KILLING FAMILY OF TENSORS IN CLASSICAL GRAVITATIONAL THEORIES

A THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES
OF
MIDDLE EAST TECHNICAL UNIVERSITY

ÇAĞATAY MENEKAY

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR
THE DEGREE OF MASTER OF SCIENCE
IN
PHYSICS

JULY 2013

Approval of the thesis:

KILLING FAMILY OF TENSORS IN CLASSICAL GRAVITATIONAL THEORIES

submitted by **ÇAĞATAY MENEKAY** in partial fulfillment of the requirements for the degree of **Master of Science in Physics Department, Middle East Technical University** by,

Prof. Dr. Canan Özgen
Dean, Graduate School of **Natural and Applied Sciences** ________________

Prof. Dr. Mehmet T. Zeyrek
Head of Department, **Physics** ________________

Prof. Dr. Bahtiyar Özgür Sarıoğlu
Supervisor, **Physics Department, METU** ________________

Examining Committee Members:

Prof. Dr. Atalay Karasu
Physics Department, METU ________________

Prof. Dr. Bahtiyar Özgür Sarıoğlu
Physics Department, METU ________________

Prof. Dr. Bayram Tekin
Physics Department, METU ________________

Assoc. Prof. Dr. Seçkin Kürkçüoğlu
Physics Department, METU ________________

Assoc. Prof. Dr. Hakan Cebeci
Physics Department, Anadolu University ________________

Date: ________________

I hereby declare that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.

Name, Last Name: ÇAĞATAY MENEKAY

Signature :

ABSTRACT

KILLING FAMILY OF TENSORS IN CLASSICAL GRAVITATIONAL THEORIES

Menekay, Çağatay
M.S., Department of Physics
Supervisor : Prof. Dr. Bahtiyar Özgür Sarıoğlu

July 2013, 68 pages

In this thesis, the basic properties of the Killing family of tensors (Killing vector, Killing tensors and Killing-Yano tensors) are considered. Their relationship with integrals of motions and conserved gravitational charges are also discussed. The fourth constant of motion of a test particle in Kerr spacetime and its relationship with Killing tensor are reviewed. We have done a similar analysis for the newly discovered solution of Conformal Gravity. Next, the use of Killing-Yano tensors in the procedure for defining conserved gravitational charges is discussed. Finally, a new identity is introduced, and its use in a new approach to overcome a shortcoming of the former construction is given.

Keywords: Killing Vector, Killing Tensor, Killing-Yano Tensor, Conserved Gravitational Charge, Fourth Constant of Kerr Spactime

ÖZ

KLASİK KÜTLEÇKİM TEORİLERİNDE KILLING TENSÖR AİLESİ

Menekay, Çağatay
Yüksek Lisans, Fizik Bölümü
Tez Yöneticisi : Prof. Dr. Bahtiyar Özgür Sarıoğlu

Temmuz 2013 , 68 sayfa

Bu tezde Killing tensör ailesinin (Killing vektörleri, Killing tensörleri ve Killing-Yano tensörleri) temel özellikleri incelendi. Bunların hareket sabitleri ve korunan kütleçekimsel yükler ile ilişkileri araştırıldı. Kerr uzayındaki bir test parçacığının dördüncü hareket sabiti ile Killing tensörü arasındaki ilişki incelendi. Açıkorur Kütleçekim kuramının çözümü olan yeni bir metrik için benzer bir analiz yapıldı. Daha sonra, Killing-Yano tensörü ve bununla korunan kütleçekimsel yük tanımlanması için kullanılan yöntem ele alındı. Son olarak, bulunan yeni bir özdeşlik verildi ve bunu kullanarak sözü edilen yöntemin bir eksikliğini gidermek için geliştirilen bir yaklaşım sunuldu.

Anahtar Kelimeler: Killing Vector, Killing Tensor, Killing-Yano Tensor

Dedicated to the memory of my father

ACKNOWLEDGEMENTS

I would like to express my sincere appreciation to all the people without whose never-ending help and support this thesis would never have been achieved.

First and foremost, I owe my deepest gratitude to my supervisor, Professor Özgür Sarıoğlu. I was an Industrial engineer and did not know what to do when I met him. He taught me to think scientifically, pushed me to work more systematically and built the courage to challenge myself through the whole process. He helped me take the first steps in the long journey to be a physicist. I thank him sincerely for all of his continuous guidance and encouragement all the way through.

I am also grateful to my collaborator Gökhan Alkaç for providing me with insightful discussions. He helped me almost every time when I did get stuck with something. We have made a great effort to find out an answer to the problem stated in this thesis. We have learnt many things while dealing with the problems we have faced with. I appreciate him for devoting his time and energy to my passion for learning.

I would also like to acknowledge with much appreciation the significant role of my beloved girlfriend, Sıla, who has provided wholehearted support and assistance in my struggles. I thank God for giving me the opportunity to meet her. She encouraged me and gave me the strength to overcome every obstacle I have encountered during this process. Feeling that she was always by my side was priceless. I am also forever indebted to my mother and thank her for trusting me and supporting me for the every single decision I have made so far. I must point out that I am really fortunate to have my mother giving me her unconditional love, care and understanding my persistence in my pursuit of the M.S degree in academia. I will always be glad for being her child.

Last but not least, many thanks go to Deniz Olgu Devecioğlu for answering my questions without any hesitation all the time , and for the rides we taken which help to clear my mind. I also need to express my appreciation to Kutlucan Velioğlu, my lifelong friend and homemate, for all of his support in every possible way through the years. I also thank all of my friends who provide friendship that I needed and I greatly value their belief in me.

TABLE OF CONTENTS

NOTATIONS

In this work, we consider the latin letters $a,b,c...$ runs from 0 to n and the latin letters $i,j,k..$ runs from 1 to n. Moreover some of the notations we used throughout the this thesis are given below

$$\tilde{\varepsilon}_{abcd} = \begin{cases} +1, & \text{if } a\,b\,c\,d \text{ is an even permutation of } 0\,1\,2\,3 \\ -1, & \text{if } a\,b\,c\,d \text{ is an od permutation of cde} \\ 0, & \text{otherwise} \end{cases}$$

$$\delta^{abc}_{cde} = \begin{cases} +1, & \text{if } a\,b\,c \text{ is an even permutation of } c\,d\,e \\ -1, & \text{if } a\,b\,c \text{ is an od permutation } o f c d e \\ 0, & \text{otherwise} \end{cases}$$

$$\varepsilon_{abcd} = \sqrt{g}\tilde{\varepsilon}_{abcd}$$

$$T^{[a_1,\ldots,a_n]} = \frac{1}{n!}\delta^{a_1,\ldots,an}_{b_1,\ldots,b_n} T^{b_1,\ldots,b_n}$$

$$T^{(a_1,\ldots,a_n)} = \frac{1}{n!}\sum_{permutations} T_{a_1,\ldots,a_n}$$

$$[\nabla_a, \nabla_b]T_{cd} = R_{abc}{}^{e}T_{ed} + R_{abd}{}^{e}T_{ce}$$

CHAPTER 1

INTRODUCTION

Almost every physical system has some symmetry and symmetries play an important role in physics. When one considers gravitational theories that are inherently geometric, one of the the first things that comes to one's mine is the Killing vectors, since they are the generators of the isometries of a given spacetime. Thus they are closely related to the symmetries studied in that spacetime. Their generalizations are Killing tensors and Killing-Yano tensors, which are symmetric and antisymmetric generalizations, respectively. The symmetries, which these tensors are related to, are called the hidden symmetries. Besides giving the symmetries of a given system, the Killing vectors and the Killing-Yano tensors can also be used to construct gravitational charges as we will see below. Before explaining the scheme of this work, let us give a short survey on this matter.

Historically it was the Killing vectors that were defined first and they have been studied and used extensively in the literature. They can be used to set symmetry restrictions to the solutions of gravitational theories, e.g. restricting solutions to spherically symmetric spacetimes, etc. They can also be used to build conserved canonical momenta when examining the geodesics of a given spacetime, since they correspond to the symmetries of such systems. Furthermore, conserved gravitational charges can be constructed through them as Abbott and Deser did in [1] for the solutions of Einstein's theory with a flat background or Cosmological Einstein theory with AdS background. It was later extended to the quadratic curvature theories accepting a flat or an AdS background in [2], and further extended to quadratic curvature theories with arbitrary backgrounds in [3]. There are, of course, other uses of Killing vectors, in any system in which isometries of the spacetime under consideration are important.

The relationship between the Killing tensors and separability of the Hamilton-Jacobi equation of a test particle in a given spacetime has been known since Eisenhart showed it in [4]. However, Killing tensors became more popular after Carter's discovery of the fourth constant of motion of a test particle in Kerr spacetime [5]. In this work, Carter also showed that the Hamilton-Jacobi equation in the Kerr spacetime is separable while finding this constant. In [6], Penrose and Walker showed that the fourth constant of motion and the separability of the Hamilton-Jacobi equation of a massless particle in the Kerr spacetime is related to the existence of a Killing tensor. They also expressed the Killing tensor in spinor formalism in terms of the principal null vectors for those spacetimes that are solutions of Einstein's theory and of Petrov type D. In [7] and [8], it was shown that any Killing tensor can be expressed as the square of a Killing-Yano tensor. Additionally, it was shown that this Killing-Yano tensor is closely related to the separability of the Dirac equation in a rotating background [9]. However, these are beyond the scope of this work, since we will not consider any quantum aspects here. In the literature, there are quite a number of spacetimes investigated for the separability of the corresponding Hamilton-Jacobi equation and the existence of the fourth constant of motion. The separability of the Hamilton-Jacobi equation of a test particle in Kerr-de Sitter metrics in all dimensions is given in [10], in the Vaidya spacetime these issues were discussed in [11] and in the higher dimensional Kerr-NUT-AdS spacetime these were investigated in [12], to name a few.

The Killing-Yano tensor is the most recent member in the family of all these 'Killing' objects, they were discovered in 1952 in [13]. They are related to the hidden symmetries of a given system which are meaningful mostly at the quantum level, such as the separability of the Dirac equation as mentioned previously or the exotic supersymmetries of a spinning particle system in a curved background [14], [15]. In [16], a conserved gravitational charge was constructed by using a Killing-Yano tensor in a way similar to how the gravitational charges are defined through a Killing vector. In this work transverse spacetimes with flat backgrounds were considered. There is also another work [17] where an analogous charge expression is given for transverse spacetimes with AdS backgrounds. However, in this work, the current used in definining a conserved charge has a drawback, even though it gives information about the spacetime. There are terms in the current which cannot be explained physically as mentioned in [16]. Here in an effort to solve this problem, we show that it is possible to generalize the definition given in [1] simply by changing the Killing vector with a Killing-Yano tensor to obtain a new gravitational current. On the other hand this new current definition has its own shortcomings which will be discussed in detail in Chapter 5.

The scheme of this thesis is as follows. In Chapter 2, general tools which will be needed in the development of the subject matter is given. It includes a brief review of the Hamiltonian formalism and the Hamilton-Jacobi method, Stokes' theorem and the linearization procedure which are needed in the construction of gravitational charges. Chapter 3, is of a special importance, since the ideas developed here are later generalized for the Killing tensors and the Killing-Yano tensors in a way analogous to their generalization from the Killing vectors. In this chapter we briefly describe the properties of the Killing vectors and derive some identities which are needed later. Furthermore, we show the use of Killing vectors in the study of geodesics and review the procedure to construct the gravitational charge, which is also called the ADT charge. In Chapter 4, we discuss the relationship between the integrability of the Hamiltonian and the Killing tensor. Later we review the two methods to obtain the fourth constant of motion of a particle in the Kerr spacetime and equations of motion of the particle. We also investigate a method to derive the Killing tensor from this constant. Additionally, we briefly describe two solutions, one neutral and one dyonic, of the four dimensional conformal gravity whose action contains the Weyl tensor squared and the usual Maxwell terms [18]. We will also discuss the separability of the Hamilton-Jacobi equation, the existence of a fourth constant of motion of a particle given in this spacetime, and derive its equations of motion. Finally, in this Chapter, we find the Killing tensor giving the fourth constant of motion of this spacetime. In Chapter 5, we review the method of constructing gravitational charge given in [16] and discuss its results found in [16], [17]. We also discuss the problem mentioned in [16]. Later, we introduce a new identity which makes it possible to generalize the current definition as mentioned before. Unfortunately this new current cannot be expressed as the total divergence of a totally antisymmetric tensor of rank-3, even though in principle this should be the case. Finally we discuss this new current and the possible reasons for the problem with this new current definition.

CHAPTER 2

PRELIMINARIES

This chapter is devoted to reviewing briefly the topics which are needed to develop the subject matter. First we recall the Hamiltonian formalism and the Hamilton-Jacobi method which are used to obtain equations of motion. The tools developed in this first section are repeatedly used in Chapter 4, since there the motion of a test particle is studied for finding the fourth constant of motion, equations of motion and the Killing tensor. Later we state the Stokes' theorem and define a conservation rule which it leads to. In Chapters 3 and 5, conserved gravitational charges are discussed and the procedure developed in the second section is what those conserved charges are based on. Thus it has a crucial importance. Finally the linearization process is reviewed and linearization of the Einstein tensor is derived. The methods developed in this section will be repeatedly used in Chapters 3 and 5.

2.1 Hamilton Formalism and Hamilton-Jacobi method

In this section we will give a brief review of the Hamiltonian formalism, Poisson brackets, canonical transformations and Hamilton-Jacobi method which will be used extensively in Chapter 4. We will mostly follow the discussion from the book by Landau and Lifshitz [19]. The methods given here are the classical methods which do not treat time t as a coordinate. However generalization of these methods to the relativistic case is quite straightforward. Time has to be considered as a coordinate, since it is no longer a parameter which is the same for everyone. The parametrization of the coordinates is done with an affine parameter λ, hence one also should consider the affine parameter instead of time. The use of these methods in relativistic studies can be seen at Chapters 3 and 4.

For a system with the phase space coordinates p_i, q^i and the Hamiltonian $H(p_i, q^i, t)$ the Hamilton's equations are given by

$$\dot{q}^i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q^i}, \tag{2.1}$$

where q_i are the generalized coordinates and p_i are the canonical momenta. Here a dot over a quantity represents the total time derivative as usual. The time derivative of a function $f = f(p,q,t)$ is

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \sum_k \left(\frac{\partial f}{\partial q^k}\dot{q}^k + \frac{\partial f}{\partial p_k}\dot{p}_k\right), \tag{2.2}$$

and by using the Hamilton's equations of motion the equation can be written in the form

$$\begin{aligned} \frac{df}{dt} &= \frac{\partial f}{\partial t} + \sum_k \left(\frac{\partial f}{\partial q^k}\frac{\partial H}{\partial p_k} - \frac{\partial f}{\partial p_k}\frac{\partial H}{\partial q^k}\right) \\ &= \frac{\partial f}{\partial t} + \{H, f\}, \end{aligned} \tag{2.3}$$

where $\{H, f\}$ is the Poisson bracket of the function $f = f(p,q,t)$ with the Hamiltonian. Using the above equation, we can define the Poisson bracket or Poisson commutation of two functions $f = f(p,q,t)$ and $g = g(p,q,t)$ as

$$\{f, g\} \equiv \sum_k \left(\frac{\partial g}{\partial q^k}\frac{\partial f}{\partial p_k} - \frac{\partial g}{\partial p_k}\frac{\partial f}{\partial q^k}\right). \tag{2.4}$$

There are some basic properties of the Poisson bracket which follow easily from the definition (2.4). It changes sign when the functions are interchanged,

$$\{f, g\} = -\{g, f\}. \tag{2.5}$$

Moreover it is linear

$$\{f + g, h\} = \{f, h\} + \{g, h\}, \tag{2.6}$$

and obeys the Leibniz rule of partial derivatives

$$\{fg, h\} = f\{g, h\} + \{f, h\}g. \tag{2.7}$$

It also satisfies a very important identity, which is called the Jacobi identity,

$$\{f, \{g, h\}\} + \{g, \{f, h\}\} + \{h, \{f, g\}\} = 0. \tag{2.8}$$

It is obvious from the equation (2.3) that if the function f does not depend on time t explicitly, its time derivative is directly found from its Poisson bracket with the Hamiltonian

$$\frac{df}{dt} = \{H, f\}. \tag{2.9}$$

Integrals of motion are quantities that remain constant during the motion. The Poisson bracket of an integral of motion K with the Hamiltonian H is indeed zero

$$0 = \{H, K\}. \tag{2.10}$$

The Poisson theorem also states that the Poisson bracket of two integrals of motion is again an integral of motion. In particular, this can be seen by examining the Jacobi identity of two of these constants, say K_1 and K_2 and the Hamiltonian H, i.e.

$$\{K_1, \{K_2, H\}\} + \{K_2, \{H, K_1\}\} + \{H, \{K_1, K_2\}\} = 0. \tag{2.11}$$

Since both $\{K_1, H\}$ and $\{H, K_2\}$ are zero, the last term $\{H, \{K_1, F_2\}\}$ must also be equal to zero and this proves the Poisson theorem.

Moreover, the canonical variables satisfy the following Poisson brackets

$$\{q^i, q^j\} = 0, \qquad \{p_i, p_j\} = 0, \qquad \{q^i, p_j\} = \delta^i{}_j. \tag{2.12}$$

These relations will be used in defining the canonical transformations, since the new canonical variables should satisfy these Poisson brackets.

Canonical transformations are the transformations that transform the momenta and coordinates to new momenta and coordinates which satisfy the Hamilton's equations of motion, so do the Poisson brackets which we have mentioned before. The new variables should satisfy the Hamilton's equations

$$\dot{Q}^i = \frac{\partial H'}{\partial P_i}, \qquad \dot{P}_i = -\frac{\partial H'}{\partial Q^i}, \tag{2.13}$$

where H' is the new Hamiltonian giving these equations of motion. They should also define the same system. The actions, whose variations give the relevant Hamilton's equations of motion, are

$$S = \int (\sum_i p_i \dot{q}^i - H) dt, \tag{2.14}$$

$$S' = \int (\sum_i P_i \dot{Q}^i - H') dt. \tag{2.15}$$

The integrands of these actions should differ only by a total time derivative. This equivalence leads one to the generating functions of the canonical transformations. The equivalence of the two integrands is

$$\sum_i p_i \, dq^i - H \, dt = \sum_i P_i \, dQ^i - H' \, dt + dF. \tag{2.16}$$

where F is called generating function and is a function of old and new canonical variables and time t. If we consider a generating function type F_1 depending on old coordinates q^i, new coordinates Q^i and time t, then it can be cast in the form

$$dF_1 = \sum_i p_i dq^i - \sum_i P_i dQ^i + (H' - H)\, dt. \tag{2.17}$$

Then the generating equations for F_1 becomes

$$p_i = \frac{\partial F_1}{\partial q^i}, \qquad P_i = -\frac{\partial F_1}{\partial Q^i}, \qquad H' = H + \frac{\partial F_1}{\partial t}. \tag{2.18}$$

The first generating function which has been found above depends on the old coordinates and the new coordinates. By using Legendre transformation, one can find another type of generating function which is a function of the old coordinates q^i, the new momenta P_i and time t. One just needs to rewrite the second term in the equation (2.18) as a total differential minus the differential of the momenta P_i, then one gets

$$\begin{aligned} dF &= \sum_i p_i dq^i - \sum_i d(P_i Q^i) + \sum_i Q^i dP_i + (H' - H)dt, \\ d(F + \sum_i P_i Q^i) &= \sum_i p_i dq^i + \sum_i Q^i dP_i + (H' - H)dt. \end{aligned} \tag{2.19}$$

This is called the generating function of the second kind and is labeled by F_2. The equations it gives for the old and new variables are

$$p_i = \frac{\partial F_2}{\partial q^i}, \qquad Q^i = \frac{\partial F_2}{\partial P_i}, \qquad H' = H + \frac{\partial F_2}{\partial t}. \tag{2.20}$$

One can also obtain generating functions of variables $F_3(p,P)$ or $F_4(p,Q)$ by similar steps. In the Hamilton-Jacobi formalism, the canonical transformations with a generating function of the is used as usual, and it will be shown below.

Before describing the Hamilton-Jacobi formalism, we should have a few words on complete integrability or Liouville integrability. A system with n degrees of freedom can be solved by quadratures, which means that the general solution can be obtained by more than one integration, if one is able to find n independent integrals of motion in involution. By involution, it is meant that the Poisson brackets of the integrals of motion vanish

$$\{K_i, K_j\} = 0, \tag{2.21}$$

where K_i $(i = 1, \ldots, n)$ are integrals of motion. A proof of this theorem can be found in [20]. Therefore, one needs to find n constants of motion to show that a system is completely integrable and this is what the Hamilton-Jacobi equation is actually based on.

Now we are ready to derive the Hamilton-Jacobi equation. From the action (2.14), it is easy to see that the action satisfies

$$\frac{\partial S}{\partial t} + H(q^1, ..., q^n, p_1, ..., p_n; t) = 0. \tag{2.22}$$

Moreover, if one considers the action as the generating function of the second kind, then the partial derivative of the action with respect to the generalized coordinates $\frac{\partial S}{\partial q_i}$ in (2.14) results in the canonical momenta p_i. When one changes the canonical momenta with partial derivative of the action with respect to the generalized coordinates $\frac{\partial S}{\partial q_i}$ in (2.22), one gets the Hamilton-Jacobi equation. Thus the Hamilton-Jacobi equation is

$$\frac{\partial S}{\partial t} + H(q^1, ..., q^n, \frac{\partial S}{\partial q^1}, ..., \frac{\partial S}{\partial q^n}; t) = 0. \tag{2.23}$$

Complete integrability of a system containing n degrees of freedom and time must have $n+1$ arbitrary constants since there should be $n+1$ integrations to solve the system. In the above equation, not the function S itself but only its first partial derivatives appear, therefore one of the constants will be an integration constant which is added to the general function and solves the equation (2.23) as

$$S = F(t, q^1, ..., q^n; \alpha_1, ..., \alpha_n) + C, \tag{2.24}$$

where α_i $(i = 1, ..., n)$ are arbitrary constants and C is also the integration constant added to the last integration. One can use the function $F = F(t, q^i; \alpha_i)$ as the generating function of the canonical transformations and look for a solution of the Hamilton-Jacobi equation. The generating function F is a function of the old coordinates and the new momenta, therefore we can use the equations found from the generating function of the second kind which have been derived earlier. The equations yield

$$p_i = \frac{\partial F}{\partial q^i}, \qquad \beta^i = \frac{\partial F}{\partial \alpha_i}, \qquad H' = \frac{\partial F}{\partial t} + H, \tag{2.25}$$

where β_i are the new coordinates of the system. Since F differs from the action only by a constant, it also satisfies the equation (2.22). Hence the new Hamiltonian is simply zero, $H' = 0$. The Hamilton's equations of motion for the new variables become

$$\dot{\alpha}_i = \{H', \beta_i\} = 0 \qquad \text{and} \qquad \dot{\beta}^i = \{H', \alpha^i\} = 0. \tag{2.26}$$

With the help of these equations of motion one can identify the n coordinates by the $2n$ constants which have been found. We will discuss a simple example as an application of the method after we discuss the separability of the action.

Separation of the action is of great use when solving the Hamilton-Jacobi equation since it makes easier to find the constants which are needed. Let $\Phi = \Phi(q^i, t, \frac{\partial S}{\partial q^i}, \frac{\partial S}{\partial t})$ denote the Hamilton-Jacobi equation. If there exists a coordinate q^1 in the Hamilton-Jacobi equation such that the coordinate q^1 and $\frac{\partial S}{\partial q^1}$ appear only in a combination which can be represented as $\phi = \phi(q^1, \frac{\partial S}{\partial q^1})$, then the Hamilton-Jacobi equation is separable in this coordinate and can be written as

$$\Phi\Big(q^j, t, \frac{\partial S}{\partial q^j}, \frac{\partial S}{\partial t}, \phi(q^1, \frac{\partial S}{\partial q^1})\Big) = 0, \tag{2.27}$$

where $j = 2, ..., n$. Then one looks for solutions of the action in the form of

$$S = S'(q^j; t) + S_1(q^1). \tag{2.28}$$

Substitution of this in (2.27) will result in

$$\Phi\Big(q^j, t, \frac{\partial S}{\partial q^j}, \frac{\partial S}{\partial t}, \phi(q^1, \frac{dS_1}{dq^1})\Big) = 0. \tag{2.29}$$

This equation should hold for any value of the coordinate q^1, therefore ϕ must be a constant to satisfy this condition

$$\phi(q^1, \frac{\partial S}{\partial q^1}) = \alpha_1. \tag{2.30}$$

The equation (2.27) then becomes

$$\Phi\Big(q^j, t, \frac{\partial S}{\partial q^j}, \frac{\partial S}{\partial t}, \alpha_1\Big) = 0. \tag{2.31}$$

If one is able to do these steps recursively for all the coordinates, the resulting equation can be expressed as

$$S = \sum_i S_i(q^i; \alpha_1, ..., \alpha_n) - E(\alpha_1, ..., \alpha_n)t. \tag{2.32}$$

Cyclic coordinates make it even easier to separate the action. Since momentum of a cyclic coordinate is already a constant, one does not need to bother for looking at the equation (2.23). Let q_1 be a cyclic coordinate and α_1 be its momentum, then its action can be written as

$$S = S'(q^j, t) + \alpha_1 q^1. \tag{2.33}$$

A simple example will be enlightening to clarify how the machinery developed so far is working. If one has the Hamiltonian

$$H = \frac{1}{2}\Big(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2 sin^2\theta}\Big) + U(r, \theta), \tag{2.34}$$

then it is separable if the function U has the following form

$$U(r,\theta) = a(r) + \frac{b(\theta)}{r^2}. \tag{2.35}$$

The variable ϕ is cyclic and therefore one can write the action as

$$S = S' + p_\phi \phi, \tag{2.36}$$

and the Hamilton-Jacobi equation becomes

$$\frac{1}{2m}\left[\frac{\partial S'}{\partial r}\right]^2 + a(r) + \frac{1}{2mr^2}\Big(\left[\frac{\partial S'}{\partial \theta}\right]^2 + 2mb(\theta)\Big) + \frac{1}{2mr^2 sin^2\theta} p_\phi^2 = E. \tag{2.37}$$

As we have discussed, one would look for the separable action to be of the form

$$S = p_\phi\, \phi + S_r(r) + S_\theta(\theta). \tag{2.38}$$

If the action has the desired form as above, the Hamilton-Jacobi equation yields

$$\frac{1}{2m}\left[\frac{dS_r}{dr}\right]^2 + a(r) + \frac{1}{2mr^2}\Big(\left[\frac{dS_\theta}{d\theta}\right]^2 + 2mb(\theta)\Big) + \frac{1}{2mr^2 sin^2\theta} p_\phi^2 = E. \tag{2.39}$$

The Hamilton-Jacobi equation became a separable equation and separating this equation will give another constant

$$\left[\frac{dS_\theta}{d\theta}\right]^2 + 2mb(\theta) + \frac{1}{sin^2\theta} p_\phi^2 = \beta. \tag{2.40}$$

Then the Hamilton-Jacobi equation becomes even simpler and it contains three constants

$$\frac{1}{2m}\left[\frac{dS_r}{dr}\right]^2 + a(r) + \frac{\beta}{2mr^2} = E. \tag{2.41}$$

Integrating the differential equations which have been found, one would get the action as

$$\begin{aligned} S = -Et + p_\phi \phi + \int^\theta d\theta' \sqrt{\beta - 2mb(\theta') - \frac{p_\phi^2}{\sin^2\theta'}} \\ + \int^r dr' \sqrt{2m\Big(E - a(r') - \frac{\beta}{r'^2}\Big)} \end{aligned} \tag{2.42}$$

Differentiating the action with respect to the constants which have been found and equating them to constants, one would get the general solutions of the equations of motion as we will do repeatedly in Chapter 4.

2.2 Stokes' Theorem

Stokes' theorem and the derivation of a conserved charge by employing it will be an important tool for our study of gravitational charges. In this section we will state Stokes' theorem without proving it and cast into a form which will be useful in the following chapters. Moreover we will derive a conservation rule out of Stokes' theorem for divergenceless tensor fields. We will mostly follow the discussion given in [21].

Let M be an n-dimensional compact oriented manifold with boundary ∂M and let α be an $(n-1)$ form on M. Then

$$\int_M d\alpha = \int_{\partial M} \alpha. \tag{2.43}$$

An $(n-1)$ form α can be obtained from a vector field v^a by Hodge dualization

$$\alpha_{a_1 \dots a_{n-1}} = \varepsilon_{b a_1 \dots a_{n-1}} v^b. \tag{2.44}$$

where $\varepsilon_{a_1 \dots a_n}$ is the volume element. Taking the exterior derivative of the above equation will result in an n form

$$\begin{aligned}(d\alpha)_{c a_1 \dots a_{n-1}} &= n\nabla_{[c}(\varepsilon_{|b|a_1 \dots a_{n-1}]} v^b) \\ &= n\varepsilon_{b[a_1 \dots a_{n-1}} \nabla_{c]} v^b,\end{aligned} \tag{2.45}$$

where

$$\nabla_c \varepsilon_{b a_1 \dots a_{n-1}} = 0 \tag{2.46}$$

has been used. On the other hand by using the Hodge dual, an n form can be thought of as proportional to $\varepsilon_{c a_1 \dots a_{n-1}}$. Hence,

$$\varepsilon_{b\,[a_1 \dots a_{n-1}} \nabla_{c]} v^b = h\, \varepsilon_{c a_1 \dots a_{n-1}}. \tag{2.47}$$

Contracting both sides with $\varepsilon^{c a_1 \dots a_{n-1}}$, we would get

$$\nabla_b v^b = n h. \tag{2.48}$$

Therefore, it is found that

$$(d\alpha)_{a_1 \dots a_n} = (\nabla_a v^a) \varepsilon_{a_1 \dots a_n}. \tag{2.49}$$

Then, the Stokes' theorem becomes

$$\int_M d^n x \sqrt{-g} \nabla_b v^b = \int_{\partial M} d^{n-1} y \sqrt{\gamma} n_b v^b, \tag{2.50}$$

where n_b is the normal to the hypersurface and γ is the determinant of the induced metric on that hypersurface.

There remains to obtain a conservation rule from divergenceless quantities. In what follows, we will derive the conservation rule for a vector field, however the generalization to tensor of any rank is straightforward. Let J^a be a divergence free vector field

$$\nabla_a J^a = 0. \tag{2.51}$$

Then,

$$\oint_\Sigma J^a d\Sigma_a = 0 \tag{2.52}$$

for any closed hypersurface Σ, where $d\Sigma_a = \varepsilon n_a \sqrt{\gamma} d^3 y$, and $\varepsilon = +1$ for timelike hypersurfaces and -1 for spacelike hypersurfaces. The boundary consists of two spacelike hypersurfaces and a timelike hypersurface at spatial infinity. If J^a vanishes at spatial infinity, the integral becomes

$$\int_{\Sigma_1} J^a d\Sigma_a + \int_{\Sigma_2} J^a d\Sigma_a = 0. \tag{2.53}$$

Let $n_a = n_{2a}$ for Σ_2, and $n_a = -n_{1a}$ for Σ_1, where n_{1a} and n_{2a} are both future directed. We then obtain

$$\int_{\Sigma_1} J^a d\Sigma_a = \int_{\Sigma_2} J^a d\Sigma_a. \tag{2.54}$$

Thus, we conclude that the total charge Q can be defined as

$$Q \equiv \int_\Sigma J^a n_a d\Sigma, \tag{2.55}$$

and this is independent of the hypersurface on which it is evaluated, if J^a is a divergenceless vector. Finally, note that we have found an integral representation of a conserved charge with the help of Stokes' theorem which we will be using extensively in Chapters 3 and 5.

2.3 Linearization

Linearization in gravity is the perturbation around the background spacetime . By background geometry we mean the asymptotical behavior of the spacetime under consideration. In this work the linearization method is used for the conserved gravitational charges. One tries to find information about the solution such as energy or angular momentum with the help of these charges. These quantities can be measured relative to the background in a way analogous to electrical potential where one sets the potential to be zero at infinity and measures the potential difference with respect to infinity to find the effect of the source. An example may help one to better understand the idea of background spacetime. The Schwarzschild metric differs from the Minkowski metruc by $\frac{2m}{r}$ term, and for the Sun, and it has its maximum at the surface of the Sun with a value of order $of10^{-5}$ [22]. Here the Minkowski metric is the background metric $\bar{g}_{ab}$ and the effects of the sun is the deviation h_{ab} from the background. With the help of these ideas one can decompose the metric as

$$g_{ab} = \bar{g}_{ab} + h_{ab}. \tag{2.56}$$

Since we will be dealing only with first order deviations throughout this work, we will neglect the higher order terms. Using the definition $g_{ab}g^{bc} = \delta^c{}_a$, one finds that the inverse metric is of the form

$$g^{ab} = \bar{g}^{ab} - h^{ab} + \mathscr{O}(h^2). \tag{2.57}$$

We are interested in linearizing the field equations and to do so, we first need to linearize the Christoffel symbols on which the curvature tensors and the Ricci scalar are based. The definition of the Christoffel symbol is

$$\Gamma^a{}_{bc} \equiv \frac{1}{2} g^{ad}(\partial_b\, g_{cd} + \partial_c\, g_{bd} - \partial_d\, g_{bc}). \tag{2.58}$$

To linearize the Christoffel symbol, the metric in equation (2.56) is substituted in the definition

$$\begin{aligned}
\Gamma^a{}_{bc} &= \frac{1}{2}(\bar{g}^{ad} - h^{ad})[\partial_b\,(\bar{g}_{cd} + h_{cd}) + \partial_c\,(\bar{g}_{bd} + h_{bd}) - \partial_d\,(\bar{g}_{bc} + h_{bc})] && (2.59)\\
&= \frac{1}{2}\bar{g}^{ad}(\partial_b\,\bar{g}_{cd} + \partial_c\,\bar{g}_{bd} - \partial_d\,\bar{g}_{bc}) - \frac{1}{2}h^{ad}(\partial_b\,\bar{g}_{cd} + \partial_c\,\bar{g}_{bd} - \partial_d\,\bar{g}_{bc})\\
&\quad + \frac{1}{2}\bar{g}^{ad}(\partial_b\,h_{cd} + \partial_c\,h_{bd} - \partial_d\,h_{bc}), && (2.60)
\end{aligned}$$

the first term in the first line is the Christoffel symbol of the background, $\bar{\Gamma}^a{}_{bc}$, and all the other terms which are first order in h will be considered as the linearized Christoffel symbol. Thus the Christoffel symbol becomes

$$\Gamma^a{}_{bc} = \bar{\Gamma}^a{}_{bc} + (\Gamma^a{}_{bc})_L + \mathscr{O}(h^2). \tag{2.61}$$

These labelings will be used from now on in what follows. The linearized Christoffel symbol is expressed as

$$(\Gamma^a{}_{bc})_L = \frac{1}{2}\bar{g}^{ad}(\partial_b\,h_{cd} + \partial_c\,h_{bd} - \partial_d\,h_{bc}) - \frac{1}{2}h^{ad}(\partial_b\,\bar{g}_{cd} + \partial_c\,\bar{g}_{bd} - \partial_d\,\bar{g}_{bc}). \tag{2.62}$$

This expression can be written in a simpler form if one considers to replace all the partial derivatives with covariant derivatives and to add the extra terms in the equation. The covariant derivatives are

$$\begin{aligned}\bar{\nabla}_b h_{cd} &= \partial_b h_{cd} - \bar{\Gamma}^e{}_{cb}\, h_{ed} - \bar{\Gamma}^e{}_{db}\, h_{ce} \\ \partial_b h_{cd} &= \bar{\nabla}_b h_{cd} + \bar{\Gamma}^e{}_{cb}\, h_{ed} + \bar{\Gamma}^e{}_{db}\, h_{ce},\end{aligned} \tag{2.63}$$

and since

$$\bar{\nabla}_b \bar{g}_{cd} = \partial_b \bar{g}_{cd} - \bar{\Gamma}^e{}_{cb}\, \bar{g}_{ed} - \bar{\Gamma}^e{}_{db}\, \bar{g}_{ce} = 0, \tag{2.64}$$

one has $\partial_b \bar{g}_{cd} = \Gamma^e{}_{cb}\, \bar{g}_{ed} + \Gamma^e{}_{db}\, \bar{g}_{ce}$. Substituting these into the equation (2.62), one gets

$$(\Gamma^a{}_{bc})_L = \frac{1}{2}\bar{g}^{ad}[\bar{\nabla}_b h_{cd} + \bar{\nabla}_c h_{bd} - \bar{\nabla}_d h_{bc}]. \tag{2.65}$$

The linearized Christoffel symbol is a tensor as is obvious from the above equation, while the ordinary Christoffel symbols is not a tensor. Having linearized the Christoffel symbol, one can move further and linearize the Riemann tensor. Riemann tensor is defined as

$$R^a{}_{bcd} = \partial_c \Gamma^a{}_{bd} - \partial_d \Gamma^a{}_{bc} + \Gamma^a{}_{ce}\Gamma^e{}_{bd} - \Gamma^a{}_{de}\Gamma^e{}_{bc}. \tag{2.66}$$

Substituting (2.61) into the above equation results in

$$\begin{aligned}R^a{}_{bcd} &= \partial_c\, [\bar{\Gamma}^a{}_{bd} + (\Gamma^a{}_{bd})_L] - \partial_d\, [\bar{\Gamma}^a{}_{bc} + (\Gamma^a{}_{bc})_L] \\ &\quad + [\bar{\Gamma}^a{}_{ce} + (\Gamma^a{}_{ce})_L][\bar{\Gamma}^e{}_{bd} + (\Gamma^e{}_{bd})_L] \\ &\quad - [\bar{\Gamma}^a{}_{de} + (\Gamma^a{}_{de})_L][\bar{\Gamma}^e{}_{bc} + (\Gamma^e{}_{bc})_L] + \mathscr{O}(h^2) \qquad (2.67) \\ &= \bar{R}^a{}_{bcd} + \partial_c(\Gamma^a{}_{bd})_L - \partial_d(\Gamma^a{}_{bc})_L + \bar{\Gamma}^a{}_{ce}(\Gamma^e{}_{bd})_L + (\Gamma^a{}_{ce})_L\bar{\Gamma}^e{}_{bd} \\ &\quad - \bar{\Gamma}^a{}_{de}(\Gamma^e{}_{bc})_L - (\Gamma^a{}_{de})_L\bar{\Gamma}^e{}_{bc} + \mathscr{O}(h^2). \qquad (2.68)\end{aligned}$$

Thus, the linearized Riemann tensor is

$$\begin{aligned}(R^a{}_{bcd})_L &= \partial_c(\Gamma^a{}_{bd})_L - \partial_d(\Gamma^a{}_{bc})_L + \bar{\Gamma}^a{}_{ce}(\Gamma^e{}_{bd})_L + (\Gamma^a{}_{ce})_L\bar{\Gamma}^e{}_{bd} \\ &\quad - \bar{\Gamma}^a{}_{de}(\Gamma^e{}_{bc})_L - (\Gamma^a{}_{de})_L\bar{\Gamma}^e{}_{bc}.\end{aligned} \tag{2.69}$$

The trick of rewriting the partial derivatives in terms of covariant derivatives and arranging the terms accordingly can be applied to the linearized Christoffel symbols. Once again this greatly simplifies the expression and the Riemann tensor becomes

$$(R^a{}_{bcd})_L = \bar{\nabla}_c(\Gamma^a{}_{bd})_L - \bar{\nabla}_d(\Gamma^a{}_{bc})_L. \tag{2.70}$$

Thus the explicit expression of the linearized Riemann tensor is

$$
\begin{aligned}
(R^a{}_{bcd})_L = & \frac{1}{2}\bar{\nabla}_c[\bar{g}^{ae}(\bar{\nabla}_b h_{de} + \bar{\nabla}_d h_{be} - \bar{\nabla}_e h_{bd})] \\
& - \frac{1}{2}\bar{\nabla}_d[\bar{g}^{ae}(\bar{\nabla}_b h_{ce} + \bar{\nabla}_c h_{be} - \bar{\nabla}_e h_{bc})]. \qquad (2.71)
\end{aligned}
$$

The linearized Ricci tensor can be obtained from (2.70) by contracting the indices a and c. Then, the linearized Ricci tensor with two lower indices is expressed as

$$
(R_{ab})_L = \bar{\nabla}_c(\Gamma^c{}_{ab})_L - \bar{\nabla}_b(\Gamma^c{}_{ac})_L. \qquad (2.72)
$$

For what follows we will need to raise one of the indices, however raising indices is not so straightforward for linearized terms. In order to see this, let us look at the Ricci tensor and carefully raise one of its indices

$$
\begin{aligned}
g^{ac}R_{bc} &= (\bar{g}^{ac} - h^{ac})[\bar{R}_{bc} + (R_{bc})_L + \mathscr{O}(h^2)] \\
&= \bar{g}^{ac}\bar{R}_{bc} - h^{ac}\bar{R}_{bc} + \bar{g}^{ac}(R_{bc})_L + \mathscr{O}(h^2). \qquad (2.73)
\end{aligned}
$$

Therefore, we see that raising or lowering an index will bring in an extra term to the linearized part of the tensor. The linearized part of the Ricci tensor is now

$$
(R^a{}_b)_L = -h^{ac}\bar{R}_{bc} + \bar{g}^{ac}(R_{bc})_L, \qquad (2.74)
$$

and the explicit form is

$$
(R^a{}_b)_L = -h^{ac}\bar{R}_{bc} + \frac{1}{2}\bar{g}^{ac}(-\bar{\Box}h_{bc} - \bar{\nabla}_c\bar{\nabla}_b h + \bar{\nabla}^d\bar{\nabla}_b h_{cd} + \bar{\nabla}^d\bar{\nabla}_c h_{bd}). \qquad (2.75)
$$

Contracting the indices of the linearized Ricci tensor will result in the linearized Ricci scalar as expected. Hence, the linearized Ricci scalar is explicitly

$$
R_L = \bar{\nabla}_a\bar{\nabla}_b h^{ab} - \bar{\Box}h - h^{ab}\bar{R}_{ab}. \qquad (2.76)
$$

The linearized versions of the curvature tensors and the curvature scalar have been found. We can now continue and linearize the field equations which will be needed. In this work, only General Relativity is considered for the conserved gravitational charges. Therefore linearizing the Einstein tensor will be enough. The Einstein tensor reads

$$
G_{ab} = R_{ab} - \frac{1}{2}g_{ab}R, \qquad (2.77)
$$

and its linearization is easy

$$
G_{ab} = \bar{R}_{ab} + (R_{ab})_L - \frac{1}{2}\bar{g}_{ab}\bar{R} - \frac{1}{2}h_{ab}\bar{R} - \frac{1}{2}\bar{g}_{ab}R_L + \mathscr{O}(h^2). \qquad (2.78)
$$

Thus, the linearized version of the Einstein tensor is

$$(G_{ab})_L = (R_{ab})_L - \frac{1}{2} h_{ab} \bar{R} - \frac{1}{2} \bar{g}_{ab} R_L. \tag{2.79}$$

The explicit expression in terms of the background metric $\bar{g}_{ab}$ and the deviation h_{ab} can be found by substituting the explicit versions of the related linearized terms and this will be presented in Chapter 5.

We have derived the linearized versions of the related curvature tensors and the Einstein tensor. Obviously one can use similar steps to linearize any field equations.

CHAPTER 3

KILLING VECTORS

3.1 Introduction

Killing vectors are widely used objects in relativistic theories. Killing vectors of a given metric are generators of the isometries of the geometry described by that metric. They also remain parallel along any geodesic. With the knowledge of these properties, one naturally expects Killing vectors to be of great use in the study of geodesic motion. When one considers the Lagrangian describing the motion of a particle, one can find out that Killing vectors are the symmetries of the system and lead to conserved canonical momenta analogous to cyclic coordinates in classical mechanics as will be discussed in Section 3.2. One can also try to find another conserved quantity related to the spacetime itself, if the background metric possesses globally well defined Killing vectors. This was discussed in [1] and a conserved current was obtained with the help of a Killing vector for Einstein's theory in flat background or the cosmological Einstein theory in AdS background. Moreover, this current leads to a conserved charge as discussed in Section 2.2 and it gives the ADM mass or the angular momentum of the spacetime depending on the Killing vector used in the construction of the charge. Therefore, it is found that the Killing vectors can also be used to find the ADM mass or angular momentum of a source under investigation. Later, the current was generalized to quadratic curvature theories that admit AdS or flat backgrounds in [2]. In [3], the procedure was further generalized to spacetimes with arbitrary backgrounds. These will also be discussed in detail in Section 3.3.

The Lie derivative of a tensor $T^{a_1 \cdots a_n}$ along a parametrized curve γ calculates the change of the tensor along the curve γ. Therefore it is a suitable tool to examine the properties of the

Killing vectors. Definition states that they are the generators of the isometries of the metric. Therefore the Lie derivative of the metric along the flow of the Killing vector should vanish. Let ξ^a be a vector field. Then the Lie derivative of the metric along its flow is

$$\begin{aligned} \mathscr{L}_\xi g_{ab} &= \xi^c g_{ab;c} + g_{ac}\xi^c{}_{;b} + g_{bc}\xi^c{}_{;a} \\ &= \xi_{a;b} + \xi_{b;a} \\ &= 2\xi_{(a;b)}. \end{aligned} \tag{3.1}$$

The first term in the first line vanishes due to metric compatibility. If this vector ξ^a is a Killing vector, then the above equation must vanish. Therefore, the Killing vectors satisfy

$$\xi_{(a;b)} = 0, \tag{3.2}$$

which is called the Killing equation. The vanishing of the Lie derivative of the metric along the Killing vector is important. This importance can be seen as follows. By using the covariance principle, one can choose a coordinate system such that x^1,x^2 and x^3 are all constant while $x^0 = \lambda$ along the integral curve of the Killing vector, where the integral curve is parametrized by λ. Then, the Killing vector is

$$\xi^a = \frac{Dx^a}{d\lambda} \stackrel{*}{=} \delta^a{}_0, \tag{3.3}$$

where $\stackrel{*}{=}$ means that equality holds for the desired coordinate system and here $\frac{D}{d\lambda}$ denotes absolute differentiation. Then the derivative of the metric along the integral curve of the Killing vector becomes

$$\begin{aligned} \mathscr{L}_\xi g_{ab} &\stackrel{*}{=} g_{ab,c}\xi^c + \xi^c{}_{,a} g_{bc} + \xi^c{}_{,b} g_{ba} \\ &\stackrel{*}{=} g_{ab,0} \qquad (3.4) \\ &\stackrel{*}{=} 0. \qquad (3.5) \end{aligned}$$

It implies that the metric does not depend on the coordinate x^0 in this coordinate system. Therefore they are the symmetries of the Lagrangian of the geodesic motion and conserved canonical momenta can be obtained by using them. These will be discussed in detail in the next section.

It was also stated that the Killing vectors are parallel along any geodesic. Consider a vector field u^a which is parametrized by λ such that

$$u^a = \frac{dx^a}{d\lambda}. \tag{3.6}$$

If it is a geodesic it should satisfy the geodesic equation

$$\frac{Du_a}{d\lambda} = u^b u_{a;b} = 0. \tag{3.7}$$

Also the parameter λ is called an affine parameter if the tangent vector u^a describes a geodesic. Assuming u^a is the tangent of a geodesic, then its contraction with a Killing vector should be a constant. This can be seen as

$$\begin{aligned} \frac{D}{d\lambda}(u^a \xi_a) &= u^b u_{a;b} \xi^a + \xi_{a;b} u^a u^b & (3.8) \\ &= 0. & (3.9) \end{aligned}$$

The first term in the right hand side of (3.8) vanishes due to the geodesic equation and the vanishing of the second term is due to the Killing equation (3.2).

We also need to state and prove some identities which will be necessary later. The first one is

$$R_{bc}\xi^c = -\nabla_c \nabla^c \xi_b. \tag{3.10}$$

The uncontracted Bianchi identity should be the starting point

$$R_{[abc]d} = 0. \tag{3.11}$$

As a second step, this identity should be contracted with the Killing vector ξ^d. Using the definition of the Riemann tensor, one gets

$$\begin{aligned} R_{abcd}\,\xi^d + R_{bcad}\,\xi^d + R_{cabd}\,\xi^d &= 0 \\ [\nabla_a, \nabla_b]\,\xi_c + [\nabla_b, \nabla_c]\,\xi_a + [\nabla_c, \nabla_a]\,\xi_b &= 0 \\ \nabla_a \nabla_b \xi_c - \nabla_b \nabla_a \xi_c + \nabla_b \nabla_c \xi_a - \nabla_c \nabla_b \xi_a + \nabla_c \nabla_a \xi_b - \nabla_a \nabla_c \xi_b &= 0. \end{aligned} \tag{3.12}$$

Using the Killing equation (3.2) in the last line of the equation above yields

$$\begin{aligned} \nabla_a \nabla_b \xi_c + \nabla_c \nabla_a \xi_b - \nabla_b \nabla_a \xi_c &= 0 \\ [\nabla_a, \nabla_b]\,\xi_c &= -\nabla_c \nabla_a \xi_b \\ R_{abcd}\,\xi^d &= -\nabla_c \nabla_a \xi_b. \end{aligned} \tag{3.13}$$

The final step in the proof of the identity is the contraction of the $a-c$ indices and the resulting identity is what has been looked for:

$$R_{bc}\,\xi^c = -\nabla_c \nabla^c \xi_b. \tag{3.14}$$

The second useful identity is $\mathscr{L}_\xi R = 0$. Derivation of this identity will be helpful in deriving yet another identity for Killing-Yano tensors, for which analogous steps will be used. We start by considering the third order covariant derivatives of the Killing vector

$$\nabla_a \nabla_b \nabla_c \xi_d - \nabla_b \nabla_a \nabla_c \xi_d = [\nabla_a, \nabla_b] \nabla_c \xi_d. \tag{3.15}$$

Rewriting the left hand side by using the uncontracted version of the first identity, i.e. (3.13) reads

$$-\nabla_a (R_{cdbe} \xi^e) + \nabla_b (R_{cdae} \xi^e) = R_{abce} \nabla^e \xi_d + R_{abde} \nabla_c \xi^e. \tag{3.16}$$

By expanding the derivatives, one gets

$$(\nabla_b R_{cdae} - \nabla_a R_{cdbe}) \xi^e = R_{abde} \nabla_c \xi^e + R_{abce} \nabla^e \xi_d + R_{cdbe} \nabla_a \xi^e - R_{cdae} \nabla_b \xi^e. \tag{3.17}$$

Next, contracting the $a-d$ indices in equation (3.17), one finds the expression

$$-(\nabla_b R_{ce} + \nabla^a R_{beca}) \xi^e = R_{be} (\nabla_c \xi^e) + R_{abce} (\nabla^e \xi^a) + R_{cabe} (\nabla^a \xi^e) + R_{ce} (\nabla_b \xi^e). \tag{3.18}$$

Using the contracted Bianchi identity,

$$\nabla_a R_{bec}{}^a + \nabla_b R_{eac}{}^a + \nabla_e R_{abc}{}^a = 0, \tag{3.19}$$

for the second term in the left hand side of (3.18), it becomes

$$-(\nabla_b R_{ce} + \nabla_e R_{bc} - \nabla_b R_{ec}) \xi^e = R_{be} (\nabla_c \xi^e) + R_{abce} (\nabla^e \xi^a) + R_{cabe} (\nabla^a \xi^e) + R_{ce} (\nabla_b \xi^e). \tag{3.20}$$

The first and third terms in the left hand side cancel each other, moreover the second and third terms in the right hand side also cancel each other due to the antisymmetry of the indices a and e

$$0 = \xi^e \nabla_e R_{cb} + R_{be} \nabla_c \xi^e + R_{ce} \nabla_b \xi^e.. \tag{3.21}$$

By carefully investigating (3.21), this can be written as the Lie derivative of the Ricci tensor, i.e. simply

$$\mathscr{L}_\xi R_{bc} = \xi^e \nabla_e R_{cb} + R_{be} \nabla_c \xi^e + R_{ce} \nabla_b \xi^e = 0. \tag{3.22}$$

The final step is the contraction of the free indices (3.22) to arrive at the desired identity

$$\mathscr{L}_\xi R = \xi_e \nabla^e R = R_{ab} (\nabla^a \xi^b) + R_{ab} (\nabla^a \xi^b) = 0. \tag{3.23}$$

3.2 Integrals of Motion

In the previous section, we have defined the Killing vectors and mentioned that they lead to conserved canonical momenta in the study of geodesic motion analogous to the cyclic coordinates in classical mechanics. In this section, we will discuss these features of the Killing vectors and we will give a simple example to see that they simplify calculations greatly to find equations of motion.

Answering the question “what is a symmetry” would be a good point to start the discussion. Symmetry is any transformation that leaves the system (form) invariant . The system is invariant if its Lagrangian is invariant up to a boundary term. Additionally, Noether’s theorem states that for every continuous global symmetry of a given system, one can write down a conserved quantity. For the case of classical mechanics, cyclic coordinates are the symmetries of the Lagrangian as mentioned in Section 2.1 and they lead to conserved canonical momenta. Moreover, if a change in time t does not affect the system, then the energy of the system is conserved. However, we are dealing with a relativistic theory, hence we consider time as a coordinate. Now we show how these are generalized in a relativistic theory.

In the absence of all forces other than gravity, the Lagrangian of a freely falling particle is

$$L = \frac{1}{2} g_{ab} \dot{x}^a \dot{x}^b, \tag{3.24}$$

where $\dot{x} \equiv \frac{dx}{d\lambda}$ and λ is an affine parameter. One can easily see that any transformation that leaves the metric invariant is also a symmetry of the system. It is also known that transformation along Killing vectors leaves the metric invariant, thus they are symmetries of the system. Therefore, by using Killing vectors $\xi^a_{(i)}$ one can build conserved quantities $C_i = \xi^a_{(i)} u_a$ and here i labels the Killing vector. If Killing vectors are along the coordinates then the corresponding conserved quantities can be identified as the canonical momenta of the system. A Killing vector along the time t leads to a conserved canonical momenta just like other coordinates, and the component of the momenta corresponding to the time component is energy.

We will look at an example given in [23]. Consider a spherically symmetric static spacetime whose metric reads

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2). \tag{3.25}$$

It is obvious that the metric does not depend on the coordinates t and ϕ, and the translations along them leave the metric invariant for the reasons discussed in the previous section. Therefore, one can identify the vectors

$$\xi^a_{(t)} = \delta^a{}_0 \frac{\partial}{\partial t}, \qquad \xi^a_{(\phi)} = \delta^a{}_3 \frac{\partial}{\partial \phi}, \tag{3.26}$$

as two of the Killing vectors of the spacetime. One of the Killing vectors is along the coordinate ϕ, so one can identify the conserved quantity $\Phi = \xi^a_{(\phi)} u_a$ as the angular momentum about the azimuthal angle by using our experience in classical mechanics. The other Killing vector is along the time coordinate and the conserved quantity corresponding to that vector $-E = \xi^a_{(t)} u_a$ can be interpreted as the energy of the particle. Killing vectors and their corresponding constants of motion simplify the work significantly as in the case of classical mechanics. In order to see this, one starts with the general form of the Hamilton-Jacobi equation. Here we use the relativistic formulation, therefore time t is no longer a parameter as mentioned before. Here the Hamilton-Jacobi equation is slightly different from the one given in (2.23) as we have substituted time with affine parameter λ and time considered a coordinates of spacetime. The Hamilton-Jacobi equation reads

$$\frac{\partial S}{\partial \lambda} + \frac{1}{2} g^{ab} \frac{\partial S}{\partial x^a} \frac{\partial S}{\partial x^b} = 0. \tag{3.27}$$

If there is a separable solution of the action, from the symmetries it must be of the form

$$S = m\lambda - Et + L\phi + S_r, \tag{3.28}$$

where S_r is a function of r only , when the system is on the $\theta = \frac{\pi}{2}$ plane. If one uses the action given above in the Hamilton-Jacobi equation, then one gets

$$0 = \frac{1}{2}m - \frac{1}{A(r)^2}E^2 + \frac{1}{r^2}L^2 + \frac{1}{f(r)^2}\left(\frac{\partial S_r}{\partial r}\right)^2 \tag{3.29}$$

$$\left(\frac{\partial S_r}{\partial r}\right) = \sqrt{f(r)^2\Big[\frac{1}{A(r)^2}E^2 - \frac{1}{2}m - \frac{1}{r^2}L^2\Big]}. \tag{3.30}$$

The only function to be determined is S_r and integration of the Hamilton-Jacobi equation yields

$$S_r = \int dr \sqrt{E^2 - \frac{1}{2}m - L^2}. \tag{3.31}$$

In order to find the equations of motion, one should follow the steps described in Section 2.1. As seen from the example given, the Killing vectors simplified the Hamilton-Jacobi equation greatly and made equations of motion easier to solve. Moreover, the two constants of motion related to Killing vectors and the conservation of the rest mass assures that the system is completely integrable. However, sometimes Killing vectors are not enough to see that the system is completely integrable. If this is the case, then one should look for Killing tensors which will be discussed in Chapter 4.

3.3 Abbott-Deser-Tekin Charge

The Killing vectors are known to be related to the conserved quantities of the geodesic motion, as we have mentioned in the previous section. In [1], Abbott and Deser showed that the Killing vectors can also be used to get information about the geometry itself. They found a conserved quantity as current and they defined a conserved charge by using it. They also showed that this charge gives the ADM mass or angular momentum of the geometry under consideration. In [1], they defined the charge for General Relativity with flat background or Cosmological Einstein theory with AdS background. In [2], the method was extended to quadratic curvature theories with flat or AdS backgrounds. Later, the procedure was further generalized to quadratic curvature theories with arbitrary backgrounds in [3].

We are now ready to explain the procedure and we will mostly follow [2]. For a gravitational theory with a coupled source, varying the action with respect to the metric results in the field equations. The general form of the gravitational field equation reads

$$\Phi^{ab} = \kappa T^{ab}, \tag{3.32}$$

where T^{ab} is the energy momentum tensor of the source and Φ^{ab} is the generalized Einstein tensor. We have mentioned that the procedure has also been extended to quadratic curvature theories, therefore in that case generalized Einstein tensor is of the form $\Phi^{ab} = \Phi^{ab}(g, R, R^2)$. Moreover, a gravitational theory should be diffeomorphism invariant, and this imposes the conservation of the field equations Φ^{ab} through the generalized Bianchi identity

$$\nabla_a \Phi^{ab} = 0. \tag{3.33}$$

The whole process is based on measuring the energy of the geometry relative to its asymptotical background spacetime. Therefore one needs to decompose the metric into its background $\bar{g}_{ab}$ and deviation h_{ab} as shown in Section 2.3, i.e.

$$g_{ab} = \bar{g}_{ab} + h_{ab}. \tag{3.34}$$

A number of assumptions has to be made at this point for the sake of safety. We assume that h_{ab} vanishes sufficiently rapidly at infinity. We also assume that background metric $\bar{g}_{ab}$ solves the field equations (3.32) when

$$\bar{\Phi}^{ab} = \bar{T}^{ab} = 0. \tag{3.35}$$

Next, if one linearizes the field equations (3.32), one gets

$$\bar{\Phi}^{ab} + (\Phi^{ab})_L + \mathscr{O}(h^2) = \kappa(\bar{T}^{ab} + (T^{ab})_L + \mathscr{O}(h^2)). \tag{3.36}$$

Then by moving every term in the left hand side except for the linearized field equations (at order h) to the right hand side and using the the background field equations (3.35), this equation reads

$$(\Phi^{ab})_L = \kappa(T^{ab})_L + \mathscr{O}(h^2). \tag{3.37}$$

Considering the right hand side as a new source term and renaming it as τ^{ab}, the equation (3.37) becomes

$$(\Phi^{ab})_L = \kappa\,\tau^{ab}. \tag{3.38}$$

We also need to check the conservation of the linearized field equations to see whether it is conserved or not. Start from the generalized Bianchi identity and linearize it as follows

$$\begin{aligned}
\nabla_a \Phi^{ab} &= 0 \\
\bar{\nabla}_a(\bar{\Phi}^{ab}) + \bar{\nabla}_a(\Phi^{ab})_L + (\Gamma^a{}_{ca})_L \bar{\Phi}^{cb} + (\Gamma^b{}_{ca})_L \bar{\Phi}^{ac} + \mathscr{O}(h^2) &= 0 \\
\bar{\nabla}_a(\Phi^{ab})_L + \mathscr{O}(h^2) &= 0.
\end{aligned} \tag{3.39}$$

Therefore the background covariant derivative of the linearized field equations vanishes to the desired order in h. Background covariant conservation of the linearized field equations might tempt one to use it as the current while defining the charge. However, it is not possible, since $(\Phi^{ab})_L$ is a symmetric object. As discussed in Section 2.2, one needs a totally antisymmetric object to apply Stokes' theorem. In [1], this was overcome by reducing one of the indices of it with a background Killing vector. Here the background Killing vector is assumed to be globally well defined. Then the new object becomes

$$J^a = (\Phi^{ab})_L\,\bar{\xi}_b. \tag{3.40}$$

This is a one index object and can be used safely to define a charge if it is conserved. Examining its divergence will give

$$\begin{aligned}
\bar{\nabla}_a[(\Phi^{ab})_L \bar{\xi}_b] &= [\bar{\nabla}_a(\Phi^{ab})_L]\bar{\xi}_b + (\Phi^{ab})_L \bar{\nabla}_a\bar{\xi}_b \\
&= (\Phi^{ab})_L \bar{\nabla}_{[a}\bar{\xi}_{b]} \\
&= 0. \qquad (3.41)
\end{aligned}$$

The first term in the first line is zero as shown in (3.39). The vanishing of the second term is due to the symmetry of the $(\Phi^{ab})_L$ and antisymmetry of the covariant derivative of the Killing vector.

Then one can use J^a as the current to obtain a conserved charge. The charge can be written as

$$Q^a = \int d^{n-1}x \sqrt{-g}\,(\Phi^{ab})_L \xi_b. \qquad (3.42)$$

With the help of the Poincarè lemma stated before, one then has to express the current as

$$(\Phi^{ab})_L\, \xi_b = \bar{\nabla}_b \mathscr{F}^{ab}. \qquad (3.43)$$

Substituting the new form of the current into the equation (3.42), the conserved charge becomes

$$\begin{aligned}
Q^a &= \int d^{n-1}x\sqrt{-g}\,\bar{\nabla}_b\mathscr{F}^{ab} \\
&= \int_{\partial\Sigma} d^{n-2}y\,\sqrt{\gamma}\,n_b\mathscr{F}^{ab} \\
&= \int_{\partial\Sigma} d\Sigma_b\mathscr{F}^{ab}, \qquad (3.44)
\end{aligned}$$

where $\partial\Sigma$ is the $n-2$ dimensional boundary at spatial infinity whose surface element is $\sqrt{\gamma}d^{n-2}x$ and n_b is the normal to that surface. Thus we have simplified the charge expression significantly. All one needs to do is to find the $\mathscr{F}^{ab}$ for the theory under consideration and use it in (3.44). Here, I ship the details of this since this is outside the scope of this thesis and refer the interested readers to [24] and [25].

The ADT formalism discussed in this section will be useful in Chapter 5 when we derive the conserved gravitational charge using Killing-Yano tensors. The steps taken here will be quite similar to the ones we have just reviewed.

CHAPTER 4

KILLING TENSORS

4.1 Introduction

Killing tensors are generalizations of Killing vectors to rank n totally symmetric tensors. The totally antisymmetric generalizations of the Killing vectors are called Killing-Yano tensors which will be discussed later in Chapter 5. As generalizations of Killing vectors, one would intuitively expect that Killing tensors are also somewhat related to the symmetries of the spacetime. This is indeed the case, the symmetries due to Killing tensors are called hidden symmetries. If there exists any conserved quantity that is higher than first order in momentum, we say that the theory has a “hidden symmetry”.

In [5], Carter discovered the fourth constant of motion of the Kerr spacetime using the separability of the Hamilton-Jacobi equation. Later this constant was called the Carter constant, which we will from now on mostly refer to as well. In [6], it is shown that this fourth constant is related to the Killing tensor. The relationship between separability and the Killing tensor is also discussed in [6], [26], [27] and the references within. In this work, we will only discuss the classical aspects of certain gravitational theories. So we will only examine the Carter constant, the separability of the Hamilton-Jacobi equation and its relationship with the Killing tensor.

Let us start with the properties of the Killing tensor. Let $K_{a_1 \dots a_n}$ be a Killing tensor of rank n. Being a totally symmetric tensor, it satisfies

$$K_{a_1 \dots a_n} = K_{(a_1 \dots a_n)}. \tag{4.1}$$

We have mentioned that it is a generalization of the Killing vector to rank n; thus it must obey the Killing equation to be parallel along the geodesic

$$\nabla_{(a_1}K_{a_2)\ldots a_{n+1}} = 0. \tag{4.2}$$

Having defined the properties of the Killing tensor we are ready to discuss the relationship between the Killing tensor and integrability, the Carter constant and how to derive Killing tensor from Carter constant.

4.2 Killing Tensor and Integrability

We first look at the relationship between integrability and the Killing tensor. As we have mentioned in Section 2.1, we need integrals of motion or the conserved quantities to have a complete description of the system. In Chapter 3, we have mentioned the conserved quantities derived from the Killing vector, however Killing vectors do not reveal all the symmetries of the spacetime. We may have conserved quantities higher than first order in momentum. In this section we will follow the procedure given in [28].

We start by assuming that we have a tensor $K_{a_1..a_s}$ of rank s and a quantity $\mathscr{K}$ formed from this tensor and the momenta

$$\mathscr{K} = K^{a_1\ldots a_s} p_{a_1} \cdots p_{a_s}. \tag{4.3}$$

To find its time derivative we need to consider its Poisson bracket with the Hamiltonian of the system. Considering the Lagrangian given in equation (3.24), the Hamiltonian of the system is

$$H = \frac{1}{2} p_a p_b g^{ab}. \tag{4.4}$$

From the definition given in Section 2.1, the Poisson brackets of $\mathscr{K}$ with the Hamiltonian read

$$\begin{aligned} \dot{\mathscr{K}} &= \{\mathscr{K}, H\} \\ &= \frac{\partial \mathscr{K}}{\partial q^a}\frac{\partial H}{\partial p_a} - \frac{\partial \mathscr{K}}{\partial p_a}\frac{\partial H}{\partial q^a} \\ &= \frac{\partial \mathscr{K}}{\partial q^a}\dot{q}^a - \frac{\partial \mathscr{K}}{\partial p_a}\dot{p}_a. \end{aligned} \tag{4.5}$$

We can find the quantities $\dot{q}^i$ and $\dot{p}_i$ by using their Poisson brackets

$$\dot{q}^a = \{q^a, H\} = p^a, \tag{4.6}$$

$$\dot{p}_a = \{p_a, H\} = \frac{1}{2} g^{bc}{}_{,a} p_b p_c. \tag{4.7}$$

Therefore $\mathscr{K}$ becomes

$$\dot{\mathscr{K}} = K^{a_1...a_s,c} p_c p_{a_1} ... p_{a_s} - \frac{s}{2} g^{bc}{}_{,a_s} K^{a_1...a_s} p_{a_1} ... p_{a_{s-1}} p_c p_b. \tag{4.8}$$

This can be written covariantly,

$$\dot{\mathscr{K}} = K^{a_1...a_s;c} p_c p_{a_1} ... p_{a_s} - \frac{s}{2} g^{bc}{}_{;a_s} K^{a_1...a_s} p_{a_1} ... p_{a_{s-1}} p_c p_b. \tag{4.9}$$

The covariant derivative of the metric vanishes, and all the indices in the first expression are symmetric due to the symmetry of p's. The final expression for the equation is

$$\dot{\mathscr{K}} = K_{(a_1...a_s;c)} p^c p^{a_1} ... p^{a_s}. \tag{4.10}$$

If $\mathscr{K}$ is a conserved quantity, then it is obvious that the tensor should satisfy the equation

$$K_{(a_1...a_s;c)} = 0, \tag{4.11}$$

since equation (4.10) has to be satisfied for any value of p's.

We have shown that if there is a conserved quantity formed from momenta of the system, then there exists a Killing tensor whose contraction with the momenta gives that conserved quantity.

4.3 Carter Constant

In his famous paper [5], Carter found the fourth constant of motion of the Kerr spacetime. That was an important discovery since it shows that the system of a particle orbiting around the black hole is completely integrable and therefore solvable due to the Liouville integrability. In this section we will be discussing two different methods to find the Carter constant.

4.3.1 The first method

The first method is the original one given in [5]. It starts with the Hamiltonian of a particle and assumes that it has a separable action and is based on the investigation of the Hamilton-Jacobi equation for separability. The separability of the Hamilton-Jacobi equation leads to the fourth constant. Let us start by writing the metric for the Kerr-Newmann metric in the standard form

$$\begin{aligned} ds^2 = \rho^2 d\theta^2 - 2a sin^2\theta dr d\varphi + 2dr du + \rho^{-2}[(r^2+a^2)^2 - \Delta a^2 sin^2\theta] sin^2\theta d\varphi^2 \\ - 2a\rho^{-2}(2mr - e^2) sin^2\theta d\varphi du - [1 - \rho^{-2}(2mr - e^2)] du^2. \end{aligned} \tag{4.12}$$

Here

$$\rho^2 \equiv r^2 + a^2 cos^2\theta, \tag{4.13}$$

$$\Delta \equiv r^2 - 2mr + a^2 + e^2. \tag{4.14}$$

The associated electromagnetic field tensor of the black hole is

$$F = -2e\rho^{-4}[(r^2 - a^2cos^2\theta)]dr \wedge du - 2a^2 rcos\theta sin\theta d\theta \wedge du$$
$$- a sin^2\theta(r^2 - a^2cos^2\theta)dr \wedge d\varphi + 2ar(r^2 + a^2)cos\theta sin\theta d\theta \wedge d\varphi], \tag{4.15}$$

where e is the electric charge of the black hole.

Now starting with a particle of mass μ and electric charge ε, the equation of motion of the particle can be written as

$$\frac{D^2x^a}{d\lambda^2} = \frac{\varepsilon}{\mu}F^a{}_b\frac{Dx^b}{d\lambda}. \tag{4.16}$$

The above equation is nothing but the relativistic version of the Lorentz force law. It is known from classical mechanics that Lagrangian formalism is more suitable for the study of geodesics. Hence, the Lagrangian for a relativistic charged particle is

$$L = \frac{1}{2}g_{ab}\dot{x}^a\dot{x}^b + \varepsilon A_a\dot{x}^a, \tag{4.17}$$

where

$$\dot{x}^a = \frac{dx^a}{d\lambda}. \tag{4.18}$$

The differentiation is with respect to the affine parameter λ and λ is related to the proper time with

$$\tau = \mu\lambda. \tag{4.19}$$

Defining the proper time this way guarantees that the conservation of rest mass is already imposed on the system

$$g_{ab}\dot{x}^a\dot{x}^b = -\mu^2 \leq 0. \tag{4.20}$$

The minus sign in front of μ means that massive particles follow timelike geodesics, and if the particle is massless, then it follows a null geodesic as desired for a physical system.

The Euler-Lagrange equations for the Lagrangian (4.17) gives the canonical momentum of the system

$$p_a = g_{ab}\dot{x}^b + \varepsilon A_a. \tag{4.21}$$

Having found the canonical momentum, one can easily find that the Hamiltonian of the system is

$$H = \frac{1}{2} g^{ab}(p_a - \varepsilon A_a)(p_b - \varepsilon A_b). \tag{4.22}$$

In terms of the metric tensor, the Hamiltonian is

$$H = \frac{1}{2} g_{ab}\dot{x}^a \dot{x}^b. \tag{4.23}$$

The normalization condition we have imposed in (4.23) leads to

$$H = -\frac{1}{2}\mu^2. \tag{4.24}$$

Since the Hamiltonian does not depend on the affine parameter λ explicitly, the Hamiltonian is a conserved quantity. Thus we have found the first conserved quantity of the system.

As well-known from the relativistic electromagnetic theory, the electromagnetic field 2-form is expressed by the exterior derivative of a vector potential A as $F = 2dA$, where d is the exterior derivative. Here the vector potential is not unique as it is in the classical case, and the simplest choice giving the electromagnetic 2-form (4.15) is

$$A = e\rho^{-2} r(du - a^2 sin^2\theta\, d\varphi). \tag{4.25}$$

It is straightforward to find the explicit expressions for the canonical momentum by using the result found so far. They are

$$p_u = -\Big([1 - \rho^{-2}(2mr - e^2)]\dot{u} - a\rho^{-2}(2mr - e^2)sin^2\theta\dot{\varphi} + \dot{r} + e\rho^{-2} r\Big), \tag{4.26}$$

$$p_r = \dot{u} - a\, sin^2\theta\dot{\varphi}, \tag{4.27}$$

$$p_\theta = \rho^2\dot{\theta}, \tag{4.28}$$

$$\begin{aligned} p_\varphi = &-a\rho^{-2}(2mr - e^2)sin^2\theta\dot{u} + \rho^{-2}[(r^2 + a^2)^2 - \Delta a^2 sin^2\theta]sin^2\theta\dot{\varphi} \\ &- a\, sin^2\theta\,\dot{r} - e\rho^{-2} a r\, sin^2\theta. \end{aligned} \tag{4.29}$$

If one writes down the terms in the Hamiltonian and rearranges them, then one finds that the Hamiltonian takes the form

$$H = \frac{1}{2\rho^2}\Big(\Delta p_r^2 + 2[(r^2 + a^2)\, p_u + a\, p_\varphi - 2\varepsilon e r]\, p_r + p_\theta^2 + (a\, sin\theta\, p_u + sin^{-1}\theta\, p_\varphi)^2\Big) \tag{4.30}$$

Finding the explicit expression for the Hamiltonian, we can move on to the Hamilton-Jacobi equation. However, before doing that , as a final step we should find the conserved quantities coming from the Killing vectors. As a rotating, axisymmetric spacetime, the Kerr spacetime has two Killing vectors, and these can be read directly from the metric since translations along them leave the metric invariant

$$\xi^a_{(u)} = \delta^a{}_0 \frac{\partial}{\partial u}, \qquad \xi^3_{(\varphi)} = \delta^a{}_3 \frac{\partial}{\partial \varphi}. \tag{4.31}$$

These Killing vectors also imply that the coordinates u and φ are cyclic, so that the corresponding canonical momenta are conserved as shown in Chapter 3. Let these conserved quantities be represented by

$$p_u = -E, \qquad p_\varphi = \Phi. \tag{4.32}$$

Three of the conserved quantities have been determined so far, however if the system is to be completely integrable then there should exist another conserved quantity.

If one assumes that the action is separable, then by using the conserved quantities found so far, one should be able to cast the action in the form

$$S = \frac{1}{2}\mu^2\lambda - E\,u + \Phi\,\varphi + S_\theta + S_r, \tag{4.33}$$

where S_r is only a function of r and S_θ is only a function of θ. The form of the Hamilton-Jacobi equation is similar to the one examined in Section 2.1

$$\frac{\partial S}{\partial \lambda} + \frac{1}{2} g^{ab}\Big[\Big(\frac{\partial S}{\partial x^a} - \varepsilon A_a\Big)\Big(\frac{\partial S}{\partial x^b} - \varepsilon A_b\Big)\Big] = 0. \tag{4.34}$$

Writing the explicit form of the Hamilton-Jacobi equation, one would get

$$\begin{aligned} 0 = \mu^2 + \frac{1}{2\rho^{-2}}\Big[\Delta\Big(\frac{dS_r}{dr}\Big)^2 + 2\Big(-(r^2+a^2)E + a\Phi - \varepsilon\, e\, r\Big)\frac{dS_r}{dr} \\ + \Big(\frac{dS_\theta}{d\theta}\Big)^2 + [-a\, sin\theta\, E + sin^{-1}\theta\,\Phi]^2\Big]. \end{aligned} \tag{4.35}$$

A careful investigation of (4.35) shows that the equation is separable. Just separating the two variables r and θ to the two sides of the equation results in the equation

$$\begin{aligned} a^2\mu^2 cos^2\theta + (aE\, sin\theta - \Phi\, sin^{-1}\theta)^2 + \Big(\frac{dS_\theta}{d\theta}\Big)^2 \\ = -\Delta\Big(\frac{dS_r}{dr}\Big)^2 + 2[(r^2+a^2)E - a\Phi + \varepsilon\, e\, r]\frac{dS_r}{dr} - \mu^2 r^2. \end{aligned} \tag{4.36}$$

The left hand side of the equation depends only on the variable θ, while the right hand side depends only on r. Thus it must be that both sides should be equal to a constant. Setting the constant as $\mathscr{K}$, (4.36) can be written as

$$\begin{aligned} \Big(\frac{dS_\theta}{d\theta}\Big)^2 + (aE\, sin\theta - \Phi\, sin^{-1}\theta)^2 + a^2\mu^2 cos^2\theta = \mathscr{K} \\ \Delta\Big(\frac{dS_r}{dr}\Big)^2 - 2[(r^2+a^2)E - a\Phi + \varepsilon\, e\, r]\frac{dS_r}{dr} + \mu^2 r^2 = -\mathscr{K}. \end{aligned} \tag{4.37}$$

Thus the fourth constant of motion $\mathscr{K}$ of the Kerr spacetime has been found. Having done so, it has also been shown that the geodesic motion in Kerr spacetime is a completely integrable system in the sense already discussed.

We have found the Carter constant by using the first method. We are ready to present the second method in the next subsection.

4.3.2 The second method

There is also another method which Carter developed in [29]. This method helps one find the fourth constant of motion directly from the Hamiltonian provided that it has the desired form. It states that if the Hamiltonian is of the form

$$H = \frac{1}{2}\Big(\frac{H_r + H_\theta}{U_r + U_\theta}\Big), \tag{4.38}$$

then there exists a fourth constant of motion in the form of

$$\tilde{\mathscr{K}} \equiv \frac{U_r H_\theta - U_\theta H_r}{U_r + U_\theta}. \tag{4.39}$$

Here $H_r = H_r(r, p_r)$ and $U_r = U_r(r)$ and similar conditions hold for H_θ and U_θ, i.e. $H_\theta = H_\theta(\theta, p_\theta)$ and $U_\theta = U_\theta(\theta)$. Then the Poisson bracket of H_r with the Hamiltonian becomes

$$\{H_r, H\} = \frac{1}{2}\,(H_r + H_\theta)\,\{H_r, \frac{1}{U_r + U_\theta}\}. \tag{4.40}$$

Obviously, U_θ commutes with H_r and U_r, so we need the Poisson bracket of U_r with the Hamiltonian

$$\{U_r, H\} = \frac{1}{2\,(U_r + U_\theta)}\{U_r, H_r\}, \tag{4.41}$$

and the Poisson bracket in the right hand side of the equation (4.40)

$$\{H_r, \frac{1}{U_r + U_\theta}\} = -\frac{1}{(U_r + U_\theta)^2}\frac{\partial H_r}{\partial p_r}\frac{dU_r}{dr} = \frac{1}{(U_r + U_\theta)^2}\{U_r, H_r\}. \tag{4.42}$$

Thus, the Poisson bracket which we are looking for has been found as

$$\{H_r, H\} = 2H\{U_r, H\}. \tag{4.43}$$

Following similar steps for H_θ, one finds that

$$\{H_\theta, H\} = 2H\{U_\theta, H\}. \tag{4.44}$$

Then it follows that the quantities $\tilde{\mathscr{K}_1} \equiv 2U_rH - H_r$ and $\tilde{\mathscr{K}_2} \equiv 2U_\theta H - H_\theta$ commute with the Hamiltonian:

$$\begin{aligned}\{\tilde{\mathscr{K}_1}, H\} \equiv \{2U_rH - H_r, H\} &= 2H\{U_r, H\} - \{H_r, H\} \\ &= 2H\{U_r, H\} - 2H\{U_r, H\} \\ &= 0,\end{aligned} \tag{4.45}$$

and similarly for $\{\tilde{\mathscr{K}_2}, H\} = 0$. Substituting the Hamiltonian (4.38) $in\tilde{\mathscr{K}_1}$, one gets the new representation of the Carter constant

$$\tilde{\mathscr{K}} \equiv \frac{U_rH_\theta - U_\theta H_r}{U_r + U_\theta}. \tag{4.46}$$

One can also check that $\tilde{\mathscr{K}_2}$ leads to the negative of (4.46). The Carter constant $\tilde{\mathscr{K}}$ has now been constructed and if one wants to find it for the Kerr spacetime, one should start form the Hamiltonian (4.30) of a particle in Kerr spacetime. It can be easily seen that the quantities H_r, H_θ, U_r and U_θ for the Kerr spacetime are

$$U_r = r^2, \qquad H_r = \Delta p_r^2 + 2\left[a\Phi - \left(r^2 + a^2\right)E - 2\varepsilon er\right]p_r, \tag{4.47}$$

$$U_\theta = a^2\cos^2\theta, \qquad H_\theta = p_\theta^2 + \left(aE\sin\theta - \Phi\sin^{-1}\theta\right)^2. \tag{4.48}$$

Finally the Carter constant $\tilde{\mathscr{K}}$ is found by substituting these in to the equation (4.46)

$$\mathscr{K} = \frac{r^2\left[p_\theta^2 + \left(aE\sin\theta - \Phi\sin^{-1}\theta\right)\right] - a^2\left(\Delta p_r^2 + 2\left[a\Phi - \left(r^2 + a^2\right)E - 2\varepsilon er\right]\right)\cos^2\theta}{r^2 + a^2\cos^2\theta} \tag{4.49}$$

for the case of Kerr spacetime.

In order to go further and solve the equations of motion, we need to show that these two constants are equal to each other i.e. $\mathscr{K} = \tilde{\mathscr{K}}$. To do so, we consider the constant $\mathscr{K}$ found in Section 4.3.1 first. The Hamiltonian has the form given in equation (4.38), so we can express it as

$$\begin{aligned}H &= \frac{1}{2}\left(\frac{H_r + H_\theta}{U_r + U_\theta}\right), \\ -\frac{1}{2}\mu^2 &= \frac{1}{2}\left(\frac{H_r + H_\theta}{U_r + U_\theta}\right), \\ -\mu^2(U_r + U_\theta) &= H_r + H_\theta, \\ H_\theta + \mu^2U_\theta &= -H_r - \mu^2U_r.\end{aligned} \tag{4.50}$$

The two sides of (4.50) has to be equal for every value of r and θ. Thus we get the fourth constant $\mathscr{K}$ as derived in Section 4.3.1

$$H_\theta + \mu^2 U_\theta = \mathscr{K}, \qquad H_r - \mu^2 U_r = -\mathscr{K}. \tag{4.51}$$

Substituting the H_r and H_θ of (4.51) above in (4.46), one finds

$$\begin{aligned} \tilde{\mathscr{K}} &= \frac{U_r(\mathscr{K} - \mu^2 U_\theta) - U_\theta(-\mathscr{K} - \mu^2 U_r)}{U_r + U_\theta} \\ &= \mathscr{K}. \end{aligned} \tag{4.52}$$

Hence, we have shown that both methods actually yield the same constant and one can use either method depending on which serves better for one's purposes, without losing any information about the system.

The next step to do is to find the equation of motion by using the Hamilton-Jacobi method. We have assumed that the action is separable and has the form given in equation (4.33). The fourth constant of motion $\mathscr{K}$ was derived in equation (4.37). By integrating these equations, we can find the functions S_r and S_θ as

$$S_\theta = \int^\theta \sqrt{\Theta} d\theta, \qquad S_r = \int^r \frac{P + \sqrt{R}}{\Delta} dr, \tag{4.53}$$

where

$$\Theta \equiv \mathscr{K} - (aE\,sin\theta - \Phi\, sin^{-1}\theta)^2 - a^2\mu^2 cos^2\theta \tag{4.54}$$

$$P \equiv E(a^2 + r^2) - \Phi a + \varepsilon e r \tag{4.55}$$

$$R \equiv P^2 - \Delta(\mu^2 r^2 + \mathscr{K}). \tag{4.56}$$

These help us write down the full action

$$S = \frac{1}{2}\mu^2\lambda - E\,u + \Phi\varphi + \int^\theta \sqrt{\Theta} d\theta + \int^r \frac{P + \sqrt{R}}{\Delta} dr. \tag{4.57}$$

The action can be expressed as $S = S(x^a, E, \Phi, \mu, \mathscr{K})$. Then one can consider it as the generating function of the canonical transformations described in Section 2.1, and the four constants $E, \Phi, \mu, \mathscr{K}$ becomes the new momenta P_a of the system. Next, taking partial derivatives of the action, which is also the generating function of the canonical transformations at the same time, with respect to these momenta will give the new canonical coordinates. These new coordinates are also constant as discussed in Section 2.1, and they can be set to zero since they can be compensated by integration constants which will result from the integrations in the action. By doing so, we arrive at the equations

$$\int^{\theta} \frac{d\theta}{\sqrt{\Theta}} = \int^{r} \frac{dr}{\sqrt{R}}, \tag{4.58}$$

$$\lambda = \int^{\theta} \frac{a^2 cos^2\theta d\theta}{\sqrt{\Theta}} + \int^{r} \frac{r^2 dr}{\sqrt{R}}, \tag{4.59}$$

$$u = \int^{\theta} \frac{-a(aE\, sin^2\theta - \Phi)d\theta}{\sqrt{\Theta}} + \int^{r} \frac{r^2+a^2}{\Delta}\Big(1 - \frac{P}{\sqrt{R}}\Big)dr, \tag{4.60}$$

$$\varphi = \int^{\theta} \frac{-a(aE - \Phi sin^{-2}\theta)d\theta}{\sqrt{\Theta}} + \int^{r} \frac{a}{\Delta}\Big(1 - \frac{P}{\sqrt{R}}\Big)dr. \tag{4.61}$$

By just taking the derivatives of the coordiates with respect to the affine parameter λ, we can also find the differentiated forms of the coordinates as

$$\dot{\theta} = \frac{\sqrt{\Theta}}{\rho^2}, \tag{4.62}$$

$$\dot{r} = \frac{\sqrt{R}}{\rho^2}, \tag{4.63}$$

$$\dot{u} = \frac{1}{\rho^2}[\frac{(r^2+a^2)(\sqrt{R}-P)}{\Delta} - a\,(aE\, sin^2\theta\, - \Phi)], \tag{4.64}$$

$$\dot{\varphi} = \frac{1}{\rho^2}[\frac{a(\sqrt{R}-P)}{\Delta} - a\,(aE\, - sin^{-2}\theta\Phi)]. \tag{4.65}$$

We should also note that the signs of the $\sqrt{\Theta}$ and $\sqrt{R}$ terms are not relevant. They can be either plus or minus. However once a specific choice is made for the signs, they should remain unchanged for all the following calculations.

4.4 The Carter Constant and The Killing Tensor

The Carter constant has been found in the previous section. It played a key role in determining the equations of motion for a particle in the Kerr spacetime, since it allowed the use of the Hamilton-Jacobi formalism. In this section we will find the Killing tensor corresponding to this constant.

In general, determining the Killing tensor of a given spacetime is not an easy task. In [6], Penrose and Walker investigated the relationship between the fourth constant of motion of the Kerr spacetime and the Killing tensor leading to it. As a result they developed a method by using spinor formalism to find the Killing tensor giving the fourth constant of motion. However, this method is applicable to spacetimes which are solutions of General Relativity and of Petrov type D. On the other hand, the spacetime, which the methods reviewed will be applied to, is a solution of the Conformal Gravity not General Relativity.

Another method which does this and that has been developed in [30]. The method is based on decomposing the given spacetime to hypersurfaces which are orthogonal to the Killing vectors. However this method does not work for stationary spacetimes since the Killing vector orthogonality condition does not hold for such geometries. It is useful for less complicated spacetimes such as static spacetimes. Moreover, the spacetime in question may have more than one Killing tensor and the Carter constant might be a linear combination of these Killing tensors. So there is no guarantee that the Killing tensor found will give the Carter constant. However, there is another method also given in [29] which determines the Killing tensors directly from the Carter constant. The method is actually quite easy; it just amounts to equating the Carter constant $\mathscr{K}$ to the conserved quantity found from the Killing tensor

$$\mathscr{K} = K^{ab} p_a p_b. \tag{4.66}$$

For example, in the case of the Kerr spacetime, this equation gives

$$\begin{aligned} K^{ab} p_a p_b = \frac{1}{p^2} \Big[& r^2 p_\theta^2 + r^2 \left(a^2 E^2 \sin^2\theta + \Phi \sin^{-2}\theta - 2aE\Phi \right) \\ & - a^2 \cos^2\theta \left(\Delta p_r^2 + 2 \left(a\Phi p_r - \left(r^2 + a^2 \right) E p_r \right) \right) \Big]. \end{aligned} \tag{4.67}$$

A careful examination then gives Killing tensor for the Kerr spacetime as

$$K^{ab} = \begin{bmatrix} \frac{r^2 a^2}{\rho^2} \sin^2\theta & a^2 \cos^2\theta \left(r^2 + a^2 \right) & 0 & \frac{a r^2}{\rho^2} \\ a^2 \cos^2\theta \left(r^2 + a^2 \right) & -\frac{a^2 \cos^2\theta}{\rho^2} \Delta & 0 & \frac{a^3 \cos^2\theta}{\rho^2} \\ 0 & 0 & \frac{r^2}{\rho^2} & 0 \\ \frac{a r^2}{\rho^2} & \frac{a^3 \cos^2\theta}{\rho^2} & 0 & \frac{r^2}{\rho^2} \sin^{-2}\theta \end{bmatrix}. \tag{4.68}$$

Having developed the necessary tools to find the Carter constant of motion and the relevant Killing tensor, we can now move on to the next step and apply this procedure to find the analogous objects of the recently found spacetime presented in [18].

4.5 Solution of Conformal Gravity

A theory which is invariant under conformal transformations $g_{ab} \to \Omega^2(x) g_{ab}$ where $\Omega(x)$ is a function on spacetime is called Conformal Gravity whose action contains square of the

Weyl tensor. Cosmological Einstein gravity can be obtained from the conformal gravity by coupling a specified cosmological constant to square of the Weyl tensor in the action [31]. The spacetime which we will investigate in this section is given in [18]. It is a solution of the four dimensional Conformal Gravity theory with a Maxwell field minimally coupled to it. The theory is given by the action

$$S = \int d^4x\sqrt{-g}\Big(\frac{1}{2}C^{abcd}C_{abcd} + \frac{1}{3}F_{ab}F^{ab}\Big), \tag{4.69}$$

where C_{abcd} is the Weyl (Conformal) tensor and $F = dA$. Varying the action with respect to the 1-form vector field A and the metric, one finds the following field equations

$$\nabla_a F^{ab} = 0, \tag{4.70}$$

$$(2\nabla^c\nabla^d + R^{cd})C_{acdb} + \frac{2}{3}(F_{ac}F^c{}_b - \frac{1}{4}F^2 g_{ab}) = 0, \tag{4.71}$$

respectively. In [18], there are two separate solutions found for these equations; one of them describes a dyonic rotating black hole, which has both an electric and a magnetic charge, and the other one represents a neutral rotating black hole.

The dyonic rotating black hole with electrical charge p and magnetic charge q is described by the metric

$$\begin{aligned} ds^2 &= \rho^2\left(\frac{dr^2}{\Delta_r} + \frac{d\theta^2}{\Delta_\theta}\right) + \frac{\Delta_\theta \sin^2\theta}{\rho^2}\left(adt - \left(r^2+a^2\right)\frac{d\phi}{\Xi}\right)^2 \\ &\quad -\frac{\Delta_r}{\rho^2}\left(dt - a\sin^2\theta\,\frac{d\phi}{\Xi}\right)^2, \end{aligned} \tag{4.72}$$

with the associated vector potential given by

$$A = \frac{qr}{\rho^2}\left(dt - a\sin^2\theta\,\frac{d\phi}{\Xi}\right) + \frac{p\cos\theta}{\rho^2}\left(adt - \left(r^2+a^2\right)\frac{d\phi}{\Xi}\right). \tag{4.73}$$

Here

$$\begin{aligned} \rho^2 &\equiv r^2 + a^2\cos^2\theta, \\ \Delta_r &\equiv \left(r^2+a^2\right)\left(1 - \frac{1}{3}\Lambda r^2\right) - 2mr + \frac{\left(p^2+q^2\right)r^3}{6m}, \\ \Xi &\equiv 1 + \frac{1}{3}\Lambda a^2, \\ \Delta_\theta &\equiv 1 + \frac{1}{3}\Lambda a^2\cos^2\theta. \end{aligned} \tag{4.74}$$

where m is the mass parameter and Λ is the integration constant. By setting the charge parameters p and q to zero, the metric reduces to the Kerr-AdS black hole in four dimensions which solves the cosmological Einstein theory. On the other hand, letting the rotation parameter a to be zero, the solution becomes a static metric with a massive spin-2 hair.

In [18], it was also discussed that neither the Reissner-Nordstorm nor the Kerr-Newman solutions can be embedded into the Conformal theory (4.69) due to the different fall-off behaviors of the charge parameters p and q in the metric.

The neutral solution cannot be obtained by simply setting the charge parameters to zero. As mentioned before, the dyonic solution reduces to the Kerr-AdS solution which is not a solution of the Conformal theory (4.69). To obtain the neutral solution, let p^2+q^2 be $-12m\mu$ and m be zero. It is claimed in [18] that the solution which was obtained by the described substitution and the Kerr-AdS metric are mutually exclusive within this ansatz. This can seen by substituting this ansatz to the equations of motion and letting A be zero and $\Delta_r = \Delta_r(r)$ be a general function of r. By doing so, one will find two different solutions, one of them is the new neutral solution and the other one is Kerr-AdS. The described substitution leads to the metric (4.72) where now

$$\Delta_r = (r^2+a^2)\left(1-\frac{1}{3}\Lambda r^2\right)-2\mu r^3, \tag{4.75}$$

It is not known whether the Hamilton-Jacobi equations of a particle in these new spacetimes are separable or not. The existence of a Carter constant and a Killing tensor was also unknown for these spacetimes beforehand. In this study, we will try to answer these questions. However, while seeking answers, we will not be considering these spacetimes separately. For the sake of brevity, we will only be dealing with the dyonic solution. The procedure can easily be carried on to the neutral solution by applying the ansatz used to obtain the neutral black hole.

The metric reduces to the Kerr-AdS spacetime when both p and q are set to zero, and the separability of the Kerr-AdS shown in [10]. Therefore one can expect that these new spacetimes are also separable and this is what we are going to check first. As mentioned above, we will be dealing with only the dyonic spacetime. Before examining the Hamiltonian, determining the Killing vectors of the dyonic spacetime would be a wise step to take. By examining the metric, it is obvious that the two obvious Killing vectors of the metric are the same with the Kerr spacetime. Their related momenta are also conserved in this case and we can use a similar labeling with the Kerr spacetime by setting $p_t = -E$ and $p_\phi = \Phi$.

The general form of the Hamiltonian of a charged particle was given in (4.22). Substituting the dyonic metric g_{ab} and the vector potential A_a in (4.22) will give the explicit form of the Hamiltonian of the charged particle in the dyonic spacetime. Then the Hamiltonian reads

$$H = \frac{(p_\phi \Xi + sin^2\theta\, p_t)}{\rho^2 \Delta_\theta}\Big[\varepsilon\, p\, \frac{cos\theta}{sin^2\theta} + \frac{p_t\, a}{2} + \frac{p_\phi\, \Xi}{2\, sin^2\theta}\Big] + \frac{\varepsilon^2 p^2 cos^2\theta}{2\rho^2\Delta_\theta sin^2\theta}$$
$$- \frac{\varepsilon^2 q^2 r^2}{2\rho^2\Delta_r} - \frac{(p_t(a^2+r^2) + a\, p_\phi \Xi)}{\Delta_r \rho^2}\Big[\frac{p_\phi\, \Xi\, a}{2} + \frac{p_t(a^2+r^2)}{2} - \varepsilon\, q\, r\Big]$$
$$+ \frac{\Delta_r\, p_r^2}{2\rho^2} + \frac{sin^2\theta\, \Delta_\theta\, p_\theta^2}{2\rho^2}. \tag{4.76}$$

A careful investigation of the Hamiltonian reveals that it has the desired form as in equation (4.38). Therefore, it is guaranteed that the spacetime has the fourth constant of motion, and we can safely assume that the action has the separable form

$$S = \frac{1}{2}\mu^2\lambda - E\, t + \Phi\varphi + S_r(r) + S_\theta(\theta). \tag{4.77}$$

Using the action which we have just defined, the Hamilton-Jacobi equation becomes

$$0 = \mu^2 + \frac{(\Phi\Xi - sin^2\theta\, E)}{\rho^2\Delta_\theta}\Big[2\,\varepsilon\, p\, \frac{cos\theta}{sin^2\theta} - E\, a + \frac{\Phi\Xi}{sin^2\theta}\Big] + \frac{\varepsilon^2 p^2 cos^2\theta}{\rho^2\Delta_\theta sin^2\theta}$$
$$- \frac{\varepsilon^2 q^2 r^2}{\rho^2\Delta_r} + \frac{(E(a^2+r^2) + a\Phi\Xi)}{\Delta_r\rho^2}\Big[\Phi\Xi a - E(a^2+r^2) - 2\varepsilon\, q\, r\Big]$$
$$+ \frac{\Delta_r\, (\frac{dS_r}{dr})^2}{\rho^2} + \frac{sin^2\theta\, \Delta_\theta\, (\frac{dS_\theta}{d\theta})^2}{\rho^2}. \tag{4.78}$$

It is easy to see that this equation is separable and separating it will result in

$$\mu^2 a^2 cos^2\theta + \frac{(\Phi\Xi - sin^2\theta\, E)}{\Delta_\theta}\Big[2\,\varepsilon\, p\, \frac{cos\theta}{sin^2\theta} - E\, a + \frac{\Phi\Xi}{sin^2\theta}\Big]$$
$$+ \frac{\varepsilon^2 p^2 cos^2\theta}{\Delta_\theta sin^2\theta} + sin^2\theta\, \Delta_\theta\, (\frac{dS_\theta}{d\theta})^2$$
$$= -\mu^2 r^2 + \frac{\varepsilon^2 q^2 r^2}{\rho^2\Delta_r}$$
$$- \frac{(E(a^2+r^2) + a\Phi\Xi)}{\Delta_r\rho^2}\Big[\Phi\Xi a - E(a^2+r^2) - 2\varepsilon\, q\, r\Big] - \frac{\Delta_r\, (\frac{dS_r}{dr})^2}{\rho^2}. \tag{4.79}$$

Both sides of this equation depend only on one variable, and as before they should be both equal to a constant for this to hold for any value of θ or r. Hence we have determined the fourth constant of motion of this spacetime. The fourth constant reads

$$\mathscr{K} = \mu^2 a^2 cos^2\theta + \frac{(\Phi\Xi - sin^2\theta\, E)}{\Delta_\theta}\Big[2\,\varepsilon\, p\, \frac{cos\theta}{sin^2\theta} - E\, a + \frac{\Phi\Xi}{sin^2\theta}\Big]$$
$$+ \frac{\varepsilon^2 p^2 cos^2\theta}{\Delta_\theta sin^2\theta} + sin^2\theta\, \Delta_\theta\, (\frac{dS_\theta}{d\theta})^2, \tag{4.80}$$

$$\mathscr{K} = - \frac{(E(a^2+r^2) + a\Phi\Xi)}{\Delta_r\rho^2}\Big[\Phi\Xi a - E(a^2+r^2) - 2\varepsilon\, q\, r\Big]$$
$$- \mu^2 r^2 + \frac{\varepsilon^2 q^2 r^2}{\rho^2\Delta_r} - \frac{\Delta_r\, (\frac{dS_r}{dr})^2}{\rho^2}. \tag{4.81}$$

Finding the fourth constant shows that geodesic motion of a test particle is completely integrable and the Hamilton-Jacobi method can be used to derive the equations of motion. One integrates (4.80) and (4.81) to get the S_θ and S_r, and the resulting integral representations are

$$S_\theta = \int^\theta \sqrt{\Theta} d\theta, \tag{4.82}$$
$$S_r = \int^r \sqrt{R} dr, \tag{4.83}$$

where

$$\Theta \equiv \Big(\mathscr{K} - \mu^2 a^2 cos^2\theta + Q_1(\theta) + \frac{\varepsilon^2 p^2 cos^2\theta}{\Delta_\theta sin^2\theta}\Big)\frac{1}{\Delta_\theta sin^2\theta}, \tag{4.84}$$
$$R \equiv -\mathscr{K} + \mu^2 r^2 - \frac{\varepsilon^2 q^2 r^2}{\rho^2 \Delta_r} + \frac{(E(a^2+r^2)+a\Phi\Xi)}{\Delta_r \rho^2} Q_2(r), \tag{4.85}$$
$$Q_1(\theta) \equiv \frac{(\Phi\Xi - sin^2\theta\, E)}{\Delta_\theta}\Big(2\varepsilon p\frac{cos\theta}{sin^2\theta} - Ea + \frac{\Phi\Xi}{sin^2\theta}\Big),$$
$$Q_2(r) \equiv [\Phi\Xi a - E(a^2+r^2) - 2\varepsilon q r].$$

The unknown functions in the action have been determined and the action can be expressed as

$$S = \frac{1}{2}\mu^2\lambda - Et + \Phi\varphi + \int^\theta \sqrt{\Theta} d\theta + \int^r \sqrt{R} dr. \tag{4.86}$$

Similar steps taken in the case of the Kerr spacetime case now be followed here. The equations for the coordinates are given below. These have been obtained by differentiating the action with respect to $\mathscr{K}$, μ, E and Φ, respectively,

$$\int^r \frac{dr}{\Delta_r\sqrt{R}} = \int \theta \frac{d\theta}{\Delta_\theta\, sin^2\theta\sqrt{\Theta}}, \tag{4.87}$$
$$\lambda = \int^\theta \frac{a^2 cot^2\theta}{\Delta_\theta} d\theta + \int^r \frac{r^2}{\Delta_r\sqrt{R}} dr, \tag{4.88}$$
$$t = \int^\theta \Big[a(\Phi\Xi - sin^2\theta\, E) + sin^2\theta(\frac{2\varepsilon p\, cos\theta}{sin^2\theta} - Ea + \frac{\Phi\Xi}{sin^2\theta})\Big]\frac{d\theta}{\Delta_\theta^2 sin^2\theta\sqrt{\Theta}}$$
$$+ \int^r 2[\Phi\Xi a - E(a^2+r^2-\varepsilon q r)]\frac{(a^2+r^2)}{\Delta_r^2\sqrt{R}} dr, \tag{4.89}$$
$$\varphi = \int^\theta \Big[2\varepsilon p\frac{cos\theta}{sin^2\theta} - E(a+1) + 2\frac{\Phi\Xi}{sin^2\theta}\Big]\frac{\Xi d\theta}{\Delta_\theta^2\sqrt{\Theta} sin^2\theta}$$
$$+ \int^r 2[\Phi\Xi a - E(a^2+r^2-\varepsilon q r)]\frac{a\Xi}{\Delta_r^2\sqrt{R}} dr. \tag{4.90}$$

Differentiating the above equations with respect to the affine parameter λ, one gets the velocity forms of the coordinates. The first order differentiated forms of the equations are

$$\dot{\theta} = \frac{\Delta_\theta\, sin^2\theta\, \sqrt{\Theta}}{\rho^2}, \tag{4.91}$$

$$\dot{r} = \frac{\Delta_r \sqrt{R}}{\rho^2}, \tag{4.92}$$

$$\begin{aligned} \dot{t} = &\Big[a(\Phi\Xi - sin^2\theta\, E) + sin^2\theta\big(\frac{2\,\varepsilon\, p\, cos\theta}{sin^2\theta} - Ea + \frac{\Phi\,\Xi}{sin^2\theta}\big)\Big]\frac{2}{\Delta_\theta\,\rho^2} \\ &+ [\Phi\,\Xi\,\varepsilon - E(a^2+r^2) - \varepsilon\, q\, r]\frac{2(a^2+r^2)}{\Delta_r\,\rho^2}, \end{aligned} \tag{4.93}$$

$$\begin{aligned} \dot{\varphi} = &\Big[2\varepsilon\, p\frac{cos\theta}{sin^2\theta} - E(a+1) + 2\frac{\Phi\,\Xi}{sin^2\theta}\Big]\frac{\Xi}{\Delta_\theta\,\rho^2\, sin^2\theta} \\ &+ [\Phi\,\Xi\, a - E(a^2+r^2-\varepsilon qr)]\frac{2\,\Xi\, a}{\Delta_r\rho^2}. \end{aligned} \tag{4.94}$$

We have found the equations which describe the motion of a test particle. If one has the knowledge of the initial conditions of the particle, one can in principle learn everything about the particle's trajectory by solving these equations using the given initial conditions.

Before deriving the equations of motion, we have mentioned that the Hamiltonian fits the form presented earlier in (4.38). While deriving the equations of motion, we chose to use the first method given. This was so, since it gives two separate equations (4.80) and (4.81) each containing only one coordinate θ and r, respectively. Therefore the equations are easier to solve than the Carter constant expression given by the second method, since it contains both coordinates at the same time. The second method on the other hand is more suitable for finding the Killing tensor. The first method gives two equations for the constant, and each equation contains either one of the p_r or the p_θ, so one ends up with two different Killing tensors. However, in the second method, the constant contains all four momenta in second order and better serves for finding the Killing tensor of this constant. Moreover, we will take the charge of the particle ε to be zero, since the Killing tensor gives a scalar which is second order in momenta, but the terms containing ε are either in zeroth or first order. Hence it is useful to ignore the charge. According to the formula (4.38), the functions H_r, H_θ, U_r and U_θ read

$$\begin{aligned} H_r &= \Delta_r p_r^2 - \frac{[p_t(a^2+r^2) + a\, p_\phi\, \Xi]^2}{\Delta_r}, \\ H_\theta &= \frac{(p_\phi\,\Xi + sin^2\theta\, p_t)(p_t\, a + \frac{p_\phi\,\Xi}{sin^2\theta})}{\Delta_\theta}, \\ U_r &= r^2, \\ U_\theta &= a^2\, cos^2\theta. \end{aligned} \tag{4.95}$$

The fourth constant is obtained merely by substituting these into (4.46), and the result is

$$\begin{aligned}\mathscr{K} = \frac{1}{\rho^2}\Big[&p_t^2\Big(\frac{a^2\,cos^2\theta(a^2+r^2)^2}{\Delta_r}+\frac{r^2\,a^2\,sin^2\theta}{\Delta_\theta}\Big)+r^2\,sin^2\theta\,\Delta_\theta\,p_\theta^2\\
&+p_t\,p_\phi\,\Xi\Big(\frac{r^2\,(a+1)}{\Delta_\theta}+\frac{2\,a^3\,cos^2\theta\,(a^2+r^2)}{\Delta_r}\Big)-a^2\,cos^2\theta\,\Delta_r\,p_r^2\\
&+p_\phi^2\,\Xi\Big(\frac{r^2}{\Delta_\theta\,sin^2\theta}+\frac{cos^2\theta\,a^2}{\Delta_r}\Big)\Big].\end{aligned} \tag{4.96}$$

(4.96) gives the Carter constant found by the second method, and the only thing to do is to equate $\mathscr{K}$ to $K^{ab}p_a p_b$ for finding the Killing tensor. After doing so, we find the non-vanishing components of the Killing tensor as

$$K^{tt} = \frac{a^2}{\rho^2}\Big(\frac{cos^2\theta(a^2+r^2)^2}{\Delta_r}+\frac{r^2\,sin^2\theta}{\Delta_\theta}\Big), \tag{4.97}$$

$$K^{rr} = -\frac{a^2\,cos^2\theta\,\Delta_r}{\rho^2}, \tag{4.98}$$

$$K^{\theta\theta} = \frac{r^2\,sin^2\theta\,\Delta_\theta}{\rho^2}, \tag{4.99}$$

$$K^{\varphi\varphi} = \frac{\Xi}{\rho^2}\Big(\frac{r^2}{\Delta_\theta\,sin^2\theta}+\frac{cos^2\theta\,a^2}{\Delta_r}\Big), \tag{4.100}$$

$$K^{t\varphi} = K^{\varphi t} = \frac{\Xi}{2\rho^2}\Big(\frac{r^2\,(a+1)}{\Delta_\theta}+\frac{2\,a^3\,cos^2\theta\,(a^2+r^2)}{\Delta_r}\Big). \tag{4.101}$$

In this section, we have shown that the Hamilton-Jacobi equation of the spacetime is separable and we have found the fourth constant of the spacetime, which is a rotating and charged solution of four dimensional Conformal Gravity theory (4.69). This constant proves that the geodesic motion is completely integrable and can be used to solve the equations of motion by the Hamilton-Jacobi equation. Finally we have derived the Killing tensor of the spacetime from the fourth constant of motion which we have derived.

CHAPTER 5

KILLING-YANO TENSORS

5.1 Introduction

Killing-Yano tensors are generalizations of the Killing vectors to higher rank tensors just like the Killing tensors. The difference between the Killing tensors and the Killing-Yano tensors is that, while Killing tensors are totally symmetric tensors, the Killing-Yano tensors are totally antisymmetric. As in the case of Killing tensors, being generalizations of Killing vectors, they also have to be parallel along a given geodesic. Hence, they correspond to hidden symmetries of the spinning particle and these issues are discussed extensively in [32]. We have mentioned in Chapter 4 that Killing tensors of rank-2 are related to the separability of the Hamilton-Jacobi equation and Carter constant [6]. In [7] and [8], it is argued that a Killing-Yano tensor f_{ab} can be expressed as the square root of a given Killing tensor K_{ab}; i.e. that given K_{ab} there exists an f_{ab} such that

$$K_{ab} = f_{ac} f^{c}{}_{b}. \tag{5.1}$$

This is closely related to the separability of the Dirac equation in the Kerr spacetime [9]. However, these issues are beyond the scope of this work, and here we will be dealing with another feature of the Killing-Yano tensors. Here, we will examine how they can be used in constructing new gravitational charges [16], [17].

Let us start by studying the properties of the Killing-Yano tensors. Let $f_{a_1 \ldots a_n}$ be a rank-n Killing-Yano tensor. By definition, it is totally antisymmetric; thus

$$f_{a_1 \ldots a_n} = f_{[a_1 \ldots a_n]}. \tag{5.2}$$

It must also remain parallel along the geodesics, so it satisfies

$$\nabla_{(a_1} f_{a_2)a_3\ldots a_{n+1}} = 0. \tag{5.3}$$

Equation (5.3) implies that the covariant derivative of the Killing-Yano tensor is also a totally antisymmetric object

$$\nabla_{a_1} f_{a_2\ldots a_{n+1}} = \nabla_{[a_{a1}} f_{a_2\ldots a_{n+1}]}. \tag{5.4}$$

Taking the covariant derivative of this equation, one gets

$$\nabla_a \nabla_b f_{c_1\ldots c_n} = (-1)^{n+1} \frac{n+1}{2} R^d{}_{a[bc_1} f_{c_2\ldots c_n]d}. \tag{5.5}$$

This result is given in [16]. To prove this identity, one should express $\nabla_a \nabla_b f_{c_1\ldots c_n}$ in terms of the Riemann tensor, and this should be repeated $n+1$ times by changing the indices $b, c_1, \ldots, c_n$ in cyclic order. The first order covariant derivative of the Killing-Yano tensor is totally antisymmetric (5.2). Therefore, changing the indices of $\nabla_a \nabla_b f_{c_1\ldots c_n}$ in cyclic order results in an identical expression. Next, summing all the $n+1$ equations obtained and rearranging the Riemann terms, one proves the identity. The derivation of this for the rank $n = 2$ case is given in Appendix A.2. Its generalization to arbitrary rank is straightforward.

It is also obvious that the Killing-Yano tensor is both traceless and divergenceless

$$f^c{}_{ca_1\ldots a_{n-2}} = 0, \tag{5.6}$$

$$\nabla_{a_1} f^{a_1\ldots a_n} = 0. \tag{5.7}$$

A rank-n conserved current was obtained in [16] merely by using these properties of the Killing-Yano tensors. The current is expressed as

$$j^{a_1\ldots a_n} = -\frac{n-1}{4} R_{bc}{}^{[a_1 a_2} f^{a_3\ldots a_n]bc} + (-1)^{n+1} R_c{}^{a_1} f^{a_2\ldots a_n,c} - \frac{1}{2n} R f^{a_1\ldots a_n}. \tag{5.8}$$

In [16], rank-2 case of this current was studied and a conserved charge was constructed using it for the transverse spacetimes with flat backgrounds. It was later extended for the transverse spacetimes with AdS backgrounds [17]. The charge leads to intrinsic properties of the spacetime such as ADM mass per unit length or ADM tension per unit time. Even though it leads to information about spacetime, the terms in the current j^{ab} does not have a physical interpretation as also mentioned in [16]. The current is conserved off-shell, contrary to the current defined in the ADT procedure. In the next section, the procedure developed in [16] will be reviewed and these issues will be discussed in detail.

This problem motivated us to work on gravitational charges constructed by using Killing-Yano tensors. Along the way, two important identities were obtained. The first identity is

$$R_{ac}f^{c}{}_{b}+R_{bc}f^{c}{}_{a}=0. \tag{5.9}$$

We have discovered this identity while studying the properties of the contraction of a Killing-Yano tensor with curvature tensors. Unfortunately though, we noticed that this identity was already found in [33], when we stumbled on it by chance afterwards. This identity is used in simplifying the charge expression and in proving the second identity, which we are about to mention. The second identity, to our knowledge has not been derived beforehand in spite of our careful review of the literature. It states that the contraction of the Einstein tensor with a Killing-Yano tensor is conserved

$$\nabla_{a}(G_{bc}f^{b}{}_{a})=0. \tag{5.10}$$

If one carefully examines this identity, then one can see that this is quite similar to the current used in defining the ADT charge. The only difference is that the Killing vector is replaced with the Killing-Yano tensor. Therefore, this identity might be an important step in constructing a new kind of conserved charge expression for spacetimes that possess background Killing-Yano tensors. This quantity is used as a conserved current to construct such a charge, however we have, so far, not been able to express it as a total divergence of a totally antisymmetric tensor. This approach of defining conserved charges and the problem it leads to is discussed in detail in subsection 5.2.2.

5.2 The Conserved Gravitational Killing-Yano Charge

Killing-Yano tensors were first introduced in [13] and have since been widely studied. It was mentioned in the previous section that these works were mostly on the hidden symmetries, the separability of the Dirac equation, etc. However, they had not been studied for the case of gravitational charges for a long time, even though Killing vectors were used in defining gravitational charges. Long after the discovery of the Killing-Yano tensors, it was found that they can be used in defining gravitational charges for transverse spacetimes with flat backgrounds in [16]. Later, the charge definition was extended to transverse spacetimes with AdS backgrounds in [17]. This procedure is reviewed in subsection 5.2.1. However, the current used in this procedure cannot be explained in terms of physical quantities such as the energy momentum tensor. Therefore, generalizing the current used in the ADT procedure by substituting the Killing vector with a Killing-Yano tensor is discussed in subsection 5.2.2.

5.2.1 The Gravitational Killing-Yano charge

In this section, we will review the procedure developed in [16] to define a gravitational charge for the transverse spacetimes admitting background Killing-Yano tensors. In [16], a conserved quantity was found as the contraction of a Killing tensor with a special combination of the curvature tensors and the Ricci scalar. This quantity is used as a conserved current to construct the conserved charge expression in a similar way as in the ADT procedure. The conserved rank-2 antisymmetric current is expressed as

$$j^{ab} = -\frac{1}{4}(R^{ab}{}_{cd}\, f^{cd} - 2\, f^{ac}\, R^{b}{}_{c} + 2\, f^{bc}\, R^{a}{}_{c} + f^{ab}\, R). \tag{5.11}$$

In fact a generalization of this current to rank-n was given in [16] and it is given in (5.8). However, in what follows we will only consider the rank-2 current tensor. The tensor is divergence-free as expected

$$\begin{aligned}
\nabla_a\, j^{ab} &= -\frac{1}{4}\nabla_a(R^{ab}{}_{cd}\, f^{cd} - 2R^{b}{}_{c}\, f^{ac} + 2R^{a}{}_{c}\, f^{bc} + R\, f^{ab}) \\
&= \frac{1}{4}\Big((\nabla_a R^{ab}{}_{cd}) f^{cd} + R^{ab}{}_{cd}\nabla_a f^{cd} - 2f^{ac}\nabla_a R^{b}{}_{c} \\
&\quad + 2R^{a}{}_{c}\nabla_a f^{bc} + 2(\nabla_a R^{a}{}_{c}) f^{bc} + f^{ab}\nabla_a R\Big) \\
&= \frac{1}{4}[-R^{b}{}_{acd}\nabla^a f^{cd} - 2f^{cd}\nabla^b R_{cd} + 2f^{cd}\nabla_c R^{b}{}_{d} - 2f^{dc}\nabla_d R^{b}{}_{c} \\
&\quad + 2R_{dc}\nabla^d f^{bc} + f^{bc}\nabla_c R + f^{cb}\nabla_c R] \qquad (5.12) \\
&= 0. \qquad (5.13)
\end{aligned}$$

To arrive at (5.12), we have used the identities

$$\nabla_a R_{bcd}{}^{a} = \nabla_c R_{bd} - \nabla_b R_{cd}, \qquad \nabla_a R^{a}{}_{b} = \frac{1}{2}\nabla_b R. \tag{5.14}$$

We have used the Bianchi identity $\nabla_{[a} R_{bc]de} = 0$ and contracted it to get the first identity in (5.14). Contracting the indices c and d in the first identity directly leads to the second one. In (5.12), the first term is automatically zero, thanks to the Bianchi identity $R_{[abc]d} = 0$, and the other terms simply cancel each other.

The linearization process is similar to the one described in Sections 2.3 and 3.3, hence the linearization of (5.11) is

$$(J^{ab})_L = k^{ab} = -\frac{1}{4}\Big[(R^{ab}{}_{cd})_L \bar{f}^{cd} - 2(R^b{}_c)_L \bar{f}^{ac} + 2(R^a{}_c)_L \bar{f}^{bc} + R_L f^{ab}\Big], \tag{5.15}$$

which is divergenceless. The vanishing of the divergence of k^{ab} can be seen through linearizing the Bianchi identities given in (5.14). By linearization, they become

$$\bar{\nabla}_a (R_{bcd}{}^a)_L = 2\bar{\nabla}_c (R_{bd})_L - 2\bar{\nabla}_b (R_{cd})_L, \qquad \bar{\nabla}_a (R^a{}_b)_L = \frac{1}{2}\bar{\nabla}_b R_L. \tag{5.16}$$

Therefore, the linearized current is divergenceless

$$\bar{\nabla}_a k^{ab} = 0. \tag{5.17}$$

The factor $\frac{1}{4}$ in the k^{ab} is chosen such that the rank-1 Killing-Yano conserved current yields the current identical to the one used. There is a difference between the ADT current and this one. The Killing-Yano current is not a vector but a tensor of rank-2, so it should be dealt with appropriately. We note that

$$\sqrt{-\bar{g}}\,\bar{\nabla}_a k^{ab} = \partial_a(\sqrt{-\bar{g}}\,k^{ab}). \tag{5.18}$$

Thus the charge can be expressed as

$$Q^{ab} = \int_\Sigma d^{n-2}x \sqrt{-\bar{g}}\,k^{ab}. \tag{5.19}$$

The Poincarè lemma guarantees that locally there is a totally antisymmetric tensor l^{abc} which satisfies

$$l^{abc} = l^{[abc]}, \qquad \nabla_c\, l^{abc} = k^{ab}. \tag{5.20}$$

With the help of this tensor, we can go further and write

$$Q^{ab} = \int_\Sigma d^{n-2}x \sqrt{-\bar{g}}\,k^{ab} \tag{5.21}$$

$$= \int_\Sigma d^{n-2}x\, \bar{\nabla}_c l^{abc} \tag{5.22}$$

$$= \int_{\partial\Sigma} d\Sigma_c\, l^{abc}, \tag{5.23}$$

where $\partial\Sigma$ is the $n-3$ dimensional spacelike hypersurface at spatial infinity, $\sqrt{\gamma}$ is the determinant of the induced metric on the surface and $d\Sigma_c = \sqrt{\gamma} n_c d^{n-3}y$ is its surface element with n_c normal to its surface. We have again simplified the problem significantly: The only thing to be explicitly calculated is the l^{abc} tensor and the rest is straightforward integration.

We are dealing with transverse spacetimes with flat backgrounds. Hence, finding l^{abc} is not so hard in this case. One has the equation $k^{ab} = \bar{\nabla}_c \, l^{abc}$, and all that is needed is to write down k^{ab}, and then try to express it as a total derivative as shown in the Appendix A.3. The final expression is

$$k^{ab} = \bar{\nabla}_c (3! \bar{f}^{[c|d|} \bar{\nabla}^b h^{a]}{}_d + \frac{3!}{2} \bar{f}^{[ac} \bar{\nabla}^{b]} h + \frac{3!}{2} \bar{f}^{[ab} \bar{\nabla}^{|d|} h^{c]}{}_d + \frac{3!}{2} \bar{\nabla}^d \bar{f}^{[cb} h^{a]}{}_d + h \bar{\nabla}^{[c} \bar{f}^{ab]}). \quad (5.24)$$

It is not hard to identify l^{abc} from this expression. It reads

$$l^{abc} = 3! \bar{f}^{[c|d|} \bar{\nabla}^b h^{a]}{}_d + \frac{3!}{2} \bar{f}^{[ac} \bar{\nabla}^{b]} h + \frac{3!}{2} \bar{f}^{[ab} \bar{\nabla}^{|d|} h^{c]}{}_d + \frac{3!}{2} \bar{\nabla}^d \bar{f}^{[cb} h^{a]}{}_d + h \bar{\nabla}^{[c} \bar{f}^{ab]}. \quad (5.25)$$

Killing-Yano charges are interesting, since they give intrinsic quantities about spacetime. In [16], the static string in five dimensional spacetime was studied. The string has been found to have ADM mass and ADM tension $M = \rho \, \Delta L$, $T = -\lambda \Delta t$, respectively. Here the parameter ρ stands for the mass per unit length and the other parameter λ represents the tension per unit time. In the same article, Killing-Yano charge was found as

$$Q = -\frac{(\rho - \lambda)}{3}. \quad (5.26)$$

The Killing-Yano charge Q describes both the mass per unit length and tension per unit time of the string. It is quite interesting to observe that the Killing-Yano charge carries information about the intrinsic quantities of the brane, while ADM charges give information about the extrinsic quantities. Another study which is in support of this property of the Killing-Yano charges is [17]. In this work, the spacetime called the "long Weyl rod" is examined. This spacetime has ADM mass M, and as discussed in [17] if M and L are set to be equal $M = L$, where L is the length of the rod, this metric reduces to the celebrated Schwarzschild solution. The conclusions of [17] are again similar to ones in [16] as expected, and it is found in [17] that the Killing-Yano charge of the "long Weyl rod" is

$$Q = \frac{M}{L}, \quad (5.27)$$

which again amounts to the mass per unit length. The Killing-Yano charge found in [17] also describes an intrinsic quantity of the spacetime, hence one can argue that they reveal the intrinsic properties of a given spacetime.

The procedure provides interesting results about intrinsic quantities of the spacetime. On the other hand, there is a problem, as mentioned in [16]. In ADT procedure, the current was formed by contracting a Killing vector with the energy momentum tensor, therefore it naturally leads to some information about the mass or the momentum of the solution at hand. However, here it does not depend on the energy momentum tensor and a physical interpretation of the current cannot be made. This is not quite surprising, since the current is off-shell conserved. Only Bianchi identities were used in showing its conservation as shown in (5.13). Nevertheless, it still leads to valuable information about the given spacetime. One of the motivations of this work was to better understand this elusive point.

5.2.2 An Alternative Conserved Killing-Yano Current

In the previous subsection, the gravitational charge defined in [16] was discussed. It was mentioned that the current used in defining the conserved charge cannot be interpreted physically. The first thing that comes to one's mind to overcome this problem, is to generalize the current used in the ADT procedure by replacing the Killing vector with a Killing-Yano tensor. The current becomes a second rank object

$$J^{ab} = G^{ac} f_c{}^b - G^{bc} f_c{}^a. \tag{5.28}$$

In fact, this was considered as current in previous versions of [17]. However, its conservation was not known at that time, so it was retracted in the final version. In this study, we also consider this quantity as a current after we show its conservation to overcome the problem stated in [16]. Before checking its conservation, let us try to express it in a simpler form. At the beginning of this chapter, we have given the identity

$$R_{ac} f^c{}_b + R_{bc} f^c{}_a = 0. \tag{5.29}$$

As mentioned earlier, while studying the older literature, we noticed that it was first derived in [33]. To derive this equation, one needs to act on the Killing-Yano tensor with the commutator of two covariant derivatives. This reads

$$[\nabla_a, \nabla_b] f^{ac} = R_{ab}{}^a{}_d f^{dc} + R_{ab}{}^c{}_d f^{ad}. \tag{5.30}$$

Expanding the commutation in the left hand side, and using $\nabla_a f^{ab} = 0$, one gets

$$\underbrace{\nabla_a \nabla_b f^{ac}}_{A_b{}^c} = R_{bd} f^{dc} + R_{ab}{}^c{}_d f^{ad}. \tag{5.31}$$

The left hand side has been renamed as $A_b{}^c$. Applying the identity given in (5.5) for the rank-2 Killing-Yano tensor, $A_b{}^c$ becomes

$$A_b{}^c = -\frac{1}{2}(R^d{}_{ab}{}^a f^c{}_d + R^d{}_a{}^c{}_b f^a{}_d + R^d{}_a{}^{ac} f_{bd}) \tag{5.32}$$

$$= -\frac{1}{2}(R^d{}_b f^c{}_d + R^d{}_a{}^c{}_b f^a{}_d - R^{dc} f_{bd}). \tag{5.33}$$

Substituting this in (5.31), it reads

$$-\frac{1}{2}(R^d{}_b f^c{}_d + R^d{}_a{}^c{}_b f^a{}_d - R^{dc} f_{bd}) = R_{bd} f^{dc} + R_{ab}{}^c{}_d f^{ad} \tag{5.34}$$

$$\frac{1}{2}R_{bd} f^{dc} + \frac{1}{2}R_d{}^c f^d{}_b = \underbrace{-\frac{1}{2}R^d{}_a{}^c{}_b f^a{}_d - R_{ab}{}^c{}_d f^{ad}}_{B_b{}^c}. \tag{5.35}$$

If one renames the right hand side as $B_b{}^c$, and applies the Bianchi identity $R_{[abd]c} = 0$ for the second term in the left hand side, then $B_b{}^c$ becomes

$$B_b{}^c = -\frac{1}{2}R^d{}_a{}^c{}_b f^a{}_d + R_{da}{}^c{}_b f^{ad} + R_{bd}{}^c{}_a f^{ad} \tag{5.36}$$

$$= \frac{1}{2}R^d{}_a{}^c{}_b f^a{}_d + \frac{1}{2}(R_{bd}{}^c{}_a - R_{ba}{}^c{}_d) f^{ad}. \tag{5.37}$$

By using the antisymmetry of the $a-d$ indices in the last term, this equation becomes

$$B_b{}^c = \frac{1}{2}R^d{}_a{}^c{}_b f^a{}_d + \frac{1}{2}(R^c{}_{abd} + R^c{}_{dab}) f^{ad} \tag{5.38}$$

$$= \frac{1}{2}R_{da}{}^c{}_b f^{ad} - \frac{1}{2}R^c{}_{bda} f^{ad} \tag{5.39}$$

$$= 0. \tag{5.40}$$

Finally, substituting this in (5.35), one arrives at the desired identity

$$R_{bd} f^{dc} + R_d{}^c f^d{}_b = 0. \tag{5.41}$$

Moreover, this can be expressed in terms of the Einstein tensor easily. It reads

$$G^a{}_c f^{cb} + G^b{}_c f^{ca} = 0. \tag{5.42}$$

Therefore the current expression (5.28) can be written also as

$$J^{ab} = f^{ac} G_c{}^b. \tag{5.43}$$

Next, one needs to check its conservation and, to do so, one needs to examine its covariant divergence as follows

$$\begin{aligned} \nabla_a J^{ab} &= \nabla_a (f_c{}^a G^{bc}) \\ &= f_c{}^a \nabla_a G^{bc} + G^{bc} \nabla_a f_c{}^a \\ &= f_c{}^a \nabla_a G^{bc}. \end{aligned} \tag{5.44}$$

The first term in the second line vanishes due to the divergenceless property of the Killing-Yano tensor (5.7). The equation becomes

$$\nabla_a J^{ab} = -(\nabla_a G^{bc}) f_c{}^a. \tag{5.45}$$

The vanishing of the remaining term is not easy to see. In fact, we have originally proved it in a similar way used in deriving (3.23). However, after finding (5.42), it is in fact much easier. Here we will derive the conservation of it from (5.42) and the second way of deriving it will be given in Appendix A.1. Taking the divergence of (5.42), it becomes

$$\begin{aligned} \nabla_a (G^a{}_c f^{cb} + G^b{}_c f^{ca}) = 0, \\ (\nabla_a G^a{}_c) f^{cb} + G^a{}_c \nabla_a f^{cb} + (\nabla_a G^b{}_c) f^{ca} + G^b{}_c \nabla_a f^{ca} = 0. \end{aligned} \tag{5.46}$$

The first term vanishes due to the Bianchi identity. The second term is zero since the covariant derivative of the Killing-Yano tensor is totally antisymmetric and the Einstein tensor is symmetric. Vanishing of the final term is due to the divergenceless property of the Killing-Yano tensor (5.7). Finally, we obtain

$$(\nabla_a G^b{}_c) f^{ca} = 0. \tag{5.47}$$

Therefore, the problem which prevented J^{ab} from being used as a current has now been overcome with the identities shown.

As promised, we take the conserved quantity J^{ab} as current and move on to the next step of constructing the charge expression. The next step is linearizing the current in the usual manner. The current can be expanded in the usual fashion as

$$G_c{}^a f^{cb} = \kappa T_c{}^a f^{cb} \tag{5.48}$$

$$\begin{aligned} &\bar{G}_c{}^a \bar{f}^{cb} + (G_c{}^a)_L \bar{f}^{cb} + \bar{G}_c{}^a (f^{cb})_L + \mathscr{O}(h^2) \\ &\qquad = \kappa (\bar{T}_c{}^a \bar{f}^{cb} + (T_c{}^a)_L \bar{f}^{cb} + \bar{T}_c{}^a (f^{cb})_L + \mathscr{O}(h^2)). \end{aligned} \tag{5.49}$$

Here it is again assumed that the background metric $\bar{g}_{ab}$ satisfies

$$\bar{\Phi}^{ab} = \kappa \bar{T}^{ab} = 0, \tag{5.50}$$

and the deviation h_{ab} vanishes sufficiently fast at infinity. As shown in the ADT case, moving every term which is of second or higher order at the left hand side to the right hand side of the equation, one can rearrange (5.49) to write in the form

$$(G_c{}^a)_L \bar{f}^{cb} = \kappa\{(T_c{}^a) f^{cb} + \mathscr{O}(h^2)\},$$
$$(G_c{}^a)_L \bar{f}^{cb} = \kappa\, \tau^{ab}. \tag{5.51}$$

The steps to take are quite similar to the ones taken in the previous section to define the charge, although the current itself is different now. The charge in this case becomes

$$\begin{aligned} Q^{ab} &= \int_\Sigma d^{n-2}x \sqrt{-g}\,(G_c{}^a)_L \bar{f}^{cb}, \\ &= \int_\Sigma d^{n-2}x \bar{\nabla}_c\, l^{abc}, \\ &= \int_{\partial\Sigma} d\Sigma_c\, l^{abc}. \end{aligned} \tag{5.52}$$

where $\partial\Sigma$ is the $n-3$ dimensional spacelike hypersurface at spatial infinity, $\sqrt{\gamma}$ is the determinant of the induced metric on the surface and $d\Sigma_c = \sqrt{\gamma} n_c d^{n-3}y$ is its surface element with n_c normal to its surface. Again, we have reduced the problem to finding the rank-3 totally antisymmetric tensor l^{abc}. We have tried to find this rank-3 tensor for a flat background. Writing the terms explicitly in the linearized current (5.51) reads

$$(G_c{}^a)_L \bar{f}^{cb} = (R_c{}^a)_L \bar{f}^{cb} - \frac{1}{2} R_L \bar{f}^{ab}. \tag{5.53}$$

We have in fact derived the linearized terms, which are needed in (5.53), already in Section 2.3. Substituting the results found earlier, one finds

$$\begin{aligned} (G_c{}^a)_L \bar{f}^{cb} = {} & \frac{1}{2}(-\bar{f}^{cb}\bar{\nabla}_d\bar{\nabla}^d h^a{}_c - \bar{f}^{cb}\bar{\nabla}_c\bar{\nabla}^a h + \bar{f}^{cb}\bar{\nabla}^d\bar{\nabla}_c h^a{}_d + \bar{f}^{cb}\bar{\nabla}^d\bar{\nabla}^a h_{cd}) \\ & - \frac{1}{2}(-\bar{f}^{ab}\bar{\nabla}^d\bar{\nabla}_d h + \bar{f}^{ab}\bar{\nabla}^c\bar{\nabla}^d h_{cd}). \end{aligned} \tag{5.54}$$

This current can be written as a total derivative plus some extra terms,

$$\begin{aligned} (G_c{}^a)_L \bar{f}^{cb} = {} & \frac{1}{2}\bar{\nabla}_d\Big(\bar{f}^{ab}\bar{\nabla}^d h - \bar{f}^{ab}\bar{\nabla}_c h^{cd} - \bar{f}^{cb}\bar{\nabla}^d h^a{}_c - \bar{f}^{db}\bar{\nabla}^a h + \bar{f}^{db}\bar{\nabla}^c h^a{}_c \\ & + \bar{f}^{cb}\bar{\nabla}^a h^d{}_c\Big) - \frac{1}{2}(\bar{\nabla}_c\bar{f}^{ab})(\bar{\nabla}^c h) + \frac{1}{2}(\bar{\nabla}_d\bar{f}^{ab})(\bar{\nabla}_c h^{cd}) \\ & + \frac{1}{2}(\bar{\nabla}_d\bar{f}^{cb})(\bar{\nabla}^d h_c{}^a). \end{aligned} \tag{5.55}$$

After some considerable algebra for arranging the terms to make the antisymmetric property manifest, one finds

$$\begin{aligned} (G_c{}^a)_L \bar{f}^{cb} = {} & \frac{1}{2}\bar{\nabla}_d\Big(3\bar{f}^{[ab}\bar{\nabla}^{d]} h + 3\bar{f}^{[db}\bar{\nabla}^{|c|} h^{a]}{}_c + 3h_c{}^{[a}\bar{\nabla}^{|c|}\bar{f}^{bd]} - h\bar{\nabla}^{[d}\bar{f}^{ab]}\Big) \\ & - \frac{1}{2}\bar{f}^{da}\bar{\nabla}_d\bar{\nabla}^b h - \frac{1}{2}\bar{f}^{ad}\bar{\nabla}_d\bar{\nabla}_c h^{cb} - \frac{1}{2}(\bar{\nabla}_c\bar{f}^{da})(\bar{\nabla}_d h^{bc}) \\ & + \frac{1}{2}\bar{\nabla}_d(\bar{f}^{cb}\bar{\nabla}^a h_c{}^d - \bar{f}^{cb}\bar{\nabla}^d h_c{}^a). \end{aligned} \tag{5.56}$$

Unfortunately, the linearized current could not be written as the total divergence of a totally antisymmetric rank-3 current tensor l^{abc}. However, this should in principle be possible due to the Poincaré lemma. This is a serious problem, preventing us from defining a charge expression. To understand the reason of it, we have compared the two currents j^{ab} and J^{ab}. We have realized that j^{ab} can be written in terms of J^{ab} plus some extra terms. The difference between the two currents is

$$j^{ab} - J^{ab} = -\frac{1}{4}(R^{ab}{}_{cd}f^{cd} - f^{ab}R). \tag{5.57}$$

Obviously, the resulting term is also another conserved quantity! Even though it has no physical interpretation, we have tried to linearize it and checked whether one can write it as a total divergence of a totally antisymmetric rank-3 tensor l^{abc} or not. The result is again negative: This is still not possible! The problem of defining a conserved charge has become quite interesting. Even though the Poincaré lemma dictates that there should be a totally antisymmetric rank-3 tensor l^{abc}, these two conserved quantities cannot be expressed as a total divergence of such a tensor. Thus they cannot be used in defining a conserved charge. However their sum can be used in defining a charge without any problem! After spending much effort to solve this problem for a considerable amount of time, we are still not able to come up with a satisfactory answer. There might still be some undiscovered identities which would lead to the antisymmetrization of the object, or there might be a problem with the linearization process itself. Finding this elusive rank-3 totally antisymmetric tensor and discovering what information it leads to would be quite interesting. We expect to continue on these problems in the future.

CHAPTER 6

CONCLUSION

In this work, basic properties of the Killing vectors, Killing tensors and Killing-Yano tensors, and their use in gravitational theories and geodesic motion have been studied. The main focus was on the constants of motion in the study of geodesics and the conserved gravitational charges that can be defined using them.

We briefly stated the properties of the Killing vectors, and derived two identities which were later used in deriving another important one about the Killing-Yano tensors. We have seen that they correspond to the symmetries of the Lagrangian of the geodesic motion, and they lead to conserved canonical momenta as expected from Noether's theorem. After demonstrating this, we discussed the ADT charge. The ADT charge is constructed through the current which is defined by contracting a Killing vector with the energy momentum tensor of the linearized field equations, and it gives the so-called ADM mass or ADM angular momentum of the geometry depending on the Killing vector chosen in the construction. The ideas developed by using the Killing vectors is of crucial importance since they were later generalized to the Killing tensors and Killing-Yano tensors in a way analogous to their generalization from Killing vectors. Therefore, appreciation of these ideas is quite important and helpful in the following sections.

Secondly, we have seen that Killing vectors are not always enough to exploit all the constants of motion in the study of geodesics of a given spacetime. It was also shown that if a system has a conserved quantity higher than first order in momentum, then there exists a Killing tensor corresponding to it. We have discussed the fourth constant of motion for a test particle in the Kerr spacetime, which was originally derived in [5], and derived the equations of motion of a charged test particle. Then, the Killing tensor related to the fourth constant was derived by the method given in [9]. Next, as a new example, we have investigated a solution to the Conformal Gravity theory presented in [18], and used the tools, which was developed earlier, to find the fourth constant of motion and equations of motion of a charged particle in this geometry. Later, we also found the Killing tensor related to the fourth constant of motion of this new spacetime.

Finally, we have discussed the Killing-Yano tensors. We were interested in the gravitational Killing-Yano charges which are constructed similarly to the ADT charge. We have discussed the method given in [16] and shown that it gives information about the internal quantities about spacetime, such as ADM mass per unit length or ADM tension per unit time, for transverse spacetimes with flat backgrounds. However, there are terms whose presence in the current j^{ab} cannot be physical, as mentioned in [16]. We have instead considered another choice for the current, one obtained by direct generalization of the ADT current by replacing the Killing vector with a Killing-Yano tensor:

$$J^{ab} = G_{ac} f^{c}{}_{b}. \tag{6.1}$$

Fortunately, we have shown that J^{ab} is indeed a conserved current and it can be considered in defining conserved charge. However, a new problem arose while defining the charge, and unfortunately, we were not able to write the linearized version of this current as a total divergence of a totally antisymmetric rank-3 tensor. For the time being, this still remains an open problem and we intend to come back to this issue in the near future.

REFERENCES

[1] L. Abbott and S. Deser, "Stability of gravity with a cosmological constant," *Nucl. Phys. B* **195** no. 1, (1982) 76 . http://www.sciencedirect.com/science/article/pii/0550321382900499.

[2] S. Deser and B. Tekin, "Energy in generic higher curvature gravity theories," *Phys.Rev.* **D67** (2003) 084009, arXiv:hep-th/0212292 [hep-th].

[3] D. O. Devecioglu and O. Sarioglu, "Conserved Killing charges of quadratic curvature gravity theories in arbitrary backgrounds," *Phys.Rev.* **D83** (2011) 021503, arXiv:1010.1711 [hep-th].

[4] L. P. Eisenhart, *Riemannian Geometry*. Princeton University Press, Princeton, NJ, 1949.

[5] B. Carter, "Global structure of the Kerr family of gravitational fields," *Phys. Rev.* **174** (1968) 1559. http://link.aps.org/doi/10.1103/PhysRev.174.1559.

[6] M. Walker and R. Penrose, "On quadratic first integrals of the geodesic equations for type 22 spacetimes," *Commun. Math. Phys.* **18** no. 4, (1970) 265. http://dx.doi.org/10.1007/BF01649445.

[7] R. Floyd, *The dynamics of Kerr fields*. PhD thesis, London University, 1973.

[8] R. Penrose, "Naked singularities," *Ann. N.Y. Acad. Sci.* **224** no. 1, (1973) 125. http://dx.doi.org/10.1111/j.1749-6632.1973.tb41447.x.

[9] B. Carter, "Killing tensor quantum numbers and conserved currents in curved space," *Phys. Rev. D* **16** (1977) 3395. http://link.aps.org/doi/10.1103/PhysRevD.16.3395.

[10] M. Vasudevan, K. A. Stevens, and D. N. Page, "Separability of the Hamilton-Jacobi and Klein-Gordon equations in Kerr-de Sitter metrics," *Class.Quant.Grav.* **22** (2005) 339. arXiv:gr-qc/0405125 [gr-qc].

[11] B. Ramachandra, "The Carter constant and Petrov classification of the Vaidya–Einstein–Kerr spacetime," *Gen. Relativ. Gravit.* **41** no. 12, (2009) 2757. http://dx.doi.org/10.1007/s10714-009-0805-y.

[12] T. Oota and Y. Yasui, "Separability of Gravitational Perturbation in Generalized Kerr-NUT-de Sitter Spacetime," *Int.J.Mod.Phys.* **A25** (2010) 3055–3094, arXiv:0812.1623 [hep-th].

[13] K. Yano, "Some remarks on tensor fields and curvature," *Ann. of Math* **55** no. 2, (1952) pp. 328. http://www.jstor.org/stable/1969782.

[14] F.A. Berezin and M.S. Marinov, "Particle spin dynamics as the Grassmann variant of classical mechanics," *Annals of Phys.* **104** no. 2, (1977) 336. http://www.sciencedirect.com/science/article/pii/0003491677903359.

[15] L. Brink, S. Deser, B. Zumino, P. D. Vecchia, and P. Howe, "Local supersymmetry for spinning particles," *Phys. Lett B* **64** no. 4, (1976) 435. http://www.sciencedirect.com/science/article/pii/0370269376901155.

[16] D. Kastor and J. Traschen, "Conserved gravitational charges from Yano tensors," *JHEP* **0408** (2004) 045, arXiv:hep-th/0406052 [hep-th].

[17] H. Cebeci, O. Sarioglu, and B. Tekin, "Gravitational charges of transverse asymptotically AdS spacetimes," *Phys.Rev.* **D74** (2006) 124021, arXiv:hep-th/0611011 [hep-th].

[18] H.-S Liu, H. Lü, "Charged Rotating AdS Black Hole and Its Thermodynamics in Conformal Gravity," *JHEP* **1302** (2013) 139, arXiv:1212.6264 [hep-th].

[19] L. D. Landau and E. M. Lifshitz, *Mechanics*. Butterworth-Heinemann, Butterworth-Heinemann, 3rd ed., 1976.

[20] V. I. Arnold, *Mathematical Methods of Classical Mechanics*. Springer, Berlin, 2nd ed., 1989.

[21] R. M. Wald, *General Relativity*. University Of Chicago Press, USA, 1st ed., 1984.

[22] A. Papapetreu, *Lectures on General Relativity*. Reidel Pub. Cor, Boston, 1st ed., 1974.

[23] E. Poisson, *A Relativist's Toolkit*. Cambridge University Press, Cambridge, 1st ed., 2004.

[24] D. Devecioglu, "Conserved charges of quadratic curvature gravity theories in arbitrary backgrounds," Master's thesis, METU, 2010.

[25] I. Güllü, "Conserved charges in asymptotically (anti)-de Sitter spacetime," Master's thesis, METU, 2005.

[26] S. Benenti and M. Francaviglia, "Remarks on certain separability structures and their applications to General Relativity," *Gen. Relativ. Gravit.* **10** no. 1, (1979) 79. http://dx.doi.org/10.1007/BF00757025.

[27] S. Benenti, "Intrinsic characterization of the variable separation in the Hamilton-Jacobi equation," *J. Math. Phys.* **38** no. 12, (1997) 6578. http://link.aip.org/link/?JMP/38/6578/1.

[28] V. Frolov and A. Zelnikov, *Introduction to Black Hole Physics*. Oxford University Press, USA, 1st ed., 2011.

[29] B. Carter, "Republication of: Black hole equilibrium states," *Gen. Relativ. Gravit.* **41** no. 12, (2009) 2873. http://dx.doi.org/10.1007/s10714-009-0888-5.

[30] D. Garfinkle and E. Glass, "Killing Tensors and Symmetries," *Class.Quant.Grav.* **27** (2010) 095004, arXiv:gr-qc/1003.0019 [gr-qc].

[31] J. Maldacena, "Einstein Gravity from Conformal Gravity," arXiv:1105.5632 [hep-th].

[32] O. P. Santillan, “Hidden symmetries and supergravity solutions,” *J. Math. Phys.* **53** (2012) 043509, `arXiv:1108.0149 [hep-th]`.

[33] C. Collinson *Tensor N. S* no. 28, (1974) 173.

APPENDIX A

IDENTITIES

A.1 Conservation of J^{ab}

In Section 5.2.2, we have shown that the current J^{ab} is conserved by using the first identity (5.42). There, we also noted that we first showed its conservation by using a similar approach to derive (3.23). Here we will derive it with the second approach. Therefore, one starts with

$$\nabla_a \nabla_b \nabla_c f_{de} - \nabla_b \nabla_a \nabla_c f_{de} = [\nabla_a, \nabla_b] \nabla_c f_{de} \tag{A.1}$$

$$\begin{aligned}
\nabla_a \nabla_b \nabla_c f_{de} - \nabla_b \nabla_a \nabla_c f_{de} &= R_{abc}{}^{k} \nabla_k f_{de} + R_{abd}{}^{k} \nabla_c f_{ke} + R_{abe}{}^{k} \nabla_c f_{dk} \\
\nabla_a \nabla_b \nabla_c f_{de} - \nabla_b \nabla_a \nabla_c f_{de} &= -\frac{1}{2} \nabla_a (R^k{}_{bcd} f_{ek} + R^k{}_{bec} f_{dk} + R^k{}_{bde} f_{ck}) \\
&\quad + \frac{1}{2} \nabla_b (R^k{}_{acd} f_{ek} + R^k{}_{aec} f_{dk} + R^k{}_{ade} f_{ck}).
\end{aligned} \tag{A.2}$$

Applying (5.5) to the left hand side of this equation, it becomes

$$\begin{aligned}
LHS = &-\nabla_a (R^k{}_{bcd} f_{ek}) - \nabla_a (R^k{}_{bec} f_{dk}) - \nabla_a (R^k{}_{bde} f_{ck}) \\
&+ \nabla_b (R^k{}_{acd} f_{ek}) + \nabla_b (R^k{}_{aec} f_{dk}) + \nabla_b (R^k{}_{ade} f_{ck}).
\end{aligned} \tag{A.3}$$

Now at the left hand side of this equation one only keeps the derivatives of the Riemann tensors, and moves all the other terms to the right hand side. Doing so, the right and the left hand sides are given as

$$\begin{aligned}
LHS = &-(\nabla_a R^k{}_{bcd}) f_{ek} - (\nabla_a R^k{}_{bec}) f_{dk} - (\nabla_a R^k{}_{bde}) f_{ck} \\
&+ (\nabla_b R^k{}_{acd}) f_{ek} + (\nabla_b R^k{}_{aec}) f_{dk} + (\nabla_b R^k{}_{ade}) f_{ck},
\end{aligned} \tag{A.4}$$

$$\begin{aligned}
RHS = &\, 2R_{abc}{}^{k} \nabla_k f_{de} + 2R_{abd}{}^{k} \nabla_c f_{ke} + 2R_{abe}{}^{k} \nabla_c f_{dk} \\
&+ R^k{}_{bcd} \nabla_a f_{ek} + R^k{}_{bec} \nabla_a f_{dk} + R^k{}_{bde} \nabla_a f_{ck} \\
&- R^k{}_{acd} \nabla_b f_{ek} - R^k{}_{aec} \nabla_b f_{dk} - R^k{}_{ade} \nabla_b f_{ck}.
\end{aligned} \tag{A.5}$$

Now contracting the a and c indices, one gets

$$
\begin{aligned}
LHS &= -(\nabla_a R^k{}_b{}^a{}_d) f_{ek} - (\nabla_a R^k{}_{be}{}^a) f_{dk} - (\nabla_a R^k{}_{bde}) f^a{}_k - (\nabla_b R^k{}_d) f_{ek} \\
&\quad + (\nabla_b R^k{}_e) f_{dk} + (\nabla_b R^k{}_{ade}) f^a{}_k, && \text{(A.6)} \\
RHS &= 2R_b{}^k \nabla_k f_{de} + 2R^a{}_{bd}{}^k \nabla_a f_{ke} + 2R^a{}_{be}{}^k \nabla_a f_{dk} \\
&\quad + R^k{}_b{}^a{}_d \nabla_a f_{ek} + R^k{}_{be}{}^a \nabla_a f_{dk} + R^k{}_{bde} \nabla_a f^a{}_k \\
&\quad + R^k{}_d \nabla_b f_{ek} - R^k{}_e \nabla_b f_{dk} - R^{ka}{}_{de} \nabla_b f_{ak}. && \text{(A.7)}
\end{aligned}
$$

Doing another contraction of the b and e indices, this equation reads

$$
\begin{aligned}
LHS &= -(\nabla_a R^{kba}{}_d) f_{bk} + (\nabla_a R^{ka}) f_{dk} - (\nabla_a R^k{}_d) f^a{}_k - (\nabla_b R^k{}_d) f^b{}_k \\
&\quad + (\nabla_b R^{kb}) f_{dk} + (\nabla_b R^k{}_{ad}{}^b) f^a{}_k, && \text{(A.8)} \\
RHS &= 2R_b{}^k \nabla_k f_d{}^b + 2R^{ab}{}_d{}^k \nabla_a f_{kb} - 2R^{ak} \nabla_a f_{dk} \\
&\quad + R^{kba}{}_d \nabla_a f_{bk} - R^{ka} \nabla_a f_{dk} \\
&\quad + R^k{}_d \nabla_b f_{ek} - R^{kb} \nabla_b f_{dk} - R^{ka}{}_d{}^b \nabla_b f_{ak}. && \text{(A.9)}
\end{aligned}
$$

By using the symmetry of the Ricci tensor and the antisymmetry of the $\nabla_a f_{bc}$ term, the contraction of both pieces yield zero. Moreover, we have $(\nabla_a f_{bc}) R^{abcd} = 0$, since $R^{[abc]d} = 0$. Therefore, the RHS vanishes and one has the LHS left only :

$$
-(\nabla_a R^{kba}{}_d) f_{bk} + (\nabla_a R^{ka}) f_{dk} - (\nabla_a R^k{}_d) f^a{}_k - (\nabla_b R^k{}_d) f^b{}_k + (\nabla_b R^k{}_e) f_{dk} + (\nabla_b R^k{}_{ade}) f^a{}_k = 0. \tag{A.10}
$$

Finally, if one uses the contracted Bianchi identity and the conservation of the Einstein tensor, one finds

$$
\begin{aligned}
2\nabla_a R_{cb} f^{ac} + \nabla_c R f_b{}^c &= 0 && \text{(A.11)} \\
\nabla_a (R_{bc} - \frac{1}{2} g_{bc} R) f^{ab} &= 0, && \text{(A.12)} \\
\nabla_a (G_{bc}) f^{ab} &= \nabla_a (G_{bc} f^{ab}), \\
&= 0. && \text{(A.13)}
\end{aligned}
$$

A.2 Second covariant derivative of the Killing-Yano tensor

In [16], the identity (5.5) was given for the Killing-Yano tensor of arbitrary rank. Since it was used repeatedly, we better give its proof. We will only give its proof for the rank two tensor, however generalization to higher ranks can be obtained following similar steps. One needs to start with

$$\underbrace{[\nabla_a, \nabla_b] f_{cd} + [\nabla_c, \nabla_a] f_{bd} + [\nabla_b, \nabla_c] f_{ad}}_{\equiv RHS} = R_{abc}{}^e f_{ed} + R_{abd}{}^e f_{ce} + R_{cab}{}^e f_{ed}$$
$$+ R_{cad}{}^e f_{be} + R_{bca}{}^e f_{ed} + R_{bcd}{}^e f_{ae}. \tag{A.14}$$

The right hand side of this equation can be written as

$$\begin{aligned} RHS &= \nabla_a \nabla_b f_{cd} - \nabla_b \nabla_a f_{cd} + \nabla_c \nabla_a f_{bd} - \nabla_a \nabla_c f_{bd} \\ &\quad + \nabla_b \nabla_c f_{ad} - \nabla_c \nabla_b f_{ad}, \\ &= 2(\nabla_a \nabla_b f_{cd} + \nabla_b \nabla_c f_{ad} - \nabla_c \nabla_b f_{ad}), \\ &= 2(\nabla_a \nabla_b f_{cd} + R_{bca}{}^e f_{ed} + R_{bcd}{}^e f_{ae}). \end{aligned} \tag{A.15}$$

Substituting this in (A.14) will result in

$$\begin{aligned} \nabla_a \nabla_b f_{cd} = \frac{1}{2}(&R_{abc}{}^e f_{ed} + R_{abd}{}^e f_{ce} + R_{cab}{}^e f_{ed} + R_{cad}{}^e f_{be} \\ &- R_{bca}{}^e f_{ed} - R_{bcd}{}^e f_{ae}). \end{aligned} \tag{A.16}$$

Repeating analogous calculations for $\nabla_a \nabla_c f_{db}$ and $\nabla_a \nabla_d f_{bc}$, one gets

$$\begin{aligned} \nabla_a \nabla_c f_{db} = \frac{1}{2}(&R_{acd}{}^e f_{eb} + R_{acb}{}^e f_{de} + R_{dac}{}^e f_{eb} + R_{dab}{}^e f_{ce} \\ &- R_{cda}{}^e f_{eb} - R_{cdb}{}^e f_{ae}) \end{aligned} \tag{A.17}$$

$$\begin{aligned} \nabla_a \nabla_d f_{bc} = \frac{1}{2}(&R_{adb}{}^e f_{ec} + R_{adc}{}^e f_{be} + R_{bad}{}^e f_{ec} + R_{bac}{}^e f_{de} \\ &- R_{dba}{}^e f_{ec} - R_{dbc}{}^e f_{ae}). \end{aligned} \tag{A.18}$$

If one adds all the three terms at left hand side, and uses the total antisymmetry of the Killing-Yano tensor, one finds

$$\nabla_a \nabla_b f_{cd} + \nabla_a \nabla_c f_{db} + \nabla_a \nabla_d f_{bc} = 3 \nabla_a \nabla_b f_{cd}. \tag{A.19}$$

By writing the left hand side, one gets the desired identity for the rank-2 Killing-Yano tensor:

$$\nabla_a \nabla_b f_{cd} = \frac{1}{2}(R^e{}_{acb} f_{de} + R^e{}_{abd} f_{ce} + R^e{}_{adc} f_{be}). \tag{A.20}$$

Generalizing this identity to a Killing-Yano tensor of arbitrary rank is straightforward. Following analogous steps and using the totally antisymmetric structure of the first covariant derivative of the Killing-Yano tensor, one arrives at (5.5).

A.3 Killing-Yano current 3-form derivation

In subsection 5.2.1, the gravitational charge constructed through a Killing-Yano tensor, and k^{ab} was written as a covariant divergence of another totally antisymmetric tensor. Here, we derive this relation. The calculations are quite straightforward. First, one just needs to substitute the linearized terms found in Section 2.3 in the linearized current (5.15). Then one gets

$$
\begin{aligned}
k^{ab} &= \bar{f}^{cd}\,(R_{cd}{}^{ab})_L - 2\bar{f}^{ac}\,(R^b{}_c)_L + 2\bar{f}^{bc} R^a{}_c + \bar{f}^{ab} R_L \qquad (A.21)\\
&= \frac{1}{2}\bar{f}(\bar{\nabla}^a\bar{\nabla}_d h^b{}_c - \bar{\nabla}^b\bar{\nabla}_d h^a{}_c - \bar{\nabla}^a\bar{\nabla}_c h^b{}_d + \bar{\nabla}^b\bar{\nabla}_c h^a{}_d)\\
&\quad - \bar{f}^{ac}(-\bar{\nabla}^d\bar{\nabla}_d h^b{}_c - \bar{\nabla}_c\bar{\nabla}^b h + \bar{\nabla}^d\bar{\nabla}_c h^b{}_d + \bar{\nabla}^d\bar{\nabla}^b h_{cd})\\
&\quad + \bar{f}^{bc}(-\bar{\nabla}^d\bar{\nabla}_d h^a{}_c - \bar{\nabla}_c\bar{\nabla}^a h + \bar{\nabla}^d\bar{\nabla}_c h^a{}_d + \bar{\nabla}^d\bar{\nabla}^a h_{cd})\\
&\quad + \bar{f}^{ab}(\bar{\nabla}^c\bar{\nabla}^d h_{cd} - \bar{\nabla}^d\bar{\nabla}_d h), \qquad (A.22)\\
&= \frac{1}{2}\bar{\nabla}_d\Big(\bar{f}^{cd}\bar{\nabla}^a h^b{}_c - \bar{f}^{cd}\bar{\nabla}^b h^a{}_c - \bar{f}^{dc}\bar{\nabla}^a h^b{}_c + \bar{f}^{dc}\bar{\nabla}^b h^a{}_c + 2\bar{f}^{ac}\bar{\nabla}^d h^b{}_c\\
&\quad + 2\bar{f}^{ad}\bar{\nabla}^b h - 2\bar{f}^{ad}\bar{\nabla}^c h^b{}_c - 2\bar{f}^{ac}\bar{\nabla}^b h^d{}_c - 2\bar{f}^{bc}\bar{\nabla}^d h^a{}_c - 2\bar{f}^{bd}\bar{\nabla}^a h\\
&\quad + 2\bar{f}^{bd}\bar{\nabla}^c h^a{}_c + 2\bar{f}^{bc}\bar{\nabla}^a h_c{}^d + 2\bar{f}^{ab}\bar{\nabla}^c h^d{}_c - 2\bar{f}^{ab}\bar{\nabla}^d h\Big)\\
&\quad - \bar{\nabla}_d\bar{f}^{ac}\bar{\nabla}^d h^b{}_c + \bar{\nabla}_d\bar{f}^{ac}\bar{\nabla}^b h_c{}^d + \bar{\nabla}_d\bar{f}^{bc}\bar{\nabla}^d h^a{}_c - \bar{\nabla}_d\bar{f}^{bc}\bar{\nabla}^a h_c{}^d\\
&\quad - \bar{\nabla}_d\bar{f}^{ab}\bar{\nabla}^c h_c{}^d + \bar{\nabla}_d\bar{f}^{ab}\bar{\nabla}^d h. \qquad (A.23)
\end{aligned}
$$

This is quite a long expression, however it simplifies greatly. Here the background is flat and therefore its Riemann tensor is identically zero. Therefore, if one considers (A.20), one sees that second covariant derivatives of the Killing-Yano tensors in this background vanish. By using this, if one carefully examines the equations and regroups them patiently, then one gets

$$k^{ab} = \bar{\nabla}_c\Big(3!\bar{f}^{[c|d|}\bar{\nabla}^b h^{a]}{}_d + \frac{3!}{2}\bar{f}^{[ac}\bar{\nabla}^{b]}h + \frac{3!}{2}\bar{f}^{[ab}\bar{\nabla}^{|d|}h^{c]}{}_d + \frac{3!}{2}\bar{\nabla}^d\bar{f}^{[cb}h^{a]}{}_d + h\bar{\nabla}^{[c}\bar{f}^{ab]}\Big). \qquad (A.24)$$