ON THE NILPOTENT LENGTH OF A FINITE GROUP WITH A FROBENIUS GROUP OF AUTOMORPHISMS

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ABSTRACT

ON THE NILPOTENT LENGTH OF A FINITE GROUP
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Let G be a finite group admitting a Frobenius group FH of automorphisms with kernel F and complement H. Assume that the order of G and FH are relatively prime and $C_{C_G(F)}(h) = 1$ for every $1 \neq h \in H$. It is proved in this thesis that $f(G) \leq f([G, F]) + 1$ and $f([G, F]) = f(C_{[G,F]}(H))$ where $f(G)$ denotes the nilpotent length of the group G.

Keywords: solvable group, automorphism, Frobenius group, nilpotent length

ÖZ

FROBENİUS OTOMORFİZMA GRUBUNA SAHİP OLAN
SONLU BİR GRUBUN NİLPOTENT UZUNLUĞU

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Sonlu bir G grubu; çekirdeği F ve komplementi H olan bir FH Frobenius otomorfizma grubunun etkisini kabul ediyor olsun. G ve FH gruplarının mertebelerinin aralarında asal olduğunu varsayalım ve $C_{C_G(F)}(h) = 1$ eşitliğinin H'in birimden farklı her elemanı için sağlandığını varsayalım. Bu tezde, belirtilen koşulları sağlayan bir (G, FH) çifti için, $f(G) \leq f([G, F]) + 1$ ve $f([G, F]) = f(C_{[G,F]}(H))$ olduğunu ispatladık. Burada $f(G)$, G'nin nilpotent uzunluğunu göstermektedir.

Anahtar Kelimeler: çözülebilir grup, otomorfizma, Frobenius grup, nilpotent uzunluğu

Firstly to God
and then
to my lovely parents

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CHAPTER 1

INTRODUCTION

A well-known result of J.Thompson says that a finite group G admitting an automorphism α of prime order p acting fixed-point-freely on G must be nilpotent, a direct product of its Sylow subgroups ([24]). The proof of this result is comparatively easy in the case of a solvable group G and can be given by reffering to some basic results of representation theory ([8]). The main contribution of Thompson was the reduction of the general case to this solvable group situation.

Let now G be a finite group and let A act on G by automorphisms. We let

$$C_G(A) = \{g \in G : g^a = g \text{ for all } a \in A\}.$$

and say that A acts fixed-point-freely on G if $C_G(A) = 1$. If A is a group of prime power order p^n, then letting P be a Sylow p-subgroup of the semi-direct product GA containing A, we see that $P \cap G \lhd P$ and hence $Z(P) \cap G \neq 1$ if $P \cap G \neq 1$. In this case we get $Z(P) \cap G \leq C_G(A)$. So if A acts fixed-point-freely, then $(|G|, |A|) = 1$.

In general, the fixed-point-free action of A on G does not imply $(|G|, |A|) = 1$. If this condition is satisfied and A acts fixed-point-freely on G, then the classification of finite simple groups implies that G is solvable ([16]). If A is not of prime order, G does not need to be nilpotent, but in some cases it can be determined in a precise way how much G deviates from being nilpotent. What we mean is the following:

For a solvable group G, we define the Fitting subgroup $F(G)$ of G as the maximal, nilpotent, normal subgroup of G. It is a characteristic subgroup and G is nilpotent if and only if $G = F(G)$. Furthermore, in a solvable group G, $F(G) \neq 1$ if $G \neq 1$. Thus we define recusively

$$\begin{aligned} F_0(G) &= 1, \\ F_i(G)/F_{i-1}(G) &= F(G/F_{i-1}(G)) \quad \text{for} \quad i \geq 1, \\ F_n(G) &= G. \end{aligned}$$

Then the ascending sequence (which is called the Fitting series of G)

$$F_0(G) \leq F_1(G) \leq F_2(G) \leq \ldots \leq F_i(G) \leq \ldots$$

terminates in G if G is solvable. So there exists a smallest nonnegative integer n with $F_n(G) = G$. This integer is called the nilpotent length (or Fitting height) of G and is denoted by $f(G)$. This invariant of G will be used as a measure of deficiency of G from being nilpotent.

We have the following conjecture:

CONJECTURE *Let G be a solvable group and $A \leq Aut(G)$ with $(|G|, |A|) = 1$. If $C_G(A) = 1$, then $f(G) \leq k(A)$. Here $k(A)$ is the number of primes dividing $|A|$ counted with multiplicities. In the case of solvable A, $k(A)$ is also the composition length of A.*

First results in this direction are due to Dade and Kurzweil, who improved the exponential bound in a previous result of Thompson to a linear bound ([12]). Berger in [13] verified the conjecture in the case that A is nilpotent and has no section isomorphic to the wreath product of two groups of order p. Later this is generalized by Turull to the case when A is supersolvable group satisfying some other special conditions ([25]).

Apart from these, without the coprimeness condition, for the case where A is nilpotent and $C_G(A) = 1$ a bound for the nilpotent length of G was shown by Dade in 1969 ([14]). However, without the coprimeness condition, there is very little known. And we should note that some coprimeness condition is necessary in general, since it was shown by Bell and Hartley in 1990, that any finite non-nilpotent solvable group A can act fixed-point-freely on a solvable group G with arbitrarily large nilpotent length when G and A have non-coprime orders ([15]).

In recent years, Khukhro handled the case where $A = FH$ is a Frobenius group of automorphisms of G with kernel F and complement H such that the kernel acts fixed-point-freely on G, that is $C_G(F) = 1$. Notice that $C_G(F) = 1$ implies $C_G(A) = 1$. It is also worth noting that since F, being a Frobenius kernel, is nilpotent, the condition $C_G(F) = 1$ implies the solvability of G, by a theorem in [17] based on the classification of finite simple groups.

Under the conditions that $(|G|, |H|) = 1$ and $C_G(F) = 1$, Khukhro proved in 2011 that the nilpotent length of G is equal to the nilpotent length of $C_G(H)$. Namely, he proved the following (see [1]):

Let G be a finite group admitting a Frobenius group of automorphisms FH with kernel F and complement H. Suppose that $C_G(F) = 1$ and $(|G|, |H|) = 1$. Then,

(i) $F_k(C_G(H)) = F_k(G) \cap C_G(H)$ *for all* k,
(ii) *the nilpotent length of G is equal to the nilpotent length of $C_G(H)$.*

One year later, in [2], he obtained the same result without the coprimeness condition, $(|G|, |H|) = 1$.

In fact, the additional action of the Frobenius complement H on G suggests another approach to the study of G. By an application of Clifford's theorem, all FH-invariant elementary abelian sections of G are free F_pH-modules (for various p), see Lemma 1.3. in [2]. Therefore, it is natural to expect that many properties or parameters of G should be close to the corresponding properties or parameters of $C_G(H)$, possibly also depending on H. Several results of this nature were obtained recently under certain additional conditions some of which are unavoidable; (see [18,19,20,21,22,23])

In this work, we handled the case where $A = FH$ is a Frobenius group with kernel F and complement H acting coprimely on G, and weakened the fixed point freeness condition of F on G in Khukhro's results slightly . Mainly, replacing the condition $C_G(F) = 1$ by the condition that $C_{C_G(F)}(h) = 1$ for every $1 \neq h \in H$, we obtained the following:

Theorem 3.3.1. Let G be a finite solvable group admitting a Frobenius group of automorphisms FH of coprime order with kernel F and complement H so that $C_{C_G(F)}(h) = 1$ for every $1 \neq h \in H$. Then $f([G, F]) = f(C_{[G,F]}(H))$ and $f(G) \leq f([G, F]) + 1$.

It should be noted that the condition $C_{C_G(F)}(h) = 1$ for every $1 \neq h \in H$ implies $C_G(FH) = 1$ and due to the coprime action, G is a solvable group by the classification of finite simple groups ([16] Theorem 1.48).

Additionally, we proved the following lemma as a key result in proving the Theorem 3.3.1:

Lemma 3.2.1. Let Q be a normal q-subgroup of a group having a complement FH which is a Frobenius group with kernel F and complement H so that $C_{C_Q(F)}(h) = 1$ for every $1 \neq h \in H$. Assume further that $|FH|$ is not divisible by q and Q is of class at most 2. Let V be a $kQFH$-module on which $[Q, F]$ acts nontrivially where k is a field with characteristic not dividing $|QFH|$. Then we have

$$Ker(C_{[Q,F]}(H) \text{ on } C_V(H)) = Ker(C_{[Q,F]}(H) \text{ on } V).$$

It should be noted that this lemma is of independent interest, as well.

The outline of the thesis is as follows:

Section 1 and 2 of Chapter 2 give the necessary preparation from the group theory and representation theory. Most of them are well known results which will be referred throughout the thesis.

Chapter 3 includes the proof of a technical lemma which is known in foklore and our original results; namely the key lemma pertaining the main result and the proof of the main theorem of this thesis. The method of proving in our results, in general, is deducing a contradiction over a series of steps, from the assumption of the existence of a counterexample.

Throughout the thesis all groups are finite. The notation and terminology are mostly standard and can be seen in Appendix A.

CHAPTER 2

PRELIMINARIES

2.1 Group Theoretical Part

In this section we present some necessary group theoretical preliminaries. Most of them are well known results whose proofs can be seen in the given references written in the proof part which follows the result. However, we will give the results, whose proofs can not be seen in any references, with their proofs.

Definition 2.1.1 ***(Frobenius action)*** *Let H and F be finite groups and suppose that H acts on F via automorphisms. The action of H on F is said to be Frobenius if and only if $C_F(h) = 1$ for all $1 \neq h \in H$ (if and only if $C_H(f) = 1$ for all $1 \neq f \in F$).*

Definition 2.1.2 ***(Frobenius complement, Frobenius kernel)*** *A finite group H is said to be a Frobenius complement if it has a Frobenius action on some nonidentity group F, and similarly, a finite group F is called a Frobenius kernel if it admits a Frobenius action by some nonidentity finite group H.*

Definition 2.1.3 ***(Frobenius group)*** *Let $G = F \rtimes H$. If the conjugation action of H on F is Frobenius then the group G is called a Frobenius group with kernel F and complement H.*

Lemma 2.1.4 *Let H and F be finite groups, and suppose there is a Frobenius action of H on F. Then $|F| \equiv 1 \mod |H|$, and hence $|F|$ and $|H|$ are coprime.*

Proof. [3] 6.1.

Corollary 2.1.5 *Let H and F be finite groups, and suppose there is a Frobenius action of H on F. Let E be an H-invariant normal subgroup of F. Then the induced action of H on F/E is Frobenius.*

Proof. [3] 6.2.

Theorem 2.1.6 *Let F be a normal subgroup of a finite group G, and suppose that H is a complement for F in G. The following are then equivalent.*

(1) The conjugation action of H on F is Frobenius.

(2) $H \cap H^g = 1$ for all elements $g \in G - H$.

(3) $C_G(h) \subseteq H$ for all nonidentity elements $h \in H$.

(4) $C_G(x) \subseteq F$ for all nonidentity elements $x \in F$.

Proof. [3] 6.4.

Theorem 2.1.7 *Let F be a Frobenius kernel. Then F is nilpotent.*

Proof. [3] 6.24.

Theorem 2.1.8 *If G is a group, then G' is a normal subgroup of G and G/G' is abelian. If N is a normal subgroup of G, then G/N is abelian if and only if $G' < N$.*

Proof. [10] II.7.8.

Definition 2.1.9 *(**Lower central series**) The terms $\gamma_i(G)$ of the lower central series of a group G are defined recursively as follows: $\gamma_1(G) = G$, $\gamma_{k+1}(G) = [\gamma_k(G), G]$ for each $k \geq 1$.*

Definition 2.1.10 *(**Upper central series**) The terms $\zeta_i(G)$ of the upper central series of a group G are defined recursively as follows: $\zeta_1(G) = Z(G)$ is the center of G, and $\zeta_{k+1}(G)$ is the full inverse image of $Z(G/\zeta_k(G))$ in G for each $k \geq 1$.*

We write $\gamma_i = \gamma_i(G)$ and $\zeta_i = \zeta_i(G)$ to simplify the notation.

Theorem 2.1.11 *For a group G, the following are equivalent:*

(a) $\gamma_{c+1} = 1$;

(b) G has a central series of length c;

(c) $[g_1, g_2, ..., g_{c+1}] = 1$ for all $g_i \in G$.

(d) $\zeta_c = G$

Proof. [5] 3.9.(a),(b),(c) and (d) in § 3.2.

Definition 2.1.12 *If a group G satisfies one of the conditions in Theorem 2.1.11 then G is called nilpotent of class at most$\leq c$; the least such number c is called the nilpotency class of G. In particular, G is nilpotent of class 2 if $G' \leq Z(G)$.*

Theorem 2.1.13 *Every finite p-group is nilpotent.*

Proof. [10] II.7.2.

Theorem 2.1.14 *Let G be a finite nilpotent group and let N be a nonidentity normal subgroup of G. Then $N \cap Z(G) \neq 1$. In particular if G is nontrivial, then $Z(G) \neq 1$.*

Proof. [3] 1.19

Theorem 2.1.15 *Let G be a finite group. G is nilpotent if and only if $G' \leq \Phi(G)$ holds. More generally; for $N \trianglelefteq G$. N is nilpotent if and only if $N' \leq \Phi(G)$ holds.*

Proof. [4] III.3.11.

Theorem 2.1.16 *Let G be a finite p-group. Then,*

i) $\Phi(G) = G'G^p$, where $G^p = \langle g^p : g \in G \rangle$. Moreover, $G/\Phi(G)$ is elementary abelian and $\Phi(G)$ is the smallest normal subgroup of G that has an elementary abelian factor group.

ii) If $N \trianglelefteq G$, then $\Phi(G/N) = \Phi(G)N/N$.

iii) If $N \trianglelefteq G$, then $\Phi(N) \leq \Phi(G)$

Proof. [4] for i) and ii) III.3.14.a and c. and for iii) III.3.3.b

Definition 2.1.17 *If U and V are subgroups of G and if $UV = G$, we call V a supplement to U in G; if further $U \cap V = 1$, then V is said to be a complement to U in G.*

Theorem 2.1.18 *In any group G the Frattini subgroup equals the set of nongenerators of G.*

Proof. [7] 5.2.12.

Theorem 2.1.19 ***(Schur-Zassenhaus Theorem)*** *Let N be a normal subgroup of a finite group G and $(|N|, |G : N|) = 1$. Then N is complemented in G, that is there exists a subgroup H of G such that $G = NH$ and $N \cap H = 1$. And if either N is solvable or G/N is solvable, then all complements for N in G are conjugate.*

Proof. [3] 3.8. and 3.12.

Definition 2.1.20 *Let G be a finite group and N be a subgroup of G. Then N is a Hall subgroup of G if* $(|G|, |G : N|) = 1$.

Definition 2.1.21 *(**π-separable**) A finite group G is said to be π-separable, where π is some set of primes, if there exists a normal series*

$$1 = N_0 \leq N_1 \leq ... \leq N_r = G$$

such that each factor N_i/N_{i-1} is either a π-group or a π'-group.

Definition 2.1.22 *(**π-number, π-element, π-group**) If π is a nonempty set of primes, a π-number is a positive integer whose prime divisors belong to π. An element of a group is called a π-element if its order is a π-number, that is the prime divisors of its order belong the set π. And should every element of a group be a π-element, the group is called a π-group.*

Theorem 2.1.23 *Let the finite group G be π-separable. Then every π-subgroup is contained in a Hall π-subgroup of G and any two Hall π-subgroups are conjugate in G.*

Proof. [7] 9.1.6.

Lemma 2.1.24 *The following commutator formulae hold for any elements a, b, c in any group*

$$[ab, c] = [a, c]^b [b, c].$$

Proof. [5] 1.11.(b)

Lemma 2.1.25 *Let H and K be subgroups of a group G. Then $K \subseteq N_G(H)$ if and only if $[H, K] \leq H$. In particular, H is a normal subgroup of G if and only if $[H, G] \leq H$.*

Proof. [3] 4.3.

Lemma 2.1.26 *Let H and K be subgroups of a group G. Then H and K normalize $[H, K]$, or equivalently, H and K are normal subgroups of $\langle H, K \rangle$, the subgroup generated by H and K.*

Proof. [3] 4.1.

Lemma 2.1.27 ***(Three Subgroup Lemma)*** *Let X, Y and Z be subgroups of an arbitrary group G, and let N be a normal subgroup of G. Suppose that $[X, Y, Z] \leq N$ and $[Y, Z, X] \leq N$. Then $[Z, X, Y] \leq N$.*

Proof. [3] 4.10.

Lemma 2.1.28 *Let A act via automorphisms on an abelian group G and assume that A and G are finite and that $(|G|, |A|) = 1$. Then $G = C_G(A) \times [G, A]$.*

Proof. [3] 4.34.

Lemma 2.1.29 *Let A and G be finite groups. Let A act via automorphisms on G and suppose that $(|G|, |A|) = 1$ and that one of A or G is solvable. Then $G = C_G(A)\,[G, A]$.*

Proof. [3] 4.28.

Lemma 2.1.30 *Let A act via automorphism on G, where A and G are finite groups, and suppose that $(|G|, |A|) = 1$. Then $[G, A, A] = [G, A]$.*

Proof. [3] 4.29.

Theorem 2.1.31 *Let A act via automorphisms on G, where A and G are finite groups, and write $C = C_G(A)$. Let $H \subseteq G$ be A-invariant subgroup, and suppose that $(|A|, |H|) = 1$ and that one of A or H is solvable. Then the A-invariant left cosets of H in G and the A-invariant right cosets of H in G are exactly those cosets of H that contain elements of C.*

Proof. [3] 3.27.

Theorem 2.1.32 *Let A act via automorphisms on G, where A and G are finite groups, and let $N \lhd G$ be A-invariant. Assume that $(|A|, |N|) = 1$ and that at least one of A or N is solvable. Writing $\overline{G} = G/N$, we have*

$$C_{\overline{G}}(A) = \overline{C_G(A)},$$

Proof. [3] 3.28.

Definition 2.1.33 ***(Fitting subgroup)*** *For a solvable group G, the Fitting subgroup of G denoted by $F(G)$ is defined as the subgroup generated by all normal nilpotent subgroups of G. So, for a nilpotent group $F(G) = G$.*

Theorem 2.1.34 *(**Fitting's Theorem**) Suppose that M and N are normal nilpotent subgroups of a group G with respective nilpotent classes c and d. Then, MN is nilpotent of class at most $c + d$.*

Proof. [7] 5.2.8.

Lemma 2.1.35 *When G is a finite solvable group, $F(G)$ is the largest normal nilpotent subgroup of G and it is nontrivial.*

Proof. By the Fitting's Theorem 2.1.34, we see that $F(G)$ of a finite solvable group G is the largest normal nilpotent subgroup. It is also nontrivial since it contains the smallest nontrivial element of the derived series of G: Recall that the derived series of a finite solvable group G is as follows.

$$G = G^{(0)} \triangleright G^{(1)} = G' \triangleright \ldots \triangleright G^{(n-1)} \triangleright (G^{(n-1)})' = G^{(n)} = 1$$

where $(G^{(n-1)})' = 1$ implies that $G^{(n-1)}$ is abelian and hence nilpotent. It is also normal in G. So, $1 \neq G^{(n-1)} \leq F(G)$.

Definition 2.1.36 *(**Fitting series, Nilpotent length (or Fitting height)**) The Fitting series of G is defined by*

$$\begin{aligned} F_0(G) &= 1, \\ F_i(G)/F_{i-1}(G) &= F(G/F_{i-1}(G)) \quad for \quad i \geq 1, \\ F_n(G) &= G. \end{aligned}$$

and the nilpotent length (or the Fitting height) of G is the least integer n such that $F_n(G) = G$. Throughout the thesis the nilpotent length of a solvable group G is denoted by $f(G)$.

But in general, a Fitting series for a group is is known to be a subnormal series with nilpotent quotients. In other words, a finite sequence of subgroups including both the whole group and the trivial group, such that each is a normal subgroup of the previous one, and such that the quotients of successive terms are nilpotent groups. And the nilpotent length of a group is defined to be the smallest possible length of a Fitting series, if one exists. And also, the Fitting series defined above is known to be the upper Fitting series of G. It is an ascending nilpotent series, at each step taking the maximal possible subgroup, therefore $F_1(G) = F(G)$. And note that, $f(G) = 1$ if and only if G is nilpotent.

Proposition 2.1.37 *Let H be a solvable group of Fitting length n. Then $f(H/F_i(H)) = n - i$ for all $1 \leq i < n$.*

Proof. Consider the Fitting series of H, which is

$$1 \triangleleft F(H) \triangleleft F_2(H) \triangleleft \ldots \triangleleft F_i(H) \triangleleft F_{i+1}(H) \triangleleft \ldots \triangleleft F_n(H) = H$$

and let $\overline{H} = H/F_i(H)$. Then we have

$$1 = \overline{F_i(H)} \lhd \overline{F_{i+1}(H)} \lhd \ldots \lhd \overline{F_n(H)} = \overline{H}$$

Now $F(\overline{H}) = F(H/F_i(H)) = F_{i+1}(H)/F_i(H) = \overline{F_{i+1}(H)}$.

Now suppose that $F_{j-1}(\overline{H}) = \overline{F_{j-1+i}(H)}$. Then,

$$\begin{aligned} F_j(\overline{H})/F_{j-1}(\overline{H}) &= F(\overline{H}/F_{j-1}(\overline{H})) = F(\overline{H}/\overline{F_{j-1+i}(H)}) \\ &= F(H/F_i(H)/F_{j-1+i}(H)/F_i(H)) \\ &\cong F(H/F_{j-1+i}(H)) = F_{j+i}(H)/F_{j-1+i}(H) \\ &\cong (F_{j+i}(H)/F_i(H))/(F_{j-1+i}(H)/F_i(H)) \\ &= \overline{F_{j+i}(H)}/\overline{F_{j-1+i}(H)} \end{aligned}$$

and hence $F_j(\overline{H}) = \overline{F_{j+i}(H)}$ as

$$\overline{F_{j+i}(H)}/\overline{F_{j-1+i}(H)} \leq F_j(\overline{H})/F_{j-1}(\overline{H}) = F(\overline{H}/F_{j-1}(\overline{H})).$$

Thus by induction we get that $F_j(\overline{H}) = \overline{F_{j+i}(H)}$, for all $1 \leq j \leq n-i$. Hence

$$F_{n-i}(\overline{H}) = \overline{F_{n-i+i}(H)} = \overline{F_n(H)} = \overline{H}$$

and $f(\overline{H}) = n - i$.

Definition 2.1.38 *(B-**tower**, Definition 1.1. [9])*

Let GB be a finite group with $G \lhd GB$. We say that a sequence of B-invariant subgroups of G, $(\hat{P}_i)$, $i = 1, ..., h$, is a B-tower of G if the following are satisfied:

(1) $\pi(\hat{P}_i) = \{p_i\}$ consists of a single prime for $i = 1, ..., h$;

(2) $\hat{P}_i$ normalizes $\hat{P}_j$, for $i \leq j$;

(3) We set $P_h = \hat{P}_h$ and $P_i = \hat{P}_i/C_{\hat{P}_i}(P_{i+1})$, $i = 1, ..., h-1$, and P_i is not trivial for $i = 1, ..., h$;

(4) $p_i \neq p_{i+1}$, $i = 1, ..., h-1$.

Here, the positive integer h is called the height (or length) of the tower. And when $B = 1$, we simply call $\hat{P}_1, \ldots, \hat{P}_h$ a tower of G instead of a $\{1\}$-tower of G.

Definition 2.1.39 *(**irreducible** B-**tower**, Definition 1.2. [9]) Let GB be a finite group with $G \lhd GB$. We say that a B-tower $(\hat{P}_i)$, $i = 1, \ldots, h$, of G is irreducible if the following are satisfied.*

(5) $\Phi(\Phi(P_i)) = 1$, $\Phi(P_i) \subseteq Z(P_i)$ *and if* $p_i \neq 2, exp(P_i) = p_i$ *for* $i = 1, \ldots, h$ *and* $\hat{P}_{i-1}$ *centralizes* $\Phi(P_i)$, $i = 2, ..., h$;

(6) P_1 *is elementary abelian;*

(7) *There exists* H_i *an elementary abelian subgroup of* P_{i-1} *normalized by* B *such that* $[H_i, P_i] = P_i$ *for* $i = 2, \ldots, h$;

(8) *If* $Q \subseteq \hat{P}_i$ *for some* i, Q *is normalized by* $\hat{P}_{i-1} \ldots \hat{P}_1 B$ *and its image in* P_i *is not contained in* $\Phi(P_i)$, *then* $Q = \hat{P}_i$.

Theorem 2.1.40 *If* GB *is a finite group where* G *is a solvable group with* $G \lhd GB$ *and* $(|G|, |B|) = 1$. *Then the following hold:*
(i) There always exists a B*-tower (and hence an irreducible* B*-tower)* $\hat{P}_1, \ldots, \hat{P}_n$ *in* G *where* n *is equal to the nilpotent length of* G. *If* $G = [G, B]$, *then we can assume that* $[P_1, B] = P_1$.
(ii) The nilpotent length of G *is the maximum of the heights of* B*-towers in* G.

Proof. [9]

Theorem 2.1.41 *Let* G *be a finite solvable group. If* $X \leq G$, *then the nilpotent length of* X *is less than or equal to the nilpotent length of* G, *that is* $f(X) \leq f(G)$.

Proof. Let $f(X) = m$. By Theorem 2.1.40, we know that there always exists a tower of X and the nilpotent length of X is the maximum of the heights of towers in X. Then $m = f(X)$ implies that there is a tower of height m of X. Obviously, every tower of X is also a tower of G. It follows that $f(G) \geq m$.

Theorem 2.1.42 *Let* $G = HK$ *be a finite solvable group such that* $H \lhd G$, *then the nilpotent length of* G *is less than or equal to the sum of the nilpotent lengths of* H *and* K, *that is* $f(G) \leq f(H) + f(K)$.

Proof. Let $f(H) = m$ and $f(K) = n$. Consider the following normal series of H, where each of the factors are nilpotent.

$$1 < F(H) = F_1(H) < F_2(H) < F_3(H) < \ldots < F_m(H) = H.$$

Since, for each $i = 1, \ldots, m$, $F_i(H) char H \lhd G$ we have also $F_i(H) \lhd G$. Now, we may extend the series above in the following way:

$$1 < F(H) = F_1(H) < F_2(H) < \ldots < F_m(H) = H < HF(K) < HF_2(K) < \ldots < HF_n(K) = HK = G.$$

Obviously, $HF_i(K) \lhd G = HK$ for each $i = 1, \ldots, n$. And each of the factors in the extended part of the series are also nilpotent. To see this, consider any factor group, $HF_{i+1}/HF_i(K)$. Then we have

$$\begin{aligned} HF_{i+1}(K)/HF_i(K) &= HF_i(K)F_{i+1}(K)/HF_i(K) \\ &\cong F_{i+1}(K)/HF_i(K) \cap F_{i+1}(K) \\ &\cong F_{i+1}(K)/F_i(K)(H \cap F_{i+1}(K)) \\ &\cong (F_{i+1}(K)/F_i(K))/(F_i(K)(H \cap F_{i+1}(K))/F_i(K)) \end{aligned}$$

where the last factor group obtained above by employing the Isomorphism theorems and the Dedekind's rule is nilpotent since $F_{i+1}(K)/F_i(K)$ is nilpotent by definition giving that $HF_{i+1}(K)/HF_i(K)$ is also nilpotent.

As a result, we have seen that the new series extended from the first one is a normal series of G, where each of the factors are nilpotent. And, since the nilpotent length of G is the smallest length of such series by Definition 2.1.36, we get $f(G) \leq m + n = f(H) + f(K)$.

Theorem 2.1.43 *Let G be a group on which B acts and $\hat{P}_1 \ldots \hat{P}_\ell$ be a sequence of B-invariant subgroups of G where $\hat{P}_i$ is a p_i-group for each $i = 1, \ldots, \ell$ with $p_i \neq p_{i+1}$ and $\hat{P}_j$ normalizes $\hat{P}_i$ for each i, j with $i \geq j$. Set $P_i = \hat{P}_i / C_{\hat{P}_i}(P_{i+1})$ for each $i = 1, \ldots, \ell$ and $\hat{P}_\ell = P_\ell$. Then $\hat{P}_1, \ldots, \hat{P}_\ell$ contains a B-tower of height ℓ of G in the sense of Definition 2.1.38 if and only if $[[\hat{P}_1, \hat{P}_2], \ldots, \hat{P}_\ell] \neq 1$.*

Proof. ($\Leftarrow$:) Assume $[[\hat{P}_1, \hat{P}_2], \ldots, \hat{P}_\ell] \neq 1$. Tower definition (1), (2), (4) of Definition 2.1.38 are already assumed for $\hat{P}_1, \hat{P}_2, \ldots, \hat{P}_\ell$. We need to show

$$P_i = \hat{P}_i / C_{\hat{P}_i}(P_{i+1}) \neq 1$$

for all i. Assume not. Then

$$C_{\hat{P}_i}(P_{i+1}) = \hat{P}_i \text{ for some } i \text{ and hence } [\hat{P}_i, P_{i+1}] = 1$$

implying

$$[\hat{P}_i, \hat{P}_{i+1}] \leq C_{\hat{P}_{i+1}}(P_{i+2})$$

due to Definition 2.1.38.(3). Then

$$[\hat{P}_i, \hat{P}_{i+1}, P_{i+2}] = 1$$

which implies

$$[\hat{P}_i, \hat{P}_{i+1}, \hat{P}_{i+2}] \leq C_{\hat{P}_{i+2}}(P_{i+3})$$

by Definition 2.1.38.(3). Repeating the same procedure, we obtain

$$[\hat{P}_i, \hat{P}_{i+1}, \hat{P}_{i+2}, \ldots, P_\ell] = 1 = [\hat{P}_i, \hat{P}_{i+1}, \hat{P}_{i+2}, \ldots, \hat{P}_\ell].$$

But then,

$$1 \neq [\hat{P}_1, \hat{P}_2, \ldots, \hat{P}_i, \hat{P}_{i+1}, \hat{P}_{i+2}, \ldots, \hat{P}_\ell] \leq [\hat{P}_i, \hat{P}_{i+1}, \hat{P}_{i+2}, \ldots, \hat{P}_\ell] = 1$$

giving us a contradiction and hence (3) of Definition 2.1.38 is also satisfied.

($\Rightarrow$:) Assume $\hat{P}_1, \ldots, \hat{P}_\ell$ contains a B-tower of height ℓ, that is $\hat{R}_1, \ldots, \hat{R}_\ell$ is a B-tower such that $\hat{R}_i \leq \hat{P}_i$. Assume also that, $[[\hat{P}_1, \hat{P}_2], \ldots, \hat{P}_\ell] = 1$. Then

$$[\hat{R}_1, \ldots, \hat{R}_\ell] = 1 \quad \text{and hence} \quad [\hat{R}_\ell, [\hat{R}_1, \ldots, \hat{R}_{\ell-1}]] = 1.$$

As $[\hat{R}_1, \ldots, \hat{R}_{\ell-1}] \leq \hat{R}_{\ell-1}$, we get

$$[\hat{R}_1, \ldots, \hat{R}_{\ell-1}] \leq C_{\hat{R}_{\ell-1}}(\hat{R}_\ell).$$

This implies that

$$[[\hat{R}_1, \ldots, \hat{R}_{\ell-2}], \hat{R}_{\ell-1}/C_{\hat{R}_{\ell-1}}(\hat{R}_\ell)] = 1,$$

that is $[\hat{R}_1, \ldots, \hat{R}_{\ell-2}] \leq C_{\hat{R}_{\ell-2}}(R_{\ell-1})$ as $[\hat{R}_1, \ldots, \hat{R}_{\ell-2}] \leq \hat{R}_{\ell-2}$ and $R_{\ell-1} = \hat{R}_{\ell-1}/C_{\hat{R}_{\ell-1}}(\hat{R}_\ell)$.

In a similiar way as above we have

$$[[\hat{R}_1, \ldots, \hat{R}_{\ell-3}], \hat{R}_{\ell-2}/C_{\hat{R}_{\ell-2}}(R_{\ell-1})] = 1,$$

i.e $[\hat{R}_1, \ldots, \hat{R}_{\ell-3}] \leq C_{\hat{R}_{\ell-3}}(R_{\ell-2})$ as $[\hat{R}_1, \ldots, \hat{R}_{\ell-3}] \leq \hat{R}_{\ell-3}$ and $R_{\ell-2} = \hat{R}_{\ell-2}/C_{\hat{R}_{\ell-2}}(R_{\ell-1})$.

Inductively, we have

$$[\hat{R}_1, \ldots, \hat{R}_i] \leq C_{\hat{R}_i}(R_{i+1})$$

for each $i \geq 1$. Consequently,

$$[\hat{R}_1, R_2] = 1,$$

that is $C_{\hat{R}_1}(R_2) = \hat{R}_1$ which is impossible by part (3) of tower Definition 2.1.38.

Theorem 2.1.44 ***(Feit-Thompson Theorem)*** *Every finite group of odd order is solvable.*

Theorem 2.1.45 *If FH is a group acting on the group V via automorphisms such that $F \lhd FH$ and $(|F|, |H|) = 1$, then $C_{FH}(V) = C_F(V)C_H(V)$.*

Proof. $C_F(V)$ is FH-invariant and so $C_F(V) \trianglelefteq C_{FH}(V)$. Then

$$C_{FH}(V)/C_F(V) = C_{FH}(V)/F \cap C_{FH}(V) \cong C_{FH}(V)F/F \leq FH/F \cong H.$$

Thus $C_F(V)$ is a normal Hall subgroup of $C_{FH}(V)$. By Schur-Zassenhaus Theorem 2.1.19 $C_F(V)$ is complemented in $C_{FH}(V)$, that is there is a subgroup $K \leq C_{FH}(V)$ such that $C_{FH}(V) = C_F(V)K$. Now $(|F|, |H|) = 1$ implies that either F or H is solvable by Feit-Thompson theorem 2.1.44 and hence K is contained in a conjugate of H in FH by 2.1.23. Then there exists $x \in F$ such that $K \leq C_{H^x}(V) = (C_H(V))^x$ as $V^x = V$. So,

$$|C_{FH}(V)| = |C_F(V)|\,|K| \leq |C_F(V)|\,|(C_H(V))^x| = |C_F(V)|\,|C_H(V)|.$$

On the other hand, $C_F(V)C_H(V) \subseteq C_{FH}(V)$. Thus we have $C_{FH}(V) = C_F(V)C_H(V)$.

Theorem 2.1.46 *Let $G \lhd GA$ with $(|G|, |A|) = 1$ where $G = HK$ such that $H \lhd G$ and H, K are A-invariant. Then $C_G(A) = C_H(A)C_K(A)$.*

Proof. We have $C_{G/H}(A) = C_G(A)H/H \cong C_G(A)/C_H(A)$.

On the other hand,

$$\begin{aligned} C_{G/H}(A) &= C_{HK/H}(A) \cong C_{K/K\cap H}(A) = C_K(A)K \cap H/K \cap H \cong C_K(A)/C_K(A) \cap (K \cap H) \\ &= C_K(A)/C_{K\cap H}(A). \end{aligned}$$

Then, $|C_G(A)/C_H(A)| = |C_K(A)/C_{K\cap H}(A)|$ implies

$$|C_G(A)| = \frac{|C_H(A)||C_K(A)|}{|C_{K\cap H}(A)|} \leq |C_H(A)|\,|C_K(A)|.$$

From $C_H(A)C_K(A) \leq C_G(A)$ the result follows.

Corollary 2.1.47 *Let A act on G, where A and G are finite groups, and assume that $(|G|, |A|) = 1$ and that the action of A on G is faithful. Then the induced action of A on $G/\Phi(G)$ is faithful.*

Proof. [3] 3.30.

Corollary 2.1.48 *Let A act on G, where A and G are finite groups, and assume that $(|G|, |A|) = 1$. If the induced action of A on the Frattini factor group $G/\Phi(G)$ is trivial than the action of A on G is trivial.*

Proof. [3] 3.29.

2.2 Representation Theoretical Part

In this section we will present a series of definitions, theorems and lemmas from the representation theory which will be used throughout the thesis. As in the first section, most of them are well known results whose proofs can be seen in the given references written in the proof part which follows the result. However, we will give the results, whose proofs can not be seen in any references, with their proofs.

Definition 2.2.1 *(**k-algebra** A) Let k be a field. A k-algebra A is a vector space over k endoved with a further binary operation (called multiplication) so that it is also a ring with identity and which satisfies*

$$k(ab) = (ka)b = a(kb)$$

The most important example of a k-algebra which we will consider in the definition of free kA-module in this thesis is the group algebra kA whose definition is as follows.

Definition 2.2.2 *(**Group algebra** kA) Let A be a finite group (written multiplicatively) and k a field. The underlying set of the group algebra kA consists of all formal linear combinations $\sum_{a\in A} k_a a$ with scalars $k_a \in k$. The set kA is then viewed as a vector space over k with $\{a : a \in A\}$ as a basis, and multiplication is defined on kA by extending the multiplication on A bilinearly. Thus:*

$$(\textstyle\sum_{a\in A} k_a a)(\sum_{b\in A} \ell_b b) = \sum_{a\in A}(\sum_{x\in A} k_x \ell_{x^{-1}a})a$$

The associativity of this binary operation follows from the associativity of multiplication in A, and the rest of the algebra axioms are obvious from the definition. Clearly, $dim_k(kA) = |A|$.

Definition 2.2.3 *(**Regular orbit**) Let a finite group G act on a set X. For $x \in X$, the orbit of x in G is called regular if $Stab_G(x) = 1$, that is the length of the orbit of x is $|G|$.*

Definition 2.2.4 ***(Regular representation)*** *A group G may be viewed as a group acting on itself by group multiplication, then we call the associated permutation representation $\alpha : G \rightarrow Sym(G)$, the regular representation of G. It is clear that it is transitive and hence $Stab_G(g) = 1$, for each $g \in G$, the identity subgroup.*

Definition 2.2.5 ***(Free kA-module, k:any field, A:any group)*** *For a group A and a field k, a free kA-module of dimension n is a direct sum of n copies of the group algebra kA, each of which can be regarded as a vector space over k of dimesion $|A|$, with a basis $\{v_g : g \in A\}$ labeled by elements of A on which A acts in a regular representation, that is $v_g h = v_{gh}$.*

Theorem 2.2.6 *Let M be a free kA-module. Then,*

(i) $M = \bigoplus_{a \in A} Na$ where N is a subspace of M and Na's are regularly permuted by A.

(ii) $C_M(A) = \{\sum_{a \in A} na \quad | \quad n \in N\}$

Proof. i) is due to the definition of a free kA-module. Since by definition, there is a basis B_M of M as a vector space over k consisting of regular orbits under the action of A. Choose one element in each of these orbits and let them span a subspace N. To see explicitly what N is, suppose

$$B_M = \{m_{11}, m_{21}, \ldots, m_{r1}\} \cup \{m_{12}, m_{22}, \ldots, m_{r2}\} \cup \ldots \cup \{m_{1s}, m_{2s}, \ldots, m_{rs}\}$$

where $r = |A|$ since the length of each orbit is equal to $|A|$.

Suppose also that $A = \{1, \ldots, a_r\}$. Since each of the sets above are regular A-orbits, then take the first element from each set and write B_M as follows:

$$B_M = \{m_{11}, m_{11}a_2, \ldots, m_{11}a_r\} \cup \{m_{12}, m_{12}a_2, \ldots, m_{12}a_r\} \cup \ldots \cup \{m_{1s}, m_{1s}a_2, \ldots, m_{1s}a_r\}$$

Now, $N = \langle m_{11}, m_{12}, \ldots, m_{1s} \rangle$, where the elements in this spanning set are also linearly independent, since they were taken from the basis elements of M. So, in fact, they form a basis for N and N becomes a subspace of M. We can write B_M then as follows:

$$B_M = \{m_{11}, m_{12}, \ldots, m_{1s}\} \cup \{m_{11}a_2, m_{12}a_2, \ldots, m_{1s}a_2\} \cup \ldots \cup \{m_{11}a_r, m_{12}a_r, \ldots, m_{1s}a_r\}$$

So, we have

$$M = N \oplus Na_2 \oplus \ldots \oplus Na_r$$

where $\langle m_{11}a_i, m_{12}a_i, \ldots, m_{1s}a_i \rangle$ is a basis for Na_i, for $i = 1, \ldots, r$. And, obviously the set of these subspaces $\{N, Na_2, \ldots, Na_r\}$ is regularly permuted by A.

And proof of ii) is as follows:

Set $S = \{\sum_{a\in A} na \quad | \quad n \in N\}$. We'll show firstly that $S \leq C_M(A)$. So, take $\sum_{a\in A} na \in S$. Then, for any $a' \in A$,

$$(\sum_{a\in A} na)a' = \sum_{a\in A}(na)a' = \sum_{a\in A} n(aa') = \sum_{a\in A} na$$

implying that $\sum_{a\in A} na \in C_M(A)$ and hence $S \leq C_M(A)$.

To show the reversed inequality, $C_M(A) \leq S$, we'll use (i), that is $M = \bigoplus_{a\in A} Na$ for a subspace N. Now, let $m = \sum_{i=1}^{\ell} n_i a_i \in C_M(A)$, where n_i's are from N and $a_i \in A$ for all $i = 1, ..., \ell$ with $a_1 = 1$. Then, for $a_2^{-1} \in A$ we have $ma_2^{-1} = m$, more explicitly,

$$\begin{aligned}(n_1 + n_2a_2 + ... + n_\ell a_\ell)a_2^{-1} &= n_1a_2^{-1} + n_2a_2a_2^{-1} + ... + n_\ell a_\ell a_2^{-1}\\ &= n_1a_2^{-1} + n_2 + ... + n_\ell a_\ell a_2^{-1}\\ &= n_1 + n_2a_2 + ... + n_\ell a_\ell\end{aligned}$$

Since the sum above is direct, we get $n_1 = n_2$.

Similarly for $a_3^{-1} \in A$, $ma_3^{-1} = m$, more explicitly,

$$\begin{aligned}(n_1 + n_2a_2 + ... + n_\ell a_\ell)a_3^{-1} &= n_1a_3^{-1} + n_2a_2a_3^{-1} + n_3a_3a_3^{-1} ... + n_\ell a_\ell a_3^{-1}\\ &= n_1a_3^{-1} + n_2a_2a_3^{-1} + n_3 + ... + n_\ell a_\ell a_3^{-1}\\ &= n_1 + n_2a_2 + ... + n_\ell a_\ell\end{aligned}$$

and we get just as above this time $n_1 = n_3$.

Continuing this way with the inverses of $a_4, a_5, ...a_\ell$, we'll get $n_1 = n_2 = ... = n_\ell = n \in N$ and hence $m = n + na_2 + ... + na_\ell = \sum_{a\in A} na$ for some $n \in N$, which shows that $m \in S$, as desired.

Definition 2.2.7 ***(irreducible)*** *A nonzero module over a ring with identity whose only submodules are the module itself and the zero module is called irreducible.*

Definition 2.2.8 ***(completely reducible)*** *Let V be a kG-module. V is called a completely reducible kG-module if it is a direct sum of its irreducible submodules.*

Theorem 2.2.9 ***(Clifford's theorem)*** *Let V be an irreducible kG-module where k is a field and G is a group and let $Q \lhd G$. Then $V|_Q$ is completely reducible. More precisely, V is the direct sum of Q-invariant subspaces V_i, $1 \leq i \leq r$, which satisfy the following conditions:*

(i) $V_i = X_{i1} \oplus X_{i2} \oplus \ldots \oplus X_{it}$, where each X_{ij} is an irreducible Q-submodule, $1 \leq i \leq r$, t is independent of i, and $X_{ij}, X_{i'j'}$ are isomorphic Q-submodules if and only if $i = i'$.

(ii) For any Q-submodule U of V, we have $U = U_1 \oplus U_2 \oplus \ldots \oplus U_r$ where $U_i = U \cap V_i$, $1 \leq i \leq r$. In particular, any irreducible Q-submodule of V lies in one of the V_i.

(iii) For x in G, the mapping $\pi(x) : V_i \to V_i x, 1 \le i \le r$, is a permutation of the set $S = \{V_1, V_2, ..., V_r\}$ and π induces a transitive permutation representation of G on S. Furthermore, $QC_G(Q)$ is contained in the kernel π.

The subspaces V_i $1 \le i \le r$, are often referred to as the Wedderburn components of V with respect to Q.

Proof. [8] 3.4.1

Definition 2.2.10 ***(homogeneous)*** *Let R be a ring and M be an R-module. If M is a direct sum of isomorphic irreducible R-modules, then M is homogeneous.*

Remark 2.2.11 *The Wedderburn components obtained as a result of the Clifford's Theorem are homogeneous.*

Theorem 2.2.12 ***(Maschke's theorem)*** *Let V be a kG-module and assume that either k is of characteristic 0 or of characteristic relatively prime to $|G|$. Then V is completely reducible.*

Proof. [8] 3.3.1

Theorem 2.2.13 *Let G be an abelian group of order n and F a field which contains a primitive n^{th} root of unity. Then every irreducible representation of G over F is linear, that is G acts by scalars on V, the representation space for G. More precisely, for any $g \in G$, there exists λ_g, a scalar depending on g such that*

$$wg = \lambda_g w \quad \textit{for any} \quad w \in V.$$

Proof. [8] 3.2.4

Proposition 2.2.14 *Let G be a group, k be a field and V be a kG-module. Let $\overline{k}$ be an extension of k and $\overline{V} = V \otimes_k \overline{k}$. If $H \le G$, then $C_{\overline{V}}(H) = C_V(H) \otimes_k \overline{k}$. In particular, $C_{\overline{V}}(x) = C_V(x) \otimes_k \overline{k}$ for any $x \in G$.*

Proof. $\overline{V} = V \otimes_k k$ is a left $\overline{k}G$-module, where $(ag)(v \otimes x) = gv \otimes ax$, for any $a, x \in \overline{k}, g \in G$, $v \in V$. Let $\{v_1, \ldots, v_n\}$ be a basis for V over k and let B be a basis for $\overline{k}$ over k. Then we have

$$\overline{V} = V \otimes_k (\textstyle\sum_{b \in B} bk) = \sum_{b \in B}(V \otimes_k bk) = \sum_{b \in B}(V \otimes_k b)$$

Let now $\overline{v} \in C_{\overline{V}}(H)$. Then there exist pairwise different elements $b_1, \ldots, b_s$ in B such that $\overline{v} = \sum_{i=1}^{s} x_i \otimes b_i$ with suitable $x_i \in V$, $i = 1, \ldots, s$ and we have $h\overline{v} = \sum_{i=1}^{s} hx_i \otimes b_i = \sum_{i=1}^{s} x_i \otimes b_i$ for any $h \in H$.

Because of the direct sum decomposition $\overline{V} = \sum_{b \in B}(V \otimes_k b)$, the above equation yields that $hx_i \otimes b_i = x_i \otimes b_i$. Since we are considering the tensor product of two vector spaces, the equation $hx_i \otimes b_i - x_i \otimes b_i = (hx_i - x_i) \otimes b_i = 0$ gives that $hx_i - x_i = 0$ for any $i = 1, \ldots, s$; that is $x_i \in C_V(h)$ for any $h \in H$. Thus we get $\overline{v} \in C_V(H) \otimes_k \overline{k}$. Obviously we also have $C_V(H) \otimes_k \overline{k} \leq C_{\overline{V}}(H)$. Hence $C_{\overline{V}}(H) = C_V(H) \otimes_k \overline{k}$.

Theorem 2.2.15 ***(Deuring-Noether theorem**, Theorem B.(5.24)) Let $\overline{k}$ be an extension field of a field k, and let V and W be kG-modules such that $W \otimes_k \overline{k}$ is isomorphic with a direct summand of $V \otimes_k \overline{k}$. Then W is isomorphic with a direct summand of V. In particular, if $W \otimes_k \overline{k} \cong V \otimes_k \overline{k}$, then $W \cong V$.*

Proof. [4] VII, 1.21.

CHAPTER 3

MAIN RESULT

In this chapter, we shall state and prove our original results Theorem 3.3.1 and Lemma 3.2.1. It should be noted that the bulk of the proof of Theorem 3.3.1 centers around Lemma 3.2.1 which is of independent interest, too. Therefore we present Lemma 3.2.1 as the key result in proving Theorem 3.3.1. However, before giving the proof of these results, we will prove a very crucial lemma which is known in folklore.

3.1 A technical lemma

In this section, we consider a finite Frobenius group FH of linear transformations with kernel F and complement H, and prove a lemma which guarantees the presence of free H-modules under an additional condition by the use of Clifford's Theorem 2.2.9.

Lemma 3.1.1 *Suppose that V is a vector space over a field k of characteristic p admitting a finite Frobenius group FH of linear transformations with kernel F and complement H. If $C_V(F) = 0$, then $V|_H$ is a free kH-module.*

Proof. We give a proof over a series of steps:

(1) We may assume that $(|F|, p) = 1$.

Since FH is a Frobenius group, we already know that F is nilpotent due to Theorem 2.1.7. So, $F = F_p \times F_{p'}$ where F_p is a Sylow p-subgrpup of F.

V, being FH-invariant, is F-invariant, too. Therefore we can observe that $C_V(F_{p'})$ is F_p-invariant. To see this, take $v \in C_V(F_{p'})$, $a \in F_p$. Then, we need to check if $va \in C_V(F_{p'})$ or not. Pick $a' \in F_{p'}$. We have,

$$(va)a' = v(aa') \overset{(*)}{=} v(a'a) = (va')a = va$$

showing that $va \in C_V(F_{p'})$ as desired. It should be noted that the equality (*) holds since F_p and F'_p commute.

We show next that $C_V(F_{p'}) = 0$: If $C_V(F_{p'}) \neq 0$, then it is a nontrivial p-group on which the p-group F_p acts. Then F_p would have nontrivial fixed points on $C_V(F_{p'})$, that is $C_{C_V(F_{p'})}(F_p) \neq 0$ due to Theorem 2.1.14, and this would imply that

$$C_{C_V(F_{p'})}(F_p) = C_V(F) \neq 0$$

contradicting the hypothesis. (Note that, here, V is considered as a p-group since $char k = p$ implies $|V| = p^{dim V}$.) Hence $C_V(F_{p'}) = 0$.

It follows now from $C_V(F_{p'}) = 0$ that the pair $(V, F_{p'}H)$ satisfies the hypothesis of the lemma as $F_{p'}H$ is a Frobenius group with Frobenius kernel $F_{p'}$ and complement H. Hence we may assume that F is a p'-group, that is $(|F|, p) = 1$.

(2) We may assume that k is algebraically closed:

We extend the ground field k to its algebraic closure $\overline{k}$, so that our vector space V can also be extended to $\overline{V} = V \otimes_k \overline{k}$ which is regarded as a $\overline{k}FH$-module. Note that, we still have

$$C_{\overline{V}}(F) = C_V(F) \otimes_k \overline{k} = 0 \otimes_k \overline{k} = 0$$

by Proposition 2.2.14.

Therefore the action of FH on $\overline{V}$ satisfies the hypothesis of the lemma. If one proves that $\overline{V}$ satisfies also the conclusion of the lemma, that is $\overline{V}$ is a free $\overline{k}H$-module, then Deuring-Noether theorem 2.2.15 implies that V is also a free kH-module and the proof will be done. Therefore we may assume that $k = \overline{k}$ and so $V = \overline{V}$.

(3) Any factor U in an unrefinable series of the kFH-module V is a direct sum of homogeneous kF-modules $W_1, \ldots, W_t$ permuted transitively by H and each of which is a nontrivial kF-module.

Consider an unrefinable series of kFH-submodules, that is,

$$V = U_1 > U_2 > \ldots > U_\ell > U_{\ell+1} = 0$$

in which each factor U_i/U_{i+1} is an irreducible kFH-module.

Since $(|F|, p) = 1$, it is obvious that $(|F|, |V|) = 1$ which gives

$$C_{U_i/U_{i+1}}(F) = C_{U_i}(F)U_{i+1}/U_{i+1}, \text{ for every } i = 1, \ldots, t.$$

due to Theorem 2.1.32. But,

$$C_{U_i}(F) \subseteq C_V(F) = 0$$

which gives $C_{U_i/U_{i+1}}(F) = 0$.

Now, take any factor $U = U_i/U_{i+1}$, from the above series. We have:

(a) U is an irreducible kFH-module.
(b) $F \lhd FH$.
(c) $C_U(F) = 0$.

We apply Clifford's Theorem 2.2.9 to U with respect to F. That is U is a completely reducible kF-module, more precisely, U is the direct sum of its Wedderburn components W_i, that is,

$$U|_F = W_1 \oplus \ldots \oplus W_t,$$

where W_i's for $i = 1, ..., t$ are homogeneous kF-modules transitively permuted by FH. Let W_i, W_j be any two Wedderburn components. Then there exists $fh \in FH$ with $f \in F$ and $h \in H$ such that $W_i^{fh} = W_j$. In fact, since W_i is F-invariant,

$$W_i^{fh} = (W_i^f)^h = W_i^h = W_j,$$

This shows that W_i's are transitively permuted by H.

Also, none of the W_i's is a trivial kF-module, as implied by

$$C_{W_i}(F) \leq C_U(F) = 0$$

for each $i = 1, \ldots, t$.

(4) The case $H_1 \neq 1$ can not happen.

Let $H_1 \neq 1$. We observe $[H_1, Z(F/C_F(W_1))] = 1$:

Since W_1 is a homogeneous F-module, it is also a homogeneous $F/C_F(W_1)$-module. Note that $F/C_F(W_1)$ is nontrivial as $C_V(F) = 0$. We already know that, being a Frobenius kernel, F is nilpotent by Theorem 2.1.7 and hence $F/C_F(W_1)$ is nilpotent. So, by Theorem 2.1.14,

$$Z(F/C_F(W_1)) \neq 1$$

and is represented on W_1 by scalars due to Theorem 2.2.13. That is, for $z \in Z(F/C_F(W_1))$, there is $\lambda_z \in k$ such that

$$wz = \lambda_z w$$

for all $w \in W_1$.

Also note that W_1 is H_1-invariant since $H_1 = Stab_H(W_1)$. Now take any $[z, h_1] \in [Z(F/C_F(W_1)), H_1]$ and $w \in W_1$. Then

$$\begin{aligned} w[z,h_1] &= w(z^{-1}h_1^{-1}zh_1) \\ &= (\lambda_z^{-1}w)h_1^{-1}zh_1 \\ &= (\lambda_z^{-1}wh_1^{-1})zh_1 \\ &= \lambda_z(\lambda_z^{-1}wh_1^{-1})h_1 \\ &= \lambda_z\lambda_z^{-1}((wh_1^{-1})h_1) \\ &= w(h_1^{-1}h_1) \\ &= w \end{aligned}$$

implying that $[z, h_1] \in C_F(W_1)$ and hence $[H_1, Z(F/C_F(W_1))]$ is trivial, that is

$$C_{F/C_F(W_1)}(H_1) \neq 1.$$

But,

$$1 \neq C_{F/C_F(W_1)}(H_1) = C_F(H_1)C_F(W_1)/C_F(W_1),$$

by Theorem 2.1.4 and Theorem 2.1.32 as $(|F|, |H|) = 1$.

Now, $1 \neq C_F(H_1)C_F(W_1)/C_F(W_1)$ implies that

$$C_F(H_1) \neq 1.$$

This contradicts with the fact that FH_1 is a Frobenius group and we have

$$C_F(H_1) \leq C_F(h) = 1$$

for all $1 \neq h \in H_1$.

(5) Consider the case $H_1 = Stab_H(W_1) = 1$ then the lemma follows.

Set $H_1 = Stab_H(W_1)$ in the action of H on $\Omega = \{W_1, \ldots, W_t\}$, the set of Wedderburn components of U. Assume that $H_1 = 1$ and $B_1 = \{w_1, \ldots, w_s\}$ be any k-basis of W_1. Now, we claim that there is a basis of U which is the union of several regular H-orbits and hence U is a free kH-module: Obviously, as $H_1 = 1$ and H acts transitively on Ω,

$$|\Omega| = \left|W_1^H\right| = |H : H_1| = |H|.$$

Let $H = \{1 = h_1, h_2, \ldots, h_t\}$. Now,

$$W_1^H = \Omega = \{W_1, W_1^{h_2} \dots, W_1^{h_t}\}$$

and so

$$U|_F = W_1 \oplus W_1^{h_2} \dots \oplus W_1^{h_t}.$$

We consider now the sets

$$\begin{aligned} B_1 &= \{w_1, \dots, w_s\} \\ B_2 &= \{w_1^{h_2}, \dots, w_s^{h_2}\} \\ &\vdots \\ B_t &= \{w_1^{h_t}, \dots, w_s^{h_t}\} \end{aligned}$$

Here, each B_i is a basis for the subspace $W_1^{h_i}$ for $i = 1, \dots, t$.

Let $B = \bigcup_{i=1}^{t} B_i$. Clearly, B is a basis for U due to the direct sum decomposition of $U|_F$ and it can also be written as

$$B = \{w_1, w_1^{h_2}, \dots, w_1^{h_t}\} \cup \{w_2, w_2^{h_2}, \dots, w_2^{h_t}\} \cup \dots \cup \{w_s, w_s^{h_2}, \dots, w_s^{h_t}\}$$

where each set now above is a regular H-orbit. Hence, U becomes a free kH-module by Definition 2.2.5 as desired.

Since an extension of a free kH-module by a free kH-module is again a free kH-module, and since each factor in the series considered in part (3) is a free kH-module. We conclude that V is also a free kH-module, which completes the proof. □

3.2 The key lemma pertaining the main result

In this section we establish Lemma 3.2.1 which is crucial in proving Theorem 3.3.1 and of independent interest, too.

Lemma 3.2.1 *Let Q be a normal q-subgroup of a group having a complement FH which is a Frobenius group with kernel F and complement H so that $C_{C_Q(F)}(h) = 1$ for every $1 \neq h \in H$. Assume further that $|FH|$ is not divisible by q and Q is of class at most 2. Let V be a $kQFH$-module on which $[Q, F]$ acts nontrivially where k is a field with characteristic not dividing $|QFH|$. Then we have*

$$Ker(C_{[Q,F]}(H) \ \ on \ \ C_V(H)) = Ker(C_{[Q,F]}(H) \ \ on \ \ V).$$

Proof. We set $K = Ker(C_{[Q,F]}(H) \text{ on } C_V(H))$ and whenever it is appropriate we'll write K instead of $Ker(C_{[Q,F]}(H) \text{ on } C_V(H))$ to simplify the notation.

Note that we already have

$$K \geq Ker(C_{[Q,F]}(H) \text{ on } V)$$

Therefore we need only to prove

$$K \leq Ker(C_{[Q,F]}(H) \text{ on } V).$$

We will proceed by induction on $dim(V) + |QFH|$ over a sequence of steps, that is we suppose that the proposition is false and consider a minimal (of least order) counterexample (V, QFH) for which

$$K \neq Ker(C_{[Q,F]}(H) \text{ on } V), \text{ that is } K \nleq Ker(C_{[Q,F]}(H) \text{ on } V).$$

We should note firstly that $C_V(H)$ is $C_Q(H)$-invariant which can be seen as follows: For this, it is enough to see that

$$[C_V(H), C_Q(H)] \leq C_V(H)$$

Firstly, we have $[C_V(H), C_Q(H)] \leq [V, Q] \leq V$. So, we only need to show that

$$[C_V(H), C_Q(H), H] = 1$$

But, this can be seen easily by the Three Subgroup Lemma 2.1.27, since we have both

$$[C_Q(H), H, C_V(H)] = [[C_Q(H), H], C_V(H)] = [1, C_V(H)] = 1$$

and

$$[H, C_V(H), C_Q(H)] = [[H, C_V(H)], C_Q(H)] = [1, C_Q(H)] = 1.$$

Claim 1. We have $Q = [Q, F]$ and hence $C_Q(F) \leq Q' \leq Z(Q)$.

Proof. We may assume that $[Q, F]$ acts nontrivially on V, because otherwise

$$C_{[Q,F]}(H) = Ker(C_{[Q,F]}(H) \text{ on } C_V(H)) = Ker(C_{[Q,F]}(H) \text{ on } V),$$

as claimed. We will refer this assumption again in Claim 3, for U_i and in Claim 8, for V_1. If $[Q, F]$ is a proper subgroup of Q, then we consider the action of the group $[Q, F]FH$ on V. Let us check if this satisfies the hypothesis of the lemma. Obviously, since $C_{C_{[Q,F]}(F)}(h) \leq C_{C_Q(F)}(h)$ we have $C_{C_{[Q,F]}(F)}(h) = 1$ for each $1 \neq h \in H$. Also $[Q, F]$ being a subgroup of Q is still a q-group and satisfies the divisibility conditions like Q. Furthermore,

$$[Q,F]' \leq Q' \cap [Q,F] \leq Z(Q) \cap [Q,F] \leq Z([Q,F]).$$

Then it follows by the induction assumption that

$$Ker(C_{[Q,F]}(H) \text{ on } C_V(H)) = Ker(C_{[Q,F]}(H) \text{ on } V)$$

which is not the case. Therefore, $Q = [Q,F]$.

We are left to show that $C_Q(F) \leq Q'$ only, in order to prove Claim 1:

Now, Q/Q' is an abelian F-invariant group such that $(|Q/Q'|, |F|) = 1$. By Theorem 2.1.28,

$$Q/Q' = [Q/Q', F] \times C_{Q/Q'}(F).$$

On the other hand $[Q,F] = Q$ implies

$$[Q/Q', F] = [Q,F]Q'/Q' = Q/Q',$$

and so $C_{Q/Q'}(F) = 1$. It follows now by Theorem 2.1.32

$$C_{Q/Q'}(F) = C_Q(F)Q'/Q' = 1$$

implying that $C_Q(F) \leq Q'$. We have already $Q' \leq Z(Q)$ since Q is of class at most 2.

Claim 2. We may assume that k *is algebraically closed.*

Proof. We extend the ground field k to its algebraic closure $\overline{k}$, so that V can also be extended to $\overline{V} = V \otimes_k \overline{k}$ which is regarded as a $\overline{k}QFH$-module. If we show that

$$Ker(C_Q(H) \text{ on } \overline{V}) = Ker(C_Q(H) \text{ on } C_{\overline{V}}(H)),$$

then we have

$$Ker(C_Q(H) \text{ on } V) = Ker(C_Q(H) \text{ on } C_V(H))$$

because $C_{\overline{V}}(H) = C_V(H) \otimes_k \overline{k}$. Therefore we may assume that $\overline{k} = k$.

Claim 3. V *is an irreducible* QFH*-module and* Q *acts faithfully on* V*, that is* $Ker(Q$ *on* $V) = 1$.

Proof. Firstly, we'll prove that Q acts faithfully on V. Suppose not. Set $\overline{Q} = Q/Ker(Q \text{ on } V)$ and consider the action of the group $\overline{Q}FH$ on V. $[Q,F] = Q$ gives

$$[\overline{Q},F]=[Q/Ker(Q \text{ on } V),F] \;=\; [Q,F]Ker(Q \text{ on } V)/Ker(Q \text{ on } V)$$
$$= QKer(Q \text{ on } V)/Ker(Q \text{ on } V)=\overline{Q}.$$

that is $\overline{Q}=[\overline{Q},F]$. Moreover, for any $N \trianglelefteq Q$ we have

$$(Q/N)' = Q'N/N \leq Z(Q)N/N \leq Z(Q/N)$$

and hence $(\overline{Q})' \leq Z(\overline{Q})$.

If $\left|\overline{Q}\right| < |Q|$, by induction applied to the action of $\overline{Q}FH$ on V, we get

$$Ker(C_{\overline{Q}}(H) \text{ on } C_V(H)) = Ker(C_{\overline{Q}}(H) \text{ on } V).$$

As $C_{\overline{Q}}(H)=\overline{C_Q(H)}$ due to Theorem 2.1.32, we have

$$Ker(\overline{C_Q(H)} \text{ on } C_V(H)) = Ker(\overline{C_Q(H)} \text{ on } V). \quad (*)$$

This gives

$$Ker(C_Q(H) \text{ on } C_V(H)) = Ker(C_Q(H) \text{ on } V) \quad (**)$$

which is a contradiction.

How equation $(*)$ gives equation $(**)$ is as follows:

We already have, as stated before,

$$\text{K}=Ker(C_Q(H) \text{ on } C_V(H)) \geq Ker(C_Q(H) \text{ on } V).$$

But if

$$K \nleq Ker(C_Q(H) \text{ on } V).$$

then there exists $x \in K - Ker(Q \text{ on } V)$ and hence

$$\overline{1} \neq \bar{x} = xKer(Q \text{ on } V) \in Ker(\overline{C_Q(H)} \text{ on } C_V(H)) = Ker(\overline{C_Q(H)} \text{ on } V) = \overline{1},$$

a contradiction.

So, $\overline{Q}=Q$ implying that $Ker(Q \text{ on } V)=1$, that is Q acts faithfully on V. We show next that V is irreducible as a QFH-module. Recall that, $Q=[Q,F]$ by Claim 1. By Maschke's Theorem

2.2.12, V is completely reducible as a QFH-module, that is V is a direct sum of irreducible QFH-submodules. Suppose $V = U_1 \oplus \ldots \oplus U_m$ for irreducible QFH-submodules U_i, $i = 1, \ldots, m$. Note that, Q acts nontrivially on U_i, $i = 1, \ldots, m$, as shown in Claim 1. By induction applied to the action of QFH on U_i we get

$$Ker(C_Q(H) \text{ on } C_{U_i}(H)) = Ker(C_Q(H) \text{ on } U_i) \text{ for each } i = 1, \ldots, m.$$

On the other hand, $C_V(H) = \bigoplus_{i=1}^{m} C_{U_i}(H)$ because

$$(w_1, ..., w_t) \in C_V(H) \Leftrightarrow (w_1, ..., w_t)^h = (w_1^h, ..., w_t^h) = (w_1, ..., w_t) \Leftrightarrow w_i^h = w_i \text{ for all } i \Leftrightarrow w_i \in C_{U_i}(H)$$

for all i. So,

$$\begin{aligned} K &= Ker(C_Q(H) \text{ on } C_V(H)) = \bigcap_{i=1}^{m} Ker(C_Q(H) \text{ on } C_{U_i}(H)) = \bigcap_{i=1}^{m} Ker(C_Q(H) \text{ on } U_i) \\ &= Ker(C_Q(H) \text{ on } V) \end{aligned}$$

which is nothing but the claim of the lemma.

Claim 4. FH *acts faithfully on* V, *that is* $C_{FH}(V) = 1$.

Proof. Note that

$$C_{FH}(V) = C_F(V)C_H(V)$$

by Theorem 2.1.45. Suppose first that $C_H(V) \neq 1$. Then the group $FC_H(V)$ is Frobenius with kernel F and complement $C_H(V)$, and hence $[F, C_H(V)] = F$. In order to obtain a contradiction, we'll employ two times the Three Subgroup Lemma 2.1.27 as in the following:

We observe first that $[F, V] \neq 1$: If $[F, V] = 1$, we get

$$[F, V, Q] = [[F, V], Q] = [1, Q] = 1$$

Since V is Q-invariant we also get

$$[V, Q, F] = [[V, Q], F] \leq [V, F] = 1$$

So, by the Three Subgroup Lemma we obtain

$$[Q, F, V] = [[Q, F], V] = [Q, V] = 1,$$

which is not the case. Hence we have $[F, V] \neq 1$.

Since $F = [F, C_H(V)]$, we write $[F, V] = [[F, C_H(V)], V]$.

Now,

$$[C_H(V), V, F] = [[C_H(V), V], F] = [1, F] = 1.$$

Consider next $[V, F, C_H(V)]$. Since V is F-invariant, we have $[V, F] \leq V$ and hence we get

$$[V, F, C_H(V)] = [[V, F], C_H(V)] \leq [V, C_H(V)] = 1.$$

Now, it follows by the Three Subgroup Lemma that $[F, C_H(V), V] = 1$ and hence

$$[F, V] = [[F, C_H(V)], V] = [F, C_H(V), V] = 1$$

which is not the case as shown above. Thus we have $C_H(V) = 1$.

We'll prove next that $C_F(V) = 1$: Assume not. We can see that Q centralizes $C_F(V)$ as follows:

We have

$$[V, C_F(V), Q] = [[V, C_F(V)], Q] = [1, Q] = 1$$

Next, consider $[Q, V, C_F(V)]$. Since V is Q-invariant, we have

$$[Q, V, C_F(V)] = [[Q, V], C_F(V)] \leq [V, C_F(V)] = 1.$$

resulting in

$$[C_F(V), Q, V] = [[C_F(V), Q], V] = 1$$

by the Three Subgroup Lemma 2.1.27. However, we know that Q is F-invariant. Therefore, $[C_F(V), Q]$ is a subgroup of Q which centralizes V. On the other hand, we know that Q is faithful on V by Claim 3, that is $C_Q(V) = 1$ implying that $[C_F(V), Q] = 1$.

Since Q centralizes $C_F(V)$, we can consider the action of the group $(QFH)/C_F(V)$ on V. To simplify the notation let $\overline{F} = F/C_F(V)$. Note that

$$[Q, \overline{F}] = [Q, F] = Q$$

is nontrivial on V and $C_{C_Q(\overline{F})}(h) = 1$ for each $1 \neq h \in H$ as $C_Q(\overline{F}) = C_Q(F)$. We may apply induction to the action of $Q\overline{F}H$ on V and get

$$Ker(C_Q(H) \text{ on } C_V(H)) = Ker(C_Q(H) \text{ on } V).$$

which is a contradiction. Hence $C_F(V) = 1$.

Claim 5. Let $\Omega = \{W_1, ..., W_\ell\}$ *be the set of* Q*-homogeneous components of* V. *Then* H *acts transitively on the set of all* F*-orbits in* Ω.

Proof. We use the notation W^F for the F-orbit containig the Q-homogeneous component W. By Clifford's Theorem 2.2.9,

$$V|_Q = W_1 \oplus ... \oplus W_\ell$$

where W_i's are Q-homogeneous components of V, for each $i = 1, ..., \ell$ and FH acts transitively on the set Ω of these components. Let W_i^F and W_j^F be two F-orbits. By the transitive action of QFH on Ω, there exists $qfh \in QFH$ where $q \in Q$, $f \in F$, $h \in H$ such that $(W_i)^{qfh} = W_j$. Now $W_j = (W_i^q)^{fh} = (W_i)^{fh}$ as W_i is Q-invariant. We'll observe that $(W_j^F)^{h^{-1}} = W_i^F$: To see this let $W_j^x \in W_j^F$ where $x \in F$. Then

$$W_j^{xh^{-1}} = (W_i^{fh})^{xh^{-1}} = W_i^{fhxh^{-1}} = W_i^{fx^{h^{-1}}} \in W_i^F \text{ as } fx^{h^{-1}} \in F.$$

And to show the converse part, let $W_i^x \in W_i^F$ where $x \in F$. Then since $W_i = W_j^{h^{-1}f^{-1}}$ due to the assumption above, we have

$$W_i^x = (W_j^{h^{-1}f^{-1}})^x = W_j^{h^{-1}f^{-1}x} = W_j^{h^{-1}f_k hh^{-1}} = (W_j^{f_k^h})^{h^{-1}} \in (W_j^F)^{h^{-1}}$$

where $f_k = f^{-1}x$ and $f_k^h \in F$.

This proves that H acts transitively on the set of F-orbits in Ω.

Claim 6. Let Ω_1 *be an* F*-orbit on* Ω *and* $H_1 = Stab_H(\Omega_1)$. *Then* H_1 *is a nontrivial subgroup of* H.

Proof. We can easily verify that for every $W_i \in \Omega$, $Stab_H(W_i) \leq Stab_H(W_i^F)$: To see this it is enough to show that if $y \in Stab_H(W_i)$, that is if $y \in H$ is such that $W_i^y = W_i$, then we have $(W_i^F)^y = W_i^F$.

Let $W_i^f \in W_i^F$. Now, $W_i^{fy} \in (W_i^F)^y$ and

$$W_i^{fy} = W_i^{yy^{-1}fy} = W_i^{yf^y} = W_i^{f^y} \in W_i^F$$

This shows that $(W_i^F)^y \leq W_i^F$.

Conversely, let $W_i^f \in W_i^F$. Then

$$W_i^f = W_i^{yf} = W_i^{yfy^{-1}y} = (W_i^{f^{y^{-1}}})^y \in (W_i^F)^y$$

giving us $W_i^F \leq (W_i^F)^y$. Thus we have $(W_i^F)^y = W_i^F$ for any $y \in Stab_H(W_i)$. In fact, we have

$$Stab_H(W_i) \leq Stab_H(W_i^F),$$

as claimed.

Let $\Omega_1 = W_1^F$ and assume that $H_1 = Stab_H(W_1^F) = 1$. Since H transitiviely permutes the F-orbits in Ω by Claim 5 for any $i \in \{1, ..., \ell\}$, there exists $h \in H$ suct that $W_i^F = (W_1^F)^h$ and hence

$$Stab_H(W_i^F) = Stab_H(W_1^F)^h = (\ Stab_H(W_1^F))^h = H_1^h = 1 \quad \text{since} \ H_1 = 1$$

It follows that $Stab_H(W_i) = 1$, for each $i = 1, ..., \ell$, since $Stab_H(W_i) \leq Stab_H(W_i^F)$ as shown above.

Now,

$$\left|W_i^H\right| = |H : Stab_H(W_i) = |H|$$

for each $i = 1, ..., \ell$. Next, set $X = \sum_{h \in H} W_i^h$, the sum of the elements in the H-orbit containing W_i, that is in W_i^H. Actually this sum is direct, that is $X = \bigoplus_{h \in H} W_i^h$

Now, we'll show that $P = \left\{\sum_{h \in H} v^h \mid v \in W_i\right\} \leq C_X(H)$: Take $\sum_{h \in H} v^h \in P$, and $\bar{h} \in H$. Then,

$$(\textstyle\sum_{h \in H} v^h)^{\bar{h}} = \sum_{h \in H} v^{h\bar{h}} = \sum_{h \in H} v^h.$$

Since $P \leq X$ we have $P \leq C_X(H)$, as desired.

On the other hand, as $X \leq V$, we have $C_X(H) \leq C_V(H)$ and since

$$K = Ker(C_Q(H) \ \text{ on } \ C_V(H))$$

we have $[K, C_X(H)] = 1$. And this result leads us to the fact that K is trivial on W_i, for each $i = 1, ..., \ell$: To show this explicitly, consider

$$P = \left\{\textstyle\sum_{h \in H} v^h \mid v \in W_i\right\}.$$

Clearly, $[K, P] \leq [K, C_X(H)] = 1$. Also K normalizes W_i since $K \leq Q$ and W_i is Q-invariant.

Let $v \in W_i$ and $H = \{1, h_2, \ldots, h_m\}$ such that $\sum_{h \in H} v^h \in P$. Then for each $k \in K$, since $[K, P] = 1$ we have

$$v + v^{h_2} + \ldots + v^{h_m} = \textstyle\sum_{h \in H} v^h = (\sum_{h \in H} v^h)^k = \sum_{h \in H} v^{hk} = v^k + v^{h_2 k} + \ldots + v^{h_m k}$$

Since $v^k \in W_i$, we must have $v^k = v$ due to the direct sum decomposition $X = \bigoplus_{h\in H} W_i^h$. It follows that K is trivial on W_i, for each $i = 1, ..., \ell$ and hence on $V = \bigoplus_{i=1}^{\ell} W_i$. This contradicts the fact that $C_Q(V) = 1$ as stated in Claim 3. Thus, H_1 is a nontrivial subgroup of H, as desired.

Claim 7. H_1 stabilizes exactly one element W of Ω_1 and all the remaining orbits of H_1 on Ω_1 are of length $|H_1|$.

Proof. Recall that $\Omega_1 = W_1^F$. To simplify the notation set $W = W_1$. Let also $S = Stab_{FH_1}(W)$ and $F_1 = F \cap S$. By considering the action of FH_1 on W, we have $\left|W^{FH_1}\right| = |FH_1 : S|$. On the other hand, as $H_1 = Stab_H(\Omega_1)$ we have

$$W^{FH_1} = (W^F)^{H_1} = \Omega_1^{H_1} = \Omega_1 = W^F,$$

and hence,

$$|\Omega_1| = \left|W^F\right| = |F : Stab_F(W) = |F : F_1|.$$

Thus $|FH_1 : S| = |F : F_1|$, that is $\frac{|F|}{|F_1|} = \frac{|F||H_1|}{|S|}$ giving $|H_1| = \frac{|S|}{|F_1|}$. Recall that $(|F|, |H|) = 1$ since FH is a Frobenius group, and hence we have also $(|F_1|, |H_1|) = (|F_1|, \frac{|S|}{|F_1|}) = 1$ since $F_1 \leq F$ and $H_1 \leq H$.

Let S_1 be a complement of F_1 in S, the existence of which is guaranteed by the Schur-Zassenhaus Theorem 2.1.19. Then we have $|S_1| = \frac{|S|}{|F_1|} = |H_1|$. We'll observe now that S contains a conjugate of H_1:

We have $S = F_1 S_1$ with $(|S_1|, |F_1|) = 1$ and also S_1 is a Hall subgroup of S due to Definition 2.1.20. On the other hand, H_1 is a Hall subgroup of FH_1 with $|S_1| = |H_1|$. Recall,

$$S = Stab_{FH_1}(W) \leq FH_1$$

which implies then that S_1 is like H_1, a Hall subgroup of FH_1.

It is a fact that in π-separable groups, Hall subgroups are conjugate which was seen as Theorem 2.1.23. Since FH_1 is a π-separable group due to Definition 2.1.21, $H_1^g = S_1$ for some $g \in FH_1$. That is, S contains a conjugate of H_1. W.l.o.g. we may assume that $H_1 \leq S$, since

$$H_1^g \leq S \Rightarrow H_1 \leq S^{g^{-1}} = (Stab_{FH_1}(W))^{g^{-1}} = Stab_{FH_1}(W^{g^{-1}}) = Stab_{FH_1}(W_j)$$

for some $j \in \{1, ..., \ell\}$. We have the last equality due to the transitive action of FH on Ω by Claim 5. In fact, W_j is still an element of $\Omega_1 = W^F$ since $H_1 = Stab_H(\Omega_1)$ and $g^{-1} \in FH_1$. So we can work with $W^{g^{-1}}$ and $S^{g^{-1}}$ instead of W and S, respectively. That is w.l.o.g. we may assume that $H_1 \leq S$, that is $H_1 \leq Stab_{FH_1}(W)$ which implies that W is H_1-invariant.

We observe next that W is the only element in Ω_1 which is stabilized by H_1 : To see this let $x \in F$ and $1 \neq h \in H_1$ such that $(W^x)^h = W^x$ holds. Then,

$$(W^x)^h = W^{xh} = W^x \Rightarrow W^{xhx^{-1}} = W \Rightarrow W^{h^{-1}xhx^{-1}} = W^{[h,x^{-1}]} = W \Rightarrow W^{[h,x^{-1}]} = W$$

since $H_1 \leq S$ and $h \in H_1$. Then,

$$[h, x^{-1}] \in F_1 \Rightarrow h^{-1}xhx^{-1} \in F_1 \Rightarrow h^{-1}xh \in F_1x \Rightarrow F_1x = F_1x^h = (F_1x)^h,$$

By theorem 2.1.31 there exists $g \in F_1x$ such that $g \in C_F(h)$. So $g \in F_1x \cap C_F(h)$. Now $F_1x = F_1g$. As $C_F(h) = 1$ for every $1 \neq h \in H$ we get $g = 1$ implying $F_1x = F_1$, that is $x \in F_1$. So, if $(W^x)^h = W^x$ holds for $1 \neq h \in H_1$, then $W^x = W$, that is W is the only element in Ω_1 which is stabilized by H_1.

It remains to show that all H_1-orbits other than $\{W\}$ on Ω_1 are of length $|H_1|$:

For this purpose, we'll show first that $Stab_{H_1}(W^f) = 1$ for each $1 \neq f \in F$.

If $1 \neq h_1 \in H_1$ and $f \in F - F_1$ such that $W^{fh_1} = W^f$, then $W^{fh_1f^{-1}} = W$ and hence $W^{h_1^{-1}fh_1f^{-1}} = W$, that is $[h_1, f^{-1}] \in F_1 = Stab_F(W)$. As in the above argument $(F_1f)^{h_1} = F_1f$ implies that there exists $g \in C_F(h_1)$ by Theorem 2.1.31 such that $F_1f = F_1g$. Since $C_F(h_1) = 1$ we get $F_1f = F_1$ which is not the case. Hence the length of any H_1-orbit containing W^f with $f \in F - F_1$ is $|H_1 : Stab_{H_1}(W^f) = |H_1|$, as desired.

Claim 8. F acts transitively on Ω and hence we have $H = H_1$.

Proof. We have already shown in Claim 5 that H acts transitively on $B = \{\Omega_i : i = 1, ..., s\}$, the collection of F-orbits on Ω. So, $s = |H : H_1|$, where $H_1 = Stab_H(\Omega_1)$. If $s = 1$, then there is only one F-orbit on Ω, which means that F acts transitively on Ω. So, suppose that $s > 1$. Then, $H_1 = Stab_H(\Omega_1)$ is a proper subgroup of H. Set now for each $i = 1, \ldots, s$

$$V_i = \bigoplus_{W \in \Omega_i} W$$

Obviously $dimV_1 + |QFH_1| < dimV + |QFH|$. And note that, FH_1 is also a Frobenius group with kernel F and complement H_1 satisfying all the conditions in the hypothesis, since $H_1 \leq H$. Moreover, V_1, being a direct sum of some of the homogeneous Q-components in Ω, is Q-invariant. V_1 is also FH_1-invariant since it is the sum of the components in the F-orbit Ω_1 and $H_1 = Stab_H(\Omega_1)$. Note also that, Q acts nontrivially on V_1 as shown in Claim 1. So, applying induction to the action of QFH_1 on V_1, we obtain

$$Ker(C_Q(H_1) \text{ on } C_{V_1}(H_1)) = Ker(C_Q(H_1) \text{ on } V_1)$$

And as $C_Q(H) \leq C_Q(H_1)$,

$$Ker(C_Q(H) \text{ on } C_{V_1}(H_1)) = Ker(C_Q(H) \text{ on } V_1).$$

Let now $x_1, \ldots, x_s$ be a complete set of coset representatives of H_1 in H. That is $H = \cup_{i=1}^{s} H_1x_i$. Then we have

$$C_V(H) = \{u^{x_1} + u^{x_2} + ... + u^{x_s} \quad | \quad u \in C_{V_1}(H_1)\}$$

To show this equality, let

$$A = \{u^{x_1} + u^{x_2} + ... + u^{x_s} \quad | \quad u \in C_{V_1}(H_1)\}$$

Take $\sum_{i=1}^{s} u^{x_i} \in A$. Then for any $h \in H$, obviously, $(\sum_{i=1}^{s} u^{x_i})^h = \sum_{i=1}^{s} u^{x_i h}$.

Now, $h = h_1 x_j$ for $h_1 \in H_1$ and $x_j \in \{x_1, ..., x_s\}$, since $H = \bigcup_{i=1}^{s} H_1 x_i$. So,

$$(\textstyle\sum_{i=1}^{s} u^{x_i})^h = \sum_{i=1}^{s} u^{x_i h} = \sum_{i=1}^{s} u^{x_i h_1 x_j} = \sum_{i=1}^{s} u^{\tilde{h}_i x_{k_i} x_j},$$

where $\tilde{h}_i \in H_1$ and $x_{k_i} \in \{x_1, ..., x_s\}$. The last equality holds because for each i, $x_i h_1 \in H$ and hence there are $\tilde{h}_i \in H_1$ and $x_{k_i} \in \{x_1, ..., x_s\}$ such that $x_i h_1 = \tilde{h}_i x_{k_i}$. Then, we have

$$(\textstyle\sum_{i=1}^{s} u^{x_i})^h = \sum_{i=1}^{s} u^{\tilde{h}_i x_{k_i} x_j} \overset{(*)}{=} \sum_{i=1}^{s} u^{x_{k_i} x_j} \overset{(**)}{=} \sum_{i=1}^{s} u^{x_i}$$

The equality (*) holds since $\tilde{h}_i \in H_1$ and $u \in C_{V_1}(H_1)$. The equality (**) holds because for each j, $\{x_1 x_j, ..., x_s x_j\}$ is also a complete set of coset representatives of H_1 in H. So, we obtained that

$$(\textstyle\sum_{i=1}^{s} u^{x_i})^h = \sum_{i=1}^{s} u^{x_i} \quad \text{for any} \quad h \in H,$$

that is $A \leq C_V(H)$.

Before showing $C_V(H) \leq A$, we observe first that

$$V = \bigoplus_{i=1}^{s} V_1^{x_i}.$$

Since H is transitive on the set B of all F-orbits by Claim 5, we have

$$B = \{\Omega_1^h \mid h \in H\}.$$

For each $h \in H$, $h = h_1 x_j$ for some $h_1 \in H$ and $j \in \{1, \ldots, s\}$. Thus

$$\Omega_1^h = \Omega_1^{h_1 x_j} = \Omega_1^{x_j}$$

as $H_1 = Stab_H(\Omega_1)$. It follows that $B = \{\Omega_1^{x_i} \mid i = 1, \ldots, s\}$. Obviously for $i \neq j$, $\Omega_1^{x_i} \neq \Omega_1^{x_j}$. We know that $\Omega = \bigcup_{j=1}^{s} \Omega_1^{x_j}$. Then, $V = \oplus_{W \in \Omega} W$ gives

$$V = \bigoplus_{W \in \bigcup_{j=1}^{s} \Omega_1^{x_j}} W = \bigoplus_{i=1}^{s} (\bigoplus_{W \in \Omega_1} W)^{x_i}$$

Thus we have $V = \bigoplus_{i=1}^{s} V_1^{x_i}$ as $V_1 = \bigoplus_{W \in \Omega_1} W$.

To show $C_V(H) \leq A$, let $w \in C_V(H)$. Since $V = \bigoplus_{i=1}^{s} V_1^{x_i}$ and $w \in V$, we can write

$$w = u_1 + u_2^{x_2} + ... + u_s^{x_s} \quad \text{with} \quad u_i \in V_1, \quad \text{for each} \quad i = 1, \ldots, s.$$

As $w \in C_V(H)$, for each $j = 1, \ldots, s$, we have $w^{x_j^{-1}} = w$. More explicitly,

$$\begin{aligned} w &= u_1 + u_2^{x_2} + ... + u_s^{x_s} \\ &= (u_1 + u_2^{x_2} + ... + u_s^{x_s})^{x_j^{-1}} \\ &= u_1^{x_j^{-1}} + u_2^{x_2 x_j^{-1}} + \ldots + u_j + \ldots + u_s^{x_s x_j^{-1}} \end{aligned}$$

Since the sum above is direct, letting $x_1 = 1$, we see that $u_1 = u_j$.

It follows that $u_1 = u_2 = ... = u_s = u \in V_1$ and so $w = u + u^{x_2} + ... + u^{x_s}$ for $u \in V_1$. We want now $u \in C_{V_1}(H_1)$, that is $u^{h_1} = u$ for any $h_1 \in H_1$.

Since $h_1 \in H_1 = Stab_H(W_1^F) \leq H$ and $w \in C_V(H)$, we have $w^{h_1} = w$, more explicitly,

$$\begin{aligned} (u + u^{x_2} + ... + u^{x_s})^{h_1} &= u^{h_1} + u^{x_2 h_1} + ... + u^{x_s h_1} \\ &= u + u^{x_2} + ... + u^{x_s} \end{aligned}$$

Recall that $V_1 = \bigoplus_{W \in \Omega_1} W$ and $H_1 = Stab_H \Omega_1)$. Then obviously V_1 is H_1-invariant and so $u^{h_1} \in V_1$ in the above equations. Due to the direct sum decomposition above we get $u^{h_1} = u$, as desired. This establishes the equality $C_V(H) = \{\sum_{i=1}^{s} u^{x_i} \mid u \in C_{V_1}(H_1)\}$.

Recall that $K = Ker(C_Q(H)$ on $C_V(H))$. It means that K acts trivially on $C_V(H)$. We observe now that K acts trivially on $C_{V_1}(H_1)$:

We see that V_1 is Q-invariant and hence K-invariant since $K \leq Q$. On the other hand, $[K, H_1] = 1$ due to $K \leq C_Q(H)$ gives H_1 is also K-invariant. As a result, K normalizes $C_{V_1}(H_1)$.

Then, for any $\sum_{i=1}^{s} u^{x_i} \in C_V(H)$ where $u \in C_{V_1}(H_1)$, and for any $k \in K$, by using $[K, C_V(H)] = 1$, we obtain

$$\begin{aligned} (u + u^{x_2} + \ldots + u^{x_s})^k &= u^k + u^{x_2 k} + \ldots + u^{x_s k} \\ &= u + u^{x_2} + \ldots + u^{x_s} \end{aligned}$$

Now we must have $u^k = u$ as K normalizes $C_{V_1}(H_1)$ due to the direct sum decomposition $V = \bigoplus_{i=1}^{s} V_1^{x_i}$. Hence $[K, C_{V_1}(H_1)] = 1$, that is $K \leq Ker(C_Q(H)$ on $C_{V_1}(H_1))$. Since

$$Ker(C_Q(H) \text{ on } C_{V_1}(H_1)) = Ker(C_Q(H) \text{ on } V_1)$$

as observed above, we see that K is trivial on V_1. Then $[K, V_1] = 1$ implying that

$$1=[K,V_1]^{x_i}=[K^{x_i},V_1^{x_i}]=[K,V_1^{x_i}]$$

for each $i=1,\dots,s$. Note that, $K^{x_i}=K$, for each $i=1,\dots,s$ since $K\le C_Q(H)$. It follows that K centralizes $V=\bigoplus_{i=1}^{s}V_1^{x_i}$, a contradiction. Thus, $V=V_1$, $H=H_1$ and so $\Omega=\Omega_1$, that is F acts transitively on Ω.

Claim 9. K acts trivially on each member of $\Omega_1=W^F$ except W.

Proof. Let $W^f\in\Omega_1$ such that $W^f\neq W$. We have already seen in Claim 7 that $Stab_{H_1}(W^f)=1$ and in Claim 8 that $H=H_1$. Obviously, W^f is Q-invariant since for any $q_1,q_2\in Q$, such that $q_2=q_1^{f^{-1}}$ we have

$$(W^f)^{q_1}=W^{fq_1}=W^{fq_1f^{-1}f}=W^{q_1^{f^{-1}}f}=W^{q_2f}=W^f$$

where $W^{q_2}=W$ since $W=W_1$, being a Wedderburn component is Q-invariant. Again to simplify the notation set $U=W^f$. Now, set $X=\sum_{h\in H}U^h$. Applying the same argument as in the proof of Claim 6, we can see that K centralizes U as follows:

Actually the sum $X=\sum_{h\in H}U^h$ is direct, that is $X=\bigoplus_{h\in H}U^h$.

Now, we'll show that $P=\left\{\sum_{h\in H}v^h\mid v\in U\right\}\le C_X(H)$: Take $\sum_{h\in H}v^h\in P$, and $\bar{h}\in H$. Then,

$$(\textstyle\sum_{h\in H}v^h)^{\bar{h}}=\sum_{h\in H}v^{h\bar{h}}=\sum_{h\in H}v^h.$$

Since $P\le X$ we have $P\le C_X(H)$, as desired.

On the other hand, as $X\le V$, we have $C_X(H)\le C_V(H)$ and since

$$K=Ker(C_Q(H)\ \text{ on }\ C_V(H))$$

we have $[K,C_X(H)]=1$. And this result leads us to the fact that K is trivial on U: To show this explicitly, consider

$$P=\left\{\textstyle\sum_{h\in H}v^h\mid v\in U\right\}.$$

Clearly, $[K,P]\le[K,C_X(H)]=1$. Also K normalizes U since $K\le Q$ and U is Q-invariant.

Let $v\in U$ and $H=\{1,h_2,\dots,h_m\}$ such that $\sum_{h\in H}v^h\in P$. Then for each $k\in K$, since $[K,P]=1$

$$v+v^{h_2}+\ldots+v^{h_m}=\textstyle\sum_{h\in H}v^h=(\sum_{h\in H}v^h)^k=\sum_{h\in H}v^{hk}=v^k+v^{h_2k}+\ldots+v^{h_mk}$$

We know that K normalizes U, then $v^k\in U$ and we must have $v^k=v$ due to the direct sum decomposition $X=\bigoplus_{h\in H}U^h$. It follows that K is trivial on U. And since $U\neq W$ was arbitrary in Ω_1, we conclude that K acts trivially on each member of $\Omega_1=W^F$ except W.

Claim 10. $C_Q(F) = 1$.

Proof. To simplify the notation we set $\overline{Q} = Q/C_Q(W)$. Notice that $Z(\overline{Q})$ acts by scalars on W due to Theorem 2.2.13, that is for $z \in Z(\overline{Q})$, there exists $\lambda_z \in k$ such that $wz = \lambda_z w$ for each $w \in W$. Hence $[Z(\overline{Q}), F_1 H] = \overline{1}$: To see this explicitly, let $z \in Z(\overline{Q})$, $x \in F_1 H$, $w \in W$. Then for each $w \in W$

$$w[z, x] = wz^{-1}x^{-1}zx = (\lambda_z^{-1}w)(x^{-1}zx) = (\lambda_z^{-1}wx^{-1})zx = \lambda_z(\lambda_z^{-1}wx^{-1})x = w$$

Therefore $[Z(\overline{Q}), F_1 H] = \overline{1}$. In particular $[Z(Q), H] \leq C_Q(W)$ as $\overline{Z(Q)} \leq Z(\overline{Q})$. So,

$$[C_Q(F), H] \leq [Q', H] \leq [Z(Q), H] \leq C_Q(W). \quad (*)$$

Clearly $C_{C_Q(F)}(H) = 1$ due to the assumption that $C_{C_Q(F)}(h) = 1$ for every $1 \neq h \in H$.

On the other hand

$$C_Q(F) = [C_Q(F), H]C_{C_Q(F)}(H)$$

by Theorem 2.1.29. So we have $C_Q(F) = [C_Q(F), H]$. Now, $C_Q(F) \leq C_Q(W)$ from equation (*) and hence

$$C_Q(F) = C_Q(F)^f \leq C_Q(W)^f$$

for each $f \in F$. Then $V = \sum_{f \in F} W^f$ from Claim 8 implies that

$$C_Q(F) \leq \bigcap_{f \in F} C_Q(W^f) \overset{(*)}{=} C_Q(V) = 1.$$

Equality (*) can be obtained as follows: We have $V = \sum_{f \in F} W^f$. Take $q \in \bigcap_{f \in F} C_Q(W)^f$. But since we have

$$(C_Q(W))^f = C_{Q^f}(W^f)$$

we obtain $\bigcap_{f \in F} C_Q(W)^f = \bigcap_{f \in F} C_Q(W^f)$, and hence q is trivial on each W^f. Let now $v \in V$. Then $v = w_1^{f_1} + w_2^{f_2} + \ldots + w_k^{f_k}$ for $w_i \in W$, $f_i \in F$ and

$$\begin{aligned} v^q &= (w_1^{f_1} + w_2^{f_2} + \ldots + w_k^{f_k})^q \\ &= w_1^{f_1 q} + w_2^{f_2 q} + \ldots + w_k^{f_k q} \\ &= w_1^{f_1} + w_2^{f_2} + \ldots + w_k^{f_k} \\ &= v \end{aligned}$$

This argument shows that $\bigcap_{f\in F} C_Q(W^f) \le C_Q(V)$. Conversely, let $q \in C_Q(V)$, then $q \in C_Q(W^f)$ for each $f \in F$ as $C_Q(V) \le C_Q(W^f)$.

Claim 11. Final Contradiction

Recall again that $K = Ker(C_Q(H)$ on $C_V(H))$. Let $\overline{K} = KC_Q(W)/C_Q(W)$. By Claim 8, $V = \sum_{f\in F} W^f$.

On the other hand $[K, W^f] = 1$ for each $f \in F - F_1$ by Claim 9. Thus we have $[K, W] \neq 1$ because otherwise $K \le C_Q(V) = 1$, which is not the case. So $\overline{K} \neq \overline{1}$. Also we have $\overline{K} \trianglelefteq C_{\overline{Q}}(H)$ since $K = C_{C_Q(H)}(C_V(H)) \lhd C_Q(H)$ as $C_V(H)$ is normalized by $C_Q(H)$.

Since $\overline{1} \neq \overline{K} \trianglelefteq C_{\overline{Q}}(H)$ and $C_{\overline{Q}}(H)$ being a quotient group of a q-group is nilpotent, we have

$$\overline{L} = \overline{K} \cap Z(C_{\overline{Q}}(H))$$

is nontrivial by Theorem 2.1.14. It follows that $[H, \overline{Q}, \overline{L}] = 1$ by Three Subgroup Lemma stated in 2.1.27, since

$$[\overline{L}, H, \overline{Q}] = [[\overline{K} \cap Z(C_{\overline{Q}}(H)), H], \overline{Q}] = [\overline{1}, \overline{Q}] = \overline{1}$$

where the second equation comes from $[\overline{K}, H] = \overline{1}$. And also, as shown in the proof of Claim 10 we have $[Z(\overline{Q}), H] = 1$, which gives

$$[\overline{Q}, \overline{L}, H] = [[\overline{Q}, \overline{L}], H] \le [(\overline{Q})', H] = [\overline{(Q')}, H] \le [\overline{Z(Q)}, H] \le [Z(\overline{Q}), H] = \overline{1}$$

Clearly, $[C_{\overline{Q}}(H), \overline{L}] = 1$ by the definition of $\overline{L}$. As $(|Q|, |H|) = 1$, we have also $\overline{Q} = [\overline{Q}, H]C_{\overline{Q}}(H)$ due to Lemma 2.1.29. And employing Lemma 2.1.24, we have then

$$\begin{aligned}
[\overline{Q}, \overline{L}] &= [[\overline{Q}, H]C_{\overline{Q}}(H), \overline{L}] \\
&\subseteq [[\overline{Q}, H], \overline{L}]^{C_{\overline{Q}}(H)}[C_{\overline{Q}}(H), \overline{L}] \\
&= [[H, \overline{Q}], \overline{L}][C_{\overline{Q}}(H), \overline{L}] \\
&= \overline{1}
\end{aligned}$$

Thus,

$$\overline{L} \le Z(\overline{Q}).$$

Let now $z \in K$ be chosen so that $z \notin C_Q(W)$ and

$$1 \neq \bar{z} \in \overline{L} = \overline{K} \cap Z(C_{\overline{Q}}(H))$$

and consider the group $\langle z^F \rangle \leq Q$. As $\langle z^F \rangle$ is F-invariant and $C_Q(F) = 1$, we have, due to Lemma 2.1.29, $[\langle z^F \rangle, F] = \langle z^F \rangle$.

Since $z \notin C_Q(W)$ and $W \leq V$, we have $z \notin C_Q(V)$, which implies $\langle z^F \rangle \nleq C_Q(V)$. Hence, $\langle z^F \rangle$ acts nontrivially on V and $\langle z^F \rangle$ being a subgroup of Q satisfies also other conditions of Lemma 3.2.1. Now, if we consider the action of $\langle z^F \rangle FH$ on V, and if $\left|\langle z^F \rangle\right| < |Q|$, as $[\langle z^F \rangle, F] = \langle z^F \rangle$, we conclude that $Q = \langle z^F \rangle$ by induction argument. This is so, because by induction we get

$$Ker(C_{\langle z^F \rangle}(H) \text{ on } C_V(H)) = Ker(C_{\langle z^F \rangle}(H) \text{ on } V) \leq Ker(Q \text{ on } V) = 1$$

where the last equation is due to the faithful action of Q on V. On the other hand, $z \in K \cap \langle z^F \rangle$, more explicitly, writing $K = C_{C_Q(H)}(C_V(H))$ we have

$$z \in C_{C_Q(H)}(C_V(H)) \cap \langle z^F \rangle = C_{C_Q(H) \cap \langle z^F \rangle}(C_V(H)) = C_{C_{\langle z^F \rangle}(H)}(C_V(H)),$$

implying

$$z \in Ker(C_{\langle z^F \rangle}(H) \text{ on } C_V(H)) = 1,$$

a contradiction. Hence, $\left|\langle z^F \rangle\right| = |Q|$, that is $Q = \langle z^F \rangle$.

Since we have found $\overline{L} \leq Z(\overline{Q})$ and $\bar{z} \in \overline{L}$, we see that $z^f \in zC_Q(W)$ for any $f \in F_1$. This is so because of $[Z(\overline{Q}), F_1 H] = 1$ as shown in Claim 10, then we'll have $[Z(\overline{Q}), F_1] = 1$ which implies $z^{-1}z^f \in C_Q(W)$ for $z \in Z(\overline{Q})$ and $f \in F_1$. More precisely,

$$z^f \in zC_Q(W) \text{ if } f \in F_1.$$

On the other hand we know that K acts trivially on $W^{f^{-1}}$ for any $f \in F - F_1$ by Claim 9. Then K^f acts trivially on W for each $f \in F - F_1$. In particular as $z \in K$,

$$z^f \in C_Q(W) \text{ for every } f \in F - F_1.$$

So, by considering the action of $\langle z^F \rangle FH$ on V, we have

$$\begin{aligned} Q = \langle z^F \rangle &= \langle z^f \mid f \in F_1 \cup (F - F_1) \rangle \\ &= \langle z^f \mid f \in F_1 \rangle C_Q(W) \\ &= \langle z \rangle C_Q(W) \end{aligned}$$

Then, $Q' = [Q, Q]$ acts trivially on W since $Q = \langle z \rangle C_Q(W)$ implies

$$\overline{Q} = \overline{\langle z \rangle} = \langle \bar{z} \rangle \leq \overline{L} \leq Z(\overline{Q})$$

This results in $\overline{Q} = Z(\overline{Q})$ and hence $\overline{Q}$ is abelian, that is $Q' \leq C_Q(W)$ due to Theorem 2.1.8 implying Q' acts trivially on W.

Now, since Q' acts trivially on W,

$$1 = [Q', W]^f = [Q'^f, W^f] = [Q', W^f]$$

for each $f \in F$, as $Q' char Q$ gives Q' is F-invariant. It follows that

$$Q' \leq \bigcap_{f \in F} C_Q(W^f) = C_Q(V) = 1$$

in the same way as shown in Claim 10. Therefore $Q' = 1$, that is Q is abelian.

We consider now $\prod_{f \in F} z^f$. As Q is abelian, it is a well defined element of Q and for any $g \in F$,

$$(\prod_{f \in F} z^f)^g = \prod_{f \in F} z^{fg} = \prod_{f \in F} z^f$$

implying $\prod_{f \in F} z^f \in C_Q(F) = 1$.

Since Q is abelian we can write $\prod_{f \in F} z^f$ in the following way,

$$(\prod_{f \in F} z^f) = (\prod_{f \in F_1} z^f)(\prod_{f \in F - F_1} z^f).$$

Thus, with $\prod_{f \in F} z^f \in C_Q(F) = 1$ we'll have

$$1 = (\prod_{f \in F} z^f) = (\prod_{f \in F_1} z^f)(\prod_{f \in F - F_1} z^f) \overset{(*)}{=} (\prod_{f \in F_1} z^f) C_Q(W) \overset{(**)}{=} z^{|F_1|} C_Q(W)$$

Equation (*) is due to $z^f \in C_Q(W)$ for $f \in F - F_1$ and hence

$$\prod_{f \in F - F_1} z^f = C_Q(W).$$

And equation (**) is due to $z^f \in z C_Q(W)$ for $f \in F_1$ and hence

$$\prod_{f \in F_1} z^f = \prod_{f \in F_1} z C_Q(W) = z^{|F_1|} C_Q(W).$$

This gives, $1 = z^{|F_1|} C_Q(W)$ implying that the order of $\bar{z}$ divides $|F_1|$. But then, as $\bar{z} \in \overline{Q}$ and $(|\overline{Q}|, |F_1|) = 1$, we get $\bar{z} = 1$, that is $z \in C_Q(W)$ contradicting our choice of $z \notin C_Q(W)$. This completes the proof of the lemma. □

Now, we are ready to prove our main result.

3.3 Proof of the main result

In this section we present the main theorem of this thesis. Let A be a finite group that acts on the finite solvable group G by automorphisms. As indicated in Introduction chapter, there have been a lot of research to obtain information about certain group theoretical invarinats of G in terms of the action of A on G. A particular major problem is to bound the nilpotent length of G in terms of the information about the structure of A alone when $C_G(A) = 1$, that is the action of A is fixed-point-free. One of the recent results in this framework is [1] in which Khukhro handled the case where $A = FH$ is a Frobenius group with kernel F and complement H. He proved that the nilpotent lengths of G and $C_G(H)$ are the same if $C_G(F) = 1$ and $(|G|, |H|) = 1$ and later in [2], he removed the coprimeness assumption of the theorem in [1]. In this thesis, we keep the coprimeness condition but weaken the fixed-point freeness of F on G slightly, and obtain the same conclusion about the nilpotent length of G. More precisely we prove the following:

Theorem 3.3.1 *Let G be a finite solvable group admitting a Frobenius group of automorphisms FH of coprime order with kernel F and complement H so that $C_{C_G(F)}(h) = 1$ for every $1 \neq h \in H$. Then $f([G, F]) = f(C_{[G,F]}(H))$ and $f(G) \leq f([G, F]) + 1$.*

Proof. We proceed over a series of steps.

Claim 1. $C_G(F)$ is nilpotent.

Proof. By the hypothesis we know that every nontrivial element of H acts fixed-point-freely on $C_G(F)$. In particular, the same is true for each element of prime order in H. It is well known that a famous theorem due to J. Thompson [24] says the following: *"A finite group admitting a fixed-point-free automorphism of prime order is nilpotent."* Hence we have $C_G(F)$ is nilpotent.

Claim 2. $f(G) \leq f([G, F]) + 1$ holds and w.l.o.g. we may assume that $[G, F] = G$.

Proof. Since $(|G|, |F|) = 1$, we have $G = [G, F]C_G(F)$ by Theorem 2.1.29. Then we have $f(G) \leq f([G, F]) + 1$ by Theorem 2.1.42 and by Claim 1. It remains only to prove that $f([G, F]) = f(C_{[G,F]}(H))$. W.l.o.g. we may assume that $[G, F] = G$, since $[G, F, F] = [G, F]$ as $(|G|, |F|) = 1$ by Lemma 2.1.30. This establishes Claim 2.

Due to the assumption $[G, F] = G$, we will prove that $f(G) = f(C_G(H))$. Let $f(G) = n$. We proceed by induction on the order of G.

Claim 3. As $G = [G, F]$ and $(|G|, |FH|) = 1$, there exists an irreducible FH-tover $\hat{P}_1, \ldots, \hat{P}_n$ in the sense of [9] where

(a) $\hat{P}_i$ is an FH-invariant p_i-subgroup, p_i is a prime, $p_i \neq p_{i+1}$, for $i = 1, \ldots, n-1$;

(b) $\hat{P}_i$ normalizes $\hat{P}_j$, for $i \leq j$;

(c) $P_n = \hat{P}_n$ *and* $P_i = \hat{P}_i/C_{\hat{P}_i}(P_{i+1})$, *for* $i = 1, ..., n-1$, *and* P_i *is not trivial for* $i = 1, \ldots, n$;

(d) $\Phi(\Phi(P_i)) = 1$, $\Phi(P_i) \subseteq Z(P_i)$ *and if* $p_i \neq 2$, $exp(P_i) = p_i$ *for* $i = 1, \ldots, n$;

(e) $[\Phi(P_{i+1}), \hat{P}_i] = 1$ *and* $[P_{i+1}, \hat{P}_i] = P_{i+1}$ *for* $i = 1, \ldots, n-1$;

(f) $(\Pi_{j<i}\hat{P}_j)FH$ *acts irreducibly on* $P_i/\Phi(P_i)$ *for* $i = 1, \ldots, n$;

(g) $P_1 = [P_1, F]$.

Proof. The existence of an irreducible FH-tower $\hat{P}_1, \ldots, \hat{P}_n$ is due to Theorem 2.1.40.i). Note that (a), (b), (c) and (d) follow directly from Definition 2.1.38 and 2.1.39. As for (f) and (g), (f) is the weaker form of (8) in Definition 2.1.39 whose proof can be found in Lemma 1.4 of [9] and (g) is due to Theorem 2.1.40.i). Finally, the first part of (e) can be found in Definition 2.1.39.(5) and the second part of (e) comes from (f), (b) and the first part of (e) as follows. Since (b) says $\hat{P}_i$ normalizes $\hat{P}_j$ for $i < j$ we have

$$[P_{i+1}, \hat{P}_i] = [\hat{P}_{i+1}/C_{\hat{P}_{i+1}}(P_{i+2}), \hat{P}_i] \leq \hat{P}_{i+1}/C_{\hat{P}_{i+1}}(P_{i+2}) = P_{i+1}$$

and hence

$$[P_{i+1}/\Phi(P_{i+1}), \hat{P}_i] = [P_{i+1}, \hat{P}_i]\Phi(P_{i+1})/\Phi(P_{i+1}) \leq P_{i+1}/\Phi(P_{i+1})$$

and also $[P_{i+1}/\Phi(P_{i+1}), \hat{P}_i]$ is $(\Pi_{j\leq i}\hat{P}_j)FH$-invariant. But from (f) we know that $P_{i+1}/\Phi(P_{i+1})$ is $(\Pi_{j\leq i}\hat{P}_j)FH$-irreducible. We get then $[P_{i+1}/\Phi(P_{i+1}), \hat{P}_i]$ is either equal to $P_{i+1}/\Phi(P_{i+1})$ or to 1. If $[P_{i+1}/\Phi(P_{i+1}), \hat{P}_i] = P_{i+1}/\Phi(P_{i+1})$ then

$$[P_{i+1}/\Phi(P_{i+1}), \hat{P}_i] = [P_{i+1}, \hat{P}_i]\Phi(P_{i+1})/\Phi(P_{i+1}) = P_{i+1}/\Phi(P_{i+1})$$

and hence $[P_{i+1}, \hat{P}_i]\Phi(P_{i+1}) = P_{i+1}$ implying $[P_{i+1}, \hat{P}_i] = P_{i+1}$ due to 2.1.18.

On the other hand, if $[P_{i+1}, \hat{P}_i]\Phi(P_{i+1}) = \Phi(P_{i+1})$ then $[P_{i+1}, \hat{P}_i] \leq \Phi(P_{i+1})$ implying

$$[P_{i+1}, \hat{P}_i, \hat{P}_i] \leq [\Phi(P_{i+1}), \hat{P}_i] = 1$$

and $[P_{i+1}, \hat{P}_i] = 1$ since $[P_{i+1}, \hat{P}_i, \hat{P}_i] = [P_{i+1}, \hat{P}_i]$ due to 2.1.30. But this is impossible, since then

$$C_{\hat{P}_i}(P_{i+1}) = \hat{P}_i$$

and we get $P_i = \hat{P}_i/C_{\hat{P}_i}(P_{i+1}) = 1$ which is a contradiction to (c). Hence, we have only

$$[P_{i+1}, \hat{P}_i] = P_{i+1}$$

for $i = 1, \ldots, n-1$ giving the second part of (e).

Claim 4. $[P_i, F] \nleq \Phi(P_i)$ *for each* $i = 1, \ldots, n$.

Proof. First observe that, part (e) of Claim 3 can be stated also as follows:

$$(e)\quad [\Phi(P_{i+1}), P_i] = 1 \quad and \quad [P_{i+1}, P_i] = P_{i+1} \quad for \quad i = 1, \ldots, n-1.$$

If there exists i such that $[P_i/\Phi(P_i), F] = 1$ then $[P_i, F] \leq \Phi(P_i)$ and hence

$$[P_i, F, P_{i-1}] \leq [\Phi(P_i), P_{i-1}] = 1.$$

And also,

$$[P_{i-1}, P_i, F] = [[P_{i-1}, P_i], F] = [P_i, F] \leq \Phi(P_i).$$

So by the Three subgroup Lemma 2.1.27 we get $[F, P_{i-1}, P_i] \leq \Phi(P_i)$ which implies that

$$[P_{i-1}, F] \leq C_{P_{i-1}}(P_i/\Phi(P_i)).$$

On the other hand, as $P_{i-1} = \hat{P}_{i-1}/C_{\hat{P}_{i-1}}(P_i)$ by Claim 3, part (c), we have $C_{P_{i-1}}(P_i) = 1$, that is the action of P_{i-1} on P_i is faithful. By employing Corollary 2.1.47, we see that $C_{P_{i-1}}(P_i/\Phi(P_i)) = 1$, too and hence $[P_{i-1}, F] = 1$.

Continuing this way we get $P_1 = [P_1, F] = 1$ which is a contradiction to Claim 3, part (c).

Claim 5. Set $X = \hat{P}_1 \cdots \hat{P}_n$. *Then* $f(X) = f(G)$.

Proof. Since $\hat{P}_n, \ldots, \hat{P}_1$ is also an FH-tower in X of height n we have $f(X) \geq n$ by Theorem 2.1.40.(ii) and hence by Theorem 2.1.41, since $X \leq G$, $n \leq f(X) \leq f(G) = n$ giving $f(X) = f(G)$.

Claim 6. $X = [X, F]$ *and so* F *is not contained in* $Ker(FH$ *on* $X)$.

Proof. We only need to see that $X = [X, F]$: To see this recall first that for each $i = 1, \ldots, n$, $[P_i, F] \nleq \Phi(P_i)$ due to Claim 4. Now consider the group

$$[P_i, F]^{P_{i-1} \cdots P_1} \Phi(P_i)/\Phi(P_i) \leq P_i/\Phi(P_i).$$

It is $P_{i-1} \cdots P_1 FH$-invariant and so by the irreducibility of $P_i/\Phi(P_i)$ as an $P_{i-1} \cdots P_1 FH$-module, it is either trivial or equal to $P_i/\Phi(P_i)$. If the former holds,

$$[P_i, F]^{P_{i-1} \cdots P_1} \leq \Phi(P_i)$$

and so $[P_i, F] \leq \Phi(P_i)$, which is not the case. Thus

$$[P_i, F]^{P_{i-1}\cdots P_1}\Phi(P_i) = P_i$$

and so $[P_i, F]^{P_{i-1}\cdots P_1} = P_i$. Then,

$$P_i = [P_i, F]^{P_{i-1}\cdots P_1} \leq \langle [P_i, F], P_{i-1}\cdots P_1 \rangle \leq [P_i P_{i-1}\cdots P_1, F]$$

due to at the first step $P_1 = [P_1, F]$ and then $P_i \leq [P_i P_{i-1}\dots P_1, F]$ for each $i \geq 2$. Hence, we have

$$X \leq [X, F], \quad \text{that is} \quad X = [X, F].$$

Claim 7. *$FH/Ker(FH \text{ on } X)$ is a Frobenius group of automorphisms of the group X with kernel $F/Ker(F \text{ on } X)$ and complement H.*

Proof. We know that $Ker(FH \text{ on } X) = C_{FH}(X) = C_F(X)C_H(X)$ by Theorem 2.1.45, that is

$$Ker(FH \text{ on } X) = Ker(F \text{ on } X)Ker(H \text{ on } X).$$

And we can see that $H_1 = Ker(H \text{ on } X) = 1$ as follows: Suppose not. By Claim 6,

$$[X, F, H_1] = [[X, F], H_1] = [X, H_1] = 1.$$

And also, we have

$$[H_1, X, F] = [[H_1, X], F] = [1, F] = 1$$

which gives $[F, H_1, X] = 1$ by the Three Subgroup Lemma 2.1.27. Since FH_1 is a Frobenius group, we have $[F, H_1] = F$. Hence

$$1 = [F, H_1, X] = [[F, H_1], X] = [F, X] = X$$

which is a contradiction.

So, we must have $Ker(H \text{ on } X) = 1$. It follows that

$$FH/Ker(FH \text{ on } X) = (F/Ker(F \text{ on } X))H$$

is a Frobenius group of automorphims of X with kernel $F/Ker(F$ on $X)$ and complement H by Corollary 2.1.5.

Claim 8. $X = G$.

Proof. If X is a proper subgroup in G, it is also a finite solvable group and by Claim 7 it admits a Frobenius group of automorphisms $(F/Ker(F$ on $X))H$ with kernel $(F/Ker(F$ on $X)$ and complement H. Obviously, we have also $C_{C_X(F)}(h) = 1$ for every $1 \neq h \in H$. Furthermore, $|X| < |G|$. Then, by induction we have $f(X) = f(C_X(H))$ and so due to Claim 5, we get

$$f(G) = f(C_X(H)).$$

On the other hand

$$f(C_X(H)) \leq f(C_G(H))$$

and hence $f(G) \leq f(C_G(H))$ giving $f(G) = f(C_G(H))$. So, we must have $X = G$.

Claim 9. $\overline{G} = G/F(G)$ *is a nontrivial group and* $f(\overline{G}) = n - 1 = f(C_{\overline{G}}(H))$.

Proof. If $G = F(G)$ then $f(G) = 1$. Note that $C_G(H) \neq 1$: To see this, for any i consider $V = P_i/\Phi(P_i)$. Clearly V is FH-invariant and $[V, F] \neq 0$ due to Claim 4. Note that, here, we consider V as a vector space over a field k of characteristic p_i.

Also, since $(|V|, |F|) = 1$ we write

$$V = [V, F] \times C_V(F)$$

due to Theorem 2.1.28. By Theorem 2.1.46 we have

$$C_V(F) = C_{[V,F]}(F) \times C_{C_V(F)}(F) = C_{[V,F]}(F) \times C_V(F)$$

and hence we get that

$$C_{[V,F]}(F) = 0.$$

Now, consider $[V, F]$ acted on by FH. By using Lemma 3.1.1, we see that $[V, F]|_H$ is free. Then $[V, F]|_H$ contains the trivial H-module and hence $C_{[V,F]}(H) \neq 0$. Then, $C_V(H) \neq 0$. Writing it explicitly in multiplicative notation we see that

$$C_{P_i/\Phi(P_i)}(H) = C_{P_i}(H)\Phi(P_i)/\Phi(P_i) \neq 1$$

where the first equality is due to Theorem 2.1.32. And this implies that $C_{P_i}(H) \neq 1$ for all $i = 1, \ldots, n$ which gives $C_G(H) \neq 1$, since

$$C_G(H) = C_X(H) = C_{\hat{P}_1}(H) C_{\hat{P}_2}(H) \cdots C_{\hat{P}_n}(H)$$

due to Claim 8 and Theorem 2.1.46. Then, $C_G(H)$ is nontrivial nilpotent subgroup of G giving $f(C_G(H)) = 1$, and the lemma follows. Therefore, we assume that $\overline{G}$ is a nontrivial group.

Finally, since $F(G)$, being a characteristic group of G is FH-invariant, we may consider the induced action of FH on $\overline{G}$. Now, being a quotient group of G, $\overline{G}$ is also a finite solvable group which admits FH as a Frobenius group of automorphisms of coprime order with kernel F and complement H. Also, due to Theorem 2.1.32, $C_{C_G(F)}(h) = 1$ for every $1 \neq h \in H$ implies

$$C_{C_{\overline{G}}(F)}(h) = C_{\overline{C_G(F)}}(h) = \overline{C_{C_G(F)}(h)} = \overline{1}$$

for every $1 \neq h \in H$. And, since $\left|\overline{G}\right| < |G|$, by induction we have $f(\overline{G}) = f(C_{\overline{G}}(H))$ and hence $f(C_{\overline{G}}(H)) = n-1$ by Theorem 2.1.37. *Claim 10. Consider the sequence* $C_{\overline{\hat{P}}_{n-1}}(H), \ldots, C_{\overline{\hat{P}}_2}(H), C_{\overline{\hat{P}}_1}(H)$ *of subgroups of* $C_{\overline{G}}(H)$. *Then*

$$Y = [C_{\hat{P}_{n-1}}(H), \ldots, C_{\hat{P}_1}(H)] \nleq F(G) \cap \hat{P}_{n-1}$$

and hence $C_{\hat{P}_{n-1}}(H), \ldots, C_{\hat{P}_2}(H), C_{\hat{P}_1}(H)$ *is an* FH*-tower of* $C_G(H)$.

Proof. Since $\overline{G} = \overline{\hat{P}_{n-1} \cdots \hat{P}_1} = \overline{\hat{P}}_{n-1} \cdots \overline{\hat{P}}_1$, we have $C_{\overline{G}}(H) = C_{\overline{\hat{P}}_{n-1}}(H) \cdots C_{\overline{\hat{P}}_2}(H) C_{\overline{\hat{P}}_1}(H)$ by Theorem 2.1.46. We know $f(C_{\overline{G}}(H)) = n-1$ by Claim 9. Obviously,

$$Y = [C_{\hat{P}_{n-1}}(H), \ldots, C_{\hat{P}_1}(H)] \leq [\hat{P}_{n-1}, \ldots, \hat{P}_1] \leq \hat{P}_{n-1}.$$

Assume that $Y \leq F(G)$. Then

$$\begin{aligned} \overline{Y} &= [\overline{C_{\hat{P}_{n-1}}(H)}, \ldots, \overline{C_{\hat{P}_1}(H)}] \\ &= [C_{\overline{\hat{P}}_{n-1}}(H), \ldots, C_{\overline{\hat{P}_1(H)}}] = 1 \end{aligned}$$

and so $C_{\overline{\hat{P}}_{n-1}}(H), \ldots, C_{\overline{\hat{P}_1(H)}}$ does not contain an FH-tower of length $n-1$ and so the maximum of the lengths of FH-towers in $C_{\overline{G}}(H)$, namely $f(C_{\overline{G}}(H))$ is strictly less than $n-1$, a contradiction. So,

$$Y = [C_{\hat{P}_{n-1}}(H), \ldots, C_{\hat{P}_1}(H)] \nleq F(G) \cap \hat{P}_{n-1}.$$

On the other hand, $Y \nsubseteq F(G)$ gives that $Y \neq 1$ and hence $C_{\hat{P}_{n-1}}(H), \ldots, C_{\hat{P}_2}(H), C_{\hat{P}_1}(H)$ is an FH-tower of $C_G(H)$ due to Theorem 2.1.43.

Claim 11. $[Y, C_{\hat{P}_n}(H)] = 1$, *that is* $Y \leq Ker(C_{\hat{P}_{n-1}}(H)$ *on* $C_{\hat{P}_n}(H))$.

Proof. Consider the sequence $C_{\hat{P}_n}(H), C_{\hat{P}_{n-1}}(H), \ldots, C_{\hat{P}_2}(H), C_{\hat{P}_1}(H)$. If $[Y, C_{\hat{P}_n}(H)] \neq 1$ then

$$C_{\hat{P}_n}(H), C_{\hat{P}_{n-1}}(H), \ldots, C_{\hat{P}_2}(H), C_{\hat{P}_1}(H)$$

is an FH-tower in $C_G(H)$ due to Theorem 2.1.43 and so $f(C_G(H)) \geq n$ (in fact $= n$ by Theorem 2.1.40) and the theorem follows. Hence we have

$$[Y, C_{\hat{P}_n}(H)] = 1,$$

On the other hand, it can be seen that Y is a subgroup of $C_{\hat{P}_{n-1}}(H)$ due to Lemma 2.1.25, part (b) of Claim 3 and TSL as noted before the Claim 1 in the proof of Lemma 3.2.1. Therefore, we have

$$Y \leq Ker(C_{\hat{P}_{n-1}}(H) \text{ on } C_{\hat{P}_n}(H)).$$

Claim 12. Final Contradiction.

Proof. We first observe that $Y \nleq C_{\hat{P}_{n-1}}(\hat{P}_n)$ as follows. Note that, since $C_{\hat{P}_{n-1}}(\hat{P}_n) \lhd G$ and also, being a p_{n-1} group, it is nilpotent, we have

$$C_{\hat{P}_{n-1}}(\hat{P}_n) \subseteq F(G).$$

Thus if $Y \leq C_{\hat{P}_{n-1}}(\hat{P}_n)$, then $Y \subseteq F(G)$ which is not the case by Claim 10.

On the other hand, by Corollary 2.1.48,

$$Ker(C_{\hat{P}_{n-1}}(H) \text{ on } \hat{P}_n/\Phi(\hat{P}_n)) = Ker(C_{\hat{P}_{n-1}}(H) \text{ on } \hat{P}_n)$$

Hence, by the above observation,

$$Y \nsubseteq Ker(C_{\hat{P}_{n-1}}(H) \text{ on } \hat{P}_n/\Phi(\hat{P}_n)).$$

Thus we have

$$Ker(C_{\hat{P}_{n-1}}(H) \text{ on } C_{\hat{P}_n/\Phi(\hat{P}_n)}(H)) \overset{(*)}{\neq} Ker(C_{\hat{P}_{n-1}}(H) \text{ on } \hat{P}_n/\Phi(\hat{P}_n)).$$

This inequality holds since Y is inside the left hand side of the inequality by Claim 11.

Note also that, we have

$$\hat{P}_{n-1} = [\hat{P}_{n-1}, F]C_{\hat{P}_{n-1}}(F)$$

by Lemma 2.1.29 implying

$$C_{\hat{P}_{n-1}}(H) = C_{[\hat{P}_{n-1},F]}(H)C_{C_{\hat{P}_{n-1}}(F)}(H)$$

due to Theorem 2.1.46. As

$$C_{C_{\hat{P}_{n-1}}(F)}(H) \leq C_{C_{\hat{P}_{n-1}}(F)}(h) \leq C_{C_G(F)}(h) = 1$$

by our assumption, we get $C_{[\hat{P}_{n-1},F]}(H) = C_{\hat{P}_{n-1}}(H)$. Moreover, we see the following:

- $[P_{n-1}, F] \neq 1$, since $[P_{n-1}, F] \nleq \Phi(P_{n-1})$ due to Claim 4.
- $[P_{n-1}, F]$ acts nontrivially on $\hat{P}_n/\Phi(\hat{P}_n)$, due to Corollary 2.1.48, since $[P_{n-1}, F] \leq P_{n-1}$ and $C_{P_{n-1}}(\hat{P}_n) = 1$, which is due to $P_{n-1} = \hat{P}_{n-1}/C_{\hat{P}_{n-1}}(P_n)$. Note that $\hat{P}_n = P_n$ by Claim 3, part (c).
- $|FH|$ is not divisible by p_n and p_{n-1} as $(|G|, |FH|) = 1$.
- By Theorem 2.1.16.i) and Claim 3, part (d), we have

  $$P'_{n-1} \leq \Phi(P_{n-1}) \leq Z(P_{n-1}),$$

  which implies that P_{n-1} is of class at most 2 by Definition 2.1.12.
- $\hat{P}_n/\Phi(\hat{P}_n)$ is $P_{n-1}FH$-invariant and can be considered as a vector space over a field k of characterisitic p_n. So, $\hat{P}_n/\Phi(\hat{P}_n)$ is an $k\hat{P}_{n-1}FH$-module, where k is a field of characteristic not dividing $|P_{n-1}FH|$.
- Due to Theorem 2.1.32 and $C_{C_{\hat{P}_{n-1}}(F)}(h) = 1$ for every $1 \neq h \in H$ as seen above, we have

  $$\begin{aligned} C_{C_{P_{n-1}}(F)}(h) &= C_{C_{(\hat{P}_{n-1}/C_{\hat{P}_{n-1}}(P_n))}(F)}(h) \\ &= C_{C_{\hat{P}_{n-1}}(F)C_{\hat{P}_{n-1}}(P_n)/C_{\hat{P}_{n-1}}(P_n)}(h) \\ &= C_{C_{\hat{P}_{n-1}}(F)}(h)C_{\hat{P}_{n-1}}(P_n)/C_{\hat{P}_{n-1}}(P_n) \\ &= \overline{1} \end{aligned}$$

Now the action of the group $P_{n-1}FH$ on $\hat{P}_n/\Phi(\hat{P}_n)$ satisfies the hypothesis of the Lemma 3.2.1 and $C_{[\hat{P}_{n-1},F]}(H) = C_{\hat{P}_{n-1}}(H)$. By letting $V = \hat{P}_n/\Phi(\hat{P}_n)$, $Q = P_{n-1}$, we apply Lemma 3.2.1 and get

$$Ker(C_{\hat{P}_{n-1}}(H) \text{ on } C_{\hat{P}_n/\Phi(\hat{P}_n)}(H)) = Ker(C_{\hat{P}_{n-1}}(H) \text{ on } \hat{P}_n/\Phi(\hat{P}_n))$$

which is a contradiction to the inequality (*) above. This completes the proof. □

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APPENDIX A

LIST OF SYMBOLS

$F(G)$	the Fitting subgroup of G.
$F_i(G)$	the i^{th} term of the Fitting series of G.
$f(G)$	the nilpotent length of G.
$cl(G)$	the nilpotency class of G.
$exp(G)$	the exponent of G.
$C_M(N)$	the centralizer of N in M,
	i.e. $C_M(N) = \{m \in M \mid n^m = n$ for all $n \in N\}$.
$\Phi(G)$	the Frattini subgroup of G.
$Z(G)$	the center of G.
$[A, B]$	the commutator of A and B.
G'	the commutator subgroup $[G, G]$ of G.
$G^{(i+1)}$	the i^{th} term of the derived series of G, which is defined by $(G^{(i)})'$.
H^K	the normal closure of H in K
	i.e. $H^K = \langle h^k \mid h \in H, k \in K \rangle$.
$A \otimes_F B$	the tensor product of A and B over F.
$Stab_G(a)$	the stabilizer of a in G.

VITA

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