BACKWARD STOCHASTIC DIFFERENTIAL EQUATIONS AND FEYNMAN-KAC FORMULA IN THE PRESENCE OF JUMP PROCESSES

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ABSTRACT

BACKWARD STOCHASTIC DIFFERENTIAL EQUATIONS AND FEYNMAN-KAC FORMULA IN THE PRESENCE OF JUMP PROCESSES

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Backward Stochastic Differential Equations (BSDEs) appear as a new class of stochastic differential equations, with a given value at the terminal time T. The application area of the BSDEs is conceptually wide which is known only for forty years. In financial mathematics, El Karoui, Peng and Quenez have a fundamental and significant article called "*Backward Stochastic Differential Equations in Finance*" (1997) which is taken as a groundwork for this thesis. In this thesis we follow the following steps: Firstly, the principal theorems of BSDEs driven by Brownian motion are proved. Later, an application to partial differential equations (PDEs) is presented i.e. generalization of Feynman-Kac formula. Moreover, the studies of Situ in 1997 and his book entitled with "*Theory of Stochastic Differential Equations with Jumps and Applications*" provide us a framework to prove explicitly the main theorems of BSDEs in the presence of jumps. Afterward, Feynman-Kac formula for general Lévy processes is proven. Lastly, the results are concluded by some applications in financial mathematics.

Keywords: backward stochastic differential equations, Feynman-Kac formula, option pricing, hedging portfolios, jump processes

ÖZ

SIÇRAMA SÜREÇLERİNİN VARLIĞINDA GERİYE DOĞRU STOKASTİK DİFERANSİYEL DENKLEMLER VE FEYNMAN-KAC FORMÜLÜ

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Geriye Doğru Stokastik Diferansiyel Denklemler (GSDDler) bitiş zamanındaki değeri verilen yeni bir stokastik diferansiyel denklem sınıfı olarak ortaya çıkmıştır. GSDDlerin son kırk yıldır bilinmelerine rağmen uygulama alanı gittikçe genişlemektedir. Bu teze temel oluşturan El Karoui, Peng ve Quenez'e ait "*Backward Stochastic Differential Equations in Finance*" (1997) isimli makale finansal matematikte son derece önemli bir yer tutmaktadır. Tezin işleniş şekli aşağıdaki aşamalardan oluşmaktadır: Öncelikle, Brown hareketi ile oluşturulan GSDDler için temel teoremler ispatlanmıştır. Daha sonra, kısmi diferansiyel denklemlere (KDDlere) uygulama olan Feynman-Kac formülü incelenmiştir. Ayrıca, sıçramaların varlığında GSDDlerin ana teoremlerini açık şekilde ispatlamamız için Situ'nun 1997 yılındaki çalışmaları ve "*Theory of Stochastic Differential Equations with Jumps and Applications*" başlıklı kitabı bize yol göstermiştir. Sonrasında, genel Lévy süreçleri için Feynman-Kac formülü ispatlanmıştır. Son olarak, finansal matematikte bazı uygulamalar yapılmıştır.

Anahtar Kelimeler: geriye doğru stokastik diferansiyel denklemler, Feynman-Kac formülü, opsiyon fiyatlama, korunma portföyleri, sıçrama süreçleri

To My Big Family

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TABLE OF CONTENTS

CHAPTER 1

INTRODUCTION

Backward stochastic differential equations (BSDEs) are one of the interesting research areas with increasing activity because of their connections with theoretical economics, nonlinear partial differential equation (PDE) theory, stochastic optimization problems and particularly, mathematical finance. The theory of finding the replicating portfolio and pricing contingent claims is typically considered in terms of a linear BSDE in mathematical finance. This thesis organized as a careful and explicit study about existence and uniqueness of BSDEs, comparison of solutions, their application in finance and connection to PDE with basic Brownian motion case and more advanced case with jumps. So as to provide a comprehensible thesis about BSDEs, the order of a brief literature review, the difference between forward stochastic differential equations (SDEs or FSDEs) and backward stochastic differential equations (BSDEs), the stream of the thesis is given in the introduction.

BSDEs were firstly defined and expressed linearly by Bismut [4] in 1973. They become more popular after the general concept of BSDEs were considered by Pardoux and Peng [23] in 1990. They proved the existence and uniqueness of adapted processes as a pair (Y, Z), such that

$$dY_t = -f(t, Y_t, Z_t)dt + {Z_t}^* dW_t, \quad Y_T = \xi \tag{1.1}$$

for given uniformly Lipschitz adapted stochastic process $f : [0, T] \times \mathbb{R}^d \times \mathbb{R}^{n\times d} \mapsto \mathbb{R}^d$ (called *generator* or *driver*), d-dimensional Brownian motion W and square integrable terminal condition ξ. The contributions of Pardoux and Peng to development of BSDEs are really rich and valued such as introducing comparison theorem (Peng [25], 1992) and generalization of Feynman-Kac formula (Pardoux and Peng [24], 1992). In 1997, El Karoui and Quenez [10] studied on crucial applications in the theory of mathematical finance with BSDEs taking parts in hedging and non-linear pricing theory. In the same year, El Karoui, Peng and Quenez [9] advanced their studies and wrote their worthy article which constructs the basis of this thesis. BSDEs driven by a Brownian motion and a Poisson point process is studied by Situ [26] in 1997 and his book *Theory of Stochastic Differential Equations with Jumps and Applications* [27] includes a chapter about BSDEs which is also closely followed to study about BSDEs with jumps in this thesis.

Recently, it is more interesting and sophisticated to work with BSDEs instead of

SDEs. A standard SDE is given as

$$dY_t = -f(t, Y_t, Z_t)dt + Z_t{}^*dW_t, \quad Y_0 = \xi$$

which can be solved under growth, Lipschitz conditions and bounded expectation of the terminal value's square ($\mathbb{E}[\xi^2] < +\infty$). Furthermore, this SDE has an adapted unique solution by the help of SDE theory [17]. However, unlike standard forward differential equations when the terminal value of the contingent claim is known, adaptedness problem for the solution arises. In other words, solving a SDE backwards becomes more complicated since the given terminal value $Y_T = \xi$ leads ξ to be $\mathcal{F}_T$-measurable. Then significantly, Y_t's will become $\mathcal{F}_T$-measurable. The equivalent integral form of equation (1.1)

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s{}^*dW_s, \tag{1.2}$$

shows that Z_s is $\mathcal{F}_T$-measurable ($Z_s \in \mathcal{F}_T$, $s \in [t, T]$) then the integral with respect to Brownian motion ($\int_t^T Z_s{}^*dW_s$) would have no meaning in normal SDE theory even with bounded and Lipschitz f and Z. Luckily, by the help of the financial experiences, the most famous backward problems of finding the replicating portfolio or the price of a contingent claim indicate the existence of a pair of random processes (Y_t, Z_t) satisfying the equation (1.1) where Y_t, Z_t could be considered as a wealth process and a hedging portfolio respectively. This distinct is the main motivation to work with BSDEs and their applications.

One of our major target is to have more realistic market assumptions since it would be more valuable. Working with multidimensional model and allowing jumps are the tools of this thesis so as to have more realistic and incomplete markets. Additionally, for practical reasons it is good to have multidimensional models in financial applications. In order to address broad research scholars, this thesis is divided into two parts mainly from theoretical aspects. The ones being interested in BSDEs in finance and their connection to PDEs in Brownian case could utilize the chapters 2,3. The other ones being attracted by BSDEs in the presence of jumps could utilize the last chapters 4,5.

The organization of this thesis can be described as follows:

After a general explanatory introduction about BSDE, in Chapter 1; the theory of BSDE including existence and uniqueness of a solution, comparison of solutions, are stated and clearly proved in Chapter 2. Also in Chapter 2, pricing and hedging contingent claims as an application of BSDEs in finance are explained.

In Chapter 3, forward-backward stochastic differential equations (FBSDEs) are introduced and by the help of FBSDEs the generalization of Feynman-Kac theorem is driven in the Brownian case.

The financial experiences show that the diffusion models are not strong enough to capture the empirical properties of asset returns, represent the main features of option prices and provide suitable tools for hedging and risk management.

Although the mathematical calculations become hard to derive, working with jump processes gives more practical results.

In Chapter 4, the structure of BSDEs with jumps are introduced, the existence and uniqueness theorem is proven and the comparison theorem is stated. In addition, the model in finance, pricing and hedging contingent claims are considered.

As a difference from the most of researches done by using orthonormalized compensated power-jump processes (called *Teugels martingales*), in Chapter 5 the generalized Feynman-Kac formula is proven by the help of FBSDEs for the general Lévy processes. Finally, the thesis is concluded by a brief summary of all work done and potential future studies.

CHAPTER 2

BACKWARD STOCHASTIC DIFFERENTIAL EQUATIONS

Both from the theory and application sides, the backward stochastic differential equations (BSDEs) are widely studied after they were introduced by Bismut [4] in 1973. As of imposing research area, many scientists contribute to evolve BDSEs in many fields during forty years as of today. One of the substantial resources of this thesis is the article of El Karoui, Peng and Quenez [9] which enlarges the earlier studies in finance. In this chapter, a detailed study of the theory and arguments will be done by closely following this article. Firstly, the space and dynamics of a BSDE are defined. Later, by some priori estimates the spread between the solutions of two BSDEs is going to be stated, from which the results of existence and uniqueness will be derived. In particular, the classical one-dimensional linear BSDEs in finance is studied and the comparison theorem is stated. Lastly, a BSDE model in finance is discussed.

For ease of use we fix the notation as follows

- For $x \in \mathbb{R}^d$, $|x|$ denotes its Euclidean norm.
- For $x \in \mathbb{R}^d$, $\langle x, y \rangle$ denotes the inner product.
- An $n \times d$ matrix will be considered as an element $y \in \mathbb{R}^{n \times d}$.
- For $y \in \mathbb{R}^{n \times d}$, the Euclidean norm is given by $|y| = \sqrt{\text{trace}(yy^*)}$.
- For $y,\ z \in \mathbb{R}^{n \times d}$, the inner product is given by $\langle y, z \rangle = \text{trace}(yz^*)$.
- For the rest of the paper "*" is used to denote transpose matrix.

Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and an $\mathbb{R}^n$-valued Brownian motion W, we consider

- $\{(\mathcal{F}_t); t \in [0, T]\}$, the filtration generated by the Brownian motion W and argumented and $\mathcal{P}$ the σ-field of predictable sets of $\Omega \times [0, T]$.
- $\mathcal{F} = \sigma\{\mathcal{W}_t : t \geq 0\}$, is the sigma algebra generated by the Brownian motion.

- $\mathbb{L}_T^2(\mathbb{R}^d)$, the space of all $\mathcal{F}_T$-measurable random variables $X : \Omega \mapsto \mathbb{R}^d$ satisfying $\|X\|^2 = \mathbb{E}(|X|^2) < +\infty$.
- $\mathbb{H}_T^2(\mathbb{R}^d)$, the space of all predictable processes $\phi : \Omega \times [0,T] \mapsto \mathbb{R}^d$ such that $\|\phi\|^2 = \mathbb{E}\int_0^T |\phi_t|^2 dt < +\infty$.
- $\mathbb{H}_T^1(\mathbb{R}^d)$, the space of all predictable processes $\varphi : \Omega \times [0,T] \mapsto \mathbb{R}^d$ such that $\mathbb{E}\sqrt{\int_0^T |\varphi_t|^2 dt} < +\infty$.
- For $\beta > 0$ and $\phi \in \mathbb{H}_T^2(\mathbb{R}^d)$, $\|\phi\|_\beta^2$ denotes $\mathbb{E}\int_0^T e^{\beta t}|\phi_t|^2 dt$. $\mathbb{H}_{T,\beta}^2(\mathbb{R}^d)$ denotes the space $\mathbb{H}_T^2(\mathbb{R}^d)$ endowed with the norm $\|\cdot\|_\beta$.

For notational simplicity we may use $\mathbb{L}_T^2(\mathbb{R}^d) = \mathbb{L}_T^{2,d}$, $\mathbb{H}_T^2(\mathbb{R}^d) = \mathbb{H}_T^{2,d}$, $\mathbb{H}_T^1(\mathbb{R}^d) = \mathbb{H}_T^{1,d}$ and $\mathbb{H}_{T,\beta}^2(\mathbb{R}^d) = \mathbb{H}_{T,\beta}^{2,d}$.

2.1 Existence and Uniqueness of Backward Stochastic Differential Equations

Before introducing the dynamics of a BSDE, it is important to emphasize that the existence and uniqueness firstly proven by Pardoux and Peng [23]. However, El Karoui, Peng found a shorter proof by using the difference between the solutions of two BSDEs [10] which is also clearly stated in this work.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and consider the BSDE

$$-dY_t = f(t, Y_t, Z_t)dt - Z_t^* dW_t, \quad Y_T = \xi, \tag{2.1}$$

or, equivalently,

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s^* dW_s, \tag{2.2}$$

where

- the terminal value $\xi > 0$ is an $\mathcal{F}_T$-measurable random variable, which maps Ω onto $\mathbb{R}^d$,
- the generator f maps $\Omega \times \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^{n\times d}$ onto $\mathbb{R}^d$ and it is $\mathcal{P} \bigotimes \mathcal{B}^d \bigotimes \mathcal{B}^{n\times d}$-measurable. Note that, $\mathcal{B}^d$ denotes Borel-measurable sets in $\mathbb{R}^d$, likewise $\mathcal{B}^{n\times d}$ denotes Borel-measurable sets in $\mathbb{R}^{n\times d}$.

Definition 2.1. The pair (Y, Z) is said to be a *solution* if it satisfies the equation (2.1) where $\{Y_t : t \in [0,T]\}$ is a continuous $\mathbb{R}^d$-valued adapted process and $\{Z_t; t \in [0,T]\}$ is an $\mathbb{R}^{n\times d}$-valued predictable process in $\mathbb{H}_T^{2,n\times d}$.

Definition 2.2. A function f is *uniformly Lipschitz* if there exists constant $C > 0$ such that

$$|f(\omega, t, y_1, z_1) - f(\omega, t, y_2, z_2)| \leq C(|y_1 - y_2| + |z_1 - z_2|) \quad \forall (y_1, z_1),\ \forall (y_2, z_2) \in \mathbb{R}^2.$$

Definition 2.3. The pair (f, ξ) are called *standard parameters* of the BSDE if $\xi \in \mathbb{L}_T^{2,d}$, $f(\cdot, 0, 0) \in \mathbb{H}_T^{2,d}$ and f is uniformly Lipschitz.

In the following proposition, some useful inequalities obtained in order to use in the proof of existence and uniqueness theorem.

Proposition 2.1. *Let (f^1, ξ^1), (f^2, ξ^2) be two standard parameters of the BSDE and (Y^1, Z^1), (Y^2, Z^2) be two square-integrable solutions. Let C be a Lipschitz constant for f^1 and put $\delta Y_t = Y_t^1 - Y_t^2$, $\delta Z_t = Z_t^1 - Z_t^2$ and $\delta_2 f_t = f^1(t, Y_t^2, Z_t^2) - f^2(t, Y_t^2, Z_t^2)$. For any (λ, μ, β) such that $\mu > 0$, $\lambda^2 > C$ and $\beta \geq C(2 + \lambda^2) + \mu^2$, it follows that*

$$\|\delta Y\|_\beta^2 \leq T \left[e^{\beta T} \mathbb{E}(|\delta Y_T|^2) + \frac{1}{\mu^2} \|\delta_2 f\|_\beta^2 \right], \tag{2.3}$$

$$\|\delta Z\|_\beta^2 \leq \frac{\lambda^2}{\lambda^2 - C} \left[e^{\beta T} \mathbb{E}(|\delta Y_T|^2) + \frac{1}{\mu^2} \|\delta_2 f\|_\beta^2 \right]. \tag{2.4}$$

Proof. Let $(Y, Z) \in \mathbb{H}_T^{2,d} \times \mathbb{H}_T^{2,n\times d}$ be a solution of our BSDE which is

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds - \int_t^T Z_s^* dW_s,$$

by using triangle inequality and taking the supremum we get,

$$\begin{aligned} |Y_t| &\leq |\xi| + \left| \int_t^T f(s, Y_s, Z_s) ds \right| + \left| \int_t^T Z_s^* dW_s \right|, \\ \sup_{t\in[0,T]} |Y_t| &\leq |\xi| + \int_0^T |f(s, Y_s, Z_s)| ds + \sup_{t\in[0,T]} \left| \int_t^T Z_s^* dW_s \right|. \end{aligned}$$

We need all of the components to belong $\mathbb{L}_T^{2,1}$ in order to have bounded $|Y_t|$.

$$\begin{aligned} \mathbb{E}\left[\sup_{t\in[0,T]} \left| \int_t^T Z_s^* dW_s \right|^2 \right] &= \mathbb{E}\left[\sup_{t\in[0,T]} \left| \int_0^T Z_s^* dW_s - \int_0^t Z_s^* dW_s \right|^2 \right], \\ &\leq \mathbb{E}\left[\sup_{t\in[0,T]} 2\left(\left| \int_0^T Z_s^* dW_s \right|^2 + \left| \int_0^t Z_s^* dW_s \right|^2 \right) \right], \\ &= 2\mathbb{E}\left[\left| \int_0^T Z_s^* dW_s \right|^2 \right] + 2\mathbb{E}\left[\sup_{t\in[0,T]} \left| \int_0^t Z_s^* dW_s \right|^2 \right], \\ &= \mathbb{E}\left[\int_0^T |Z_s|^2 ds \right] + 2\mathbb{E}\left[\sup_{t\in[0,T]} \left| \int_0^t Z_s^* dW_s \right|^2 \right]. \end{aligned} \tag{2.5}$$

by Itô isometry. It follows by Burkholder-Davis-Gundy inequalities (see Appendices, Lemma A.1) and quadratic variation of the Brownian motion,

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left|\int_t^T Z_s^* dW_s\right|^2\right] \leq 2\mathbb{E}\left[\int_0^T |Z_s|^2 ds\right] + 2d_1\mathbb{E}\left[\int_0^T |Z_s|^2 ds\right] \leq \infty.$$

Hence, $\sup_{t\in[0,T]} |\int_t^T Z_s^* dW_s|^2 \in \mathbb{L}_T^{2,1}$. Moreover, (f, ξ) are standard parameters given, $|\xi| + \int_0^T |f(s, Y_s, Z_s)|ds$ belongs to $\mathbb{L}_T^{2,1}$. Then $\sup_{s\leq T} |Y_s| \in \mathbb{L}_T^{2,1}$ and we have shown the boundedness.

Remember that the Itô formula applied to $f(t, x_t)$ is as follows:

$$\begin{aligned} f(T, x_T) = f(t, x_t) + \int_t^T \frac{\partial f}{\partial s}(s, x_s)ds + \int_t^T \frac{\partial f}{\partial x_s}(s, x_s)dx_s \\ + \frac{1}{2}\int_t^T \frac{\partial^2 f}{\partial x_s^2}(s, x_s)d\langle x, x\rangle(s). \end{aligned} \tag{2.6}$$

We will apply Itô formula (2.6) from t to T to the semimartingale $e^{\beta t}|\delta Y_t|^2$, by taking

$$f(t, x) = e^{\beta t}|x|^2,$$

then

$$\frac{\partial f}{\partial t}(t, x) = \beta e^{\beta t}x^2, \quad \frac{\partial f}{\partial x}(t, x) = 2e^{\beta t}x, \quad \frac{\partial^2 f}{\partial x^2}(t, x) = 2e^{\beta t}.$$

Let us rewrite the given conditions which will be used in the following calculations,

$$\begin{aligned} \delta Y_t &= Y_t^1 - Y_t^2, \\ \delta Z_t &= Z_t^1 - Z_t^2, \\ -dY_t^1 &= f^1(t, Y_t^1, Z_t^1)dt - (Z_t^1)^* dW_t, \\ -dY_t^2 &= f^2(t, Y_t^2, Z_t^2)dt - (Z_t^2)^* dW_t. \end{aligned}$$

After replacing x with $|\delta Y_t|$ in the Itô formula (2.6), we get

$$\begin{aligned} e^{\beta T}|\delta Y_T|^2 = e^{\beta t}|\delta Y_t|^2 + \int_t^T \beta e^{\beta s}|\delta Y_s|^2 ds + \int_t^T 2e^{\beta s}|\delta Y_s| d|\delta Y_s| \\ + \frac{1}{2}\int_t^T 2e^{\beta s} d\langle|\delta Y|, |\delta Y|\rangle(s), \end{aligned}$$

with the following quadratic variation;

$$d\langle|\delta Y|, |\delta Y|\rangle(t) = |\delta Z_t|^2 dt.$$

which implies to

$$e^{\beta t}|\delta Y_t|^2 + \beta\int_t^T e^{\beta s}|\delta Y_s|^2 ds + \int_t^T e^{\beta s}|\delta Z_s|^2 ds$$

$$
\begin{aligned}
&= e^{\beta T}|\delta Y_T|^2 - 2\int_t^T e^{\beta s}|\delta Y_s| d|Y_s^1 - Y_s^2|, \\
&= e^{\beta T}|\delta Y_T|^2 - 2\int_t^T e^{\beta s}|\delta Y_s||dY_s^1 - dY_s^2|, \\
&= e^{\beta T}|\delta Y_T|^2 - 2\int_t^T e^{\beta s}|\delta Y_s|\Big|\Big(f^2(s,Y_s^1,Z_s^1) - f^1(s,Y_s^2,Z_s^2)\Big)ds \\
&\quad + \Big((Z_s^1)^* - (Z_s^2)^*\Big)dW_s\ \Big|, \\
&= e^{\beta T}|\delta Y_T|^2 + 2\int_t^T e^{\beta s}|\delta Y_s|\Big|\Big(f^1(s,Y_s^1,Z_s^1) - f^2(s,Y_s^2,Z_s^2)\Big)ds \\
&\quad - \Big((Z_s^1)^* - (Z_s^2)^*\Big)dW_s\ \Big|, \\
&= e^{\beta T}|\delta Y_T|^2 + 2\int_t^T e^{\beta s}\langle \delta Y_s, f^1(s,Y_s^1,Z_s^1) - f^2(s,Y_s^2,Z_s^2)\rangle ds \\
&\quad - 2\int_t^T e^{\beta s}\langle \delta Y_s, \Big((Z_s^1)^* - (Z_s^2)^*\Big)dW_s\rangle, \\
&= e^{\beta T}|\delta Y_T|^2 + 2\int_t^T e^{\beta s}\langle \delta Y_s, f^1(s,Y_s^1,Z_s^1) - f^2(s,Y_s^2,Z_s^2)\rangle ds \\
&\quad - 2\int_t^T e^{\beta s}\langle \delta Y_s, \delta Z_s^* dW_s\rangle. \qquad (2.7)
\end{aligned}
$$

We will use the Lipschitz condition below and the difference of the generator functions for the second solution

$$
\big|f^1(\omega,t,y_1,z_1) - f^1(\omega,t,y_2,z_2)\big| \le C\big(|y_1-y_2| + |z_1-z_2|\big) \quad \forall (y_1,z_1),\ \forall (y_2,z_2),
$$

$$
\delta_2 f_t = f^1(t,Y_t^2,Z_t^2) - f^2(t,Y_t^2,Z_t^2),
$$

for the calculations of $\big|f^1(s,Y_s^1,Z_s^1) - f^2(s,Y_s^2,Z_s^2)\big|$ to get two necessary inequalities; primarily,

$$
\begin{aligned}
\big|f^1(s,Y_s^1,Z_s^1) - f^2(s,Y_s^2,Z_s^2)\big| &= \big|f^1(s,Y_s^1,Z_s^1) - f^2(s,Y_s^2,Z_s^2) \pm f^1(s,Y_s^2,Z_s^2)\big|, \\
&= \big|f^1(s,Y_s^1,Z_s^1) + \delta_2 f_s - f^1(s,Y_s^2,Z_s^2)\big|, \\
&\le \big|f^1(s,Y_s^1,Z_s^1) - f^1(s,Y_s^2,Z_s^2)| + \big|\delta_2 f_s|, \\
&\le C\big(|\delta Y_s| + |\delta Z_s|\big) + |\delta_2 f_s|.
\end{aligned}
$$

Besides let us show now, $2y(Cz+t) \le Cz^2/\lambda^2 + t^2/\mu^2 + y^2(\mu^2 + C\lambda^2)$ where $\mu > 0$, $\lambda^2 > C$.

$$
0 \le \left(\frac{z}{\lambda} - y\lambda\right)^2 = \frac{z^2}{\lambda^2} + y^2\lambda^2 - 2zy,
$$

$$
C2zy \le \left(\frac{z^2}{\lambda^2} + y^2\lambda^2\right)C = C\frac{z^2}{\lambda^2} + Cy^2\lambda^2, \qquad (2.8)
$$

additionally,

$$0 \leq \left(\frac{t}{\mu} - y\mu\right)^2 = \frac{t^2}{\mu^2} + y^2\mu^2 - 2ty,$$

$$2ty \leq \frac{t^2}{\mu^2} + y^2\mu^2. \tag{2.9}$$

By adding side by side the equations (2.8) and (2.9), we get

$$2y(Cz + t) \leq C\frac{z^2}{\lambda^2} + \frac{t^2}{\mu^2} + y^2(\mu^2 + C\lambda^2). \tag{2.10}$$

This inequality will imply the following, by taking expectation in equation (2.7); also $\sup_{s\leq T}|Y_s| \in \mathbb{L}_T^{2,1}$ leads to $e^{\beta s}\delta Z_s \delta Y_s \in \mathbb{H}_T^{1,n}$ and $\int_t^T e^{\beta s}\langle \delta Y_s, \delta Z_s^* dW_s\rangle$ is $\mathbb{P}$-integrable with zero expectation, then

$$\begin{aligned}
&\mathbb{E}[e^{\beta t}|\delta Y_t|^2] + \beta\mathbb{E}\left[\int_t^T e^{\beta s}|\delta Y_s|^2 ds\right] + \mathbb{E}\left[\int_t^T e^{\beta s}|\delta Z_s|^2 ds\right],\\
&\leq \mathbb{E}[e^{\beta T}|\delta Y_t|^2] + 2\mathbb{E}\left[\int_t^T e^{\beta s}\langle \delta Y_s, C\big(|\delta Y_s| + |\delta Z_s|\big) + |\delta_2 f_s|\rangle ds\right],\\
&= \mathbb{E}[e^{\beta T}|\delta Y_t|^2] + \mathbb{E}\left[\int_t^T e^{\beta s}\langle 2\delta Y_s, C\big(|\delta Y_s| + |\delta Z_s|\big) + |\delta_2 f_s|\rangle ds\right].
\end{aligned}$$

Here by using $\langle 2\delta Y_s, C(|\delta Y_s| + |\delta Z_s|) + |\delta_2 f_s|\rangle$ part, we get

$$\begin{aligned}
\langle 2\delta Y_s, C\big(|\delta Y_s| + |\delta Z_s|\big) + |\delta_2 f_s|\rangle &\leq 2C\langle \delta Y_s, |\delta Y_s|\rangle + 2C\langle \delta Y_s, |\delta Z_s|\rangle + 2\langle \delta Y_s, |\delta_2 f_s|\rangle\\
&= 2C|\delta Y_s|^2 + 2|\delta Y_s|\big(C|\delta Z_s| + |\delta_2 f_s|\big).
\end{aligned}$$

After defining $y := |\delta Y_s|$, $C := C$, $z := |\delta Z_s|$ and $t := |\delta_2 f_s|$ in equation (2.10), we turn back expectations

$$\begin{aligned}
&\mathbb{E}[e^{\beta t}|\delta Y_t|^2] + \beta\mathbb{E}\left[\int_t^T e^{\beta s}|\delta Y_s|^2 ds\right] + \mathbb{E}\left[\int_t^T e^{\beta s}|\delta Z_s|^2 ds\right],\\
\leq\ & \mathbb{E}[e^{\beta T}|\delta Y_T|^2] + \mathbb{E}\Bigg[\int_t^T e^{\beta s}\Bigg(2C|\delta Y_s|^2 + C\frac{|\delta Z_s|^2}{\lambda^2} + |\delta Y_s|^2(\mu^2 + \lambda^2 C)\\
&+ \frac{|\delta_2 f_s|^2}{\mu^2}\Bigg) ds\Bigg],\\
=\ & \mathbb{E}[e^{\beta T}|\delta Y_T|^2] + \mathbb{E}\Bigg[\int_t^T e^{\beta s}\Bigg(|\delta Y_s|^2\big(C(2 + \lambda^2) + \mu^2\big) + C\frac{|\delta Z_s|^2}{\lambda^2}\\
&+ \frac{|\delta_2 f_s|^2}{\mu^2}\Bigg) ds\Bigg].
\end{aligned} \tag{2.11}$$

By using $C(2+\lambda^2)+\mu^2 \leq \beta$ and $\frac{C}{\lambda^2} \leq 1$, we continue with

$$
\begin{aligned}
&\leq \quad \mathbb{E}[e^{\beta T}|\delta Y_T|^2] + \mathbb{E}\bigg[\int_t^T e^{\beta s}\bigg(|\delta Y_s|^2\beta + |\delta Z_s|^2 + \frac{|\delta_2 f_s|^2}{\mu^2}\bigg)ds\bigg], \\
&= \quad \mathbb{E}[e^{\beta T}|\delta Y_T|^2] + \beta\ \mathbb{E}\bigg[\int_t^T e^{\beta s}|\delta Y_s|^2 ds\bigg] + \mathbb{E}\bigg[\int_t^T e^{\beta s}|\delta Z_s|^2 ds\ \bigg] \\
&\qquad + \frac{1}{\mu^2}\ \mathbb{E}\bigg[\int_t^T e^{\beta s}|\delta_2 f_s|^2 ds\bigg].
\end{aligned} \tag{2.12}
$$

After doing cancellations, we have

$$
\mathbb{E}[e^{\beta t}|\delta Y_t|^2] = \mathbb{E}[e^{\beta T}|\delta Y_T|^2] + \frac{1}{\mu^2}\mathbb{E}\bigg[\int_t^T e^{\beta s}|\delta_2 f_s|^2 ds\bigg].
$$

Finally, we obtain the upper-bound of the β-norm of the process δY by integration,

$$
\|\delta Y\|_\beta^2 \leq T\ \bigg[\ e^{\beta T}\mathbb{E}[|\delta Y_T|^2] + \frac{1}{\mu^2}\|\delta_2 f\|_\beta^2\ \bigg].
$$

Now, by using inequality (2.11) we will find the bound for the norm of the process δZ,

$$
\begin{aligned}
&\mathbb{E}[e^{\beta t}|\delta Y_t|^2] + \beta\ \mathbb{E}\bigg[\int_t^T e^{\beta s}|\delta Y_s|^2 ds\bigg] + \mathbb{E}\bigg[\int_t^T e^{\beta s}|\delta Z_s|^2 ds\bigg], \\
&\leq \mathbb{E}[e^{\beta T}|\delta Y_T|^2] + \beta\ \mathbb{E}\bigg[\int_t^T e^{\beta s}|\delta Y_s|^2 ds\bigg] + \frac{C}{\lambda^2}\mathbb{E}\bigg[\int_t^T e^{\beta s}|\delta Z_s|^2 ds\bigg] \\
&+ \frac{1}{\mu^2}\mathbb{E}\bigg[\int_t^T e^{\beta s}|\delta_2 f_s|^2 ds\bigg].
\end{aligned}
$$

The second terms cancel each other and it implies to

$$
\begin{aligned}
\bigg(1-\frac{C}{\lambda^2}\bigg)\mathbb{E}\bigg[\int_t^T e^{\beta s}|\delta Z_s|^2 ds\bigg] \quad &\leq \quad -\mathbb{E}[e^{\beta t}|\delta Y_t|^2] + \mathbb{E}[e^{\beta T}|\delta Y_T|^2] \\
&\qquad + \frac{1}{\mu^2}\mathbb{E}\bigg[\int_t^T e^{\beta s}|\delta_2 f_s|^2 ds\bigg], \\
&\leq \quad +\mathbb{E}[e^{\beta T}|\delta Y_T|^2] + \frac{1}{\mu^2}\mathbb{E}\bigg[\int_t^T e^{\beta s}|\delta_2 f_s|^2 ds\bigg].
\end{aligned} \tag{2.13}
$$

in other way,

$$
\mathbb{E}\bigg[\int_t^T e^{\beta s}|\delta Z_s|^2 ds\bigg] \leq \bigg(\frac{\lambda^2}{\lambda^2 - C}\bigg)\ \bigg\{\mathbb{E}[e^{\beta T}|\delta Y_T|^2] + \frac{1}{\mu^2}\ \mathbb{E}\bigg[\int_t^T e^{\beta s}|\delta_2 f_s|^2 ds\bigg]\bigg\}.
$$

Similarly, by integration, β-norm of the process δZ is bounded:

$$\|\delta Z\|_\beta^2 \leq \left(\frac{\lambda^2}{\lambda^2 - C}\right) \left[e^{\beta T}\mathbb{E}[|\delta Y_T|^2] + \frac{1}{\mu^2}\|\delta_2 f\|_\beta^2 \right].$$

□

Remark 2.1. • An upper-bound for the β-norm of δY can be found after replace T by $\inf(T, [\beta - C(2+\lambda^2) - \mu^2]^{-1})$.

• By classical results it can be proven similarly that

$$\mathbb{E}[\sup_{t\leq T} |\delta Y_t|^2] \leq K\mathbb{E}\left[|\delta Y_T|^2 + \int_0^T |\delta_2 f_t|^2 dt\right],$$

for a positive constant K only depending on T. [9]

Now all the tools are ready to prove the existence and uniqueness theorem. The following theorem is stated by Pardoux-Peng [23] but here the paper of El Karoui, Peng, Quenez [9] is followed and the theorem is explicitly proven by using a contraction map.

Theorem 2.2. *Let (f, ξ) be standard parameters, then there exists a unique pair $(Y, Z) \in \mathbb{H}_T^{2,d} \times \mathbb{H}_T^{2,n\times d}$ solving (2.1).*

Proof. Let (f, ξ) be the standard parameters. Hence, $f(\cdot, 0, 0) \in \mathbb{H}_T^{2,d}$, $\xi \in \mathbb{L}_T^{2,d}$ and f is uniformly Lipschitz. Also let M be

$$M_t := \xi + \int_0^T f(s, y_s, z_s)ds - \int_t^T Z_s^* dW_s.$$

The solution (Y, Z) is defined by considering the continuous square-integrable martingale $\mathbb{E}\big[\xi + \int_0^T f(s, y_s, z_s)ds|\mathcal{F}_t\big]$.

$$\begin{aligned} M_t = \mathbb{E}[M_T|\mathcal{F}_t] &= \mathbb{E}\left[\xi + \int_0^T f(s, y_s, z_s)ds - \int_T^T Z_s^* dW_s|\mathcal{F}_t\right] \\ &= \mathbb{E}\left[\xi + \int_0^T f(s, y_s, z_s)ds|\mathcal{F}_t\right]. \end{aligned}$$

By the martingale representation theorem (MRT, see Appendices, Theorem A.4) of the Brownian motion [17] there exists a unique integrable process $Z \in \mathbb{H}_T^{2,n\times d}$ such that

$$M_t = M_0 + \int_0^t Z_s^* dW_s.$$

We introduce the adapted and continuous process Y by

$$Y_t := M_t - \int_0^t f(s, y_s, z_s)ds.$$

When we put M_t in the above equation, Y is also given as

$$\begin{aligned}
Y_t &= M_0 + \int_0^t Z_s^* dW_s - \int_0^t f(s, y_s, z_s)ds, \\
&= \xi + \int_0^T f(s, y_s, z_s)ds - \int_0^T Z_s^* dW_s + \int_0^t Z_s^* dW_s - \int_0^t f(s, y_s, z_s)ds, \\
&= \xi + \int_t^T f(s, y_s, z_s)ds - \int_t^T Z_s^* dW_s. \\
Y_t &= \mathbb{E}[Y_t|\mathcal{F}_t], \quad \text{since } Y_t \in \mathcal{F}_t, \\
&= \mathbb{E}\bigg[\xi + \int_t^T f(s, y_s, z_s)ds|\mathcal{F}_t\bigg] - \mathbb{E}\bigg[\int_t^T Z_s^* dW_s|\mathcal{F}_t\bigg], \\
&= \mathbb{E}\bigg[\xi + \int_t^T f(s, y_s, z_s)ds|\mathcal{F}_t\bigg] - \int_t^t Z_s^* dW_s, \\
&= \mathbb{E}\bigg[\xi + \int_t^T f(s, y_s, z_s)ds|\mathcal{F}_t\bigg].
\end{aligned}$$

Thus, the square-integrability of Y follows from the above assumptions.

Assume that (y^1, z^1), (y^2, z^2) are two elements of $\mathbb{H}^{2,d}_{T,\beta} \times \mathbb{H}^{2,n\times d}_{t,\beta}$ and let (Y^1, Z^1) and (Y^2, Z^2) be the associated solutions;

$$Y_t^1 = \xi + \int_t^T f(s, Y_s^1, Z_s^1)ds - \int_t^T Z_s^{1*} dW_s,$$

$$Y_t^2 = \xi + \int_t^T f(s, Y_s^2, Z_s^2)ds - \int_t^T Z_s^{2*} dW_s,$$

$$\delta Y_t = Y_t^1 - Y_t^2 = \int_t^T \big(f(s, Y_s^1, Z_s^1) - f(s, Y_s^2, Z_s^2)\big)ds - \int_t^T (Z_s^{1*} - Z_s^{2*})dW_s.$$

It is obvious that $\delta Y_T = 0$ which implies $\mathbb{E}[|\delta Y_T|] = 0$. We will use $\mathbb{E}[|\delta Y_T|] = 0$ and apply the equation (2.3) of Proposition 2.1 with $C = 0$ and $\mu^2 = \beta$, then

$$\begin{aligned}
\|\delta Y\|_\beta^2 &\leq T\left[e^{\beta T}\mathbb{E}[|\delta Y_T|^2] + \frac{1}{\mu^2}\|\delta_2 f\|_\beta^2\right], \\
&= T\left[0 + \frac{1}{\beta}\|\delta_2 f\|_\beta^2\right], \\
&= \frac{T}{\beta}\|f(t, y_t^1, z_t^1) - f(t, y_t^2, z_t^2)\|_\beta^2, \\
&= \frac{T}{\beta}\,\mathbb{E}\bigg[\int_0^T e^{\beta s}|f(s, y_s^1, z_s^1) - f(s, y_s^2, z_s^2)|^2 ds\bigg].
\end{aligned}$$

By Proposition 2.1 equation (2.4) its obtained that

$$\begin{aligned}
\|\delta Z\|_\beta^2 &\le \frac{\lambda^2}{\lambda^2 - C}\left[e^{\beta T}\mathbb{E}[|\delta Y_T|^2] + \frac{1}{\mu^2}\|\delta_2 f\|_\beta^2 \right], \\
&= \frac{\lambda^2}{\lambda^2 - 0}\left[0 + \frac{1}{\beta}\|\delta_2 f\|_\beta^2 \right], \\
&= \frac{1}{\beta}\|f(t, y_t^1, z_t^1) - f(t, y_t^2, z_t^2)\|_\beta^2, \\
&= \frac{1}{\beta}\,\mathbb{E}\left[\int_0^T e^{\beta s}|f(s, y_s^1, z_s^1) - f(s, y_s^2, z_s^2)|^2 ds\right].
\end{aligned}$$

Before adding $\|\delta Y\|_\beta^2$ and $\|\delta Z\|_\beta^2$, let us remind that f is Lipschitz with constant c

$$\big|f(t, y_t^1, z_t^1) - f(t, y_t^2, z_t^2)\big| \le c\big(|\delta y| + |\delta z|\big).$$

By taking squares and applying $(a+b)^2 \le 2(a^2+b^2)$ here and we obtain

$$\big|f(t, y_t^1, z_t^1) - f(t, y_t^2, z_t^2)\big|^2 \le c^2\big(|\delta y| + |\delta z|\big)^2 \le 2c^2\big(|\delta y|^2 + |\delta z|^2\big).$$

Now we add $\|\delta Y\|_\beta^2$, $\|\delta Z\|_\beta^2$ and take $C = c^2$, then

$$\begin{aligned}
\|\delta Y\|_\beta^2 + \|\delta Z\|_\beta^2 &\le \frac{T+1}{\beta}\,\mathbb{E}\left[\int_0^T e^{\beta s}\big|f(s, y_s^1, z_s^1) - f(s, y_s^2, z_s^2)\big|^2 ds\right], \\
&\le \frac{T+1}{\beta}\,\mathbb{E}\left[\int_0^T e^{\beta s} 2C\big(|\delta y|^2 + |\delta z|^2\big) ds\right], \\
&\le \frac{2(T+1)C}{\beta}\,\mathbb{E}\left[\int_0^T e^{\beta s}|\delta y|^2 ds + \int_0^T e^{\beta s}|\delta z|^2 ds\right]. \\
\|\delta Y\|_\beta^2 + \|\delta Z\|_\beta^2 &\le \frac{2(T+1)C}{\beta}\,\big[\|\delta y\|_\beta^2 + \|\delta z\|_\beta^2\big]. \qquad (2.14)
\end{aligned}$$

The Banach fixed-point theorem (see Appendices, Theorem A.5) is used for the mapping Φ. Note that, this mapping Φ is from $\mathbb{H}_{T,\beta}^{2,d} \times \mathbb{H}_{T,\beta}^{2,n\times d}$ onto $\mathbb{H}_{T,\beta}^{2,d} \times \mathbb{H}_{T,\beta}^{2,n\times d}$, which maps the processes (y, z) onto the solution (Y, Z) of the BSDE, $\Phi : (y, z) \mapsto (Y, Z)$ with generator $f(t, y_t, z_t)$; i.e.,

$$Y_t = \xi + \int_t^T f(s, y_s, z_s) ds - \int_t^T Z_s^* dW_s.$$

The mapping Φ is a contraction from $\mathbb{H}_{T,\beta}^{2,d} \times \mathbb{H}_{T,\beta}^{2,n\times d}$ onto itself, if we choose $\beta > 2(1+T)C$ and there exists a fixed point, which is the unique continuous solution of the BSDE. $\square$

When the contraction condition is satisfied, the related iterative sequence of Y converges almost surely to the solution of the BSDE with the following corollary.

Corollary 2.3. *Let β be constant such that $2(T+1)C < \beta$. Let (Y^k, Z^k) be the sequence defined recursively by $(Y_0 = 0, Z_0 = 0)$ and*

$$-dY_t^{k+1} = f(t, Y_t^k, Z_t^k)dt - (Z_t^{k+1})^* dW_t, \quad Y_T^{k+1} = \xi. \tag{2.15}$$

Then the sequence (Y^k, Z^k) converges to (Y, Z), $d\mathbb{P} \otimes dt$ a.s. when $k \to +\infty$ in $\mathbb{H}_{t,\beta}^{2,d} \times \mathbb{H}_{T,\beta}^{2,n\times d}$.

Proof. Assume (Y^k, Z^k) be the sequence defined recursively as in (2.15). Then by (2.14),

$$\begin{aligned}
\|Y^{k+1} - Y^k\|_\beta^2 + \|Z^{k+1} - Z^k\|_\beta^2 &= \|\delta Y^k\|_\beta^2 + \|\delta Z^k\|_\beta^2, \\
&\leq \left(\frac{2(T+1)C}{\beta}\right) \big[\|\delta Y^{k-1}\|_\beta^2 + \|\delta Z^{k-1}\|_\beta^2\big], \\
&\leq \left(\frac{2(T+1)C}{\beta}\right)^2 \big[\|\delta Y^{k-2}\|_\beta^2 + \|\delta Z^{k-2}\|_\beta^2\big], \\
&\leq \dots, \\
&\leq \left(\frac{2(T+1)C}{\beta}\right)^k \big[\|\delta Y^{0}\|_\beta^2 + \|\delta Z^{0}\|_\beta^2\big], \\
&\leq \left(\frac{2(T+1)C}{\beta}\right)^k \big[\|Y^1 - Y^0\|_\beta^2 + \|Z^1 - Z^0\|_\beta^2\big].
\end{aligned}$$

$$\|Y^{k+1} - Y^k\|_\beta^2 + \|Z^{k+1} - Z^k\|_\beta^2 \leq \epsilon^k \; K.$$

where $\epsilon := (\frac{2(T+1)C}{\beta}) < 1$ and $K := \|Y^1 - Y^0\|_\beta^2 + \|Z^1 - Z^0\|_\beta^2$. Then while $k \to \infty$ the geometric series converges

$$\sum_k \|Y^{k+1} - Y^k\|_\beta^2 + \sum_k \|Z^{k+1} - Z^k\|_\beta^2 < +\infty.$$

□

Remark 2.2. It is possible to consider the norm $\|\sup_{s\in[0,T]} |Y_s^k - Y_s| \;\|_2$ instead of $\|Y\|_\beta$ as it is remarked in [9]; consequently the $\sup_{s\in[0,T]} |Y_s^k - Y_s|$ converges $\mathbb{P}$ a.s. to 0 which provides to work with different norms.

2.1.1 Linear Backward Stochastic Differential Equation

A BSDE is called *linear backward stochastic differential equation* (LBSDE), if it is generated by linear functions and defined as follows;

$$-dY_t = [\varphi_t + Y_t\beta_t + Z_t^*\gamma_t]dt - Z_t^* dW_t, \quad Y_T = \xi.$$

Later we will use a LBSDE to specify the integrability properties of the solution of the standard pricing problem in Theorem 2.8.

Proposition 2.4. *Assume that (β, γ) is a bounded $(\mathbb{R}, \mathbb{R}^n)$-valued predictable process, φ is an element of $\mathbb{H}_T^{2,1}$ and ξ is an element of $\mathbb{L}_T^{2,1}$. Also assume that, Γ_s^t is the adjoint process defined for $s \geq t$ by the forward linear stochastic differential equation,*

$$d\Gamma_s^t = \Gamma_s^t\big[\beta_s ds + \gamma_s^* dW_s\big], \quad \Gamma_t^t = 1. \tag{2.16}$$

Then the LBSDE

$$-dY_t = [\varphi_t + Y_t\beta_t + Z_t^*\gamma_t]dt - Z_t^* dW_t, \quad Y_T = \xi, \tag{2.17}$$

has unique solution (Y, Z) in $\mathbb{H}_{T,\beta}^{2,1} \times \mathbb{H}_{T,\beta}^{2,n}$ and Y_t is given by the closed formula

$$Y_t = \mathbb{E}\Big[\xi\Gamma_T^t + \int_t^T \Gamma_s^t\varphi_s ds|\mathcal{F}_t\Big] \ \mathbb{P} \ a.s., \tag{2.18}$$

in particular, if ξ and φ are nonnegative, the process Y is nonnegative. If, in addition, $Y_0 = 0$, then $Y_t = 0$ a.s. for any t, $\xi = 0$ a.s. and $\varphi_t = 0$ $d\mathbb{P} \otimes dt$ a.s.

Proof. Our first aim is to show (f, ξ) are the standard parameters. $\xi \in \mathbb{L}_T^{2,n}$ is given and f is uniformly Lipschitz by the below inequality,

$$\begin{aligned}
\big|f(w,t,y_1,z_1) - f(w,t,y_2,z_2)\big| &= |\varphi_t + \beta_t y_1 + z_1^*\gamma_t - \varphi_t - \beta_t y_2 - z_2^*\gamma_t|, \\
&= \big|\beta_t(y_1 - y_2) + (z_1 - z_2)^*\gamma_t\big|, \\
&\leq C\Big(|y_1 - y_2| + \big|(z_1 - z_2)^*\big|\Big),
\end{aligned}$$

where $|\beta_t| < K_1$, $|\gamma_t| < K_2$ and $C = \max\{K_1, K_2\}$.

The (f, ξ) pair are standard parameters as β and γ are given as bounded processes and the linear generator $f(t, y, z) = \varphi_t + \beta_t y + \gamma_t^* z$ is uniformly Lipschitz. By Theorem 2.2 there exists a unique solution (Y, Z) of the linear BSDE associated with (f, ξ) and the solution is square-integrable. By standard calculations, the solution of SDE (2.16) is found. We apply Itô formula (2.6) with $f(x) = \log x$,

$$\begin{aligned}
\log\Gamma_s^t &= \log\Gamma_t^t + \int_t^s \frac{1}{\Gamma_s^t} d\Gamma_s^t - \frac{1}{2}\int_t^s \Big(\frac{1}{\Gamma_s^t}\Big)^2 d\langle\Gamma_s^t, \Gamma_s^t\rangle, \\
&= \log 1 + \int_t^s \frac{1}{\Gamma_s^t}\Gamma_s^t[\beta_s ds + \gamma_s^* dW_s] - \frac{1}{2}\int_t^s |\gamma_s^*|^2 ds, \\
&= \int_t^s \beta_s ds + \int_t^s \gamma_s^* dW_s - \frac{1}{2}\int_t^s |\gamma_s^*|^2 ds,
\end{aligned}$$

then the solution will be

$$\Gamma_s^t = \exp\Big\{\int_t^s \beta_s ds + \int_t^s \gamma_s^* dW_s - \frac{1}{2}\int_t^s |\theta_s^*|^2 ds\Big\}.$$

When we consider the starting time from 0, we get

$$\Gamma_t := \Gamma_t^0 = \exp\Big\{ \int_0^t \beta_s ds + \int_0^t \gamma_s^* dW_s - \frac{1}{2}\int_0^t |\gamma_s^*|^2 ds \Big\}.$$

It follows from Novikov's condition (see Appendices, Theorem A.7) that for any square-integrable contingent claim $\mathbb{E}(\Gamma_T^2) < +\infty$, $\mathbb{E}(\Gamma_T \xi) < +\infty$.

$$\begin{aligned}
d(\Gamma_t Y_t) &= Y_t d\Gamma_t + \Gamma_t dY_t + d[\Gamma, Y](t), \\
&= Y_t \Gamma_t \big[\beta_t dt + \gamma_t^* dW_t\big] + \Gamma_t\big[- \varphi_t dt - \beta_t Y_t dt - Z_t^* \gamma_t dt + Z_t^* dW_t\big], \\
&\quad + \Gamma_t Z_t^* \gamma_t dt, \\
&= \Gamma_t Y_t \beta_t dt + \Gamma_t Y_t \gamma_t^* dW_t - \Gamma_t \varphi_t dt - \Gamma_t Y_t \beta_t dt - \Gamma_t Z_t^* \gamma_t dt + \Gamma_t Z_t^* dW_t, \\
&\quad + \Gamma_t Z_t^* \gamma_t dt, \\
&= \Gamma_t Y_t \gamma_t^* dW_t - \Gamma_t \varphi_t dt + \Gamma_t Z_t^* dW_t, \\
&= \Gamma_t (Z_t^* - Y_t \gamma_t^*) dW_t - \Gamma_t \varphi_t dt.
\end{aligned}$$

Then is is seen that $\sup_{s\leq T} |Y_s|$ and $\sup_{s\leq T} |\Gamma_s|$ belong to $\mathbb{L}_T^{2,1}$ and $\sup_{s\leq T} |Y_s| \times \sup_{s\leq T} |\Gamma_s|$ belongs to $\mathbb{L}_T^{1,1}$. Hence, $\Gamma_t Y_t + \int_0^t \Gamma_s \varphi_s ds$ is uniformly integrable martingale and equal to the conditional expectation of its terminal value.

$$\Gamma_t Y_t + \int_0^t \Gamma_s \varphi_s ds = \mathbb{E}\Big[\Gamma_T \xi + \int_0^T \Gamma_s \varphi_s ds \mid \mathcal{F}_t\Big].$$

If we consider $Y_0 = 0$,

$$\begin{aligned}
\Gamma_0 Y_0 + \int_0^0 \Gamma_s \varphi_s ds &= \mathbb{E}\Big[\Gamma_T \xi + \int_0^T \Gamma_s \varphi_s ds \mid \mathcal{F}_0\Big] \\
0 &= \mathbb{E}\Big[\Gamma_T \xi + \int_0^T \Gamma_s \varphi_s ds\Big],
\end{aligned} \tag{2.19}$$

then the nonnegative variable $\Gamma_T \xi + \int_0^T \Gamma_s \varphi_s ds$ has 0 expectation. Therefore $\xi = 0$, $\mathbb{P}$ a.s., $\varphi_t = 0$, $d\mathbb{P} \otimes dt$ a.s. and $Y = 0$ a.s. Particularly, if ξ and φ are nonnegative, Y_t is nonnegative. □

2.2 Comparison Theorem

The interplay between the solutions could affect the decisions and choices. Also, this differences should reflect the real world as expected by which the comparison theorem becomes important. For the forward SDEs the same diffusion functions is needed so as to confront the solutions [13]. However, the conditions are more relaxed for the BSDEs. For one-dimensional BSDEs, comparison of solutions are first acquired by [25]. In this section, we will state a corollary from our main article [9] which provides us a sufficient condition for the nonnegative solution of BSDE. Later, we will state the comparison theorem again from our main article and give a precise, detailed proof.

Corollary 2.5. *If $f(t,0,0) \geq 0$ $d\mathbb{P} \otimes dt$ a.s. and $\xi \geq 0$ a.s., then $Y \geq 0$ $\mathbb{P}$ a.s. Moreover, if $Y_t = 0$ on a set $A \in \mathcal{F}_t$, then $Y_s = 0$, $f(s,0,0) = 0$ on $[t,T] \times A$, $\xi = 0$ a.s. on A and $d\mathbb{P} \otimes ds$ a.s.*

Theorem 2.6. *(Comparison Theorem). If (f^1, ξ^1) and (f^2, ξ^2) are two standard parameters of BSDEs with the square-integrable solutions (Y^1, Z^1) and (Y^2, Z^2) respectively, in addition*

- $\xi^1 \geq \xi^2$ $\mathbb{P}$ *a.s.*
- $\delta_2 f_t = f^1(t, Y_t^2, Z_t^2) - f^2(t, Y_t^2, Z_t^2) \geq 0$, $d\mathbb{P} \otimes dt$ *a.s.*

then we have $Y_t^1 \geq Y_t^2$ almost surely for any time t.

Besides if it is given that, $Y_0^1 = Y_0^2$, then $\xi^1 = \xi^2$, $f^1(t, Y_t^2, Z_t^2) = f^2(t, Y_t^2, Z_t^2)$, $d\mathbb{P} \otimes dt$ a.s. and $Y^1 = Y^2$ a.s. More generally if $Y_t^1 = Y_t^2$ on a set $A \in \mathcal{F}_t$, then $Y_s^1 = Y_s^2$ a.s. on $[t,T] \times A$, $\xi^1 = \xi^2$ a.s. on A and $f^1(s, Y_s^2, Z_s^2) = f^2(s, Y_s^2, Z_s^2)$ on $A \times [t,T]$ $d\mathbb{P} \otimes ds$ a.s.

Proof. Here we have two BSDEs,

$$-dY_t = f^1(t, Y_t, Z_t)dt - Z_t^* dW_t; \quad Y_T = \xi^1,$$

$$-dY_t = f^2(t, Y_t, Z_t)dt - Z_t^* dW_t; \quad Y_T = \xi^2,$$

so that the solutions (Y^1, Z^1), (Y^2, Z^2) will satisfy these equations;

$$-dY_t^1 = f^1(t, Y_t^1, Z_t^1)dt - (Z_t^1)^* dW_t; \quad Y_T^1 = \xi^1,$$

$$-dY_t^2 = f^2(t, Y_t^2, Z_t^2)dt - (Z_t^2)^* dW_t; \quad Y_T^2 = \xi^2.$$

Define $\delta Y = Y^1 - Y^2$, $\delta Z = Z^1 - Z^2$, subtract $-dY_t^2$ from $-dY_t^1$;

$$\begin{aligned}
-dY_t^1 + dY_t^2 &= f^1(t, Y_t^1, Z_t^1)dt - (Z_t^1)^* dW_t - f^2(t, Y_t^2, Z_t^2)dt + (Z_t^2)^* dW_t, \\
-d(Y_t^1 - Y_t^2) &= f^1(t, Y_t^1, Z_t^1)dt - (Z_t^1)^* dW_t \\
&\quad - f^2(t, Y_t^2, Z_t^2)dt + (Z_t^2)^* dW_t \pm f^1(t, Y_t^2, Z_t^2)dt, \\
-d(\delta Y_t) &= f^1(t, Y_t^1, Z_t^1)dt - f^1(t, Y_t^2, Z_t^2)dt - f^2(t, Y_t^2, Z_t^2)dt \\
&\quad + \delta_2 f_t dt - (\delta Z_t)^* dW_t \pm f^1(t, Y_t^2, Z_t^1)dt,
\end{aligned}$$

and $\delta Y_T = Y_T^1 - Y_T^2 = \xi^1 - \xi^2$. We define

$$\Delta_y f^1(t) := \begin{cases} (f^1(t, Y_t^1, Z_t^1) - f^1(t, Y_t^2, Z_t^1))/(Y_t^1 - Y_t^2) & \text{if } Y_t^1 - Y_t^2 \neq 0, \\ 0 & \text{if otherwise.} \end{cases}$$

Also,

$$\Delta_z f^{1,i}(t) := \begin{cases} (f^1(t, Y_t^2, \tilde{Z}_t^{i-1}) - f^1(t, Y_t^2, \tilde{Z}_t^i))/(Z_t^{1,i} - Z_t^{2,i}) & \text{if } Z_t^{1,i} - Z_t^{2,i} \neq 0, \\ 0 & \text{if otherwise.} \end{cases}$$

where $\tilde{Z}^i$ is the vector such that the associated components of Z^2 are the first i components and the associated components of Z^1 are the last $n-i$ components i.e.,

$$\tilde{Z}_t^i = (Z_t^{2,1}, \ldots, Z_t^{2,i}, Z_t^{1,i+1}, \ldots, Z_t^{1,n}).$$

The processes $\Delta_y f^1$ and $\Delta_z f^{1,i}$ are bounded since by assumption the generator f^1 is uniformly Lipschitz with respect to (y, z). We turn back and see that our equation becomes,

$$-d(\delta Y_t) = \Delta_y f^1(t)\delta Y_t dt + \Delta_z f^1(t)^* \delta Z_t dt + \delta_2 f_t dt - (\delta Z_t)^* dW_t. \tag{2.20}$$

By proposition 2.4, the LBSDE (2.20) has unique solution $(\delta Y, \delta Z)$ such that

$$\delta Y_t = \mathbb{E}\Big[(\xi^1 - \xi^2)\Gamma_T^t + \int_t^T \Gamma_s^t \delta_2 f_s ds | \mathcal{F}_t\Big], \tag{2.21}$$

where Γ_s^t is the adjoint process defined for $s \geq t$ by the forward linear stochastic differential equation

$$d\Gamma_s^t = \Gamma_s^t\big[\Delta_y f^1(s)ds + \Delta_z f^1(s)^* dW_s\big], \quad \Gamma_t^t = 1.$$

The given condition $\xi^1 \geq \xi^2$, $\delta_2 f \geq 0$ with Γ_s^t then $\delta Y_t \geq 0$ a.s. for any time t. Also, if $_t^1 = Y_t^2$ on a set $A \in \mathcal{F}_t$, then $\xi^1 = \xi^2$, $\delta_2 f_s = 0$, $d\mathbb{P} \otimes ds$ on $A \times [t, T]$ and $f^1(s, Y_s^1, Z_s^1) = f^2(s, Y_s^2, Z_s^2)$ a.s. on $A \times [t, T]$ which completes the proof. □

The comparison theorem explains naturally the option pricing facts in a financial market. It tells that a bigger contingent claim ξ_1 makes the option price Y_t^1 bigger at the present time.

2.3 The Model in Finance

In this section, we will deal with pricing and hedging problems in a complete market by using the previous results. Our aim is to find a unique solution of a LBSDE which is the fair price in the market.

It is natural to gain some interest when an individual lends money by buying a riskless asset. Thus, let the riskless asset price for continuous-time model be

$$dP_t^0 = P_t^0 r_t dt, \tag{2.22}$$

where r_t is the short rate. For instance, this riskless asset can be assumed as a bond. Furthermore, we assume there are number of n risky securities (for example stocks) for continuous-time where $i = 1, 2, ..., n$ has the following

$$dP_t^i = P_t^i \left[b_t^i dt + \sum_{j=1}^{n} \sigma_t^{i,j} dW_t^j \right], \tag{2.23}$$

remember $W = (W^1, W^2, ..., W^n)^*$ is a standard Brownian motion on $\mathbb{R}^n$, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. $\mathbb{P}$ is said to be the probability measure. Right-continuous filtration $(\mathcal{F}_t\ ;\ 0 \leq t \leq T)$ which is usually generated by the Brownian motion W, gives the information structure and "*" means transpose.

Example 2.1. If the number of risky assets is taken as $n = 2$, then there exist $n + 1 = 3$ assets. Then the dynamics of the assets have the following form:

$$\begin{aligned} dP_t^0 &= P_t^0\ r_t dt, \\ dP_t^1 &= P_t^1\ [b_t^1 dt + \sigma_t^{1,1} dW_t^1 + \sigma_t^{1,2} dW_t^2], \\ dP_t^2 &= P_t^2\ [b_t^2 dt + \sigma_t^{2,1} dW_t^1 + \sigma_t^{2,2} dW_t^2], \end{aligned}$$

where the coefficients satisfy some regularity conditions.

2.3.1 Hypothesis (A)

- The interest (short) rate r is a predictable and bounded process. Moreover, it is typically nonnegative due to the fact that the pay-off is nonnegative.
- The stock appreciation rate (drift term) $b = (b^1, b^2, ..., b^n)^*$ is a column vector of predictable and bounded processes.
- The volatility $\sigma = (\sigma^{i,j})$ is a $n \times n$ matrix of predictable and bounded processes. σ_t has full rank a.s. for all $t \in [0, T]$ and the inverse matrix σ^{-1} is also bounded process.
- There exists θ vector called a risk premium such that it is predictable and bounded process and

  $$b_t - r_t \mathbf{1} = \sigma_t \theta_t, \quad d\mathbb{P} \otimes dt \text{ a.s.},$$

  where $\mathbf{1}$ denotes the vector with all components being 1.

By the assumptions for r_t, b_t, σ_t and the number of risky assets being equal to the number of source of randomness, we get the necessary condition for a continuous market to be complete.

It is assumed that the market prices are not affect by purchases of the small investors. In addition, the decisions can only be based on the current information $(\mathcal{F}_t)$. Here π_t^i is the amount of the wealth V_t to invest in the i-th risky asset where $i = 1, 2, ...n$ at the time $t \in [0, T]$. $\pi_t^0 = V_t - \sum_{i=1}^n \pi_t^i$ defines the amount of wealth of the enterpriser to invest in the riskless asset.

Definition 2.4. A strategy (V, π) is called *self-financing* if the wealth process $V_t = \sum_{i=0}^{n} \pi_t^i$ satisfies the following equality

$$V_t = V_0 + \int_0^t \sum_{i=0}^{n} \pi_t^i \frac{dP_t^i}{P_t^i}. \tag{2.24}$$

Proposition 2.7. *A strategy is self-financing if the wealth process satisfies the linear stochastic differential equation (LSDE)*

$$\begin{aligned} dV_t &= r_t V_t dt + \pi_t^*(b_t - r_t \mathbf{1})dt + \pi_t^* \sigma_t dW_t, \\ &= r_t V_t dt + \pi_t^* \sigma_t [dW_t + \theta_t dt]. \end{aligned}$$

Here see that $W_t + \int_0^t \theta_s ds$ is a Brownian motion under risk neutral probability measure $\mathbb{Q}$. (See Appendices, Theorem A.6)

Proof. Get $\frac{dP_t^i}{P_t^i}$ from (2.22) and (2.23) then replace in (2.24) and differenciate.

$$\begin{aligned} V_t &= V_0 + \int_0^t \pi_s^0 r_s ds + \int_0^t \sum_{i=1}^{n} \pi_s^i \left[b_s^i ds + \sum_{j=1}^{n} \sigma_s^{i,j} dW_s^j \right], \\ dV_t &= \pi_t^0 r_t dt + \sum_{i=1}^{n} \pi_t^i \left[b_t^i dt + \sum_{j=1}^{n} \sigma_t^{i,j} dW_t^j \right], \\ &= \left[V_t - \sum_{i=1}^{n} \pi_t^i \right] r_t dt + \sum_{i=1}^{n} \pi_t^i \left[b_t^i dt + \sum_{j=1}^{n} \sigma_t^{i,j} dW_t^j \right], \\ &= \left[r_t V_t - \sum_{i=1}^{n} \pi_t^i r_t + \sum_{i=1}^{n} \pi_t^i b_t^i \right] dt + \sum_{i=1}^{n} \sum_{j=1}^{n} \pi_t^i \sigma_t^{i,j} dW_t^j, \\ &= r_t V_t dt + \sum_{i=1}^{n} \pi_t^i (b_t^i - r_t) dt + \sum_{i=1}^{n} \sum_{j=1}^{n} \pi_t^i \sigma_t^{i,j} dW_t^j. \end{aligned}$$

In matrix notation and we use the last assumption of 2.3.1 Hypothesis (A)

$$\begin{aligned} dV_t &= r_t V_t dt + \pi_t^*(b_t^i - r_t \mathbf{1})dt + \pi_t^* \sigma_t dW_t, \\ &= r_t V_t dt + \pi_t^* \sigma_t \theta_t dt + \pi_t^* \sigma_t dW_t, \\ &= r_t V_t dt + \pi_t^* \sigma_t \left[dW_t + \theta_t dt \right]. \end{aligned}$$

$\square$

In order to understand the ideas better let us give a simple example.

Example 2.2. Let us assume there are two number of risky assets (i.e. $n = 2$),

$$\begin{aligned}
\frac{dP_t^0}{P_t^0} &= r_t dt, \\
\frac{dP_t^1}{P_t^1} &= [b_t^1 dt + \sigma_t^{1,1} dW_t^1 + \sigma_t^{1,2} dW_t^2], \\
\frac{dP_t^2}{P_t^2} &= [b_t^2 dt + \sigma_t^{2,1} dW_t^1 + \sigma_t^{2,2} dW_t^2].
\end{aligned}$$

Then the self-financing strategy satisfies

$$V_t = V_0 + \int_0^t \pi_s^0 \frac{dP_s^0}{P_s^0} + \int_0^t \pi_s^1 \frac{dP_s^1}{P_s^1} + \int_0^t \pi_s^2 \frac{dP_s^2}{P_s^2},$$

by replacing $\frac{dP_t^i}{P_t^i}$ for i=0,1,2; we get

$$\begin{aligned}
V_t =& V_0 + \int_0^t \pi_s^0 r_s ds + \int_0^t \pi_s^1 [b_s^1 ds + \sigma_s^{1,1} dW_s^1 + \sigma_s^{1,2} dW_s^2] \\
&+ \pi_s^2 [b_s^2 ds + \sigma_s^{2,1} dW_s^1 + \sigma_s^{2,2} dW_s^2].
\end{aligned}$$

After several derivations, it is obtained

$$\begin{aligned}
dV_t &= \pi_t^0 r_t dt + \pi_t^1 [b_t^1 dt + \sigma_t^{1,1} dW_t^1 + \sigma_t^{1,2} dW_t^2] + \pi_t^2 [b_t^2 dt + \sigma_t^{2,1} dW_t^1 + \sigma_t^{2,2} dW_t^2], \\
&= \pi_t^0 r_t dt + \sum_{i=1}^{2} \pi_t^i [b_t^i dt + \sum_{j=1}^{2} \sigma_t^{i,j} dW_t^j],
\end{aligned}$$

If we replace $\pi_t^0 = V_t - \sum_{i=1}^2 \pi_t^i$,

$$\begin{aligned}
dV_t &= (V_t - \sum_{i=1}^{2} \pi_t^i) r_t dt + \sum_{i=1}^{2} \pi_t^i [b_t^i dt + \sum_{j=1}^{2} \sigma_t^{i,j} dW_t^j], \\
&= (r_t V_t - \sum_{i=1}^{2} \pi_t^i r_t + \sum_{i=1}^{2} \pi_t^i b_t^i) dt + \sum_{i=1}^{2} \sum_{j=1}^{2} \pi_t^i \sigma_t^{i,j} dW_t^j, \\
&= r_t V_t dt + \sum_{i=1}^{2} \pi_t^i (b_t^i - r_t) dt + \sum_{i=1}^{2} \sum_{j=1}^{2} \pi_t^i \sigma_t^{i,j} dW_t^j.
\end{aligned}$$

The above term can also be written as follows:

$$dV_t = r_t V_t dt + \begin{bmatrix} \pi_t^1 & \pi_t^2 \end{bmatrix} \begin{bmatrix} b_t^1 - r_t \\ b_t^2 - r_t \end{bmatrix} dt + \begin{bmatrix} \pi_t^1 & \pi_t^2 \end{bmatrix} \begin{bmatrix} \sigma_t^{1,1} & \sigma_t^{1,2} \\ \sigma_t^{2,1} & \sigma_t^{2,2} \end{bmatrix} \begin{bmatrix} dW_t^1 \\ dW_t^2 \end{bmatrix},$$

which is

$$\begin{aligned}
dV_t &= r_t V_t dt + \pi_t^* (b_t - r_t \mathbf{1}) dt + \pi_t^* \sigma_t dW_t, \\
&= r_t V_t dt + \pi_t^* \sigma_t \theta_t dt + \pi_t^* \sigma_t dW_t, \\
&= r_t V_t dt + \pi_t^* \sigma_t [dW_t + \theta_t dt].
\end{aligned}$$

The model (called Merton model) can be generalized by adding c_t (the positive rate of consumption at time t), then the LSDE becomes

$$dV_t = r_t V_t dt - c_t dt + \pi_t^* \sigma_t [dW_t + \theta_t dt].$$

The cumulative amount of consumption between 0 and t as C_t is introduced as

$$C_t = \int_0^t c_s ds.$$

See [9] for further details.

Definition 2.5. A strategy (V, π) is called *trading strategy* if the wealth process $V_t = \sum_{i=0}^n \pi_t^i$ and portfolio process $\pi_t = (\pi_t^1, \pi_y^2, ..., \pi_t^n)^*$ satisfies the following

$$dV_t = r_t V_t dt + \pi_t^* \sigma_t [dW_t + \theta_t dt], \qquad \int_0^T |\sigma_t^* \pi_t|^2 dt < +\infty, \mathbb{P} \text{ a.s.} \tag{2.25}$$

Definition 2.6. A strategy is said to be *feasible* if the constraint of nonnegative wealth holds:

$$V_t \geq 0 \quad t \in [0, T], \ \mathbb{P} \text{ a.s.}$$

Definition 2.7. A strategy (V, π, C) is called *superstrategy* if it satisfies

$$dV_t = r_t V_t dt - dC_t + \pi_t^* \sigma_t [dW_t + \theta_t dt], \qquad \int_0^T |\sigma_t^* \pi_t|^2 dt < +\infty, \mathbb{P} \text{ a.s.}, \tag{2.26}$$

here V is the wealth process(or the market value), π is the portfolio process, C_t is the cumulative consumption process and C_t is an increasing, right-continuous, adapted process with $C_0 = 0$

Definition 2.8. A superstrategy is called *feasible* if the constraint of nonnegative wealth holds:

$$V_t \geq 0 \quad t \in [0, T], \ \mathbb{P} \text{ a.s.}$$

2.4 Pricing and Hedging Positive Contingent Claims in Diffusion Models

According to the arbitrage-free pricing principle of a positive contingent claim, if the price of the claim is started as initial endowment and it is invested in the number of $n + 1$ assets, then the value of the portfolio must be just enough to guarantee ξ at time T, where ξ ,the $\mathcal{F}_T$-measurable random variable, is the contingent claim settled at time T.

Now, some definitions preserving the presentation of [13] is given so as to obtain a closed formula for the fair price.

Definition 2.9. Let $\xi \geq 0$ be a positive contingent claim,

1. A feasible self-financing strategy (V,π) (resp. (V,π,C)) such that $V_T=\xi$ is called a *hedging strategy* against ξ (resp. a *superhedging strategy*).

2. If the class of hedging strategies (resp. superhedging strategies) $\mathcal{H}(\xi)$ (resp. $\mathcal{H}'(\xi)$) against ξ is nonempty, then ξ is called *hedgeable* (resp. *superhedgeable*).

3. The *fair price* X_0 (resp. *upper price* X_0') at time 0 of hedgeable (resp. superhedgeable) claim ξ is the smallest initial endowment needed to hedge ξ; i.e.,
$$X_0=\inf\{x\geq 0;\exists(V,\pi)\in\mathcal{H}(\xi)\ \textit{such that}\ V_0=x\},$$
$$X_0'=\inf\{x\geq 0;\exists(V,\pi,C)\in\mathcal{H}'(\xi)\ \textit{such that}\ V_0=x\}.$$

If 2.3.1 Hypothesis (A) is satisfied, for any square-integrable claim $\xi\geq 0$, then the space $\mathcal{H}(\xi)$ becomes nonempty and the market is called as *complete*. It means that every contingent claim in the complete financial market can be hedged [9]. Moreover, the fair price is the market value of a hedging strategy in $\mathcal{H}(\xi)$ [13], as proved in the following theorem.

Theorem 2.8. *Assume 2.3.1 Hypothesis (A) holds and $\xi\geq 0$ be a square-integrable contingent claim. Then there exists a hedging strategy (X,π) against ξ such that*
$$dX_t=r_tX_tdt+\pi_t^*\sigma_t\theta_tdt+\pi_t^*\sigma_tdW_t,\quad X_T=\xi,\tag{2.27}$$
and the fair price and the upper price of the claim is the market value X.

Moreover, if we assume that $(H_s^t;s\geq t)$ is the deflator started at time t defined as following:
$$dH_s^t=-H_s^t[r_sds+\theta_s^*dW_s],\quad H_t^t=1.\tag{2.28}$$

Then
$$X_t=\mathbb{E}[H_T^t\xi|\mathcal{F}_t],\ \ a.s.\tag{2.29}$$

Proof. Let us find the solution of the SDE (2.28) of deflator H by applying Itô formula (2.6) to $f(x)=\log(x)$ where $s\geq t$, then
$$\begin{aligned}\log H_s^t &= \log H_t^t+\int_t^s\frac{1}{H_s^t}dH_s^t-\frac{1}{2}\int_t^s\Big(\frac{1}{H_s^t}\Big)^2d\langle H_s^t,H_s^t\rangle,\\ &= \log 1-\int_t^s\frac{1}{H_s^t}H_s^t[r_sds+\theta_s^*dW_s]-\frac{1}{2}\int_t^s|\theta_s^*|^2ds,\\ &= -(\int_t^s r_sds+\int_t^s\theta_s^*dW_s+\frac{1}{2}\int_t^s|\theta_s^*|^2ds),\end{aligned}$$
then the solution is started at time t
$$H_s^t=\exp\Big\{-\int_t^s r_sds-\int_t^s\theta_s^*dW_s-\frac{1}{2}\int_t^s|\theta_s^*|^2ds\Big\}.$$

Here, we can show the solution started at time 0 as follows

$$H_t := H_t^0 = \exp\Big\{ -\int_0^t r_s ds - \int_0^t \theta_s^* dW_s - \frac{1}{2}\int_0^t |\theta_s^*|^2 ds \Big\}.$$

Since r and θ are bounded processes, it follows Novikov's condition (see Appendices, Theorem A.7) that $\mathbb{E}(H_T^2) < +\infty$ and $\mathbb{E}(H_T\xi) < +\infty$ for any square-integrable contingent claim.

$$\begin{aligned} d(H_tX_t) =& X_t dH_t + H_t dX_t + d\langle H, X\rangle_t \\ =& - X_tH_t[r_tdt + \theta_t^* dW_t] + H_t(r_tX_tdt + \pi_t^*\sigma_t\theta_tdt + \pi_t^*\sigma_t dW_t) \\ & - H_t\pi_t^*\sigma_t\theta_tdt \\ =& - H_tX_tr_tdt - H_tX_t\theta_t^*dW_t + H_tX_tr_tdt + H_t\pi_t^*\sigma_t\theta_tdt \\ & + H_t\pi_t^*\sigma_tdW_t - H_t\pi_t^*\sigma_t\theta_tdt \\ =& - H_tX_t\theta_t^*dW_t + H_t\pi_t^*\sigma_tdW_t \\ =& H_t(\pi_t^*\sigma_t - X_t\theta_t^*)dW_t. \end{aligned}$$

We realize that there is no drift term in the above equality which proves it is a martingale. After defining the continuous adapted process X from;

$$H_tX_t = \mathbb{E}[H_T\xi|\mathcal{F}_t]$$

Then we define U_t as

$$U_t := H_t(\pi_t^*\sigma_t - X_t\theta_t^*).$$

Which implies that

$$\begin{aligned} H_t^{-1}U_t &= \pi_t^*\sigma_t - X_t\theta_t^*, \\ H_t^{-1}U_t + X_t\theta_t^* &= \pi_t^*\sigma_t, \\ (H_t^{-1}U_t + X_t\theta_t^*)\sigma_t^{-1} &= \pi_t^*, \\ \pi_t &= (\sigma_t^*)^{-1}(H_t^{-1}U_t + X_t\theta_t). \end{aligned}$$

By martingale representation theorem for the Brownian motion [17] H_tX_t can be represented as a stochastic integral such that

$$H_tX_t = \mathbb{E}(H_T\xi) + \int_0^t U_s^* dW_s, \qquad \int_0^T |U_t|^2 dt < +\infty \text{ a.s.}$$

We can find (X, π) satisfying the linear BSDE (2.27) by Itô's lemma. Since the processes X and H are continuous, θ is bounded, we can show $\int_0^T |\sigma_t^*\pi_t|^2dt < +\infty$ a.s. So (X, π) is a hedging strategy against ξ with $X_0 = \mathbb{E}(H_T\xi)$.

Lastly, X_0 (resp. X_t) is the upper price (resp. the smallest superhedging strategy). Let (X, φ, C) be a superhedging strategy against ξ. Again using Itô's lemma for the product of the RCLL semimartingale V and the continuous semimartingale H and using (2.26), we have that $(H_tV_t)_{t\in[0,T]}$ is a positive local supermartingale with decomposition $dH_tV_t = -H_tdC_t + (U_t^V)^*dW_t$, where $U_t^V = H_t[-V_t\theta_t + (\sigma_t^*)\varphi_t]$. Thus, $(H_tV_t)_{t\in[0,T]}$ is a positive supermartingale by Fatou's lemma [14] and

$$H_tV_t \geq \mathbb{E}[H_TV_T|\mathcal{F}_t] = H_tX_t, \quad V_0 \geq \mathbb{E}(H_T\xi) = X_0. \tag{2.30}$$

□

Remark 2.3. The fair price of the claim ξ has the property in equation (2.29). The fair price can be calculated as the expectation of the discounted asset pricess such that

$$X_t = \mathbb{E}_{\mathbb{Q}}[e^{-\int_t^T r_s ds}\xi|\mathcal{F}],$$

under the risk neutral probability measure $\mathbb{Q}$ where the Radon-Nikodym derivative with respect to $\mathbb{P}$ on $\mathcal{F}_T$ is given by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\Big\{-\int_0^T \theta_s^* dW_s - \frac{1}{2}\int_0^T |\theta_s|^2 ds\Big\}.$$

We realize that $\mathbb{Q}$ is well defined as a probability measure since, by assumption, θ is bounded. Moreover, $\mathbb{Q}$ is a martingale measure; that is, the discounted wealth processes are $\mathbb{Q}$-local martingales.

CHAPTER 3

BACKWARD STOCHASTIC DIFFERENTIAL EQUATION APPLICATION TO PDE

In this chapter, crucial connection between backward stochastic differential equations (BSDEs) and classical partial differential equations (PDEs) for the continuous case are concerned from [8] and [9] which are closely followed. As first a coupled forward-backward system is consider. Later, a lucid proof is given for the solution of BSDE included in the forward-backward stochastic differential equation (FBSDE) is Markovian (in the sense that depending only on the solution of the forward stochastic differential equation (SDE) part of the system). Additionally, the generalization of Feynman-Kac formula is proven. Lastly, the chapter is concluded by a proposition remarking that Delta hedging will be the strategy in order to replicate the claims in the Brownian case.

3.1 Forward-Backward Stochastic Differential Equations

In this section, the BSDEs whose parameters (f, ξ) with Markovian standard parameters are considered. Initially, a new system of equations including SDE and BSDE is defined.

3.1.1 The Model for FBSDEs

Given a probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ and $\mathbb{R}^n$-valued Brownian motion W, for any $(t, x) \in [0, T] \times \mathbb{R}^p$, we have the stochastic price process on $[0, T]$:

$$\begin{aligned} dP_s &= b(s, P_s)ds + \sigma(s, P_s)dW_s, \quad t \leq s \leq T \\ P_s &= x, \quad 0 \leq s \leq t. \end{aligned} \tag{3.1}$$

The solution of the SDE (3.1) will be denoted $\{P_s^{t,x}, 0 \leq s \leq T\}$. Then we consider the BSDE coming from the state of the forward equation,

$$\begin{aligned} -dY_s &= f(s, P_s^{t,x}, Y_s, Z_s)ds - Z_s^* dW_s \\ Y_T &= \psi(P_T^{t,x}). \end{aligned} \tag{3.2}$$

The solution of the BSDE (3.2) will be denoted $\{(Y_s^{t,x}, Z_s^{t,x}),\ \ \leq s \leq T\}$. The dynamics (3.1) and (3.2) form the system called *forward-backward stochastic differential equation* (FBSDE) and the solution is denoted by $\{(P_s^{t,x}, Y_s^{t,x}, Z_s^{t,x}),\ 0 \leq s \leq T\}$.

Let $T > 0$, $b(\cdot,\cdot) : [0,T] \times \mathbb{R}^p \mapsto \mathbb{R}^p$, $f(\cdot,\cdot,\cdot,\cdot) : [0,T] \times \mathbb{R}^p \times \mathbb{R}^d \times \mathbb{R}^{n\times d} \mapsto \mathbb{R}^d$, $\sigma(\cdot,\cdot) : [0,T] \times \mathbb{R}^p \mapsto \mathbb{R}^{p\times n}$ and $\psi(\cdot) : \mathbb{R}^p \mapsto \mathbb{R}^d$ be measurable functions satisfying the following Lipschitz conditions on the coefficients: there exists constant $C > 0$ such that

$$\begin{aligned} |b(t,x) - b(t,y)| + |\sigma(t,x) - \sigma(t,y)| &\leq C(1 + |x - y|), \\ |f(t,x,y_1,z_1) - f(t,x,y_2,z_2)| &\leq C(|y_1 - y_2| + |z_1 - z_2|). \end{aligned} \tag{3.3}$$

Lastly, in order to have unique solutions for the equations (3.1) and (3.2) assume that, there exists constant C satisfying the growth condition, which is

$$\begin{aligned} |b(t,x)| + |\sigma(t,x)| &\leq C(1 + |x|), \\ |f(t,x,y,z)| + |\psi(x)| &\leq C(1 + |x|^p) \end{aligned} \tag{3.4}$$

for any real $p \geq 1/2$. Here, the existence and uniqueness of FBSDEs is not proven since it follows from existence and uniqueness of SDEs and BSDE's. However, it could be found in [1].

3.1.2 Markov Properties of FBSDEs

A.A. Markov derived Markov property for discrete time in 1906. Intuitively, the random variables are said to Markovian when their future values after some time t do not influence by the history of the process before that time t. This property is crucial for the pricing of options. To illustrate, when an asset price is Markovian and this asset is also the underlying asset of the option, then the option price is only dependent to the price of underlying asset at time t.

Definition 3.1. When the terminal condition is only depending on the state of the forward diffusion and the driver of BSDE is only depending on the uncertainty through the state of Markov process (more specially through the state of diffusion process) then the parameters of the BSDE is said to be *Markovian*.

Here with the above assumptions, we will show that the solution of FBSDE at time s, $(P_s^{t,x}, Y_s^{t,x}, Z_s^{t,x})$, is Markovian (i.e. all these processes only depend on s and $P_s^{t,x}$).

Lemma 3.1. *Let ϕ is a continuous bounded $\mathbb{R}^d$-valued function and $\mathcal{B}_e^d$ be the filtration on $\mathbb{R}^d$ generated by the functions*

$$\mathbb{E}\left[\int_t^T \phi(s, P_s^{t,x})ds\right].$$

Then for any f, ψ in $\mathcal{B}_e^d$ such that

$$\mathbb{E}\Big[\int_0^T |f(s,P_s^{t,x})|^2 ds\Big] < +\infty, \quad \mathbb{E}\big[|\psi(P_T^{t,x})|^2\big] < +\infty$$

the process $Y_s^{t,x} = \mathbb{E}[\psi(P_T^{t,x} + \int_s^T f(r,P_r^{t,x})dr|\mathcal{F}_s]$ admits a continuous version given by $Y_s^{t,x} = m(s,P_s^{t,x})$, where $m(t,x) = \mathbb{E}[\psi(P_T^{t,x}) + \int_t^T f(r,P_r^{t,x})dr]$ is $\mathcal{B}_e^d$-measurable. Moreover, $\int_t^s f(r,P_r^{t,x})dr + Y_s^{t,x}$ is an additive square-integrable martingale which yield the following representation

$$\int_t^s f(r,P_r^{t,x})dr + Y_s^{t,x} = \int_t^s \delta(r,P_r^{t,x})^*\sigma(r,P_r^{t,x})dW_r, \quad t \le s \le T, \ \mathbb{Q} \ a.s.,$$

where $\delta(t,x) \in \mathcal{B}([0,T]) \otimes \mathcal{B}_e(\mathbb{R}^{p\times d})$.

Proof. The proof can be done by the similar way in the proof of existence uniqueness theorem 2.2. Introduce m as the following,

$$m(t,P_s^{t,x}) = \psi(P_T^{t,x}) + \int_t^T f(r,P_r^{t,x})dr - \int_s^T (Z_r^{t,x})^* dW_r, \tag{3.5}$$

where $Z_r^{t,x} = \sigma(r,P_r^{t,x})^*\delta(r,P_r^{t,x})$, $\delta(t,x) \in \mathcal{B}([0,T]) \otimes \mathcal{B}_e(\mathbb{R}^{p\times d})$ and $\sigma(t,x) \in \mathcal{B}([0,T]) \otimes \mathcal{B}_e(\mathbb{R}^{p\times n})$.

The solution (Y,Z) is defined by considering the continuous version m of the square-integrable martingale $\mathbb{E}[\psi(P_T^{t,x}) + \int_t^T f(r,P_r^{t,x})dr|\mathcal{F}_s]$. So as to find it we use $m(t,P_s^{t,x})$ in (3.5),

$$\begin{aligned} m(t,P_s^{t,x}) &= \mathbb{E}\big[m(T,P_T^{t,x})|\mathcal{F}_s\big], \\ &= \mathbb{E}\Big[\psi(P_T^{t,x}) + \int_t^T f(r,P_r^{t,x})dr - \int_T^T (Z_r^{t,x})^* dW_r|\mathcal{F}_s\Big], \\ &= \mathbb{E}\Big[\psi(P_T^{t,x}) + \int_t^T f(r,P_r^{t,x})dr|\mathcal{F}_s\Big]. \end{aligned}$$

By the martingale representation theorem (Appendices, Theorem A.4), there exists a unique integrable process $Z \in \mathbb{H}_T^{2,n\times d}$ such that

$$m(t,P_s^{t,x}) = m(t,P_t^{t,x}) + \int_t^s (Z_r^{t,x})^* dW_r.$$

Here, the adapted and continuous process Y is introduced by

$$Y_s^{t,x} = m(t,P_s^{t,x}) - \int_t^s f(r,P_r^{t,x})dr. \tag{3.6}$$

When we apply martingale representation theorem in the above equation (3.6), then insert $m(t, P_t^{t,x})$ according to equation (3.5), Y is also given as

$$\begin{aligned}
Y_s^{t,x} &= m(t, P_t^{t,x}) + \int_t^s (Z_r^{t,x})^* dW_r - \int_t^s f(r, P_r^{t,x}) dr, \\
&= \psi(P_T^{t,x}) + \int_t^T f(r, P_r^{t,x}) dr - \int_t^T (Z_r^{t,x})^* dW_r + \int_t^s (Z_r^{t,x})^* dW_r \\
&\quad - \int_t^s f(r, P_r^{t,x}) dr, \\
&= \psi(P_T^{t,x}) + \int_s^T f(r, P_r^{t,x}) dr - \int_s^T (Z_r^{t,x})^* dW_r \\
Y_s^{t,x} &= \mathbb{E}[Y_s^{t,x} | \mathcal{F}_s], \quad \text{since } Y_s^{t,x} \in \mathcal{F}_s, \\
&= \mathbb{E}\Big[\psi(P_T^{t,x}) + \int_s^T f(r, P_r^{t,x}) dr | \mathcal{F}_s\Big] - \mathbb{E}\Big[\int_s^T (Z_r^{t,x})^* dW_r | \mathcal{F}_s\Big], \\
&= \mathbb{E}\Big[\psi(P_T^{t,x}) + \int_s^T f(r, P_r^{t,x}) dr | \mathcal{F}_s\Big] - \int_s^s (Z_r^{t,x})^* dW_r, \\
&= \mathbb{E}\Big[\psi(P_T^{t,x}) + \int_s^T f(r, P_r^{t,x}) dr | \mathcal{F}_s\Big]. \qquad (3.7)
\end{aligned}$$

From the above assumptions $m(t, x) = \mathbb{E}[\psi(P_T^{t,x}) + \int_t^T f(r, P_r^{t,x}) dr]$ is $\mathcal{B}_e$-measurable, $\int_t^s f(r, P_r^{t,x}) dr + Y_s^{t,x}$ is an additive square-integrable martingale which can be represented as

$$\int_t^s f(r, P_r^{t,x}) dr + Y_s^{t,x} = \int_t^s \delta(r, P_r^{t,x})^* \sigma(r, P_r^{t,x}) dW_r, \quad t \le s \le T, \ \mathbb{Q} \text{ a.s.}$$

□

Theorem 3.2. *For the deterministic functions $m(t, x) \in \mathcal{B}([0, T]) \otimes \mathcal{B}_e(\mathbb{R}^d)$ and $\delta(t, x) \in \mathcal{B}([0, T]) \otimes \mathcal{B}_e(\mathbb{R}^{p\times d})$, such that the solution $(Y^{t,x}, Z^{t,x})$ is of BSDE (3.2) is*

$$Y_s^{t,x} = m(s, P_s^{t,x}), \quad Z_s^{t,x} = \sigma^*(s, P_s^{t,x}) \delta(s, P_s^{t,x}), \quad t \le s \le T, \ d\mathbb{Q} \otimes ds \ a.s.$$

Moreover, for $s \ge t$ the solution $(Y_s^{t,\chi}, Z_s^{t,\chi})$ is $(m(s, P_s^{t,\chi}), \sigma^*(s, P_s^{t,\chi}) \delta(s, P_s^{t,\chi}))$ $d\mathbb{Q} \otimes ds$ a.s. for any $\mathcal{F}_t$-measurable random variable $\chi \in \mathbb{L}^2(\mathbb{R}^p)$.

Proof. Let $(Y^{(t,x),k}, Z^{(t,x),k})$ be the sequence defined recursively by $(Y^{(t,x),0} = 0, Z^{(t,x),0} = 0)$ and

$$-dY_s^{k+1} = f(s, P_s^{t,x}, Y_s^k, Z_s^k) ds - (Z_s^{k+1})^* dW_s, \quad Y_T^{k+1} = \psi(P_T^{t,x}).$$

Then the sequence $(Y^{(t,x),k}, Z^{(t,x),k})$ converges to $(Y^{t,x}, Z^{t,x})$, $d\mathbb{Q} \otimes ds$ a.s. Here, $(Y^{t,x}, Z^{t,x})$ is the unique square-integrable solution of the BSDE by Theorem 2.2

and Corollary 2.3. Here, remark that $\sup_{s\in[t,T]}|Y_s^{(t,x),k} - Y_s^{(t,x)}|$ converges $\mathbb{Q}$ a.s. to zero. By Lemma 3.1, it is concluded by recursion that there exists $m_k, \delta_k \in \mathcal{B}_e$ such that

$$Y_s^{(t,x),k} = m_k(s, P_s^{t,x}), \quad Z_s^{(t,x),k} = \sigma(s, P_s^{t,x})^* \delta_k(s, P_s^{t,x}).$$

We replace

$$\limsup_{k\to+\infty} m_k^i(s,x) = m^i(s,x), \quad \limsup_{k\to+\infty} \delta_k^{i,j}(s,x) = \delta^{i,j}(s,x),$$

where $m = (m^i)_{1\le i\le d}$ and $\delta = (\delta^{i,j})_{1\le i\le p, 1\le j\le d}$. Notice that from the a.s. convergence of the sequence $(Y^{(t,x),k}, Z^{(t,x),k})$ to $(Y^{t,x}, Z^{t,x})$, it follows that $\mathbb{Q}$ a.s., $\forall s \in [t,T]$,

$$\begin{aligned}
m^i(s, P_s^{t,x}) &= (\limsup_{k\to+\infty} m_k^i)(s, P_s^{t,x}) = \limsup_k (m_k^i(s, P_s^{t,x})) = \lim_{k\to+\infty} Y_s^{i,(t,x),k} \\
&= Y_s^{i,(t,x),k}, \\
\delta^{i,j}(s, P_s^{t,x}) &= (\limsup_{k\to+\infty} \delta_k^{i,j})(s, P_s^{t,x}) = \limsup_k (\delta_k^{i,j}(s, P_s^{t,x})) = \lim_{k\to+\infty} Z_s^{i,j,(t,x),k} \\
&= Z_s^{i,j,(t,x),k}.
\end{aligned}$$

Consequently, $m(s, P_s^{t,x}) = Y_s^{t,x}$; $\delta(s, P_s^{t,x}) = Z_s^{t,x}$ $d\mathbb{Q}\otimes ds$ a.s. $\square$

3.2 Feynman-Kac Formula

For Markovian standard parameters, the solutions of the FBSDEs give us a generalization of the Feynman-Kac formula for nonlinear PDEs as stated by Peng [25]. By Feynman-Kac formula one can find classical solution for some PDEs. Here a detailed proof of the generalized Feynman-Kac formula from [8] and our main article [9].

Theorem 3.3. *Let v be a $\mathcal{C}^{1,2}$ function defined on $[0,T]\times\mathbb{R}^d$. If $\forall(t,x)\in[0,T]\times\mathbb{R}^d$, v satisfies*

$$\begin{aligned}
v(T,x) &= \psi(x), \quad \forall x\in\mathbb{R}^d, \\
\partial_t v(t,x) &+ \mathcal{L}v(t,x) + f(t,x,v(t,x),\sigma(t,x)^*\partial_x v(t,x)) = 0,
\end{aligned} \tag{3.8}$$

where $\partial_x v(t,x)$ is the gradient of v and $\mathcal{L}_{(t,x)}$ is the infinitesimal generator such that

$$\mathcal{L}_{(t,x)} = \sum_{i,j} a_{ij}(t,x)\partial^2_{x_ix_j} + \sum_i b_i(t,x)\partial_{x_i}, \quad a_{ij} = \frac{1}{2}[\sigma\sigma^*]_{ij},$$

also there exists a constant C such that, for each (s,x),

$$|v(s,x)| + |\sigma(s,x)^*\partial_x v(s,x)| \le C(1+|x|),$$

then $(Y_s^{t,x}, Z_s^{t,x}) = (v(s,P_s^{t,x}), \sigma(s,P_s^{t,x})^*\partial_x v(s,P_s^{t,x}))$, $t \le s \le T$ *is the unique solution of BSDE (3.2) with standard parameters* (f,ψ).

Conversely, if the discounted asset prices are given as martingales and $(Y_s^{t,x}, Z_s^{t,x}) = (v(s,P_s^{t,x}), \sigma(s,P_s^{t,x})^*\partial_x v(s,P_s^{t,x}))$, $t \le s \le T$ *is the unique solution of BSDE (3.2) with standard parameters* (f,ψ) *then the associated PDE can be found as the equation (3.8).*

Proof. Assume that P_s is d-dimensional price process and $\partial_t v(s,P_s^{t,x})$ is the partial derivative with respect to time variable, $\partial_x v(s,P_s^{t,x})$ is the gradient of v and $\partial_x^2 v(s,P_s^{t,x})$ is $(d \times d)$ Hessian matrix of v with respect to P_s. Then we apply Itô formula (2.6) to $v(s,P_s^{t,x})$,

$$dv(s,P_s^{t,x}) = \partial_t v(s,P_s^{t,x})ds + [\partial_x v(s,P_s^{t,x})]^* dP_s^{t,x} + \frac{1}{2}[dP_s^{t,x}]^* \partial_x^2 v(s,P_s^{t,x}) dP_s^{t,x},$$

$$\begin{aligned}
&= \partial_t v(s,P_s^{t,x})ds + [\partial_x v(s,P_s^{t,x})]^*[b(s,P_s^{t,x})ds + \sigma(s,P_s^{t,x})dW_s] \\
&+ \frac{1}{2}[dP_s^{t,x}]^* \partial_x^2 v(s,P_s^{t,x}) dP_s^{t,x}, \\
&= \partial_t v(s,P_s^{t,x})ds + [\partial_x v(s,P_s^{t,x})]^* b(s,P_s^{t,x})ds + [\partial_x v(s,P_s^{t,x})]^*\sigma(s,P_s^{t,x})dW_s \\
&+ \frac{1}{2}[dP_s^{t,x}]^* \partial_x^2 v(s,P_s^{t,x}) dP_s^{t,x}, \\
&= \partial_t v(s,P_s^{t,x})ds + [\partial_x v(s,P_s^{t,x})]^* b(s,P_s^{t,x})ds + [\partial_x v(s,P_s^{t,x})]^*\sigma(s,P_s^{t,x})dW_s \\
&+ \frac{1}{2}\sum_{i=1}^{p}\sum_{j=1}^{p}\left[dP_{js}^{t,x}\frac{\partial^2 v(s,P_s^{t,x})}{\partial x_i \partial x_j}\right]dP_{is}^{t,x},
\end{aligned}$$

notify that W_i and W_j are independent ($W_i \perp W_j$ when $i \neq j$,

$$\begin{aligned}
&= \partial_t v(s,P_s^{t,x})ds + [\partial_x v(s,P_s^{t,x})]^* b(s,P_s^{t,x})ds + [\partial_x v(s,P_s^{t,x})]^*\sigma(s,P_s^{t,x})dW_s \\
&+ \frac{1}{2}\sum_{i=1}^{p}\sum_{j=1}^{p}[(\sigma\sigma^*)_{ij}(s,P_s^{t,x})]\frac{\partial^2 v(s,P_s^{t,x})}{\partial x_i \partial x_j}ds, \\
&= \partial_t v(s,P_s^{t,x})ds + [\partial_x v(s,P_s^{t,x})]^* b(s,P_s^{t,x})ds + [\partial_x v(s,P_s^{t,x})]^*\sigma(s,P_s^{t,x})dW_s \\
&+ \sum_{i,j=1}^{p}\frac{1}{2}[(\sigma\sigma^*)_{ij}(s,P_s^{t,x})]\frac{\partial^2 v(s,P_s^{t,x})}{\partial x_i \partial x_j}ds, \\
&= \big[\partial_t v(s,P_s^{t,x}) + \mathcal{L}_{(t,x)} v(s,P_s^{t,x})\big]ds + [\partial_x v(s,P_s^{t,x})]^*\sigma(s,P_s^{t,x})dW_s. \qquad (3.9)
\end{aligned}$$

We introduce the following

$$\begin{aligned}
r(s,P_s^{t,x})v(s,P_s^{t,x}) &:= \partial_t v(s,P_s^{t,x}) + \mathcal{L}_{(t,x)} v(s,P_s^{t,x}), \\
dM_s &:= [\partial_x v(s,P_s^{t,x})]^*\sigma(s,P_s^{t,x})dW_s, \\
L_s &:= e^{\int_s^T r(m,P_m^{t,x})dm} v(s,P_s^{t,x}).
\end{aligned}$$

The equation (3.9) gives us Ornstein-Uhlenbeck (OU) Process,

$$dv(s, P_s^{t,x}) = r(s, P_s^{t,x})v(s, P_s^{t,x})ds + dM_s.$$

Then the OU process is solved by the help of L_s,

$$\begin{aligned}
dL_s &= -r(s, P_s^{t,x})e^{\int_s^T r(m,P_m^{t,x})dm}v(s, P_s^{t,x})ds \\
&+ e^{\int_s^T r(m,P_m^{t,x})dm}\big[r(s, P_s^{t,x})v(s, P_s^{t,x})ds + dM_s\big], \\
&= e^{\int_s^T r(m,P_m^{t,x})dm}dM_s. \\
\int_s^T dY_s &= \int_s^T e^{\int_s^T r(m,P_m^{t,x})dm}dM_s \\
L_T &= L_s + \int_s^T e^{\int_s^T r(m,P_m^{t,x})dm}dM_s \\
\psi(P_T^{t,x}) &= L_s + \int_s^T e^{\int_s^T r(m,P_m^{t,x})dm}dM_s
\end{aligned}$$

and L_s is a martingale such that ,

$$\psi(P_T^{t,x}) = e^{\int_s^T r(m,P_m^{t,x})dm}v(s, P_s^{t,x}) + \int_s^T e^{\int_s^T r(m,P_m^{t,x})dm}dM_s$$

$$\mathbb{E}\Big[e^{-\int_s^T r(m,P_m^{t,x})dm}\psi(P_T^{t,x})|P_s^{t,x} = x\Big] = v(s, P_s^{t,x})$$

and it is the solution of boundary value problem. Also, equation (3.9) is a BSDE with the following

$$\begin{aligned}
-dv(s, P_s^{t,x}) = f(s, P_s^{t,x}, v(s, P_s^{t,x}), [\sigma(s, P_s^{t,x})]^*\partial_x v(s, P_s^{t,x}))ds \\
-[\partial_x v(s, P_s^{t,x})]^*\sigma(s, P_s^{t,x})dW_s,
\end{aligned} \tag{3.10}$$

with $v(T, P_T^{t,x}) = \psi(P_T^{t,x})$.

Thus, $(Y_s, Z_s) = (v(s, P_s^{t,x}), [\sigma(s, P_s^{t,x})]^*\partial_x v(s, P_s^{t,x})),\ s \in [0, T]$ is equal to unique solution of BSDE (3.2).

On the contrary, let the discounted asset prices be martingales then the associated PDE which leads to an analogue between BSDE and PDE. Moreover, assume that the unique solution of BSDE (3.2) is given as $(Y_s^{t,x}, Z_s^{t,x}) = (v(s, P_s^{t,x}), \sigma(s, P_s^{t,x})^*\partial_x v(s, P_s^{t,x})),\ t \le s \le T$. Without loss of generality we consider the time between $[0, t]$ instead of $[t, s]$. We have the following derivative for the discounted asset prices,

$$\begin{aligned}
d(e^{-\int_0^t r_s ds}v(t, P_t)) =& -r_t e^{-\int_0^t r_s ds}v(t, P_t)dt + e^{-\int_0^t r_s ds}d(v(t, P_t)) \\
=& e^{-\int_0^t r_s ds}\big[-r_t v(t, P_t) \\
& - f(t, P_t, v(s, P_t), \sigma(s, P_t)^*\partial_x v(s, P_t)\big]dt \\
& + e^{-\int_0^t r_s ds}[\partial_x v(s, P_t)]^*\sigma(s, P_t)dW_t
\end{aligned} \tag{3.11}$$

which leads dt term to be zero, hence

$$- r_t v(t, P_t) = f(t, P_t, v(s, P_t), \sigma(s, P_t)^* \partial_x v(s, P_t). \qquad (3.12)$$

Additionally, we define

$$f(t, x) = e^{-\int_0^t r_s ds} v(t, x),$$
$$\frac{\partial f}{\partial t}(t, x) = -r_t e^{-\int_0^t r_s ds} v(t, x) + e^{-\int_0^t r_s ds} \frac{\partial v}{\partial t}(t, x),$$
$$\frac{\partial f}{\partial x}(t, x) = e^{-\int_0^t r_s ds} \frac{\partial v}{\partial x}(t, x), \quad \frac{\partial^2 f}{\partial x^2}(t, x) = e^{-\int_0^t r_s ds} \frac{\partial^2 v}{\partial x^2}(t, x).$$

Now Itô formula (2.6) is applied to $f(t, P_t) = e^{-\int_0^t r_s ds} v(t, P_t)$,

$$\begin{aligned} e^{-\int_0^t r_s ds} v(t, P_t) =& e^0 v(0, P_0) + \int_0^t \big[- r_s e^{-\int_0^s r_u du} v(s, P_s) + e^{-\int_0^s r_u du} \frac{\partial v}{\partial t}(s, P_s) \big] ds \\ &+ \int_0^t e^{-\int_0^s r_u du} \left[\frac{\partial v}{\partial x}(s, P_s) \right]^* \big[b(s, P_s) ds + \sigma(s, P_s) dW_s \big] \\ &+ \frac{1}{2} \int_0^t e^{-\int_0^s r_u du} [\sigma(s, P_s)]^* \frac{\partial^2 v}{\partial x^2}(s, P_s) \sigma(s, P_s) ds \\ =& v(0, P_0) + \int_0^t e^{-\int_0^s r_u du} \big[- r_s v(s, P_s) + \partial_t v(s, P_s) \\ &+ [\partial_x v(s, P_s)]^* b(s, P_s) + \frac{1}{2} [\sigma(s, P_s)]^* \partial_x^2 v(s, P_s) \sigma(s, P_s) \big] ds \\ &+ \int_0^t e^{-\int_0^s r_u du} [\partial v_x(s, P_s)]^* \sigma(s, P_s) dW_s. \end{aligned}$$

Since it is martingale the dt term should be equal to zero then the inside of the integral should be equal to zero which leads to

$$-r_s v(s, P_s) + \partial_t v(s, P_s) + [\partial_x v(s, P_s)]^* b(s, P_s) + \frac{1}{2} [\sigma(s, P_s)]^* \partial_x^2 v(s, P_s) \sigma(s, P_s) = 0.$$

By using equation (3.12) $\forall x \in \mathbb{R}^d$ the given PDE has the boundary value $v(T, x) = \psi(x)$ and the equation

$$\partial_t v(t, x) + \mathcal{L} v(t, x) + f(t, x, v(t, x), \sigma(t, x)^* \partial_x v(t, x)) = 0.$$

□

Proposition 3.4. *For the BSDE given in equation (3.10) with the unique solution* $(Y_s^{t,x}, Z_s^{t,x}) = (v(s, P_s^{t,x}), \sigma(s, P_s^{t,x})^* \partial_x v(s, P_s^{t,x}))$, $t \leq s \leq T$, *the replicating portfolio gives the delta hedging.*

Proof. We remark that the market value process for self financing trading strategy (v, π) satisfies

$$dv_s = r_s v_s ds + \pi_s^* \sigma(s, P_s) \big[dW_s + \theta_s ds \big], \quad \int_0^T |\sigma_s^* \pi_s|^2 ds < +\infty \ \mathbb{Q} \text{ a.s.},$$

$$-dv_s = [-r_s v_s - \pi_s^* \sigma(s, P_s)\theta_s]ds - \pi_s^* \sigma(s, P_s)dW_s,$$
$$f(s, P_s, v_s, Z_s) = -r_s v_s - \pi_s^* \sigma(s, P_s)\theta_s,$$
$$Z_s = [\pi_s^* \sigma(s, P_s)]^* = \sigma(s, P_s)^* \pi_s,$$

(v_s, Z_s) is the solution of BSDE, where v_s is wealth process and Z_s is hedging strategy, $b_s - r_s\mathbf{1} = \sigma(s, P_s)\theta_s$,

$$Z_s = \sigma(s, P_s)^* \pi_s = \sigma(s, P_s)^* \frac{\partial v_s}{\partial x}$$

where we get $\pi_s = \partial_x v_s$ the Delta Hedging. $\square$

Throughout the second and third chapters the continuous models in the BSDE theory is worked with. The existence and uniqueness of BSDEs is verified and a closed formula for pricing and hedging contingent claims for LBSDEs is given. Moreover, FBSDE system is considered and the general Feynman-Kac formula is proven.

CHAPTER 4

BACKWARD STOCHASTIC DIFFERENTIAL EQUATIONS IN THE PRESENCE OF JUMPS

The Brownian case for the backward stochastic differential equations (BSDEs) is the foundation and focus of many researches. However, the weaknesses of diffusion models makes the models less realistic. So as to make our study deeper, we will place jumps in our model for the next chapters. In this chapter, mainly [27] is followed, the existence and uniqueness theorem is proven by using previously stated lemmas. Furthermore, the comparison theorem is stated without proof. Pricing-hedging contingent claims are applied theoretically to finance at the end of this section.

4.1 Motivation

The diffusion models, in other words the models using Brownian motion has some weaknesses to preserve the empirical properties of assets prices, represent the main features of option prices modeling and provide realistic tools for hedging and risk management. The below table compares the diffusion and jump models [7].

Facts to be modeled	**Diffusion models**	**Jump models**
Discontinuous sudden moves in prices	Continuous large volatilities	Discontinuous sudden moves in prices
The asset returns has high volatility	Nonlinear volatility is needed	Realistic volatility
Options are risky	Risks can be reduced to risk-free return	Options are risky
Some risks can not be hedged	Perfect hedges	Perfect hedges do not exist
Markets are incomplete	Markets are complete	Markets are incomplete

In spite of more advanced calculations, the practical benefits make us study the BSDE in the presence of jumps.

The more general model of BSDEs in $\mathbb{R}^n$ with jumps can be defined as

$$\begin{aligned} -dY(t) &= f\big(t, Y(t), Z(t), \gamma(t,z)\big)dt - Z(t)^* dW(t) - \int_{\mathbb{R}^m} \gamma(t,z)\widetilde{M}(dt,dz); \\ Y(T) &= \xi, \end{aligned}$$

where $\xi \geq 0$ and which has an additional compensated Poisson random measure $\widetilde{M}$ on $\mathbb{R}^m$ such that $\widetilde{M}(dt,dz) = M(dt,dz) - \nu(dz)dt$ where $\nu(\cdot)$ is a σ-finite counting measure, $M(dt,dz)$ is the Poisson counting measure on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Here we recall and modify some notations.

- For $x \in \mathbb{R}^n$, $|x|$ denotes its Euclidean norm.
- For $x \in \mathbb{R}^n$, $\langle x, y \rangle$ denotes the inner product.
- An $d \times n$ matrix will be considered as an element $y \in \mathbb{R}^{d\times n}$.
- For $y \in \mathbb{R}^{d\times n}$, the Euclidean norm is given by $|y| = \sqrt{\mathrm{trace}(yy^*)}$.
- For $y,\ z \in \mathbb{R}^{d\times n}$, the inner product is given by $\langle y, z \rangle = \mathrm{trace}(yz^*)$.
- "$*$" is used to denote transpose matrix.
- "c" is used to denote continuous part of the given process.

Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and an $\mathbb{R}^d$-valued Brownian motion W and $\mathbb{R}^m$-valued Poisson Process N, we consider

- $\{(\mathcal{F}_t); t \in [0,T]\}$, the filtration generated by W and N.
- $\mathbb{L}^2_T(\mathbb{R}^n)$, the space of all $\mathcal{F}_T$-measurable random variables $X : \Omega \mapsto \mathbb{R}^n$ satisfying $\|X\|^2 = \mathbb{E}(|X|^2) < +\infty$.
- $\mathbb{H}^2_T(\mathbb{R}^{n\times d})$, the space of all predictable processes $\phi : \Omega \times [0,T] \mapsto \mathbb{R}^{n\times d}$ such that $\mathbb{E}\int_0^T |\phi(t)|^2 dt < +\infty$.
- $\mathbb{F}^2_T(\mathbb{R}^{n\times m})$, the space of all predictable processes $\varphi : \Omega \times [0,T] \times \mathbb{R}^m \mapsto \mathbb{R}^{n\times m}$ such that $\mathbb{E}\int_0^T \int_{\mathbb{R}^m} |\varphi(t,z)|^2 \nu(dz)dt < +\infty$.

For simplicity we may use $\mathbb{L}^2_T(\mathbb{R}^n) = \mathbb{L}^{2,n}_T$, $\mathbb{H}^2_T(\mathbb{R}^{n\times d}) = \mathbb{H}^{2,n\times d}_T$, $\mathbb{F}^2_T(\mathbb{R}^{n\times m}) = \mathbb{F}^{2,n\times m}_T$.

4.2 Existence and Uniqueness of Backward Stochastic Differential Equations

In this thesis, the general form of Lévy processes is used instead of Lévy-Itô decomposition (i.e. separating small and big jumps) or Lévy-Khinchin representation since square-integrable Lévy processes are studied (i.e. $\mathbb{E}[|\nu(t)|^2] < \infty$).

Compensated Poisson measure $\widetilde{M}$ is introduced by subtracting from M its intensity measure:

$$\widetilde{M}(dt, dz) = M(dt, dz) - \nu(dz)dt.$$

The Lévy-Itô representation theorem states that there exists a_1 and $\sigma \in \mathbb{R}$ such that

$$\begin{aligned}
\nu(t) &= a_1 t + \sigma W(t) + \int_0^t \int_{|z|<1} z\widetilde{M}(ds, dz) + \int_0^t \int_{|z|\geq 1} zM(ds, dz) \\
&= a_1 t + \sigma W(t) + \int_0^t \int_{|z|<1} z\widetilde{M}(ds, dz) + \int_0^t \int_{|z|\geq 1} z\big(\widetilde{M}(ds, dz) + \nu(dz)dt\big) \\
&= a_1 t + \sigma W(t) + \int_0^t \int_{\mathbb{R}} z\widetilde{M}(ds, dz) + t\int_{|z|\geq 1} z\nu(dz) \\
&= a_t + \sigma W(t) + \int_0^t \int_{\mathbb{R}} z\widetilde{M}(ds, dz) \qquad (4.1)
\end{aligned}$$

where $a = a_1 + \int_{|z|\geq 1} z\nu(dz)$ and at time $t = 0$, no jump occurs. Then it becomes natural to define the stochastic differential equations of the form

$$dX(t) = \mu(t, X(t))dt + \sigma(t, X(t))dW(t) + \int_{\mathbb{R}} \gamma(t, X(t), z)\widetilde{M}(dt, dz)$$

for $\mathbb{R}$-valued deterministic functions μ, σ and γ satisfying certain growth conditions [22]. From this view, the BSDE can be generalized to the following differential or integral form.

$$\begin{aligned}
-dY(t) &= f\big(t, Y(t), Z(t), \gamma(t, z)\big)dt - Z(t)^* dW(t) - \int_{\mathbb{R}^m} \gamma(t, z)\widetilde{M}(dt, dz); \\
Y(T) &= \xi, \qquad (4.2)
\end{aligned}$$

or an equivalent formulation is

$$\begin{aligned}
Y(t) &= \xi + \int_t^T f\big(s, Y(s), Z(s), \gamma(s, z)\big)ds - \int_t^T Z(s)^* dW(s) \\
&\quad - \int_t^T \int_{\mathbb{R}^m} \gamma(s, z)\widetilde{M}(ds, dz), \qquad (4.3)
\end{aligned}$$

where

- The terminal value is an $\mathcal{F}_T$-measurable random variable, $\xi : \Omega \mapsto \mathbb{R}^n$.
- The generator (driver) f maps $\Omega \times \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^{d\times n} \times \mathbb{R}^{n\times m}$ onto $\mathbb{R}^n$ and is $\mathcal{P} \bigotimes \mathcal{B}^n \bigotimes \mathcal{B}^{d\times n} \bigotimes \mathcal{B}^{n\times m}$-measurable and $\mathcal{B}^n$ denotes Borel-measurable sets in $\mathbb{R}^n$, $\mathcal{B}^{d\times n}$ denotes Borel-measurable sets in $\mathbb{R}^{d\times n}$ likewise $\mathcal{B}^{n\times m}$ denotes Borel-measurable sets in $\mathbb{R}^{n\times m}$.
- $W = (W^1, W^2, ..., W^d)^*$ is a standard Brownian motion on $\mathbb{R}^d$.

- $\widetilde{M} = (\widetilde{M}^1, \widetilde{M}^2, ..., \widetilde{M}^m)^*$ is a Poisson martingale measure on $\mathbb{R}^m$ such that $\widetilde{M}(dt, dz) = M(dt, dz) - \nu(dz)dt$ where $\nu(\cdot)$ is a σ-finite counting measure, $M(dt, dz)$ is the Poisson counting measure.

Definition 4.1. A triple (Y, Z, γ) is said to be the *solution* of BDSE (4.2) if $(Y(t), Z(t), \gamma(t, z)) \in \mathbb{H}_T^{2,n} \times \mathbb{H}_T^{2,d\times n} \times \mathbb{F}_T^{2,n\times m}$.

The existence and uniqueness of a solution of the BSDE's with jumps is proven by the following lemmas and theorem from [27]. Gronwall lemma will be useful to show our solution has bounded elements.

Lemma 4.1. *Let $f(t)$ be a non-negative random variable such that $f(t) \leq \alpha g(t) + \int_t^T \beta(s)f(s)ds$ $\forall t \geq 0$ where $\alpha > 0$ is a constant and $\beta(s) \geq 0$ then*

$$f(t) \leq \alpha g(t) + \alpha \int_t^T \exp\Big\{ \int_t^s \beta(r)dr \Big\} \beta(s)g(s)ds.$$

Proof. For all $t \geq 0$, let us define $u(t) := \int_t^T \beta(s)f(s)ds$ then $u(t) \geq 0$ since $\beta(s)$, $f(s) \geq 0$ and $h(t) := f(t) - \alpha g(t) - u(t) \leq 0$. By adding and subtracting $\int_0^T \beta(s)f(s)ds$ to $u(t)$ the differential is found as

$$\begin{aligned} du(t) &= -\beta(t)f(t) \\ &= -\beta(t)\Big(h(t) + \alpha g(t) + u(t)\Big) \\ &= -\beta(t)\Big(h(t) + \alpha g(t)\Big) - \beta(t)u(t), \end{aligned}$$

with u(T) = 0. The the solution of the differential equation is

$$u(t) = \int_t^T \exp\Big\{ \int_t^s \beta(r)dr \Big\} \beta(s) \Big(h(s) + \alpha g(s)\Big) ds.$$

Hence one finds that

$$u(t) \leq \int_t^T \exp\Big\{ \int_t^s \beta(r)dr \Big\} \beta(s)\alpha g(s)ds.$$

(The best general reference here for the solution of the differential equation is [6]). $\square$

Lemma 4.2. *Let $(Y(t), Z(t), \gamma(t, z))$ be a solution of the BSDE (4.2), $\mathbb{E}|\xi|^2 < \infty$,*

$$\langle Y, f\big(t, Y, Z, \gamma\big)\rangle \leq c_1(t)\big(1 + |Y| + |Y|^2\big) + c_2(t)|Y| \Bigg(|Z^*| + \Big(\int_{\mathbb{R}^m} |\gamma(z)|^2 \nu(dz) \Big)^{\frac{1}{2}} \Bigg), \tag{4.4}$$

where $c_1(t)$, $c_2(t) \geq 0$ are deterministic such that

$$\int_0^T c_1(t)dt + \int_0^T \big(c_2(t)\big)^2 dt \leq \infty.$$

Then

$$\mathbb{E}\Big[\sup_{t\in[0,T]}|Y(t)|^2+\int_0^T\big(|Z(t)^*|^2+\int_{\mathbb{R}^m}|\gamma(t,z)|^2\nu(dz)\big)dt\Big]\le k_0<\infty,$$

with a constant k_0 only depending on deterministic constants $c_1,\ c_2$ and $\mathbb{E}|\xi|^2$.

Proof. Let $Y(t)$ be given by the equation (4.3). When the multidimensional Itô Formula (see Appendices, Theorem A.10) is applied to $|Y(t)|^2$,

$$\begin{aligned}|Y(T)|^2=|Y(t)|^2&-\int_t^T\langle 2|Y(s)|,f(s,Y(s),Z(s),\gamma(s,z))\rangle ds\\&+\int_t^T\langle 2|Y(s)|,Z(s)^*dW(s)\rangle+\frac{1}{2}\int_t^T 2|Z(s)^*|^2ds\\&+\int_t^T\int_{\mathbb{R}^m}\big\{|Y(s)+\gamma(s,z)|^2-|Y(s)|^2-\langle 2|Y(s)|,\gamma(s,z)\rangle\big\}\nu(dz)ds\\&+\int_t^T\int_{\mathbb{R}^m}\big\{|Y(s^-)+\gamma(s,z)|^2-|Y(s^-)|^2|\big\}\widetilde{M}(ds,dz).\end{aligned}$$

Here $\widetilde{M}(dt,dz)=M(dt,dz)-\nu(dz)dt$ is replaced and by necessary calculations

$$\begin{aligned}|Y(T)|^2=|Y(t)|^2&-\int_t^T\langle 2|Y(s)|,f(s,Y(s),Z(s),\gamma(s,z))\rangle ds\\&+\int_t^T\langle 2|Y(s)|,Z(s)^*dW(s)\rangle+\int_t^T|Z(s)^*|^2ds\\&+\int_t^T\int_{\mathbb{R}^m}|\gamma(s,z)|^2M(ds,dz)+\int_t^T\int_{\mathbb{R}^m}\langle 2|Y(s^-)|,\gamma(s,z)\widetilde{M}(ds,dz)\rangle.\end{aligned}$$

After changing places of variables accordingly, we have

$$\begin{aligned}&|Y(t)|^2+\int_t^T|Z(s)^*|^2ds+\int_t^T\int_{\mathbb{R}^m}|\gamma(s,z)|^2M(ds,dz)\\&=|Y(T)|^2+2\int_t^T\langle|Y(s)|,f(s,Y(s),Z(s),\gamma(s,z))\rangle ds\\&-2\int_t^T\langle|Y(s)|,Z(s)^*dW(s)\rangle-2\int_t^T\int_{\mathbb{R}^m}\langle|Y(s^-)|,\gamma(s,z)\widetilde{M}(ds,dz)\rangle.\end{aligned}\tag{4.5}$$

Since

$$\begin{aligned}\mathbb{E}\Big[\int_t^T|Y(s)Z(s)^*|^2ds\Big]^{1/2}&\le\mathbb{E}\Big[\int_0^T|Y(t)Z(t)^*|^2dt\Big]^{1/2}\\&\le 2\mathbb{E}\sup_{t\in[0,T]}|Y(t)|^2+2\mathbb{E}\int_0^T|Z(t)^*|^2dt<\infty,\end{aligned}$$

$\int_t^T \langle |Y(s)|, Z(s)^* dW(s)\rangle$ and $\int_t^T \int_{\mathbb{R}^m} \langle |Y(s^-)|, \gamma(s,z)\widetilde{M}(ds,dz)\rangle$ are martingales. We take expectation of equation (4.5)

$$
\begin{aligned}
&\mathbb{E}\Big[|Y(t)|^2 + \int_t^T |Z(s)^*|^2 ds + \int_t^T \int_{\mathbb{R}^m} |\gamma(s,z)|^2 M(ds,dz)\Big] \\
=&\mathbb{E}\big[|Y(T)|^2\big] + \mathbb{E}\Big[2\int_t^T \langle |Y(s)|, f(s,Y(s),Z(s),\gamma(s,z))\rangle ds\Big].
\end{aligned}
$$

The equation (4.4) gives us

$$
\begin{aligned}
&\mathbb{E}\Big[|Y(t)|^2 + \int_t^T |Z(s)^*|^2 ds + \int_t^T \int_{\mathbb{R}^m} |\gamma(s,z)|^2 M(ds,dz)\Big] \\
\leq&\mathbb{E}\big[|Y(T)|^2\big] + 2\int_t^T c_1(s)ds + \mathbb{E}\Big[2\int_t^T c_1(s)|Y(s)|ds\Big] \\
+&\mathbb{E}\Big[2\int_t^T c_1(s)|Y(s)|^2 ds\Big] \\
+&\mathbb{E}\Big[2\int_t^T c_2(s)|Y(s)||Z(s)^*|ds\Big] \\
+&\mathbb{E}\Big[2\int_t^T c_2(s)|Y(s)|\Big(\int_{\mathbb{R}^m} |\gamma(s,z)|^2\nu(dz)\Big)^{\frac{1}{2}} ds\Big].
\end{aligned}
$$

Here the following three inequalities are used and they can be found in Appendices, Lemma A.3

$$
\begin{aligned}
&2\int_t^T c_1(s)|Y(s)|ds \leq \int_t^T c_1(s)|Y(s)|^2 ds + \int_t^T c_1(s)ds, \\
&2\int_t^T c_2(s)|Y(s)||Z(s)^*|ds \leq 2\int_t^T \big(c_2(s)\big)^2|Y(s)|^2 ds + \frac{1}{2}\int_t^T |Z(s)^*|^2 ds, \\
&2\int_t^T c_2(s)|Y(s)|\Big(\int_{\mathbb{R}^m} |\gamma(s,z)|^2\nu(dz)\Big)^{\frac{1}{2}} ds \leq 2\int_t^T \big(c_2(s)\big)^2|Y(s)|^2 ds \\
&\qquad + \frac{1}{2}\int_t^T \int_{\mathbb{R}^m} |\gamma(s,z)|^2\nu(dz)ds.
\end{aligned}
$$

By using these equations

$$
\begin{aligned}
&\mathbb{E}\left[|Y(t)|^2+\int_t^T |Z(s)^*|^2 ds+\int_t^T\int_{\mathbb{R}^m}|\gamma(s,z)|^2 M(ds,dz)\right]\\
\leq&\mathbb{E}\big[|Y(T)|^2\big]+2\int_t^T c_1(s)ds+\mathbb{E}\left[\int_t^T c_1(s)|Y(s)|^2ds+\int_t^T c_1(s)ds\right]\\
+&\mathbb{E}\left[2\int_t^T c_1(s)|Y(s)|^2ds\right]\\
+&\mathbb{E}\left[2\int_t^T \big(c_2(s)\big)^2|Y(s)|^2ds+\frac{1}{2}\int_t^T |Z(s)^*|^2ds\right]\\
+&\mathbb{E}\left[2\int_t^T \big(c_2(s)\big)^2|Y(s)|^2ds+\frac{1}{2}\int_t^T\int_{\mathbb{R}^m}|\gamma(s,z)|^2\nu(dz)ds\right],
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&\mathbb{E}\left[|Y(t)|^2+\int_t^T |Z(s)^*|^2 ds+\int_t^T\int_{\mathbb{R}^m}|\gamma(s,z)|^2 M(ds,dz)\right]\\
=&\mathbb{E}\big[|Y(T)|^2\big]+3\int_t^T c_1(s)ds\\
+&\mathbb{E}\left[3\int_t^T c_1(s)|Y(s)|^2ds\right]+\mathbb{E}\left[4\int_t^T \big(c_2(s)\big)^2|Y(s)|^2ds\right]\\
+&\mathbb{E}\left[\frac{1}{2}\int_t^T |Z(s)^*|^2ds\right]+\mathbb{E}\left[\frac{1}{2}\int_t^T\int_{\mathbb{R}^m}|\gamma(s,z)|^2\nu(dz)ds\right].
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
&\mathbb{E}\left[|Y(t)|^2+\frac{1}{2}\int_t^T |Z(s)^*|^2 ds+\frac{1}{2}\int_t^T\int_{\mathbb{R}^m}|\gamma(s,z)|^2 M(ds,dz)\right]\\
\leq&\mathbb{E}\big[|Y(T)|^2\big]+3\int_t^T c_1(s)ds+\mathbb{E}\left[\int_t^T\Big(3c_1(s)+4\big(c_2(s)\big)^2\Big)|Y(s)|^2ds\right].
\end{aligned}
$$

Let a constant $\alpha=\mathbb{E}\big[|Y(T)|^2\big]+3\int_t^T c_1(s)ds>0$, then

$$
\begin{aligned}
&\mathbb{E}\left[|Y(t)|^2+\frac{1}{2}\int_t^T |Z(s)^*|^2 ds+\frac{1}{2}\int_t^T\int_{\mathbb{R}^m}|\gamma(s,z)|^2 M(ds,dz)\right]\\
\leq&\alpha+\int_t^T\Big(3c_1(s)+4\big(c_2(s)\big)^2\Big)\mathbb{E}\big[|Y(s)|^2\big]ds.
\end{aligned}
$$

We apply Gronwall's inequality with $\beta(s):=3c_1(s)+4\big(c_2(s)\big)^2\geq 0$, $g(t)=1$ and conclude

$$
\begin{aligned}
&\mathbb{E}\Big[|Y(t)|^2+\frac{1}{2}\int_t^T|Z(s)^*|^2ds+\frac{1}{2}\int_t^T\int_{\mathbb{R}^m}|\gamma(s,z)|^2M(ds,dz)\Big]\\
\leq&\alpha+\alpha\int_t^T\exp\Big\{\int_t^s\beta(r)dr\Big\}\beta(s)ds=k_0<\infty.
\end{aligned}
$$

Moreover,

$$
\sup_{t\geq 0}\mathbb{E}\big[|Y(t)|^2\big]\leq k_0<\infty.
$$

□

Before moving to the theorem of existence and uniqueness, the existence and uniqueness for a special case of BSDE is proven. Thus, the variables in the generating function f are narrowed to a simple form such that

$$
Y(t)=\xi+\int_t^T f(w,s)ds-\int_t^T Z(s)^*dW(s)-\int_t^T\int_{\mathbb{R}^m}\gamma(s,z)\widetilde{M}(ds,dz);\quad t\geq 0. \tag{4.6}
$$

Then, the following theorem can be obtained.

Theorem 4.3. *If ξ is $\mathcal{F}_T$-measurable and $\mathbb{R}^n$ valued, $f(w,t)$ is $\mathcal{F}_t$-adapted and $\mathbb{R}^n$-valued such that*

$$
\mathbb{E}|\xi|^2<\infty,\quad \mathbb{E}\Big[\int_0^T|f(w,s)|ds\Big]^2<\infty,
$$

then the equation (4.6) has a unique solution.

Proof. Let $Y(t)=\mathbb{E}\Big[\xi+\int_t^T f(s)ds|\mathcal{F}_t\Big]$. Note that, it exists and it is well defined since

$$
\mathbb{E}\Big[|\xi+\int_t^T f(s)ds|^2\big|\mathcal{F}_t\Big]<\infty\quad\forall t\geq 0.
$$

We realize that

$$
Y(0)=\mathbb{E}\Big[\xi+\int_0^T f(s)ds|\mathcal{F}_0\Big]=\mathbb{E}\Big[\xi+\int_0^T f(s)ds\Big].
$$

In addition, for all $t\geq 0$

$$
E[Y(t)|\mathcal{F}_t]=Y(t).
$$

We define $L(t):=\mathbb{E}\Big[\xi+\int_0^T f(s)ds|\mathcal{F}_t\Big]$ which is square-integrable martingale which can be seen by taking conditional expectation and using tower property, for $s\leq t$

$$
\mathbb{E}[L(t)|\mathcal{F}_s]=\mathbb{E}\Big[\mathbb{E}\big[\xi+\int_0^T f(s)ds|\mathcal{F}_t\big]\big|\mathcal{F}_s\Big]=\mathbb{E}\Big[\xi+\int_0^T f(s)ds|\mathcal{F}_s\Big]=L(s).
$$

Also, by dividing integral into two parts $L(t)$ can be written in terms of $Y(t)$;

$$\begin{aligned} L(t) &= \mathbb{E}\Big[\xi + \int_0^T f(s)ds|\mathcal{F}_t\Big] = \mathbb{E}\Big[\xi + \int_0^t f(s)ds + \int_t^T f(s)ds|\mathcal{F}_t\Big] \\ &= Y(t) + \int_0^t f(s)ds. \end{aligned} \tag{4.7}$$

It is obvious that $L(0) = Y(0)$ and

$$L(T) = Y(T) + \int_0^T f(s)ds = \xi + \int_0^T f(s)ds. \tag{4.8}$$

By martingale representation theorem (see Appendices, Theorem A.4)

$$\begin{aligned} L(t) &= L(0) + \int_0^t Z(s)^* dW(s) + \int_0^t \int_{\mathbb{R}^m} \gamma(s,z)\widetilde{M}(ds,dz) \\ &= Y(0) + \int_0^t Z(s)^* dW(s) + \int_0^t \int_{\mathbb{R}^m} \gamma(s,z)\widetilde{M}(ds,dz), \quad t \geq 0. \end{aligned} \tag{4.9}$$

Now we have

$$L(T) = Y(0) + \int_0^T Z(s)^* dW(s) + \int_0^T \int_{\mathbb{R}^m} \gamma(s,z)\widetilde{M}(ds,dz). \tag{4.10}$$

Taking into account (4.8) and (4.10), it is found

$$\xi + \int_0^T f(s)ds = Y(0) + \int_0^T Z(s)^* dW(s) + \int_0^T \int_{\mathbb{R}^m} \gamma(s,z)\widetilde{M}(ds,dz).$$

By dividing integrals into two parts from 0 to t and t to T,

$$\begin{aligned} \xi + \int_0^t f(s)ds + \int_t^T f(s)ds = Y(0) + \int_0^t Z(s)^* dW(s) + \int_t^T Z(s)^* dW(s) \\ + \int_0^t \int_{\mathbb{R}^m} \gamma(s,z)\widetilde{M}(ds,dz) + \int_t^T \int_{\mathbb{R}^m} \gamma(s,z)\widetilde{M}(ds,dz). \end{aligned}$$

The integrals form t to T are taken on the right hand side of the equation,

$$\begin{aligned} &\xi + \int_t^T f(s)ds - \int_t^T Z(s)^* dW(s) - \int_t^T \int_{\mathbb{R}^m} \gamma(s,z)\widetilde{M}(ds,dz) \\ =&Y(0) + \int_0^t Z(s)^* dW(s) + \int_0^t \int_{\mathbb{R}^m} \gamma(s,z)\widetilde{M}(ds,dz) - \int_0^t f(s)ds. \end{aligned}$$

By equations (4.7) and (4.9) it follows

$$\xi+\int_t^T f(s)ds-\int_t^T Z(s)^*dW(s)-\int_t^T\int_{\mathbb{R}^m}\gamma(s,z)\widetilde{M}(ds,dz)$$
$$=L(t)-\int_0^t f(s)ds=Y(t).$$

Thus, $(Y(t),Z(t),\gamma(t,z))$ satisfies the BSDE (4.6). By using the fact that $(a+b+c)^2\le 3(a^2+b^2+c^2)$ and $(a+b)^2\le 2(a^2+b^2)$;

$$\begin{aligned}
[Y(t)]^2\le&3\Big[\xi^2+\Big(\int_t^T f(s)ds\Big)^2+\Big(\int_t^T Z(s)^*dW(s)\\
&+\int_t^T\int_{\mathbb{R}^m}\gamma(s,z)\widetilde{M}(ds,dz)\Big)^2\Big]\\
\le&3\Big[\xi^2+\int_t^T\big(f(s)\big)^2ds+\Big(\int_t^T Z(s)^*dW(s)\\
&+\int_t^T\int_{\mathbb{R}^m}\gamma(s,z)\widetilde{M}(ds,dz)\Big)^2\Big]\\
\le&3\xi^2+3\int_t^T\big(f(s)\big)^2ds+3\cdot 2\Big(\int_t^T Z(s)^*dW(s)\Big)^2\\
&+3\cdot 2\Big(\int_t^T\int_{\mathbb{R}^m}\gamma(s,z)\widetilde{M}(ds,dz)\Big)^2\\
\le&3\xi^2+3\int_0^T\big(f(s)\big)^2ds+6\int_0^T|Z(s)^*|^2ds\\
&+6\int_0^T\int_{\mathbb{R}^m}|\gamma(s,z)|^2\nu(dz)ds.
\end{aligned}\tag{4.11}$$

After taking expectation,

$$\begin{aligned}
\mathbb{E}\big[\sup_{t\in[0,T]}|Y(t)|^2\big]\le&\mathbb{E}\Big[3\xi^2+3\int_0^T\big(f(s)\big)^2ds\\
&+6\int_0^T|Z(s)^*|^2ds+6\int_0^T\int_{\mathbb{R}^m}|\gamma(s,z)|^2\nu(dz)ds\Big]\le\infty.
\end{aligned}$$

Let $(Y^1(t),Z^1(t),\gamma^1(t,z))$ and $(Y^1(t),Z^1(t),\gamma^1(t,z))$ be two solutions such that

$$\begin{aligned}
Y^1(t)&=\xi+\int_t^T f(w,s)ds-\int_t^T Z^1(s)^*dW(s)-\int_t^T\int_{\mathbb{R}^m}\gamma^1(s,z)\widetilde{M}(ds,dz),\\
Y^2(t)&=\xi+\int_t^T f(w,s)ds-\int_t^T Z^2(s)^*dW(s)-\int_t^T\int_{\mathbb{R}^m}\gamma^2(s,z)\widetilde{M}(ds,dz).
\end{aligned}$$

We need the difference between solutions for the uniqueness which is defined as,

$$\begin{aligned}\Delta Y(t) &:= Y^1(t) - Y^2(t),\\ \Delta Z(t) &:= Z^1(t) - Z^2(t),\\ \Delta\gamma(t,z) &:= \gamma^1(t,z) - \gamma^2(t,z).\end{aligned}$$

Therefore,

$$\Delta Y(t) = -\int_t^T \Delta Z(s)^* dW(s) - \int_t^T \int_{\mathbb{R}^m} \Delta\gamma(s,z)\widetilde{M}(ds,dz),$$

by multidimensional Itô formula (see Appendices, Theorem A.10),

$$\begin{aligned}|\Delta Y(T)|^2 = 0 = |\Delta Y(t)|^2 + \int_t^T \langle 2|\Delta Y(s)|, \Delta Z(s)^* dW(s)\rangle + \frac{1}{2}\int_t^T 2|\Delta Z(s)^*|^2 ds\\ + \int_t^T \int_{\mathbb{R}^m} \Big\{\big|\Delta Y(s^-) + \Delta\gamma(s,z)\big|^2 - |\Delta Y(s^-)|^2 - 2|\Delta\gamma(s,z)||\Delta Y(s^-)|\Big\}\nu(dz)ds\\ + \int_t^T \int_{\mathbb{R}^m} \Big\{\big|\Delta Y(s^-) + \Delta\gamma(s,z)\big|^2 - |\Delta Y(s^-)|^2\Big\}\widetilde{M}(ds,dz).\end{aligned}$$

Hence,

$$\begin{aligned}|\Delta Y(t)|^2 + \int_t^T |\Delta Z(s)^*|^2 ds + \int_t^T \int_{\mathbb{R}^m} \big|\Delta\gamma(s,z)\big|^2 \nu(dz)ds\\ = -2\int_t^T \langle |\Delta Y(s)|, \Delta Z(s)^* dW(s)\rangle\\ - \int_t^T \int_{\mathbb{R}^m} \Big\{\big|\Delta\gamma(s,z)\big|^2 + 2|\Delta Y(s^-)||\Delta\gamma(s,z)|\Big\}\widetilde{M}(ds,dz).\end{aligned}$$

Note that the uniqueness follows as

$$0 \le \mathbb{E}\Big[|\Delta Y(t)^2| + \int_t^T |\Delta Z(s)^*|^2 ds + \int_t^T \int_{\mathbb{R}^m} |\Delta\gamma(s,z)|^2 \nu(dz)ds\Big] \le 0,$$

with $\Delta Y(t) = 0$, $\Delta Z(t) = 0$ and $\Delta\gamma(t,z) = 0$. □

Now the theorem of the existence and uniqueness of a solution of BSDE with Lipschitzian coefficients can be stated. [27] is utilized for the sketch for the proof and it is given a detailed comprehensible proof by using the contraction principle.

Theorem 4.4. *We suppose that $f(t,Y,Z,\gamma) : [0,T]\times\mathbb{R}^n\times\mathbb{R}^{d\times n}\times\mathbb{F}^2(\mathbb{R}^{n\times m}) \mapsto \mathbb{R}^n$ is a $\mathcal{F}_t$-adapted and measurable process such that $\mathbb{P}$ a.s. and there exists $c_1(t)$,*

$c_2(t)$ non-negative deterministic functions such that $\int_0^T c_1(t)dt + \int_0^T \big(c_2(t)\big)^2 dt < +\infty$ such that

$$\begin{aligned}|f(t,Y^1,Z^1,\gamma^1) - f(t,Y^2,Z^2,\gamma^2)| \le & c_1(t)|Y^1 - Y^2| + c_2(t)\Big[|(Z^1 - Z^2)^*| \\ & + \Big(\int_{\mathbb{R}^m} |\gamma^1 - \gamma^2|^2 \nu(dz)\Big)^{1/2}\Big],\end{aligned} \tag{4.12}$$

$$|f(t,Y,Z,\gamma)| \le c_1(t)(1+|Y|) + c_2(t)\Big[1 + |Z^*| + \Big(\int_{\mathbb{R}^m} |\gamma|^2 \nu(dz)\Big)^{1/2}\Big], \tag{4.13}$$

$$\mathbb{E}|\xi|^2 < +\infty, \quad \xi \in \mathcal{F}_T, \tag{4.14}$$

then for any $t \ge 0$ the BSDE (4.3) admits a unique solution in $[0,T]$.

Proof. Let us briefly describe the proof. Firstly, the Banach space $\widetilde{\mathcal{B}}$ is introduced. A solution of BSDE (4.3) belonging to that Banach space is used in order to find a fixed point for a contraction mapping at the end. Later, a simpler form BSDE by fixing some variables is considered since we can comment on the existence and uniqueness of its solution. Next, the difference between BSDEs is taken into account for the simpler case produced. By the help of the assumptions an inequality being suitable to apply Gronwall lemma is obtained. Then, the norm of difference of fixed variables are bounded the difference of variables which can be the solution of BSDE (4.3). Finally, a function Φ is introduced which maps the fixed variables onto the solution then Φ turns out a contraction from $\widetilde{\mathcal{B}}$ to $\widetilde{\mathcal{B}}$. As a result, this mapping has a fixed point being the unique solution of BSDE (4.3).

For all $t \ge 0$, $(Y(t), Z(t), \gamma(t,z)) \in \widetilde{\mathcal{B}} = \mathbb{H}_T^{2,n} \times \mathbb{H}_T^{2,d\times n} \times \mathbb{F}_T^{2,n\times m}$, let

$$\begin{aligned}\|(Y(t), Z(t), \gamma(t,z))\|_S^2 = \sup_{t\in[0,T]} e^{-\zeta A(t)} \mathbb{E}\Big[& |Y(t)|^2 + \int_t^T |Z(s)^*|^2 ds \\ & + \int_t^T \int_{\mathbb{R}^m} |\gamma(s,z)|^2 \nu(dz) ds\Big],\end{aligned}$$

where $\zeta \ge 0$ is a constant and $A(t) = \int_t^T \big(c_1(s) + 2(c_2(s))^2\big) ds$.

New norm space is introduced as

$$\mathcal{H} = \{(Y(t), Z(t), \gamma(t,z)) \in \widetilde{\mathcal{B}} : \|(Y(t), Z(t), \gamma(t,z))\|_S < \infty\},$$

which is a Banach space.

Let the adapted process Y be

$$Y(t) = \mathbb{E}\Big[\xi + \int_t^T f(s, Y(s), Z(s), \gamma(s,z)) ds | \mathcal{F}_t\Big].$$

Moreover, we define the square-integrable martingale

$$L(t) := \mathbb{E}\Big[\xi + \int_0^T f(s, Y(s), Z(s), \gamma(s,z))ds|\mathcal{F}_t\Big].$$

Then, $L(t)$ can be written in terms of $Y(t)$:

$$L(t) = Y(t) + \int_0^t f(s, Y(s), Z(s), \gamma(s,z))ds. \tag{4.15}$$

By martingale representation theorem (see Appendices, Theorem A.4)

$$\begin{aligned} L(t) &= L(0) + \int_0^t Z(s)^* dW(s) + \int_0^t \int_{\mathbb{R}^m} \gamma(s,z)\widetilde{M}(ds,dz) \\ &= Y(0) + \int_0^t Z(s)^* dW(s) + \int_0^t \int_{\mathbb{R}^m} \gamma(s,z)\widetilde{M}(ds,dz), \quad t \geq 0. \end{aligned} \tag{4.16}$$

At time T, we consider the equations (4.15),(4.16) and get

$$\begin{aligned} \xi + \int_0^T f(s, Y(s), Z(s), \gamma(s,z))ds = Y(0) &+ \int_0^T Z(s)^* dW(s) \\ &+ \int_0^T \int_{\mathbb{R}^m} \gamma(s,z)\widetilde{M}(ds,dz). \end{aligned}$$

After dividing integrals, doing necessay rearrangements and replacing the equations (4.15),(4.16); we see $(Y(t), Z(t), \gamma(t,z))$ satisfies the given BSDE

$$\begin{aligned} Y(t) &= \xi + \int_t^T f\big(s, Y(s), Z(s), \gamma(s,z)\big)ds - \int_t^T Z(s)^* dW(s) \\ &\quad - \int_t^T \int_{\mathbb{R}^m} \gamma(s,z)\widetilde{M}(ds,dz), \end{aligned} \tag{4.17}$$

Now, for every t, any fixed $(\overline{Y}^1(t), \overline{Z}^1(t), \overline{\gamma}^1(t,z)),\ \overline{Y}^2(t), \overline{Z}^2(t), \overline{\gamma}^2(t,z)) \in \mathcal{H}$ placed in the generator function f, by Theorem 4.3 there exists unique solutions $(Y^1(t),\ Z^1(t), \gamma^1(t,z))$ and $(Y^2(t), Z^2(t), \gamma^2(t,z))$ of the following simpler form BSDE for $i = 1, 2$ and $t \geq 0$;

$$\begin{aligned} Y^i(t) =& \xi + \int_t^T f(s, \overline{Y}^i(s), \overline{Z}^i(s), \overline{\gamma}^i(s,z))ds - \int_t^T Z^i(s)^* dW(s) \\ &- \int_t^T \int_{\mathbb{R}^m} \gamma^i(s,z)\widetilde{M}(ds,dz). \end{aligned}$$

As defined before, we recall;

$$\begin{aligned} \Delta Y(t) &= Y^1(t) - Y^2(t), \\ \Delta Z(t) &= Z^1(t) - Z^2(t), \\ \Delta\gamma(t,z) &= \gamma^1(t,z) - \gamma^2(t,z). \end{aligned}$$

Likewise,

$$\begin{aligned}\Delta\overline{Y}(t) &= \overline{Y}^1(t) - \overline{Y}^2(t),\\ \Delta\overline{Z}(t) &= \overline{Z}^1(t) - \overline{Z}^2(t),\\ \Delta\overline{\gamma}(t,z) &= \overline{\gamma}^1(t,z) - \overline{\gamma}^2(t,z).\end{aligned}$$

Then the associated BDSEs are the following,

$$\begin{aligned}Y^1(t) =& \xi + \int_t^T f(s,\overline{Y}^1(s),\overline{Z}^1(s),\overline{\gamma}^1(s,z))ds - \int_t^T Z^1(s)^* dW(s)\\ &- \int_t^T\int_{\mathbb{R}^m} \gamma^1(s,z)\widetilde{M}(ds,dz),\end{aligned}$$

$$\begin{aligned}Y^2(t) =& \xi + \int_t^T f(s,\overline{Y}^2(s),\overline{Z}^2(s),\overline{\gamma}^2(s,z))ds - \int_t^T Z^2(s)^* dW(s)\\ &- \int_t^T\int_{\mathbb{R}^m} \gamma^2(s,z)\widetilde{M}(ds,dz),\end{aligned}$$

$$\begin{aligned}\Delta Y(t) =& Y^1(t) - Y^2(t)\\ =& \int_t^T \big(f(s,\overline{Y}^1(s),\overline{Z}^1(s),\overline{\gamma}^1(s,z)) - f(s,\overline{Y}^2(s),\overline{Z}^2(s),\overline{\gamma}^2(s,z))\big)ds\\ &- \int_t^T \Delta Z(s)^* dW(s) - \int_t^T\int_{\mathbb{R}^m} \Delta\gamma(s,z)\widetilde{M}(ds,dz).\end{aligned}$$

Here, Itô formula (see Appendices, Theorem A.10) is applied to $f(x) = |x|^2$,

$$\begin{aligned}&|\Delta Y(T)|^2 = |\Delta Y(t)|^2\\ &- \int_t^T \langle 2|\Delta Y(s)|, f(s,\overline{Y}^1(s),\overline{Z}^1(s),\overline{\gamma}^1(s,z)) - f(s,\overline{Y}^2(s),\overline{Z}^2(s),\overline{\gamma}^2(s,z))\rangle ds\\ &+ \int_t^T \langle 2|\Delta Y(s)|, \Delta Z(s)^* dW(s)\rangle + \frac{1}{2}\int_t^T 2|\Delta Z(s)^*|^2 ds\\ &+ \int_t^T\int_{\mathbb{R}^m} \Big\{|\Delta Y(s^-) + \Delta\gamma(s,z)|^2 - |\Delta Y(s^-)|^2\\ &\qquad\qquad - 2|\Delta Y(s^-)||\Delta\gamma(s,z)|\Big\}\nu(dz)ds\\ &+ \int_t^T\int_{\mathbb{R}^m} \Big\{|\Delta Y(s^-) + \Delta\gamma(s,z)|^2 - |\Delta Y(s^-)|^2\Big\}\widetilde{M}(ds,dz).\end{aligned}$$

It can be rewriten as

$$
\begin{aligned}
&|\Delta Y(t)|^2 + \int_t^T |\Delta Z(s)^*|^2 ds + \int_t^T \int_{\mathbb{R}^m} |\Delta\gamma(s,z)|^2 \nu(dz) ds \\
&= \int_t^T \langle 2|\Delta Y(s)|, f(s, \overline{Y}^1(s), \overline{Z}^1(s), \overline{\gamma}^1(s,z)) - f(s, \overline{Y}^2(s), \overline{Z}^2(s), \overline{\gamma}^2(s,z))\rangle ds \\
&\quad - \int_t^T \langle 2|\Delta Y(s)|, \Delta Z(s)^* dW(s)\rangle \\
&\quad - \int_t^T \int_{\mathbb{R}^m} \Big\{ |\Delta\gamma(s,z)|^2 + 2|\Delta Y(s^-)||\Delta\gamma(s,z)| \Big\} \widetilde{M}(ds,dz).
\end{aligned}
$$

By considering martingale terms, the expectation is taken

$$
\begin{aligned}
&\mathbb{E}\Big[|\Delta Y(t)|^2 + \int_t^T |\Delta Z(s)^*|^2 ds + \int_t^T \int_{\mathbb{R}^m} |\Delta\gamma(s,z)|^2 \nu(dz) ds \Big] \\
&= 2\mathbb{E}\Big[\int_t^T \langle |\Delta Y(s)|, f(s, \overline{Y}^1(s), \overline{Z}^1(s), \overline{\gamma}^1(s,z)) - f(s, \overline{Y}^2(s), \overline{Z}^2(s), \overline{\gamma}^2(s,z))\rangle ds \Big].
\end{aligned}
$$

Here, the following inequality is used by the Lipschitz assumption (4.12)

$$
\begin{aligned}
&|f(t, \overline{Y}^1(t), \overline{Z}^1(t), \overline{\gamma}^1(t,z)) - f(t, \overline{Y}^2(t), \overline{Z}^2(t), \overline{\gamma}^2(t,z))| \\
&\leq c_1(t)|\Delta\overline{Y}(t)| + c_2(t)\Big[|\Delta\overline{Z}(t)^*| + \Big(\int_{\mathbb{R}^m} |\Delta\overline{\gamma}(t,z)|^2 \nu(dt) \Big)^{1/2} \Big].
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\mathbb{E}\Big[|\Delta Y(t)|^2 + \int_t^T |\Delta Z(s)^*|^2 ds + \int_t^T \int_{\mathbb{R}^m} |\Delta\gamma(s,z)|^2 \nu(dz) ds \Big] \\
&\leq 2\mathbb{E}\Bigg[\int_t^T \langle |\Delta Y(s)|, c_1(s)|\Delta\overline{Y}(s)| \\
&\qquad + c_2(s)\Big[|\Delta\overline{Z}(s)^*| + \Big(\int_{\mathbb{R}^m} |\Delta\overline{\gamma}(s,z)|^2 \nu(dz) \Big)^{1/2} \Big] \rangle ds \Bigg].
\end{aligned}
$$

Now, we use the following inequalities stated in Appendices, Lemma A.3,

$$
\begin{aligned}
&2c_1(s)|\Delta Y(s)||\Delta\overline{Y}(s)| \leq 2|\Delta Y(s)|^2 + \frac{1}{2}|\Delta\overline{Y}(s)|^2, \\
&2\int_t^T c_2(s)|\Delta Y(s)||\Delta\overline{Z}(s)^*| ds \leq 2\int_t^T (c_2(s))^2 |\Delta Y(s)|^2 ds + \frac{1}{2}\int_t^T |\Delta\overline{Z}(s)^*|^2 ds, \\
&2\int_t^T c_2(s)|\Delta Y(s)| \Big(\int_{\mathbb{R}^m} |\Delta\overline{\gamma}(s,z)|^2 \nu(dz) \Big)^{\frac{1}{2}} ds \leq 2\int_t^T (c_2(s))^2 |\Delta Y(s)|^2 ds \\
&\qquad + \frac{1}{2}\int_t^T \int_{\mathbb{R}^m} |\Delta\overline{\gamma}(s,z)|^2 \nu(dz) ds.
\end{aligned}
$$

These inequalities implies that

$$
\begin{aligned}
&\mathbb{E}\Big[|\Delta Y(t)|^2+\int_t^T|\Delta Z(s)^*|^2ds+\int_t^T\int_{\mathbb{R}^m}|\Delta\gamma(s,z)|^2\nu(dz)ds\Big]\\
\le&2\mathbb{E}\Big[\int_t^T c_1(s)|\Delta Y(s)|^2ds\Big]+\frac{1}{2}\mathbb{E}\Big[\int_t^T c_1(s)|\Delta\overline{Y}(s)|^2ds\Big]\\
&+2\mathbb{E}\Big[\int_t^T\big(c_2(s)\big)^2|\Delta Y(s)|^2ds\Big]+\frac{1}{2}\mathbb{E}\Big[\int_t^T|\Delta\overline{Z}(s)^*|^2ds\Big]\\
&+2\mathbb{E}\Big[\int_t^T\big(c_2(s)\big)^2|\Delta Y(s)|^2ds\Big]+\frac{1}{2}\mathbb{E}\Big[\int_t^T\int_{\mathbb{R}^m}|\Delta\overline{\gamma}(s,z)|^2\nu(dz)ds\Big]\\
=&\frac{1}{2}\mathbb{E}\Big[\int_t^T c_1(s)|\Delta\overline{Y}(s)|^2ds+\int_t^T|\Delta\overline{Z}(s)^*|^2ds+\int_t^T\int_{\mathbb{R}^m}|\Delta\overline{\gamma}(s,z)|^2\nu(dz)ds\Big]\\
&+2\mathbb{E}\Big[\int_t^T\big(c_1(s)+2(c_2(s))^2\big)|\Delta Y(s)|^2ds\Big].
\end{aligned}
$$

Here we can use Gronwall's inequality by Lemma 4.1 with $f(t)=\mathbb{E}\Big[|\Delta Y(t)|^2+\int_t^T|\Delta Z(s)^*|^2ds+\int_t^T\int_{\mathbb{R}^m}|\Delta\gamma(s,z)|^2\nu(dz)ds\Big]$, the constant $\alpha=1/2$ and $g(t)=\mathbb{E}\Big[\int_t^T c_1(s)|\Delta\overline{Y}(s)|^2ds+\int_t^T|\Delta\overline{Z}(s)^*|^2ds+\int_t^T\int_{\mathbb{R}^m}|\Delta\overline{\gamma}(s,z)|^2\nu(dz)ds\Big]$ and $\beta(t)=2\big(c_1(t)+2(c_2(t))^2\big)$ then

$$
\begin{aligned}
&\mathbb{E}\Big[|\Delta Y(t)|^2+\int_t^T|\Delta Z(s)^*|^2ds+\int_t^T\int_{\mathbb{R}^m}|\Delta\gamma(s,z)|^2\nu(dz)ds\Big]\\
\le&\frac{1}{2}\mathbb{E}\Big[\int_t^T c_1(s)|\Delta\overline{Y}(s)|^2ds+\int_t^T|\Delta\overline{Z}(s)^*|^2ds+\int_t^T\int_{\mathbb{R}^m}|\Delta\overline{\gamma}(s,z)|^2\nu(dz)ds\Big]\\
&+\frac{1}{2}\int_t^T e^{\int_t^s 2\left(c_1(r)+2(c_2(r))^2\right)dr}2\big(c_1(s)+2(c_2(s))^2\big)\mathbb{E}\Big[\int_s^T c_1(r)|\Delta\overline{Y}(r)|^2dr\\
&+\int_s^T|\Delta\overline{Z}(r)^*|^2dr+\int_s^T\int_{\mathbb{R}^m}|\Delta\overline{\gamma}(r,z)|^2\nu(dz)dr\Big]ds.
\end{aligned}
$$

As we define before $A(t):=\int_t^T\{c_1(s)+2(c_2(s))^2\}ds$ and we take $\theta=2$, which implies

$$
\begin{aligned}
&\mathbb{E}\Big[|\Delta Y(t)|^2+\int_t^T|\Delta Z(s)^*|^2ds+\int_t^T\int_{\mathbb{R}^m}|\Delta\gamma(s,z)|^2\nu(dz)ds\Big]\\
&\le\theta^{-1}\mathbb{E}\Big[\int_t^T c_1(s)|\Delta\overline{Y}(s)|^2ds+\int_t^T|\Delta\overline{Z}(s)^*|^2ds+\int_t^T\int_{\mathbb{R}^m}|\Delta\overline{\gamma}(s,z)|^2\nu(dz)ds\Big]\\
&\quad+\int_t^T e^{\theta(A(t)-A(s))}\big(c_1(s)+2(c_2(s))^2\big)\mathbb{E}\Big[\int_s^T c_1(r)|\Delta\overline{Y}(r)|^2dr\\
&\qquad+\int_s^T|\Delta\overline{Z}(r)^*|^2dr+\int_s^T\int_{\mathbb{R}^m}|\Delta\overline{\gamma}(r,z)|^2\nu(dz)dr\Big]ds. \qquad (4.18)
\end{aligned}
$$

Our aim is to reach the norms defined at the beginning of the proof. For this reason, we multiply the inequality (4.18) by $\exp\{-\zeta A(t)\}$ where $\zeta \geq 0$ and $A(t)$ is decreasing,

$$\begin{aligned}
&e^{-\zeta A(t)}\mathbb{E}\Big[|\Delta Y(t)|^2 + \int_t^T |\Delta Z(s)^*|^2 ds + \int_t^T \int_{\mathbb{R}^m} |\Delta\gamma(s,z)|^2 \nu(dz)ds\Big] \\
&\leq \theta^{-1} e^{-\zeta A(t)}\mathbb{E}\Big[\int_t^T c_1(s)|\Delta\overline{Y}(s)|^2 ds + \int_t^T |\Delta\overline{Z}(s)^*|^2 ds \\
&\quad + \int_t^T \int_{\mathbb{R}^m} |\Delta\overline{\gamma}(s,z)|^2 \nu(dz)ds\Big] \\
&\quad + e^{-\zeta A(t)} \int_t^T e^{\theta(A(t)-A(s))}\big(c_1(s) + 2(c_2(s))^2\big)\mathbb{E}\Big[\int_s^T c_1(r)|\Delta\overline{Y}(r)|^2 dr \\
&\quad + \int_s^T |\Delta\overline{Z}(r)^*|^2 dr + \int_s^T \int_{\mathbb{R}^m} |\Delta\overline{\gamma}(r,z)|^2 \nu(dz)dr\Big]ds. \qquad (4.19)
\end{aligned}$$

As a last step to get the norms we need the followings,

$$\begin{aligned}
e^{-\zeta A(t)}\mathbb{E}\Big[\int_t^T c_1(s)|\Delta\overline{Y}(s)|^2 ds\Big] &\leq \sup_{t\in[0,T]} e^{-\zeta A(t)}\mathbb{E}\big[|\Delta\overline{Y}(t)|^2\big]\int_0^T c_1(s)ds \\
&= \sup_{t\in[0,T]} e^{-\zeta A(t)}\mathbb{E}\big[|\Delta\overline{Y}(t)|^2\big]\widetilde{k}_0, \qquad (4.20)
\end{aligned}$$

where $\widetilde{k}_0 = \int_0^T c_1(s)ds$.

$$\begin{aligned}
&e^{-\zeta A(t)} \int_t^T e^{\theta(A(t)-A(s))}\big(c_1(s) + 2(c_2(s))^2\big)\mathbb{E}\Big[\int_s^T c_1(r)|\Delta\overline{Y}(r)|^2 dr\Big]ds \\
&\leq \sup_{t\in[0,T]} e^{-\zeta A(t)}\mathbb{E}\big[|\Delta\overline{Y}(t)|^2\big]\widetilde{k}_0 \int_t^T e^{(\theta+\zeta)(A(t)-A(s))}\big(c_1(s) + 2(c_2(s))^2\big)ds \\
&= \sup_{t\in[0,T]} e^{-\zeta A(t)}\mathbb{E}\big[|\Delta\overline{Y}(t)|^2\big]\widetilde{k}_0\widetilde{k}_1, \qquad (4.21)
\end{aligned}$$

where $\widetilde{k}_1 = \int_t^T e^{(\theta+\zeta)(A(t)-A(s))}\big(c_1(s) + 2(c_2(s))^2\big)ds$.

$$\begin{aligned}
&e^{-\zeta A(t)} \int_t^T e^{\theta(A(t)-A(s))}\big(c_1(s) + 2(c_2(s))^2\big)\mathbb{E}\Big[\int_s^T |\Delta\overline{Z}(r)^*|^2 dr\Big]ds \\
&\leq \sup_{t\in[0,T]} e^{-\zeta A(t)}\mathbb{E}\Big[\int_t^T |\Delta\overline{Z}(s)^*|^2 ds\Big] \int_t^T e^{(\theta+\zeta)(A(t)-A(s))}\big(c_1(s) + 2(c_2(s))^2\big)ds \\
&= \sup_{t\in[0,T]} e^{-\zeta A(t)}\mathbb{E}\Big[\int_t^T |\Delta\overline{Z}(s)^*|^2 ds\Big]\widetilde{k}_1, \qquad (4.22)
\end{aligned}$$

$$
\begin{aligned}
&e^{-\zeta A(t)}\int_t^T e^{\theta(A(t)-A(s))}\big(c_1(s)+2(c_2(s))^2\big)\mathbb{E}\big[\int_s^T\int_{\mathbb{R}^m}|\Delta\overline{\gamma}(r,z)|^2\nu(dz)dr\big]ds\\
&\le \sup_{t\in[0,T]} e^{-\zeta A(t)}\mathbb{E}\big[\int_t^T\int_{\mathbb{R}^m}|\Delta\overline{\gamma}(s,z)|^2\nu(dz)ds\big]\int_t^T e^{(\theta+\zeta)(A(t)-A(s))}\big(c_1(s)\\
&\quad +2(c_2(s))^2\big)ds\\
&= \sup_{t\in[0,T]} e^{-\zeta A(t)}\mathbb{E}\big[\int_t^T\int_{\mathbb{R}^m}|\Delta\overline{\gamma}(s,z)|^2\nu(dz)ds\big]\widetilde{k}_1. \qquad (4.23)
\end{aligned}
$$

We take supremum of equation (4.19) after implying the inequalities (4.20), (4.21), (4.22) and (4.23);

$$
\begin{aligned}
&\sup_{t\in[0,T]} e^{-\zeta A(t)}\mathbb{E}\Big[|\Delta Y(t)|^2+\int_t^T|\Delta Z(s)^*|^2ds+\int_t^T\int_{\mathbb{R}^m}|\Delta\gamma(s,z)|^2\nu(dz)ds\Big]\\
&\le \theta^{-1}\sup_{t\in[0,T]} e^{-\zeta A(t)}\mathbb{E}\Big[|\Delta\overline{Y}(t)|^2\widetilde{k}_0+\int_t^T|\Delta\overline{Z}(s)^*|^2ds\\
&\qquad +\int_t^T\int_{\mathbb{R}^m}|\Delta\overline{\gamma}(s,z)|^2\nu(dz)ds\Big]\\
&\quad +\sup_{t\in[0,T]} e^{-\zeta A(t)}\mathbb{E}\Big[|\Delta\overline{Y}(t)|^2\widetilde{k}_0\widetilde{k}_1+\big(\int_t^T|\Delta\overline{Z}(s)^*|^2ds\big)\widetilde{k}_1\\
&\qquad +\big(\int_t^T\int_{\mathbb{R}^m}|\Delta\overline{\gamma}(s,z)|^2\nu(dz)ds\big)\widetilde{k}_1\Big].
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\|(\Delta Y(t),\Delta Z(t),\Delta\gamma(t,z))\|_S^2\\
&\le(\theta^{-1}\widetilde{k}_0+\widetilde{k}_0\widetilde{k}_1)\left\|(\Delta\overline{Y}(t),\Delta\overline{Z}(t),\Delta\overline{\gamma}(t,z))\right\|_S^2. \qquad (4.24)
\end{aligned}
$$

The Banach fixed-point theorem (in Appendices, Theorem A.5) is used for the mapping Φ. Note that, this mapping Φ is from $\mathbb{H}_T^{2,n}\times\mathbb{H}_T^{2,d\times n}\times\mathbb{F}_T^{2,n\times m}$ onto $\mathbb{H}_T^{2,n}\times\mathbb{H}_T^{2,d\times n}\times\mathbb{F}_T^{2,n\times m}$, which maps the stochastic processes $(\overline{Y},\overline{Z},\overline{\gamma})$ onto the solution (Y,Z,γ) of the BSDE i.e. $\Phi:(\overline{Y},\overline{Z},\overline{\gamma})\mapsto(Y,Z,\gamma)$ with generator f; i.e.,

$$
\begin{aligned}
Y(t)=&\xi+\int_t^T f\big(s,\overline{Y}(s),\overline{Z}(s),\overline{\gamma}(s,z)\big)ds-\int_t^T Z(s)^*dW(s)\\
&-\int_t^T\int_{\mathbb{R}^m}\gamma(s,z)\widetilde{M}(ds,dz).
\end{aligned}
$$

Choosing ζ accordingly we see this mapping Φ is a contraction from $\mathbb{H}_T^{2,n}\times\mathbb{H}_T^{2,d\times n}\times\mathbb{F}_T^{2,n\times m}$ onto itself and that there exists a fixed point, which is the unique solution of the BSDE. □

4.3 Comparison Theorem with Jumps

The comparison theorem states the natural fact that the bigger contingent claim leads to bigger option price at present time. In this section, we only state the comparison theorem and refer the reader to [27] for a deeper discussion.

Theorem 4.5. *Let ξ^1 and $\xi^2 \in \mathcal{F}_T$ and let (Y^1, Z^1, γ^1) and (Y^2, Z^2, γ^2) be the associated solutions of the BSDEs with jumps. We suppose that*

- $\xi^1 \geq \xi^2$ $\mathbb{P}$ *a.s.*
- $f^1(t, \omega, Y^2, Z^2, \gamma^2) - f^2(t, \omega, Y^2, Z^2, \gamma^2) \geq 0$ $d\mathbb{P} \otimes dt$, *a.s.*
- $|f^2(t, \omega, Y^1, Z^1, \gamma^1) - f^2(t, \omega, Y^2, Z^2, \gamma^2)| \leq k_0(|Y^1 - Y^2| + |(Z^1 - Z^2)^*|) + \int_{\mathbb{R}^m} |C(t, \omega, z)||\gamma^1(t, z) - \gamma^2(t, z)|\nu(dz)$, *where $k_0 \geq 0$ is constant and let $C(t, \omega, z)$ to satisfy the condition*

$$|C_{jt}(\omega, z)| \leq 1,$$

for $j = 1, 2, ...m$ and $C(t, z) \in \mathbb{F}_T^{2,1\times m}$.

Then for any time $t \in [0, T]$ we have $Y^1 \geq Y^2$, $\mathbb{P}$ a.s..

The proof can be found in [27].

4.4 The Model in Finance

In this section, we apply the theory explained in this chapter to finance. Firstly, we build the model for a complete market. Later, we deal with pricing problems in order to find a fair price being the unique solution of the relevant BSDE.

We have the riskless asset price

$$dP^0(t) = P^0(t)r(t)dt, \tag{4.25}$$

where $r(t)$ is the short rate. We can assume this riskless asset as a bond. In addition, n risky assets for discontinuous time where $i = 1, 2, ..., n$ has the following

$$\begin{aligned} dP^i(t) &= P^i(t^-)[b^i(t)dt + \sum_j^d \sigma^{ij}(t)dW_t^j + \sum_{j=1}^m \rho^{ij}(t)d\widetilde{N}^j(t)] \\ P^i(0) &= P^i(0), \quad i = 1, 2, ..., n \end{aligned} \tag{4.26}$$

where $W = (W^1, W^2, ..., W^d)^*$ is a d-dimensional standard Brownian motion on $\mathbb{R}^d$ and $\widetilde{N} = (\widetilde{N}^1, \widetilde{N}^2, ..., \widetilde{N}^m)^*$ is an m-dimensional centralized Poisson process on $\mathbb{R}^m$, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. $\mathbb{P}$ is said to be probability measure. Right-continuous filtration $(\mathcal{F}_t; 0 \leq t \leq T)$ gives the information structure such that quadratic variation of W and $\widetilde{N}$ is zero.

4.4.1 Hypothesis (B)

2.3.1 Hypothesis (A) in Chapter 2 is partially valid such that

- The interest (short) rate r is a predictable and bounded process. Moreover, it is usually non-negative due to the fact that the pay-off is non-negative.
- The stock appreciation rate (drift term) $b = (b^1, b^2, ..., b^n)^*$ is a column vector of predictable and bounded processes.
- The volatility $\sigma = (\sigma^{i,j})$ is a $n \times d$ matrix of predictable and bounded processes. σ_t does not need to have full rank a.s. for all $t \in [0, T]$.
- There exist u and θ constructing risk premium such that they are predictable and bounded processes and

$$b(t) - r(t)\mathbf{1} = \sigma(t)u(t) + \int_{\mathbb{R}^m} \rho(t)\theta(t,z)\nu(dz), \quad d\mathbb{P} \otimes dt \text{ a.s.},$$

  where $\mathbf{1}$ denotes the vector with all components being 1.

Additionally, we have the following assumptions

- The jump size process $\rho = (\rho^{ik}) \in \mathbb{R}^{d \times m}$ is a matrix of predictable and bounded processes.
- $\widetilde{N}(t) = \widetilde{M}((0,T], \mathbb{R}^m) = M((0,T], \mathbb{R}^m) - t\nu(\mathbb{R}^m)$ where $\widetilde{M}((0,T], \mathbb{R}^m)$ is the counting measure generated by m-dimensional centralized Poisson point process and for simplicity we take $\nu(\mathbb{R}^m) = 1$. Poisson point process has the following properties.
  1. Independent increment property that is, if $0 < t_1 < t_2 < ... < t_n$ then all increments $\widetilde{N}(t_i) - \widetilde{N}(t_{i-1})$ are independent for $i = 1, 2, ..., n$.
  2. Stationary increment property that is, the distribution of $\widetilde{N}(t+h) - \widetilde{N}(t)$ does not depend on t.
  3. It is impossible to occur two or more jumps at the same time that is, $\mathbb{P}\{\widetilde{N}(t+h) - \widetilde{N}(t) \geq 2\} = 0$ a.s.
  4. $\mathbb{E}[\widetilde{N}^i = t]$, for $i = 1, 2, ...m$.

Under these assumptions the market is dynamically complete if the number of risky assets equals to number of source of randomness i.e. $n = d + m$. The completeness is motivated by [5]. In this section, the case of constraints on the portfolio is not dealt with. Let us start with the recall of the following definition:

Definition 4.2. A strategy (V, π) is called *self-financing* if the wealth process $V(t) = \sum_{i=0}^{n} \pi^i(t)$ satisfies the following equality

$$V(t) = V(0) + \int_0^t \sum_{i=0}^{n} \pi^i(t) \frac{dP^i(t)}{P^i(t)}, \tag{4.27}$$

where $\pi^i(t)$ is the amount of the wealth $V(t)$ to invest in the i-th risky asset where $i = 1, 2, ...n$ at the time $t \in [0, T]$. $\pi^0(t) = V(t) - \sum_{i=1}^n \pi^i(t)$ defines the amount of wealth of the enterpriser to invest in the riskless asset.

Proposition 4.6. *A strategy is self-financing if the wealth process satisfies the linear stochastic differential equation (LSDE)*

$$dV(t) = r(t)V(t)dt + \pi(t)^*(b(t) - r(t)\mathbf{1})dt + \pi(t)^*\sigma(t)dW(t) + \pi(t)^*\rho(t)d\widetilde{N}(t). \tag{4.28}$$

Proof. Get $\frac{dP^0(t)}{P^0(t)}$ from (4.25) and $\frac{dP^i(t)}{P^i(t)}$ from (4.26) then replace in self-financing wealth process (4.28) and differentiate, which is done for n=2 in the following example. □

Here see that $d\widetilde{N}(t) = d\widetilde{M}((0, T], \mathbb{R}^m) = dM((0, T], \mathbb{R}^m) - dt\ \nu(\mathbb{R}^m)$ is a martingale with the intensity function $\mathbb{E}[dM((0, T], \mathbb{R}^m)] = dt\ \nu(\mathbb{R}^m) = dt$, $\widetilde{M}(dt, dz) + \theta(t, z)\nu(dz)dt$ is a Poisson measure and $dW(t) + u(t)dt$ is a Brownian motion under risk neutral probability measure $\mathbb{Q}$.

Example 4.1. Take the number of risky assets is as $n = 2$,

$$\begin{aligned} \frac{dP^0(t)}{P^0(t)} &= r(t)dt, \\ \frac{dP^1(t)}{P^1(t^-)} &= [b^1(t)dt + \sigma^1(t)dW(t) + \rho^1(t)d\widetilde{N}(t)], \\ \frac{dP^2(t)}{P^2(t^-)} &= [b^2(t)dt + \sigma^2(t)dW(t) + \rho^2(t)d\widetilde{N}(t)]. \end{aligned}$$

The self-financing strategy satisfies

$$V(t) = V(0) + \int_0^t \pi^0(s)\frac{dP^0(s)}{P^0(s)} + \int_0^t \pi^1(s)\frac{dP^1(s)}{P^1(s)} + \int_0^t \pi^2(s)\frac{dP^2(s)}{P^2(s)},$$

by replacing $\frac{dP^i(t)}{P^i(t)}$ for i=0,1,2 we get

$$\begin{aligned} V(t) =& V(0) + \int_0^t \pi^0(s)r(s)dt + \int_0^t \pi^1(s)[b^1(s)ds + \sigma^1(s)dW(s) + \rho^1(s)d\widetilde{N}(s)] \\ &+ \int_0^t \pi^2(s)[b^2(s)ds + \sigma^2(s)dW(s) + \rho^2(s)d\widetilde{N}(s)]. \end{aligned}$$

The derivation yields,

$$\begin{aligned} dV(t) &= \pi^0(t)r(t)dt + \pi^1(t)[b^1(t)dt + \sigma^1(t)dW(t) + \rho^1(t)d\widetilde{N}(t)] \\ &\quad + \pi^2(t)[b^2(t)dt + \sigma^2(t)dW(t) + \rho^2(t)d\widetilde{N}(t)] \\ &= \pi^0(t)r(t)dt + \sum_{i=1}^{2} \pi^i(t)[b^i(t)dt + \sigma^i(t)dW(t) + \rho^i(t)d\widetilde{N}(t)], \end{aligned}$$

by replacing $\pi^0(t) = V(t) - \sum_{i=1}^{2} \pi^i(t)$,

$$\begin{aligned}
dV(t) &= (V(t) - \sum_{i=1}^{2} \pi^i(t))r(t)dt + \sum_{i=1}^{2} \pi^i(t)[b^i(t)dt + \sigma^i(t)dW(t) + \rho^i(t)d\widetilde{N}(t)] \\
&= (r(t)V(t) - \sum_{i=1}^{2} \pi^i(t)r(t) + \sum_{i=1}^{2} \pi^i(t)b^i(t))dt + \sum_{i=1}^{2} \pi^i(t)[\sigma^i(t)dW(t) \\
&\quad + \rho^i(t)d\widetilde{N}(t)] \\
&= r(t)V(t)dt + \sum_{i=1}^{2} \pi^i(t)(b^i(t) - r(t))dt + \sum_{i=1}^{2} \pi^i(t)[b^i(t)dt + \sigma^i(t)dW(t) \\
&\quad + \rho^i(t)d\widetilde{N}(t)].
\end{aligned}$$

In matrix notation,

$$\begin{aligned}
dV(t) = r(t)V(t)dt + \begin{bmatrix}\pi^1(t) & \pi^2(t)\end{bmatrix} \begin{bmatrix} b^1(t) - r(t) \\ b^2(t) - r(t) \end{bmatrix} dt \\
+ \begin{bmatrix}\pi^1_t & \pi^2_t\end{bmatrix} \begin{bmatrix} \sigma^1(t) \\ \sigma^2(t) \end{bmatrix} dW(t) + \begin{bmatrix}\pi^1_t & \pi^2_t\end{bmatrix} \begin{bmatrix} \rho^1(t) \\ \rho^2(t) \end{bmatrix} d\widetilde{N}(t),
\end{aligned}$$

which is

$$\begin{aligned}
dV(t) =& r(t)V(t)dt + \pi(t)^*(b(t) - r(t)\mathbf{1})dt + \pi(t)^*\sigma(t)dW(t) + \pi(t)^*\rho(t)d\widetilde{(t)} \\
=& r(t)V(t)dt + \pi(t)^*\sigma(t)\big[dW(t) + u(t)\big]dt \\
&+ \pi(t)^*\rho(t)\big[d\widetilde{N}(t) + \int_{\mathbb{R}^m} \theta(t,z)\nu(dz)dt\big].
\end{aligned}$$

Definition 4.3. A strategy (V, π) is called *trading strategy* if the wealth process $V_t = \sum_{i=0}^{n} \pi^i_t$ and portfolio process $\pi_t = (\pi^1_t, \pi^2_y, ..., \pi^n_t)^*$ satisfies the following

$$\begin{aligned}
dV(t) =& r(t)V(t)dt + \pi(t)^*\sigma(t)\big[dW(t) + u(t)\big]dt \\
&+ \pi(t)^*\rho(t)\big[d\widetilde{N}(t) + \int_{\mathbb{R}^m} \theta(t,z)\nu(dz)dt\big].
\end{aligned}$$

where $\int_0^T |\sigma(t)^*\pi(t)|^2 dt < +\infty$, $\int_0^T |\sigma(t)^*\rho(t)|^2 dt < +\infty$, $\mathbb{P}$ a.s.

Definition 4.4. A strategy is said to be *feasible* if the wealth process is nonnegative

$$V(t) \geq 0 \quad t \in [0,T],\ \mathbb{P} \text{ a.s.}$$

4.4.2 Pricing and Hedging for the Models with Jumps

The practical financial experience shows us the solution of BSDEs help to hedging and option pricing. Let us introduce a fair price as first.

Definition 4.5. Let $\xi \geq 0$ be a positive contingent claim,

1. A feasible self-financing strategy (V, π) is said to be a *hedging strategy* against ξ if $V(T) = \xi$.

2. ξ is said to be *hedgeable* if the class of hedging strategies $\mathcal{H}(\xi)$ is nonempty.

3. X_0 being the smallest initial endowment needed to hedge ξ is said to be the *fair price* at time 0 of hedgeable claim ξ if,

$$X_0 = \inf\{x \geq 0; \exists (V, \pi) \in \mathcal{H}(\xi) \textit{ such that } V_0 = x\},$$

If 4.4.1 Hypothesis (B) is assumed, then for the square-integrable claim $\xi \geq 0$, $\mathcal{H}(\xi)$ has an element making it nonempty and the market is said to be *complete market*. It means that every contingent claim in the complete financial market can be hedged [9]. Moreover, the fair price is the market value of a hedging strategy in $\mathcal{H}(\xi)$ [13] which is proved in the following theorem.

Theorem 4.7. *We suppose that 4.4.1 Hypothesis (B) is satisfied and $\xi \geq 0$ is a square-integrable contingent claim. Then there exists a hedging strategy against (X, π) against ξ such that*

$$\begin{aligned} dX(t) =& r(t)X(t)dt + \pi(t)^*\sigma(t)(b(t) - r(t)\mathbf{1})dt + \pi(t)^*\sigma(t)dW(t) \\ &+ \int_{\mathbb{R}^m} \pi(t)^*\rho(t,z)\widetilde{M}(dt,dz), \\ X_T =& \xi, \end{aligned} \tag{4.29}$$

and the market value X is the fair price of the claim.

Moreover, if we assume that $(H^t(s); s \geq t)$ is the deflator started at time t; that is,

$$dH^t(s) = -H^t(s^-)[r(s)ds + u(s)^*dW(s) + \int_{\mathbb{R}^m} \theta(s,z)\widetilde{M}(dt,dz)], \quad H^t(t) = 1. \tag{4.30}$$

Then

$$X(t) = \mathbb{E}[H^t(T)\xi|\mathcal{F}_t], \ \textit{a.s.} \tag{4.31}$$

Proof. Let us find the solution of the SDE (4.30) of deflator H by applying Itô formula to $f(x)=\log(x)$ where $s\geq t$, then

$$\begin{aligned}
\log H^t(s) =& \log H^t(t) + \int_t^s \frac{1}{H^t(s)} H^t(s)^c - \frac{1}{2}\int_t^s \Big(\frac{1}{H^t(s)}\Big)^2 d\langle H^t(s)^c, H^t s^c\rangle, \\
&+ \int_t^s \int_{\mathbb{R}^m} \Big[\log\big(H^t(s) - H^t(s)\theta(s,z)\big) - \log(H^t(s)) \\
&\quad + \frac{1}{H^t(s)} H^t(s)\theta(s,z)\Big]\nu(dz)ds \\
&+ \int_t^s \int_{\mathbb{R}^m} \Big[\log\big(H^t(s^-) - H^t(s^-)\theta(s,z)\big) - \log(H^t(s^-))\Big]\widetilde{M}(ds,dz) \\
=& \log 1 + \int_t^s \frac{1}{H^t_s}\big[-H^t(s)[r(s)ds + u(s)^* dW(s)]\big] - \frac{1}{2}\int_t^s |u(s)^*|^2 ds, \\
&+ \int_t^s \int_{\mathbb{R}^m} \Big[\log\Big(\frac{H^t(s)(1-\theta(s,z))}{H^t(s)}\Big) + \theta(s,z)\Big]\nu(dz)ds \\
&+ \int_t^s \int_{\mathbb{R}^m} \log\Big(\frac{H^t(s^-)(1-\theta(s,z))}{H^t(s^-)}\Big)\widetilde{M}(ds,dz) \\
=& -\Big(\int_t^s r(s)ds + \int_t^s u(s)^* dW(s) + \frac{1}{2}\int_t^s |u(s)^*|^2 ds \\
&- \int_t^s \int_{\mathbb{R}^m} [\log(1-\theta(s,z)) + \theta(s,z)]\nu(dz)ds \\
&- \int_t^s \int_{\mathbb{R}^m} \log(1-\theta(s,z))\widetilde{M}(ds,dz)\Big),
\end{aligned}$$

then the solution is started at time t

$$\begin{aligned}
H^t(s) = \exp\Big\{ &- \int_t^s r(s)ds - \int_t^s u(s)^* dW(s) - \frac{1}{2}\int_t^s |u(s)^*|^2 ds \\
&+ \int_t^s \int_{\mathbb{R}^m} [\log(1-\theta(s,z)) + \theta(s,z)]\nu(dz)ds \\
&+ \int_t^s \int_{\mathbb{R}^m} \log(1-\theta(s,z))\widetilde{M}(ds,dz)\Big\}.
\end{aligned}$$

Here, the solution started at time 0 can be shown as follows

$$\begin{aligned}
H(t) := H^0(t) = \exp\Big\{ &- \int_0^t r(s)ds - \int_0^t u(s)^* dW(s) - \frac{1}{2}\int_0^t |u(s)^*|^2 ds \\
&+ \int_0^t \int_{\mathbb{R}^m} [\log(1-\theta(s,z)) + \theta(s,z)]\nu(dz)ds \\
&+ \int_0^t \int_{\mathbb{R}^m} \log(1-\theta(s,z))\widetilde{M}(ds,dz)\Big\}.
\end{aligned}$$

Since r, u and θ are bounded processes, it follows Novikov's condition (see Appendices, Theorem A.9) that $\mathbb{E}[(H(T))^2] < +\infty$ and $\mathbb{E}[H(T)\xi] < +\infty$ for any square-integrable contingent claim.

$$
\begin{aligned}
d(H(t)X(t)) =& X(t^-)dH(t) + H(t^-)dX(t) + d\langle X, H\rangle(t) \\
=& - X(t^-)H(t^-)[r(t)dt + u(t)^* dW_t + \int_{\mathbb{R}^m} \theta(s,z)\widetilde{M}(dt,dz)] \\
&+ H(t^-)[r(t)X(t)dt + \pi(t)^*(b(t) - r(t)\mathbf{1})dt + \pi(t)^*\sigma(t)dW(t) \\
&+ \int_{\mathbb{R}^m} \pi(t)^*\rho(t,z)\widetilde{M}(dt,dz)] \\
&- H(t)\pi(t)^*\sigma(t)u(t)dt + \int_{\mathbb{R}^m} -H(t)\pi(t)^*\rho(t,z)\theta(t,z)\nu(dz)dt \\
&- \int_{\mathbb{R}^m} H(t^-)\pi(t)^*\rho(t,z)\theta(t,z)\widetilde{M}(dt,dz) \\
=& - H(t)X(t)u(t)^* dW(t) - H(t^-)X(t^-)\int_{\mathbb{R}^m} \theta(t,z)\widetilde{M}(dt,dz) \\
&+ H(t)\pi(t)^*\sigma(t)dW(t) + \int_{\mathbb{R}^m} \pi(t)^*\rho(t)\widetilde{M}(dt,dz) \\
&- \int_{\mathbb{R}^m} H(t^-)\pi(t)^*\rho(t,z)\theta(t,z)\widetilde{M}(dt,dz) \\
&+ H(t)\pi(t)\Big(b(t) - r(t)\mathbf{1} - \sigma(t)u(t) - \int_{\mathbb{R}^m} \rho(t)\theta(t,z)\nu(dz)\Big)dt.
\end{aligned}
$$

The given process $u(t)$ and $\theta(t,z) \leq 1$ in the Theorem A.9 (see Appendices) should satisfy that

$$
b(t) - r(t)\mathbf{1} = \sigma(t)u(t) + \int_{\mathbb{R}^m} \rho(t)\theta(t,z)\nu(dz).
$$

Then the Theorem A.9 can be applied with well defined $Z(t)$ and the equivalent measure on $\mathcal{F}_T$ such that

$$
\frac{d\mathbb{Q}}{d\mathbb{P}} = Z(T),
$$

with the Poisson measure $\widetilde{M}_{\mathbb{Q}}(dt,dz) = \widetilde{M}(dt,dz) + \theta(t,z)\nu(dz)dt$ and the Brownian motion $dW_{\mathbb{Q}}(t) = dW(t) + u(t)dt$ under probability measure $\mathbb{Q}$ [22]. Hence,

$$
\begin{aligned}
d(H(t)X(t)) =& - H(t)X(t)u(t)^* dW(t) - H(t^-)X(t^-)\int_{\mathbb{R}^m} \theta(t,z)\widetilde{M}(dt,dz) \\
&+ H(t)\pi(t)^*\sigma(t)dW(t) + \int_{\mathbb{R}^m} \pi(t)^*\rho(t)\widetilde{M}(dt,dz) \\
&- \int_{\mathbb{R}^m} H(t^-)\pi(t)^*\rho(t,z)\theta(t,z)\widetilde{M}(dt,dz)
\end{aligned}
$$

We realize that there is no drift term in the above equality which proves it is a martingale. After defining the continuous adapted process X from;

$$
H(t)X(t) = \mathbb{E}[H(T)\xi|\mathcal{F}_t]
$$

(X, π) satisfying the linear BSDE (4.29) can be found by Itô's lemma. Since the processes X and H are continuous, u and θ are bounded, it can be shown that $\int_0^T |\sigma_t^* \pi_t|^2 dt < +\infty$ a.s. So (X, π) is a hedging strategy against ξ with $X_0 = \mathbb{E}(H_T \xi)$. $\square$

This theorem is a result of existence and uniqueness of BSDE's and self-financing markets. Moreover, the existence of such $X(t)$ helps to price the contingent claim at time t.

Remark 4.1. The fair price of the claim ξ has the property in equation (4.31). The fair price is calculated as the expectation of the discounted asset prices such that

$$X(t) = \mathbb{E}_{\mathbb{Q}}[e^{-\int_t^T r(s)ds} \xi | \mathcal{F}],$$

under the risk neutral probability measure $\mathbb{Q}$ where the Radon-Nikodym derivative with respect to $\mathbb{P}$ on $\mathcal{F}_T$ is given by

$$\begin{aligned} \frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\Big\{ & -\int_0^T u(t)^* dW(t) - \frac{1}{2}\int_0^T |u(t)^*|^2 dt \\ & - \int_0^T \int_{\mathbb{R}^m} [\log(1-\theta(t,z)) + \theta(t,z)] \nu(dz) dt \\ & - \int_0^T \int_{\mathbb{R}^m} \log(1-\theta(t,z)) \widetilde{M}(dt,dz) \Big\}. \end{aligned} \tag{4.32}$$

We realize that $\mathbb{Q}$ is well defined as a probability measure. Moreover, $\mathbb{Q}$ is a martingale measure; that is, the discounted wealth processes are $\mathbb{Q}$-local martingales.

CHAPTER 5

BSDE APPLICATION TO PDIE

In this chapter, the forward-backward stochastic differential equations (FBSDEs) is considered. Later, the generalization of Feynman-Kac formula with jumps is presented. For a system of FBSDE in low dimensions we can get at least a numerical solution (to control problems) with partial differential integral equation (PDIE) techniques. However, when the dimension increases, the numerical schemes for PDIE becomes more inefficient (from the computational point of view). The analogue of Feynman-Kac formula provides more accurate framework for numerical applications by using probabilistic approach in the simulations. There are many articles concerned with BSDEs and Feynman-Kac formula (for instance [19], [3] and [11]) but they complete the market by using orthonormalized compensated power-jump processes (called *Teugels martingales*). The main difference of this thesis is proving a generalized Feynman-Kac formula which is constructed in the unified frame of this thesis.

5.1 Forward-Backward Stochastic Differential Equations with Jumps

5.1.1 The Model for FBSDE

Let $(\Omega, \mathcal{F}, \mathbb{Q})$ be a probability space with $\mathbb{R}^d$-valued Brownian motion W and $\mathbb{R}^m$-valued compensated Poisson measure $\widetilde{M}$. For any given $(t, x) \in [0, T] \times \mathbb{R}^p$, the forward stochastic price process on $[0, T]$ is;

$$\begin{aligned} dP(s) &= b(s, P(s))ds + \sigma(s, P(s))dW(s) + \int_{\mathbb{R}^m} \rho(s, P(s^-), z)\widetilde{M}(ds, dz), \\ &\qquad\qquad\qquad\qquad\qquad\qquad t \leq s \leq T \\ P(s) &= x, 0 \leq s \leq t. \end{aligned} \tag{5.1}$$

The solution of SDE (5.1) will be denoted $\{P^{t,x}(s),\ 0 \leq s \leq T\}$. Then the associated BSDE is

$$\begin{aligned} -\,dY(s) &= f(s, P^{t,x}(s), Y(s), Z(s), \gamma(s,z))ds - Z(s)^* dW(s) \\ &\quad - \int_{\mathbb{R}^m} \gamma(s,z)\widetilde{M}(ds,dz), \quad t \leq s \leq T \\ Y(T) &= \psi(P^{t,x}(T)). \end{aligned} \tag{5.2}$$

The solution of the BSDE (5.2) is denoted $\big\{(Y^{t,x}(s), Z^{t,x}(s), \gamma^{t,x}(s)),\ 0 \leq s \leq T\big\}$. The combined equations (5.1) and (5.2) helps to construct FBSDEs with jumps. The solution of that system is shown as $\big\{(P^{t,x}(s), Y^{t,x}(s), Z^{t,x}(s), \gamma^{t,x}(s)),\ 0 \leq s \leq T\big\}$.

The existence and uniqueness of FBSDE is a result of the existence and uniqueness of underlying SDE and BSDE. Thus, we have the following assumptions:

1. Let $T > 0$, $b(\cdot,\cdot) : [0,T] \times \mathbb{R}^p \mapsto \mathbb{R}^p$, $\sigma(\cdot,\cdot) : [0,T] \times \mathbb{R}^p \mapsto \mathbb{R}^{p\times d}$, $\rho(\cdot,\cdot,\cdot) : [0,T] \times \mathbb{R}^p \times \mathbb{R}^m \mapsto \mathbb{R}^{p\times m}$ be $\mathcal{F}_t$-adapted and jointly measurable, furthermore ρ is $\mathcal{F}_t$-predictable such that $\mathbb{Q}$ a.s.

   - $|b(t,X)| \leq c_0(t)(1 + |x|)$,
   - $|\sigma(t,X)|^2 + \int_{\mathbb{R}^m} |\rho(t,X,z)|^2 \nu(dz) \leq c_0(t)(1 + |x|^2)$,
   - $|b(t,X) - b(t,Y)| \leq c_0(t)|X - Y|$,
   - $|\sigma(t,X) - \sigma(t,Y)|^2 + \int_{\mathbb{R}^m} |\rho(t,X,z) - \rho(t,Y,z)|^2 \nu(dz) \leq c_0(t)|X - Y|^2$ where there exists a non-negative deterministic function $c_0(t) \geq 0$ such that $\int_0^T c_0(t)dt < +\infty$ and $\mathbb{E}|x|^2 < \infty$.

2. Let $T > 0$, $f(\cdot,\cdot,\cdot,\cdot,\cdot) : [0,T] \times \mathbb{R}^p \times \mathbb{R}^n \times \mathbb{R}^{d\times n} \times \mathbb{F}_T^{2,n\times m} \mapsto \mathbb{R}^n$, $\psi(\cdot) : \mathbb{R}^p \mapsto \mathbb{R}^d$ be $\mathcal{F}_t$-adapted and measurable such that $\mathbb{Q}$ a.s.

   - $|f(t,Y^1,Z^1,\gamma^1) - f(t,Y^2,Z^2,\gamma^2)| \leq c_1(t)|Y^1 - Y^2| + c_2(t)\Bigg[|(Z^1 - Z^2)^*| + \Big(\int_{\mathbb{R}^m} |\gamma^1 - \gamma^2|^2 \nu(dz)\Big)^{1/2}\Bigg]$,
   - $|f(t,Y,Z,\gamma|) \leq c_1(t)|Y^1 - Y^2| + c_2(t)\Bigg[1 + |Z^*| + \Big(\int_{\mathbb{R}}^m |\gamma|^2 \nu(dz)\Big)^{1/2}\Bigg]$ where there exist two deterministic functions $c_1(t)$, $c_2(t) \geq 0$ such that $\int_0^T c_1(t)dt + \int_0^T (c_2(t))^2 dt < +\infty$.

In order to be in the safe side all the existence and uniqueness assumptions is considered for both SDE and BSDE. The assumptions may be weakened in the further studies but the result is more important for now. Moreover, the researches are interested in weakening the assumptions can found more in [27].

5.2 Feynman-Kac Formula with Jumps

In this section the relationship between these FBSDEs with jumps and PDIEs is studied by generalized Feynman-Kac Formula.

Theorem 5.1. *Let v be a $\mathcal{C}^{1,2}$ function defined on $[0,T]\times\mathbb{R}^n$. If $\forall(t,x)\in[0,T]\times\mathbb{R}^n$, v satisfies*

$$v(T,x)=\psi(x),\quad \forall x\in\mathbb{R}^n,$$

$$\begin{aligned}\partial_t v(t,x) &+ \mathcal{L}v(t,x)+\int_{\mathbb{R}^m}\Big\{v(t,x+\rho(t,x,z))-v(t,x)\\ &\quad -[\partial_x v(t,x)]^*\rho(t,x,z)\Big\}\nu(dz)\\ &+ f(t,x,v(t,x),\sigma(t,x)^*\partial_x v(t,x),[\partial_x v(t,x)]^*\rho(t,x,z))=0, \quad (5.3)\end{aligned}$$

where $\partial_x v(t,x)$ is the gradient of v and $\mathcal{L}_{(t,x)}$ is the infinitesimal generator such that

$$\mathcal{L}_{(t,x)}=\sum_{i,j}a_{ij}(t,x)\partial^2_{x_ix_j}+\sum_i b_i(t,x)\partial_{x_i},\quad a_{ij}=\frac{1}{2}[\sigma\sigma^*]_{ij},$$

also there exists non-negative and non-random c_0, c_1 and c_2 satisfying the above existence and uniqueness assumptions (1. and 2.) of FBSDE for each (s,x), then $(Y^{t,x}(s),Z^{t,x}(s),\gamma^{t,x}(s,z))$ is the unique solution of BSDE (5.2) where $t\le s\le T$ and

$$\begin{aligned}Y^{t,x}(s)=v(s,P^{t,x}(s)),\quad Z^{t,x}(s)&=\sigma(s,P^{t,x}(s))^*\partial_x v(s,P^{t,x}(s))\\ \gamma^{t,x}(s,z)&=\big[\partial_x v(s,P^{t,x}(s^-))\big]^*\rho(s,P^{t,x}(s^-),z)\end{aligned}\quad (5.4)$$

$\mathbb{Q}$ *a.s.*

Conversely, if the unique solution of FBSDE is $(Y^{t,x}(s),Z^{t,x}(s),\gamma^{t,x}(s,z)) = \Big(v(s,P^{t,x}(s)),\quad \sigma(s,P^{t,x}(s))^*\partial_x v(s,P^{t,x}(s)),\quad \big[\partial_x v(s,P^{t,x}(s^-))\big]^*\rho(s,P^{t,x}(s^-),z)\Big)$ where $t\le s\le T$, in addition if the discounted asset prices are given as martingales then the associated PDIE can be found as the equation (5.3).

Proof. Assume that $P(s)$ is n-dimensional price process, $\partial_t v(s,P^{t,x}(s))$ is the partial derivative with respect to time variable, $\partial_x v(s,P^{t,x}(s))$ is the gradient of v and $\partial^2_x v(s,P^{t,x}(s))$ is $(d\times d)$ Hessian matrix of v with respect to $P(s)$. Then apply Itô formula (Appendices, Theorem A.10) to $v(s, P^{t,x}(s))$,

$$
\begin{aligned}
dv(s,P^{t,x}(s)) &= \partial_t v(s,P^{t,x}(s))ds + [\partial_x v(s,P^{t,x}(s))]^* dP^{t,x}(s)^c \\
&+ \frac{1}{2}[dP^{t,x}(s)^c]^* \partial_x^2 v(s,P^{t,x}(s)) dP^{t,x}(s)^c \\
&+ \int_{\mathbb{R}^m} \Big\{ v(s,P^{t,x}(s)+\rho(s,P^{t,x}(s),z)) - v(s,P^{t,x}(s)) \\
&\qquad - \big[\partial_x v(s,P^{t,x}(s))\big]^* \rho(s,P^{t,x}(s),z) \Big\} \nu(dz)ds \\
&+ \int_{\mathbb{R}^m} \Big\{ v(s,P^{t,x}(s^-)+\rho(s,P^{t,x}(s^-),z)) - v(s,P^{t,x}(s^-)) \Big\} \widetilde{M}(ds,dz).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
dv(s,P^{t,x}(s)) &= \partial_t v(s,P^{t,x}(s))ds + [\partial_x v(s,P^{t,x}(s))]^* \big[b(s,P^{t,x}(s))ds \\
&+ \sigma(s,P^{t,x}(s))dW(s)\big] \\
&+ \frac{1}{2}\big[b(s,P^{t,x}(s))ds + \sigma(s,P^{t,x}(s))dW(s)\big]^* \partial_x^2 v(s,P^{t,x}(s)) \big[b(s,P^{t,x}(s))ds \\
&+ \sigma(s,P^{t,x}(s))dW(s)\big] \\
&+ \int_{\mathbb{R}^m} \Big\{ v(s,P^{t,x}(s)+\rho(s,P^{t,x}(s),z)) - v(s,P^{t,x}(s)) \\
&- \big[\partial_x v(s,P^{t,x}(s))\big]^* \rho(s,P^{t,x}(s),z) \Big\} \nu(dz)ds \\
&+ \int_{\mathbb{R}^m} \Big\{ v(s,P^{t,x}(s^-)+\rho(s,P^{t,x}(s^-),z)) - v(s,P^{t,x}(s^-)) \Big\} \widetilde{M}(ds,dz),
\end{aligned}
$$

then

$$
\begin{aligned}
dv(s,P^{t,x}(s)) &= \partial_t v(s,P^{t,x}(s))ds + [\partial_x v(s,P^{t,x}(s))]^* b(s,P^{t,x}(s))ds \\
&\quad + [\partial_x v(s,P^{t,x}(s))]^* \sigma(s,P^{t,x}(s))dW(s) \\
&+ \frac{1}{2}\big[b(s,P^{t,x}(s))ds + \sigma(s,P^{t,x}(s))dW(s)\big]^* \partial_x^2 v(s,P^{t,x}(s)) \big[b(s,P^{t,x}(s))ds \\
&\quad + \sigma(s,P^{t,x}(s))dW(s) \\
&+ \int_{\mathbb{R}^m} \Big\{ v(s,P^{t,x}(s)+\rho(s,P^{t,x}(s),z)) - v(s,P^{t,x}(s)) \\
&\quad - \big[\partial_x v(s,P^{t,x}(s))\big]^* \rho(s,P^{t,x}(s),z) \Big\} \nu(dz)ds \\
&+ \int_{\mathbb{R}^m} \Big\{ v(s,P^{t,x}(s^-)+\rho(s,P^{t,x}(s^-),z)) - v(s,P^{t,x}(s^-)) \Big\} \widetilde{M}(ds,dz).
\end{aligned}
$$

Note $W_i \perp W_j$ when $i \neq j$,

$$\begin{aligned}
dv(s,P^{t,x}(s)) &= \partial_t v(s,P^{t,x}(s))ds + [\partial_x v(s,P^{t,x}(s))]^* b(s,P^{t,x}(s))ds \\
&+ [\partial_x v(s,P^{t,x}(s))]^* \sigma(s,P^{t,x}(s))dW(s) \\
&+ \frac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{d}[(\sigma\sigma^*)_{ij}(s,P^{t,x}(s))]\frac{\partial^2 v(s,P^{t,x}(s))}{\partial x_i \partial x_j}ds \\
&+ \int_{\mathbb{R}^m}\Big\{v(s,P^{t,x}(s)+\rho(s,P^{t,x}(s),z)) - v(s,P^{t,x}(s)) \\
&- \big[\partial_x v(s,P^{t,x}(s))\big]^* \rho(s,P^{t,x}(s),z)\Big\}\nu(dz)ds, \\
&+ \int_{\mathbb{R}^m}\Big\{v(s,P^{t,x}(s^-)+\rho(s,P^{t,x}(s^-),z)) - v(s,P^{t,x}(s^-))\Big\}\widetilde{M}(ds,dz),
\end{aligned}$$

by the necessary arrangements,

$$\begin{aligned}
dv(s,P^{t,x}(s)) &= \partial_t v(s,P^{t,x}(s))ds + [\partial_x v(s,P^{t,x}(s))]^* b(s,P^{t,x}(s))ds \\
&+ [\partial_x v(s,P^{t,x}(s))]^* \sigma(s,P^{t,x}(s))dW(s) \\
&+ \sum_{i,j=1}^{d}\frac{1}{2}[(\sigma\sigma^*)_{ij}(s,P^{t,x}(s))]\frac{\partial^2 v(s,P^{t,x}(s))}{\partial x_i \partial x_j}ds \\
&+ \int_{\mathbb{R}^m}\Big\{v(s,P^{t,x}(s)+\rho(s,P^{t,x}(s),z)) - v(s,P^{t,x}(s)) \\
&- \big[\partial_x v(s,P^{t,x}(s))\big]^* \rho(s,P^{t,x}(s),z)\Big\}\nu(dz)ds, \\
&+ \int_{\mathbb{R}^m}\Big\{v(s,P^{t,x}(s^-)+\rho(s,P^{t,x}(s^-),z)) - v(s,P^{t,x}(s^-))\Big\}\widetilde{M}(ds,dz)
\end{aligned}$$

which is

$$\begin{aligned}
dv(s,P^{t,x}(s)) = \Big[&\partial_t v(s,P^{t,x}(s)) + \mathcal{L}_(t,x)v(s,P^{t,x}(s)) \\
&+ \int_{\mathbb{R}^m}\Big\{v(s,P^{t,x}(s)+\rho(s,P^{t,x}(s),z)) - v(s,P^{t,x}(s)) \\
&- \big[\partial_x v(s,P^{t,x}(s))\big]^* \rho(s,P^{t,x}(s),z)\Big\}\nu(dz)\Big]ds \\
&+ [\partial_x v(s,P^{t,x}(s))]^* \sigma(s,P^{t,x}(s))dW(s) \\
&+ \int_{\mathbb{R}^m}\Big\{v(s,P^{t,x}(s^-)+\rho(s,P^{t,x}(s^-),z)) - v(s,P^{t,x}(s^-))\Big\}\widetilde{M}(ds,dz). \qquad (5.5)
\end{aligned}$$

The followings are defined:

$$\begin{aligned}
r(s,P^{t,x}(s))v(s,P^{t,x}(s)) &:= \partial_t v(s,P^{t,x}(s)) + \mathcal{L}_(t,x)v(s,P^{t,x}(s)) \\
&+ \int_{\mathbb{R}^m}\Big\{v(s,P^{t,x}(s)+\rho(s,P^{t,x}(s),z)) - v(s,P^{t,x}(s)) \\
&- \big[\partial_x v(s,P^{t,x}(s))\big]^* \rho(s,P^{t,x}(s),z)\Big\}\nu(dz),
\end{aligned}$$

$$
\begin{aligned}
dL(s) :=& [\partial_x v(s, P^{t,x}(s))]^* \sigma(s, P^{t,x}(s)) dW(s) + \int_{\mathbb{R}^m} \Big\{ v(s, P^{t,x}(s^-) \\
&+ \rho(s, P^{t,x}(s^-), z)) - v(s, P^{t,x}(s^-)), \Big\} \widetilde{M}(ds, dz) \\
K(s) :=& e^{\int_s^T r(m, P^{t,x}(m)) dm} v(s, P^{t,x}(s)).
\end{aligned}
$$

The equation (5.5) gives that

$$
dv(s, P^{t,x}(s)) = r(s, P^{t,x}(s)) v(s, P^{t,x}(s)) ds + dL(s).
$$

Similar to Brownian case $K(s)$ can be found as the following,

$$
K(s) = K(T) - \int_s^T e^{\int_s^T r(m, P^{t,x}(m)) dm} dL(u)
$$

$L(s)$ is a martingale and $K(T) = \psi(P^{t,x}(T))$, then it becomes

$$
\begin{aligned}
& e^{-\int_s^T r(m, P^{t,x}(m)) dm} K(s) \\
= & e^{-\int_s^T r(m, P^{t,x}(m)) dm} \psi(P^{t,x}(T)) - e^{-\int_s^T r(m, P^{t,x}(m)) dm} \int_s^T e^{\int_s^T r(m, P^{t,x}(m)) dm} dL(u)
\end{aligned}
$$

After taking expectation we get,

$$
\mathbb{E}\big[e^{-\int_s^T r(m, P^{t,x}(m)) dm} \psi(P^{t,x}(T)) \mid P^{t,x}(s) = x\big] = v(s, P^{t,x}(s))
$$

is the solution of boundary value problem. Also, we have the following BSDE by the given PDIE (5.3).

$$
\begin{aligned}
-dv(s, P^{t,x}(s)) =& f\big(s, P^{t,x}(s), v(s, P^{t,x}(s)), \sigma(s, P^{t,x}(s))^* \partial_x v(s, P^{t,x}(s)), \\
& \big[\partial_x v(s, P^{t,x}(s))\big]^* \rho(s, P^{t,x}(s), z)\big) ds \\
& - \big[\partial_x v(s, P^{t,x}(s))\big]^* \sigma(s, P^{t,x}(s)) dW(s) \\
& - \int_{\mathbb{R}^m} \big[\partial_x v(s, P^{t,x}(s^-))\big]^* \rho(s, P^{t,x}(s^-), z) \widetilde{M}(ds, dz),
\end{aligned} \tag{5.6}
$$

with $v(T, P^{t,x}(T)) = \psi(P^{t,x}(T))$. Thus,

$$
(v(s, P^{t,x}(s)), \sigma(s, P^{t,x}(s))^* \partial_x v(s, P^{t,x}(s)), \big[\partial_x v(s, P^{t,x}(s^-))\big]^* \rho(s, P^{t,x}(s^-), z))
$$

is equal to unique solution of BSDE, where $s \in [0, T]$.

Reversely, if it is supposed that the discounted asset prices are martingales and the unique solution of BSDE (5.2) is given as the equations (5.4) then the discounted asset prices have the following derivative,

$$
\begin{aligned}
d(e^{-\int_0^t r(s) ds} v(t, P(t))) =& -r(t) e^{-\int_0^t r(s) ds} v(t, P(t)) dt + e^{-\int_0^t r(s) ds} d(v(t, P(t))) \\
=& e^{-\int_0^t r(s) ds} \big[- r(t) v(t, P(t)) \\
& - f\big(t, P(t), v(t, P(t)), \sigma(t, P(t))^* \partial_x v(t, P(t)), [\partial_x v(t, P(t))]^* \rho(t, P(t), z)\big)\big] dt \\
& + e^{-\int_0^t r(s) ds} [\partial_x v(t, P(t))]^* \sigma(t, P(t)) dW(t) \\
& + e^{-\int_0^t r(s) ds} \int_{\mathbb{R}^m} [\partial_x v(t, P(t))]^* \rho(t, P(t), z) \widetilde{M}(dt, dz)
\end{aligned} \tag{5.7}
$$

which leads dt term to be zero, hence

$$
\begin{aligned}
&-r(t)v(t,P(t))\\
&\qquad = f\big(t,P(t),v(t,P(t)),\sigma(t,P(t))^*\partial_x v(t,P(t)),[\partial_x v(t,P(t))]^*\rho(t,P(t),z)\big).
\end{aligned} \tag{5.8}
$$

Additionally, we define

$$
\begin{aligned}
f(t,x) &= e^{-\int_0^t r(s)ds}v(t,x),\\
\frac{\partial f}{\partial t}(t,x) &= -r(t)e^{-\int_0^t r(s)ds}v(t,x)+e^{-\int_0^t r(s)ds}\frac{\partial v}{\partial t}(t,x),\\
\frac{\partial f}{\partial x}(t,x) = e^{-\int_0^t r(s)ds}\frac{\partial v}{\partial x}(t,x), &\quad \frac{\partial^2 f}{\partial x^2}(t,x) = e^{-\int_0^t r(s)ds}\frac{\partial^2 v}{\partial x^2}(t,x).
\end{aligned}
$$

Now Itô formula (see Appendices, Theorem A.10) is applied to $f(t,P(t)) = e^{-\int_0^t r(s)ds}v(t,P(t))$,

$$
\begin{aligned}
d[e^{-\int_0^t r(s)ds}v(t,P(t))] &= -r(t)e^{-\int_0^t r(s)ds}v(t,P(t))dt + e^{-\int_0^t r(s)ds}\frac{\partial v}{\partial t}(t,P(t))dt\\
&+ e^{-\int_0^t r(s)ds}\left[\frac{\partial v}{\partial x}(t,P(t))\right]^* dP(t)^c + \frac{1}{2}e^{-\int_0^t r(s)ds}[dP(t)^c]^*\frac{\partial^2 v}{\partial x^2}(t,P(t))dP(t)^c\\
&+ \int_{\mathbb{R}^m} e^{-\int_0^t r(s)ds}\Bigg[v(t,P(t)+\rho(t,P(t),z))-v(t,P(t))\\
&\qquad - \left[\frac{\partial v}{\partial x}(t,P(t))\right]^*\rho(t,P(t),z)\Bigg]\nu(dz)dt\\
&+ \int_{\mathbb{R}^m} e^{-\int_0^t r(s)ds}\Big[v(t,P(t^-)+\rho(t,P(t^-),z))-v(t,P(t^-))\Big]\widetilde{M}(dt,dz),
\end{aligned}
$$

which is

$$
\begin{aligned}
d[e^{-\int_0^t r(s)ds}v(t,P(t))] &= -r(t)e^{-\int_0^t r(s)ds}v(t,P(t))dt + e^{-\int_0^t r(s)ds}\frac{\partial v}{\partial t}(t,P(t))dt\\
&+ e^{-\int_0^t r(s)ds}\left[\frac{\partial v}{\partial x}(t,P(t))\right]^*\Big[b(t,P(t))dt+\sigma(t,P(t))dW(t)\Big]\\
&+ \frac{1}{2}e^{-\int_0^t r(s)ds}\sum_{i=1}^d\sum_{j=1}^d[(\sigma\sigma^*)_{ij}(t,P(t))]\frac{\partial^2 v(t,P(t))}{\partial x_i\partial x_j}dt\\
&+ \int_{\mathbb{R}^m} e^{-\int_0^t r(s)ds}\Bigg[v(t,P(t)+\rho(t,P(t),z))-v(t,P(t))\\
&\qquad - \left[\frac{\partial v}{\partial x}(t,P(t))\right]^*\rho(t,P(t),z)\Bigg]\nu(dz)dt\\
&+ \int_{\mathbb{R}^m} e^{-\int_0^t r(s)ds}\Big[v(t,P(t^-)+\rho(t,P(t^-),z))-v(t,P(t^-))\Big]\widetilde{M}(dt,dz),
\end{aligned}
$$

Hence,

$$
\begin{aligned}
d[e^{-\int_0^t r(s)ds}v(t,P(t))] &= e^{-\int_0^t r(s)ds}\Big\{ - r(t)v(t,P(t)) + \frac{\partial v}{\partial t}(t,P(t)) \\
&+ \left[\frac{\partial v}{\partial x}(t,P(t))\right]^* b(t,P(t)) + \frac{1}{2}\sum_{i,j=1}^{d}[(\sigma\sigma^*)_{ij}(t,P(t))]\frac{\partial^2 v(t,P(t))}{\partial x_i \partial x_j} \\
&+ \int_{\mathbb{R}^m}\Bigg[v(t,P(t)+\rho(t,P(t),z)) - v(t,P(t)) \\
&\qquad - \left[\frac{\partial v}{\partial x}(t,P(t))\right]^* \rho(t,P(t),z)\Bigg]\nu(dz)\Big\}dt \\
&+ e^{-\int_0^t r(s)ds}\left[\frac{\partial v}{\partial x}(t,P(t))\right]^* \sigma(t,P(t))dW(t) \\
&+ e^{-\int_0^t r(s)ds}\int_{\mathbb{R}^m}\Big[v(t,P(t^-)+\rho(t,P(t^-),z)) - v(t,P(t^-))\Big]\widetilde{M}(dt,dz).
\end{aligned}
$$

Since the discounted asset prices are martingales, the dt term should be equal to zero. Therefore,

$$
\begin{aligned}
&- r(t)v(t,P(t)) + \partial_t v(t,P(t)) + \mathcal{L}v(t,P(t)) \\
&+ \int_{\mathbb{R}^m}\Big[v(t,P(t)+\rho(t,P(t),z)) - v(t,P(t)) - \big[\partial_x v(t,P(t))\big]^*\rho(t,P(t),z)\Big]\nu(dz) \\
&= 0.
\end{aligned}
\tag{5.9}
$$

By using equation (5.8) the PDIE with the boundary value $v(T,x) = \psi(x)$, $\forall x \in \mathbb{R}^n$ is found:

$$
\begin{aligned}
\partial_t v(t,x) + \mathcal{L}v(t,x) + \int_{\mathbb{R}^m}&\Big\{v(t,x+\rho(t,x,z)) - v(t,x) \\
&- [\partial_x v(t,x)]^*\rho(t,x,z)\Big\}\nu(dz) \\
&+ f(t,x,v(t,x),\sigma(t,x)^*\partial_x v(t,x),[\partial_x v(t,x)]^*\rho(t,x,z)) = 0.
\end{aligned}
\tag{5.10}
$$

□

Feynman-Kac formula and the related PDIE have an importance in finance: they provide an analogue of famous Black-Sholes partial differential equation and is used for the purpose of option pricing in a multidimensional Lévy market.

Example 5.1. For the sake of simplicity, suppose that there is one riskless asset with the return $r(t)$ and one risky asset having 1-dimensional compensated Poisson process with intensity λ as source of randomness with the following dynamics:

$$
\begin{aligned}
dP^0(t) &= P^0(t)r(t)dt \\
dP^1(t) &= P^1(t^-)[b(t)dt + \rho(t)d\widetilde{N}(t)],
\end{aligned}
$$

where $\rho(t)d\widetilde{N}(t) = \int_{\mathbb{R}} \rho(t,z)\widetilde{M}(dt,dz)$ and $\widetilde{M}$ is the compensated Poisson random measure.

The portfolio is defined by the wealth process $v(t,P(t))$ and number of assets $\eta(t)$. Previously, the portfolio is considered by the wealth process $v(t,P(t))$ and amount of money invested $\pi(t)$ but there is one-to-one correspondence between $\eta(t)$ and $\pi(t)$ by definition i.e. $\eta^i(t) := \frac{\pi^i(t)}{P^i(t)}$ for $i = 0,1$. Moreover, the wealth process can be written as

$$v(t,P(t)) = \eta^0(t)P^0(t) + \eta^0(t)P^1(t). \tag{5.11}$$

The equation (5.11) is differentiated,

$$\begin{aligned}
dv(t,P(t)) &= \eta^0(t)dP^0(t) + \eta^1(t)dP^1(t) \\
&= \eta^0(t)P^0(t)r(t)dt + \eta^1(t)P^1(t^-)[b(t)dt + \rho(t)d\widetilde{N}(t)], \\
&= [v(t,P(t)) - \eta^1(t)P^1(t)]r(t)dt + \eta^1(t)P^1(t)b(t)dt \\
&\quad + \eta^1(t)P^1(t^-)\rho(t)d\widetilde{N}(t) \\
&= \big[v(t,P(t))r(t) + \eta^1(t)P^1(t)\big(b(t) - r(t)\big)\big]dt + \eta^1(t)P^1(t^-)\rho(t)d\widetilde{N}(t).
\end{aligned}$$

We assume $d\widetilde{N}_{\mathbb{Q}}(t) = d\widetilde{N}(t) + \lambda\theta(t)dt$ is compensated Poisson process under the risk neutral probability measure $\mathbb{Q}$ with intensity λ where the Radon-Nikodym derivative with respect to $\mathbb{P}$ on $\mathcal{F}_T$ is given by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\Big\{ - \int_0^T \lambda[\log(1-\theta(t)) + \theta(t)]dt - \int_0^T \log(1-\theta(t))\widetilde{N}(dt)\Big\}.$$

Under risk neutral probability measure $\mathbb{Q}$, the price process becomes

$$\begin{aligned}
dP^1(t) &= P^1(t)b(t)dt + P^1(t^-)\rho(t)d\widetilde{N}(t) \\
&= P^1(t)b(t)dt + P^1(t^-)\rho(t)\big[d\widetilde{N}_{\mathbb{Q}}(t) - \lambda\theta(t)dt\big] \\
&= P^1(t)\big[b(t) - \lambda\rho(t)\theta(t)\big]dt + P^1(t^-)\rho(t)d\widetilde{N}_{\mathbb{Q}}(t), \\
&= P^1(t)r(t)dt + P^1(t^-)\rho(t)d\widetilde{N}_{\mathbb{Q}}(t),
\end{aligned} \tag{5.12}$$

and wealth process becomes

$$\begin{aligned}
dv(t,P(t)) =& \big[v(t,P(t))r(t) + \eta^1(t)P^1(t)\big(b(t) - r(t)\big)\big]dt + \eta^1(t)P^1(t^-)\rho(t)d\widetilde{N}(t) \\
=& \big[v(t,P(t))r(t) + \eta^1(t)P^1(t)\big(b(t) - r(t)\big)\big]dt \\
& + \eta^1(t)P^1(t^-)\rho(t)\big[d\widetilde{N}_{\mathbb{Q}}(t) - \lambda\theta(t)dt\big] \\
=& \big[v(t,P(t))r(t) + \eta^1(t)P^1(t)\big(b(t) - r(t) - \lambda\theta(t)\rho(t)\big)\big]dt \\
& + \eta^1(t)P^1(t^-)\rho(t)d\widetilde{N}_{\mathbb{Q}}(t) \\
=& v(t,P(t))r(t)dt + \eta^1(t)P^1(t^-)\rho(t)d\widetilde{N}_{\mathbb{Q}}(t)
\end{aligned} \tag{5.13}$$

where $b(t) - \lambda\rho(t)\theta(t) = r(t)$.

Then associated with the equation (5.12) the FBSDE becomes,

$$-dv(t, P^1(t)) = f(t, P^1(t), v(t, P^1(t)), \eta^1(t)P^1(t)\rho(t))dt - \eta^1(t)P^1(t^-)\rho(t)d\widetilde{N}_{\mathbb{Q}}(t)),$$
$$v(T, P^1(T)) = \psi(P^1(T)) \tag{5.14}$$

where $f(t, P^1(t), v(t, P^1(t)), \eta^1(t)P^1(t)\rho(t)) = -v(t, P^1(t))r(t)$.

Under risk neutral probability measure $\mathbb{Q}$, the discounted asset price is martingale. Hence, to find the related PDIE it is introduced that

$$\begin{aligned} g(t,x) =& e^{-\int_0^t r(s)ds} v(t,x), \\ \frac{\partial g}{\partial t}(t,x) = -r(t)e^{-\int_0^t r(s)ds}&v(t,x) + e^{-\int_0^t r(s)ds}\frac{\partial v}{\partial t}(t,x), \\ \frac{\partial g}{\partial x}(t,x) = e^{-\int_0^t r(s)ds}\frac{\partial v}{\partial x}(t,x), &\quad \frac{\partial^2 g}{\partial x^2}(t,x) = e^{-\int_0^t r(s)ds}\frac{\partial^2 v}{\partial x^2}(t,x). \end{aligned}$$

Here Itô formula (Appendices Theorem A.10) is applied to the function $g(t, P^1(t)) = e^{-\int_0^t r(s)ds}v(t, P^1(t))$,

$$\begin{aligned} e^{-\int_0^t r(s)ds}v(t, P^1(t)) &= e^0 v(0, P^1(0)) + \int_0^t \big[- r(s)e^{-\int_0^s r(u)du}v(s, P^1(s)) \\ &+ e^{-\int_0^s r(u)du}\frac{\partial v}{\partial t}(s, P^1(s))\big]ds + \int_0^t e^{-\int_0^s r(u)du}\frac{\partial v}{\partial x}(s, P^1(s))\big(P^1(s)b(s)ds\big) \\ &+ \int_0^t \int_{\mathbb{R}} e^{-\int_0^s r(u)du}\Big[v(s, P^1(s) + \rho(s,z)) - v(s, P^1(s)) \\ &- \rho(s,z)\frac{\partial v}{\partial x}(s, P^1(s))\Big]\nu(dz)ds \\ &+ \int_0^t \int_{\mathbb{R}} e^{-\int_0^s r(u)du}\big[v(s, P^1(s^-) + \rho(s,z)) - v(s, P^1(s^-))\big]\widetilde{M}_{\mathbb{Q}}(ds,dz), \end{aligned}$$

leading to

$$\begin{aligned} e^{-\int_0^t r(s)ds}v(t, P^1(t)) &= v(0, P^1(0)) + \int_0^t e^{-\int_0^s r(u)du}\Bigg[- r(s)v(s, P^1(s)) \\ &+ \partial_t v(s, P^1(s)) + \partial_x v(s, P^1(s))P^1(s)b(s, P^1(s)) \\ &+ \int_{\mathbb{R}} \Big[v(s, P^1(s) + \rho(s,z)) - v(s, P^1(s)) - \rho(s,z)\partial_x v(s, P^1(s))\Big]\nu(dz)\Bigg]ds \\ &+ \int_0^t e^{-\int_0^s r(u)du}\big[v(s, P^1(s^-) + \rho(s)) - v(s, P^1(s^-))\big]\widetilde{N}_{\mathbb{Q}}(ds) \end{aligned}$$

Since it is martingale the dt term should be equal to zero then the inside of the integral should be equal to zero which leads to

$$\begin{aligned} &- r(s)v(s, P^1(s)) + \partial_t v(s, P^1(s)) + [\partial_x v(s, P^1(s))]P^1(s)b(s, P^1(s)) \\ &+ \int_{\mathbb{R}} \Big[v(s, P^1(s) + \rho(s,z)) - v(s, P^1(s)) - \rho(s,z)\partial_x v(s, P^1(s))\Big]\nu(dz) = 0. \end{aligned}$$

Finally, the related PDIE is found as

$$
\begin{aligned}
& f(t, P^1(t), v(t, P^1(t)), \eta^1(t)P^1(t)\rho(t)) + \partial_t v(s, P^1(s)) \\
& + [\partial_x v(s, P^1(s))]P^1(s)b(s, P^1(s)) \\
& + \int_{\mathbb{R}} \Big[v(s, P^1(s) + \rho(s,z)) - v(s, P^1(s)) - \rho(s,z)\partial_x v(s, P^1(s))\Big]\nu(dz) = 0, \\
& v(T, P^1(T)) = \psi(P^1(T)).
\end{aligned} \tag{5.15}
$$

By Feynman-Kac formula, the solutions of the PDIE (5.15) and FBSDE system (5.12), (5.14) coincide.

Example 5.2. The boundary value problem in the domain $[0,T] \times \mathbb{R}^2$ given with the following PDIE,

$$
\begin{aligned}
& \frac{\partial F(t,x)}{\partial t} + rx\frac{\partial F(t,x)}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 F(t,x)}{\partial x^2} + \int_{\mathbb{R}} \Big\{F(t, x + \rho(z)x) \\
& \quad - F(t,x) - \left[\frac{\partial F(t,x)}{\partial x}\right]^* \rho(z)x\Big\}\nu(dz) = rF(t,x) \\
& F(T,x) = \sqrt{x},
\end{aligned}
$$

where r, σ are constants and $\rho:\ \mathbb{R} \mapsto \mathbb{R}$ is an $\mathcal{F}_t$-predictable under risk neutral probability measure $\mathbb{Q}$. Simply, by the Feynman-Kac Theorem 5.1 the forward-backward stochastic representation can be found as

$$
\begin{aligned}
dP(t) &= rP(t)dt + \sigma P(t)dW(t) + \int_{\mathbb{R}} \rho(z)P(t^-)\widetilde{M}(dt,dz) \\
P(0) &= m
\end{aligned}
$$

for any given $m \in \mathbb{R}^2$. In addition,

$$
\begin{aligned}
-dY(t) &= f(t, Y(t))dt - Z(t)^* dW(t) - \int_{\mathbb{R}} \gamma(t,z)\widetilde{M}(dt,dz) \\
Y(T) &= \sqrt{P(T)},
\end{aligned}
$$

where $f(t, Y(t)) = -rY(t)$ and the solution is

$$
\begin{aligned}
Y(t) = F(t, P(t)), \quad Z(t) = [\sigma P(t)]^* \partial_x F(t, P(t)) \\
\gamma(t,z) = \big[\partial_x F(t, P(t^-))\big]^* \rho(z)P(t^-).
\end{aligned}
$$

By Remark 4.1 the solution can be found as,

$$
Y(t) = \mathbb{E}_{\mathbb{Q}}\big[e^{-r(T-t)}\sqrt{P(T)}|\mathcal{F}\big].
$$

CHAPTER 6

CONCLUSION

It is a pleasure and imposing experiment to study on the BSDEs. In this thesis, the BSDE theory, its relation with financial and PDE theory are considered from the academic rules and ethics framework. It provides me interest to investigate additionally demonstrates the connections of interdisciplinary areas. The work done is the comprehension of BSDE theory and its application to financial mathematics and PDE theory. The paper "*Backward Stochastic Differential Equations in Finance*" of El Karoui, Peng, and Quenez [9] is closely followed for the theory without jumps. Moreover, the notation of the model including jumps is introduced by following the book "*Theory of Stochastic Differential Equations with Jumps and Applications*" of Situ [27]. The FBSDE system with jumps is used to generalize Feynman-Kac formula for the models in the incomplete markets. Especially, Feynman-Kac formula is considered for complete markets by the help of Teugel martingales in the other references. This thesis encourages me to study about Malliavan calculus in order to see its usage for BSDEs for future studies.

As a summary, initially the story of BSDEs is stated and they are introduced. Next, the fundamental definitions and theorems for the Brownian case are studied. The existence and uniqueness of the solution of a BSDE from [9] which is necessarily done for BSDE theory together with H. Sevda Nalbant [18] is restated and explicitly proven. Later, the special case of BSDE with linear generator (LBSDE) is examined and it is give an elaborate proof with a closed formula for the solution. Moreover, by the comparison theorem the fact that bigger contingent claim makes the option price bigger is proven. As an application, the BSDEs in finance and Feynman-Kac formula are practiced. By the Feynman-Kac formula a duality is provided for BSDE and PDE. In order to broaden our study, the jumps are enabled in our model and the theory from the beginning is study over which is also intentionally done for the more advanced reader. The individuals are interested in the BSDEs in the presence of jump processes can benefit from the last chapters of the thesis. As a source, [27] is closely followed. The existence and uniqueness, comparison, theorems are considered. Significantly, the generalization of Feynman-Kac formula in the presence of jumps is given for the incomplete markets in our unified notation, which is usually done with completion for Lévy processes.

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APPENDIX A

SOME USEFUL INEQUALITIES, DEFINITIONS AND THEOREMS

A.1 Some Useful Inequalities

Lemma A.1. *(The Burkholder-Davis-Gundy Inequality) Let M be a continuous martingale and $\langle M\rangle_T$ is its quadratic variation process. Define $M_t^{\max} = \max_{0\leq s\leq t}|M_s|$ then for any $m>0$, there exist universal positive constants c_m, d_m depending only on m, such that*

$$c_m\mathbb{E}\big[\langle M\rangle_\tau^m\big] \leq \mathbb{E}\big[(M_\tau^{\max})^2\big] \leq d_m\mathbb{E}\big[\langle M\rangle_\tau^m\big]$$

holds for any stopping time τ.

(Reference [13])

Lemma A.2. *(Doob's Martingale Inequality) Let M be a positive submartingale then for any $p>1$,*

$$\mathbb{E}\big[\sup_{0\leq s\leq t} M(s)^p\big] \leq q^p\mathbb{E}[M(t)^p]$$

holds $\frac{1}{p}+\frac{1}{q}=1$.

(Reference [2])

Lemma A.3. *Let $a(t), b(t), c(t) > 0$ for any t, then we have the following inequalities;*

$$2\int_t^T a(s)b(s)ds \leq \int_t^T a(s)b(s)^2ds + \int_t^T a(s)ds,$$
$$0 \leq \int_t^T a(s)[b(s)-1]^2ds.$$

$$2\int_t^T a(s)b(s)c(s)ds \leq 2\int_t^T a(s)^2b(s)^2ds + \frac{1}{2}\int_t^T c(s)^2ds,$$
$$0 \leq \int_t^T \Big(\sqrt{2}a(s)b(s) - \frac{\sqrt{2}}{2}c(s)\Big)^2ds.$$

A.2 Martingale Representation Theorem (MRT)

Theorem A.4. *Suppose M_t is a square integrable martingale, with respect to the filtration $\mathcal{F}_t$ for all $t \geq 0$. Then there exists adapted process Z_t such that $\mathbb{E}[\int_0^T Z_s^2 ds] < \infty$ for $t \geq 0$ and*

$$M_t = M_0 + \int_0^t Z_s dW_s \quad a.s.$$

(Reference [17])

A.3 Banach Fixed Point Theorem

Theorem A.5. *Let $\mathcal{H}$ be a Banach space with S-norm, where $\mathcal{H} \neq \emptyset$ and $f : \mathcal{H} \mapsto \mathcal{H}$ be a contraction on $\mathcal{H}$ (i.e. there is a real number c such that $c \in (0,1)$ and for any $x, y \in \mathcal{H}$; $\|f(x) - f(y)\|_S \leq c\,\|x - y\|_S$). Then f has only one fixed point.*

(Reference [15])

A.4 Girsanov Theorem

Theorem A.6. *Let θ_t be an adapted process satisfying $\mathbb{E}^{\mathbb{P}}\left[e^{\frac{1}{2}\int_0^T \theta_t^2 dt}\right] < \infty$ (Appendices, Theorem A.5 Novikov's condition), W_t be d-dimensional Brownian motion under probability measure $\mathbb{P}$ and Y_t be an $\mathbb{R}^d$-valued Itô process such that*

$$dY_t = \theta_t dt + dW_t, \quad 0 \leq t \leq T.$$

Define

$$Z_t := e^{[-\int_0^t \theta_s^* dW_s - \frac{1}{2}\int_0^t \theta_s^2 ds]}.$$

Then the process $Y_t = W_t + \int_0^t \theta_s ds$ is a Brownian motion under the probability $\mathbb{Q}$ defined by $d\mathbb{Q} = Z(T)d\mathbb{P}$.

(Reference [20])

A.5 Novikov's Condition

Theorem A.7. *Let X be a d-dimensional vector of measurable, adapted processes on a probability space $(\omega, \mathcal{F}, \mathbb{P})$, with d-dimensional Brownian motion W, which* *satisfies*

$$\mathbb{P}\Big[\int_0^T (X_t^i)^2 dt < \infty\Big] = 1, \quad 1 \le i \le d, \quad 0 \le T < \infty.$$

If $\mathbb{E}\big[e^{\frac{1}{2}\int_0^T X_t^2 dt}\big] < \infty$, *then* $Z_t(X)$ *defined as follows is a martingale;*

$$Z_t(X) := e^{[-\int_0^t X_s^* dW_s - \frac{1}{2}\int_0^t X_s^2 ds]}.$$

(Reference [13])

A.6 Poisson and Lévy Processes: definitions and properties

Definition A.1. The process $(N(t),\ t \ge 0)$ defined by

$$N(t) = \sum_{n\ge 1} 1_{t \ge T_n}$$

is called a *Poisson process with intensity* λ, if $(\tau_i)_{i\ge 1}$ is a sequence of independent exponential random variables with parameter λ and $T_n = \sum_{i=1}^n \tau_i$.

Proposition A.8. *Let* $N(t)$, $t \ge 0$ *be a Poisson process then it has the following properties:*

1. *For any* $t > 0$, $N(t)$ *is almost surely finite.*
2. *For any* ω, *the sample path* $t \mapsto N(w, t)$ *is piecewise constant and increases by jump size* 1.
3. *The sample paths* $t \mapsto N(t)$ *are right continuous with left limit (i.e. cadlag).*
4. *For any* $t > 0$, $N(t^-) = N(t)$ *with probability* 1.
5. $N(t)$ *is continuous in probability:*

$$N(s) \xrightarrow{\mathbb{P}} N(t) \text{ as } s \to t, \quad \forall t > 0.$$

6. *For any* $t > 0$, $N(t)$ *follows a Poisson distribution with parameter* λt*:*

$$\mathbb{P}(N(t) = n) = e^{-\lambda t}\frac{(\lambda t)^n}{n!}, \quad \forall n \in \mathbb{N}.$$

7. *The characteristic function of* $N(t)$ *is given by*

$$\mathbb{E}[e^{iuN(t)}] = \exp\{\lambda t(e^{iu} - 1)\}, \quad \forall u \in \mathbb{R}.$$

8. *N(t) has independent increments (i.e.* $N(t_n) - N(t_{n-1}), ..., N(t_2) - N(t_1)$, $N(t_1)$ *are independent random variables, for any* $t_1 < ... < t_n$*).*

9. *The increments of N are stationary (i.e. $N(t) - N(s)$ has the same distribution as $N(t-s)$, for any $t > s$).*

10. *$N(t)$ has the Markov property:*

$$\mathbb{E}[f(N(t))|N(u), u \leq s] = \mathbb{E}[f(N(t))|N(s)], \quad \forall t > s.$$

Definition A.2. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $E \subset \mathbb{R}^m$ and μ a given (positive) Radon measure on (E, ξ). An integer valued random measure M is called a *Poisson random measure* on E with intensity measure μ:

$$M : \Omega \times \xi \mapsto \mathbb{N}$$

such that

1. For $\omega \in \Omega$, $M(\omega; t, \cdot)$ is an integer-valued Radon measure on E.
2. For each measurable set $A \in E$, $M(\cdot; t, A) = M(t, A)$ is a Poisson random variable with parameter $\mu(A)$.
3. The variables $M(t, A_1), ..., M(t, A_n)$ are independent for disjoint measurable sets $A_1, ..., A_n \in \xi$.

Definition A.3. A right continuous with left limit (i.e. cadlag) stochastic process $Y(t)$, $t \geq 0$ on $(\Omega, \mathcal{F}, \mathbb{P})$ with values in $\mathbb{R}^m$ such that $Y(0) = 0$ is called a *Lévy process* if it has the following properties:

1. Independent increments: for every sequence of times $t_0, ..., t_n$ the random variables $Y(t_n) - Y(t_{n-1}), ..., Y(t_1) - Y(t_0), Y(t_0)$ are independent.
2. Stationary increments: the law of $Y(t+h) - Y(t)$ does not depend on t.
3. Stochastic continuity: $\forall \epsilon > 0$, $\lim_{h \to 0} \mathbb{P}(|Y(t+h) - Y(t)| \geq \epsilon) = 0$.

Definition A.4. Let $Y(t)$, $t \geq 0$ be a Lévy process on $\mathbb{R}^m$. The measure ν on $\mathbb{R}^m$ defined by:

$$\nu(A) = \mathbb{E}[\#\{t \in [0,1] : \Delta Y(t) \neq 0, \ \Delta Y(t) \in A\}], \quad A \in \mathcal{B}(\mathbb{R}^m)$$

is called the *Lévy measure* of Y : $\nu(A)$ is the expected number of jumps per unit time whose size belongs to A.

(References [7], [16])

A.7 Girsanov Theorem for Lévy Processes

Theorem A.9. *Let $\theta(t, z) \leq 1$, $z \in \mathbb{R}^m$ and u(t) be an adapted processes where $t \in [0, T]$ such that*

$$\int_0^T \int_{\mathbb{R}^m} \left\{ |\log(1 + \theta(t, z)| + (\theta(t, z))^2 \right\} \nu(dz) dt < \infty,$$

$$\int_0^t \big(u(s)\big)^2 ds < \infty.$$

Let the following exists for all $t \in [0,T]$:

$$Z_t := \exp\Big\{ -\int_0^t u(s)dW_s - \frac{1}{2}\int_0^t (u(s))^2 ds$$
$$+ \int_0^t \int_{\mathbb{R}^m} [\log(1-\theta(s,z)) + \theta(s,z)]\nu(dz)ds + \int_0^t \int_{\mathbb{R}^m} \log(1-\theta(s,z))\widetilde{M}(ds,dz)\Big\}.$$

Define $\mathbb{Q}$ as an equivalent measure of $\mathbb{P}$ such that

$$d\mathbb{Q} = Z(T)d\mathbb{P}.$$

Additionally, suppose that $Z(t)$ satisfies the Novikov condition [12]. Then the process $\widetilde{M}_{\mathbb{Q}}(dt,dz) = \widetilde{M}(dt,dz) + \theta(t,z)\nu(dz)dt$ is a Poisson measure and the process $dW_{\mathbb{Q}}(t) = dW(t) + u(t)dt$ is a Brownian motion under probability measure $\mathbb{Q}$.

(Reference [22])

A.8 Itô Formula for Multidimensional Case with Jumps

Theorem A.10. *Let $X(t) \in \mathbb{R}^d$ be an Itô-Lévy process of the form*

$$dX(t) = b(w,t)dt + \sigma(w,t)dW(t) + \int_{\mathbb{R}^m} \gamma(w,t,z)\widetilde{M}(dt,dz) \tag{A.1}$$

where $b:\ \Omega \times [0,T] \mapsto \mathbb{R}^d$, $\sigma:\ \Omega \times [0,T] \mapsto \mathbb{R}^{d\times n}$, $\gamma:\ \Omega \times [0,T] \times \mathbb{R}^m \mapsto \mathbb{R}^{d\times m}$ are adapted processes such that the integral exists. Here $W(t)$ is an n-dimensional Brownian motion and

$$\begin{aligned}\widetilde{M}(dt,dz)^* &= \Big(\widetilde{M}_1(dt,dz_1), ..., \widetilde{M}_m(dt,dz_m)\Big)\\ &= \Big(M_1(dt,dz_1) - \mathbb{E}\Big[\widetilde{M}_1(dt,dz_1)\Big]\nu_1(dz_1), ...,\\ &\qquad M_m(dt,dz_m) - \mathbb{E}\Big[\widetilde{M}_m(dt,dz_m)\Big]\nu_m(dz_m)\Big)\end{aligned}$$

where $\{M_j\}$'s are independent Poisson random measures with Lévy measures ν_j coming from m independent (one-dimensional) Lévy processes $\eta_1, \eta_2, ..., \eta_m$. Note that the k-th column $\gamma^{(k)}$ of $d\times m$ matrix $\gamma = [\gamma_{ij}]$ depend on z only through the k-th coordinate z_k i.e.

$$\gamma^{(k)} = \gamma^{(k)}(w,t,z_k), \quad z^* = (z_1, ..., z_m),$$

$z \in \mathbb{R}^m$. The components of equation (A.1) has the following form

$$dX_i(t) = b_i(w,t)dt + \sum_{j=1}^{n} \sigma_{ij}(w,t)dW_j(t)$$
$$+ \sum_{j=1}^{m} \int_{\mathbb{R}} \gamma_{ij}(w,t,z)\widetilde{M}_j(dt,dz_j), \quad 1 \le i \le d.$$

To find Itô formula, take $f : [0,T] \times \mathbb{R}^d \mapsto \mathbb{R}$ such that $f \in \mathcal{C}^{1,2}$. Define $Y(t) = f(t,X(t))$, then $Y(t)$ is again an Itô-Lévy process and

$$\begin{aligned} dY(t) &= \frac{\partial f}{\partial t}dt + \sum_{i=1}^{d} \frac{\partial f}{\partial x_i}\Big(b_i dt + \sigma_i dW(t)\Big) + \frac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{d}(\sigma\sigma^*)_{ij}\frac{\partial^2 f}{\partial x_i \partial x_j}dt \\ &+ \sum_{k=1}^{m}\int_{\mathbb{R}}\Big\{f(t,X(t)+\gamma^{(k)}(t,z_k)) - f(t,X(t)) \\ &- \sum_{i=1}^{d}\gamma_i^{(k)}(t,z_k)\frac{\partial f}{\partial x_i}(t,X(t))\Big\}\nu_k(dz_k)dt \\ &+ \sum_{k=1}^{m}\int_{\mathbb{R}}\big\{f(t,X(t^-)+\gamma^{(k)}(t,z_k)) - f(t,X(t^-))\big\}\widetilde{M}_k(dt,dz_k), \end{aligned} \tag{A.2}$$

where $\gamma^{(k)} \in \mathbb{R}^m$ is column k of the $d \times m$ matrix γ and $\gamma_i^{(k)} = \gamma_{ik}$ is the coordinate i of $\gamma^{(k)}$.

(Reference [21])

A.9 Quadratic covariation of Itô-Lévy Processes

Proposition A.11. *Assume that*

$$dX_i(t) = b_i(t,\omega)dt + \sigma_i(t,\omega)dW(t) + \int_{\mathbb{R}} \rho_i(t,z)\widetilde{N}(dt,dz); \quad i = 1,2$$

are two Itô-Lévy processes. Then by the Itô formula (Appendices Theorem A.10) we have

$$d(X_1(t)X_2(t)) = X_1(t^-)dX_2(t) + X_2(t^-)dX_1(t) + \sigma_1(t)\sigma_2(t)dt$$
$$+ \int_{\mathbb{R}} \rho_1(t,z)\rho_2(t,z)N(dt,dz).$$

Thus, in this case the quadratic covariation is

$$[X_1,X_2](t) = \int_0^t \sigma_1(s)\sigma_2(s)ds + \int_0^t \int_{\mathbb{R}} \rho_1(s,z)\rho_2(s,z)N(ds,dz)$$
$$= \int_0^t \big[\sigma_1(s)\sigma_2(s) + \int_{\mathbb{R}} \rho_1(s,z)\rho_2(s,z)\nu(dz)\big]ds$$
$$+ \int_0^t \int_{\mathbb{R}} \rho_1(s,z)\rho_2(s,z)\widetilde{N}(ds,dz). \tag{A.3}$$

(Reference [21])