ASYMPTOTIC INTEGRATION OF DYNAMICAL SYSTEMS

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ABSTRACT

ASYMPTOTIC INTEGRATION OF DYNAMICAL SYSTEMS

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In almost all works in the literature there are several results showing asymptotic relationships between the solutions of

$$x'' = f(t, x) \tag{0.1}$$

and the solutions 1 and t of $x'' = 0$. More specifically, the existence of a solution of (0.1) asymptotic to $x(t) = at + b$, $a, b \in \mathbb{R}$ has been obtained.

In this thesis we investigate in a systematic way the asymptotic behavior as $t \to \infty$ of solutions of a class of differential equations of the form

$$(p(t)x')' + q(t)x = f(t, x), \quad t \geq t_0 \tag{0.2}$$

and

$$(p(t)x')' + q(t)x = g(t, x, x'), \quad t \geq t_0 \tag{0.3}$$

by the help of principal $u(t)$ and nonprincipal $v(t)$ solutions of the corresponding homogeneous equation

$$(p(t)x')' + q(t)x = 0, \quad t \geq t_0. \tag{0.4}$$

Here, $t_0 \geq 0$ is a real number, $p \in C([t_0, \infty), (0, \infty))$, $q \in C([t_0, \infty), \mathbb{R})$, $f \in C([t_0, \infty) \times \mathbb{R}, \mathbb{R})$ and $g \in C([t_0, \infty) \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$.

Our argument is based on the idea of writing the solution of $x'' = 0$ in terms of principal and nonprincipal solutions as $x(t) = av(t) + bu(t)$, where $v(t) = t$ and $u(t) = 1$.

In the proofs, Banach and Schauder's fixed point theorems are used. The compactness of the operator is obtained by employing the compactness criteria of Riesz and Avramescu.

The thesis consists of three chapters. Chapter 1 is introductory and provides statement of the problem, literature review, and basic definitions and theorems.

In Chapter 2 first we deal with some asymptotic relationships between the solutions of (0.2) and the principal $u(t)$ and nonprincipal $v(t)$ solutions of (0.4). Then we present existence of a monotone positive solution of (0.3) with prescribed asimptotic behavior.

In Chapter 3 we introduce the existence of solution of a singular boundary value problem to the Equation (0.2).

Keywords: dynamical system, differential equation, asymptotic integration, principal and nonprincipal solutions, fixed point theory.

ÖZ

DİNAMİK SİSTEMLERİN ASİMPTOTİK İNTEGRASYONU

Ertem, Türker
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Tez Yöneticisi : Prof. Dr. Ağacık Zafer

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Literatürde yer alan çalışmaların hemen hemen hepsinde

$$x'' = f(t, x) \tag{0.5}$$

denkleminin çözümleri ile $x'' = 0$ denkleminin çözümleri 1 ve t arasında asimptotik ilişkileri gösteren sonuçlar vardır. Yapılan çalışmalarda özel olarak (0.5) denkleminin $x(t) = at+b$, $a, b \in \mathbb{R}$ fonksiyonuna asimptotik olan bir çözümünün varlığı gösterilmiştir.

Biz bu tezde,

$$(p(t)x')' + q(t)x = f(t, x), \quad t \geq t_0 \tag{0.6}$$

ve

$$(p(t)x')' + q(t)x = g(t, x, x'), \quad t \geq t_0 \tag{0.7}$$

tipinde bir sınıf denklemin çözümlerinin sonsuz civarında asimptotik davranışını, ilgili

$$(p(t)x')' + q(t)x = 0, \quad t \geq t_0 \tag{0.8}$$

homojen denkleminin küçük (recessive/principal) ve büyük (dominant/nonprincipal) çözümleri yardımıyla daha sistematik bir şekilde inceledik. Burada $t_0 \geq 0$ verilen bir reel sayı, p fonksiyonu $C([t_0, \infty), (0, \infty))$ sınıfından, q fonksiyonu $C([t_0, \infty), \mathbb{R})$ sınıfından, f fonksiyonu $C([t_0, \infty)\times\mathbb{R}, \mathbb{R})$ sınıfından ve g fonksiyonu da $C([t_0, \infty) \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ sınıfındandır.

Argümanımız temel olarak, $x(t) = at + b$ fonksiyonunun $x'' = 0$ denkleminin bir çözümü olduğu ve bu çözümün $x(t) = av(t) + bu(t)$ şeklinde büyük $v(t) = t$ ve küçük $u(t) = 1$ çözümleri cinsinden yazılabilmesi gerçeğine dayanmaktadır.

Yapılan ispatlarda Banach ve Schauder Sabit Nokta Teoremleri kullanılmıştır. Schauder Sabit Nokta Teoremi kullanılarak yapılan ispatlarda, operatörün kompaktlığını göstermeye ihtiyaç duyulduğunda Avramescu Lemması, M. Riesz Teoremi gibi kompaktlık kriterleri kullanılmıştır.

Tez üç bölümden oluşmaktadır. Birinci bölüm giriş niteliğindedir ve bu bölümde problemin ifadesi, literatür taraması ve temel tanım teoremler verilmektedir.

İkinci bölümde ilk olarak (0.6) denkleminin çözümleri ile (0.8) denkleminin küçük $u(t)$ ve büyük $v(t)$ çözümleri arasında elde edilen asimptotik ilişkileri verdik. Daha sonra (0.7) denkleminin belli bir asimptotik gösterimde monoton pozitif bir çözümünün varlığına yönelik sonucu verdik.

Üçüncü bölümde (0.6) denklemi için bir singüler sınır değer probleminin çözümünün varlığını gösterdik.

Anahtar Kelimeler: dinamik sistem, diferansiyel denklem, asimptotik integrasyon, küçük ve büyük çözümler, sabit nokta teorisi.

To the memory of my father,
Mustafa ERTEM

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Please appreciate my excitement, if it will seem to be emotional, what I will be writing here. This work is one of my three biggest and most significant successes which I have achieved up to now. Let me please to express my feelings freely.

Since the first day of my PhD at METU I felt like a member of a big, warm family. Now, with the same feeling, I am starting a new period in my life. I am grateful to all the members of METU Mathematics family, academic and administrative. I would like to thank, especially, all lecturers, whom I took courses from. And, of course, all my friends.

When I started to my PhD, I was living in a very hard period in my life. I was feeling that I was very far from my great goals. During this period, when I was being lost in my chaotic inner world, when I had no expectations from life, your interest, your patience, your guidance were like a lighthouse, which always showed to me the direction I should go. With your motivation and your encouragement, I was able to do this job. You have a heart of gold, Professor. Today I emerged again. I have new goals to achieve in the rest of my life. I would like to express my deepest gratitude to you my advisor, Professor Ağacık Zafer. I will always follow your way in my life, Sir.

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During three years of my PhD, I studied as a project assistant and I received support from The Scientific and Technological Research Council of Turkey (TÜBİTAK). I would like to thank those who have contributed.

I began my graduate studies at Hacettepe University in 1998. While I was a masters student of him at the Department of Mathematics, I can say that, I became passionate about doing of research and learning. I saw them as a way of life, and it still is the same. I would like to thank Professor Varga Kalantarov.

The other two biggest successes of myself were to be born and to steal the heart of my ex-wife. She was my biggest supporter, who encouraged me to start a PhD at METU. She also supported the first one and a half year at METU, financially. I want to thank ”my Raccoon” Duygu Ertem, with whom I shared for a long and beautiful period of my life.

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CHAPTER 1

INTRODUCTION AND PRELIMINARIES

1.1 Statement of the Problem

Consider a second order nonhomogeneous differential equation

$$x'' = \frac{1}{t^3}$$

having the general solution of the form

$$x(t) = \frac{1}{2t} + c_1 t + c_2, \quad c_1, c_2 \in \mathbb{R}.$$

If we let $t \to \infty$ we see that solution is asymptotic to a line $c_1 t + c_2$. In other words, for each real numbers a and b there is a solution $x(t)$ of the equation with the representation

$$x(t) - (at + b) = o(1), \quad t \to \infty.$$

Note that the line $at + b$ is a solution of the corresponding homogeneous differential equation $x'' = 0$. There are many articles which concerning the existence of solutions satisfying similar asymptotic representations of the second order nonlinear differential equations

$$x'' = f(t, x)$$

or

$$x'' = g(t, x, x').$$

In this thesis our aim is to study the asymptotic integration problem for the nonlinear equation

$$(p(t)x')' + q(t)x = f(t, x), \quad t \geq t_0, \tag{1.1}$$

or

$$(p(t)x')' + q(t)x = g(t, x, x'), \quad t \geq t_0, \tag{1.2}$$

in connection with the nonprincipal and principal solutions of the corresponding linear homogeneous equation

$$(p(t)x')' + q(t)x = 0 \tag{1.3}$$

where t_0 is a fixed nonnegative real number, $p \in C([t_0, \infty), (0, \infty))$, $q \in C([t_0, \infty), \mathbb{R})$, $f \in C([t_0, \infty) \times \mathbb{R}, \mathbb{R})$, and $g \in C([t_0, \infty) \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$. Later, we will give the definitions of the above mentioned principal and nonprincipal solutions of the homogeneous equation (1.3) and talk about their existence.

1.2 Literature review

To the best of our knowledge the first work concerning the asymptotic integration problem is due to Caligo [1] who considered the simple linear second-order equation

$$x'' + q(t)x = 0 \tag{1.4}$$

where $q \in C([0,\infty),\mathbb{R})$ and proved by using an integral equation approach that if

$$|q(t)| \le \frac{l}{t^{2+\rho}}, \quad l > 0, \quad \rho > 1,$$

then the solutions of (1.4) are asymptotically linear, i.e., they satisfy

$$x(t) = at + b + o(1), \quad t \to \infty \tag{1.5}$$

where a and $b \in \mathbb{R}$.

In 1942 Boas, Boas and Levinson [2] considered second order linear nonhomegeneous differential equations of the form

$$x'' + q(t)x = r(t)$$

and they showed under the conditions

$$\int_0^\infty t|q(t)|dt < \infty, \quad \int_0^\infty r(t)dt < \infty$$

that these equations have solutions satisfying the same representation as in Caligo's result. Here the condition on the function q is weaker than that in Caligo's result. So Boas et al. obtained the same result for the homogeneous equations under weaker conditions.

In 1942 Haupt [3] considered the $n-$th order linear equation

$$x^{(n)} + q_{n-1}(t)x^{(n-1)} + \cdots + q_0(t)x = h(t)$$

where h and q_m, $m = 0,1,2,\cdots n-1$ are taken from the set $C([a,\infty),\mathbb{R})$ for some given $a > 0$ and showed that the limits

$$\lim_{t\to\infty}(n-1-m)!t^{m+1-n}x^{(m)}(t), \quad m = 0,1,2,\cdots n-1$$

are exist and equal for all solutions of the equation under the conditions

$$\int_a^\infty h(t)dt < \infty, \quad \int_a^\infty t^{n-1-m}q_m(t)dt < \infty, \quad m = 0,1,2,\cdots n-1.$$

In 1947 Bellman [4] considered the $n-$th order linear equation

$$x^{(n)} + r_1(t)x^{(n-1)} + \cdots + r_n(t)x = 0$$

and proved using the Gronwall-Bellman inequality that the limits

$$\lim_{t\to\infty} x^{(n-1)}(t)$$

are exist for all solutions of the equation under the conditions

$$\int_0^\infty t^{k-1}|r_k(t)|dt < \infty \quad k = 1,2,\cdots n.$$

In 1957 Bihari [5] considered the second order nonlinear differential equation of the form

$$x'' + q(t)f(x) = 0 \tag{1.6}$$

and he proved that this equation has solution for any given real numbers a and b with the representation (1.5) under the below conditions on the functions q and f:

$$\int_0^\infty tq(t)dt < \infty, \quad |f(x)| \le tg\left(\frac{|x|}{t}\right)$$

$$\int_0^\infty \frac{1}{g(x)}dx = \infty, \quad g(x) > 0, \quad \forall x > 0.$$

In general, there are no functions q and f as in (1.6) that will correspond to the nonhomonegenous equation given by Boas et al. [2]. So the result obtained by Bihari does not contain the result obtained in [2]. But, the homogeneous equation given by Caligo in [1] can be written as (1.6) with $f(x) = x$ and in this case Bihari's conditions are weaker than that in Caligo's study, that is, Bihari showed the same result under weaker conditons.

In 1963 Trench [6] considered the differential equation of the form

$$x'' = (f(t) + g(t))x. \tag{1.7}$$

He showed that if the general solution of the equation

$$y'' = f(t)y \tag{1.8}$$

is known and if the integral

$$\int_0^\infty |g(t)|z(t)dt < \infty, \qquad z(t) = \max\left\{|y_1(t)|^2, |y_2(t)|^2\right\}$$

is convergent, where y_1 and y_2 are linearly independent solutions of (1.8), then for each $a, b \in \mathbb{R}$ the equation (1.7) has a solution $x(t)$ with the representation

$$x(t) = \alpha(t)y_1(t) + \beta(t)y_2(t)$$

where

$$\alpha(t) \to a, \ \ \beta(t) \to b \ \text{ as } \ t \to \infty.$$

In 1963 Hale and Onuchic [7] considered the equation

$$x'' + f(t, x, x') = 0, \ \ t \ge t_0 > 0 \tag{1.9}$$

and they showed that under the conditions

$$|f(t, x, y)| \le h(t)F(|x|, |y|), \quad \int_{t_0}^\infty h(t)F(\beta, \beta t^{-1})dt < \infty, \ \ \beta > 0$$

the equation (1.9) has a solution $x(t)$ with the representations

$$x(t) = a + o(1), \quad t \to \infty$$

and

$$x'(t) = o(1/t), \quad t \to \infty.$$

In 1963 Hartman and Onuchic [8] considered the same equation under the conditions

$$|f(t, at + b + u, a + v)| \le g(t),$$
$$\text{where } |u|, |v| \le \rho \text{ for some } \rho > 0 \text{ and } \int_{t_0}^{\infty} tg(t)dt < \infty$$

and showed that (1.9) these equations has a solution satisfying the representations

$$x(t) = at + b + o(1), \quad x'(t) = a + o(1), \quad t \to \infty.$$

In 1964 Waltman [9] consdired the differential equation

$$x'' + q(t)x^{2m+1} = 0, \quad m \ge 0 \tag{1.10}$$

under the condition

$$\int^{\infty} |q(t)|t^{2m+1}dt < \infty,$$

and showed that the equation (1.10) has a solution satisfying

$$x(t) = at + b + o(1), \quad t \to \infty.$$

In 1967 Cohen [10] considered second the order nonlinear differential equation

$$x'' + f(t, x) = 0. \tag{1.11}$$

He showed that if f is differentiable with respect to x and if

$$f_x(t, x) > 0, \quad (t, x) \in D = [0, \infty) \times \mathbb{R},$$
$$|f(t, x(t))| \le f_x(t, 0)|x(t)|, \quad \text{on } D$$

with the integral satisfying

$$\int^{\infty} t f_x(t, 0)dt < \infty,$$

then for each b and nonzero a there is a solution $x(t)$ of the equation (1.11) satisfying

$$x(t) = at + b, \quad t \to \infty.$$

Later, several others worked on the same problem and obtained quite important results: Brauer, Wong [11], Coffman, Wong [12], Hartman [13], Kusano, Trench [14, 15], Kusano, Naito, Usami [16].

Recently, Mustafa and Rogovchenko [17] considered the equation

$$x'' + f(t, x, x') = 0, \tag{1.12}$$

for $t \ge t_0 \ge 1$. They proved that, if

$$|f(t, x, y)| \le h(t)\left[p_1\left(\frac{|x|}{t}\right) + p_2(|y|)\right]$$

where $h \in C((0,\infty),(0,\infty))$, $p_1, p_2 \in C((0,\infty),(0,\infty))$, nondecreasing functions satisfying

$$\int_{t_0}^{\infty} sh(s)ds < \infty$$

and

$$\int_{t_0}^{\infty} \frac{1}{p_1(s) + p_2(s)} ds = \infty,$$

then for any given number a there is a solution of equation (1.12) satisfying

$$x(t) = at + o(t), \quad t \to \infty.$$

In 2003 Yin [18] considered the same equation for $t \geq 0$. He showed that the equation has a monotone positive solution $x(t)$ defined on the interval $[0,\infty)$ satisfying

$$\lim_{t\to\infty} \frac{x(t)}{t} = c$$

under the conditions

$$\int_{0}^{\infty} F(t, 2ct, 2c)dt < c, \quad c > 0$$

and

$$|f(t, x, x')| \leq F(t, |x|, |x'|)$$

where F is nondecreasing with respect to its second and third arguments.

Lipovan [19] obtained the asymptotic representation (1.5) for solutions of more general second-order nonlinear equations

$$x'' = f(t, x), \quad t \geq 1 \tag{1.13}$$

when the function f satisfies

$$|f(t, x)| \leq h_1(t) g\left(\frac{|x|}{t}\right) + h_2(t) \tag{1.14}$$

for some continuous functions h_1 and h_2 such that

$$\int_{1}^{\infty} sh_i(s)ds < \infty, \quad i = 1, 2. \tag{1.15}$$

Mustafa and Rogovchenko [20] improved the result of Lipovan by showing that (1.13) has a solution $x(t)$ such that

$$x(t) = at + o(t^{1-\mu}), \quad t \to \infty, \quad \mu \in [0, 1) \tag{1.16}$$

provided that f satisfies (1.14) with

$$\int_{t_0}^{\infty} t^{\mu} h_i(t)dt < \infty, \quad i = 1, 2.$$

In 2006, Mustafa and Rogovchenko [21] considered a ”singular boundary value problem of the form”

$$x'' + f(t, x) = 0, \quad t \geq t_0 \geq 1$$

$$x(t_0) = x_0, \quad x_0 \in \mathbb{R},$$

$$x(t) = at + o(t^{1-\mu}), \quad t \to \infty$$

where $a > 0$ and $\mu \in [0, 1)$. They showed that the problem has a unique solution, under the conditions

$$\alpha(t) \leq \int_t^\infty f(s, x(s))ds \leq \beta(t), \quad x \in X$$

and

$$|f(t, x_1(t)) - f(t, x_2(t))| \leq \frac{k(t)}{t}|x_1(t) - x_2(t)|, \quad x_1, x_2 \in X$$

where

$$\int_{t_0}^\infty k(t)dt \leq 1 - \mu$$

and X is defined as

$$X = \{C([t_0, \infty); \mathbb{R})| \quad \alpha(t) \leq x'(t) - a \leq \beta(t), \qquad t \geq t_0, \quad x(t_0) = x_0\}$$

where

$$\alpha(t) \leq \beta(t), \quad \beta(t) = o(t^{-\mu}), \quad t \to \infty.$$

In functional differential equations Grammatikopoulos [22] and Philos [23], in the application to partial differential equations Zhao [24] and Constantin [25] gave important contributions to the literature with regards to using asymptotic integration.

The proofs in the recent papers based on obtaining an equivalent integral equation and using fixed point theorems.

For an excellent survey of almost all results up to 2007, we refer the reader in particular to a recent paper by Agarwal et al. [26]. Further results can be found in the monographs by Bellman [27], Coppel [28], Brauer [29], Eastham [30], Agarwal et al. [31], Kiguradze and Chanturia [32].

1.3 Factorizations

Consider the equation (1.3). Let p be a strictly positive continuous function and q be a continuous function on some interval $I \subset \mathbb{R}$. Define $\mathbb{D}$ as

$$\mathbb{D} := \{x : x \text{ and } px' \text{ continuously differentiable on I}\}.$$

$\mathbb{D}$ is a linear space with usual addition and multiplication. Define a second order formally self-adjoint linear operator

$$Lx(t) = (p(t)x'(t))' + q(t)x(t), \quad t \in I$$

on the vector space $\mathbb{D}$, cf. [33].

Theorem 1.3.1 (Polya factorization) [33] *Assume that $Lx = 0$ has a positive solution u on some interval $J \subset I$. Then L can be written as*

$$Lx(t) = \frac{1}{u(t)}\left[p(t)u^2(t)\left(\frac{x(t)}{u(t)}\right)'\right]', \quad \forall x \in \mathbb{D}, \quad \forall t \in J.$$

Theorem 1.3.2 (Trench factorization) [33] *Assume $Lx = 0$ has a positive solution on $[a, b) \subset I$, where $-\infty < a < b \leq \infty$. Then there is a positive solution v of $Lx = 0$ such that*

$$Lx(t) = \frac{1}{v(t)}\left[p(t)v^2(t)\left(\frac{x(t)}{v(t)}\right)'\right]', \quad \forall x \in \mathbb{D}, \quad \forall t \in J$$

and

$$\int_a^b \frac{1}{p(t)v^2(t)}dt < \infty.$$

These factorizations are especially useful when converting (1.1) and (1.2) into an integral equation, which we will be doing in proving our results.

It is well-known that (see [33, 34]), if the second order linear equation (1.3) has an eventually positive solution or equivalently (1.3) is nonoscillatory at infinity, then there exist two special linearly independent solutions u and v of (1.3) called, respectively, the principal and nonprincipal solutions. The principal solution u is unique up to a constant multiple, and any solution v that is linearly independent of u is a nonprincipal solution. The solutions u and v have the following useful properties:

$$\lim_{t\to\infty} \frac{u(t)}{v(t)} = 0; \tag{1.17}$$

$$\int_{t_*}^{\infty} \frac{1}{p(t)u^2(t)}dt = \infty, \quad \int_{t_*}^{\infty} \frac{1}{p(t)v^2(t)}dt < \infty, \quad t_* \geq 0; \tag{1.18}$$

$$\frac{p(t)v'(t)}{v(t)} > \frac{p(t)u'(t)}{u(t)}, \quad t \geq t_*, \tag{1.19}$$

where $t_* \geq 0$ is sufficiently large.

Factorizations and the existence of principal and nonprincipal solutions for impulsive differential equations are obtained by Özbekler and Zafer [35]. Factorizations and the existence of those solutions are well-known for difference equations and time scale calculus, cf. [36, 37].

We should note that $x(t) = at + b$ appearing (1.5) is a solution of the unperturbed equation $x'' = 0$, and this solution can be written in the form $x(t) = av(t) + bu(t)$, where $v(t) = t$ and $u(t) = 1$ are nonprincipal and principal solutions, respectively of $x'' = 0$. Indeed, this observation has been our motivation to study the asymptotic integration problem for (1.1) and (1.2) in connection with the nonprincipal and principal solutions of the corresponding linear equation (1.3).

1.4 Basic Definitions and Theorems

Let us denote by $\mathbb{K}$ either of the fields $\mathbb{R}$ or $\mathbb{C}$.

Linear spaces [38] A *linear space* V over $\mathbb{K}$ is a nonempty set V, in which an addition $+ : V \times V \to V$ and scalar multiplication $\cdot : \mathbb{K} \times V \to V$ with the following properties are defined:

1. $(V, +)$ is an Abelian group with the zero element 0.

2. $\lambda(x + y) = \lambda x + \lambda y$ for all $\lambda \in \mathbb{K}$ and $x, y \in V$.

3. $(\lambda + \mu)x = \lambda x + \mu x$ for all $\lambda, \mu \in \mathbb{K}, x \in V$.

4. $(\lambda\mu)x = \lambda(\mu x)$ for all $\lambda, \mu \in \mathbb{K}, x \in V$.

5. $1x = x$ for all $x \in V$.

The elements of V are called *vectors*.

Linear subspaces [38] A nonempty subset W of a $\mathbb{K}-$vector space V is called a *linear subspace* of V if $\lambda x + \mu y \in W$ for all $x,\ y \in W$ and $\lambda,\ \mu \in \mathbb{K}$.

Convex set [38] A subset M of a $\mathbb{K}-$vector space V is said to be *convex* if $\lambda\, x + (1 - \lambda)\, y \in M$ for all $x,\ y \in M$ and $\lambda \in [0, 1]$.

Metric spaces [38] Let X be a set. A *metric* on X is a function $d : X \times X \to [0, \infty)$ with the following properties:

(M1) $d(x, y) = d(y, x)$ for all $x, y \in X$ (symmetry).

(M2) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$ (triangle inequality).

(M3) $d(x, y) = 0$ if and only if $x = y$.

A *metric space* (X, d) is a nonempty set X on which a metric d is given.

Cauchy sequences [38] Suppose d is a metric on a set X. A sequence $\{x_n\}$ in X is a *Cauchy sequence* if to every $\epsilon > 0$ there corresponds an integer N such that $d(x_m, x_n) < \epsilon$ whenever $m > N$ and $n > N$.

Convergent sequences [38] A sequence $\{x_n\}$ in a metric space $X = (X, d)$ said to *converge* or to be *convergent* if there is an $x \in X$ such that

$$\lim_{n\to\infty} d(x_n, x) = 0.$$

Complete metric spaces [38] A metric space X is said to be *complete* if every Cauchy sequence in X is convergent.

Continuous map [38] Let X and Y be metric spaces. A map $f : X \to Y$ is said to be *continuous at a point* $x_0 \in X$ if for every $\epsilon > 0$, there is a $\delta > 0$, such that $d(f(x), f(x_0)) < \epsilon$ for all $x \in X$ satisfying $d(x, x_0) < \delta$. A map f said to be *continuous* if it is continuous at every point of X.

Uniformly continuous map [38] Let X and Y be metric spaces. A map $f : X \to Y$ is said to be *uniformly continuous* if, for each given $\epsilon > 0$, there exists a $\delta > 0$, such that $d(f(x), f(y)) < \epsilon$ for all $x, y \in X$ satisfying $d(x, y) < \delta$.

Bounded set [38] If d is a metric on a set X, a set $M \subset X$ is said to be $d-$bounded if there is a number $M < \infty$ such that $d(x, y) \leq L$ for all x and y in M.

Relatively compact set [38] A subset M of a metric space X is said to be *relatively compact* if its closure $\overline{M}$ is compact.

Precompact set [38] Let X be a metric space and $M \neq \emptyset$ be a subset of X. Then M is *precompact* (in the induced metric) if and only if for every $\epsilon > 0$ there are finitely many $x_1, x_2, ... \in X$, such that $\cup_{j=1}^{n} U_\epsilon(x_j)$.

Theorem 1.4.1 [38] *For a subset M of a complete metric space X the following statements are equivalent:*

1. *M is relatively compact.*
2. *M is precompact.*
3. *Every sequence in M contains a subsequence which is convergent in X.*

Normed Spaces [38] Let V be a $\mathbb{K}-$vector space. A *norm* on V is a function $\|\cdot\| : V \to [0, \infty)$ with the following properties:

(N1) $\|\lambda x\| = |\lambda| \, \|x\|$ for all $\lambda \in \mathbb{K}$, $x \in V$.

(N2) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in V$ (triangle inequality).

(N3) $\|x\| = 0$ holds only if $x = 0$.

A *normed space* $(V, \|\cdot\|)$ is a $\mathbb{K}-$vector space V, on which a norm is defined.

If V is a normed space, then it follows from the properties (N1)-(N3) of the norm that a metric d is defined on V by

$$d(x, y) := \|x - y\|, \quad x, y \in V.$$

This metric is called the *canonical metric* of the normed space V.

Banach Spaces [38] A *Banach space* is a normed space which is complete under its canonical metric.

Compact and completely continuous operators An operator F on a normed space is *compact* if it maps every bounded set into a relatively compact set. A continuous compact operator is called *completely continuous*, cf. [39, 40].

We also need to use Lebesgue dominated convergence theorem in the proofs.

Theorem 1.4.2 [41] *If $\{f_n\}$ is a sequence of measurable functions on a measurable set E such that $f_n(t) \to f(t)$ as $n \to \infty$ a.e. on E and $|f_n(t)| \leq g(t)$ a.e. on E, where g is an itegrable function on E, then*

$$\int_E f d\mu = \lim_{n\to\infty} \int_E f_n d\mu.$$

1.4.1 Fixed Point Theorems

We used the following Banach fixed point theorem and Schauder's fixed point theorem in the proofs.

Theorem 1.4.3 [42] *Let (X, d) be a complete metric space and let $F : X \to X$ be a contraction with Lipschitzian constant L. Then F has a unique fixed point $u \in X$. Furthermore, for any $x \in X$ we have*

$$\lim_{n\to\infty} F^n(x) = u$$

with

$$d(F^n(x), u) \leq \frac{L^n}{1 - L} d(x, F(x)).$$

Theorem 1.4.4 [43] *Let C be a closed bounded convex subset of a normed linear space E. Then every compact continuous map $F : C \to C$ has at least one fixed point.*

1.4.2 Compactness Criteria

We need the following compactness criterion for L_p spaces due to M. Riesz and J. D. Tamarkin (see [44, 45, 46, 47]).

Lemma 1.4.5 *Let $\Omega \subset \mathbb{R}^n$. A set $M \subset L_p(\Omega)$ is compact if*

(a) there exists a number $B > 0$ such that $\|f\|_{L_p(\Omega)} \leq B$ for all $f \in M$;

(b) $\|\tau_h f - f\|_{L_p(\Omega)} \to 0$ as $h \to 0$, where

$$(\tau_h f)(x) := f(x_1 + h, x_2 + h, ..., x_n + h), \quad x \in \Omega.$$

Let

$$V := \left\{ h \in C\left([t_0, \infty), \mathbb{R}\right) \mid \lim_{t\to\infty} h(t) \text{ exists} \right\}.$$

It is known that V is a Banach space with usual supremum norm (see [48]). We need also the next compactness criteria for the subsets of V due to C. Avramescu:

Lemma 1.4.6 [49] *Assume that a subset $N \subset V$ satisfies the following conditions:*

(1) N is uniformly bounded: There exists an $L > 0$ such that $|h(t)| \leq L$ for all $t \geq t_0$ and for all $h \in N$.

(2) N is equicontinuous: For all $\epsilon > 0$ there exists a $\delta_\epsilon > 0$ such that

$$|t_1 - t_2| < \delta_\epsilon \quad \Rightarrow \quad |h(t_1) - h(t_2)| < \epsilon$$

for all $t_1,\ t_2 \geq t_0$ and for all $h \in N$.

(3) N is equiconvergent: For all $\epsilon > 0$ there exists a $t_\epsilon > t_0$ such that $|h(t) - h(s)| < \epsilon$ for all $t,\ s \geq t_\epsilon$ and for all $h \in N$.

Then, N is relatively compact in V. Coversely, if a set $N \subset V$ is relatively compact in V, then it satisfies (1)–(3).

CHAPTER 2

NONLINEAR DIFFERENTIAL EQUATIONS

2.1 Asymptotic Integration

2.1.1 Introduction

In this chapter we study the asymptotic integration problem for a general class of second-order differential equations of the form

$$(p(t)x')' + q(t)x = f(t, x), \quad t \geq t_0, \tag{2.1}$$

where t_0 is a fixed nonnegative real number, $p \in C([t_0, \infty), (0, \infty))$, $q \in C([t_0, \infty), \mathbb{R})$ and $f \in C([t_0, \infty) \times \mathbb{R}, \mathbb{R})$.

For clarity and comparison purposes we also restate the results as corollaries for the special case

$$x'' = f(t, x), \quad t \geq 1. \tag{2.2}$$

It is clear that if u is a given principal solution of (1.3) which is positive for $t \geq t_1$ for some $t_1 > t_*$, then one can take the set

$$\left\{u(t),\ v(t) = u(t) \int_{t_1}^{t} \frac{1}{p(s)u^2(s)}\, ds\right\} \tag{2.3}$$

as a fundamental set of solutions of (1.3), where v is a nonprincipal solution. Furthermore, if v is a given nonprincipal solution of (1.3) which is positive for $t \geq t_1$ for some $t_1 > t_*$, then a close look at the proof of Polya factorization reveals that

$$\left\{u(t) = v(t) \int_{t}^{\infty} \frac{1}{p(s)v^2(s)}\, ds,\ v(t)\right\} \tag{2.4}$$

becomes a fundamental set of solutions of (1.3), where u is principal solution.

2.1.2 Main Results

In the sequel we make the standing hypothesis that the unperturbed equation (1.3) has an eventually positive solution and denote by $\{u, v\}$ a set of principal and nonprincipal solutions introduced in (2.3) and (2.4). We will state and prove four theorems concerning the asymptotic integration of solutions. The results in the special case (2.2) will be stated as corollaries.

Theorem 2.1.1 *Let u and v be principal and nonprincipal solutions of (1.3) given by (2.3) and $g \in C([0,\infty),[0,\infty))$, h_1, $h_2 \in C([t_1,\infty),[0,\infty))$. Suppose that*

$$|f(t,x)| \leq h_1(t) g\left(\frac{|x|}{v(t)}\right) + h_2(t), \quad t \geq T \tag{2.5}$$

for some $T > t_1$, and

$$\int_T^\infty v(s)h_i(s)ds < \infty, \quad i = 1,2. \tag{2.6}$$

Then for any given $a, b \in \mathbb{R}$ there is a solution $x(t)$ of equation (2.1) satisfying

$$x(t) = av(t) + bu(t) + o(u(t)), \quad t \to \infty. \tag{2.7}$$

Proof. Let a, $b \in \mathbb{R}$ be given. Define

$$y(t) := x(t) - av(t).$$

Then from (2.1) we have

$$(p(t)y')' + q(t)y = f(t, y + av(t)). \tag{2.8}$$

It suffices to show that (2.8) has a solution $y(t)$ so that

$$y(t) = bu(t) + o(u(t)), \quad t \to \infty.$$

Define

$$M := \max_{0 \leq \eta \leq |a|+|b|+1} |g(\eta)|. \tag{2.9}$$

In view of (1.17) and (2.6), we may choose $T_1 > T$ large enough so that $u(t) \leq v(t)$,

$$\int_{T_1}^\infty v(s)h_1(s)ds < \frac{1}{2M}, \quad \int_{T_1}^\infty v(s)h_2(s)ds < \frac{1}{2}. \tag{2.10}$$

Consider the linear space

$$Y = \left\{ y \in C([T_1,\infty),\mathbb{R}) \middle| \quad \frac{|y(t)|}{v(t)} \leq M_y, \quad t \geq T_1 \right\}.$$

It is easy to check that Y is a Banach space with norm

$$||y|| = \sup_{T_1 \leq t < \infty} \frac{|y(t)|}{v(t)}.$$

Let K be the set given by

$$K := \{y \in Y| \quad ||y - bu|| \leq 1\}.$$

It is easy to show that K is a closed, bounded, convex, and nonempty subset of Y. Define the operator $F : K \to Y$ by

$$(Fy)(t) = bu(t) + u(t) \int_t^\infty f(\tau, y(\tau) + av(\tau))u(\tau) \int_t^\tau \frac{1}{p(s)u^2(s)} ds d\tau, \quad t \geq T_1.$$

It is easy to check that each fixed point of F is a solution of (2.8). We will use Schauder's fixed point theorem to show that F has a fixed point.

F maps K into K: For each $y \in K$ we have

$$\begin{aligned}
\frac{|(Fy)(t) - bu(t)|}{v(t)} &\leq \frac{|(Fy)(t) - bu(t)|}{u(t)} \\
&\leq \int_t^\infty |f(\tau, y(\tau) + av(\tau))| u(\tau) \int_t^\tau \frac{1}{p(s)u^2(s)} ds d\tau \\
&\leq M \int_t^\infty v(\tau) h_1(\tau) d\tau + \int_t^\infty v(\tau) h_2(\tau) d\tau \\
&\leq M \int_{T_1}^\infty v(\tau) h_1(\tau) d\tau + \int_{T_1}^\infty v(\tau) h_2(\tau) d\tau \\
&\leq 1,
\end{aligned}$$

where (2.3), (2.5), and (2.10) are used. Taking the supremum we see that $FK \subset K$.

F is continuous: Let $\{y_n\}_{n=1}^\infty \subset K$ be an arbitrary sequence converging to $y \in K$. For each $n \in \mathbb{N}$, in view of (2.3) and (2.5) we have that

$$\begin{aligned}
\frac{|(Fy_n)(t) - (Fy)(t)|}{v(t)} &\leq \frac{|(Fy_n)(t) - (Fy)(t)|}{u(t)} \\
&\leq \int_t^\infty g_n(\tau) u(\tau) \int_t^\tau \frac{1}{p(s)u^2(s)} ds\, d\tau \\
&\leq \int_t^\infty g_n(\tau) v(\tau) d\tau \\
&\leq 2M \int_{T_1}^\infty h_1(\tau) v(\tau) d\tau + 2 \int_{T_1}^\infty h_2(\tau) v(\tau) d\tau,
\end{aligned}$$

where

$$g_n(\tau) = |f(\tau, y_n(\tau) + av(\tau)) - f(\tau, y(\tau) + av(\tau))|.$$

By applying Lebesgue's dominated convergence theorem we obtain from (2.11) that $Fy_n \to Fy$ as $n \to \infty$.

F is completely continuous: Let $\{y_n\}_{n=1}^\infty \subset K$ be an arbitrary sequence. We need to show that there exist $w \in K$ and a subsequence $\{y_{n_k}\}_{k=1}^\infty$ so that $Fy_{n_k} \to w$ as $k \to \infty$. We will use Lemma 1.4.5 to show the existence of such a function w.

Define

$$f_n(\tau) := f(\tau, y_n(\tau) + av(\tau)) u(\tau) \int_t^\tau \frac{1}{p(s)u^2(s)} ds.$$

Since $\{Fy_n\}_{n=1}^\infty \subset K$, it follows that

$$\|f_n\|_{L_1([T_1,\infty))} \leq 1, \quad n \geq 1,$$

i.e., the condition (a) of Lemma 1.4.5 holds. To see that (b) is also satisfied, we first define

$$(\tau_h f)(\tau) = f(\tau + h).$$

Using (2.5) we estimate that

$$\begin{aligned}
\int_{T_1}^\infty |(\tau_h f_n)(\tau) - f_n(\tau)| d\tau &\leq \int_{T_1}^\infty |f_n(\tau + h)| d\tau + \int_{T_1}^\infty |f_n(\tau)| d\tau \\
&\leq \int_{T_1+h}^\infty |f_n(\tau)| d\tau + \int_{T_1}^\infty |f_n(\tau)| d\tau \\
&\leq \int_{T_1}^\infty 2|f_n(\tau)| d\tau \\
&\leq \int_{T_1}^\infty 2(Mh_1(\tau) + h_2(\tau)) v(\tau) d\tau
\end{aligned}$$

By Lebesgue's dominated convergence theorem, we obtain from the above inequality that

$$\|\tau_h f_n - f_n\|_{L_1([T_1,\infty))} \to 0 \quad \text{as } h \to 0.$$

Now an application of Lemma 1.4.5 shows that there exists a subsequence $\{f_{n_k}\}$ so that

$$\left\| f_{n_k} - z \right\|_{L_1([T_1,\infty))} \to 0 \quad \text{as } k \to \infty$$

for some $z \in L_1([T_1, \infty))$.

If we define

$$w(t) := bu(t) + u(t)\int_t^\infty z(\tau)d\tau,$$

then we see that

$$\frac{|(Fy_{n_k})(t) - w(t)|}{v(t)} \leq \int_{T_1}^\infty |f_{n_k}(\tau) - z(\tau)|d\tau.$$

Taking the supremum and applying Lebesgue's dominated convergence theorem, we get the complete continuity of F.

It follows from the Schauder's fixed point theorem that the operator F has a fixed point $y \in K$, that is,

$$y(t) = bu(t) + u(t)\int_t^\infty f(\tau, y(\tau) + av(\tau))u(\tau)\int_t^\tau \frac{1}{p(s)u^2(s)}dsd\tau.$$

To show the asymptotic representation (2.7), we start with the following estimate:

$$\begin{aligned}
|y(t) - bu(t)| &\leq u(t)\int_t^\infty |f(\tau, y(\tau) + av(\tau))|u(\tau)\int_t^\tau \frac{1}{p(s)u^2(s)}dsd\tau \\
&\leq u(t)\int_t^\infty |f(\tau, y(\tau) + av(\tau))|u(\tau)\int_{t_1}^\tau \frac{1}{p(s)u^2(s)}dsd\tau \\
&\leq u(t)\int_t^\infty v(\tau)(Mh_1(\tau) + h_2(\tau))d\tau,
\end{aligned}$$

where (2.3) and (2.5) are used. Dividing by $u(t)$ and taking limit as $t \to \infty$ we see that

$$y(t) - bu(t) = o(u(t)), \quad t \to \infty. \tag{2.11}$$

In view of

$$x(t) = y(t) + av(t) \tag{2.12}$$

and the fact that v is a solution of (1.3), we have

$$Lx = Ly + aLv = f(t, y + a\,v(t)) = f(t, x),$$

i.e, x is a solution of (2.1). Using (2.11) and (2.12) we obtain the asymptotic representation (2.7).

■

Corollary 2.1.2 *If (1.14) and (1.15) hold, then for any given $a, b \in \mathbb{R}$ there is a solution $x(t)$ of (2.2) satisfying*

$$x(t) = at + b + o(1), \quad t \to \infty.$$

Theorem 2.1.3 *Let u and v be principal and nonprincipal solutions of (1.3) given by (2.4). Suppose that (2.5) holds along with the following being true;*

$$\int_T^\infty u(s)h_i(s)ds < \infty, \quad i = 1, 2, \tag{2.13}$$

$$\limsup_{t\to\infty} \frac{u(t)}{v(t)} \int_T^t v(s)h_i(s)ds = 0, \quad i = 1, 2. \tag{2.14}$$

Then for any given $a, b \in \mathbb{R}$ there is a solution $x(t)$ of (2.1) satisfying

$$x(t) = av(t) + bu(t) + o(v(t)), \quad t \to \infty. \tag{2.15}$$

Proof. Let $a, b \in \mathbb{R}$ be given. Define

$$y(t) := x(t) - bu(t). \tag{2.16}$$

Then equation (2.1) becomes

$$(p(t)y')' + q(t)y = f(t, y + bu(t)). \tag{2.17}$$

Now we need to show that equation (2.17) has a solution $y(t)$ satisfying

$$y(t) = av(t) + o(v(t)), \quad t \to \infty.$$

Let M be as in (2.9). Without loss of generality we can choose T_1 so large that

$$\int_{T_1}^\infty u(s)h_1(s)ds < \frac{1}{2M}, \quad \int_{T_1}^\infty u(s)h_2(s)ds < \frac{1}{2}, \tag{2.18}$$

hold with $v(t) \geq u(t) > 0$, $t \geq T_1$. Consider the linear space

$$Y = \left\{ y \in C([T_1, \infty), \mathbb{R}) \middle| \quad \frac{|y(t)|}{v(t)} \leq M_y, \quad t \geq T_1 \right\}.$$

Y is a Banach space with the norm defined by

$$\|y\| = \sup_{T_1 \leq t < \infty} \frac{|y(t)|}{v(t)}.$$

Let K be the set given by

$$K := \{y \in Y| \quad \|y - av\| \leq 1\}.$$

It is easy to show that K is a closed, bounded, convex, and nonempty subset of Y. Define an operator $F : K \to Y$ by

$$(Fy)(t) = av(t) - u(t) \int_{T_1}^t f(\tau, y(\tau) + bu(\tau))v(\tau)d\tau - v(t) \int_t^\infty f(\tau, y(\tau) + bu(\tau))u(\tau)d\tau, \quad t \geq T_1.$$

It is easy to check that each fixed point of F is a solution of (2.17). To show that F has a fixed point, again we will use the Schauder's fixed point theorem.

F maps K into K: For each $y \in K$ we have

$$\begin{aligned}
\frac{|(Fy)(t) - av(t)|}{v(t)} &= \Big| -\int_t^\infty \frac{1}{p(s)v^2(s)} ds \int_{T_1}^t f(\tau, y(\tau) + bu(\tau))v(\tau)d\tau \\
&\quad - \int_t^\infty f(\tau, y(\tau) + bu(\tau))u(\tau)d\tau \Big| \\
&\le \int_{T_1}^t |f(\tau, y(\tau) + bu(\tau))| v(\tau) \int_t^\infty \frac{1}{p(s)v^2(s)} ds d\tau \\
&\quad + \int_t^\infty |f(\tau, y(\tau) + bu(\tau))| u(\tau)d\tau \\
&\le \int_{T_1}^\infty |f(\tau, y(\tau) + bu(\tau))| v(\tau) \int_\tau^\infty \frac{1}{p(s)v^2(s)} ds d\tau \\
&\le M \int_{T_1}^\infty u(\tau)h_1(\tau)d\tau + \int_{T_1}^\infty u(\tau)h_2(\tau)d\tau \\
&\le 1
\end{aligned}$$

where (2.4), (2.5), and (2.18) are used. Taking the supremum on left-hand side we get $FK \subset K$.

F is continuous: Now let $\{y_n\}_{n=1}^\infty \subset K$ be an arbitrary sequence converging to $y \in K$. In order to see that F is continuous we need to show that $Fy_n \to Fy, \quad n \to \infty$. For each $n \in \mathbb{N}$, in view of (2.4) and (2.5) we have that

$$\begin{aligned}
\frac{|(Fy_n)(t) - (Fy)(t)|}{v(t)} &\le \int_t^\infty \frac{1}{p(s)v^2(s)} ds \int_{T_1}^t g_n(\tau)v(\tau)d\tau + \int_t^\infty g_n(\tau)u(\tau)d\tau \\
&\le \int_{T_1}^\infty g_n(\tau)u(\tau)d\tau \\
&\le 2M \int_{T_1}^\infty h_1(\tau)u(\tau)d\tau + 2\int_{T_1}^\infty h_2(\tau)u(\tau)d\tau.
\end{aligned}$$

where

$$g_n(\tau) = |f(\tau, y_n(\tau) + bu(\tau)) - f(\tau, y(\tau) + bu(\tau))|.$$

Applying Lebesgue dominated covergence theorem we get the continuity of F.

F is completely continuous: Let $\{y_n\}_{n=1}^\infty \subset K$ be an arbitrary sequence. Consider the corresponding sequence

$$\begin{aligned}
(Fy_n)(t) &= av(t) - v(t) \int_{T_1}^t f(\tau, y_n(\tau) + bu(\tau))v(\tau) \int_t^\infty \frac{1}{p(s)v^2(s)} ds d\tau \\
&\quad - v(t) \int_t^\infty f(\tau, y_n(\tau) + bu(\tau))u(\tau)d\tau.
\end{aligned}$$

Define a sequence $\{f_n^1\}_{n=1}^\infty$ by

$$f_n^1(\tau) := f(\tau, y_n(\tau) + bu(\tau))v(\tau) \int_t^\infty \frac{1}{p(s)v^2(s)} ds.$$

Since $\{Fy_n\}_{n=1}^\infty \subset K$ we have

$$\left\| f_n^1 \right\|_{L_1([T_1,\infty))} \le 1, \quad \forall n.$$

By the following estimate

$$\begin{aligned}\int_{T_1}^{\infty} |f_n^1(\tau+h) - f_n^1(\tau)|d\tau &\leq \int_{T_1}^{\infty} |f_n^1(\tau+h)|d\tau + \int_{T_1}^{\infty} |f_n^1(\tau)|d\tau \\ &\leq \int_{T_1+h}^{\infty} |f_n^1(\tau)|d\tau + \int_{T_1}^{\infty} |f_n^1(\tau)|d\tau \\ &\leq \int_{T_1}^{\infty} 2|f_n^1(\tau)|d\tau \\ &\leq 2M\int_{T_1}^{\infty} h_1(\tau)u(\tau)d\tau + 2\int_{T_1}^{\infty} h_2(\tau)u(\tau)d\tau\end{aligned}$$

and using Lebesgue dominated convergence theorem we get

$$\left\|\tau_h f_n^1 - f_n^1\right\|_{L_1([T_1,\infty))} \to 0, \quad h \to 0,$$

where $(\tau_h f)(x) = f(x+h)$. Thus according to M. Riesz Theorem there exists a subsequence $\{f_{n_k}^1\}_{n=1}^{\infty}$ so that

$$\left\|f_{n_k}^1 - w^1\right\|_{L_1([T_1,\infty))} \to 0, \quad k \to \infty$$

for some $w^1 \in L_1([T_1,\infty))$.

Consider now the subsequence $\{y_{n_k}\}_{k=1}^{\infty}$ and define a sequnece $\{f_{n_k}^2\}_{k=1}^{\infty}$ by

$$f_{n_k}^2(\tau) := f(\tau, y_{n_k}(\tau) + bu(\tau))v(\tau)\int_{\tau}^{\infty} \frac{1}{p(s)v^2(s)}ds.$$

By the similar estimates as above there exists a subsequence $\{f_{n_{k_l}}^2\}_{l=1}^{\infty}$ so that

$$\left\|f_{n_{k_l}}^2 - w^2\right\|_{L_1([T_1,\infty))} \to 0, \quad k \to \infty$$

for some $w^2 \in L_1([T_1,\infty))$.

Now define

$$z(t) := av(t) - v(t)\int_{T_1}^{t} w^1(\tau)d\tau - v(t)\int_{t}^{\infty} w^2(\tau)d\tau.$$

Then we have

$$\frac{|(Fy_{n_{k_l}})(t) - z(t)|}{v(t)} \leq \int_{T_1}^{\infty} |f_{n_{k_l}}^1(\tau) - w^1(\tau)|d\tau + \int_{T_1}^{\infty} |f_{n_{k_l}}^2(\tau) - w^2(\tau)|d\tau.$$

Taking the supremum on the left handside and applying Lebesgue dominated convergence theorem we get the complete continuity of F.

Now applying the Schauder fixed point theorem we can see that F has a fixed point $y \in K$, that is, there exists a $y \in K$ so that

$$y(t) = av(t) - u(t)\int_{T_1}^{t} f(\tau, y(\tau) + bu(\tau))v(\tau)d\tau - v(t)\int_{t}^{\infty} f(\tau, y(\tau) + bu(\tau))u(\tau)d\tau, \quad t \geq T_1.$$

Finally, we show that (2.15) holds. We start with the following estimate:

$$\begin{aligned}|y(t) - av(t)| &\leq u(t)\int_{T_1}^{t} |f(\tau, y(\tau) + bu(\tau))|v(\tau)d\tau + v(t)\int_{t}^{\infty} |f(\tau, y(\tau) + bu(\tau))|u(\tau)d\tau \\ &\leq u(t)\int_{T_1}^{t}(Mh_1(\tau) + h_2(\tau))v(\tau)d\tau + v(t)\int_{t}^{\infty}(Mh_1(\tau) + h_2(\tau))u(\tau)d\tau,\end{aligned}$$

where (2.4) and (2.5) are used. Dividing by $v(t)$ and using (2.14) it is easy to see that

$$y(t) - av(t) = o(v(t)), \quad t \to \infty. \tag{2.19}$$

In view of

$$x(t) = y(t) + bu(t) \tag{2.20}$$

and the fact that u is a solution of (1.3), we have

$$Lx = Ly + bLu = f(t, y + b\,u) = f(t, x),$$

i.e, x is a solution of (2.1). Using (2.19) and (2.20) we obtain the asymptotic representation (2.15).

■

Remark 2.1.4 *If $v'(t) \neq 0$, $t \geq T$, then the condition (2.14) can be replaced with*

$$\limsup_{t\to\infty} \frac{u'(t)}{v'(t)} \int_T^t v(s)h_i(s)ds = 0, \quad i = 1, 2. \tag{2.21}$$

Corollary 2.1.5 *Let (1.14) hold. If*

$$\int_1^\infty h_i(s)ds < \infty, \quad i = 1, 2$$

then for any given $a, b \in \mathbb{R}$ there is a solution $x(t)$ of (2.2) satisfying

$$x(t) = at + b + o(t), \quad t \to \infty.$$

Theorem 2.1.6 *Let u and v be principal and nonprincipal solutions of (1.3) given by (2.3). Suppose that (2.5) holds and*

$$|f(t, x_1) - f(t, x_2)| \leq \frac{k(t)}{v(t)}|x_1 - x_2|, \quad t \geq T \tag{2.22}$$

where $k \in C([t_1, \infty), [0, \infty))$. Suppose further that

$$\int_T^\infty u(s)k(s)ds < \infty \tag{2.23}$$

and

$$\frac{1}{p(t)u^2(t)} \int_t^\infty u(s)h_i(s)ds \leq \beta(t), \quad t \geq T, \quad i = 1, 2, \tag{2.24}$$

where $\beta \in C([t_1, \infty), [0, \infty))$ and

$$\int_T^t \beta(s)ds = o((v(t))^\mu), \quad t \to \infty, \quad \mu \in (0, 1). \tag{2.25}$$

Then for any given $a, b \in \mathbb{R}$ there is a solution $x(t)$ of (2.1) satisfying

$$x(t) = av(t) + bu(t) + o(u(t)(v(t))^\mu), \quad t \to \infty \tag{2.26}$$

Proof. Let M be as in (2.9), and T_1 large enough so that (2.18) holds and

$$\int_{T_1}^{\infty} u(s)k(s)ds < \mu.$$

Consider the space of functions

$$X = \left\{ x \in C([T_1,\infty),\mathbb{R}) \middle| \quad \frac{|x(t) - av(t) - bu(t)|}{v(t)} \le 1, \quad \forall t \ge T_1 \right\}.$$

It can be shown that X is a complete metric space with the metric

$$d(x_1, x_2) = \sup_{t\ge T_1} \frac{1}{v(t)} |x_1(t) - x_2(t)|, \quad x_1, x_2 \in X.$$

Define an operator F on X by

$$(Fx)(t) = -u(t) \int_{T_1}^{t} \frac{1}{p(s)u^2(s)} \int_{s}^{\infty} u(\tau) f(\tau, x(\tau)) d\tau ds + av(t) + bu(t)$$

Note that F is well defined by the conditions of the theorem. We will use the Banach contraction principle to show that F has a fixed point.

Let $x \in X$. In view of (2.3) and (2.5), we see that

$$\begin{aligned}
|(Fx)(t) - av(t) - bu(t)| &\le u(t) \int_{T_1}^{t} \frac{1}{p(s)u^2(s)} \int_{s}^{\infty} u(\tau) |f(\tau, x(\tau))| d\tau ds \\
&\le u(t) \int_{T_1}^{t} \frac{1}{p(s)u^2(s)} \int_{T_1}^{\infty} u(\tau) |f(\tau, x(\tau))| d\tau ds \\
&\le u(t) \int_{T_1}^{t} \frac{1}{p(s)u^2(s)} \int_{T_1}^{\infty} u(\tau) \left(h_1(\tau) g\left(\frac{|x(\tau)|}{v(\tau)} \right) + h_2(\tau) \right) d\tau ds \\
&\le u(t) \int_{T_1}^{t} \frac{1}{p(s)u^2(s)} \int_{T_1}^{\infty} u(\tau)(Mh_1(\tau) + h_2(\tau)) d\tau ds \\
&\le u(t) \int_{T_1}^{t} \frac{1}{p(s)u^2(s)} ds \\
&\le v(t),
\end{aligned}$$

i.e., $FX \subset X$.

Let $x_1, x_2 \in X$. Using (2.22) and (2.23), we have

$$\begin{aligned}
|(Fx_1)(t) - (Fx_2)(t)| &\le u(t) \int_{T_1}^{t} \frac{1}{p(s)u^2(s)} \int_{s}^{\infty} u(\tau) |f(\tau, x_1(\tau)) - f(\tau, x_2(\tau))| d\tau ds \\
&\le u(t) \int_{T_1}^{t} \frac{1}{p(s)u^2(s)} \int_{s}^{\infty} u(\tau) \frac{k(\tau)}{v(\tau)} |x_1(\tau) - x_2(\tau)| d\tau ds \\
&\le d(x_1, x_2) u(t) \int_{T_1}^{t} \frac{1}{p(s)u^2(s)} \int_{s}^{\infty} u(\tau) k(\tau) d\tau ds \\
&\le d(x_1, x_2) u(t) \int_{T_1}^{t} \frac{1}{p(s)u^2(s)} \int_{T_1}^{\infty} u(\tau) k(\tau) d\tau ds \\
&\le \mu d(x_1, x_2) v(t).
\end{aligned}$$

This implies that

$$d(Fx_1, Fx_2) \le \mu d(x_1, x_2),$$

i.e., F is a contraction mapping.

According to the Banach contraction principle F has a unique fixed point x and this fixed point is the solution of equation (2.1).

Finally, since

$$\begin{aligned} |x(t) - av(t) - bu(t)| &\leq u(t) \int_{T_1}^{t} \frac{1}{p(s)u^2(s)} \int_{s}^{\infty} u(\tau)|f(\tau, x(\tau))| d\tau ds \\ &\leq u(t) \int_{T_1}^{t} \frac{1}{p(s)u^2(s)} \int_{s}^{\infty} u(\tau)(Mh_1(\tau) + h_2(\tau)) d\tau ds \\ &\leq (M+1)u(t) \int_{T_1}^{t} \beta(s) ds, \end{aligned}$$

where (2.5), (2.23) and (2.24) are employed, we see that this solution satisfies (2.26). ■

Corollary 2.1.7 *Let (1.14) hold and*

$$|f(t, x_1) - f(t, x_2)| \leq \frac{k(t)}{t}|x_1 - x_2|, \quad t \geq 1, \tag{2.27}$$

where $k \in C([1, \infty), [0, \infty))$. Suppose further that

$$\int_{1}^{\infty} k(s) ds < \infty$$

and

$$\int_{t}^{\infty} h_i(s) ds \leq \beta(t), \quad t \geq 1, \quad i = 1, 2,$$

where $\beta \in C([1, \infty), [0, \infty))$ with

$$\int_{1}^{t} \beta(s) ds = o(t^{\mu}), \quad t \to \infty, \quad \mu \in (0, 1).$$

Then for any given $a, b \in \mathbb{R}$ there is a solution $x(t)$ of (2.2) satisfying

$$x(t) = at + b + o(t^{\mu}), \quad t \to \infty.$$

Theorem 2.1.8 *Let u and v be principal and nonprincipal solutions of (1.3) given by (2.4). Suppose that (2.5) and (2.22) hold. Suppose further that*

$$\int_{T}^{\infty} v(s)k(s) ds < \infty \tag{2.28}$$

and

$$\frac{1}{p(t)v^2(t)} \int_{t}^{\infty} v(s)h_i(s) ds \leq \beta(t), \quad t \geq T, \quad i = 1, 2, \tag{2.29}$$

where $\beta \in C([t_1, \infty), [0, \infty))$ and

$$\int_{t}^{\infty} \beta(s) ds = o((u(t))^{\mu}), \quad t \to \infty, \quad \mu \in (0, 1). \tag{2.30}$$

Then for any given $a, b \in \mathbb{R}$ there is a solution $x(t)$ of (2.1) satisfying

$$x(t) = av(t) + bu(t) + o(v(t)(u(t))^{\mu}), \quad t \to \infty \tag{2.31}$$

Proof. Let M be as in (2.9), and T_1 large enough so that (2.10) holds and

$$\int_{T_1}^{\infty} v(s)k(s)ds < \mu.$$

We also take (X, d) the same metric space as in the proof of previous theorem.

Define an operator F on X by

$$(Fx)(t) = v(t)\int_t^{\infty} \frac{1}{p(s)v^2(s)} \int_s^{\infty} v(\tau)f(\tau, x(\tau))d\tau ds + av(t) + bu(t).$$

It is easy to see that F is well defined by the conditions of the theorem. We will use the Banach fixed point theorem to show that F has a fixed point.

Let $x \in X$. In view of (2.4) and (2.5), we see that

$$\begin{aligned}
|(Fx)(t) - av(t) - bu(t)| &\leq v(t)\int_t^{\infty} \frac{1}{p(s)v^2(s)} \int_s^{\infty} v(\tau)|f(\tau, x(\tau))|d\tau ds \\
&\leq v(t)\int_t^{\infty} \frac{1}{p(s)v^2(s)} \int_{T_1}^{\infty} v(\tau)|f(\tau, x(\tau))|d\tau ds \\
&\leq v(t)\int_t^{\infty} \frac{1}{p(s)v^2(s)} \int_{T_1}^{\infty} v(\tau)\left(h_1(\tau)g\left(\frac{|x(\tau)|}{v(\tau)}\right) + h_2(\tau)\right)d\tau ds \\
&\leq v(t)\int_t^{\infty} \frac{1}{p(s)v^2(s)} \int_{T_1}^{\infty} v(\tau)(Mh_1(\tau) + h_2(\tau))d\tau ds \\
&\leq v(t)\int_t^{\infty} \frac{1}{p(s)v^2(s)}ds \\
&\leq v(t),
\end{aligned}$$

i.e., $FX \subset X$.

Let $x_1, x_2 \in X$. Using (2.22) and (2.28), we have

$$\begin{aligned}
|(Fx_1)(t) - (Fx_2)(t)| &\leq v(t)\int_t^{\infty} \frac{1}{p(s)v^2(s)} \int_s^{\infty} v(\tau)|f(\tau, x_1(\tau)) - f(\tau, x_2(\tau))|d\tau ds \\
&\leq v(t)\int_t^{\infty} \frac{1}{p(s)v^2(s)} \int_s^{\infty} v(\tau)\frac{k(\tau)}{v(\tau)}|x_1(\tau) - x_2(\tau)|d\tau ds \\
&\leq d(x_1, x_2)v(t)\int_t^{\infty} \frac{1}{p(s)v^2(s)} \int_s^{\infty} v(\tau)k(\tau)d\tau ds \\
&\leq d(x_1, x_2)v(t)\int_t^{\infty} \frac{1}{p(s)v^2(s)} \int_{T_1}^{\infty} v(\tau)k(\tau)d\tau ds \\
&\leq \mu d(x_1, x_2)v(t) \\
&\leq v(t)\int_t^{\infty} \frac{1}{p(s)v^2(s)}ds \\
&\leq \mu d(x_1, x_2)v(t).
\end{aligned}$$

This implies that

$$d(Fx_1, Fx_2) \leq \mu d(x_1, x_2),$$

i.e., F is a contraction mapping.

By Banach fixed point theorem F has a unique fixed point x. It is easy to see that x is a solution of equation (2.1).

Finally, since

$$\begin{aligned}|x(t) - av(t) - bu(t)| &\le v(t)\int_t^\infty \frac{1}{p(s)v^2(s)}\int_s^\infty v(\tau)|f(\tau, x(\tau))|d\tau ds \\ &\le v(t)\int_t^\infty \frac{1}{p(s)v^2(s)}\int_s^\infty v(\tau)(Mh_1(\tau) + h_2(\tau))d\tau ds \\ &\le (M+1)v(t)\int_t^\infty \beta(s)ds,\end{aligned}$$

where (2.5), (2.29) and (2.30) are employed, we see that this solution satisfies (2.31). ■

Corollary 2.1.9 *Let (1.14) and (2.27) hold. Suppose further that*

$$\int_1^\infty sk(s)ds < \infty$$

and

$$\frac{1}{t^2}\int_t^\infty sh_i(s)ds \le \beta(t), \quad t \ge 1, \quad i = 1, 2,$$

where $\beta \in C([1,\infty),[0,\infty))$ *with*

$$\int_t^\infty \beta(s)\,ds = o(1), \quad t \to \infty.$$

Then for any given $a, b \in \mathbb{R}$ *there is a solution* $x(t)$ *of (2.2) satisfying*

$$x(t) = at + b + o(t), \quad t \to \infty.$$

2.1.3 Examples

In this section we give three examples to illustrate the results. The examples are constructed in such a way that the principal and nonprincipal solutions are easy to calculate.

Example 2.1.10 Consider the nonlinear differential equation

$$(tx')' - \frac{1}{t}x = \frac{x^3 \sin x}{t^4(1+x^2)} + te^{-t}, \quad t \ge 1. \tag{2.32}$$

The corresponding linear equation is

$$(tx')' - \frac{1}{t}x = 0, \quad t \ge 1.$$

In correlation with the definitions given in Section 2.1.1 and Theorem 2.1.1 let $t_1 = 1$ and $T = 2$. It is easy to see that

$$p(t) = t,\ f(t,x) = \frac{x^3 \sin x}{t^4(1+x^2)} + te^{-t},\ u(t) = \frac{1}{2t},\ \text{and } v(t) = t - \frac{1}{t}.$$

We take

$$h_1(t) = \frac{1}{t^3},\ h_2(t) = te^{-t},\ g(x) = x.$$

Clearly,

$$|f(t,x)| \leq \frac{|x|}{t^4} + te^{-t} \leq h_1(t)g\left(\frac{|x|}{v(t)}\right) + h_2(t), \quad t \geq 2,$$

$$\int_2^\infty v(s)h_1(s)ds = \int_2^\infty \left(s - \frac{1}{s}\right)\frac{1}{s^3}ds = \frac{11}{24} < \infty,$$

and

$$\int_2^\infty v(s)h_2(s)ds = \int_2^\infty \left(s - \frac{1}{s}\right)se^{-s}ds = 7e^{-2} < \infty.$$

Since all conditions of Theorem 2.1.1 are satisfied, we may conclude that for any given real numbers a, b there exists a solution $x(t)$ of (2.32) such that

$$x(t) = a\left(t - \frac{1}{t}\right) + \frac{b}{2t} + o\left(\frac{1}{2t}\right), \quad t \to \infty.$$

Example 2.1.11 We consider the nonlinear differential equation

$$(t^2x')' - 2x = \frac{\ln t}{1+x^2} + \frac{x^2}{1+t^2}, \quad t \geq 1. \tag{2.33}$$

The corresponding linear equation is

$$(t^2x')' - 2x = 0, \quad t \geq 1.$$

Let $t_1 = T = 1$. Notice that

$$p(t) = t^2,\ f(t,x) = \frac{\ln t}{1+x^2} + \frac{x^2}{1+t^2},\ u(t) = \frac{1}{3t^2},\ \text{and } v(t) = t.$$

If we take

$$h_1(t) = 1,\ h_2(t) = \ln t,\ g(x) = x^2,$$

then we see that

$$|f(t,x)| \leq \frac{x^2}{t^2} + \ln t = h_1(t)g\left(\frac{|x|}{t}\right) + h_2(t),$$

and

$$\int_1^\infty u(s)h_i(s)ds = \frac{1}{3} < \infty, \quad \frac{u(t)}{v(t)}\int_1^t v(s)h_i(s)ds \leq \frac{1}{t}, \quad i = 1, 2.$$

and

$$\int_1^\infty u(s)h_2(s)ds = \int_1^\infty \frac{1}{3s^2}\ln s\,ds = \frac{1}{3} < \infty.$$

Since all conditions of Theorem 2.1.3 are satisfied, for any given real numbers a and b there exists a solution $x(t)$ of (2.33) such that

$$x(t) = at + \frac{b}{3t^2} + o(t), \quad t \to \infty.$$

Note that Theorem 2.1.1 fails to apply here since

$$\int_1^\infty v(s)h_1(s)ds = \int_1^\infty s\,ds = \infty.$$

Example 2.1.12 We consider the nonlinear differential equation

$$(t\sqrt{t}x')' - \frac{1}{2\sqrt{t}}x = \frac{x^3}{t^3(t+x^2)} + t^\nu, \quad t \geq 1, \quad \nu \leq -\frac{5}{2}. \tag{2.34}$$

Let $t_1 = 1$ and $T = 2$. Then

$$p(t) = t\sqrt{t}, \quad f(t,x) = \frac{x^3}{t^3(t+x^2)}, \; u(t) = \frac{2}{3t}, \text{ and } v(t) = \sqrt{t} - \frac{1}{t}.$$

We take

$$h_1(t) = \frac{1}{t^3}\left(\sqrt{t} - \frac{1}{t}\right), \; h_2(t) = t^\nu, \; g(x) = x, \; k(t) = \frac{3}{2}h_1(t), \; \beta(t) = \frac{1}{t^2}.$$

Clearly,

$$|f(t,x)| \leq \frac{|x|}{t^3} + t^\nu = h_1(t)g\left(\frac{|x|}{v(t)}\right) + h_2(t), \quad t \geq 2,$$

$$|f(t,x_1) - f(t,x_2)| \leq \frac{3}{2t^3}|x_1 - x_2| = \frac{k(t)}{v(t)}|x_1 - x_2|, \quad t \geq 2,$$

$$\int_2^\infty u(s)k(s)ds = \int_2^\infty \left(s^{-7/2} - s^{-5}\right)ds < \infty,$$

$$\frac{1}{p(t)u^2(t)}\int_2^\infty u(s)h_i(s)ds \leq \frac{1}{t^2} = \beta(t),$$

and

$$\int_2^t \frac{1}{s^2}ds = o\left((v(t))^\mu\right), \quad t \to \infty, \quad \mu \in (0,1).$$

We may conclude by Theorem 2.1.6 that for any given real numbers a, b there exists a solution $x(t)$ of (2.34) such that

$$x(t) = a\left(\sqrt{t} - \frac{1}{t}\right) + b\frac{2}{3t} + o\left(\frac{2}{3t}\left(\sqrt{t} - \frac{1}{t}\right)^\mu\right), \quad t \to \infty.$$

2.2 Monotone Positive Solutions

2.2.1 Introduction

The problem of existence of monotone positive solutions for equations of the form

$$x'' = f(t,x,x'), \quad t \geq 0$$

has been studied by several authors since such equations are often encountered in studying mathematical modeling of real-life problems.

Among numerous works we choose to mention about the work of Yin [18], who proved that if f satisfies the inequality

$$|f(t,x,y)| \leq F(t,|x|,|y|)$$

where $F \in C([0,\infty)\times[0,\infty)\times[0,\infty),[0,\infty))$ is nondecreasing with respect to its second and third variables for each fixed $t \in [0,\infty)$ and satisfying

$$\int_0^\infty F(t,2ct,2c)dt < c$$

for some $c > 0$, then the above equation has a monotone positive solution $x(t)$ such that $x(0) = 0$ and

$$\lim_{t\to\infty} \frac{x(t)}{t} = c. \tag{2.35}$$

Yin's proof is based on arguments developed by Constantin [25] for second-order equations of the form

$$x'' = h(t,x), \quad t \geq 0,$$

where the same limit conclusion as in (2.35) was obtained.

We consider the second-order nonlinear differential equation of the from

$$(p(t)x')' + q(t)x = f(t,x,x'), \qquad t \geq 0, \tag{2.36}$$

where $p \in C([0,\infty),(0,\infty))$, $q \in C([0,\infty),\mathbb{R})$ and $f \in C([0,\infty)\times\mathbb{R}\times\mathbb{R},\mathbb{R})$. Motivated by the above works, our aim is to prove that under some reasonable conditions there is a monotone positive solution $x(t)$ of (2.36) such that

$$\lim_{t\to\infty} \frac{x(t)}{v(t)} = c, \tag{2.37}$$

where $v(t)$ is a nonprincipal solution of the corresponding homogeneous equation (1.3).

We assume that (1.3) has a positive solution, i.e., it is nonoscillatory. Then, for any given nonprincipal solution $v(t)$ of (1.3) we may write in view of (1.18) that

$$u(t) = v(t)\int_t^\infty \frac{1}{p(s)v^2(s)}ds. \tag{2.38}$$

We will also assume that

$$v(t) > 0, \quad v'(t) > 0, \quad \text{and} \quad u'(t) \geq 0. \tag{2.39}$$

for all $t \geq t_0$ for some $t_0 \geq 0$.

Note that if $p(t) = 1$ and $q(t) = 0$, then we may take $u(t) = 1$ and $v(t) = t$. In this case (2.39) holds and the asymptotic representations (2.35) and (2.37) coincide.

2.2.2 A Compactness Criterion

This subsection is devoted to a compactness result which we will need in the proof of our next main result.

Lemma 2.2.1 *Let* (2.39) *hold, and denote*

$$Y = \left\{ y \in C^1([t_0,\infty),\mathbb{R}) \mid \lim_{t\to\infty} y(t) \;\; \textit{and} \;\; \lim_{t\to\infty} z(t) \;\; \textit{exist} \right\} \tag{2.40}$$

where

$$z(t) = y(t) + \frac{v(t)}{v'(t)} y'(t). \tag{2.41}$$

Then the set Y, which is endowed with usual linear operations is a Banach space with norm

$$\|y\|_Y = \max\left\{\sup_{t\ge t_0} |y(t)|\,,\ \sup_{t\ge t_0} |z(t)|\right\}.$$

Proof. It is easy to check that Y is a normed space. Let $\{y_n\}_{n=1}^{\infty}$ be a Cauchy sequence in Y. Then it is a Cauchy sequence in V. Therefore there exists a $y \in V$ such that $y_n \to y$ in V as $n \to \infty$. Define z_n by

$$z_n(t) := y_n(t) + \frac{v(t)}{v'(t)} y_n'(t).$$

Then $\{z_n\}_{n=1}^{\infty}$ is also a Cauchy sequence in V. So there exists a $z \in V$ such that $z_n \to z$ in V as $n \to \infty$. Clearly,

$$z_n \to z \ \text{ in } \ C([t_0, t^*], \mathbb{R}), \ \ \forall\, t^* \ge t_0.$$

From this we have

$$w_n := v' z_n \to v' z =: w \ \text{ in } \ C([t_0, t^*], \mathbb{R}), \ \ \forall\, t^* \ge t_0.$$

By the definition of w_n we also can write

$$v(t)y_n(t) - v(t_0)y_n(t_0) = \int_{t_0}^{t} (v(s)y_n(s))' ds = \int_{t_0}^{t} w_n(s) ds.$$

Because of uniform convergence, by letting $n \to \infty$ we see that

$$v(t)y(t) - v(t_0)y(t_0) = \int_{t_0}^{t} w(s) ds, \quad t \in [t_0, t^*].$$

This implies that $y \in C^1([t_0, \infty), \mathbb{R})$, since t^* is arbitrary. Then we can write

$$w(t) = v'(t)y(t) + v(t)y'(t), \quad z(t) = y(t) + \frac{v(t)}{v'(t)} y'(t).$$

Hence $y \in Y$, and by the definitions of $\|\cdot\|_V$ and $\|\cdot\|_Y$ we obtain $y_n \to y$ in Y. Therefore Y is complete. ■

Lemma 2.2.2 *Let* (2.39) *hold and Y be as in* (2.40). *Assume that a subset $E \subset Y$ satisfies the following conditions:*

(i) E is uniformly bounded: There exists an $L > 0$ such that

$$|y(t)| \le L \ \text{ and } \ |z(t)| \le L, \ \ \forall t \ge t_0, \ \ \forall y \in E.$$

(ii) E is equicontinuous: $\forall\, \epsilon > 0\ \exists\, \delta_\epsilon > 0$ such that

$$|t_1 - t_2| < \delta_\epsilon \ \ \Rightarrow \ \ |y(t_1) - y(t_2)| < \epsilon \ \text{ and } \ |z(t_1) - z(t_2)| < \epsilon$$

for all $t_1,\ t_2 \ge t_0$ and for all $y \in E$.

(iii) E is equiconvergent: $\forall\, \epsilon > 0\ \exists\, t_\epsilon > t_0$ *such that*

$$|y(t) - y(s)| < \epsilon \ \ \textit{and} \ \ |z(t) - z(s)| < \epsilon$$

for all $t,\ s \geq t_\epsilon$ *and for all* $y \in E$.

Then, E is relatively compact in Y. Conversely, if a set $E \subset Y$ is relatively compact in Y, then it satisfies (i)–(iii).

Proof.

Let $\{y_n\}_{n=1}^{\infty}$ be a sequence in E. Then by Lemma 1.4.6 there exists a subsequence $\{y_{n_k}\}_{k=1}^{\infty}$ such that $y_{n_k} \to y$ as $k \to \infty$ for some $y \in V$. Define z_n as in proof of Lemma 2.2.1. Then by Lemma 1.4.6 again there exists a subsequence $\{z_{n_{k_l}}\}_{l=1}^{\infty}$ such that $z_{n_{k_l}} \to z$ as $k \to \infty$ for some $z \in V$. Without loss of generality, we may write that $y_n \to y$ and and $z_n \to z$ as $n \to \infty$. It is clear that $z_n \to z$ in $C([t_0, t^*], \mathbb{R})$ for all $t^* \geq t_0$. Define w_n as in proof of Lemma 2.2.1. Thus $w_n \to w$ in $C([t_0, t^*], \mathbb{R})$. Now by following the same procedure as in the proof of Lemma 2.2.1, we get $y \in Y$ and $y_n \to y$ in Y. Therefore E is relatively compact in Y.

Assume now that $E \subset Y$ is relatively compact in Y. Let $\{y_n\}_{n=1}^{\infty}$ be an arbitrary sequence in E. Then there exits a subsequence $\{y_{n_k}\}_{n=1}^{\infty}$ so that $y_{n_k} \to y$ in Y as $k \to \infty$, for some $y \in Y$. Then by the definitions of the norms $\|\cdot\|_Y$ and $\|\cdot\|_V$ we have $y_{n_k} \to y$ in V as $k \to \infty$, that is, E is relatively compact in V. Define the set

$$M = \left\{ z \in V \middle| \ \exists\, y \in E \ \text{ such that } \ z(t) = y(t) + \frac{v(t)}{v'(t)} y'(t) \right\}.$$

Let $\{z_n\}_{n=1}^{\infty}$ be an arbitrary sequence in M. Consider the sequence $\{\phi_n\}_{n=1}^{\infty}$, where

$$z_n(t) = \phi_n(t) + \frac{v(t)}{v'(t)} \phi_n^{'}(t).$$

Then there exists a subsequence $\{\phi_{n_l}\}_{l=1}^{\infty}$ which is converging to some $\phi \in Y$, that is,

$$\left\|\phi_{n_l} - \phi\right\|_Y = \max\left\{ \sup_{t \geq t_0} |\phi_{n_l}(t) - \phi(t)|,\ \sup_{t \geq t_0} |z_{n_l}(t) - z(t)| \right\} \to 0 \ \text{ as } \ l \to \infty,$$

and hence

$$\left\|z_{n_l} - z\right\|_V = \sup_{t \geq t_0} |z_n(t) - z(t)| \to 0 \ \text{ as } \ l \to \infty.$$

Thus M also is relatively compact in V. According to Lemma 1.4.6 we conclude that (i)–(iii) are fullfilled. ■

2.2.3 The main result

Theorem 2.2.3 *Let* (2.39) *hold, and asssume that*

$$|f(t, x, y)| \leq F(t, |x|, |y|), \quad t \geq t_0, \quad x, y \in \mathbb{R}, \tag{2.42}$$

where $F \in C([t_0,\infty)\times[t_0,\infty)\times[t_0,\infty),[0,\infty))$ is nondecreasing with respect to its second and third arguments for each fixed t and for which

$$\int_{t_0}^{\infty} u(t)F(t,2cv(t),2cv'(t))dt < c, \quad c > 0 \tag{2.43}$$

and

$$\frac{u'(t)}{v'(t)}\int_{t_0}^{t} v(\tau)F(\tau,2cv(\tau),2cv'(\tau))d\tau \to \infty, \quad \text{as} \quad t\to\infty \tag{2.44}$$

are satisfied. Then (2.36) has a monotone positive solution $x(t)$ defined on $[t_0,\infty)$ which satisfies the asymptotic property (2.37).

Proof.

By the hypotheses (2.42) and (2.43) there exists a $\delta \in (0,c)$ so that

$$\int_{t_0}^{\infty} u(t)F(t,(2c-\delta)v(t),(2c-\delta)v'(t))dt \le c-\delta. \tag{2.45}$$

Let

$$K := \left\{y\in Y \middle| \, \delta \le y(t) \le 2c-\delta, \;\; \delta \le z(t) \le 2c-\delta\right\}$$

Obviously K is closed, bounded, convex and nonempty. Define an operator T on the set K by the formula

$$(Ty)(t) := c - \frac{u(t)}{v(t)}\int_{t_0}^{t} v(\tau)g(\tau)d\tau - \int_{t}^{\infty} u(\tau)g(\tau)d\tau, \tag{2.46}$$

where for simplicity

$$g(\tau) := f(\tau, y(\tau)v(\tau),(y(\tau)v(\tau))').$$

T maps K into K. Indeed, we may write from (2.38) and (2.46) that

$$\begin{aligned}
|(Ty)(t)-c| &\le \frac{u(t)}{v(t)}\int_{t_0}^{t} v(\tau)|g(\tau)|d\tau + \int_{t}^{\infty} u(\tau)|g(\tau)|d\tau \\
&= \int_{t}^{\infty}\frac{1}{p(s)v^2(s)}ds\int_{t_0}^{t} v(\tau)\,|g(\tau)|\,d\tau + \int_{t}^{\infty} u(\tau)\,|g(\tau)|\,d\tau.
\end{aligned}$$

Then, in view of (2.42) and (2.45),

$$\begin{aligned}
|(Ty)(t)-c| &\le \int_{t_0}^{t} u(\tau)\,|g(\tau)|\,d\tau + \int_{t}^{\infty} u(\tau)\,|g(\tau)|\,d\tau \\
&= \int_{t_0}^{\infty} u(\tau)\,|g(\tau)|\,d\tau \\
&\le \int_{t_0}^{\infty} u(\tau)F(\tau,(2c-\delta)v(\tau),(2c-\delta)v'(\tau))d\tau \\
&\le c-\delta.
\end{aligned}$$

Next setting

$$(Sy)(t) = (Ty)(t) + \frac{v(t)}{v'(t)}(Ty)'(t)$$

we see in a similar manner that

$$\begin{aligned}
|(Sy)(t)-c| &\leq \frac{u'(t)}{v'(t)}\int_{t_0}^{t} v(\tau)|g(\tau)|d\tau + \int_{t}^{\infty} u(\tau)|g(\tau)|d\tau \\
&\leq \frac{u(t)}{v(t)}\int_{t_0}^{t} v(\tau)\,|g(\tau)|\,d\tau + \int_{t}^{\infty} u(\tau)\,|g(\tau)|\,d\tau \\
&\leq c-\delta
\end{aligned}$$

where we have used (1.19) and (2.39).

T is compact. It is enough to prove that $T(K)$ is relatively compact in Y. To do this we show that $T(K)$ satisfies the hypotheses (i)–(iii) of the Lemma 2.2.2.

(i) $T(K)$ *is uniformly bounded.* By the estimates above we have

$$|(Ty)(t)| \leq 2c-\delta \text{ and } |(Sy)(t)| \leq 2c-\delta, \quad \forall t \geq t_0, \quad \forall y \in K.$$

(ii) $T(K)$ *is equicontinuous.* Le $\epsilon > 0$ be given. In view of (1.18) and (2.45), there exist $t_\epsilon \geq t_0$ and $t_\epsilon^{'} \geq t_\epsilon$ such that

$$\int_{t_\epsilon}^{\infty} u(\tau)G(\tau)d\tau < \frac{\epsilon}{6} \text{ and } \int_{t_0}^{t_\epsilon} v(\tau)G(\tau)\int_{t_\epsilon^{'}}^{\infty}\frac{1}{p(s)v^2(s)}dsd\tau < \frac{\epsilon}{2},$$

where $G(\tau) := F(\tau,(2c-\delta)v(\tau),(2c-\delta)v'(\tau))$. Since v is increasing and u is nondecreasing, and (1.17) and (1.19) hold, there exists $t_\epsilon^{''} \geq t_\epsilon$ such that

$$\frac{u'(t)}{v'(t)}\int_{t_0}^{t_\epsilon} v(\tau)G(\tau)d\tau < \frac{\epsilon}{6}, \quad t \geq t_\epsilon^{''}.$$

Rewrite the operator T as

$$(Ty)(t) = c - \int_{t_0}^{\infty} u(\tau)g(\tau)d\tau + \int_{t_0}^{t} v(\tau)g(\tau)\int_{\tau}^{t}\frac{1}{p(s)v^2(s)}dsd\tau.$$

Let $t_1,\ t_2 \geq t_0$ be arbitrary. Without loss of generality assume that $t_2 \geq t_1$. Then,

$$\begin{aligned}
&|(Ty)(t_1)-(Ty)(t_2)| = \\
&\quad \Big|\int_{t_0}^{t_1} v(\tau)g(\tau)\int_{\tau}^{t_1}\frac{1}{p(s)v^2(s)}dsd\tau - \int_{t_0}^{t_2} v(\tau)g(\tau)\int_{\tau}^{t_2}\frac{1}{p(s)v^2(s)}dsd\tau \\
&\quad + \int_{t_0}^{t_1} v(\tau)g(\tau)\int_{\tau}^{t_2}\frac{1}{p(s)v^2(s)}dsd\tau - \int_{t_0}^{t_1} v(\tau)g(\tau)\int_{\tau}^{t_2}\frac{1}{p(s)v^2(s)}dsd\tau\Big| \\
&\quad \leq \int_{t_0}^{t_1} G(\tau)v(\tau)\int_{t_1}^{t_2}\frac{1}{p(s)v^2(s)}dsd\tau + \int_{t_1}^{t_2} G(\tau)v(\tau)\int_{\tau}^{t_2}\frac{1}{p(s)v^2(s)}dsd\tau. \qquad (2.47)
\end{aligned}$$

Define

$$M_1 := \max_{t_0 \leq s \leq t_\epsilon}\frac{1}{p(s)v^2(s)}, \quad M_2 := \max_{t_\epsilon \leq s \leq t_\epsilon^{'}}\frac{1}{p(s)v^2(s)}, \quad M_3 := \max_{t_0 \leq \tau \leq t_\epsilon} G(\tau)u(\tau).$$

If $t_0 \leq t_1 \leq t_2 \leq t_\epsilon$, then

$$\begin{aligned}
\int_{t_0}^{t_1} G(\tau)v(\tau)\int_{t_1}^{t_2}\frac{1}{p(s)v^2(s)}dsd\tau &\leq \int_{t_0}^{t_\epsilon} G(\tau)v(\tau)\int_{t_1}^{t_2}\frac{1}{p(s)v^2(s)}dsd\tau \\
&\leq M_1|t_1-t_2|\int_{t_0}^{t_\epsilon} G(\tau)v(\tau)d\tau \qquad (2.48)
\end{aligned}$$

and

$$\int_{t_1}^{t_2} G(\tau)v(\tau)\int_{\tau}^{t_2}\frac{1}{p(s)v^2(s)}dsd\tau \le \int_{t_1}^{t_2} G(\tau)v(\tau)\int_{\tau}^{\infty}\frac{1}{p(s)v^2(s)}dsd\tau$$
$$= \int_{t_1}^{t_2} G(\tau)u(\tau)d\tau \le M_3|t_1-t_2|. \tag{2.49}$$

If $t_\epsilon \le t_1 \le t_2 \le t_\epsilon^{'}$, then

$$\int_{t_0}^{t_1} G(\tau)v(\tau)\int_{t_1}^{t_2}\frac{1}{p(s)v^2(s)}dsd\tau \le \int_{t_0}^{t_\epsilon} G(\tau)v(\tau)\int_{t_1}^{t_2}\frac{1}{p(s)v^2(s)}dsd\tau$$
$$+\int_{t_\epsilon}^{t_1} G(\tau)v(\tau)\int_{t_1}^{t_2}\frac{1}{p(s)v^2(s)}dsd\tau$$
$$\le M_2|t_1-t_2|\int_{t_0}^{t_\epsilon} G(\tau)v(\tau)d\tau + \int_{t_\epsilon}^{\infty} G(\tau)u(\tau)d\tau \tag{2.50}$$

and

$$\int_{t_1}^{t_2} G(\tau)v(\tau)\int_{\tau}^{t_2}\frac{1}{p(s)v^2(s)}dsd\tau \le \int_{t_1}^{t_2} G(\tau)v(\tau)\int_{\tau}^{\infty}\frac{1}{p(s)v^2(s)}dsd\tau$$
$$\le \int_{t_\epsilon}^{\infty} G(\tau)u(\tau)d\tau. \tag{2.51}$$

If $t_\epsilon^{'} \le t_1 \le t_2$, then

$$\int_{t_0}^{t_1} G(\tau)v(\tau)\int_{t_1}^{t_2}\frac{1}{p(s)v^2(s)}dsd\tau \le \int_{t_0}^{t_\epsilon} G(\tau)v(\tau)\int_{t_1}^{t_2}\frac{1}{p(s)v^2(s)}dsd\tau$$
$$+\int_{t_\epsilon}^{t_1} G(\tau)v(\tau)\int_{t_1}^{t_2}\frac{1}{p(s)v^2(s)}dsd\tau$$
$$\le \int_{t_0}^{t_\epsilon} G(\tau)v(\tau)\int_{t_\epsilon{'}}^{\infty}\frac{1}{p(s)v^2(s)}dsd\tau + \int_{t_\epsilon}^{\infty} G(\tau)u(\tau)d\tau \tag{2.52}$$

and

$$\int_{t_1}^{t_2} G(\tau)v(\tau)\int_{\tau}^{t_2}\frac{1}{p(s)v^2(s)}dsd\tau \le \int_{t_1}^{t_2} G(\tau)v(\tau)\int_{\tau}^{\infty}\frac{1}{p(s)v^2(s)}dsd\tau$$
$$\le \int_{t_\epsilon}^{\infty} G(\tau)u(\tau)d\tau. \tag{2.53}$$

Now let us consider the operator S. Clearly,

$$|(Sy)(t_1)-(Sy)(t_2)| = \Big| -\frac{u'(t_1)}{v'(t_1)}\int_{t_0}^{t_1} v(\tau)g(\tau)d\tau + \frac{u'(t_2)}{v'(t_2)}\int_{t_0}^{t_2} v(\tau)g(\tau)d\tau$$
$$-\int_{t_1}^{\infty} u(\tau)g(\tau)d\tau + \int_{t_2}^{\infty} u(\tau)g(\tau)d\tau\Big|$$
$$\le \left|\frac{u'(t_1)}{v'(t_1)} - \frac{u'(t_2)}{v'(t_2)}\right|\int_{t_0}^{t_1} v(\tau)|g(\tau)|d\tau + \frac{u'(t_2)}{v'(t_2)}\int_{t_1}^{t_2} v(\tau)|g(\tau)|d\tau + \int_{t_1}^{t_2} u(\tau)|g(\tau)|d\tau$$
$$\le \left|\frac{u'(t_1)}{v'(t_1)} - \frac{u'(t_2)}{v'(t_2)}\right|\int_{t_0}^{t_1} v(\tau)G(\tau)d\tau + 2\int_{t_1}^{t_2} u(\tau)G(\tau)d\tau \tag{2.54}$$

If $t_0 \le t_1 \le t_2 \le t_\epsilon$, then

$$\left|\frac{u'(t_1)}{v'(t_1)} - \frac{u'(t_2)}{v'(t_2)}\right| \int_{t_0}^{t_1} v(\tau)G(\tau)d\tau \le \left|\frac{u'(t_1)}{v'(t_1)} - \frac{u'(t_2)}{v'(t_2)}\right| \int_{t_0}^{t_\epsilon} v(\tau)G(\tau)d\tau \tag{2.55}$$

and

$$\int_{t_1}^{t_2} G(\tau)u(\tau)d\tau \le M_3|t_1 - t_2|. \tag{2.56}$$

Note that since u'/v' is uniformly continuous on $[t_0, t_\epsilon]$, there exists a $\rho_1 > 0$ such that if $|t_1 - t_2| < \rho_1$,where $t_0 \le t_1,\ t_2 < t_\epsilon$, then

$$\left|\frac{u'(t_1)}{v'(t_1)} - \frac{u'(t_2)}{v'(t_2)}\right| \int_{t_0}^{t_\epsilon} v(\tau)G(\tau)d\tau < \frac{\epsilon}{3}. \tag{2.57}$$

If $t_\epsilon \le t_1 \le t_2 \le t_\epsilon^{''}$, then

$$\begin{aligned}\left|\frac{u'(t_1)}{v'(t_1)} - \frac{u'(t_2)}{v'(t_2)}\right| \int_{t_0}^{t_1} v(\tau)G(\tau)d\tau &\le \left|\frac{u'(t_1)}{v'(t_1)} - \frac{u'(t_2)}{v'(t_2)}\right| \int_{t_0}^{t_\epsilon} v(\tau)G(\tau)d\tau \\ &\quad + 2\max\left\{\frac{u'(t_1)}{v'(t_1)},\ \frac{u'(t_2)}{v'(t_2)}\right\} \int_{t_\epsilon}^{t_1} v(\tau)G(\tau)d\tau \\ &\le \left|\frac{u'(t_1)}{v'(t_1)} - \frac{u'(t_2)}{v'(t_2)}\right| \int_{t_0}^{t_\epsilon} v(\tau)G(\tau)d\tau + 2\int_{t_\epsilon}^{\infty} G(\tau)u(\tau)d\tau \end{aligned} \tag{2.58}$$

and

$$\int_{t_1}^{t_2} G(\tau)u(\tau)d\tau \le \int_{t_\epsilon}^{\infty} G(\tau)u(\tau)d\tau.$$

Since u'/v' is uniformly continuous on $[t_\epsilon, t_\epsilon^{''}]$, there exists a $\rho_2 > 0$ such that if $|t_1 - t_2| < \rho_2$,where $t_\epsilon \le t_1,\ t_2 < t_\epsilon^{''}$, then

$$\left|\frac{u'(t_1)}{v'(t_1)} - \frac{u'(t_2)}{v'(t_2)}\right| \int_{t_0}^{t_\epsilon} v(\tau)G(\tau)d\tau < \frac{\epsilon}{3}. \tag{2.59}$$

Lastly, if $t_\epsilon^{''} \le t_1 \le t_2$, then

$$\begin{aligned}\left|\frac{u'(t_1)}{v'(t_1)} - \frac{u'(t_2)}{v'(t_2)}\right| \int_{t_0}^{t_1} v(\tau)G(\tau)d\tau &\le 2\max\left\{\frac{u'(t_1)}{v'(t_1)},\ \frac{u'(t_2)}{v'(t_2)}\right\} \int_{t_0}^{t_\epsilon} v(\tau)G(\tau)d\tau \\ &\quad + 2\max\left\{\frac{u'(t_1)}{v'(t_1)},\ \frac{u'(t_2)}{v'(t_2)}\right\} \int_{t_\epsilon}^{t_1} v(\tau)G(\tau)d\tau \\ &\le 2\max\left\{\frac{u'(t_1)}{v'(t_1)},\ \frac{u'(t_2)}{v'(t_2)}\right\} \int_{t_0}^{t_\epsilon} v(\tau)G(\tau)d\tau + 2\int_{t_\epsilon}^{\infty} G(\tau)u(\tau)d\tau \end{aligned} \tag{2.60}$$

and

$$\int_{t_1}^{t_2} G(\tau)u(\tau)d\tau \le \int_{t_\epsilon}^{\infty} G(\tau)u(\tau)d\tau. \tag{2.61}$$

Let

$$\delta_\epsilon = \min\left\{\frac{\epsilon}{2M_1\int_{t_0}^{t_\epsilon} G(\tau)v(\tau)d\tau},\ \frac{\epsilon}{3M_3},\ \frac{\epsilon}{2M_2\int_{t_0}^{t_\epsilon} G(\tau)u(\tau)d\tau},\ \rho_1,\ \rho_2\right\}.$$

In view of (2.47)–(2.61), we see that if $|t_1 - t_2| < \delta_\epsilon$, then

$$|(Ty)(t_1) - (Ty)(t_2)| < \epsilon \text{ and } |(Sy)(t_1) - (Sy)(t_2)| < \epsilon$$

for all $t \geq t_0$ and for all $y \in Y$. Thus $T(K)$ is equicontinuous.

(iii) $T(K)$ *is equiconvergent.* Fix $\epsilon > 0$, and take $t_\epsilon^{'}$ and $t_\epsilon^{''}$ as above. Let $t_1,\ t_2 \geq \max\{t_\epsilon^{'},\ t_\epsilon^{''}\}$. Without loss of generality assume that $t_2 \geq t_1$. Then we have

$$\begin{aligned}
|(Ty)(t_1) - (Ty)(t_2)| &\leq \int_{t_0}^{t_1} G(\tau)v(\tau)\int_{t_1}^{t_2}\frac{1}{p(s)v^2(s)}dsd\tau + \int_{t_1}^{t_2} G(\tau)v(\tau)\int_{\tau}^{t_2}\frac{1}{p(s)v^2(s)}dsd\tau \\
&\leq \int_{t_0}^{t_\epsilon} G(\tau)v(\tau)\int_{t_\epsilon^{'}}^{\infty}\frac{1}{p(s)v^2(s)}dsd\tau + 2\int_{t_\epsilon}^{\infty} G(\tau)u(\tau)d\tau < \epsilon
\end{aligned}$$

and

$$\begin{aligned}
|(Sy)(t_1) - (Sy)(t_2)| &\leq \left|\frac{u'(t_1)}{v'(t_1)} - \frac{u'(t_2)}{v'(t_2)}\right|\int_{t_0}^{t_1} v(\tau)G(\tau)d\tau + 2\int_{t_1}^{t_2} u(\tau)G(\tau)d\tau \\
&\leq 2\max\left\{\frac{u'(t_1)}{v'(t_1)},\ \frac{u'(t_2)}{v'(t_2)}\right\}\int_{t_0}^{t_\epsilon} v(\tau)G(\tau)d\tau + 4\int_{t_\epsilon}^{\infty} G(\tau)u(\tau)d\tau < \epsilon
\end{aligned}$$

for all $y \in K$. Thus $T(K)$ is equiconvergent. In conclusion, $T(K)$ satisfies all the hypotheses of Lemma (2.2.2) and so T is a compact operator.

T *is continuous.* Fix $\epsilon > 0$ and $y_1,\ y_2 \in K$. Let $t_* \geq t_0$ be so large that

$$\int_{t_*}^{\infty} u(\tau)G(\tau)d\tau < \frac{\epsilon}{3}.$$

Define

$$m_1 = \min_{t_0 \leq \tau \leq t_*} v'(t), \quad M_1 = \max_{t_0 \leq \tau \leq t_*} v'(t).$$

Since f is uniformly continuous on $[t_0, t_*] \times [\delta v(t_0), (2c - \delta)v(t_*)] \times [\delta m_1, (2c - \delta)M_1]$, there exists a $\rho > 0$ such that for all $\tau \in [t_0, t_*]$ and $r_1,\ r_2 \in [\delta v(t_0), (2c - \delta)v(t_*)]$ with $|r_1 - r_2| < \rho$ and for all $s_1,\ s_2 \in [\delta m_1, (2c - \delta)M_1]$ with $|s_1 - s_2| < \rho$ we have

$$|g_1(\tau) - g_2(\tau)| \leq \frac{\epsilon}{3(t_* - t_0)u(t_*)},$$

where

$$g_i(\tau) = f(\tau, v(\tau)y_i(\tau), (v(\tau)y_i(\tau))'), \quad i = 1, 2.$$

Let

$$\gamma = \min\left\{\frac{\rho}{2v(t_*)},\ \frac{\rho}{2M_1}\right\}.$$

Then if $\|y_1 - y_2\| < \gamma$, by (2.42) and (2.43) we have

$$\begin{aligned}
|(Ty_1)(t) - (Ty_2)(t)| &\leq \frac{u(t)}{v(t)}\int_{t_0}^{t} v(\tau)|g_1(\tau) - g_2(\tau))|d\tau + \int_{t}^{\infty} u(\tau)|g_1(\tau) - g_2(\tau)|d\tau \\
&\leq \int_{t_0}^{\infty} u(\tau)|g_1(\tau) - g_2(\tau)|d\tau \\
&\leq \int_{t_0}^{t_*} u(\tau)|g_1(\tau) - g_2(\tau)|d\tau + \int_{t_*}^{\infty} u(\tau)|g_1(\tau) - g_2(\tau)|d\tau \\
&\leq u(t_*)(t_* - t_0)\frac{\epsilon}{3u(t_*)(t_* - t_0)} + 2\int_{t_*}^{\infty} u(\tau)G(\tau)d\tau \\
&< \epsilon
\end{aligned}$$

Also,

$$\begin{aligned}
|(Sy_1)(t)-(Sy_2)(t)| &\le \frac{u'(t)}{v'(t)}\int_{t_0}^{t} v(\tau)|g_1(\tau)-g_2(\tau)|d\tau + \int_t^{\infty} u(\tau)|g_1(\tau)-g_2(\tau)|d\tau \\
&\le \frac{u(t)}{v(t)}\int_{t_0}^{t} v(\tau)|g_1(\tau)-g_2(\tau)|d\tau + \int_t^{\infty} u(\tau)|g_1(\tau)-g_2(\tau)|d\tau \\
&< \epsilon.
\end{aligned}$$

Therefore,

$$\|Ty_1 - Ty_2\| = \max\left\{\sup_{t\ge t_0}|(Ty_1)(t)-(Ty_2)(t)|,\ \ \sup_{t\ge t_0}|(Sy_1)(t)-(Sy_2)(t)|\right\} < \epsilon$$

whenever $\|y_1 - y_2\| < \gamma$, which means that T is continuous.

According to Schauder's fixed point theorem, T has a fixed point in K, say $Ty = y$. Define

$$x(t) = v(t)y(t), \quad t \ge t_0.$$

We can write

$$x(t) = cv(t) - u(t)\int_{t_0}^{t} v(\tau)f(\tau, x(\tau), x'(\tau))d\tau - v(t)\int_t^{\infty} u(\tau)f(\tau, x(\tau), x'(\tau))d\tau.$$

It is not difficult to see that x is a solution of (2.36), and by the definition of K,

$$x(t) \ge \delta > 0$$

and

$$z(t) = y(t) + \frac{v(t)}{v'(t)}y'(t) = \frac{x'(t)}{v'(t)} \ge \delta > 0.$$

One can also easily check that

$$\lim_{t\to\infty}\frac{x(t)}{v(t)} = c.$$

The proof is complete. ■

We end this section by posing two open problems: (1) Find conditions (if any) under which (2.36) has a monotone positive solution $x(t)$ satisfying

$$\lim_{t\to\infty}\frac{x(t)}{u(t)} = c.$$

(2) Is there also a monotone positive solution of (2.36) which is asymptotic to $au(t) + bv(t)$ for some real numbers a and b as $t \to \infty$?

2.2.4 An Example

Example 2.2.4 *Consider the second order nonlinear differential equation*

$$\left(\frac{1}{t}x'\right)' + \frac{3}{4t^3}x = \frac{x'^3\,(x^2 + \sin t)}{(1 + x'^2)t^6}, \quad t \ge 1. \tag{2.62}$$

We can easily check that

$$u(t) = t^{1/2}, \quad v(t) = t^{3/2}$$

are principal and nonprincipal solutions of

$$\left(\frac{1}{t}x'\right)' + \frac{3}{4t^3}x = 0,$$

and that

$$f(t,x,y) = \frac{y^3\,(x^2+\sin t)}{(1+y^2)t^6}$$

satisfies the estimate

$$|f(t,x,y)| \le \frac{|y|(|x|^2+1)}{t^6} =: F(t,|x|,|y|).$$

Also, the function F is nondecreasing with respect to its second and third arguments for each fixed t and

$$\int_{t_0}^{\infty} u(t)F(t,2cv(t),2cv'(t))dt = \int_1^{\infty} \frac{3c\,(4c^2t^3+1)}{t^5}dt < c$$

$$\limsup_{t\to\infty} \frac{u'(t)}{v'(t)} \int_{t_0}^{t} v(\tau)F(\tau,2cv(\tau),2cv'(\tau))d\tau = \limsup_{t\to\infty} \frac{1}{3t}\int_1^t \frac{3c\,(4c^2\tau^3+1)}{\tau^4}d\tau = 0$$

hold with $c = 1/7$. Since all conditions of Theorem 2.2.3 are satisfied, we may conclude that (2.62) has a monotone positive solution $x(t)$ satisfying

$$\lim_{t\to\infty} \frac{x(t)}{t^{3/2}} = \frac{1}{7}.$$

Below are the graphs of the solution $x = x(t)$ and the function $7x(t)/t^{3/2}$ that confirm our finding in this example. The graphs are obtained by employing Mathematica software.

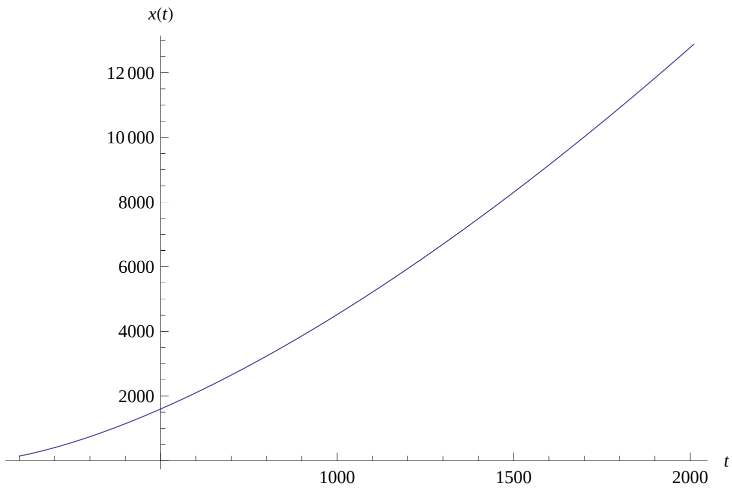

Figure 2.1: Graph of the solution $x = x(t)$.

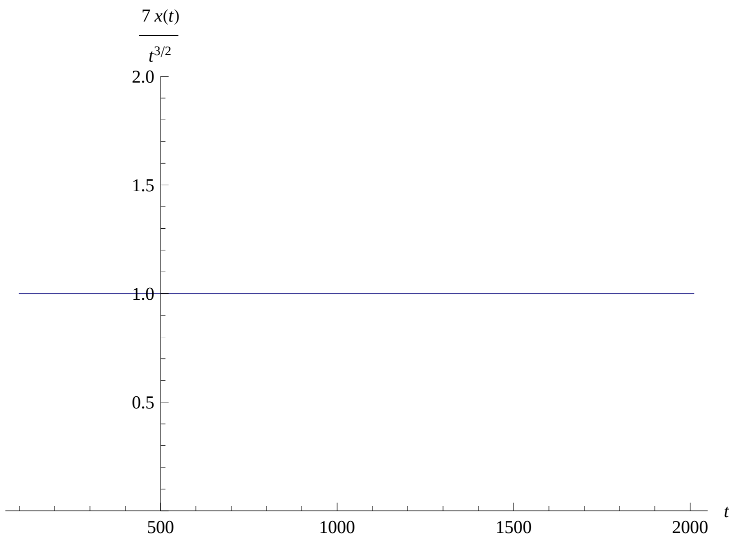

Figure 2.2: Graph of the function $7x(t)/t^{3/2}$.

CHAPTER 3

NONLINEAR SINGULAR BOUNDARY VALUE PROBLEMS

3.1 Introduction

Boundary value problems on half-line occur in various applications such as in the study of the unsteady flow of a gas through semi-infinite porous medium, in analyzing the heat transfer in radial flow between circular disks, in the study of plasma physics, in an analysis of the mass transfer on a rotating disk in a non-Newtonian fluid, etc. More examples and a collection of works on the existence of solutions of boundary value problems on half-line for differential, difference and integral equations may be found in the monographs [50, 51] For some works and various techniques dealing with such boundary value problems (we may refer to [52, 53, 54, 55] and the references cited therein).

In this chapter by employing principal and nonprincipal solutions we introduce a new approach to study nonlinear boundary problems on half-line of the form

$$(p(t)x')' + q(t)x = f(t, x), \quad t \geq t_0, \tag{3.1}$$

$$x(t_0) = x_0, \tag{3.2}$$

$$x(t) = a\, v(t) + b\, u(t) + o(r(t)), \quad t \to \infty, \tag{3.3}$$

where a and b are any given real numbers, u and v are, respectively, principal and nonprincipal solutions of

$$(p(t)x')' + q(t)x = 0, \quad t \geq 0 \tag{3.4}$$

and $p \in C([0, \infty), (0, \infty))$, $q \in C([0, \infty), \mathbb{R})$ and $f \in C([0, \infty) \times \mathbb{R}, \mathbb{R})$.

We will show that the problem (3.1)–(3.3) has a unique solution in the case when

$$r(t) = o(u(t)(v(t))^{\mu}) \tag{3.5}$$

and

$$r(t) = o(v(t)(u(t))^{\mu}), \tag{3.6}$$

where $\mu \in (0, 1)$ is arbitrary but fixed real numbers.

The nonlinear boundary value problem (3.1)–(3.3) is also closely related to asymptotic integration of second order differential equations. Indeed, there are several important works in the literature, see [1, 2, 3, 5, 7, 19, 20, 26, 30, 32], dealing with mostly the asymptotic integration of solutions of second order nonlinear equations of the form

$$x'' = f(t, x).$$

The authors are usually interested in finding conditions on the function $f(t, x)$ which guarantee the existence of a solution asymptotic to a linear function of the form

$$x(t) = a\,t + b, \quad t \to \infty. \tag{3.7}$$

We should point out that $u(t) = 1$ and $v(t) = t$ are principal and nonprincipal solutions, respectively, of the corresponding unperturbed equation

$$x'' = 0,$$

and the function $x(t)$ in (3.7) can be written as

$$x = a\,v(t) + b\,u(t).$$

Note that $v(t) \to \infty$ as $t \to \infty$ but $u(t)$ is bounded in this special case. It turns out such information is crucial in investigating the general case. Our results will be applicable whether or not $u(t) \to \infty$ ($v(t) \to \infty$) as $t \to \infty$.

3.2 Existence of Solutions

Let u be a principal solution of (3.4). Without loss of generality we may assume that $u(t) > 0$ if $t \geq t_1$ for some $t_1 \geq 0$. It is easy to see that

$$v(t) = u(t) \int_{t_1}^{t} \frac{1}{p(s)u^2(s)}\, ds \tag{3.8}$$

is a nonprincipal solution of (3.4), which is strictly positive for $t > t_1$.

Theorem 3.2.1 *Let $t_0 > t_1$. Assume that the function f satisfies*

$$|f(t, x)| \leq h_1(t)g(|x|) + h_2(t), \quad t \geq t_0 \tag{3.9}$$

and

$$|f(t, x_1) - f(t, x_2)| \leq \frac{k(t)}{v(t)}|x_1 - x_2|, \quad t \geq t_0, \tag{3.10}$$

where $g \in C([0, \infty), [0, \infty))$ is bounded; h_1, h_2, $k \in C([t_0, \infty), [0, \infty))$. Suppose further that

$$\int_{t_0}^{\infty} u(s)k(s)ds \leq \mu \tag{3.11}$$

and

$$\frac{1}{p(t)u^2(t)} \int_{t}^{\infty} u(s)h_i(s)ds \leq \beta(t), \quad t \geq t_0, \quad i = 1, 2 \tag{3.12}$$

hold for some $\beta \in C([t_0, \infty), [0, \infty))$ such that

$$\int_{t_0}^{t} \beta(s)ds = o((v(t))^{\mu}), \quad t \to \infty. \tag{3.13}$$

If either

$$v(t) \to \infty, \quad t \to \infty \tag{3.14}$$

or

$$b = \frac{x_0}{u(t_0)} - a \int_{t_1}^{t_0} \frac{1}{p(s)u^2(s)}ds, \tag{3.15}$$

then there is a unique solution $x(t)$ of (3.1)–(3.3), *where r is given by* (3.5).

Proof. Denote by M the supremum of the function g over $[0,\infty)$. Let X be a space of functions defined by

$$X = \{x \in C([t_0,\infty),\mathbb{R}) | \quad |x(t)| \le l_1 v(t) + l_2 u(t), \quad \forall t \ge t_0\},$$

where

$$l_1 = (M+1)p(t_0)u^2(t_0)\beta(t_0) + |a|$$

and

$$l_2 = \frac{|x_0|}{u(t_0)} + |a| \int_{t_1}^{t_0} \frac{1}{p(s)u^2(s)} ds.$$

Note that X is a complete metric space with the metric d defined by

$$d(x_1,x_2) = \sup_{t\ge t_0} \frac{1}{v(t)} |x_1(t) - x_2(t)|, \quad x_1, x_2 \in X.$$

Define an operator F on X by

$$\begin{aligned}(Fx)(t) &= -u(t)\int_{t_0}^{t} \frac{1}{p(s)u^2(s)} \int_s^\infty u(\tau) f(\tau, x(\tau)) d\tau ds + av(t) \\ &\quad + \left[\frac{x_0}{u(t_0)} - a\int_{t_1}^{t_0} \frac{1}{p(s)u^2(s)} ds\right] u(t).\end{aligned}$$

In view of conditions (3.9) and (3.12) we see that F is well defined. Next we show that $FX \subset X$. Indeed, let $x \in X$, then

$$\begin{aligned}|(Fx)(t)| &\le u(t)\int_{t_0}^{t} \frac{1}{p(s)u^2(s)} \int_s^\infty u(\tau)|f(\tau, x(\tau))| d\tau ds + |a|v(t) + l_2 u(t) \\ &\le u(t)\int_{t_0}^{t} \frac{1}{p(s)u^2(s)} \int_{t_0}^\infty u(\tau)|f(\tau, x(\tau))| d\tau ds + |a|v(t) + l_2 u(t) \\ &\le u(t)\int_{t_0}^{t} \frac{1}{p(s)u^2(s)} \int_{t_0}^\infty u(\tau)(h_1(\tau)g(|x(\tau)|) + h_2(\tau)) d\tau ds + |a|v(t) + l_2 u(t) \\ &\le u(t)\int_{t_0}^{t} \frac{1}{p(s)u^2(s)} \int_{t_0}^\infty u(\tau)(Mh_1(\tau) + h_2(\tau)) d\tau ds + |a|v(t) + l_2 u(t) \\ &\le (M+1)p(t_0)u^2(t_0)\beta(t_0)u(t)\int_{t_0}^{t} \frac{1}{p(s)u^2(s)} ds + |a|v(t) + l_2 u(t) \\ &\le l_1 v(t) + l_2 u(t),\end{aligned}$$

which means that $Fx \in X$.

Using (3.8), (3.10), and (3.11) we also see that

$$\begin{aligned}|(Fx_1)(t) - (Fx_2)(t)| &\le u(t)\int_{t_0}^{t} \frac{1}{p(s)u^2(s)} \int_s^\infty u(\tau)|f(\tau, x_1(\tau)) - f(\tau, x_2(\tau))| d\tau ds \\ &\le u(t)\int_{t_0}^{t} \frac{1}{p(s)u^2(s)} \int_s^\infty u(\tau)\frac{k(\tau)}{v(\tau)}|x_1(\tau) - x_2(\tau)| d\tau ds \\ &\le d(x_1,x_2)u(t)\int_{t_0}^{t} \frac{1}{p(s)u^2(s)} \int_s^\infty u(\tau)k(\tau) d\tau ds \\ &\le d(x_1,x_2)u(t)\int_{t_0}^{t} \frac{1}{p(s)u^2(s)} \int_{t_0}^\infty u(\tau)k(\tau) d\tau ds \\ &\le d(x_1,x_2)v(t)\int_{t_0}^{\infty} u(\tau)k(\tau) d\tau \\ &\le \mu d(x_1,x_2)v(t),\end{aligned}$$

where $x_1, x_2 \in X$ are arbitrary. This implies that F is a contraction mapping.

Thus according to Banach contraction principle F has a unique fixed point x. It is not difficult to see that the fixed point solves (3.1) and (3.2). It remains to show that $x(t)$ satisfies (3.3) as well. It is not difficult to show that

$$\begin{aligned}
|x(t) - av(t) - bu(t)| &\le u(t)\int_{t_0}^{t} \frac{1}{p(s)u^2(s)} \int_{s}^{\infty} u(\tau)|f(\tau, x(\tau))|d\tau ds + |c|u(t) \\
&\le u(t)\int_{t_0}^{t} \frac{1}{p(s)u^2(s)} \int_{s}^{\infty} u(\tau)(Mh_1(\tau) + h_2(\tau))d\tau ds + |c|u(t) \\
&\le (M+1)u(t)\int_{t_0}^{t} \beta(s)ds + |c|u(t),
\end{aligned}$$

where

$$c = \frac{x_0}{u(t_0)} - a\int_{t_1}^{t_0} \frac{1}{p(s)u^2(s)}ds - b.$$

If (3.14) is satisfied, then in view (3.13) and the above inequality, we easily obtain (3.3). In case (3.15) holds, then $c = 0$ and hence we still have (3.3). ∎

From Theorem 3.2.1 we deduce the following Corollary.

Corollary 3.2.2 *Assume that the function f satisfies* (3.9) *and*

$$|f(t, x_1) - f(t, x_2)| \le \frac{k(t)}{t}|x_1 - x_2|, \quad t \ge t_0,$$

where $k \in C([t_0, \infty), [0, \infty))$. Suppose further that

$$\int_{t_0}^{\infty} k(s)ds \le \mu; \quad \int_{t}^{\infty} h_i(s)ds \le \beta(t), \quad t \ge t_0, \quad i = 1, 2$$

for some $\mu \in (0, 1)$ and $\beta \in C([t_0, \infty), [0, \infty))$, where

$$\int_{t_0}^{t} \beta(s)ds = o(t^{\mu}), \quad t \to \infty.$$

Then for each $a, b \in \mathbb{R}$ the boundary value problem

$$\begin{aligned}
&x'' = f(t, x), \quad t \ge t_0, \\
&x(t_0) = x_0, \\
&x(t) = at + b + o(t^{\mu}), \quad t \to \infty
\end{aligned}$$

has a unique solution.

Let v be a nonprincipal solution of (3.4). Without loss of generality we may assume that $v(t) > 0$, if $t \ge t_2$ for some $t_2 \ge 0$. It is easy to see that [34, 33]

$$u(t) = v(t)\int_{t}^{\infty} \frac{1}{p(s)v^2(s)}ds \tag{3.16}$$

is a principal solution of (3.4) which is strictly positive. Take t_2 large enough so that

$$\int_{t}^{\infty} \frac{1}{p(s)v^2(s)}ds \le 1, \quad t \ge t_2.$$

Then from (3.16), we have $v(t) \ge u(t)$ for $t \ge t_2$, which is needed in the proof of the next theorem.

Theorem 3.2.3 *Let $t_0 \geq t_2$. Assume that the function f satisfies* (3.9) *and* (3.10). *Suppose further that*

$$\int_{t_0}^{\infty} v(s)k(s)ds \leq \mu \tag{3.17}$$

and

$$\frac{1}{p(t)v^2(t)} \int_t^{\infty} v(s)h_i(s)ds \leq \beta(t), \quad t \geq t_0, \quad i = 1, 2 \tag{3.18}$$

hold for some $\beta \in C([t_0, \infty), [0, \infty))$ such that

$$\int_{t_0}^{t} \beta(s)ds = o((u(t))^{\mu}), \quad t \to \infty. \tag{3.19}$$

If either

$$u(t) \to \infty, \quad t \to \infty \tag{3.20}$$

or

$$a = \frac{x_0}{v(t_0)} - b \int_{t_0}^{\infty} \frac{1}{p(s)v^2(s)} ds, \tag{3.21}$$

then there is a unique solution $x(t)$ of (3.1)–(3.3), *where r is given by* (3.6).

Proof. Denote by M the supremum of the function g over $[0, \infty)$, as in the proof of the previous theorem. Define a function set X by

$$X = \{x \in C([t_0, \infty), \mathbb{R}) | \quad |x(t)| \leq l_1 v(t) + l_2 u(t), \quad \forall t \geq t_0\},$$

where

$$l_1 = (M + 1)p(t_0)u(t_0)v(t_0)\beta(t_0) + \frac{|x_0|}{v(t_0)} + |b| \int_{t_0}^{\infty} \frac{1}{p(s)v^2(s)} ds \quad \text{and} \quad l_2 = |b|.$$

Again, X is a complete metric space with the metric d given by the formula as in the proof of the previous theorem. Define an operator F on X by

$$\begin{aligned}
(Fx)(t) &= -v(t) \int_{t_0}^{t} \frac{1}{p(s)v^2(s)} \int_s^{\infty} v(\tau) f(\tau, x(\tau)) d\tau ds \\
&\quad + \left[\frac{x_0}{v(t_0)} - b \int_{t_0}^{\infty} \frac{1}{p(s)v^2(s)} ds \right] v(t) + bu(t).
\end{aligned}$$

In view of conditions (3.9) and (3.18) we see that F is well defined. Now, let $x \in X$ be arbitrary. Then

$$\begin{aligned}
|(Fx)(t)| &\leq v(t) \int_{t_0}^{t} \frac{1}{p(s)v^2(s)} \int_s^{\infty} v(\tau)|f(\tau, x(\tau))| d\tau ds + |d_1| v(t) + l_2 u(t) \\
&\leq v(t) \int_{t_0}^{\infty} \frac{1}{p(s)v^2(s)} \int_{t_0}^{\infty} v(\tau)|f(\tau, x(\tau))| d\tau ds + |d_1| v(t) + l_2 u(t) \\
&\leq v(t) \int_{t_0}^{\infty} \frac{1}{p(s)v^2(s)} \int_{t_0}^{\infty} v(\tau)(h_1(\tau)g(|x(\tau)|) + h_2(\tau)) d\tau ds + |d_1| v(t) + l_2 u(t) \\
&\leq v(t) \int_{t_0}^{\infty} \frac{1}{p(s)v^2(s)} \int_{t_0}^{\infty} v(\tau)(Mh_1(\tau) + h_2(\tau)) d\tau ds + |d_1| v(t) + l_2 u(t) \\
&\leq (M + 1)p(t_0)u(t_0)v^2(t_0)\beta(t_0)v(t) \int_{t_0}^{\infty} \frac{1}{p(s)v^2(s)} ds + |d_1| v(t) + l_2 u(t) \\
&\leq l_1 v(t) + l_2 u(t),
\end{aligned}$$

where

$$d_1 = \frac{x_0}{v(t_0)} - b \int_{t_0}^{\infty} \frac{1}{p(s)v^2(s)} ds.$$

Thus we see that $Fx \in X$.

Using (3.10), (3.16), and (3.17) we also see that

$$\begin{aligned}
|(Fx_1)(t) - (Fx_2)(t)| &\leq v(t) \int_{t_0}^{t} \frac{1}{p(s)v^2(s)} \int_{s}^{\infty} v(\tau)|f(\tau, x_1(\tau)) - f(\tau, x_2(\tau))| d\tau ds \\
&\leq v(t) \int_{t_0}^{t} \frac{1}{p(s)v^2(s)} \int_{s}^{\infty} v(\tau) \frac{k(\tau)}{v(\tau)} |x_1(\tau) - x_2(\tau)| d\tau ds \\
&\leq d(x_1, x_2) v(t) \int_{t_0}^{t} \frac{1}{p(s)v^2(s)} \int_{s}^{\infty} v(\tau) k(\tau) d\tau ds \\
&\leq d(x_1, x_2) v(t) \int_{t_0}^{t} \frac{1}{p(s)v^2(s)} \int_{t_0}^{\infty} v(\tau) k(\tau) d\tau ds \\
&\leq d(x_1, x_2) v(t) \int_{t_0}^{\infty} v(\tau) k(\tau) d\tau \\
&\leq \mu d(x_1, x_2) v(t),
\end{aligned}$$

where $x_1, x_2 \in X$ are arbitrary. This implies that F is a contraction mapping.

Thus according to Banach contraction principle F has a unique fixed point x. It is easy to check that the fixed point solves (3.1) and (3.2). It remains to show that $x(t)$ satisfies (3.3) as well. It is not difficult to show that

$$\begin{aligned}
|x(t) - av(t) - bu(t)| &\leq v(t) \int_{t_0}^{t} \frac{1}{p(s)v^2(s)} \int_{s}^{\infty} v(\tau)|f(\tau, x(\tau))| d\tau ds + |d_2| v(t) \\
&\leq v(t) \int_{t_0}^{t} \frac{1}{p(s)v^2(s)} \int_{s}^{\infty} v(\tau)(Mh_1(\tau) + h_2(\tau)) d\tau ds + |d_2| v(t) \\
&\leq (M+1) v(t) \int_{t_0}^{t} \beta(s) ds + |d_2| v(t),
\end{aligned}$$

where

$$d_2 = \frac{x_0}{v(t_0)} - b \int_{t_0}^{\infty} \frac{1}{p(s)v^2(s)} ds - a.$$

If (3.20) is satisfied, then in view (3.19) and the above inequality we easily obtain (3.3). In case (3.21) holds, then $d_2 = 0$ and hence we still have (3.3).

■

Corollary 3.2.4 *Assume that the function f satisfies* (3.9) *and* (3.10). *Suppose further that*

$$\int_{t_0}^{\infty} s k(s) ds \leq \mu; \quad \frac{1}{t^2} \int_{t}^{\infty} s h_i(s) ds \leq \beta(t), \quad t \geq t_0, \quad i = 1, 2$$

for some $\mu \in (0, 1)$ and $\beta \in C([t_0, \infty), [0, \infty))$, where

$$\int_{1}^{t} \beta(s) ds = o(1), \quad t \to \infty.$$

If for any given $a, b \in \mathbb{R}$ the condition (3.21) *holds then the boundary value problem*

$$\begin{aligned}
x'' &= f(t, x), \quad t \geq t_0, \\
x(t_0) &= x_0, \\
x(t) &= at + b + o(t), \quad t \to \infty
\end{aligned}$$

has a unique solution.

3.3 An Example

Example 3.3.1 *Consider the boundary value problem*

$$(tx')' = \frac{1}{t^2}\arctan x + t^\nu, \quad t \ge t_0, \quad \nu < -2, \tag{3.22}$$

$$x(t_0) = x_0, \tag{3.23}$$

$$x(t) = a\ln t + b + o((\ln t)^\mu), \quad t \to \infty. \tag{3.24}$$

where $t_0 > t_1 = 1$ and $\mu \in (0,1)$ are chosen to satisfy

$$\frac{1+\ln t_0}{t_0} \le \mu. \tag{3.25}$$

Note that since

$$\lim_{t_0\to\infty}\frac{1+\ln t_0}{t_0} = 0$$

for any given $\mu \in (0,1)$, there is a t_0 such that (3.25) *holds.*

Comparing with the boundary value problem (3.1)–(3.3) we see that $p(t) = t$, $q(t) = 0$, and $f(t,x) = (1/t^2)\arctan x + t^\nu$. The corresponding linear equation becomes

$$(tx')' = 0, \quad t \ge t_0.$$

Clearly, we may take

$$u(t) = 1 \quad \textit{and} \quad v(t) = \ln t.$$

Let

$$h_1(t) = \frac{1}{t^2}, \quad h_2(t) = t^\nu, \quad g(x) = \arctan x, \quad k(t) = \frac{\ln t}{t^2}, \quad \beta(t) = \frac{1}{t^2}.$$

Then it is easy to check that

$$|f(t,x)| \le \frac{1}{t^2}\arctan|x| + t^\nu = h_1(t)g(|x|) + h_2(t),$$

$$|f(t,x_1) - f(t,x_2)| \le \frac{1}{t^2}|x_1 - x_2| = \frac{k(t)}{v(t)}|x_1 - x_2|,$$

$$\int_{t_0}^\infty k(s)ds = \int_{t_0}^\infty \frac{\ln s}{s^2}ds = \frac{1+\ln t_0}{t_0} \le \mu \quad \textit{by } (3.25)$$

$$\frac{1}{t}\int_t^\infty h_1(s)ds \le \frac{1}{t}\int_t^\infty \frac{1}{s^2}ds = \frac{1}{t^2} = \beta(t), \quad t \ge t_0,$$

$$\frac{1}{t}\int_t^\infty h_2(s)ds = -\frac{t^\nu}{\nu+1} \le \beta(t), \quad t \ge t_0,$$

$$\int_{t_0}^t \beta(s)ds = \int_{t_0}^t \frac{1}{s^2}ds = \frac{1}{t_0} - \frac{1}{t} = o((\ln t)^\mu), \quad t \to \infty, \quad \mu \in (0,1),$$

and

$$v(t) = \ln t \to \infty, \quad t \to \infty,$$

i.e., all the conditions of Theorem 3.2.1 are satisfied. Therefore we may conclude that if (3.25) holds, then the boundary value problem (3.22)–(3.24) has a unique solution.

Furthermore, we may also deduce that there exist solutions $x_1(t)$ and $x_2(t)$ such that

$$x_1(t) = 1 + o((\ln t)^\mu), \quad t \to \infty$$

and

$$x_2(t) = \ln t + o((\ln t)^\mu), \quad t \to \infty.$$

by taking $(a,b) = (0,1)$ and $(a,b) = (1,0)$, respectively.

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