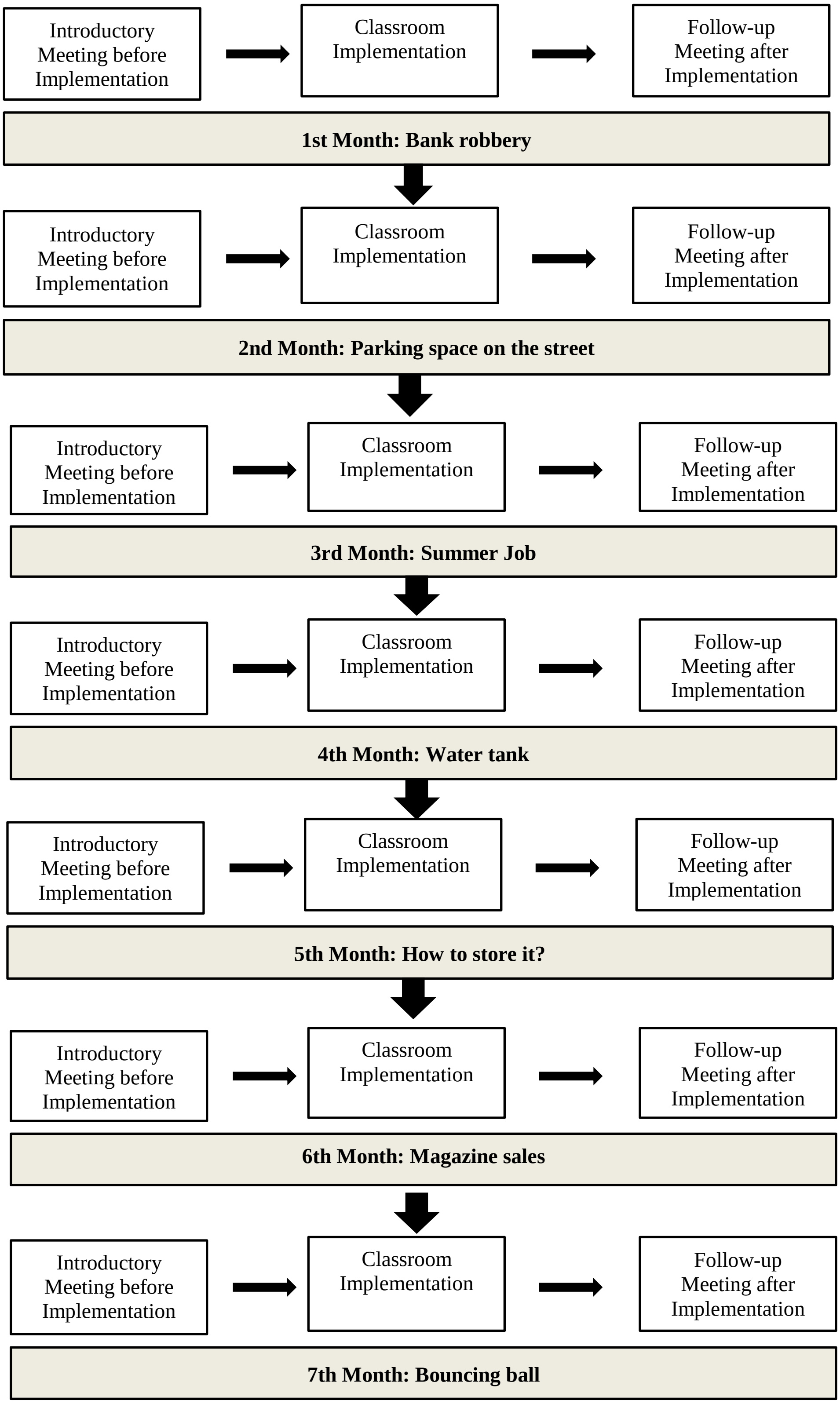


Figure 3.2. A schematic representation of the professional development program

insights into the nature of model-eliciting activities and their use mathematics education.

As mentioned previously, design of the professional development program was adapted from the multitier professional development program design. The implementation of the program was composed of seven similar one-month periods, which included three stages: (i) Introductory meeting before classroom implementation of the MEA, (ii) Classroom implementation of the MEA, (iii) Follow-up meeting after classroom implementation of the MEA. Each stage wascyclically repeated for each month (See Figure 3.2 for a schematic representation of the program). The details of the three stages in one-month periods are explained below.

3.6.1 Introductory meeting before the implementation of the MEA

First stage of the one-month period started with an introductory meeting. Before this introductory meeting, teachers were required to make a preparation. Firstly, teachers worked on the given MEA individually and wrote their own solution approach. Also they individually filled in a pre-implementation STS that included their expectations and predictions about students' solution approaches (e.g., various solution methods, mathematical ideas underlying these methods, mistakes.). They brought their documents to the introductory meeting.

At the beginning of the introductory meeting, teachers expressed their thoughts about the given MEA, underlying mathematical concepts, and some issues related to the implementation of the activity. Then, each teacher shared her/his solution approach with other teachers and all teachers discussed these approaches. Afterwards, they were encouraged to look at the problem from students' perspectives and to share their expectations/predictions of students' solution approaches, mathematical ideas, mistakes, difficulties, etc. Teachers expressed what they noted down in their STS related to these issues. All introductory meetings were audiotaped and videotaped and all documents were collected.

3.6.2 Implementation of the MEA in the classrooms

Teachers generally had approximately a ten day period to implement the MEA in their classrooms. In the introductory meeting, teachers had already decided on grade levels they would implement the MEA according to their schedule. Although each MEA had a recommended grade level, teachers did not have to strictly follow these recommendations. On some occasions, two teachers from the same school implemented the same MEA in the different grade levels.

Before the implementation, teachers generally informed their classrooms about the activity. At the beginning of the implementation, students formed the groups themselves. After a brief introduction of the MEA by the teacher, activity sheets including problem statement and necessary materials were distributed to students. Students worked on the problems in groups of 3 or 4 during one class period (i.e., 50 minutes). During the activity, the teacher circulated around the classroom by closely observing the students. She sometimes asked the students to explain what they tried to do and answered students' questions. In the second class period, all groups shared their solution approaches with other groups in a poster session. At the end of the activity, students' all written works, i.e. worksheets, posters, were collected. All classroom implementations were audiotaped and videotaped. Each group's work was also audiotaped.

3.6.3 Follow-up meeting after the implementation of the MEA

After the implementation of the MEA by two teachers from each school, copies of all students' works accompanied by post-implementation STSs were distributed to the teachers. All teachers had all students' works collected from both schools. Before the follow-up meeting, teachers were supposed to examine the students' works in depth individually and note down different solution approaches, their mistakes, etc. To help them express their observations about these issues while investigating students' works, post-implementation STS were provided. However, they did not have to strictly use these forms. They brought students' written works and related documents to the meeting with them. Researcher also prepared for the meeting by carefully analyzing students' works and kept notes of some ideas about students' works in order to discuss in the meetings.

At the beginning of the meeting, teachers who implemented the MEA in their classrooms shared their experiences of implementation and general ideas about students' solution approaches. Then each teacher expressed what they had noted down in their STS relating to the associated MEA. When sharing their observations on students' solutions, they showed one of the groups' written works to exemplify their claims. Other teachers also closely examined such examples and shared their opinions about them. Each teacher had a chance to share what she thought about students' works. Sometimes, they preferred to share students' work that they found interesting or unexpected. On some occasions, they shared their confusion about a group's solution and encouraged other teachers to explain that solution. In these situations, teachers discussed on that particular group's work and exchanged their ideas. These discussions enabled teachers to both express their thoughts about students' mathematical thinking and revise them as necessary.

Researcher moderated the discussions in these meetings. She had a facilitator role in these meetings and encouraged teachers to investigate students' written works in several ways. All follow-up meetings were audiotaped and videotaped. Additionally, all documents including STS teachers individually filled after the investigations of students' works and their personal notes related to the works were collected. At the end of the meeting, teachers were reminded about the next MEA and necessary documents were distributed.

This one month period continued cyclically throughout the study. As in multitier professional development program design, in this design of professional development program, students' various ways of mathematical thinking were revealed through a series of model-eliciting activities. This part of the design is associated with the first-tier of the multitier design. Also, teachers investigated students' mathematical thinking both individually and collectively and hade a chance to test, revise, and refine their ways of thinking about students' thinking. This component can be associated with the second-tier of the multitier design. Moreover, the researcher investigated both students' mathematical thinking and teachers' thinking about their students' thinking. Throughout the meetings and interviews, she repeatedly tested, revised and refined her ways of thinking about teachers' ability to notice their students' thinking, which is associated with the third-tier of the multitier program design.

3.7 Data Collection

For this study, multiple forms of data from the second, fifth, and sixth implementations were collected to answer the research questions. The intent was to collect data from the beginning, the middle and the end of the study to gain insight into whether there was a shift in teachers' noticing over time. The main data sources were audio and videotaped focused group meetings, one-to-one interviews conducted after the meetings, field notes from the observations of the implementations, and collected documents such as students' papers showing their responses to the MEAs, STSs filled by teachers, and teachers' personal notes. In the following sections, these various forms of data collection will be presented briefly.

3.7.1 Focus group meetings

Introductory and follow-up meetings with all participating teachers in each school were focus group meetings and they were the main data sources for the study. These group interviews lasted approximately 45 minutes and the subject of interest was ways of students' mathematical thinking. Researcher fostered discussions among the teacher participants about ways of students thinking revealed through modeling problems. She posed questions such as "What do you think about underlying students' mathematical thinking?" that fostered teachers to elaborate their statements and to prompt discussions. Samples of questions that the researcher posed during the meetings can be seen in Appendix D.

The main reason behind using focus group meetings as a data collection method was that participant teachers stimulated each other to express their views about students works (Bogdan & Biklen, 2006). According to Bogdan and Biklen, one of the limitations of focus group meeting methods is that some participants can hesitate about articulating their views in a collective environment. This situation was also valid for this study in some of the meetings. In some occasions, because one of the teachers in the group talked too much and dominated the discussion, other teachers could not express their thoughts comfortably. To overcome this limitation, one-to-one, in-depth interviews were also conducted with the teachers who rarely engaged in the group discussions in the meeting

3.7.2 One-to-one interviews

Immediately after the follow-up meetings, one-to-one interviews were conducted with the teachers. These interviews lasted approximately an hour. The purpose of these interviews was to foster teachers who did not talk so much in the group discussions to individually express their thoughts on students' works. These interviews also gave teachers a chance to examine students' solution approaches more closely and revise/refine their thoughts on students' solutions that they expressed in the meeting. Teachers brought their STS and students' worksheets to the interviews with them. Depending on teachers' preference, teachers talked about groups' works in general or at times about some of the groups' works that they found noteworthy. Researcher prompted teachers to talk about and elaborate on what they noticed on students work by asking questions similar to those asked in the meetings. All interviews were audiotaped and videotaped.

Additionally, semi-structured interviews were conducted with teachers before and after the implementations of the MEAs. They lasted approximately 20 minutes. In pre-implementation interviews, teachers were asked about their predictions and expectations about students' ways of thinking. On the other hand, in post-implementation interviews, teachers expressed their observations on ways of students' thinking that they made in the implementations. Each interview was audiotaped.

3.7.3 Observations

During the implementation of the MEAs, researcher made participant observations.

Creswell (2003) described various types of observations where researcher's role varied from a complete participant to a complete observer. Researcher adopted a participant observer role. All students were aware of her role as a researcher. She took field notes on students' solution processes while observing. Especially in the poster session, while students were presenting their solution approach with detailed explanations of their thinking process, researcher also kept field notes on underlying thinking process. These field notes were used by the researcher while analyzing students' works so that she could make more sense of teachers' comments during the meetings and the interviews.

She also observed the conversations between the teacher and students and took field notes on teachers' comments on students' solutions. Sometimes, teachers shared their observations on students' works they made while circulating the classroom with the researcher. Researcher also kept notes of this kind of informal conversations with teachers related to students' works. These entire field notes were used as supplements to data collected through other sources (Bogdan & Biklen, 2007).

3.7.4 Documents

Instructional tools such as MEAs and STS are also considered as data collection tools upon their implementation in the study. According to Schorr and Lesh (2003), besides their role of instructional tools in teachers' and students' learning environments, they also provided researchers with rich data on their ways of thinking. In this study, the implementation of MEAs provided data on students' mathematical thinking in the context of a real life problem situation. These ranged from 10th grade to 11th grades. They included differing solution strategies to MEAs. On some occasions, the same MEA was implemented in both 10th and 11th grade levels. In Appendix E, samples of students' solutions for the three MEAs used in the study can be seen. Similarly, STSs yielded data on what teachers notice in students' works. A sample of STS can be seen in Appendix B. Hence, documents used as other data sources consisted of students' written works showing their responses to the MEAs, STSs filled by teachers before and after the implementations, and teachers' personal notes related to their solution approaches to MEAs and their observations about students' works.

3.8 Data Analysis

In this study, data consisted of videotape and audiotape recordings of meetings and interviews, students' written works including their responses to modeling problems, teachers' personal notes on what they noticed while investigating students' works, and student thinking sheets (STSs). Once the data collection was done, all videotaped and audiotaped data were transcribed and prepared for analysis. The researcher carefully read all transcriptions more than once to engage herself fully with the data.

Students' written works had already been analyzed by the researcher during the data collection processes. Since the researcher had a role of a facilitator in the meetings, she analyzed students' responses carefully before coming to the meeting and took notes on some aspects of students' solution approaches that were worthy of consideration. After the data collection processes, during the in-depth analysis of data from the meeting and interviews, students 'works were carefully examined once more.

Besides generic processes of qualitative data analysis, the researcher attempted to create a more specific data analysis method to answer the research questions. As Creswell (2003) states "Researchers need to tailor the data analysis beyond more generic approaches" (p.191). Similarly, Yıldırım and Şimşek (2008) point out that

> Every qualitative study has specific features and requires specific new approaches during its analysis. That is why the qualitative researchers need to make a data analysis plan in light of existing qualitative data analysis approaches, taking features of the study and the collected data into consideration. (p. 221)

To decide on the most appropriate method of data analysis in terms of the purpose of the study, relevant literature on teacher noticing was reviewed. After the identification of commonly used, central coding categories for investigating teachers' noticing, the researcher analyzed how these categories were used to make sense of data. The decisions regarding the data analysis for this study were taken throughout this process, which will be explained in the following sections.

3.8.1 Identification of the coding categories and sub-codes used in the study

At the beginning of data analysis process, a draft code list was created. While creating this code list, literature on teacher noticing was considered. As discussed in the literature review section, studies on noticing use various different conceptualizations of this construct (Jacobs, Lamb, & Philipp, 2010; Sherin & van Es, 2009; van Es & Sherin, 2002; van Es, 2011). However, they generally point out two main sub-dimensions of noticing: (a) "Attending to particular events in an instructional setting" and (b) "Making sense of events in an instructional setting" (Sherin, Jacobs, &Philipp, 2011, p.5). Sherin and van Es (2009) called these two sub dimensions "Selective attention" and "Knowledge-based reasoning", respectively (p.22). In this study, this terminology was used. Moreover, there are some other studies that particularly focus on teachers' noticing of students' mathematical thinking (Jacobs, Lamb, & Philipp, 2010; van Es, 2011). By considering these different conceptualizations of teachers' noticing and the nature of data collected in this study a coding list was created. To clarify the meaning and scope of each code,

Table 3.4. The List of Coding Categories Used in Data Analysis

Dimensions of Noticing	Sub-dimensions	Coding Categories
Selective Attention	Topic	Mathematical aspects
		Non-mathematical aspects
	Level of detail	General features
		Substantial details
Knowledge-based Reasoning	Analiytic stance	Describe
		Evaluate
		Interpret
		Make sense of solution

Note. Some of the names of sub-dimensions and coding categories were quoted form literature (Sherin & van Es, 2009; van Es, 2011) all the data were pre-analyzed by using the code list. After this pre-analysis process, necessary revisions were made to the code list and the last version of the list was formed (See Table 3.4).

In this study, selective attention was considered as what teachers paid attention to in investigating students' written works. In other words, what they perceived in students' works as worth commenting on. There were two sub dimensions of selective attention called topic and level of detail. "Topic" referred to whether teachers paid attention to mathematical aspects of the work, i.e. solution approach, or non-mathematical aspect such as content of the reports not related with mathematics, the quality of the pictures, whether students expressed their solution in their reports in a tidy manner, or who the group members were. "Level of detail" referred to whether teachers attended to general features of the students' solutions (e.g., underpinning mathematical topic) or their comments included substantial details of the solutions.

Knowledge-based reasoning was considered as how teachers reasoned about what they attended to in investigating students' written works. In the coding categories of knowledge-based reasoning, "Describe" referred to whether teachers recounted (sometimes with direct quotations) what they attended to while investigating students' responses. "Evaluate" referred to whether teachers made judgments about what was good or bad. "Interpret" referred to whether teachers made inferences about underlying meaning of what they attended to while investigating students' responses. When teachers attempted to analyse students' solutions to better understand, these segments were coded as "Make sense of solution". After creating this code list and clarifying the meaning and scope of each code through pre-analysis of data, in-depth analysis of data began.

When the work on transcribed data started, first idea units were determined. In relevant studies, idea units were defined as "segments in which a particular idea was discussed. In other words, each time the conversation shifted to a new issue, it was coded as another idea unit" (Sherin & van Es, 2009, p. 23). This principle was also used to identify idea units in this study. In the data from both meetings and interviews, teachers' statements on a group of students' work were considered as an idea unit. Afterwards, each idea unit was coded in terms of both selective attention and knowledge-based reasoning by using the code list. Although data from the meetings were different from the interview data, to segment meeting data the same principle was used. Each teacher's statements on a group of students in the conversations were coded in terms of two dimensions. Table 3.5 portrays these coding processes. This table included two main dimensions of noticing and coding categories of these dimensions. This table included 12 idea units and each idea unit was characterized in terms of each of the coding categories. It showed coding processes of both meeting and interview data and they were created for each teacher for three investigations.

The coding process was an interpretive one, and the reliability of the coding needed to be documented. In that context, an expert in mathematics education was asked to code the data as well, in order to check the inter-rater consistency of the coding. A randomly selected interview transcript of each of the four teachers was read and coded independently by the second coder. Then a comparison of the coding by the researcher and the second coder was made. 80 % agreement of coding on these pieces of data was reached. Discrepancies between the codes assigned by two coders were discussed and a consensus on these codes was reached.

Following the coding processes, each idea units was also characterized in terms of three levels of selective attention and knowledge-based reasoning for the purpose of revealing any developmental shifts if they existed. For this step, a framework adopted from van Es (2011) was used. How this framework was created and how it was useful in characterizing teachers' levels of noticing is described in the next section.

Table 3.5. A Sample Table for Analyzing Teachers' Nature and Level of Noticing in a Meeting and Interview

	SELECTIVE ATTENTION							KNOWLEDGE-BASED REASONING				
DIMENSIONS	**Attend to non-mathematical aspect**	**Comments toward general features of the solution**	**Discrepancy**	**Confusion because of its complexity**	**Attend to mathematical aspect**	**Comments toward entire strategy with substantial detail**	**NOTICING LEVEL**	**Description**	**Evaluation**	**Interpretation**	**Engage in make sense of responses**	**NOTICING LEVEL**
G#1					(M)(I)	(M)(I)	FOC(M)-FOC(I)		(M)	(I)	(M)(I)	MIX(M)-FOC(I)
G#2	(I)		(I)		(M)	(M)(I)	FOC(M)-MIX(I)	(M)(I)		(I)	(M) (I)	MIX (M)-FOC (I)
G#3	(I)		(I)			(I)	MIX(I)		(I)	(I)		MIX(I)
G#4					(I)	(I)	FOC(I)			(I)	(I)	FOC(I)
G#5					(I)	(I)	MIX(I)	(I)		(I)		MIX(I)
G#6		(M)	(M)		(M)(I)	(I)	BASE(M)-FOC(I)	(M)		(I)	(M)(I)	MIX(M)-FOC(I)
G#7	(M)				(M)(I)	(M)(I)	MIX(M)-FOC(I)		(M)	(M)(I)	(M)(I)	MIX(M)-FOC(I)
G#8	(I)	(I)			(M)(I)	(M)	FOC(M)-MIX(I)		(I)	(M)(I)	(M)	FOC(M)-MIX(I)
G#9	(I)		(I)		(I)		BASE(I)	(I)	(I)			BASE(I)
G#10					(I)	(I)	FOC(I)			(I)	(I)	FOC(I)
G#11					(M)(I)	(M)(I)	MIX(M)-MIX(I)	(M)	(I)	(I)		BASE(M)-MIX(I)
G#12		(I)			(M)(I)	(M)	FOC(M)-MIX(I)	(I)	(I)	(M)	(M)	FOC(M)-BASE(I)
BROAD							FOC(M)-MIX(I)					MIX(M)-FOC(I)

Note. M = meeting; I = interview; BASE = baseline level; MIX = mixed level; FOC = focused level.

Table 3.6. Framework for Analyzing Teachers' Noticing of Students' Mathematical Thinking

	Baseline Level	Mixed Level	Focused Level
Selective Attention	Attend to non-mathematical aspects of the work	Attend to mathematical aspects of the work	Attend to mathematical aspects of the work
	Comments towards general features of the solution	Comments towards general features of the solution	Comments towards entire solution with substantial details
	Discrepancy between responses and teachers' comments on that	Begin to attend to details	
	Confusion because of its complexity		
Knowledge-based Reasoning	Making descriptive and/or evaluative comments	Making descriptive and/or evaluative comments & Provide interpretive comments	Attempt to make sense of the solution approach Provide interpretive comments
		Making descriptive and evaluative comments & Begin to make sense of the solution	

3.8.2 Use of the framework for analyzing and presenting teachers' noticing

Van Es (2011) proposed a framework for investigating developmental trajectory of teachers' noticing of students' mathematical thinking. In this study, a similar framework was formed by the researcher with slight modifications. These modifications were necessary because, as opposed to using video excerpts from the classroom as in van Es' study, in this study students written works were used as practice-based materials around which teachers were provided with professional development experiences. This framework enabled the researcher to determine each teacher's levels of selective attention and knowledge-based reasoning at a group level.

Table 3.6 shows the components of the framework. This framework was formed by using the coding categories (see Table 3.4) and three levels of selective attention and knowledge-based reasoning were identified. First level is called baseline level. In baseline level selective attention, teachers pay attention to non-mathematical aspects of students' works and they focus on general features of the solution approaches. They do not refer to details of the solutions. These two features were considered as two robust evidence of baseline level selective attention. Moreover, Jacobs et al. (2010) consider two more indicators of lack of evidence of teachers' attention to children's solution strategies.

First one is a discrepancy between students' solutions and teachers' comments on them. Second one is teachers' confusion about solutions due to their complexity. In this framework, these two dimensions were also used as indicators of baseline level selective attention. If an idea unit had two or more of these characterizations, it was labeled as baseline level selective attention whatever it was meeting or interview data (see Table 3.5). In baseline level knowledge-based reasoning, teachers' statements on students' works are highly descriptive and evaluative. If an idea unit had one or both of these characterizations, it was labeled as baseline level knowledge-based reasoning whether it was meeting or interview data (see Table 3.5).

Another level is called focused level. In focused level selective attention, teachers attend to mathematical aspects of students' works and their comments include referencs to mathematically significant aspects of the solution approaches. If an idea unit had both of these characterizations, it was labeled as focused level selective attention in both meeting and interview data. In focused level knowledge-based reasoning, teachers attempt to reason about students' solutions in an effort to understand underlying meaning and make interpretive comments on what they attend to. If an idea unit had both of these characterizations, it was labeled as focused level knowledge-based reasoning (see Table 3.5).

A third level which can be conceptualized between a baseline and focused level is called mixed level. When an idea unit has characteristics from both baseline level and focused level, then it is labeled as mixed level. For example, in Table 3.5, teacher's selective attention to a group's solution (e.g. G#2) was labeled as a mixed level since while he attended to details of the solution (i.e., an indicator of a focused level), he also attended to non-mathematical aspect of the work and there was a discrepancy between students' solution and his comment on that (i.e., indicators of a baseline level). On some occasions, even though a teacher's comments on a group's solution referred to mathematical aspects and included some key aspects of this solution, they were labeled as mixed level rather than focused level (e.g. G#5; G#11 in Table 3.5). In this case, the teacher does not have a holistic approach in terms of his focus on all important details and steps of students' solutions as one would expect at a focused level. Similarly, a teacher's knowledge-based reasoning in investigating a group's solution was labeled as a mixed level (e.g. G#1 in Table 3.5) since while he engaged in making sense of the solution (i.e., an indicator of a focused level), he made evaluative comments rather than making interpretations (i.e., an indicator of a baseline level).

Once the coding of data was completed, the researcher obtained tables indicating teachers' nature and levels of selective attention and knowledge-based reasoning for each group, in each of the meetings and interviews. As the next step, for every interview and meeting, teachers were assigned a level for their selective attention and knowledge-based reasoning. This was done by calculating the percentages of each of the three levels, baseline, mixed or focused, and assigning the level having the highest percentage as the teacher's broad level in that particular meeting or interview. Thus a teacher with the distribution of selective attention at a baseline level for 50% of the groups, at a mixed level for 30% of the groups and a focused level for 20% of the groups during an interview, was labelled to be at a baseline level for this interview (see Tables 4.7 and 4.8). Yet, these broad level labels were only used as guidelines and the actual distribution of levels according to groups were taken into consideration while interpreting the data. Broad levels of selective attention and knowledge-based reasoning were used at analyzing the developmental trajectories of the teachers, in terms of the shifts that took place in their noticing.

3.9 Validity and Reliability Considerations for the Study

Validity and reliability are the pillars of a scientifically acceptable work. However, they are considered in different ways for different approaches to research. As discussed previously, this study adopted a qualitative approach. Unlike in quantitative studies, issues of reliability (i.e. stability or consistency of responses over time) and generalizability (i.e. the extent to which findings can be generalized to other settings) do not carry major importance for qualitative studies (Creswell, 2003). Qualitative studies target providing rich descriptions of the context and the participants of the study and the main aim is making sense of what is going on in that particular setting. Hence, it is the reader who needs to make considerations about how generalizable findings from the study are to other settings. In this study, the methodology and results chapters provide rich information about the context and how results are reached in order to give the readers such opportunities.

On the other hand, validity, i.e. whether findings from a study are accurate, is an important consideration for qualitative studies. This idea is generally referred to as trustworthiness in qualitative research (Hammersley, 1992) and researchers are expected to provide justifications and detailed explanations for the findings that they present as a result of their analysis. There are various suggestions for ensuring trustworthiness of qualitative studies (Creswell, 2003). The following strategies were implemented in this study for this purpose.

Data were collected through various methods (focus group meetings, one-to-one interviews, observations) in order to allow inquiry of a phenomenon from different angles. This process, referred to as triangulation in research (Cohen et al., 2007) is used to ensure that premature interpretations are avoided by capturing the complexity of human experiences.

Rich descriptions of the data and how the data were analyzed are presented in the following chapters so that the readers are given opportunities to make sense of researcher's reasoning and claims (Creswell, 2003). Findings are discussed in detail, in a context of extracts taken from the meetings and interviews.

A considerable part of the analysis conducted for the study went through debriefing by another researcher in the field of mathematics education (Creswell, 2003). Various issues were discussed and negotiated. This allowed the researcher to review her analysis and sense making process.

Throughout the year, the researcher spent a considerable amount of time in the research setting, and engaged in both formal and informal encounters with the participants. This allowed the researcher to base her interpretations on her rich experiences of the setting and teachers' approaches.

Reliability was also addressed in this study through conducting intercoder agreement checks, as briefly expressed before. Creswell (2007) recommends intercoder agreement checks particularly in the case of highly interpretive coding processes. In this study interrater reliability was 80 %, which is recommended by Miles and Huberman (1994).

.

CHAPTER 4

RESULTS

In this chapter four teachers' nature and development of noticing throughout the study will be presented in light of the findings from the three investigations where data were collected through meetings and interviews. Noticing will be discussed in terms of selective attention and knowledge-based reasoning. While the shifts in noticing will be investigated through the sequence of teachers' comments on students' ways of thinking, the results will also be presented with an eye on the links between the two components of noticing: selective attention and knowledge-based reasoning.

4.1 Nature and Level of Teachers' Noticing in the First Investigation

4.1.1 Nature and level of teachers' selective attention

4.1.1.1 Findings from the first meetings

In the first noticing meetings conducted in each school, Selda's and Kutay's selective attention was at a baseline level while investigating groups' works. Rana's and Ayfer's selective attention was also at a baseline level. Contrary to other three teachers, Rana's selective attention to half of the groups' works was at a mixed level, when her levels of attention were considered at a group level. In what follows, these findings will be elaborated.

In the first meeting, Kutay's and Selda's selective attention was at a baseline level because their comments on the groups' solution approaches were mainly general and did not refer to the details of the solutions. Furthermore, some of their comments were focused on a range of issues on groups' works not related to mathematical aspects of the work. Also, Selda described a group's solution approach in a manner that was inconsistent with the students' solution. Following is a presentation of examples from the meeting supporting these findings.

In the following extract quoted from the first meeting, Selda expressed her observation of groups' works:

> **Selda:** I investigated in four categories. In one I thought they made the drawings and tried to place the cars parallel to the road. All the groups did this. But there is one group which tried to solve only by using this idea of being parallel. In the second one they tried to make use of trigonometric ratios and park the cars with an angle but they could not reach a solution. There are groups that used only Sin10 and worked with no other angle. In the third solution strategy, they tried to find the number of cars through the areas occupied by the cars and the parking spaces. And there is also one where they tried to find the solution by using the Pythagorean Theorem.

In this extract, Selda referred to different solution approaches by describing mathematical topics that these approaches were based on. However, these statements were so general that they did not include any reference to details of these solutions. Such comments from the first meeting were considered as indicators of her baseline level attention. Similarly, in the first meeting, Kutay's comments on groups' works were also very general and did not include any reference to details of the works. He made comments such as "there were groups that tried to solve it by using areas", "there was a group that tried to place them perpendicularly", but he did not elaborate on how students solved the problem by using these methods.

Furthermore, as mentioned before, some of Kutay's and Selda's comments were based on a range of issues on groups' works, rather than mathematical aspects of the works. These comments did not include any reference to solution approaches. They focused on other aspects of the works such as content of the reports not related with mathematics and appearance of the sheets. For instance, some groups had provided explanations related to their failure to solve the problem in their reports. Such explanations did not have any mathematical content. Kutay and Selda paid attention to these explanations and stated that such explanations were interesting for them:

Kutay: They also wrote on the worksheets that…erm… they even expressed that they did not understand the question at first. I mean, they wrote that they first looked at the question for about 10 minutes in confusion and then they produced some ideas.
Selda: Yes, their power of expression there is very good.

They also made statements on issues relating to the appearance of the sheets ranging from the quality of the pictures students drew for modeling the problem to whether students expressed their solution in their reports in a tidy manner. In the following quote from the first meeting, Selda and Kutay made some comments on whether students wrote their solution report clearly and in a tidy manner and how well the students drew the pictures they used to model the problem:

Selda: But they made a very nice drawing. For example cars are very neatly drawn, sentences are beautiful.
Kutay: They also understood the question well. I mean, they left spaces…I mean while parking the cars, that area is important. They did not attach the cars to each other.
Selda: There's also a "thank you" at the end. That's a very meticulous group.

Rana's and Ayfer's levels of attention in the first meeting were similar to Kutay's and Selda's levels, with some slight differences. Ayfer's attention was at a baseline level because her remarks did not contain mathematically significant aspects of the group's solution approaches. Moreover, some of her comments were focused on various issues other than the mathematical aspects. Similarly, Rana's attention was at a baseline level when all her comments on all the groups' works were considered. However, her attention to half of the groups' works was at a mixed level. While investigating these groups' works, Rana attempted to pay attention to details of students' solution approaches after the researcher prompted teachers to examine the solutions more carefully.

In the following discussion on a group' solution as shown in Figure 4.1, Rana's and Ayfer's comments were considered as evidence of their baseline level attention with respect to common features of this level:

Researcher: Here, what kind of a solution approach did the students present?
Hande: Even though they could not get the correct solution, this is an exemplary

paper in terms of details in the solution.

Rana: At least they have an order in their presentation of ideas. It looks nice. For example among the sheets I held in my hand, I first investigated this one because it seems to have an order. But to investigate some other sheets that seem to have produced a rather less orderly solution was difficult. Either the solution is right or wrong presentation is important.

…

Researcher: What have they done as the solution approach?

Ayfer: They made use of trigonometric ratios.

Hande: Yes, trigonometric ratios are used.

Rana: And angle values, I mean, trigonometric values of angles are used. They progressed through assumptions.

Ayfer: Yeah

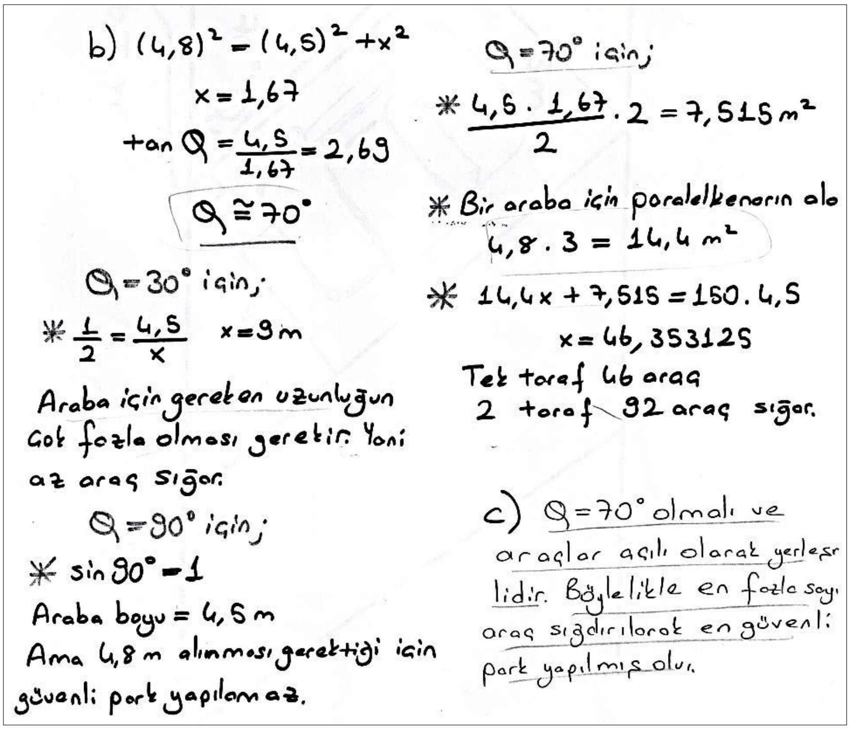


Figure 4.1. A sample of students' works in Parking Spaces on the Street activity

At the beginning of the dialogue, after the researcher's prompt for teachers to explain the solution approach, Rana made a comment on appearance of the sheets and whether students expressed their solution in their report in a tidy manner. These statements provided evidence of her baseline level attention to the work. Further in the dialogue, after the researcher's second prompt, Rana and Ayfer attempted to describe the solution with broad terms such as "trigonometric values of angles are used" or "they made use of trigonometric ratios" that did not point to key aspects of the solution. Actually, in this students' work, one of the noteworthy parts of the solution was that students formed an equation to balance the sum of the spaces occupied by the cars and the useless space at the sides with the total area of the parking space in order to find how many cars would fit into the parking space. Despite the researcher's prompt, teachers did not pay attention to this noteworthy part of the solution.

4.1.1.2 Findings from the first interviews

In the first noticing interviews, Kutay's, Selda's and Ayfer's selective attention was at a baseline level. On the other hand, Rana's selective attention was at a mixed level. When teachers' selective attention was considered at a group level, Selda's and Ayfer's selective attention to most groups' works was at a baseline level. Different from Selda's and Ayfer's comments, Kutay's comments on some groups' works had elements of a mixed or a focused level selective attention. While Kutay's attention in investigating half of the groups' works was at a baseline level, his attention in investigating other groups' works was at a mixed or a focused level. Similarly, Rana's attention while investigating most groups' works had elements of a mixed level or a focused level. In the following paragraphs, these findings will be elaborated with examples from the first interview.

One of the reasons for Selda's baseline level attention in the first interview was that she predominantly commented on superficial features of groups' solution approaches. In other words, her comments on groups' works did not refer to details of the solutions. For instance, in Figure 4.2, there is one of the solution approaches adopted by a group. In this work, although students could not solve the problem and find the correct answer, they endeavored to adopt a solution approach. Two aspects of this approach were mathematically significant. First of all, they placed the cars with an angle and in a way it would not spill onto the street and modelled the givens accordingly and obtained a right triangle from this model. They analyzed whether cars would spill onto the road when placed with specific angles. In their solution, they showed the side length properties in 30-60-90 triangles. They used these length properties to see whether the car would spill onto the road when placed with a 30 degree angle. Following are Selda's comments on this solution that did not contain any reference to these aspects. Her comments focused on students' use of sin30 and the 30-60-90 triangle:

> **Selda:** They used only Sin30 too. These thought about the 30-60-90 triangle, nothing more. Yes, they only used 30-60-90. They must have tried to place with a 30-degree angle. But this is it.

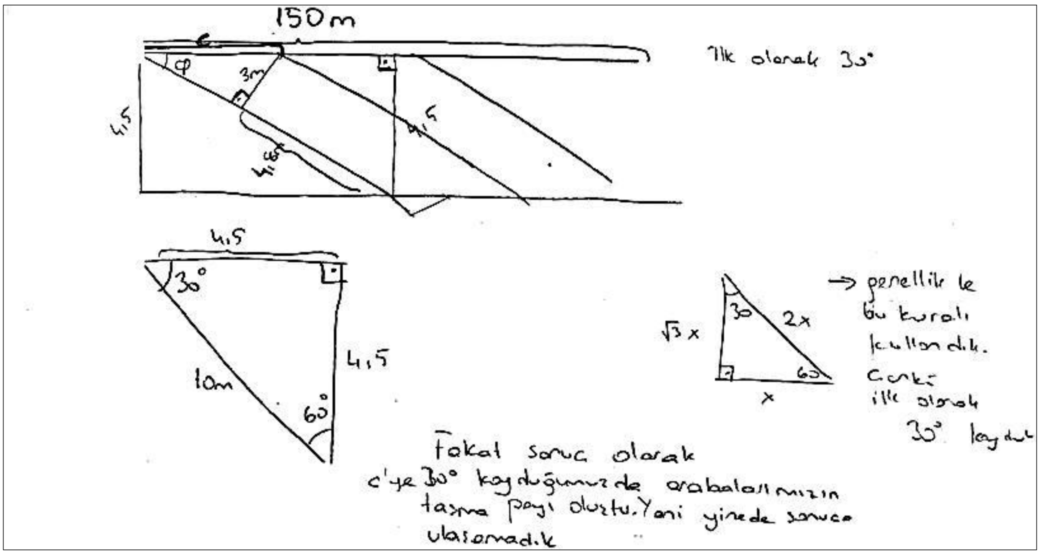


Figure 4.2. A sample of students' works in Parking Spaces on the Street activity

Furthermore, besides focusing on superficial features of groups' solutions, Selda often expressed that she could not make sense of what students tried to do. In the interview, she often did not endeavor to carefully analyze students' works since she was confused about what they did. This was considered as another prevalent feature of Selda's baseline level attention. In the following excerpt, Selda's statements had these two prevalent features of her baseline level selective attention:

> **Selda:** They tried to make use of trigonometry. I did not get how they found 44. They tried to do something more reasonable but how did they find 88 cars? But they are doing direct division.
> **Researcher:** Yes, what did they do here?

Selda: For example cotangent 58, where did that come from? Cotangent 58, I did not get this at all. Where did this 58 come from? I mean, when there is not even a sinus here…I did not get this at all. They used quite a lot of trigonometry, right or wrong, completely used trigonometry. But I did not understand very much what they did.

Here, despite the researcher’s prompt to get her to explain what students did, her remarks primarily consisted of a general description of the solution approach, referring to the topic of the solution, i.e. trigonometry and a specific value they used in the solution, cotangent 58. She did not describe the progress or steps within the solution. Also she frequently stated her confusion about the solution. Additionally, Selda’s comments on some groups’ works included reference to aspects of the works not related to mathematics. This was also considered as an element of a baseline level attention. In the following quote, her statements on a group’s work included this kind of reference. Besides general features of the solution, she paid attention to whether students wrote their report in a tidy manner. Therefore this quote provided further evidence of her baseline level attention.

Selda: G5 used that too. They explained quite a lot. They wrote Sin10. They found the parallel case, that 31 cars could park etc. Reasonable. Theirs was different. They expressed it very neatly. I thought they approached it systematically since they wrote very neatly.

Likewise, Ayfer’s selective attention in the first interview was at a baseline level because she tended to comment on general features of groups’ works. Although she implemented this activity in her own classroom and thus she had a chance to closely observe students’ solution process, her statements included references to general features of the solution approaches. She made comments such as “this group thought about both ways” or “these guys used trigonometric ratios” that did not include references to details of the entire solutions. For example, in the following extract Ayfer commented on a group’s solution, which is illustrated in Figure 4.3:

Ayfer: They found the sum of the areas of the triangles, and subtractec that from the total area. Well, in fact, I don’t think this is a correct method.
Researcher: Why?
Ayfer: Well, not a mistake but, not a practical way. Because every car’s dimensions are different. If you put a small car, then a large car next to it and then you’d have more space left here. You don’t have the chance to connect it here. The space here is

different and the one there is different. Therefore, use of areas is not a good approach in my opininon.

In this work, students first modeled the problem by separating the total parking space into shapes that they can calculate the areas of. They formed an equation by subtracting the dead areas from the total parking space and equating that to the space to be occupied by cars, where the number of cars was the unknown (See Figure 4.3). In her comments on this solution, Ayfer just pointed out students' use of areas of triangles and did not give any reference to the equation and how students produced this equation. As a result, this extract provided robust evidence of her baseline level selective attention.

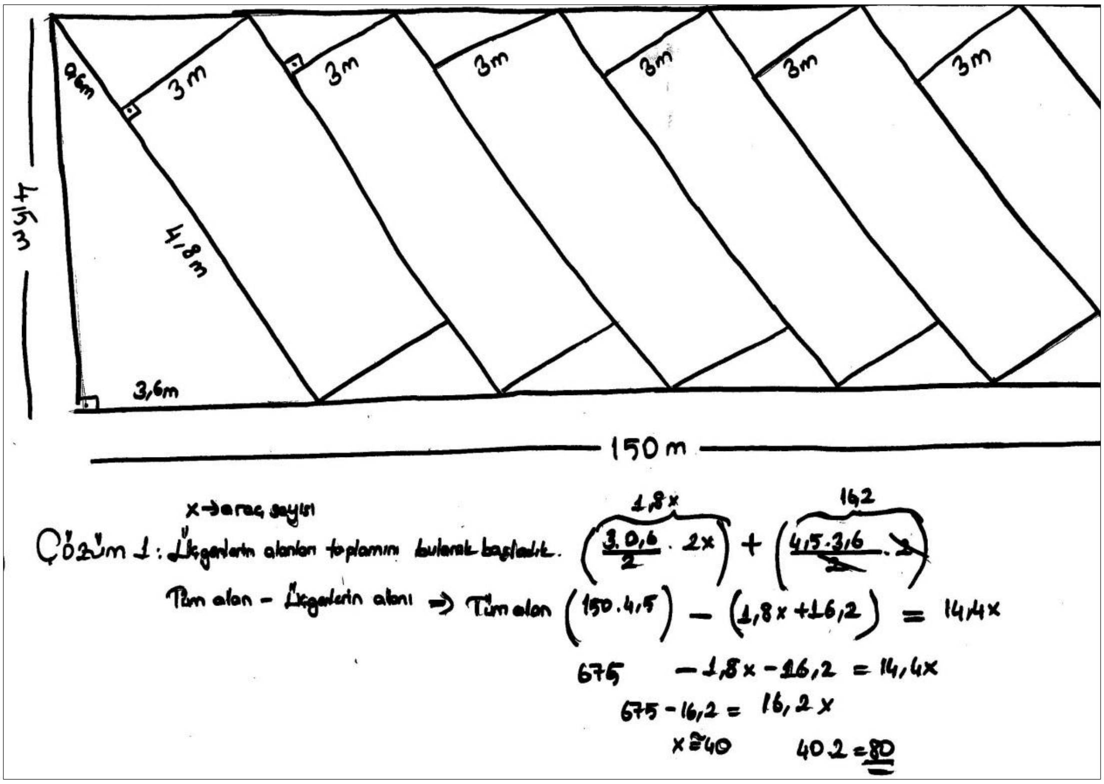


Figure 4.3. A sample of students' works in Parking Spaces on the Street activity

Like Selda's focus on non-mathematical aspects, on some occasions, Ayfer was interested in a range of issues that were not related to the solution approach. For example, she particularly paid attention to who the group members were since she did the implementation in her class and the groups consisted of her students. These kinds of comments were regarded as evidence of a baseline level attention. In the following quote, her remarks on a group's work contained this kind of comments, in addition to a general restatement of the solution such as "this group tried both ways". Therefore it provided further evidence of her baseline level selective attention:

> **Ayfer:** This group tried both ways. But again 4.8. In fact, Filiz and the likes are very good students in this class. I was their teacher last year. Who else was in this group? Özlem, hmm…there are good students in this group. Let's put that aside, at least they tried two ways.

Like Selda's and Ayfer's levels of attention, Kutay's selective attention was at a baseline level in the first interview. While investigating half of the groups' works, his attention was at a baseline level because he made general comments that did not refer to details of the solutions. For example, in the following quote, Kutay's comments on a group's solution provided evidence of his baseline level attention to this solution:

> **Kutay:** They didn't place geometrically anyway. Look…classical.
> **Researcher.** What did they do?
> **Kutay:** They did here…the sinus here, they tried to build it on the right triangle here, only on the right triangle. But in fact they needed to divide it here, they needed to see the dimensions of the cars here. Since they couldn't see that, they found some trigonometric ratios here, but they found them without thinking what it would be good for.

In this solution approach, students found the projection of the length of the space a car would occupy along the road, by using the useless space in front of the car in the model formed. And they calculated different values by using trigonometric ratios of various angles such as 10, 20, 30, and 40. Kutay's description of this solution approach such as "they found some trigonometric ratios here" did not point to this aspect of the solution.

Similarly, his comments on another group's work (see Figure 4.2) provided robust evidence of his baseline level attention while investigating the work.

> **Kutay:** For this one, they have some easy angle values. We saw somewhere 45, here they have 30 and 60. In fact the area here, is half the area of the rectangle there but

> since the car came partly there, it didn't…They used Pythagorean but they couldn't do it since they took it as a fixed triangle.

This comment had features of a baseline level attention because of two reasons. Firstly, as mentioned in one of the previous paragraphs, mathematically significant aspect of this solution approach was that they analyzed whether the car would spill onto the road by when there was a 30 degree parking angle by using a 30-60-90 triangle. Kutay's comment on this solution did not point to this aspect of the solution. Secondly, there was a discrepancy between his description of the solution and students' real solution. For example, he stated that students used Pythagorean theorem in their solution. However, they did not use it. This discrepancy was also considered as evidence of a baseline level attention.

In the first interview, in some cases, Kutay paid attention to issues other than mathematical aspects of groups' works. While investigating some groups' works, he attended to content of the reports not related with mathematics and appearance of the sheets. He was also interested in who the group members were since he did the implementation in his class and the groups consisted of his students. The following comment has manifestations of both of these issues:

> **Kutay:** These guys have made a big confession about what they did and did not. They can in fact be good social science students. In fact, not in a science like mathematics. There are even very talented kids in this group. They ordered the drawings very well. Look, how nicely they drew the parallel cars. These are students who could do well in social sciences or arts.

As mentioned before, Kutay's statements on some groups' work had elements of both a baseline and focused level, point to a dual nature of his selective attention to this group's work. These chunks of statements were labeled as a mixed level. On some occasions, his statements provided evidence of a focused level attention. Following is a presentation of examples from the interview supporting these findings.

> **Kutay:** For example, what did these guys do? Hmm, they tried to calculate the dead areas, calculated the things over there. That 1.7 here, but they made a mistake…1.7 and there is a 1.7 here, so there's two of them. They subtracted 3.4 from 150. That's their mistake. What did they say… on a side, 35 cars. And they tried to find those

> things from Pythagorean theorem but the thing is always wrong, they didn't place it on the figure. They took that part as 3 from the right isosceles triangle. They took it as 45 degrees. Taking it as 45 degrees directly and building their solution on 45 degrees. I know this kid. He got the highest mark from me in the math exam.

Here Kutay was trying to describe the steps in the students' solution which were accompanied by his efforts to understand what they did. He was trying to understand how they used specific values they referred to in their solution .However, towards the end of the extract, he was commenting on who the student was and his experience in mathematics, rather than aspects of the solution. Therefore, he was focusing on aspects that were not directly related to mathematical elements in the solution. Such a dual nature of attention to certain elements can be described as a mixed level of selective attention.

While investigating students' works, Rana's selective attention to many groups' solutions was at mixed level because she tended to comment on details of the solutions. However, there were two reasons for why these comments were labeled as a mixed level rather than a focused level. Firstly, these kinds of comments contained the mixture of evidence from both a focused level and a baseline level selective attention. A typical example of this coexistence of levels can be seen in the following extract.

> **Rana:** Then in part b the same mistake again, they carried all of 4.8 here. Then, from trigonometric values, they decided that the angle should be approximately 70 degrees. Taking the angle as 70, they found c. They did the operations by finding c. I mean, their common mistake here is 4.8. They said if ϴ is greater than 70, that is if it's 90 degrees, it would spill out. Since they took 4.8 wrong, they couldn't see that it would spill out for 70 degrees.

Here, Rana described the solution method by giving references to the steps in the progress of the solution, which can be considered as evidence of a focused level selective attention. On the other hand, there were some discrepancies between her description of the solution and students' real solution, which is considered as evidence of a baseline level selective attention. She stated that students found the length of c by using the angle of 70. However, rather than attempting to find the length of c, students had formed an equation by using the areas to find out the number of cars (see Figure 4.1). Her comments, having elements of both a baseline and focused level, point to a dual nature of her selective attention to this group's work. In such cases, the level of selective attention was labelled as a mixed level, which is between a baseline and focused levels.

Secondly, considering why Rana's comments were labeled as a mixed level rather than a focused level, while investigating a group's solution, Rana focused on students' thinking, yet not on all aspects or all elements of the solution in some occasions. While such selective attention is beyond a baseline level, it does not have a holistic approach in terms of its focus on important details and steps of students' solutions as one would expect at a focused level. Therefore such cases of selective attention are labelled as a mixed level. In the following excerpt, Rana's comments on a group's solution, as shown in Figure 4.4, can be considered as an example of her mixed level selective attention in terms of aforementioned nature of the mixed level

> **Rana:** They said alpha, right? Beta, 90, alpha and here it is beta. They called here beta. Then I'm trying to understand what they did. Here it's x, 4.5 minus x, OK. Here it's 3, that is 4.8 times x divided by 3. They took K as sinus, right? Here they took it wrong. They took the angle values wrong.

In this statement, she relatively focused on some details of the solution such as rate of similarity that students formed by using the similar triangles. However, rather than attending to the entire solution method, she paid attention to a part of the method. For example, she did not point to how students found the number of cars after finding the area a car parked with an angle would occupy in the parking space.

4.1.2 Nature and level of teachers' knowledge-based reasoning:

4.1.2.1 Findings from the first meetings

In the first noticing meetings conducted in each school, teachers' knowledge-based reasoning was at a baseline level. However, when teachers' comments were studied separately for each group, teachers' levels of knowledge-based reasoning were found to be at a mixed level while investigating some groups' works, often through the researcher's prompts. Even so, the most dominant characteristic of all teachers' comments on students' works in the first noticing meeting was that they were generally descriptive and/or evaluative in nature. These findings are elaborated in the following paragraphs.

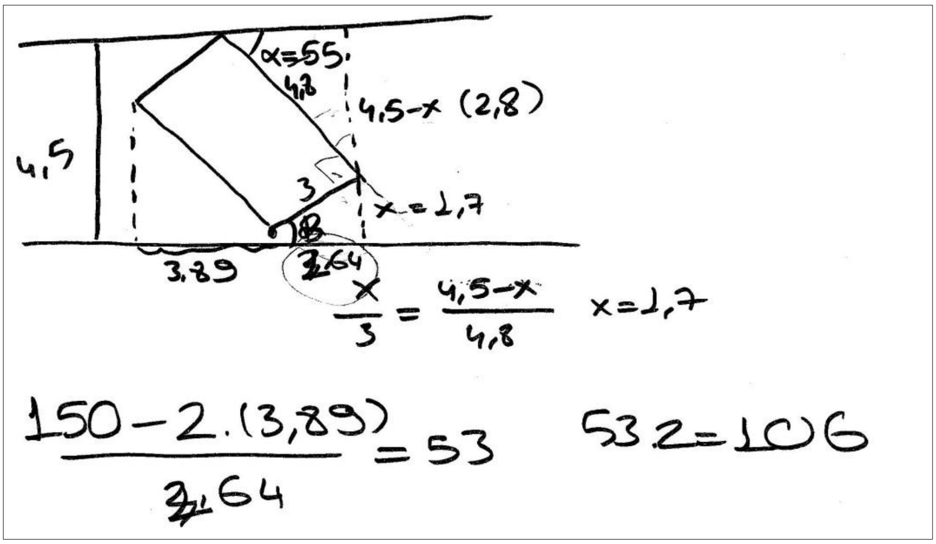


Figure 4.4. A sample of students' works in Parking Spaces on the Street activity

Accordingly, in the first noticing meetings, Kutay and Selda predominantly described or evaluated what they noticed, which provided evidence of their baseline level knowledge-based reasoning. Kutay implemented this activity in his own classroom and therefore he had better knowledge of students' solution processes. However, in the following extract quoted form the meeting, he made statements about students' solution approaches that were dominantly evaluative in nature:

> **Kutay:** Well, they did some things about the problem but all their logic was wrong. There was no group that fitted their drawing well and used the trigonometric values. Two of my groups tried to do some things. But even what these two groups did was nonsense. I mean, they tried to do something mathematically but it was nonsense

Kutay was trying to highlight students' failure to solve the problem. While doing that, he used words such as wrong, nonsense which indicate an evaluative disposition. On top of such evaluative comments, he often expressed what students should have done, but what they did not. This can be seen in the following extract:

Kutay: They did not do the drawing and indicating the unknowns on the figure. Some of them took these all as 4.8. Unless they show the unknowns and the height on the figure, they can't solve it.

Like Kutay's comments, Selma's comments on groups' works were mostly descriptive and/or evaluative in nature. For example, in the extract that was presented previously, she simply described different solution approaches with general terms such as "they used trigonometric ratios" or "there is a group that made use of areas, they found by using the areas covered by cars". She also made evaluative comments on some aspects of students' works such as "their expressions are powerful" or "they made a very nice drawing".

The following excerpt also consisted of Kutay's and Selda's statements that exhibited features of a baseline level reasoning. At the beginning of the conversation, the researcher prompted teachers to elaborate on their comments on the solution by asking a specific question. In response to this prompt, Kutay made an evaluative comment by highlighting what students should have done but what they did not. Despite another teacher's endeavor to attract teachers' attention to a different aspect of the solution, Kutay continued to make evaluative comments by highlighting a mistake that students made. Similarly, at a baseline level, Selda described the solution with general terms such as "this group made use of Pythagorean Theorem".

Researcher: Why did they find x here?
Kutay: In fact they did not think about that… the height thing…they did not calculate it in the figure. They only calculated the c and left it there.
Kaan: They did it in your group…11.
Kutay: No, they took it really wrong.
Selda: That group made use of Pythagorean theorem.
Kaan: They have it on the last page of that group's work. They placed the car with an angle. And they also considered the way car went out of the area.
Kutay: But that group took 4.8 wrong. While placing values on the figure, they took 4.8 as one side of the parallelogram.

Similar to Kutay's and Selda's comments, Rana's and Ayfer's comments on students' works were predominantly descriptive and/or evaluative in the first meeting. Rana primarily described what she noticed while investigating groups' works and made evaluative comments on some of these works. These kinds of comments gave evidence of her baseline level knowledge-based reasoning. In some cases, as well as making descriptive and evaluative comments, she also attempted to make sense of the groups' solutions, which was considered as a feature of a focused level knowledge-based reasoning. Due to this dual nature of their comments on some groups' works, her knowledge-based reasoning was labeled as a mixed level.

A typical example indicating the descriptive nature of Rana's comments on the groups' works was a comment she made at the beginning of the meeting. She made a descriptive comment that oversimplified students' various solution approaches. She also highlighted a mistake that almost all students made:

> **Rana:** There was not a different method, they were all parallel to our anticipations. I mean, they both wrote the trigonometric ratios by turning them into right triangles. And did it by dividing the total area by the area of the parallelogram. A common mistake made by almost all the students is that they took the hypotenuse as 4.8.

Ayfer's comments on groups' works in the first meeting were also mainly descriptive and/or evaluative in nature. Like Rana, through the researcher's prompt, she endeavored to make sense of some group's works. However, she also made descriptive and evaluative comments on these particular groups' works, which was the reason why her knowledge-based reasoning was labeled as a mixed level.

Following is an example that contained evidence of both Rana's and Ayfer's baseline level knowledge-based reasoning in the first noticing meting. The group's solution, as illustrated in Figure 4.1 and the extracts from the discussion of teachers on this solution were already presented in the previous section. In this excerpt, Rana's first comment on this solution method was evaluative in nature: "In this group every student analyzed especially part a beautifully". Despite the researcher's prompt by asking a general question to invite the teachers to explain students' solution, Rana continued to make an evaluative comment on the work: "At least there is an order when they are defending their ideas. It looks nice. For example, among the papers I held in my hand, I first analyzed this because this one has an order." In spite of the researcher's second prompt, Rana and Ayfer sustained describing the solution with general terms such as "They made use of trigonometric values of angles." These statements exhibited features of a baseline level reasoning since they only displayed description and evaluation.

Rana's and Ayfer's statements on another group's solution provided further evidence of their baseline level reasoning in the first meeting because they consisted of primarily descriptive remarks. For example, Rana made statements on this group's solution such as "This group has considered both assumptions" and "The most noteworthy feature of this group's work is that they tried both approaches. They tried both trigonometry and areas." These remarks were predominantly descriptive in nature and they oversimplified the solution approach. Further in the discussion, Ayfer made the following remarks:

> **Ayfer:** And there was also that problem. They answered part a where it was asking what would happen if they were parallel to the road. Then they focused on…the thing…they started working on this angle. They thought the answer would come out of that angle".

This comment had an interpretive dimension because there was an attempt to make sense of students' failure. She offered a probable reason for why students could not find the correct answer. Yet, this claim was not grounded in students' works. She implemented this activity in her own classroom and therefore she had better knowledge of students' solution process. Her comments were grounded in this contextual knowledge about the group of students.

As mentioned previously, although Rana's and Ayfer's statements on groups' works were mostly descriptive and/or evaluative, they engaged in a close analysis of students' solutions, through the researcher's prompts. In these incidents, the researcher's role was crucial. By using general or specific prompts, she encouraged teachers to reason about the solutions and make sense of underlying mathematical thinking.

Following is an example of how teachers engaged in making sense of a group's solution approach following initial descriptive comments. This was a unique solution in comparison with the other groups' solutions (see Figure E1). In this solution approach, students placed the cars on the drawing so that they would not spill onto the road. They used the parallelograms formed by considering the unused areas in front of and behind the cars and taking both the long and short sides as the base, they used the area formulas twice and equated the areas they found for the two cases. They obtained a second order equation with the unknown x. From the x value which is the solution of the equation, they tried to obtain the tangent of the angle and thus tried to find the parking angle of the car.

In the excerpt below, Ayfer first made descriptive comments with general terms and judgmentally referred to the use of angle. After the researcher's specific questions such as "According to what did they form this equation?" or "In the drawn figure, which is the parallelogram exactly?" Rana and Ayfer attempted to make sense of the solution and finally they comprehended this unique solution approach. These statements combined, point to both baseline and focused level knowledge-based reasoning, hence a mixed level.

> **Ayfer:** They tried to find teta from areas. They used tangent 90. But tangent 90 is undefined.
> …
> **Researcher:** According to what did they form this equation?
> **Ayfer:** They used the area of the parallelogram.
> **Hande:** Ok, but what is that equation? Did they not write it from the tangents? That's what I thought.
> **Rana:** c times 4.5.
> **Hande:** Opposite over…they wrote the sinus, sorry. Opposite…yes because if we call it sinus θ, opposite over hypotenuse is three over c. They wrote the sinus, right? Sinus.
> **Rana:** Yes, 3 over c is equal to 4.5 over x plus 4.8.
> **Ayfer:** Yeah, x plus 4.8.
> **Researcher:** In the drawn figure, which is the parallelogram exactly?
> **Hande:** That one friends, that parallelogram is in fact there. That shape.
> **Ayfer:** Yes, yes.
> **Rana:** If we write the area for that parallelogram?
> **Hande:** Well…It is long side times height.
> **Ayfer:** c times 4.5.
> **Rana:** The right side of the equation is x plus4.8 times 3. They thought about both sides.

Another example of how teachers made sense of students' works through the researcher's prompts can be seen in the following dialogue where Rana and Ayfer engaged in an in-depth analysis of students' solution. This was a sense-making effort by the teachers. However, this analysis was interrupted by Ayfer's evaluative comment that highlighted the common mistake almost all students made. This interruption brought the sense making process to an end before teachers went through the entire solution. The extract shows the dual nature of the comments, i.e. both evaluative and working within an inquiry to make sense:

> **Researcher:** There is a solution related to the area on the right hand side. What exactly did they do, how did they form the equation?
> **Hande:** They wrote the calculations about the area of the parallelogram.
> **Rana:** 1.67 times 4.5 divided by two and multiplied by two.
> **Funda:** Well, they calculated by Pythagorean theorem, to find what x should be.
> **Rana:** There, they thought about 1.67 as the…
> **Ayfer:** They must have thought about this as the base.
> **Rana:**… base there at the bottom. The height is thought as 4.5 and this is 1.67.
> **Hande:** Did they find it by using the similarity of triangles?
> **Funda:** They used proportions to decide which value of tangent gives the equality.
> **Ayfer:** Then, when they placed the 4.8 there on the figure that way, they got the answer wrong

4.1.2.2 Findings from the first interviews

In the first noticing interviews conducted after the first noticing meetings, three of the four teachers' knowledge-based reasoning was at a baseline level. Kutay's knowledge-based reasoning was at a mixed level. When it was considered at a group level, however, Rana's knowledge-based reasoning also had features of a mixed level or even a focused level without the researcher's prompt. Similarly, Ayfer's knowledge-based reasoning had features of a mixed level through the researcher's prompts. In this section, these findings are shared.

In the first noticing interview, Selda's knowledge-based reasoning was at a baseline level because, her remarks on students' works were highly descriptive and/or evaluative in nature. As in the first noticing meeting, in most cases, she recounted what she noticed with evaluative comments. In the following excerpt quoted from the first interview, her comments on a group's work had features of a baseline level knowledge-based reasoning because it consisted of restatements of students' works with judgmental remarks:

> **Selda:** They did correct drawings and placed them in a parallel way. This group used only Sin30. They thought about 30-60-90 triangle. They did not have much here. Yes, they used only 30-60-90. They must have wanted to place them with 30 degree angles. But that is all, there is not much here. Angles are not calculated at all. They only placed cars in a parallel way. There is not even a single angle value. They only explained qualitatively.

In this excerpt, Selda simply recounted what she observed in the group's solution approach. She also oversimplified the solution by offering an evaluative commentary such as "but that is all, there is not much here". She also assessed the solution by comparing it with the correct solution in her mind by saying "angles are not calculated at all". Consequently, these statements provided robust evidence of her baseline level knowledge-based reasoning.

Likewise, in the following extract, Selda's statement on another group's work was highly evaluative. While she recounted what she noticed such as "they called it the observation angle" or "this group finished at Sin10", she primarily evaluated student's works, i.e., pictures, writing:

> **Selda:** For example they placed the cars with perspective drawing. Straight and inverted. This is something many students would not do. Horizontally but windows, tyres…I mean, they thought about everything. Here, they are placed in the same way. They called it the observation angle. I found this interesting. This group finished at Sin10. 10^{th} graders' power of expression is very good and clear. 11^{th} graders seemed to be in confusion.

Although Selda's remarks on groups' solutions were predominantly descriptive and evaluative in the first noticing interview, she rarely attempted to carefully analyze and make sense of students' solution approaches either through the researcher's prompt, or spontaneously. For example, while investigating a group's work, she first mentioned the group's work by saying: "Here they only found the areas. That means they found the parking area of the parking region and the areas occupied by the cars and divided them. They found 31.25 again because they thought about placing them in a parallel way". These comments were descriptive in nature and they described the solution in a highly general way. When Selda looked at this group's work secondly further in the interview, the researcher attracted her attention to another solution approach students endeavored to adopt. Selda sustained making predominantly descriptive and evaluative commentaries on the approach. In the extract below, she used sentences describing what she noticed such as “they wrote the sinus and found something” or “they took the square of that number”. Moreover, she adopted a judgmental approach while restating what she noticed:

> **Selda:** I didn’t look at that part very much because it was not possible for them to obtain a result from a single equation with two unknowns, they got stuck anyway. Now, they wrote sinus, and they found something. They got the value for 9 digits but what was this worth? Their operations can’t be understood well. And they also did them a little bit complicated. There was a lot of scribbling so I couldn’t interpret that part well. This is the same equation; they took the square of that number. They obtained the values to many digits. I think it was a loss of time

After the researcher prompted her to elaborate on her remarks by asking a question such as “What did students generally try to do and why couldn’t they succeed?”, Selda attempted to carefully examine the same group’s work for the third time. In the following excerpt, she made interpretive comments on what students’ actual difficulty was in solving the problem:

> **Selda:** Besides finding it wrong, I think they didn’t think about where they would use it. That is, they couldn’t set a target. They found sin10 and sin20 but they didn’t know where to use them. They reasoned in a way that since they were given a trigonometric table, they thought they needed to use it. But they couldn’t decide where and how they would use it.

Like Selda’s, Ayfer’s knowledge-based reasoning was at a baseline level in the first noticing interview because her statements on group’s solution were mainly descriptive and/or evaluative. In the following extract, for example, Ayfer’s remarks on the solution were predominantly descriptive and evaluative. She literally described what she noticed in the work. She oversimplified the solution by using general terms in her descriptions, such as “they thought about both approaches” or “here there is more area”. Also she adopted a judgmental stance such that she often highlighted the mistake that students made and the incorrect answer students found:

> **Ayfer:** They divided into areas here. But the mistake is taking that length as 4.8. In fact they thought about both approaches. But the problem is that 4.8. After that they found wrong results all the time. They made use of trigonometric ratios, in fact all of them did this. It is almost on every paper. Here there is more area..

Besides Ayfer’s tendency to make word by word descriptions of what she noticed in groups’ works in the first noticing interview, she primarily adopted a judgmental stance while investigating the works. Her following statements included characteristics of her evaluative stance:

> **Ayfer:** Let me have a look what they did here. They found the sum of triangular areas, then subtracted them from the total area. Well, I think area is not a correct approach. Well, the area…you indicate the area that the cars will occupy. Because each car has different dimensions. For example if you put a small car and a big car next to it, and you occupy less area. Trying to solve it using areas is not a solution approach that would yield a result.

This solution approach was illustrated in Figure 4.3. Accordingly, this group of students developed a solution approach for this problem that no other group thought of. Similarly, this excerpt was previously used to provide evidence of Ayfer’s baseline level selective attention. These statements are considered once more because they also provided robust evidence of her judgmental stance. In this dialogue, she began by making literal descriptions of the approach. Then she suddenly stopped describing and assessed the solution as ineffective in solving the problem. As exemplified in this excerpt, making quick judgments and conclusive claims about the effectiveness of the solution approach was also considered as robust evidence of Ayfer’s baseline level knowledge-based reasoning.

Similar to Selda’s, although Ayfer’s remarks on groups’ solutions continue to be predominantly descriptive and evaluative in nature, she rarely attempted to make sense of students’ solution approaches through the researcher’s prompt. For example, in the extract below, her statements had features of both a baseline level and a focused level knowledge-based reasoning. Therefore, these following statements were labeled as a mixed level. On the one hand, these statements involved evidence of Ayfer’s judgmental stance that she generally adopted while investigating students’ works. For example, at the beginning of the dialogue, her comments focused on the effectiveness and correctness of the solution. Like Selda, she assessed the effectiveness of the solution approach by considering whether students’ solution fitted with a correct solution that she already had in mind. Moreover, further in the dialogue, she made a remark such as “but I didn’t get where the mistake is”. This remark was also considered as evidence of her judgmental stance because, it demonstrated her effort to detect a mistake in this solution that she claimed to exist. Hence, her evaluation oriented comments provided evidence of her baseline level knowledge-based reasoning.

> **Ayfer:** Yes, since they thought about ϴ, that's more correct. They took this length correctly. 4.8 plus x. Ratio of opposite over adjacent. 3 times x plus 4.8. Yes, base times height.. here, they did base times height here. They saw the area of the triangle that way. But…did the take the correct height? What they called as c is this length…
> **Researcher:** Height of which one?
> **Ayfer:** Now, the height for this base, that's OK. Then they must have thought this as the base. But they can't think of that as the base. It should have been this.
> **Researcher:** Did they write the area formula based on the triangle?
> **Ayfer:** Did they write the area of the parallelogram? Hmm, they wrote the area of the parallelogram. OK, this is the area of the parallelogram. Yes. Base times, height…base times height. They found c. But where is the mistake, I couldn't get that.
> **Researcher:** Did you think this had a mistake?
> **Ayfer:** From here, they can find x. Tangent 3 divided by x, they said that would give ϴ. Ohhh, they couldn't solve the second order equation here, that's why. If they solved the equation, they would reach the ϴ and obtain something..

Besides her judgmental stance, these statements also contained evidence of Ayfer's focused level knowledge-based reasoning in that she engaged in making sense of the solution, through both her own initiative and the researcher's specific prompt. In this case, the researcher stimulated her to carefully analyze the solution by posing a specific question such as "did they write the area formula based on the triangle?"

In the first noticing interview, Rana's knowledge-based reasoning was labeled as a baseline level because she mainly made descriptive and evaluative comments on group's works. For example, in the following excerpt, she remarked on a group's solution approach:

> **Rana:** They parked 62 parallel cars. Then, in part b there is the same mistake, they carried all...4.8 here. Then they decided the angle to be approximately 70 degree by using trigonometric values. Taking the angle as 70, they found c and did the operations. I guess they are not referring to the angle here. Are they? They didn't give angle values. They used the areas, they said that there were some empty spaces. They said if ϴ is greater than 70, for example 90, then it would spill out. Since they took 4.8 wrongly, they didn't see that it would spill out even for 70 degrees.

These statements had features of a baseline level knowledge-based reasoning because they consisted of predominantly descriptive comments. She literally described what she noticed in the work. Also, like Selda and Ayfer, Rana assessed the solution by comparing it with the correct solution in her mind, which can be seen in her comments such as “Are they referring to angles? They didn’t give angle values”. This evaluative comment can be considered as a prevalent feature of a baseline level knowledge-based reasoning.

Similarly, another excerpt from the interview contained evidence of her baseline level knowledge-based reasoning. In this excerpt, besides judgmental statements, Rana made primarily descriptive statements on the group’s work (see Figure 4.3). She also made the following comment:

> **Rana:** On this sheet, the sum of triangular areas is found. 3 times 0.6 divided by 2 times x times 2. Then they found the areas of those two triangles. They summed the areas of this triangle and that one. A different mistake on this sheet was, if I’m not mistaken, they assumed a triangle would be formed when they did the drawing there. We thought it could also be a trapezoid. That was the mistake in this solution. But the logic is good; we appreciated their reasoning and approach..

In these statements, Rana first recounted what she noticed in the work. Then she suddenly stopped recounting and made a judgmental comment that was similar to the previous one. That is, she assessed the solution by comparing it with the correct solution in her mind. She expressed what students should have done but what they did not. She ended her statements with another evaluative comment on the work.

Although Rana’s comments on groups’ works were mainly descriptive and evaluative in the first noticing interview, while investigating some groups’ works, she engaged in in-depth analysis of solutions and made interpretive comments on them. Sometimes, her attempts to make sense of a solution resulted in interpretive statements related to this solution, which was considered as evidence of her focused level knowledge-based reasoning. All these in-depth analyses of solutions were initiated by Rana without the researcher’s prompt. Nevertheless, although she engaged in making sense of the solution, this attempt did not always result in an interpretive statement. In this incident, her knowledge-based reasoning was labeled as mixed-level. In what follows, these findings were elaborated with examples of each case. In the interview, Rana attempted to carefully analyze the group solution as illustrated in Figure 4.4. She made an effort to make sense of the solution:

> **Rana:** They said alpha, beta…90…alpha, and here it is beta. They wrote beta here. I'm trying to get what they did next. Here it's x, 4.5 minus x, OK. Here it's 3, that one's the ratio of 4.8 times x to 3. They took K as sinus, right? Here they took it wrong; they took the angle values wrong.

Although, at the beginning, she attempted to make sense of the solution, she stopped investigating suddenly and made an evaluative comment that pointed to the mistake students made. As opposed to this incident, at times, Rana's effort to make sense of a solution resulted in interpretive remarks. In the following excerpt, for example, she initiated an in-depth analysis of the group's solution illustrated in Figure E1. At the beginning of this dialogue, she started with generally describing and recounting the solution such as "Here they called x, they said 3 meters, 4.8. If we find x from the area formula of parallelograms, we can find ϴ from the tangent formula. Why we take all of the width of the road is that we are trying to find the greatest ϴ…" Then she initiated an in-depth analysis of the solution by asking a question about it:

> **Rana**: Could they find ϴ as a value in such an equation? In fact, you need to try, right?1.5 parenthesis, 3 over 2…Hmm, they said 1.5, they did it correctly, I got it wrong. They did it correctly. There is 1.5, and 3 over 2, yes. Ok, let's go on from here. 4.8…9.6, isn't it?9.6.
> **Researcher:** Yes, they multiplied by two.
> **Rana:** They took the square of it, 4x squared plus 38.4 plus 96…yes, this is the equation. They found that as 9.6 squared. Can this equation be solved?
> **Researcher:** You can look at the delta.
> **Rana:** To find its delta…according to the ϴ value they would find from this, they can find c and when they can calculate the number of cars from c, they would probably see that a lower number of cars from the amount found in part a can be parked.
> **Researcher:** Right.
> **Rana:** Probably the c here would be…c would be greater than 4.8?
> **Researcher:** It would be greater so that a lower number of cars would fit.
> **Rana:** Yes. The approach is correct, nice. For example, just like you said earlier, this sheet is a sheet we can accept as closest to the correct answer.

In the excerpt above, following her specific question on the equation that students produced, she attempted to in-depth analyze the solution. During this analysis, the researcher also engaged in the analysis process. She played a role of a participant and occasionally shared her opinions to stimulate the teacher to proceed with her analysis. At the end of this analysis, Rana made an interpretive comment on what conclusions the students would draw if they pursued the solution method that they initially adopted. She finished her investigation with an evaluative comment on the solution. However, as opposed to some evaluative comments, this comment followed an in-depth analysis of the solution. Hence, this excerpt provided robust evidence of her focused level knowledge-based reasoning.

In the first noticing meeting, as opposed to the other teachers, Kutay's knowledge-based reasoning was at a mixed level. General characteristics of Kutay's mixed level knowledge-based reasoning in the interview was that his remarks on a group's work generally contained a mix of elements of both a baseline level and focused level knowledge-based reasoning. That is to say, Kutay's statements on a group's work usually included description, evaluation, and interpretation at the same time. In contrast to what the other teachers did, he often made interpretive comments on students' solution approaches. Despite this dominant pattern, his statements on some groups' works were predominantly descriptive and evaluative, which were considered as a baseline level knowledge-based reasoning. On some occasions, he made a noteworthy effort to carefully analyze the groups' solution approaches, generally without the researcher's prompt. His endeavors to carefully analyze solution approaches resulted in interpretive comments, which was considered as evidence of a focused level knowledge-based reasoning. In the following paragraphs, these findings are presented via examples.

Kutay's statements on students' solution approaches, that were considered as evidence of his mixed level knowledge-based reasoning, mainly included a mix of descriptive, evaluative and interpretive comments. The following two paragraphs had prevalent features of this mixed level knowledge-based reasoning.

> **Kutay:** This group again placed the cars. But, I don't know why, they don't divide this car from there. They placed beautifully that 4.8. They need to calculate this part. Or they need to calculate that part using the angle values here. But if I'm not mistaken, just like the previous one, they tried Pythagorean Theorem on the big triangle. No. So that it is easier. Yeah, this is something kids do. If they can't deal with an angle, they try by taking round angles and they give that as the answer if it is

parallel to the answer. For them, they have easy angle values, for example we saw 45 somewhere. Here we have 30 and 60. That's why they used these angles.

Kutay: What did they say, on a lane, we can fit 35 cars. They tried to do those things from Pythagorean theorem but the thing is always wrong. I mean, they couldn't place it on the figure. These calculations here are always wrong. Here, they took as a 3 meter isosceles right triangle, took it as 45 degrees. Students have that logic. In angles…the two values are what…for example which is the right triangle having the greatest area: isosceles right triangle. These guys used that logic but their solution was wrong.

In the first paragraph, Kutay started with judgmental statements on the solution approach. Like the other three teachers, he made these judgmental statements by highlighting what students should have done but what they did not. Then he made an interpretive comment on what he noticed in the solution. By using his prior experience of students' ways of thinking, he speculated on the reason why students preferred to use well-known angles in their solution. Likewise, in the second example, he made descriptive and mainly evaluative comments on another group's solution. Similar to the first excerpt, this excerpt included an example of Kutay speculating on the reason why students used the 45 degrees in their solution. Consequently, these two excerpts exemplify Kutay's statements on students' works that involved description, evaluation, and interpretation at the same time.

As mentioned previously, on some occasions, Kutay endeavored to carefully analyze and make sense of students' solution approaches. In these kind of sense-making activities, Kutay made many interpretive comments on the solution approach. In the first interview, Kutay engaged in such sense-making activities twice. One of them was prompted by the researcher. The second was initiated by Kutay himself.

Following is an example of Kutay's endeavor to make sense of a solution approach, without the researcher's prompt. Rather than literally recounting what he noticed in the work, he reasoned about the meaning of students' solution approach. He speculated about the underlying student thinking. He also made evaluative comments but this comment followed making sense of the solution. Hence, this excerpt provided robust evidence of a focused level knowledge-based reasoning.

Kutay: They wrote something like "the most". That is 87, 90. Hmm, this group might have done, as the angle increases…I mean it spills out when it is 90 degrees. They might have thought, "well it spills out a little, so I should check 87 degrees" and then they might have thought whether it would just touch the line. They started from the end, in fact they formed a nice reasoning here. Maybe by thinking that spilling out…Could they have thought if I decrease it by 3 degrees, maybe the part that spills out would fit in. They have an interesting approach here. When everyone else started from small angles, they started from the big angles. Because they thought about the part which spills out when you park perpendicularly and in terms of how much it would fit in as you decrease the angle.

Similarly, in the following dialogue, Kutay engaged in a sense-making activity through the researcher's specific prompts:

Kutay: Here they found sinus 10s, that one is c. Did they call this length x? They did not indicate well. It's not very clear what they called x, probably this length, because that is the only unknown there. I mean, they tried to calculate this length. I think there is a logical mistake here. They should have drawn the car this way, and then drawn that.
Researcher: What did they not calculate here?
Kutay: Most of them took the distance from this point to there as 4.8.
Researcher: Is it also this way in this sheet?
Kutay: We need to look at that operation. Sinus 10, c/x, x is this length. What did this group calculate? Since they called it 3/x, they must have thought this angle is alpha. Oh, that can't be 3/x. No this ration is wrong. They called a different length as x.
Researcher: I think they said it for c because they divided 150 by x too. I think they didn't call it c, they called it x.
Kutay: Hmm, they took it as 3/x. I get it.
…..
Researcher: On the last page of their solution, there is a solution approach.
Kutay: Here there is Pythagorean. They used Pythagorean theorem. They took 4.8 here and they used Pythagorean theorem with this other triangle. In fact, with an equation that can be set up from this triangle, a value that would yield the maximum angle can be calculated. You can calculate the a here by using the angle, you can have a single unknown equation. From the solution of that, if they find the value of x over there, they could have the chance to find that angle by using the sinus or tangent of the value there.

In this extract, particularly in the first part of the dialogue, Kutay attempted to carefully analyze the solution approach through the researcher's specific prompts. At the beginning, he described what he noticed. Then he made a quick evaluation and concluded that this approach was wrong. For the purpose of stimulating him to elaborate on his claim, the researcher posed specific questions about the solution such as "What did they not calculate here?" or "Is it also this way in this sheet?" In response to these questions, Kutay more carefully analyzed the solution. Further in the dialogue, the researcher also played a role of participant and shared her opinions to stimulate the teacher to proceed with her analysis. In the second part of the extract, what can be seen is Kutay engaging in an effort to make sense of students' solution and how it could be carried further, by using the Pythagorean Theorem and trigonometric ratios. This took place in response to the researcher's prompt on the approach underlying the solution.

4.1.3 Summary of findings from the first investigation

In the first noticing meetings conducted in each school, teachers' selective attention was at a baseline level (See Table 4.1 and Table 4.2). The most common indicator of teachers' baseline level attention was that they described groups' solution approaches with broad terms that did not include references to mathematically significant aspects of the solutions. Compared to the other teachers' baseline level attention to most groups' works in the meeting, Rana's attention to half of the groups' works was at a mixed level since she began to pay more attention to the specifics of the solution as a response to the researcher's prompts. Moreover, the other indicator of teachers' baseline level attention was that some of their comments on groups' works included references to a range of issues on the works, rather than mathematical aspects of the solutions, such as content of the reports not related with mathematics, the quality of the pictures, whether students expressed their solution in their reports in a tidy manner, or who the group members were.

Similarly, in the first meeting, teachers' knowledge-based reasoning was at a baseline level (See Table 4.1 and Table 4.2). The most robust indicator of teachers' baseline level reasoning was that their comments on groups' works were highly descriptive and/or evaluative in nature and did not include any interpretation at all. On some occasions, teachers endeavored to carefully analyze the solution approach and make sense of what they attended to in the solution. Such kinds of analyzing, sense making, and questioning processes were considered as robust evidence of teachers' focused level knowledge-based reasoning. However, they engaged in such kinds of reasoning processes generally through the researcher's prompts in the first meeting.

Findings from the first interviews with Selda and Ayfer supported the previous findings from the first meeting. Selda's and Ayfer's selective attention was at a baseline level in the first interviews. The dominant characteristic of their selective attention was that, while investigating most groups' works, they described the solution approaches with broad terms, missing any reference to specifics of the solutions. Furthermore, their statements on some groups' works included references to non-mathematical aspects of the works. For example, Selda attended to whether students wrote their report in a tidy manner. Ayfer particularly paid attention to who the group members were since she did the implementation. Moreover, there were some minor indicators of their baseline level attention. For instance, Selda often expressed that she could not make sense of what students tried to do and did not endeavor to carefully examine students' works in such circumstances. Selda's and Ayfer's knowledge-based reasoning was also at a baseline level in the first interviews. The most robust indicator of their baseline level reasoning was that their statements on almost all groups' works were highly descriptive and/or evaluative in nature. They almost never made interpretive comments on what they attended to in the works. Table 4.1 and Table 4.2 demonstrate that they engaged in making sense of a few groups' works after the researcher prompted them to carefully analyze the works. However, these prompted reasoning processes did not result in interpretations of the solution approaches.

Some findings from the first interviews with Kutay and Rana confirmed the previous findings from the first meeting. However, there were also some new findings. Like his level of attention in the meeting, Kutay's selective attention was at a baseline level in the first interview. However, when the distribution of his level of attention according to groups was considered, his selective attention to half of the groups was at a mixed, or rarely, at a focused level (See Table 4.1). It means that, he began to pay attention to key aspects of these groups' solutions. Similar to his focus on non-mathematical aspects of the works in the meeting, his comments on some groups' works continued to include references to non-mathematical aspects such as content of the reports not related with mathematics, the quality of the pictures, whether students expressed their solution in their reports in a tidy manner, or who the group members were.

Unlike his baseline level knowledge-based reasoning in the first meeting, Kutay's knowledge-based reasoning was at a mixed level in the first interview. The distribution of his levels of reasoning according to groups is ranging from a baseline level to a focused level (See Table 4.1). It means that, in investigating some groups' works, his statements on some groups' works were descriptive and/or evaluative. These were mainly sheets where groups did not manage to produce a particular approach that would take them to a result. On the other hand, he also endeavored to make sense of a few groups' solution approaches and this sense making processes resulted in interpretations on the solutions, which provided robust evidence of a focused level reasoning. In these incidents, the richness of the solution presented on the sheets and the questions and prompts from the researcher to guide Kutay during the dialogues were decisive. However, his statements on many groups' works had elements of a mixed level reasoning, that is why his overall level was labeled as mixed. The most dominant indicator of his mixed level reasoning was that while he primarily made descriptive and/or evaluative comments on groups' solutions, he also made interpretations on what he attended to in the solutions.

Compared to her baseline level selective attention in the first meeting, Rana's attention was at a mixed level in the first interview. As opposed to her tendency towards making literal descriptions of solutions with broad terms in the meeting, Rana paid more attention to specifics of the solutions in the interview. Similarly, different from her attention to non-mathematical aspects of the groups' works in the meeting, in the interview, she only focused on the solution approaches. Additionally, she spontaneously endeavored to focus on specifics, rather than as a response to the researcher's prompts as in the meeting. While Rana's mixed level attention to many groups' works was beyond a baseline level in the interview, it does not have a holistic approach in terms of its focus on all important details and steps of students' solutions as one would expect at a focused level. Her selective attention to a few groups' solutions was at a focused level in the interview (See Table 4.2). In these cases, she referred to substantial details of the solutions. In such groups' works, students offered unique approaches not adopted by the other groups. Either spontaneously or through the researcher's prompts, Rana endeavored to focus on and referred to the mathematically significant aspects of these unique solutions.

Findings from the interview confirmed Rana's baseline level knowledge-based reasoning in the meeting with slight changes.She primarily described what she attended to in the groups' works with judgmental comments on them. As mentioned earlier, her descriptions included references more to specifics of the solutions. Yet, often, her descriptive comments were not accompanied with interpretive comments. Her interpretive comments ensued from only her attempt to carefully analyze and make sense of a few groups' solution approaches. Such kinds of reasoning process and accompanying interpretations were the most robust indicators of her focused level reasoning.

4.2 Nature and Level of Teachers' Noticing in the Second Investigation

4.2.1 Nature and level of teachers' selective attention

4.2.1.1 Findings from the second meetings

Out of the four teachers in the study, one teacher from each school, Selda and Rana, attended the second noticing meeting conducted in each school. Data analysis revealed that there were shifts in these two teachers' levels of selective attention. Both Selda's and Rana's selective attention shifted from a baseline level to a mixed level.

Selda's selective attention to each group's work was at a mixed level apart from their analysis of one group. Her comments contained elements of both a baseline and a focused level attention. For example, while she maintained referring to the quality of the pictures or whether students expressed their solution in their reports in a tidymanner, her endeavor to point to mathematically significant aspects of some

Table 4.1. Kutay’s and Selda’s Levels of Selective Attention and Knowledge-based Reasoning in the First Noticing Interviews and Meetings

Groups			G#1	G#2	G#3	G#4	G#5	G#6	G#7	G#8	G#9	G#10	G#11	LEVEL
Kutay	**SA**	**Meet**	Base	-	-	-	Base	-	-	-	-	-	Base	BASELINE
		Inter	Base	Base	Base	Mıx(TI)	Base	Foc(RI)	Mıx(TI)	Base	Foc(TI)	Mıx(TI)	Mıx(TI)	BASELINE
	KBR	**Meet**	Base	-	-	Mıx(TI)	Base	Mıx(TI)	-	-	-	-	Base	BASELINE
		Inter	Base	Mıx(TI)	Mıx(TI)	Base	Base	Foc(RI)	Mıx(TI)	Base	Foc(TI)	Mıx(TI)	Mıx(TI)	MIXED
Selda	**SA**	**Meet**	Base	-	-	-	Base	Base	-	-	-	Base	Base	BASELINE
		Inter	Base	-	Base	Base	Base	Base	Base	Base	-	Base	Base	BASELINE
	KBR	**Meet**	Base	-	-	-	Base	Mıx(OTI)	-	-	-	Base	Base	BASELINE
		Inter	Mıx(RI)	-	Base	Base	Base	Mıx(RI)	Base	Base	-	Base	Base	BASELINE

Note. RI = Researcher initiated; TI = Teacher initiated; Base = Baseline level; Mıx = Mixed level; Foc = Focused level.

Table 4.2. Rana's and Ayfer's Levels of Selective Attention and Knowledge-based Reasoning in the First Noticing Interviews and Meetings

Groups			G#1	G#2	G#3	G#4	G#5	G#6	G#7	G#8	G#9	G#10	LEVEL
Rana	**SA**	**Meet**	Base	Base	Mıx(RI)	Base	Mıx(RI)	Mıx(OTI)	Base	-	Mıx(RI)	-	BASELINE
		Inter	Mıx(RI)	Base	Mıx(TI)	Mıx(TI)	Mıx(TI)	Mıx(TI)	Mıx(TI)	Foc(RI)	Foc(TI)	Foc(TI)	MIXED
	KBR	**Meet**	Base	Base	Mıx(RI)	Base	Mıx(RI)	Mıx(OTI)	Base	-	Mıx(RI)	-	BASELINE
		Inter	Base	Base	Mıx(TI)	Base	Base	Base	Mıx(TI)	Foc(RI)	Foc(TI)	Foc(TI)	BASELINE
Ayfer	**SA**	**Meet**	Base	Base	Base	Base	Base	-	Base	-	Mıx(RI)	-	BASELINE
		Inter	-	Base	Mıx(TI)	Base	Base	-	Base	Base	Mıx(TI)	Base	BASELINE
	KBR	**Meet**	Base	Base	Mıx(RI)	Base	Mıx(RI)	-	Base	-	Mıx(RI)	-	BASELINE
		Inter	-	Base	Mıx(RI)	Base	Base	-	Base	Mıx(RI)	Mıx(RI)	Base	BASELINE

Note. RI = Researcher initiated; TI = Teacher initiated; Base = Baseline level; Mıx = Mixed level; Foc = Focused level groups' solutions gave evidence of a focused level attention. Even then, in some cases, although her descriptions mainly involved references to details of the solutions, there were some discrepancies between her description of the solution and students' actual solution. For example, although a group's solution method was correct according to the researcher's interpretation of the solution, Selda claimed it to be incorrect. This discrepancy was also considered as evidence of a baseline level attention. Following is an example of Selda's comments on a group's solution that provided evidence of her mixed level attention.

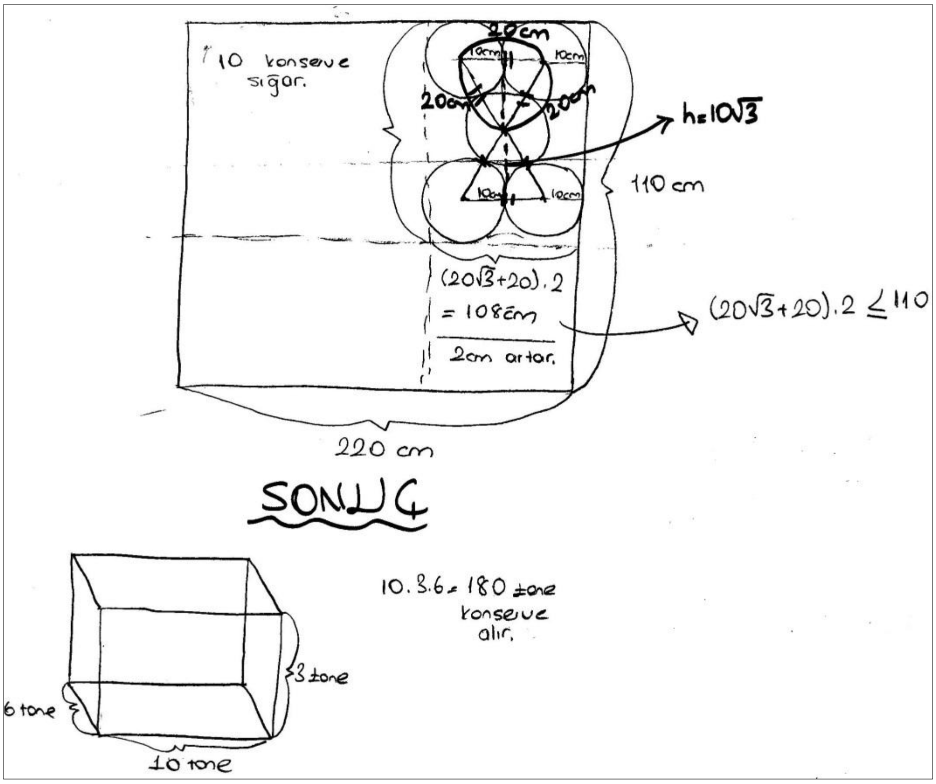


Figure 4.5. A sample of students' works in how to store it activity

In this solution approach, students had started to align the first row of the lids on long side of the bottom surface of the box. Then they aligned the second row in a way that it is set so that the lids are aligned with the spaces between the lids in the first row (see Figure 4.5). Then they formed equilateral triangles by using the line segments that joined the centers of the three tangent circles. By using the length of the altitudes of the equilateral triangles and radii, they calculated how much space three rows of the lids occupied horizontally. They regarded these three rows of the lids as one row and thought that how many rows they could align horizontally in the bottom surface of the box. Additionally, they expressed this situation through an inequality. However, they made a mistake in calculating the number of lids in each row (i.e., 10 lids in each row) and they found the wrong answer. In the following quote from the meeting, Selda commented on this solution:

> **Selda:** But I did not understand that…at the bottom 20 square root 2plus 20 times 2 is smaller than 110. How did they find it? I also did not understand 6 times 10 times 3 is 180. Where did they get this from is a question mark. That's what I wrote here.
> **Melda:** From there to there they placed 10. Here there are 6 of them.
> **Selda:** Okay, they thought 10.
> **Melda:** Yes, 10 of them. They expressed here, 6 of them. On the top, they ordered 3.
> **Selda:** But their drawing here is nice.

In this dialogue, Selda first expressed her confusion about a mathematically significant aspect of the solution method (i.e., inequality) and asked other teachers to make it clearer. She completed her statement with an evaluative comment on the drawings. This statement primarily consisted of the elements of mixed level selective attention. On the one hand she began to pay attention to the details of the solution, on the other she expressed her confusion about the solution. Through another teacher's help, she noticed details of this part of the solution. Also, she continued to focus on an issue other than the mathematical aspects of the solution. Therefore, this statement can be considered as an example of her mixed level selective attention.

Similar to Selda's selective attention, when Rana's comments were analyzed at a group level, her attention to each group's works was at a mixed level. One dominant characteristic of Rana's comments was that although they contained reference to some mathematically significant details of the solutions, without researcher's prompts, they did not refer to all aspects of the entire solution as one would expect at a focused level. That was the main reason for labeling her attention as a mixed level. For example, in the following quote, Rana paid attention to a mathematically noteworthy aspect of a solution. She pointed to whether students had started to align the first row of the lids on long side or short side of the bottom surface of the box, which affected the answer students obtained. However, her statements did not contain reference to other key aspects of the entire solution approach. For instance she did not elaborate on how this first stage of the solution affected students' solution processes:

> **Rana:** But what attracted my attention in the reports, what I thought was missing was that, when they were placing 11-10, they did not explain that they started from the longer side.
> **Researcher:** Were there also groups that started from the shorter sides?
> **Rana:** Yes, two groups did that. R8 and F4 placed them by fitting onto the short side.

Following is another example of Rana's mixed level attention due to the aforementioned reason. In Figure 4.6, a group's solution is illustrated. This group found the length of the diagonal from the given side lengths and calculated how many could be placed by diving this length by the diameter of the lids. After they found 11, they decided about the number of lids that could be placed on both sides of the diagonal in a symmetrical fashion. However, they found the number of lids by paying attention to the drawings they made, rather than doing calculations about how many lids would exactly fit on such parallel rows. In the meeting, Rana highlighted this solution approach and made statements on that.

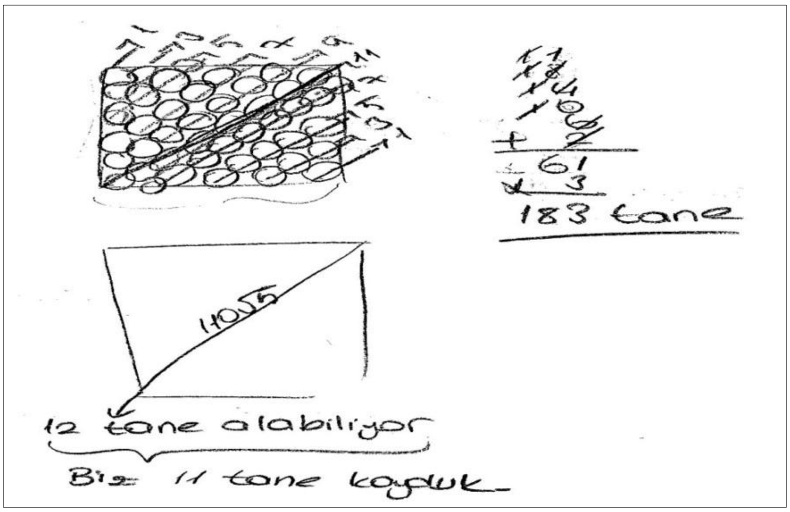


Figure 4.6. A sample of students' works in how to store it activity

Rana: In another solution, they placed them onto the diagonal. I even made a note like they calculated the diagonal by using the Pythagorean Theorem Group F7 did that.

Further in the discussion, the researcher prompted Rana to elaborate on the solution by saying "What do you think about this solution method?" Following this prompt, teachers attempted to discuss the solution. In the following extract from this discussion, it can be seen that, Rana began to attend to this solution more carefully and pointed to a mathematically important aspect of the solution. The fact that the group considered the region to be a square was an important step in that particular group's solution:

Rana: Can there be such a placement?
Funda: They found the diagonal as 110 root 5 from Pythagorean theorem. I mean, we had their calculations here.
Rana: It seems to me like they treated it as a square region.
Funda: Yes.
Hande: Because when it is a rectangle and you place them along the diagonals, boxes would go out from the sides.
Rana: Yes.

As opposed to first noticing meeting, in this meeting, Rana generally responded to the researcher's prompts for teachers to more closely analyze some groups' solutions that were worthy of attention. In order to stimulate a discussion, researcher encouraged teachers to turn their attention to the details of a solution. Rana primarily took these prompts into account and attempted to pay attention to the details. For instance, in the following extract from the meeting, researcher initiated a discussion by highlighting a group's solution as illustrated in Figure 4.7.

Researcher: Can you look at this group's solution? They could not find the solution but what did they try to adopt as an approach?
Rana: Here, they made calculations based on the distance between the first row and the second row. They drew a distance between the circle on the third row and said it was 14.
Hande: They took the distance between these two circles as 14.
Rana: They said it is 17 above. Here they took it as 14.Here it should have been 17 too. They made a mistake there. It's nonsense. They could not express it during the implementation either.

In this excerpt, following the researcher's question, Rana began to pay more attention to the details of the solution including the distances they used and values they gave and described them specifically. However, she did not sustain her attention to the details and eventually she finished by making evaluative comments on the solution and students' failure, with expressions such as "this is nonsense". Therefore, these statements contained elements of a mixed level selective attention

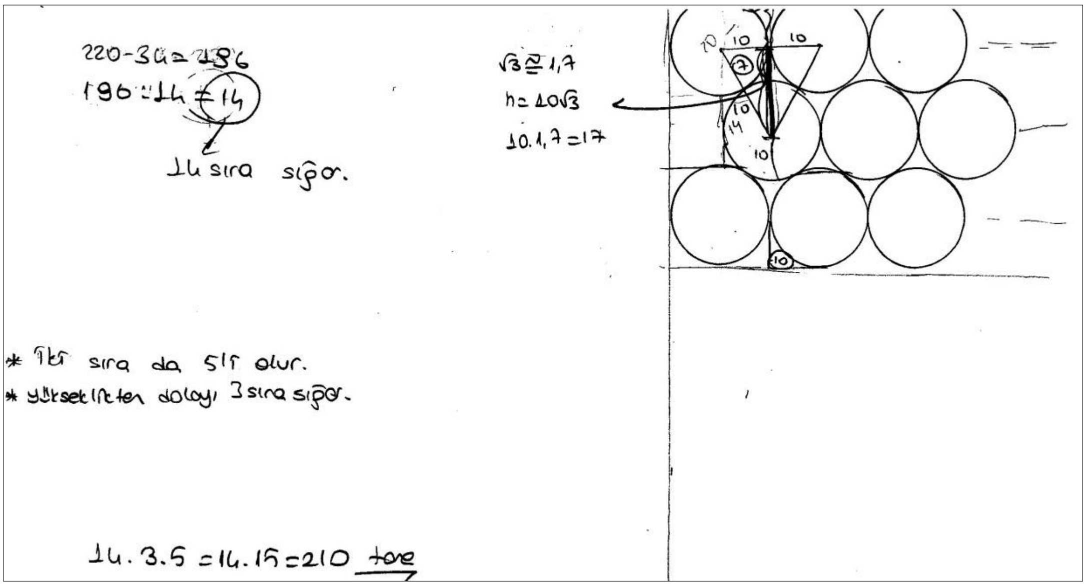


Figure 4.7. A sample of students' works in how to store it activity

4.2.1.2 Findings from the second interviews

Kutay and Selda, whose selective attention was at a baseline level in the first noticing interview, shifted to a mixed level in the second noticing interview. On the other hand, Rana's selective attention shifted from a mixed level to a baseline level. The level of Ayfer's selective attention did not change and it continued to be at a baseline level. When Kutay's and Selda's selective attention were considered at a group level in the second noticing interview, it can be said that both teachers' selective attention to each group's work ranged from a baseline level to a focus level, while it was predominantly at a mixed level. Compared with the first noticing interview, while nature of Kutay' selective attention changed slightly, there was a considerable shift in nature of Selda's selective attention in the second noticing interview. As in the second meeting, majority of Selda's comments on groups' work provided evidence of a progress in her selective attention.

In the second noticing interview, Kutay's selective attention to the three groups' works was at a baseline level. A common feature of these groups' work was the lack of production of a solution approach. In their written works, only some attempts to solve the problem stood out. For example, one of these groups tried to use the concept of volume to solve the problem, besides finding an answer to the first, and a relatively easy part of the problem. But they did not manage to come up with an answer. Kutay commented on this group's work as "they did our standard solution, found 165, and then they did not force themselves. Standard, there's nothing to say about this". But he did not refer to the approach students endeavored to adopt: the concept of volume.

While investigating students' works, Kutay primarily attended to mathematical aspects of the works, rather than other aspects such as the appearance or unrelated content of the report, except for one group. More importantly, he predominantly paid attention to the details of groups' solution approaches either through the researcher's prompts or spontaneously. A difference between such cases was that when he attended to them through researcher's prompts, he could pay attention to the entire strategy with substantial details. This was considered as a focused level selective attention.

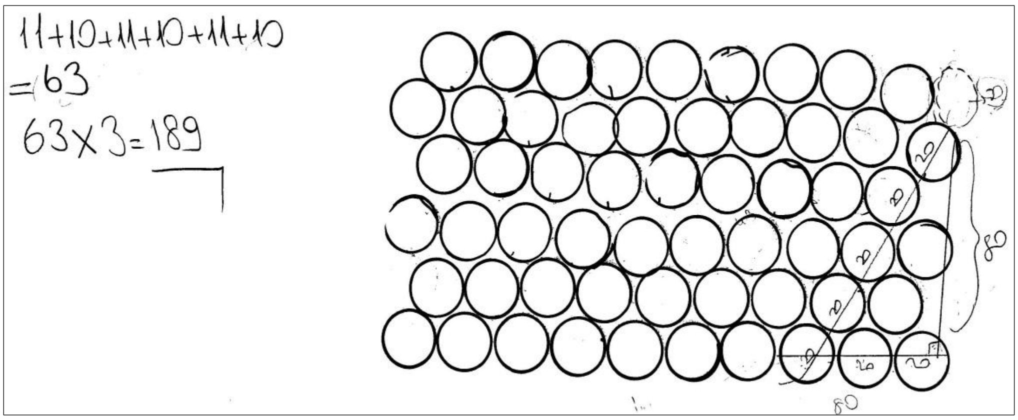


Figure 4.8. A sample of students' works in how to store it activity

In the following excerpt from the second interview, Kutay's remarks on a solution approach provided evidence of his mixed level selective attention. In this solution approach, as illustrated in Figure 4.8, the mathematically noteworthy part was that in order to estimate how many rows of lids they can place at the bottom surface of the box, students attempted to form a 3–4–5 right triangle. While at the beginning of the dialogue Kutay pointed to some aspects of the solution, he did not mention about this mathematically significant detail of the solution. He even claimed that students used trial and error strategy to find the number of rows, which did not actually take place according to researcher's observation. These can be regarded as evidence of his mixed level selective attention to this group's work, without the researcher's prompt.

> **Kutay:** This group made the most detailed solution. They tried to solve it through volume. And that wasn't enough for them, so they tried to calculate it by changing the base of the prism. They also found the 10-11 sequence but without using an equilateral triangle. How did they find it? If they had the lids drawn to scale, they might have tried or took a hint from somewhere.
> **Researcher:** They don't have the solution?
> **Kutay:** No. Well, they drew it here. They might have thought like, well we're making the 10-11 sequence. They tried to find something using the left overs from here but I think they couldn't calculate the number of rows. They found one through trial and error and multiplied by 3.

As mentioned above, when the researcher prompted Kutay to carefully examine a solution approach by asking questions related to the solution such as "how did they find this value?" or "how did they find the number of rows?", he considered these prompts and began attending to substantial details of the solution. This was considered as a focused level selective attention. For example, following is a segment from the second interview where Kutay commented on a group's solution as shown in Figure E2.

> **Kutay:** M3 tried a lot and formed the equilateral triangle. It's nice to see that they formed this approach. They calculated the height. They also thought about the +10 gaps at the bottom and top. And also, they tried to place them starting from the short side.
> **Researcher:** How did they calculate that 12 of them would fit?
> **Kutay:** They must have found by dividing 12. They divided 220 by 34. They took root3 as 1.7. They found the space to be taken by 6.47 cylinders by dividing them. They multiplied 2 and 6.47 and found 12.94. Then they took 12 as the integer.
> **Researcher:** Then how did they find 34?

> **Kutay:** 20 times root3. They multiplied 1.7 by 20. They found 34 from that. Then, they divided 220 by 34, in fact they did the operation you did here. Without formulating, they did that.

In this extract, Kutay started with describing the solution by giving references to details such as which side of the bottom surface of the box students had started to place the first row of lids. Following the researcher's question on a mathematically important part of the solution such as how students calculated the number of lids on the long side of the bottom surface of the box, Kutay began to focus on the details of the solution. Further in this investigation process, through the researcher's prompt, he continued to attend to all substantial details of the entire solution, which can be considered as a focused level selective attention. In this kind of incidents, when a teacher attended to a group's solution at a focused level through the researcher's prompt, the researcher's role was crucial. Besides acting as the researcher, she switched between roles of a facilitator and a participant. She either promoted the teacher to attend to aspects of the solution worthy of attention by asking some specific questions or she also engaged in the investigation process with the teachers and shared her ideas.

Compared to the first noticing interview, there was a shift in nature of Selda's selective attention in the second noticing interview. Although Selda's comments on some groups' works maintained containing elements of a baseline level selective attention, the most dominant characteristic of her comments was that they included references to some details of the solutions. This shift suggested that she began to make progress in her selective attention. Another noteworthy aspect of Selda's selective attention is that, often, she paid attention to details of a group's solution without the researcher's prompt. For instance, in the following extract quoted from the second noticing interview, Selda's statement about the group's solution, illustrated in Figure 4.8, provided evidence of her attention to some details of the solution. However, since she did not comment on the entire strategy with substantial details, her selective attention to this solution was labeled as a mixed level, rather than a focused level:

> **Selda:** They thought of turning the box. They also found 189 by compressing. But these students did not make use of knowledge about equilateral triangles or circles. Here they made a drawing. But this should be 60, not 50. In fact, they could draw one more here. Because in the solution it says 11-10 but they drew 9-10. I did not understand why they did so.

In this statement, Selda noticed some details of the solution such as the idea of turning the box around or the incorrect number of lids that were fitted into the bottom surface of the box in students' drawing. Yet, she did not attend to the right triangle that students produced on the drawing, which was the most noteworthy aspect of the solution.

Selda's selective attention to several groups' works was at a focused level, even without the researcher's prompt. Following is an example of Selda's focused level selective attention to a group's solution (see Figure 4.5). As stated previously, in the second noticing meeting, this solution approach had been discussed by the teachers, and Selda's selective attention to this group was at a mixed level. In the interview, however, her comments exhibited features of a focused level selective attention:

> **Selda:** They found 180. Hmm…180 and they fit 6 here. And over here 10. Yes, they said it increases by 2 cm because they took 110. 20 root 3 plus 20 x2. They must have divided 110 into 2 equal pieces. They must have placed 3 for each 55. They constructed an inequality here. When making that base placement, they must have thought that they should find something smaller than 110. 40 root 3 plus 40. Let's multiply 1.7 by 4. Does that match? 68. Plus 40. 108. Yes, it increases by 2 cm. This is the only group that used an inequality. That's why it was different.

When Rana and Ayfer's selective attention were considered at a group level in the second noticing interview, Rana's selective attention was primarily at a baseline level. Compared with the first noticing interview, Rana's selective attention was very unstable in the second noticing interview. Although it was predominantly at a baseline level, in some occasions, her comments on various groups' works contained elements of a focused level selective attention. There was no considerable shift in Ayfer's selective attention from the first interview to the second.

As opposed to the first noticing interview, majority of Rana's comments on groups' works exhibited features of a baseline level selective attention. Accordingly, she continued to focus on a range of issues other than mathematical aspects. For example, she was interested in who the group members were because she did the implementation in her class and the groups consisted of her students. Some of her comments about groups' work depended on who the students were. She made comments such as "Did Berkay say this? Berkay drew it very well"; "Tarik did this solution. Normally, he's a student who is not interested in the lesson very much". She also made comments on the appearance of groups' reports.

Another manifestation of her baseline level selective attention was her comments on general features of the solutions, missing any reference to details. For instance, her comments on a group's solution as shown in Figure 4.10, provided evidence of her baseline level selective attention. Although this solution was discussed in the second meeting before the interview and she paid attention to the details of the solution through the researcher's prompt, in the interview, she did not give any reference to the details of the solution. She just offered her general impression:

> **Rana:** This group had a different idea. I actually liked that this student approached it this way. He tried and did it. He's a student who is not interested in the lesson much. It was nice, that he thought like this. Yes, this is a kind of placement and it is an alternative for our 176. But it is nice that he thought about it, found the isosceles triangles and solved it correctly. This group has such a character.

On the other hand, Rana's selective attention to two groups' solution was at a focused level in the second interview. For example in one of these group's solution, as illustrated in Figure E3, two details of the solution were worthy of attention. One of them was the side of the bottom surface of the box on which students had started to place the first row of the lids. Here, students had started to place the first row of the lids on the long side. Second important aspect of the solution was the inequality that students used to find the number of rows they could fit into the bottom surface of the box.

In the following extract, Rana's description of the solution contained aforementioned details of the solution. In her statement, she first pointed to the side that students had started to align first row of the lids. Then she referred to the inequality that students used and explained how students formed this inequality by referring to the drawing. These statement exhibited evidence of her focused level selective attention to this group's solution.

> **Rana:** This group drew it neatly. They fitted them along the long side. First they fitted 11 of them along the long side, then they placed 10 by shifting them so they fit in the gaps. Then they showed this in the following way: Here they found an inequality. This was 220, that was 110. Here they found it as 10 root 3. 10 root 3, 10 root 3, 10 root 3. They calculated 5 times root 3 plus 10 and 10. It should have been 110. In fact here, if they said n times 10 root 3, they would find a formula. But since they drew the diagram straight away, they said 50.

Like in the first noticing interview, Ayfer's selective attention was at a baseline level in the second selective attention. Her selective attention to most groups' works was at a baseline level because her comments on these works included elements of a baseline level selective attention. Her comments predominantly focused on general features of the solution. Furthermore, there were some discrepancies between her description of some solutions and students' actual solutions.

Following is an example of her comments on a group's solution that provided evidence of her baseline level selective attention. In this solution approach, unlike the other groups' solutions, students placed the circles along the short side of the box by leaving 2.5 cm gaps between two circles (see Figure 4.9). Upon placing the first row, they fitted the circles in the second row so that a circle would correspond to each gap in the first row. They continued with this structure. In order to find how many rows they need to place, they calculated the minimum number of rows that would allow them to place the required number of circles. In the following excerpt, Ayfer commented on the solution:

> **Ayfer:** Here they formed isosceles triangles. They found that gap as 2.5 but they did not do the operations accordingly. Here, there is a similar situation. This is in fact a good group, they thought about the third way but could not apply it. But they did a few things…If they did this one too, they would see what the third one would give, that it would give a different result. Which one is that? One of Rana's groups. That's a typical thing.

In this statement, Ayfer began with a reference to the isosceles triangles that students sketched in their works. Then she pointed to the gap between the circles but she did not continue to describe other aspects of the solution. She continued to offer her general impression on the solution. As a result, her statement on this group's solution provided evidence of her baseline level selective attention.

Although Ayfer's selective attention was predominantly at a baseline level in the second interview, she also attempted to pay attention to mathematically important aspects of several groups' solutions through the researcher's prompt. For instance, following extract shows her comments on the group's solution (see Figure E3):

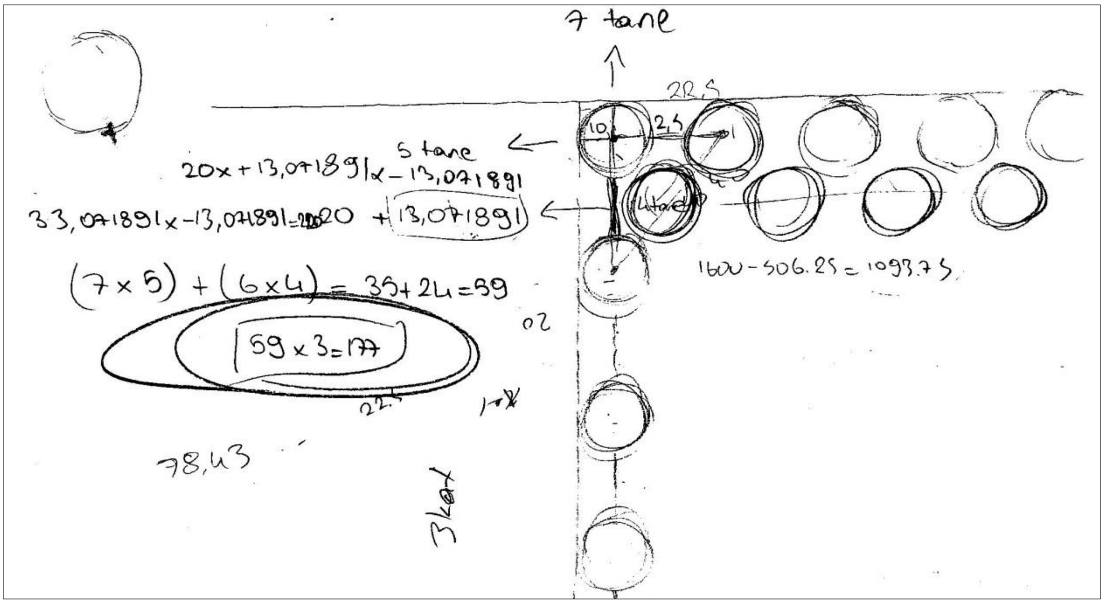


Figure 4.9. A sample of students' works in how to store it activity

> **Researcher:** What exactly did they do here?
> **Ayfer:** This group fitted along the short side, and did the operations by fitting along the long side. They formed an inequality. They calculated the 10 root 3's and used inequalities. And they reached the correct answer.
> **Researcher:** How did they form the inequality?
> **Ayfer:** They summed the 10 root 3's and added the 10 cm gaps, and wrote that it should be smaller than or equal to 110. Because the short side is 110. But I could not make sense of this, I mean, what they did later, for what purpose they used the inequality.

In the extract above, Ayfer pointed to some mathematically important aspects of the solution through the researcher's prompts. Here researcher used two types of prompts, that is a general prompt and a specific prompt. First she prompted Ayfer to describe the solution by using a general question such as “What exactly did they do here?” After this prompt, Ayfer referred to two important points related to the solution such as the side that students had started to align the first row of the lids and the inequality that was used. Then the researcher asked her a more specific question about the solution: “How did they form the inequality?” In response, Ayfer described how students produced the inequality. However, she completed this conversation by expressing her confusion about the solution. Therefore, these statements can be considered as an example of a researcher initiated mixed level selective attention. .

4.2.2 Nature and level of teachers’ knowledge-based reasoning

4.2.2.1 Findings from the second meetings

As mentioned before, only two of the four teachers attended this meeting. Compared to their baseline level knowledge-based reasoning in the first noticing meeting, Selda’s and Rana’s levels of knowledge-based reasoning shifted to a mixed level in the second noticing meeting. Shifts in Selda’s knowledge-based reasoning were especially noteworthy. While she sustained making comments that were descriptive and/or evaluative in nature, as opposed to the first meeting, she initiated and engaged in various sense making processes in the second meeting. Similarly, unlike the first noticing meeting, Rana took the initiative and initiated some of the sense making processes in the second meeting. Furthermore, compared to the first meeting, she made more interpretations on solution approaches. In this section, these findings will be elaborated.

Selda’s knowledge-based reasoning was at a mixed level in the second meeting because she often endeavored to carefully analyze and make sense of students’ solution approaches. Although she almost never made interpretive comments on students’ thinking, she made a considerable effort to make sense of solution approaches, which is a prevalent feature of a focused level knowledge-based reasoning. Sometimes she engaged in a sense-making activity through the researcher’s prompts. For instance, in the following extract from the second meeting, in response to the researcher's questions, Selda endeavored to comprehend the solution approach as illustrated in Figure 4.5:

> **Selda:** This group found 189 anyway.
> **Researcher:** How did they find 189?
> **Selda:** They drew it one by one, it's good that they drew one by one like this. But if you paid attention the drawings are 10 by 9, they are not 11 to 10.
> **Researcher**: How did this group find that they could fit 6 rows horizontally?
> **Selda**: They did not explain how they found this, they used root3 too. It's not clear how they found it.
> **Kaan:** They drew a figure, made the height.
> **Melda:** Yes, they drew a height there, but I thought whether they should have drawn a tangent.
> **Selda**: They calculated this part. They called it a 10-8-6 triangle.
> **Kaan:** Would this side be 80?
> **Selda**: 10-8-6 triangle. But how did they see that these were equivalent, that there was a right triangle? They took this side too big. This side needs to be 50, not 60. The result, 8, is not meaningful. It's interesting that the distance from that point is 100, because the base is 50, not 60. And the line that formed a side of the triangle does not pass through the centers of the circles. They took the chords as 20 even though they did not go through the centers. I think they found the number of rows intuitively.

In this excerpt, the researcher first posed a general question in order to prompt teachers to explain the solution approach. Selda responded to this prompt by making a comment that was evaluative and descriptive in nature. She mentioned in detail how they made the solution and expressed that she approved it. Then, the researcher posed a specific question to prompt teachers to analyze the solution more carefully by saying "How did this group find that they could fit 6 rows horizontally?". In response to this prompt, another teacher, Kaan, pointed to a detail of the solution: drawing the height. Then, at the end of the dialogue, Selda analyzed the solution more carefully and reasoned about the correctness of numerical values in the solution. Consequently, in this excerpt, many of Selda's remarks had elements of a focused level reasoning. However, some remarks were descriptive and evaluative in nature rather than interpretive, which was a prevalent feature of a baseline level knowledge-based reasoning. Such chunks of comments were considered as evidence of her mixed level knowledge-based reasoning.

Following is another excerpt that exemplified Selda's mixed level knowledge-based reasoning:

Selda: But I did not understand that 20 root 3 plus 20 times 2 is smaller than 110 at the bottom. How did they find it? And I also did not understand, 6 times 10 times 3 is 180. Where did that come from?
Melda: From here to there they placed 10 that way and here 6 of them.
Selda: Hmm, 10 of them there.
Melda: Yes, 10 of them. They indicated there, 6 of them. At the top, they placed 3 of them.
Selda: Why use an inequality? Why did they feel the need to write an inequality?
Melda: This calculation needs to be smaller than 110.
Selda: Oh, is this a height?
Melda: Yes, I think it is a height.
Selda: Hmm, I think I didn't get that before.
Kaan: Yes, there…One is 10 root 3, two of them 20 root 3 plus 20.
Selda: But not two of them. They said 6 rows. Why times 2?
Kaan: 3 by 3. Now all of these triples have become, 20 root 3 plus 20. Here, there will be another row of three. Then they said times 2, 108. They copied the same here.
Selda: But not the same, they will copy the following. If they copy the same on top of each other…
Melda: It will be like this. There'll be 3 of them. That means two more of itself. That is 108, which is a smaller value than 110.

In this dialogue, Selda initiated a discussion on the solution approach by posing a specific question related to a detail of the solution, in order to make it clearer. Her question prompted teachers in the meeting to carefully analyze the solution and explain what students did. Further in the discussion, Selda sustained posing specific questions about some other details of the solution that confused her. In response to her questions, teachers collectively endeavored to reason about what students did. As opposed to previous discussion, this time, Selda took the initiative and prompted a collaborative effort to make sense of the solution approach. Despite her effort to make sense of the solution, indicating a focused level knowledge-based reasoning, her comments were simply restatements of what the students did. Such comments were descriptive in nature. Consequently, like the previous example, the level of knowledge-based reasoning was labeled as a mixed level.

Analysis of the second noticing meeting revealed that, Rana's knowledge-based reasoning was at a mixed level because her statements on groups' works had elements of both a baseline and a focused level knowledge-based reasoning. In the second meeting, she continued to make predominantly descriptive and evaluative comments on some groups' solution approaches. This is a prevalent feature of a baseline level reasoning. Nonetheless, she also endeavored to make sense of many groups' approaches and some interpretations occurred, which provided evidence of a focused level reasoning. Because these statements were a mixture of both a baseline level and focused level knowledge-based reasoning, they were considered as a mixed level.

The following is a common example of how Rana mainly commented on students' works during the second meeting at a mixed level reasoning. She responded to the researcher's prompts and endeavored to make sense of solution approaches. While she sustained making descriptions of what she noticed, in response to the researcher's prompts, she reasoned about the solution like in the following discussion:

> **Rana:** As a second solution, they placed along the diagonal. That is, they did it by placing along this diagonal. One group even calculated the length of the diagonal from Pythagorean theorem.
> **Researcher:** What kind of a solution approach did the ones placing along the diagonal adopt?
> **Rana:** But they couldn't get an answer.
> **Funda:** One group did. They found 183.
> **Researcher:** Well, can the diagonal be placed this way?
> …
> **Rana:** Can there be such a placement?
> Researcher: That's exactly what I was trying to ask, can there be a placement on the diagonal?
> **Funda:** They found the diagonal as 10 root 5, well, they had the calculations here.
> **Rana:** It seemed to me that they did the operations as if it was a square region.
> **Hande:** Because when it is a rectangle, and you place along the diagonal the boxes would spill out from the sides.

In the discussion above, Rana began with literally describing what she noticed in the students' work. Then the researcher prompted teachers to explain the entire solution rather than describing it by using general terms such as "they placed along the diagonal". Rana did not respond to this question and made en evaluative, result-oriented remark. Then researcher prompted teachers once more by posing a specific question on the solution approach. Further in the discussion, Rana repeated the researcher's question and prompted other teachers to reason about the deficiency in the solution approach. Rana's endeavor to make sense of the solution resulted in her interpretive comment on the correctness of the approach. Existence of early descriptive and evaluative comments followed by interpretations, lead to the labelling of this episode as a mixed level.

Besides the commonly occurring mixed level reasoning, Rana’s comments on one group’s work was at a baseline level. The following excerpt includes Rana’s comments about that group’s work:

> **Rana:** In R6, can we go to R6? This student, Tarık, tried hard and found it. Normally he has very little interest in the lesson. He thought the same way as I did but I did not interfere. He found it himself. And I gave him support to be honest, Ok it might be right or wrong. At the end of the day he thought of such a thing and reached a conclusion, that’s the good thing.
> **Hande:** Yeah, he even calculated the gaps among the boxes.
> **Rana:** Absolutely, he calculated the gaps, drew right triangles.
> **Researcher:** How did he find 4 and 5?I mean, 6 rows of 4 boxes, 7 rows of 5 boxes?
> **Funda:** Well he left it at 177 anyway.
> **Researcher:** Well, how did he decide that 5 of them will be placed and 4 of them will be placed in the second row?
> **Rana:** He probably did not write this.

At the beginning of the dialogue above, Rana first attracted teachers’ attention to a solution approach. Then she commented on the solution. Her comments were predominantly evaluative. She expressed her appreciation of the solution. She also made descriptions by using general terms such as “calculated the gaps” or “drew right triangles”. Further in the dialogue, despite the researcher’s specific questions on the solution, she did not respond to these prompts and did not attempt to make sense of the solution. As a result, her statements in this excerpt had prevalent elements of a baseline level knowledge-based reasoning.

On the contrary to the baseline level reasoning about one group’s work, Rana’s knowledge-based reasoning occurred to be at a focused level for one of the groups. Rather than making descriptive and evaluative comments, she reasoned about a group’s solution and made interpretations on the solution approach.

> **Rana:** Yes this group found 180, they placed it by fitting along the short side.
> **Hande:** Yes, I also thought the solution had a mistake.
> **Rana:** Should it be thought as a mistake, or a solution strategy?
> **Funda:** But they placed it along the short side.
> **Hande:** But they found the result differently.

> **Rana:** In this question, you could reach an answer by trial and error; I think we should think about it as a step towards reaching the solution. If we call this a mistake, then we should also think about why the students who placed along the long side did not think about the short side. Because can they know that you can't fit in more than 189 when you place along the short side.
> **Hande:** You're right, they couldn't have known.
> **Rana:** We should not see it as a mistake but rather see it as steps along the solution of the problem.

In the discussion quoted above, while other teachers made quick judgmental statements on a part of students' solution approach, Rana first questioned whether it should have been considered as incorrect. Thus, she prompted a discussion on this issue. While other teachers continued to make judgmental remarks such as "but they found the result differently", Rana made an interpretive comment on why this approach should not have been considered as incorrect. As a result, as opposed to the previous two discussions, Rana's statements in this one predominantly exhibited features of a focused level knowledge-based reasoning.

4.2.2.2 Findings from the second interviews

In the second noticing interview, Kutay's and Selda's knowledge-based reasoning were at a mixed level. Although, Kutay's knowledge-based reasoning remained at a mixed level, when it was considered at a group level, there was a shift in his level of knowledge-based reasoning from the first interview to the second one. Similarly, compared to her baseline level knowledge-based reasoning in the first interview, Selda's level of knowledge-based reasoning proceeded to a mixed level in the second interview. On the other hand, there was no considerable shift in Rana's and Ayfer's levels of knowledge-based reasoning from the first interview to the second one. Their knowledge-based reasoning remained at a baseline level. When Rana's knowledge-based reasoning was considered at a group level, however, her remarks on some groups' work had prevalent features of a focused level knowledge-based reasoning. These findings will next be discussed.

As opposed to his highly evaluative and descriptive comments on group's works in the first interview, Kutay generally made interpretive comments on groups' solution. While he adopted a judgmental stance in the first interview, he rarely made judgmental statements on groups' works and adopted an interpretive stance in the second interview. Moreover, he mainly endeavored to carefully analyze and make sense of students' solutions, without the researcher's prompts. He mostly took the initiative for carefully analyzing the solutions. For example, in the following extract from the second interview, Kutay endeavored to make sense of a group's solution approach:

> **Kutay:** They did the solution everyone else did. But 12, they said we placed 1 more. I looked and looked but couldn't make sense of it. I didn't get what they meant. I wonder if they placed them on their side. There would be a 10 cm gap, maybe they put on their side to fill that gap? That is, I didn't understand according to what they found this? They said there were 11 left, but we placed 14 of them. They didn't find a number 14, I didn't get this. But I think the logic they followed was this…the placing one more one less approach. But it's not clear where that 14 is coming from. They found the volume, they tried to do something. When they said 1 out of 12, if we assume it is 220, it should have been 1 out of 11.
> **Researcher:** Do they mean for 11, 1 more?
> **Kutay:** I couldn't get this part. Could they have thought horizontally? If they placed the boxes horizontally, for a length of 220 they could place 7. 7…14…then they should place 3 rows, 21 like that. 2 rows of 7…7…14…then they placed horizontally to the remaining 10 cm gap. They might have thought like that, but I don't know.

Initially, he reasoned about the solution and made an effort to comprehend the meaning of what students did. He endeavored to interpret what he noticed in the work. He often speculated on the underlying students' thinking by saying "maybe they put on their side to fill that gap" or "I think the logic they followed was this". Further in the dialogue he continued to reason about the solution. At the end, he speculated on students' ways of thinking that the solution was based on. Thus, Kutay's statements in this extract provided evidence of his focused level knowledge-based reasoning, without the researcher's prompt.

Similarly, in the excerpt below, Kutay remarked upon another group's solution. He attempted to make sense of some details of the solution that were unclear for him. For instance, initially, he reasoned about how students found out the 11-10 sequence and got the correct answer without making the necessary calculations. Besides 11-10 sequence, he also questioned how students found out the number of rows. Like

Kutay's remarks in the previous excerpt, his following statement also had features of a focused level knowledge-based reasoning, without the researcher's prompt:

> **Kutay:** This group found the 10-11 logic without making operations and reached the correct result. But how did they find it without making calculations? I wonder whether they filled the lids on a one-to-one scale? That is very interesting, is it possible to find something by placing the lids? This is what we should try and see. Here, they said we tried as in the figure by placing the lids in the gaps. I mean, this could be a reason to fit the lids in the gaps in between but why 11 to 10 and why 6 rows? Why 3 from this and 3 from that? If that was one to one scale and they placed it, I would accept but with that scale they used…there is no such box. I don't think they placed the lids with such sensitivity.

While Kutay's remarks on some groups' solution approach had features of only focused level knowledge-based reasoning, like in the previous two extracts, some of them had elements of both a baseline level and a focused level knowledge-based reasoning. His remarks on various groups' works had this kind of dual nature and therefore labeled as a mixed level. Following is an example of this kind of statements.

> **Kutay:** This group tried with scaling. They found 25x3=75, I think they made an error on scaling. With the area they took, they have scaled down by 3/20. I mean, 20 cm… its diameter 20 cm…this one's diameter 3 cm, they drew something according to that. They tried something but there's an error in the calculation.
> **Researcher:** What can you say about the mathematical thinking here?
> **Kutay:** Well, as a logic it is correct. 20…3…they will scale it down with a 3/20 ratio but they didn't scale down the sides by 3/20. Well that 220 needs to be scaled down by 3/20. That is, it needs to be divided by 3/20. Since there are mistakes in the calculations there, the result turned out to be much smaller.
> **Researcher:** Why did they think about doing such a scaling?
> **Kutay:** In order to fit the lids in their hands to the paper in front of them. That is probably from their geography knowledge. There is scaling in maps. They did the scaling for diameters but they didn't calculate the scaling for the sides. Or maybe they couldn't. I don't know.

In this excerpt, Kutay's first statements on the solution were descriptive and evaluative in nature. He described what he noticed in the students' work and evaluated the correctness of the approach. Further in the dialogue, although the researcher prompted him to make comments on underlying mathematical thinking, he sustained a judgmental stance and expressing what students should have done but what they did not. These statements had elements of a baseline level knowledge-based reasoning. At the end, in response to the researcher's specific question on students' thinking, Kutay made interpretive comments about it. He speculated on what prior knowledge students benefitted from while solving the problem. These interpretive comments were prevalent elements of focused level knowledge-based reasoning. Consequently, these statements provided evidence of a mixed level knowledge-based reasoning.

Analysis of the second noticing interview data revealed that, compared to a baseline level knowledge-based reasoning in the first interview, Selda's level of knowledge-based reasoning proceeded to a mixed level. In the following excerpt from the first meeting, Selda attempted to make sense of a solution approach through the researcher's prompts:

> **Selda:** Here they did a different placement; they used words such as "we need to fill in the gaps". Yes, they did compression. But 11 of them…to reach the result 176…They have this expression in some reports. They wrote "I found 165, but where will I fit 11". Then they targeted filling up the gaps. They have 176+3 there, but I didn't get clearly where they found it from. This group also wanted to make use of volumes, but the problem in such a calculation, there would be gaps. They are not thinking about that.
> **Researcher:** Can we look at the explanation there, did they explain it?
> **Selda:** For 12 of them, we placed an extra tin. In 176, we placed 14. We needed 11 of them anyway. Well, why is it 1 out of 12? If they said 1 out of 11. Probably they are trying to say 11+1.
> **Researcher:** Did they reach such a conclusion by making calculations?
> **Selda:** They tried to form a ratio. In fact, they tried to show some effort to reach 176. I mean, by going from the result, it looks like they tried to find it. Because they are trying to work their way backwards from the result. They are dividing 176, they are saying, we need to place one extra for sure.

In this excerpt, initially, Selda literally described what she noticed in the students' work. Also she made an evaluative remark on a solution approach students endeavored to adopt and she assessed it as inaccurate. These statements had features of a baseline level knowledge-based reasoning. After the researcher suggested looking at students' explanations in their report, Selda made an effort to make sense of what she noticed. Also she speculated on the meaning of what students did. Further in the conversation, the researcher prompted her to elaborate on her interpretation by saying "Did they reach such a conclusion by making calculations?". Following this question, Rana made an inference about students' ways of thinking that the solution approach was based on. Selda's such remarks exhibited prevalent features of a focused level knowledge-based reasoning. As a result, this excerpt contained evidence of Selda' mixed level knowledge-based reasoning.

Likewise, following is another example of Selda's endeavor to make sense of a group's solution approach through the researcher's prompts:

> **Selda:** This group found 180, because they fitted 6 here. Here, 10. They made a different placomment, but it's not a correct one. But they managed to use their knowledge about circles and equilateral triangles. Yes, they said it would increase by 2 cm since they took 110. 20 root 3 plus 20 times 2. They divided 110 into 2 equal pieces. For every 55, they placed 3 rows.
> **Researcher:** Why did the students form this inequality?
> **Selda:** I think because of thing… They are probably thinking we need to find a thing smaller than 110 while doing the placement on the base. 40 root 3 plus 40. Let's multiply 1.7 and 4 and see whether that gives the right amount…68, plus 40…108. Yes, it increases by 2 cm. But this seemed different to me, because this is the only group thinking about inequalities.

Selda's statements in this conversation had features of a mixed level knowledge-based reasoning because they included elements of both a baseline level and a focused level knowledge-based reasoning. For instance, at the beginning, Selda's remarks were primarily descriptive and evaluative in nature. She first described what she noticed in students' works. Also, she made some evaluative remarks such as "They managed to use their knowledge about circles and equilateral triangles". All these statements were prevalent features of a baseline level knowledge-based reasoning. After the researcher prompted her to comment on underlying students thinking by asking a specific question, Selda attempted to speculate on students' ways of thinking. Then she checked whether her speculation was consistent with the students' actual solution. These statements can be considered as evidence of a focused level knowledge-based reasoning. That is why Selda's statements in this extract were labeled as a mixed level knowledge-based reasoning.

Although Selda's knowledge-based reasoning was primarily at a mixed level in the second noticing interview, her remarks on a group's solution had prevalent elements of a focused-level knowledge-based reasoning. Following is an example of this kind of remarks:

> **Selda:** They did a standard solution. They tried the 10-11 solution. They tried to find the 176 by working back from the result. There is a 10-11 here, but according to what is it 10-11? In fact since they did many 10-11, they went all the way to the result 95x3. By doing different 10-11 placements, they are trying to find 176. But with 10 to 11…for example they are not all 10 or all 11. Where did you find 10 11 from? Because they must have rarely thought placing in between. Because there is all 10-11 on this sheet. Now they are multiplying…3 times 10, 2 times 11 then multiply by 3…152…not what we want, let's increase. Let's put another 11. This is what they thought..

In this excerpt, Selda spontaneously attempted to make sense of the group's solution approach and speculated on the solution approach. Following her description of what she noticed in the solution, she reasoned about how students found out 10-11 sequencing. Further, she speculated on students' ways of thinking that their solution approach was based on.

The nature of Rana's knowledge-based reasoning in the second interview was similar to the nature of her reasoning in the first interview. In both interviews, her knowledge-based reasoning was unstable. While her remarks on some groups' solutions had features of a baseline level knowledge-based reasoning, her comments on some other groups' solution provided robust evidence of a mixed level or even focused level knowledge-based reasoning. Nonetheless, because her comments having features of a baseline level were more than the other kinds, her reasoning was labeled as a baseline level in both interviews. For example, as in the first interview, Rana made statements on some groups' solution approach that had prevalent features of a baseline level knowledge-based reasoning. Following are two examples of these kinds of statements on two groups' solutions in the second interview:

> **Rana:** This was nice in terms of presentation; this group's work was especially good in terms of its explanation. They explained it in detail, they drew some figures, they made this representation. Well, they expressed it by saying 20 minus our income. This is what this group did.

> **Rana:** This group found 165. We don't have a problem here. What did they do? They placed 11 on the long side and then 10 and calculated the height. They thought about this as a whole. As in the standard placement, two rows take 40 cm, they calculated that there would be a 30 cm gap. When they are place by compressing, they said the area it would occupy on the horizontal is 20+10 root 3. They thought, in this placement, there would be a profit of 20 – 10 root 3. Since they thought they

could fit one more into that remaining area, they placed 63 boxes in the horizontal area. In the vertical, they would fit 3 rows, so they found the cupboard as 189.

In the first quote above, Rana adopted a judgmental stance in that she made evaluative comments on the group's work. Also she literally described what she noticed in the work. Therefore, her statements on this group had elements of a baseline level knowledge-based reasoning. Correspondingly, in the second quote, Rana only made descriptive comments on the solution. She simply recounted what students did. Accordingly, these excerpts provided evidence of a baseline level knowledge-based reasoning.

On the other hand, on some occasions, her remarks on a group's solution approach exhibited features of both a baseline level and a focused level knowledge-based reasoning. The following excerpt included these kinds of statements:

> **Rana:** The students caught an inequality here. They fitted them along the long side. On the long side…they first placed 11 of them side by side, then they fitted 10 into the gaps in between. Then again 11, again 10, again 11 all together 6 rows. Then they showed it like this. Here it was 220, there 110. They found this length 10 root 3. 10 root 3, 10 root 3, 10 root 3. 5 times root 3 plus 10 and 10. That's what they calculated. It should make 110. In fact here, if they said n times 10 root 3, they would find a formula. But since they drew it directly, they wrote 50 straight away.
> **Researcher:** Then why did they use the inequality?
> **Rana:** In fact, as I told this should have been n, then it would have been a clearer formula. They got rid of n is smaller than or equal to 20, they got 90 n times 10 root 3. Instead of 90 n is smaller or equal to 9 root 3, when they wrote 1.7, they would see that n is 5 by dividing 9 by 1.7. But instead of using the formula, they drew it directly. They did it accordingly.

In this excerpt, Rana initially recounted students' solution approach. Then she attempted to interpret the approach in terms of the underlying mathematical idea. She speculated on what the solution would look like if it was structured in a formal way in comparison with what the students did. She interpreted the students' approach as conjecturing about the number of rows by the help of the drawings they made. This chunk of statements was an example of how an interpretive comment followed descriptive comments that referred to the details of the solution. Following the researcher's specific question on a detail of the solution, Rana elaborated on her interpretive comments and speculated on what conclusions the students would draw if they expressed their solution approach in a more structured way that she just offered. These remarks had prevalent features of a focused level knowledge-based reasoning. As a result, this extract was considered as an example of her mixed level knowledge-based reasoning.

Similar to the previous excerpt, following excerpt had elements of a focused level knowledge-based reasoning. In this excerpt, after Rana spontaneously endeavored to engage in in-depth analysis of a solution approach, she used an interpretive stance to comment on what she noticed in the work:

> **Rana:** If you wish, we can look at this more carefully because I didn't get it in the meeting. Well, my observation is that: what did they do, how many in a row did they start with? I think they started with the short side. They said both rows would have 5, I mean, they set out thinking they would place 5-5, OK.
> …
> **Rana:** It would have been 20 between these two centres. There it's 17, here it's also 17 they thought. 17…17…34. They thought of the distance between the two centres, well…this became the mid point. 17…17…34…there it's 10, here it's 10…20. Then 14 left for this length. That's how they found the 14.
>
> **Rana:** In fact I think they would have thought how many n's there are. I mean, in that approach they would do, how many n are there. 20 of them. That is, in order to find how many rows there are, they would write an inequality like $n \times 20 + m \times 14 \leq 220$ They would try to find something from there. Well, in fact they would find a relationship between n and m. For example, when it ends like this, if the thing I called n is 1,2,3,4, then what I called m would be 3. They would write such an inequality.

At the beginning, Rana endeavored to deeply analyze the solution approach, without the researcher's prompt. While investigating the solution, she tried to make sense of what she observed. Following making sense of what students tried to do, she suggested a method for improving the group's solution in order to make the solution more structured. Indeed, this interpretation can be considered as an evaluation because she implied what students should have done but what they did not.

Unlike other three teachers, there was no considerable shift in the nature of Ayfer's knowledge-based reasoning from the first interview to the second one. Like her tendency to make descriptive and evaluative comments on group's work without any interpretation in the first interview, she continued to literally recount what she noticed in the works and to judgmental statements on them in the second interview.

She rarely endeavored to make sense of students' solution as a response to the researcher's prompts. For example, the following two excerpts quoted from the second noticing interview. The reason for quoting these two dialogues was that both had prevalent features of a baseline level knowledge-based reasoning. Furthermore, Ayfer's statements in the second conversation were connected to her statement in the first one in some respects:

> **Ayfer:** This group found 180. But they didn't calculate +20. They only used the 10 root 3's. This is why they have a 180 here.
> **Researcher:** Then, does this group have a difference from the others, in terms of its approach?
> **Ayfer:** No no, the same. Not as an approach. That 10 root 3 they have at the end, they did the operations without calculating the two radii. Then they got 180 since they didn't have the 20. There is nothing else here.
>
> **Ayfer:** And, in this group, they calculated this as 10 root 3 and then they went on. But in fact it is there, which should be 10 root 3. They calculated 10 root 3 like that, that's where the mistake came from.
> **Researcher:** Is that why they found 180?
> **Ayfer:** Yes
> **Researcher:** Compared with the previous group that found the same answer, does this group have a difference in terms of their approach?
> **Ayfer:** Well, they have a difference. In that group they calculated the total from 10 root 3, to see how many 10 root 3's there are. They didn't calculate the 10's at the beginning. Here, they took 10 root 3 not as this distance but that one. I mean they found the 10 root 3 but they took it as the value of the wrong distance.

In both paragraphs, Ayfer adopted a judgmental stance in that her comments mainly focused on the correctness of the solution approaches and she often highlighted mistakes that she claimed to exist in the solutions. In the first paragraph, although the researcher prompted her to elaborate on her statement about the solution by asking a question by saying "does this group have a difference from the others, in terms of its approach?", Ayfer sustained her emphasis on a deficiency in the solution. Hence this chunk of statements in the first excerpt provided evidence of a baseline level knowledge-based reasoning. Correspondingly, in the second paragraph, she continued to make judgmental statements on the correctness of the solution approach. She did not describe what students tried to do. Like in the previous paragraph, she only highlighted a deficiency in the solution that she claimed to exist. The researcher prompted her once more to elaborate on her comments by asking her to compare this solution with the previous one. In response to this prompt, however, Ayfer continued to refer to the deficiencies of these two solutions. Hence, her evaluative oriented comments on these two solution approaches provided evidence of her baseline level knowledge-based reasoning. She did not respond to the researcher's specific prompts.

In the second interview, Ayfer rarely attempted to reason about a group's solution approach. While she continued to make descriptive and evaluative comments on the solution, in response to the researcher's prompt, she attempted to carefully analyze the solution and speculate on underlying students' thinking.

In the following excerpt, for instance, Ayfer initially recounted what she noticed in the work, in respond to the researcher's question such as "what exactly did they do here?". Then the researcher posed more specific question by saying "how did they construct the inequality?" Following this specific prompt, Ayfer first explained how students produced the inequality. Then she questioned the reason of students' attempt to produce an inequality. This reasoning process resulted in an interpretive comment on a probable reason for producing an inequality. As a result, this chunk of statements on this solution approach had features of a mixed level knowledge-based reasoning.

> **Researcher:** What exactly did they do here?
> **Ayfer:** This group placed them along the short side, then did the operations by placing along the long side. They did the inequality. They calculated the 10 root 3's and used inequalities. And they got the correct result.
> **Researcher:** How did they construct the inequality?
> **Ayfer:** Now, they summed the 10 root 3's and added the 10 cm gaps. They said that should be smaller than or equal to 110. Because the short side is 110. But I could not understand what they did next. I didn't get for what purpose they used the inequality. Well, what is the purpose? 85 is smaller than or equal to 90, yes. They found it would be 10-11, 10-11, 3 of them. Hmm, they looked to see whether it would fit one box or not. We have 175 of them. That's why they constructed the inequality.

4.2.3 Summary of findings from the second investigation

Only two of the four teachers, Selda and Rana, attended the second noticing meeting. Both teachers did the implementation in their own classes and the groups consisted of their own students. Compared to her baseline level attention and reasoning in the first meeting, Selda's selective attention and the accompanying knowledge-based reasoning proceeded to a mixed level in the second meeting. Contrary to the first meeting, Selda clearly endeavored to carefully analyze and make sense of what students did either spontaneously or as a response to the researcher's prompt (See Table 4.3). Although she maintained making primarily descriptive and/or evaluative comments on groups' works, these sense-making, reasoning processes prompted her to both pay more attention to key aspects of the solutions and make preliminary interpretations on specifics of the solutions.

Findings from the second interview conducted after the meeting also confirmed these findings. As opposed to the first interview, Selda considerably attempted to carefully analyze and make sense of solution approaches rather than expressing her confusion due to a complexity of the approaches. While the researcher's prompts encouraged her sense making efforts, she often took the initiative for attempting to make sense of what students did (See Table 4.3). As a result, her descriptions involved references to details of the students' solution approaches, which provide evidence of a progress in her selective attention in the second interview. However such kind of efforts to carefully analyze groups' work did not resulted in interpretations. Although she made preliminary interpretations on solutions following her analyses, her statements were predominantly descriptive and/or evaluative in nature. She sustained a judgmental disposition towards the works in a way that she often assessed the solutions by comparing them with a correct solution in her mind. She often criticized students for not expressing the steps of their solutions in a clear and tidy manner.

Compared with her baseline level attention and reasoning, Rana's selective attention and accompanying knowledge-based reasoning shifted to mixed level in the second meeting (See Table 4.4). The most dominant characteristic of her mixed level selective attention was that she clearly paid more attention to specifics of the solution approaches. Additionally, she mainly focused on the solutions rather than non-mathematical aspects of the work. She only once highlighted particular students while making comments on the sheets where she personally liked the solutions. Similar to Selda's endeavor to reason about the solutions, Rana considerably attempted to carefully analyze and make sense of groups' solution approaches in the meeting. In some cases, she endeavored to make sense of a solution in response to the researcher's prompt for teachers to carefully analyze the solution. In such incidents, she only described what she observed during the sense-making process. However, on some occasions, she initiated a sense-making process by questioning about a groups' solution approach. In such cases, where she took the initiative and endeavored to make sense of a solution, she made interpretations on what she reasoned about in the solution. Other than these particular incidents, she had a tendency to highly descriptive comments on groups' solution approaches.

Findings from the second interview with Rana supported mainly the findings from the second meeting with some small differences. A discrepancy between findings from the meeting and interview was related to the direction of the shift in Rana's levels of attention from the first investigation to the second. Analysis of data from the meetings revealed that Rana's selective attention shifted from a baseline level in the first meeting to a mixed level in the second meeting. This shift pointed to a progress in Rana's selective attention to groups' works. On the other hand, analysis of data from the first and second interviews revealed an opposite direction. Namely, Rana's selective attention shifted from a mixed level in the first interview to a baseline level in the second interview. This discrepancy between findings from the meetings and interviews highlight a difference in the dynamics of noticing interviews and noticing meetings.

As opposed to her mixed level attention and reasoning in the second meeting, Rana's selective attention and knowledge-based reasoning were at a baseline level in the second interview. While Rana's levels were labeled as mixed for all groups in the meeting, Table 4.4 highlights that the distribution of her levels of attention and accompanying reasoning according to groups ranged from a baseline level to a focused level in the interview. The most common indicators of her baseline level attention and reasoning were that she made highly descriptive and/or evaluative comments on some groups' solution approaches. Moreover, they did not include any references to key aspects of the solution. Some of these evaluative comments even pointed to non-mathematical aspects of the works.

On the other hand, Rana's statements on some groups' solutions provided robust evidence of a focused level attention and accompanying focused level reasoning in the interview. Without the researcher's prompt, she spontaneously attempted to carefully analyze these solutions (See Table 4.4). Her sense-making processes enabled her to attend to mathematically significant aspects of these solutions and resulted in making interpretive comments on them. Such solution approaches that Rana attended to and reasoned about at a focused level were generally unique approaches not adopted by the other groups. Analysis of data from the interview also revealed that, Rana's endeavor to make sense of the solution ended in interpretive comments provided that she initiated the sense making process rather than as a response to the researcher's prompt. This finding was also supported by findings from the meeting.

Analysis of data from the second interview with Kutay revealed that there was a gradual progress in his selective attention and accompanying knowledge-based reasoning from the first interview to the second interview. The most dominant characteristic of Kutay's selective attention and knowledge-based reasoning in the second interview was that, rather than a judgmental disposition towards students' solution approaches, as in the first interview, he spontaneously endeavored to make sense of what students offered as solution approaches (See Table 4.3). His references to specifics of the solutions were a robust indicator of his high level attention. With the researcher's specific prompts for him to turn his attention to key aspects of a solution that deserved attention, Kutay attended to the solution at a focused level and engaged in making sense of what he observed. During the sense-making processes, he often speculated on underpinning student thinking that the solution was based on and made interpretive comments. As in Rana's case, Kutay was at a focused level while engaging in a study of sheets having rich and elaborate solutions.

Unlike other three teachers, there was no shift in nature and level of Ayfer's selective attention and knowledge-based reasoning from the first interview to the second one. Her level of selective attention and knowledge-based reasoning remained at a baseline level (See Table 4.4). The most dominant indicator of her baseline level attention was that her comments on most groups' works did not include references to key aspects of solution approaches. She highlighted general features of the solutions. Another common indicator of her baseline level attention was that there were some discrepancies between her descriptions of some groups' solutions and students' actual solutions. Similar to her judgmental disposition while investigating the groups' works in the first interview, she sustained making highly evaluative comments on solutions in the second interview. She rarely endeavored to make sense of students' solution as a response to the researcher's prompts.

4.3 Nature and Level of Teachers' Noticing in the Third Investigation

4.3.1 Nature and level of teachers' selective attention

4.3.1.1 Findings from the third meetings

Analysis of third noticing meetings conducted in each school revealed that Kutay's and Selda's selective attention, which was at a baseline level in the first noticing meeting and at a mixed level in the second meeting, proceeded to a focused level in the third meeting. Hence, it can be said that, there was a gradual development in their levels of selective attention from the first meeting to the third. On the other hand, Rana's selective attention, which was at a baseline level in the first meeting and at a mixed level in the second meeting, shifted back to a baseline level. However, when Rana's selective attention was considered at a group level, there were some shifts in her selective attention. Ayfer's selective attention remained at a baseline level. These issues will be discussed in the following paragraphs.

In the third noticing meeting, Kutay and Selda's comments on groups' solutions had predominantly characteristics of a focused level selective attention. Unlike other two

Table 4.3. Kutay's and Selda's Levels of Selective Attention and Knowledge-based Reasoning in the Second Noticing Interviews and Meetings

Groups			G#1	G#2	G#3	G#4	G#5	G#6	G#7	G#8	G#9	G#10	G#11	G#12
Kutay	**SA**	**Meet**	-	-	-	-	-	-	-	-	-	-	-	-
		Inter	Mıx(TI)	Base	Foc(RI)	Base	Mıx(TI)	Foc(TI)	Foc(RI)	Base	Mıx(TI)	Mıx(TI)	Foc(TI)	Mıx(TI)
	KBR	**Meet**	-	-	-	-	-	-	-	-	-	-	-	-
		Inter	Foc(TI)	Base	Foc(RI)	Mıx(TI)	Mıx(TI)	Mıx(TI)	Foc(TI)	Base	Mıx(TI)	Mıx(RI)	Foc(TI)	Mıx(TI)
Selda	**SA**	**Meet**	-	-	Mıx(TI)	-	Mıx(RI)	Mıx(TI)	Base	-	-	-	-	-
		Inter	Mıx(RI)	Base	Base	Mıx(TI)	Mıx(TI)	Foc(TI)	Mıx(RI)	Foc(TI)	Foc(TI)	Base	Mıx(RI)	Foc(TI)
	KBR	**Meet**	-	-	Mıx(TI)	-	Mıx(RI)	Mıx(TI)	Mıx(RI)	-	-	-	-	-
		Inter	Mıx(RI)	Base	Mıx(TI)	Base	Base	Mıx(TI)	Mıx(RI)	Base	Foc(TI)	Base	Mıx(RI)	Mıx(TI)

Note. RI = Researcher initiated; TI = Teacher initiated; Base = Baseline level; Mıx = Mixed level; Foc = Focused level.

Table 4.4. Rana's and Ayfer's Levels of Selective Attention and Knowledge-based Reasoning in the Second Noticing Interviews and Meetings

Groups			**G#1**	**G#2**	**G#3**	**G#4**	**G#5**	**G#6**	**G#7**	**G#8**	**G#9**	**G#10**	**G#11**	**LEVEL**
Rana	**SA**	**Meet**	-	Mıx(TI)	Mıx(RI)	Mıx(RI)	-	Mıx(TI)	Mıx(RI)	-	-	-	Mıx(TI)	MIXED
		Inter	Foc(TI)	Mıx(TI)	Base	Foc(TI)	Base	Base	Base	Base	Mıx(RI)	-	-	BASELINE
	KBR	**Meet**	-	Foc(TI)	Mıx(RI)	Mıx(RI)	-	Mıx(TI)	Base	-	-	-	Mıx(RI)	MIXED
		Inter	Foc(TI)	Mıx(TI)	Base	Foc(TI)	Base	Mıx(TI)	Base	Base	Base	-	-	BASELINE
Ayfer	**SA**	**Meet**	-	-	-	-	-	-	-	-	-	-	-	**-**
		Inter	Mıx(RI)	Base	Base	-	Base	-	Mıx(RI)	Base	Mıx(TI)	Base	Base	BASELINE
	KBR	**Meet**	-	-	-	-	-	-	-	-	-	-	-	**-**
		Inter	Mıx(RI)	Base	Base	-	Base	-	Mıx(RI)	Base	Base	Base	Base	BASELINE

Note. RI = Researcher initiated; TI = Teacher initiated; Base = Baseline level; Mıx = Mixed level; Foc = Focused level.

noticing meetings, in this meeting, both teachers only focused on mathematical aspects of the students' works. Additionally, they took the initiative and prompted teachers in the meeting to pay attention to the details of the solution. When one of those two teachers attracted attention to a detail of a solution in this meeting, the other also paid attention to that particular detail. Following are two examples that provided evidence of Kutay's and Selda's focused level selective attention and how they took the initiative and prompted teachers' selective attention in the meeting.

A group of students developed a solution method for this problem that no other group thought of (see Figure E4). In this method, students started with calculating the incomes in case of regularly increasing amounts of sale price. Then they realized that the incomes were the same when the sale price was 7,5 and 8 liras. Then they decided on looking at the incomes from the sale prices that were between these prices, 7,5 and 8 by reducing the amount of raise. Then they realized again that the incomes were the same when the sale price was 7,7 and 7,8 liras. Then they looked at the incomes from the sale prices that were between these prices, 7,7 and 7,8 by reducing the raise even more. The mathematically significant aspect of this solution was that students intuitively attempted to find out the relation between the sale price of the magazine and the income in case of different prices, without using a parabola. In the following dialogue, teachers made comments on this solution approach:

> **Kutay:** Also that group K5 attracted my attention.
> **Selda:** They did 1 kuruş, 5 kuruş and 10 kuruş increases in prices. Yes it's interesting. Because they increased the interval in tenths and hundredths. They divided an interval into ten, and then divided one of them again by ten.
> **Melda:** They found it by narrowing down. They used a method of compression.
>
> **Kutay:** Yes. Here they tried to find the price intervals when you make 1 kuruş, 5 kuruş and 10 kuruş increases in prices. Because of that at 7.75 they must have…they looked at the 7.76, 7.74 interval. And they found the highest value at 7.75. They found 150,156.
> **Melda:** By doing this, they are trying to find the highest profit.
> **Selda:** And since it is equal at 150.
> **Kutay:** They are trying to find the vertex.
> **Selda:** Yes.
> **Kutay:** Until which point is it rising and after which point is it going down again?
> **Selda:** Yes, exactly where.

In this excerpt above, firstly, Kutay attracted teachers' attention to this solution approach. He implemented this activity in his own classroom and thus he had a chance to closely observe students' solution process. Selda then engaged in the discussion and made a comment by referring details of the solution. Further in the dialogue Kutay described how students got the new sale price that provided the maximum income. At the end of the conversation, Kutay made comments that included references to a mathematically important detail of the solution, which is students' attempt to find the vertex of the parabola produced intuitively. As a result, Kutay's and Selda's comments, in this excerpt, had characteristics of a focused level selective attention.

Further in the meeting, Selda attracted teachers' attention to a solution approach as shown in Figure E5. Selda's and Kutay's comments on this solution also provided evidence of a focused level selective attention because they commented on the entire strategy with substantial details:

> **Selda:** On top of his, I also noted M8. They drew a parabola but a linear parabola.
> **Kutay:** They used the logic of vertex there.
> **Selda:** And also…after 7 they used 7.5, 7.6, 7.7 and they reached 7.75 but without taking a value between 7.75 and 8they concluded that it is the highest value.
> **Kutay:** Without starting to decrease…yes they decided without seeing its start of decrease.
> **Selda:** Hmm, how did they know that? Without taking 7.75 or 7.80.
> **Kutay:** But maybe here the 8 in that graph might show that they tried. Look, they saw that it is decreasing at 8.
> **Selda:** Yes I looked at that too, but here they did not do such a thing.
> **Kutay:** They did not use the arithmetic mean either. I said they found it with the values from the table.

Here, Selda first attended to the list of incomes in case of different prices that students calculated according to 0.5 lira increases in the price. She noticed that students found the optimum sale price ensuring the maximum profit without realizing the pattern of change in the incomes in case of different prices. Further in the dialogue, Kutay attracted Selda's attention to the graphs and commented on students' thinking. His comments also included references to some other details of the solution. Like the previous extract, this conversation between the two teachers can be considered as evidence of their focused level selective attention to this group's work.

In another solution approach, students considered two factors while determining the new sale price of the magazine. They considered both the number of the people who would buy the magazine and the income that the owner of the magazine would earn. Then they expressed the relations among these variables as parabolas (see Figure E6). To form these graphs, students used the linear relationship between the amount of the raise to determine the new sale price of the magazine and the number of people they would give up buying the magazine because of the determined raise. They also expressed the relation between the sale price of the magazine and the income by using the vertex form of a parabola's equation. Additionally, they attempted to find coefficient of the x^2 by substituting a point on the graph into the equation. Nonetheless, they did not use this equation for the purpose of finding the new sale price. They found the new sale price by calculating the arithmetic mean (i.e., average) of the 7, 5 and 8 liras.

In the meeting, there was a long discussion on this solution approach initiated by the one of the other teachers in the group. Selma actively participated in this discussion and attempted to make sense of the solution strategy by collaboratively working with the other teachers. For instance, in the dialogue below, Selma engaged in a sense making activity, in which teachers tried to find out how students got the result, 7,75.

> **Kutay:** Did they find 7.75 as the arithmetic mean?
> **Selda:** This is the group that used the equation of the parabola there. They found it from the equation of a parabola whose vertex is known.
> **Kutay:** This one found 7.75 by arithmetic mean.
> **Mahmut:** No, no, they found it by drawing the parabola.
> **Kutay:** Ok, they drew the parabola but how did they find 7.75 from the parabola? They found it using the arithmetic mean.
> **Selda:** Yes, that is not very clear anyway. Yes, yes, they added the two. Yes…all of them by using the arithmetic mean.
> **Kutay:** They did not find the symmetry axis or anything

4.3.1.2 Findings from the third interviews

In the third noticing meeting, while Kutay's and Rana's selective attention was at a mixed level, Selda's and Ayfer's selective attention was at a baseline level. Compared to his baseline level attention in the first interview, Kutay's level of attention proceeded to a mixed level in the third interview. When his level of attention was considered at a group level, there was a gradual development in Kutay's level of attention from the first interview to the third one. Shift in Selda's level of attention from the first interview to the third one can be considered as cyclic. She was initially at a baseline level which shifted into a mixed level in the second interview. Her selective attention was once again at a baseline level in the third interview. However when the two baseline levels were compared, it can be seen that her attention levels were baseline for all the groups in the first interview but for only half of the groups in the final interview. This shift in her progress in selective attention was considered as occurring in a cycle since it shifted from baseline, to mixed and back to baseline, yet not exactly shifting back to how it initially was.

Rana's selective attention was at a mixed level in the third interview, similar to her mixed level attention in the first one. In the second interview, although Rana's comments on some groups' works provided evidence of a mixed or a focused level attention, her attention was at a baseline level. Ayfer's selective attention remained at a baseline level from the first interview to the final interview.

Kutay's comments on groups' works had features of a mixed level or a focused level because they generally included references to the details of the solution approaches. His comments primarily focused on mathematical aspects of the students works. While investigating some groups' works, he was interested in who the group members were because he did the implementation in his class and the groups consisted of his students. Therefore, he had better knowledge of students and their solution process.

Similarly, while investigating some groups' works, Selda's statements had elements of a baseline level attention. Such statements did not include references to the mathematically significant details of the solutions. Additionally, in some of these statements, she expressed her confusion about solution approaches. She pointed to lack of students' explanations for their solution process. Such kinds of comments provided further evidence of a baseline level attention. On the other hand, Selda's comments on some other groups' works had features of a mixed level attention. Such kinds of comments contained references to some key aspects of the solution.

However, they did not refer to all aspects of the entire solution as one would expect at a focused level. Following is a presentation of examples from the interview supporting these findings regarding Kutay's and Selda's selective attention.

In the following excerpt, Selda's statements on a group's solution approach had features of a baseline level attention:

> **Selda:** This group is finding a vertex and this is an equation like the one you formed (referring to the researcher's solution). I appreciated this group anyway. Good, because they found numerical values.

The students Selda referred to, presented a unique solution on their sheet (see Figure 4.10). They formed an equation between the product of sale price and sales of the magazine, and the total profit where the number of increases in the price was the variable. This was the only group of students who adopted such an approach in this class. They also formed a table showing key values and drew the graph for this second order equation using the points from the table. While Selda was commenting on this detailed solution, she referred to finding the vertex and forming equality similar to the one formed by the researcher. She did not elaborate on the steps within students' solution.

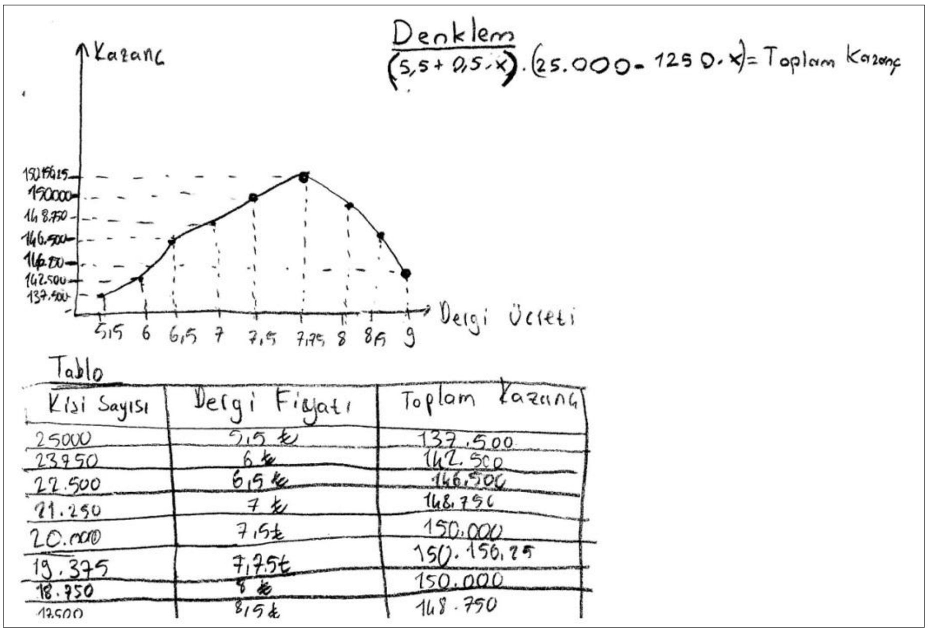


Figure 4.10. A solution approach to Magazine sales activity

In the following two paragraphs, Kutay's and Selda's statements on a group's solution approach were quoted:

> **Kutay:** They used the normal values. But the biggest mistake they made is that, they stopped when they came to a point…to 7.5 lira. The reason why they stopped is that: "we are now not making a loss". According to their guess, their estimation. If we make an increase in prices now, we'll lose many readers. We don't really have a target of making profit, we never have such a target. That's why they stopped at 7.5. Their approach is correct until 7.5 but they didn't do after 7.5. They said this is enough for us.
>
> **Selda:** This group is K1, 10th grade I think. They said "to find a sale with a little profit. We found, we thought it should be 7.5". Well, how did you think about this? 7.5? I couldn't understand from the sheet. 6.25, 6.50 they found these but… I couldn't get how they found 7.5. Without drawing the graph. They didn't say it's the vertex. Well, they tried to do 6 months, 9 months etc…they found it but there was no explanation about 7.5 in this sheet.

Kutay's statements in this quote carry characteristics of both a baseline and a focused level selective attention. He was focusing on students' steps within their solution by looking at the values they analysed on a table. This helped him to engage in a sense making process towards what students did. However, within such a focused attention, his statements contained inconsistencies with students' solution. For example Kutay claimed that students were not interested in making a profit but rather trying to find a sale price where they would avoid loss and reach as many customers as possible. This was not in accordance with what the students wrote on their sheets. Students clearly indicated that they were trying to maximize their profit. This inconsistency between Kutay's statements and students' work pointed to a baseline level attention. Additionally, at the beginning of the quote, Kutay expressed his curiosity about who the students were and stated that his meaning making process would depend on this information had he known who the students were. This was also considered as further evidence of his baseline level attention. As a result such chunks of statements had features of a mixed level selective attention.

On the other hand, Selda's statements on the same solution provided further evidence of her baseline level attention. For example, she expressed her confusion about the work. Rather than carefully examining what students offered as a solution approach, she often expressed that she could not realize how students found 7,5. Furthermore, her remarks did not include any reference to how students solve the problem. She paid attention to some values that students calculated, however she did not describe how they found them. Consequently, such chunk of statements provided evidence of her baseline level attention.

Following is another example of Kutay's mixed level attention:

> **Kutay:** In this group's work, the direction of the parabolas were different. In fact, they have the parabola approach. Because that one is the formula for writing the equation of a parabola with a given vertex. They found it with the vertex and wrote the data. That equation gives us. I mean, it gives us the equation we wrote by using r and k. But here, they took the axes differently. But they used the proportionality approach by the thing…if it's this much here, then that much there. But their approach to forming an equation is different from the other. The others used the table to form the equation. But these guys used the values of the vertex to form the equation..

In this extract, Kutay was once again trying to describe the solution by noticing some key aspects of the solution. One was about characteristics of parabolas such as the direction of the arms of the parabola, labels of the axes and use of vertex form of the equation of parabola. Furthermore he pointed to the difference of this approach to forming the equation compared to other groups' approaches. These statements can be considered as containing elements of a focused level attention. However, this extract from his interview also displays his mixed level attention, but for different reasons from what was discussed in the previous extract. Here, there was not an inconsistency between his remarks and students' work and he was only focusing on the mathematical work rather than who the students were. Yet, he did not attend to some mistakes students made during the solution such as using wrong variables while graphing and forming the equation. Besides students were trying to find a value that would maximize the profit but also keep the number of customers as high as possible. They expressed it clearly in their sheets. But Kutay did not refer to this key feature of the solution approach in his comments. These are characteristics of a baseline level attention. Existence of such evidence for baseline level, when taken together with the evidence for elements of a focused level once again showed a mixed level selective attention.

Likewise, Selda's comments on the same group's work had elements of a mixed level attention:

> **Selda:** The drawing of the parabola seemed different to me because they perceived the X and Y axes differently. It's starting from 148.000. Here they put the total sales from the first price and here they put the number of people. But they started from 26.750. I mean, we normally started from 25.000. In fact they have gone further up. But anyway, it was different. They tried to write the equation of a parabola. From x-r squared plus k. They also found the 7.75 as vertex

In this extract, Selda pointed to some key aspects of the solution. For instance, she pointed out some unique characteristics of the parabola students drew such as the direction of the arms of the parabola and labels of the axes. She also paid attention to the values of specific points on the parabola. Furthermore, she referred to the vertex form of the equation of parabola. These remarks showed that, unlike a baseline level attention, she focused more on some key aspects of the solution. However, she maintained describing what she attended to with broad terms. For example, despite highlighting the equation of parabola, she did not elaborate on some details about this equation. Also, like Kutay did, she did not attend to some mistakes students made during the solution such as using wrong variables while graphing and forming the equation. These are characteristics of a baseline level attention. Existence of such evidence for baseline level, when taken together with the evidence for elements of a focused level showed a mixed level selective attention.

Following is another example of Selda's mixed level attention to a group's solution:

> **Selda:** This group's approach was nice. They took 5.5-8.5 interval by halves. Then they took 7.5-8 interval. And even that wasn't enough, again….Theirs is very logical, for example they don't need to draw a parabola here. Because, unconsciously, they used the parabola idea. By reducing the ratios, they are trying to fit it into that interval and they saw it clearly.

These statements exhibited various features of a focused-level selective attention. For example, they included reference to students' incremental analysis towards finding the vertex, which was done intuitively rather than by using the equation of the parabola. She also attended to the mathematical idea underlying the solution. However, she did not mention how students found the income for amounts not given in the problem, while they were incrementally finding the mid values. Students assumed a linear relationship between the increase in prices and the number of customers deciding not to buy the magazine. This is one of the important ideas in the solution that Selda did not refer to. So her statements all together, did not point to her attention to all important details of the solution. That was the reason why it was labelled as a mixed level attention.

As mentioned previously, in the third interview, Kutay's statements on some groups' works had features of a focused level attention. In such kind of statements, Kutay described the solution by referring to all substantial details of the solution. In the following excerpt, for example, Kutay's statements on a group's solution approach provide evidence of a focused level attention:

> **Kutay:** This group also calculated how much loss there would be compared with the initial situation. I mean, they took 6 TL, and 137,500. They are looking at the differences among the incomes. There's a loss of 6375…What price did they use here? Since they said 2.55, is that 8.05?Yes, they came up to 8.05. The loss here increased quite a lot. They did the following: They wrote the difference between the initial income and the income after the increase as loss. I mean, in terms of money. Sorry, when I mean loss, the loss in readers in person. These are number of people, 2500, 3750, 6375 decided not to buy. Here, compared to the initial situation, 5000 more, 8750 more, 11250 more, and again 12500 more. Then it started to fall. It became 12431. From that, they decided on the price as 8.

This solution approach was shown in Figure E7. Kutay's statements on this solution included references to details of the entire solution. For example, he highlighted that students' attempt to write the profit in reference to the total income with the initial sale price was a unique approach not adopted by the other groups. This is an indicator of Kutay's attention to the approach used by the group. Besides, by going over the profits and decrease in sales one by one, he followed every single step students took towards the solution and consequently explained how students decided on the new sale price.

Similarly, the following quote gives further evidence of Kutay's focused level attention:

> **Kutay:** They distorted the symmetry in fact. They moved onto the previous one here, there is not a value here between them. 148,750. In their solution, there is not symmetry in theirs. This group also did a different mistake in calculation. What about the other side? 148,750 isn't here.
> **Researcher:** 7,5…didn't they do 6,5? Well, no 6,5. Ohh, they did 6.5 here.
> **Kutay:** Is there 146, 250 ? Oh, yes. Is there 148,750 ? That isn't there. From 7.75 to here, what do they need? 75. 0.75 from that…It should be 7. Or it should go up to 6.50. In fact, this group has thing…they went up by 50s, and then narrowed down the intervals. But I think here they made a mistake in calculations. Look, 146 146 146 The reason why they stopped is that they dropped to 145. That's why they didn't go on. They found it like they found the vertex. Maybe, if they didn't make that operational mistake, and they had higher values, they would have continued.

In this quote, Kutay is analysing students' solution with a feature of the mathematical topic in his mind. Paying attention to the symmetry of parabolas according to its symmetry axis, he was trying to find symmetrical pairs of points on the table students formed while they were studying the list of incomes for various sale prices. Through focusing on the details of the solution, he also detected a computational mistake students made within their solution.

Rana's comments on groups' works mainly focused on mathematically significant aspects of the solutions and thus had elements of a mixed or a focused level attention. However, while investigating some groups' works, her comments on solution approaches had features of a baseline level attention. On some occasions, especially when she thought the group adopted a solution approach similar to the solution provided by the researcher, she was also interested in who the group members were. She did not implement this activity, yet she taught the group of students before and knew the students well. On the other hand, Ayfer's remarks did not refer generally to mathematically significant features of the solutions. Her selective attention to almost all groups' works was at a baseline level.

In the following quote, Rana's comments on a group's solution approach give evidence of a baseline level attention:

> **Rana:** First, we thought like this. We found our total income. They found the income. Even if the number of readers decreased, we found that the profit would increase up to a point because of the increase in prices. We found it increased up to a certain point. They drew the graph for profit and loss and the profit and price graphs. And they decided that it should be 7.5. I think it is…with the previous sheet…isn't

> it? They did it with a table and drawing the graph without using the word parabola, once again.

Here, the solution she was talking about had detailed explanation of what the group did and both tables and graphs indicating the steps within the solution. In the graph drawn by the group, the students showed clearly that there was a parabolic relationship between the sale price and the income by referring to various data points and the distances between them. Yet Rana claimed that students did not use a parabola in their solution because they did not use the word parabola. This was an indication that she was attending to the mathematical term itself rather than their conceptual manifestations, e.g. features of a parabolic graph.

Ayfer's comments on the same group's solution also provided evidence of her baseline level attention. Similar to Rana's comments, Ayfer's remarks focused on superficial indicators of what the students did. This claim will be discussed in light of her comments in the following extract:

> **Ayfer:** Hmm, here again 5 5.6. They drew the graph, and saw that it was a parabola. They decided. I was in general expecting that from the groups. They didn't surprise me much..
> **Researcher:** How did they decide that it is a parabola?
> **Ayfer:** Well, roughly they must have thought it should be a parabola when they saw it. Not thinking that it would be a parabola at the beginning. Am I clear? Without thinking about things like whether these parts are linear or anything. They saw that it increased and then decreased. So they must have said "Ok, that's a parabola and this is the vertex".

Ayfer did not refer to the details of the solution. Since she did not attend to the details that showed students' thinking describing how they decided that the relationship was parabolic, i.e. their analysis of the data points, she oversimplified their decision that the graph was parabolic. Ayfer's baseline level attention incident took place even after researcher's prompt for her to analyse students' decision about drawing a parabola.

In this interview there was also evidence that Rana engaged with students' works with a mixed level attention. Following is an example of this kind of attending to student work:

> **Rana:** (looking at the sheet silently for a while) When they calculated the income they would have when the price is increased and the number of readers changed, they also caught the equality of 7.5 and 8. They wrote the same thing. What attracted my attention in that paper was that: Not linear but a parabolic change this graph would display, they wrote. They used the Word parabolic. That's what attracted my attention in that sheet. But when preparing their report, they didn't make something more mathematical in their final report. But at least they made a mathematical operation. And they conjectured that it could be a parabola.

Here, Rana examined students' display of data points and as a result of this analysis she expressed that students made sense of the relationship between the sale price and income. She referred to the equivalence of incomes for certain data points. These were some indicators of her focused attention to central features of the solution. But there was a discrepancy between her explanation of students' use of the term parabola and how students actually used the term in their explanation. In their solution, students claimed that the relationship between the increase in prices and the number of people who decide to give up buying the magazine could not be linear as given in the question and it had to have a parabolic nature. This explanation by the students did not refer to the parabolic relationship already existing between the variables. However, as in the previous extract discussing Rana's baseline level in the interview, she attributed students' use of the term parabola to the existing parabolic relationship without paying attention to students' written explanations. This discrepancy is a prevalent feature of a baseline level. And the coexistence of the focused and baseline levels showed a mixed level attention.

> **Rana:** Taner, instead of calculating one by one, thought about how he would get it. Was the initial price 5 lira? When we make it 5.5, that many people. Then we make an addition to 5.5, and reduce this one here. By saying that in every addition, we're selling the same amount and calling the addition as x, they formed an equation. Then they thought what is our aim? Profit. What shall we do to that profit? It should be more than the price, more then the profit now. They also said it can be equal. So they thought, I should at least get this amount. They said that it is a second order equation and that the graph would be a parabola. They thought the vertex would give the maximum value. The vertex is 9/2. With 9/2 you need a 2.25increase, and that increase gives a price of 7.75.

In the extract above, Rana described the solution approach step by step, with substantial details. First she commented on why this group of students preferred to write an inequality to solve the problem rather than calculating various incomes to spot the maximum profit. Then she explained both sides of the inequality students formed and discussed the meaning of the terms. She also elaborated on their choice of smaller than or equal to sign in the inequality. Next she summarized how students decided on using the vertex for finding the maximum income. She finished her explanations by checking the result students found. All these statements point to a focused level attention.

The following quote is from the interview with Ayfer about the same group's work, discussed for Rana above.

> **Ayfer:** Well, here, they wrote the formula straight away. By writing the formula, they sorted it out. Look, this one is very nice, I'm putting a star here. Yes, they wrote the formula from the beginning abd sorted it out. They also found the vertex.

She described the solution with broad terms which did not refer to specifics of the solution discussed by Rana. For example Ayfer did not mention about how students produced the inequality and how they determined the maximum income by using the vertex of the parabola. In opposition to Rana's detailed focused attention, Ayfer's statements on the solution gave robust evidence of a baseline level.

4.3.2 Nature and level of teachers' knowledge-based reasoning

4.3.2.1 Findings from the third meetings

There was a gradual development in Kutay's and Selda's knowledge-based reasoning from the first noticing meeting to the third meeting. Compared to his baseline level knowledge-based reasoning in the first meeting, Kutay's reasoning proceeded to a mixed level in the third meeting. Likewise, compared to her baseline level knowledge-based reasoning in the first meeting and her mixed level reasoning in the second meeting, Selda's reasoning proceeded to a mixed level in the third meeting. Although it looks as if Selda's reasoning in the second and third meetings were at the same level, there is a considerable shift in the nature of her reasoning when it is considered at a group level.

The shift in levels of Rana's knowledge-based reasoning from the first meeting to the third can be described as cyclical between a baseline level and a mixed level. Her reasoning was at a baseline level in the first meeting and proceeded to a mixed level in the second meeting. But it did not continue to be at a mixed level or did not shift to a focused level in the third meeting. On the contrary it shifted back to a baseline level. However, when it was considered as a group level, some shifts in her reasoning stood out. For example, her statements on some groups' work had featured of a focused level knowledge-based reasoning. In contrast to other three teachers, there was no considerable shift in Ayfer's knowledge-based reasoning from the first meeting to the third one. In this section, these findings will be presented.

As mentioned previously, there were noteworthy shifts in both Kutay's and Selda's knowledge-based reasoning from the first meeting to the third one. As opposed to their comments on students' works in the first meeting that were predominantly descriptive and evaluative, they tended to make interpretations on what they noticed in the works and often endeavored to reason about and make sense of what students did. Furthermore, they often took responsibility for initiating a discussion on students' solution approaches that all teachers in the meeting engaged in.

The following discussion on a group's solution approach provided robust evidence of their focused level knowledge-based reasoning in that they carefully analyzed the approach and made interpretive comments on what they noticed:

> **Kutay:** Did they find 7.75 as the arithmetic mean?
> **Selda:** This is the group that used the equation of the parabola there. They found it from the equation of a parabola whose vertex is known.
> **Kutay**: This one found 7.75 by arithmetic mean.
> **Mahmut:** No no, they found it by drawing the parabola.
> **Kutay:** Ok, they drew the parabola but how did they find 7.75 from the parabola? They found it using the arithmetic mean.
> **Selda:** Yes, that is not very clear anyway. Yes, yes, they added the two. Yes…all of them by using the arithmetic mean.
> **Kutay:** They did not find the symmetry axis or anything.

In this excerpt, Kutay initiated a discussion on how students found the correct answer of the problem by asking "did they find 7.75 as the arithmetic mean?" Following

Kutay's prompt, teachers reasoned about this issue. Selda also attempted to make sense of how students found the answer. After a reasoning process, teachers reached a consensus on the issue. Further in the conversation on the same group's solution approach, Kutay once more initiated a discussion on the parabola in the solution by posing a question. Teachers, including Selda, engaged in the discussion and attempted to make sense of the parabola. During the sense-making process, both Kutay and Selda made interpretive comments on whether the parabola was correct or not, mathematically:

> **Kutay:** In fact I did not have the opportunity to analyze the first thing…I mean, whether such a parabola can be found here or not? But its arms are looking the other way. The same parabola as this (pointing to the one below) but arms looking right. Probably because they did not plot the data points to scale so they did not get the shape of a parabola.
> **Mustafa:** This one drew the total money and the total number of people.
> **Kutay:** I mean the arms looking down or up, turned into arms looking right or left for this parabola here.
> **Selda:** Twisted the axis…
> **Kutay:** Since they did not plot the points to scale, it is not drawn well. I mean, one of them…
> **Meral:** But the axes are different.
> **Selda:** No, they took the axes differently.
> **Mustafa:** They took the axes differently.
> **Kutay:** No, what I mean is that, it looks like the y equals x squared parabola.
> **Selda:** Yeah, like that.
> **Kutay:** and, like, x equals root y parabola.
> **Selda:** They did not wrap it around the axis exactly.
> **Kutay:** x equals y squared. Scales are different in fact, a parabola like that…
> **Selda:** Yeah, it would be like that.
> **Kutay: …**comes out.
> **Selda:** Yeah, they couldn't wrap it around the axis.
> **Kutay:** I mean, both graphs are correct in my opinion, reasonable from a mathematical view.

Similarly, Selda initiated a discussion on another group's solution by posing a question about the solution that was unclear for her. Following Selda's prompt, Kutay and Selda endeavored to analyze the solution deeply to comprehend students' ways of thinking. Specifically, they reasoned about how students concluded that 7,75 was the highest value without taking a value between 7,75 and 8. Further in the dialogue, following the sense-making process, Selda and Kutay made interpretive comments. They offered an interpretation on students' mathematical thinking that their solution was based on:

Selda: Well, naturally as the axis was not taken as symmetrical, the graph came out this way. And also, it looks like they are using direct proportionality all along the way. If this is increased by a unit, there it is 100; 2 units here, 200 there…
Kutay: Yes, with that logic they formed the equation…to make it shorter they formed the equation.
Selda: That's why it does not lead up to a parabola. I mean, since the values in between are skipped, it comes out as linear.

These excerpts had prevalent features of a focused-level knowledge-based reasoning. On the other hand, their comments on sole group's solutions had features from both a baseline level and a focused level knowledge-based reasoning. These chunks of statements were labeled as a mixture level reasoning. In this excerpt, Selda initially made literal description and evaluation of the solution by saying "they did 1 kuruş, 5 kuruş and 10 kuruş increases in prices. Yes it's interesting. Because they increased the interval in tenths and hundredths. They divided an interval into ten, and then divided one of them again by ten". These literal descriptions exhibited elements of a baseline level knowledge-based reasoning. After the researcher prompted them by asking a specific question such as "What is the aim of the students by doing this" Kutay and Selda offered interpretations of the solution approach.

Following excerpt also provided evidence of Kutay's and Selda's mixed level knowledge-based reasoning:

Selda: That parabola is a little bit different. I did not get what they tried to do there.
Mustafa: This group decreased the number of people in the graph.
Selda: Well, the one at the top seems to have a logic, but what does the bottom one mean?
Kutay: They thought about both possibilities. The variables in both graphs are the same. TL.
Researcher: How did this group find 7.75?
Selda: For example they didn't make a table.
Kutay: Again, they used the thing…arithmetic mean. They found by arithmetic mean.
Selda: But they also explained quite a bit. Their working style is different.
Kutay: Finally they must have thought they need to use the vertex. The difference is 1 between them, and 0.25 between 7.5 and 8.
Selda: I think they thought as if it's the arithmetic mean. Then why did they draw this?
Kutay: No, they could not decide whether the arms are facing downwards or facing upwards. With the data, you can't get that…if they paid attention that wouldn't happen. But they messed it up a bit.
Selda: Yes, that's why it looked different to me.

In this discussion, Kutay's and especially Selda's statements on the solution approach had elements of a baseline level and a focused level knowledge-based reasoning. On the one hand, Selda tried to make sense of the parabolic graphs that students used in their solution. Also she speculated on underlying students' thinking such as "I think they thought as if it's the arithmetic mean". On the other hand, she often made judgmental statements on the solution by saying "that parabola is a little bit different" or "For example they didn't make a table". As a result, Selda's statements on this group's solution provided evidence of a mixed level knowledge-based reasoning. Similarly, although Kutay's comments were generally interpretive in nature, he also made evaluative remarks such as "they messed it up a bit"

In the third noticing meeting, Rana commented on a few groups' works. She expressed that she did not have enough time to examine the students' works because she got them late. Rana's knowledge-based reasoning in the meeting was labeled as a baseline level because she made descriptive and evaluative comments on some group's solution. For instance, in the following extract, she first offered a literal description of what she noticed in a group's work by saying "this group used the word parabola". Indeed, this comment has a dual meaning in that it also evaluative in nature. She assessed students' use of "parabola" word by judging it against the correct solution in her mind. Further in the dialogue, she criticized the group by "not taking the explanation seriously".

> **Rana:** This group used the word parabola. I don't know whether they did it deliberately, knowing its meaning but they used the word.
> **Ayfer:** Yes. In fact they did the solution but they couldn't interpret their solution. I mean, in order to reach the result, they left it unexplained. There's no interpretation on the paper.
> **Rana:** There was not a formal attitude. I mean, did you read this? They wrote"as a standard promotion strategy we preferred small numbers, we chose 4.99". I thought they were not taking it seriously.

In the excerpt above, Ayfer's statements were also evaluative in nature. She criticized the solution approach because she claimed that students could not interpret what they did. Ayfer's knowledge-based reasoning was also labeled as a baseline level. However, unlike Rana, almost all her comments on groups' works were predominantly evaluative and descriptive. The following excerpt provided robust evidence of her baseline level reasoning in the meeting. Her comments were highly evaluative and judgmental in nature:

> **Ayfer:** For example I liked A5 very much. A5 found the solution by actually doing mathematical operations but not drawing any graphs or anything.
> **Hande:** Yes, for example they even calculated how many people would buy the magazines. Only they did this calculation.
> **Ayfer:** They also did their explanations well, a group of direct reasoning (referring to the group name students chose: "direct reasoning")

Although Rana's knowledge-based reasoning was labeled as a baseline level, her comments on some groups' solution approach had elements of a focused level. For instance, in the excerpt below, she initiated a discussion on a group's solution approach and reasoned about the solution:

> **Rana:** I wonder whether they first drew this parabola when preparing the report, or they did this and then presented it as a report. Or else, they wrote this and then drew the graph at the back…
> **Funda:** No, I think they must have drawn the graph first.
> **Rana:** Here it looks like this is written first but maybe they first drew the graph and then they decided to look at the values.
> **Hande:** I think it's the opposite. After they found these values by four operations, they must have decided that it's a parabola when they plotted the points on the graph. They wrote the equation to find the vertex.
> **Rana:** That is, they approved the result they found here once again by doing that. They checked their result.
> **Hande:** Yes, that's what I think.

In the discussion above, Rana's question on the solution prompted teachers to speculate on and interpret students' solution processes. Teachers, including Rana, endeavored to make sense of the stages that students went through and which stage was used for the purpose of checking the answer they already got. Therefore, this excerpt provided evidence of a focused-level knowledge-based reasoning.

Likewise, the following excerpt contained evidence of Ayfer's baseline level reasoning and Rana's focused level reasoning:

Ayfer: It's Adem's paper but he did not draw it correctly…I mean, did not place the values well onto the graph. There's a problem here. Under normal circumstances, it should have been a parabola.
Hande: Well, they used the profits for doing this…in fact it's like this: the reason why Adem did it correctly is that they used the profits for the solution. They took the differences between the case of 5.5 lira as a selling price and 6 lira as a selling price, as the basis of their progress.
Rana: They formed a relationship between the number of readers and price. Above, they wrote something like x times y is greater than…In fact I thought like they were thinking about the area under the plot in the graph. Thinking of it as a linear graph and then…
Funda: Like, as the price goes up, the number of buyers would go down…like that…
Ahmet: Inverse proportion.
Rana: There's an inverse proportion.

At the beginning of the following discussion, Ayfer made evaluative comments on a graph that a group of students used in their solution. She stated what it should have been rather than what it really was. She claimed that this graph should have been parabolic rather than linear. Following Ayfer's claim, teachers, including Rana, made an effort to make sense of the graph. Then, Rana made an interpretation by linking a formula that she noticed and the graph drawn. She highlighted that the students have thought about the area under the curve in the graphical representation of the inequality. This helped other teachers to question why students drew this graph as a part of the solution.

4.3.2.2 Findings from the third interviews

In the third noticing interview, Kutay's reasoning was at a focused level. There was a gradual development in Kutay's knowledge-based reasoning from the first noticing interview to the third interview. Compared to his baseline level knowledge-based reasoning in the first interview and his mixed level reasoning in the second interview, Kutay's reasoning proceeded to a focused level in the third interview. On the other hand, the other teachers' knowledge-based reasoning was at a baseline level in the third interview. The shift in the levels of Selda's reasoning from the first meeting to the third one seemed as cycling between a baseline level and a mixed level. Although it seemed that her level of reasoning shifted back to a baseline level in the third interview, when it was considered at a group level, Selda's comments on some groups' work provided evidence of a mixed level reasoning.

There was no considerable shift in both Rana's and Ayfer's levels of knowledge-based reasoning from the first interview to the third one. Similar to their baseline level reasoning in the second and third interview, their knowledge-based reasoning was at a baseline level in the third meeting. Although they primarily offered descriptive and evaluative comments on groups' solution approaches, their comments on a few groups' approaches had elements of a mixed level reasoning. Next, these findings will be elaborated.

Kutay's knowledge-based reasoning was at a focused level because he predominantly made interpretive comments on groups' solution approaches. Furthermore, he mainly endeavored to make sense of what he noticed in the students' works. As opposed to his comments that were highly descriptive and evaluative in the first interview, in the third meeting, he adopted an interpretive stance and often attempted to reason about solution approaches. In the following excerpt quoted from the third interview, Kutay's statements on a group's solution approach had prevalent features of a focused level knowledge-based reasoning:

> **Kutay:** This group made a table of income but how did they know that they needed to stop at 7.75? Since the following values are not shown on the table, it is not clear how they knew this was the maximum price. They drew the graph linear, it shows as 8 on the graph. It also shows as 7.5. Actually the figure is not symmetrical. I would make a comment like is it possible that they thought about the symmetrical of this and found as 7.75 but since it is not exactly symmetrical, how did they make this decision? They drew the graph as a line, typical student approach. When you take points, how would you connect them? With a line. But at least they showed the 8 here. Maybe we could think about that if the shape was symmetrical. But it is not symmetrical here. I would say that's how they found it, if it was symmetrical.

In this excerpt, Kutay endeavored to make sense of how students got the correct answer without calculating all necessary values to realize the parabolic change among the values. Further in the excerpt, he questioned whether students used the symmetrical structure of a parabola to find the answer. Also he made an interpretive comment on the parabola that students drew by using his teaching experience with students by saying "they drew the graph as a line, typical student approach. When you take points, how would you connect them? With a line" As a result, rather than making literal descriptions of the solution, Kutay reasoned about underlying ways of thinking of the students' and made interpretive comments on that. Therefore, this chunk of statements provided evidence of his focused level knowledge-based reasoning. Similarly, the following extract provided robust evidence of a focused level knowledge-based reasoning in that he primarily made interpretive statements on a group's solution:

> **Kutay:** This was the group that did the equation. In fact, I don't think this group thought about the logic of equation from the logic of forming a parabola equation. I think they tried to form an equation to make the operations on this table shorter. I don't think they thought this was an equation of a parabola and then tried to write a parabola equation. I think this equation came out of that reasoning, because they called the number of increases in price x. We find the parabola equation when we multiply it anyway.

In this extract, Kutay made highly interpretive comments on the solution approach. For example by referring to students' efforts to shorten the process of making the calculations, he made interpretive claims about why students needed to form an equation. Additionally, he pointed out what students referred to as x. This can be considered as another interpretive comment since it focuses on how students produced the equation mathematically. Following is another example of Kutay's interpretive stance that he adopted during the third noticing interview:

> **Kutay:** But this group used the logic of ration and proportionality. I mean, if there are 1250 people for 50 kuruş…now the price was 5.5…let's add 2.25 and make it 7.75. How many would it be from 2.25 TL. That is, with the difference in between, they thought about how much loss would there be and tried to form a proportion. I mean, do you remember the approach of forming an equation from the table done by a group? This group used a similar logic. If this group spent a bit more time, they would find that equation.

At the beginning, Kutay was trying to figure out the meaning of students' solution steps as to what their target is and he mentioned which mathematical concept they used by referring to proportions. He then made a comparison between two different groups' solution approaches, which was also considered as evidence of a focused level reasoning besides his interpretive comments. But, maybe most importantly, he was trying to speculate on the mathematical concepts underpinning students' intuitive approach by referring to the "equation forming logic". In addition to adopting an interpretive stance, Kutay often attempted to make sense of what he noticed in the groups' works, which was also considered as evidence of his focused level knowledge-based reasoning. While he generally took responsibility for his sense making process, he also responded to the researcher's prompts to do that.

In first example below, Kutay spontaneously endeavored to make sense of the solution. Here Kutay was questioning how students might have reached the correct solution without using the symmetrical structure of parabolas. He did this by referring to what kind of an explanation he would expect to see in students' solution. Similarly in the second excerpt below, Kutay was trying to make sense of whether students used the idea of arithmetic mean of corresponding points to find the vertex. He speculated on what alternative methods students might have utilized, e.g. looking at the drawings, in case they did not use the arithmetic mean idea. However this time the inquiry was initiated by the researcher's question rather than occurring spontaneously.

> **Kutay:** What did this group do? I think they did our standard table. But through which reasoning did they reach 7.75? I wonder if they reached by symmetry again. I mean, if they showed these like that, it would be much…erm…For example if they showed which is matched with which…if they showed 8.5 and 7, 9 and 6.5 and told the mean is always 7.75. If they wrote, this one and this one is 7.75, I would understand. I don't know how they found this without any such operations. I think it has been guessed since they said mid values. I understand those who found by trying but not the ones who found without even trying. They wrote that they tried, they wrote they tried the values in between but I don't know how they decided it was the highest. Maybe it was higher for 7.76.
> **Researcher:** OK, which mathematical idea might be underlying the thought of taking the arithmetic mean of those two values.
> **Kutay:** But maybe they caught that this value is the half of the sum of those two here. Or that they are going by same increments and they took the average of those two here 150-150 and said could it be the arithmetic mean of the two. I don't know what they thought about when they caught it but if they caught it well and drew the figure neatly, they might have caught it from the figure. But they formed the table and then drew the figure, the fact that 7.75 would be the vertex…There is ambiguity there. Without moving onto the parabola it is interesting that they found it. But I don't know if they did it as arithmetic mean.

As opposed to Kutay's focused level knowledge-based reasoning, Selda's reasoning in the third noticing interview was at a baseline level. She predominantly made descriptive and evaluative comments on students' solution approaches. For example, the following two quotes provided robust evidence of her baseline level reasoning in the interview:

> **Selda:** This group, they also analysed the 7.5-8 interval. For example this one seemed more meaningful to me, because they saw that it's increasing between 8.5-9, that's meaningful. They are finding a vertex and this equation is similar to the one you constructed (referring to the researcher's solution). I appreciated this group anyway.
> **Selda:** The drawing of the parabola seemed different to me because they perceived the x and y axis differently. It's starting from 148.000. Here they put the total sales from the first price and here they put the number of people. But they started from 26.750. I mean, we normally started from 25.000. In fact they have gone further up. But anyway, it was different. They tried to write the equation of a parabola. From x-r squared plus k. They also found the 7.75 as vertex.

In the first excerpt, Selda simply recounted what she noticed in the solution. She literally described that when calculating the income for the range of various sale prices, students could realize the required sale price for the maximum income. Then she pointed out the equality they she noticed in the solution. She also made evaluative comments in that she expressed her appreciation of the solution. Thus, this excerpt exhibited features of a baseline level knowledge-based reasoning.

In the second excerpt, Selda started with an evaluative comment on the parabola that students drew. She expressed that this parabola was interesting for her. She recounted what she noticed in the parabola. Then she pointed to the equality and literally described it. At the end, she highlighted that students found the correct answer. These statements were highly descriptive in nature. Thus this second excerpt was also considered as robust evidence of Selda's baseline level reasoning, in the third interview.

Although Selda's comments on many groups' solution approaches were descriptive and evaluative in nature, only once, she made an interpretive comment on a group solution:

> **Selda:** This group's approach was nice. They took 5.5-8.5 interval by halves. Then they took 7.5-8 interval. And even that wasn't enough, again….Theirs is very logical, for example they don't need to draw a parabola here. Because, unconsciously, they used the parabola idea. By reducing the ratios, they are trying to

> fit it into that interval and they saw it clearly. Well, they could have drawn a parabola but this is even better. Here, they are totally displaying their own idea. In parabolas, they are using the knowledge we teach them.

In this excerpt, after briefly explaining what students did, Selda made an inference about the underlying mathematical thinking that students' intuitive solution approach was based on. She speculated that students unconsciously used the concept of parabola in their intuitive solution. Further in the dialogue, she made evaluative comments on this approach in that she expressed her appreciation of this kind of solution approach. Because this chunk of statements included elements of both a baseline level and a focused level reasoning, it was labeled as a mixed level.

Likewise, on some occasions, besides some interpretive comments, she attempted to make sense of students' solution approaches, without the researcher's prompt. In the following excerpts, she was questioning how students determined 7,75 as the optimum sale price for the maximum income without considering any other sale price. Then she speculated that they used the arithmetic mean idea while determining 7,75.

> **Selda:** Here they found the proportional values. This one is the same too. "We thought about this last" they said. How did you think about that? I mean, why did they multiply it with 7.80, for example? Why did they think about 7.75 straight away? I think, most probably, they found it always from arithmetic mean.

> **Selda:** Yes, that one also with 5.5…now, they did something for the interval between 5.5 and 7. Then, they didn't do the interval between 7 and 9 normally but, they felt the need to do that. When they looked at this, it looks like they thought about calculating the middle values but not for the right interval. Then, I would understand that. If they didn't do it, I would understand. But why did they pick that among all the values. I mean, that the interval was like that…Well, maybe like that, after this point they saw that it was decreasing and said let's not analyse the intervals. But I think they should have analysed here and this one too. Then they found the result 7.75 but did they find it with a reason, I cannot comprehend.

At the beginning of the third interview, Rana repeated what she previously expressed in the meeting, that she did not have enough time to carefully examine the student's works because she only got the worksheets late. In the interview, Rana's knowledge-based reasoning was at a baseline level because she predominantly made descriptive and evaluative comments on solution approaches. She described what students did by almost direct quotations from the students' work sheets. She criticized many group's works as "not taking it seriously" and as lacking mathematical aspect. Rana's comments that had these features were labeled as baseline level reasoning.

Following is an example of these kinds of statements from Rana:

> **Rana:** In this class' sheets, it looked like they didn't take it very seriously. For example one group wrote things like we thought a product sold for 499, instead of 500 lira would be more attractive, things like that. Consequently, they said they were aiming both high number of people and high price. When sold for 6 lira, probably it would be 235 lira less. "We decided on the price to be 5.99" : a report based on no mathematical idea. When they calculated the income they would obtain with the number of people and the increased price at the back pages, this group thought about the equality between 7.5 and 8 liras. This was what attracted my attention on this paper. By saying "not linear, a parabolic relationship would be seen" they used the word parabolic, and parabola.

This excerpt had prevalent features of a baseline level knowledge-based reasoning in that the statements were highly judgmental and consisted of literal description of what students expressed in their worksheets. Initially, Rana criticized this work as "not taking it seriously" and lacking a mathematical approach. She described what students did by reading what students expressed in their report. Then she literally described students' use of the word "parabola" without making any interpretation about how the concept of parabola was used in the solution. This comment can be also considered as evaluative in nature because here, Rana assessed the use of the word parabola by judging it against the correct solution in her mind.

Similarly, Rana's comments on another group's work also provided evidence of her baseline level reasoning:

> **Rana:** This group calculated the annual profit. They found how much profit would keep the magazine on the plus. "By making a table we saw that 7.5 lira and 8 lira bring the same profit. So we were indecisive between 2 and 2.5 lira increase. But the prices were going up first and then decreasing. We tried to find the maximum of this profit" But definitely not a mathematical approach. They did not use the word parabola and thought about it as increasing and decreasing. And they decided to take the value in between: 2.25.

In this excerpt, the nature of Rana's reasoning is very similar to the previous one. She first literally described what she noticed in the worksheet by using direct quotations. Then she criticized students' solution approach for a lack of mathematical base. Then, like in the first excerpt, she evaluated the approach by the use of the word "parabola". This judgmental stance and literally describing what was observed are two prevalent features of baseline-level reasoning. As a result, this excerpt was considered further evidence of Rana's baseline-level reasoning.

Besides these findings, Rana's statements on a few groups' works exhibited features of a mixed level reasoning. That is, they included elements of both a baseline level and a focused level reasoning. In these statements, while Rana sustained making descriptive and evaluative comments, she also made interpretations of what she noticed in the students' works. In what follows, an excerpt from the interview providing evidence of a mixed level reasoning will be presented.

> **Rana:** Again they made 5 kuruş increases between 7.5 and 8 lira. When calculating this, they also broke 1250 people into smaller pieces? In order to calculate this, 7.5…when 20000 people, 10875. How did they find this? They divided 1250 by 10. What does it give you when you divide by 10? I mean, for every 5 kuruş, sales decrease by 125 people, that's the proportion they formed. Is that correct? We can't know that. Because this is a statistical thing.
> ….
> **Rana:** Since this is a parabolic curve, that means there is a second order function involved, then what is this a function of. By saying these they tried to turn it into a function. This was what I really liked. I like the way that they said "We solved the problem like this but we wonder whether there is an equation that we can use". I congratulate this group because of that. Their approach is nice. That's how they approached the question.
> **Researcher:** For what purpose did they use the equation?
> **Rana:** They evaluated that the vertex is 7.75 once again by using this equation. In fact this was some sort of checking their work. They supported their answer..

At the beginning of this excerpt, Rana explained the solution approach. Then she spontaneously endeavored to make sense of how students found the decrease in the number of people in the case of 5 kuruş raise in the sale price. After sense making, she was questioning the correctness of this kind of proportioning. These reasoning processes exhibited features of a focused-level reasoning. Further in the dialogue, she first highlighted students' attempt to produce an equation of parabolic function and expressed her appreciation of this attempt. These comments were highly descriptive and evaluative in nature. These were prevalent features of a baseline level reasoning. In response to the researcher's specific question on the underlying students' thinking, she speculated that students produced the equation to support their result which was found intuitively. This interpretive comment was evidence of a focused level reasoning. As a result, because this chunk of statements included elements of both a baseline level and a focused level reasoning, it was considered as an example of Rana's mixed level knowledge-based reasoning.

Like Rana, Ayfer's knowledge-based reasoning was at a baseline level in the third interview. Ayfer's statements on many groups' works were predominantly descriptive and evaluative in nature. For example, she commented on a group's solution by saying "Here they wrote the formula straight away. They found the result by using the formula. Yes, they wrote the formula from the beginning, and sorted it out. They also found the vertex. I like it a lot." In these comments, Ayfer simply recounted what she noticed in the students' work by referring to the formula that they produced. She evaluated the solution by expressing her appreciation of the group's attempt to put a formula in their solution and getting the correct answer. As a result, these statements provided robust evidence of her baseline-level reasoning. Similarly, following extract is also had evidence of her baseline level reasoning:

> **Ayfer:** Here they calculated the income for 5.5, 6, 6.5, 8.5….Tables are made but the graph isn't drawn. They decided without drawing the graph. They wrote things like the increase isn't linear but parabolic. I mean, they saw that the increase wasn't linear, they decided that it was parabolic. Well, they did certain things but their comments are strange.

At the beginning, Ayfer literally described what she noticed in the work by saying "Here they calculated the income for 5.5, 6, 6.5, 8.5". Then, she offered comments that were highly evaluative in nature in terms of two aspects. Firstly, Ayfer assessed the solution by comparing it with the correct solution in her mind. For example, she highlighted that students did not draw a graph that they were supposed to do according to her point of view. Similarly, she pointed to students' realizing a parabolic change rather than linear, which was consistent with the correct solution in her mind. Secondly, she criticized students' explanations as absurd. Hence, these statements exhibited prevalent features of a baseline level reasoning.

Despite this dominant pattern, Ayfer statements on a few groups' works exhibited features of a mixed level reasoning. In these statements, while Ayfer maintained making descriptive or evaluative comments, she also endeavored to make sense of what students did. In the following excerpt, Ayfer's comments on a group's works had elements of a mixed level reasoning:

> **Ayfer:** Here they gave values. Then, to be able to decide which one, they gave values from the interval.
> **Researcher:** How did they calculate the incomes for the values from the interval, for example for an increase of 0.1 kuruş?
> **Ayfer:** They did the thing…by finding the number of people and multiplying. For every 50 kuruş…hmm, I think they calculated proportionally there. Look, if 1250 people go for every 50 kuruş, in 5 kuruş, how many people go for 5 kuruş…from that they decided 125 people, from ratio and proportionality. I think here they looked at the thing…but look they used it for the formula. But is their formula correct? What did they do? A times x minus…I think they made a mistake in the formula. I think they tried to write a times x minus x_1, x minus x_2 equals y.
> **Researcher:** Why did they construct the equation?
> **Aylin:** Now, they tried to make this zero. They're saying if a point is zero…We use this formula if the points intersecting the parabola are known. But this would start from zero, there is a problem here. But still, in general they reached somewhere roughly as a result. And these…Well, if such a question was asked in the exam and we were looking for a clear answer, then what they did would not really have much significance. They would not be able to get the full credit. Look, here someone did it too. Look, they all started from 0.

At the beginning, Ayfer described students' attempt to find the values of income for raises less than 50 kuruş. Then, as a response to the researcher's question on how students calculated these incomes for raises less than 50 kuruş, she attempted to make sense of this part of solution. Following a sense making process, she realized that students used proportions to find the values of income. These statements had features of a focused level reasoning.

Further in the dialogue, she highlighted a parabolic equation that students produced and attempted to understand whether this equation was correct or not. These statements had a dual nature. On the one hand, Ayfer endeavored to understand what students did. On the other hand, the motivation behind her endeavor seemed to assess the correctness of the equation. Following the researcher's prompt: "why did they construct the equation?" she first attempted to carefully analyze the solution. However, she suddenly stopped analyzing the solution and evaluated it as incorrect.

Her further statements on the solution were highly evaluative. She expressed that if students offered this kind of solution approach in an examination; their solution would be assessed as worthless. As a result, because this chunk of statements had elements of both a baseline level and a focused level reasoning, it was considered as evidence of a mixed level knowledge-based reasoning.

4.3.3 Summary of Findings from the Third Investigation

In Table 4.7, a gradual progress in Kutay's levels of selective attention and knowledge-based reasoning from the first meeting to the third one can be seen. Compared to his baseline level attention and reasoning in the first meeting, Kutay's selective attention proceeded to a focused level and his knowledge-based reasoning proceeded to a mixed level in the third meeting. These findings were also confirmed by the findings from the interview with Kutay (see Table 4.8). Analysis of interview data revealed that a gradual progress occurred in Kutay's attention and reasoning over time. Compared to his baseline level attention in the first interview, Kutay's selective attention proceeded to a mixed level in the second and third interviews. Similarly, compared to his mixed level reasoning in the first interview, Kutay's knowledge-based reasoning proceeded to a focused level in the third interview.

Likewise, a gradual development occurred in Selda's selective attention and knowledge-based reasoning from the first meeting to the third one. Compared to her baseline level attention and reasoning in the first meeting, Selda's selective attention shifted to a mixed level in the second meeting and a focused level in the third meeting (see Table 4.7). Findings from the interviews also supported the finding that there was a gradual progress in Selda's selective attention and reasoning over time. Yet, the progress revealed by analysis of interview data was not as sharp as the progress revealed by analysis of data from the meetings. Throughout the three interviews, the direction of shift in Selda's selective attention and accompanying reasoning seemed to be occurring in a cycle since it shifted from baseline, to mixed and back to baseline, yet not exactly to how it initially was (see Table 4.8). In fact, when the second and third interviews are compared, even though the second one is labelled as mixed and the third one as baseline, the distributions of the attention and

Table 4.5. Kutay's and Selda's Levels of Selective Attention and Knowledge-based Reasoning in the Second Noticing Interviews and Meetings

Groups			**G#1**	**G#2**	**G#3**	**G#4**	**G#5**	**G#6**	**G#7**	**G#8**	**G#9**	**G#10**	**G#11**	**G#12**	**LEVEL**
Kutay	**SA**	**Meet**	Foc(OTI)	Foc(OTI)	-	-	-	Base	Mıx(OTI)	Foc(OTI)	-	-	Mıx(OTI)	Foc(TI)	FOCUSED
		Inter	Foc(TI)	Mıx(TI)	Mıx(TI)	Foc(RI)	Mıx(RI)	Foc(TI)	Foc(TI)	Mıx(TI)	Base	Foc(TI)	Mıx(TI)	Mıx(TI)	MIXED
	KBR	**Meet**	Mıx(OTI)	Mıx(OTI)	-	-	-	Mıx(TI)	Mıx(OTI)	Foc(RI)	-	-	Base	Foc(TI)	MIXED
		Inter	Foc(TI)	Foc(TI)	Mıx(TI)	Foc(RI)	Mıx(RI)	Foc(TI)	Foc(TI)	Mıx(TI)	Base	Foc(TI)	Mıx(TI)	Base	FOCUSED
Selda	**SA**	**Meet**	Foc(TI)	Foc(TI)	-	Base	-	Mıx(OTI)	Mıx(TI)	Foc(OTI)	-	-	-	Foc(OTI)	FOCUSED
		Inter	Mıx(TI)	Mıx(TI)	Base	Base	Mıx(TI)	Base	Base	Mıx(TI)	Base	Base	Mıx(RI)	Base	BASELINE
	KBR	**Meet**	Foc(TI)	Foc(TI)	-	Base	-	Mıx(OTI)	Mıx(TI)	Mıx(RI)	-	-	-	Foc(OTI)	MIXED
		Inter	Mıx(TI)	Mıx(TI)	Base	Base	Base	Base	Base	Mıx(TI)	Mıx(TI)	Mıx(RI)	Base	Base	BASELINE

Note. RI = Researcher initiated; TI = Teacher initiated; Base = Baseline level; Mıx = Mixed level; Foc = Focused level.

Table 4.6. Rana’s and Ayfer’s Levels of Selective Attention and Knowledge-based Reasoning in the Third Noticing Interviews and Meetings

Groups			G#1	G#2	G#3	G#4	G#5	G#6	G#7	G#8	G#9	LEVEL
Rana	**SA**	**Meet**	Base	-	-	Base	-	-	Foc(TI)	-	Mıx(TI)	BASELINE
		Inter	Foc(TI)	Base	Mıx(TI)	Mıx(TI)	Base	Mıx(RI)	Foc(TI)	Foc(TI)	-	MIXED
	KBR	**Meet**	Base	-	-	Base	-	-	Foc(TI)	-	Mıx(RI)	BASELINE
		Inter	Mıx(TI)	Base	Base	Base	Base	Base	Mıx(RI)	Mıx(TI)	-	BASELINE
Ayfer	**SA**	**Meet**	Base	-	-	Base	-	-	Mıx(TI)	-	Base	BASELINE
		Inter	Mıx(RI)	Base	Base	Base	Base	Base	Base	Base	Base	BASELINE
	KBR	**Meet**	Base	-	-	Base	-	-	Mıx(OTI)	-	Base	BASELINE
		Inter	Mıx(RI)	Base	Base	Base	Base	Mıx(TI)	Mıx(RI)	Base	Mıx(TI)	BASELINE

Note. RI = Researcher initiated; TI = Teacher initiated; Base = Baseline level; Mıx = Mixed level; Foc = Focused level.

reasoning according to groups were much alike in those two interviews (See Table 4.3 and Table 4.5).

The most dominant characteristic of Kutay's and Selda's selective attention and knowledge-based reasoning in the third meeting was that they considerably endeavored to make sense of groups' solution approaches. As opposed to previous two meetings, in the third meeting, Kutay and Selda actively engaged in sense-making processes initiated by one of them, rather than as a response to the researcher's prompt. More importantly, they spontaneously decided on groups' solutions that were worthy of attention to discuss on. Although the researcher occasionally continued to pose specific questions on some parts of the solutions to stimulate a discussion, generally they took responsibility to lead discussions on students' solutions. Usually, one of two teachers initiated a discussion by questioning about mathematically significant details of a solution, and the other turned his/her attention to these details and endeavored to make sense of them. Such kind of collaborative effort to make sense of solution approaches prompted teachers to pay more attention to specifics of the solutions and speculate on underpinning student thinking, which were robust indicators of their high level selective attention and accompanying knowledge-based reasoning.

Kutay's high level selective attention and corresponding high level knowledge-based reasoning were also confirmed by findings from the interview with slight changes. While his level of attention in the interview was slightly lower than the level in the meeting, his level of reasoning was slightly higher than the level in the meeting. Despite these slight changes, the interview results also provided robust evidence of his high level attention and reasoning in the third investigation, compared to the previous two investigations. In the third interview, the most common indicators of Kutay's high level selective attention was that his statements on most groups' solutions included references to the specifics of the solutions. In some occasions, he explained the entire solution approach with substantial details, which was considered as evidence of a focused level attention. In most of these incidents, he took the initiative and attempted spontaneously, rather than through the researcher's prompt

(See Table 4.5). In some occasions, he was also interested in who the group members were because he did the implementation in his class and the groups consisted of his students. However, contrary to especially first interview, he did not focus on non-mathematical aspects of groups' works.

The most robust indicator of Kutay's focused level knowledge-based reasoning in the third interview was that he considerably endeavored to make sense of what students offered as a solution approach. More importantly, his statements on what he attended to in the solutions were predominantly interpretive. Unlike his judgmental disposition in the first interview, he rarely made evaluative comments on groups' works. He mainly tried to make sense of the solutions rather than criticizing them. He often made interpretive comments related to the solution or underpinning student thinking. Despite Kutay mostly took the initiative for his sense-making process and making interpretations, occasionally, the researcher's specific prompts for him to stimulate his reasoning processes made a contribution to his focused level knowledge-based reasoning in the third interview.

As mentioned previously, findings from the interviews also pointed to a gradual progress in Selda's selective attention and corresponding knowledge-based reasoning throughout the study. However this progress was not as sharp as the progress in her attention and reasoning throughout the meetings. There was a discrepancy between findings from the third meeting and the third interview conducted following the meeting. Although compared to her baseline level attention in the first interview Selda began to pay more attention to specifics of the solutions, her attention to details of the solution in the third interview was not as high as it was in the third meeting. Similarly, compared to her tendency towards literally recounting what she observed in the groups' works in the first interview, in the third interview, she began attempting to make sense of underlying student thinking the solutions were based on. Nonetheless, as opposed to her considerable endeavor to discuss groups' solutions in the meeting, in the interview, she had a tendency to literally describe the solutions with a judgmental disposition. At the beginning of the interview, she expressed that she was bored and tired of thinking what the students might have thought for the solutions. She claimed that she was happy as long as the solution was correct, and that she preffed such a thinking.

Table 4.6 demonstrates that Rana’s and Ayfer’s selective attention and accompanying knowledge-based reasoning were at a baseline level in the third meeting. As opposed to Kutay’s and Selda’s progess, there was no a gradual progress in Rana’s attention and reasoning throughout the meetings. The shift in her progress in attention and corresponding reasoning was considered as occurring in a cycle since it shifted from baseline, to mixed and back to baseline, yet not exactly to how it initially was (see Table 4.7). At the beginning of the meeting, Rana expressed that she did not have enough time to carefully examine the student’s works because she could get the worksheets late. Thus, throughout the meeting, she generally preferred to keep silent or to briefly recount what she attended with direct quotation. She also criticized many groups’ works for lacking a mathematical base and posited that she found the groups as not taking the work as seriously.

However, on some occasions, she actively engaged in sense making processes that were initiated by the researcher’s prompt or another teacher’s question about specifics of a solution and made interpretive comments. These were particular solutions that were unique and rich in mathematical content. She initiated a discussion once, on such kind of solution approach by speculating on the underlying student thinking. Throughout the discussion, her statements provided robust indicators of a focused level selective attention and knowledge-based reasoning.

Rana’s high level attention and corresponding high level reasoning in investigating mathematically rich solution approaches were also confirmed by findings from the interview. In the third interview, Rana’s selective attention and knowledge-based reasoning in investigating these particular solutions had features of a focused level (see Table 4.8). She explained these solutions by referring substantial details of the solutions and also made interpretations on what she attended to, besides descriptive and evaluative comments. Occasionally, the researcher’s specific prompts also stimulated her to make interpretations on the solutions or underlying student thinking.

Another finding that was supported by findings from the meeting was Rana's tendency to make literal descriptions. In the interview, while she was investigating the other groups' solutions except the mathematically rich ones, although her attention was still at a high level, her knowledge-based reasoning was at a baseline level since she literally described what students did. Furthermore, following a literal description of these solutions, she criticized them for a lack of mathematical basis, as she expressed in the meeting.

Ayfer's selective attention and knowledge-based reasoning remained as baseline level from the first meeting to the third one (see Table 4.7). Her comments on groups' solutions provided indicators of a baseline level attention and reasoning. For instance, in the meeting, she described groups' works with broad terms. She sustained her judgmental disposition in describing them. Occasionally, she attracted other teachers' attention to a unique solution by referring a key aspect of the solution. Yet, her further statements on the solution had features of a baseline level attention such as having a discrepancy between her statements and what students did.

These findings were confirmed by findings from the interview with Ayfer. Like her baseline level attention and reasoning in the meeting, Ayfer's selective attention and knowledge-based reasoning were at a baseline level in the interview (see Table 4.8). The most dominant characteristic of her baseline level reasoning was that she literally described what she observed with a judgmental disposition. She occasionally endeavored to make sense of a few groups' solutions, as a response to the researcher's prompt. However, these sense making processes generally resulted in evaluative comments, rather than interpretations.

Table 4.7. Broad Levels of Teachers' Selective Attention and Knowledge-based Reasoning in the Three Noticing Meetings

	FIRST NOTICING MEETING		SECOND NOTICING MEETING		THIRD NOTICING MEETING	
	Selective Attention	Knowledge-Based Reasoning	Selective Attention	Knowledge-Based Reasoning	Selective Attention	Knowledge-Based Reasoning
Kutay	BASELINE [100%B]	BASELINE [60%B, 40%M]	-	-	FOCUSED [14%B, 29%M, 57%F]	MIXED [29%B, 42%M, 29%F]
Selda	BASELINE [100%B]	BASELINE [80%B, 20%M]	MIXED [25%B, 75%M]	MIXED [100%M]	FOCUSED [14%B, 29%M, 57%F]	MIXED [14%B, 43%M, 43%F]
Rana	BASELINE [50%B, 50%M]	BASELINE [50%B, 50%M]	MIXED [100%M]	MIXED [17%B, 83%M]	BASELINE [50%B, 25%M, 25%F]	BASELINE [50%B, 25%M, 25%F]
Ayfer	BASELINE [86%B, 14%M]	BASELINE [57%B, 43%M]	-	-	BASELINE [75%B, 25%M]	BASELINE [75%B, 25%M]

Note. B = Baseline; M = Mıxed; F = Focused

Table 4.8. Broad Levels of Teachers' Selective Attention and Knowledge-based Reasoning in the Three Noticing Interviews

	FIRST NOTICING INTERVIEW		SECOND NOTICING INTERVIEW		THIRD NOTICING INTERVIEW	
	Selective Attention	**Knowledge-Based Reasoning**	**Selective Attention**	**Knowledge-Based Reasoning**	**Selective Attention**	**Knowledge-Based Reasoning**
Kutay	BASELINE [46%B, 36%M, 18%F]	MIXED [36%B, 46%M, 18%F]	MIXED [25%B, 42%M, 33%F]	MIXED [17%B, 50%M, 33%F]	MIXED [8%B, 50%M, 42%F]	FOCUSED [17%B, 33%M, 50%F]
Selda	BASELINE [100%B]	BASELINE [78%B, 22%M]	MIXED [25%B, 42%M, 33%F]	MIXED [42%B, 50%M, 8%F]	BASELINE [58%B, 33%M, 9%F]	BASELINE [50%B, 42%M, 8%F]
Rana	MIXED [10%B, 60%M, 30%F]	BASELINE [50%B, 20%M, 30%F]	BASELINE [56%B, 22%M, 22%F]	BASELINE [56%B, 22%M, 22%F]	MIXED [24%B, 38%M, 38%F]	BASELINE [62%B, 13%M, 25%F]
Ayfer	BASELINE [75%B, 25%M]	BASELINE [63%B, 37%M]	BASELINE [67%B, 33%M]	BASELINE [78%B, 22%M]	BASELINE [89%B, 11%M]	BASELINE [56%B, 44%M]

Note. B = Baseline; M = Mıxed; F = Focused

CHAPTER 5

CONCLUSIONS & DISCUSSION

The main purpose of this study was to investigate nature and development of secondary mathematics teachers' ability to notice students' ways of mathematical thinking in the context of a professional development program, which is based on multitier professional development program. The main research question and sub-questions were:

How does secondary teachers' noticing of students' mathematical thinking evolve during a professional development program designed with models and modeling perspective?

1. How does secondary teachers' selective attention of students' mathematical thinking evolve during a professional development program designed with models and modeling perspective?

2. How does secondary teachers' knowledge-based reasoning of students' mathematical thinking evolve during a professional development program designed with models and modeling perspective?

There were two main findings of this study. Firstly, throughout the professional development program, there was a gradual development in three of the four teachers' attention to and reasoning about students' mathematical thinking revealed through modeling problems. This finding pointed out that professional noticing of student mathematical thinking is an expertise that can be learned by teachers through professional support. This finding is also parallel to findings from the literature

(Goldsmith & Seago, 2011; Jacobs, Lamb, & Philipp, 2010; Sherin & Han, 2004; Sherin & van Es, 2009; Star & Strickland, 2007; van Es, 2011; van Es & Sherin, 2008). Secondly, similar to what has been found by van Es and Sherin (2008) and van Es (2011), teachers went through different developmental paths. These previous studies also pointed out which factors could direct teachers to different developmental paths. Next, an elaboration of discussion on these two findings and possible factors influencing developmental paths in noticing will be presented in light of the relevant literature.

Data analysis on what teachers attended to revealed that, early on in the study (in the first investigation) teachers' attention to groups' works were at a baseline level. Rather than focusing on specifics of various solution approaches, teachers described what they saw in students' worksheets with broad terms that did not refer to details of the solutions. Additionally, they paid attention to non-mathematical aspects of students' works, which was another common indicator of a lack of focused attention to students' mathematical thinking revealed from their responses to modeling problems. In the first meeting, since teachers had a tendency to share their general impression on students' solution approaches, the researcher, as the facilitator, played a crucial role in encouraging teachers to note and articulate details of the solutions. The findings of relevant research studies confirmed these findings (Goldsmith & Seago, 2011; Kazemi & Franke, 2004; Sherin & Han, 2004; Sherin & van Es, 2009; van Es, 2011; van Es & Sherin, 2008). In their studies of teachers' noticing on students' mathematical thinking while watching a video excerpt from a classroom environment, Sherin and van Es (2009), van Es and Sherin, (2008) and van Es, (2011) found that in the early meetings teachers paid attention to range of issues other than mathematical ideas raised by students in the class such as climate in the classroom or classroom management. They reinforced these findings with the noticing interviews conducted after the meetings. In the early noticing interviews, teachers sustained referring to a range of issues other than students' mathematical thinking in the video clip (Sherin & van Es, 2009; van Es, 2011; van Es & Sherin, 2008). These findings are parallel to the findings of this study in a way that teachers in the early meetings and interviews paid attention to non-mathematical aspects of students' written works such as the quality of pictures that students drew or whether students expressed their solutions in their reports in a tidy manner (Goldsmith & Seago, 2011; Jacobs, Lamb, & Philipp, 2010; Kazemi & Franke, 2004; Star & Strickland, 2007).

There might be various reasons of teachers' low level attention to details of students' mathematical thinking in the early meetings and interviews. Firstly, at the beginning of the study, teachers were not familiar with examining such kind of students' written works that included a variety of unexpected mathematical ideas generally expressed in an untidy manner as a response to a non-routine task. Although examining students' works is a key component of teaching, in their routine teaching practice, they are accustomed to assessing students' examination sheets rarely including unexpected solution/ideas and usually expressed in a tidy way. Also their purpose of examining these sheets is to assign a grade by checking the extent to which students' responses are consistent with a previously prepared rubric. In Kazemi and Franke's (2004) study, for example, teachers expressed their difficulty in examining students' responses to non-routine tasks because of their unfamiliarity with such kinds of student work. Therefore, this might be an explanation for teachers' low level attention in the early meetings in this study.

Additionally, at the beginning of the study, they were also unfamiliar with investigation of students' works as they did during the noticing meetings. Although at the workshop they were informed about and had experiences in investigation of students' responses to modeling problems, earlier in the main study, teachers did not completely make sense of what they were expected to do with students' written works. The researcher's role as a facilitator particularly in the first meeting was central in teachers' understanding of the primary purpose of investigating students' responses. She endeavored to form a norm that the aim of examining students' responses was to try to understand students' various solution approaches and to find clues about students' mathematical thinking, rather than assessing them for grading as in their routine practice. This central role of the researcher in creating such a norm, particularly at the beginning, has been highlighted by other researchers (e.g.

Blythe, Allen, & Powell, 1999; Goldsmith & Seago, 2011; van Es, 2011; van Es & Sherin, 2008).

Although the previously mentioned factors probably had influence on teachers' baseline level attention to students' responses earlier in the study, they are not enough to explain alone the meaning of this finding. Relevant literature points out some other crucial factors that may help to explain the reason for teachers' baseline level attention to specifics of responses that were worthy of consideration in the first investigation. For example, studies underscored teachers have a tendency to make evaluative comments on students' works when they first deal with students' works (Chamberlin, 2005; Goldsmith & Seago, 2011; Kazemi & Franke, 2004). Similarly, Blythe, Allen, and Powell (1999) and Rodgers (2002) claim that, as opposed to what is generally assumed, making specific, detailed, careful description is much more difficult than making interpretation and/or evaluation. This claim is consistent with the findings of this study. In this study, teachers made primarily evaluative comments at the beginning, which can be considered as a reason of their difficulty in attending to mathematical ideas embedded in students' works.

According to Blythe et al. (1999), teachers' tendency to make quick judgments on students' works and lack of ability to attend to specifics of these works is mainly related to pressures of teaching profession. She claimed that since teachers feel the pressure from a curriculum that needs to be covered, they take on a responsibility to quickly identify any student outcome as good or bad, and make corrections whenever necessary. Similar findings were revealed from this study. Although participant teachers in this study had years of teaching experience, their descriptions of students' solution approaches were often very general, missing any reference to details of the solutions.

On the other hand, the findings from the interviews confirmed that if teachers slow down and attempt to see the details, they pay more attention to specifics of students' ways of thinking. For example, Kutay's and Rana's levels of selective attention in the first interview were higher than their levels in the meeting. In the interview, they were provided with more time to concentrate on students’ responses and talk about them. This finding pointed out the importance of time that teachers spend in examining students’ works. However, teachers had a tendency to not spend so much time on carefully examining students’ outcomes because of the previously mentioned pressures of teaching profession. Similarly, in the early phases of the study, teachers expressed that they did not spend so much time at examining students’ sheets. They indicated that they used five minutes breaks between classes in the school for this purpose rather than sparing time for it at home after school. An implication of these findings is the importance of providing teachers with time and opportunities at school where they can feel a sense of collaboration with researchers and other teachers. Such professional development programs might work better when organized and conducted to include teachers’ examinations of materials at the workplace, rather than as after school activities.

Similar to teachers’ baseline level selective attention in the first meeting, their levels of knowledge-based reasoning were at a baseline level. The most common characteristics of teachers’ baseline level knowledge-based reasoning were the following: Teachers had a tendency to only recount what they saw in students’ sheets by almost direct quotations. They did not attempt to make sense of what they saw unless the researcher prompted them to reason about what they saw. Furthermore, they primarily made evaluative comments on students’ responses. Previous studies revealed that in the early phase of the video-club meetings, teachers made primarily descriptive and evaluative comments on what they saw in the video excerpt without assigning a meaning to what they observed (Sherin & Han, 2004; Sherin & van Es, 2009; van Es, 2011; van Es & Sherin, 2008). The findings of this study are parallel to findings of these previous studies on teacher noticing. All of these findings reinforce the claim that teachers are inclined to quickly attach their personal judgments to students’ works without an attempt to make sense of student thinking reflected by their works (Blythe et al., 1999, Rodgers, 2002). As mentioned previously, the researcher’s role particularly in the first meeting was crucial to be a role model for teachers and to create a norm that the main aim of investigating students’ responses was to understand their mathematical thinking rather than assessing them. With this purpose in her mind, the researcher often forced teachers to make sense of what students put in their work either by asking specific questions related to the key aspects of the solutions or by modeling how to engage in making sense of the solutions. On some occasions, the researcher's such attempts to prompt teachers to try to understand what students did served its purpose, yet not every time the researcher tried to do this.

Until now, the findings from early phase of the study were discussed in light of the relevant literature. To sum up, all findings point out that attending to specifics of students' mathematical thinking and making sense of what is attended to, that is professional noticing of students' thinking, is a crucial component of teaching expertise. Teachers do not have this expertise inherently and even years of experience with students alone are not enough to acquire this expertise (Sherin, Jacobs, & Philipp, 2011). As a result, teachers need professional support to develop their noticing skills. Research has shown that professional noticing of students' thinking is an expertise that can be learned through both teaching experience and professional support (Jacobs, et al., 2010).

5.1 Development in Teachers' Noticing of Students' Thinking

Another finding that is supported by what has been reported in the literature is the influence of in-service professional development programs on developing teachers' noticing. Previous studies revealed that some kind of intervention for such purposes resulted in various changes in teachers' noticing patterns (Goldsmith & Seago, 2011; Kazemi & Franke, 2004; Sherin & Han, 2004; Sherin & van Es, 2009; van Es, 2011; van Es & Sherin, 2008). Likewise, despite slight discrepancies between the findings from the meetings and the following interviews in terms of the extent to which they revealed these shifts, overall findings of this study indicated a gradual progress in teachers' noticing skills from the first investigation to the third. This can be taken as a supporting evidence for the claim that noticing can be learned through professional support (Ball & Cohen, 1999; Jacobs, Lamb, & Philipp, 2010).

As previously discussed, teachers' levels of attention and accompanying reasoning were mainly at a baseline level at the onset of the study. Without the researcher's effort to prompt teachers to concentrate on the specifics of students' responses and make sense of what they see, teachers had a tendency to make general comments not including reference to the details of the solutions. Furthermore, their comments generally consisted of literal descriptions of what they observed and/or were highly evaluative. They rarely attempted to make sense of the responses when the researcher urged them to do that. However, over time, some shifts occurred in what teachers attended to and how they reasoned about what they observed. Although there were some differences in how and the extent to which these shifts occurred in teachers' noticing, overall findings point out a shift in teachers' noticing skills.

As opposed to their baseline level selective attention and knowledge-based reasoning in the first investigation, teachers' levels of attention and accompanying reasoning proceeded to a mixed level or even a focused level in the second and third investigations. Firstly, rather than being interested in non-mathematical aspects of the students' works, teachers gradually started to focus their attention on what students offered as a solution approach. Most importantly, over time, teachers began to attend to students' responses more carefully and referred in their discussions to specifics of the responses that were worthy of consideration. However, as teachers' attention had shifts, they did not consistently operate at a focused level of attending to details of the solution, and were at a mixed level while investigating in many groups' work. Jacobs et al. (2010) pointed out this phenomena stating that teachers labelled to be at a particular level could show variations in their levels of attention to students' thinking and "can lose focus and miss a strategy or not fully understand a particular aspect of a strategy" (p.179). The factors potentially influencing this kind of variation will be discussed in the following sections.

The finding of this study that teachers' ability to attend to specifics of students' mathematical thinking can be improved by professional support is parallel to what has been found in the literature. For example, the findings of Sherin and van Es' extensive studies on mathematics teachers' noticing has indicated that teachers learned to pay focused attention to ideas that students raise throughout a video-based professional development approaches (Sherin & Han, 2004; Sherin & van Es, 2005, 2009; van Es, 2011; van Es & Sherin, 2002, 2008). Furthermore, there were various other studies that reached similar findings (Fernández, Llinares, &Valls, 2011; Rodgers, 2002; Star, Lynch, & Perova, 2011, Star & Strickland, 2007). Studies that selected students' mathematical thinking as a particular focus for teachers' noticing report similar findings (Goldsmith & Seago, 2011; Jacobs, Lamb, & Philipp, 2010; Jacobs, Lamb, Philipp, & Schappelle, 2011; Kazemi & Franke, 2004).

Data analysis also revealed that, similar to the shift in what teachers attended to in students' works over time, how they reasoned about what they attend to changed throughout the study. As mentioned previously, they began to endeavor to make sense of what they observed in investigation of students' responses. As opposed to a judgmental disposition towards students' solution approaches in the first investigation, they made efforts to understand before making conclusive remarks. For example, at the beginning, Kutay's statements on students' works were highly judgmental. He tended to assess students' responses by comparing it with the correct solution in his mind. Nonetheless, in the second and third investigation he quitted making highly evaluative comments and endeavored to make sense of what students offered as a solution approach. Likewise, he was considerably more interpretive in his analysis of students' responses. Similar findings were obtained for two other teachers, Rana and Selda. Although, their developmental patterns were different from Kutay's, there were similar shifts in their knowledge-based reasoning throughout the study. For example, even though Selda did not completely quit investigating students' solutions by assessing the extent to which they were compatible with the correct solution in her mind, she began to engage in a careful analysis of the responses. In the same way, in contrast to Rana's tendency to literally describe what she attended to in students' responses in the first investigation without interpretation, over time, she began to question key elements of the solutions. When she spontaneously engaged in such kind of questioning, rather than through researcher's prompts, this reasoning process usually resulted in interpretive comments. Different from the progress in these three teachers' noticing of students' thinking, there was not a noteworthy shift in Ayfer's noticing. These previous findings supported what has been previously reported by other relevant studies on teacher noticing (Fernández, Llinares, &Valls, 2011; Goldsmith & Seago, 2011; Jacobs, Lamb, & Philipp, 2010; Kazemi & Franke, 2004; van Es, 2011; van Es & Sherin, 2008).

5.2 Developmental Paths

This study also revealed that teachers' learning of noticing skills can follow different developmental paths. This finding is also parallel to findings of relevant studies (van Es, 2011; van Es & Sherin, 2008). Research on development of teaching expertise highlights this phenomenon. For example, in her research on transitions and trajectories for studies of expertise Lajoie (2003) stated that expertise can be reached in several ways. Similarly, in their research on teaching and teacher development from models and modeling perspective, Doerr and Lesh (2003) pointed out that:

> These characteristics of expertise in a complex and ill-structured knowledge domain such as teaching suggest that there is not a single image of a good teacher nor is there a single trajectory in becoming a good teacher. Rather, expertise in teaching varies considerably across individuals and across settings as well as within a given individual and within a particular context. (p. 127)

Similarly, different backgrounds of teachers in terms of their beliefs, knowledge and experiences influence their learning trajectories of noticing expertise (Goldsmith & Seago, 2011; Jacobs, Lamb, & Philipp, 2010; van Es & Sherin, 2008). Van Es and Sherin (2008) also refer to the importance of relevation of such kinds of different trajectories of teacher change for researchers and teacher educators whose work relates to teacher learning.

Similarly, in this study, different developmental paths that teachers displayed throughout the study provided clues about which components of the in-service training context had influences on shifts in teachers' noticing skills. In the following paragraph, shifts in teachers' noticing levels will be reviewed by using van Es and Sherin's (2008) classification of three different developmental paths, i.e. direct path, cyclical path, and incremental path. In the next section, the probable influence of the training on teachers' different developmental patterns will be discussed in light of the relevant literature.

Findings of the study indicated that Kutay's selective attention and corresponding knowledge-based reasoning developed gradually throughout the study. In van Es and Sherin's study, such kind of development was referred to as a direct path among the three paths in the development of learning to notice. Similar to teachers following what has been labelled as a direct path as learning to notice in van Es and Sherin's study, in the first investigation Kutay's levels of attention and reasoning were at a baseline level. Then, in the second investigation, he started trying to carefully analyze the responses and adopted an interpretive disposition. He sustained this disposition in the third investigation (See Tables 4.7 and 4.8). Like Kutay's direct path, findings from the meetings revealed that Selda's learning to notice trajectory can also be regarded as an example of a development along the direct path (See Tables 4.7). However, findings from the interview showed that Selda's learning to notice trajectory followed a cyclical path because it seems that her level of noticing cycled between a baseline level and a mixed level (See Tables 4.8). Table 4.7 highlights that, similar to Selda's cyclic path, Rana's learning to notice trajectory can also be labeled as cyclical path since Rana's level of noticing shifted from baseline, to mixed and back to baseline, yet not exactly to how it initially was. According to Tables 4.7 and 4.8, Ayfer's selective attention and knowledge-based reasoning remained at a baseline level throughout the study during all the meetings and interviews.

5.3 Factors that Influence Teachers' Development in Different Paths

The findings discussed above indicate that the implementations carried out through the professional development program can bring some changes in teachers' ability to notice students' mathematical thinking. Even though Ball (1997) suggested that analysing students' work is one of the most effective professional development approaches for developing teachers' skills for understanding and interpreting students' different ways of thinking, the context in which professional development materials are presented to teachers is also important (Ball & Cohen,1999; Goldsmith

& Seago, 2011; NCTM, 2003). Next there will be a discussion of the dimensions of the implementation carried out within this professional development program that could have resulted in shifts in teachers' ability to notice. Most of these dimensions have been presented in the literature as features of a successful professional development program targeting teachers' learning (Ball & Cohen, 1999; Hawley & Valli, 1999; NCTM, 2003). According to the data analysis carried out within this study and results from other studies previously reported in the literature, various dimensions of this program, that might have promoted shifts in teachers' noticing of students' ways of thinking are: a) Thought-revealing activities (i.e. MEAs) used in the study, b) Meetings carried out as group work and the cooperative problem solving environment provided for the teachers, c) The role of the researcher as the facilitator, and d) School based context of the in-service training. In the following sections, these dimensions will be discussed with their potential influences on teachers' professional development in noticing.

5.3.1 Modelling problems and students' solutions

One of the basic components of this professional development program based on professional development principles of models and modelling theory is the use of model-eliciting activities. One of the most prominent features of these activities is that they provide opportunities for students to create their own unique solutions in an open ended thinking environment, rather than using problems with a predetermined correct solution and having strict formulas and steps to be followed. Through this feature, they elicit students' different ways of thinking and provide opportunities for teachers to analyse and make sense of students' thinking (Lesh & Doerr, 2003). In light of the findings of this study, it can be claimed that such non-routine problems provide teachers with a rich environment in which they can focus on students' ways of thinking. This point has been highlighted by previous studies (Chamberlin, 2002, 2004, 2005; Doerr & Lesh, 2003; Hjalmarson, 2004; Koellner-Clark & Lesh, 2003; Schorr & Koellner-Clark, 2003).

These findings may also have some implications for supporting improvement in teachers' knowledge and skills of using modelling and applications problems in their teaching practice. Integrating modelling activities into teaching practice is not a skill that would spontaneously develop through the education given in teacher training programs. Teachers need learning environments that would develop them in this direction (Ball & Cohen, 1999; Niss, Blum, & Galbraith, 2007). In order for teachers to integrate modelling into their teaching practice, they first need to notice mathematical ideas that are revealed through MEAs. In light of the findings of this study, it might be speculated that promoting teachers' noticing of students' mathematical thinking may also provide teachers with skills that are likely to be helpful in promoting students' different ways of thinking in implementation of modelling problems in their classrooms.

5.3.2 Collaborative learning environment

One of the central features of the design of teachers' investigation of students' works is that it is directing teachers to work in groups in a cooperative learning environment. Group work, one of the basic practical outcomes of social learning theories, has been often emphasised in studies on teacher training and professional development programs (Blythe, Allen, & Powell, 1999; Doerr & Lesh, 2003; Hawley &Valli, 1999; NCTM, 1991). Blythe et al. (1999) contended that teachers' analysis of students work in groups brought more effective outcomes than individual analysis. They claimed that if teachers worked in solidarity and exchanged opinions with each other, they would take more accurate decisions about students' ways of thinking. Similarly NCTM (2003) and Ball and Cohen (1999) expressed that practice based professional development programs need to promote teachers' cooperative work.

The findings of the study also support previous findings that emphasise the contribution of cooperation and group work to development of teachers' noticing of students' mathematical thinking (Chamberlin, 2004, 2005; Kazemi & Franke, 2004; Koellner-Clark & Lesh, 2003; Zawojewski, Chamberlin, Hjalmarson, & Lewis, 2008). In the meetings and the preparation stages before the meetings, teachers' continuous mutual exchange of ideas influenced them positively in both cognitive and affective dimensions. In the interviews and meetings conducted, teachers often expressed this positive effect. The fact that teachers did group work and exchanged ideas with each other during both their work before the implementations on modelling problems and analysing and trying to make sense of students' solution sheets produced for those problems, made this process more efficient for them and contributed to their professional development (Chamberlin, 2002, 2005). A good example of that can be seen in the mini research studies done by the teachers in the last meetings for analysing and making sense of student solutions and the underlying mathematical ideas. Such mini research studies, started with a question from one of the teachers from the group, developed with the contribution of the other teachers in a constant exchange of ideas and provided a rich learning environment for the teachers to think, analyse and interpret students' ways of thinking.

5.3.3 The role of the researcher as the facilitator

Findings also point out the crucial role of the researcher as a facilitator in stimulating teachers' noticing of students' mathematical thinking. The critical role of the facilitator in promoting teachers' noticing is also supported by findings from previous studies (Blythe, Allen, & Powell, 1999; Goldsmith & Seago, 2011; Sherin & van Es, 2009; van Es, 2011; Zawojewski, Chamberlin, Hjalmarson, & Lewis, 2008). Researcher interfered with the flow of the meetings occasionally when she felt necessary, in order to make the discussion richer. In the meetings and the face-to-face interviews researcher gave some directions by using different methods to make the teachers focus on students' ways of thinking. Such facilitating actions of the researcher can be summarised as: a) asking the teachers to summarise the solution approach, b) asking them to provide details about the solution approach, c) trying to enable the teachers to make sense of the solution by giving clues, d) attracting attention to the report, e) asking questions. With such directions from the researcher, teachers understood how they could analyse the sheets and made more detailed comments. The researcher's crucial role in establishing a norm that the aim of investigating students' responses was to understand their mathematical thinking rather than assessing them was also emphasised in literature (Ball & Cohen, 1999; NCTM, 2003; van Es, 2011).

As mentioned in the results chapter, teachers were inclined to make explanations about students' solutions in the first meetings with general comments, without going into much detail even though they spotted various different solution approaches. In such cases, the fact that the researcher interfered by questions such as "Can we think about this solution strategy a bit more?", "What did the student try to do here?", "Do you think this is a good strategy?" in order to get the teachers to think more deeply on these solution approaches, might be claimed to have changed teachers' approaches while interpreting students' solution sheets. These directions have encouraged the teachers to come to the following meetings having analysed students' solution sheets in more detail and enriched the content of the discussions in the meetings.

5.3.4 School based context of the in-service training

One of the basic principles of the in-service professional development program carried out within this study was keeping the teachers in their own school environment, with their own students and colleagues while trying to conduct training sessions that would contribute to their professional development. Previous studies showed that when teachers were asked to engage in short term in-service training programs in environments dissimilar to their own settings and conditions and requiring the teachers to spend extra time beyond their working hours, they showed resistance against the knowledge and skills expected to be developed by them (Hargreaves, 1995). Similarly, according to Koellner-Clark and Lesh (2003) one of the most important issues to be considered for contributing to teachers' professional development is keeping them in their own school and classroom settings. In this context, the designed professional development program having such features was one of the reasons why the it progressed smoothly with success and the teachers were motivated in the course of the program. Also, research on teachers' noticing highlights that when teachers work on artifacts that are obtained from their own classroom, professional development programs have more influence on change in their practice (Borko, Jacobs, Eiteljorg, &Pittman, 2008). Since the materials used for discussion were produced in teachers' classrooms within the context of the study, this can be considered as another factor that might have contributed to the shifts in teachers' noticing as targeted by the program.

5.4 Limitations of the Study

There were some limitations of the study. Some of these limitations were already pointed out in the literature, which will be discussed in light of relevant literature below:

Two participant teachers, Kutay and Ayfer did not attend the meetings in the second investigations. Although interviews were conducted with them after the meeting, interview data did not sufficient to completely investigate their nature and level of noticing during the second investigations.

There were some limitations related to the design of teachers' investigation of students' works that made the analysis of data process difficult for the researcher in some ways. Firstly, although there was a structure of the process of teachers' investigations of students written works (i.e., multitier professional development design) and the recommended principles of teachers' investigation of students' works were considered by the researcher while designing the process, there were some deficiencies in the design of teachers' investigation. For example, Allen (1998) and Blythe, Allen, and Powell (1999) recommend that the purpose of teachers' investigation of students works should be clear and specific. Although, occasionally, the researcher reminded the teachers that their purpose of investigation was to create individual student thinking sheets including their observations about some critical aspects of students' ways of thinking (i.e., various solution strategies, underlying mathematical ideas, difficulties, mistakes, misconceptions), what they wrote on their sheets did not reflect completely their thoughts and observations. Teachers expressed that they did not like to write their thoughts on the sheet and they preferred to express their thoughts verbally in the meetings or interviews. Furthermore, in the pre-analysis of the sheets during data collection process the researcher found some clues that teachers copied some parts of their work on their sheets from other teachers. Moreover, some teachers noted down their thoughts on some other pieces of paper rather than the sheet. As a result, in this study, student thinking sheets were not used as efficiently as palanned. Similarly, such limitations of using student thinking sheets have also been indicated in previous studies (Chamberlin, 2002; Hallagan, 2003). Secondly, since the purpose of this study was not to investigate the nature and development of a group of teachers' ability to notice students' mathematical thinking, the interviews should have been conducted before the meetings. This design would be better to understand the nature and development of each teacher's noticing throughout the study. While analyzing the interview data, it was difficult for the researcher to recognize whether teacher's statements reflected his/her real thoughts or he/she simply repeated what another teacher said in the meeting before the interview. Also, if the interviews were conducted before the meeting, it would be possible to find the potential influence of group discussions on each teacher's ability to notice.

Findings of the study revealed that, as the facilitator, the researcher's prompts had potentially influence on shifts in teachers' noticing (such as posing specific and/or general questions to encourage teachers to inquire into students' thinking, etc). Researcher used these prompt spontaneously with a purpose in her mind that. As van Es (2011) stated, the purpose of the researcher was to establish a norm that the main aim of investigating students' responses was to understand their mathematical thinking rather than assessing them. However, some critical points related to using these prompts such as what kind of prompts would be used for which purpose, when and how often they would be used during the investigation were not determined at the beginning of the study. This also made data analysis process difficult for the researcher. For example, on some occasions, it was very difficult to understand and analyze teachers' reasoning processes. Hence, although there were some advantages of using prompts spontaneously, it is also suggested that some specific prompts for specific purpose should be determined before as recommended by Jacobs, Lamb, and Philipp (2010). Likewise, when and how often these prompts would be used should be identified at the beginning of the study. For this purpose, some kind of "protocols" can be used as recommended by Blythe, Allen, and Powell (1999).

Another limitation of the study was related to the use of students' written works and a video clip in the professional development program. Although MEAs that were used in the study were thought-revealing activities and they were very useful to reveal students' mathematical thinking, some groups' works did not include mathematically significant solution approaches. As a result, these groups' works were not useful for such kind of a study exploring teachers' noticing of students' thinking. Such kind of non-stimulating works made labeling the level of teachers' selective attention and knowledge-based reasoning difficult for the researcher. Therefore, it is not necessary to use all samples of students' written works and the researcher should be selective about samples of students' works to use in teachers' investigation processes. It may be a useful dimension of the study if teachers determine which samples of works should be investigated and explain their criteria for selection. This can be considered as an element of their noticing skills. The importance of selecting classroom-based artifacts containing rich elements of students' ways of mathematical thinking in terms of stimulating teachers' investigation has also been highlighted previously in studies (Jacobs, Lamb, & Philipp, 2010; van Es & Sherin, 2008).

5.5 Suggestions for Further Research

As mentioned previously, Jacobs et al. (2010) conceptualized professional noticing of children's mathematical thinking as three interrelated skills. These were (a) "Attending to children's strategies", (b) "Interpreting children's mathematical understanding", and (c) "Deciding how to respond on the basis of children's understanding" (p.173). This study focused on the first two dimensions. However, the last dimension of this conceptualization is also important for teachers to reflect on what they learned about students' mathematical thinking and thus equipping them with the skill of building their instructions on what they learned about students' thinking. As a result, studying this third dimension of noticing needs to be subject of further research.

This study revealed that the professional development design that was adopted from the multitier professional development program design (Koellner Clark & Lesh,

2003) promoted development of teachers' noticing of students' mathematical thinking. Although the findings of teachers' different developmental patterns gave clues about dimensions of this design that have a potential to influence the shifts in teachers' noticing, that was not a research question, hence an aim of this study. For example, findings provided clues that teachers' collective work on investigating students' responses stimulated their ability to attend to and make sense of students' responses (Kazemi & Franke, 2004). Such probable influences of components of professional development design, which have been discussed in this final chapter can inspire further research in this area.

The relationship between two dimensions of teachers' noticing, i.e. selective attention and knowledge-based reasoning, can be another significant research question that the findings of this research raise. Some studies point out that teachers' selective attention is directly related to their knowledge-based reasoning (van Es, 2011). This relationship can also be the subject of further investigations.

REFERENCES

Allen, D. (1998). Introduction: Learning from students' works. In D. Allen (Ed.), *Assessing student learning: From grading to understanding* (pp. 1-20). New York, NY: Teachers College Press.

Ball, D. L. (1997). What do students know? Facing challenges of distance, context, and desire in trying to hear children. In B. J. Biddle, T. L. Good, & I. F. Goodson (Eds.*), International handbook of teachers and teaching* (pp. 769-818). Dordrecht, Netherlands: Kluwer Academic Publishers.

Ball, D. L., & Cohen, D. K. (1999). Developing practice, developing practitioners: Toward a practice-based theory of professional education. In L. Darling-Hammond & G. Sykes (Eds.), *Teaching as the learning profession: Handbook of policy and practice* (pp. 3-32). San Francisco, CA: Jossey-Bass Inc.

Ball, D. L., Thames, M. D., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education, 59* (5), 389-407.

Barnett, C. (1991). Building a case-based curriculum to enhance the pedagogical content knowledge of mathematics teachers. *Journal of Teacher Education*, *42*(4), 263-272.

Barnett, C. (1998). Mathematics teaching cases as a catalyst for informed strategic inquiry. *Teaching and Teacher Education, 14*(1), 81-93.

Bergqvist, T. (2005). How students verify conjectures: Teachers' expectations. *Journal of Mathematics Teacher Education, 8*(2), 171-191.

Blythe, T., Allen, D., & Powell, B. S. (1999). *Looking together at student work: A companion guide to assessing student learning.* New York: Teachers College Press.

Bogdan, R. C., & Biklen, S. K. (2006). *Qualitative research for education: An introduction to theories and methods* (5th ed.). Boston: Pearson Education Group.

Borasi, R., Fonzi, J., Smith, C. F., & Rose, B. J. (1999). Beginning the process of rethinking mathematics instruction: A professional development program. *Journal of Mathematics Teacher Education, 2*(1), 49-78.

Borko, H., Jacobs, J., Eiteljorg, E., & Pittman, M. E. (2008). Video as a tool for fostering productive discussions in mathematics professional development. *Teaching and Teacher Education, 24*, 417-436.

Borko, H., & Putnam, R. T. (1995). Expanding a teacher's knowledge base: A cognitive psychological perspective on professional development. In T. Guskey & M. Huberman (Eds.), *Professional development in education: New paradigms and practices* (pp. 35-65). New York: Teachers College Press.

Carpenter, T. P., Fennema, E., & Frank, M. L. (1996). Cognitively guided instruction: A knowledge base for reform in primary mathematics instruction. *The Elementary School Journal, 97*(1), 3-20.

Carpenter, T. P., Fennema, E., Peterson, P. L., Chiang, C. P., & Loef, M. (1989). Using knowledge of children's mathematics thinking in the classroom teaching: An experimental study. *American Educational Research Journal, 26*(4), 499-531.

Chamberlin, M. T. (2002). *Teacher investigations of students' work: The evolution of teachers' social processes and interpretations of students' thinking*. Unpublished doctoral dissertation, Purdue University, Indiana.

Chamberlin, M. T. (2004). Design principles for teacher investigations of student work. *Mathematics Teacher Education and Development, 6,* 61-72.

Chamberlin, M. T. (2005). Teachers' discussions of students' thinking: Meeting the challenge of attending to students' thinking. *Journal of Mathematics Teacher Education, 8,* 141-170.

Cochran-Smith, M., & Lytle, S. (1999). Relationships of knowledge and practice: Teacher learning in communities. In A. Iran-Nejad & P. D. Pearson (Eds.), *Review of research in education* (Vol. 24, pp. 249-305). Washington, DC: American Educational Research Association.

Cohen, L., Manion, L., & Morrison, K. (2007). *Research methods in education* (6th ed.). London: Routledge.

Cooney, T. J. (1999). Conceptualizing teachers' ways of knowing. *Educational Studies in Mathematics, 38*, 163-187.

Creswell, J. W. (2003). *Research design: Qualitative, quantitative, and mixed methods approaches* (2nd ed.). Thousand Oaks, CA: Sage Publication Inc.

Doerr, H. M. (2007). What knowledge do teachers need for teaching mathematics through applications and modelling? In W. Blum, P. Galbraith, H. Henn, & M. Niss (Eds.), *Modelling and applications in mathematics education: The 14th ICMI study* (pp. 69-78). New York: Springer.

Doerr, H. M., & Lesh, R. (2003). A modeling perspective on teacher development. In R. Lesh & H. M. Doerr (Eds.), *Beyond constructivism: Models and modeling perspectives on mathematics problem solving, learning, and teaching* (pp. 125-140). Mahwah, NJ: Lawrence Erlbaum Associates Inc.

Elmore, R. F. (2002a). *Bridging the gap between standards and achievement: The imperative for professional development in education*. Washington, DC: Albert Shanker Institute.

Fennema, E., Franke, M. L. (1992). Teachers' knowledge and its impact. In D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 147-164). New York: Macmillan.

Fennema, E., Franke, M. L., Carpenter, T. P., & Carey, D. A. (1993). Using children's mathematical knowledge in instruction. *American Educational Research Journal, 30*(3), 555-583.

Fernández, C., Llinares, S., & Valls, J. (2011). Development of prospective mathematics teachers' professional noticing in a specific domain: Proportional reasoning. *Proceedings of the 35th Conference of the International Group for the Psychology of Mathematics Education, TURKEY, 2,* 329-336.

Gearhart, M., & Saxe, G. B. (2004). When teachers know what students know: Integrating mathematics assessment. *Theory Into Practice, 43*, 304-313.

Goldsmith, L. T., & Seago, N. (2011). Using classroom artifacts to focus teachers' noticing: Affordances and opportunities. In M. G. Sherin, V. R. Jacobs, & R. A. Philipp (Eds.), *Mathematics teacher noticing: Seeing through teachers' eyes* (pp. 169-187). New York: Routledge.

Goodwin, C. (1994). Professional vision. *American Anthropologist, 96,* 606-633.

Grossman, P. L. (1990). *The making of a teacher: Teacher knowledge and teacher education*. New York: Teachers College Press.

Hadjidemetriou, C., & Williams, J. S. (2002). Teachers' pedagogical content knowledge: Graphs, from a cognitivist to a situated perspective. *Proceedings of the 26th Conference of the International Group for the Psychology of Mathematics Education. UK*, 57-64.

Guskey, T. R. (1989). Attitude and perceptual change in teachers. *International Journal of Educational Research*, *13*(4), 439-453.

Guskey, T. R. (2002). Professional development and teacher change. *Teachers and Teaching: Theory and Practice, 8,* 381-391.

Hallagan, J. E. (2003). *Teachers' models of student responses to middle school algebraic tasks*. Unpublished doctoral dissertation, Syracuse University, New York.

Hargreaves, A. (1995). Development and desire: A postmodern perspective. In T. R. Guskey & M. Huberman (Eds.), *Professional development in education: New paradigms and practices* (pp. 9-34). New York: Teachers College Press.

Hawley, W. D., & Valli, L. (1999). The essentials of effective professional development: A new consensus. In L. Darling-Hammond & G. Sykes (Eds.), *Teaching as the learning profession: Handbook of policy and practice* (pp. 3-32). San Francisco: Jossey-Bass.

Hjalmarson, M. A. (2004). *Designing presentation tools: A window into mathematics teacher practice*. Unpublished doctoral dissertation, Purdue University, Indiana.

Jacobs, V. R., Lamb, L. C., & Philipp, R. A. (2010). Professional noticing of children's mathemtical thinking. *Journal for Research in Mathematics Education, 41*, 169-202.

Jacobs, V. R., Lamb, L. C., Philipp, R. A., & Schappelle, B. P. (2011). Deciding how to respond on the basis of children's understandings. In M. G. Sherin, V. R. Jacobs, & R. A. Philipp (Eds.), *Mathematics teacher noticing: Seeing through teachers' eyes* (pp. 97-116). New York: Routledge.

Katims, N., & Tolbert, C. F. (1998). Accomplishing new goals for instruction and assessment through classroom-embedded professional development. In L. Leutzinger (Ed.), *Mathematics in the middle* (pp. 55-64). Reston, VA: National Council of Teachers of Mathematics.

Kazemi, E., & Franke, M. L. (2004). Teacher learning in mathematics: Using student work to promote collective inquiry. *Journal of Mathematics Teacher Education, 7*, 203-235.

Kieran, C. (2007). Learning and teaching algebra at the middle school through college levels. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 707-762). Charlotte, NC: Information Age Publishing.

Knapp, M. S. (2003). Professional development as a policy pathway. *Review of Research in Education, 27*, 109-157.

Koca, S. A. Ö., Yaman, M., & Şen, A. I. (2005). Öğretmen adaylarının etkin öğrenme-öğretme ortamı hakkındaki görüşlerinin farklı yöntemler kullanılarak tespit edilmesi. *Hacettepe Üniversitesi Eğitim Fakültesi Dergisi, 29*, 117-126.

Koellner-Clark, K., & Lesh, R. (2003). A modeling approach to describe teacher knowledge. In R. Lesh & H. M. Doerr (Eds.), *Beyond constructivism: Models and modeling perspectives on mathematics problem solving, learning, and teaching* (pp. 159-174). Mahwah, NJ: Lawrence Erlbaum Associates Inc.

Lajoie, S. (2003). Transitions and trajectories for studies of expertise. *Educational Researcher, 32*(8), 21-25.

Lampert, M., & Ball, D. L. (1998). *Teaching, multimedia, and mathematics: Investigations of real practice.* New York: Teachers College Press.

Lave, J., & Wenger, E. (1991). *Situated learning: Legitimate peripheral participation*. Cambridge, England: Cambridge University Press.

Lerman, S. (2001). A review of research perspectives on mathematics teacher education. In F. L. Lin & T. J. Cooney (Eds.), *Making sense of mathematics teacher education* (pp.33-52). Dordrecht, The Netherlands: Kluwer Academic Publishers.

Lesh, R., & Doerr, H. M. (Eds.). (2003). *Beyond constructivism: Models and modeling perspectives on mathematics problem solving, learning, and teaching*. Mahwah, NJ: Lawrence Erlbaum Associates Inc.

Lesh, R., Hoover, M., Hole, B., Kelly, A., & Post, T. (2000). Principles for developing thought-revealing activities for students and teachers. In R. Lesh & A. Kelly (Eds.), *Handbook of research design in mathematics and science education* (pp. 591-646). Mahwah, NJ: Lawrence Erlbaum Associates, Publishers.

Little, J. W. (2004). "Looking at student work" in the United States: A case of competing impulses in Professional development. In C. Day & J. Sachs (Eds.), *International handbook on the continuing Professional development of teachers* (pp. 94-118). Berkshire, England: Open University Press.

Llinares, S., & Valls, J. (2010). Prospective primary mathematics teachers' learning from on-line discussions in a virtual video-based environment. *Journal of Mathematics Teacher Education, 13*(2), 177-196.

Marshall, C., & Rossman, G. B. (2006). *Designing qualitative research* (4th ed.). Thousand Oaks, CA: Sage Publication Inc.

Masingila, J. O., & Doerr, H. M. (2002). Understanding preservice teachers' emerging practices through their analyses of multimedia case study of practice. *Journal of Mathematics Teacher Education, 5*, 235-263.

Mason, J. (2002). *Researching your own practice: The discipline of noticing.* London: RoutledgeFalmer.

Nathan, M. J., & Koedinger, K. R. (2000a). An investigation of teachers' beliefs of students' algebra development. *Cognition and Instruction*, *18*(2), 209-237.

Nathan, M. J., & Koedinger, K. R. (2000b). Teachers' and researchers' beliefs about the development of algebraic reasoning. *Journal for Research in Mathematics Education, 31*(2), 168-190.

Nathan, M. J., & Petrosin, A. (2003). Expert blind spot among preservice teachers. *American Educational Research Journal*, *40*(4), 905-928.

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics*. Reston, VA: Author.

National Council of Teachers of Mathematics. (1991). *Professional standards for teaching mathematics*. Reston, VA: Author.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: Author.

National Council of Teachers of Mathematics. (2001). *Practice-based professional development for teachers of mathematics*. Reston, VA: Author.

Niss, M., Blum, W., & Galbraith, P. L. (2007). Introduction. In W. Blum, P. Galbraith, H. Henn, & M. Niss (Eds.), *Modelling and applications in mathematics education: The 14th ICMI study* (pp. 3-32). New York: Springer.

Petrou, M., & Goulding, M. (2011). Conceptualising teachers' mathematical knowledge in teaching. In T. Rowland & K. Ruthven (Eds.), *Mathematical knowledge in teaching* (pp. 9-25). New York: Springer.

Remillard, J. T. (2000). Can curriculum materials support teachers' learning? Two fourth-grade teachers' use of a new mathematics text. *The Elementary School Journal*, *100*, 331-350.

Rodgers, C. R. (2002). Seeing student learning: Teacher change and the role of reflection. *Harvard Educational Review, 72,* 230-253.

Rosaen, C., Lundeberg, M., Cooper, M., Fritzen, A., & Terpstra, M. (2008). Noticing noticing: How does investigation of video cases change how teachers reflect on their experiences? *Journal of Teacher Education*, *59,* 347–360.

Saxe, G. B., Gearhart, M., & Nasir, N. S. (2001). Enhancing students' understanding of mathematics. A study of three contrasting approaches to Professional support. *Journal of Mathematics Teacher Education, 4,* 55-79.

Schifter, D. (2001). Learning to see the invisible. What skills and knowledge are needed to engage with students' mathematical ideas? In T. Wood, B. S. Nelson, & J. Warfield (Eds.), *Beyond classical pedagogy: Teaching elementary school mathematics* (pp. 109-134). Mahwah, NJ: Erlbaum.

Schorr, R. Y., & Koellner-Clark, K. (2003). Using a modeling approach to analyze the ways in which teachers consider new ways to teach mathematics. *Mathematical Thinking and Learning, 5*(2&3), 191–210.

Schorr, R. Y., & Lesh, R. (2003). A modeling approach for providing teacher development. In R. Lesh & H. M. Doerr (Eds.), *Beyond constructivism: Models and modeling perspectives on mathematics problem solving, learning, and teaching* (pp. 141-157). Mahwah, NJ: Lawrence Erlbaum Associates Inc.

Sherin, M. G., & Han, S. Y. (2004). Teacher learning in the context of a video club. *Teaching and Teacher Education, 20,* 163-183.

Sherin, M. G., Jacobs, V. R., & Philipp, R. A. (Eds.) (2011). *Mathematics teacher noticing: Seeing through teachers' eyes.* New York: Routledge.

Sherin, M. G., & van Es, E. A. (2005). Using video to support teachers' ability to notice classroom interactions. *Journal of Technology and Teacher Education, 13*(3), 475-491.

Sherin, M. G., & van Es, E. A. (2009). Effects of video club participation on teachers' Professional vision. *Journal of Teacher Education, 60*(1), 20-37.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher, 15*(2), 4-14.

Sowder, J. T. (2007). The mathematical education and development of teachers. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 157-223). Charlotte, NC: Information Age Publishing.

Sowder, J. T., Armstrong, B., Lamon, S., Simon, M., Sowder, L., & Thompson, A. (1998). Educating teachers to teach multiplicative structures in the middle grades. *Journal of Mathematics Teacher Education, 1*, 127-155.

Star, J. R., Lynch, K., & Perova, N. (2011). Using video to improve preservice mathematics teachers' abilities to attend to classroom features. In M. G. Sherin, V. R. Jacobs, & R. A. Philipp (Eds.), *Mathematics teacher noticing: Seeing through teachers' eyes* (pp. 117-133). New York: Routledge.

Star, J. R., & Strickland, S. K. (2008). Learning to observe: Using video to improve preservice mathematics teachers' ability to notice. *Journal of Mathematics Teacher Education, 11*, 107–125.

van Es, E. A. (2011). A framework for learning to notice student thinking. In M. G. Sherin, V. R. Jacobs, & R. A. Philipp (Eds.), *Mathematics teacher noticing: Seeing through teachers' eyes* (pp. 134-151). New York: Routledge.

van Es, E. A., & Sherin, M. G. (2002). Learning to notice: Scaffolding new teachers' interpretations of classroom interactions. *Journal of Technology and Teacher Education*, *10*(4), 571–596.

van Es, E. A., & Sherin, M. G. (2006). How different video club designs support teachers in "learning to notice." *Journal of Computing in Teacher Education, 22,* 125-135.

van Es, E. A., & Sherin, M. G. (2008). Mathematics teachers' "learning to notice" in the context of a video club. *Teaching and Teacher Education, 24,* 244–276.

Wallach, T., & Even, R. (2005). Hearing students: The complexity of understanding what they are saying, showing, and doing. *Journal of Mathematics Teacher Education, 8,* 393-417.

Wilson, S. M., & Berne, J. (1999). Teacher learning and the acquisition of professional knowledge: An examination of research on contemporary professional development. *Review of Research in Education, 24,* 173-209.

Yıldırım, A., & Şimşek, H. (2008). *Sosyal bilimlerde nitel araştırma yöntemleri* (7. Baskı). Ankara: Seçkin Yayıncılık.

Zawojewski, J., Chamberlin, M. T., Hjalmarson, M. A., & Lewis, C. (2008). Developing design studies in mathematics education Professional development. In A. E. Kelly, R. A. Lesh, & J. Y. Baek (Eds.), *Handbook of design research methods in education: Innovations in science, technology, engineering, and mathematics learning and teaching* (pp. 219-245). New York: Routledge.

Zoest, L. R., & Bohl J. V. (2005). Mathematics teacher identity: a framework for understanding secondary school mathematics teachers’ learning through practice. *Teacher Development, 9*(3), 315-345.

APPENDIX A

DESIGN PROCEDURE OF MEAs

Before development of MEAs, the mathematical topics in the secondary mathematics curriculum that modeling problems would be based were determined. Then, a literature review was carried out for the purpose of determining the features and design principles of these kinds of modeling activities. An evaluation form was created by using the principles that were described Lesh, et al. (2000). In light of these principles, first versions of the activities were created by pairs of project members. At the end of the process of creating the first versions, second versions of the activities were formed after all activities were examined, revised and refined by the project team.

According to the schedule of the project, in the first six months, approximately 60 modeling activities were developed. In the second six months, these activities were piloted at a private school. In this pilot study, all modeling problems were reviewed by five secondary mathematics teachers. Some problems were implemented in the classrooms by the teachers. At the end of this second six month period, all modeling problems had been already revised by the project team in light of feedbacks from the pilot study.

APPENDIX B

STUDENT THINKING SHEET

EXPLANATION: The aim of this sheet is to help teachers in eliciting students' ways of mathematical thinking while solving modelling activities. These sheets, which are prepared separately for each activity, will be later assembled into a leaflet. These leaflets will guide pre-service and practicing teachers in application of the activities. In order for the sheet to serve that purpose, it is crucial that you think in depth about every single dimension listed below and express your ideas in a detailed way. Thanks for your contribution.
Name of the Activity: **Classroom of the Implementation:** **Implementing Teachers Name-Surname:**
Student Solution Strategy–1
Student Solution Strategy –2
Student Solution Strategy –3
Student Solution Strategy –4

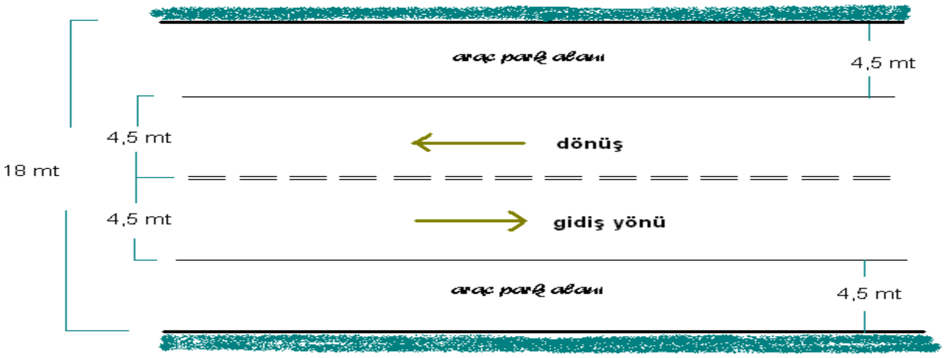
<table>
<tr><th>Strategies</th><th>Mathematical concept/ skill/process</th><th>Mistakes/ Misconceptions</th></tr>
<tr><td>Solution strategy–1</td><td></td><td></td></tr>
<tr><td>Solution strategy –2</td><td></td><td></td></tr>
<tr><td>Solution strategy –3</td><td></td><td></td></tr>
<tr><td>Solution strategy –4</td><td></td><td></td></tr>
<tr><td colspan="3">Please write any other thoughts you want to express regarding this sheet.</td></tr>
</table>

APPENDIX C

MODEL-ELICITING ACTIVITIES

Parking Spaces on the Street

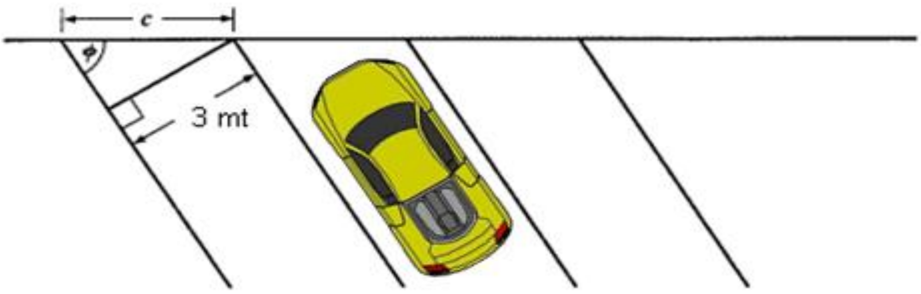
A city planner needs your help for designing the parking spaces in front of houses on two sides of a two-way street. City planners aim to come up with designs that maximise the number of cars that can park. Parking spaces lie on the two sides of the street for 150 meters. The total width of the street is 18 meters as seen in the drawing below. Traffic should flow on both directions on single lanes and cars should be able to park on both sides. Each lane should be 4.5 meters wide including the lines drawn on the street. A space 3 meters wide and 4.8 meters long including the lines, should be allocated for each car so that they can park safely. This space can be parallel to the road, or designed with an angle as in Figure 2. But in case of an angled design, the cars should not extend beyond the space onto the road.



You are asked to design parking spaces either parallel or lying with an angle to the road so that a maximum number of cars can park on that 150 meter-long part of the road.

a. Calculate how many cars can park on the street in case of parallel parking to the road.

b. Calculate how many cars can park on the street in case of angled parking to the road. You can use the drawing below for your calculations.



c. You have made calculations for different design options. Which design would you suggest to the city planner so that a maximum number of cars can park on that street? Explain and justify your suggestion.

Note: This activitiy has been adapted from Swetz, F., & Hartzler, J. S. (1991). *Mathematical modeling in the secondary school curriculum: A resource guide of classroom exercises*. Reston, VA: NCTM. This work has been carried out in a research project supported by the Scientific and Technological Research Council of Turkey (TÜBITAK) under grant number 110K250.

How to store it?

A company producing cans, needs a short term storage unit for the cylinder shaped cans being produced. The company is trying to sort this out with the minimum cost. The cans that need to be stored are all right circular cylinder shaped, having a base radius of 10cm and a height of 30cm. The company is planning to store 175 cans for a duration of 2 months.

There are 3 different sized storage boxes that the company can use. Each box has a height of 100cm, and the rent prices are given below according to the base dimensions of the boxes.

Width (cm)	Length (cm)	Monthly Rent (TL)
110	110	100
110	220	150
110	330	200

1. If you were the owner of the company, which storage box would you use and in what way, in order to minimize the storage costs?

2. The company might need to store different numbers of cans in future productions. Would it be suitable for the company to always use the same kind of storage box? What would you suggest?

For security reasons, it is important that the boxes are stored perpendicularly.

Note: This activitiy has been adapted from Swetz, F., & Hartzler, J. S. (1991). *Mathematical modeling in the secondary school curriculum: A resource guide of classroom exercises*. Reston, VA: NCTM. This work has been carried out in a research project supported by the Scientific and Technological Research Council of Turkey (TÜBITAK) under grant number 110K250.

Magazine Sales

Mathematical Thought magazine, issued every three months with approximately 25,000 sales per issue, has a sale price of 5.5 TL. But increases in publication and paper costs made an increase in the sale price inevitable. In order to understand the negative effect of this price increase on sales of the magazine, a survey has been made. According to this, it is anticipated that for every 50 kuruş increase in the price, 1,250 readers will decide not to buy the magazine. If you were the board members of the magazine, what would you decide as the new sale price of the magazine?

Note: This activitiy has been adapted from Aydın, N., & Erbaş, A. K. (2008). *Matematik 10: Ders kitabı.* Ankara: Aydın Yayıncılık. This work has been carried out in a research project supported by the Scientific and Technological Research Council of Turkey (TÜBITAK) under grant number 110K250.

APPENDIX D

THE LIST OF QUESTIONS USED AS PROMPTS

General Prompts:

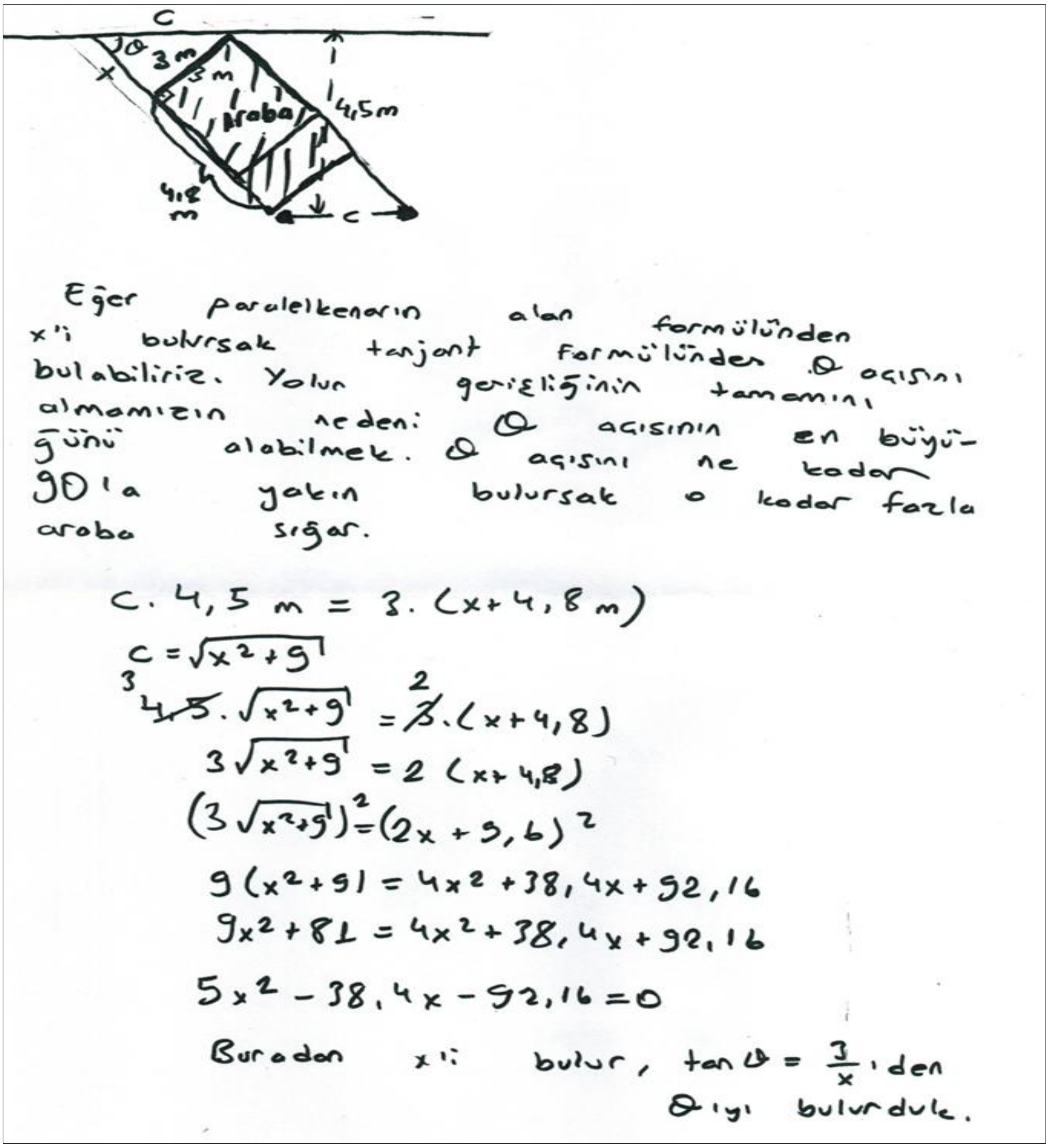
1. What do you think about underlying students' mathematical thinking?
2. How can we summarize what students tried to do here?
3. How can we explain the two methods?
4. What did they do here?
5. Is there a different solution approach in this solution sheet?
6. What is the difference of this group's solution approach from the other group's?
7. Which mathematical idea attracted your attention here?

Specific Prompts:

1. How did they write the first equation?
2. What was the reason that they couldn't make progress?
3. How did they find this value?
4. Did they use the idea of parabola here?
5. How did they form this inequality? Did they use that inequality to reach the solution?
6. How did they reach that conclusion?
7. First they found the ratio of 4.5 to 4.8 and tried to find θ. Then what did they do?

APPENDIX E

SAMPLES OF STUDENTS' WORKS



Eğer paralelkenarın alan formülünden x'i bulursak tanjant formülünden θ açısını bulabiliriz. Yolun genişliğinin tamamını almamızın nedeni: θ açısının en büyüğünü alabilmek. θ açısını ne kadar 90'a yakın bulursak o kadar fazla araba sığar.

$$c \cdot 4,5\,m = 3 \cdot (x + 4,8\,m)$$

$$c = \sqrt{x^2+9}$$

$$4,5 \cdot \sqrt{x^2+9} = 3 \cdot (x+4,8)$$

$$3\sqrt{x^2+9} = 2\,(x+4,8)$$

$$(3\sqrt{x^2+9})^2 = (2x+9,6)^2$$

$$9(x^2+9) = 4x^2 + 38,4x + 92,16$$

$$9x^2 + 81 = 4x^2 + 38,4x + 92,16$$

$$5x^2 - 38,4x - 92,16 = 0$$

Buradan x'i bulur, $\tan\theta = \frac{3}{x}$'den θ'yı bulurduk.

Figure E1. A sample of students' works in parking spaces on the street activity

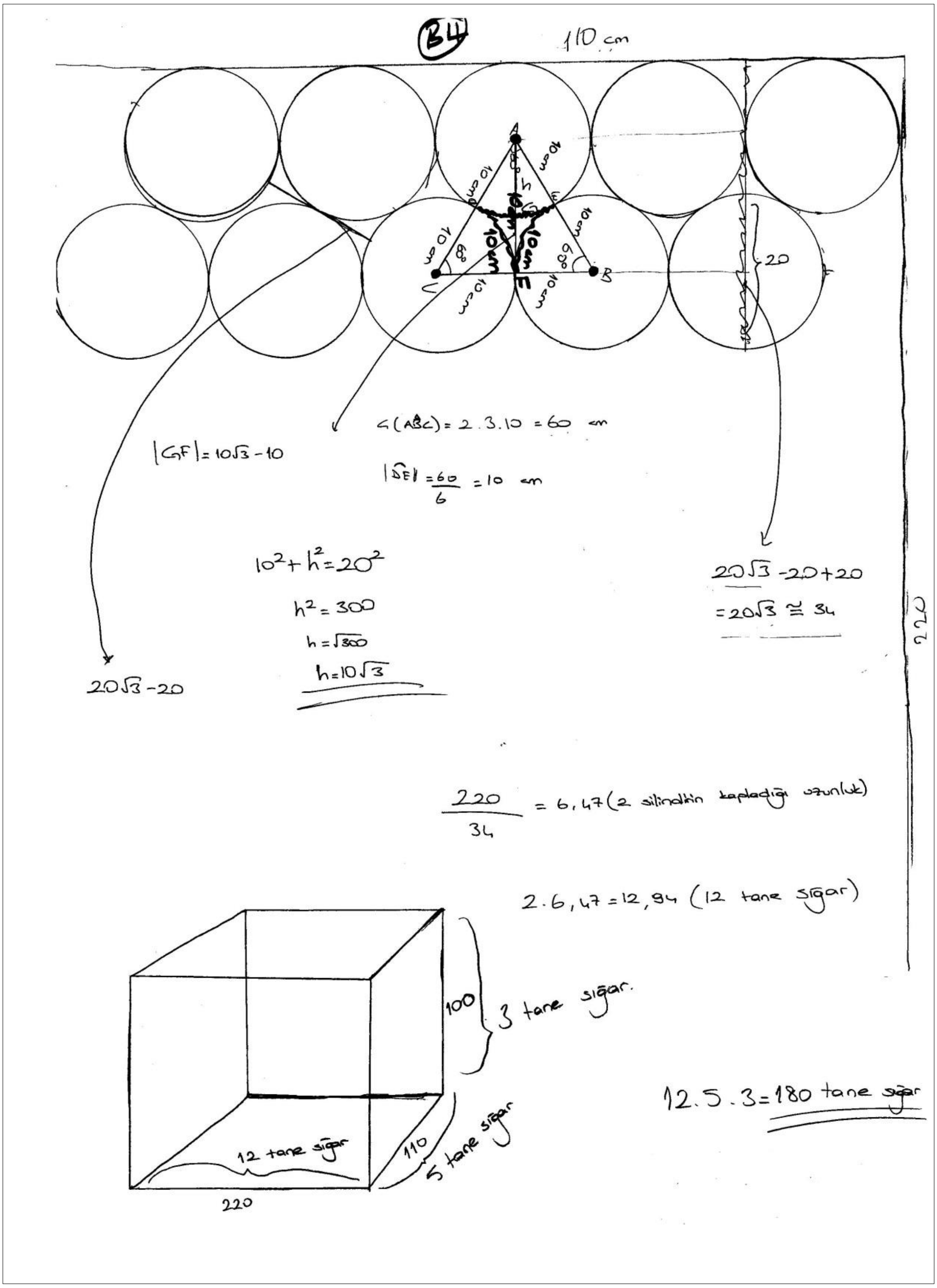


Figure E2. A sample of students' works in how to store it activity

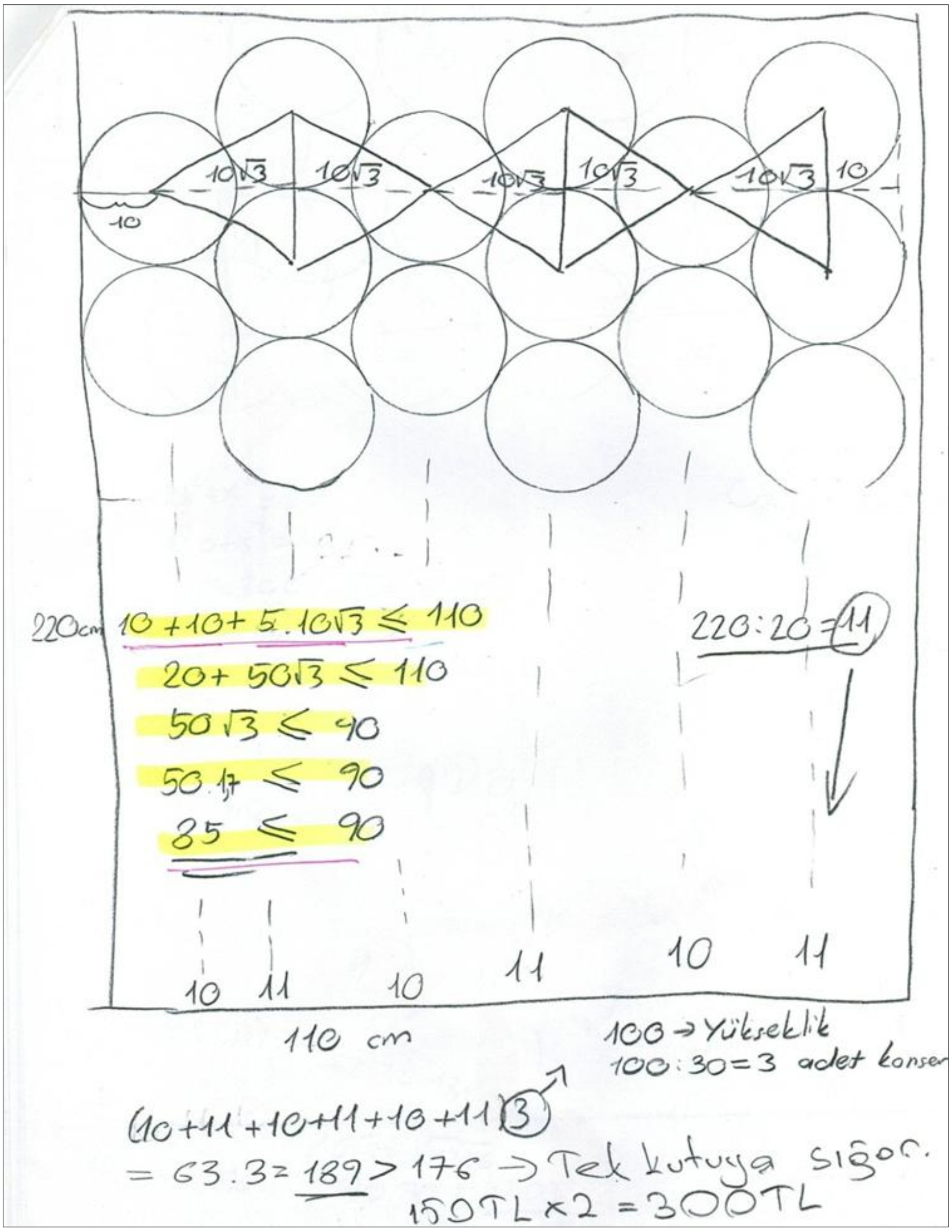
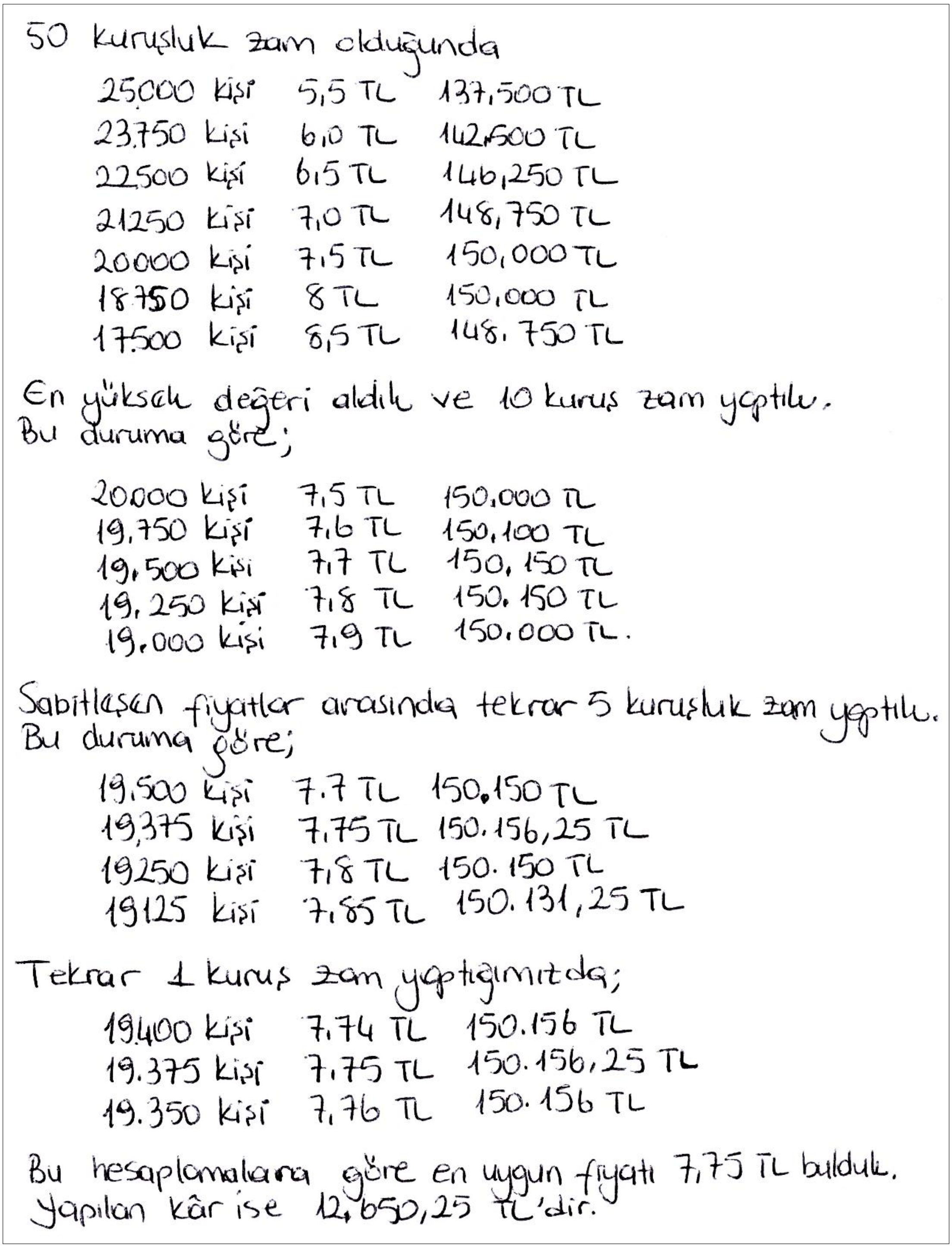


Figure E3. A sample of students' works in how to store it activity


50 kuruşluk zam olduğunda

25000 kişi	5,5 TL	137,500 TL
23750 kişi	6,0 TL	142,500 TL
22500 kişi	6,5 TL	146,250 TL
21250 kişi	7,0 TL	148,750 TL
20000 kişi	7,5 TL	150,000 TL
18750 kişi	8 TL	150,000 TL
17500 kişi	8,5 TL	148.750 TL

En yüksek değeri aldık ve 10 kuruş zam yaptık.
Bu duruma göre;

20000 kişi	7,5 TL	150,000 TL
19.750 kişi	7,6 TL	150,100 TL
19,500 kişi	7,7 TL	150,150 TL
19,250 kişi	7,8 TL	150.150 TL
19,000 kişi	7,9 TL	150,000 TL.

Sabitleşen fiyatlar arasında tekrar 5 kuruşluk zam yaptık.
Bu duruma göre;

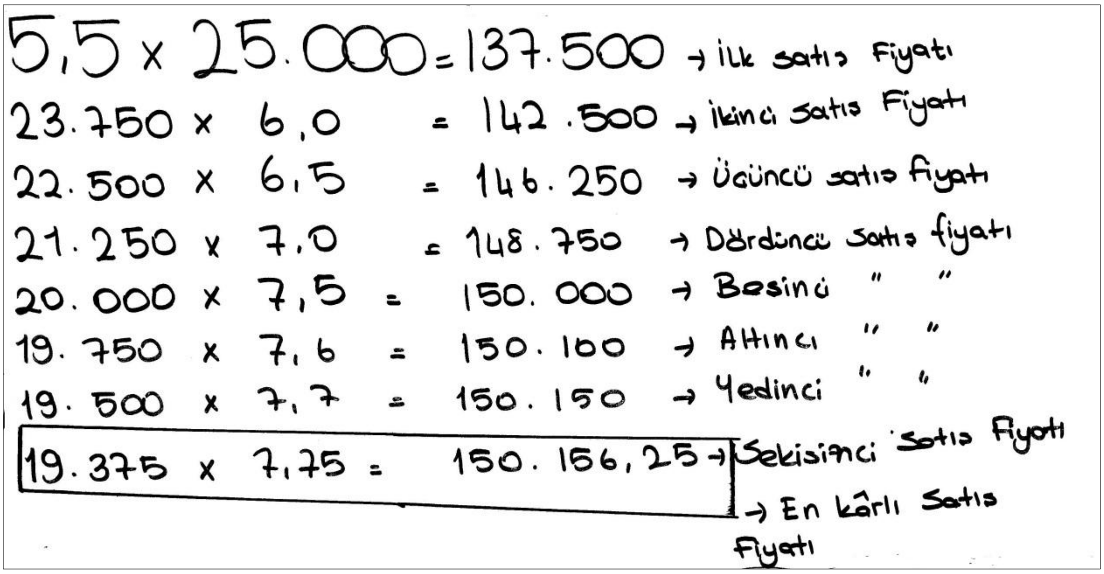
19.500 kişi	7.7 TL	150.150 TL
19,375 kişi	7,75 TL	150.156,25 TL
19250 kişi	7,8 TL	150.150 TL
19125 kişi	7,85 TL	150.131,25 TL

Tekrar 1 kuruş zam yaptığımızda;

19400 kişi	7,74 TL	150.156 TL
19.375 kişi	7,75 TL	150.156,25 TL
19.350 kişi	7,76 TL	150.156 TL

Bu hesaplamalara göre en uygun fiyatı 7,75 TL bulduk.
Yapılan kâr ise 12,650,25 TL'dir.

Figure E4. A sample of students' works in magazine sales activity



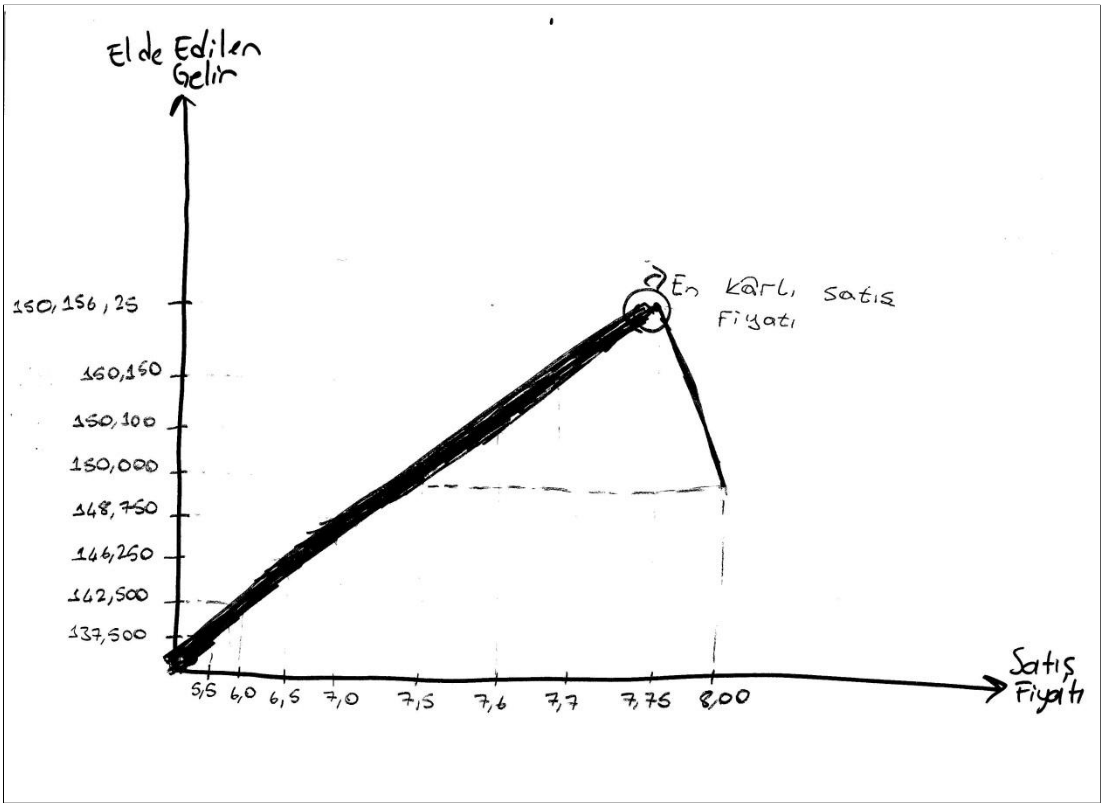


Figure E5. A sample of students' works in magazine sales activity

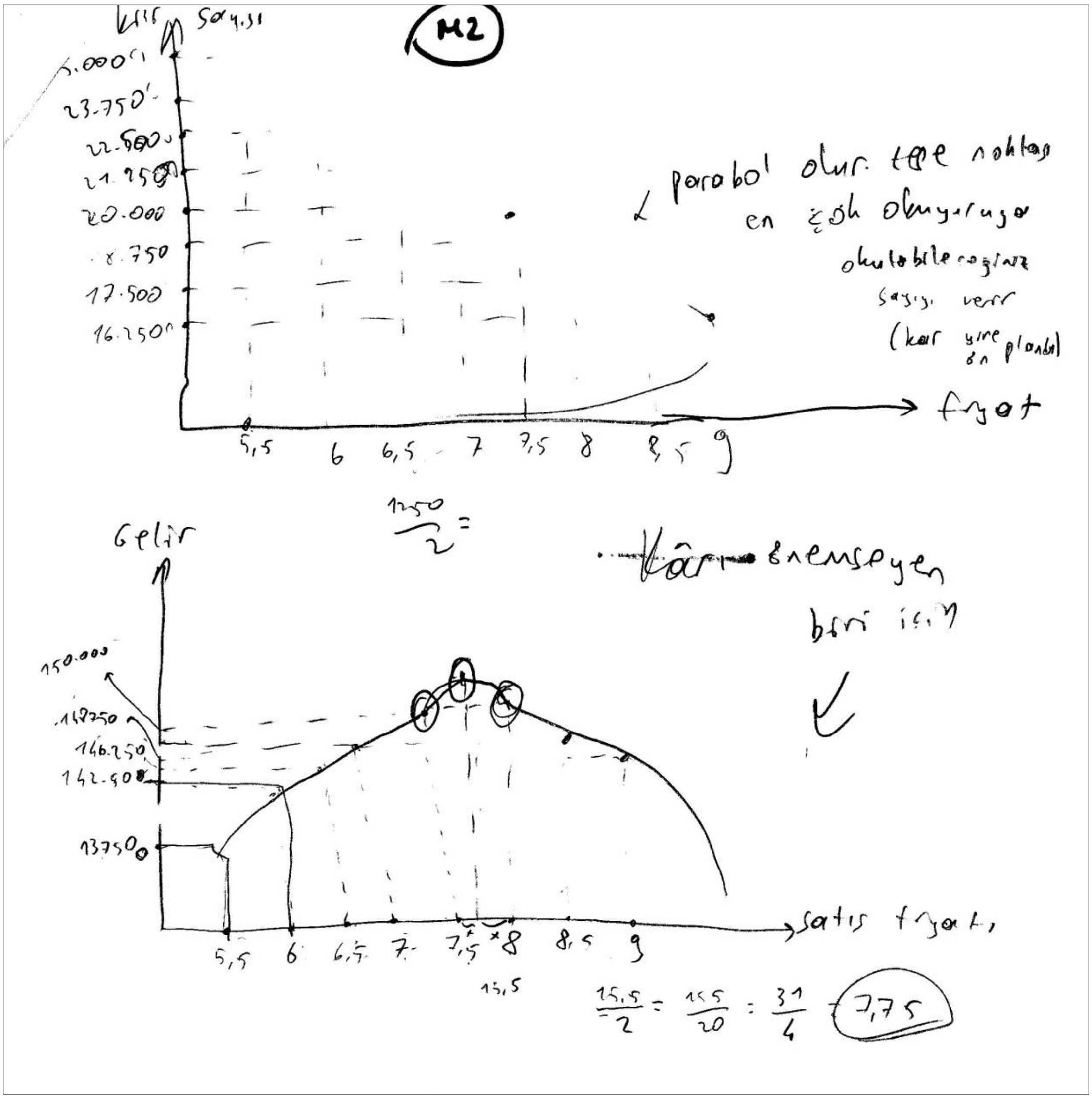


Figure E6. A sample of students’ works in magazine sales activity

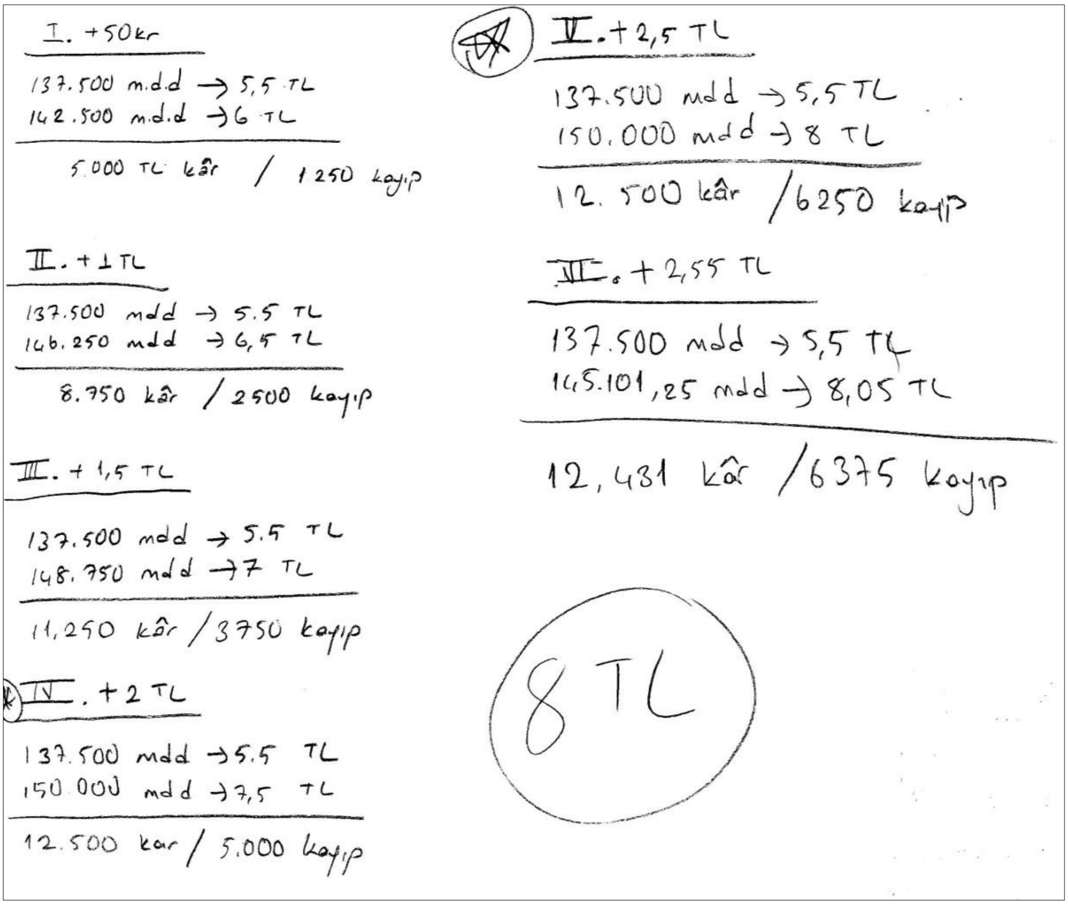


Figure E7. A sample of students' works in magazine sales activity

CURRICULUM VITAE

PERSONAL INFORMATION

Surname, Name: Baş, Sinem
Nationality: Turkish (TC)
Date and Place of Birth: 1 October 1981, Mersin
Marital Status: Single
email: bashsinem@gmail.com

EDUCATION

Degree	Institution	Year of Graduation
BS	Hacettepe Uni. Secondary Science and Mathematics Education Dept.	2005

FOREIGN LANGUAGES

English

PUBLICATIONS

1. Bas, S., Didis, M. G., Erbas, A. K., Cetinkaya, B., Cakıroğlu, E., Alacacı, C. (2013). *Teachers as investigators of students' written work: Does this approach provide an opportunity for professional development?* Paper presented at Eighth Congress of the European Society for Research in Mathematics Education, Antalya, TURKEY.

2.Bas, S. (2012, August). *Secondary school mathematics teachers' interpretations of students' responses to modeling problems.* Paper presented at the Sixth Yerme Summer School, Faro, PORTUGAL.

3.Lesh, R., Bas, S., Ader, E., Ozel, S., Sriraman, B., Aygün, B., &Turker, B. (2011). *Models and Modeling.* Discussion group session at the 35th Conference of the International Group for the Psychology of Mathematics Education, Ankara, TURKEY.

4.Bas, S., Erbas, A. K., & Cetinkaya, S. (2011). Ogretmenlerin dokuzuncu sinif ogrencilerinin cebirsel dusunme yapilariyla ilgili bilgileri [Teachers' knowledge about ninth grade students' ways of algebraic thinking]. *Egitim ve Bilim—Education and Science, 35*(159), 41–55

5.Bas, S., Erbas, A. K., & Saglam, Y. (2011). Investigating preservice mathematics teachers' modeling processes. In B. Ubuz (Ed.*), Proceedings of the 35th Conference* *of the International Group for the Psychology of Mathematics Education, TURKEY, 1*, 258.

6.Didis, M. G., Bas, S., & Erbas, A. K. (2011). Students' reasoning in quadratic equations with one unknown. In M. Pytlak, T. Rowland, & E. Swoboda (Ed.), *Proceedings of the Seventh Congress of the European Society for Research in Mathematics Education, POLAND,*479-489.

7.Bas, S., Çetinkaya, B., Yildirim, U., & Erbaş, A. K. (2009). *Pre-service mathematics teachers' development of mathematical models: Motion with simple pendulum*. Paper presented at the 14th International Conference on the Teaching of Mathematical Modelling and Applications, Hamburg, GERMANY.

8.Bas, S., Erbas, A. K., & Cetinkaya, B. (2008). *Dokuzuncu sınıf öğrencilerinin cebirsel örüntüler konusundaki düşünme yapılarıyla ilgili öğretmen bilgilerinin modellenmesi.* Poster session presented at the *Eighth National Science and Mathematics Education Congress, Bolu, TURKEY.*

9.Cetinkaya, B., Sen, A., & Bas, S. (2008). Integrating mathematical modeling and technology in teaching and learning mathematics. *Proceedings of the Eighth International Educational Technology Conference, TURKEY,* 241-246.

HOBBIES

Swimming, Jogging