TEACHERS' KNOWLEDGE OF CONTENT AND STUDENTS ABOUT THE FUNCTION CONCEPT AND ITS INTERRELATION WITH STUDENT LEARNING OUTCOMES IN VOCATIONAL HIGH SCHOOLS

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ABSTRACT

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The present study aimed to identify the teachers' knowledge of content and students in vocational high schools, and investigate the patterns of interrelation between teachers' knowledge of content and students, and their students' learning outcomes with respect to one of the most challenging concepts in secondary school mathematics curriculum, the function concept. To achieve the former, a questionnaire was administered to 42 teachers to identify their knowledge of content and students about the function concept. For the latter, two case studies were carried out.

The results showed that most teachers perceive the function as set a correspondence and were aware of the univalence requirement of functions. However, most were not fully aware of the different representations of a function and ignored the arbitrary nature of functions. The teachers' knowledge of student difficulties in the function concept varied in terms of content and quality.

The data suggested some evidence of the teachers' knowledge of content and students about the function concept and student learning outcomes. Interactions made between the teachers' conceptions of the function and understanding of the univalence requirement of functions, and student learning outcomes. As to relating a domain and range to its graph, identifying two equal functions, and locating pre-images, images, and (pre-image, image) pairs on the axes in the graphs, no interaction was found. Also, the study revealed that the teachers' knowledge of content and students about the function concept influenced their instructional practices. The teaching experiences in the class interacted with student learning outcomes.

Keywords: Vocational High Schools, Secondary Mathematics Teachers, Knowledge of Content and Students, the Function Concept, Student Learning Outcomes

ÖZ

MESLEK LİSELERİNDEKİ MATEMATİK ÖĞRETMENLERİNİN FONKSİYON KAVRAMINA DAİR ALAN VE ÖĞRENCİ BİLGİSİ İLE BUNUN ÖĞRENCİ ÇIKTILARIYLA İLİŞKİSİ

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Bu çalışmanın amacı meslek liselerindeki matematik öğretmenlerinin ortaöğretim matematik müfredatının en zorlayıcı kavramlarından biri olan fonksiyon kavramına ilişkin alan ve öğrenci bilgisini belirlemek ve öğretmenlerin bu konudaki alan ve öğrenci bilgisi ile öğrencilerinin öğrenme çıktıları arasındaki ilişkiyi incelemektir. Çalışmanın ilk amacını gerçekleştirmek için 42 öğretmene, öğretmenlerinin fonksiyon kavramı konusundaki alan ve öğrenci bilgisi belirleyen bir anket uygulanmıştır. Çalışmanın diğer amacı için iki durum çalışması yürütülmüştür.

Çalışmanın bulguları birçok öğretmenin fonksiyonu bir eşleme olarak algıladığını ve fonksiyonun tek değerlilik (univalence) özelliğinin farkında olduğunu göstermiştir. Ancak birçok öğretmen bir fonksiyonun farklı gösterimlerinin tam olarak farkında değildir ve fonksiyonun keyfilik (arbitrariness) özelliğini göz ardı etmiştir. Öğretmenlerin öğrencilerin fonksiyon kavramıyla ilgili güçlüklerine dair bilgisi ise içerik ve kalite olarak değişiklik göstermiştir.

Durum çalışmaları, öğretmenlerin fonksiyon kavramına ait alan ve öğrenci bilgisi ile öğrencilerin öğrenme çıktıları arasında etkileşim olduğunu göstermiştir. Etkileşim öğretmenlerin fonksiyon kavramı algısı ve fonksiyonun tek değerlilik özelliği ile öğrencilerin öğrenme çıktıları arasında olmuştur. Tanım ve değer kümesi verilen fonksiyonun grafiğini bulma, eşit iki fonksiyonu belirleme ve görüntüsünü, ters görüntüsünü ve bunlardan oluşan sıralı ikilileri eksenler üzerine işaretlemede ise etkileşim görülmemiştir. Ayrıca, çalışmada öğretmenlerin fonksiyon kavramına dair alan ve öğrenci bilgisinin öğretimlerini etkilediği ortaya çıkmıştır. Sınıftaki öğretim etkinlikleri öğrencilerin öğrenme çıktılarıyla etkileşim göstermiştir.

Anahtar Kelimeler: Meslek Liseleri, Ortaöğretim Matematik Öğretmenleri, Öğretmenin Alan ve Öğrenci Bilgisi, Fonksiyon Kavramı, Öğrenme Çıktıları

To my father

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CHAPTER 1

INTRODUCTION

In contemporary societies, mathematics is considered a backbone for science and technology developments. As a result, educators and governments have paid considerable attention to the quality of mathematics teaching. Mathematics curriculum and teaching methods have long been debated. Many countries have monitored their students' science and mathematics performance by comparing their students to those in other countries through surveys (e.g., Program for International Student Assessment-PISA). Results from these comparative studies have shown that Turkish educational policy and its educational system have some shortcomings, especially about the quality of mathematics and science education. In fact, vocational high school students obtained significantly lower mathematics achievement scores than general high school students in these international studies (Alacacı & Erbaş, 2010; OECD, 2004).

After graduating from a general or vocational and technical high school, students take a university entrance exam to obtain acceptance for a higher educational program. The examination comprises multiple-choice test items and intends to measure students' knowledge and competencies in five basic fields: Turkish, Social Sciences, Mathematics, Science, and Foreign Language (European Commission [EC], 2011). The results of the students' university entrance exam scores, also, showed that students who study in vocational schools scored significantly lower than those who study in other schools (e.g., Berberoğlu & Kalender, 2005; Köse, 1996). As mastery of mathematics is important for high school students, it is necessary to explore factors behind student learning.

A growing body of documents has reported that teachers play a crucial role in students' learning of mathematics (Lerman, 2001; Mewborn, 2001, 2003; National Council of Teachers of Mathematics [NCTM], 2000; National Mathematics Advisory Panel, 2008). They commonly claimed that teachers' knowledge matters (Ball, Lubienski, & Mewborn, 2001). What teachers should know and understand, and teachers' knowledge of learning processes in both elementary and secondary level mathematics have become a focus of interest for educators (Ball, 1997; Fennema & Franke, 1992; Shulman, 1986). However, there is no a clear consensus on what knowledge will enable teachers to teach effectively. For instance, mathematics teachers' knowledge of mathematical content often has been recognized as a crucial component of teacher knowledge (Grossman & Schoenfeld, 2005; Mewborn, 2003). Researchers have conducted a number of studies on teachers' knowledge of mathematics, which is often measured by the number of college mathematics courses completed or a score obtained on a standardized test (e.g., Monk, 1994). In general, no significant relationship between teachers' knowledge of mathematics and student achievement has been found (Hill & Ball, 2004). Some researchers and teacher educators have claimed that knowing the subject well may not be solely sufficient for teaching (American Council on Education, 1999; Ball, Thames, & Phelps, 2008). In addition to knowledge of mathematics content per se, teachers' knowledge should include "the ways of representing and formulating the subject that make it comprehensible to others" (Shulman, 1986, p.9). This statement underscores the importance of understanding the subject matter for teaching and clearly shows that having a mastery of the subject alone is not enough for teaching (RAND, Mathematics Study Panel, 2003).

In 1985, Shulman introduced the term pedagogical content knowledge (PCK), a special domain of teacher knowledge that links content, students, and pedagogy to the teaching and teacher education research lexicon (RAND, Mathematics Study Panel, 2003). Shulman (1986) defined PCK as “the particular form of content knowledge that embodies the aspects of content most germane to its teachability” (p. 9) and suggested that for a particular subject area, it includes “… an understanding of what makes the learning of specific concepts easy or difficult” (p. 9). This notion of PCK necessitates a body of knowledge of common student conceptions, preconceptions and also student difficulties with particular ideas (An, Kulm, & Wu, 2004; Ball et al., 2008; Shulman, 1986).

Ever since the concept of PCK was introduced, it has been widely used in framing and describing research and practice in teacher education in many fields of education; there have been quite a few attempts to explore what teachers know in this domain (An et al., 2004; Gess-Newsome, 1999; Graeber & Tirosh, 2008; Grossman, 1990; Hill, Ball, & Schilling, 2008; Marks, 1990). Despite these attempts, PCK has not been universally conceptualized (Berry, Loughran, & van Driel, 2008). For about two decades, in mathematics education area, Ball et al. (2008) focused on the aspects of the coherent theoretical framework of Shulman (1986, 1987) that needed improvement. Building on the promise of PCK, the research group headed by Deborah Ball from the University of Michigan developed a theoretical framework and set of measurement instruments for the assessment of school teachers’ mathematical knowledge for teaching. Ball and her colleagues concluded that Shulman’s PCK could be divided into knowledge of content and teaching (KCT) and knowledge of content and students (KCS).

According to Ball et al. (2008), KCS is an amalgam of a particular mathematical idea or procedure and familiarity with what students frequently think or do. For instance, when choosing an example, teachers need to predict what students will find interesting, or when assigning a task, they need to anticipate what students are likely to do and whether the students will find it easy or difficult. For Ball et al. (2008), in teaching, KCS is used to attend to the specific content as well as something particular about learners, but central to KCS is the knowledge of student difficulties about particular mathematical content. Research confirms that teacher knowledge of students' common conception, difficulties, and errors about a subject area, which is one of the primary elements of KCS, is critical for student learning (An et al., 2004; Even & Tirosh, 2002; Graeber, 1999; Hill et al., 2008; Park & Oliver 2008). However, the specific ways in which such knowledge of teachers affects their students' learning are not widely understood (Even & Tirosh, 1995; Hill et al., 2008; Tirosh, 2000).

1.1 Statement of the Problem

The present study dealt with KCS of mathematics teachers in technical and industrial vocational high schools and its impact on student learning, particularly on the learning of one of the least understood concepts in secondary school mathematics curriculum, the function concept. The study had two major purposes: (1) to identify the KCS of mathematics teachers with respect to the function concept, and (2) to examine the patterns of interrelation between KCS of a mathematics teacher and his/her students' learning outcomes regarding the function concept. In line with these aims, two research questions are formulated:

1. As to the function concept, what is the extent to which mathematics teachers have knowledge of:
    a. content;
    b. student difficulties?

2. How do mathematics teachers' knowledge of content and students and the student learning outcomes interrelate as regards the function concept?

1.2 Significance of the Study

This study contributes to three fields of the related literature. To begin with, PCK represents knowledge that is "uniquely the province of teachers, their own special form of professional understanding" (Shulman, 1987, p.8) and includes areas of student difficulty. Teachers' knowledge of students' difficulties, common conceptions and misconceptions about particular mathematical content is critical in student learning (An et al., 2004; Australian Education Council, 1990; Even & Tirosh, 2002; Graeber, 1999; Hill et al., 2008; Marks, 1990; Park & Oliver 2008). On the other hand, how such teacher knowledge affects student learning is less understood (Hill, Rowan, & Ball, 2005; Hill et al., 2008; Tirosh, 2000). This study attempted to provide evidence of the patterns of the interrelation between teachers' KCS pertaining to the function concept, and their students' emergent knowledge of the concept. The study intended to provide suggestions on what the mathematics education community can do to enhance teachers' KCS, especially on how to develop this kind of knowledge so that teachers may be aware of their students' difficulties and help them to overcome them.

The study examined teachers' KCS with respect to the function concept, one of the central and essential concepts in mathematics curriculum (Harel & Dubinsky, 1992; NCTM, 1989, 2000; Selden & Selden, 1992; Vinner, 1992). Despite its importance in mathematics, literature review on student understanding of the function concept has indicated that students have difficulties (e.g., Akkoç, 2006; Clement, 2001; Dubinsky & Harel, 1992; Eisenberg, 1991; Hatısaru & Çetinkaya, 2011; Hatısaru & Erbaş, 2013; Lambertus, 2007; Leinhardt, Zaslavsky, & Stein, 1990; Markovits, Eylon, &

Bruckheimer, 1988; Talim ve Terbiye Kurulu Başkanlığı [TTKB], 2011; Nilklad, 2004; Özmantar, Bingölbali, & Akkoç, 2010; Sajka, 2003; Tall & Bakar, 1991; Tall & Vinner, 1981; Vinner, 1983, 2002; Vinner & Dreyfus, 1989). In order to facilitate student learning, teachers themselves need to have complete knowledge of the functions (Howald, 1998) and be aware of their students' difficulties (Even & Tirosh, 1995). On the other hand, research that focus on prospective teachers' knowledge on functions has shown that many prospective teachers' knowledge is generally insufficient. In fact, they lack a genuine understanding of the function concept (Agarwal, 2006; Bolte, 1993; Ebert, 1994; Even, 1989, 1990, 1993; Hacıömeroğlu, 2006; Hansonn, 2006; Karahasan, 2010; McGehee, 1990; You, 2006; Wilson, 1992). Thus, it would be beneficial to examine in-service teachers' knowledge of functions and how their knowledge influences their students' learning. Nevertheless, few research studies (Duah-Agyeman, 1999; Hitt, 1998; Howald, 1998; Lloyd & Wilson, 1998; Norman, 1992) have attempted to examine experienced in-service teachers' understanding of functions, and even fewer about mathematics teachers' in vocational high schools. The current study extensively analyzed experienced in-service teachers' KCS of the function concept. The study built on previous research in two major ways: (1) it provided greater depth of information concerning a large sample of experienced in-service secondary teachers' KCS about the function concept in vocational schools than has previously been available, (2) it went beyond identifying the teachers' KCS of the function concept, and described the interrelation between teachers' KCS of the function concept and their student learning outcomes of this concept. A better understanding of experienced mathematics teachers' KCS about the function concept contributes to our understanding of the current state of the issue and raised the question of whether teachers' KCS about the function concept enables them to help students' learning of this concept.

Review of the related literature has revealed that students who attend vocational education are underachievers in natural and social science courses

(Bottom & Korcheck, 1989; Usul, Eroğlu & Akın, 2007; Yörük, Dikici & Uysal, 2002), and in mathematics (Adams, 2001; Berberoğlu & Kalender, 2005; Green, 1998; Lewis, 2000; Köse, 1996; Scarpello, 2005). The researcher of this study taught mathematics in a technical and industrial vocational high school for five years. Based on her own experience, also, she believes that mathematics learning has been unsuccessful for the majority of students who attend vocational high schools. As mastery of mathematics is important for high school students, she thinks, it is necessary to explore factors that could lead to a better understanding of what is necessary for teaching and learning mathematics in these schools. Her personal mathematics experiences led her to believe that teachers who are better prepared and highly motivated would affect students' performance. In line with this belief, she has become aware of her own tendency to focus on the teacher knowledge and its effect on student learning. The researcher acknowledges that teachers' content knowledge has an effect on student learning. However, she has met teachers who have a deep understanding and knowledge of subjects but cannot maintain their students' learning, simply because they seem to lack one part of the particular mathematical knowledge for teaching, KCS. She personally believes that KCS is vital to student learning, such that if it is not there, teachers may not be able to pay attention to students' difficulties, correct students' existing misconceptions about subject matters properly, and lead them to succeed in learning mathematics. These perspectives influenced the researcher to conduct this study and provide tangible evidences of vocational mathematics teachers' KCS about a specific mathematical subject. It was hoped that the study about knowledge of teachers in technical and industrial vocational high schools will help the mathematics education community to better understand the factors behind difficulties of students.

1.3 Definition of Terms

Following are the terms commonly used in the present study.

Pedagogical content knowledge: Pedagogical content knowledge is “the particular form of content knowledge that embodies the aspects of content most germane to its teachability” (Shulman, 1986, p. 9). It includes “an understanding of what makes specific topics easy or difficult for a certain group of learners” (Shulman, 1986, p. 9).

Knowledge of content and students (KCS): The knowledge that is blending of a particular mathematical idea or procedure and familiarity with what students frequently think or do. One of the main components of KCS is knowledge of student difficulties about particular mathematical content (Ball et al., 2008).

Knowledge of content: Knowledge about the subject and its structure (Shulman, 1986).

Student learning outcomes: What students will know, be able to do or be able to demonstrate at the completion of instructional units on functions. The present study expected that at the completion of 9th grade instructional units on functions, the students should be able to demonstrate an understanding of the function concept; demonstrate an understanding of the essential features of functions (arbitrariness and univalence); relate a domain and range to its graph, and vice versa; identify two equal functions; and locate pre-image, image, and (pre-image, image) pairs on the axes in the graphs.

Vocational and technical education: Secondary education for any occupation, which follows the compulsory primary education and requires a high school diploma. Its main objective is to prepare students for higher education and employment in accordance with their fields by means of varied curriculum and several institutions (EC, 2010).

Technical and industrial vocational high schools: Secondary schools implementing vocational and technical curriculum of education for four years. Students who graduated from elementary schools are entitled to attend vocational high schools. They aim at preparing students both for tertiary education and employment (EC, 2010, 2011).

CHAPTER 2

REVIEW OF LITERATURE

This study explores KCS of mathematics teachers who teach in technical and industrial vocational high schools and its impact on student learning outcomes as regards the function concept. The purpose of this chapter is to provide a review of research concerning the concept of teacher knowledge, knowledge and understanding of teachers and students about the function concept, as well as to provide the relationship between teacher knowledge and student learning.

2.1 Teacher Knowledge

To date, quite a few attempts at identifying the full extent of teachers' knowledge have been made (Mayer & Marland, 1997). On the basis of an in-depth study of one secondary teacher, Elbaz (1983) made such an effort (as cited in Mayer & Marland, 1997). Elbaz (1983) proposed five categories of teacher knowledge: knowledge of curriculum, knowledge of students, knowledge of instruction, knowledge of subject matter, and knowledge of self. A similar set of hypothetical domains of teacher knowledge were offered by Shulman (1986). He introduced the term "missing paradigm" and criticized the literature of research on teaching, in which the central questions were unasked. According to him, the importance was on teachers' classroom managements, planning lessons, organizing activities, and assessing students' general understanding. What were missed were the questions about the content. In

Shulman's (1987) view, the teacher should be able to combine subject-matter understanding and pedagogical skills for different kinds of students, different themes, different pedagogical purposes, and different levels of difficulty. S/he should flexibly respond to the capacities of the students, and the difficulty and character of the subject matter. S/he should understand how to organize the work s/he is teaching, frame it for teaching, and divide it appropriately for activities as well as assessments. Accordingly, he called for the study of three types of content understanding and their impact on classroom practice: subject content knowledge, pedagogical knowledge, and pedagogical content knowledge.

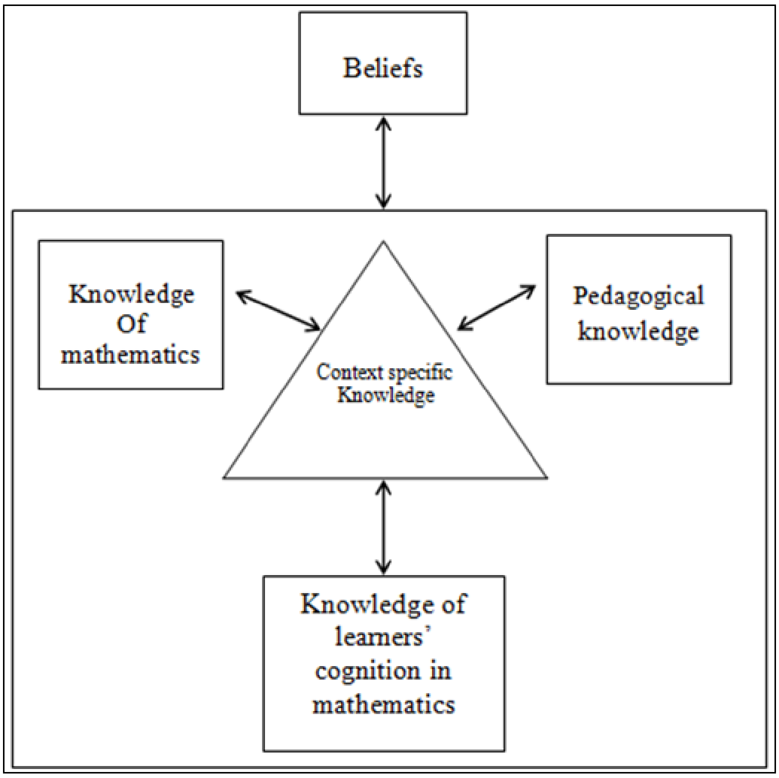


Figure 2.1: Teachers' knowledge: Developing in context (Fennema & Franke, 1992, p.162)

In addition to these categorizations, in the research on the teacher knowledge, several frameworks and models of teachers' knowledge have emerged. Fennema and Franke (1992) stated that teachers' knowledge includes knowledge of pedagogy, as well as understanding the underlying process of the concepts, being able to interpret these concepts for teaching, understanding students' thinking, and being able to assess students' learning to make instructional decisions. They put forth a model which includes the components of teacher knowledge of the content of mathematics, knowledge of pedagogy, knowledge of students' cognition, and teachers' beliefs (see Figure 2.1).

Fennema and Franke (1992) identified that the knowledge of the content of mathematics includes teacher knowledge of the concept, procedures, and problem solving process in the domain that they teach; knowledge of pedagogy includes teacher knowledge of teaching procedures, planning, organization, management, and motivation; and knowledge of students' cognition includes knowledge of students' thinking, learning, difficulties, and successes. According to them, in terms of definition and relationships with other components, each of these components requires further study. However, the study of these components must not be performed out of context in order to be able to gain an insight of teacher knowledge.

Ball, Lubienski, and Mewborn (2001) asked questions about how mathematics teachers' knowledge influences the teachers' effectiveness. They found that advanced mathematical understanding mostly contributed little to teacher effectiveness. Referencing Monk (1994), they reported that teachers' knowledge gained in mathematics pedagogy courses contributes more to students' learning (p. 24). According to the researchers, that is, teachers' thinking skills that influence their pedagogical reasoning and their articulateness as teachers support their students' achievement. In that sense, they draw a theoretical distinction between subject-matter knowledge, content knowledge, and the knowledge needed for teaching a specific subject, pedagogical content knowledge, which is discussed in the next section.

2.1.1 Pedagogical Content Knowledge

Pedagogical content knowledge (PCK) is an important characteristic of teacher knowledge (Even & Tirosh, 1995). Shulman (1986) defined PCK as the knowledge "…which goes beyond knowledge of subject matter per se to the dimension of subject-matter knowledge for teaching" (p. 9). He suggested that for a particular subject area, PCK includes:

> the most regularly taught topics in one's subject area, the most useful forms of representation of those ideas, the most powerful analogies, illustrations, examples, explanations, and demonstrations… including an understanding of what makes the learning of specific concepts easy or difficult: the conceptions and preconceptions that students of different ages and backgrounds bring with them to the learning of those most frequently taught topics and lessons (Shulman, 1986, p. 9).

Beyond Shulman's original formulation, researchers have expanded and elaborated the characterization of PCK through case studies of teachers with different academic majors (Graeber & Tirosh, 2008). In a study of a high school English teacher, for instance, Grossman (1988) developed an expanded definition of PCK, based on four central components: knowledge of students' understanding, curriculum, instructional strategies, and purposes for teaching. On the basis of his in-depth study of eight 5th grade teachers, Marks (1990) proposed four components of PCK: "subject matter for instructional purposes, students' understanding of the subject matter, media for instruction in the subject matter (i.e., texts and materials), and instructional processes for the subject matter" (p. 4). Marks (1990) detailed the notion of students' understanding including student learning process, typical understandings, common errors and difficulties (p. 5). In the same years, Carpenter, Fennema, Peterson, and Carey (1988) described PCK as blending knowledge of content, students' thinking, and also instructional strategies.

To date, PCK has been widely used in framing and describing research and practice in different fields of teacher education, including mathematics (Hill et al., 2008). For instance, Magnusson, Krajcik, and Borko (1999) conceptualized PCK for science teaching as consisting of five components: "(a) orientations toward science teaching, (b) knowledge and beliefs about science curriculum, (c) knowledge and beliefs about students' understanding of specific science topics, (d) knowledge and beliefs about assessment in science, and (e) knowledge and beliefs about instructional strategies for teaching science" (p. 97). For Magnusson et al. (1999), the first component of PCK refers to teachers' knowledge and beliefs about the purposes and aims for teaching the content at a certain grade level. The second represents knowledge that distinguishes the content specialists from the pedagogue. The third refers to the knowledge teachers should possess about students in order to support their learning of a specific topic. It includes two categories of knowledge: requirements for learning specific science concepts and areas of science that students have difficulty. The fourth component refers to knowledge of teachers about the aspects of students' learning that are important within a particular unit of study. The last refers to teachers' knowledge of particular strategies that are helpful for assisting students' understanding of both general approaches to science teaching and specific science concepts. This component of PCK includes two categories: knowledge of subject-specific strategies, and knowledge of topic-specific strategies (p. 96-115).

In the same field, Park and Oliver (2008) rethought the conceptualization of PCK based on their descriptive research findings. According to them, scholars have conceptualized PCK by identifying the components that form PCK and considered PCK as a combination of those components. For Park and Oliver (2008), with respect to the components that form PCK, some differences occurred among the researchers, but most researchers agreed on two key components of PCK: "(a) knowledge of instructional strategies incorporating representations of subject matter, and responses to specific learning difficulties and (b) student conceptions with respect to that subject matter" (p. 264). Along with these statements, the researchers identified five components of PCK for science teaching: "(a) orientations to science teaching, (b) knowledge of students' understanding in science, (c) knowledge of science curriculum, (d) knowledge of instructional strategies and representations for teaching science, and (e) knowledge of assessments of science learning" (p. 264). They, also, aimed to show how this new conceptualization helps educators to understand teachers as professionals through a multiple case study. The results indicated that students had an important impact on PCK development, and students' misconceptions played a significant role in shaping PCK.

An et al. (2004) defined PCK for mathematics teaching as the knowledge of effective teaching which includes three components: knowledge of content, knowledge of curriculum, and knowledge of teaching. According to the researchers, as NCTM stated, knowledge of content consists of mathematical content knowledge; knowledge of curriculum consists of selecting and using appropriate textbooks and materials, understanding the goals of textbooks and curricula; and knowledge of teaching includes knowing students' thinking, planning instruction, and understanding the modes of presenting instruction. For them, all three parts of PCK are very important to effective teaching, but knowledge of teaching is the main component of PCK. In this process, knowing students' thinking is most critical. They developed the following set of categories for knowing students' thinking: addressing students' misconception, engaging students in math learning, building on students' math ideas, and promoting students' mathematical thinking. They have broadened Shulman's (1987) original designation and presented a network of pedagogical content knowledge (see Figure 2.2). To them, there is an interactive relationship among the three components of PCK, and knowledge of content and curriculum contribute to strength knowledge of teaching.

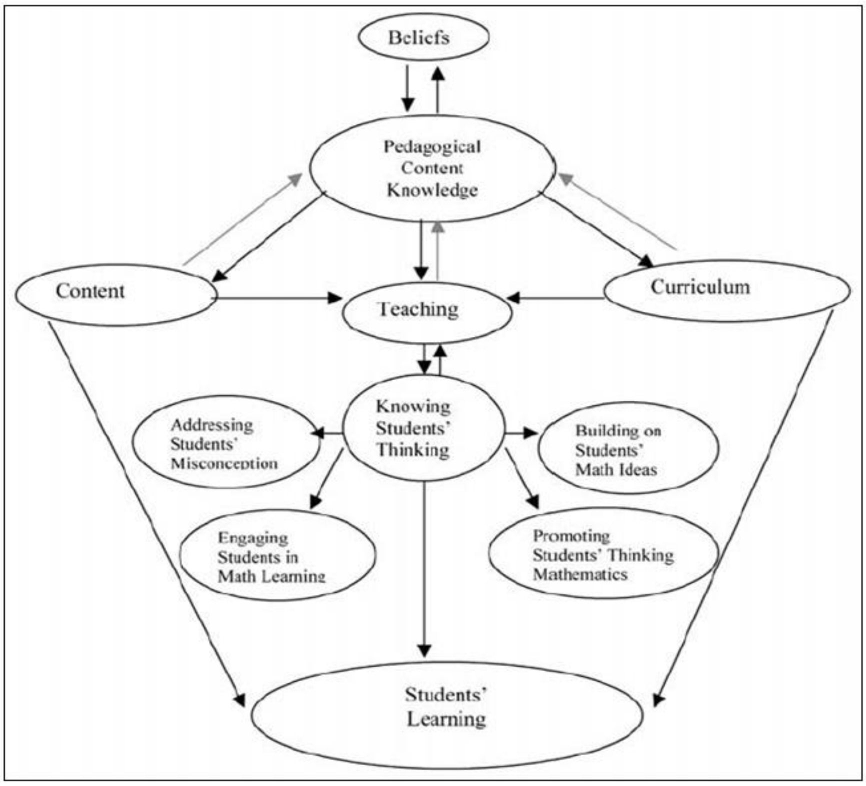


Figure 2.2: The network of pedagogical content knowledge (An, Kulm, & Wu, 2004, p.147)

In mathematics education area, Ball, Thames, and Phelps (2008) identified mathematical knowledge needed to teachers in teaching tasks, what they have called mathematical knowledge for teaching.

2.1.2 Mathematical Knowledge for Teaching

Mathematical knowledge for teaching (MKT) is defined as "the mathematical knowledge that teachers use in classrooms to produce instruction and student growth" (Hill et al., 2008, p. 374). Besides content knowledge, the concept encompasses knowledge of the connections among ideas, knowledge of the representations, knowledge of the student's thinking, and knowledge of the common student difficulties with particular ideas (Ball et al., 2001). Ball et al. (2001) stated:

> …such knowledge is not something a mathematician would have by virtue of having studied advanced mathematics. Neither would it be part of a high school social studies teacher's knowledge by virtue of having teaching experience. Rather, it is knowledge special to the teaching of mathematics. (p. 448)

Ball and colleagues developed and validated measures of domains of MKT that is presented in Figure 2.3 (Ball et al., 2005; Hill, Schilling, & Ball, 2004). They wrote items in different categories and conducted these items with large groups of teachers to test the items (Hill & Ball, 2004; Rowan, Schilling, Ball, & Miller, 2001). Based on these analyses, Ball et al. (2008) hypothesized that Shulman's content knowledge could be subdivided into common content knowledge (CCK) and specialized content knowledge (SCK). They defined CCK as the mathematical knowledge and skill used in settings other than teaching. That is, this knowledge is used in a wide variety of settings. In other words, it is not unique to teaching. Hill and Ball (2004) gave some examples of CCK such as: "…being able to compute 35x25 accurately, identifying what power of 10 is equal to 1, solving word problems satisfactorily, and so forth" (p. 333). Contrary to CCK, SCK is the mathematical knowledge and skill unique to teaching. Ball et al. (2008) stated SCK is mathematical knowledge not typically needed for purposes other than teaching. Hill et al. (2004) reported that SCK "…is used in the course of different sorts of tasks-choosing representations, explaining, interpreting student responses, assessing student understanding, analyzing student difficulties, evaluating the correctness and adequacy of curriculum materials" (p. 16).

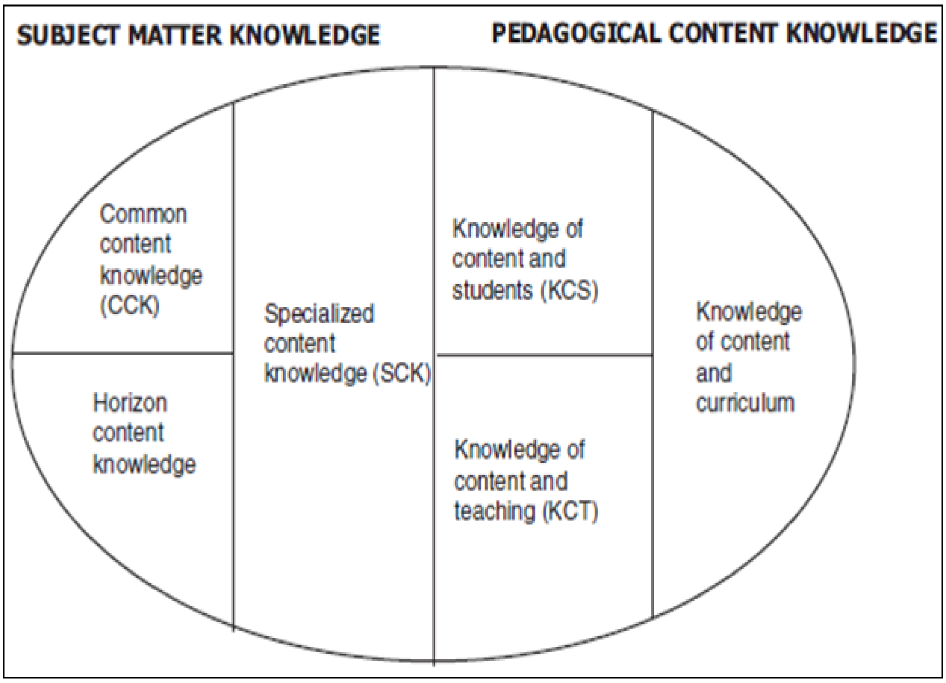


Figure 2.3: Domains of mathematical knowledge for teaching (Ball, Thames, & Phelps, 2008, p. 403)

Regarding PCK, Ball et al. (2008) hypothesized that it could be divided into two domains. The first one is knowledge of content and teaching (KCT), which combines knowing about teaching and knowing about mathematics. The second one is knowledge of content and students (KCS), which is discussed in the following section. For example:

> …recognizing a wrong answer is common content knowledge (CCK), whereas sizing up the nature of an error, especially an unfamiliar error, typically requires nimbleness in thinking about numbers, attention to patterns, and flexible thinking about meaning in ways that are distinctive

> of specialized content knowledge (SCK). In contrast, familiarity with common errors and deciding which of several errors students are most likely to make are examples of knowledge of content and students (KCS) (Ball et al., 2008, p. 401).

The present study focused on secondary mathematics teachers' KCS, a primary element in Shulman's PCK.

2.1.2.1 Knowledge of Content and Students

Hill et al. (2008) defined knowledge of content and students (KCS) as content knowledge that connects with knowledge of how students think about, know, or learn a particular content. Ball et al. (2008) added that KCS is an amalgam that involves a particular mathematical idea or procedure and familiarity with what students frequently think or do. For instance, when choosing an example, teachers need to guess what students will find interesting or when giving a task, teachers need to be aware of what students are likely to do it, and whether they will find it easy or difficult. According to Ball et al. (2008), in teaching, KCS is used that involve attending to the specific content as well as something particular about students, but one of the crucial components of KCS is knowledge of students' difficulties and misconceptions with respect to a specific mathematical content.

Earlier studies mentioned the importance of having knowledge of students about particular academic content (e.g., Carpenter et al., 1988; Maurer, 1987; Nesher, 1987). Maurer (1987) stated that it is important for teachers to know there are systematic errors many students commit. The teachers should be familiar with the most common types of those errors and look for them in the classroom. Maurer (1987) asserted, instead of just focus on the answers students produce, teachers should focus on what students are doing and why. Similarly, Nesher (1987) claimed that teachers should be aware of students' mistakes in order to learn about students' understanding and then connect the new knowledge to the student's previous conceptual framework. Carpenter et al. (1988), from a different view point, stressed that short-term computational goals may be achieved without attending to students' knowledge, but achieving higher level goals could be related to teachers' attempts to understand students' thinking.

Later studies supported this view. Even and Tirosh (2002) discussed what one might mean by teacher knowledge about students. They reported that students build their knowledge of mathematical concepts different from what is expected. Therefore, for teachers, it is important to be aware students' limited conceptions and misconceptions. A more recent study by An et al. (2004) claimed that "teachers should be able to identify students' misconceptions and be able to correct misconceptions by probing questions or using various tasks" (p. 169). If teachers enter the classroom without valuing student thinking, they will not apt to use knowledge of students' current understanding to make instructional decisions (Graeber, 1999).

The empirical studies cited above suggest that having knowledge of student including students' mistakes, misconceptions and difficulties about a specific academic content is essential for teachers. It is perceived that such knowledge significantly contributes to the teachers' instruction (Even & Tirosh, 2002) and influences what students learn from instruction (NCTM, 2000). The following sections presented a review on functions in secondary school mathematics as well as what is known about the difficulties students experience in learning the concept of function and teachers' familiarity of those student difficulties.

2.2 Functions in School Mathematics

The concept of function is one of the central underlying concepts in mathematics (Vinner, 1992) and it is essential in school mathematics (Dreyfus & Eisenberg, 1982; Harel & Dubinsky, 1992; NCTM, 1989, 2000; Selden & Selden, 1992). According to Dreyfus and Eisenberg (1982):

> The reason for this may be found in its unifying nature. For example, in many school curricula the function concept ties algebra, trigonometry, and geometry together. More than that, it appears and reappears like a thread throughout school mathematics from grade 1 (e.g., addition as a function from RxR to R) to grade 12 (e.g., calculus) (p. 361).

In the vision of secondary school mathematics, NCTM (2000) standards point that in grades 9 through 12 all students should, for instance, generalize patterns using explicitly and recursively defined functions; understand relations and functions and select, convert flexibly among, and use various representations for them; and interpret representations of functions of two variables. According to Markovits, Eylon, and Bruckheimer (1986), students should be able to: (1) classify relations into functions and non-functions, (2) give examples of relations which are functions, and of relations which are not, (3) (for a given function) identify pre-images, images and (pre-image, image) pairs, (4) find the image of a given pre-image and vice-versa, (5) identify identical functions, (6) transfer from one representation to another, (7) identify functions satisfying some given constraints, and (8) give examples of functions satisfying some given constraints (p. 181). Also, the Common Core State Standards (2010) provide a clear understanding of what students are expected to learn with respect to the function concept.

In the Turkish school mathematics program (TTKB, 2011) functions are first introduced in 9th grade (aged 15/16). In the 9th grade mathematics program, the learning areas are organized as follows: Logic, Sets, Relation-Function-Operation, and Numbers. The function topic, as is seen, follows the Sets topic, and the function concept is defined based on set theory. The objectives belonging to each topic are itemized, which is followed by related activities and details. With respect to functions, it is stated that students should be able to:

- Define the concept of function.
- Drawing its diagram, identify the domain, range, and image of the function.
- State equality of functions.
- Explain the types of functions.
- Explain the composition of functions through examples.
- Find the inverse of a function.
- Find the inverse from a graph.
- Locate the images of some given pre-images on the axes in graphs and vice versa.
- Interpret the behavior of the function in the given intervals.
- Find $f+g$, $f-g$, $f.g$, and f/g, derived from f and g functions defined from R to R. (TTKB, 2011, p. 67-68)

The 9th grade mathematics textbook (Milli Eğitim Bakanlığı [MEB], 2012) has been prepared in accordance with the program. In the textbook, the function topic is presented under the following subheadings: Functions, Types of Function, and Linear Function. This study focused on the Functions subheading. Two activities are used in this subheading of the book to introduce the function concept. Some visual representations are given in the first activity. These visuals highlight that function is a mechanism that converts inputs into outputs. In the second activity, a relation which corresponds four customers at a restaurant with dishes is defined, and thus it is highlighted that a function is a special relation which corresponds the elements in a set to those in another. After this activity, an example based on a set correspondence is provided, wherein two conditions are defined: (1) All the elements in Set A is to be corresponded to those in Set B and (2) Each element in Set A is to be corresponded to only one element in Set B. The relation realizing the first and the second conditions in the example is called as function. It is also underlined that each subset of $A \times B$ is a relation, and thus function is also a relation, yet not all relations are a function. In fact, ultimately it is stated that the set of customers in the second activity is called as the domain of the function, and the set of the dishes is called as the range of the function, and the set of dishes eaten is called as the image set of the function. The function concept is defined based on the set approach. The definition is exemplified through mostly the set correspondence representation. Types of function are defined by exemplification through the same representation. The process of finding the domain, range and image sets of functions, the equality of functions, and locating pre-images, images, and (pre-image, image) pairs on the axes in graphs are explained through examples.

Yavuz and Baştürk (2011) compared the Turkish curriculum and that accompanying textbooks with the French ones in terms of functions. This comparison pointed at serious limitations of the Turkish curriculum and the textbook. For instance, as regards the function concept, in the Turkish curriculum, the interaction between the Cartesian product, relation and function is emphasized. The objectives as to each concept are itemized, which is followed by related activities and details. In addition, basic function concepts are generally presented with functions defined in infinite sets, and then the focus solely shifted onto real numbers or infinite intervals, which are the immediate subsets of real numbers. The solution methods for certain problem types are provided (e.g., the vertical line test, the horizontal line test). Various representations of a function are considered one by one, and different objectives were specified for each representation. The French curriculum, however, indicates that function can be shown by different representations, and defined by means of the variable concept. It also draws attention to function and non-function situations in daily life, requiring the students to think about the word ‘function’ in daily life. The curriculum gives examples for students and has them define these examples in infinite sets, thereby preparing them for possible future concepts. The curriculum does not formulate any solution or technique regarding the objectives, leaving them entirely to the teachers’ initiative. Fewer but more comprehensive objectives were accompanied by short explanations. Apparently, they drew the general framework, leaving ample space for the teachers to do their own planning. In addition, the program includes the student mistakes and misconceptions that educational research on functions commonly reveals, which is not included in the Turkish program.

Yavuz and Baştürk (2011) pointed out that, in the Turkish textbook, only activities were utilized during the introduction to the concept of function. The necessary definitions, theorems, and explanations were given as end-notes after the exercises. Function was not defined or explained in any way whatsoever. Diagrams and algebraic representations dominated the introduction of the subject by 80%. On the other hand, little use of graphical representation was made, which was solely for visual purposes rather than the internalization of the concept. Activities that function as transition between different representations of the function were also included. However, there was no trace of any information about what to take into consideration in these transitions, what remains constant, and what changes. In a way, the role of these transitions in the internalization of the function concepts was overlooked. The book does not involve any interdisciplinary exercises or activities. By contrast, in the French textbooks, such headings as ‘Introduction’, ‘Methods and Examples with Solutions’ were used. What is more, the representations of the function were defined under separate headings and explained through examples. The representations were used in a balanced way. It was even observed that some textbooks refer more to graphical representations than algebraic representations. Explanations about the transitions between representations and what to focus on during these transitions were placed under related headings. Also included were activities and questions that involve real data and examine functional relations so as to highlight the uses of the function concept in different fields and help students internalize it (e.g., the distance taken by the taximeter relation).

As a result of analysis of both national and international contexts, the present study expected that at the completion of 9th grade instructional units on the function concept, the students should be able to: (a) demonstrate an understanding of the function concept, (b) demonstrate an understanding of the essential features of functions, (c) relate the domain and range to its graph, and vice versa, (d) identify two equal functions, and (e) locate pre-image, image, and (pre-image, image) pairs on the axes in the graphs.

2.3 Students' Understanding of the Function Concept

When students learn the concept of function, they pass through some stages: "…first, they learn that a function is composed of three sub-concepts: domain, range, and the rule of correspondence. Then, they learn that functions can be represented in several forms, such as arrow diagrams, verbal, graphical and algebraic representations" (Markovits et al., 1986, p. 179). Appropriate forms that the three sub-concepts take are shown in Table 2.1. Then, students learn that the same function may be represented by each of the above representations "… so they have to learn to translate a given function from one representation to another, dealing with the three sub-concepts and with two representations simultaneously… the students then go on to study linear and, … quadratic functions" (Markovits et al., 1986, p. 180). In this sense, it is reasonable to assume that if students develop a good understanding of the function concept, they can comprehend the knowledge of those concepts that are related to the concept of function (Hansson, 2006; Markovits et al., 1986). However, numerous studies that have been focused on students' understanding of functions have converged on similar conclusions: many students have difficulties in learning the concept of function. The present study described students' difficulties under the following three subsections: the definition of function, the essential features of function, and the concepts that are related to function.

Table 2.1: Representations of a function and its components (Markovits, Eylon, & Bruckheimer, 1986, p. 180)

representation / sub-concept	Verbal	Arrow diagrams	Algebraic	Graphical
Domain	verbal or mathematical notation	a curve enclosing the members of the domain	verbal or mathematical notation	the horizontal (x) axis or thereof
Range	verbal or mathematical notation	a curve enclosing the members of the range	verbal or mathematical notation	the vertical (y) axis or parts thereof
Rule of correspondence	verbal	arrows	formula	a set of points in the coordinate system

The definition of function: In several research studies, college and high school students were asked to provide a definition for the function concept (e.g., Breidenbach, Dubinsky, Hawks, & Nichols, 1992; Clement, 2001; Hatısaru & Erbaş, 2013; Lambertus, 2007; Nilklad, 2004; Tall & Bakar, 1991; Vinner, 1983; Vinner & Dreyfus, 1989). These studies suggested that students' understanding of functions appear to be too weak. In an earlier study, Vinner (1983) examined images and definitions that junior high school students have for the function concept. By a questionnaire, 146 students in grade 10 and 11 were asked in their opinion what a function is. Four main categories were distinguished from students' definitions. That is, students gave the textbook definition, they described the function as a rule of correspondence, they identified function as an algebraic statement, an equation etc., or they described the function as a graph or the symbols $y = f(x)$. To the same end, the same questions were asked to 271 college students and 36 junior high school teachers (Vinner & Dreyfus, 1989). Refining the categories of Vinner (1983), Vinner and Dreyfus (1989) classified the students' definition of a function into six categories as the following. According to Vinner and Dreyfus (1989), and also Vinner (1983), many of the participants' definitions and even the images were quite lower than the expected level.

> **Correspondence:** A function is any correspondence between two sets that assigns to every element in the first set exactly on element in the second set (the Dirichlet-Bourbaki definition).
> **Dependence Relation:** A function is a dependence relation between two variables (y depends on x).
> **Rule:** A function is a rule. A rule is expected to have some regularity, whereas a correspondence may be "arbitrary".
> **Operation:** A function is an operation or a manipulation (one acts on a given number, generally by means of algebraic operations in order to get its image).
> **Formula:** A function is a formula, an algebraic expression, or an equation.
> **Representation:** The function is identified, in a possibly meaningless way, with one of its graphical or symbolic representations (p. 359-360).

Similar results were appeared on later studies. Clement (2001), for instance, asked 35 high school students to furnish their own definition of a function. According to Clement (2001), only four students could give a definition of function, which recognizes that every element in the domain must be mapped to exactly one element in the range. Students' perceptions mostly were either an image of a machine (that is when numbers are entered, numbers are produced) or an image of a graph that passes the vertical line test. Tall and Bakar (1991) asked a group of twenty-eight high school students (aged 16/17):

“Explain in a sentence or so what you think a function is, if you can give a definition of a function then do so” (p. 2). Tall and Bakar (1992) stated that in the study none of the students could give satisfactory definitions but all gave explanations such as: “A function is like an equation which has variable inputs, processes the inputted number and gives an output.” and “A series of calculations to determine a final answers, to which you have submitted a digit” (p. 2).

In a recent study, Hatısaru and Erbaş (2013) investigated technical and industrial vocational high school students’ understanding of the function concept and their ability to classify relations as functions and non-functions. In the study, 130 tenth grade students (16-17 years old) were asked to define the concept of function in their own words. Results of this study showed that rather than providing a mathematical definition many students defined the function concept as ‘function’. To illustrate, one student stated that “what an instrument or a tool can do is called a function” and another one wrote that “function reminds me of feature, for example, the features of a car remind me of its functions” (p. 872). Many 9th grade students (aged 14/15) from Markovits et al.’s (1988) study gave linear examples when they asked to give examples of functions.

The essential features of function: The definition of function clearly suggests, every element in the domain should be paired with one but only one element in the range (univalence requirement) and a function does not necessarily realize the correspondence between the elements of the two sets through an arithmetical or algebraic rule (arbitrariness) (Even, 1990, 1993). Many students cannot recognize these features that are necessary for determining whether a relation defines a function or not (Akkoç, 2006; Akkoç & Tall, 2005; Clement, 2001; Leinhardt et al., 1990; Slavit, 1997; Vinner, 1983; Vinner & Dreyfus, 1989). Accordingly, they succeed with items that are considered prototypes of functions such as graphs, algebraic statements, and set correspondences (Akkoç, 2006), but they run into difficulty when the given function is less familiar (Tall & Bakar, 1991). For instance, many students have difficulty in many-to-one correspondence. They often require that the elements of two sets be in a one-to-one correspondence (Markovits et al., 1986, 1988; Vinner, 1983). Also, many have difficulty with piecewise defined functions. They often consider that functions given by more than one rule are not functions (Markovits et al., 1986; Vinner, 1983; Vinner & Dreyfus, 1989). Many students often assume that functions must consist of algebraic symbols relating x and y (Clement, 2001; Tall & Bakar, 1991; Sfard, 1992). Thus, they do not view constant functions as functions (Markovits et al., 1986). Many students, also, possess inaccurate ideas about graphs of functions (Markovits et al., 1986; Vinner, 1983; Vinner & Dreyfus, 1989). They are unwilling to consider irregular graphs as graphs of functions (Leinhardth et al., 1990), and often believe that graphs of functions should be continuous and graphs of functions exhibit a linear pattern (Markovits et al., 1986).

The concepts that are related to function: The concept of function has an extensive set of sub-concepts (Dreyfus & Eisenberg, 1982; Markovits et al., 1986). Many students often do not identify and describe various characteristics of functions and concepts which are related to the concept of function. They have difficulty in identifying domain and range of functions. In addition, many have difficulty in understanding that the set of images may be a subset of the range (Markovits et al., 1986, 1988). In the graphical representation, many students do not appreciate that the x axis represents the domain and the y axis the range, whereas the point on the graph represent (pre-image, image) pairs (Markovits et al., 1988).

Review of the literature on students’ understanding of the function concept, however, has revealed that, with few exceptions (Even & Tirosh, 1995; Postelnicu, 2011; You, 2006), there is a more emphasis on documenting the kinds of difficulties and misconceptions that students exhibit than teachers’ familiarity of student difficulties. By contrast, the present study investigated mathematics teachers' knowledge of student difficulties in the function concept. Teachers' knowledge of student about a specific mathematical topic is strongly related to teachers' content knowledge of this topic (Australian Education Council, 1990; Even & Tirosh, 1995). Accordingly, the study turned from student understanding to the aspects of content knowledge of functions.

2.4 The Aspects of Content Knowledge of Functions

Norman (1992) reported a number of general aspects of teachers' content knowledge of function, among which are: (1) exemplification and characterization of functions, (2) the ability to use functions in a variety of ways and contexts, and (3) the expression of functional reasoning (p. 217). According to Norman (1992), exemplification and characterization of functions are indicators of one's breadth and depth of understanding of functions and include the ability to define, exemplify, and characterize functions. That is, teachers should exhibit an understanding of formal definition of function; identify and describe various characteristics of functions (such as continuity) and related concepts (such as domain and range); recognize conditions which are necessary and sufficient for determining the functionality of a relation; and provide counterexamples to a given false generalization about functions. To Norman (1992), other important part of the knowledge of function is application of functions. Teachers should relate to applications of the concept in a variety of situations and construct situations in which functions are essential components or which can be described by functions. The other aspect of teachers' understanding of functions is functional reasoning. According to Norman (1992) teachers should deduce properties or generalizations related to functions; analyze and interpret mathematical situations involving graphical or algebraic presentations of functionally related information; communicate about functional situations; and use functions to extend their knowledge about a mathematical concept, process, or situation.

Even (1989) identified six very important facets of content knowledge of function. Based on research on related literature, in a later study, Even (1990) proposed a theoretical framework of subject matter knowledge for teaching the function concept. The framework consists of seven aspects. Each of these aspects is described as follows:

Essential features-what is a function? With reference to Freudenthal (1983), Even (1989, 1990) considered *arbitrariness* and *univalence* to be the essential features of the concept of function. The arbitrary nature of functions is implicit in the definition of function. It refers both the relationship between the two sets on which the function is defined and the sets themselves. That is, functions do not have to be defined on any specific set of objects, in particular sets of numbers. Furthermore, functions do not have to be described by any specific expression or described by a graph with any particular shape (Even, 1990).

Contrary to the arbitrary nature of functions, the univalance requirement is explicitly stated in the definition of function. Univalance requirement is the cornerstone of function and so important for the understanding of the concept (Even, 1990). Also, it helps to distinguish between relations that represent functions and non-functions (Even, 1989). In almost every text, this requirement is emphasized and usually, it is presented to students as one of the most important characteristics of functions (Even, 1990).

Different representations of functions: According to Even (1990), functions appear and behave in different ways. With reference to Freudenthal (1983), she pointed to the different labels functions have in mathematics such as: mapping, permutation, operation, etc. For Even (1990), many functions have specific names and use specific notations such as: trigonometric functions; sin, cos, tan, etc., and exponential (exp) and logarithmic (log) functions. In addition to having various classes of functions, the same function can appear in different representations (Common Core State Standards, 2010; Even, 1989, 1990; Selden & Selden, 1992). The most common representations are graphs, verbal rules, algebraic expressions, set correspondences, and sets of ordered pairs. These representations play an important role in the understanding of the concept. That is, a more complete understanding of functions means understanding it in different representations (Even, 1989).

Alternative ways of approaching functions: Even (1989, 1990) proposed two different ways of approaching functions: *point-wise* approach and *global* approach. Reading values from a given graph or dealing with discrete point of the function are examples of a point-wise approach to functions. According to Even (1990), it is not hard to learn point-wise approach. However, this approach is not appropriate for all situations. There are times when one has to consider the function in a global way, and look at the behavior of the function. The global approach to functions is more powerful than point-wise approach.

The strength of the concept-the inverse function and the composition of functions: According to Even (1990), in addition to the typical algebraic operation (addition, subtraction, etc.) functions can also be converted and inverted. Even (1990) asserted that compose and invert functions help with create new functions and study of differentials and integrals. However, the inverse function and the composition of functions cannot be understood in one simplistic way only (Even, 1990). Understanding the inverse function, for instance, requires understanding the formal mathematical definition of the function (Even, 1989).

Basic repertoire-functions of the high school curriculum: Even (1990) reported that every high school teachers should have a basic repertoire of functions including linear, quadratic and general polynomial; exponential and logarithmic; trigonometric and rational functions.

Knowledge and understanding of the function concept: For Even (1989), teachers' knowledge of functions must go beyond procedural knowledge. Teachers' knowledge should rely more on conceptual knowledge and meanings. Rich relationships characterize conceptual knowledge.

Analyses of the works of Even (1990, 1993) and Norman (1992), Lloyd and Wilson (1998) provided the parts of knowledge that have been identified as essential for teaching the function concept as following: definition and image of the function concept, repertoire of functions in the high school curriculum, the importance and use of functions in varying contexts, and multiple representations and connections among them.

It has been reported that the works that cover every aspect of the concept of function will give a comprehensive picture of many aspects of the concept, but these works will be very general and miss a lot of details of each aspect. Researchers, therefore, may choose to concentrate on only one aspect and study it deeply or choose a manageable of the important aspects of the concept and try to illustrate a general picture including details on these aspects (Even, 1989). The present study chose to concentrate on what a function is and attempted to provide teachers' KCS on this aspect.

2.5 Teachers' Knowledge of Content and Students about the Function Concept

The following two sections provide a review of literature on both pre-service and in-service teachers' content knowledge of functions and also teachers' knowledge of student difficulties about the function concept.

2.5.1 Teachers' Content Knowledge of the Function Concept

Current research literature on pre-service teachers' content knowledge of functions has shown that pre-service teachers' knowledge tends to be weak; such that they lack a deep, integrated understanding of the function concept. Even (1989), for instance, described pre-service teachers' knowledge and understanding about mathematical functions and pointed to some of the limitations of their conceptions. In this research, six aspects of teachers' subject matter knowledge (what is a function?, different representations of functions, inverse function and composition of functions, functions of the high school curriculum, different ways of approaching functions: point-wise, interval-wise, global and as entities, and different kinds of knowledge and understanding of function and mathematics), and two aspects of their PCK (teaching toward different kinds of knowledge and understanding of functions and mathematics, and students' mistakes) were studied. Data were collected in two phases. An open-ended questionnaire was administered to 152 participants first. Then, interviews were conducted with ten participants. According to Even (1989), the results showed that the participants had several misunderstanding about what a function is. Many of them had a limited and old concept image of a function, i.e. functions to always be represented by an equation. Most of the participants knew about the univalent property of functions, however, they ignored the arbitrary feature of the function concept and did seem to expect functions to be defined on numbers only. Some expected all functions to be continuous, and some expected graphs of functions need to be nice. Most of them could not make good connections between different representations of a function.

Similar results have been replicated in later studies. McGehee (1990), for example, used the questionnaire Even (1989) had used to examine three components of prospective teachers' knowledge about functions: concept definition, concept identification, and concept representation. The questionnaire was administered to 19 prospective teachers. Based on their responses to the questionnaire, six participants were chosen for the interview phase. By examining the questionnaire and the interview responses, McGehee (1990) concluded that:

> …prospective teachers had developed procedural knowledge and instrumental understanding. There were several inconsistencies in subjects' responses which indicated that many subjects did not have a strong relational understanding of the function concept… There was evidence that the prospective teachers in this study did not always apply a definition to identify examples of functions from a list of 14 items… Their definitions for students reflected a procedural method for identifying examples of functions rather than the essential feature of functions (p. 169-171).

Breidenbach, Dubinsky, Hawks, and Nichols (1992) investigated pre-service teachers' understanding of functions. The researchers implemented the study with 62 pre-service teachers in a one-semester course on discrete mathematics. The participants were asked to respond to the question what a function is. Then, they were asked to give examples of a function. Participants' responses to the first question were grouped into three main categories: pre-function, action, and process. A pre-function response was "…one in which it appears that the student does not have very much of a function concept at all." (p. 252) Responses such as "I don't know" or "a mathematical equation with variables" were assigned as pre-function. An action response was "…that emphasized the act of substituting numbers for variables and calculating to get a number…" (p. 252). Responses such as "a function is something that evaluates an expression in terms of x" or "a function is a combination of operations used to derive an answer" were assigned as action. In a process response "…the input, transformation, and output were present, integrated and fairly general" (p. 252). Responses such as "a function is some sort of input being processed, a way to give some sort of output" or "a function as an operation that accepts a given value and returns a corresponding value" (p. 252) were assigned as process. Breidenbach et al. (1992) reported that of all, 40% of the participants' ways of thinking about functions were pre-function, 24% of the participants' were action, and 14% of the participants' were process. The researchers added that all of 26% participants' choice of examples indicated a pre-function conception, 67% participants' were action conception, and 3.7% participants' were process conception.

Bolte (1993) explored the extent of pre-service secondary mathematics teachers' subject matter knowledge and PCK of functions, and described how they envision applying their content knowledge of functions in various classroom situations. The components of subject matter knowledge that were studied were definition and identification of functions, facility with different representational forms, and knowledge of the inverse function. Like Even's (1989) and McGehee's (1990) studies, this study also was completed in two phases. Phase I consisted of two concept maps and a survey of function concept. A group of 17 prospective teachers participated in this phase. Phase II consisted of two interviews. The first one focused on envisioned application of content knowledge of functions within classroom situation, while the second explored teachers' content knowledge. Based on their level of success in Phase I tasks, eight prospective teachers participated in the second phase of the study. Bolte (1993) stated that several general trends were evident when analyzing teachers' responses. Teachers used the vertical line test, for instance, to discriminate between examples and non-examples of functions when the relations were given graphically. Some did not consider the functions defined on a discrete set of numbers to be function and justified their decision as "it is only a list of points, not a function or rule" (p. 112). Some correctly identified the discrete sets as functions, but "…based their decisions on a graph in which the points were joined to form a continuous curve that passed the vertical line test" (p. 112). The researcher added that when the teachers listed examples of functions students would encounter in the secondary school curriculum, they preferred algebraic and graphical representations. Only two gave examples of real world situations. In a similar vein, no participants suggested a real world situation as an alternate form of functions. Analyses of the concept maps constructed by these seventeen participants indicated that:

> There was a marked difference in the extent and organization of knowledge of functions exhibited by the 17 original participants. On the one hand, virtually all participants were familiar with the vertical line test, comfortable with the terms domain and range, and familiar with linear and quadratic functions; they also identified routine graphical functions with relatively high proficiency. On the other hand, identifying less routine functions, distinguishing between univalence and one-to-one, and lack of familiarity with logarithmic and exponential functions presented serious difficulties for a number of participants (Bolte, 1993, p. 251).

In the study, participants who possessed more degrees of integration of their content knowledge about functions were considered for the interview sample. Bolte (1993) reported that interview analyses showed how a selected group of prospective teachers thought about functions. That is, most agreed that functions are an important topic for secondary mathematics students. Six of them provided a well-defined, accurate definition of a function. Four provided a valid justification for whether each relation from survey defined a function. Most of them had an adequate knowledge of graphical and algebraic representations of functions within the secondary curriculum including linear and quadratic functions. Most were aware of arrow diagrams and tables as different forms of representing functions. According to Bolte (1993), however, for most prospective teachers, the procedural aspects of working with functions were less difficult than conceptual aspects.

Hacıömeroğlu (2006) examined two prospective secondary mathematics teachers' subject matter knowledge and PCK of the concept of functions, and also the relationship between them. During six weeks of data collection, the prospective teachers participated in tasks including addressing different features of the function concept, organizing the use of different representations, and depicting mathematical problems. Even's (1990) framework was used to analyse and assess the participants' subject matter knowledge and pedagogical content knowledge. The results showed that the prospective secondary mathematics teachers mentioned univalence feature of the function concept, but they did not have much knowledge on the arbitrary nature of functions. They excessively used the vertical line test to determine whether a relation is a function or not. They tried to transform the relations given in different representations (e.g. an equation or a table) into graphical representation to determine their functionality by using vertical line test. On the other hand, they failed to give further explanations on why the test worked or what it meant to fail the test. In addition, they experienced difficulty when determining functions that given verbally. The researcher found the participants' subject matter knowledge weak. He added that the participants' weak knowledge resulted in weak PCK and also inappropriate organization of lesson plans. In a similar vein, Agarwal (2006) and Karahasan (2010) reported pre-service teachers' lack of knowledge in their studies which explored pre-service secondary mathematics teachers' content knowledge and PCK of functions.

The part of research on in-service teachers' content knowledge of functions has shown that, like pre-service teachers, in-service teachers' content knowledge of functions is less developed than desired. For instance, Duah-Agyeman (1999) described the understanding of mathematical functions held by a selected group of in-service secondary teachers (grades 8-12), in light of Even's (1990) subject matter knowledge of functions and Vinner and Dreyfus's (1989) categories for the definition of functions. A group of 11 teachers, whose teaching experience ranging from five to twenty-five years, were participated in the study. Qualitative research tools such as observations, review of written tasks, interviews, and researcher's notes were used to collect data. Regarding Even's (1990) theoretical model of understanding functions, the study reported that in participants' definition of functions they either accepted or rejected the essential features of functions. From their definitions alone, five accepted the essential features of functions. One accepted the univalence feature, and her explanation of a function to a student where she used two sets of different elements other than sets of numbers was evident for her acceptance of the arbitrary feature. On the other hand, two participants failed to accept the arbitrary feature of functions, and one did not allude to them at all in the definition. The researcher reported that while their definitions not have indicated so, however, the rest of the tasks indicated their familiarity with the univalence feature. Also, their concept image of the vertical line test gave an indication of their acceptance of this feature. The participants' responses indicated an acceptance of arbitrary feature of functions, when they required giving real life examples of functional relationships. When they determined whether given graphs were functions, they commonly used the vertical line test.

Norman (1992) conducted in-depth interviews with eight mathematics teachers. The teachers were first asked to provide informal meaning for function and then a formal, mathematical definition. After having given a formal definition, they were asked to explain how the particular informal examples given earlier reflected the formal definition. Then, they asked to identify the functionality or non-functionality of various algebraic expressions, numerical data sets, graphical representations, and physical situations. Teachers were then requested to describe how they typically introduce the function concept for their students. Finally, the teachers were asked to interpret application problems. Norman (1992) stated that the teachers generally favored informal definitions of function and graphical representations. They often exhibited a single concept fixation when interpreting functions and had not build strong connections between their informal definitions of function and what they view as the formal mathematical definition. According to the researcher, the results of the study did not showed "a uniformly deep understanding of the function concept among the participating teachers, in spite of the fact that each had had considerable experience in teaching the concept, as well as having worked with functions at a fairly sophisticated mathematical level" (p. 229).

Hitt (1998) stated that like secondary school students, mathematics teachers make mistakes when they work on the function concept. The study looked for these mistakes that committed by mathematics teachers. A series of 14 questionnaires were prepared and implemented to 30 secondary level mathematics teachers. One questionnaire, for instance, presented the teachers with 26 curves. The participants were asked to indicate whether the graphical representations define a function and required a reason for their responses. A group of 29 teachers said that the graph presented left hand side in Figure 2.4 did not represent the graph of a function. Most grounded it on the vertical line test, while few explained it by the definition of function.

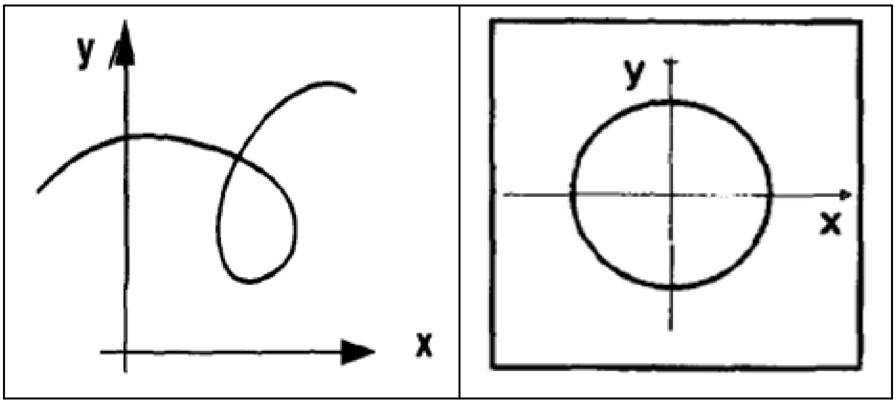


Figure 2.4: Errors and abstentions linked to conic curves (Hitt, 1998, p.126)

When they were shown conic curves like one of them presented right hand side in Figure 2.4, none of them used the definition of function or explicitly used a vertical line in their reasoning. Most of them stated that the shown conic defined a function. According to Hitt (1998), teachers' conception of the function concept was that functions represented with analytic expressions. Cooney (1992) reported a similar result. In a survey, 200 experienced secondary teachers were asked to write a question that could reveal students' thorough understanding of functions. Many teachers wrote a question that requires students to solve equations (as cited in Cooney, 1999).

In the same study, through a questionnaire, the teachers were presented with four different definitions of the function concept. They were asked to decide whether the definition(s) given was correct or incorrect and classify them in preference of teaching. Hitt (1998) reported that a group of 18 teachers gave their definition in terms of a correspondence and ten in terms of ordered pairs, none in terms of a relation between variables. From the teaching perspective, the teachers' preference was mostly correspondences (n=14) and sets of ordered pairs (n=13).

In the study, another questionnaire was designed to detect the teachers' possible mistakes when identifying equal functions. The teachers were asked whether $f(x) = 2$ was equal to $g(x) = \sqrt{4}$ for all $x \in R$. Their responses were mostly satisfactory. However, they made mistakes when they worked on the sub-concepts of the function concept. In one questionnaire, they were asked to identify some points on the graph of a function. They were successfully located the points in arrow diagrams, but only some of the teachers identified the points on graphical representation (e.g., see Figure 2.5).

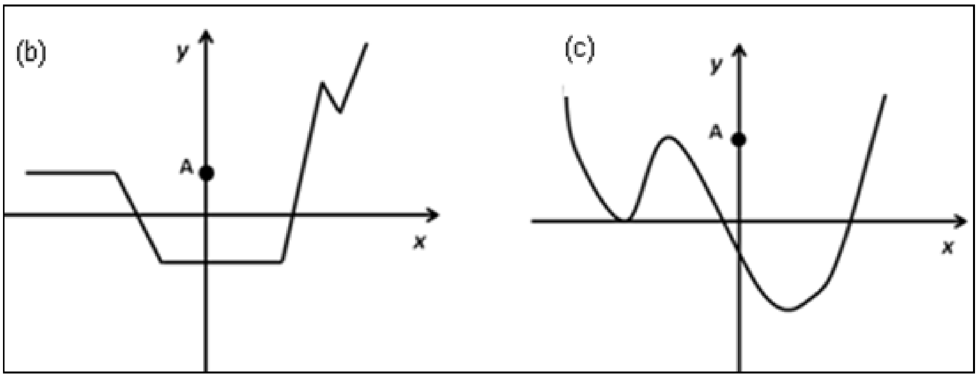


Figure 2.5: Sub-concepts-Domain (Hitt, 1998, p. 130)

Different from above studies, Stein, Baxter, and Leinhardt (1990) examined connections between an experienced (18 years) fifth grade teacher's (Mr. Gene) understanding of functions and graphing and his classroom instruction. Data were collected through observations, an interview, and a card sort task. In the study, a mathematics educator also was interviewed and given the card sort task. His knowledge was used to represent an expert knowledge. The interview consisted of a request for a definition of function, comments regarding the importance of functions and graphing in mathematics, a series of open-ended questions about the topics of functions and graphing, and their instruction at the elementary level. The card sort task consisted of 20 cards. A mathematical relationship is depicted on each card. The participants were asked to categorize the cards into group and comment on each group. After one grouping, they were asked to categorize the cards in a different way.

Mr. Gene's subject matter knowledge divided into three sections: definitions of function, purpose of teaching and graphing, and organization of knowledge. The comparative analysis between the two participants' definitions of function showed that Mr. Gene's subject matter knowledge had lack key ideas. Both Mr. Gene's and the mathematics educator's definition included two essential features of the function concept: "two interrelated entities" and "one entity depends on the other" (p. 646). Some additional essential features of the concept, however, referenced only by the mathematics educator such as the two entities may be related to one another with or without a rule, or each element of the first set can be related to one and only one element on the second set. Mr. Gene's reason for teaching functions was arithmetic in nature. He suggested that functions and graphs should be taught because, "…graphs can be used to check the answers to function machine problems" (p. 648).

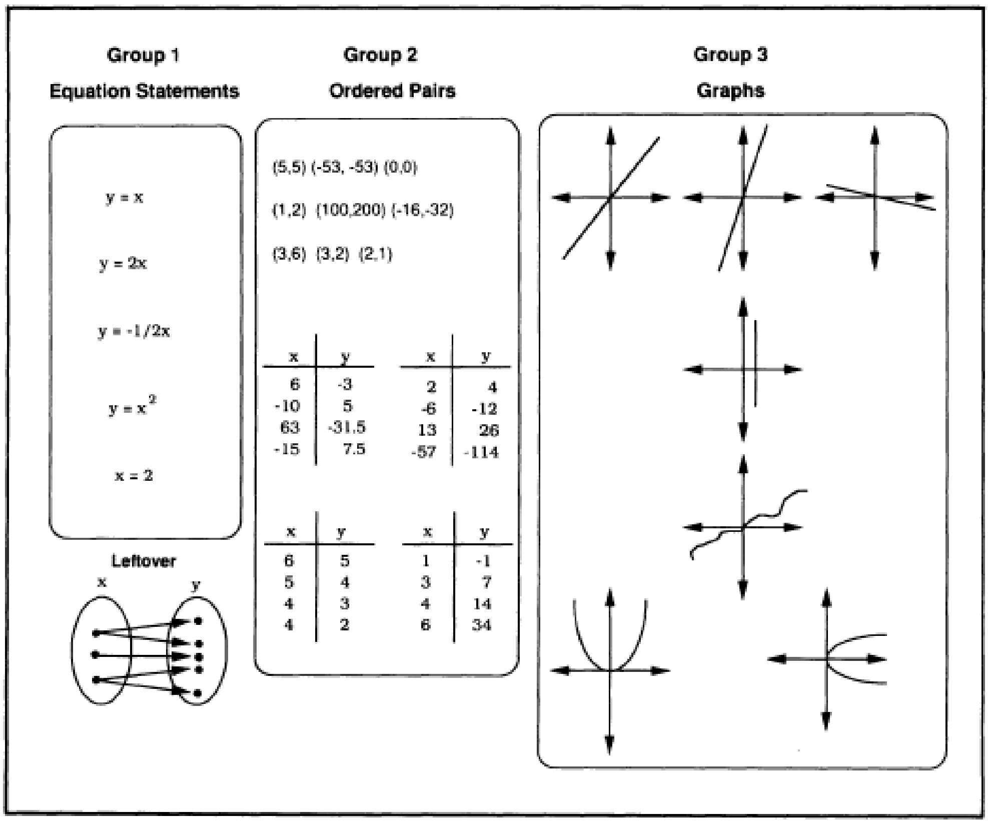


Figure 2.6: Groups into which Mr. Gene sorted the cards (Stein, Baxter, & Leinhardt, 1990, p. 648)

For Stein et al. (1990), the comparative analysis between their card sorts revealed a significant difference in how they organized their knowledge of functions and graphing, and also suggested that the idea that mathematical relations can be formed in different way was missing in Mr. Gene’s knowledge. He used the sole the criterion of representational format of relations. His only arrangement was grouping all equations together, all ordered pairs together, and all graphs together (see Figure 2.6). He did not view algebraic equations and graphs as alternate ways of representing functions. As for the mathematics educator, he “… sorted the cards twice. Both of her arrangements began with the distinction between functions and non-functions” (p. 649) (see Figure 2.7).

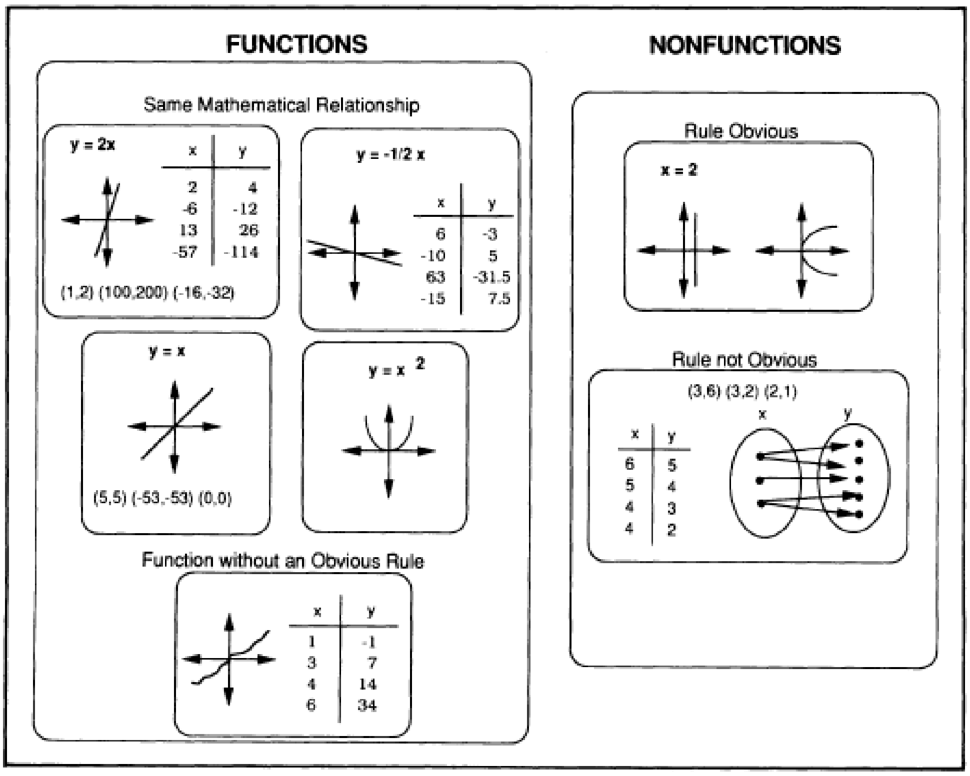


Figure 2.7: Groups into which the mathematics educator sorted the cards (Stein, Baxter, & Leinhardt, 1990, p. 650)

Also, Howald (1998) examined the extent and organization of experienced secondary mathematics teachers' subject matter and PCK of functions and characterized how their conceptions are applied in the classroom. The participants' years of teaching experience was with an average of 15.4 years. The study consisted of two phases: a survey study of the extent and organization of 20 teachers' knowledge of functions and two case studies that examined the way this knowledge manifests itself in the classroom. Data were collected through a survey of function knowledge, an interview based on the survey, concept maps, card sort tasks, and classroom observations. The results of the study showed that compared to pre-service teachers, experienced in-service teachers had a better understanding of functions, but some did not demonstrate a deep understanding of the function concept. Four teachers, for instance, provided a correspondence definition, but they were not able to apply this definition when identifying functions. According to Howald (1998), many teachers had not built a strong connection between their formal definition of function and criteria for identification of functions. This group of teachers had difficulty identifying functions in situations described verbally. In the study, also, the teachers were asked to give an alternate definition of a function. Most provided a modern definition of function (the correspondence definition) and indicated that the modern definition would be the definition used with students. Most teachers provided an analogy for functions that they would use with functions. For Howald (1998), the quality of the aspects they shared with the function concept was varied. To illustrate, some analogies highlighted only the fact that the machine, such as a meat grinder, changed the input. These analogies put little regard for other aspects of functions, such as the dependence of the output on the input.

2.5.2 Teachers' Knowledge of Student Difficulties in the Function Concept

Although considerable work has been completed on teachers' content knowledge of functions fewer studies have focused on one of the essential aspects of teachers' PCK about the function concept, knowledge of student difficulties. One of these studies (Even, 1989) described two aspects of pre-service teachers' PCK about functions: teaching toward different kinds of knowledge and understanding of functions and mathematics, and students' mistakes. According to Even (1989), the results showed that when defining the concept of function for students, the participants tended to use old definition of the concept. For the participants, "a very popular illustration for students what a function is, was to describe it as a machine or as a black box" (p. 229). For the graph of functions, many prospective teachers chose to use vertical line test to explain students what a function is, but without relating it to definition of the function concept. Concerning knowledge of students' mistakes, Even (1989) stated that the participants did seem to be aware of the common misconceptions that students have about functions, but the participants who had limited knowledge of functions had difficulties providing explanations for students' mistakes. It is concluded that "subject matter content knowledge seemed to be related to the explanations provided" (p. 231).

Bolte (1993) explored the extent of pre-service secondary mathematics teachers' PCK of functions, and described how they envision applying their content knowledge of functions in various classroom situations. The components of PCK were definitions suggested for use with students, ability to generate varied examples and non-examples, and potential use of different representational forms. The study found that most pre-service teachers were able to identify students' mistakes in graphing rational functions, determining the domain and range of functions, and the inverse functions. According to Bolte (1993), however, for most prospective teachers, analyzing sources of students' mistakes was more difficult for them.

Ebert (1994) investigated PCK of four prospective secondary mathematics teachers pertaining to functions and graphs. The researcher explored prospective teachers' knowledge of students' conceptions of functions and graphs through a vignette tasks and the follow-up interview. The content of each vignette was related to definition and notation of the function concept, the composition of functions, and the inverse of function. In this task the participants were asked to response some scenarios of students' misconceptions about functions and graphs that described in the literature. In examining the data, two dimensions emerged—whether the prospective teachers exhibited any misconceptions themselves and the quality of their responses to the student's conceptions and/or misconceptions. Ebert (1994) reported that on these vignettes, their responses revealed that two of them (Penny and Beth) experienced misconceptions about core conceptions concerning the concept of function, the composition of functions, and the inverse function, while two (Sam and Mark) indicated strong conceptual understanding of functions and graphs. According to Ebert (1994), Penny's misconceptions were as pervasive. In her case, she did not discern the students' dilemma. Furthermore, in some cases, her responses would have confirmed the misconceptions. Her "…procedural admonitions represented the standard pedagogical response rather than a genuine invitation for the students to engage in mathematical discourse" (p. 330). Beth's misconceptions were not as pervasive and for the majority of the vignettes, Beth's responses indicated an appreciation for instructional strategies that would enable students to build upon their own conceptions of function. To Ebert (1994), other two prospective teachers' responses indicated their own strong conceptual understanding of functions and graphs. Their comments revealed that they gave importance to engage students in mathematical discourse and sense-making. For Ebert (1994), "this task provided the initial opportunity for them to reveal how their knowledge would relate to their responses to hypothetical students" (p. 329).

Even and Tirosh (1995) concentrated on prospective teachers' knowledge of students about a different mathematical domain and their understanding of possible reasons of students' responses. They gave illustrations in functions and undefined mathematical operations (e.g., $4/0$, $0/0$). The researchers reported that many pre-service teachers found it difficult to explain students' way of thinking in those two mathematics domains, and added that many teachers did not try to examine the students' way of thinking also they found it difficult to explain why the students reacted that way.

These results were consistent with You's (2006) study of two aspects of prospective teachers' PCK about linear functions: knowledge of students' conceptions and misconceptions, and teaching strategies for helping students' misconceptions. The results of the study showed that most of the pre-service teachers did not perform well on understanding students' misconceptions with respect to linear functions. In addition, the teachers "were not able to provide effective strategies if they did not know the nature and sources of students' mistakes. With limited knowledge of students' misconceptions, their strategies tended to be general and not mathematics content specific" (p. 147).

You (2006) indicated that there may be several reasons why prospective teachers had problems with identifying and solving students' misconceptions. Firstly, the prospective teachers' experience with students is very limited. Sánchez and Llinares's (2003) study supported this view and claimed teacher education should include knowledge of understanding of students a particular mathematical subject and their common difficulties. Also, Cha's (1999) study, which searched the nature of pre-service teachers' knowledge of functions and their preferred definitions of function for teaching, mentioned that pre-service teachers' PCK was evolving as a result of their learning experiences. The vertical line test definition, for example, was very popular before the unit, but not so after the functions unit. Some pre-service teachers indicated that "…they would teach the vertical line test not as a type of definition but as a necessary (though still insufficient) elementary test for determining function" (p. 182). Secondly, the prospective teachers may have problems with the particular topic themselves, thus, "it is hard to imagine teachers could help students' misconceptions if they themselves had trouble with the topic" (p. 147).

In previous studies, with few exceptions, in-service teachers' knowledge of students with respect to functions has been ignored. Postelnicu (2011) identified secondary school students' difficulties about linear functions and assessed their teachers' understanding of the nature of the difficulties experienced by the students. The study employed a large group of Grades 8-10 students (n=1561) enrolled in mathematics courses from Pre-Algebra to Algebra II and their mathematics teachers (n=26). A Mini-Diagnostic Test (MDT) and a Ranking Questionnaire for Teachers (RQT) about linearity and linear functions were designed. All participants completed MDT test to rank the problems by perceived difficulty and comment on the nature of the difficulties. The teachers were completed RQT to comment on the nature of their students' difficulties. Interviews were conducted with 40 students and 20 teachers. The researcher reported that the teachers' written comments on the RQT and explanations to interview questions revealed their weak understanding of student difficulties with important topics like slope or rate of change. As the researcher pointed out:

> … with respect to students' and teachers' assessments of the nature of problem difficulty, students from Group 1 [students enrolled in courses above their grade level] were better able to describe the mathematical nature of their difficulties than were their teachers. Teachers' assessment of problem difficulty was based on the "look of the problem," that is, student familiarity with the problem, number of sub-problems in the problem, and format of the problem (e.g., extended response, requiring explanations). Without prompting, teachers did not attend to the mathematical nature of actual student difficulties. When prompted by the researcher during interviews, teachers commented on student difficulties regarding the correct application of an algorithm, a formula, or the reading of a graph, and not on conceptual difficulties like conceiving of slope as a measure of steepness in a geometric context. …. "I beat that

> topic to death," commented one teacher, surprised that students still had difficulties with slope after being taught for a long period of time (8-10 weeks). (p. 226)

Teachers' knowledge of students about different mathematical content was also examined. Watson, Callingham, and Donne's (2008) study, for instance, asked 1205 students to respond two chance and data problems involving proportional reasoning, and a group of 44 teachers to suggest interventions for four typical incomplete or inappropriate student responses. Results showed that it is difficult for these teachers to know what questions to ask students or to generate discussion without directly tell the students the answer. Asquith, Stephend, Knuth, and Alibali's (2007) study with 20 middle school teachers reported that teachers mostly could not identify students' misconceptions about core algebraic concepts and variables. Erbaş (2004) investigated two in-service secondary teachers' knowledge of student difficulties and teaching practice in algebra. The study made the distinction of knowing as 'knowing-about' and 'knowing-to'. The study reported that both teachers showed an awareness of student difficulties concerning 'knowing-that', but their knowledge was limited as to 'knowing-why' and 'knowing-how'. The other aspects of the teachers' limited knowledge were "their lack of arithmetical and geometrical knowledge base, lack of motivation, lack of experience with nontraditional curricula, lack of practice in similar type of problems, carelessness, and inability to understand and apply definitions" (p. 262). According to the study, the teachers' such limited knowledge might have limited their PCK accordingly. In addition, the study reported that the both teachers' teaching practice were dependent on the textbook at the stages of planning lessons, assigning homework, and assessing students' learning. To the study, such dependence might negatively impact on teachers' acquisition of student thinking.

Cunningham (2005) searched a different aspect of the issue. According to Cunningham (2005) students have difficulties with transferring among algebraic, numeric, and graphic representations. The researcher surveyed algebra teachers (n=28) to determine the amount of instructional time they dedicate to different types of transfer problems. Results indicated that the teachers gave little time to solve the problems that the students experience the most difficulty.

Some studies revealed that pre-service teachers' unfamiliarity with student difficulties results from their lack of content knowledge (e.g., Halim & Meerah, 2002; Kılıç 2008; You, 2006). In a study with 12 secondary science trainee teachers, Halim and Meerah (2002) explored trainee teachers' awareness of students' misconceptions about the physics concepts in lower secondary school science. The study found that the teachers' knowledge of students namely was depended on their understanding of the content knowledge. The researchers reported that the majority of twelve trainee teachers had problems in understanding the scientific ideas themselves, and those teachers who gave incorrect answers were less likely to be aware of students' misconceptions. You's (2006) study reported that pre-service teachers' weak knowledge of linear functions limited their understanding of students' misconceptions but who performed better at the representation flexibility tended to understand students' misconceptions and their sources better.

2.6 Interrelation between Teachers' Knowledge and Student Learning

Over the years, educational researchers have investigated many factors considered to affect student learning. A growing body of research reported the teacher as one of the most important school factors influencing student learning (Darling-Hammond, 2000; Lerman, 2001; Wright, Horn, & Sanders, 1997). The effectiveness of teachers rests on the knowledge needed for teaching that they possess (NCTM, 2000). For decades, therefore, an extensive body of research has focused on to identify the relations among teachers' knowledge and student learning (Mewborn, 2003).

The relations have been explored by different approaches. Earlier studies was mainly quantitative; they aimed to demonstrate the impact of teachers' knowledge on student achievement by using variables such as teachers' educational level, years of teaching experience, number of undergraduate mathematics or mathematics education courses. In general, no significant relationship between these teacher variables and student achievement has been found (Begle, 1979; Monk, 1994; Schoen, Cebulla, Finn, & Fi, 2003). Begle (1979) sought the relation between students' mathematics achievement and teachers' mathematical knowledge by using proxy variables including teachers' mathematics credit beginning with calculus, credits in mathematics, and majoring or minoring in mathematics. No evidence was found in this study to suggest a significant positive relation between teachers' mathematical knowledge and students' achievement. Indeed, Monk (1994) found that the relationship between the number of mathematics courses a teacher had taken and student achievement was not linear, i.e. the effect of mathematics courses on student achievement diminished beyond five or more courses. In their studies, both Begle (1979) and Monk (1994) found evidences to suggest relationship between the number of mathematics education courses and student achievement in secondary school level. That is, the number of mathematics education courses was more positively correlated with students' achievement gains that the total number of mathematics courses.

Another example of a study in this tradition is Wayne and Youngs's (2003) review of 21 studies. The researchers found that when students' SES was controlled, degrees in mathematics significantly interact with high school students' achievement but the same interaction was not present with elementary school students. Similarly, the National Mathematics Advisory Panel (2008) found that teachers having majors in mathematics had a positive effect on high school students’ learning but this result was not applicable to elementary students.

Another body of research used direct measures of teachers’ knowledge. In one of studies involving this approach, Rowan, Chiang, and Miller (1997) hypothesized that a teacher’s specific subject matter knowledge has effect on students’ performance. The researchers used the National Education Longitudinal Study of 1988 (NELS: 88) data and tested the effects of teachers on student achievement in mathematics. To assess teachers’ subject matter knowledge, two measures were used: (1) teachers’ responses to a single-item mathematics questionnaire, and (2) whether a teacher majored in mathematics at the undergraduate and/or graduated level or not. Also, the effects of teachers’ ability, motivation, and work situations on students’ achievement were studied. The results showed that students’ achievement in mathematics was directly affected by teachers’ knowledge of subject matter. The effects the researchers found were small, but they found two additional considerations suggest that they are important. First, the effects size of the teaching variables on students’ achievement was found statistically significant especially for students in schools with high percentages of students who were low achievers. Second, teaching ability and motivation had larger effects on students’ achievement in just schools where students entered with lower level of achievement.

Recently, Hill et al. (2005) investigated the impact of teachers’ mathematical knowledge for teaching on student learning. In this study, however, teachers’ mathematical knowledge was assessed via a questionnaire focusing on the specialized mathematical knowledge used in teaching mathematics. A total of 1190 first and 1773 third grade students, and a group of 334 first and 365 third grade teachers participated in the study. Student data were obtained from student assessments and parent interviews. The results showed that the effects of teachers’ mathematical knowledge for teaching on the 1st and 3rd grade students were positive. The researchers concluded that more teaching-based measurements are related to student achievement.

These results showed that some factors that are not related to teachers' mathematical content knowledge play in teachers' effectiveness. Another line of research concerning teachers' effectiveness has, therefore, focused on exploring teachers' knowledge in teaching environment. This view claimed that mathematical knowledge for teaching is different from mathematical content knowledge. It comes into being in class and it goes beyond teachers' scores on mathematics courses and/or mathematics tests. Accordingly, most of the work done in this line has been qualitative in nature. Several of them focused on interrelation between teachers' lack of mathematical knowledge and quality of their instruction.

In the context of functions, also, Sánchez and Llinares (2003) identified four pre-service teachers' ways of knowing the subject matter and teaching on their hypothetical presentation of subject matter for teaching. The results showed that these four prospective teachers' ways of knowing the subject matter had influence on the way they tried to represent the subject matter to the students. Similarly, Kahan, Cooper, and Bethea (2003) explored the relationship of teachers' mathematical content knowledge (MCK) and their teaching. A total of sixteen pre-service secondary mathematics teachers participated in the study. All participants studied Advanced Calculus, Linear Algebra, and Abstract Algebra and some had previous field experience (through a tutoring program). The data collected through a MCK test, assigned lesson plans, observation of the pre-service teachers during their teaching experiences, and transcripts that show number and level of mathematical courses students took. The MCK test assesses the teachers in three major areas - number, algebra/functions, and geometry. Test items test students' factual knowledge, conceptual understanding, and ability to apply that knowledge. The results showed that MCK plays a role in preparing appropriate lessons. The top scorers for the

MCK and transcripts rank produced strong lesson plans. Scorers near the bottom on the MCK and transcripts rank produced weaker lesson plans. MCK is a factor in recognizing and seizing teachable moments and enhances the possibilities for the teachers, but a lack of MCK narrows the scope of what is possible for teaching.

Similar results were found in the studies with secondary school mathematics teachers. Stein et al. (1990) aimed to describe and analyze the teaching of functions and graphs in the elementary grades. Through a detailed analysis of a teacher's knowledge and classroom lessons, the study suggested that "limited, poorly organized teacher knowledge often leads to instruction characterized by few, if any, conceptual connections, less powerful representations, and over routinized student responses" (p. 659). According to the researchers, lack of deep subject matter knowledge "led to the narrowing of instruction in three ways: (a) the lack of provision of groundwork for future learning in this area, (b) overemphasis of a limited truth, and (c) missed opportunities for fostering meaningful connections between key concepts and representations" (p. 659).

Lloyd and Wilson (1998) claimed that teachers' instructional practices are closely related to their beliefs about mathematics, teaching, and the knowledge of their students. Based on this assumption, the researchers investigated a secondary school level mathematics teacher's beliefs and instruction about function. The study found that the teacher's beliefs contributed to his instruction; that is he made conceptual connections, used powerful representations, and conducted fruitful discussions. He emphasized to use of multiple representations to conceptualize dependence patterns in data.

Several other studies have described an intimate relationship between mathematics teaching and student learning (e.g., Hofacker, 2006; Nilklad, 2004). Hofacker (2006) explored differences between college algebra students' understanding of linear and exponential functions based on the type of instruction they received. Students in the control group (n=75) were taught from a traditional perspective which primarily used lecture methods, whereas students in the experimental group (n=95) were thought from a contemporary perspective which focused on working in a discovery-based environment. The study found that students in the contemporary group had a more connected and flexible understanding of the content. Nilklad's (2004) study with 24 college students and their instructor indicated that the instruction supported students' understanding of functions such that, after completing the course, their definitions of a function improved toward a more formal definition, they had a better understanding of multiple representations, and the application of functions to real world situations. However, the instruction did not encourage solving mathematical problems in multiple ways. The study reported that, perhaps this reason, the students' algebraic reasoning abilities did not seem to progress as much.

Also, teachers' knowledge of student thinking has been recognized an important aspect of teacher knowledge and consequently student learning. An et al. (2004) asserted that knowledge of student thinking helps teachers to enrich their practice and teach mathematics effectively. As a part of a project entitled Cognitively Guided Instruction (CGI), studies have been conducted to determine whether the knowledge of students' thinking about a specific topic makes a difference in both instructional decisions of teachers and their students' learning (Fennema & Franke, 1992). The findings of these studies have indicated that teachers' knowledge of students' thinking may have an important influence on beliefs and instruction of teachers (Carpenter, Fennema, Peterson, Chiang, & Loef, 1989), also on student learning (Carpenter et al., 1988). Based on this series of studies, Fennema and Franke (1992) asserted that knowledge of how students think and learn is crucial for teachers. They noted that they believe the knowledge about student thinking do influence what teachers do in the classroom and the learning of students improve as a result of this knowledge.

On the other hand, the relationship among teacher knowledge, teaching practices, and student learning is not straightforward. Teacher beliefs and several other factors could mediate the effect of teachers' knowledge on teaching practices and student learning. Ebert (1994) investigated how prospective teachers' knowledge and beliefs about the students and content related to their instructional activities including planning lesson, teaching simulations, and reflecting on teaching. The researcher reported that a prospective teacher's knowledge; beliefs about students, learning mathematics, and mathematics; and knowledge and understanding of students' conceptions about a particular mathematical content had an impact on his or her instructional activities.

2.7 Summary

The above discussion shows that the process of learning is influenced by teachers. To be a teacher requires an extensive and organized body of knowledge (Shulman, 1986). Research confirms what makes up the crucial knowledge needed to teach subject matter effectively is teachers' content knowledge and PCK which includes knowledge of student difficulties in a particular academic content. The empirical studies cited above have revealed two facts: (a) the experienced secondary mathematics teachers' understanding of function is less developed than desired, and (b) teachers' knowledge of student difficulties in functions and how it influences student learning have been ignored. These results motivate further studies of in-service teachers' knowledge of functions, teachers' knowledge of student difficulties in functions, and their effect on student learning. The current study is built on these previous research in two major ways: (a) using a large sample, it provided information on in-service secondary teachers' KCS of the function concept at a much greater depth than has previously been available, and (b) it described the interrelation between teachers' KCS about the function concept and student learning of this concept.

CHAPTER 3

METHODOLOGY

To reiterate, the present study aimed to identify the KCS of mathematics teachers in technical and industrial vocational high schools, and discover the patterns that suggest potential relationships between teachers' KCS of the function concept and their students' learning outcomes of this concept. The study is based on teachers' responses to the function concept questionnaire and a follow-up interview task, observations of teachers' and students' interactions during teacher-designed instructional units on functions, and students' responses to the function concept test and teacher-designed exam questions. The methodology to accomplish the study was outlined below. The methodology section included the design of the study, the context of the study, the selection of the research site and sampling, instruments, data collection, data analysis, and the trustworthiness and ethical considerations.

3.1 Design of the Study

Research design is a logical plan for the route from the initial research questions to a set of conclusions about these questions (Yin, 2009). According to Creswell (2008), research design involves the intersection of philosophical worldviews (e.g., postpositive, pragmatic), selected strategies of inquiry (e.g., qualitative strategies, quantitative strategies), and research methods (e.g., statistical analysis, instrument based questions, open-ended questions, interview data). The worldviews, the strategies, and the methods together generally contribute to a research design that is either qualitative, or quantitative, or mixed methods. For Creswell (2008), these three approaches are not as discrete as they appear, but a study tends to be more qualitative than quantitative, and vice versa. Mixed methods incorporate elements of both qualitative and quantitative methods. To him, three factors affect a choice of one approach over another for the design of a study: the research problem, the personal experiences of the researcher, and the audience (s) for whom the report will be written.

Creswell's (2008) first factor that affects a choice of research approach focuses on the nature of the research problem. According to him, a quantitative approach is best to investigate problems that call for the identification of factors that influence an outcome or understanding the predictors of the outcomes. On the other hand, it is best to adopt a qualitative approach to investigate problems that call for understanding a concept or phenomenon. The second factor relies on the researchers' own personal training and experiences. In quantitative approach, there are carefully worked out procedures and rules, while qualitative approach allows room for researcher-designed frameworks. Individuals trained in technical, scientific writing, statistic, and statistical programs, therefore, may choose to use the quantitative design, whereas individuals who prefer to write in a literary way or enjoy conducting personal interviews or making observations would most likely opt for the qualitative design. The third factor focuses on the audience(s). To Creswell (2008), the researchers write for audiences such as journal editors, journal readers, or graduate committees that will accept their research. The experiences of these audiences with qualitative, quantitative, or mixed methods studies can shape the decision made about this choice.

Examining Creswell's (2008) factors, a qualitative approach was appropriate for the present study. That is, the first research question of this study focused on to identify technical and industrial vocational high school mathematics teachers' KCS about the function concept, and the second focused on to illuminate interrelation between teachers' KCS and students' learning outcomes of this concept. Although teachers' KCS and the relationship between their KCS and students' learning could have been investigated through quantitative approaches, it was thought that the extent of teachers' KCS and the patterns of interrelation could be better addressed through a qualitative approach. Moreover, the researcher of the present study is one who likes to observe the dynamics of human relations and the environment, and share her observations in a detailed way. She is also keen on talking to people, and probing the intricacies in communication. That is, her personal skills and life experience are specifically tailored for interviews and observations which opt for the qualitative design. Finally, she is willing to see how research, involving teachers, their students, and interaction between them, evolves rather than testing hypothesis. That is why, qualitative data collection and analysis techniques were primarily used in the study. Another major factor, the audience of the research, also influenced the general choice of research design. As the classroom is a living entity, with teachers, students, and the teaching material actively interacting, and as it is not very meaningful to quantify results in a teaching-learning atmosphere, without looking into what actually takes place behind the statistical data, qualitative approach has so far dominated the educational research. The present study is no exception because the researcher here desires to provide a vivid picture of what happens in the mathematical classroom within the scope of the study, thus resorts to qualitative research techniques.

There are different types of qualitative research. For instance, Creswell (2008) presented five approaches: ethnography, grounded theory, case study, phenomenological research, critical research. Denzin and Lincoln (2005) suggested five research strategies: case study, ethnography, grounded theory, life and narrative approaches, participatory research, and clinical research.

Merriam (2009) introduced critical research, qualitative case study, phenomenology, ethnography, grounded theory, and narrative analysis. According to Yin (2009), the first and most important condition differentiating these various research methods is the type of research questions being asked. For Yin (2009), basic types of questions are ‘who’, ‘what’, ‘where’, ‘how’, and ‘why’ questions (p. 9). When the focus questions of the study are formulated, a further distinction among the research methods is the extent of the researcher’s control over and access to actual behavioral events.

The first research question of the present study was an examination of what teachers know about the function concept and students’ difficulties of this concept. To answer this question, a qualitative survey (Jansen, 2010) was conducted. Yin’s (2009) conditions address that a case study research should be preferred for the second research question. Firstly, the second question was an examination of the ways teachers’ knowledge interrelates with student learning. Although the relationships between teacher knowledge and student learning can be investigated successfully through a correlational research, the how’s of the phenomena were better addressed through a case study research. Secondly, since the teachers’ knowledge impacts student learning in the class, classroom observations were necessary to provide a detailed description of the interrelations between teachers’ knowledge and student learning. The case study relies on direct observation of the events being studied and interviews of the persons involved in the events (Yin, 2009). Instead of selecting control variables in an attempt to arrive at cause and effect relationships, it was felt that a case study exploration was needed. This design allowed for a broad view of experienced mathematics teachers’ KCS of the function concept and for the careful selection of cases to provide insight into how a teacher’s KCS of the function concept contribute to the students’ learning in the classroom

Several writers have found it useful to further differentiate case studies (Merriam, 2009). According to Yin (2009) “a primary distinction in designing case studies is between single- and multiple-case designs" (p. 47). This means that the researcher needs to make a decision whether a single case or multiple cases are going to be used to answer the research questions (Baxter & Jack, 2008; Yin, 2009). For Yin (2009), the single-case study is an appropriate design under several circumstances. For instance, it is appropriate to conduct a single-case study when the case represents the critical case in testing a well-formulated theory, where the case represents an extreme case or a unique case, when the case is the representative or typical, when the case is the revelatory case, or when the same single case is studied at two or more different points in time. On the other hand, the same study may contain more than a single case (Yin, 2009) and the same issue may be illustrated in more than one case (Creswell, 2007). When this occurs, the study uses a multiple-case design (Yin, 2009). To Yin (2009), "the rationale for multiple-case designs derives directly from your understanding of literal and theoretical replications" (p. 59). In a multiple case-design, the researcher examines several cases to understand the similarities and differences between the cases, and s/he analyzes within each setting and across settings (Baxter & Jack, 2008).

The case study research methods also can be based on their function or characteristics (Hancock & Algozzine, 2006). Stake (1995) identifies case studies as intrinsic, instrumental, and collective. The intrinsic case study is employed when the researcher is interested in a particular case itself. According to Stake (1995), an intrinsic case study is "undertaken because of an intrinsic interest in, for example, this particular child, clinic, conference, or curriculum" (p. 445). On the other hand, an instrumental case study "is examined mainly to provide insight into an issue or redraw a generalization. The case is of secondary interest. It plays a supportive role, and it facilitates our understanding of something else" (p. 437). A collective case study is similar in nature to a multiple case study (Merriam, 2009). Yin (2003) categorizes case studies as explanatory, exploratory, and descriptive. According to Hancock and Algozzine (2006), an exploratory case study seeks “to define research questions of a subsequent study or to determine the feasibility of research procedures” (p. 33), an explanatory case study seeks “to establish cause-and-effect relations” (p. 33), and a descriptive case study attempts “to present a complete description of a phenomenon within its context” (p. 33).

The second phase of the current study investigated students’ learning outcomes of two teachers whose KCS level about the function concept were varied, then a multiple (or collective) case study would be indicated. One of the most important components of case study is its unit(s) of analysis. The unit of analysis (case) may be individuals, small groups, organizations, relationships, decisions, programs, partnerships etc. (Yin, 2009). According to Yin (2009), determining the case has plagued many researchers at the outset of cases studies. Baxter and Jack (2008) suggested that “asking yourself the following questions can help to determine what your case is: do I want to ‘analyze’ the individuals? Do I want to ‘analyze’ a program? Do I want to ‘analyze’ the process?” (p. 545). For the researchers, answering these questions can be effective strategies to delineate the case. This study wanted to analyze the interrelations between two teachers’ KCS of the function concept and their students’ emergent knowledge of this concept. The patterns of relationship between each teacher’s KCS of the function concept and the changes which occur in his or her students’ understanding of this concept observed and analyzed. Thus, the case was the interrelations between the teachers’ KCS and students’ learning outcomes as to the function concept.

Once the researcher has determined what his/her case will be, s/he should place boundaries on the case (Baxter & Jack, 2008). The case may be bounded by time and activity or by place and time (Stake, 1995). This study examined two teachers’ KCS about the function concept and their students’ learning outcomes within around three months of the instructional units on functions.

3.2 Context of the Study

In the Turkish educational system, secondary education encloses all general, vocational and technical education schools offering a minimum four-year education including 9th through 12th grades. General education schools is comprised of seven types of schools: General High School, Anatolian High School, Science High School, Social Sciences High School, Anatolian Teacher High School, Fine Arts High School, and Sports High School. There are nearly thirty different types of vocational and technical education schools. The most common are as follows: Technical and Industrial Vocational High School, Girls Technical and Vocational High School, Hotel and Tourism Vocational High School, Commerce Vocational High School, Theology High School, and Health Vocational High School. The former prepares students for higher education, and the latter for higher education and employment (EC, 2010, 2011).

All of these schools implement curricula developed by the Ministry of National Education and employ the following common general educational courses in the 9th grade: (a) language, literature and art courses (e.g., Turkish literature, foreign languages, fine arts), (b) social sciences courses (e.g., history, geography, philosophy), and (c) mathematics and natural sciences courses (e.g., mathematics, physics, chemistry). The students are allocated to branches in the 10th grade in both general education and vocational and technical education; and in higher grades, branch specific courses are offered (EC, 2010, 2011). All teachers, including mathematics teachers in both general and vocational high schools, receive the same university education.

After graduating from a general or vocational and technical high school, students take a university entrance exam to obtain acceptance for a higher educational program. The examination comprises multiple-choice test items and intends to measure students’ knowledge and competencies in five basic fields: Turkish, Social Sciences, Mathematics, Science, and Foreign Language (EC, 2010, 2011). Some studies point to low performance of students particularly from vocational and technical high schools (Berberoğlu & Kalender, 2005; Köse, 1996) and their scores were prone to get worse year by year (Berberoğlu & Kalender, 2005).

3.3 Participants

A total of 31 female, 11 male experienced mathematics teachers who are currently teaching in technical and industrial vocational high schools in Ankara participated in the first phase of the study. The length of teaching experience of the teachers ranged from two to 25 years, with an average of 12.7 years. A total of 28 teachers held a bachelor's degree in mathematics and 14 in mathematics education. In this group, two held a master's degree in mathematics education, three in mathematics, two in educational sciences, and one was pursuing a doctorate degree in mathematics education. None of the teachers had participated in any in-service or special training related to mathematics education, or the function concept.

The second phase of the study involves two case studies from this group. Of the 42 teachers, 13 agreed to be observed in their classroom. On the basis of the questionnaire results, which assess teachers' KCS of the function concept, these 13 teachers were categorized into three distinct levels: strong, intermediate, and weak. There were two teachers in the strong group with KCS scores ranging from 125 to 190, three in the intermediate group with KCS scores ranging from 90 to 124, and eight in the weak group with KCS scores ranging from 40 to 89 (see Appendix E for the scoring rublic).

To investigate the patterns of interrelation between KCS of a mathematics teacher and his or her students' learning outcomes, one teacher from the strong and one from the weak group from the same technical and industrial vocational high school were selected. Two selections were made from the far end groups, rather than from the mid group, so that it would be easier to differentiate the contribution of teachers' KCS to student learning. The teachers were referred to as Fatma and Ali (pseudonyms). A total of 59 ninth grade students of the two teachers participated in the study.

The teachers from the same school were selected for (greater) ease of access to the data source. The school administration stated that there are 1815 students and 159 teachers in the school. The students are from the middle-income families. The majority of the parents are high-school graduates. The rate of enrolment to university is low among these students just like in other vocational high schools. However, it is unique in that it is more successful in terms of social, cultural, and sports activities. For instance, in the academic year while the study was carried out, the school won a world silver medal in athletics.

3.4 Instruments

To gather and triangulate information on teachers' KCS as regards the function concept and examine the patterns of interrelation between KCS of a mathematics teacher and his or her students' learning outcomes of this concept, a variety of instruments were used: 1) a questionnaire on teachers' KCS pertaining to the function concept; 2) a follow-up interview including a card sort task; and 3) a test on student learning outcomes of the function concept. Each of these instruments is presented below.

3.4.1 The Function Concept Questionnaire

This questionnaire was developed to collect data on KCS of mathematics teachers regarding the function concept (see Appendix A). The process began with literature review. Additionally, the related literature was examined extensively for possible questions to assess teachers' KCS of the function concept. A total of fifteen questions were identified and translated in Turkish.

One of the main components of KCS is knowledge of student difficulties about particular mathematical content (Hill et al., 2008). When developing the questionnaire, therefore, an attempt was made to use questions based on students' difficulties and limited conceptions of the function concept. Although the questions are mainly based on students' difficulties and limited conceptions about the function concept, they also helped reveal teachers' content knowledge. In most cases, the teachers were presented with a hypothetical student's mistake and asked to decide whether the student was right or wrong. The teachers were asked to give their reasons if they decided the student was right. It is assumed that teachers' explanations would reveal their content knowledge. If the teachers decided the student was wrong, they were asked what they thought the student had in mind when he answered the question that way. It is assumed that teachers' explanations would address their knowledge of student difficulties about this concept.

The questionnaire was modified several times. During this process, language experts were consulted for increased clarity. Experts in mathematics and mathematics education were consulted for the validity of the questionnaire. Assessment and evaluation experts were consulted for ensuring face validity. Also, the questionnaire was piloted with six secondary mathematics teachers from two different technical and industrial vocational high schools. The time needed to complete the questionnaire was better estimated.

Table 3.1: The function concept questionnaire items

Two aspects of teachers' knowledge	Essential sub-concepts	Item
Content knowledge of the function concept	Conceptions of the function concept	3, 4, 5, 8, 9
	The essential features of functions (arbitrariness and univalence requirement)	10, 15
	Relating image and range to its graph	11
	Identifying two equal functions	12
	Locating pre-images, images, and (pre-image, image) pairs on the axes in graphs	13, 14
Knowledge of student difficulties	Students' conceptions of the function concept	1, 2
	Potential areas of difficulties for students in mastering function concepts	6,7
	Students' difficulty with the essential features of functions	10
	Students' difficulty in relating domain and range of a function to its graph	11
	Students' difficulty in identifying two equal functions	12
	Students' such limited conception that every function need to be linear	15

Table 3.1 shows the questionnaire items by possible sub-concepts they shed light upon. Appendix B describes the rationale for inclusion of each of 15 questionnaire items and gives their sources.

3.4.2 Follow-up Interview

Information gathered from the questionnaire was sufficient for a general description of two teachers' KCS of the function concept but was limited in other ways and sometimes hard to interpret. Thus, the second phase included a follow-up interview with two teachers (see Appendix C) so that these difficulties would be overcome.

The interview was used to clarify the answers to the questionnaire and at the same time to develop a more accurate and detailed picture of the teachers' KCS of the function concept by asking teachers questions which were related to the questionnaire but which required longer or more detailed responses. Based primarily on Even (1989, 1993) and Bolte (1993), the interview consisted of three parts that together addressed teachers' KCS of the function concept. Part one included questions that did not appear on the questionnaire. Questions were based on classroom situations and focused on clarifying concepts and procedures for students as well as identifying and analyzing students' misconceptions about the function concept. Part two was a review of teachers' responses to selected questionnaire items. In this part, the teachers were asked to reflect on their thinking when answering items, and to explain and clarify their answers to the questionnaire. They were probed in non-uniform ways. The non-uniform probing included questions that were based on the specific responses each teacher provided on the questionnaire. This was meant to clarify ambiguous answers and to discover specific points that seemed important. Part three included a card sort task (Stein et al., 1990). This part of the interview was considered essential after the analysis of the questionnaire. In this part, the interviewees were presented a card sort tasks. The card sort task consisted of a stack of 20 cards. According to Stein et al. (1990), this activity was designed to provide the opportunity for teachers to categorize the cards based on a variety of criteria. The cards differed along several dimensions including the representational format in which the mathematical relations were depicted and whether or not the mathematical relationships were functions. For Stein et al. (1990), it was possible to come up with a variety of grouping arrangements by using any combination of these dimensions:

> For example, a subject could categorize the cards into groups based on representational format, that is, placing all the graphs together, all the equations together, all the tables together, etc. At a somewhat more sophisticated level, the subject could pull together those cards depicting the same mathematical relationship regardless of representational format, or could place together the instances of functions vs. non-functions. Various other arrangements were also possible based on these and other dimensions that the subject may have found relevant (p. 643).

In the interview, each teacher was asked to categorize the cards into groups and to give a description of each group. After one arrangement was completed, the teacher was asked to sort the cards again in a different way.

As with the questionnaire, the interview went through several phases of modification and piloting. Mathematics and mathematics education experts were consulted. The piloting was done with three mathematics teachers from different technical and industrial vocational high schools. During the piloting, attention was given to the clarity of the questions asked, the quality of the answers given, and the time needed to complete the interview.

Additional data for describing teachers' KCS of the function concept were provided by the observation of teachers' functions instructional unit.

3.4.3 The Function Concept Test

Since the study proposed to find the patterns of interrelation between KCS of a mathematics teacher and his or her students' learning outcomes of the function concept, it was necessary to assure that the instruments were in fact describing the same thing. To this end, an attempt was made to use the same instruments for teachers and students. The questionnaire that had been used to assess teachers' KCS of the function concept was modified to be administered to the students at the end of teachers' functions instructional unit to explore students' learning outcomes of the function concept.

Table 3.2: The function concept test items

	Essential sub-concepts	**Item**
Students' learning outcomes	Conceptions of the function concept	1, 2, 5, 6
	The essential features of functions (arbitrariness and univalence)	7, 13, 14
	Relating a domain and range to its graph	8
	Identifying two equal functions	9
	Locating pre-images, images, and (pre-image, image) pairs on the axes in graphs	10, 11, 12

The function concept test (see Appendix D) mainly included the same questions that were asked to the teachers. To elicit students' understanding of the function concept more deeply, however, three questions from Markovits et al., (1988) added the test (question 12, 14, and 15). Table 3.2 shows the questionnaire items by possible function concepts they shed light upon.

3.4.4 Participant Observation

It is in the classroom that student learning is affected. Therefore, classroom observations were critical to provide a detailed description of the interrelations between teachers' KCS of the function concept and their students' learning outcomes. Classroom observations and audio taping began the day the function concept was introduced for each case study in two days a week. They continued until when the primary aspects of the function concept were no longer the focus of instruction. For tentative trends, field notes and audiotapes were reviewed daily to establish an ongoing focus on the observations. Neither of the two teachers involved in the case studies used written lesson plans. However, informal interviews before and after each class were probed into the teachers' planning for the day and objectives for the lesson. Student handouts and exams were collected. Field notes, audiotapes of the classes, and student exams contributed to a characterization of the role teachers' KCS of the function concept impacted their students' learning.

3.5 Procedure and Data Collection

Data collection for this study was conducted from April 2012 to February 2013. Data were collected in two phases. The administration of phase one questionnaire took place between April 2012 and June 2012. During this phase, data contributing to a description of the teachers' KCS of the function concept was collected from a group of 42 teachers using the questionnaire. Phase two consisted of case studies involving interviews and classroom observations of two teachers from phase one. Data collection in this phase took place between November 2012 and February 2013.

3.5.1 Phase 1: The Function Concept Questionnaire

Initially 20 technical and industrial vocational high schools in the various regions of Ankara were specified. The group leaders of the teachers of mathematics in each school were contacted in person. In a brief meeting (10 to 15 minutes), the study and questionnaire were described. In each school, with the help of the group leaders, other mathematics teachers were contacted in person. This process brought the total sample of 42 teachers who agreed to participate in the study. An appointment was made with each teacher. The appointment was made on the same day for practical reasons in the case that a few teachers from the same school would do the questionnaire.

In the study, while some teachers showed interest in the study and accepted to respond to the questionnaire, some were quite reluctant for different reasons. Some excuses were as follows: “I do not have enough time”, “I do not ask such problems in the class as mathematics achievement of the students in the vocational high schools is very low”, “Our students cannot solve such problems”, “I do not ask these kinds of questions”, “Functions are taught in the 9th grade, but I have not taught these classes for a few years”, “I have taught students in vocational high schools for 2 years. I do not know what they may think”, “I cannot know what the students think”, and “The questionnaire seems to measure the teacher's knowledge. Why? Is it to be sent to the Ministry?” In addition, some teachers stated that the students in vocational high schools have difficulties even in basic mathematical operations; therefore, the study must have been carried out in the other schools such as general high schools, because a valid result would not be obtained from the vocational high schools. An initial concern was that only those comfortable and confident in their knowledge of functions would volunteer to participate. Fortunately, this did not appear to be the case because the analysis of the questionnaires showed that some of participating teachers’ KCS of the function concept seemed very low.

Therefore, it may be concluded that the teachers did not necessarily participate in the study according to their KCS of the function concept.

3.5.2 Phase 2: The Interview

Within the group of 42, thirteen teachers were willing to participate in the second phase of the study. Two teachers were identified within this group of thirteen to participate in case studies, focusing on the interrelation between their KCS of the function concept and their students' learning outcomes. The selection of these two cases was based on their KCS of the function concept as demonstrated on the questionnaire (see Participants section for detailed information).

Both teachers, Ali and Fatma, completed the interview two weeks before they began the unit on functions instruction. This gave the researcher an opportunity to examine the interview data before the observations began. The interviews were held in the school after their classes. The interview focused on teacher's KCS of the function concept (see the previous section). It took the teachers approximately one and half hours to complete the interview.

3.5.3 Phase 2: Classroom Observations

A synthesis of the data from the questionnaire and the interviews for the two teachers provided a detailed characterization of their KCS of the function concept. This characterization formed the basis of the observational study of the interrelation between their KCS of the function concept and their students' learning outcomes.

The major topics in Ali's course were Logic, Sets, Relation-Function-Operation, and Numbers. In order to minimize the influence of researcher in the classroom, the researcher attended three classes prior to the instruction on functions unit. Except these classes, a total of 18 classes were observed and audio taped in Ali's classroom. During this time, the class was studying 'Functions' including 'Defining a Function, the Domain and Range of a Function, Types of Functions, Linear Functions, the Inverse Function, Basic Operations on Functions, the Composition of Functions, and Reading Graph of a Function'. The class met on Monday morning and again on Thursday morning, each time for 80 minutes. After each class, Ali and the researcher discussed the overall reaction of the class and the objectives of the next session.

The major topics in Fatma's course were the same as Ali's course. With the same aim, the researcher attended four classes prior to the instruction on functions unit. Except these four classes, a total of 18 class periods were observed and audio taped in Fatma's classroom. During this time, the class was studying 'Functions' including the same subheadings as studied in Ali's class. The class met on Wednesday morning and again on Friday afternoon, each time for 80 minutes. After each class Fatma and the researcher discussed the reaction of the class and the objectives of the next class.

Throughout the observations, the researcher kept fields notes describing classroom activities including a record of all the board work. During these periods, the student works and each teacher's interaction with students were also observed and noted.

3.5.4 Phase 2: Student Artifacts

The function concept test was given to the students at the end of the teachers' functions instructional unit to explore the student outcomes. For the other aspect of data to describe students' emergent knowledge of the function concept, artifacts of student works were gathered. Both teachers provided students' exams.

3.6 Data Analysis

The primary goal of this study was to identify mathematics teachers' KCS of the function concept, and examine the patterns of interrelation between KCS of a mathematics teacher and his/her students' learning outcomes regarding the function concept. In order to enhance the trustworthiness of the findings, multiple data collection methods, including a questionnaire, a follow-up interview, and classroom observations were used. The purpose of the data collection was not to rank the extent of the participants' knowledge. Therefore, the analysis of the data was predominantly qualitative in nature.

To answer the first research question, the responses to the function concept questionnaire of 42 teachers were analyzed to identify their KCS. The questionnaire was designed to reveal the teachers' both knowledge of content, and knowledge of student difficulties regarding the function concept. Therefore, the data were unitized on the bases of teachers' knowledge of the content, and knowledge of student difficulties to identify response patterns. The key aspects of the function concept structuring the teachers' content knowledge of this concept were as follows: conceptions of the function concept, essential features of functions, relating an image and range to its graph, identifying two equal functions, and locating pre-images, images, and (pre-image, image) pairs on the axes in the graphs. The student difficulties structuring the teachers' knowledge of students were as follows: students' conceptions of the function concept, potential areas of difficulties for students in mastering function concepts, students' difficulty with the essential features of functions, students' difficulty in relating a domain and range to its graph, students' difficulty in identifying two equal functions, and students' such limited conception as that every function needs to be linear. Throughout the analysis process, the data were integrated and summarized for the trends of participants' KCS of the function concept.

In order to gain an impression of the responses, a total of fifteen questionnaires were surveyed. For each questions, a preliminary categories of responses were created. The definitions and explanations of the function concept given by the participants were coded into the categories described by Vinner and Dreyfus (1989): correspondence, operation, and representation. They were coded as correspondence if there seemed to be some reference to mapping, pairing, or relation between the elements of two sets. They were coded as operation if there seemed to be some reference to an operation or manipulation. They were coded as representation if there seemed to be some reference to symbolic representations of functions (e.g., $y = f(x)$, graphs). The exemplifications of the function concept given by the participants were coded into the same categories as the definition of the function concept. The explanations for alternate representational forms for functions given by the participants were classified into two general categories: representations and notations. They were coded as representations if there were some reference to different representations of functions (e.g., set correspondences, graphs, algebraic expressions). They were coded as notations if there were different function notations such as $x \to y$, $f(x) = y$, $f : x \to y$. The justifications of whether a relation defines a function given by the participants were examined with regard to the arbitrariness and univalence properties of functions. The possible reasons for the student's way of thinking were characterized within the scope of student difficulties and limited conceptions as described in the literature (see Table

3.1). Descriptions of teachers' responses were written. The academic background information (e.g. the length of teaching experience) obtained through the questionnaire was recorded directly.

These preliminary categories painstakingly were discussed with a mathematics professor. New categories that were considered important in relation to what was known about the function concept and student limited conceptions and difficulties were added, and similar categories were combined. At the end of this process, a coding scheme was created (see Appendix F). This scheme was used to analyze all the questionnaires. The mathematics professor was consulted when faced with ambiguous cases.

To increase the reliability of the analyses, also, the mathematics professor was given the coding scheme and asked to use it to code a total of ten questionnaires. The researcher and the mathematics professor coded those ten questionnaires independently from each other. There was consensus on 82 out of 90 statements (91.11%).

To answer the second question, the data obtained from the function concept questionnaire implemented to two teachers, their interview sessions, the classroom observations, and the function concept test of 59 students as well as the exam results were analyzed and described. The interviews aimed at clarifying each of the two teacher's comments on the questionnaire items and getting a more detailed picture of their KCS of the function concept. The initial analysis of the interviews began with transcribing audio records, and then listening to audio records, and editing the transcripts. The subsequent steps were reading each interview transcript and analyzing the interview based on the specific concepts addressed by the questionnaire (see Table 3.1). Any evidence related to these concepts was recorded and later used to discover patterns, relationships, and contradictions.

The initial analysis of instruction involved transcribing the audiotapes of 18-lesson unit on functions and taking notes on each teacher's instruction soon after each observation. However, the main analysis of instruction occurred right during the analysis process. The set of lessons began with an introductory set of lessons that provided the concept of function including finding the domain, range and image of functions, relating the domain and image to its graph, and identifying two equal functions. Then types of functions were introduced, followed by a two lessons devoted to the linear function and the graphing of simple linear functions. Following several lessons devoted to the composition and inverse of functions. The final set of lessons focused on the basic operations on functions and locating pre-images, images, and (pre-image, image) pairs on the axes in the graphs. From among this extended group of lessons, five of Ali's and six of Fatma's lessons, all of which introduced and dealt with the function concept, were identified for detailed analysis. The analysis began with reading and dividing lessons into segments. More specifically, instructional episodes that suggested the teachers' KCS about the function concept were identified. The key aspects that contributed to the design of the function concept questionnaire (see Table 3.1) formed the foundation for analyzing the teacher's KCS of the function concept. Then, a content analysis was performed on those segments of the lessons. The main activities and procedures presented during these segments were summarized.

A total of 59 students' responses to the function concept test and their performance in the exams were analyzed to describe their learning outcomes. The students' responses were analyzed with respect to their ability to: (a) demonstrate an understanding of mathematical definition of the function concept, (b) identify relations that define functions, (c) relate the domain and range to its graph, and vice versa, (d) identify two equal functions, and (e) locate pre-image, image, and (pre-image, image) pairs on the axes in the graphs.

After the major analyses were completed, summary cases for each teacher were written. Each case described the teacher's KCS of the function concept based on their comments on the questionnaire items and interview questions, their teaching practice, and their students' learning outcomes of the function concept.

During the analysis and assessment of the data, participants were contacted face-to-face to confirm their responses on the questionnaire, interviews, and observations. The participants were handed a summary of the research findings, and they were asked to give consent to it.

3.7 Trustworthiness

All research is concerned with producing credible knowledge (Merriam, 2009). The credibility of quantitative researches and findings can be established by addressing construct validity, internal validity, external validity, and reliability (Kidder & Judd, 1986, as cited in Yin, 2009; Merriam, 2009). Construct validity refers to identifying the correct operational measures for the concepts studied (Kidder & Judd, 1986, as cited in Yin, 2009). Internal validity deals with the question of how the research findings match reality. External validity is concerned with the extent to which the findings of a study can be applied to broader situations. Finally, reliability refers to the extent to which research findings can be replicated (Merriam, 2009). Since their nature and the main purpose are different, the strategies for rigor in qualitative studies necessarily differ from those of quantitative research (Krefting, 1990; Merriam, 2009; Morse, Barrett, Mayan, Olson, & Spiers, 2002; Shenton, 2004). A common term used to describe validity in qualitative research is trustworthiness (Gay, Mills, & Airasian, 2006). Lincoln and Guba (1985) proposed four criteria to establish the trustworthiness of a qualitative study, which have been used by qualitative researchers for a number of years (Krefting, 1990; Shenton, 2004):

credibility (in preference to internal validity), transferability (in preference to external validity), dependability/consistency (in preference to reliability), and confirmability (in preference to objectivity). Lincoln and Guba (1985) suggested specific strategies for dealing with each of these criteria. For instance, the common strategies used to ensure credibility are member checking, prolonged engagement and persistent observation, triangulation, and peer examination. The strategies used to ensure dependability are triangulation, code-recode procedure, peer examination, researcher's position, and the audit trail. A common strategy used to establish confirmability is triangulation. Finally, the most common strategies enhancing transferability are the use of rich, tick description and utmost attention given careful attention to selecting the study sample. However, "not all qualitative research can be assessed with the same strategies" (Krefting, 1990; p. 214). The specific methods and procedures used in the present study to establish trustworthiness are as follows:

Full access to the research site and prolonged engagement: According to Lundy (2008), "the use of prolonged engagement allows the research study to go farther in the investigation of certain phenomena that cannot be adequately explored with short-term study designs" (p. 691). In order to effectively engage participants on a prolonged basis, the researcher must find a way to gain entry and establish a trust relationship with the respondents (Lundy, 2008). For Lundy (2008), "it is only at this point that the researcher can effectively explore, analyze, and interpret the data derived from the fieldwork of the research study" (p. 692). As the researcher in the present study had taught at high schools for a long time in the past, she had easy entry to the field. She spent approximately eight months in the field. In the meantime, she established relationships with the teachers, built rapport, identified possible data collection sources, and developed strategies to obtain quality data. She achieved relations with both teachers and other personnel in the school based on good communication and rapport, which enabled her to have access to the right data sources.

Triangulation: Triangulation is the process of using multiple methods, data collection methods, and data sources to obtain a more complete picture of what is being studied (Gay et al, 2006; p. 405). In the present study, data from multiple sources are triangulated. Data collected through the questionnaire, observations, and interviews are compared and cross-checked (Merriam, 2009).

Peer examination: Peer examination refers to external review or examination of the research process (Cresswell, 1998). Merriam (2009) stated that "certainly there's a sense in which all graduate students have this process built into their theses or dissertation committee, since each member of the committee read and comments on the findings" (p. 220). Such an examination of review, on the other hand, can also be done by a colleague. A colleague could be asked to scan some of the raw data and assess whether the findings are plausible based on the data (Merriam, 2009). Several stages of this research were shared by other researchers and modified according to their feedback. The thesis supervising committee also provided regular feedback on the methodology of the study, data collection process and tools, and the data analysis procedures as the study progresses. Last but not least, the study was presented at an international conference, where it received invaluable feedback.

Maximum variation sampling: One of the strategies used to enhance transferability of a qualitative research is using maximum variation sampling (Merriam, 2009), which involves looking for outlier cases to see whether the main patterns still hold (Miles & Huberman, 1994; Patton, 1990). The researcher reached a total of 42 teachers from 18 different schools. From among the thirteen teachers who agreed to proceed to the second phase of the study, two were selected, one with inadequate, and one with good KCS about the function concept. These two teachers were the participants in the case studies.

Member checking: Member checking "is the single most important way of ruling out the possibility of misinterpreting the meaning of what participants say and do…, as well as being an important way of identifying your own biases and misunderstanding of what you observed" (Maxwell, 2005; p. 111). The idea here is that the researcher solicits feedback on the emerging research findings to the informants to ensure that the researcher has accurately translated their viewpoints into data (Krefting, 1990; Merriam, 2009). Member checking can be done through several ways. For example, summaries of taped interviews can be played to informants for their responses, they can be asked to comment on a draft of the analytical codes (Lincoln & Guba, 1985), or they may be asked to read the transcripts of the interviews or observations (Shenton, 2004). In addition, at the conclusion of the study, a final member check may be done with the key informants to ensure that the interpretation of the data reflects the experience accurately (Lincoln & Guba, 1985). In the present study, the transcriptions of the interviews and the class observations were sent to the participants by email. A summary of the findings obtained from the questionnaire, interviews, and observations were translated into Turkish, and shared with both teachers. They were asked to assess the correctness of the findings, which were about themselves. At the end of the study, meetings with each of two teachers were arranged, wherein findings about the effect of teachers' KCS on students' learning were orally reported. They were also asked to comment on whether the findings are realistic or not.

Rich, thick description: Rich and thick descriptions allow transferability (Creswell, 1998; Lincoln & Guba, 1985). This provides such description they can contextualize the study and determine the extent to which their situations match the research context (Merriam, 2009). It was also taken into consideration in the present study; a detailed summary of each stage of the study was provided to the readers.

Researcher's position: This term refers to reflecting on the self as researchers. That is, it is critically assessing the effectiveness of the human as a tool (Lincoln & Guba, 2000, p. 183). The researchers need to explain their biases, dispositions, and assumptions with respect to the study to be undertaken. Such a clarification allows the reader to better understand how the individual researcher might have arrived at the particular interpretation of the data (Merriam, 2009). The researcher of this study taught mathematics in high schools for ten years (five years in a general high school and five years in a technical and industrial vocational high school). This makes her both an insider of the research context and a colleague, which is a considerable advantage for the researcher. She could enter the schools and communicate with the teachers easily. However, many teachers did not want to participate in the study indicating that vocational high school students are low performers and, thus, a study conducted in this context would not be fruitful. The researcher stated that she knows that the majority of vocational high school students experience difficulties in learning mathematics and believes that the factors affecting student learning must be searched. Her aim, therefore, is to understand how teachers' knowledge of students about a particular mathematical content affects students' learning.

Among 42, two teachers (Ali and Fatma) participated in the second phase of the study which consists of interviews and classroom observations. The researcher met Fatma two years ago. She had an interview with her and also observed her two classes for an assignment. Since she confidence the researcher, she accepted to be observed in the class. With the help of Fatma, the researcher convinced Ali to accept to participate in the study. In time, the researcher developed a good relationship with two teachers. As colleague, they discussed the quality of mathematics education in vocational schools and the possible reasons of vocational students' difficulties in learning mathematics (e.g., the content and quality of curriculum and textbook, prior knowledge, attitude towards learning, school and family factor) whenever it was possible.

However, the main role of the researcher was to investigate the interrelation between the teachers' knowledge and student learning outcomes.

3.8 Ethical Consideration

The researcher is fully aware of ethical considerations. She submitted the research proposal and the data collection tools to the office of ethics of the university. Upon obtaining the necessary permission, the researcher applied to the Ministry of National Education before conducting the study. During the implementation of the function concept questionnaire, first, the school administrations were informed of the study in general, and then the teachers were approached. The teachers similarly were informed of the aim of the study and the questionnaire. Personal information of the teachers who agreed to fill in the questionnaire was kept confidential. The two teachers who were involved in the second phase of the study were briefed on the procedure, especially the class observations and interviews. The teachers were asked to consent to the audio recording of the interviews and class observations. The teachers introduced the researcher and her aim to the students. Information about the teachers and their school was kept confidential. It was only shared with the supervisor and co-supervisor of the study. The teachers were attached pseudonames, and the data was presented by these pseudonames. Similarly, the students' names were not given out. The data collected was only used in this study. The students expressed their contentment about the fact that scientific research is carried out in the field of mathematics education. The teachers also were pleased to have participated in the study. Ali, for example, was happy to increase his awareness of the function concept after reading the findings of the study. Similar to Ali, Fatma stated that her participation in the study helped her develop professionally and she would like to participate, if possible, in research groups.

CHAPTER 4

RESULTS

The results are presented in three sections. The first section summarizes the results obtained from the instrument administered to the 42 mathematics teachers in technical and industrial vocational high schools. This data reveals KCS of mathematics teachers about the function concept within the secondary mathematics curriculum. It also provides the reader with a background for the remaining two sections, the presentation of case studies of two individuals selected from those 42 teachers. The second section summarizes the results obtained from the function concept questionnaire, the follow-up interview with the teacher in Case 1, and observation of his class. The narrative provides an in-depth description of the teacher's content knowledge of the function concept, his knowledge of student difficulties about this concept, and his student learning outcomes on the function concept test and exams. The final section presents results for the teacher in Case 2 in a parallel organization.

4.1 Teachers' Knowledge of Content and Students about the Function Concept

Teachers' KCS was described on their responses on the function concept questionnaire (see Appendix A). These descriptions are presented under two headings in the following sections.

4.1.1 Teachers’ Content Knowledge of the Function Concept

The key aspects of the function concept structuring the participants’ content knowledge of this concept were as follows: conceptions of the function concept, the essential features of functions (arbitrariness and univalence), relating a domain and range to its graph, identifying two identical functions, and locating pre-images, images, and (pre-image, image) pairs on the axes in the graphs.

Conceptions of the function concept: The comments the participants made revealed that they perceive functions as a correspondence between two sets. When they were asked to write how the students should define and exemplify function (question 3 and 4), they mostly (n=29) stated that students should define function in terms of a set correspondence. Participants generally came up with such definitions as:

> The relation that corresponds each element of Set A with only one element of Set B, given that Set A and Set B are not empty sets.
>
> There should be a domain and range; no element should remain un-corresponded in the domain.
>
> Provided that no element remains un-corresponded in Set A and that no one element in Set A corresponds with more than one element in Set B, the relation drawn from Set A to Set B.

Defining function, one participant used the following analogy:

> Cars are going from city A to city B; however, these cars have certain characteristics: (1) all people in city A should be taken to city B, (2) arriving at city B, these people should remain as a whole. That is, they may be transformed, but they need to go to the destination as one whole.

Some (n=9), however, thought that students should define function as an operation involving “the mechanism of transforming one element to another depending on the type of function defined” or “a conveyer converting an element into another element by exposing it to various processes”. One participant thought that students do not need to define function. Having defined function, most participants (n=22) stated that the students should give examples in the form of set correspondences. For example, a participant said that students should be able to “establish a relation pairing two sets”, while another one stated that students should be able to give examples “by making more use of a set correspondence demonstrating the domain, range and image sets of a function”. Some participants (n=6) who believe that students should give examples through a set correspondence made analogies such as the following:

> A group of students on a trip will be accommodated in hotels, and in these hotels, they will be placed in rooms. Some of these rooms may remain vacant, but each student will definitely be given a room. A student cannot be given two rooms at once.
>
> Set A is the set of children, and Set B is the set of mothers; then, (i) no element will be left out in Set A because every child has a mother, (ii) a child cannot have more than one mother.

For some (n=14), however, students should give algebraic examples (e.g., $f(x) = 2x + 5$), and for some others (n=6), they should “define a particular function, and indicate what is to come out of elements that they have chosen themselves” or “olives (input) processed in a factory coming out of the factory as olive oil (output)”. Only one participant thought students should give graphical examples. And another pointed to the fact that students should be able to exemplify functions “not only by explaining its rules but also through diagrams and in terms of ordered pairs”.

When the participants were asked to create an item that could reveal their students’ understanding of mathematical functions (question 5), most of them (n=25) gave some relations and asked whether the relations define functions or not. Their perceptions of the function concept also became evident, i.e. most of them (n=13) wrote relations in terms of a set correspondence. Few participants wrote relations in terms of a set of ordered pairs (n=3), an algebraic expression (n=3), or a graphical representation (n=4). Some participants (n=8) asked questions which required evaluating functions at specific points (e.g., " $f(x)=3x-5$, $f(3)=?$", " $A=\{1,2,3\}, B=\{3,5,6\}$ and $f: A \rightarrow B, f(x)=2x+1$, $f(2)=?$"). Two other participants drew attention to explanations like "I would give examples from daily life by using the logic of function" and "like the most important functions we use in daily life". However, it was not clear what kind of function conditions these participants implied.

When a student asked whether a function could be represented in different forms, the participants were asked how they would respond to the student's question (question 9). Their comments revealed that few participants (n=15) view algebraic expressions, set of ordered pairs, and graphs as alternate ways of expressing functions. Even fewer (n=3) stressed that a function could be represented in the form of tables or verbal statements. Sixteen participants provided explanations pertaining to notations: "It can be expressed as in $x \rightarrow y,\ f(x)=y,\ f: x \rightarrow y$" or "It can have a variety of representations; $f, p, \cos, \log$". Also included were these statements: "I would tell them that I have given them all possible representations, and there is no other", "There are other representations, which I will show some other time", and "My students would not ask because they are vocational school students".

The essential features of functions: The participants were asked to comment on a situation in which the student labeled each of the relations represented by a set correspondence, a graph, a set of ordered pairs, a verbal statement, and algebraic expressions (see Figure 4.1) as non-function, and if they think the student is right, explain the reason (question 10).

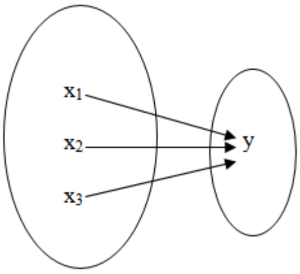
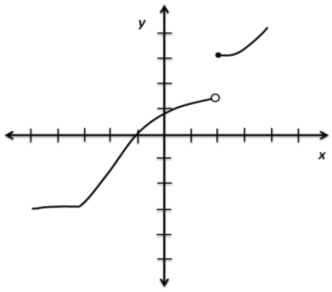
(a) 	(b) 
(c) $f: R \to R$ $x \to \begin{cases} -3x^3+3, & if\ x \geq 0 \\ 5, & if\ x < 0 \end{cases}$	(d) A correspondence which corresponds all positive numbers to 1, all negative numbers to -1, and 0 to 3.
(e) $y = 4$	(f) $\{(1,4),(2,5),(3,9)\}$

Figure 4.1: The relations in question 10

Most participants used the univalence property of function in their comments, especially when determining whether the set correspondence relation is a function. However, some participants did seem to experience the same difficulties as students do. To exemplify, for two participants, the student's response to item (a) is correct. In one's explanation "it is correct because domain and range are not clear". A few (n=5) regarded the student's response to item (b) correct. One justified this by saying that "… the graph must be connected" and another by stating that "it is not a function; an element is left out in the domain". Four participants considered the student's response to item (c) correct. In particular, one stated that " $f(x)$ is not defined". One participant thought the student's response to item (d) is correct, stating that "the domain and range should be given". The participants' comments revealed that some do not use the arbitrariness characteristics of functions. In fact, these participants are in the opinion that a function does necessarily make correspondence between the elements of two sets through an arithmetical or algebraic rule. To illustrate, for most (n=21), the student's response to item (f) is right. They commonly stated that the representation "is not a function as there is no equation or a determining condition". Also, a group of participants found the response to item (e) correct, stating that "because of $A = ?, B = ?, f = ?$ and lack of domain, range, and expression, it is not a function".

Similarly, a group of participants (n=16) did not use the arbitrariness characteristics of functions when they comment on the student's response in question 15. In this question, the participants were asked: "Assume that you have asked your students to give an example to the function graph that runs through points A and B in Figure 4.2 and that the student has produced a graph like the one in Figure 4.3.

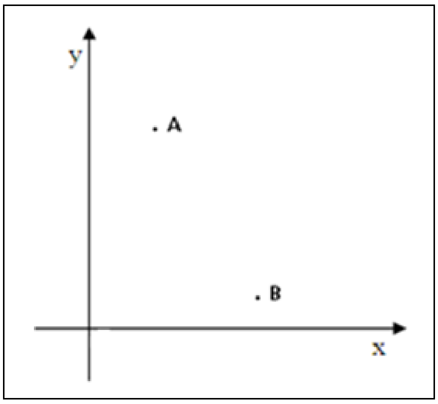


Figure 4.2: Figure 1 in question 15

Assume that you have also asked 'whether it is possible to draw a function graph that runs through points A and B or not', and the students said, 'No'. How do you think the student should have responded? Please specify".

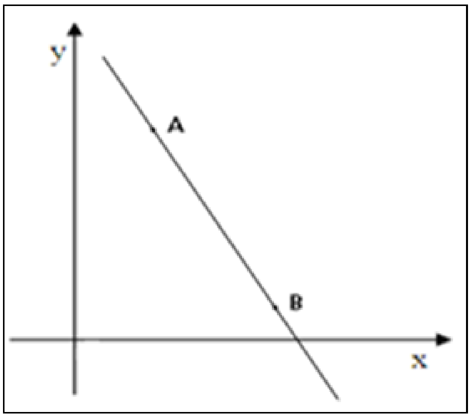


Figure 4.3: Figure 2 in question 15

Relating a domain and range to its graph: The participants were asked to comment on a situation in which a student had been asked to identify the graph/s that represent/s a function whose domain is $\{x: 2 \le x \le 6\}$ and whose range is $\{y: -1 \le y \le 4\}$ (see Figure 4.4). The comments they made have revealed that most accurately identify the function graphs whose domain and range were given. However, some participants experienced difficulty in relating the image and range of a function to its graph. To illustrate, fourteen participants found the student's response to item (a) incorrect. Some participants (n=4) did not provide any justification, while some (n=8) provided explanations such as:

> Domain is between 2 and 6, but piecewise function with x value between 3 and 4 occurred.

Domain x axis varies between 2 and 6, range y axis varies between -1 and 4. The $4, 5, 6$ elements in the domain are left out.

What kind of numbers x and y are and how they are defined is not given.

Assume that you have asked your students to identify the graph/s which represent/s a function whose domain is $\{x: 2 \leq x \leq 6\}$ and whose range is $\{y: -1 \leq y \leq 4\}$. A student marked the item (a)/(b)/(c).

Is it correct? If so, explain why.

Is it wrong? If so, what do you think has caused the mistake?

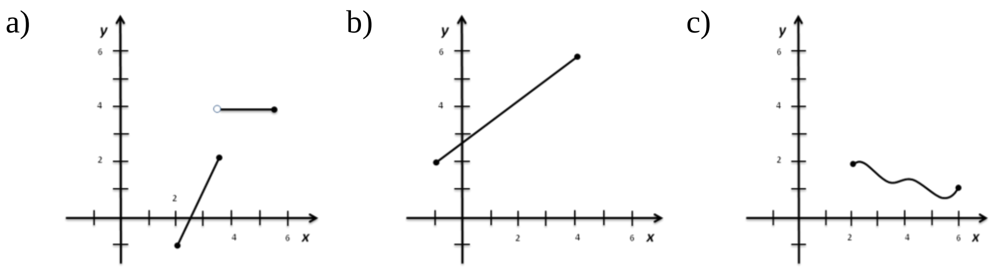


Figure 4.4: Question 11

Without considering its domain and range, one participant viewed the student's response to item (b) as correct explaining that "vertical lines intersect the graph at one point". Also, most (n=36) experienced difficulty in identifying the function graph whose image set is the immediate subset of the range. Indeed, 24 participants found the student's response to item (c) incorrect. As indicated by one participant "domain is okay but the range does not satisfy", and by another "range is not accurately shown in the graph". Some (n=12) believed the student's response was correct. However, their comments revealed that they focus on whether the graph is a function or not, rather than on its domain and range. Three participants provided the following explanations: “Each x in the domain correspond only one y in the range”, “Because no element of the domain is left out, and they have only one image”, and “Vertical lines intersect the graph at one point”. Only few participants (n=2) contended that the function, the graph of which is given, lies within the domain and range limits.

Identifying two equal functions: The participants were asked to comment on a situation in which a student had been asked to identify the function/s which is equal to $f: N \to N$, $f(x) = 4x + 6$ (see Figure 4.5). The comments they made have revealed that, while identifying two equal functions, most participants consider the domain and range, as well as whether the elements in the domain have identical images or not. In the question, they mostly accurately stated that the student’s responses to item (a), (b), and (c) are incorrect; however, that to item (d) is correct. As a result, they thought that the function $g: R \to R, g(x) = 4x + 6$ and $h: N \to N, h(x) = 2x + 3$, and the graph in item (c) do not equal to the function f. On the other hand, some participants’ comments marked their lack of understanding in identifying equality of functions. Two participants, for instance, indicated that the student’s response to item (a) is correct. One supported this by saying “because $R \cap N = N$”, and the other “s/he considers the set of natural and real numbers, because $N \subset R$”. For few other participants (n=4) the student’s response to item (b) is correct. One said “it is correct because both functions have equal domain and range, only that $f(x)$ is as twice as $h(x)$”.

As for function $f: N \to N$, $f(x) = 4x + 6$, assume that you have asked your students to identify which function/s equal/s to f. A student identifies that the function in item (a)/(b)/(c)/(d) equals to f.

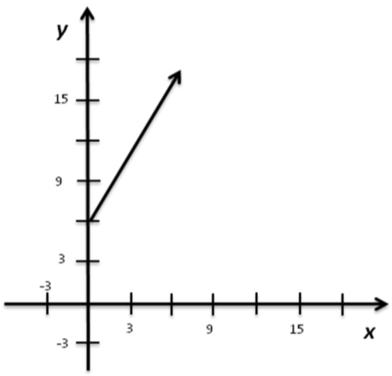
Is it correct? If so, explain why.

Is it wrong? If so, what do you think has caused the mistake?

a) $g: R \to R$ ve $g(x) = 4x + 6$

b) $h: N \to N$ ve $h(x) = 2x + 3$

c)



d)

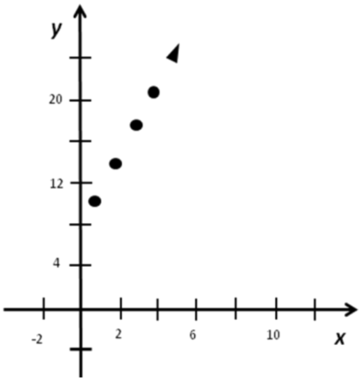
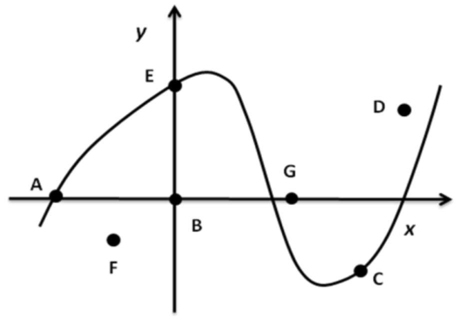


Figure 4.5: Question 12

Locating pre-images, images, and (pre-image, image) pairs on the axes in graphs: For a given function represented by graph, most of the participants experienced difficulty in identifying pre-images, images, and (pre-image, image) pairs. In question 13 (see Figure 4.6), most (n=30) believed that the student's responses are wrong, but only eight participants responded to this question accurately.



Assume that, as regards the function graph above, you asked the following:

a) Which points represent an element of the domain?
b) Which element represents an element of the range?
c) Which points represent (pre-image, image) pairs?
d) Which points do not represent (pre-image, image) pairs?

A student came up with the following responses:

a) A, E, C, b) E, B, G, c) B, G, and d) F, D.

Are the students' responses right or wrong? If wrong, how do you think the student should have responded?

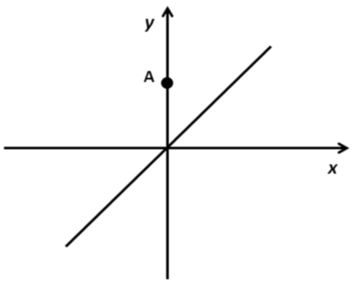
Figure 4.6: Question 13

A total of twelve gave a correct response to items (a) and (b), incorrect response to items (c) and (d). Ten of them responded to the question all in a wrong way. Such are among the wrong answers: "(a) A, B; (b) C, E; (c) B, E; (d) C, G" and "(a) A; (b) C, E; (c) $(A, B), (B, E)$; (d) E, B". Two of the participants claimed that domain and range need to be given for the question to be tackled with. These are two participants' opinions: "Abscissas and ordinates are not clear, so the domain and range is not definite", and "The domain and range should be well-defined; indeed, any response to such a shape would be wrong".

Also, many participants experienced difficulty to identify pre-images of A on the graphs in question 14 (see Figure 4.7).

Assume that, for each of the function below, you have asked your students to locate the pre-images of point on the graphs and that one of your students has responded as follows. Please mark what you think about the overall response and specify.

(a)

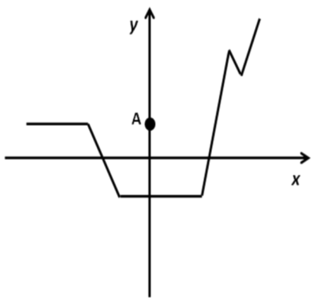


The student's response: $\{0\}$

Correct Incorrect

Specify:

(b)

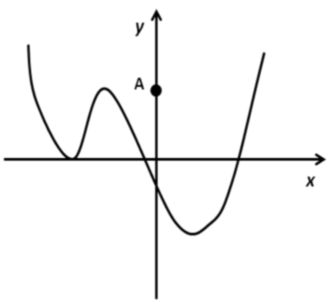


The student's response:

Correct Incorrect

Specify:

(c)

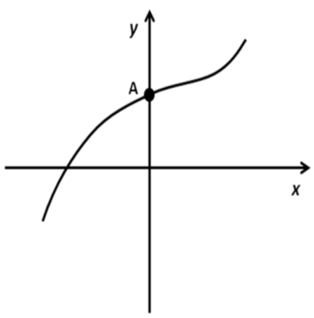


The student's response:

Correct Incorrect

Specify:

(d)



The student's response:

Correct Incorrect

Specify:

Figure 4.7: Question 14

For 30 participants, for instance, the student's response to item (a) is wrong. Some of these (n=15) did not specify their opinions. Some (n=15), however, justified why the student's response is wrong in a variety of ways (e.g., "a number that corresponds the function to A does not exist", "A, is not a point that belongs to the function", "the image should be along A"). Only, six participants contend that the pre-image of A must be a positive real number. One participant pointed at it on the graph, indicating that "the pre-image of A is $m \in R^+$". Similarly, a total of 13 participants did not respond to item (b). Some did not give any explanation. Four participants made the comment that "it is not a function, because it is not one-to-one". Only two participants indicated that the response could well be correct and highlighted that still an indefinite number of real numbers exist, the image of which is point A. Also, some (n=4) participants asserted that graph (c) "…is not one-to-one". Only a group of participants (n=6) asserted that there are four real numbers whose image can be A. The participants outperformed in item (d). As regards the pre-image of A, most stated that "the answer to this question is obvious, 0".

4.1.2 Teachers' Knowledge of Student Difficulties in the Function Concept

The student difficulties structuring the participants' knowledge of students were as follows: students' conceptions of the function concept, potential areas of difficulties for students in mastering function concepts, students' difficulty with the essential features of functions, students' difficulty in relating a domain and range of a function to its graph, students' difficulty in identifying two equal functions, and students' misconceptions that every function needs to be linear.

Students' conceptions of the function concept: The participants generally stated that the students define function (question 1) as a correspondence (n=13) with such statements as "set A corresponds to set B; a number from A goes to B, no element remains in A; an element can remain in B" or as an operation (n=19) with such statements as "function is a kind of factory, with its input and output", "assigning a value to x to find y". Some participants (n=6), however, thought the students identify function with its symbolic representations (e.g., "$f : A \to B, x \to y = f(x)$" or "$f : R \to R$, $x \to f(x) = 2x+1$"). Most participants (n=20) stated that students exemplify function (question 2) in algebraic terms (e.g., $f(x) = x+1$, $f(x) = x$) . For some (n=14), they give examples in terms of set correspondences. For example, one participant stated that students give examples to function by "identifying children as domain and mothers as range (every child does have a mother)". For few (n=6), the examples students give relate to the meaning of function as operation (e.g. fruit juice squeezer, factory machines, or bread factory).

Potential areas of difficulties for students in mastering function concepts: When the participants were asked what aspect(s) of the study of functions they think cause students the most/little difficulty (question 6 and 7), some (n=24) believed that it is difficult for students to learn the composition of functions. Most of these participants did not indicate which aspects of functions are difficult for students. However, some (n=7) stated that students particularly have difficulty in finding a function based on a given function in composite operation (e.g., if $f(x) = x + 2$ and $fog(x) = x^3$, what is $f(x)$). Some participants (n=21) thought that students have difficulty in learning the inverse function. Some others (n=18) stressed that students cannot comprehend the basic concepts of functions. As a matter of fact, they stated that students confuse the "domain, range and image of function, and which relations are actually functions" and "the fact that one element does not have more than one image".

Some participants (n=5) reported that students have difficulty in doing operations with functions. They suggested that a possible reason for this problem is that students do not do basic mathematical operations. One wrote: "The students have difficulty in doing operations by evaluating values rather than the subtopics. The multiplicity of four operations does not appeal to them". Some participants (n=5) thought students have difficulty in solving questions about functions. Some question types students have the greatest difficulty with, as specified by the participants, are:
" $f(2x-1)=4-5x \Rightarrow f(5x)=?$ $f(x)=ax+b$ $f(2)=3, f(1)=2 \Rightarrow f(7)=?$ "
Most participants (n=33) believed that students easily learn to evaluate algebraic functions at specific points.

Students' difficulty with the essential features of functions: The participants were asked to comment on a situation in which a student labeled each of the relations represented by a set correspondence, a graph, a set of ordered pairs, a verbal statement, and algebraic expressions as non-function, and if they think the student is wrong, comment on the possible reasons for the student's way of thinking (question 10). The student's response to each item is wrong. It was observed that the participants have knowledge that is of varying quality and content. To be more specific, many participants (n=20) ascribed the student's way of thinking in item (a) to the fact that the image of the elements in the domain set is unique. One explained this by the following words:

As they all go to the same element, it probably confused the student because the students have the tendency to take each element to a different element; that is, they always consider the function to be one-to-one.

For some participants (n=8), the student thought this way because s/he has not thoroughly understood the definition of function. One participant, for example, said that "all elements in the domain have the same image, s/he does not know the definition of function very well", and another "it meets the function criteria; s/he has not grasped the function property". Some participants (n=3) stated that the student has thought this way because s/he does not comprehend constant functions.

In the opinion of most participants (n=18), the student did not perceive the graph in item (b) as a function because the graph is disconnected. Some (n=5), however, stated that the shape is somewhat unfamiliar to students, which caused the student's response. Five participants said the student does not understand the subject of piecewise function, and two said s/he does not understand the function graphs at all. And three said the student does not know the interval concept. The participants commented on this as follows: "The student does not know interval concepts", and "If the parallels drawn to y axis intersect the graph at one point, then it is a function. It could be because points included or not included". One participant suggested this: "If vertical lines were drawn, each element would have an equivalent on the y axis, so the student probably mixed this up".

As regards the following relation $f: R \to R, x \to \begin{cases} -3x^3+3, if\ x \geq 0 \\ 5, \quad\quad\quad if\ x < 0 \end{cases}$ some participants (n=12) believed that its being piecewise influenced the student's approach. One specified the reason: "The student may have thought that the function should be given by one rule only". Some participants (n=13) suggested that the student does not understand the piecewise function, which is why s/he gave this response. Others (n=4) thought the student does not identify the representation as a function because of having difficulty with constant functions. For one participant, "the student probably did not perceive the $y=5$ expression as a function because it has not involve x". For another, "he might be confused because images for each value smaller than 0 is 5". One other participant also indicated that "he may not have seen this as a function as different values come out when zero is assigned to x".

Almost all participants thought the student's response to the relation in item (d), which is "A correspondence which corresponds all positive numbers to 1', all negative numbers to -1, and 0 to 3", is incorrect. They justified the student's way of thinking in a variety of ways. For instance, some participants (n=4) stated that the student does not understand the concept of function, some (n=10) believed that s/he does not understand piecewise function, and some (n=5) wrote that s/he has trouble with constant functions. One specified the reason as "s/he might be confused because the image of positive and negative numbers all equaled to the same number". Some participants (n=3) suspected that the student has a problem with numbers. Indeed, one believed that "as regards 'all positive numbers', the fractional equations could have misled him. That 0 is neither positive nor negative may not have made sense to the student" and another "he can consider 0 to be positive". Some participants (n=3) suspected that the student has a problem with graphs. One commented as "this is wrong; it is a function, he may not be able to imagine its graph". Only three participants expressed the possibility that the student does not regard this as a function, because the relation is given in the form of a verbal statement. In the opinion of one participant "because verbal expressions are used; students do not know the verbal expression of function".

The relation in item (e) favored success. A group of participants (n=19) thought that the student is mistaken, because the relation does not involve x. However, some participants (n=13) ascribed the student's mistake to his failure to understand constant functions. Two other participants drew attention to the possibility that the student confused $y = 4$ with $x = 4$.

Students' difficulty in relating a domain and range to its graph: The participants were presented an imaginary situation in which the students were asked to identify the function graph/s, with the domain $\{x : 2 \le x \le 6\}$ and the range $\{y : -1 \le y \le 4\}$, among the given ones, and asked to evaluate the correctness of a student's response. When the participants think the student is wrong, they were asked what they think has caused the mistake (question 11). The student's response to item (b) is wrong. Most participants correctly viewed the student's response to this item as incorrect. Among the group who thought the response is incorrect, only 28 suggested possible reasons for the student's way of thinking. Some (n=13) claimed that the student has mixed domain with range, while some (n=10) claimed the student has not grasped to relate the image and range of a function to its graph well. Others explained problem in relation with "the number line, lack of knowledge on coordinate system", "a possible misconception about linear function", and "inability to read the domain and range on the graph."

Students' difficulty in identifying two equal functions: The participants were asked to evaluate a student's responses to the function that is identical to function $f: N \to N,\ f(x) = 4x + 6$. Similarly, when they think the student is wrong, they were asked what they think has caused the mistake (question 12). The student's answer to item (a), (b), and (c) are wrong, while to function (d) is right. Most of the participants did seem to be aware of students' limited conceptions in relation to ignoring the domain and range when identifying two equal functions. For many (n=40), the student's response to item (a) is incorrect. It is discussed by some (n=17) that the student disregarded the domain and range of the function and responded as such because the rule of the functions is identical. All participants found the student's response to item (c) incorrect. For some (n=13), the student disregarded the domain and range of the function. One participant stated "incorrect because $f: R \to R$; the student took no notice of it is N", and another, "domain of f is N but, domain of the graph is R, he does not perceive the difference". Some of them (n=12) expressed that, like function f, the value of the function in the graph for 0 is 6, thus the student considered that the functions are equal. In the opinion of one participant "f should be comprised of dots; it cannot be linear. The student could have found an image by assigning values, like $f(0)=6$”. Some participants (n=6), however, believed that the student does not have complete mastery of R and N, which may account for his/her thinking.

When domains of functions are the same, on the other hand, participants thought that students ignore the function. To illustrate, for most participants (n=38), the student’s response to item (b) is not correct. Some of them (n=8) believed that the student give the particular response because both functions has identical domain and range. One specifically said: “The student considers the domain and range without considering the function”. Some of them (n=21) thought that function f is twice as function g, and it might explain the way the student approached the question. One participant speculated that “the student assumed that half of a function would probably be identical to the function itself” and another “he might have thought that $f(x)/2=h(x)$”.

Students’ such limited conception that every function needs to be linear: When a student gave a linear example to the function that runs through points A and B and said there is no other function that pass through these points, the participants were asked whether the student’s answer is correct, and if they think it is not, they were asked what they think has caused the mistake (question 15). Most of the participants did seem to be aware of students’ limited conceptions that every function needs to be linear. For many participants (n=18), the student only considered the linear functions, thus responded in this way. Few participants (n=4) came up with different reasons for this: “The student does not know the function types”; “He cannot think broad about the function subject”; “He could not really grasp the concept of function and does not know the graphical-function connection”. For some (n=8), the reason for the student’s thinking is the idea that any two distinct points are incident with just one line. One worded this as follows: “He might have responded this way leaving from the axiom that any two distinct points are incident with just one line”.

4.2 The Case of Ali

On the basis of the questionnaire results, which assess teachers' KCS of the function concept, Ali was categorized into weak group with his low score. He held a bachelor's degree in mathematics. He had got pedagogical formation for four months before he began teaching. He had 14 years of teaching experience. For 12 years, he had been teaching in the technical and industrial vocational high school where the study was applied. He had not taken any courses, or attended seminars or in-service training in mathematics education and teaching functions.

The study has revealed that Ali thinks a mathematics teacher should have complete mastery of his or her field. This is important in dealing with questions students may ask in class. However, there may of course be some subjects that the teacher does not know very well or have forgotten. The teacher should be comfortable at such times. Taking a look at the subject will suffice to remember the subject. He also believed that a mathematics teacher should be able to predict which aspects of a subject the students will find difficult or make mistakes at. If the teacher can do so, he or she can spend more time on these issues in class. For him, the teacher can develop his or her mathematics knowledge by solving questions. The teacher's knowledge of students, however, will develop in the actual class. The questions students frequently ask in class, for example, may help him or her understand what they find difficult.

Ali did not make lesson plans. He thought there was no need to make lesson plans because the course books explains the topics anyway; so the main source of his 9th grade instruction was the course book (MEB, 2012), which has some limitations (see Section 2.2 for detailed information). He presented the major concepts just as instructed in the book.

4.2.1 Ali's Knowledge of Content and Students about the Function Concept

The descriptions of Ali's KCS were developed from his responses on the function concept questionnaire (see Appendix A), the follow-up interview including a card sort task (see Appendix C), and classroom observations. These descriptions are presented under two headings in the following sections.

4.2.1.1 Ali's Content Knowledge of the Function Concept

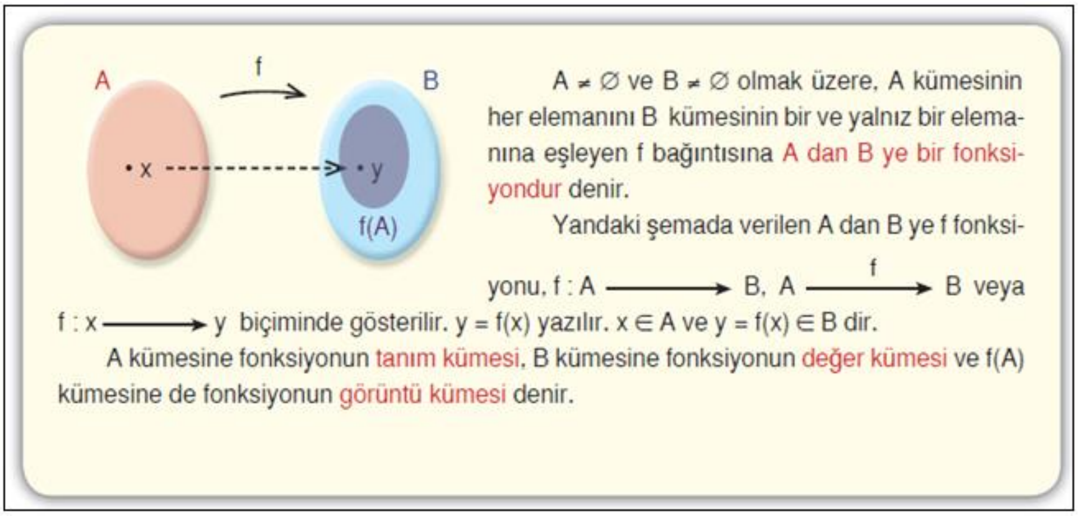
The key aspects of the function concept structuring Ali's content knowledge of this concept were as follows: conceptions of the function concept, the essential features of functions (arbitrariness and univalence), relating a domain and range to its graph, identifying two equal functions, and locating pre-images, images, and (pre-image, image) pairs on the axes in the graphs.

Conceptions of the function concept: The comments Ali made both in the function concept questionnaire and in the follow-up interview revealed that he perceives function as operation, a mechanism converting input into output. Therefore, he mostly used algebraic functions.

When he was asked how the students should define and exemplify function; he stated that students should define function as "the identification of what each element of a set corresponds to as required" and that, having defined a function, they should give examples so as to find the value of some elements. In the interview, he said, as it is more important to find the image of the domain elements of a function, students should define and exemplify function like this. He thought that students can cite examples to function from daily life such as the process of grinding coffee beans into coffee, but it is more important to specify function in a way to evaluate function for required elements so as to cope up with the exam questions and certain operations relating to the compound functions or the inverse function. When asked how he himself defines function, he responded as follows:

> I haven't taught this topic for the last few years, so I am not pretty sure how to define it or what to do with it. I will have a look at it at the weekend, but an example that occur to me now is cattle's going into the slaughterhouse only to leave it as bacon and ham. Then, I would present the mathematical definition, and demonstrate how to find the numbers and letters according to the given rule.

As he stated in the interview, in the class, he explained the function concept as transformation of living and nonliving objects into a different entity under a certain process. He likened this to the grinding of wheat at the mill to make flour, or to the baking of flour at the baker's to make bread. He drew the attention to the fact that *a process* should exist. For example, coffee beans are put into the machine, exposed to a process, end up in powder form. Or in another example the livestock goes to the slaughterhouse to go through a process, and the outcome is sausages, ham, and bacon. However, as stated in the course book, he provided set theoretic definition of the function concept (see Figure 4.8).



$A \neq \emptyset$ ve $B \neq \emptyset$ olmak üzere, A kümesinin her elemanını B kümesinin bir ve yalnız bir elemanına eşleyen f bağıntısına A dan B ye bir fonksiyondur denir.

Yandaki şemada verilen A dan B ye f fonksiyonu, $f : A \longrightarrow B$, $A \xrightarrow{f} B$ veya $f : x \longrightarrow y$ biçiminde gösterilir. $y = f(x)$ yazılır. $x \in A$ ve $y = f(x) \in B$ dir.

A kümesine fonksiyonun tanım kümesi, B kümesine fonksiyonun değer kümesi ve f(A) kümesine de fonksiyonun görüntü kümesi denir.

[Provided that $A \neq \emptyset$ and $B \neq \emptyset$, the relation f is called a function from Set A to Set B if every element in A corresponds to exactly one element in B under f. The function f, from A to B given in the scheme is represented by $f : A \to B$, $A \xrightarrow{f} B$, or $f : x \to y$. It is written as $y = f(x)$. $x \in A$ and $y \in B$. Set A is called the domain of the function, set B is called the range of the function, and set $f(A)$ is called the image of the function.]

Figure 4.8: The function concept (MEB, 2012, p. 75)

In case of a student's inquiry of whether a function has different representations or not, he added that, he would clarify this for the students by "… specifying examples from all representations". This comment did not give sufficient clues about his knowledge of different representations of a function. To probe this, in the interview, he was required to specify a typical example from mathematics class to function and to state whether there are other ways of representing this function. The example he provided was "$f(x) = 2x + 5$". As covered similarly in the course book, he stated that he could show it as a machine converting 1 to 7, and 2 to 9. Apart from this, he pointed out that he could show the function in the form of a set diagram. In the same interview, he was handed a series of cards. He was supposed to predict how the students would group and define the cards. He stated that they would group the cards in the form of graphs, ordered pairs, and equations. When asked whether and how the students would try a variation of these, he said they would not do any other type of grouping. It was also inquired how he himself expects the students to group the cards but, he did not want to group the cards. His comments revealed that he did not seem to view set of ordered pairs, tables, and graphs as alternate ways of expressing functions.

The essential features of functions: It was clear that Ali was aware of the univalence property of functions, and he used it functionally while deciding whether the relations given in the form of set correspondences are functions or not. For him, in question 10, the student's response to the relation in item (a) is wrong. Thus, he correctly associated relation as a function. In the class, he clearly addressed univalence property of functions. He used the activity in the course book (MEB, 2012; p. 74). In this activity, a relation wherein four customers at a restaurant are corresponded to dishes is given. Two conditions of the relation are as follows: (1) Everybody will eat and (2) One dish for one person only. Given the set of customers at the restaurant as:
$A = \{Yasar,\ Soner,\ Okan,\ Hakan\}$, and the set of dishes as:
$B = \{Kebab,\ Beans, Meatball,\ Spinach,\ Potatoes,\ Celery,\ Stuffedpeper\}$
a relation (β_1) is defined (see Figure 4.9).

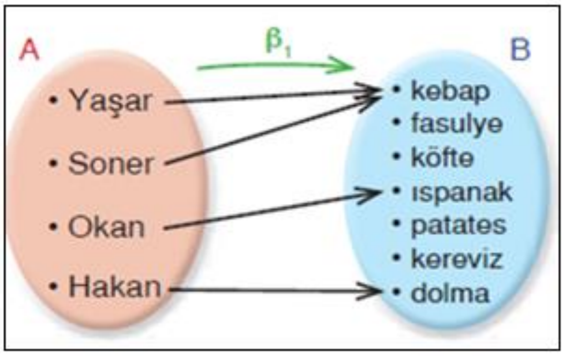


Figure 4.9: Relation β_1 (MEB, 2012, p. 74)

As regards relation β_1, it is asked that "(a) Is there anyone who ate more than one kind of dish?, (b) Is there anyone who did not eat at all?" The same two questions are also asked for relations $\beta_2, \beta_3, \beta_4$ (see Figure 4.10).

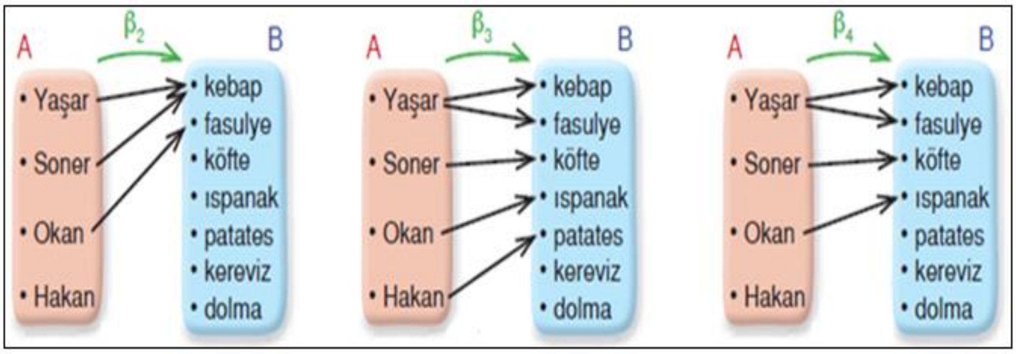


Figure 4.10: Relations $\beta_2, \beta_3, \beta_4$ (MEB, 2012, p. 74)

Ali concluded that relation β_1 meets the conditions given. For relations $\beta_2, \beta_3, \beta_4$, he stated that relation β_2 does not meet condition (1) because Hakan does not eat; relation β_3 does not meet condition (2) because Yaşar eats two dishes, and relation β_4 does not meet either condition because Hakan does not eat and Yaşar eats two dishes.

Finishing this activity, he solved a similar example in the course book (see Figure 4.11). Two conditions were met in the example: “(1) All the elements in Set A will be corresponded to those in Set B and (2) Each element in Set A is to be corresponded to only one element in Set B”. He announced that relation β_1 meets conditions (1) and (2); relation β_2 does not meet condition (1), and relation β_3 does not meet either condition. He announced that a relation that meets these two conditions is called as *function*.

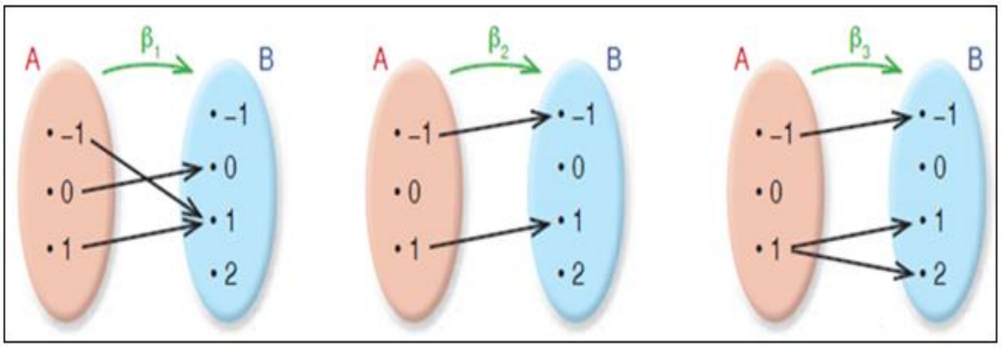


Figure 4.11: Relations $\beta_1, \beta_2, \beta_3$ (MEB, 2012, p. 75)

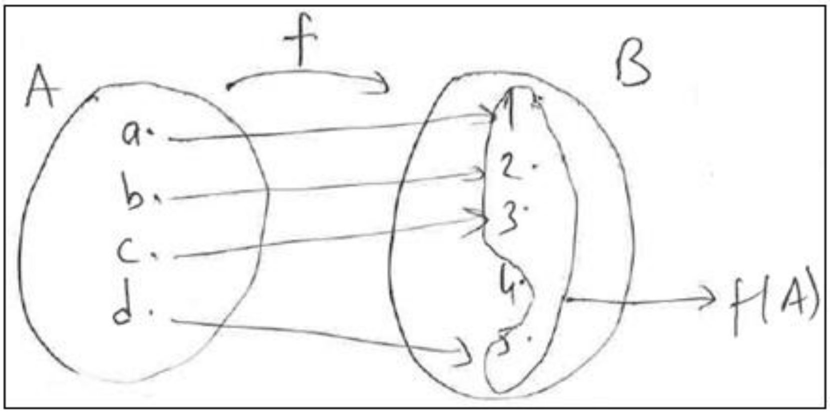


Figure 4.12: Excerpt from Ali's instruction

He resorted to a few more examples to underline the univalence property of functions (see Figure 4.12). He asked if a new relation which forms as a result of corresponding element d of Set A to also 4 would be a function or not. He likened the fact that an element in the domain cannot have two images to the fact that two arrows cannot come from an element in Set A to Set B. In another example (see Figure 4.13), he stressed that all the elements in the domain can correspond to the same element in the range.

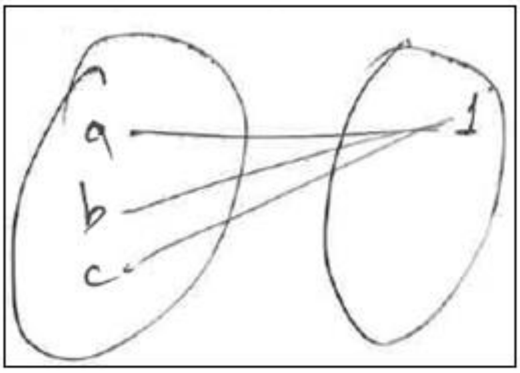


Figure 4.13: Excerpt from Ali's instruction

Although Ali was aware of the univalence property of functions, he could not use it functionally while deciding whether the relations given in the form of set of ordered pairs, algebraic expressions, and graphs are functions or not. For him, in question 10, the student's response to items (b), (d) and (e) are wrong, while those to (c) and (f) are correct. Thus, he correctly associated items (b), (d), and (e) with a function, whereas he incorrectly did not identify items (c) and (f) as a function.

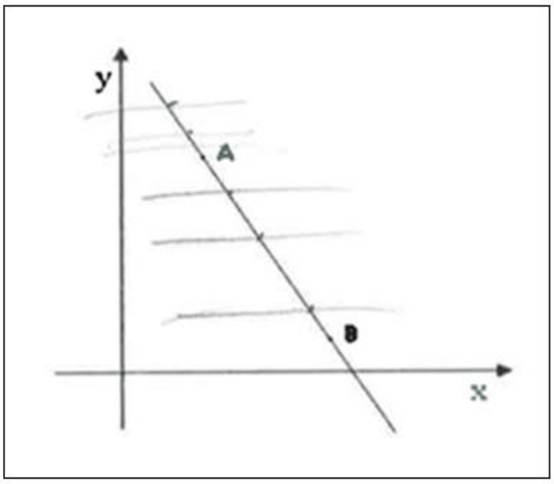


Figure 4.14: Ali's response to question 15

As regards the fact that the relation in item (d) is a function, he provided the following explanation: "Every element has an image". While justifying that the graph in item (b) is a function, he used the vertical line test: "If the lines drawn parallel to y axis intersect the graph at one point, then it is a function". He justified functionality of $y = 4$ with the vertical line test again stating that "lines drawn parallel to y axis intersect at a single point", but he often confused the vertical line test and the horizontal line test. For instance, trying to decide whether the graph drawn by the student in question 15 is a function or not, he resorted to horizontal line test (see Figure 4.14), while he used the vertical line test to answer the interview question 6 (see Figure 4.15).

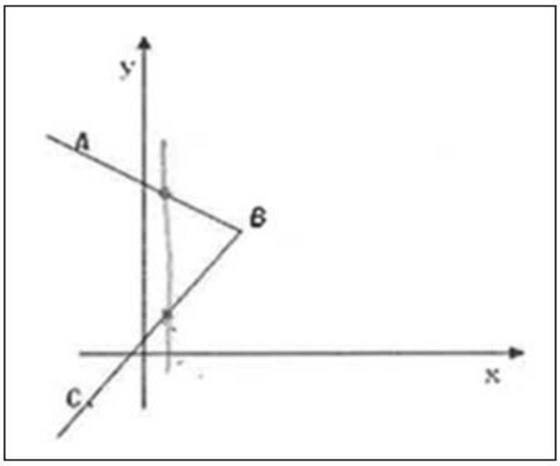


Figure 4.15: Ali's response to the interview question 6

During the interview, he said he faintly remember the test. This is illustrated in the following dialog:

> **Researcher:** How do you decide whether a given relation is a function or not?
> **Ali**: For it to be a function, no element should remain un-corresponded in the domain. In other words, all the coffee beans, or cattle should be involved, or all the given numbers should be given a corresponding value. If it is a graph, I will draw parallel lines and see at how many points it corresponds. There is a test, which I hardly remember, but I will check it up in the book. As far as I remember, if the vertical lines intersect at a point, then it is a function.

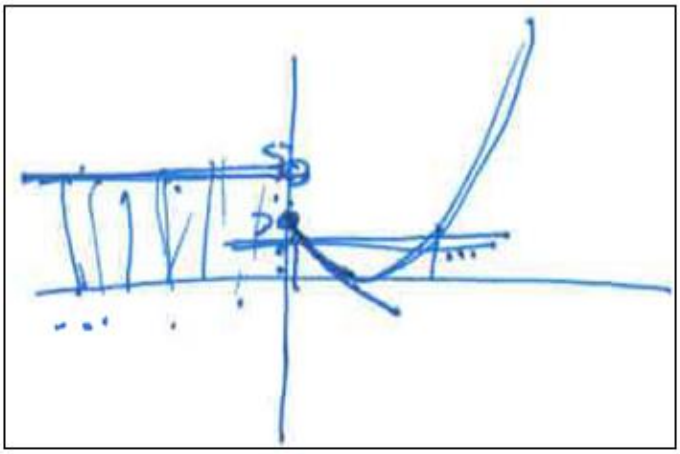


Figure 4.16: Ali’s response to question 10-item (c) in the interview

He had accounted for that the fact that the relation in item (c), $f : R \to R$ $x \to \begin{cases} -3x^3 + 3, & if \;\; x \geq 0 \\ 5, & if \;\; x < 0 \end{cases}$, is not a function with the following words: “It has two different images”. In the interview, he was asked how he had identified the two different images of the function; he drew a graph and resorted to the horizontal line test (see Figure 4.16). He stated: “When we draw its graph, two images will emerge from here, from 1 or 2. This is not a function because it intersects here at two points”.

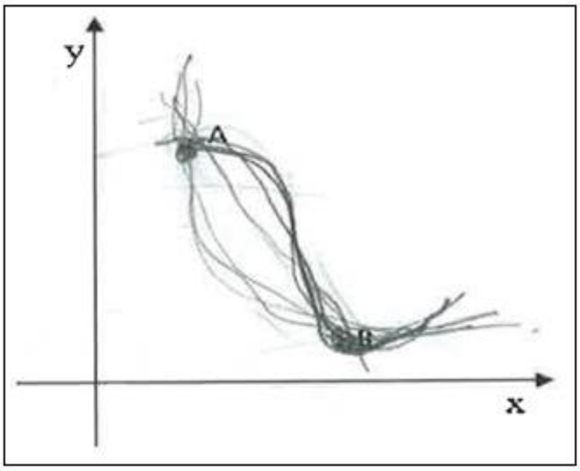


Figure 4.17: Ali's response to question 15 in the interview

According to Ali, a function must do the correspondence between the elements of two sets by means of an arithmetical or algebraic rule. As a matter of fact, he said the relation in item (f), $\{(1,4),(2,5),(3,9)\}$, is not a function, providing the following explanation: "The image set is not rule based". Also, in the interview, he made mention of a curve graph that passes through points A and B (see Figure 4.17) and implied that it is not clear whether the curve he drew is a function or not:

> It can also be like this [draws a curve], but this does not tell anything about whether it is a function or not. I guess it is when we think of it in a graphical representation. I mean it can be drawn like this [the curve drawn by the student] or like this [the curve he drew]. It cannot be drawn in any other way actually.

Moreover, as can be seen in his following words, he thought only piecewise function graph passing through points A, B, C in Figure 4.15 can be drawn:

> [Draws a line parallel to y axis] We draw this. Having two images, this is not a function. What kind of a thing can he draw here? [Thinks] Could be piecewise function ... [tries to draw a piecewise function]. Only

piecewise function is possible. No other way. What do we have usually? Connecting these three points is easiest. If you give me two points, I cannot think of any other way of connecting them. Even if I can, usually it is drawn like this.

Relating a domain and range to its graph: Ali accurately related a domain and range to its graph in the questionnaire (question 11). He checked the x axis for the domain interval and y axis for the range interval. However, his comments both in the questionnaire and in the interview revealed that he had difficulty identifying the function graph whose image set is the immediate subset of the range. For him, the student's response to graph (c) was incorrect, but he did not elaborate why. During the interview, he focused on whether the graph is a function or not, rather than on its domain and range. He applied the horizontal line test to decide functionality of the graph (see Figure 4.18).

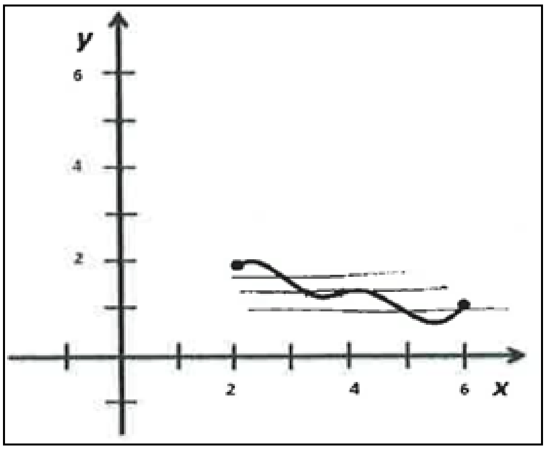


Figure 4.18: Ali's response to question 11-item (c) in the interview

This is illustrated in the following dialog:

Researcher: You say the student's response to item (c) is wrong; can you explain how you made this decision?

> **Ali:** This is not supposed to be a function, for when we draw horizontal lines [draws lines parallel to the x axis], both will have the same image! Therefore, it cannot be a function! It is after all the most important thing we teach to students!

In the class, as stated in the course book (MEB, 2012, p. 76), he announced that "the vertical projection of points on the graph onto x axis forms the domain, and that on the y axis forms the image set of the function".

Identifying two equal functions: Identifying the two equal functions, Ali took into consideration the domain of the function and the equality of the images of domain elements. In the questionnaire, he correctly viewed that the student's responses to items (b) and (c) are wrong. His comments revealed he thought that function $h: N \to N,\ h(x) = 2x+3$ and graph (c) are not equal to function $f: N \to N$, $f(x) = 4x+6$. He ascribed the fact that h is not equal to f to that same elements have the different images.

He indicated that the student's responses to items (a) and (d) are correct. Therefore, he assumed that function $g: R \to R,\ g(x) = 4x+6$ and graph (d) are equal to function $f: N \to N,\ f(x) = 4x+6$. His justification to the fact that graph (d) is equal to f is as follows: "Every element has an image. The elements which are not natural numbers do not have an image". He justified that g is equal to f as "… because $N \subset R$". However, he corrected this in the interview and indicated that g is not equal to f. According to him, the reason for this was the image of a natural number under f and the image of a real number under g may not be the same. He noted that:

> I should have said that they are not equal. I probably thought like this. As N is the subset of R, I can find an image of every element that I take from N but, under f, I cannot find image of every element that I take from g because here [f] I should take natural number. Every natural

number has an image under g but a number I take from here [g] may not have image here [f].

In the class, he defined equality of two functions as following: “If image of an element under two different functions are equal, then the functions are equal”. He used the example in the course book (see Figure 4.19). He found the image of domain elements for both functions and compared the images to show the equality of them. Because each element in the domain has an equal image, he indicated that f and g are equal functions. He highlighted that “equal functions will take the elements from the domain to the same value”.

ÖRNEK

A = {–4, –1, 1, 3} ve B = {7, 9, 115, 199} olmak üzere f : A → B, f(x) = x⁴ + x³ + 7 ve g : A → B, g(x) = 13x² + x – 5 fonksiyonları veriliyor. f ve g fonksiyonlarının eşitliğini gösterelim.

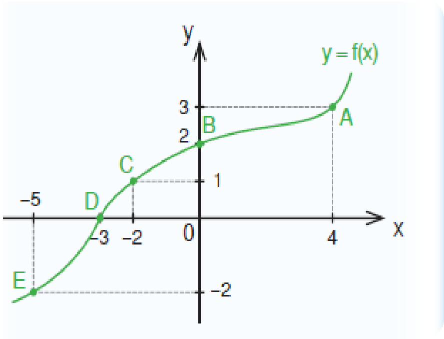
[EXAMPLE: Given that $A=\{-4,-1,1,3\}$ and $B=\{7,9,115,199\}$, function $f:A\rightarrow B, f(x)=x^4+x^3+7$ and $g:A\rightarrow B, g(x)=13x^2+x-5$ are given. Let us show the equality of f and g.]

Figure 4.19: Classroom activity I (MEB, 2012, p. 76)

Locating pre-images, images, and (pre-image, image) pairs on the axes in graphs: It was observed that Ali could not read the graphs of functions properly. In question 13, he pointed out that there are some mistakes with the student’s response to the question, yet, he did not specify these mistakes. In the question wherein he was asked to indicate the correct answers, he could only identify the elements of the domain correctly. He could not correctly identify the range elements and the points which indicate or which do not indicate (pre-image, image) pairs. He provided vague responses during the interview, often made mistakes answering the question. For example, he stated that point G on the x axis is an element of the range like E and B. According to him, "the elements of the range could also be E, B, G. Why? It corresponds to 0, or when this is 3, it is 0 to 3". He stated the elements that indicate (pre-image, image) pairs are those above the function. However, it did not occur to him that C is a point of this sort; on the contrary, he thought B is a point like this because of the coordinates $(0,0)$. He hesitated while identifying the elements which do not indicate (pre-image, image) pairs, examined the question a few times, and could not produce a clear response.

In question 14, he found the student's response to item (a) correct, and claimed that "the image of 0 is point A". He marked the student's response to items (b), (c) and (d) as wrong but did not specify its reasons. During the interview, how he identified whether the student's responses are correct/incorrect, and how he identified pre-images of images on a given graph were investigated. It was found out that he did this by reversing the axes. To be more specific, trying to identify the pre-image of point A on graph (a), he reversed the axes and, similarly to what he indicated in the questionnaire, he said "I reversed it like this, the point with the image A is 0". As the graphs (b) and (c) are not one-to-one, he had difficulty reversing the axes. As a result, he could not determine the pre-image of point A.

In the class, he used an activity in the course book (see Figure 4.20). This activity provides a graph and stages questions about the graph. Firstly, the coordinates of points A, B, C, D, E are asked. Secondly, the explanation of $A(x, y) = A(4,3)$, $x = 4 \wedge y = 3$, $f(x) = y \Rightarrow f(4) = 3$ is provided, and it is asked to fill in the dotted spaces. Thirdly, considering that $f(x) = y \Leftrightarrow f^{-1}(y) = x$, it is again asked to fill in the dotted spaces.



Yanda f: R $\longrightarrow$ R, y = f(x) fonksiyonunun grafiği verilmiştir.

Grafik üzerinde işaretlenen A, B, C, D ve E noktalarının koordinatlarını noktalı yerlere yazınız.

A(4 , 3) B(.. , ..) C(.. , ..)

D (..,..) E (..,..)

[Graph of function $f : R \to R, y = f(x)$ is given. Write the coordinates of points A, B, C, D, and E in the blanks.]

A(x , y) = A(4 , 3), x = 4 $\wedge$ y = 3 ve f(x) = y den f(4) = 3 olduğuna göre aşağıdaki noktalı yerleri doldurunuz.

f(x) = y	f(0) = ...	...... = 1
f(−3) = ...	f(−5) = ...	(fof)(−5) = ...

[If $A(x, y) = A(4,3), x = 4 \wedge y = 3$ and $f(x) = y$, then $f(4) = 3$. Fill in the blanks below.]

---

f(x) = y $\Leftrightarrow$ f^{-1}(y) = x olduğunu göz önüne alarak aşağıdaki noktalı yerleri doldurunuz.

$f^{-1}(1)$ = ...	$f^{-1}(...)$ = 0	$f^{-1}(...)$ = 4
$f^{-1}(...)$ = −5	$f^{-1}(2)$ = ...	$(f^{-1} o f^{-1})(1)$ = ...

[Considering $f(x) = y \Leftrightarrow f^{-1}(y) = x$, fill in the blanks below.]

Figure 4.20: Classroom activity II (MEB, 2012, p. 109)

Ali stated that:

> $y = f(x)$ means the graph showing what a point on the x axis corresponds to on the y axis! Anyway we have been doing this all the time so far, haven't we? We were writing the x value to where it belongs in the function, and finding the value that corresponds to it. When finding the y value that corresponds to x value on the graph, you move from x towards the graph. The distance to y axis at that point is the value you are looking for.

While finding the pre-images, Ali used the method of reversing the axes just as he did in the interview:

> What if it gets the other way around? I mean, what if it is f^{-1}? Imagine, if it is $(3,2)$, then the reverse is $(2,3)$. In other words, you will take the first component from y and second from x. It says $f^{-1}(1)$, so what is this? The easy way is like this! This is your x, and this is your y. You reverse it like this, and when you do it, it will have been x any more, and this will have been y. So, the only thing you will do is move the graph sidewise, make x, y, and make y, x.

4.2.1.2 Ali's Knowledge of Student Difficulties in the Function Concept

The student difficulties structuring Ali's knowledge of students were as follows: students' conceptions of the function concept, potential areas of difficulties for students in mastering function concepts, students' difficulty with the essential features of functions (arbitrariness and univalence), students' difficulty in relating a domain and range to its graph, students' difficulty in identifying two equal functions, and students' such limited conception that every function needs to be linear. The descriptions of his knowledge of student difficulties were developed from his responses on the function concept questionnaire (see Appendix A), the follow-up interview (see Appendix C), and classroom observations.

Students' conceptions of the function concept: Ali thought that the students generally identify functions in the way he himself does, i.e. as an operation. He pointed out that the students define function as, "the transformation of an object into a new entity" (question 1) and cite examples to function like, "the production of hams and sausages from cattle in the slaughterhouse" (question 2).

Potential areas of difficulties for students in mastering function concepts: Ali was unaware of the potential areas of difficulties for students in mastering function concepts. For him, "it is more difficult for students to comprehend graphs and find the domain of a function" (question 6) and "it is easy for students to find the inverse of a function and the composition of functions as they act within certain rules" (question 7). For most of his students it was more difficult to learn the inverse of a function and the composition of functions, and it was easy to learn relations.

Students' difficulty with the essential features of functions: Ali did not seem to have a deep insight into the students' difficulties and limited conceptions about the function concept including difficulties in many-to-one correspondence, difficulties with functions represented by a disconnected graph, difficulties with functions given by more than one rule, difficulties with the verbal representation of functions, difficulties with the set notation of functions, and students' such limited conception that a function must include some algebraic formula. He generally thought students' inability to comprehend the subject is responsible for their mistakes, or he pointed at other factors that are remarkably different from those discussed in the related literature. To Ali, for instance, in question 10, the student did not perceive the representation in item (d) as a function because s/he might take "0 as positive", s/he did not consider the representation in item (e) as a function because "the student might not have comprehended the constant function", and s/he did not identify the representation in item (a) as a function because "the student might not remember the domain and range".

To better understand the extent to which the students have comprehended the concept of function, Ali said: "I would give an element that has more than one image and have the students check the result" (question 5). Clearly, he intended to understand students' understanding of the univalence requirement of functions. On the other hand, classroom observations revealed that Ali's main goal was to equip students with procedural skills. He did not dwell on the meaning of concepts at all. He mostly presented the related rules, formulas, and when possible, practical tips with respect to the function. He did not attach much importance to whether the students learn the concept of function or not. For him, the students would anyway be told that the questions include functions. Knowing how to do the operations, therefore, was so much the more crucial for them. He did not mention that a function can be shown by different representations such as a graph, an arrow diagram, or a set of ordered pairs. While the students were copying down on their notebooks the definition of function in the course book, he said functions can be named with different letters like $h,\ g$. He provided examples which require that functions are shown by sets of ordered pairs and arrow diagrams. One of these examples was: "Given $A=\{-2,0,2,3\}$, $B=\{-3,0,1,4,5,7\}$, $f: A \rightarrow B$ and $f(x)=2x+1$, show the function by means of a list and a map, and find domain and range of f". He found the image of domain elements. Showing these in the form of a set of ordered pairs and an arrow diagram, he denoted the range and image sets (see Figure 4.21). However, he did not make any mention of that these are the two different representations of the same function. Also, he did not examine functionality of such varied relations.

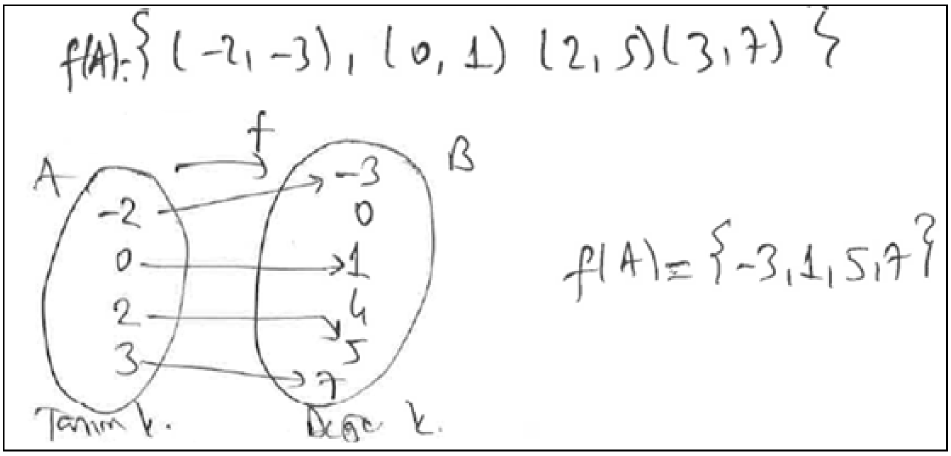
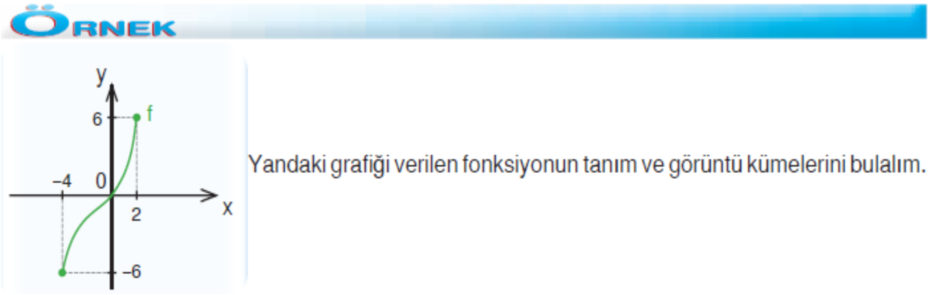


Figure 4.21: Excerpt from Ali's instruction

Students' difficulty in relating a domain and range to its graph: Ali correctly viewed the student's response to graph (b) in question 11 as incorrect and claimed that the student had mixed domain with range. However, he incorrectly viewed the student's response to graph (c) as incorrect and said that this graph does not represent a function of the type required by the question. His comments in the interview showed that, in fact, he himself experienced difficulty in relating a domain and range to its graph which image set is only a subset of the range. In the class, he used an example in the course book (see Figure 4.22) and, as stated in the book, he provided quite a complex explanation to find the domain and range of a function given in graphical representation: "The vertical projection of points on the graph onto x axis forms the domain, and that on the y axis forms the image set of the function. Thus, the domain of the function above is $[-4,2]$ and the image set is $[-6,6]$ range".



[EXAMPLE: Let us find the domain and image of the function with the given graph.]

Figure 4.22: Solution of the example (MEB, 2012, p. 76)

Students' difficulty in identifying two equal functions: Ali also experienced difficulty in identifying two equal functions. In question 12, he indicated that function $g: R \to R, g(x) = 4x + 6$ is equal to function $f: N \to N, f(x) = 4x + 6$ because $N \subset R$. He corrected this in the follow-up interview. For him, the student's response to items (b) and (c) are wrong. He provided a vague explanation for the possible reason for the student's way of thinking to item (b). That is, he stated that since the images of the same pre-images are different, the students considered that function $h: N \to N$, $h(x) = 2x + 3$ is equal to f. He expressed that "the student could have substituted the given values" thus; s/he considered that graph (c) is equal to f.

Students' such limited conception that every function needs to be linear: In fact, it was noticed that in relation to the functions that pass through two points, say A and B, Ali experienced the same difficulty held by high school students. In the questionnaire, he did not answer question 15. In the interview, he stated that only a linear function or a parabolic curve can be drawn through A and B, and he did not provide much for the possible reasons for the student’s way of thinking. Also, he thought that only a piecewise linear function graph passing through points A, B, and C on the plane can be drawn.

4.2.2 Ali’s Students’ Learning Outcomes of the Function Concept

A total of 33 students were enrolled in Ali’s course. The students’ elementary school GPA average was 63.33 and first semester mathematics grade point average was 37.96. Ali warned that his students perform poorly in mathematics classes, and they do not really like mathematics. He said that few of the students are interested in mathematics. This minority believe that knowledge of mathematics will help them in other courses. They think that mathematics will help them deal with problems of daily life more easily and will be part of any exam in the future. However, most students think that mathematics will not be useful for them and that they have to learn it just because it is a course to complete. Ali added, since mathematics is made an elective course, therefore, the tendency have been that majority of the students will not choose mathematics with the fear that they will fail in the course or with the conception that mathematics is not useful. Classroom observations revealed that many of his students had indeed certain difficulties with basic mathematical operations.

The function concept test (see Appendix D) and two teacher-designed exams provided the data for examining Ali’s student learning outcomes of the function concept including the students’ understanding of the essential features of functions; relating a domain and range to its graph, and vice versa; identifying two equal functions; and locating pre-image, image, and (pre-image, image) pairs on the axes in the graphs.

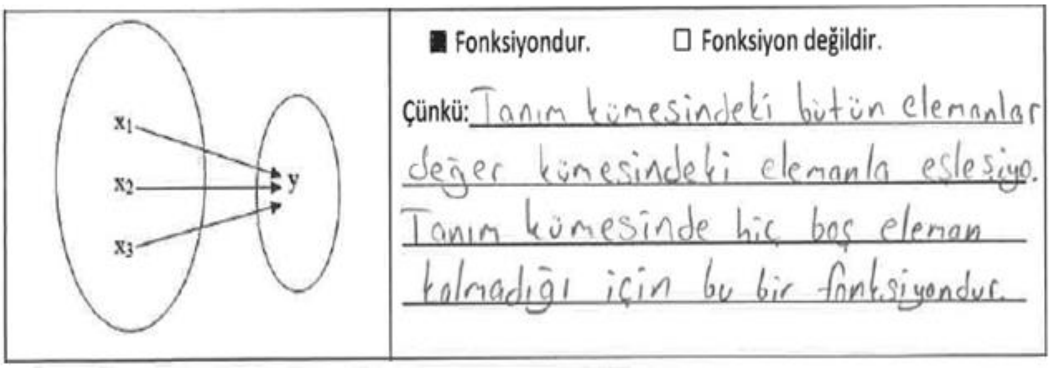
Conceptions of the function concept: The results of the function concept test and the exams revealed that at the completion of instructional units on functions, Ali's students' understanding of function was fairly limited. A few of them (n=5) defined function as a correspondence (question 1). Here are some of the sample definitions: "The concept of function is one of the most important concepts in mathematics. It is a relation that correspond every element of a set to an element in a second set", "It is a unit corresponding all elements of set A to only one element". Some students (n=4) wrote certain subtopics that relate to functions. In one student's words "functions are of different kinds such as constant function, inverse function". Many others viewed function as mathematical operations. Here are some of the statements: "Function is the A, B statements of an equation", "Function is a mathematical operation. It is no use in daily life, but essential for success in the exams", "Functions are operations that comprised of mathematical operations".

To give an example to function (question 2), most of them (n=10) indicated algebraic functions (e.g., "$f: R \to R$, $f(x) = 2x + 11$", "$A = \{1,2,3,4,5\}, B = \{2,3,7,6\}$, $f: A \to B$, $f(x) = 3x - 2$"). Some provided correspondences (n=7). However, the correspondences cited by most of them (n=5) are not define a function (e.g., "Ali-Pear, Ali-Banana, Ali-Cherries", "Mehmet-Apple, Mehmet-Apricot, Mehmet-Cherries"). Some students (n=4) gave examples in the meaning of operation (e.g., "Function likes the meat becoming bacon, or tomatoes becoming tomato paste").

Most students (n=25) did not respond to the question that inquires different representations of a function (question 6). Seven of them expressed that they did not know the answer. Five of them said there are no other representations of function, while another five said there are; yet they did not explain their responses. Four stated that a function can be shown by letters like f. None of them made any mention of such representations as a set correspondence, a graph, or a set of ordered pairs.

The essential features of functions: When identifying functionality of such varied relations the students generally did not take into consideration the essential features of functions (question 7). Their limited perceptions as regards the function concept became apparent while they were deciding whether relations in different representations define functions or not. Thinking that functions are equations that need to be solved, most (n=24) indicated that $y = 4$ does not define a function in their following words: “Nothing to solve” or “No operation to do”. Most (n=20) associated the relation $f : R \to R, x \to \begin{cases} -3x^3 + 3, & if \ \ x \geq 0 \\ 5, & if \ \ x < 0 \end{cases}$ with a function. Also, for some students, it defines a function because it needs to be solved. One made such a comment as “we find the unknown, which is a function”. Similarly, some students (n=12) thought that the relation $\{(1,4), (2,5), (3,9)\}$ does not define a function because not a certain rule dominates it. For some, “the square of 1 should be 2, the square of 2 should be 4, and the square of 3 should be 9” (here, it is seen that some students thought the square of 1 is 2). Most students (n=17) thought that the graph of a function must be nice, so they did not consider item (e) as a function because of the disconnectedness in the graph. Some of the justifications were as follows: “All lines should go through a plane, but they do not” and “They do not unite and they do not go straight”.

Unlike with other forms, the students were more successful with the set correspondence representation. Some of them (n=9) expressed that the representation in item (d) defines a function and pointed to the univalence property of functions (e.g., see Figure 4.23). On the other hand, most of them (n=22) thought the relation “a correspondence which corresponds all positive numbers to 1, all negative numbers to -1, and 0 to 3” defines a function just because of the word *correspondence* included.



[All elements of the domain assign to the range element. Since no domain elements remain unassigned, this is a function.]

Figure 4.23: Response of Student #6 to question 7

When asked to draw function graphs that pass through points A and B on the plane (question 13), the majority of students thought that the function graphs that pass through these two points should be linear. Without considering the arbitrariness feature of functions, most (n=22) drew line graphs (e.g., see Figure 4.24) and indicated that a certain number of graphs (e.g., one, two…) passing these two lines can be drawn. When asked to draw function graphs that pass from several points like A, B, C, D, E, F on the plane (question 14), none of the students considered the univalence feature of functions. Accordingly, almost all of them connected the dots randomly (e.g., see Figure 4.25).

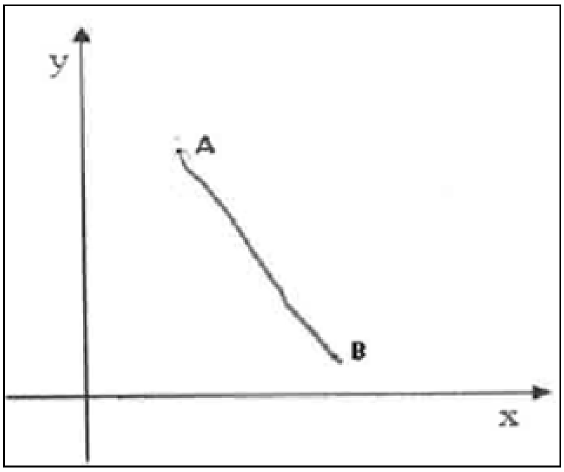


Figure 4.24: Response of Student #7 to question 13

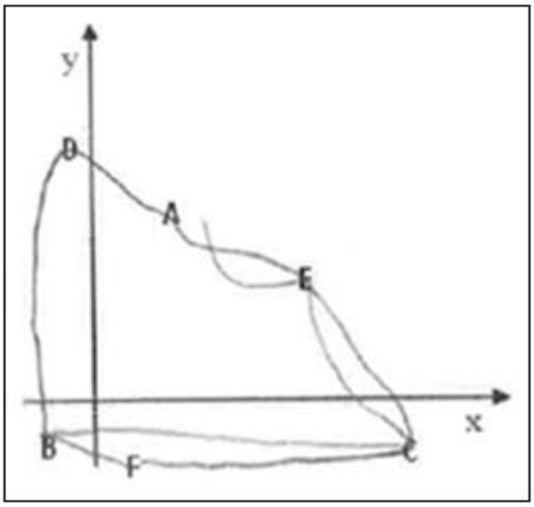


Figure 4.25: Response of Student #7 to question 14

Relating a domain and range to its graph: When the students were asked to identify the function graph/s, with the domain $\{x : 2 \leq x \leq 6\}$ and range $\{y : -1 \leq y \leq 4\}$, among the given ones, and specify why (question 8), only some of the students knew the correct answer. It became evident in their comments that most students answered the question without knowing why they did it in a certain way. Nine students thought the correct answer is item (a). While some students did not provide any comment, one student said the points are appropriate; one said he did not know the reason, and four said there are two functions in the graph. Five of them thought that the correct answer is item (b). One of them said the points are suitable, and two said they did not know the reason. A total of 12 students thought the correct answer is item (c). Again some students did not provide any explanation. Three students stated that they did not know the answer; three said domain and range are suitable, and one said the graph is nice.

Identifying two equal functions: While identifying the function that is equal to function $f: N \rightarrow N,\ f(x) = 4x + 6$ (question 9) the students did not take into consideration the domain and the range, but only made their decisions based on the representational form of the functions. Therefore, they did not mark the appropriate items which bear the graphs. The majority of them (n=25) marked items (a) and (b) just because they are algebraic functions and they involve x. Those who marked item (a) stated that the functions are same only that the letters (f and g) are different, while those who marked item (b) stated that sets are same only that h is half of f.

Locating pre-images, images, and (pre-image, image) pairs on the axes in graphs: The students did not demonstrate an understanding of reading graphs of functions. The majority of them wrote the domain and the range elements, and the (pre-image, image) pairs of a function whose graph is given randomly, without knowing why they did so (question 10). Similarly, most of the students could not comprehend the meaning of locating the image or the pre-image of a point, say point A, on the axes in a graph (questions 11 and 12). Some drew the symmetrical image of the given graph according to different lines (e.g., see Figures 4.26) and most of them marked one of the items that present the graph (e.g., see Figure 4.27).

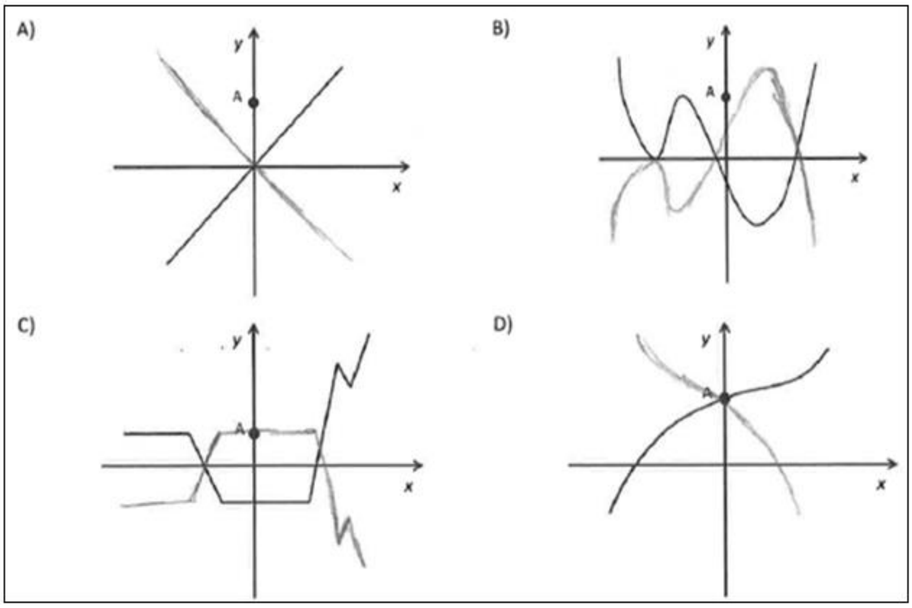


Figure 4.26: Response of Student #15 to question 11

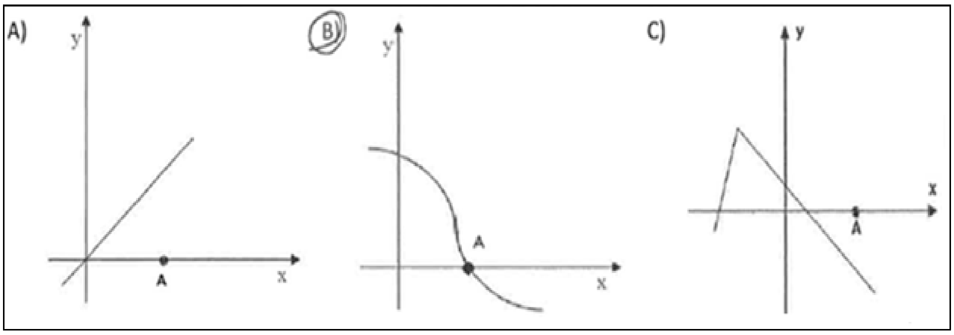


Figure 4.27: Response of Student #1 to question 12

Similar results appeared in the second teacher-designed exam. One of the questions in this exam required the identification of the image and the pre-image of some elements in a function whose graph is given (see Figure 4.28). Most of the students (n=17) did not respond to the question while most others made calculation mistakes. Only two students made the question properly.

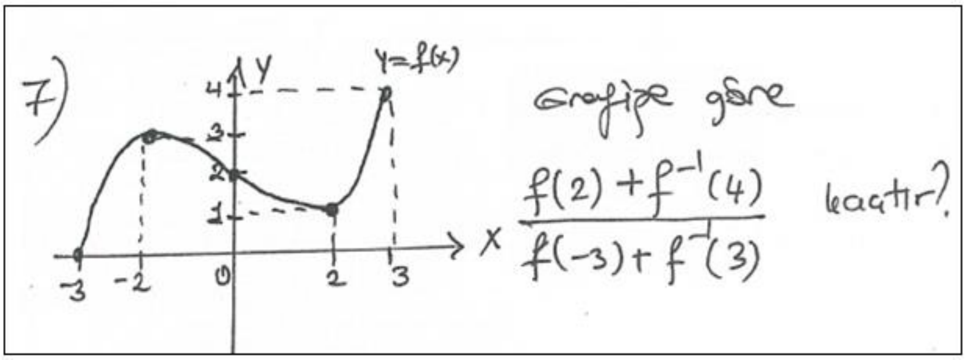


Figure 4.28: Response of Student #9 to the second exam question 7

Also, students could not find the pre-image or the image of some elements under algebraic functions. The study revealed that many students had certain difficulties with basic mathematical operations. In question 15, some (n=10) students wrote the images or the pre-images haphazardly. Nine of them found the image of 2 and the image of 1, yet they made mistakes in the operation trying to find out the pre-image of -26. They failed to find the image of $-\frac{1}{2}$. The rest of the group (n=13) did not respond to the question. A few of them indicated that they did not know the answer.

Similar results emerged in the first exam. One of the questions in this exam required the students to identify the images of some elements under an algebraic function ($f : R \rightarrow R, f(x) = 5x - 4 \Rightarrow f(2) = ?, f(-3) = ?$). Only two students succeeded in doing this question. Most made various calculation mistakes (e.g., see Figures 4.29). In the same exam, only one student could correctly do the following question: “ $f : R \rightarrow R,\ f(3) = 6,\ f(6) = 3 \Rightarrow f(1) = ?$”. Most others could not do it.

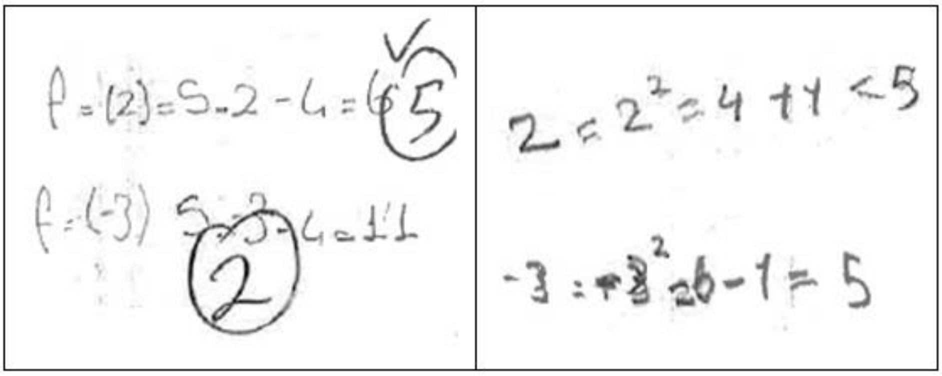


Figure 4.29: Response of Student #5 and #24 to the first exam question 10

4.3 The Case of Fatma

On the basis of the questionnaire results, which assess teachers' KCS of the function concept, Fatma was categorized into strong group with her high score. She had a bachelor degree in mathematics education. She had 25 years of teaching experience. She had been teaching in the technical and industrial vocational high school where the study was applied for the last 15 years. Like Ali, she did not receive any training on mathematics education or the teaching of functions.

The study has revealed that Fatma thinks a mathematics teacher should have complete mastery of the field. This is important in explaining the rationale of the mathematical facts and concepts. However, it is not enough for the teacher to have good knowledge of mathematics. The teacher should also know how to teach it. S/he should continuously refresh his/her knowledge and teaching skills. For Fatma, also, it is very important that the teacher is aware of his/her students' thinking about a particular content, and what kind of difficulties they have about it. If the teacher has such awareness, s/he can know how to teach that content. Nevertheless, Fatma thinks that it is quite difficult to understand how the students think because they cannot express clearly their ways of mathematical thinking. She said, therefore, she generally finds it difficult to pinpoint the areas of difficulty for students.

Fatma pointed out that she made lesson plans regularly in the first ten years of her teaching profession. However, once she had gained enough of experience, she quit making lesson plans. Classroom observations revealed that she mostly made mental lesson plans. She sometimes spontaneously constructed the concepts related to function based on how the class progresses. She generally prepared the exercises herself, though she also sometimes used the ones in the course book or in some other source books.

4.3.1 Fatma's Knowledge of Content and Students about the Function Concept

As so Ali, the descriptions of Fatma's KCS were developed from her responses on the function concept questionnaire (see Appendix A), the follow-up interview including a card sort task (see Appendix C), and classroom observations. These descriptions are presented under two headings in the following sections.

4.3.1.1 Fatma's Content Knowledge of the Function Concept

The key aspects of the function concept structuring Fatma's content knowledge of this concept were as follows: conceptions of the function concept, the essential features of functions (arbitrariness and univalence), relating a domain and range to its graph, identifying two equal functions, and locating pre-images, images, and (pre-image, image) pairs on the axes in the graphs.

Conceptions of the function concept: It was evident in her responses to both the function concept questionnaire and follow-up interview Fatma perceives function as a set correspondence. When she was asked how the students should define and exemplify function, she said they should define it as “a rule that takes one element to another” and illustrate it as in “ $f(x)=3x+1$”. Indeed, Fatma was in the opinion that students can also define function likening it to an object’s going through a process and transformation of it into a different object (e.g. transformation of cloth into a school uniform). However, according to her, this “does not exactly define [it]; only that this roughly describes it. Yes, could be, it reminds of function, but does not give a thorough definition of it”. She said, if the students define function as a rule taking an element to another element or to itself, it means that they have comprehended the essence. In this case, they will also know that no element will be left in the domain and no element in the range would have two images.

Her perceptions also became evident in the class. To illustrate function, she talked about functions of things, citing various examples in the immediate environment. She said, for example, the pencil case is for keeping the pens, and the telephone is for transmitting voice. She also brought up the meaning of function as an operation. She used the analogies of a washing machine changing dirty clothes into clean ones, and soil transforming seed into tomatoes. However, she mentioned that the mathematical definition of function is different from these and defined function as a special relation between two sets. As stated in the course book (MEB, 2012, p. 75), she provided the set theoretic definition of the function concept, and said that a function from Set A to Set B is a relation, and just like a relation, it is a subset of Set AxB. She linked certain phrases in the definition with the conditions in the activity:

> Let’s see what the definition says! The relation that connects all the elements in Set A to one and only one element in Set B is called a function. What we mean by ‘no elements will be left unmatched’ is indicated by this phrase in the definition: ‘All elements in Set A’. As we

said, nobody will remain hungry. That is actually what is meant by 'all the elements in Set A!' Remember the meaning of 'all', we covered it when we studied the 'Logic' earlier. It means all of something. It is not ok even if one of them is missing. Therefore, the word 'all' here is not used just like that randomly. 'One and only one element in Set B' refers to, remember what we said about it, that they will eat only one dish.

It was found that, unlike Ali, Fatma views algebraic expressions, set of ordered pairs, and graphs as alternate ways of expressing functions. However, rather than taking the tables as an alternate representation of functions, she uses them to facilitate the transition between algebraic expressions and graphs. In the event that a student asks her if the same function has different representations, she pointed out that she would explain it to the student by "using different letters, $f, g, h...$ by showing it with a diagram. $f: A \to B,\ f(x) = y$". To probe this issue in the interview, she was asked to give an example to a function that the students can come across in the class, and show whether there are other forms of showing this function. She gave the example of $f(x) = 3x+1$. This, she said, could also be shown with $y = 3x+1$ or $f: x \to 3x+1$. Just as in the interview with Ali, Fatma was given a series of cards. She was asked to imagine how the students would group and identify these cards. She stated that the students would automatically group the cards according to their representational format. That is, they would form three groups: all the graphs together, all the equations together, and all the set of ordered pairs and tables together. The set correspondence relation (Card 13) would be left behind as it is one of a kind among these. When asked how the students would go grouping in a different way, she thought that a second grouping could be in the form of an algebraic expression, its graph, and algebraic expression in the form of a solution set, which was created like a set of ordered pairs or a table. Therefore, she grouped $y = x$ and $\{(5,5),(-53,-53),(0,0)\}$ cards and the card with the $y = x$ graph on it (see Figure 4.30). When she was asked what kind of a grouping she would expect of students, she said she would like to see such a grouping. However, she did not consider functions and non-functions as a way of sorting the cards.

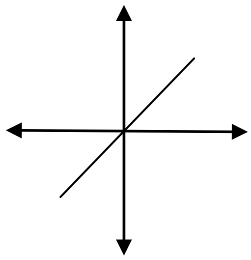
4	5	10
$y = x$	$(5,5)\ (-53,-53)\ (0,0)$	

Figure 4.30: The kind of grouping Fatma would expect of students

In the class, she announced that functions can be written in different ways, i.e. $f : A \to B$, $f(x) = y$, $A \xrightarrow{f} B$, $x \xrightarrow{f} y$, $f : x \to y$. She stressed that a function can be represented in the form of an algebraic expression, a set correspondence, a set of ordered pairs, and a graph. She wrote a function, $f : \{3,4,5\} \to Z,\ f(x) = 2x + 3$, and showed it on a set correspondence and a set of ordered pairs. Also, she represented it by drawing its graph. Besides the set correspondence and algebraic statement representations of functions, she frequently used the set of ordered pairs representation.

The essential features of functions: It was clear that Fatma was aware of the essential features of functions, and she used it functionally while identifying functionality of such varied relations. She correctly found all the responses of the student to question 10 wrong. Therefore, she thought all the relations are functions. She frequently emphasized the univalence property of function during the interview. To exemplify, she was asked how she illustrates the function concept to students; drawing an analogy, she stressed the univalence property of function:

> I draw analogies between functions and matches. What I say is that a player in team A cannot mark two opponents at the same time. How can he mark two people, one is here when the other is there? If two opponents run to the opposite directions, where will he go? If a player marks two other players, then it is not a function, it is a relation. If the opposite is true, or two players mark one, I say it is now a function.

She analyzed functionality of some set correspondence relations (e.g., see Figure 4.31) in the class. In the first relation, she said, not all domain elements correspond to range elements, while in the second one domain element corresponds to more than one range elements. She concluded, since these relations do not meet the univalence condition, they do not define functions.

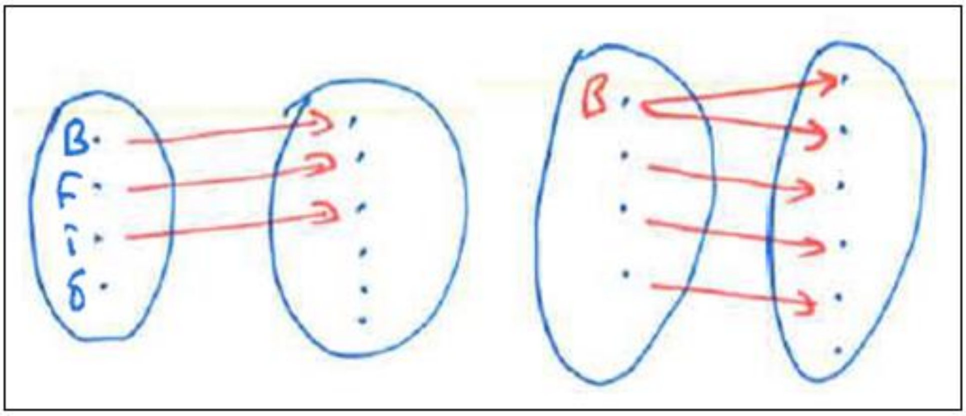


Figure 4.31: Excerpt from Fatma’s instruction

Also, her comments revealed that she is in the opinion that a function does not necessarily make correspondence between the elements of two sets through an arithmetical or algebraic rule. To illustrate, in question 15, she found the student’s response incorrect. She stated that an infinite number of graphs passing through points A and B can be drawn. In the interview, she provided the following explanation:

Researcher: How did you decide that infinite function passing through points A and B can be drawn?
Fatma: It is possible to draw any type of graph passing through points A and B. After all it is infinite; we can draw as many as we wish.
Researcher: Ok, how do you decide?
Fatma: If it is to pass through these two points, it is infinite; draw as many as you want as long as you meet the function rule!
Researcher: Can you give a few examples?
Fatma: Sine function is drawn; cosine function is drawn; tangent is drawn, whatever you want is drawn. What were we doing to it, the vertical line test! According to the vertical line test, any kind of function, with or without a rule that does not pass through two points can be drawn.

$A=\{1,2,3\}, B=\{a,b,c\}$ kümeleri veriliyor. $f:A\rightarrow B$ fonksiyonunun liste biçimindeki ifadesi aşağıdakilerden hangisi olabilir?

A) $\{(1,a),(2,a),(3,a),(3,b)\}$

B) $\{(1,a),(1,b),(1,c)\}$

C) $\{(a,1),(b,2),(c,3)\}$

D) $\{(1,a),(2,b),(3,a)\}$

E) $\{(3,a),(2,b),(1,c),(2,c),(3,b)\}$

[Given $A=\{1,2,3\}$ and $B=\{a,b,c\}$, which of the followings is the list form of function $f:A\rightarrow B$?]

Figure 4.32: Question 2 in the test

Taking attention to the essential properties of functions, through a test she handed out in the class, she analyzed whether the relations that are in the form of set of ordered pairs, algebraic expressions, and graphs are functions or not. One of the questions in the test asked the expression of function in the form of a set of ordered pairs (see Figure 4.32). Upon one student's question, she stated that all the elements in the domain can have the same image:

> **A student:** Madam, can we use 'a' twice?
> **Fatma:** Yes! Two can go to one point, even three or more of them can go to one point. That depends on the characteristics of the function. We have to look at the first set to see if this is a function or not! The elements in the first set will absolutely go to the elements in the second set, and it will happen only once.

N , doğal sayılar kümesi olmak üzere, aşağıdakilerden hangisi N den N ye bir fonksiyondur?

A) $f(x) = 4x - 3$

B) $f(x) = \dfrac{3x+6}{2}$

C) $f(x) = \sqrt{x}$

D) $f(x) = \dfrac{3x+6}{3}$

E) $f(x) = \dfrac{x+1}{x}$

[Provided that N is the natural numbers set, which of the followings is a function from N to N ?]

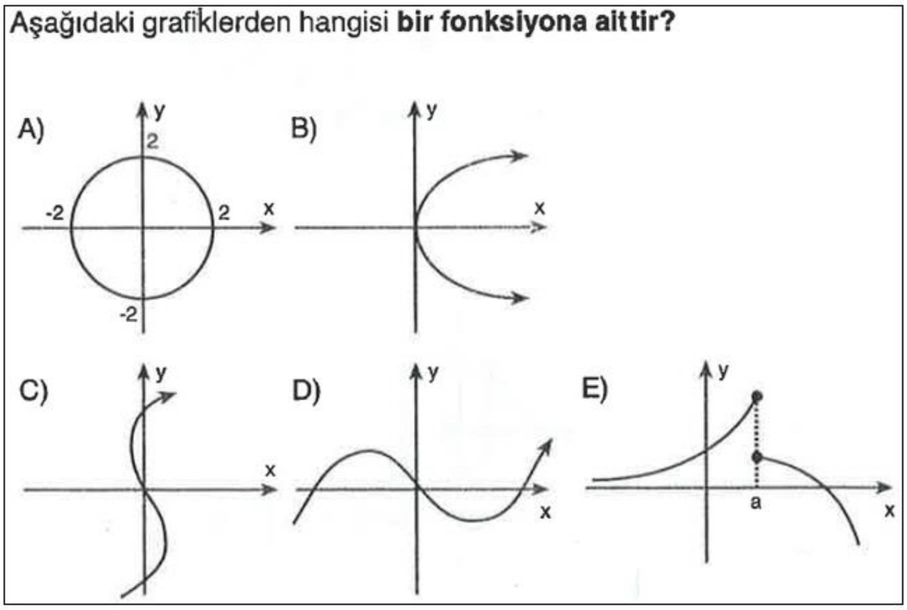
Figure 4.33: Question 3 in the test

One of the other questions asked functionality of some algebraic expressions (see Figure 4.33). While analyzing these expressions, she usually used the method of proof by contradiction. In item (d), on the other hand, she made a generalization:

> **Fatma:** In other words, when you give natural numbers, is the result a natural number again? Or is it minus, or a fractional number? [Reminds the set of natural numbers] Let's look at item (a)! I want you to think what we can write in the place of x.
> **A student:** 3!
> **Fatma:** Ok let's say 3; the result is 9, is it a natural number?
> **Some students:** Yes.
> **Fatma:** What else!
> **A student:** 0!
> **Fatma:** For 0, the result is -3, is it a natural number?
> **A few students:** No!
> **Fatma:** $x = 0 \Rightarrow f(x) = -3 \notin N$. That is, f is not a function from N to N. Write it here next to it! One more thing, for $x \geq 1$ the result would be a natural number. Only for 0, not a natural number, that is 0 does not have an image. Then, we say it is not a function!
> ... [Explains the other items in the same way]
> **Fatma:** Item (d)?
> **A student:** You are on the right track. For 0, it is 2. For 1, it is 3.
> **Fatma:** Let's make a generalization! If we take it to the 3 parenthesis, then this will end up being $x+2$. If x is a natural number, then what will $x+2$ be, again a natural number?
> **A few students:** Yes!
> **Fatma:** Then, for $\forall x \in N$, $x+2 \in N$. All numbers have an image, it is a function!

One of the other questions asked functionality of some graphs (see Figure 4.34). She used the vertical line test to identify whether the graphs define functions. She likened the vertical line test to combing downwards:

> To understand if there is a function in the graph or not, we draw lines parallel to y axis. This is called the vertical line test! Ok, let me explain. You comb the graph with a comb from top to bottom. The pointed ends of the comb should touch the graph within the domain interval once. If they touch it twice even at one point, then it is not a function. Let's give one example for this (see Figure 4.35). [Draws parallel lines to y axis] It passes from two points; it is not a function!



[Which of the following graphs belongs to a function?]

Figure 4.34 Question 8 in the test

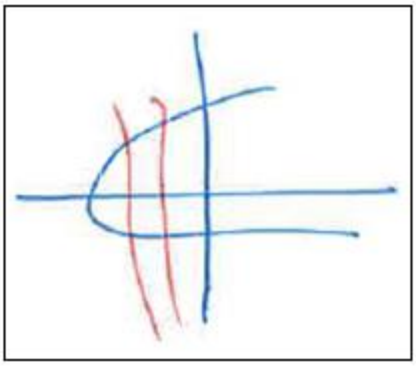


Figure 4.35: Excerpt from Fatma’s instruction

In the follow-up interview, however, she had stated that if the students have understood the definition of the function well, there is no need for such practical rules. In other words, she associated the vertical line test with the definition of function:

> **Researcher:** To help students identify which relations are actually functions, how do you explain the topic?
> **Fatma:** If it is to be a function, the elements in the first set should be in the first component and only for once. After thoroughly understanding function, what if comes a graph? After drawing the graph, I give the vertical line test. I ask if an element x has two images, does it go to two places. For example, does 1 go to both 3 and 5, check it out! This means if the line drawn parallel to y axis intersects the function at two points, then x corresponds to 3 and 5. But the vertical line test works at the graph, no need to use it apart from it. The definition of function exists. If the students has complete knowledge, complete understanding, of the definition of function, it also happens without you telling them. You draw the graph, and ask if it is a graph. The students check whether x goes to two elements. Even if he does not know the vertical line test, it speaks for itself.

Relating a domain and range to its graph: Fatma accurately identified the function graphs whose domain and range are given, and also the function graph whose image set is the immediate subset of range. In question 11, she correctly stated that the student's responses to items (a) and (c) are correct. As regards item (a), she justified her decision by saying that "the borders are correct", implying that the domain and range of the function given in the graph are the given intervals. She marked the response to item (b) as wrong, so she thought the graph does not meet the given domain and range. She added that "x values starts from -1, when in fact they should start from 2". As regards item (c), she said "the student's response is correct because the function is within borders". It is indicative of the fact that, different from many other participants of the first phase of the study, she realized the image set of the function is the subset of the range. Her knowledge also became apparent in the class; she announced that the domain of the function is the interval on the x axis, and the range is the interval on the y axis. What is more, she clearly drew attention to the relation between range and image and addressed that in some cases the image set is a subset of the range (see Figure 4.36): "There is an indefinite number of elements in Set B, set of integers, but our image set is comprised of three elements only. The image set, Set $f(A)$, only consists of y's".

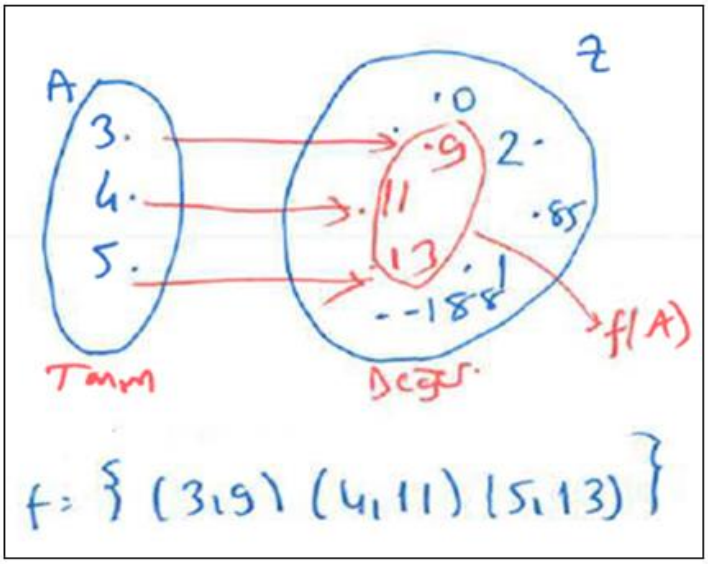
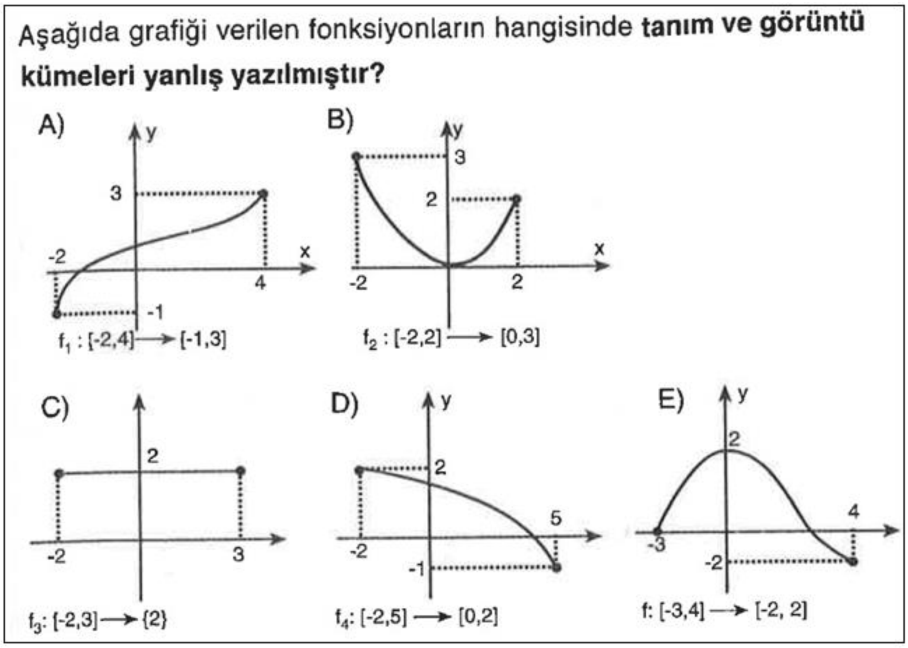


Figure 4.36: Excerpt from Fatma's instruction

Through some other questions in the test (e.g., see Figure 4.37), she wanted to dwell more on the relating the domain and range of a function with its graph. However, the disinterest and misbehaviors of students disrupted the discussion environment, so she just covered these issues superficially.



[For which of the following graphs the domain and the image are written incorrect?]

Figure 4.37: Question 10 in the test

Identifying two equal functions: While identifying the two equal functions, Fatma considered the domain and range, as well as whether the elements in the domain have identical images or not. In question 12, she accurately stated that the student's responses to items (a), (b), and (c) are incorrect; however, that to item (d) is correct. As a result, she thought that function $g: R \to R,\ g(x) = 4x + 6$, function $h: N \to N,\ h(x) = 2x + 3$, and graph (c) do not equal to the function $f: N \to N$, $f(x) = 4x + 6$. For her, function g does not equal to f because "domain and range are different", and h does not equal to f because "natural numbers do not give the same points". According to her, graph (c) does not equal to f because "domain and range do not form a piece of line. There should be points with intervals in between". With these words, she highlighted that the domain and range of the function which equals to f should be N.

She explained the equality of functions in the class as follows, making no reference to another topic. After the class, she said that she forgot to make more reference to the equality of functions.

> Two functions which take the domain elements to the same range elements, although they have different rules, are called as equal functions. Assume that there are two functions, both of them take, for example, 1 to 3, 0 to 9, and 7 to 15. Their rules are different, but when you run the operations, they take the same numbers to the same numbers. Do you see what I mean? There is a similar example in the book. Let's have a look at that, or maybe we will come across other examples in the exercises.

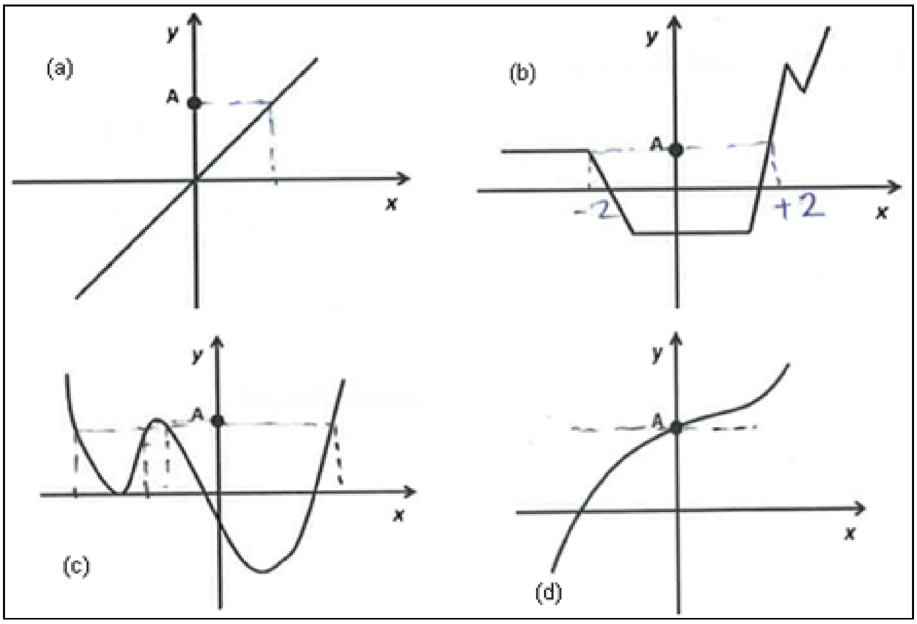


Figure 4.38: Fatma's response to question 14

Locating pre-images, images, and (pre-image, image) pairs on the axes in graphs: It was observed that Fatma read the graphs of functions properly. She indicated that the student's responses to question 13 were incorrect. In the question wherein she was asked to indicate her opinion about the correct answer, differently from the many other participants in the first phase of the study, she provided the right answer. She pointed out that the domain elements are those on the x axis (A, B, G), and the range elements are those on the y axis (E, B). She thought that the elements on the graph represent (pre-image, image) pairs. Also, she claimed that the student's responses are wrong in question 14. She accurately located the pre-image of point A on each graph in each item as Figure 4.38 presents. To be more specific, on graph (a), she said "the pre-image of A is on the x^+ axes". About graph (c), she implied that point A has four pre-images: "There are four x values whose images are A". She jotted this on the graph. On graph (d), she said "for A, $(0, A)$ is the only point". She expressed that the pre-image of point A on the graph (b) can be $\{-2, 2\}$, and also marked this on the graph. However, apart from these, she failed to notice that point A has indefinite negative pre-images.

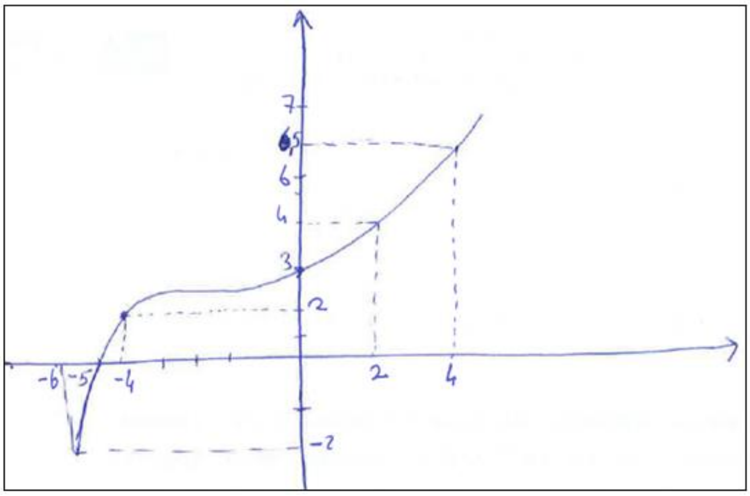


Figure 4.39: Excerpt from Fatma's instruction

In the class, she stated that the first component in these pairs $(-6,-2),(-5,0),(-4,2)$, $(0,3),(2,4),(4,6.5)$ (see Figure 4.39) is x and the second one is y. She made it clear that, when finding the image of a point, a vertical line should be drawn from that point to the function. To find $f(4)$, for example, she demonstrated this: "I have found 4. To find where it intersects the graph, I am drawing a vertical, then $f(4)=6{,}5$". She explained finding the pre-image based on the following condition: $f(x)=y \Rightarrow f^{-1}(y)=x$. To be more specific, she said "if $f^{-1}(6{,}5)$ is asked, I am looking for this $[6{,}5]$ value in the y axis. There it is! I am looking to see where it intersects the graph".

4.3.1.2 Fatma's Knowledge of Student Difficulties in the Function Concept

The student difficulties structuring Fatma's knowledge of students were as follows: students' conceptions of the function concept, potential areas of difficulties for students in mastering function concepts, students' difficulty with the essential features of functions (arbitrariness and univalence), students' difficulty in relating a domain and range to its graph, students' difficulty in identifying two equal functions, and students' such limited conception that every function needs to be linear. The descriptions of her knowledge of student difficulties were developed from her responses on the function concept questionnaire (see Appendix A), the follow-up interview (see Appendix C), and classroom observations.

Students' conceptions of the function concept: Fatma thought that students define function as "machine" (question 1) and cite examples to function as "rules" (question 2). She stated that, when she first introduces the topic, she mostly gives linear examples with coefficient and constant term. She believed that the students too will give similar examples such as "$f(x)=5$" or

“ $f(x) = x + 5$ ”. In the interview, she added that students cannot actually define mathematical concepts like the concept of function.

Potential areas of difficulties for students in mastering function concepts: Fatma was aware of the potential areas of difficulties for students in mastering function concepts. For her, students learn composition of functions and inverse of a function more difficult. Also, they have difficulty in doing operations (question 6). But they can learn the types of function more easily (question 7). Some of her students reported, for them, it is easier to learn types of functions (e.g., one-to-one function, identity function), but it is more difficult to learn the inverse of a function and the composition of functions.

Students' difficulty with the essential features of functions: Fatma was in the opinion that it is absolutely very important for students to know the concept of function. She thinks that full comprehension of function will help students learn the topics in the advanced stages more easily, and more importantly, transfer their knowledge to the essence of mathematics or to a level that is beyond numbers. She did seem to have more deep insight into the students' difficulties and limited conceptions about functions including difficulties in many-to-one correspondence, difficulties with functions represented by a disconnected graph, difficulties with functions given by more than one rule, difficulties with the verbal representation of functions, difficulties with the set notation of functions, and students' limited conception that a function must include some algebraic formula. In question 10, she ascribed the student's way of thinking to the representation in item (a) to the fact that the image of the elements in the domain set is unique. In the opinion of her, the student did not perceive the graph in item (b) as a function because the graph is disconnected. As regards to the representation in item (c), she believed that “the student may have thought that the function should be given by one rule only”. She justified why the student did not regard the representation in item (e) as a function by saying that “since it does not involve x, the student may think so”. In the interview, she ascribed the fact that the student did not regard the representation in item (d) as a function to his or her being unfamiliar to the verbal expression of piecewise function. Similarly, because, she thought, the students are not familiar to the set of ordered pairs representation of function, they did not take the following representation $\{(1,4),(2,5),(3,9)\}$ as a function. In the interview, she represented this relation in the form of arrow diagram, by the help of which, she said, the students would easily recognize that the relation is a function.

To better understand the extent to which the students have comprehended the concept of function, she said she would ask the students such questions as examining functionality of graphs or "$f: A \rightarrow B, A=\{0,1,2,3\}$, $f(x)=x+1 \Rightarrow B=?$" (question 5). She justified her choice in her following words: "They should be able to identify whether a graph or a set of ordered pairs is a function or not. If we are to start the topic of drawing graphs, they must draw this function's graph. Also, they must evaluate the function at specific points". Moreover, she stated the following:

> The students should make a connection! This function is written in the set parentheses [in the form of a set of ordered pairs], it was a relation, now it is a function; then $f(x)=\ldots$, this is also a function; and when you draw its graph, this is also a function! The students must connect them, and know that they all are the same thing.

Her knowledge of students as to the function concept became also evident in the class. She viewed function as a special relation, and just like a relation, it is a subset of the set of Cartesian product. She stressed, since it is a relation, a function can be represented in the form of sets of ordered pairs but also in algebraic expressions, set correspondences, and graphs. She analyzed whether the relations that are given in these representations define functions or not. When the relation was given graphically, she used the vertical line test to decide whether the graph is a function, but she stated that if the students have understood the definition of the function well, there is no need for such practical rules. Additionally, she linked the test with the definition of function.

Students' difficulty in relating a domain and range to its graph: Fatma accurately related a domain and range to its graph, and also, different from many other participants, she realized the 'into function'. In question 11, she correctly stated that the student's responses to items (a) and (c) are correct, whereas to item (b) is wrong. She expressed that the student may have mixed the domain with range. Her knowledge became apparent in the class. She described the domain, range, and image set of function; and explained the relation between them. She carefully addressed that the in many cases the image set could be a subset of the range.

Students' difficulty in identifying two equal functions: When identifying two equal functions, Fatma considered the domain and range sets, and also whether the pre-images have the same images. She indicated that the student's response to item (a), (b), and (c) in question 12 are wrong. She claimed that the student had neglected the domain and range when identifying functions identical to f. In the class, she announced that two different functions which take the same pre-images to the same images are called as equal functions. However, as she said, she forgot to make further reference to equality of functions.

Students' such limited conception that every function needs to be linear: Since, she thinks, an infinite number of graphs could pass through points A and B, she found the student's response to question 15 as incorrect. She claimed that students think that any two distinct points are incident with just one line. The student, therefore, might draw a straight line. Because of the students' disinterest, she gave little time on the graphical representation of function. The graphs she used were mostly constant and linear graphs.

4.3.2 Fatma's Students' Learning Outcomes of the Function Concept

A group of 26 students were enrolled in Fatma's course. The students' elementary school GPA average was 66.58 and first semester mathematics grade point average was 43.48. Fatma regretted to tell that her students have a negative attitude towards mathematics. They believe that they will not be able to learn mathematics. She said that the students usually consider mathematics as the arithmetic operations only and that mathematics is actually unnecessary for them.

Classroom observations revealed that indeed some of Fatma's students did not like mathematics at all. These students did not make any sense of what the teacher taught them, and they even sometimes made fun of it. They apparently get bored during classes, and thus displayed behavioral problems. As a result, the class was sometimes interrupted and the class atmosphere was disrupted. Consequently, they miss the opportunity to learn important things.

The function concept test (see Appendix D) and two teacher-designed exams provided the data for examining Fatma's student learning outcomes of the function concept including the students' understanding of the essential features of functions; relating a domain and range to its graph, and vice versa; identifying two equal functions; and locating pre-image, image, and (pre-image, image) pairs on the axes in the graphs.

Conceptions of the function concept: The results of the function concept test and the exams revealed that at the completion of instructional units on functions, compare to Ali's students, Fatma's student learning outcomes were slightly better. The majority of students (n=16) defined function as a set correspondence (question 1). For example, one student said: "When the elements in Set A correspond to those in Set B, it is a function". Another student stated: "Function is denoted by f. It has two stages. The first is listing, and the second is diagram, that is a set! Each arrow should go to the alternative once only; one alternative can receive two arrows". Some (n=4) defined function with its set correspondence and set of ordered pairs representation. For instance, one student said: "In my opinion, the function concept is forming ordered pairs, and then to show them in ordered pairs and present them in diagrams" and another wrote: "It is the representation of each element in lists and diagrams".

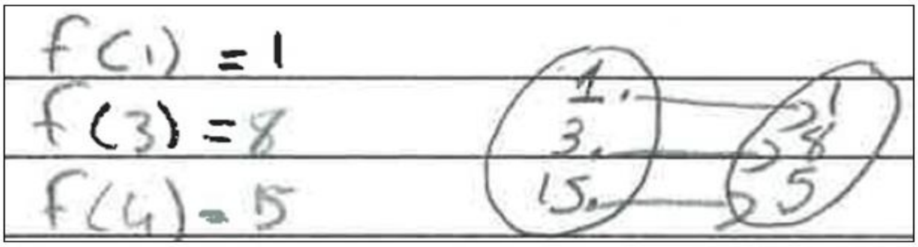


Figure 4.40: Response of Student #7 to question 2

Having defined functions, most students (n=11) also illustrated the function as a set correspondence (question 2). One of them used the analogy of a restaurant and stated: "Imagine that we are in a restaurant and everyone will order one dish. We can order the same dish as someone else does", and others drew diagrams (e.g., see Figure 4.40). Some examples (n=5) involved ordered pairs (e.g., "$A=\{a,b,c,d\},\ B=\{1,2,3,4\},\ f:\{(a,1),(b,2),(c,3),(d,4)\}$"). Few (n=1) referred to algebraic functions.

When students were asked whether a function has different representations or not (question 6), some students (n=10) mentioned the set correspondence, set of ordered pairs, and graphical representation of function (e.g., see Figure 4.41). Three students wrote the notations used to name a function (e.g.,

“ $f : A \rightarrow B$, $f : R \rightarrow N$ ”) and two mentioned the types of function (e.g., one-to-one function, onto function, constant function, linear function).

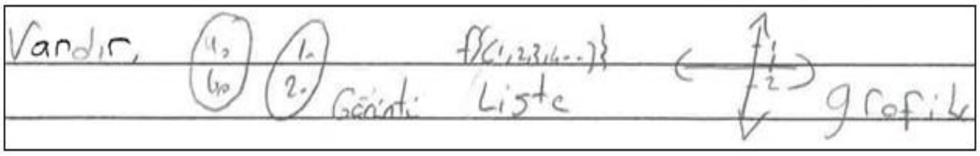


Figure 4.41: Response of Student #7 to question 6

The essential features of functions: When deciding whether a relation that is given in the form of a set correspondence defines a function or not, most of Fatma’s students regarded the univalence property of functions. To illustrate, in question 7, most of them (n=22) labeled the representation in item (d) as a function. Some of them claimed that “all are used and only once” and “all ate something and ate once only”.

In the first exam, the students were asked the following question: “Provided that $A = \{a, b, c\}, B = \{x, y, z, t\}$, write a function from Set A to Set B, and show it on a diagram”. Similarly, most students (n=18) took into account the univalence property of functions, and they wrote, first of all, a function in the form of a set of ordered pairs, then they showed it on a diagram (e.g., see Figure 4.42).

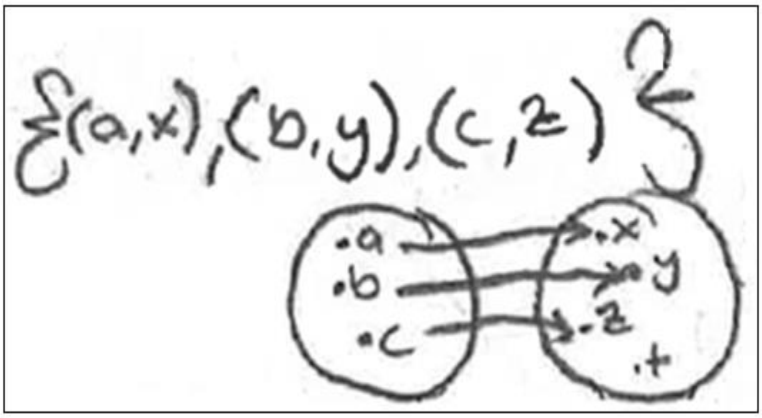


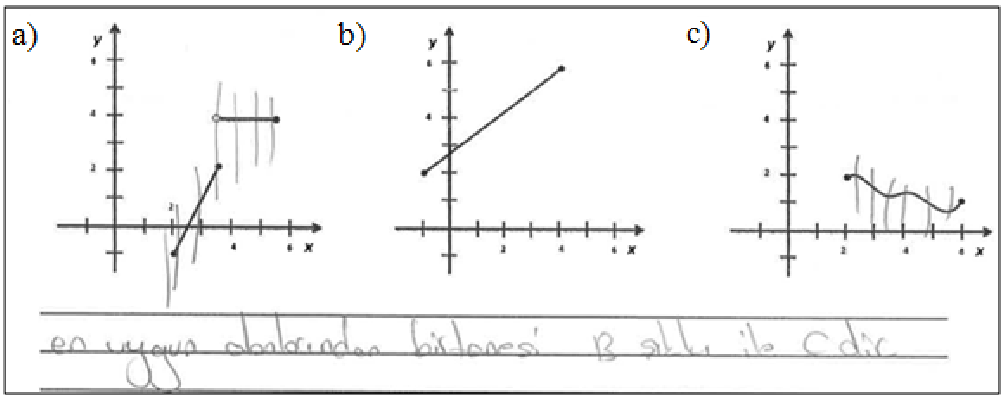
Figure 4.42: Response of Student #16 to the first exam question 2

While they were deciding functionality of the other relations, however, the students' perceptions of function became apparent. That is, most students considered function as a set of ordered pairs or as a set correspondence. Since the following relation $y=4$ does not represented in the form of a set of ordered pairs, half of them (n=13) thought that it is not a function. One of them expressed this in the following words: "A pair is a condition for a function". The majority (n=23) considered the following relation $\{(1,4),(2,5),(3,9)\}$ as a function just because it is represented in the form of a set of ordered pairs. In particular, two said that "the operation is done and the pairs formed" and "they are written in the form of a listing". Similarly, some stated that the graph in item (e) is not a function by the same rationale: "It should be in the form of a list or set".

When asked to draw function graphs that pass through points A and B on the plane (question 13), like students of Ali, none of the students considered the arbitrariness feature of functions. The majority of students thought that the function graphs that pass through these two points should be linear and indicated that one or two graphs passing these two lines can be drawn. When asked to draw function graphs that pass through several points like

A, B, C, D, E, F on the plane (question 14), similar to Ali's students, none of the students considered the univalence requirement of functions; accordingly, almost all of them connected the dots randomly.

Relating a domain and range to its graph: Most students did not take into consideration the domain and range interval while identifying the graph of a function, although these two were given. Most (n=10), for instance, did not respond question 8 that require them to relate the domain and range to its graph. Some (n=5) marked a graph but did not provide an explanation. Some salient points in other students' justifications are as follows. Many students (n=6) marked the graph in item (b) and explained this by saying that the points are suitable. One marked the graph in item (a) and wrote that there are two functions on the graph. One applied the vertical line test (see Figure 4.43) but eliminated the graph in item (a) and thought that the graph (b) and (c) are suitable.



[One of the most appropriate are b and c]

Figure 4.43: Response of Student #11 to question 8

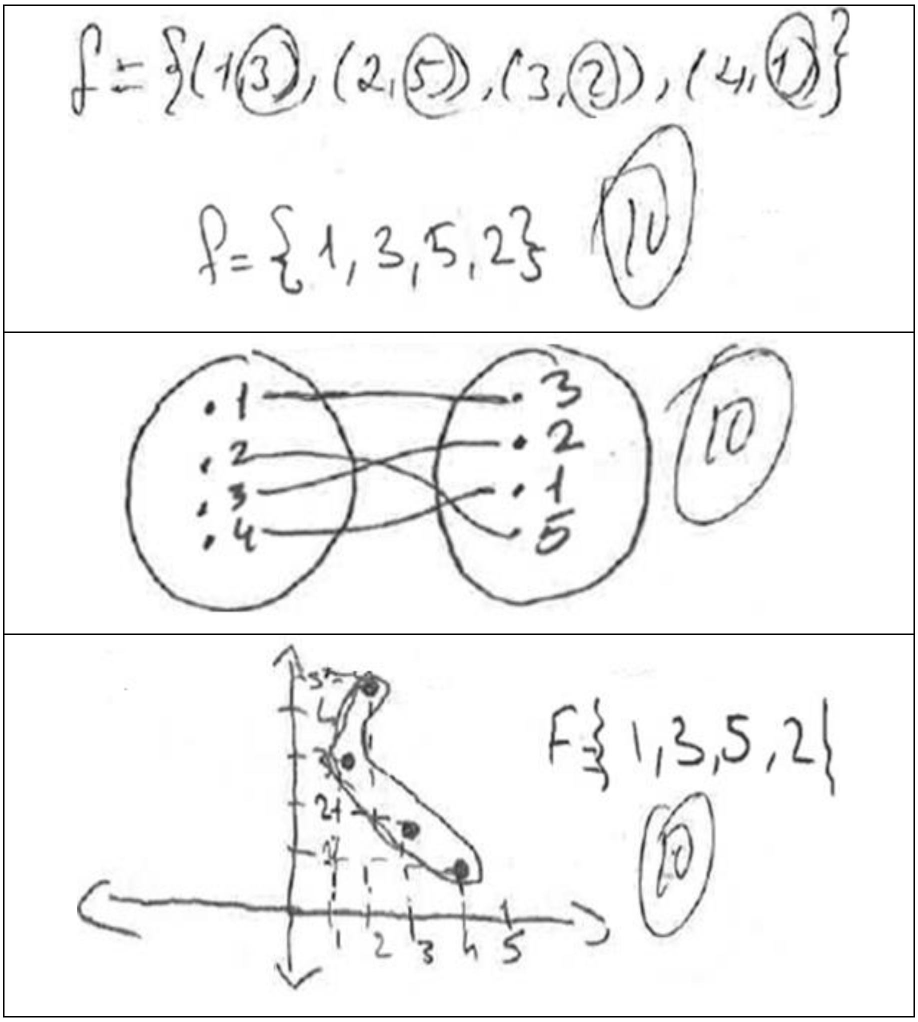


Figure 4.44: Response of Student #11, #9, and #7 to the first exam question

It was a different case when the function was presented in a set of ordered pairs rather than in a graph. To illustrate, in the first exam, the students were asked to find the image set of a function that is represented in a set of ordered pairs: "Function $f = \{(1,3), (2,5), (3,2), (4,1)\}$ is given. Find the image set of f". Many students (n=22) had correctly identified the image set of the function. Some (n=12) had showed it on a set, some others (n=8) on an arrow diagram, and two on a graph (e.g., Figure 4.44).

Identifying two equal functions: The students did not consider the domain and range of the function while identifying the function which is equal to function $f: N \to N$, $f(x) = 4x + 6$ (question 9). Some students (n=5) stated that function $g: R \to R$, $g(x) = 4x + 6$ is equal to f because the functions are the same, and some students (n=3) stated that function $h: N \to N$, $h(x) = 2x + 3$ is equal to f because the sets are the same. Different from Ali's students, some of Fatma's students marked the graphs in item (c) and (d), but they did not justify their answers.

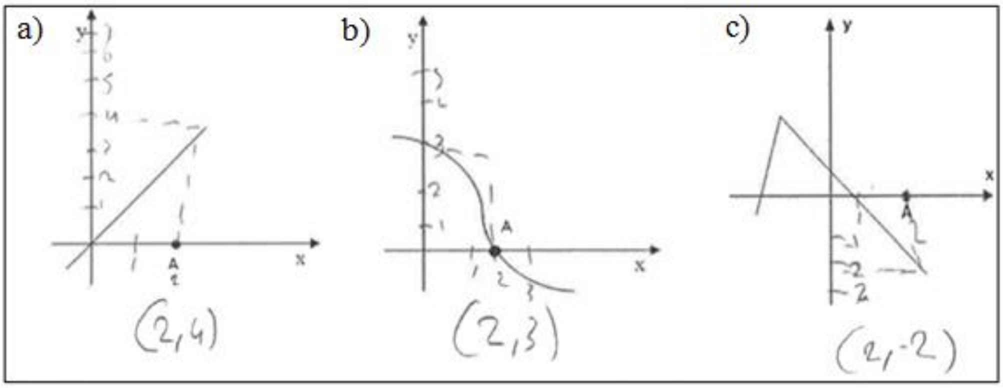


Figure 4.45: Response of Student #7 to question 12

Locating pre-images, images, and (pre-image, image) pairs on the axes in graphs: Just as Ali's students, Fatma's students could not comprehend what it means to locate the pre-image (question 11) and the image (question 12) of a point, say point A, on the axes in a graph. Like some of Ali's students, some drew the symmetrical image of point A according to the x axis, and some (n=13) marked one of the items that include the graph. Few students (n=4) correctly attempted to locate image of A on the axes in the graphs (e.g., see Figure 4.45).

The students were more successful in identifying the images and pre-images of some elements under algebraic functions. In question 15, for instance, most could find the images of some elements under an algebraic function. They were asked the following question in the first exam:

"$A = \{-3,-2,-1,0,1,2\}, B = \{-1,0,1,2,3,4,5,8,9\}$, $f(x) = x^2 - 1, f : A \to B$, write f in the form of a list". Although some made calculation mistakes, most of them (n=18) did the question correctly. One of the correct responses is shown in Figure 4.46.

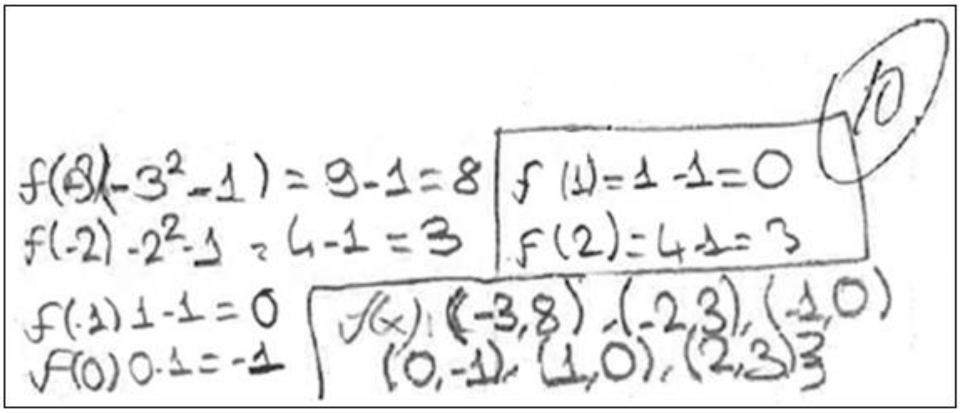


Figure 4.46: Response of Student #3 to the first exam question 3

4.4 Interrelation between Teachers' Knowledge of Content and Students about the Function Concept and Student Learning Outcomes

The data analyses revealed particular strands of the complex interrelation between the teachers' KCS about the function concept and student learning outcomes. The results suggested some evidences that the teachers' KCS about the function concept interacts with student learning outcomes. Interactions could be made between the teachers' conceptions of the function concept and understanding of the essential features of functions, and student learning outcomes. On the other hand, with respect to relating a domain and range to its graph, identifying two equal functions, and locating pre-images, images, and (pre-image, image) pairs on the axes in the graphs, no interaction was found. The study revealed that some learning outcomes develop independently of the teachers' KCS about the function concept. The teachers' instructional practices played a mediating role in the relationship between their KCS about the function concept and student learning outcomes. In addition, different factors came into play in students' learning.

Conceptions of the function concept: The results showed that Ali perceives function as an operation. He views algebraic expressions as a way of expressing functions but not sets of ordered pairs, tables, and graphs. In the class, he provided set theoretic definition of the function concept. Since he was in the opinion that it is more important for students to be able to evaluate functions for specific points so as to cope up with the exam questions and certain operations relating to the compound functions or the inverse functions, he mostly worked on algebraic functions. He frequently carried out class activities geared towards finding the image of the domain elements or the element itself the image of which is given. Different representations of a function were not much emphasized in his teaching practice.

On the other hand, Fatma sees functions as special relations between two sets. She thinks that functions can also be identified as operations. However, because of the univalence requirement of the function, this does not exactly define it. Unlike Ali, she views algebraic expressions, set of ordered pairs, and graphs as alternate ways of expressing functions. She was in the opinion that it is absolutely very important for students to know the concept of function. For her, if the students grasp the function, they will also know the sub-concepts of the function. Also, she stated that if the students have understood the concept of the function well, there is no need for such practical rules (e.g., vertical line test). She thinks that full comprehension of functions will help students learn the topics in the advanced stages more easily, and more importantly, transfer their knowledge to the essence of mathematics or to a level that is beyond numbers. According to her, however, students define function as a machine or algebraic statements. Actually, they cannot actually define mathematical concepts like the concept of function.

When they were done with the instructional units on functions, the majority of Ali’s students viewed functions as formulas. For instance, when the students were asked to define the concept of function in their own words, many students defined functions as mathematical operations. To give an example to function, most students either wrote algebraic functions or listed certain subtopics that relate to functions. When they were asked how functions and equations are related to each other, again many students wrote that a function is an equation because both need to be solved. None of the students were aware of the different representations of functions. On the other hand, the majority of Fatma’s students defined and illustrated function as a correspondence between two sets. Some regarded functions as a set of ordered pairs. Many students were aware of the set correspondence, set of ordered pairs, and graphical representation of a function.

The essential features of functions: Both Ali and Fatma were aware of the univalence property of functions. They always resorted to this requirement of functions while determining functionality of set correspondence relations. However, it seemed that Ali experienced difficulty in applying the univalence property of functions. He used the vertical line test for determining functionality of graphs and transformed the algebraic statements into graphs to decide functionality of them by using the test. However, he often confused the vertical line test and the horizontal line test. In his instruction, he did not mention the test. Different from Ali, Fatma could apply the univalence requirement of functions to analyze functionality of relations in the form of set of ordered pairs, algebraic expressions, and graphs. She used vertical line test for the graphs and justified the test by the univalence requirement. In the class, she used the test and likened it to combing a graph downwards.

When the students were done with the instructional units on functions, many of students in both groups demonstrated some awareness of the univalence requirement of function. Most of them considered this property of functions to decide whether the given set correspondence representation in item (d) in question 7 is a function or not. In relation to the set of ordered pairs, however, the students' outcomes were differed. Fatma's students could write a function from a Set A to a Set B, and write it, first of all, in the form of a set of ordered pairs, and then show it on an arrow diagram. Also, the majority of them considered the following set of ordered pairs $\{(1,4),(2,5),(3,9)\}$ as a function and some clearly mentioned the univalence requirement of function. Few also transferred the relation to an arrow diagram. However, most of Ali's students thought that the set of ordered pairs does not define a function because not a certain rule dominates it, i.e. as Ali thought. In addition, the papers that belong to Ali's students did not bear any results pertaining to the vertical line test as they were not familiar with it. Some students in Fatma's group, however, resorted to this test in some questions about the graphs.

The study indicated that some student learning outcomes developed independently of the teachers' KCS about the function concept. To illustrate, Fatma was well aware of the essential features of functions and mostly resorted these features of functions to decide functionality of relations. Indeed, instead of the essential features of functions, many of her students demonstrated their perceptions to identify whether the given relations define functions or not (question 7). Viewing functions as sets of ordered pairs, for instance, some students thought that the representation $y=4$ is not a function because it is not represented in the form of a set of ordered pairs. Some ascribed the relation

$f: R \rightarrow R,\ x \rightarrow \begin{cases} -3x^3+3, & if\ x \geq 0 \\ 5, & if\ x < 0 \end{cases}$ with a function because "ordered pairs will come out in the solution". Some students stated that the graph in item (e) is not a function because it is not in the form of a list or a set. Most students thought that the function graph that passes through two points on the plane like point A and B should be linear. Furthermore, when they were asked to draw a function graph that passes from several points like A, B, C, D, E, F on the plane, almost all of them connected the dots randomly.

Relating a domain and range to its graph: The study suggested no relation between the teachers' KCS in relation to relating a domain and range to its graph and student learning outcomes. That is, Ali had difficulty in relating the graph whose image set is the immediate subset of the range, but he had appreciated that in a graph the x axis represents the domain and the y axis represents the range. Fatma demonstrated a deep understanding of relations a domain and range to its graph, and vice versa. Furthermore, different from Ali, and also many other participants of the first phase of the study, set did not experience difficulty in relating the graph whose image set is the immediate subset of the range.

At the completion of instructional units on functions, it was found that most students in both groups did not take into consideration the domain and range interval while identifying the graph of the function. Some of the students in both groups used the sole criterion of representational format to relate a domain and range to its graph. That is, when they were asked to identify the function graph/s, with the domain $\{x: 2 \leq x \leq 6\}$ and range $\{y: -1 \leq y \leq 4\}$, among the given ones, they thought the correct answer is the piece-wise graph and said there are two functions in the graph.

The results showed that the teachers' instructional practices played a mediating role in the relationship between their knowledge and student learning outcomes. The students in both groups were provided less number of opportunities to learn relating a domain and range to its graphs, and vice versa. By solving a few questions, Ali's students were given the opportunity to learn that the vertical projection of points on the graph onto x axis forms the domain, and the vertical projection on the y axis forms the image set of the function. Similarly, by solving a test item, Fatma's students learned that the x axis of the graph is the domain, and the y axis is the range. The interval on the x axis between the start and end points of the function is the domain of the function, the interval on the y axis between the start and end points is the image set. Fatma's students had more opportunities to discover the image set of a set of ordered pairs. When the function was presented in a set of ordered pairs rather than in a graph many of her students had correctly identified the image set. Some had showed it on a set, others on an arrow diagram, and two of them on a graph.

Identifying two equal functions: The teaching experiences in the class also interacted with student learning of identifying two equal functions. Fatma demonstrated an understanding of identifying two equal functions. She took into consideration the domain and range, as well as whether the elements in the domain have identical images or not while identifying the two equal functions. However, since she forgot, she did not much emphasis on the equality of functions in the class. Identifying the two equal functions, Ali took into consideration the domain of the function and the equality of the images of domain elements. Solving two questions, he provided his students, of the two given sets, for the members of the domain they can compare the members of the range. If they are equal, they can decide that the two functions are equal.

When the students were asked such a question (question 9), more than half of Fatma's students did not answer the question or marked an item without an explanation. On the other hand, some students stated that function $g: R \rightarrow R, g(x) = 4x + 6$ is equal to $f: N \rightarrow N,\ f(x) = 4x + 6$ because the functions are the same, and some stated that function $h: N \rightarrow N,\ h(x) = 2x + 3$ is equal to f because the sets are the same. Most of Ali's students used the criterion of representational format and marked the items which bear algebraic statements just because they involve x like f. Some thought that function $g: R \rightarrow R$, $g(x) = 4x + 6$ is equal of f, and stated the functions are the same just the letters (N and R) are different. None of his students marked the appropriate items which bear the graphs.

Locating pre-images, images, and (pre-image, image) pairs on the axes in graphs: Similarly, the teaching experiences in the class interacted with student learning outcomes of reading graphs. In addition, a factor contributory to student learning outcomes was evident: attitude towards learning. Ali seemed to have difficulties in reading the graphs properly. Consequently, the graphical representation of function was not much addressed in his instruction. Unlike Ali, Fatma read the graphs properly. She appreciated that the domain elements are those on the x axis, the range elements are those on the y axis, and the elements on the graph represent (pre-image, image) pairs whereas the element not on the graph do not. She wanted to raise a class discussion on reading graph, finding the domain and range of a function with a given graph, and finding the increasing and decreasing intervals of the function, and she wanted to dwell more on these concepts. However, the disinterest and misbehaviors of her students disrupted the discussion environment, so she just covered these issues superficially. Consequently, her students missed the opportunity to learn important things.

By the time the students finished the instructional units on functions, none of them in both groups demonstrated an understanding of reading graphs of functions. The majority of the students could not accurately write the domain and range elements of a function whose graph is given. When the students were asked to locate the pre-images of images on the axes in a graph, and vice versa, most of them marked one of the items that present the graph. And some students drew the symmetrical image of the given graph according to different lines. Just few of Fatma's students correctly attempted to locate the images of pre-images on the y axis in the graphs, and few could effectively identify the domain and range points in the graphical representation.

CHAPTER 5

CONCLUSION AND DISCUSSION

The major purposes of the present study was to identify KCS of mathematics teachers in technical and industrial vocational high schools, and investigate the patterns of interrelation between KCS of mathematics teachers and student learning outcomes with respect to one of the least understood concepts in secondary school mathematics curriculum, the function concept. Two research questions are formulated:

1. As to the function concept, to what extent do mathematics teachers have knowledge of:
   a. content;
   b. student difficulties?
2. How do mathematics teachers' knowledge of content and students and the student learning outcomes interrelate as regards the function concept?

To address the first research question about teachers' KCS of the function concept, a questionnaire was administered to 42 mathematics teachers in technical and industrial vocational high schools. For the second research question, i.e. concerning the interrelation between teachers' KCS of the function concept and their students' learning outcomes of this concept, case studies of two teachers were carried out. For this purpose, two teachers were selected from the 42 teachers. Interviews and classroom observations were performed to capture how teachers' KCS and student learning outcomes interrelate. Organized around the research questions, this chapter concludes and discusses the main findings of the study. The chapter also addresses the implication of the study, makes recommendations for further study, and finally states limitations of the study.

5.1. Teachers' Knowledge of Content and Students about the Function Concept

The teachers' KCS was discussed under two headings: their content knowledge and knowledge of student difficulties about the function concept

5.1.1 Teachers' Content Knowledge of the Function Concept

The results showed that the majority of teachers view functions as correspondences. The teachers were asked to elaborate how they believe students should define function and what type of examples they should use to illustrate it. Some teachers thought that students should define function as an operation and some believed that they should give algebraic examples to functions. However, the majority of them were in the opinion that students should give the set correspondence definition and give examples based on it. Accordingly, more than half of them described function as a correspondence between two sets. They mostly provided an accurate set correspondence definition of function, and some provided analogies for functions. Few analogies put regard for operation aspect of function (e.g., "olives processed in a factory coming out of the factory as olive oil"), while most highlighted the univalence requirement of function. One teacher, for instance, wrote: "A group of students on a trip will be accommodated in hotels, and in these hotels, they will be placed in rooms. Some of these rooms may remain vacant, but each student will definitely be given a room. A student cannot be given two rooms at once". The teachers were asked what kind of a question they would ask the students so that it would be a good indicator of whether they have understood the function concept; most cited the types of questions that involve relations, i.e. the teachers would show the students some relations and ask them to identify whether they are functions or not. In contrast with the secondary pre-service teachers from Bolte's (1993) study, who preferred to list examples of algebraic functions and graphical representations that students would encounter in the secondary school curriculum, again the teachers commonly established relations reflecting set correspondences. These findings differ from some studies (Bolte, 1993; Even, 1993). As cited in Conney (1999), Cooney (1992) stated that some teachers' conceptions of function were closely related to the concept of algebraic expressions. Even (1993) reported that many pre-service teachers had an old (dependence relation, operation, or formula) concept image of a function and expected functions to always be represented by an equation. Bolte (1993) found that "only two of the 17 participants gave a well-defined accurate, modern definition [a set correspondence definition] of the function concept" (p. 255). A possible reason for this difference is that the set correspondence definition has been popular as the formal definition of the function concept in school curricula of many countries, and Turkish school curriculum is no exception to this. Consequently, the in-service trainers, who have associated the function with set correspondence for many years, develop the perception of function as a set correspondence Not incidentally, several research focusing on experienced in-service trainers have revealed similar results (e.g., Duah-Agyeman, 1999; Hitt, 1998; Howald, 1998). Many in-service teachers' definition from Duah-Agyeman's (1999) study, for instance, was categorized as correspondence. In Hitt's (1998) study, of all thirty teachers, eighteen gave their definition in terms of rule of correspondence from the teaching perspective. Experienced in-service teachers from Howald's (1998) study also provided a modern definition of function when they were asked to. That is, like teachers in the current study, most teachers from

Howald's (1998) study provided the correspondence definition of function and made analogies for functions that they would use with functions. In stark contrast to the teacher in the study of Howald (1998), whose analogies highlighted only the fact that the machine but put little regard for other aspects of functions, the teachers in the present study put greater regard for the univalence requirement of functions.

It has been well understood that for the teachers of the present study, a common way to consider functions was through the idea of set correspondences. This was clearly appeared in the teachers' comments on the questionnaire. To illustrate, when asked whether a function has different representations or not, most mentioned the representation of function as in a set correspondence, while fewer mentioned the graphical representation, and even fewer mentioned the table or verbal representation. Also, when asked to elaborate how they believe students should define function and what type of examples they should use to exemplify it; only one teacher indicated that students should give examples in the form of graphs. And one teacher believed that students should be able to cite examples as in sets of ordered pairs. None of them mentioned functional situations in the real world. It seemed such representations had played a minor role in their conceptions. These results confirm previous results (Bolte, 1993; Even, 1989; Norman, 1992; Stein et al., 1990). Bolte's (1993) study reported that the majority of pre-service secondary teachers were aware of arrow diagrams as alternate ways of forming functions but "not one suggested a real world situation as an alternate representation of a function" (p. 144). Stein et al.'s (1990) study with an experienced middle school teacher revealed that the teacher failed to see tables and graphs as different representations of the same function. Even's (1989) study found that many pre-service teachers experienced difficulty in making connections among the tabular, algebraic, and graphical representations of a function. Indeed, understanding those representations for functions and the relationships among them play an important role to develop a rich concept of functions

(Cunningham, 2005; Eisenberg, 1992; Lloyd & Wilson, 1998; Norman, 1993; Sierpinska, 1992). It is believed that the teachers with limited awareness of the different representations of the function are bound to be even more limited themselves in helping students perceive these representations and establish connection between them.

A relation can be accepted as a function provided that it meets two conditions. First, as the function of definition clearly suggests, every element in the domain should be paired with one but only one element in the range (the univalence requirement). Second, a function does not necessarily realize the correspondence between the elements of the two sets through an arithmetical or algebraic rule; this correspondence could well be arbitrary, and the elements of domain and range could be any object (Even, 1990, 1993). When a student identified each of the relations given in such varied forms as 'non-function', the teachers were asked to evaluate the accuracy of the student's response with the necessary justifications. It was observed in their justifications that the teachers took into account the univalence property of function especially in identifying whether the relation given a set correspondence is a function. For the relations that are represented in the other forms most teachers did not always apply the definition of the function concept to decide functionality of them. For the relation that was given graphical, some applied the vertical line test. For the relations that were given in verbal or algebraic statements, some transformed the relations into graphs to determine their functionality by using the vertical line test. Considering the arbitrariness property of the function concept, it was found that, especially in their discussion on the relation that was represented as a set of ordered pairs, most teachers had the belief that functions must do the correspondence between the elements of two sets by means of an arithmetical or algebraic rule. Accordingly, half of the teachers stated that the relation $\{(1,4),(2,5),(3,9)\}$ is not a function because it is not defined based on a particular rule. And, some thought that the relation $y=4$ is not a function because a specific rule is not given. Also, the teachers were presented that the student had thought only one function graph containing two points, say point A and B, could be drawn, and asked how they would correct such a response if they thought it is wrong. Similarly, by referring to specific examples of functions, many stated several functions as curves and parabolas passing through points A and B could be drawn. The teachers' awareness of the univalence requirement but ignorance of the arbitrary nature of function is fairly consistent with those identified in the related literature (Duah-Agyeman, 1999; Even, 1989, 1993, Hacıömeroğlu, 2006). However, in contrast to the pre-service teachers in the study of Even (1989), who seemed to expect functions to be defined on numbers only, some teachers in the present study commented that the elements in a function's domain and range can be objects different from numbers (e.g., cars, mothers and their children, olives, students, hotel rooms).

The study found that most of the teachers gave satisfactory responses to the questionnaire items that were designed to reveal the teachers' knowledge of identifying the two equal functions and relating a domain and range to its graph. On the contrary, pertaining to identifying the function graph whose image set is the immediate subset of the range most experienced the same difficulty that secondary school students from Markovits et al.'s (1986) study had experienced. Also, the majority of teachers seemed to find it difficult to decide the points which indicate, or do not indicate, (pre-image, image) pairs in the graphical representation. Some of the teachers commented that points on the curve represent (pre-image, image) pairs and points not on the curve do not. However, like students from Markovits et al.'s (1986) study, a considerable number of teachers were not be able to identify (pre-image, image) pairs. Many teachers also did seem to experience difficulty in locating pre-images of images on the x axis in graphs when the functions are not one-to-one. These findings are in concert with Hitt's (1998) findings that found in-service teachers gave satisfactory responses to the questionnaire items that were designed to detect possible weakness of the teachers when comparing two functions but exhibited errors when they confound the other sub-concepts of the function concept. That is, teachers from Hitt's (1998) study successfully found the pre-image of an image in arrow diagrams but, like the teachers from the present study, only some of them located the pre-images of images on the x axis in graphs. Similarly, in Hitt's (1998) study a "greater difficulty is found when the functions are not one-to-one" (p. 131).

The other studies have indicated that teaching experience does not affect teachers' content knowledge of functions (Lucus, 2006) and even experienced mathematics teachers, like the teachers from this study, commit mistakes when they carry out a task related to functions (Even, 1989; Hitt, 1998). Teachers' limited conceptions and difficulties "have been observed across several countries, including France, England, Israel, Poland and the United States" (Selden & Selden, 1992, p. 5)

5.1.2 Teachers' Knowledge of Student Difficulties in the Function Concept

The current study revealed that the majority of teachers' identifications about what aspect(s) of the study of functions cause students the most/little difficulty were consistent with those identified in the research literature. That is, many teachers thought that the students have difficulty in comprehending the composition of functions and finding the inverse of a function but, they comprehend how to evaluate algebraic functions at specific points relatively more easily. The identifications of some teachers about the students' conceptions of the function concept were also consistent with those identified in the research literature. Some teachers stated that students cannot define mathematical concepts such as function. Some pointed out that the students define function as in "the bluetooth function of the cel phones, or something like that" or "a series of functions, like the function of dishwasher" and give examples to function as in "the function of television, car, or computer". Some teachers stated that the students generally identify functions as an operation (e.g., "Function is a kind of factory, with its input and output"). Some thought the students identify functions with its symbolic representations (e.g., " $f: R \to R, \quad f(x) = 2x + 1$"). However, the majority of teachers thought that the students generally identify functions in the way they themselves do, i.e. as a correspondence between two sets.

In the study, the teachers were presented students' some difficulties and limited conceptions about functions that have been identified in the literature including difficulties in many-to-one correspondence, difficulties with functions represented by a disconnected graph, difficulties with functions given by more than one rule, difficulties with the verbal representation of functions, difficulties with the set notation of functions, and students' limited conception that a function must include some algebraic formula. In the teachers' explanations, it became evident that some teachers quite precisely identify what kind of problems students experience pertaining to the set correspondence representation of function and their possible mental reasons. Teachers' knowledge of students' difficulties and limited conceptions concerning the graphical representation, verbal statement, algebraic expressions, and set of ordered pairs representations of function, varied in terms of content and quality. Some teachers' diagnosis of student mistakes and their possible mental reasons was quite precise. On the other hand, many other teachers thought students' inability to comprehend the subject is responsible for their mistakes, or they pointed at other factors that are remarkably different from those discussed in the related literature. These findings corroborate of other studies which has reported teachers' limited knowledge of students' difficulties pertaining to different mathematical subjects (Asquith et al., 2007; Even & Tirosh, 1995; Kılıç, 2008; Postelnicu, 2011; You, 2006; Watson et al., 2008).

The study revealed that the teachers' content knowledge of functions did seem to help them effectively determine the difficulties students experience in relation to the function concept and their mental reasons. The teachers who were more successful particularly in the set correspondence representation, for instance, were aware of the types and causes of the student mistakes. This also occurred in students' difficulty in relating the domain and range of a function to its graph, and students' difficulty in identifying identical functions. Indeed, most teachers who had no difficulty with identifying the two equal functions indicated that students disregard the domain and range when identifying the equal functions. In contrast, some teachers who were observed to have limited knowledge related to deciding the graph of the function the domain and range of which is given also displayed limited ability to determine the student mistakes and their causes. The factor of teachers' content knowledge in the kinds of explanations provided for students' mistakes were fairly consistent with those identified in the research literature (Bolte, 1993; Ebert, 1994; Even, 1989; Even & Tirosh, 1995; Halim & Meerah, 2002; Kılıç, 2008; Tirosh, 2000; You, 2006).

5.2. Interrelation between Teachers' Knowledge of Content and Students about the Function Concept and Student Learning Outcomes

The second phase of the study showed complex interrelation between the teachers' KCS about the function concept and student learning outcomes. The data suggested some evidence of the teachers' KCS about the function concept and student learning outcomes. Interactions made between the teachers' conceptions of the function concept and understanding of the essential features of functions, and student learning outcomes. As to relating a domain and range to its graph, identifying two equal functions, and locating pre-images, images, and (pre-image, image) pairs on the axes in the graphs, no interaction was found. Also, the study revealed that the teachers' KCS of the function concept influenced their instructional practices. The teaching experiences in the class interact with student learning outcomes. In addition, different factors came into play in student learning.

The data revealed that Ali perceives functions as operations, the mechanisms converting inputs into outputs. He did not seem to view set of ordered pairs, tables, and graphs as alternate ways of expressing functions. He was aware of the univalence property of function and also the idea that the elements in a function's domain and range could be any object (e.g., numbers, letters, animals, coffee beans, wool). However, he could not always use those properties of function. Also, he did not seem to have a deep insight into the difficulties and limited conceptions students have in relation to the function concept. He generally failed to identify the student's difficulties about this concept. He ascribed some mistakes to student's failure to comprehend the subject or to causes different from those indicated in the related literature. On the other hand, Fatma sees functions as special relations between two sets. She views algebraic expressions, set of ordered pairs, and graphs as alternate ways of expressing functions. Like Ali, she was aware of the univalence property of function and also the idea that the elements in a function's domain and range could be any object (e.g., numbers, letters, cloths, seed, and fabric). Yet, different from Ali, she was in the opinion that a function does not necessarily make the correspondence between the elements of two sets through an arithmetical or algebraic rule. Also, unlike Ali, she could more precisely identify the difficulties students experience about the function concept, and their reasons. Indeed, the content and quality of her knowledge of student difficulties and limited conceptions was greater than Ali and also some of the participants involved in the first phase of the study. When the students were done with the instructional units on functions, the majority of Ali's students viewed functions as formulas. None of the students were aware of the different representations of functions. However, the majority of Fatma's students defined and illustrated function as a correspondence between two sets. Some regarded function as a set of ordered pairs. Many students were aware of the set of ordered pairs representation of a function. Additionally, Fatma's students demonstrated more awareness of the univalence requirement of function, whereas many of Ali's students were not.

The data revealed that the diverse classroom instruction interacts with student learning outcomes. That is, the students in both groups were provided less number of opportunities to learn the sub-concepts of function and spent little time on the graphical representation of function with which students have more difficulty. By the time the students finished the instructional units on functions, most of them in both groups had a fairly limited understanding. Evidently, most of Ali's students just randomly chose answers to the questions that were designed to detect their understanding of concepts related to functions. The criteria that some other students used to respond to the questions were definitely not up to the desired level. Some students, for example, used the sole criterion of representational format to relate a domain and range to its graph and also to decide the equality of two functions. As to Fatma's students, most did not take into consideration the domain and range interval while identifying the graph of a function. Instead, they retrieved their knowledge of linear functions. Some marked a graph but did not provide a justification. None of the students in both groups demonstrated an understanding of reading graphs of functions. The majority of the students could not accurately write the domain and range elements of a function whose graph is given. When the students were asked to locate the pre-images of images on the axes in a graph, and vice versa, most of them marked one of the items that present the graph. And upon hearing in the class that a function and its inverse graph are reflections of each other across the line $y = x$, some students incorrectly transferred their knowledge and drew the symmetrical image of the given graph according to different lines.

Actually, there appears to be consensus among researchers that teaching practice affects student performance. In the US, the NCTM (2000) standards stated that “students learn mathematics through the experiences that teachers provide. Thus, students’ understanding of mathematics, their ability to use it to solve problems, and their confidence in, and disposition toward, mathematics are all shaped by the teaching they encounter in school” (p. 16-17). Similarly, several studies have described an intimate relationship between student learning and mathematics teaching (e.g., Lloyd & Wilson, 1998; Nilklad, 2004; Schoen et al., 2003). Hofacker’s (2006) study found that students in the contemporary group, who were thought from a contemporary perspective which focused on working in a discovery-based environment, had a more connected and flexible understanding of the content. Nilklad’s (2004) study reported, since the instruction did not encourage solving mathematical problems in multiple ways, the students’ algebraic reasoning abilities did not seem to progress as much.

The study revealed that the teachers’ KCS about the function concept influenced the quality of their instructional practices. Ali placed varying degrees of emphasis on some concepts in his instruction, and he did not address some concepts related to functions. In his instruction, he mostly dwelled on the procedural aspects of functions rather than the conceptual aspects. On the other hand, unlike Ali’s class time, Fatma’s instructional practices involved more varied analogies, more detailed and diverse explanations, and more acts of relations. Her instruction was relatively base on conceptual aspects of functions as well as procedural aspects. Different from Ali, her instruction addressed major concepts related to functions.

The factor of teachers’ knowledge in the quality of their teaching practice is consistent with those identified in the research literature. Sánchez and Llinares’s (2003) study, for instance, showed that the four prospective teachers’ ways of knowing the subject matter had influence on the way they tried to represent the subject matter to the students. Kahan, et al.’s (2003) study reported MCK is a factor in recognizing and seizing teachable moments and enhances the possibilities for the teachers, but a lack of MCK narrows the scope of what is possible for teaching. Similarly, Stein, Baxter, and Leinhardt's (1990) study suggested that lack of deep subject matter knowledge lead to narrow the instruction in somehow.

In addition, the study found that the students' attitude towards learning mathematics was a contributory factor to their learning outcomes. Similarly, several studies reported that students who attend vocational education are less motivated (Yörük et al., 2002), more reluctant towards learning (Binici & Arı, 2004; Şahin & Fındık, 2008), and not keen on academic subjects (Lewis, 2000).

5.3 Implications

The present study highlights the KCS of mathematics teachers in technical and industrial vocational high schools, and the patterns of interrelation between KCS of mathematics teachers and student learning outcomes regarding the function concept. The results have practical and methodological implications for several parties: mathematics teachers at the high school level (especially in technical and industrial vocational high schools), the mathematics educators, and the policy makers.

Six aspects of content knowledge of function were identified as very important for secondary mathematics teachers: (a) essential features-what a function is, (b) different representations of functions, (c) alternative approaches to functions, (d) the strength of the concept-the inverse function and the composition of functions, (e) basic repertoire-functions of the high school curriculum, and (f) knowledge and understanding of the function concept (Even, 1990). This study chose to concentrate on what a function is and shed light on teachers' KCS on this aspect. The identified aspect can serve as a starting point for a discussion of what secondary mathematics teachers need to know about functions to teach them effectively.

It is reported that knowledge of student difficulties about particular mathematical content, which is one of the primary elements of KCS (Ball et al., 2008), is critical for student learning (Hill et al., 2008). However, the present study revealed that the teachers' KCS regarding the function concept is not sufficient, which might impede effective student learning. The study pointed at two major factors that might adversely influence student learning: the teachers' limited content knowledge and low awareness of students' difficulties in the function concept. Secondary mathematics teachers should be concerned about these findings.

The findings indicate some link between teachers' KCS about the function concept and student learning outcomes. Policy makers can use the results to design in-service training programs that develop teachers' KCS about particular mathematical contents. In addition, in the case of functions, it is essential to review and improve the major limitations in the curriculum and the textbook (see Section 2.2). The program and textbooks should address the student mistakes and misconceptions that educational research on functions commonly reveals.

Both recent national and international (e.g., TTKB, 2011; NCTM, 2000) reform recommendations have emphasized that it is important for mathematics teachers to provide student-centered instruction for their students. Also, research has demonstrated (e.g., Schoen et al., 2003; Wood & Sellers, 1997) that reform-oriented instruction is positively related to increased student achievement. However, for teachers, embracing a student-centered instruction should be based on in-depth descriptions of instructional practices and student learning outcomes of teaching experiences. Indeed, it is exactly what this study provides. That is, in his instruction, Ali mostly had the students copy down the definitions, explanations, and the text in the course book on their notebooks. In learning process, the students were passive most of the time. Accordingly, many of them seemed to daydream; their participation in the lesson was not beyond taking notes and listening passively; their mathematical communication with each other and also with the teacher was quite limited. Fatma, on the other hand, intended to have students internalize the concepts and their meanings. To this end, she carried out question-and-answer sessions, encouraging students to think critically, question, and be active during class. Especially some of the students seemed to be enjoying the class, and actively participating. Some students' responses to the questions in the function test, not incidentally, had more frequent traces of Fatma's student-oriented teaching methodology. These results can help secondary mathematics teachers to understand the importance of embracing more student-centered approaches to teaching mathematics.

In the present study, it was observed that many teachers indicated that vocational high school students are low performers and, thus, a study conducted in this context would not be fruitful. The teachers involved in the second phase of the study had different views about it. That is, Fatma admitted that the vocational high school students have difficulty in doing the basic arithmetic calculations. In her opinion, one reason for this is that the mathematical content is given in the order of 'Logic', 'Sets', 'Functions', and 'Numbers'. She believed that students should first be taught 'Numbers'. Ali thought that a major objective of mathematics education in vocational high schools should be to teach the basic mathematical concepts. For him, to teach students many additional topics does not make much sense. The otherwise would be more difficult both for students and teachers. It was observed that Fatma expected more of her students as regards their mathematical learning, and designed her teaching accordingly, which ultimately produced higher student output. In brief, if mathematics teachers in vocational high schools have higher expectations about students' performance in mathematics and design their teaching accordingly, this may result in improved attitude and learning in mathematics.

The findings of the study indicated that both Fatma and Ali believe in the importance of teachers' understanding the way students think about a certain mathematics subject or the difficulties they experience with it. Nevertheless, they had limited awareness of how to identify students' difficulties. For instance, Ali thought that it is up to the students whether a teacher knows their way of thinking or not. He said, if the students raise questions during class, he can see what they have or they have not understood. Fatma believed that if the students express clearly their ways of mathematical thinking, the teacher will understand how the students think. Indeed, the teachers could be achieved the knowledge of students' weaknesses and strengths through grading students' homework (An, Kulm, Wu, Ma, & Wang, 2006). However, it seemed that the teachers' purpose of assigning and checking homework was not actually understanding students' thinking. Ali's aim was to have students open their books at home and study, rather than identify how much they have learned. For him, the accuracy was the focus; he said he checks the accuracy of the results and tracks which questions they have generally failed to do. For Fatma, the purpose of assigning homework was twofold: to make students take on responsibility and to have them revise what they have learned in class. She stated that when doing homework-check, she looks at whether students have done their homework or not. She asks the students which questions they could not do, and solve these questions herself on the board. To sum up, it seems that teachers should improve their skills of identifying students' strengths and weaknesses.

Latest research on teacher education has suggested that the professional development experiences seem to influence teachers' instructional practice strongly (Schoen et al., 2003). Through professional development programs, teachers gain the ability to make effective and appropriate instructional decisions. Then, teachers need to "seek out high-quality professional development opportunities that fit their learning needs" (NCTM, 2000, p. 373). However, in drastic agreement with those of other studies (OECD, 2009), the results showed that almost 71% of teachers in the current study had not participated in any professional development activities. The second part of the study showed that Ali had low awareness of professional development and Fatma was in the opinion that teachers have limited professional development opportunities. In conclusion, teachers should be made aware of the purpose and effect of the professional development. Also, more varied professional development activities should be organized.

The study also revealed that the technical and industrial vocational high school students have markedly limited capability of doing the basic mathematical operations. The difficulties that the students have in arithmetic operations could cause difficulties in many subjects of mathematics (Markovits et al., 1988), as well as in functions. The teachers in these schools should be aware of this weakness and seek ways to overcome this problem.

The methodological implications of this study relate to the combined use of a qualitative survey (Jansen, 2010) and a case study. This combination proved to be fruitful for the purpose of the study. The use of a qualitative survey study was helpful in eliciting a relatively large number of responses, which provided a general picture of the teachers' KCS about the function concept. The case studies not only clarified this picture adding more details, but also revealed the patterns of interrelation between teachers' KCS and their students' learning outcomes about the function concept.

5.4 Recommendations for Future Studies

The present study contributes to the understanding of the complex interrelation between teachers' KCS and student learning outcomes about a particular mathematical content, the concept of function. There is much more to be learned about this interrelation. An important question to investigate is whether and what changes occur in learning outcomes of students with different academic backgrounds. Therefore, this study should be replicated with different student populations (e.g., students in general high schools). Information gathered from these studies will help understand how students' academic backgrounds mediate the contributions of teachers' KCS to students' emergent knowledge about particular mathematical content.

This study suggested that teachers' KCS regarding the function concept in technical and industrial vocational high schools is limited. KCS of teachers in the other type of schools needs to be identified also. To this end, data from participants in those schools should be added to this study. Information gathered from these studies will help understand whether students had an impact on teacher knowledge (Park & Oliver, 2008).

This study was limited to one of the aspects of functions, what a function is. More aspects of functions should be added to expand the scope of the study. A more complete picture of teachers' KCS about functions will be depicted from these studies.

Further studies should extend into which changes in teacher knowledge have the greatest potential to influence student learning outcomes about a particular mathematical content. They should, for example, investigate which factors are more influential in student learning (e.g., teachers' knowledge of content, teachers' knowledge of teaching, teachers' pedagogical content knowledge, or teachers' beliefs about mathematics and mathematics learning and teaching).

Thus, far more aspects of teacher knowledge should be added to expand the scope of the study.

The study indicated relationship between two teachers' KCS about functions and their instructional practices. That is, Ali's limited KCS of the function concept influenced his instructional practices; he placed varying degrees of emphasis on some concepts in his instruction, and he did not address some concepts related to functions. On the other hand, it was observed that Fatma had a more comprehensive KCS; unlike Ali's class time, her instructional practices involved more varied analogies, more detailed and diverse explanations, and more acts of relations. To this end, the question of whether and how teachers' KCS impacts on their teaching need to be investigated.

The 9th grade mathematics program has been revised in June 2013 and the 9th grade mathematics textbook has been prepared in accordance with the program. It is necessary to examine the organization of functions in the new program and textbook, and to investigate student learning of the function concept around this new organization.

5.5 Limitations of the Study

As any study does, the study has some limitations. In some sense, the selection of participants, the data collection instruments and procedures, and the researcher herself limited the results of the study.

The selection of the participants was one of the obvious limitations to the study. A total of 42 teachers volunteered to participate in the first phase of the study. From this group, 13 volunteered to participate in the second phase. Two teachers were identified from this group for the case studies. The teachers who volunteered might have had different mathematical backgrounds, experiences, and beliefs from those teachers who did not volunteer.

As part of the study, 42 teachers were identified from eighteen different technical and industrial vocational high schools in one of the districts of Ankara. The study also included an in-depth description of a particular setting of two teachers and their students from this group. The results of this study may not apply to other settings, different groups of teachers, or students.

Another limitation was directly related to the data collection instruments. The function concept questionnaire may have been influenced by the researcher's beliefs and biases. The questionnaire also may have been biased towards specific teacher groups. For instance, some items might have favored teachers who have more conceptual knowledge about functions. The results of the present study, therefore, may not have represented a typical understanding of the function concept of secondary mathematics teachers although each item of the questionnaire was discussed with a mathematics professor to minimize this problem. Also, the function test may have been influenced by the researcher's beliefs and biases. To combat this problem, the students' learning outcomes on the function concept test and teacher-designed exams were discussed together. Still, the results may not be representative of the understanding of typical secondary students.

The researcher herself also was a limitation in this study. Although, to minimize this, the researcher attended a few classes prior to the instruction on the function units, her presence still might have had an effect on the instructional practices of two teachers. Since they knew the purposes of the study, the explanations that they gave, the examples that they used, and the questions that they posed to the students may have been affected.

A final limitation of the study involved the researcher's biases. The data were collected and analyzed by the researcher. Her beliefs and background might have led to unintended biases in the data collection and the data analysis. To minimize this effect and strengthen the results of the study, various data collection techniques were used (e.g., a questionnaire, interviews, classroom observations). Also, a mathematics professor assisted the researcher in the development of the instruments and analyzing the data. Still, the results should be interpreted with caution.

REFERENCES

Adams, P. C. (2001). *A comparative analysis of the academic achievement of vocational education and college preparatory students*. (Unpublished doctoral dissertation). South Carolina State University, Orangeburg.

Agarwal, S. (2006). *The nature of pre-service secondary mathematics teachers' knowledge of mathematics for teaching of functions*. (Unpublished doctoral dissertation). State University of New York, Buffalo.

Akkoç, H. (2006). Fonksiyon kavramının çoklu temsillerinin çağrıştırdığı kavram görüntüleri. *Hacettepe Üniversitesi Eğitim Fakültesi Dergisi, 30,* 1-10.

Akkoç, H., & Tall, D. (2005). A mismatch between curriculum design and student learning: the case of the function concept. In D. Hewitt, & A. Noyes (Ed.), *Proceedings of the sixth British Congress of Mathematics Education* held at the University of Warwick, (pp. 1-8).

Alacaci, C., & Erbaş, A. K. (2010). Unpacking the inequality among Turkish schools: Finding from PISA 2006. *International Journal of Educational Development, 30*(2), 182-192.

American Council on Education. (1999). *To touch the future: Transforming the way teachers are taught.* Washington, DC: Author.

An, S., Kulm, G., & Wu, Z. (2004). The pedagogical content knowledge of middle school, mathematics teachers in China and the U.S. *Journal of Mathematics Teacher Education, 7,* 145-172.

An, S., Kulm, G., Wu, Z., Ma, F., & Wang, L. (2006). A comparative study on mathematics teachers' beliefs and their impact on the teaching practice between the U.S. and China. In F. K.S. Leung, K. D. Graf, & F. J. Lopez-Real (Ed.), *Mathematics education in different cultural* *traditions: A comparative study of East Asia and the West, The 13th ICMI Study* (p. 449-464). New York, NY: Springer.

Asquith, P., Stephen, A. C., Knuth, E. J., & Alibali, M. W. (2007). Middle school mathematics teachers' knowledge of students' understanding of core algebraic concepts: Equal sign and variable. *Mathematical Thinking and Learning, 9*(3), 249-272.

Australian Education Council (1990). *A national statement on mathematics for Australian schools*. Southern Regional Education Board, Publication Orders Department, 592 10th St. N.W., Atlanta, GA 30318-5790. (ERIC No. ED428947).

Ball, D. L. (1997). Developing mathematics reform: What don't we know about teacher learning-but would make good working hypotheses. In S.N. Friel, & G.W. Bright (Ed.), *Reflecting on our work: NSF teacher enhancement in K-5 (pp. 27-45)*. Lanham, NY: University Press of America.

Ball, D. L., Lubienski, S. T., & Mewborn, D. S. (2001). Research on teaching mathematics: The unsolved problem of teachers' mathematical knowledge. In V. Richardson (Ed.), *Handbook of research on teaching* (pp. 433-456). Washington, DC: American Educational Research Association.

Ball, D.L., Bass, H., Hill, H., & Schilling, S. (2005). *Developing measures of mathematical knowledge for teaching*. Presentation at Teachers Development Group Leadership Seminar, Mathematics Professional Development, Portland, OR.

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education, 59*(5), 389-407.

Baxter, P., & Jack. S. (2008). Qualitative case study methodology: Study design and implementation for novice researchers. *The Qualitative Report, 13*(4), 544-559.

Begle, E. (1979). *Critical variables in mathematics education: Findings from a survey of the empirical literature*. Washington, DC: The Mathematical Association of America and the National Council of Teachers of Mathematics.

Berberoğlu, G., & Kalender, İ. (2005). Öğrenci başarısının yıllara, okul türlerine, bölgelere göre incelenmesi: ÖSS ve PISA analizi. *Eğitim Bilimleri ve Uygulama, 4*(7), 21–35.

Berry, A., Loughran, J. J., & Van Driel, J. H. (2008). Revisiting the roots of pedagogical content knowledge. *International Journal of Science Education, 30*, 1271-1279.

Binici, H., & Arı, N. (2004). Mesleki ve teknik eğitimde arayışlar. *Gazi Üniversitesi Eğitim Fakültesi Dergisi, 24* (3), 383-396.

Bolte, L. A. (1993). *Preservice teachers' content knowledge of students: Status, organization, and envisioned application*. (Unpublished doctoral dissertation). University of Missouri, Columbia.

Bottoms, G., & Presson, A. (2000). *Finishing the job: Improving the achievement of vocational students*. Southern Regional Education Board, Publication Orders Department, 592 10th St. N.W., Atlanta, GA 30318-5790. (ERIC No. ED462559).

Breidenbach, D., Dubinsky, E., Hawks, J., & Nichols, D. (1992). Development of the process conception of function. *Educational Studies in Mathematics, 23*(3), 247-285.

Carpenter, T. P., Fennema, E., Peterson, P. L., & Carey, D. A. (1988). Teachers' pedagogical content knowledge of students' problem solving in elementary arithmetic. *Journal for Research in Mathematics Education, 19*, 29-37.

Carpenter, T. P., Fennema, E., Peterson, P. L., Chiang, C. P., & Loef, M. (1989). Using knowledge of children' mathematics thinking in classroom teaching: An experimental study. *American Educational Research Journal, 26*, 499-531.

Cha, I. (1999). *Prospective secondary teachers' conceptions of function: mathematical and pedagogical understanding*. (Unpublished Doctorial Dissertation). The University of Michigan, USA.

Clement, L. (2001). What do students really know about functions? *The Mathematics Teacher, 94*(9), 745-748.

Cooney, T. (1999). Conceptualizing teachers' ways of knowing. *Educational Studies in Mathematics, 38,* 163-187.

Creswell, J. W. (1998). *Qualitative inquiry and research design: Choosing among five traditions*. Thousand Oaks, CA: Sage Publications.

Creswell, J. W. (2007). *Qualitative inquiry and research design* (2nd ed.) Thousand Oaks, CA: Sage Publications.

Creswell, J. W. (2008). *Research design: Qualitative, quantitative, and mixed methods approaches*. Thousand Oaks, CA: Sage Publications.

Cunningham, R. F. (2005). Algebra teachers' utilization of problems requiring transfer between algebraic, numeric, and graphic representations. *School, Science and Mathematics 105*(2), 73-81.

Darling- Hammond, L. (2000). Teacher quality and student achievement: A review of state policy evidence. *Education Policy Analysis Archives 8*(1), 1-44.

Denzin, N.K., & Lincoln, Y.S. (2005). *The SAGE Handbook of Qualitative Research* (3rd. ed.). Thousand Oaks, CA: Sage Publications.

Dreyfus, T., & Eisenberg, T. (1982). Intuitive functional concepts: A baseline study on intuitions. *Journal for Research in Mathematics Education, 13,* 360–380.

Duah-Agyeman, J. K. (1999). *Secondary mathematics teachers understanding of mathematical functions*. (Unpublished doctorial dissertation). The Graduate School of Syracuse University, New York.

Dubinsky, E., & Harel, G. (1992). The nature of the process conception of function. In G. Harel & E. Dubinsky (Ed.), *The concept of function: Aspects of epistemology and pedagogy: MAA Notes Vol. 25* (pp. 85-106). Washington, DC: Mathematical Association of America.

Ebert, C. L. (1994). *Assessment of prospective secondary teachers' pedagogical content knowledge about functions and graphs*. (Unpublished doctoral dissertation). The University of Delaware, Newark.

Eisenberg, T. (1991). Functions and associated learning difficulties. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 140-152). Dodrecht: Kluwer Academic.

Eisenberg, T. (1992). On the development of a sense for functions. In G. Harel, & E. Dubinsky (Ed.), *The concept of function: Aspects of epistemology and pedagogy: MAA Notes Vol. 25* (pp. 153-174). Washington, DC: Mathematical Association of America.

Erbaş, A. K. (2004).*Teachers' knowledge of student thinking and their instructional practices in algebra*. (Unpublished doctoral dissertation). The University of Georgia, Athens.

European Commission, (2010). Structures of education and training systems in Europe. Turkey, 2009/10 Edition. Retrieved from the Web November 01, 2010. http://bit.ly/1aELd0Z

European Commission, (2011). Structures of education and training systems in Europe. Turkey, 2010/11 Edition. Retrieved from the Web September 11, 2011. http://bit.ly/1aELd0Z

Even, R. D. (1989). *Prospective secondary mathematics teachers' knowledge and understanding of mathematical functions*. (Unpublished doctoral dissertation). Michigan State University, Michigan.

Even, R. (1990). Subject matter knowledge for teaching and the case of functions. *Educational Studies in Mathematics, 21,* 521-554.

Even, R. (1993). Subject-matter knowledge and pedagogical content knowledge: Prospective secondary teachers and the function concept. *Journal for Research in Mathematics Education, 24*(2),94-116.

Even, R., & Tirosh, D. (1995). Subject-matter knowledge and the knowledge about students as sources of teacher presentations of the subject-matter. *Educational Studies in Mathematics, 29*(1), 1-20.

Even, R., & Tirosh, D. (2002). Teacher knowledge and understanding of students' mathematical learning. In L. English (Ed.), *Handbook of international research in mathematics education* (pp. 219-240). Mahwah, NJ: Lawrence Erlbaum Associates.

Fennema, E., & Franke, M. L. (1992). Teachers' knowledge and its impact. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 147-164). New York: Macmillan Publishing Company.

Gay, L. R., Mills, G. E., & Airasian, P. (2006). *Educational research: Competencies for analysis and applications.* New Jersey, Pearson Merrill Prentice Hall.

Gess-Newsome, J. (1999). Pedagogical content knowledge: An introduction and orientation. In J. Gess-Newsome, & N. G. Lederman (Ed.), *Examining pedagogical content knowledge* (pp. 3-17). Dordrecht: Kluwer Academic Publishers.

Graeber, A. (1999). Forms of knowing mathematics: What preservice teachers should learn. *Educational Studies in Mathematics*, *38*, 189-208.

Graeber, A., & Tirosh, D. (2008). Pedagogical content knowledge: Useful concept or elusive Notion. In P. Sullivam, & T. Wood (Ed.), *The international handbook of mathematics teacher education* (pp. 117-132). Rotterdam: Sense Publishers.

Green, A. (1998). Core skills, key skills and general culture: In search of the common foundation in vocational education. *Evaluation and Research in Education, 12* (1), 23-43.

Grossman, P. L. (1988). *A study of contrast: Sources of pedagogical content knowledge for secondary English*. (Unpublished doctoral dissertation). Standford University, Standford.

Grossman, P. (1990). *The making of a teacher: Teacher knowledge and teacher education*. New York: Teachers College Press.

Grossman, P., & Schoenfeld, L. (2005). Teaching subject matter. In L. Darling-Hammond, & J. Bransford (Ed.), *Preparing teachers for a changing worlds* (pp. 201-231). San Francisco: Jossey- Bass.

Hacıömeroğlu, G. (2006). *Prospective secondary teachers' subject matter knowledge and pedagogical content knowledge of the concept of functions*. (Unpublished doctoral dissertation). Florida State University, Florida.

Halim, L., & Meerah, S. M. (2002). Science trainee teachers' pedagogical content knowledge and its influence on physics teaching. *Research in Science & Technological Education, 20*, 215-225.

Hancock, D. R., & Algozzine, B. (2006). *Doing case study research: A practical guide for beginning researchers.* New York: Teachers College Press.

Hansson, O. (2006). *Preservice teachers' view on the concept of function: A study including the utilization of concept maps*. (Unpublished doctoral dissertation). Luleå University of Technology, Luleå.

Harel, G., & Dubinsky, E. (1992). *The concept of function: Aspects of epistemology and pedagogy*. Washington, DC: Mathematical Association of America.

Hatısaru, V., & Çetinkaya, B. (2011). Meslek lisesi öğrencilerinin doğrusal ve sabit fonksiyon ile bunların grafiksel gösterimine ilişkin algıları. *Eskişehir Osmangazi Üniversitesi Sosyal Bilimler Dergisi, 11*(1) 67-89.

Hatısaru, V., & Erbaş, A. K. (2013). Endüstri meslek lisesi öğrencilerinin fonksiyon kavramını anlama düzeylerinin incelenmesi. *Kastamonu Eğitim Dergisi, 21*(3), 865-882.

Hill, H. C., & Ball, D. L. (2004). Learning mathematics for teaching: Results from California's mathematics professional development institutes. *Journal for Research in Mathematics Education, 35*(5), 330-351.

Hill, H. C., Ball, D. L., & Schilling, S. G. (2008). Unpacking pedagogical content knowledge: Conceptualizing and measuring teachers' topic-specific knowledge of students. *Journal for Research in Mathematics Education, 39*, 372-400.

Hill, H. C., Rowan, B., & Ball, D. L. (2005). Effects of teachers' mathematical knowledge for teaching on student achievement. *American Educational Research Journal, 42*(2), 371-406.

Hill, H. C., Schilling, S. G., & Ball, D. L. (2004). Developing measures of teachers' mathematics knowledge for teaching. *The Elementary School Journal, 105*(1), 12-30.

Hitt, F. (1998). Difficulties in the articulation of different representations linked to the concept of function. *Journal of Mathematical Behavior, 17*(1), 123-134.

Hofacker, E. B. (2006). *The effect of different college algebra courses on students' understanding of linear and exponential function concepts*. (Unpublished doctoral dissertation). The University of Minnesota, Minneapolis.

Howald, C. L. (1998). *Secondary teachers' knowledge of functions: Subject matter knowledge, pedagogical content knowledge, and classroom practice*. (Unpublished doctoral dissertation). University of Iowa, Iowa.

Jansen, H. (2010). The logic of qualitative survey research and its position in the field of social research methods [63 paragraphs]. Forum Qualitative Sozialforschung /Forum: Qualitative Social Research, 11(2), Art. 11. Retrieved from the Web Julu 07, 2013. http://nbn-resolving.de/urn:nbn:de:0114-fqs1002110.

Kahan, J.A., Cooper, D.A., & Bethea, K.A. (2003). The role of mathematics teachers' content knowledge in their teaching: A framework for research applied to a study of student teachers. *Journal of Mathematics Teacher Education, 6,* 223-252.

Karahasan, B. (2010). *Preservice secondary mathematics teachers' pedagogical content knowledge of composite and inverse functions*. (Unpublished doctoral dissertation). Middle East Technical University, Ankara.

Kılıç, H. (2011). Preservice secondary mathematics teachers' knowledge of students. *Turkish Online Journal of Qualitative Inquiry, 2*(2), 17-35.

Köse, M. R. (1996). Üniversiteye giriş ve liselerimiz. *Hacettepe Üniversitesi Eğitim Fakültesi Dergisi, 15*, 51-60.

Krefting, L. (1990). Rigor in qualitative research: The assessment of trustworthiness. *The American Journal of Occupational Therapy, 45*(3), 214-222.

Lambertus, A. J. (2007). *Students' understanding of the function concept: Concept images and concept definitions*. (Unpublished master thesis). North Carolina State University, Raleigh.

Leinhardt, G., Zaslavsky, O., & Stein, M. K. (1990). Functions, graphs and graphing: Tasks, learning, and teaching. *Review of Educational Research, 60,* 1-64.

Lerman, S. (2001). A review of research perspectives on mathematics teacher education. In F. L. Lin & T. J. Cooney (Ed.), *Making Sense of Mathematics Teacher Education* (pp. 33-52). Kluwer Academix Publisher.

Lewis, M. V. (2000). Vocational education and the dilemma of education. *Journal of Vocational Education Research, 25*(4), 575-584.

Lincoln, Y. S., & Guba, E. G. (1985). *Naturalistic inquiry*. Thousand Oaks, CA: Sage Publications.

Lincoln, Y. S., & Guba, E. G. (2000). Paradigmatic controversies, contradictions, and emerging confluences. In N. K. Denzin & Y. S. Lincoln (Ed.), *Handbook of qualitative research* (2nd ed., pp. 163-188). Thousand Oaks, CA: Sage Publications.

Lloyd, G. M., & Wilson, M. R. (1998). Supporting innovation: The impact of a teacher's conceptions of functions on his implementation of a reform curriculum. *Journal for Research in Mathematics Education, 29*(3), 248-274.

Lucus, C. A. (2006). Is subject matter knwoledge affected by experience? The case of composition of functions. In Novotná, J., Moraová, H., Krátká, M. & Stehlíková, N. (Ed.). *Proceedings 30th Conference of the International Group for the Psychology of Mathematics Education, 4,* (pp. 97-104). Prague: PME.

Lundy, K. (2008). Prolonged engagement. In L. Given (Ed.), *The SAGE encyclopedia of qualitative research methods* (pp. 691-693). Thousand Oaks, CA: Sage Publications.

Magnusson, S., Krajcik, L., & Borko, H. (1999). Nature, sources and development of pedagogical content knowledge. In J. Gess-Newsome, & N. G. Lederman (Ed.), *Examining pedagogical content knowledge* (pp. 95-132). Dordrecht, The Netherlands: Kluwer.

Markovits, Z., Eylon, B. S., & Bruckheimer, M. (1986). Functions today and yesterday. *For the Learning of Mathematics, 6*(2), 18-24.

Markovits, Z., Eylon, B. S., & Bruckheimer. M. (1988). Difficulties students have with the function concept. In A. Coxford (Ed.), *The ideas of algebra, K-12, 1988 yearbook of the National Council of Teachers of Mathematics* (pp. 43-60). Reston, VA: NCTM.

Marks, R. (1990). Pedagogical content knowledge: From a mathematical case to a modified conception. *Journal of Teacher Education, 41*(3), 3-11.

Maurer, S. B. (1987). New knowledge about errors and new views about learners: What they mean to educators and more educators would like to know. In A. H. Schoenfeld (Ed.), *Cognitive science and mathematics education* (pp. 71-98). New Jersey: Lawrence Erlbaum Associates, Inc.

Maxwell, J. A. (2005). *Qualitative research design: An interactive approach* (2nd ed.). Thousand Oaks,CA: Sage Publications.

Mayer, D., & Marland, P. (1997). Teachers' knowledge of students: A significant domain of practical knowledge? *Asia-Pacific Journal of Teacher Education*, *25*, 17-34.

McGehee, J. J. (1990). *Prospective secondary teachers' knowledge of the function concept*. (Unpublished doctoral dissertation). The University of Texas, Austin.

Merriam, S. B. (2009). *Qualitative research: A guide to design and implementation*. San Francisco, CA: Jossey-Bass.

Mewborn, D. (2001). Teachers content knowledge, teacher education, and their effects on the preparation of elementary teachers in United States. *Mathematics Education Research Journal, 3*, 28-36.

Mewborn, D.S. (2003). Teaching, teachers' knowledge, and their professional development. In J. Kilpatrick, W.G. Martin, & D. Schifter (Ed.), *A research companion to the principles and standards for school mathematics* (pp. 45-52). Reston, VA: NCTM.

Miles, M. B., & Huberman, A. M. (1994). *Qualitative data analysis: An expanded source book.* Thousand Oaks,CA: Sage Publications.

Milli Eğitim Bakanlığı. (2012). *Ortaöğretim matematik 9. sınıf ders kitabı.* İstanbul: Devlet Kitapları.

Monk, D. (1994). Subject area preparation of secondary mathematics and science teachers and student achievement. *Economics of Education Review, 13*(2), 125-145.

Morse J. M., Barrett M., Mayan M., Olson K. & Spiers J. (2002). Verification strategies for establishing reliability and validity in qualitative research. *International Journal of Qualitative Methods, 1*(2), 1-19.

National Council of Teachers of Mathematics (1989). *Curriculum and evaluation standards for school mathematics*. Reston, VA: NCTM.

National Council of Teachers of Mathematics (2000). *NCTM principles and standards for school mathematics*. Reston, VA: NCTM.

National Governors Association Center for Best Practices & Council of Chief State School Officers. (2010). *Common core state standards for mathematics.* Washington, DC: Authors.

National Mathematics Advisory Panel (2008). *Foundations for success. The final report of the National Mathematics Advisory Panel.* Washington, DC: U.S. Departmet of Education.

Nesher, P. (1987). Towards an instructional theory: the role of student's misconceptions. *For The Learning of Mathematics, 7*(3), 115-132.

Nilklad, L. (2004). *College algebra students' understanding and algebraic thinking and reasoning with functions*. (Unpublished doctoral dissertation). Oregon State University, Corvallis.

Norman, F. A. (1993). Intergating research on teachers' knowledge of functions and their graphs. In T.A. Romberg, E., Fennema, & T.P. Carpenter (Ed.), *Integrating research on the graphical representation of function* (pp.159-187). Hillsdale, NJ: Lawrence Erlbaum Associates.

Norman, F. A. (1992). Teachers' mathematical knowledge of the concept of functions. In G. Harel & E. Dubinsky (Ed.), *The concept of function:*

Aspects of epistemology and pedagogy: MAA Notes Vol. 25 (pp. 215-235). Washington, DC: Mathematical Association of America.

OECD, (2004). *Learning for tomorrow's world: First results from PISA 2003.* Paris: OECD

OECD. (2009). *Creating effective teaching and learning environments: First results from TALIS.* Paris: OECD.

Özmantar, M. F., Bingölbali, E., & Akkoç, H. (2010). *Matematiksel kavram yanılgıları ve çözüm önerileri*. Ankara: PEGEM Akademi.

Park, S., & Oliver, J. S. (2008). Revisiting the conceptualization of pedagogical content knowledge (PCK): PCK as a conceptual tool to understand teachers as professionals. *Research in Science Education, 38*(3), 261-284.

Patton, M.Q. (1990). *Qualitative evaluation and research methods* (2nd ed.). Newbury Park, CA: Sage Publications.

Postelnicu, V. (2011). *Student difficulties with linearity and linear functions and teachers' understanding of student difficulties*. (Unpublished doctoral dissertation). Arizona State University, Arizona.

RAND Mathematics Study Panel. (2003). *Mathematical proficiency for all students*. Santa Monica, CA: RAND.

Rowan, B., Chiang, F. S., & Miller, R. J. (1997). Using research on employee's performance to study the effects of teachers on student achievement. *Sociology of Education, 70,* 256-284.

Rowan, B., Schilling, S. G., Ball, D. L., & Miller, R. (2001). Measuring teachers' pedagogical content knowledge in surveys: An exploratory study. (Research Note S-2). Ann Arbor, MI: Consortium for Policy Research in Education, Study of Instructional Improvement, University of Michigan.

Sajka, M. (2003). A secondary school student understands of the concept of function: A case study. *Educational Studies in Mathematics, 53*, 229-254.

Sánchez, V., & Llinares, S. (2003). Four student teachers' pedagogical reasoning on functions. *Journal of Mathematics Teacher Education, 6*, 5-25.

Scarpello, G. V. (2005). *The effect of mathematics anxiety on the course and career choice of high school vocational-technical education students*. (Unpublished doctoral dissertation). Drexel University, Philadelphia.

Selden, A., & Selden, J. (1992). Research perspectives on conceptions of functions: Summary and overview. The concept of functions: Aspects of epistemology and pedagogy. In G. Harel & E. Dubinsky (Ed.), *The concept of function: Aspects of epistemology and pedagogy: MAA Notes Vol. 25* (pp. 1-66). Washington, DC: Mathematical Association of America.

Sfard, A. (1992). Operational origins of mathematical objects and the quandary of reification-the case of function. In G. Harel & E. Dubinsky (Ed.), *The concept of function: Aspects of epistemology and pedagogy: MAA Notes Vol. 25* (pp. 59-84). Washington, DC: Mathematical Association of America.

Shenton, A. (2004). Strategies for ensuring trustworthiness in qualitative research projects. *Education for Information, 22,* 63-75.

Shulman, L. S. (1986). Those who understand: Knowledge and growth in teaching. *Educational Researcher, 15*(2), 4-14.

Shulman, L. S. (1987). Knowledge and teaching: Foundations of the new reform. *Harvard Educational Review, 57*(1), 1-22.

Sierpinska, A. (1992). On understanding the notion of functions. In G. Harel & E. Dubinsky (Ed.), *The concept of function: Aspects of epistemology and pedagogy: MAA Notes Vol. 25* (pp. 25-38). Washington, DC: Mathematical Association of America.

Slavit, D. (1997). An alternate route to the reification of function. *Educational Studies in Mathematics, 33*(3), 259-282.

Stake, R. E. (1995). *The art of case study research.* Thousand Oaks, CA: Sage Publications.

Stein, M. K., Baxter, J. A., & Leinhardt, G. (1990). Subject-matter knowledge and elementary instruction: A case from functions and graphing. *American Educational Research Journal, 27*(4), 639-663.

Şahin, İ., & Fındık, T. (2008). Türkiye'de mesleki ve teknik eğitim: Mevcut durum, sorunlar ve çözüm önerileri. *TSA 12*(3), 65-86.

Talim ve Terbiye Kurulu Başkanlığı. (2011). *Ortaöğretim matematik (9, 10, 11 ve 12. sınıflar) dersi öğretim programı ve kılavuzu.* Ankara: Milli Eğitim Bakanlığı.

Tall, D., & Bakar, M. (1991). Students' mental prototypes for functions and graphs. In F. Furinghetti (Ed.) *Proceedings of the 15th Conference of the International Group for the Psychology of Mathematics Education, Assisi. Vol. 1* (pp. 104-111).

Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics with special reference to limits and continuity. *Educational Studies in Mathematics, 12,* 151-169.

Tirosh, D. (2000). Enhancing prospective teachers' knowledge of children's conceptions: The case of division of fractions. *Journal for Research in Mathematics Education, 31*(1), 5-25.

Usul, H., Eroglu, H., & Akın, O. (2007). Meslek liseleri ve meslek yüksek okullarındaki eğitim süreçleri arasındaki uyum sorununun analizi ve ticaret meslek lisesi örneği. *Selçuk Üniversitesi Karaman İ. İ. B. F. Dergisi, 9*(12), 235-243.

Vinner, S. (1983). Concept definition concept image and the notion of function. *International Journal for Mathematics Education in Science and Technology, 14*(3), 293-305.

Vinner, S. (1992). The function concept as a prototype for problems in mathematics learning. In G. Harel & E. Dubinsky (Eds.), *The concept of function: Aspects of epistemology and pedagogy: MAA Notes Vol. 25* (pp. 195-213). Washington, DC: Mathematical Association of America.

Vinner, S. (2002). The role of definitions in the teaching and learning of mathematics. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 65-79). Kluwer Academic Publishers.

Vinner, S., & Dreyfus, T. (1989). Images and definitions for the concept of function. *Journal for Research in Mathematics Education, 20*(4), 356-366.

Watson, J., Callingham, R., & Donne, J. (2008). Proportional reasoning: Student knowledge and teachers' pedagogical content knowledge. Proceedings of the 31st Annual Conference of the Mathematics Education Research Group of Australasia M. Goos, R. Brown, & K. Makar (Ed.), MERGA, pp. 563-571.

Wayne, A. J., & Young, P. (2003). Teacher characteristics and student achievement gains: A review. *Review of Educational Research, 73*(1), 89-122.

Wilson, M. R. (1992). *A study of three preservice secondary mathematics teachers' knowledge and beliefs about mathematical functions*. (Unpublished doctoral dissertation). University of Georgia, Georgia.

Wood, T., & Sellers, P. (1997). Deepening the analysis: Longitudinal assessment of a problem-centered mathematics program. *Journal for Research in Mathematics Education, 28*(2), 163-186.

Wright, S. P., Horn, S. P., & Sanders, W.L. (1997). Teacher and classroom context effects on student achievement: Implications for teacher evaluation. *Journal of Personnel Evaluation in Education, 11*, 57-67.

Yavuz, İ., & Baştürk, S. (2011). Ders kitaplarında fonksiyon kavramı: Türkiye ve Fransa örneği. *Kastamonu Eğitim Dergisi, 9*(1), 199-220.

Yin, R. K. (2003). *Case study research: Design and methods* (3rd ed.). Thousand Oaks, CA: Sage publications.

Yin, R. K., (2009). *Case study research: Design and methods* (4th ed.). Thousand Oaks, CA: Sage publications.

You, Z. (2006). *Preservice teachers' knowledge of linear functions within multiple representation modes* (Unpublished doctoral dissertation). Texas A&M University, Texas.

Yörük, S. Dikici, A., & Uysal, A. (2002). Bilgi toplumu ve Türkiye'de mesleki eğitim. *Fırat Üniversitesi Sosyal Bilimler Dergisi*, *24(3)*, 229-312.

APPENDIX A

THE FUNCTION CONCEPT QUESTIONNAIRE

Dear Colleagues;

The present questionnaire was designed to identify teachers' awareness of the difficulties high school students have in the function concept. It does not aim to measure teachers' knowledge. The teachers' views and experience on the topic is crucial for the richness and effectiveness of the study. The responses to the questions will not be shared in a way to reveal the identities of the participants. I appreciate your contribution.

Vesife HATISARU
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Personal Information:

1. Gender: ☐ Female ☐ Male

2. Which faculty did you graduate from?
   Faculty of Education ☐
   Faculty of Arts and Science ☐
   Other ☐
   Please specify: ____

3. (for graduates of Faculty of Education) Did you receive any pedagogical formation?

☐Yes ☐No

4. If yes, please specify how long?

_______ month/year

5. Did you do/Are you doing a post-graduate study? If yes, please specify the type and name of the programme.

☐Master ☐Doctorate ____

6. How long have you been teaching?

_______ year/s

7. Have you taken an in-service training or attended any conferences, activities on mathematics education? If yes, please indicate their title and content briefly.

____

____

____

8. Would you like to participate in the second phase of the study which includes interviews and classroom observations?

☐ Yes ☐ No

Question 1: Imagine that you have asked your students to define the concept of function in their own words. How do you think they will define function? Please give a few definitions.

Question 2: Imagine that you have asked your students to give some examples of functions. What kind of examples do you think they will give for functions? Please specify a few of them.

Question 3: How do you think the students should define the function concept? Please specify the definition/s.

Question 4: What kind of examples do you think the students should give for function? Please specify these examples.

Question 5: Imagine that you want to write a question that could reveal your students' understanding of functions, what kind of an item would you generate?

Question 6: What aspect(s) of the study of functions do you think cause students the most difficulty? Please explain by giving examples.

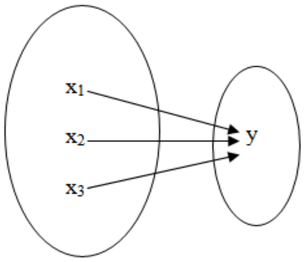
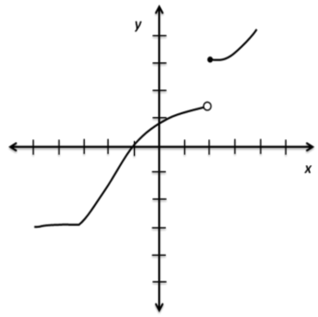
Question 7: What aspect(s) of the study of functions do you think cause students little difficulty? Please explain by giving examples.

Question 8: Assume that one of your students asks how functions and equations relate to each other, how would you respond?

Question 9: Assume that one of your students have inquired whether a function can be shown in different ways. How would you respond to the student?

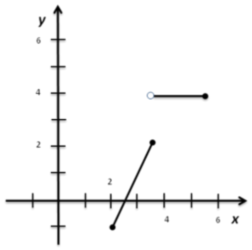
Question 10: Assume that you asked your students to identify the representations below, and a student marked all of them as non-functions. Please answer these questions for each case:

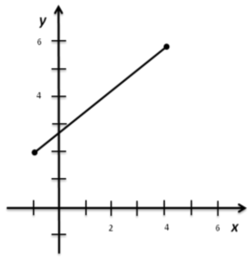
a) Is the student right? Why?
b) Is the student wrong? What do you think may have caused the mistake?

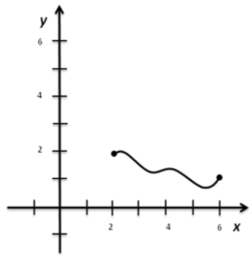
(a) 	(b) 
(c) $f : R \to R$ $x \to \begin{cases} -3x^3 + 3, & if\ \ x \geq 0 \\ 5, & if\ \ x < 0 \end{cases}$	(d) A correspondence which corresponds all positive numbers to 1, all negative numbers to -1, and 0 to 3.
(e) $y = 4$	(f) $\{(1,4), (2,5), (3,9)\}$

Question 11: Assume that you have asked your students to identify the graph/s which represent/s a function whose domain is $\{x : 2 \le x \le 6\}$ and whose range is $\{y : -1 \le y \le 4\}$. A student marked the graph (a)/(b)/(c). Please answer these questions for each case:

a) Is it correct? If so, explain why.
b) Is it wrong? If so, what do you think has caused the mistake?

a) 

b) 

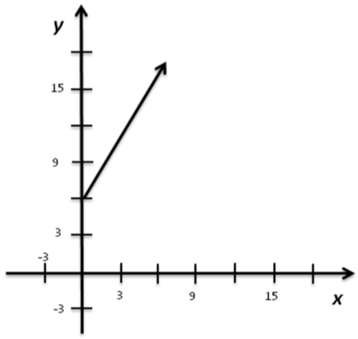
c) 

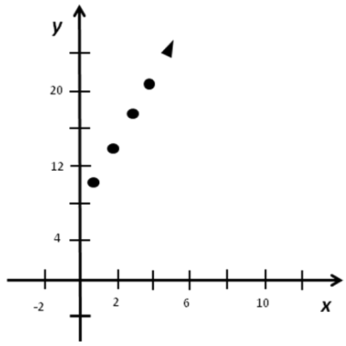
Question 12: As for $f : N \to N$, $f(x) = 4x + 6$, assume that you have asked your students to identify which item/s equal/s to f. A student identifies that item (a)/(b)/(c)/(d) equals to f. Please answer these questions for each case:

a) Is it correct? If so, explain why.
b) Is it wrong? If so, what do you think has caused the mistake?

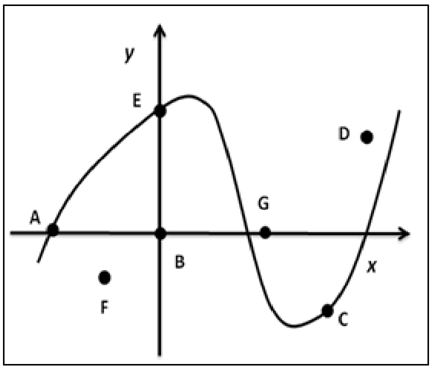
a) $g : R \to R$ ve $g(x) = 4x + 6$

b) $h : N \to N$ ve $h(x) = 2x + 3$

c) 

d) 

Question 13: Assume that, as regards the graph below, you asked the following:



a) Which points represent an element of the domain?
b) Which element represents an element of the range?
c) Which points represent (pre-image, image) pairs?
d) Which points do not represent (pre-image, image) pairs?

A student came up with the following responses:

a) A, E, C b) E, B, G

c) B, G d) F, D.

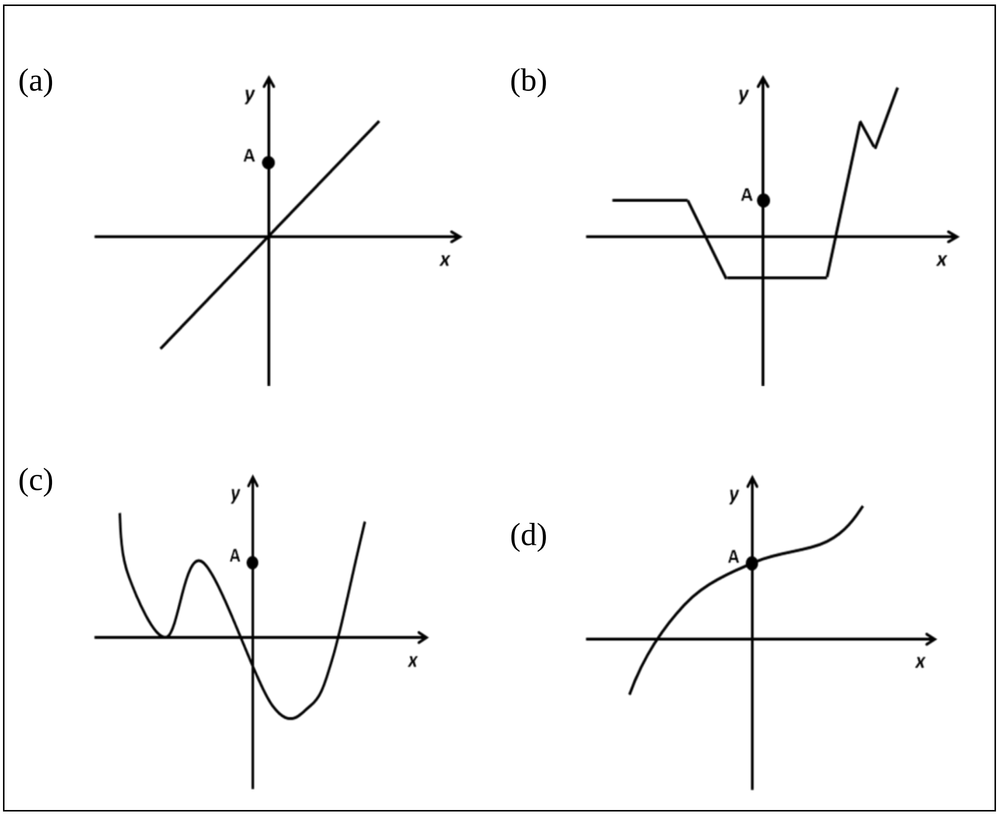
Are the student's responses right or wrong? If wrong, how do you think the student should have responded?

Question 14: Assume that you have asked your students to locate the pre-images of point A on each graph below and that one of your students has responded as follows.

a) A, E, C b) E, B, G

c) B, G d) F, D.

For each case, are the students' responses right or wrong? If wrong, how do you think the student should have responded?



Question 15: Assume that you have asked your students to give an example of a graph of a function that runs through points A and B (see Figure 1). A student has drawn the one in Figure 2. You have also asked if there is another answer the student said, 'No'. How do you think the student should have responded? Please specify.

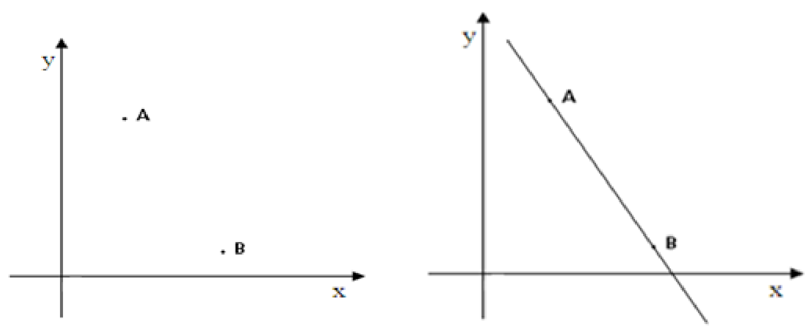


Figure 1

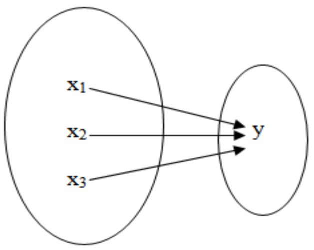
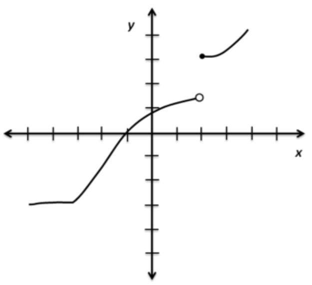
Figure 2

APPENDIX B

SOURCES OF EACH QUESTION AND ITS RATIONALE FOR INCLUSION

	The rationale for inclusion of the question
Question 1: (Source: Clement, 2001; Tall & Bakar, 1991; Vinner 1983; Vinner & Dreyfus, 1989)	In several studies, students were asked to define the concept of function (Clement, 2001; Tall & Bakar, 1991; Vinner, 1983). It has been found that students' understanding of functions appears either to be too narrowly focused or to include erroneous assumptions (Clement, 2001).
Question 2: (Source: Breidenbach et al., 1992)	Students' understanding of the function concept does seem to be weak. When they were asked to give examples of a function, their choice of examples indicated that their thinking about functions is expressions (Breidenbach et al., 1992). In addition, students often have the misconception that every function should be linear (Markovits et al., 1988).
Question 3: (Source: Cooney, 1992, as cited in Cooney, 1999; Even, 1989; Vinner & Dreyfus, 1989)	Prospective teachers' understanding of the function concept does seem to be weak. Their conceptions are similar to students' conceptions as described in the literature (Even, 1993). Also, in-service teachers' conceptions of the function are not consistent with contemporary characterizations of the concept (Cooney, 1992, as cited in Cooney, 1999).

Question 5: (Source: Cooney, 1992, as cited in Cooney, 1999)	Many teachers' conception of function does seem to be related to the concept of equation (Cooney, 1992, as cited in Cooney, 1999; Even, 1993) and revealed a strong computational orientation (Cooney, 1992, as cited in Cooney, 1999).
Question 6: (Source: Bolte, 1993)	In the literature some aspects of functions are identified to be the most difficult for students. For instance, students exhibit errors when they work with the composition of functions and the inverse function (Bolte, 1993).
Question 7: (Source: Bolte, 1993)	In the literature some aspects of functions are identified to be the least difficult for students such as graphing points, operating on functions except composing them, and evaluating functions at specific points (Bolte, 1993).
Question 8: (Source: Even, 1989)	Most prospective teachers think that all functions can be formed by using a formula (Even, 1989).
Question 9: (Source: Bolte, 1993)	Different representations of functions do seem to play a minor role in some prospective teachers' conceptions (Bolte 1993).

Question 10:	Many students cannot apply definition of the function concept to a specific representation (Vinner, 2002). When students have to determine whether given relations are functions, they succeed with items that are considered prototypes of functions and non-functions such as graphical, algebraic, and set correspondence (Akkoç, 2006).
a)  (Source: Hitt, 1998)	Students have difficulty with many-to-one correspondences. They often believe that the elements of two sets be in a one-to-one (Markovits, et al., 1988; Vinner, 1983).
b)  (Source: Even, 1989; Bolte, 1993)	Students' conceptions of the graphs of functions are limited (Markovits et al., 1986; Tall & Bakar, 1991; Vinner, 1983; Vinner & Dreyfus, 1989). They have difficulties with functions represented by a disconnected graph. Often they identify only linear graphs as graphs of functions (Leinhardt et al., 1990).
c) $f: R \to R$ $x \to \begin{cases} -3x^3+3, & \text{if } x \geq 0 \\ 5, & \text{if } x < 0 \end{cases}$ (Source: Markovits et al., 1986)	Students have difficulties with functions defined piecewise. They often think that functions given by more than one rule are not functions (Markovits et al., 1986; Vinner, 1983).

d) A correspondence which corresponds all positive numbers to 1, all negative numbers to -1, and 0 to 3. (Source: Even, 1989)	Students run into difficulty when the function given is not a prototype. The verbal representation of functions, for instance, is most difficult to students, and even to some pre-service teachers (Bolte, 1993).
e) $y = 4$ (Source: Tall & Bakar, 1991)	Most students often assume that functions must involve x (Markovits et al., 1986). They often believe that a function must include some algebraic formulas (Clement, 2001).
f) $\{(1,4), (2,5), (3,9)\}$ (Source: Even, 1989; Bolte, 1993)	It is reported that even some prospective teachers do not apply mathematical definition of the function concept to a set of ordered pairs representation of functions and do not consider the functions defined on a discrete set of number as functions (Even, 1989).
Question 11: (Source: Markovits et al., 1986)	In graphical representation, many students have difficulty in identifying domain and range of functions, and vice versa. In addition, they have difficulty in understanding that the set of images may be a subset of the range (Markovits et al., 1986)

Question 12: (Source: Markovits et al., 1988)	Most students ignore the domain and range when they identify two identical functions (Markovits et al., 1988).
Question 13: (Source: Markovits et al., 1988)	Many students do not appreciate that in the graphical representation the x axis represents the domain and the y axis the range, whereas the point on the graph represent (pre-image, image) pairs (Markovits et al., 1988).
Question 14: (Source: Hitt, 1998)	Like many students at secondary schools (Markovits et al., 1986) many teachers have difficulty in identifying domain and image set in the graphical representation of the function (Hitt, 1998).
Question 15: (Source: Even, 1989; Markovits et al., 1986)	Many students ignore the arbitrary nature of functions. Most are prone to graph the function as a relation that represents a linear pattern (Markovits et al., 1988). They often have the misconception that every function is a linear function (Markovits et al., 1988). Also, many prospective teachers ignore the arbitrary nature of function (Even, 1989).

APPENDIX C

FOLLOW-UP INTERVIEW

Dear Colleague;

The present interview was designed to clarify the answers to the function concept questionnaire and at the same time to develop a detailed picture of the teachers' KCS of the function concept. The interview will last about one hour. The responses to the questions will not be shared in a way to reveal the identities of the participants. Thank you in advance for your participation.

Part I

1. How do you define the concept of function?

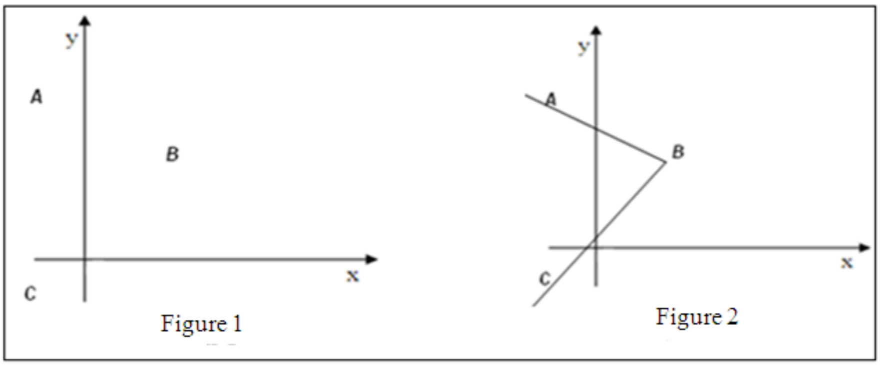
2. Could you give an alternate definition of the function concept?

3. Could you compare this definition with your first definition? Are they consistent with each other? Do they mean the same thing?

4. How do you explain the concept of function to your students? Which of those definitions do you use? Or, would you give them a different definition? What examples do you use?

5. How do you teach your students to decide functionality of relations? Explain by giving examples.

6. Assume that you have asked your students to give an example of a graph of a function that runs through points A, B, and C (see Figure 1), and a student draw the one in Figure 2. What do you think about this answer? Is the student's response right? If yes, are there any other correct answers? If no, how do you think the student should have responded? What do you think has caused the mistake?



7. How important for students to know the concept of function?

8. How important for students to learn about functions?

<u>Part II</u>: A review of Ali's responses to the function concept questionnaire:

 # 3&4: Why is it important for you the students define and examplify the concept of function like that? What types of other examples can students give to functions?

 # 5: What is your aim to ask that question? Why is it important for you the students solve that question? Could you state an alternative(s) question(s)?

 # 8: How are functions and equations related to one another? Are all functions equations? Are all equations functions? Are all functions can be represented in the form of equations? Explain by giving examples.

 # 9: Give an example of functions students encounter in mathematics class. Can you represent this function in a different way? Please, state the all possible different ways?

- In your response, you have stated that you would give examples to all representations. Could you please give some examples?

 # 10: You have stated that the function in item (c) has two images. Also, the representation in item (f) does not define a function. Could you explain it?

 # 11: You have stated that the student's responses to item (a) and (c) are wrong. Could you justify it?

 # 12: You have stated that the student's response to item (a) is correct. Could you explain how did you decide it?

13: Could you explain how did you decide to the item (b)?
 # 14: How do you decide the student's response is wrong?
#15: How do you answer this question?

<u>Part II</u>: A review of Fatma's responses to the function concept questionnaire

 # 3&4: Why is it important for you the students define and exemplify the concept of function like that? What types of other examples can students give to functions?

 # 5: What is your aim to ask that question? Why is it important for you the students solve that question? Could you state an alternative(s) question(s)?

 # 8: Are all equations equal to "0"? How are functions and equations related to one another? Are all functions equations? Are all equations functions? Are all functions can be represented in the form of equations? Explain by giving examples.

 # 9: Give an example of functions students encounter in mathematics class. Can you represent this function in a different way? Please, state the all possible different ways?

 # 10: What fo you mean in items (d) ve (f)?

#11: You have stated that the student's response to item (b) is incorrect. What do you think has caused the mistake?

#12: You have stated that the student's responses to item (a), (b), and (c) are incorrect. What do you think has caused the mistake?

 # 15: How did you decide that an infinite number of graphs of functions can be drawn passing through A? Could you please give some examples?

<u>Part III</u>: Card Sort Activity

Assume that you have given your students a stack of 20 cards and asked them to group the cards. How do you think the students would group the cards? How would they group the cards in a different way(s)? How do you think the student should group the cards?

Table C.1: Cards

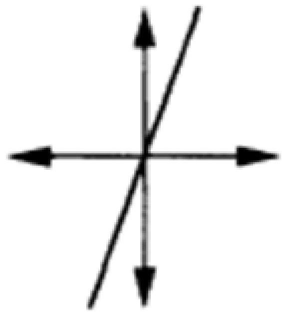
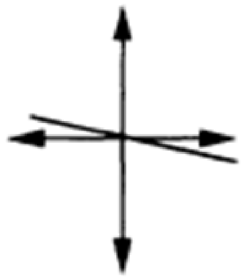
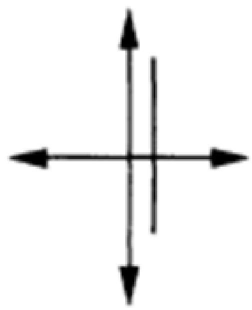
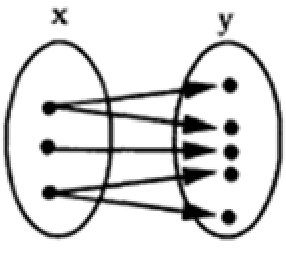
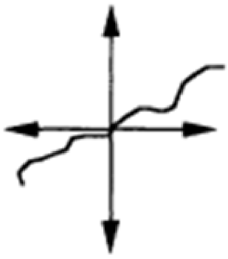
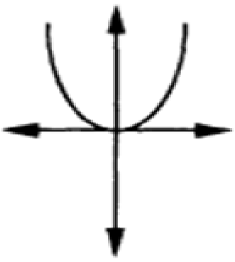
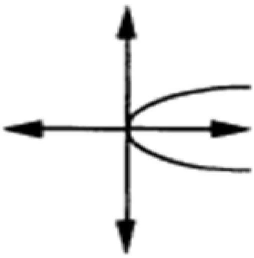
1 (3,6) (3,2) (2,1)	**2** x | y 2 | 4 -6 | -12 13 | 26 -57 | -114	**3** 
4 $y = x$	**5** (5,5) (-53, -53) (0,0)	**6** 
7 x | y 6 | -3 -10 | 5 63 | -31.5 -15 | 7.5	**8** $y = -1/2x$	**9** 

Table C.1. Cards (continued)

<table>
<tr>
<td>10
</td>
<td>11
$x = 2$</td>
<td>12
$y = 2x$</td>
</tr>
<tr>
<td>13
</td>
<td>14
</td>
<td>15
</td>
</tr>
<tr>
<td>16
<table><tr><th>x</th><th>y</th></tr><tr><td>6</td><td>5</td></tr><tr><td>5</td><td>4</td></tr><tr><td>4</td><td>3</td></tr><tr><td>4</td><td>2</td></tr></table></td>
<td>17
<table><tr><th>x</th><th>y</th></tr><tr><td>1</td><td>-1</td></tr><tr><td>3</td><td>7</td></tr><tr><td>4</td><td>14</td></tr><tr><td>6</td><td>34</td></tr></table></td>
<td>20
</td>
</tr>
<tr>
<td>18
(1,2) (100,200) (-16,-32)</td>
<td>19
$y = x^2$</td>
<td></td>
</tr>
</table>

APPENDIX D

THE FUNCTION CONCEPT TEST

Dear Students;
This test was designed to identify your understanding of the function concept. Your responses will be used in a research study. Please read the questions carefully and try to answer all questions.
I appreciate your contribution.

Vesife HATISARU
Middle East Technical University
Faculty of Education
Department of Secondary Science and Mathematics Education

Question 1: Define the concept of function in your own words.

Question 2: Give some examples of functions.

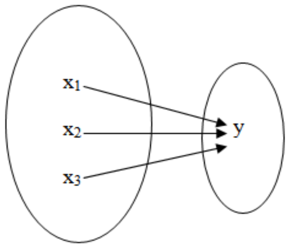
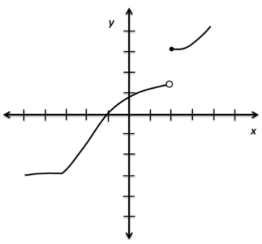
Question 3: What aspect(s) of the study of functions is the most difficult for you? Please, explain by giving examples.

Question 4: What aspect(s) of the study of functions is little difficult for you? Please, explain by giving examples.

Question 5: How functions and equations relate to each other? Please, explain by giving examples.

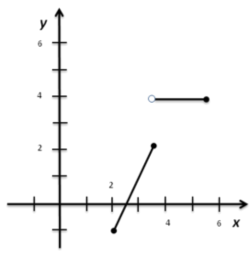
Question 6: Do you think a function can be shown in different ways? If so, please explain by giving examples.

Question 7: Identify functionality of each representation below. Please, explain your answer.

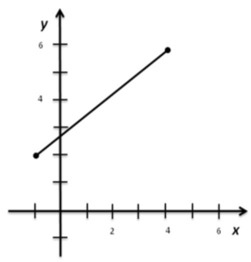
<table>
<tr><td>(a) A correspondence which corresponds all positive numbers to 1, all negative numbers to -1, and 0 to 3.</td><td>(b) $y = 4$</td></tr>
<tr><td>(c) $f: R \to R$
$x \to \begin{cases} -3x^3 + 3, & \text{if } x \geq 0 \\ 5, & \text{if } x < 0 \end{cases}$</td><td>(d) 
</td></tr>
<tr><td>(e) 
</td><td>(f) $\{(1,4), (2,5), (3,9)\}$</td></tr>
</table>

Question 8: Identify the graph/s which represent/s a function whose domain is $\{x : 2 \leq x \leq 6\}$ and whose range is $\{y : -1 \leq y \leq 4\}$. Please, explain your answer.

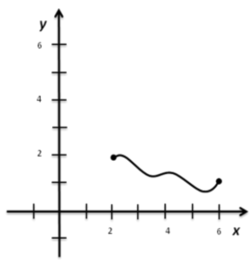
a)



b)



c)

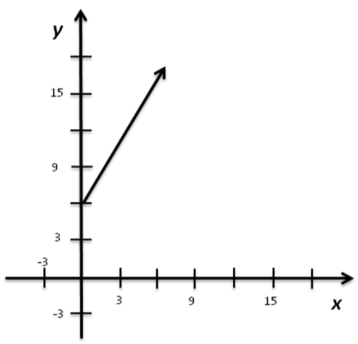


Question 9: Identify the function(s) which equal(s) to $f : N \to N$, $f(x) = 4x + 6$. Please, explain your answer.

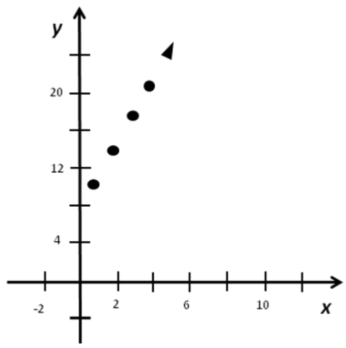
a) $g : R \to R$ ve $g(x) = 4x + 6$

b) $h : N \to N$ ve $h(x) = 2x + 3$

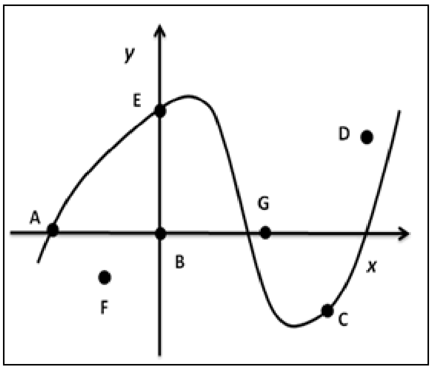
c)



d)

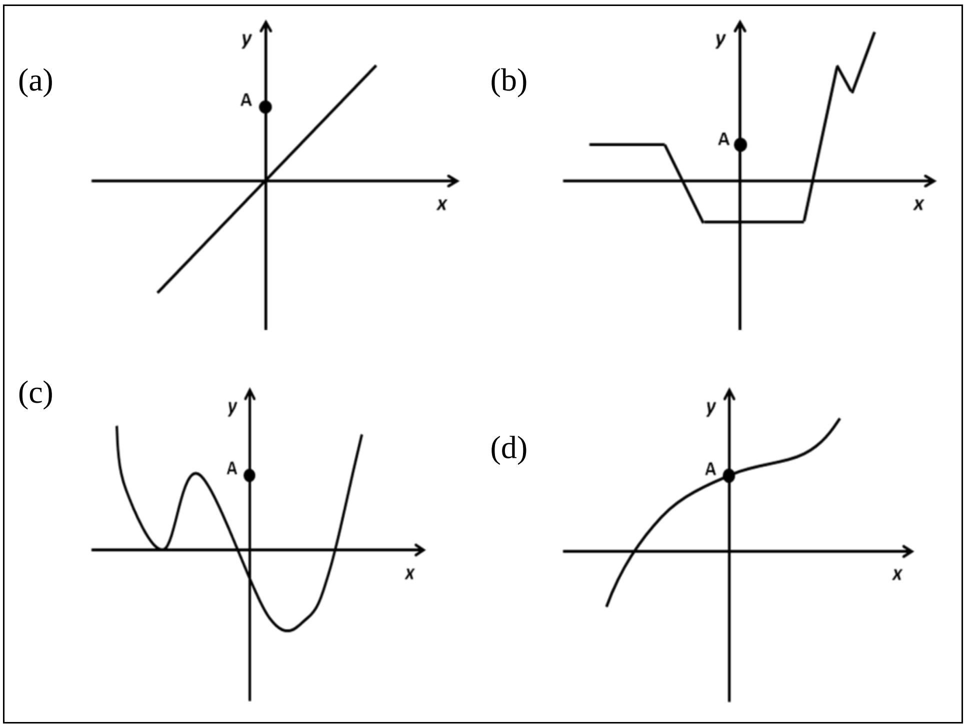


Question 10: As regards the graph below, please answer the following questions:

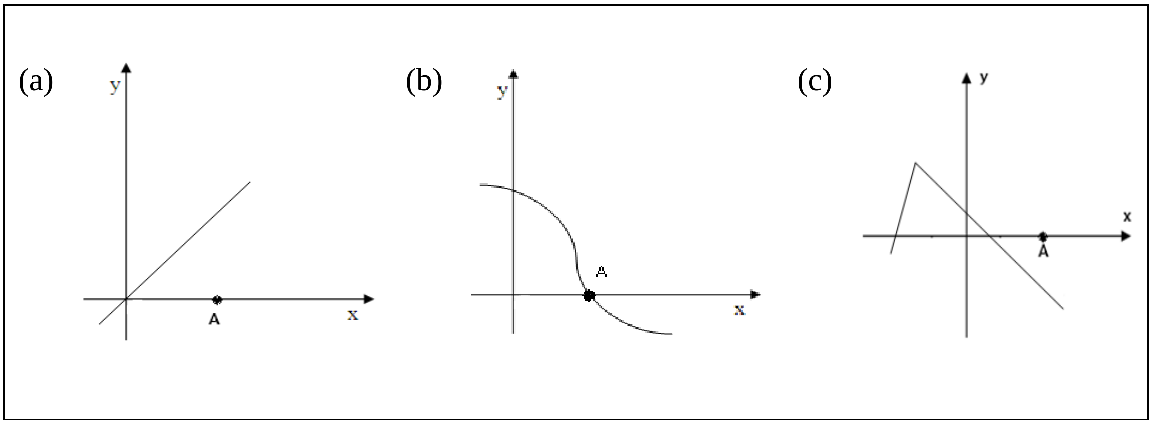


a) Which points represent an element of the domain?
b) Which element represents an element of the range?
c) Which points represent (pre-image, image) pairs?
d) Which points do not represent (pre-image, image) pairs?

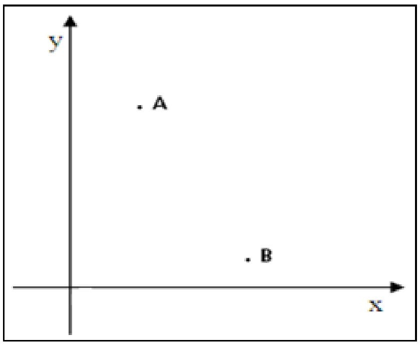
Question 11: Locate the pre-images of point A on each graph below.



Question 12: Locate the images of point A on each graph below.



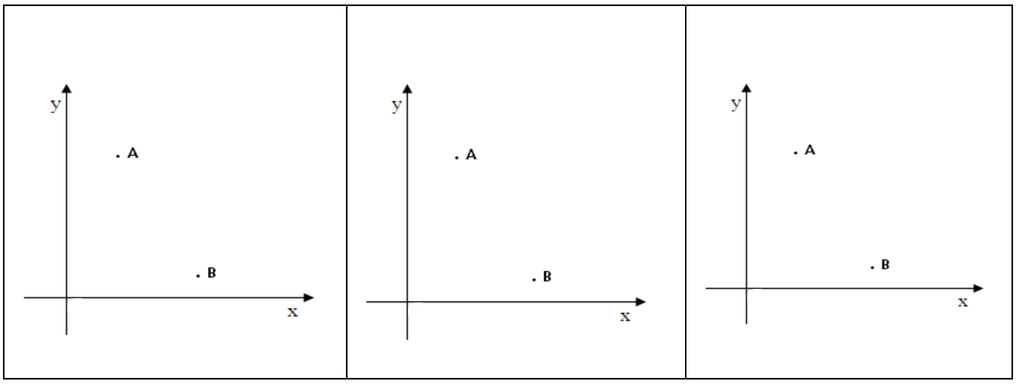
Question 13: Draw a graph of a function that runs through points A and B below.



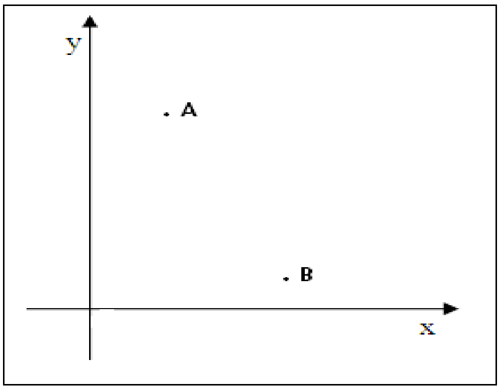
How many different such functions that can be drawn?

A) 0 D) more than 2 but fewer than 10
B) 1 E) more than 10 but not infinite
C) 2 F) infinite

Please, give some of those the graphs of functions below.



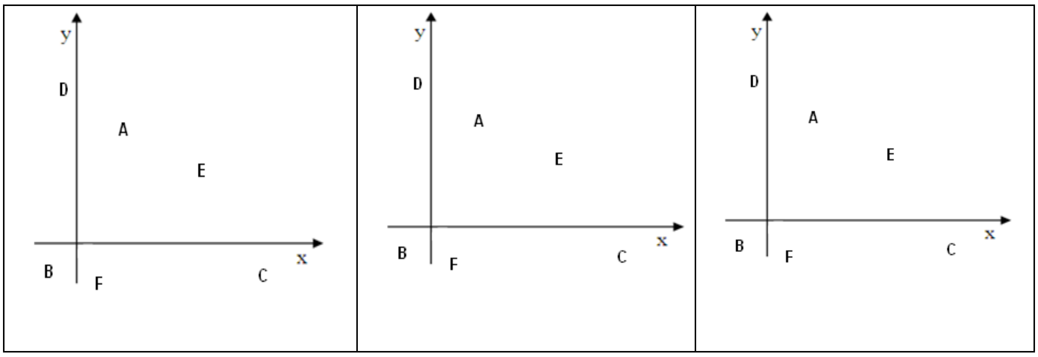
Question 14: Draw a graph of a function that runs through points A, B, C, D, E, F below.



How many different such functions that can be drawn?

D) 0	D) more than 2 but fewer than 10
E) 1	E) more than 10 but not infinite
F) 2	F) infinite

Please, give some of those the graphs of functions below.



Question 15: For the function $f: R \to R$, $f(x) = 4x + 6$, fill in the blanks.

A) $f(2) = \_\_\_$ B) $f(\_\_\_) = 10$ C) $f(\_\_\_) = -26$ D) $f(-\frac{1}{2}) = \_\_\_$

For the function $g: R \to R$, $g(x) = -7$, fill in the blanks.

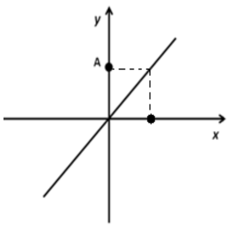
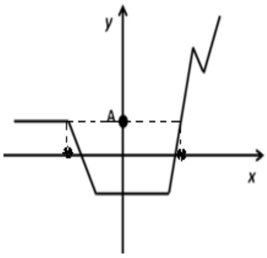
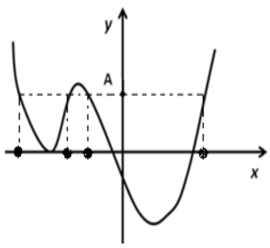
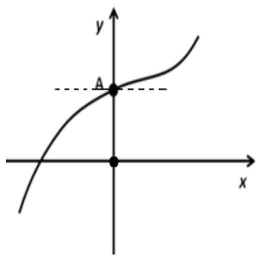
A) $g(4) = \_\_\_$ B) $g(-7) = \_\_\_$ C) $g(\_\_\_) = 0$ D) $g(\_\_\_) = -7$

APPENDIX E

SCORING RUBLIC

#1&2	Students' conceptions of the function concept information were recorded directly.			
#3&4	Teachers' conceptions of the function concept information were recorded directly.			
#5	Determining functionality of such varied relations (e.g., set correspondences, graphs, algebraic expressions, or sets of ordered pairs **20 points**	Determining functionality of relations a set correspondence **10 points**	Evaluating functions at specific points (e.g., if $f(x)=3x-5$, find $f(2)$). **5 points**	Specific questions (e.g., $f(3x-1)=4x+2$, find $f(x)$, $f(4-x)$. **5 points**
#6&7	The potential areas of functions studying for students information was recorded directly.			

#8	An equation states a condition on a single quantity. A function expresses a relationship between two quantities **20 points**	They are the same. Here is a function: $f(x) = 3x + 4$; here is an equation: $3x + 4 = 0$ **10 points**	No answer **0 point**	
#9	Different representations **20 points**	Notations **10 points**	No answer **0 point**	
#10	Difficulty in many-to-one correspondence Difficulties with functions represented by a disconnected graph Difficulties with functions given by more than one rule Difficulties with the verbal representation of functions A function must include some algebraic formula Difficulties with the set notation of functions			**10 points** **10 points** **10 points** **10 points** **10 points** **10 points**
#11	The student's answer is correct (with a true justification) **10 points**	The student's answer is wrong (with a true justification) **10 points**	The student's answer is correct (with a true justification) **10 points**	

#12	The student's answer is wrong (with a true justification) **10 points**	The student's answer is wrong (with a true justification) **10 points**	The student's answer is wrong (with a true justification) **10 points**	The student's answer is correct (with a true justification) **10 points**
#13	Pre-images: A, B, G **10 points**	Images: B, E **10 points**	(pre-image, image) pairs: A, E, C **10 points**	Other pairs: B, D, F, G **10 points**
#14	 **10 points**	 **10 points**	 **10 points**	 **10 points**
#15	The student's answer is wrong, infinitely many graphs **20 points**	The student's answer is wrong, parabolic curves **10 points**		

APPENDIX F

CODING SCHEME

Teachers’ Content Knowledge of the Function Concept

Conceptions of the function concept			
Coding		Meaning	Examples
#3	Correspondence	Indicates that the function is defined by giving some reference to mapping, pairing, or relation between the elements of two sets.	“The connection that corresponds each element of Set A with only one element of Set B, given that A and B are not empty sets.” “There should be a domain and range; no element should remain un-corresponded in the domain.”
	Operation	Indicates that the function is defined by giving some reference to an operation or manipulation.	“The mechanism of transforming one element to another depending on the type of function defined.” “A conveyer converting an element into another element by exposing it to various processes.”

#4	Correspondence	Indicates that the function is exemplified by giving some reference to mapping, pairing, or relation between the elements of two sets.	"Establish a relation pairing two sets." "Set A is the set of children, Set B is the set of mothers; then, (i) no element will be left out in Set A because every child has a mother, (ii) a child cannot have more than one mother."
	Operation	Indicates that the function is exemplified by giving some reference to an operation or manipulation.	"Defining a particular function, and indicating what is to come out of elements that they have chosen themselves." "Olives processed in a factory coming out of the factory as olive oil."
	Algebraic expressions	Indicates that the function is exemplified by giving algebraic statements.	"$y = 2x + 3$, $y = \frac{x+5}{2x+7}$, $y = \sin x$."
	Graphs	Indicates that the function is exemplified by giving graphs.	
	Verbal meaning	Indicates that the function is exemplified by its word meaning.	"A person can lose his or her life functions." "A tool can have different functions." "The gadgets we use such as computer, telephone have functions."

#5	Evaluating functions at specific points	Indicates that giving an algebraic function, it is asked to evaluate the function at a specific point.	“ $f(x)=3x-5$, $f(3)=?$ ” “ $A=\{1,2,3\}$, $B=\{3,5,6\}$, and $f: A \rightarrow B$, $f(x)=2x+1$, $f(2)=?$ ”
	Examining functionality of relations	Indicates that giving some relations, it is asked to examine functionality of the relations.	
	Specific questions	Indicates that specific questions are asked.	“ $f(3x-1)=4x+2 \Rightarrow f(x)=?, f(4-x)=?$ ”
#8	Relation	Indicates that functions are identified as a relationship between two quantities.	“Function is a rule/representation depicting the relation between two variables and is the mathematical form of the relation between the dependent and independent variables.” “In function, x yields different output according to the input.”
	A condition on a single quantity	Indicates that equations are identified as a condition on a single quantity.	“Equation is a representation established for finding the unknown part from among the other parts in the equality, and has a solution set.” “Equation is a condition of equality, and it has a set of solutions.”

	Representation	Indicates that functions and equations are identified by their representational forms.	"Equations are like $x-2=3$, and functions are like $f(x)=3x+5$." "Functions have domain and range, whereas equations have solution set."
#9	Different representations	Indicates that there is some reference to different representations of a function (e.g., graphs, sets of ordered pairs).	"We can possibly show functions by means of set of ordered pairs, graphs, or their rules." "Graphical illustrations, schematic diagrams, and listing are possible."
	Notations	Indicates that there is some reference to the notations of functions.	"It can be expressed as in $x \to y$, $f(x)=y$, $f: x \to y$." "It can have a variety of representations such as $f, p, \cos, \log$."
		Essential features of functions	
#10	Univalence requirement	Indicates that there is some reference to the univalence requirement of functions, i.e. every element in the domain should be paired with one but only one element in the range.	"All elements in the domain have the same image (like the constant function), it meets the function criteria."

	Representational form of functions	Indicates that there is no reference to the essential features of functions.	"It is correct [item-a], because domain and range are not clear." "It is not a function [item-b]; an element is left out in the domain." "It is correct [item-e] because there is the $y = 4$ line, $f: R \to R, x \to f(x) = 4$ should be for it to be a function." "The student is right; what is the domain of this function [item-f]? Is there an element left out?"
		Relating image and range to its graph	
#11	Correct	Indicates that the image and range are correctly related to its graph.	"It is correct as the domain and range is fulfilled." "It is correct as the domain and range meet the given conditions."
	Incorrect	Indicates that the image and range are related to its graph incorrectly.	"Domain is between 2 and 6, but piecewise function with x value between 3 and 4 occurred." "Domain, x axis vary between 2 and 6. Range, y axis varies between -1 and 4. The $4, 5, 6$ elements in the domain are left out."
	Ambiguous responses	Indicates that unclear responses are given.	"It is not known from which sets x and y are selected."

	Coding	Meaning	Examples
		Identifying two equal functions	
#12	Correct	Indicates that the two equal functions are accurately identified.	"Domain N and Domain R of the graphs are diffferent [item-c]." "It is correct; every element has an image; element which are not natural numbers have no image [item-d]."
	Incorrect	Indicates that the two equal functions are identified incorrectly.	"Because $N \subset R$ [item-a]." "Correct [item-b], because both functions have equal domain and range, only that $f(x)$ function is as twice as $h(x)$ function." "Correct [item-b] because these two rules are bound to produce the same output for the same input."
		Locating pre-images, images, and (pre-image, image) pairs on the axes in graphs	
#13	Correct	Indicates that the pre-images, images, and (pre-image, image) pairs are accurately located on the axes in graphs.	"(a) A, B, G (b) E, B (c) A, E, C (d) F, B, G, D"

	Incorrect	Indicates that the pre-images, images, and (pre-image, image) pairs are located on the axes in graphs incorrectly.	"(a) A (b) C, E (c) $(A,B),(B,E)$ (d) E, B"
	Domain and range is to be given.	Indicates that it is stated that the domain and range need to be given for the question to be tackled with.	"Abscissa and ordinates are not clear, so the domain and range is not definite." "Domain and range should be well-defined; indeed, any response to such a shape would be wrong."
		Locating pre-images and images on the axes in graphs	
#14	Correct	Indicates that the pre-images and images are accurately located on the axes in graphs.	See Appendix E
	Incorrect	Indicates that the pre-images and images are located on the axes in graphs incorrectly.	"Because there is no one-to-one, it is not a function graph." "The points with image A are not clear."
	Domain and range is to be given.	Indicates that it is stated that the domain and range need to be given.	"We do not know the domain points, and we do not know the function equation."

		Essential features of functions	
#15	Arbitrariness	Indicates that there is some reference to the arbitrary nature of functions, i.e. a function does not necessarily realize the correspondence between the elements of the two sets through an arithmetical or algebraic rule.	“From A and B, I can draw an indefinite number of functions which do not violate the rules of a function. To double-check this, I would utilize the vertical line test”, “An indefinite number of functions that pass through A and B can be drawn; they do not have to be necessarily linear.”
	Specific functions		“Parabolic curves or various curves could be drawn.” “Second or third degree parabolas, curves could be drawn.”
	No reference to the arbitrariness	Indicates that there is no reference to the arbitrary nature of functions.	“Right, because a single line can pass through any two points.” “It is true because when lines parallel to y axis are drawn, they intersect with the axis at one point only. Then, it applies to this situation.”

Teachers' Knowledge of Common Student Conceptions and Misconceptions about the Function Concept

	Students' conceptions of the function concept		
	Coding	**Meaning**	**Examples**
#1	Correspondence	Indicates that the teachers think students identify function as a correspondence.	"Set A corresponds to Set B; a number from Set A goes to Set B, no element remains in Set A; an element can remain in Set B." "All elements in the domain go to the image set only once."
	Operation	Indicates that the teachers think students identify function as an operation.	"Function is a kind of factory, with its input and output." "Assigning a value to x to find y." "A transformation operation, i.e. functions are like machines, making carpets by using threads."
	Representation	Indicates that the teachers think students identify function with its symbolic representations.	$f: A \to B, x \to y = f(x)$
	Verbal meaning	Indicates that the teachers think students identify function with its word meaning.	The blue-tooth functions of the mobile phones or something like that. A series of functions like the function of dishwasher.

#2	Algebraic expressions	Indicates that the teachers think students exemplify function in algebraic terms.	"$f(x)=3x+1,\ f(x)=x^2-2x+5$." "$f:R\to R,\ f(x)=2x+4$."
	Set correspondence	Indicates that the teachers think students give examples in terms of set correspondence.	"Identifying children as domain and mothers as range, every children have a mother."
	Operation	Indicates that the teachers think students give examples in terms of set correspondence.	"e.g., fruit juice squeezer, factory machines, bread factory, or cattle going into a slaughterhouse, and hams and sausages going out."
	Functions of things	Indicates that the teachers think students give to functions relate to the 'daily use' meaning of function.	"The running of a machine and serving its function." "The function of a telephone, the function of a teacher in class."
	The potential areas of functions studying for students information was recorded directly.		
#6&7	The potential areas of functions studying for students information was recorded directly.		

	Students’ difficulty with the essential features of functions	
#10	There is some reference to the following student difficulties; Difficulty in many-to-one correspondence Difficulties with functions represented by a disconnected graph Difficulties with functions given by more than one rule Difficulties with the verbal representation of functions A function must include some algebraic formula Difficulties with the set notation of functions	“As they all go to the same element, it probably confused the student because the students have the tendency to take each element to a different element; that is, they always consider the function to be one-to-one [item-a]).” “The student came up with a wrong answer because the graph was clearly disconnected [item-b].” “The student may have thought that the function should be equivalent to a single expression only [item-c].” “He probably could not understand the expression, and that they cannot think of the verbal expressions mathematically [item-d].” “He cannot understand that it is a function because he does not see a statement with x, he would succeed if it was put in $f(x)=4$ [item-e].”
	Students’ mistakes are referenced to their inability to comprehend the subject. Ambiguous responses	“The student had though this way because he did not comprehend constant function.” “He did not understand the function graphs at all.” “The student does not know interval concepts [item-b]” “He might be confused because there are five images for each value smaller than 0.” “He can consider 0 to be positive.”

	Students' difficulty in relating domain and range of a function to its graph	
#11	There is some reference to students' difficulty in relating domain and range of a function to its graph.	"The students had mixed domain with range."
	Students' mistakes are referenced to their inability to comprehend the subject.	"The student had not grasped the domain and range well." "Lack of knowledge on the number line and the coordinate system", "A possible misconception about linear function." "Inability to read the domain and range on the graph."
	Students' difficulty in identifying two equal functions	
#12	There is some reference to students' difficulty in identifying two equal functions.	"The student considers the domain and range without looking at the function." "He is mistaken probably because the domain and range are the same."
	Students' mistakes are referenced to their inability to comprehend the subject.	"He did not know the definition and equality of function, so approached the question the way he did." "The student does not know the function at all."

Students’ misconceptions that every function need to be linear		
#15	There is some reference to students’ difficulty in identifying two equal functions.	“The students respond to this with the belief that any two distinct points are incident with just one line.” “He might have responded this way leaving from the axiom that any two distinct points are incident with just one line.”
	Students’ mistakes are referenced to their inability to comprehend the subject.	“He could not really grasp the concept function and does not know the graphical-function connection.” “He did not understand the subject.”

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WORK EXPERIENCE

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October 2013-Present	Turkish National Agency for LLP & YiA Programmes	Expert
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Advanced English, Basic French

PUBLICATIONS

Journal Articles

1. Hatısaru, V. (2013). İlköğretim öğrencilerinin dinleme temelli test performansları ile okuma temelli test performanslarının karşılaştırılması. Dicle Üniversitesi Ziya Gökalp Eğitim Fakültesi Dergisi, 21(2), 107-119.

2. Hatısaru, V. & Erbaş, A. K. (2013). Endüstri meslek lisesi öğrencilerinin fonksiyon kavramı anlama düzeylerinin incelenmesi. Kastamonu Eğitim Dergisi, 21(3), 865-882.

3. Hatısaru, V. & Çetinkaya, B. (2011). Endüstri meslek lisesi öğrencilerinin doğrusal ve sabit fonksiyon ile bunların grafiksel gösterimine ilişkin algıları. Eskişehir Osmangazi Üniversitesi Sosyal Bilimler Dergisi, 11(1), 67-89.

4. Hatısaru, V. & Küçükturan, A. G. (2011). Probleme dayalı öğrenme yönteminin endüstri meslek lisesi 9. sınıf öğrencilerinin matematik dersi başarısına etkisi. Çağdaş Eğitim Dergisi, 36(382), 29-38.

Proceedings

1. Hatısaru, V., & Erbaş, A. K. (2012). Matematik öğretiminde matematik tarihinin yeri: Türk, Portekiz, İspanyol ve Fransız matematik öğretmenlerinin görüşleri. X. Ulusal Fen Bilimleri ve Matematik Eğitimi Kongresi, Niğde Üniversitesi, Niğde.

2. Hatısaru, V., & Erbaş, A. K. (2012). Matematik eğitiminde endüstri meslek liselerinde yaşanan sorunlar ve çözüm önerileri. X. Ulusal Fen Bilimleri ve Matematik Eğitimi Kongresi, Niğde Üniversitesi, Niğde.

3. Hatısaru, V., & Erbas, A. K. (2010). Students' perceptions of the concept of function: The case of Turkish students attending technical and industrial vocational high schools. World Conference on Educational Sciences, Bahçeşehir University, İstanbul.

4. Hatısaru, V. (2009). Matematik öğretimi nasıl olmalı? Sınıf tekrarı yapan öğrencilerle örnek uygulama. 8. Matematik Sempozyumu, TOBB Üniversitesi, Ankara.

5. Hatısaru, V. (2009). Ortaöğretim öğrencilerinin matematiğe yönelik düşünceleri: Kompozisyon yazma uygulaması. 8. Matematik Sempozyumu, TOBB Üniversitesi, Ankara.

6. Hatısaru, V., & Küçükturan, A. G. (2009). Problem-based learning in vocational and technical education: Sample scenario. World Conference on Educational Sciences, Near East University, North Cyprus.

7. Hatısaru, V. & Küçükturan, A. G. (2009). 9th grade industrial vocational high school students' views on problem-based learning experiences. World Conference on Educational Sciences, Near East University, North Cyprus.

Paper presented

1. Hatısaru, V. (2012). Investigating the interrelationships between teachers' pedagogical content knowledge and student achievement within vocational high school context. The 35th Annual Conference of the Mathematics Education Research Group of Australasia (MERGA), Nanyang Technological University (NTU), Singapore.

2. Hatısaru, V., & Çetinkaya, B. (2012). İlköğretim öğrencilerinin matematik ve matematikçi algılarının incelenmesi. X. Ulusal Fen Bilimleri ve Matematik Eğitimi Kongresi, Niğde Üniversitesi, Niğde.

3. Hatısaru, V., Erbaş, A. K., & Çetinkaya, B. (2011). Using history of mathematics in teaching mathematics: teachers' views. 35th Conference of the International Group for the Psychology of Mathematics Education, Middle East Technical University, Ankara.

4. Hatısaru, V., & Çetinkaya, B. (2011). Assessing students' listening skills: A case study. 35th Conference of the International Group for the Psychology of Mathematics Education, Middle East Technical University, Ankara.

5. Hatısaru, V., & Çetinkaya, B. (2010). Meslek lisesi öğrencilerinin fonksiyon kavramını algılamaları. IX. Ulusal Fen Bilimleri ve Matematik Eğitimi Kongresi, Dokuz Eylül Üniversitesi, İzmir.

6. Hatısaru, V., & Çetinkaya, B. (2009). Endüstri meslek lisesi öğrencilerinin doğrusal ve sabit fonksiyon kavramına ait bilgileri. 18. Ulusal Eğitim Bilimleri Kurultayı, Ege Üniversitesi Eğitim Fakültesi, İzmir.

7. Hatısaru, V. (2009). Anlatım yönteminin güçlendirilmesi: Örnek bir matematik etkinliği uygulaması. 18. Ulusal Eğitim Bilimleri Kurultayı, Ege Üniversitesi Eğitim Fakültesi, İzmir.

8. Hatısaru, V. (2009). Probleme dayalı öğrenme yöntemi. Eğitiminin Değişen Yüzü Yeni Paradigmalar. Özel Arı Okulları, Ankara.

9. Hatısaru, V. (2008). Faktöriyel Kavramı. Ortaöğretim Öğrencileri I. Ulusal Matematik Sempozyumu, Milli Eğitim Vakfı Ankara Özel Okulları, Ankara.

Poster Presentations

1. Hatısaru, V., & Ribeiro, M. (2011). Turkish and Portuguese vocational high school students' perceptions on algebra. 35th Conference of the International Group for the Psychology of Mathematics Education, Middle East Technical University, Ankara.

2. Hatısaru, V., & Aksu, M. (2010). Matematik dersinde iki farklı öğretim uygulaması: Türkiye ve İngiltere örneği (Poster Bildiri). IX. Ulusal Fen Bilimleri ve Matematik Eğitimi Kongresi, Dokuz Eylül Üniversitesi, İzmir.

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