COVARIANT SYMPLECTIC STRUCTURE AND CONSERVED CHARGES OF NEW MASSIVE GRAVITY

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submitted by **GÖKHAN ALKAÇ** in partial fulfillment of the requirements for the degree of **Master of Science in Physics Department, Middle East Technical University** by,

Prof. Dr. Canan ÖZGEN ________________
Dean, Graduate School of **Natural and Applied Sciences**

Prof. Dr. Mehmet ZEYREK ________________
Head of Department, **Physics**

Prof. Dr. Bahtiyar Özgür SARIOĞLU ________________
Supervisor, **Physics Department, METU**

Examining Committee Members:

Prof. Dr. Atalay KARASU
Physics Department, METU ________________

Prof. Dr. Bahtiyar Özgür SARIOĞLU
Physics Department, METU ________________

Prof. Dr. Bayram TEKİN
Physics Department, METU ________________

Assoc. Prof. Dr. Seçkin KÜRKÇÜOĞLU
Physics Department, METU ________________

Dr. İsmet YURDUŞEN
Mathematics Department, Hacettepe University ________________

Date: ________________

I hereby declare that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.

Name, Last Name: GÖKHAN ALKAÇ

Signature :

ABSTRACT

COVARIANT SYMPLECTIC STRUCTURE AND CONSERVED CHARGES OF NEW MASSIVE GRAVITY

Alkaç, Gökhan
M.S., Department of Physics
Supervisor : Prof. Dr. Bahtiyar Özgür SARIOĞLU

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We show that the symplectic current obtained from the boundary term, which arises in the first variation of a local diffeomorphism invariant action, is covariantly conserved for any gravity theory described by that action. Therefore, a Poincaré invariant two-form can be constructed on the phase space, which is shown to be closed without reference to a specific theory. Finally, we show that one can obtain a charge expression for gravity theories in various dimensions, which plays the role of the Abbott-Deser-Tekin charge for spacetimes with nonconstant curvature backgrounds, by using the diffeomorphism invariance of the symplectic two-form. As an example, we calculate the conserved charges of some solutions of new massive gravity and compare the results with previous works.

Keywords: symplectic two-form, conserved charges

ÖZ

NEW MASSIVE GRAVITY TEORİSİNİN KOVARYANT SİMPLEKTİK YAPISI VE KORUNAN YÜKLERİ

Alkaç, Gökhan

Yüksek Lisans, Fizik Bölümü

Tez Yöneticisi : Prof. Dr. Bahtiyar Özgür SARIOĞLU

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Lokal ve difeomorfizmler altında değişmez olan bir eylemin birinci varyasyonunda beliren sınır teriminden elde edilen simplektik akımın, bu tür bir eyleme sahip herhangi bir kütleçekimi kuramı için kovaryant biçimde korunduğu gösterildi. Bu sayede Poincaré dönüşümleri altında değişmez olacak biçimde faz uzayında tanımlanabilen simplektik iki-formun kapalı olduğu belli bir kurama bağlı kalmaksızın ispatlandı. Ayrıca, simplektik iki-formun difeomorfizmler altındaki değişmezliği kullanılarak korunan yükler için bir ifade elde edilebileceği ve bu ifadenin eğriliği sabit olmayan arka-plana sahip uzay-zamanlar için Abbott-Deser-Tekin (ADT) yüküne denk olduğu gösterildi. Son olarak, New Massive Gravity kuramının bazı çözümleri için korunan yükler hesaplandı ve daha önceki çalışmalarla karşılaştırıldı.

Anahtar Kelimeler: simplektik iki-form, korunan yükler

To Merve

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CHAPTER 1

INTRODUCTION

In canonical formulation of gravity theories one faces a major obstacle at the very beginning. The definition of momenta, which seems to be needed in order to construct the phase space of the theory, requires an explicit choice of time coordinate. The Arnowitt-Deser-Misner (ADM) formalism of General Relativity [1] is an important example, where spacetime is decomposed into spacelike hypersurfaces parametrized by the time coordinate. The induced metric of the spacelike hypersurface and its conjugate momentum are used as dynamical variables to put field equations into the canonical form. Moreover, the Hamiltonian of General Relativity obtained through this way can be used to calculate the conserved charges. Doing the same for higher curvature theories is obviously much harder to perform.

In this thesis, we attack this problem in a different way. It was shown in [2, 3, 4] that it is possible to construct the phase space from the solutions of the classical equations instead of choosing coordinates and momenta. The symplectic two-form defined on the phase space contains all the relevant properties of the phase space in an invariant manner, which can be used for the geometric quantization of the theory under consideration. What is more important for our purposes is that the diffeomorphism invariance of the symplectic two-form leads to a closed expression for the conserved charges of solutions of the theory, when the diffeomorphisms are restricted to be the isometries of the background spacetime. The most important result proved in this work is that the conserved charges obtained through this procedure is equivalent to the Abbott-Deser-Tekin (ADT) [5, 6, 7] charge definition for quadratic curvature gravity theories on constant curvature backgrounds.

Computation of conserved charges by employing this procedure was first given for Topologically Massive Gravity (TMG) [8] in [9]. Here, we will work on a three-dimensional gravity theory, called New Massive Gravity (NMG) [10, 11], which is obtained by the addition of a particular higher curvature term ($\alpha R^2 + \beta R_{ab}^2$ with $8\alpha + 3\beta = 0$) to the usual Einstein-Hilbert action. This combination makes the theory tree-level unitary [12] but not renormalizable [13]. It is a valuable model for us to study because of its interesting solutions with AdS_3 and arbitrary backgrounds.

The organization of this thesis is as follows: Chapter 2 starts with the definition of the fundamental objects on the phase space and the construction of the symplectic two-form for a scalar field theory. Then, the symplectic structure of General Relativity is constructed. The main novelty here is that the diffeomorphism invariance of the symplectic two-form can be established, which is also related to the conserved charges. In Chapter 3, we consider a generic local gravity action and some general remarks about the procedure are made. Chapter 4 is devoted to the study of theories arising from the action

$$I = \int \mathrm{d}^D x \, \sqrt{|g|} \Big(\frac{1}{\kappa}(R + 2\Lambda_0) + \alpha R^2 + \beta R_{ab}^2 \Big).$$

Finally, the energy and angular momentum of some solutions of NMG are computed in Chapter 5 using the formulas derived in the previous one.

The results of this thesis has already been published in [14]. Chapter 2 is included to discuss the basic ideas from previous works better and all the other chapters are mainly based on [14].

Conventions used throughout are: The signature of the metric is $(-, +, \cdots, +)$. The Riemann tensor is defined through $[\nabla_a, \nabla_b]V_c = R_{abc}{}^d V_d$ and $R_{ab} = R^c{}_{acb}$. For the symmetrization and antisymmetrization of tensors, the factors and signs are chosen so that, e.g. $T_{(ab)} \equiv \frac{1}{2}(T_{ab} + T_{ba})$, $T_{[ab]} \equiv \frac{1}{2}(T_{ab} - T_{ba})$.

CHAPTER 2

CONSTRUCTION OF THE SYMPLECTIC TWO-FORM AND ITS RELATION TO CONSERVED CHARGES

At first glance, it might seem impossible to develop the canonical formulation of a geometrical theory because manifest Poincaré invariance is lost by the choice of an explicit time coordinate, which is necessary to define momenta conjugate to coordinates. However, it was shown in [2, 3, 4] that one can construct phase space without defining the momenta and therefore the canonical formulation can be developed in a way that preserves Poincaré invariance manifestly. To demonstrate the basic ideas, we start with a very simple case, i.e. a scalar field theory . In what follows, we will employ the construction and notation of [3].

2.1 Scalar Field Theory

For a theory with N degrees of freedom, one can introduce coordinates q^i and momenta p_i, where $i = 1, \ldots, N$. As is well known, the Poisson bracket of any two functions $A(q, p)$ and $B(q, p)$ is defined by

$$[A, B] = \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q^i} - \frac{\partial A}{\partial q^i}\frac{\partial B}{\partial p_i}. \tag{2.1}$$

A two-form on the phase space can be defined as

$$\omega = \mathrm{d}p_i \wedge \mathrm{d}q^i. \tag{2.2}$$

When $q^i$and p_i are combined in a new variable Q^I (where $I = 1, \ldots, 2N$ and $Q^i = p_i$ for $i \leq N$, $Q^i = q^{i-N}$ for $i > N$), ω can also be written as

$$\omega = \frac{1}{2}\,\omega_{IJ}\,\mathrm{d}Q^I \wedge \mathrm{d}Q^J. \tag{2.3}$$

The components of ω can be seen as an antisymmetric $2N \times 2N$ matrix ω_{IJ} with the only non-vanishing elements given by $\omega_{i,i+N} = -\omega_{i+N,i} = 1$. This matrix is invertible, and the Poisson bracket of the functions $A(Q)$ and $B(Q)$ can be written using its inverse ω^{IJ}:

$$[A, B] = \omega^{IJ} \frac{\partial A}{\partial Q^I} \frac{\partial B}{\partial Q^J}. \tag{2.4}$$

It is easy to see that (2.1) and (2.4) coincide. The advantage of using the definitions (2.3) and (2.4) is that it allows one to describe the fundamental features of ω in an invariant way.

ω is a two-form on the phase space Z of the theory under consideration with coordinates ps and qs. It is closed

$$\mathrm{d}\omega = \frac{1}{2}\, \partial_{\mathrm{K}} \omega_{\mathrm{IJ}}\, \mathrm{dQ^K} \wedge \mathrm{dQ^I} \wedge \mathrm{dQ^J} = 0, \tag{2.5}$$

since its components are all constant. It is also non-degenerate, i.e. the matrix $\omega_{IJ}(z)$ is invertible at each point $z \in Z$. However, the converse is more important for our purposes. Let ω be any two-form on the phase space Z of a classical theory. Darboux's theorem states that if ω is closed ($\mathrm{d}\omega = 0$) and non-degenerate, it is possible to introduce local coordinates on Z such that ω can be put in the form (2.2). A non-degenerate closed two-form on Z is called a *symplectic two-form*. In order to obtain the canonical formulation of a theory, one can simply construct a symplectic two-form on the classical phase space of the theory ,instead of choosing ps and qs, because it includes all the essential properties of the phase space.

Now, the problem is to find a new way to construct the phase space, which can be achieved by making use of the fact that classical solutions of a physical theory are in one-to-one correspondence with the initial values of ps and qs. We define our phase space as the space of solutions of the classical equations of motion to maintain the covariance. One can always pick a coordinate system and work with the ps and qs with their initial values corresponding to the classical solutions but there is no need to do this and to destroy the manifestation of Poincaré invariance.

For the construction of the symplectic two-form on this phase space Z, we first need to discuss the fundamental objects on Z, which are functions, tangent vectors and differential forms. Exterior derivatives on Z must also be defined to obtain the differential forms of rank higher than 1. Let us consider a scalar field theory derived from the following action

$$S = \int \mathrm{d}^D x \left(\frac{1}{2} \partial_a \phi\, \partial^a \phi - V(\phi) \right), \tag{2.6}$$

whose first variation gives

$$\delta S = -\int \mathrm{d}^D x \, (\Box\phi + V'(\phi))\, \delta\phi + \int \mathrm{d}^D x \, \partial_a(\underbrace{\partial^a \phi \, \delta\phi}_{\Lambda^a}), \tag{2.7}$$

where $\Box \equiv \partial_a \partial^a$ and $V'(\phi) = \frac{\mathrm{d}V}{\mathrm{d}\phi}$. Therefore, the field equations are

$$\Box\phi + V'(\phi) = 0, \tag{2.8}$$

and a point in Z is a solution of the classical equations (2.8).

In order to define functions on Z, consider a function ϕ, which is a solution of (2.8). When it is evaluated at a spacetime point x, the result is a real number $\phi(x)$. We can define a function on Z, denoted by $\phi(x)$, as the mapping from the function ϕ to the number $\phi(x)$.

For the tangent vectors at a point on Z, we consider a small variation in ϕ. If $\tilde{\phi} = \phi + \delta\phi$ preserves the field equations (2.8) to the first order in the variation, $\delta\phi$ is a vector at the point ϕ on Z. Therefore, vectors are the solutions of

$$[\Box + V''(\phi)]\, \delta\phi = 0. \tag{2.9}$$

A one-form on Z maps a solution $\delta\phi$ of (2.8) into $\delta\phi(x)$, the number obtained by evaluating $\delta\phi$ at a spacetime point x. One can generalize this to define p-forms as "wedge functions" of the one-forms $\delta\phi(x)$

$$\Omega = \int \mathrm{d}x_1 \cdots \mathrm{d}x_p \, \Theta(x_1, \cdots, x_p)\, \delta\phi(x_1) \wedge \cdots \wedge \delta\phi(x_p), \tag{2.10}$$

where $\Theta(x_1, \cdots, x_p)$ is a zero-form, a function of p-tuple of spacetime points $x_1, \cdots, x_p$. The last term in (2.10) is the wedge product of one-forms $\delta\phi(x_i)$, which is anti-commuting as usual

$$\delta\phi(x_1) \wedge \delta\phi(x_2) = -\,\delta\phi(x_2) \wedge \delta\phi(x_1). \tag{2.11}$$

We use the following exterior derivative operator which maps p-forms to $(p+1)$-forms

$$\delta\Omega = \int \mathrm{d}x_{p+1}\, \mathrm{d}x_1 \cdots \mathrm{d}x_p \, \frac{\delta\Theta(x_1, \cdots, x_p)}{\delta\phi(x_{p+1})}\, \delta\phi(x_{p+1}) \wedge \delta\phi(x_1) \wedge \cdots \wedge \delta\phi(x_p), \tag{2.12}$$

where $\frac{\delta\Theta(x_1, \cdots, x_p)}{\delta\phi(x_{p+1})}$ is the functional derivative of Θ with respect to $\delta\phi(x)$. Note that the exterior derivative of (2.12) is

$$\begin{aligned} \delta^2\Omega &= \int \mathrm{d}x_{p+2}\, \mathrm{d}x_{p+1}\, \mathrm{d}x_1 \cdots \mathrm{d}x_p \, \frac{\delta^2\Theta(x_1, \cdots, x_p)}{\delta\phi(x_{p+2})\delta\phi(x_{p+1})}\, \delta\phi(x_{p+2}) \wedge \delta\phi(x_{p+1}) \wedge \delta\phi(x_1) \wedge \cdots \wedge \delta\phi(x_p) = 0, \\ &= 0 \end{aligned} \tag{2.13}$$

since the functional derivative is symmetric and the wedge product is anti-symmetric in $\delta\phi(x_{p+1})$ and $\delta\phi(x_{p+2})$, and thus this operator obeys the Poincaré lemma ($\delta^2 = 0$). The exterior derivative of the wedge product of a p-form A and q-form B is

$$\delta(A \wedge B) = \delta A \wedge B + (-1)^p A \wedge \delta B, \tag{2.14}$$

since the operator moves through p one-forms and each gives a factor of -1. Thus, it also obeys the (modified) Leibniz rule and can be safely used as an exterior derivative operator on the phase space Z.

Now, we are ready to define a symplectic two-form using the "symplectic current" obtained from the boundary term in the first variation of the action (2.7)

$$J^a = -\delta\Lambda^a = \delta\phi \wedge \partial^a \delta\phi. \tag{2.15}$$

It is easy to show that J^a is conserved in its spacetime dependence

$$\partial_a J^a = \partial_a \delta\phi \wedge \partial^a \delta\phi + \delta\phi \wedge \Box\delta\phi = 0. \tag{2.16}$$

The first term in (2.16) vanishes by the anticommutation of wedge products

$$\partial_a \delta\phi \wedge \partial^a \delta\phi = \eta^{ab} \partial_a \delta\phi \wedge \partial_b \delta\phi = \partial^a \delta\phi \wedge \partial_a \delta\phi = -\partial_a \delta\phi \wedge \partial^a \delta\phi = 0. \tag{2.17}$$

For the second term in (2.16), one should also use (2.9)

$$\delta\phi \wedge \Box\delta\phi = -V''(\phi)\, \delta\phi \wedge \delta\phi = V''(\phi)\, \delta\phi \wedge \delta\phi = 0. \tag{2.18}$$

Since J^a is a conserved current on the spacetime manifold, its integral over a spacelike hypersurface Σ is independent of the choice of Σ and is Poincaré invariant

$$\omega = \int_\Sigma \mathrm{d}\Sigma_a\, J^a = \int \mathrm{d}\Sigma_a\, \delta\phi \wedge \partial^a \delta\phi. \tag{2.19}$$

In order to prove this, we consider the integral of a covariantly conserved vector over a spacetime with the metric g_{ab}, where the integration over the Minkowski spacetime is a special case

$$\int \mathrm{d}^D x \sqrt{|g|}\, \nabla_a J^a = \oint_{\Sigma'} \mathrm{d}\Sigma_a J^a = 0. \tag{2.20}$$

Here Σ' is any closed hypersurface. Choosing Σ' as the composition of two spacelike hypersurfaces Σ_1 and Σ_2, which extend to infinity, and a $(D-1)$-cylinder at infinity, on which J^a vanishes, leads to

$$\int_{\Sigma_1} \mathrm{d}\Sigma_a J^a + \int_{\Sigma_2} \mathrm{d}\Sigma_a J^a = 0. \tag{2.21}$$

Since we have used Stokes' theorem in (2.20), the normal vector of Σ_1 is past directing and that of Σ_2 is future directing (outward normal vectors of Σ'). Using the future directing normal vectors for both changes the sign of the integration over Σ_1 and we have

$$\int_{\Sigma_1} \mathrm{d}\Sigma_a J^a = \int_{\Sigma_2} \mathrm{d}\Sigma_a J^a. \tag{2.22}$$

Therefore, the result of this integration is independent of the choice of the hypersurface over which the integration is performed and it is also Poincaré invariant.

Note also that ω is closed as a result of the Poincaré lemma

$$\delta\omega = \int_{\Sigma} \mathrm{d}\Sigma_a\, \delta^2\phi \wedge \partial^a\delta\phi + \delta\phi \wedge \partial^a\delta^2\phi = 0. \tag{2.23}$$

Choosing Σ as the surface defined by $t = 0$ with a normal vector $n^a = \delta^a_t$ gives

$$\begin{aligned} \omega &= \int_{\Sigma} \mathrm{d}\Sigma_a\, J^a = \int_{\Sigma} \mathrm{d}^3x\, n_a \delta\phi \wedge \partial^a\delta\phi = \int_{\Sigma} \mathrm{d}^3x\, n^a \delta\phi \wedge \partial_a\delta\phi \\ &= \int_{\Sigma} \mathrm{d}^3x\, \delta\phi \wedge \delta\dot{\phi}, \end{aligned} \tag{2.24}$$

where $\dot{\phi} \equiv \frac{\partial\phi}{\partial t}$. This expression coincides with the standard definition (2.2) of the symplectic two-form (apart from a minus sign). Therefore, we conclude that ω in (2.19) is the symplectic two-form of the scalar field theory given by the action (2.6). Our next aim is to apply the same procedure to General Relativity.

2.2 General Relativity

In this section, we follow closely [3] and expand the computation given there to find the symplectic two-form of General Relativity. We start with the Einstein-Hilbert action

$$S = \int \mathrm{d}^D x \sqrt{|g|}\, R, \tag{2.25}$$

and its first variation

$$\delta S = \int \mathrm{d}^D x \sqrt{|g|} \left(R_{ab} - \frac{1}{2} g_{ab} R \right) \delta g^{ab} + \int \mathrm{d}^D x \sqrt{|g|}\, \partial_a (\sqrt{|g|} g^{bc} \delta\Gamma^a_{bc} - \sqrt{|g|} g^{ab} \delta\Gamma^c_{bc}). \tag{2.26}$$

The field equations and the boundary term read

$$\begin{aligned} \mathcal{G}_{ab} &= R_{ab} - \frac{1}{2} g_{ab} R = 0 \Rightarrow R_{ab} = 0, \\ \Lambda^a &= \sqrt{|g|} g^{bc} \delta\Gamma^a_{bc} - \sqrt{|g|} g^{ab} \delta\Gamma^c_{bc}. \end{aligned} \tag{2.27}$$

This time, the symplectic current is given by

$$J^a = -\frac{\delta\Lambda^a}{\sqrt{|g|}} = \delta\Gamma^a_{bc} \wedge (\delta g^{bc} + \frac{1}{2}g^{bc}\,\delta\ln|g|) - \delta\Gamma^c_{bc} \wedge (\delta g^{ab} + \frac{1}{2}g^{ab}\,\delta\ln|g|), \tag{2.28}$$

where $\delta\ln|g| = g^{ab}\delta g_{ab} = -g_{ab}\delta g^{ab}$ and

$$\delta\Gamma^a_{bc} = \frac{1}{2}g^{ad}(\nabla_b\delta g_{cd} + \nabla_c\delta g_{bd} - \nabla_d\delta g_{bc}). \tag{2.29}$$

We need to compute the covariant divergence of this current and show that . The terms with $\delta\ln|g|$ are

$$\begin{aligned} A \quad = \quad & \underbrace{\frac{1}{2}g^{bc}\,\nabla_a\delta\Gamma^a_{bc} \wedge \delta\ln|g|}_{X} + \underbrace{\frac{1}{2}g^{bc}\,\delta\Gamma^a_{bc} \wedge \nabla_a\delta\ln|g|}_{B} - \underbrace{\frac{1}{2}g^{ab}\,\nabla_a\delta\Gamma^c_{bc} \wedge \delta\ln|g|}_{C} \\ & -\underbrace{\frac{1}{2}g^{ab}\,\delta\Gamma^c_{bc} \wedge \nabla_a\delta\ln|g|}_{D}. \end{aligned} \tag{2.30}$$

From (2.29) we obtain

$$\begin{aligned} \delta\Gamma^a_{ba} \quad &= \quad \frac{1}{2}g^{ad}(\nabla_b\delta g_{ad} + \nabla_a\delta g_{bd} - \nabla_d\delta g_{ba}) = \frac{1}{2}g^{ad}\,\nabla_b\delta g_{ad} \\ &= \quad \frac{1}{2}\,\nabla_b\delta\ln|g|. \end{aligned} \tag{2.31}$$

Using this we can write

$$\begin{aligned} B \quad &= \quad g^{bc}\,\delta\Gamma^a_{bc} \wedge \delta\Gamma^d_{ad}, \\ D \quad &= \quad -g^{ab}\,\delta\Gamma^c_{bc} \wedge \delta\Gamma^d_{ad} = 0. \end{aligned} \tag{2.32}$$

The term C can be written with the help of $\delta R_{ab} = \nabla_c\delta\Gamma^c_{ab} - \nabla_a\delta\Gamma^c_{bc}$ as

$$\begin{aligned} C \quad &= \quad -\frac{1}{2}g^{ab}(\nabla_c\delta\Gamma^c_{ab} - \delta R_{ab}) \wedge \delta\ln|g| \\ &= \quad -\underbrace{\frac{1}{2}g^{ab}\,\nabla_c\delta\Gamma^c_{ab} \wedge \delta\ln|g|}_{X} + \frac{1}{2}g^{ab}\,\delta R_{ab} \wedge \delta\ln|g|. \end{aligned} \tag{2.33}$$

Terms X cancel each other and we get

$$A = g^{bc}\,\delta\Gamma^a_{bc} \wedge \delta\Gamma^d_{ad} + \frac{1}{2}g^{ab}\,\delta R_{ab} \wedge \delta\ln|g|. \tag{2.34}$$

The terms that do not contain the factor $\delta\ln|g|$ are

$$E = \underbrace{\nabla_a\delta\Gamma^a_{bc} \wedge \delta g^{bc}}_{X} + \underbrace{\delta\Gamma^a_{bc} \wedge \nabla_a\delta g^{bc}}_{F} + \underbrace{\nabla_a\delta\Gamma^c_{bc} \wedge \delta g^{ab}}_{G} + \underbrace{\delta\Gamma^c_{bc} \wedge \nabla_a\delta g^{ab}}_{H}. \tag{2.35}$$

For the terms F and H, we need a relation between the covariant derivative of the inverse metric and the variation of the Christoffel symbols. We first write (2.29) as

$$\begin{aligned} 2g_{ae}\,\delta\Gamma^a_{bc} &= \nabla_b\delta g_{ce} + \nabla_c\delta g_{be} - \nabla_e\delta g_{bc}, \\ 2g_{ac}\,\delta\Gamma^a_{be} &= \nabla_b\delta g_{ec} + \nabla_e\delta g_{bc} - \nabla_c\delta g_{be}, \end{aligned} \tag{2.36}$$

which together yield

$$\begin{aligned} \nabla_b\delta g_{ec} &= g_{ae}\,\delta\Gamma^a_{bc} + g_{ac}\,\delta\Gamma^a_{be}, \\ g^{ef}g^{ch}\,\nabla_b\delta g_{ec} &= g^{ef}g^{ch}\left(g_{ae}\,\delta\Gamma^a_{bc} + g_{ac}\,\delta\Gamma^a_{be}\right), \\ -\nabla_b\delta g^{fh} &= g^{ch}\,\delta\Gamma^f_{bc} + g^{cf}\,\delta\Gamma^h_{bc}. \end{aligned} \tag{2.37}$$

The desired relations can be written as

$$\begin{aligned} \nabla_a\delta g^{bc} &= -g^{db}\,\delta\Gamma^c_{ad} - g^{dc}\,\delta\Gamma^b_{ad}, \\ \nabla_a\delta g^{ab} &= -g^{db}\,\delta\Gamma^a_{ad} - g^{da}\,\delta\Gamma^b_{ad}. \end{aligned} \tag{2.38}$$

Thus, we find that

$$\begin{aligned} F &= -g^{db}\,\delta\Gamma^a_{bc}\wedge\delta\Gamma^c_{ad} - g^{dc}\,\delta\Gamma^a_{bc}\wedge\delta\Gamma^d_{ad} = 0 \\ H &= \underbrace{g^{db}\,\delta\Gamma^c_{bc}\wedge\delta\Gamma^a_{ad}}_{=0} + g^{da}\,\delta\Gamma^c_{bc}\wedge\delta\Gamma^b_{ad} = g^{da}\,\delta\Gamma^c_{bc}\wedge\delta\Gamma^b_{ad} \\ G &= -(\nabla_c\delta\Gamma^c_{ab} - \delta R_{ab})\wedge\delta g^{ab} = -\underbrace{\nabla_c\delta\Gamma^c_{ab}\wedge\delta g^{ab}}_{X} + \delta R_{ab}\wedge\delta g^{ab}. \end{aligned} \tag{2.39}$$

Terms X again cancel and

$$E = g^{da}\,\delta\Gamma^c_{bc}\wedge\delta\Gamma^b_{ad} + \delta R_{ab}\wedge\delta g^{ab}.$$

Finally

$$\begin{aligned} \nabla_a J^a &= A + E = \underbrace{g^{bc}\,\delta\Gamma^a_{bc}\wedge\delta\Gamma^d_{ad} + g^{da}\,\delta\Gamma^c_{bc}\wedge\delta\Gamma^b_{ad}}_{=0} + \frac{1}{2}g^{ab}\,\delta R_{ab}\wedge\delta\ln|g| + \delta R_{ab}\wedge\delta g^{ab} \\ &= \frac{1}{2}g^{ab}\,\delta R_{ab}\wedge\delta\ln|g| + \delta R_{ab}\wedge\delta g^{ab} \end{aligned} \tag{2.40}$$

which clearly vanishes on-shell ($R_{ab} = 0 \Rightarrow \delta R_{ab} = 0$). As a result, the following Poincaré invariant two-form can be defined

$$\begin{aligned} \omega &= \int_\Sigma \mathrm{d}\Sigma_a\,\sqrt{|g|}\,J^a, \\ &= \int_\Sigma \mathrm{d}\Sigma_a\,\sqrt{|g|}\left[\delta\Gamma^a_{bc}\wedge(\delta g^{bc} + \frac{1}{2}g^{bc}\,\delta\ln|g|) - \delta\Gamma^c_{bc}\wedge(\delta g^{ab} + \frac{1}{2}g^{ab}\,\delta\ln|g|)\right], \end{aligned} \tag{2.41}$$

whose exterior derivative is

$$\delta\omega = \int_\Sigma \mathrm{d}\Sigma_a\,(\delta\sqrt{|g|} \wedge J^a + \sqrt{|g|}\,\delta J^a), \tag{2.42}$$

where $\delta\sqrt{|g|} = \frac{1}{2}\sqrt{|g|}\delta\ln|g|$. The second term is

$$\delta J^a = \frac{1}{2}\,\delta g^{bc} \wedge \delta\Gamma^a_{bc} \wedge \delta\ln|g| - \frac{1}{2}\,\delta g^{ab} \wedge \delta\Gamma^c_{bc} \wedge \delta\ln|g|, \tag{2.43}$$

which is equal to

$$\begin{aligned} -\frac{1}{2}J^a \wedge \delta\ln|g| &= -\frac{1}{2}\,\delta\Gamma^a_{bc} \wedge \delta g^{bc} \wedge \delta\ln|g| - \underbrace{\frac{1}{4}g^{bc}\,\delta\Gamma^a_{bc} \wedge \delta\ln|g| \wedge \delta\ln|g|}_{=0} \\ &\quad +\frac{1}{2}\,\delta\Gamma^c_{bc} \wedge \delta g^{ab} \wedge \delta\ln|g| + \underbrace{\frac{1}{4}g^{ab}\,\delta\Gamma^c_{bc} \wedge \delta\ln|g| \wedge \delta\ln|g|}_{=0}, \\ &= \frac{1}{2}\,\delta g^{bc} \wedge \delta\Gamma^a_{bc} \wedge \delta\ln|g| - \frac{1}{2}\,\delta g^{ab} \wedge \delta\Gamma^c_{bc} \wedge \delta\ln|g|. \end{aligned} \tag{2.44}$$

Since J^a is a two-form, $\delta\sqrt{|g|} \wedge J^a = J^a \wedge \delta\sqrt{|g|} = \frac{1}{2}\sqrt{|g|}J^a \wedge \delta\ln|g|$, and therefore (2.42) vanishes.

The final task is to show the gauge invariance of ω in the space of classical solutions Z and in the quotient space $\bar{Z} = Z/G$, where G denotes the group of diffeomorphisms ($x^a \to x^a + \xi^a$). The first one is trivial since all the objects in ω are tensors. For the latter, we should check the behavior of ω under the following transformation

$$\delta g_{ab} \to \delta g_{ab} + \nabla_a\xi_b + \nabla_b\xi_a, \tag{2.45}$$

where we assume that ξ is asymptotically a Killing vector field at the boundary of the space-time ($\nabla_a\xi_b + \nabla_b\xi_a = 0$ at infinity). We need the transformation of the following quantities

$$\begin{aligned} \delta g^{ab} &\to \delta g^{ab} - \nabla^a\xi^b - \nabla^b\xi^a, \\ \delta\ln|g| &\to \delta\ln|g| + 2\nabla_d\xi^d, \end{aligned} \tag{2.46}$$

which can be obtained using $\delta g^{ab} = -g^{ab}g^{cd}\,\delta g_{cd}$ and $g^{ab}\,\delta g_{ab} = \delta\ln|g|$, and also $\delta\Gamma^a_{bc}$. Inserting (2.45) into (2.29) gives

$$\begin{aligned} \delta\Gamma^a_{bc} &\to \frac{1}{2}g^{ad}\,[\nabla_b(\delta g_{cd} + \nabla_c\xi_d + \nabla_d\xi_c) + \nabla_c(\delta g_{bd} + \nabla_b\xi_d + \nabla_d\xi_b) - \nabla_d(\delta g_{bc} + \nabla_b\xi_c + \nabla_c\xi_b)] \\ &= \delta\Gamma^a_{bc} + \frac{1}{2}(\nabla_b\nabla_c\xi^a + \nabla_b\nabla^a\xi_c + \nabla_c\nabla_b\xi^a + \nabla_c\nabla^a\xi_b - \nabla^a\nabla_b\xi_c - \nabla^a\nabla_c\xi_b). \end{aligned} \tag{2.47}$$

Using

$$\begin{aligned}
\nabla_b \nabla_c \xi^a &= \nabla_c \nabla_b \xi^a + R^a_{\ dbc} \xi^d, \\
\nabla_b \nabla^a \xi_c &= \nabla^a \nabla_b \xi_c + R_{cdb}{}^a \xi^d, \\
\nabla_c \nabla^a \xi_b &= \nabla^a \nabla_c \xi_b + R_{bdc}{}^a \xi^d
\end{aligned} \tag{2.48}$$

in (2.47) gives

$$\delta\Gamma^a_{bc} \to \delta\Gamma^a_{bc} + \frac{1}{2}\left[2\nabla_c \nabla_b \xi^a + \underbrace{(R^a_{\ dbc} + R_{cdb}{}^a + R_{bdc}{}^a)}_{=R^a_{\ dbc}+R^a_{\ bdc}+R^a_{\ cdb}} \xi^d \right], \tag{2.49}$$

which can be written in a simpler form with the help of

$$R^a_{\ dbc} + R^a_{\ bcd} + R^a_{\ cdb} = 0 \Rightarrow R^a_{\ dbc} + R^a_{\ cdb} = -R^a_{\ bcd} = R^a_{\ bdc}, \tag{2.50}$$

as

$$\begin{aligned}
\delta\Gamma^a_{bc} &\to \delta\Gamma^a_{bc} + R_{dc}{}^a{}_b \xi^d + \nabla_b \nabla_c \xi^a, \\
\delta\Gamma^c_{bc} &\to \delta\Gamma^c_{bc} - R_{bd} \xi^d + \nabla_c \nabla_b \xi^c.
\end{aligned} \tag{2.51}$$

Therefore, under (2.45), the symplectic current given in (2.28) transforms as

$$\begin{aligned}
J^a \;\to\; & (\delta\Gamma^a_{bc} + R_{ec}{}^a{}_b \xi^e + \nabla_c \nabla_b \xi^a) \wedge (\delta g^{bc} - \nabla^b \xi^c - \nabla^c \xi^b) \\
+ \; & \frac{1}{2} g^{bc} (\delta\Gamma^a_{bc} + R_{ec}{}^a{}_b \xi^e + \nabla_c \nabla_b \xi^a) \wedge (\delta \ln|g| + 2\nabla_d \xi^d) \\
- \; & (\delta\Gamma^c_{bc} - R_{be} \xi^e + \nabla_c \nabla_b \xi^c) \wedge (\delta g^{ab} - \nabla^a \xi^b - \nabla^b \xi^a) \\
- \; & \frac{1}{2} g^{ab} (\delta\Gamma^c_{bc} - R_{be} \xi^e + \nabla_c \nabla_b \xi^c) \wedge (\delta \ln|g| + 2\nabla_d \xi^d),
\end{aligned} \tag{2.52}$$

and the change in the symplectic current, which we denote $\triangle J^a$ to avoid a confusion, is given by

$$\begin{aligned}
\triangle J^a =\; & -\underbrace{2\delta\Gamma^a_{bc} \wedge \nabla^b \xi^c}_{F^a} + (\underbrace{g^{bc}\delta\Gamma^a_{bc}}_{E^a_1} - \underbrace{g^{ab}\delta\Gamma^c_{bc}}_{C^a_1}) \wedge \nabla_d \xi^d + R_{ec}{}^a{}_b \xi^e \wedge \delta g^{bc} + R_e{}^a \xi^e \wedge \delta \ln g \\
& + \underbrace{(\nabla_c \nabla_b \xi^a) \wedge \delta g^{bc}}_{B^a_2} + \underbrace{\frac{1}{2} \Box \xi^a \wedge \delta \ln|g|}_{A^a_1} + \underbrace{\delta\Gamma^c_{bc} \wedge (\nabla^a \xi^b + \nabla^b \xi^a)}_{C^a_2} + R_{eb} \xi^e \wedge \delta g^{ab} \\
& - \nabla_c \nabla_b \xi^c \wedge (\underbrace{\delta g^{ab}}_{B^a_1} + \underbrace{\frac{1}{2} g^{ab} \delta \ln|g|}_{A^a_2}).
\end{aligned} \tag{2.53}$$

Our aim now is to write this expression as the covariant derivative of an antisymmetric tensor plus some additional terms which vanish on-shell.

We start with A^a_1

$$
\begin{aligned}
A^a_1 \quad &\equiv \quad \frac{1}{2}\,\Box\xi^a \wedge \delta\ln|g| = \nabla_c(\frac{1}{2}\,\nabla^c\xi^a \wedge \delta\ln|g|) - \frac{1}{2}\,\nabla^c\xi^a \wedge \nabla_c\delta\ln|g| \\
&= \quad \nabla_c(\frac{1}{2}\,\nabla^c\xi^a\delta\ln|g|) - \frac{1}{2}(\nabla^c\xi^a)\delta\Gamma^b_{cb}.
\end{aligned} \tag{2.54}
$$

Its antisymmetric part comes from A^a_2

$$
\begin{aligned}
A^a_2 \quad &\equiv \quad -\frac{1}{2}g^{ab}\,\nabla_c\nabla_b\xi^c \wedge \delta\ln g = -\nabla_c(\frac{1}{2}\,\nabla^a\xi^c \wedge \delta\ln g) + \frac{1}{2}\,\nabla^a\xi^c \wedge \nabla_c\delta\ln g \\
&= \quad -\nabla_c(\frac{1}{2}\,\nabla^a\xi^c \wedge \delta\ln g) + \nabla^a\xi^c \wedge \delta\Gamma^b_{cb},
\end{aligned} \tag{2.55}
$$

so that

$$
A^a_1 + A^a_2 = \nabla_c(\underbrace{\nabla^{[c}\xi^{a]} \wedge \delta\ln g}_{Q^{ac}_1}) + (\nabla^a\xi^c - \nabla^c\xi^a) \wedge \delta\Gamma^b_{cb}. \tag{2.56}
$$

For B^a_1 and B^a_2, we have

$$
B^a_1 \equiv -\nabla_c\nabla_b\xi^c \wedge \delta g^{ab} = -\nabla_c(\nabla_b\xi^c \wedge \delta g^{ab}) + \nabla_b\xi^c \wedge \nabla_c\delta g^{ab}), \tag{2.57}
$$

$$
B^a_2 \equiv \nabla_c\nabla_b\xi^a \wedge \delta g^{bc} = \nabla_c(\nabla_b\xi^a \wedge \delta g^{bc}) - \nabla_b\xi^a \wedge \nabla_c\delta g^{bc}, \tag{2.58}
$$

which give

$$
B^a_1 + B^a_2 = \nabla_c(\underbrace{2\nabla_b\xi^{[a} \wedge \delta g^{c]b}}_{Q^{ac}_2}) + \nabla_b\xi^c \wedge \nabla_c\delta g^{ab} - \nabla_b\xi^a \wedge \nabla_c\delta g^{bc}. \tag{2.59}
$$

Now we write C^a_1 as

$$
\begin{aligned}
C^a_1 \quad &\equiv \quad -2g^{ab}\,\delta\Gamma^c_{bc} \wedge \nabla_d\xi^d + g^{ab}\,\delta\Gamma^c_{bc} \wedge \nabla_d\xi^d \\
&= \quad -\nabla_c(2g^{ab}\,\delta\Gamma^d_{bd} \wedge \xi^c) + 2g^{ab}\,\nabla_c\delta\Gamma^d_{bd} \wedge \xi^c + g^{ab}\,\delta\Gamma^c_{bc} \wedge \nabla_d\xi^d \\
&= \quad -\nabla_c(\nabla^a\delta\ln g \wedge \xi^c) + 2g^{ab}\,\nabla_c\delta\Gamma^d_{bd} \wedge \xi^c + g^{ab}\,\delta\Gamma^c_{bc} \wedge \nabla_d\xi^d
\end{aligned} \tag{2.60}
$$

We take $C^a_2 + A^a_1 + A^a_2$

$$
\begin{aligned}
C^a_2 + A^a_1 + A^a_2 &= \nabla_c Q^{ac}_1 + 2\delta\Gamma^c_{bc}\,\nabla^b\xi^a \\
&= \nabla_c Q^{ac}_1 + \nabla^b(2\,\delta\Gamma^c_{bc}\;\wedge \nabla\xi^a) - 2\,\nabla^b\delta\Gamma^c_{bc} \wedge \xi^a \\
&= \nabla_c(Q^{ac}_1 + \nabla^c\delta\ln g \wedge \xi^a) - 2\,\nabla^b\delta\Gamma^c_{bc} \wedge \xi^a.
\end{aligned} \tag{2.61}
$$

Together with C^a_1, these give

$$
\begin{aligned}
C^a_1 + C^a_2 + A^a_1 + A^a_2 \quad &= \quad \nabla_c(Q^{ac}_1 + \underbrace{2\nabla^{[c}\delta\ln g \wedge \xi^{a]}}_{Q^{ac}_3}) + 2g^{ab}\,\nabla_c\delta\Gamma^d_{bd} \wedge \xi^c + g^{ab}\,\delta\Gamma^c_{bc} \wedge \nabla_d\xi^d \\
&\quad\; -2\,\nabla^b\delta\Gamma^c_{bc} \wedge \xi^a.
\end{aligned} \tag{2.62}
$$

We need to consider the last term in $B_1^a + B_2^a$:

$$
\begin{aligned}
-\nabla_b\xi^a \wedge \nabla_c\delta g^{bc} &= -\nabla_c\xi^a \wedge \nabla_b\delta g^{bc} \\
&= -\nabla_c(\xi^a \wedge \nabla_b\delta g^{bc}) + \xi^a \wedge \nabla_c\nabla_b\delta g^{bc} \\
&= \nabla_c(2\xi^{[c} \wedge \nabla_b\delta g^{a]b}) - \nabla_c(\xi^c \wedge \nabla_b\delta g^{ab}) + \xi^a\nabla_c\nabla_b\delta g^{bc} \\
&= \nabla_c(2\xi^{[c} \wedge \nabla_b\delta g^{a]b}) + \nabla_c(g^{bd}\,\xi^c \wedge \delta\Gamma^a_{bd} + g^{ab}\,\xi^c \wedge \delta\Gamma^d_{bd}) \\
&- \xi^a \wedge \nabla_c(g^{bd}\,\delta\Gamma^c_{bd} + g^{cd}\,\delta\Gamma^b_{bd}) \\
&= \nabla_c(\underbrace{2\xi^{[c} \wedge \nabla_b\delta g^{a]b}}_{Q_4^{ac}}) + (\nabla_c\xi^c)[g^{bd}\delta\Gamma^a_{bd} + g^{ab}\delta\Gamma^d_{bd}] + g^{bd}\,\xi^c \wedge \nabla_c\delta\Gamma^a_{bd} \\
&+\ g^{ab}\,\xi^c \wedge \nabla_c\delta\Gamma^d_{bd} - g^{bd}\,\xi^a \wedge \nabla_c\delta\Gamma^c_{bd} - \xi^a \wedge \nabla^d\delta\Gamma^b_{db}.
\end{aligned}
\tag{2.63}
$$

Now, we can write $C_1^a + C_2^a + A_1^a + A_2^a + B_1^a + B_2^a + E_1^a$ as

$$
\begin{aligned}
&\sum_{i=1}^{4}\nabla_c Q_i^{ac} + 2g^{ab}\,\nabla_c\delta\Gamma^d_{bd} \wedge \xi^c + \underbrace{g^{ab}\,\delta\Gamma^c_{bc} \wedge \nabla_d\xi^d)}_{X_2^a} - 2\,\nabla^b\delta\Gamma^c_{bc} \wedge \xi^a + \nabla_b\xi^c \wedge \nabla_c\delta g^{ab} \\
&+ \underbrace{g^{bd}\,\nabla_c\xi^c \wedge \delta\Gamma^a_{bd}}_{X_1^a} + \underbrace{g^{ab}\,\nabla_c\xi^c \wedge \delta\Gamma^d_{bd}}_{X_2^a} + g^{bd}\,\xi^c \wedge \nabla_c\delta\Gamma^a_{bd} + g^{ab}\xi^c \wedge \nabla_c\delta\Gamma^d_{bd} \\
&- g^{bd}\,\xi^a \wedge \nabla_c\delta\Gamma^c_{bd} - \xi^a \wedge \nabla^d\delta\Gamma^b_{db} + \underbrace{g^{bc}\,\delta\Gamma^a_{bc} \wedge \nabla_d\xi^d}_{X_1^a}.
\end{aligned}
\tag{2.64}
$$

Terms X_1^a and X_2^a vanish. Now for the last part of the antisymmetric tensor, we take

$$
\begin{aligned}
F^a \equiv -2\,\delta\Gamma^a_{bc} \wedge \nabla^b\xi^c &= -2\frac{1}{2}g^{ad}\,(\nabla_b\delta g_{cd} + \nabla_c\delta g_{bd} - \nabla_d\delta g_{bc}) \wedge \nabla^b\xi^c \\
&= \nabla_b\delta g^{ac} \wedge \nabla^b\xi_c + \nabla_c\delta g^{ab} \wedge \nabla_b\xi^c - \nabla^a\delta g^{bc} \wedge \nabla_b\xi_c \\
&= \nabla_c\delta g^{ab} \wedge \nabla^c\xi_b + \nabla_c\delta g^{ab} \wedge \nabla_b\xi^c - \nabla^a\delta g^{bc} \wedge \nabla_b\xi_c \\
&= \nabla_c(\underbrace{\nabla^c\delta g^{ab} \wedge \xi_b - \nabla^a\delta g^{cb} \wedge \xi_b}_{Q_5^{ac}}) - \Box\delta g^{ab}\ \wedge \xi_b + \nabla_c\nabla^a\delta g^{cb} \wedge \xi_b \\
&+ \nabla_c\delta g^{ab} \wedge \nabla_b\xi^c
\end{aligned}
\tag{2.65}
$$

Then, the whole result $C_1^a + C_2^a + A_1^a + A_2^a + B_1^a + B_2^a + E_1^a + F^a$ is

$$
\begin{aligned}
&\underbrace{g^{ab}\,\nabla_c\delta\Gamma^d_{bd} \wedge \xi^c}_{A^a} + \xi^a \wedge (\nabla^b\delta\Gamma^c_{bc} - g^{bd}\,\nabla_c\delta\Gamma^c_{bd}) + \underbrace{g^{bd}\,\xi^c \wedge \nabla_c\delta\Gamma^a_{bd}}_{B^a} \\
&+ g^{ab}\,\xi^c \wedge \nabla_c\delta\Gamma^d_{bd} - \Box\delta g^{ab} \wedge \xi_b + \nabla_c\nabla^a\delta g^{cb} \wedge \xi_b.
\end{aligned}
\tag{2.66}
$$

We write A^a and B^a as

$$
A^a = g^{ab}\,\nabla_c\delta\Gamma^d_{bd} \wedge \xi^c = \frac{1}{2}g^{ab}\,\nabla_c\nabla_b\delta\ln g \wedge \xi^c = \frac{1}{2}\,\nabla_c\nabla^a\delta\ln g \wedge \xi^c, \tag{2.67}
$$

$$
\begin{aligned}
B^a = g^{bd}\,\xi^c \wedge \nabla_c \delta\Gamma^a_{bd} &= \frac{1}{2} g^{bd} g^{ae}\,\xi^c \wedge (\nabla_c \nabla_b \delta g_{de} + \nabla_c \nabla_d \delta g_{be} - \nabla_c \nabla_e \delta g_{bd}) \\
&= \frac{1}{2}\,\xi^c \wedge (-\nabla_c \nabla_b \delta g^{ab} - \nabla_c \nabla_d \delta g^{ad} - \nabla_c \nabla^a \delta \ln g) \qquad (2.68) \\
&= -\xi^c \wedge \nabla_c \nabla_b \delta g^{ab} - \frac{1}{2}\,\xi^c \wedge \nabla_c \nabla^a \ln g.
\end{aligned}
$$

with

$$
A^a + B^a = -\xi^c \wedge \nabla_c \nabla_b \delta g^{ab} - \xi^c \wedge \nabla_c \nabla^a \ln g, \qquad (2.69)
$$

and finally (2.66) becomes

$$
\begin{aligned}
& \sum_{i=1}^{5} \nabla_c Q_i^{ac} + \xi^a \wedge (\nabla^b \delta\Gamma^c_{bc} - g^{bd} \nabla_c \delta\Gamma^c_{bd}) - \xi^c \wedge \nabla_c \nabla_b \delta g^{ab} - \Box \delta g^{ab} \wedge \xi_b \\
& + \nabla_c \nabla^a \delta g^{cb} \wedge \xi_b + \nabla_c \nabla^a \delta \ln g \wedge \xi^c \\
= \quad & \sum_{i=1}^{5} \nabla_c Q_i^{ac} + g^{bd}\,\xi^a \wedge (\nabla_d \delta\Gamma^c_{bc} - \nabla_c \delta\Gamma^c_{bd}) + \nabla_c \nabla_b \delta g^{ab} \wedge \xi^c + \nabla_c \nabla^a \delta g^{cb} \wedge \xi_b \\
& - \Box \delta g^{ab} \wedge \xi_b + \nabla_c \nabla^a \delta \ln g \wedge \xi^c \\
= \quad & \sum_{i=1}^{5} \nabla_c Q_i^{ac} - g^{bd}\,\xi^a \delta R_{bd} - g^{ae} g^{bd}\,\nabla_c \nabla_b \delta g_{ed} \wedge \xi^c - g^{ae} g^{bd}\,\nabla^c \nabla_e \delta g_{ed} \wedge \xi^c \\
& - g^{ae} g^{bd}\,\nabla^c \nabla_e \delta g_{cd} \wedge \xi_b + g^{ae}\,\Box \delta g_{eb} \wedge \xi^b + g^{ae}\,\nabla_c \nabla_e \delta \ln g \wedge \xi^c \\
= \quad & \sum_{i=1}^{5} \nabla_c Q_i^{ac} - \xi^a g^{bd} \delta R_{bd} - g^{ae} g^{bd} ([\nabla_c, \nabla_b] \delta g_{ed}) \wedge \xi^c - g^{ae} g^{bd}\,\nabla_b \nabla_c \delta g_{ed} \wedge \xi^c \qquad (2.70) \\
& - g^{ae}\,\nabla^d \nabla_e \delta g_{db} \wedge \xi_b + g^{ae}\,\Box \delta g_{eb} \wedge \xi^b + g^{ae}\,\nabla_b \nabla_e \delta \ln g \wedge \xi^b \\
= \quad & \sum_{i=1}^{5} \nabla_c Q_i^{ac} - g^{bd}\,\xi^a \wedge \delta R_{bd} - g^{ae} g^{bd}\,([\nabla_c, \nabla_b] \delta g_{ed}) \wedge \xi^c \\
& - g^{ae}\,(\nabla^d \nabla_b \delta g_{ed} + \nabla^d \nabla_e \delta g_{db} - \Box \delta g_{eb} - \nabla_b \nabla_e \delta \ln g) \wedge \xi^b.
\end{aligned}
$$

For the last term, we need the following relation

$$
\begin{aligned}
\delta R_{ab} &= \nabla_c \delta\Gamma^c_{ab} - \nabla_a \delta\Gamma^c_{ac} \\
&= \frac{1}{2} g^{ad} \nabla_c (\nabla_a \delta g_{bd} + \nabla_b \delta g_{ad} - \nabla_d \delta g_{ab}) - \frac{1}{2} g^{cd} \nabla_a (\nabla_b \delta g_{cd} + \underbrace{\nabla_c \delta g_{bd} - \nabla_d \delta g_{ab}}_{=0}) \\
&= \frac{1}{2} (\nabla^d \nabla_a \delta g_{bd} + \nabla^d \nabla_b \delta g_{ad} - \Box \delta g_{ab} - \nabla_a \nabla_b \delta \ln g). \qquad (2.71)
\end{aligned}
$$

Using this, (2.70) becomes

$$
\begin{aligned}
& \sum_{i=1}^{5} \nabla_c Q_i^{ac} - g^{bd} \xi^a \wedge \delta R_{bd} - g^{ae} g^{bd} ([\nabla_c, \nabla_b] \delta g_{ed}) \wedge \xi^c - 2 g^{ae}\, \delta R_{eb} \wedge \xi^b \\
= & \sum_{i=1}^{5} \nabla_c Q_i^{ac} - g^{bd}\, \xi^a \wedge \delta R_{bd} - g^{ae} g^{bd} R_{cbe}{}^{f}\, \delta g_{df} \wedge \xi^c - g^{ae} g^{bd} R_{cbd}{}^{f}\, \delta g_{ef} \wedge \xi^c - 2 g^{ae}\, \delta R_{eb} \xi^b \\
= & \sum_{i=1}^{5} \nabla_c Q_i^{ac} - g^{bd}\, \xi^a \wedge \delta R_{bd} - R_c{}^{daf}\, \delta g_{fd} \wedge \xi^c + R_c{}^{d}{}_{df} \delta g_{af} \xi^c - 2 g^{ae}\, \delta R_{eb} \wedge \xi^b \\
= & \sum_{i=1}^{5} \nabla_c Q_i^{ac} - g^{bd}\, \xi^a \wedge \delta R_{bd} - R_e{}^{bac}\, \delta g_{bc} \wedge \xi^e - R_{cb}\, \delta g^{ab} \wedge \xi^c - 2 g^{ae}\, \delta R_{eb} \wedge \xi^b \\
= & \sum_{i=1}^{5} \nabla_c Q_i^{ac} - g^{bd}\, \xi^a \wedge \delta R_{bd} + R_{eb}{}^{a}{}_{c}\, \delta g_{bc} \wedge \xi^e - R_{cb}\, \delta g^{ab} \wedge \xi^c - 2 g^{ae}\, \delta R_{eb} \wedge \xi^b . \qquad (2.72)
\end{aligned}
$$

Finally, the change $\triangle J^a$ in the symplectic current is

$$
\begin{aligned}
\triangle J^a = & \sum_{i=1}^{5} \nabla_c Q^{ac} - g^{bd}\, \xi^a \wedge \delta R_{bd} + R_{eb}{}^{a}{}_{c}\, \delta g^{bc} \wedge \xi^e - R_{cb}\, \delta g^{ab} \wedge \xi^c - 2 g^{ae}\, \delta R_{eb} \wedge \xi^b + R_{ec}{}^{a}{}_{b}\, \xi^e \wedge \delta g^{bc} \\
& + R_e{}^{a}\, \xi^e \wedge \delta \ln g + R_{eb}\, \xi^e \wedge \delta g^{ab} \\
& = \nabla_c \mathcal{F}^{ac} + g^{bd}\, \delta R_{bd} \wedge \xi^a + 2 R_{cb}\, \xi^c \wedge \delta g^{ab} + 2 g^{ae}\, \xi^b \wedge \delta R_{eb} + R_e{}^{a}\, \xi^e \wedge \delta \ln g
\end{aligned}
\qquad (2.73)
$$

where

$$
\begin{aligned}
\mathcal{F}^{ac} \quad = \quad & 2\xi^{[c} \wedge \nabla_b \delta g^{a]b} - 2\xi_b \wedge \nabla^{[c} \delta g^{a]b} - 2\delta g^{b[c} \wedge \nabla_b \xi^{a]} - 2\xi^{[a} \wedge \nabla^{c]} \delta \ln |g| \\
& - \delta \ln |g| \wedge \nabla^{[c} \xi^{a]} . \qquad (2.74)
\end{aligned}
$$

Being the covariant derivative of an anti-symmetric tensor, the first term in (2.73) gives no contribution when inserted in the integral (2.41) for ω. Since the other terms already vanish on-shell, ω defined in (2.41) is gauge invariant, and thus is the symplectic two-form of General Relativity.

2.3 Conserved Charges

It was shown in [9] that the boundary term in (2.74) can be used to calculate conserved charges of a theory. Note that $\nabla_a(\Delta J^a) = \nabla_a \nabla_c \mathcal{F}^{ac} = 0$ from the antisymmetry of $\mathcal{F}^{ac}$. Therefore, we can try to construct a conserved charge expression from ΔJ^a. At this point, we restrict diffeomorphisms to isometries of the spacetime and linearize the metric as $g_{ab} = \bar{g}_{ab} + h_{ab}$, as a sum of the background $\bar{g}_{ab}$ and perturbation h_{ab} parts, where h_{ab} should vanish sufficiently slowly at "infinity"[1]. We identify ξ's as the Killing vectors of the background metric $\bar{g}_{ab}$. Index raising/lowering and covariant derivatives are also defined using the background metric $\bar{g}_{ab}$. The variation is identified as $\delta g_{ab} \to h_{ab}$, $\delta g^{ab} \to -h^{ab}$. Therefore, tensors in ΔJ^a are taken as the background ones whereas their variations are identified as the ones to the first order in h_{ab}. For example, we do the following replacements for the Ricci tensor: $R_{ab} \to \bar{R}_{ab}$ and $\delta R_{ab} \to (R_{ab})_L$, where the subscript L is used to denote the linearized version of a quantity. At the end, all the ξ terms are put into the right hand side of the wedge products, which are finally dropped to get a vector in spacetime. After all these identifications, we obtain a charge expression as

$$\begin{aligned} Q(\bar{\xi}) &= -\frac{1}{2}\int_{\Sigma} \mathrm{d}^{D-1}x\ \sqrt{|\sigma|}\, n_a\,(\Delta\tilde{J}^a) = -\frac{1}{2}\int_{\Sigma} \mathrm{d}^{D-1}x\ \sqrt{|\sigma|}\, n_a\,\bar{\nabla}_c Q^{ac} \\ &= -\frac{1}{2}\int_{\partial\Sigma} \mathrm{d}^{D-2}x\ \sqrt{|\sigma^{(\partial\Sigma)}|}\, n_a\, s_c\, Q^{ac}, \end{aligned} \tag{2.75}$$

where the second line follows from Stokes' theorem. Σ is an arbitrary $(D-1)$-dimensional spacelike hypersurface with induced metric σ, unit normal vector n^a and $\partial\Sigma$ is the boundary of Σ, which is a $(D-2)$-dimensional hypersurface with induced metric $\sigma^{(\partial\Sigma)}$, unit normal s^c. $\Delta\tilde{J}^a$ is the vector obtained from ΔJ^a as a result of the above mentioned identifications and

$$Q^{ac} = 2\bar{\nabla}_b h^{b[a}\bar{\xi}^{c]} - 2\bar{\nabla}^{[c}h^{a]b}\bar{\xi}_b + 2h^{b[c}\bar{\nabla}_b\bar{\xi}^{a]} + 2(\bar{\nabla}^{[c}h)\bar{\xi}^{a]} - h\bar{\nabla}^{[c}\bar{\xi}^{a]}. \tag{2.76}$$

Note that it is conserved up to a multiplicative constant and the factor $-\frac{1}{2}$ is introduced in [9] to "normalize" the results. We will give an explanation to this in the next chapter.

We now apply this result to solutions of General Relativity. Let us start with the Schwarzschild black hole described by the metric

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2), \tag{2.77}$$

with the background metric ($M \to 0$)

$$ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2), \tag{2.78}$$

which is the Minkowski spacetime in spherical coordinates. We use its Killing vector $\bar{\xi}^a = -\delta^a_t$ to calculate the energy. The timelike and spacelike normals can be found from the normalization conditions[2] $n^a n_a = -1$ and $s^a s_a = +1$ as $n^a = \delta^a_t$ and $s^a = \delta^a_r$. The measure of the

[1] This condition is put to guarantee non-zero results.

[2] Since the integral is performed over the boundary, the background metric is used in normalization ($n^a n_a = \bar{g}_{ab} n^a n^b$).

integral in (2.75) is $\sqrt{|\sigma^{(\partial\Sigma)}|} = r|\sin\theta|$. Finally, we compute the energy as

$$E_{Sch} = -\frac{1}{2}\lim_{r\to\infty}\int_0^{\pi}\int_0^{2\pi} \mathrm{d}\theta\mathrm{d}\phi\, \mathrm{r}|\sin\theta|\, Q^{tr} = M \tag{2.79}$$

as expected. Next, we consider the Kerr black hole with the metric

$$ds^2 = -\left(1 - \frac{r_s r}{\rho^2}\right)dt^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \left(r^2 + \alpha^2 + \frac{r_s r\alpha}{\rho^2}\sin^2\theta\right)d\phi^2 + \frac{2r_s r\alpha\sin^2\theta}{\rho^2}dtd\phi, \tag{2.80}$$

where

$$r_s = 2M, \quad \alpha = \tfrac{J}{M}, \quad \rho^2 = r^2 + \alpha^2\cos^2\theta, \quad \Delta = r^2 - r_s r + \alpha^2 \ . \tag{2.81}$$

The background metric is obtained by taking $M, J \to 0$

$$ds^2 = -dt^2 + \frac{\rho^2}{r^2+\alpha^2}dr^2 + \rho^2 d\theta^2 + (r^2+\alpha^2)\sin^2\theta\, d\phi^2, \tag{2.82}$$

Its Killing vectors are $\bar{\xi}^a = -\delta^a_t$, $\bar{\vartheta}^a = \delta^a_\phi$. For the normals, we first write $n^a = A\delta^a_t$ and $s^a = B\delta^a_r$ and find $A = 1$, $B = \frac{\sqrt{r^2+\alpha^2}}{\rho}$ from the normalization condition. The measure is $\sqrt{|\sigma^{(\partial\Sigma)}|} = \rho\sqrt{r^2+\alpha^2}|\sin\theta|$.Then, the energy and the angular momentum can be computed as

$$\begin{aligned} E_{Kerr} &= -\frac{1}{2}\lim_{r\to\infty}\int_0^{\pi}\int_0^{2\pi} \mathrm{d}\theta\mathrm{d}\phi\, (\mathrm{r}^2+\alpha^2)|\sin\theta|\, Q^{tr} = M, \\ J_{Kerr} &= -\frac{1}{2}\lim_{r\to\infty}\int_0^{\pi}\int_0^{2\pi} \mathrm{d}\theta\mathrm{d}\phi\, (\mathrm{r}^2+\alpha^2)|\sin\theta|\, Q^{tr} = Ma = J. \end{aligned} \tag{2.83}$$

CHAPTER 3

APPLICATION TO A LOCAL GRAVITY ACTION

So far, we have reviewed the construction of the symplectic two-form as described in [3] and the procedure to obtain the conserved charges, which was given in [9]. In this chapter, we consider a local gravity action

$$S = \int \mathrm{d}^D x \sqrt{|g|}\, \mathcal{L}(g, R, \nabla R, R^2, \cdots) \tag{3.1}$$

where $\mathcal{L}$ is a function of a metric, its Riemann tensor, its contraction and covariant derivatives. Its first variation is

$$\delta S = \int \mathrm{d}^D x \sqrt{|g|}\, \Phi_{ab}\, \delta g^{ab} + \int \mathrm{d}^D x\, \partial_a \Lambda^a(g, \delta g, \nabla \delta g \cdots). \tag{3.2}$$

Here, $\Phi_{ab} = 0$ is the field equation and Λ^a is the boundary term. Considering it as a one-form on the phase space Z, we start by taking its exterior derivative

$$\delta^2 S = \int \mathrm{d}^D x \sqrt{|g|}\, \delta\Phi_{ab} \wedge \delta g^{ab} - \frac{1}{2} \int \mathrm{d}^D x \sqrt{|g|}\, \Phi_{ab}\, \delta g^{ab} \wedge \delta \ln |g| + \int \mathrm{d}^D x\, \partial_a \delta\Lambda^a = 0, \tag{3.3}$$

which vanishes by the Poincaré lemma. The first two terms vanish on-shell and the third one gives

$$-\delta^2 S = \int \mathrm{d}^D x \sqrt{|g|}\, \nabla_a J^a = 0, \tag{3.4}$$

where

$$J^a \equiv -\delta\Lambda^a / \sqrt{|g|} \tag{3.5}$$

is the "symplectic current", which was used in the previous chapter. Since its covariant derivative vanishes on-shell ($\nabla_a J^a = 0$), the following Poincaré invariant two-form can be constructed

$$\omega = \int_\Sigma \mathrm{d}\Sigma_a \sqrt{|g|}\, J^a, \tag{3.6}$$

where Σ is a $(D-1)$-dimensional spacelike hypersurface. In order to prove that (3.5) gives the symplectic current of any theory derived from an action of the form (3.1), we need to show that the two-form given in (3.1) is closed and gauge invariant. We first consider the exterior derivative of (3.6)

$$\delta\omega = \int_\Sigma \mathrm{d}\Sigma_a \,(\delta\sqrt{|g|} \wedge J^a + \sqrt{|g|}\,\delta J^a), \tag{3.7}$$

which is exactly what we had in (2.42). This time, we will show that ω is always closed with the definition of the symplectic current given in (3.5). The exterior derivative of (3.3) reads

$$\begin{aligned}\delta^3 S &= \int \mathrm{d}^D x\,\sqrt{|g|}\,(\frac{1}{2}\,\delta\ln|g| \wedge \delta\Phi_{ab} \wedge \delta g^{ab} - \frac{1}{2}\,\delta\Phi_{ab} \wedge \delta g^{ab} \wedge \delta\ln|g|\,) \\ &\quad + \int \mathrm{d}^D x\,\sqrt{|g|}\,(\,\frac{1}{2}\,\delta\ln|g| \wedge \nabla_a J^a + \delta(\nabla_a J^a)\,) = 0. \end{aligned}\tag{3.8}$$

The first two terms cancel each other out by the antisymmetry of the wedge product. Therefore, using $\delta(\nabla_a J^a) = \nabla_a \delta J^a + \delta\Gamma^b{}_{ab} \wedge J^a$, we obtain

$$\delta^3 S = \frac{1}{2}\int \mathrm{d}^D x\,\sqrt{|g|}\,\delta\ln|g| \wedge \nabla_a J^a + \int \mathrm{d}^D x\,\sqrt{|g|}(\,\nabla_a\delta J^a + \delta\Gamma^b{}_{ab} \wedge J^a\,) = 0. \tag{3.9}$$

First, we write the last term as

$$\begin{aligned}\delta\Gamma^b{}_{ab} \wedge J^a &= \frac{1}{2}\nabla_a\delta\ln|g| \wedge J^a \\ &= \nabla_a\left(\frac{1}{2}\delta\ln|g| \wedge J^a\right) - \frac{1}{2}\delta\ln|g| \wedge \nabla_a J^a, \end{aligned}\tag{3.10}$$

and then insert into this (3.9) to get

$$\delta^3 S = \int \mathrm{d}^D x\,\sqrt{|g|}\,\nabla_a\delta J^a + \frac{1}{2}\int \mathrm{d}^D x\,\sqrt{|g|}\,\nabla_a\left(\frac{1}{2}\delta\ln|g| \wedge J^a\right) = 0. \tag{3.11}$$

This can be written with the help of Stokes' theorem in the desired form

$$\int_\Sigma \mathrm{d}\Sigma_a\,\sqrt{|g|}\,\delta J^a = -\frac{1}{2}\int_\Sigma d\Sigma_a\,\sqrt{|g|}\,J^a \wedge \delta\ln|g|, \tag{3.12}$$

which confirms $\delta\omega = 0$ for any theory derived from (3.1). Note that we have shown that ω is closed in general without the use of any field equations and that throughout no explicit calculation is required.

Our next task is to show the gauge invariance of the symplectic two-form. Transformations of some basic quantities were given in the previous chapter. Indeed, we can write a general rule for the variation of a tensor $T_{a\cdots}{}^{b\cdots}$

$$\begin{aligned}\delta T_{a\cdots}{}^{b\cdots} &\rightarrow \delta T_{a\cdots}{}^{b\cdots} + \xi^c\nabla_c T_{a\cdots}{}^{b\cdots} + T_{d\cdots}{}^{b\cdots}\,\nabla_a\xi^d + \cdots - T_{a\cdots}{}^{d\cdots}\,\nabla^b\xi_d + \cdots \\ &= \delta T_{a\cdots}{}^{b\cdots} + \mathcal{L}_\xi T_{a\cdots}{}^{b\cdots}, \end{aligned}\tag{3.13}$$

where $\mathcal{L}_\xi T_{a\cdots}{}^{b\cdots}$ denotes the Lie derivative of the tensor $T_{a\cdots}{}^{b\cdots}$ along the vector ξ. Knowing how one-forms transform, the transformations of p-forms can be obtained by substituting the expressions for one-forms and keeping the terms linear in ξ. Then, the change in the symplectic two-form is

$$\Delta\omega = \int_\Sigma d\Sigma_a \sqrt{|g|}\, \Delta J^a. \tag{3.14}$$

As we know from our work on General Relativity, if ΔJ^a can be written as the covariant derivative of an antisymmetric tensor plus some additional terms that vanish on-shell, then $\Delta\omega = 0$ and ω is gauge invariant. It was actually shown in [15] that this is true for a generic gravity theory derived from an action with local symmetries. However, we need to calculate $\Delta\omega$ explicitly to obtain the conserved charges.

For the conserved charges, we consider the transformation of (3.3) under (2.45)

$$\begin{aligned} 0 &= \int \mathrm{d}^D x \sqrt{|g|} \left(\delta\Phi_{ab} + \mathcal{L}_\xi \Phi_{ab}\right) \wedge \left(\delta g^{ab} - \nabla^a \xi^b - \nabla^b \xi^a\right) \\ &\quad -\frac{1}{2} \int \mathrm{d}^D x \sqrt{|g|}\, \Phi_{ab} \left(\delta g^{ab} - \nabla^a \xi^b - \nabla^b \xi^a\right) \wedge \left(\delta \ln |g| + 2\, \nabla_c \xi^c\right) \\ &\quad - \int \mathrm{d}^D x \sqrt{|g|}\, \nabla_a (J^a + \Delta J^a), \end{aligned} \tag{3.15}$$

Using (3.3) and the symmetry of Φ_{ab}, we get

$$\begin{aligned} 0 &= -2 \int \mathrm{d}^D x \sqrt{|g|}\, \delta\Phi_{ab} \wedge \nabla^a \xi^b + \int \mathrm{d}^D x \sqrt{|g|}\, \mathcal{L}_\xi \Phi_{ab} \wedge \delta g^{ab} - \int \mathrm{d}^D x \sqrt{|g|}\, \Phi_{ab}\, \delta g^{ab} \wedge \nabla_c \xi^c \\ &\quad + \int \mathrm{d}^D x \sqrt{|g|}\, \Phi_{ab} \nabla^a \xi^b \wedge \delta \ln g - \int \mathrm{d}^D x \sqrt{|g|}\, \nabla_a (\Delta J^a). \end{aligned} \tag{3.16}$$

Applying the identifications described in the previous chapter, all the terms, except the first and the last one, drop as a result of $\bar{\Phi}_{ab} = 0$ and (3.16) becomes

$$0 = 2 \int \mathrm{d}^D x \sqrt{|\bar{g}|}\, (\Phi_{ab})_L\, \bar{\nabla}^a \bar{\xi}^b + \int \mathrm{d}^D x \sqrt{|\bar{g}|}\, \bar{\nabla}_a (\Delta \tilde{J}^a), \tag{3.17}$$

where $\Delta\tilde{J}^a$ is the vector obtained from the two-form ΔJ^a. Since $\bar{\nabla}^a (\Phi_{ab})_L = 0$ as a result of the Bianchi identity, $(\Phi_{ab})_L\, \bar{\nabla}^a \xi^b = \bar{\nabla}^a \left((\Phi_{ab})_L\, \xi^b\right) = \bar{\nabla}_a \left(\left(\Phi^{ab}\right)_L \xi_b\right)$ and hence (3.17) can be written as

$$\int \mathrm{d}^D x \sqrt{|\bar{g}|}\, \bar{\nabla}_a \left(\left(\Phi^{ab}{}_L\right) \bar{\xi}_b\right) = -\frac{1}{2} \int \mathrm{d}^D x \sqrt{|\bar{g}|}\, \bar{\nabla}_a (\Delta \tilde{J}^a). \tag{3.18}$$

The left hand side of (3.18)

$$\begin{aligned} \bar{\nabla}_a \left(\left(\Phi^{ab}{}_L\right) \bar{\xi}_b\right) &= \bar{\nabla}_a \left(\Phi^{ab}\right)_L \bar{\xi}_b + \left(\Phi^{ab}\right)_L \bar{\nabla}_a \bar{\xi}_b \\ &= \bar{\nabla}_a \left(\Phi^{ab}\right)_L \bar{\xi}_b + \left(\Phi^{ab}\right)_L \bar{\nabla}_{(a} \bar{\xi}_{b)} = 0 \end{aligned} \tag{3.19}$$

vanishes by the Bianchi identity and the Killing equation for the background metric $(\bar{\nabla}_{(a}\bar{\xi}_{b)}) = 0$. It is the conserved current used in the ADT charge definition [5, 6, 7]. Here, we use the right hand side of (3.18) to construct the conserved charges as

$$Q_{ADT}(\bar{\xi}) = -\frac{1}{2}\int_{\Sigma} \mathrm{d}^{D-1}x\ \sqrt{|\sigma|}\, n_a\, \bar{\nabla}_c Q^{ac} = -\frac{1}{2}\int_{\partial\Sigma} \mathrm{d}^{D-2}x\ \sqrt{|\sigma^{(\partial\Sigma)}|}\, n_a\, s_c\, Q^{ac}\,, \tag{3.20}$$

where Q^{ac} is the tensor obtained from the two-form $\mathcal{F}^{ac}$ and Stokes' theorem was used with the identifications given after (2.75). This charge expression is completely equivalent to the ADT charge and explains the factor of $-\frac{1}{2}$ used in (2.75), which was introduced in [9].

CHAPTER 4

APPLICATION TO THE QUADRATIC ACTION WITH $\mathcal{L} \equiv \kappa^{-1}(R + 2\Lambda_0) + \alpha R^2 + \beta R_{ab}^2$

In this chapter, we apply the procedure described in the previous chapter to the following quadratic action

$$I = \int \mathrm{d}^D x \ \sqrt{|g|}\,\mathcal{L} \equiv \int \mathrm{d}^D x \ \sqrt{|g|}\left(\frac{1}{\kappa}(R + 2\Lambda_0) + \alpha R^2 + \beta R_{ab}^2\right), \tag{4.1}$$

where NMG is a special case ($8\alpha + 3\beta = 0$ and $D = 3$). Its first variation is

$$\delta I = \int \mathrm{d}^D x \ \sqrt{|g|}\,(\frac{1}{\kappa}\mathcal{G}_{ab} + \alpha\mathcal{A}_{ab} + \beta\mathcal{B}_{ab})\,\delta g^{ab} + \int \mathrm{d}^D x\Big(\frac{1}{\kappa}\partial_a\Lambda^a_\kappa + \alpha\partial_a\Lambda^a_\alpha + \beta\partial_a\Lambda^a_\beta\Big), \tag{4.2}$$

where

$$\begin{aligned}
\mathcal{G}_{ab} &\equiv R_{ab} - \frac{1}{2}g_{ab}R + \Lambda_0 g_{ab}, &(4.3)\\
\mathcal{A}_{ab} &\equiv 2RR_{ab} - 2\nabla_a\nabla_b R + g_{ab}(2\Box R - \frac{1}{2}R^2), &(4.4)\\
\mathcal{B}_{ab} &\equiv 2R_{acbd}R^{cd} - \nabla_a\nabla_b R + \Box R_{ab} + \frac{1}{2}g_{ab}(\Box R - R_{cd}R^{cd}). &(4.5)
\end{aligned}$$

The boundary terms

$$\begin{aligned}
\Lambda^a_\kappa &\equiv \sqrt{|g|}\Big(g^{bc}\delta\Gamma^a_{bc} - g^{ab}\delta\Gamma^c_{bc}\Big),\\
\Lambda^a_\alpha &\equiv \sqrt{|g|}\Big(2Rg^{bc}\delta\Gamma^a_{bc} - 2Rg^{ab}\delta\Gamma^c_{bc} + 2\nabla^a R\,\delta\ln|g| + 2\nabla_b R\,\delta g^{ab}\Big),\\
\Lambda^a_\beta &\equiv \sqrt{|g|}\Big(2R^{cb}\,\delta\Gamma^a_{bc} - 2R^{ab}\delta\Gamma^c_{bc} + \frac{1}{2}\nabla^a R\,\delta\ln|g| + 2\nabla_c R^a{}_b\,\delta g^{bc} - \nabla^a R_{cb}\,\delta g^{cb}\Big)
\end{aligned}$$

yield the following symplectic current

$$J^a = J^a_\kappa + J^a_\alpha + J^a_\beta, \tag{4.6}$$

with

$$J^a_\kappa = -\frac{\delta\Lambda^a_\kappa}{\sqrt{|g|}} = \delta\Gamma^a_{bc}\wedge(\delta g^{bc} + \frac{1}{2}g^{bc}\,\delta\ln|g|) - \delta\Gamma^c_{bc}\wedge(\delta g^{ab} + \frac{1}{2}g^{ab}\,\delta\ln|g|), \tag{4.7}$$

$$
\begin{aligned}
J^a_\alpha &= -\frac{\delta\Lambda^a_\alpha}{\sqrt{|g|}} = \delta\Gamma^a_{bc}\wedge(2R\,\delta g^{bc} + Rg^{bc}\,\delta\ln|g| + 2g^{bc}\,\delta R) \\
&\quad -\delta\Gamma^c_{bc}\wedge(2R\,\delta g^{ab} + Rg^{ab}\,\delta\ln|g| + 2g^{ab}\,\delta R) \\
&\quad -\delta\ln|g|\wedge\Big(\nabla_b R\,\delta g^{ab} - 2\delta(\nabla^a R)\Big) + \delta g^{ab}\wedge\Big(2\delta(\nabla_b R) - 2\nabla_b R\,\delta\ln|g|\Big), \qquad (4.8)
\end{aligned}
$$

$$
\begin{aligned}
J^a_\beta &= -\frac{\delta\Lambda^a_\beta}{\sqrt{|g|}} = \delta\Gamma^a_{bc}\wedge(R^{bc}\,\delta\ln|g| + 2\delta R^{bc}) - \delta\Gamma^c_{bc}\wedge(R^{ab}\,\delta\ln|g| + 2\delta R^{ab}) \\
&\quad +\delta\ln|g|\wedge(\frac{1}{2}\delta(\nabla^a R) - \nabla_c R^a{}_b\,\delta g^{bc} + \frac{1}{2}\nabla^a R_{cb}\,\delta g^{bc}) \\
&\quad +\delta g^{bc}\wedge\Big(2\delta(\nabla_c R^a{}_b) - \delta(\nabla^a R_{cb})\Big). \qquad (4.9)
\end{aligned}
$$

As shown in the previous chapter, it is covariantly conserved on-shell. Now, it remains to check the gauge invariance. After a tedious calculation, the change in the symplectic current can be found as

$$
\Delta J^a = \nabla_c\mathcal{F}^{ac} + 2\Phi_{bc}\,\xi^c\wedge\delta g^{ab} + \Phi^a{}_c\,\xi^c\wedge\delta\ln|g| + \Phi_{bc}\,\xi^a\wedge\delta g^{bc}, \qquad (4.10)
$$

where

$$
\mathcal{F}^{ac} = -\mathcal{F}^{ca} = \frac{1}{\kappa}\mathcal{F}^{ac}_\kappa + \alpha\mathcal{F}^{ac}_\alpha + \beta\mathcal{F}^{ac}_\beta, \qquad (4.11)
$$

with

$$
\begin{aligned}
\mathcal{F}^{ac}_\kappa &\equiv 2\xi^{[c}\wedge\nabla_b\delta g^{a]b} - 2\xi_b\wedge\nabla^{[c}\delta g^{a]b} - 2\delta g^{b[c}\wedge\nabla_b\xi^{a]} \\
&\quad -2\xi^{[a}\wedge\nabla^{c]}\delta\ln|g| - \delta\ln|g|\wedge\nabla^{[c}\xi^{a]}, \qquad (4.12)
\end{aligned}
$$

$$
\mathcal{F}^{ac}_\alpha \equiv 2R\mathcal{F}^{ac}_\kappa + 4\delta g^{b[c}\wedge\xi^{a]}\nabla_b R + 4\delta R\wedge\nabla^{[a}\xi^{c]} + 8\nabla^{[c}\delta R\wedge\xi^{a]}, \qquad (4.13)
$$

$$
\begin{aligned}
\mathcal{F}^{ac}_\beta &\equiv 2R^{b[a}\delta\ln|g|\wedge\nabla_b\xi^{c]} + 4\,g^{d[a}\delta R_{de}\wedge\nabla^{|e|}\xi^{c]} + 2\,\delta\ln|g|\wedge\xi^b\nabla^{[c}R^{a]}{}_b + 4\,\delta g^{d[a}\wedge\nabla_b\xi^{c]}R_d{}^b \\
&\quad -4\,R_e{}^{[a}\nabla_b\xi^{c]}\wedge\delta g^{be} + 4R^{bd}\xi^{[a}\wedge\delta\Gamma^{c]}{}_{bd} + 4R^{b[a}\xi^{c]}\wedge\delta\Gamma^d{}_{bd} - 4\xi^b\wedge\delta g^{d[a}\nabla^{c]}R_{db} \\
&\quad -4\xi_e\wedge\delta g^{b[c}\nabla_b R^{a]e} + 4g^{d[a}g^{c]e}\delta(\nabla_e R_{db})\wedge\xi^b - 2\xi^{[a}\wedge\nabla^{c]}\delta R + 2\delta g^{b[c}\wedge\xi^{a]}\nabla_b R \\
&\quad -2g^{bd}\xi^{[c}\wedge\nabla^{a]}\delta R_{bd} + 4g^{e[a}\xi^{c]}\wedge\nabla^b\delta R_{be} + 4g^{be}R_d{}^{[c}\xi^{a]}\wedge\delta\Gamma^d{}_{be} \\
&\quad +4R^{d[a}\delta\Gamma^{c]}{}_{bd}\wedge\xi^b. \qquad (4.14)
\end{aligned}
$$

The first term in (4.10) gives no contribution when inserted into (3.14) for sufficiently fast decaying metric variations and the other terms vanish on-shell. Following the discussion in the previous chapter, the charge expression (3.20) can be written as

$$
Q^{ac} = -Q^{ca} = \frac{1}{\kappa}Q^{ac}_\kappa + \alpha Q^{ac}_\alpha + \beta Q^{ac}_\beta, \qquad (4.15)
$$

where

$$Q_{\kappa}^{ac} \equiv 2\bar{\nabla}_b h^{b[a}\bar{\xi}^{c]} - 2\bar{\nabla}^{[c}h^{a]b}\bar{\xi}_b + 2h^{b[c}\bar{\nabla}_b\bar{\xi}^{a]} + 2(\bar{\nabla}^{[c}h)\bar{\xi}^{a]} - h\bar{\nabla}^{[c}\bar{\xi}^{a]}, \tag{4.16}$$

$$Q_{\alpha}^{ac} \equiv 2\bar{R}Q_{\kappa}^{ac} - 4\bar{\nabla}_b\bar{R}\, h^{b[c}\bar{\xi}^{a]} + 4R_L\bar{\nabla}^{[a}\bar{\xi}^{c]} + 8(\bar{\nabla}^{[c}R_L)\bar{\xi}^{a]}, \tag{4.17}$$

$$\begin{aligned} Q_{\beta}^{ac} \quad \equiv \quad & 2\bar{R}^{b[a}h\bar{\nabla}_b\bar{\xi}^{c]} + 4\,\bar{g}^{d[a}(R_{de})_L\bar{\nabla}^{|e|}\bar{\xi}^{c]} + 2\,\bar{\nabla}^{[c}\bar{R}^{a]}{}_b h\bar{\xi}^b - 4\,h^{d[a}\bar{\nabla}_b\bar{\xi}^{c]}\bar{R}_d{}^b \\ & +4\,\bar{R}^{e[a}h_{be}\bar{\nabla}^{c]}\xi^b - 4\bar{R}^{bd}(\Gamma^{[c}{}_{bd})_L\bar{\xi}^{a]} - 4\bar{R}^{b[a}(\Gamma^{|d|}{}_{bd})_L\bar{\xi}^{c]} - 4h^{d[a}\bar{\xi}^{|b|}\bar{\nabla}^{c]}\bar{R}_{db} \\ & -4h^{b[c}\bar{\xi}_e\bar{\nabla}_b\bar{R}^{a]e} + 4\bar{g}^{d[a}\bar{g}^{c]e}(\nabla_e R_{db})_L\bar{\xi}^b + 2(\bar{\nabla}^{[c}R_L)\bar{\xi}^{a]} - 2h^{b[c}\bar{\xi}^{a]}\bar{\nabla}_b\bar{R} \\ & +2\bar{g}^{bd}\bar{\nabla}^{[a}(R_{bd})_L\,\bar{\xi}^{c]} - 4\bar{g}^{e[a}\bar{\nabla}^{|b|}(R_{be})_L\bar{\xi}^{c]} - 4\bar{g}^{be}\bar{R}_d{}^{[c}(\Gamma^{|d|}{}_{be})_L\bar{\xi}^{a]} \\ & +4\bar{R}^{d[a}(\Gamma^{c]}{}_{bd})_L\bar{\xi}^b. \end{aligned} \tag{4.18}$$

This charge expression is equivalent to the ADT charge for theories with arbitrary backgrounds, which was given in [16]. While (4.16) and (4.17) are identical to their counterparts, some computation is necessary to show the equivalence of the third one. In the next chapter, we calculate the conserved charges of some solutions of NMG with the help of (4.15).

CHAPTER 5

CONSERVED CHARGES OF SOME SOLUTIONS OF NMG

Having the charge expression (4.15) at hand, we first consider solutions of NMG, which are asymptotically AdS_3 and then the ones which are not spaces of constant curvature. All these examples have been studied before in [17, 18, 19], with which the results will be compared.

5.1 The BTZ black hole

Our first example is the BTZ black hole [20] described by the metric

$$ds^2 = \Big(\frac{-2\rho}{l^2} + \frac{M}{2}\Big)dt^2 + \Big(\frac{4\rho^2}{l^2} - \frac{(M^2l^2 - J^2)}{4}\Big)^{-1} d\rho^2 - Jdtd\phi + \Big(2\rho + \frac{Ml^2}{2}\Big)d\phi^2, \tag{5.1}$$

which is a solution of NMG with

$$\kappa = 16\pi G,\ \beta = -\frac{1}{\kappa m^2},\ \Lambda_0 = \frac{1 + 4l^2m^2}{4l^4m^2},\ \alpha = -\frac{3}{8}\beta. \tag{5.2}$$

The background spacetime can be obtained by setting $M \to 0$, $J \to 0$ in (5.1)

$$ds^2 = -\frac{2\rho}{l^2}dt^2 + \frac{l^2}{4\rho^2}d\rho^2 + 2\rho d\phi^2. \tag{5.3}$$

This is AdS_3 spacetime with two globally defined Killing vectors $\bar{\xi}^a = -\delta^a_t$ and $\bar{\vartheta}^a = \delta^a_\phi$, used in the calculation of the energy and angular momentum, respectively. The timelike and spacelike normal vectors are obtained by employing the normalization conditions $n^a n_a = -1$ and $s^a s_a = +1$ as

$$n^a = -\frac{\ell}{\sqrt{2\rho}}\delta^a_t,\ \ s^a = \frac{2\rho}{\ell}\delta^a_\rho. \tag{5.4}$$

The measure of (2.75) is $\sqrt{|\sigma^{(\partial\Sigma)}|} = \sqrt{2\rho}$ and using (4.15) there gives

$$E_{BTZ} = \Big(1 - \frac{1}{2l^2m^2}\Big)\frac{M}{16G}, \qquad J_{BTZ} = \Big(1 - \frac{1}{2l^2m^2}\Big)\frac{J}{16G}. \tag{5.5}$$

These "renormalized mass and angular momentum" are identical to the ones given in [18], where ADT charge definition was used and in [19], which employed the boundary stress tensor method.

5.2 The "logarithmic" black hole in [21]

Our second example is the solution given in [21]

$$ds^2 = -\frac{4\rho^2}{\ell^2 f(\rho)}dt^2 + f(\rho)\Big[d\phi - \frac{q\ell \ln|\rho/\rho_0|}{f(\rho)}dt\Big]^2 + \frac{\ell^2 d\rho^2}{4\rho^2}, \tag{5.6}$$

where

$$f(\rho) = 2\rho + q\ell^2 \ln|\rho/\rho_0|. \tag{5.7}$$

This solves NMG with

$$\kappa = 8\pi G,\ \ \beta = -\frac{2\ell^2}{\kappa},\ \ \Lambda_0 = \frac{3}{2\ell^2}. \tag{5.8}$$

The background spacetime is taken to be AdS_3 in the form (5.3) and the same Killing vectors, normals and induced metric as in the BTZ case can be used in the computation, which yields

$$E \ = \ \lim_{\rho\to\infty}\int_0^{2\pi} \sqrt{2\rho}\, n_t\, s_\rho\, Q^{t\rho}(\bar{\xi})d\phi = \frac{2q}{G}, \tag{5.9}$$

$$J \ = \ \lim_{\rho\to\infty}\int_0^{2\pi} \sqrt{2\rho}\, n_t\, s_\rho\, Q^{t\rho}(\bar{\vartheta})d\phi = \frac{2\ell q}{G}. \tag{5.10}$$

These result coincide with the ones given in [21], which are obtained through ADT definition.

5.3 The rotating black hole in [22]

Now, we consider a stationary solution given in [22]

$$ds^2 = \Big(-N(r)F(r) + r^2K(r)^2\Big)dt^2 + \frac{dr^2}{F(r)} + 2r^2K(r)\,dt\,d\phi + r^2\,d\phi^2, \tag{5.11}$$

where

$$N(r) \ = \ \Big[1 + \frac{q\ell^2}{4H(r)}(1 - \sqrt{\Xi})\Big]^2, \tag{5.12}$$

$$F(r) \ = \ \frac{H(r)^2}{r^2}\Big[\frac{H(r)^2}{\ell^2} + \frac{q}{2}(1 + \sqrt{\Xi})H(r) + \frac{q^2\ell^2}{16}(1 - \sqrt{\Xi})^2 - 4GM\sqrt{\Xi}\Big], \tag{5.13}$$

$$K(r) \ = \ -\frac{p}{2r^2}(4GM - qH(r)), \tag{5.14}$$

$$H(r) \ = \ \Big[r^2 - 2GM\ell^2(1 - \sqrt{\Xi}) - \frac{q^2\ell^4}{16}(1 - \sqrt{\Xi})^2\Big]^{1/2}, \tag{5.15}$$

$$\Xi \ \equiv \ 1 - p^2/\ell^2, \tag{5.16}$$

with parameters

$$\Lambda_0 = \frac{1}{2\ell^2}, \quad \beta = \frac{2\ell^2}{\kappa}, \ \alpha = -\frac{3}{8}\beta, \quad \kappa = 16\pi G. \tag{5.17}$$

The rotation parameter p should lie between $-\ell \leq p \leq \ell$ and the parameter q is the "gravitational hair" where $q = 0$ gives the BTZ blackhole. The background spacetime ($q \to 0$, $M \to 0$) is AdS_3

$$ds^2 = -\frac{r^2}{l^2}dt^2 + \frac{l^2}{r^2}dr^2 + r^2 d\phi^2, \tag{5.18}$$

with the timelike and spacelike normals

$$n^a = -\frac{\ell}{r}\delta^a_t, \ s^a = \frac{r}{\ell}\delta^a_r, \ \sigma^{(\partial\Sigma)} = r^2. \tag{5.19}$$

The energy and angular momentum are found to be

$$E = \lim_{r\to\infty}\int_0^{2\pi} r\, n_t\, s_r\, Q^{tr}(\bar{\xi})d\phi = M, \tag{5.20}$$

$$J = \lim_{r\to\infty}\int_0^{2\pi} r\, n_t\, s_r\, Q^{tr}(\bar{\vartheta})d\phi = Mp. \tag{5.21}$$

Note that the parameter q does not appear in the conserved charges, which is why it is called the "gravitational hair".

5.4 Three-dimensional Lifschitz black hole

Our first example with a nonconstant curvature background is the three-dimensional Lifshitz black hole [23]

$$ds^2 = -\frac{r^6}{\ell^6}\left(1 - \frac{M\ell^2}{r^2}\right)dt^2 + \frac{\ell^2}{r^2}\left(1 - \frac{M\ell^2}{r^2}\right)^{-1}dr^2 + \frac{r^2}{\ell^2}dx^2, \tag{5.22}$$

which is a solution of NMG with

$$\Lambda_0 = \frac{13}{2\ell^2}, \quad \beta = \frac{2\ell^2}{\kappa}, \quad \alpha = -\frac{3\ell^2}{4\kappa}, \quad \kappa = 16\pi G.$$

The background metric is ($M \to 0$)

$$ds^2 = -\frac{r^6}{\ell^6}dt^2 + \frac{\ell^2}{r^2}dr^2 + \frac{r^2}{\ell^2}dx^2.$$

The timelike, spacelike normals and one-dimensional induced metric can easily be found as

$$n^a = -\frac{\ell^3}{r^3}\delta^a_t, \ s^a = \frac{r}{\ell}\delta^a_r, \ \sigma^{(\partial\Sigma)} = \frac{r^2}{\ell^2}.$$

For the energy, the timelike Killing vector $\bar{\xi}^a = -\delta^a_t$ can be employed and

$$E = \lim_{r\to\infty} \int_0^{2\pi\ell} \frac{r}{\ell}\, n_t\, s_r\, Q^{tr}(\bar{\xi}) dx = \frac{7M^2}{8G}. \tag{5.23}$$

This result agrees with the given in [16] that was calculated through the ADT procedure for arbitrary backgrounds, however it differs from the expression in [19].

5.5 The Warped AdS$_3$ black hole

Our final example is the warped AdS$_3$ black hole [18] which reads

$$ds^2 = -\mu^2 \frac{r^2 - r_0^2}{F(r)}\, dt^2 + F(r)\Big[d\phi - \frac{r + (1-\mu^2)\omega}{F(r)}\, dt\Big]^2 + \frac{1}{\mu^2\zeta^2}\, \frac{dr^2}{r^2 - r_0^2}, \tag{5.24}$$

where

$$F(r) = r^2 + 2\omega r + \omega^2(1-\mu^2) + \frac{\mu^2 r_0^2}{1-\mu^2}.$$

This is a solution of the NMG theory with

$$\kappa = 8\pi G, \quad \beta = -\frac{1}{m^2\kappa}, \quad \alpha = \frac{3}{8m^2\kappa},$$

$$\mu^2 = \frac{9m^2 + 21\Lambda_0 - 2m\sqrt{3(5m^2 - 7\Lambda_0)}}{4(m^2 + \Lambda_0)} \quad \text{and} \quad \zeta^2 = \frac{8m^2}{21 - 4\mu^2},$$

with m^2 as the NMG parameter. In order to have a causally regular black hole, μ^2 and Λ_0 must be [18]

$$0 < \mu^2 < 1 \quad \text{and} \quad \frac{35m^2}{289} \geq \Lambda_0 \geq -\frac{m^2}{21}\,.$$

The background spacetime of this black hole can be defined by taking $\omega \to 0,\ r_0 \to 0$ in (5.24)

$$ds^2 = (1-\mu^2)\, dt^2 + \frac{1}{r^2\zeta^2\mu^2}\, dr^2 - 2r\, d\phi\, dt + r^2 d\phi^2. \tag{5.25}$$

The timelike, spacelike normals and the measure is apparent considering the standard ADM form of the metric (5.25)

$$n_a = -\mu\, \delta^t_a, \quad s_a = \frac{1}{\mu r \zeta}\, \delta^r_a, \quad \sqrt{|\sigma^{(\partial\Sigma)}|} = r.$$

To find the energy, one again has to choose the timelike Killing vector as $\bar{\xi}^a = -\delta^a_t$ and for the angular momentum one has to use $\bar{\vartheta}^a = \delta^a_\phi$. Then,

$$\begin{aligned} E &= \lim_{r\to\infty} \int_0^{2\pi} r\, n_t\, s_r\, Q^{tr}(\bar{\xi}) d\phi = \frac{4\mu^2(1-\mu^2)\omega\zeta}{G(21-4\mu^2)}, \qquad (5.26) \\ J &= \lim_{r\to\infty} \int_0^{2\pi} r\, n_t\, s_r\, Q^{tr}(\bar{\vartheta}) d\phi = -\frac{\zeta}{8G(21-4\mu^2)}\Big[\frac{16 r_0^2\mu^2}{(1-\mu^2)} + \frac{(1-\mu^2)}{\mu^2}(21 - 29\mu^2 + 24\mu^4)\omega^2\Big]. \end{aligned}$$

The values for the energy and angular momentum agree with the ones given in [deniz2], however angular momentum is in conflict with the one in [18]. The discrepancy of these results, and the validity of the charge expression are discussed more explicitly in [17].

CHAPTER 6

CONCLUSIONS

In this thesis, we first reviewed the construction of the symplectic two-form and its relation to conserved charges. Then, we have shown that consideration of a generic local gravity action with the same procedure yields some general and useful facts. Using the boundary term appearing in the first variation of the action, it is always possible to obtain a covariantly conserved symplectic current, whose integration over a spacelike hypersurface gives a Poincaré invariant two-form on the phase space. It was also proved that this two-form is always closed, which is one of the conditions that must be satisfied for it to be the symplectic two-form of the theory. The other condition, the invariance under the diffeomorphisms, yields the conserved charges, which was shown to be equivalent to the extended ADT formalism for arbitrary backgrounds with at least one global Killing isometry [16].

Then, we found the symplectic two-form of the theories described by (4.1), for which NMG is a special case, and obtained a closed expression for conserved charges. The energy and the angular momentum of several solutions of NMG were calculated through that expression. The charges of black holes with AdS_3 backgrounds agree with the previous works [18, 19, 22]. Our results for black holes with non-constant backgrounds, Lifshitz and warped AdS_3 spacetimes, are in agreement with the ones given in [16, 17]. This was expected since they were calculated using the ADT procedure for arbitrary backgrounds, whose equivalence to our procedure was already proved. However, as was shown in [17], there is a discrepancy between these results and the ones computed by other means [18, 19], which necessiates further investigation regarding the validity of the charge expression for generic backgrounds. It might also be interesting to perform a covariant, geometric quantization of the generic theories arising from (4.1).

REFERENCES

[1] R. L. Arnowitt, S. Deser and C. W. Misner, *The Dynamics of General Relativity* in *Gravitation: An Introduction to Current Research*, L. Witten, Ed. (Wiley, New York, 1962), Chap. VII , pp. 227-265

[2] M. J. Gotay, in *Differential Geometric Methods in Theoretical Physics*, K. Bleuler and M. Werner, Eds., NATO Advanced Study Institutes, Series C., Vol. 250 (Kluwer Acad. Publ., Dordrecht 1988), pp. 295-305.

[3] E. Witten, *Interacting Field Theory of Open String*, Nuc. Phys. B **276** 291 (1986); C. Crnkovic and E. Witten, *Covariant Description of Canonical Formalism in Geometrical Theories* in *Three Hundred Years of Gravitation,* S. W. Hawking and W. Israel, Eds. (Cambridge University Press, 1987), pp. 676-684.

[4] G. J. Zuckerman, *Action Principle and Global Geometry* in *Mathematical Aspects of String Theory* S.T. Yau ed. (World Scientific Singapore, 1987).

[5] L. F. Abbott and S. Deser, *Stability of Gravity with A Cosmological Constant*, Nucl. Phys. B **195**, 76 (1982).

[6] S. Deser and B. Tekin, *Gravitational Energy in Quadratic Curvature Gravities*, Phys. Rev. Lett. **89**, 101101 (2002) [arXiv:hep-th/0205318].

[7] S. Deser and B. Tekin, *Energy in Generic Higher Curvature Gravity Theories*, Phys. Rev. D **67**, 084009 (2003) [arXiv:hep-th/0212292].

[8] S. Deser, R. Jackiw and S. Templeton, *Topologically Massive Gauge Theories*, Phys. Rev. Lett. **48**, 975 (1982); Annals Phys. **140**, 372 (1982), Erratum-ibid. Annals Phys. **185** (1988) 406, Annals Phys. **281** (2000) 409.

[9] C. Nazaroglu, Y. Nutku, B. Tekin, *Covariant Symplectic Structure and Conserved Charges of Topologically Massive Gravity*, Phys. Rev. D **83**, 124039 (2011) [arXiv:hep-th/1104.3404].

[10] E. A. Bergshoeff, O. Hohm and P. K. Townsend, *Massive Gravity in Three Dimensions*, Phys. Rev. Lett. **102**, 201301 (2009) [arXiv:hep-th/0901.1766].

[11] E. A. Bergshoeff, O. Hohm and P. K. Townsend, *More on Massive 3D Gravity*, Phys. Rev. D **79**, 124042 (2009) [arXiv:hep-th/0905.1259].

[12] I. Gullu and B. Tekin, *Massive Higher Derivative Gravity in D-dimensional Anti-de Sitter Spacetimes*, Phys. Rev. D **80**, 064033 (2009) [arXiv:hep-th/0906.0102].

[13] K. Muneyuki and N. Ohta, *Unitarity versus Renormalizability of Higher Derivative Gravity in 3D*, Phys. Rev. D **85**, 101501(R) (2012) [arXiv:hep-th/1201.2058].

[14] G. Alkac, D. O. Devecioglu, *Covariant Symplectic Structure and Conserved Charges of New Massive Gravity*, Phys. Rev. D **85**, 064048 (2012) [arXiv:gr-qc/1202.1905]

[15] J. Lee and R. M. Wald, *Local Symmetries and Constraints*, J. Math. Phys. **31**, 725 (1990).

[16] D. O. Devecioglu and O. Sarioglu, *Conserved Killing Charges of Quadratic Curvature Gravity Theories in Arbitrary Backgrounds*, Phys. Rev. D **83**, 021503 (2011) [arXiv:hep-th/1010.1711].

[17] D. O. Devecioglu, O.Sarioglu, *On the Thermodynamics of Lifshitz Black Holes*, Phys.Rev.D **83**, 124041 (2011) [arXiv:hep-th/1103.1993].

[18] G. Clement, *Warped AdS(3) Black Holes in New Massive Gravity*, Class.Quant.Grav. **26**, 105015 (2009) [arXiv:hep-th/0902.4634].

[19] O. Hohm and E. Tonni, *A Boundary Stress Tensor for Higher-derivative Gravity in AdS and Lifshitz Backgrounds*, JHEP **1004**, 093 (2010) [arXiv:hep-th/1001.3598].

[20] M. Banados, C. Teitelboim, J. Zanelli, *The Black hole in three-dimensional space-time*, Phys.Rev.Lett. **69**, 1849-1851 (1992) [arXiv:hep-th/9204099].

[21] G. Clement, *Black holes with a null Killing vector in new massive gravity in three dimensions*, Class. Quant. Grav. **26**, 165002 (2009) [arXiv:hep-th/0905.0553].

[22] J. Oliva, D. Tempo, R. Troncoso, *Three-dimensional Black Holes, Gravitational Solitons, Kinks and Wormholes for BHT Massive Gravity*, JHEP **0907**, 011 (2009) [arXiv:hep-th/0905.1545].

[23] E. Ayon-Beato, A. Garbarz, G. Giribet and M. Hassaine, *Lifshitz Black Hole in Three Dimensions*, Phys. Rev. D **80**, 104029 (2009) [arXiv:hep-th/0909.1347].

APPENDIX A

VARIATION OF SEVERAL TERMS

We will consider some relations which will be frequently used in the calculation of the symplectic current (Appendix C), where the formula $J^a = -\frac{\delta \Lambda^a}{\sqrt{|g|}}$ is used, and in the calculation of the exterior derivative of the symplectic current (Appendix E). The variation of the covariant derivative of a tensor of rank $(1,1)$ is

$$\begin{aligned}
\delta(\nabla_a T^{\ b}_{c}) &= \delta(\partial_a T^{\ b}_{c} + \Gamma^{\ b}_{da} T^{\ d}_{c} - \Gamma^{\ d}_{ca} T^{\ b}_{d}) \\
&= \partial_a \delta T^{\ b}_{c} + T^{\ d}_{c}\, \delta\Gamma^{\ b}_{da} + \Gamma^{\ b}_{da}\, \delta T^{\ d}_{c} - T^{\ b}_{d}\, \delta\Gamma^{\ d}_{ca} - \Gamma^{\ d}_{ca}\, \delta T^{\ b}_{d} \\
&= \nabla_a \delta T^{\ b}_{c} + T^{\ d}_{c}\, \delta\Gamma^{\ b}_{da} - T^{\ b}_{d}\, \delta\Gamma^{\ d}_{ca}.
\end{aligned}$$

Its generalization to tensors of higher ranks is obvious. Now, we can apply this rule to obtain the following

$$\delta(\nabla_a R_{cd}) = \nabla_a \delta R_{cd} - R_{ed}\, \delta\Gamma^e_{ac} - R_{ec}\, \delta\Gamma^e_{ad},$$

$$\begin{aligned}
\delta(\nabla_b \nabla_a R_{cd}) &= \nabla_b \delta(\nabla_a R_{cd}) - \nabla_e R_{cd}\, \delta\Gamma^e_{ba} - \nabla_a R_{ed}\, \delta\Gamma^e_{bc} - \nabla_a R_{ce}\, \delta\Gamma^e_{bd} \\
&= \nabla_b \nabla_a \delta R_{cd} - \nabla_b R_{ed}\, \delta\Gamma^e_{ac} - R_{ed}\, \nabla_b \delta\Gamma^e_{ac} - \nabla_b R_{ec}\, \delta\Gamma^e_{ad} - R_{ec}\, \nabla_b \delta\Gamma^e_{ad} \\
&\quad -\nabla_e R_{cd}\, \delta\Gamma^e_{ba} - \nabla_a R_{ed}\, \delta\Gamma^e_{bc} - \nabla_a R_{ce}\, \delta\Gamma^e_{bd},
\end{aligned}$$

$$\begin{aligned}
\delta(\Box R_{cd}) &= \delta(g^{ab} \nabla_b \nabla_a R_{cd}) = g^{ab}\, \delta(\nabla_b \nabla_a R_{cd}) + \nabla_b \nabla_a R_{cd}\, \delta g^{ab} \\
&= \Box \delta R_{cd} - 2\nabla^a R_{ed}\, \delta\Gamma^e_{ac} - 2R_{ec}\, \nabla^a \delta\Gamma^e_{ad} - R_{ed}\, \nabla^a \delta\Gamma^e_{ac} - R_{ec}\, \nabla^a \delta\Gamma^e_{ad} \\
&\quad -g^{ab} \nabla_e R_{cd}\, \delta\Gamma^e_{ba},
\end{aligned}$$

$$\begin{aligned}
\delta(\nabla_b \nabla_a R) &= \delta(g^{cd} \nabla_b \nabla_a R_{cd}) = g^{cd}\, \delta(\nabla_b \nabla_a R_{cd}) + \nabla_b \nabla_a R_{cd}\, \delta g^{cd} \\
&= g^{cd}\, \nabla_b \nabla_a \delta R_{cd} - \nabla_e R\, \delta\Gamma^e_{ba} - 2\nabla_b R^c_e\, \delta\Gamma^e_{ac} - 2R^c_e\, \nabla_b \delta\Gamma^e_{ac} - 2\nabla_a R^{\ c}_{e}\, \delta\Gamma^e_{bc},
\end{aligned}$$

$$
\begin{aligned}
\delta(\Box R) &= \delta(g^{ab}\nabla_b\nabla_a R) = g^{ab}\,\delta(\nabla_b\nabla_a R) + \nabla_b\nabla_a R\,\delta g^{ab} \\
&\quad g^{cd}\,\Box\delta R_{cd} - g^{ab}\nabla_e R\,\delta\Gamma^e_{ab} - 2\nabla^a R^c_e\,\delta\Gamma^e_{ac} - 2R^c_e\,\nabla^a\delta\Gamma^e_{ac} + \nabla_b\nabla_a R\,\delta g^{ab} \\
&\quad -4\nabla^a R^c_e\,\delta\Gamma^e_{ac}.
\end{aligned}
$$

APPENDIX B

FIELD EQUATIONS AND BOUNDARY TERMS

B.1 Einstein-Hilbert Term

$$
\begin{aligned}
S &= \int \mathrm{d}^D x \ \sqrt{|g|}\, R, \\
\delta S &= \int \mathrm{d}^D x \ \sqrt{|g|}\, \delta R + \int \mathrm{d}^D x\, R\, \delta(\sqrt{|g|}) = 0.
\end{aligned}
$$

The relevant terms are

$$
\begin{aligned}
\delta(\sqrt{|g|}) &= -\frac{1}{2}\sqrt{|g|} g_{ab}\, \delta g^{ab}, \\
\delta R &= \delta(g^{ab} R_{ab}) = R_{ab}\, \delta g^{ab} + g^{ab}\, \delta R_{ab}, \\
\delta R_{ab} &= \nabla_c \delta\Gamma^c_{ab} - \nabla_a \delta\Gamma^c_{bc} = \nabla_c(\delta\Gamma^c_{ab} - \delta^c_a\, \delta\Gamma^d_{bd}), \\
\delta R &= R_{ab}\, \delta g^{ab} + \nabla_c(g^{ab}\, \delta\Gamma^c_{ab} - g^{bc}\, \delta\Gamma^d_{bd}), \\
&= R_{ab}\, \delta g^{ab} + \underbrace{\nabla_a(g^{bc}\, \delta\Gamma^a_{bc} - g^{ab}\, \delta\Gamma^c_{bc})}_{=g^{ab}\,\delta R_{ab}}.
\end{aligned}
$$

The variation of the action becomes

$$
\delta S = \int \mathrm{d}^D x \ \sqrt{|g|}\, (R_{ab} - \frac{1}{2} g_{ab} R)\, \delta g^{ab} + \int \mathrm{d}^D x \ \sqrt{|g|}\, \nabla_a(g^{bc}\, \delta\Gamma^a_{bc} - g^{ab}\, \delta\Gamma^c_{bc}) = 0.
$$

Therefore, the field equations and the boundary term are

$$
\mathcal{G}_{ab} = \; = R_{ab} - \frac{1}{2} g_{ab} R \qquad \Lambda^a_\kappa = \sqrt{|g|}\Big(g^{bc}\delta\Gamma^a_{bc} - g^{ab}\delta\Gamma^c_{bc}\Big).
$$

B.2 R^2 Term

$$
\begin{aligned}
S &= \int \mathrm{d}^D x \ \sqrt{|g|}\, R^2 \\
\delta S &= \int \mathrm{d}^D x \ \sqrt{|g|}\, 2\, \delta R + \int \mathrm{d}^D x\, R^2\, \delta(\sqrt{|g|})
\end{aligned}
$$

We start with

$$\begin{aligned} R\,\delta R &= R\,\delta(g^{ab}R_{ab}) = RR_{ab}\,\delta g^{ab} + Rg^{ab}\,\delta R_{ab} \\ &= RR_{ab}\,\delta g^{ab} + R\,\nabla_a(g^{bc}\,\delta\Gamma^a_{bc} - g^{ab}\,\delta\Gamma^c_{bc}) \\ &= RR_{ab}\,\delta g^{ab} + \nabla_a(g^{bc}R\delta\Gamma^a_{bc} - g^{ab}R\delta\Gamma^c_{bc}) - g^{bc}\,\nabla_a R\delta\Gamma^a_{bc} + \nabla^b R\delta\Gamma^c_{bc}. \end{aligned}$$

The last two term should be written in the standard form ($A_{ab}\,\delta g^{ab} + \nabla_a B^a$ = contribution to the field equations + a boundary term). The first one is

$$\begin{aligned} g^{bc}\,\nabla_a R\,\delta\Gamma^a_{bc} &= \frac{1}{2}g^{ad}g^{bc}\,(\nabla_a R)(\nabla_b\delta g_{cd} + \nabla_c\delta g_{bd} - \nabla_d\delta g_{bc}) \\ &= \frac{1}{2}(\nabla^d R)\,\nabla^c\delta g_{cd} + \frac{1}{2}(\nabla^d R)\,\nabla^b\delta g_{bd} - \frac{1}{2}g^{bc}(\nabla^d R)\,\nabla_d\delta g_{bc} \\ &= (\nabla^d R)\,\nabla^c\delta g_{cd} - \frac{1}{2}g^{bc}(\nabla^d R)\,\nabla_d\delta g_{bc} \\ &= \nabla^c\left((\nabla^d R)\,\delta g_{cd}\right) - \left(\nabla^c\nabla^d R\right)\,\delta g_{cd} - \nabla_d\left(\frac{1}{2}g^{bc}(\nabla^d R)\,\delta g_{bc}\right) + \frac{1}{2}g^{bc}\,\Box R\,\delta g_{bc} \\ &= -\nabla_a\left((\nabla_b R)\,\delta g^{ab}\right) + \nabla_a\nabla_b R\,\delta g^{ab} + \nabla_a\left(\frac{1}{2}g_{bc}(\nabla^a R)\,\delta g^{bc}\right) - \frac{1}{2}g_{ab}\,\Box R\,\delta g^{ab} \\ &= \left(\nabla_a\nabla_b R - \frac{1}{2}g_{ab}\,\Box R\right)\,\delta g^{ab} + \nabla_a\left(-(\nabla_b R)\,\delta g^{ab} + \frac{1}{2}g_{bc}(\nabla^a R)\,\delta g^{bc}\right). \end{aligned}$$

Using

$$\delta\Gamma^c_{bc} = \frac{1}{2}g^{cd}(\nabla_b\delta g_{cd} + \nabla_c\delta g_{bd} - \nabla_d\delta g_{bc}) = \frac{1}{2}g^{cd}\,\nabla_b\delta g_{cd}$$

in the second one gives

$$\begin{aligned} \nabla^b R\,\delta\Gamma^c_{bc} &= \frac{1}{2}g^{cd}\nabla^b R\,\nabla_b\delta g_{cd} = \nabla_b\left(\frac{1}{2}g^{cd}(\nabla^b R)\,\delta g_{cd}\right) - \frac{1}{2}g^{cd}\,\Box R\,\delta g_{cd} \\ &= -\nabla_b\left(\frac{1}{2}g_{cd}(\nabla^b R)\,\delta g^{cd}\right) + \frac{1}{2}g_{cd}\,\Box R\,\delta g^{cd} \\ &= \nabla_a\left(-\frac{1}{2}g_{bc}(\nabla^a R)\,\delta g^{bc}\right) + \frac{1}{2}g_{ab}\,\Box R\,\delta g^{ab}. \end{aligned}$$

With the help of

$$\delta(\sqrt{|g|})R^2 = -\frac{1}{2}\sqrt{|g|}g_{ab}R^2\,\delta g^{ab},$$

the first variation of the action becomes

$$\begin{aligned} \delta S &= \int \mathrm{d}^D x\,\sqrt{|g|}\,(2RR_{ab} - 2\nabla_a\nabla_b R + 2g_{ab}\,\Box R - \frac{1}{2}g_{ab}R^2)\,\delta g^{ab} \\ &+ \int \mathrm{d}^D x\,\sqrt{|g|}\,\nabla_a\left(2g^{bc}R\delta\Gamma^a_{bc} - 2g^{ab}R\delta\Gamma^c_{bc} + 2(\nabla_b R)\delta g^{ab} - 2g_{bc}(\nabla^a R)\delta g^{bc}\right). \end{aligned}$$

Therefore

$$\begin{aligned} \mathcal{A}_{ab} &= 2RR_{ab} - 2\nabla_a\nabla_b R + g_{ab}(2\Box R - \frac{1}{2}R^2), \\ \Lambda^a_\alpha &= \sqrt{|g|}\left(2g^{bc}R\delta\Gamma^a_{bc} - 2g^{ab}R\delta\Gamma^c_{bc} + 2\nabla^a R\,\delta\ln|g| + 2\nabla_b R\,\delta g^{ab}\right) \end{aligned}$$

are the field equations and the boundary term.

B.3 R_{ab}^2 Term

$$
\begin{aligned}
S_\beta &= \int \mathrm{d}^D x \sqrt{|g|}\, R_{ab}^2, \\
\delta S_\beta &= \int \mathrm{d}^D x \Big(\sqrt{|g|} R^{ab} \delta R_{ab} + \sqrt{|g|} R_{ab} \delta R^{ab} + R_{ab} R^{ab} \delta(\sqrt{-g}) \Big).
\end{aligned}
$$

The first term to deal with is

$$
\begin{aligned}
R^{ab}\delta R_{ab} &= R^{ab}(\nabla_c \delta\Gamma^c_{ab} - \nabla_a \delta\Gamma^c_{ac}) \\
&= R^{ab}\nabla_c \delta\Gamma^c_{ab} - R^{ab}\nabla_a \delta\Gamma^c_{ac} \\
&= \nabla_c(R^{ab}\delta\Gamma^c_{ab}) - \nabla_c R^{ab}\, \delta\Gamma^c_{ab} - \nabla_a(R^{ab}\delta\Gamma^c_{ac}) + \nabla_a R^{ab}\, \delta\Gamma^c_{ac} \\
&= \nabla_a(R^{bc}\delta\Gamma^a_{bc} - R^{ab}\delta\Gamma^c_{ac}) - \nabla_c R^{ab}\, \delta\Gamma^c_{ab} + \nabla_a R^{ab}\, \delta\Gamma^c_{ac}.
\end{aligned}
$$

Again, we should write the last two term in the standard form as follows

$$
\begin{aligned}
\nabla_c R^{ab}\, \delta\Gamma^c_{ab} &= \frac{1}{2}\nabla_c R^{ab}\, g^{cd}(\nabla_a \delta g_{bd} + \nabla_b \delta g_{ad} - \nabla_d \delta g_{ab}) \\
&= \nabla^d R^{ab}\, \nabla_a \delta g_{bd} - \frac{1}{2}\nabla^d R^{ab}\, \nabla_d \delta g_{ab} \\
&= \nabla_a(\nabla^d R^{ab}\, \delta g_{bd}) - \nabla_a \nabla^d R^{ab}\, \delta g_{bd} - \nabla_d(\frac{1}{2}\nabla^d R^{ab}\, \delta g_{ab}) + \frac{1}{2}\Box R^{ab}\, \delta g_{ab} \\
&= -\nabla_a(\nabla_b R^a{}_b\, \delta g^{bc}) + \nabla_a \nabla_d R^a{}_b\, \delta g^{bd} + \nabla_d(\frac{1}{2}\nabla^d R_{ab}\, \delta g^{ab}) - \frac{1}{2}\Box R_{ab}\, \delta g^{ab} \\
&= \nabla_a(-\nabla_b R^a{}_c\, \delta g^{bc} + \frac{1}{2}\nabla^a R_{bc}\, \delta g^{bc}) + (\nabla_c \nabla_a R^c{}_b - \frac{1}{2}\Box R_{ab})\delta g^{ab},
\end{aligned}
$$

and

$$
\begin{aligned}
\nabla_a R^{ab}\, \delta\Gamma^c_{bc} &= \frac{1}{2}\nabla^b R \frac{1}{2} g^{cd}(\nabla_b \delta g_{cd} + \nabla_c \delta g_{bd} - \nabla_d \delta g_{bc}) = \frac{1}{4} g^{cd} \nabla^b R \nabla_b \delta g_{cd} \\
&= \nabla_b(\frac{1}{4} g^{cd} \nabla^b R \delta g_{cd}) - \frac{1}{4} g^{cd} \Box R\, \delta g_{cd} \\
&= -\nabla_b(\frac{1}{4} g_{cd} \nabla^b R \delta g^{cd}) + \frac{1}{4} g_{cd} \Box R\, \delta g^{cd} \\
&= -\nabla_a(\frac{1}{4} g_{bc} \nabla^a R \delta g^{bc}) + \frac{1}{4} g_{ab} \Box R\, \delta g^{ab}.
\end{aligned}
$$

Using

$$
\begin{aligned}
R_{ab}\delta R^{ab} &= R_{ab}\delta(g^{ac} g^{bd} R^{cd}) \\
&= g^{ac} g^{bd} R_{ab} \delta R_{cd} + g^{bd} R_{ab} R_{cd} \delta g^{ac} + g^{ac} R_{ab} R_{cd} \delta g^{bd} \\
&= R^{cd}\delta R_{cd} + R_a{}^d R_{cd} \delta g^{ac} + R^c{}_b R_{cd} \delta g^{bd} \\
&= R^{ab}\delta R_{ab} + 2 R^c{}_a R_{cb} \delta g^{ab},
\end{aligned}
$$

and

$$\delta(\sqrt{-g})R^2_{ab} = -\frac{1}{2}g_{ab}R_{cd}R^{cd}\delta g^{ab},$$

gives

$$\begin{aligned}\delta S &= \int d^D x\,\sqrt{-g}\Big(-2\nabla_c\nabla_a R^c{}_b + \Box R_{ab} + \frac{1}{2}g_{ab}\Box R + 2R^c{}_a R_{cb}\Big)\\ &+ \int d^D x\,\sqrt{-g}\,\nabla_a\Big(2R^{bc}\delta\Gamma^a_{bc} + 2\nabla_b R^a{}_c\delta g^{bc} - \nabla^a R_{bc}\delta g^{bc} - \frac{1}{2}g_{bc}\nabla^a R\delta g^{bc}\Big),\end{aligned}$$

from which the field equations and the boundary term can be written as

$$\begin{aligned}\mathcal{B}_{ab} &= -\nabla_c\nabla_a R^c{}_b - \nabla_c\nabla_b R^c{}_a + \Box R_{ab} + \frac{1}{2}g_{ab}\Box R + 2R^c{}_a R_{cb},\\ &=\\ \Lambda^a_\beta &\equiv \sqrt{|g|}\Big(2R^{bc}\,\delta\Gamma^a_{bc} - 2R^{ab}\delta\Gamma^c_{bc} + \frac{1}{2}\nabla^a R\,\delta\ln|g| + 2\nabla_c R^a{}_b\,\delta g^{bc} - \nabla^a R_{cb}\,\delta g^{cb}\Big).\end{aligned}$$

Employing the following useful relation

$$\begin{aligned}\nabla_c\nabla_a R^c{}_b &= \nabla_a\nabla_c R^c{}_b + R^c{}_{dca}R^d{}_b - R^d{}_{bca}R^c{}_b\\ &= \frac{1}{2}\nabla_a\nabla_b R + R_{da}R^d{}_b - R^d{}_{bca}R^c{}_b\\ &= \frac{1}{2}\nabla_a\nabla_b R + R^c{}_a R_{cb} - R_{acbd}R^{cd},\end{aligned}$$

the field equations can also be written as

$$\mathcal{B}_{ab} = 2R_{acbd}R^{cd} - \nabla_a\nabla_b R + \Box R_{ab} + \frac{1}{2}g_{ab}\Box R - \frac{1}{2}g_{ab}R_{cd}R^{cd}.$$

APPENDIX C

SYMPLECTIC CURRENT

The symplectic current, which is necessary to construct the symplectic current, will be obtained through the formula $J^a = -\frac{\delta\Lambda^a}{\sqrt{|g|}}$.

C.1 R^2 Term

We start with the boundary term

$$\Lambda^a_\alpha = 2\sqrt{|g|}Rg^{bc}\,\delta\Gamma^a_{bc} - 2\sqrt{|g|}Rg^{ab}\,\delta\Gamma^c_{bc} + 2\sqrt{|g|}(\nabla_b R)\,\delta g^{ab} + 2\sqrt{|g|}g^{ab}(\nabla_b R)\,\delta\ln|g|,$$

whose exterior derivative is

$$\begin{aligned}
\delta\Lambda^a_\alpha \quad = \quad & \sqrt{|g|}Rg^{bc}\,\delta\ln|g| \wedge \delta\Gamma^a_{bc} + 2\sqrt{|g|}g^{bc}\,\delta R \wedge \delta\Gamma^a_{bc} + 2\sqrt{|g|}R\,\delta g^{bc} \wedge \delta\Gamma^a_{bc} \\
& - \sqrt{|g|}Rg^{ab}\,\delta\ln|g| \wedge \delta\Gamma^c_{bc} - 2\sqrt{|g|}g^{ab}\,\delta R \wedge \delta\Gamma^c_{bc} - 2\sqrt{|g|}R\,\delta g^{ab} \wedge \delta\Gamma^c_{bc} \\
& + \sqrt{|g|}(\nabla_b R)\,\delta\ln|g| \wedge \delta g^{ab} + 2\sqrt{|g|}\,\nabla_b\delta R \wedge \delta g^{ab} + \underbrace{\sqrt{|g|}(\nabla^a R)\,\delta\ln|g| \wedge \delta\ln|g|}_{=0} \\
& + 2\sqrt{|g|}(\nabla_b R)\,\delta g^{ab} \wedge \delta\ln|g| + 2\sqrt{|g|}\,\nabla^a\delta R \wedge \delta\ln|g|.
\end{aligned}$$

Therefore, the symplectic current is

$$\begin{aligned}
J^a_\alpha \quad = \quad & -Rg^{bc}\,\delta\ln|g| \wedge \delta\Gamma^a_{bc} - 2g^{bc}\,\delta R \wedge \delta\Gamma^a_{bc} - 2R\,\delta g^{bc} \wedge \delta\Gamma^a_{bc} \\
& + g^{ab}R\,\delta\ln|g| \wedge \delta\Gamma^c_{bc} + 2g^{ab}\,\delta R \wedge \delta\Gamma^c_{bc} + 2R\,\delta g^{ab} \wedge \delta\Gamma^c_{bc} \\
& - (\nabla_b R)\,\delta\ln|g| \wedge \delta g^{ab} - 2\nabla_b\delta R \wedge \delta g^{ab} - 2(\nabla_b R)\,\delta g^{ab} \wedge \delta\ln|g| \\
& -2\nabla^a\delta R \wedge \delta\ln|g|
\end{aligned}$$

C.2 R_{ab}^2 Term

$$\Lambda^a_\beta = 2\sqrt{|g|}R^{bc}\,\delta\Gamma^a_{bc} - 2\sqrt{|g|}R^{ab}\,\delta\Gamma^c_{bc} + \frac{1}{2}\sqrt{|g|}(\nabla^a R)\,\delta\ln|g| + 2\sqrt{|g|}(\nabla_b R^a_c)\,\delta g^{bc} - \sqrt{|g|}(\nabla^a R_{bc})\,\delta g^{bc}$$

$$\begin{aligned}\delta\Lambda^a_\beta = {}& \sqrt{|g|}R^{bc}\,\delta\ln|g|\wedge\delta\Gamma^a_{bc} + 2\sqrt{|g|}\,\delta R^{bc}\wedge\delta\Gamma^a_{bc} - \sqrt{|g|}R^{ab}\,\delta\ln|g|\wedge\delta\Gamma^c_{bc} \\ & - 2\sqrt{|g|}\,\delta R^{ab}\wedge\delta\Gamma^c_{bc} + \underbrace{\frac{1}{4}\sqrt{|g|}(\nabla^a R)\,\delta\ln|g|\wedge\delta\ln|g|}_{=0} + \frac{1}{2}\sqrt{|g|}\delta(\nabla^a R)\wedge\delta\ln|g| \\ & + \sqrt{|g|}(\nabla_b R^a_c)\,\delta\ln|g|\wedge\delta g^{bc} + 2\sqrt{|g|}\,\delta(\nabla_b R^a_c)\wedge\delta g^{bc} - \frac{1}{2}\sqrt{|g|}(\nabla^a R_{bc})\,\delta\ln|g|\wedge\delta g^{bc} \\ & - \sqrt{|g|}\,\delta(\nabla^a R_{bc})\wedge\delta g^{bc}\end{aligned}$$

$$\begin{aligned}J^a_\beta = {}& -R^{bc}\,\delta\ln|g|\wedge\delta\Gamma^a_{bc} \underbrace{-2\,\delta R^{bc}\wedge\delta\Gamma^a_{bc}}_{A^a} + R^{ab}\,\delta\ln|g|\wedge\delta\Gamma^c_{bc} \\ & \underbrace{+2\,\delta R^{ab}\wedge\delta\Gamma^c_{bc}}_{B^a} \underbrace{-\frac{1}{2}\,\delta(\nabla^a R)\wedge\delta\ln|g|}_{C^a} - (\nabla_b R^a_c)\,\delta\ln|g|\wedge\delta g^{bc} \\ & \underbrace{-2\,\delta(\nabla_b R^a_c)\wedge\delta g^{bc}}_{D^a} + \frac{1}{2}(\nabla^a R_{bc})\,\delta\ln|g|\wedge\delta g^{bc} \underbrace{+\delta(\nabla^a R_{bc})\wedge\delta g^{bc}}_{E^a}\end{aligned}$$

We need to rewrite some terms as follows

$$\begin{aligned}A^a &= -2\,\delta(g^{bd}g^{ce}R_{de})\wedge\delta\Gamma^a_{bc} \\ &= -2g^{ce}R_{de}\,\delta g^{bd}\wedge\delta\Gamma^a_{bc} - 2g^{bd}R_{de}\,\delta g^{ce}\wedge\delta\Gamma^a_{bc} - 2g^{bd}g^{ce}\,\delta R_{de}\wedge\delta\Gamma^a_{bc} \\ &= -2R_d^{\ c}\,\delta g^{bd}\wedge\delta\Gamma^a_{bc} - 2R^b_e\,\delta g^{ce}\wedge\delta\Gamma^a_{bc} - 2g^{bd}g^{ce}\,\delta R_{de}\wedge\delta\Gamma^a_{bc} \\ &= -4R^c_d\,\delta g^{bd}\wedge\delta\Gamma^a_{bc} - 2g^{bd}g^{ce}\,\delta R_{de}\wedge\delta\Gamma^a_{bc},\end{aligned}$$

$$\begin{aligned}B^a &= 2\,\delta(g^{ad}g^{be}R_{de})\wedge\delta\Gamma^c_{bc} \\ &= 2g^{be}R_{de}\,\delta g^{ad}\wedge\delta\Gamma^c_{bc} + 2g^{ad}R_{de}\,\delta g^{be}\wedge\delta\Gamma^c_{bc} + 2g^{ad}g^{be}\,\delta R_{de}\wedge\delta\Gamma^c_{bc} \\ &= 2R_d^{\ b}\,\delta g^{ad}\wedge\delta\Gamma^c_{bc} + 2R^a_e\,\delta g^{be}\wedge\delta\Gamma^c_{bc} + 2g^{ad}g^{be}\,\delta R_{de}\wedge\delta\Gamma^c_{bc},\end{aligned}$$

$$\begin{aligned}C^a &= \frac{1}{2}\delta\ln|g|\wedge\delta(\nabla^a R) = \frac{1}{2}\delta\ln|g|\wedge\delta(g^{ab}\nabla_b R) \\ &= \frac{1}{2}(\nabla_b R)\,\delta\ln|g|\wedge\delta g^{ab} + \frac{1}{2}\delta\ln|g|\wedge\nabla^a\delta R,\end{aligned}$$

$$\begin{aligned}D^a &= -2\,\delta(g^{ad}\nabla_c R_{db})\wedge\delta g^{bc} = -2(\nabla_c R_{db})\,\delta g^{ad}\wedge\delta g^{bc} - 2g^{ad}\,\delta(\nabla_c R_{db})\wedge\delta g^{bc} \\ &= -2(\nabla_c R_{db})\,\delta g^{ad}\wedge\delta g^{bc} - 2g^{ad}\,(\nabla_c\delta R_{db} - R_{eb}\,\delta\Gamma^e_{dc} - R_{de}\,\delta\Gamma^e_{bc})\wedge\delta g^{bc} \\ &= -2(\nabla_c R_{db})\,\delta g^{ad}\wedge\delta g^{bc} - 2g^{ad}\,\nabla_c\delta R_{db}\wedge\delta g^{bc} + 2g^{ad}R_{eb}\,\delta\Gamma^e_{dc}\wedge\delta g^{bc} + 2R^a_e\,\delta\Gamma^e_{bc}\wedge\delta g^{bc},\end{aligned}$$

$$
\begin{aligned}
E^a &= \delta(g^{ad}\nabla_d R_{bc}) \wedge \delta g^{bc} = \nabla_d R_{bc}\,\delta g^{ad} \wedge \delta g^{bc} + g^{ad}\,\delta(\nabla_d R_{bc}) \wedge \delta g^{bc} \\
&= \nabla_d R_{bc}\,\delta g^{ad} \wedge \delta g^{bc} + g^{ad}\,(\nabla_d \delta R_{bc} - R_{eb}\,\delta\Gamma^e{}_{cd} - R_{ce}\,\delta\Gamma^e{}_{bd}) \wedge \delta g^{bc} \\
&= \nabla_d R_{bc}\,\delta g^{ad} \wedge \delta g^{bc} + g^{ad}\,\nabla_d \delta R_{bc} \wedge \delta g^{bc} - g^{ad} R_{eb}\,\delta\Gamma^e{}_{cd} \wedge \delta g^{bc} - g^{ad} R_{ce}\,\delta\Gamma^e{}_{bd} \wedge \delta g^{bc} \\
&= \nabla_d R_{bc}\,\delta g^{ad} \wedge \delta g^{bc} + \nabla^a \delta R_{bc} \wedge \delta g^{bc} - 2g^{ad} R_{eb}\,\delta\Gamma^e{}_{cd} \wedge \delta g^{bc}.
\end{aligned}
$$

Then, the symplectic current becomes

$$
\begin{aligned}
J^a_\beta \;=\;& -R^{bc}\,\delta\ln|g| \wedge \delta\Gamma^a{}_{bc} + R^{ab}\,\delta\ln|g| \wedge \delta\Gamma^c{}_{bc} - (\nabla_b R^a_c)\,\delta\ln|g| \wedge \delta g^{bc} + \frac{1}{2}(\nabla^a R_{bc})\,\delta\ln|g| \wedge \delta g^{bc} \\
& - 4R^c_d\,\delta g^{bd} \wedge \delta\Gamma^a{}_{bc} - 2g^{bd} g^{ce}\,\delta R_{de} \wedge \delta\Gamma^a{}_{bc} + 2R_d{}^b\,\delta g^{ad} \wedge \delta\Gamma^c{}_{bc} + 2R^a_e\,\delta g^{be} \wedge \delta\Gamma^c{}_{bc} \\
& + 2g^{ad} g^{be}\,\delta R_{de} \wedge \delta\Gamma^c{}_{bc} + \frac{1}{2}(\nabla_b R)\,\delta\ln|g| \wedge \delta g^{ab} + \frac{1}{2}\delta\ln|g| \wedge \nabla^a \delta R - 2(\nabla_c R_{db})\,\delta g^{ad} \wedge \delta g^{bc} \\
& - 2g^{ad}\,\nabla_c \delta R_{db} \wedge \delta g^{bc} \underbrace{+2g^{ad} R_{eb}\,\delta\Gamma^e{}_{dc} \wedge \delta g^{bc}}_{X^a_1} + 2R^a_e\,\delta\Gamma^e{}_{bc} \wedge \delta g^{bc} + \nabla_d R_{bc}\,\delta g^{ad} \wedge \delta g^{bc} \\
& + \nabla^a \delta R_{bc} \wedge \delta g^{bc} \underbrace{-2g^{ad} R_{eb}\,\delta\Gamma^e{}_{cd} \wedge \delta g^{bc}}_{X^a_1}
\end{aligned}
$$

Terms X^a_1 cancel each other.

APPENDIX D

COVARIANT DERIVATIVE OF THE SYMPLECTIC CURRENT

Although we have shown that the symplectic current is covariantly conserved for any theory derived from a generic local gravity action, we will show this explicitly for the action which is quadratic in Ricci scalar and Ricci tensor.

D.1 R^2 Term

The covariant derivative of the symplectic current is given by

$$\begin{aligned}
\nabla_a J^a_\alpha \;=\;& -g^{bc}(\nabla_a R)\,\delta\ln|g| \wedge \delta\Gamma^a_{bc} - g^{bc} R\,\nabla_a \delta\ln|g| \wedge \delta\Gamma^a_{bc} - \underbrace{R\,\delta\ln|g| \wedge \nabla_a \delta\Gamma^a_{bc}}_{F} \\
& - 2g^{bc}\,\nabla_a \delta R \wedge \delta\Gamma^a_{bc} - \underbrace{2g^{bc}\,\delta R \wedge \nabla_a \delta\Gamma^a_{bc}}_{B} - 2(\nabla_a R)\,\delta g^{bc} \wedge \delta\Gamma^a_{bc} - 2R\,\underbrace{\nabla_a \delta g^{bc} \wedge \delta\Gamma^a_{bc}}_{=0} \\
& - \underbrace{2R\,\delta g^{bc} \wedge \nabla_a \delta\Gamma^a_{bc}}_{D} + (\nabla^b R)\,\delta\ln|g| \wedge \delta\Gamma^c_{bc} + \underbrace{R\,\nabla^b \delta\ln|g| \wedge \delta\Gamma^c_{bc}}_{G} \\
& + \underbrace{R\,\delta\ln|g| \wedge \nabla^b \delta\Gamma^c_{bc}}_{E} + 2\nabla^b \delta R \wedge \delta\Gamma^c_{bc} + \underbrace{2\delta R \wedge \nabla^b \delta\Gamma^c_{bc}}_{A} + 2(\nabla_a R)\,\delta g^{ab} \wedge \Gamma^c_{bc} \\
& + \underbrace{2R\,\nabla_a \delta g^{ab} \wedge \Gamma^c_{bc}}_{H} + \underbrace{2R\,\delta g^{ab} \wedge \nabla_a \Gamma^c_{bc}}_{C} - (\nabla_a \nabla_b R)\,\delta\ln|g| \wedge \delta g^{ab} \\
& - \underbrace{(\nabla_b R)\,\nabla_a \delta\ln|g| \wedge \delta g^{ab}}_{L} - \underbrace{(\nabla_b R)\,\delta\ln|g| \wedge \nabla_a \delta g^{ab}}_{I} - 2\nabla_a \nabla_b \delta R \wedge \delta g^{ab} \\
& - 2\nabla_a \nabla_b \delta R \wedge \delta g^{ab} - \underbrace{2\nabla_b \delta R \wedge \nabla_a \delta g^{ab}}_{M} - 2(\nabla_a \nabla_b R)\,\delta g^{ab} \wedge \delta\ln|g| \\
& - \underbrace{2(\nabla_b R)\,\nabla_a \delta g^{ab} \wedge \delta\ln|g|}_{J} - \underbrace{2(\nabla_b R)\,\delta g^{ab} \wedge \nabla_a \delta\ln|g|}_{K} - 2\nabla_a \nabla^a \delta R \wedge \delta\ln|g| \\
& - 4\nabla^a \delta R \wedge \delta\Gamma^b_{ab},
\end{aligned}$$

where

$$A + B = 2g^{bc}\,\delta R \wedge (\nabla_c \delta\Gamma^d_{bd} - \nabla_a \delta\Gamma^a_{bc}) = 2g^{ab}\,\delta R_{ab} \wedge \delta R,$$

$$C + D = 2R\,\delta g^{ab} \wedge (\nabla_a \delta\Gamma^c_{bc} - \nabla_c \delta\Gamma^c_{ab}) = 2R\,\delta R_{ab} \wedge \delta g^{ab},$$

$$E + F = g^{ab} R\,\delta \ln|g| \wedge (\nabla_a \delta\Gamma^c_{bc} - \nabla_c \delta\Gamma^c_{ab}) = g^{ab} R\,\delta R_{ab} \wedge \delta \ln|g|,$$

$$G = 2g^{ab} R\,\delta\Gamma^d_{ad} \wedge \delta\Gamma^c_{bc} = 0,$$

$$\begin{aligned} H &= 2R\,\delta\Gamma^c_{bc} \wedge (g^{de}\,\delta\Gamma^b_{de} + g^{db}\,\delta\Gamma^e_{de}) \\ &= 2g^{de} R\,\delta\Gamma^c_{bc} \wedge \delta\Gamma^b_{de} + \underbrace{2g^{db} R\,\delta\Gamma^c_{bc} \wedge \delta\Gamma^e_{de}}_{=0} = 2g^{ab} R\,\delta\Gamma^c_{bc} \wedge \delta\Gamma^d_{ab}, \end{aligned}$$

$$\begin{aligned} I + J &= (\nabla_b R)\,\delta \ln|g| \wedge \nabla_a \delta g^{ab} = (\nabla_b R)\,(g^{de}\delta\Gamma^b_{de} + g^{db}\delta\Gamma^e_{de}) \wedge \delta \ln|g| \\ &= g^{ab}(\nabla_c R)\,\delta\Gamma^b_{de} \wedge \delta \ln|g| + (\nabla^a R)\,\delta\Gamma^b_{ab} \wedge \delta \ln|g|, \end{aligned}$$

$$K + L = (\nabla_b R)\,\nabla_a \delta \ln|g| \wedge \delta g^{ab} = 2(\nabla_b R)\,\delta\Gamma^c_{ac} \wedge \delta g^{ab},$$

$$\begin{aligned} M &= 2\nabla_b \delta R \wedge (g^{de}\delta\Gamma^b_{de} + g^{db}\delta\Gamma^e_{de}) \\ &= 2g^{ab}\,\nabla_c \delta R \wedge \delta\Gamma^c_{ab} + 2\nabla^a \delta R \wedge \delta\Gamma^b_{ab}, \end{aligned}$$

which give

$$\begin{aligned} \nabla_a J^a_\alpha = &- \underbrace{g^{bc}(\nabla_a R)\,\delta \ln|g| \wedge \delta\Gamma^a_{bc}}_{I} - \underbrace{2g^{bc} R\,\delta\Gamma^d_{ad} \wedge \delta\Gamma^a_{bc}}_{B} - \underbrace{2g^{bc}\,\nabla_a \delta R \wedge \delta\Gamma^a_{bc}}_{G} \\ &- 2(\nabla_a R)\,\delta g^{bc} \wedge \delta\Gamma^a_{bc} + \underbrace{(\nabla^b R)\,\delta \ln|g| \wedge \delta\Gamma^c_{bc}}_{K} + \underbrace{2\nabla^b \delta R \wedge \delta\Gamma^c_{bc}}_{D} \\ &+ \underbrace{2(\nabla_a R)\,\delta g^{ab} \wedge \Gamma^c_{bc}}_{M} - \underbrace{(\nabla_a \nabla_b R)\,\delta \ln|g| \wedge \delta g^{ab}}_{N} - 2\nabla_a \nabla_b \delta R \wedge \delta g^{ab} \\ &- \underbrace{2(\nabla_a \nabla_b R)\delta g^{ab} \wedge \delta \ln|g|}_{O} - 2\Box \delta R \wedge \delta \ln|g| - \underbrace{4\nabla \delta R \wedge \delta\Gamma^b_{ab}}_{E} + 2g^{ab}\,\delta R_{ab} \wedge \delta R \\ &+ 2R\,\delta R_{ab} \wedge \delta g^{ab} + g^{ab} R\,\delta R_{ab} \wedge \delta \ln|g| + \underbrace{2g^{ab} R\,\delta\Gamma^c_{bc} \wedge \delta\Gamma^d_{ab}}_{A} \\ &+ \underbrace{g^{ab}(\nabla_c R)\,\delta\Gamma^b_{de} \wedge \delta \ln|g|}_{H} + \underbrace{(\nabla^a R)\,\delta\Gamma^b_{ab} \wedge \delta \ln|g|}_{J} + \underbrace{2(\nabla_b R)\,\delta\Gamma^c_{ac} \wedge \delta g^{ab}}_{L} \\ &\underbrace{+\,2g^{ab}\,\nabla_c \delta R \wedge \delta\Gamma^c_{ab}}_{F} + \underbrace{2\nabla^a \delta R \wedge \delta\Gamma^b_{ab}}_{C}. \end{aligned}$$

This can be put into the final form using

$$A + B = 0 \qquad C + D + E = 0 \qquad F + G = 0 \qquad K + J = 0 \qquad L + M = 0,$$

$$H + I = 2g^{ab}(\nabla_c R)\,\delta\Gamma^c_{ab} \wedge \delta\ln|g| \qquad N + O = (\nabla_a\nabla_b R)\,\delta\ln|g| \wedge \delta g^{ab},$$

as

$$\begin{aligned}
\nabla_a J^a_\alpha &= -2(\nabla_a R)\,\delta g^{bc} \wedge \delta\Gamma^a_{bc} - 2\nabla_a\nabla_b\delta R \wedge \delta g^{ab} - 2\Box\delta R \wedge \delta\ln|g| \\
&\quad + 2g^{ab}\,\delta R_{ab} \wedge \delta R + 2R\,\delta R_{ab} \wedge \delta g^{ab} + g^{ab}R\,\delta R_{ab} \wedge \delta\ln|g| \\
&\quad + 2g^{ab}(\nabla_c R)\,\delta\Gamma^c_{ab} \wedge \delta\ln|g| + (\nabla_a\nabla_b R)\,\delta\ln|g| \wedge \delta g^{ab}.
\end{aligned}$$

We will show that it is equal to $\bigstar = \frac{1}{2}g^{ab}\,\delta\mathcal{A}_{ab} \wedge \delta\ln|g| + \delta\mathcal{A}_{ab} \wedge \delta g^{ab}$ which vanish on-shell. The field equation and its variation are

$$\begin{aligned}
\mathcal{A}_{ab} &= 2RR_{ab} - \frac{1}{2}g_{ab}R^2 + 2g_{ab}\Box R - 2\nabla_a\nabla_b R, \\
\delta\mathcal{A}_{ab} &= 2R_{ab}\,\delta R + 2R\,\delta R_{ab} - \frac{1}{2}R^2\,\delta g_{ab} - g_{ab}R\,\delta R + 2(\Box R)\,\delta g_{ab} \\
&\quad + 2g_{ab}\,\delta(\Box R) - 2\delta(\nabla_a\nabla_b R).
\end{aligned}$$

Using the following relations derived in Appendix A

$$\begin{aligned}
\delta(\nabla_a\nabla_b R) &= \nabla_a\nabla_b\delta R - (\nabla_c R)\,\delta\Gamma^c_{ba}, \\
\delta(\Box R) &= \delta(g^{ab}\nabla_a\nabla_b R) = (\nabla_a\nabla_b R)\,\delta g^{ab} + \Box\delta R - g^{ab}(\nabla_c R)\,\delta\Gamma^c_{ab},
\end{aligned}$$

we have

$$\begin{aligned}
\delta\mathcal{A}_{ab} &= 2R_{ab}\,\delta R + 2R\,\delta R_{ab} - \frac{1}{2}R^2\,\delta g_{ab} - g_{ab}R\,\delta R + 2(\Box R)\,\delta g_{ab} \\
&\quad + 2g_{ab}(\nabla_c\nabla_d R)\,\delta g^{cd} + 2g_{ab}\,\Box\delta R - 2g_{ab}g^{cd}(\nabla_e R)\,\delta\Gamma^e_{cd} \\
&\quad - 2\nabla_a\nabla_b\delta R + 2(\nabla_c R)\,\delta\Gamma^c_{ba},
\end{aligned}$$

through which we obtain

$$\begin{aligned}
\frac{1}{2}g^{ab}\,\delta\mathcal{A}_{ab} \wedge \delta\ln|g| &= \underbrace{R\,\delta R \wedge \delta\ln|g|}_{D} + g^{ab}R\,\delta R_{ab} \wedge \delta\ln|g| + \underbrace{\frac{1}{4}R^2\delta\ln|g| \wedge \delta\ln|g|}_{=0} \\
&\quad \underbrace{-\frac{D}{2}R\,\delta R \wedge \delta\ln|g|}_{E} + \underbrace{\Box R\,\delta\ln|g| \wedge \delta\ln|g|}_{=0} + D(\nabla_a\nabla_b R)\,\delta g^{ab} \wedge \delta\ln|g| \\
&\quad + \underbrace{D\,\Box\delta R \wedge \delta\ln|g|}_{B} - \underbrace{Dg^{cd}(\nabla_e R)\,\delta\Gamma^e_{cd} \wedge \delta\ln|g|}_{I} - \underbrace{\Box\delta R \wedge \delta\ln|g|}_{C}\,\delta\ln|g| \\
&\quad + \underbrace{g^{ab}(\nabla_c R)\,\delta\Gamma^c_{ab} \wedge \delta\ln|g|}_{H},
\end{aligned}$$

$$\begin{aligned}
\delta\mathcal{A}_{ab}\wedge\delta g^{ab} &= 2R_{ab}\,\delta R\wedge\delta g^{ab}+2R\,\delta R_{ab}\wedge\delta g^{ab}-\underbrace{\frac{1}{2}R^2\delta g_{ab}\wedge\delta g^{ab}}_{=0}+\underbrace{R\,\delta R\wedge\delta\ln|g|}_{F}\\
&+\underbrace{2\Box R\,\delta g_{ab}\wedge\delta g^{ab}}_{=0}-2(\nabla_a\nabla_b R)\,\delta g^{ab}\wedge\delta\ln|g|-\underbrace{2\,\Box\delta R\wedge\delta\ln|g|}_{A}\\
&+\underbrace{2g^{cd}(\nabla_e R)\,\delta\Gamma^e_{cd}\wedge\delta\ln|g|}_{G}-2\,\nabla_a\nabla_b\delta R\wedge\delta g^{ab}+2(\nabla_c R)\,\delta\Gamma^c_{ab}\wedge\delta g^{ab}.
\end{aligned}$$

With the following simplifications

$$\begin{aligned}
A+B+C &= (D-3)\,\Box\delta R\wedge\delta\ln|g|,\\
D+E+F &= \left(2-\frac{D}{2}\right)R\,\delta R\wedge\delta\ln|g|,\\
G+H+I &= (3-D)g^{ab}(\nabla_c R)\,\delta\Gamma^c_{ab}\delta\ln|g|,
\end{aligned}$$

★becomes

$$\begin{aligned}
\bigstar &= g^{ab}R\,\delta R_{ab}\delta\ln|g|+D(\nabla_a\nabla_b R)\,\delta g^{ab}\wedge\delta\ln|g|+2R_{ab}\,\delta R\wedge\delta g^{ab}+2R\,\delta R_{ab}\wedge\delta g^{ab}\\
&\quad-2(\nabla_a\nabla_b R)\,\delta g^{ab}\wedge\delta\ln|g|-2\,\nabla_a\nabla_b\delta R\wedge\delta g^{ab}+2(\nabla_c R)\,\delta\Gamma^c_{ab}\wedge\delta g^{ab}\\
&\quad+(D-3)\,\Box\delta R\wedge\delta\ln|g|+\left(2-\frac{D}{2}\right)R\,\delta R\wedge\delta\ln|g|+(3-D)g^{ab}(\nabla_c R)\,\delta\Gamma^c_{ab}\delta\ln|g|.
\end{aligned}$$

We need to get rid of the quantities whose coefficients depend on D. For that, consider the trace of the field equations and its first variation

$$\begin{aligned}
\mathcal{A} &= g^{ab}\mathcal{A}_{ab}=2R^2-\frac{D}{2}R^2+2D\Box R-2\Box R=\left(2-\frac{D}{2}\right)R^2+2(D-1)\Box R,\\
\delta\mathcal{A} &= 2\left(2-\frac{D}{2}\right)R\,\delta R+2(D-1)\left[(\nabla_a\nabla_b R)\,\delta g^{ab}+\Box\delta R-g^{ab}(\nabla_c R)\,\delta\Gamma^c_{ab}\right],
\end{aligned}$$

which give

$$\begin{aligned}
\frac{1}{2}\,\delta A\wedge\delta\ln|g| &= \left(2-\frac{D}{2}\right)R\,\delta R\wedge\delta\ln|g|+(D-1)(\nabla_a\nabla_b R)\,\delta g^{ab}\wedge\delta\ln|g|\\
&\quad+(D-1)\Box\delta R\wedge\delta\ln|g|-(D-1)g^{ab}(\nabla_c R)\,\delta\Gamma^c_{ab}\wedge\delta\ln|g|.
\end{aligned}$$

Finally, we write

$$\begin{aligned}
\bigstar &= \underbrace{\frac{1}{2}\,\delta A\wedge\delta\ln|g|}_{=0\,\text{(on-shell)}}-\underbrace{(\nabla_a\nabla_b R)\,\delta g^{ab}\wedge\delta\ln|g|}_{A}-\underbrace{2\Box\delta R\wedge\delta\ln|g|}_{B}+\underbrace{2g^{ab}(\nabla_c R)\,\delta\Gamma^c_{ab}\wedge\delta\ln|g|}_{C}\\
&\quad+\underbrace{g^{ab}R\,\delta R_{ab}\wedge\delta\ln|g|}_{D}+\underbrace{2R_{ab}\,\delta R\wedge\delta g^{ab}}_{H}+\underbrace{2R\,\delta R_{ab}\wedge\delta g^{ab}}_{I}-\underbrace{2\nabla_a\nabla_b\delta R\wedge\delta g^{ab}}_{F}\\
&\quad+\underbrace{2(\nabla_c R)\,\delta\Gamma^c_{ab}\wedge\delta g^{ab}}_{G},
\end{aligned}$$

which is identical to the covariant derivative of the symplectic current

$$\begin{aligned}
\nabla_a J^a_\alpha &= -\underbrace{2(\nabla_a R)\,\delta g^{bc}\wedge\delta\Gamma^a_{bc}}_{G} - \underbrace{2\nabla_a\nabla_b\delta R\wedge\delta g^{ab}}_{F} - \underbrace{2\Box\delta R\wedge\delta\ln|g|}_{B} \\
&+ \underbrace{2g^{ab}\,\delta R_{ab}\wedge\delta R}_{\tilde{H}} + \underbrace{2R\,\delta R_{ab}\wedge\delta g^{ab}}_{I} + \underbrace{g^{ab}R\,\delta R_{ab}\wedge\delta\ln|g|}_{D} \\
&+ \underbrace{2g^{ab}(\nabla_c R)\,\delta\Gamma^c_{ab}\wedge\delta\ln|g|}_{C} + \underbrace{(\nabla_a\nabla_b R)\,\delta\ln|g|\wedge\delta g^{ab}}_{A}.
\end{aligned}$$

All the terms from A to G are the same. For the last term we have

$$\delta R\wedge\delta R = g^{ab}\,\delta R_{ab}\wedge\delta R + R_{ab}\,\delta g^{ab}\wedge\delta R = 0 \Rightarrow H = \tilde{H}.$$

D.2 R_{ab}^2 Term

This time, the covariant derivative of the symplectic current is

$$
\begin{aligned}
\nabla_a J^a_\beta = & -\nabla_a R^{bc} \ \ln|g| \wedge \delta\Gamma^a_{bc} - \underbrace{2R^{bc}\,\delta\Gamma^d_{ad} \wedge \delta\Gamma^a_{bc}}_{B} - \underbrace{R^{bc}\,\delta\ln|g| \wedge \nabla_a\delta\Gamma^a_{bc}}_{K} \\
& + \frac{1}{2}\nabla^b R\,\delta\ln|g| \wedge \delta\Gamma^c_{bc} + \underbrace{2R^{ab}\,\delta\Gamma^d_{ad} \wedge \delta\Gamma^c_{bc}}_{=0} + \underbrace{R^{ab}\,\delta\ln|g| \wedge \nabla_a\delta\Gamma^a_{bc}}_{K} \\
& - (\nabla_a\nabla_c R^a{}_b)\,\delta\ln|g| \wedge \delta g^{bc} - \underbrace{2\nabla_c R^a{}_b\,\delta\Gamma^d_{ad} \wedge \delta g^{bc}}_{D} - \underbrace{\nabla_c R^a{}_b\,\delta\ln|g| \wedge \nabla_a\delta g^{bc}}_{X_1} \\
& \frac{1}{2}\Box R_{bc}\,\delta\ln|g| \wedge \delta g^{bc} + \underbrace{\nabla^a R_{bc}\,\delta\Gamma^d_{ad} \wedge \delta g^{bc}}_{E} + \underbrace{\nabla^a R_{bc}\,\delta\ln|g| \wedge \nabla_a\delta g^{bc}}_{X_2} \\
& - \underbrace{4(\nabla_a R_d{}^b)\,\delta g^{cd} \wedge \delta\Gamma^a_{bc}}_{F} - \underbrace{4R_d{}^b\,\nabla_a\delta g^{cd} \wedge \delta\Gamma^a_{bc}}_{X_3} - 4R_d{}^b\,\delta g^{cd} \wedge \nabla_a\delta\Gamma^a_{bc} \\
& - 2g^{cd}g^{be}\,\nabla_a\delta R_{de} \wedge \delta\Gamma^a_{bc} - \underbrace{2g^{cd}g^{be}\,\delta R_{de} \wedge \nabla_a\delta\Gamma^a_{bc}}_{L} + \underbrace{2(\nabla_a R_d{}^b)\,\delta g^{ad} \wedge \delta\Gamma^c_{bc}}_{D} \\
& + \underbrace{2R_d{}^b\,\nabla_a\delta g^{ad} \wedge \delta\Gamma^c_{bc}}_{X_4} + 2R_d{}^b\,\delta g^{ad} \wedge \nabla_a\delta\Gamma^c_{bc} + \underbrace{(\nabla_d R)\,\delta g^{bd} \wedge \delta\Gamma^c_{bc}}_{G} \\
& + \underbrace{2R^a{}_c\,\nabla_a\delta g^{bc} \wedge \delta\Gamma^e_{be}}_{X_5} + 2R^a{}_d\,\delta g^{bd} \wedge \nabla_a\delta\Gamma^c_{bc} + 2g^{bc}\ \nabla^d\delta R_{de} \wedge \delta\Gamma^c_{bc} \\
& + \underbrace{2g^{bc}g^{be}\ \delta R_{de} \wedge \nabla^d\delta\Gamma^c_{bc}}_{L} - \frac{1}{2}(\nabla_a\nabla_b R)\ \delta g^{ab} \wedge \delta\ln|g| - \underbrace{(\nabla_b R)\,\delta g^{ab} \wedge \delta\Gamma^c_{ac}}_{G} \\
& - \underbrace{\frac{1}{2}(\nabla_b R)\,\nabla_a\delta g^{ab} \wedge \delta\ln|g|}_{X_6} - \frac{1}{2}\,\Box\delta R \wedge \delta\ln|g| - \nabla^a\delta R \wedge \delta\Gamma^b_{ab} - 2(\nabla_a\nabla_c R_{db})\,\delta g^{ad} \wedge \delta g^{bc} \\
& - \underbrace{2(\nabla_c R_{db})\ \nabla_a\delta g^{ab} \wedge \delta g^{dc}}_{X_7} - \underbrace{2(\nabla_c R_{eb})\ \delta g^{ae} \wedge \nabla_a\delta g^{bc}}_{X_8} - 2\,\nabla^d\nabla_c\delta R_{db} \wedge \delta g^{bc} \\
& - \underbrace{2\,(\nabla_c\delta R_{db}) \wedge \delta\nabla^d\delta g^{bc}}_{X_9} + (\nabla_d R)\,\delta\Gamma^d_{bc} \wedge \delta g^{bc} + 2R^a{}_d\,\nabla_a\delta\Gamma^d_{bc} \wedge \delta g^{bc} \\
& + \underbrace{2R^a{}_d\ \delta\Gamma^d_{bc} \wedge \nabla_a\delta g^{bc}}_{X_{10}} + (\nabla_a\nabla_d R_{bc})\ \delta g^{ad} \wedge \delta g^{bc} + \underbrace{(\nabla_b R_{dc})\ \nabla_a\delta g^{ab} \wedge \delta g^{dc}}_{X_{11}} \\
& + \underbrace{(\nabla_b R_{dc})\ \delta g^{ab} \wedge \nabla_a\delta g^{bc}}_{X_{12}} + \Box\delta R_{bc} \wedge \delta g^{bc} + \underbrace{\nabla^a\delta R_{bc} \wedge \nabla_a\delta g^{bc}}_{X_{13}}.
\end{aligned}
$$

We first write

$$
X_1 = \nabla_c R^{ad}\ \delta\ln|g| \wedge \delta\Gamma^c_{ad} + \underbrace{\nabla^d R^a{}_b\,\delta\ln|g| \wedge \delta\Gamma^b_{ad}}_{H},
$$

$$X_2 = \underbrace{-\frac{1}{2}\nabla^a R^d{}_c\,\delta\ln|g| \wedge \delta\Gamma^c_{ad}}_{H} - \underbrace{\frac{1}{2}\nabla^a R_b{}^d\,\delta\ln|g| \wedge \delta\Gamma^b_{ad}}_{H},$$

$$X_3 = \underbrace{4g^{ec}R_d{}^b\,\delta\Gamma^d_{ae} \wedge \delta\Gamma^a_{bc}}_{A} + \underbrace{4R^{eb}\,\delta\Gamma^c_{ae} \wedge \delta\Gamma^a_{bc}}_{=0},$$

$$X_4 = -2g^{ea}R_d{}^b\,\delta\Gamma^b_{ae} \wedge \delta\Gamma^c_{bc} - \underbrace{2R^{ed}\,\delta\Gamma^a_{ea} \wedge \delta\Gamma^e_{dc}}_{=0},$$

$$X_5 = -2g^{db}R^a{}_c\,\delta\Gamma^c_{ad} \wedge \delta\Gamma^e_{be} - \underbrace{2R^{ad}\,\delta\Gamma^b_{ad} \wedge \delta\Gamma^e_{be}}_{B},$$

$$X_6 = \frac{1}{2}g^{ea}(\nabla_b R)\,\delta\Gamma^b_{ae} \wedge \delta\ln|g| + \underbrace{\frac{1}{2}(\nabla^e R)\,\delta\Gamma^a_{ea} \wedge \delta\ln|g|}_{C},$$

$$X_7 = 2g^{ea}(\nabla_c R_{db})\ \delta\Gamma^b_{ae} \wedge \delta g^{bc} + \underbrace{2(\nabla_c R_d{}^e)\ \delta\Gamma^a_{ea} \wedge \delta g^{dc}}_{D},$$

$$X_8 = \underbrace{2(\nabla_c R_e{}^d)\,\delta g^{ae} \wedge \delta\Gamma^c_{ad}}_{F} + 2(\nabla^d R_{eb})\,\delta g^{ae} \wedge \delta\Gamma^b_{ad},$$

$$X_9 = 2g^{ad}g^{eb}\,(\nabla_c \delta R_{db}) \wedge \delta\Gamma^c_{ae} + 2g^{ad}\,(\nabla^e \delta R_{db}) \wedge \delta\Gamma^b_{ae},$$

$$X_{10} = -\underbrace{2g^{eb}R^a{}_d\,\delta\Gamma^d_{bc} \wedge \delta\Gamma^c_{ae}}_{A} - \underbrace{2g^{ec}R^a{}_d\,\delta\Gamma^d_{bc} \wedge \delta\Gamma^b_{ae}}_{A},$$

$$X_{11} = -g^{ea}(\nabla_b R_{dc})\,\delta\Gamma^b_{ae} \wedge \delta g^{de} - \underbrace{(\nabla^e R_{dc})\,\delta\Gamma^a_{ea} \wedge \delta g^{dc}}_{E},$$

$$X_{12} = -(\nabla_d R^e{}_c)\,\delta g^{ad} \wedge \delta\Gamma^c_{ae} - \underbrace{((\nabla_d R_b{}^e)\,\delta g^{ad} \wedge \delta\Gamma^b_{ae}}_{E},$$

$$X_{13} = -\underbrace{(\nabla_d R^e{}_c)\,\delta g^{ad} \wedge \delta\Gamma^c_{ae}}_{I} - \underbrace{(\nabla_d R_b{}^e)\,\delta g^{ad} \wedge \delta\Gamma^b_{ae}}_{I},$$

$$X_{14} = -\underbrace{g^{eb}\,\nabla^a \delta R_{bc} \wedge \delta\Gamma^c_{ae}}_{J} - \underbrace{g^{ec}\,\nabla^a \delta R_{bc} \wedge \delta\Gamma^b_{ae}}_{J}.$$

We now have

$$D = 2(\nabla_c R^e_d)\,\delta g^{dc} \wedge \delta\Gamma^a_{ea} \qquad F = -2(\nabla_c R^d_e)\,\delta g^{ae} \wedge \delta\Gamma^c_{ad},$$

$$K = R^{ab}\,\delta R_{ab} \wedge \delta\ln|g| \qquad L = -2g^{cd}g^{be}\,\delta R_{be} \wedge \delta R_{be} = 0,$$

$$I = -2(\nabla_d R^e_c)\,\delta g^{ad} \wedge \delta\Gamma^c_{ae} \qquad J = -2g^{eb}\,\nabla^a \delta R_{bc} \wedge \delta\Gamma^c_{ae},$$

and $A = B = C = E = G = H = 0$. Therefore,

$$
\begin{aligned}
\nabla_a J^a_\beta = &-\underbrace{(\nabla_a R^{bc})\,\delta\ln|g| \wedge \delta\Gamma^a_{bc}}_{F} - (\nabla_a\nabla_c R^a{}_b)\,\delta\ln|g| \wedge \delta g^{bc} + \frac{1}{2}(\Box R_{bc})\,\delta\ln|g| \wedge \delta g^{bc} \\
&- \underbrace{4R_d{}^b\,\delta g^{cd} \wedge \nabla_a\delta\Gamma^a_{bc}}_{V} - 2g^{cd}g^{be}\,\nabla_a\delta R_{de} \wedge \delta\Gamma^a_{bc} + \underbrace{2R_d{}^b\,\delta g^{ad} \wedge \nabla_a\delta\Gamma^c_{bc}}_{V} \\
&+ R^a{}_d\,\delta g^{bd} \wedge \nabla_a\delta\Gamma^c_{bc} + 2g^{be}\,\nabla^d\delta R_{de} \wedge \delta\Gamma^c_{bc} - \underbrace{\frac{1}{2}(\nabla_a\nabla_b R)\,\delta g^{ab} \wedge \ln|g|}_{Y} \\
&- \underbrace{\frac{1}{2}\,\Box\delta R \wedge \delta\ln|g|}_{I} - \nabla^a\delta R \wedge \delta\Gamma^b_{ab} - \underbrace{2(\nabla_a\nabla_c R_{db})\,\delta g^{ad} \wedge \delta g^{bc}}_{M} - \underbrace{2\,(\nabla^d\nabla_c\delta R_{db}) \wedge \delta g^{bc}}_{N} \\
&+ \underbrace{(\nabla_d R)\,\delta\Gamma^d_{bc} \wedge \delta g^{bc}}_{D} + 2R^a{}_d\,\nabla_a\delta\Gamma^d_{bc} \wedge \delta g^{bc} + \underbrace{(\nabla_a\nabla_d R_{bc})\,\delta g^{ad} \wedge \delta g^{bc}}_{K} \\
&+ \underbrace{\Box\delta R_{bc} \wedge \delta g^{bc}}_{J} + \underbrace{(\nabla_c R^{ad})\,\delta\ln|g| \wedge \delta\Gamma^c_{ad}}_{F} - 2g^{ea}(\nabla_c R_{db})\,\delta\Gamma^b_{ae} \wedge \delta g^{dc} \\
&- 2g^{db}R^a{}_c\,\delta\Gamma^c_{ad} \wedge \delta\Gamma^e_{be} + \underbrace{\frac{1}{2}g^{ea}(\nabla_b R)\,\delta\Gamma^b_{ae} \wedge \delta\ln|g|}_{G} + \underbrace{2g^{ea}(\nabla_c R_{db})\,\delta\Gamma^b_{ae} \wedge \delta g^{dc}}_{S} \\
&+ \underbrace{2(\nabla^d R_{eb})\,\delta g^{ae} \wedge \nabla_a\delta\Gamma^b_{ad}}_{O} + 2g^{ad}g^{be}\,\nabla_c\delta R_{db} \wedge \delta\Gamma^c_{ae} + 2g^{ad}\nabla^e\delta R_{db} \wedge \delta\Gamma^b_{ae} \\
&- \underbrace{g^{ea}(\nabla_b R_{dc})\,\delta\Gamma^b_{ae}\delta g^{dc}}_{L} + 2(\nabla_c R_d{}^e)\,\delta g^{dc} \wedge \delta\Gamma^a_{ea} - \underbrace{2(\nabla_c R_e{}^d\,\delta g^{ae} \wedge \delta\Gamma^c_{ad}}_{R} \\
&- \underbrace{2(\nabla_d R^e{}_c\,\delta g^{ad} \wedge \delta\Gamma^c_{ae}}_{T} - 2g^{eb}\,\nabla^a\delta R_{bc} \wedge \delta\Gamma^c_{ae} + \underbrace{R^{ab}\,\delta R_{ab} \wedge \delta\ln|g|}_{A}.
\end{aligned}
$$

We need to calculate $\bigstar = \frac{1}{2}g^{ab}\,\delta\mathcal{B}_{ab} \wedge \delta\ln g + \delta\mathcal{B}_{ab} \wedge \delta g^{ab} - \frac{1}{2}\delta\mathcal{B} \wedge \delta\ln|g|$ where

$$
\mathcal{B}_{ab} = -\frac{1}{2}g_{ab}R^2_{cd} + 2R^e_bR^a_e + \frac{1}{2}g_{ab}\Box R + \Box R_{ab} - \nabla_c\nabla_a R^c{}_b - \nabla_c\nabla_b R^c{}_a
$$

and $\mathcal{B} = g^{ab}\mathcal{B}_{ab}$ is its trace. We need the following relations

$$
\begin{aligned}
\delta(-\frac{1}{2}g_{ab}g^{ce}g^{df}R_{cd}R_{ef}) &= -\frac{1}{2}R_{cd}R^{cd}\delta g_{ab} - \frac{1}{2}g_{ab}R_c{}^fR_{ef}\delta g^{ce} - \frac{1}{2}g_{ab}R^e{}_dR_{ef}\delta g^{df} - g_{ab}R^{cd}\delta R_{cd} \\
&= -\frac{1}{2}R_{cd}R^{cd}\delta g_{ab} - g_{ab}R_c{}^fR_{ef}\delta g^{ce} - g_{ab}R^{cd}\delta R_{cd},
\end{aligned}
$$

$$
\delta(2g^{ce}R_{cb}R_{ae}) = 2R_{cb}R_{ae}\delta g^{ce} + 2R_a{}^c\delta R_{cb} + 2R^e{}_b\delta R_{ae},
$$

$$
\begin{aligned}
\delta(\frac{1}{2}g_{ab}\Box R) &= \frac{1}{2}\Box R\delta g_{ab} + \frac{1}{2}g_{ab}\delta\Box R \\
&= \frac{1}{2}\Box R\delta g_{ab} + \frac{1}{2}g_{ab}(\nabla_e\nabla_c R)\delta g^{ce} + \frac{1}{2}g_{ab}\Box\delta R - \frac{1}{2}g_{ab}g^{ce}(\nabla_d R)\delta\Gamma^d_{ec},
\end{aligned}
$$

$$
\begin{aligned}
\delta(\Box R_{ab}) = &\Box\delta R_{ab} + (\nabla_c\nabla_d R_{ab})\delta g^{cd} - (\nabla^c R_{db})\delta\Gamma^d_{ac} - R_{cb}\nabla^d\delta\Gamma^c_{ad} - (\nabla^c R_{ad})\delta\Gamma^d_{bc} - R_{ac}\nabla^d\delta\Gamma^c_{bd} \\
&- g^{cd}(\nabla_e R_{ab})\delta\Gamma^e_{dc} - (\nabla^c R_{db})\delta\Gamma^d_{ac} - (\nabla^c R_{ad})\delta\Gamma^d_{bc},
\end{aligned}
$$

$$
\begin{aligned}
\delta(-\nabla_c\nabla_a R^c{}_b) = \delta(-g^{ce}\nabla_c\nabla_a R_{eb}) &= -(\nabla_c\nabla_a R_{eb})\delta g^{ce} - g^{ce}\nabla_c(\delta\nabla_a R_{eb}) + (\nabla_d R^c{}_b)\delta\Gamma^d_{ca} \\
&+ g^{ce}(\nabla_a R_{df})\delta\Gamma^d_{ce} + (\nabla_a R^c{}_d)\delta\Gamma^d_{cb} \\
&= -(\nabla_c\nabla_a R_{eb})\delta g^{ce} - \nabla^e\nabla_a\delta R_{eb}) + (\nabla^e R_{db})\delta\Gamma^d_{ea} + R_{db}\nabla^e\delta\Gamma^d_{ea} + \frac{1}{2}(\nabla_d R)\delta\Gamma^d_{ab} \\
&+ R_{ed}\nabla^e\delta\Gamma^d_{ab} + (\nabla_d R^c{}_b)\delta\Gamma^d_{ac} + g^{ce}(\nabla_a R_{db})\delta\Gamma^d_{ce} + (\nabla_a R^c{}_d)\delta\Gamma^d_{cb}.
\end{aligned}
$$

Making use of the fact that

$$
\begin{gathered}
\frac{1}{2}g^{ab}\delta(-\nabla_c\nabla_a R^c{}_b) + \frac{1}{2}g^{ab}\delta(-\nabla_c\nabla_b R^c{}_a) = g^{ab}\delta(-\nabla_c\nabla_a R^c{}_b), \\
\delta(-\nabla_c\nabla_a R^c{}_b)\wedge\delta g^{ab} + \delta(-\nabla_c\nabla_b R^c{}_a)\wedge\delta g^{ab} = 2\,\delta(-\nabla_c\nabla_a R^c{}_b)\wedge\delta g^{ab}.
\end{gathered}
$$

We understand that there is no need to calculate the variation of the last term in the field equations. Hence, we obtain

$$
\begin{aligned}
\frac{1}{2}g^{ab}\delta B_{ab}\wedge\delta\ln g = &-\underbrace{\frac{1}{4}R_{cd}R^{cd}\,\delta\ln|g|\wedge\delta\ln|g|}_{=0} - \underbrace{\frac{1}{2}\Box R_{cf}R_e{}^f\,\delta g^{ce}\wedge\delta\ln|g|}_{X_1} - \underbrace{\frac{1}{2}\Box R^{cd}\,\delta R_{cd}\wedge\delta\ln|g|}_{X_2} \\
&+ \underbrace{R_a{}^e R_c{}^a\,\delta g^{ce}\wedge\delta\ln|g|}_{X_1} + \underbrace{R^{bc}\,\delta R_{bc}\wedge\delta\ln|g|}_{X_2} + \underbrace{R^{ae}\,\delta R_{ae}\wedge\delta\ln|g|}_{X_2} \\
&+ \underbrace{\frac{1}{4}\Box R\,\delta\ln|g|\wedge\delta\ln|g|}_{=0} + \underbrace{\frac{1}{4}(\nabla_e\nabla_c R)\,\delta g^{ce}\wedge\delta\ln|g|}_{X_3} + \frac{1}{4}\,\Box\delta R\wedge\delta\ln|g| \\
&- \underbrace{\frac{1}{4}g^{ac}(\nabla_d R)\;\delta\Gamma^d_{ac}\wedge\delta\ln|g|}_{X_4} + \frac{1}{2}g^{ab}\,\Box\delta R_{ab}\wedge\delta\ln|g| + \underbrace{\frac{1}{2}(\nabla_c\nabla_d R)\,\delta g^{cd}\wedge\delta\ln|g|}_{X_3} \\
&- \underbrace{\frac{1}{2}(\nabla^c R_d{}^a)\,\delta\Gamma^d_{ac}\wedge\delta\ln|g|}_{X_5} - \underbrace{\frac{1}{2}R_c{}^a\,\nabla^d\delta\Gamma^c_{ad}\wedge\delta\ln|g|}_{X_6} - \underbrace{\frac{1}{2}(\nabla^c R_d{}^b)\,\delta\Gamma^d_{bc}\wedge\delta\ln|g|}_{X_5} \\
&- \underbrace{\frac{1}{2}R^b{}_a\,\nabla^d\delta\Gamma^c_{bd}\wedge\delta\ln|g|}_{X_6} - \underbrace{\frac{1}{2}g^{cd}(\nabla_e R)\,\delta\Gamma^e_{dc}\wedge\delta\ln|g|}_{X_4} - \underbrace{\frac{1}{2}(\nabla^c R_d{}^a)\,\delta\Gamma^d_{ac}\wedge\delta\ln|g|}_{X_5} \\
&- \underbrace{\frac{1}{2}(\nabla^c R^b{}_d)\,\delta\Gamma^d_{bc}\wedge\delta\ln|g|}_{X_5} - \underbrace{\frac{1}{2}(\nabla_c\nabla_e R)\,\delta g^{ce}\wedge\delta\ln|g|}_{X_3} - (\nabla^a\nabla^b\delta R_{ab})\wedge\delta\ln|g| \\
&- \underbrace{(\nabla^c R_d{}^b)\,\delta\Gamma^d_{cb}\wedge\delta\ln|g|}_{X_5} - \underbrace{R_d{}^b\,\nabla^e\delta\Gamma^d_{cb}\wedge\delta\ln|g|}_{X_6} - \underbrace{\frac{1}{2}g^{ab}(\nabla_d R)\,\delta\Gamma^d_{ab}\wedge\delta\ln|g|}_{X_4} \\
&- g^{ab}R_{ed}\,\nabla^e\delta\Gamma^d_{ab}\wedge\delta\ln|g| - (\nabla_d R^{cb})\,\delta\Gamma^d_{cb}\wedge\delta\ln|g| - \underbrace{\frac{1}{2}g^{ce}(\nabla_d R)\,\delta\Gamma^d_{ce}\wedge\delta\ln|g|}_{X_4} \\
&- \underbrace{(\nabla^a R^c{}_d)\,\delta\Gamma^d_{ca}\wedge\delta\ln|g|}_{X_5}
\end{aligned}
$$

where

$$X_! = (1-\frac{D}{2})R_{ae}R_c{}^a\ \delta g^{ce}\wedge\delta\ln|g| \qquad X_4 = -(\frac{3}{2}+\frac{D}{4})g^{ec}(\nabla_d R)\ \delta\Gamma^d_{ec}\wedge\delta\ln|g|,$$
$$X_2 = (2-\frac{D}{2})R^{bc}\ \delta R_{bc}\wedge\delta\ln|g| \qquad X_5 = -4(\nabla^a R^c{}_d)\ \delta\Gamma^d_{ca}\wedge\delta\ln|g|,$$
$$X_3 = \frac{D}{4}(\nabla_e\nabla_c R)\ \delta g^{ce}\wedge\delta\ln|g| \qquad X_6 = -2R_c{}^a\ \nabla^d\delta\Gamma^c_{ad}\wedge\delta\ln|g|.$$

Therefore, we have

$$\begin{aligned}\frac{1}{2}g^{ab}\delta\mathcal{B}_{ab}\wedge\delta\ln|g| = {}& \underbrace{\frac{D}{4}\Box\delta R\wedge\delta\ln|g|}_{I} + \frac{1}{2}g^{ab}\,\Box\delta R_{ab}\wedge\delta\ln|g| - \nabla^a\nabla^b\delta R_{ab}\wedge\delta\ln|g|\\
& - g^{ab}R_{ed}\,\nabla^e\delta\Gamma^d_{ab}\wedge\delta\ln|g| - (\nabla_d R^{cb})\,\delta\Gamma^d_{cb}\wedge\delta\ln|g| + (1-\frac{D}{2})R_{ae}R_c{}^a\ \delta g^{ce}\wedge\delta\ln|g|\\
& + (2-\frac{D}{2})R^{bc}\ \delta R_{bc}\wedge\delta\ln|g| + \frac{D}{4}(\nabla_e\nabla_c R)\ \delta g^{ce}\wedge\delta\ln|g|\\
& - (\frac{3}{2}+\frac{D}{4})g^{ec}(\nabla_d R)\,\delta\Gamma^d_{ec}\wedge\delta\ln|g| - 4(\nabla^a R^c{}_d)\,\delta\Gamma^d_{ca}\wedge\delta\ln|g|\\
& - 2R_c{}^a\ \nabla^d\delta\Gamma^c_{ad}\wedge\delta\ln|g|.\end{aligned}$$

The next term to be considered is

$$\begin{aligned}\delta\mathcal{B}_{ab}\wedge\delta g^{ab} = {}& R_c{}^f R_{ef}\ \delta g^{ce}\wedge\delta\ln|g| + R^{cd}\ \delta R_{cd}\wedge\delta\ln|g| + \underbrace{R_{cb}R_{ae}\ \delta g^{ce}\wedge\delta g^{ab}}_{=0}\\
& + \underbrace{2R_a{}^c\,\delta R_{cb}\wedge\delta g^{ab}}_{X_1} + \underbrace{2R^e{}_b\ \delta R_{ae}\wedge\delta g^{ab}}_{X_1} - \frac{1}{2}(\nabla_e\nabla_c R)\ \delta g^{ce}\wedge\delta\ln|g|\\
& - \frac{1}{2}\Box\delta R\wedge\delta\ln|g| + \frac{1}{2}g^{ce}(\nabla_d R)\,\delta\Gamma^d_{ce}\wedge\delta\ln|g| + \Box\delta R_{ab}\wedge\delta g^{ab}\\
& + (\nabla_c\nabla_d R_{ab})\,\delta g^{cd}\wedge\delta g^{ab} - \underbrace{(\nabla^c R_{db})\ \delta\Gamma^d_{ac}\wedge\delta g^{ab}}_{X_2} - \underbrace{R_{cb}\,\nabla^d\delta\Gamma^c_{ad}\wedge\delta g^{ab}}_{X_3}\\
& - \underbrace{(\nabla^c R_{ad})\,\delta\Gamma^d_{bc}\wedge\delta g^{ab}}_{X_2} - \underbrace{R_{ac}\,\nabla^d\delta\Gamma^c_{bd}\wedge\delta g^{ab}}_{X_3} - g^{cd}(\nabla_e R_{ab})\,\delta\Gamma^e_{dc}\wedge\delta g^{ab}\\
& - \underbrace{(\nabla^c R_{db})\,\delta\Gamma^d_{ac}\wedge\delta g^{ab}}_{X_2} - \underbrace{(\nabla^c R_{ad})\ \delta\Gamma^d_{bc}\wedge\delta g^{ab}}_{X_2} - 2(\nabla_c\nabla_b R_{ea})\,\delta g^{ce}\wedge\delta g^{ab}\\
& - 2\nabla^e\nabla_b\delta R_{ea}\wedge\delta g^{ab} + \underbrace{2(\nabla^e R_{da})\,\delta\Gamma^d_{eb}\wedge\delta g^{ab}}_{X_2} + \underbrace{2R_{da}\,\nabla^e\delta\Gamma^d_{eb}\wedge\delta g^{ab}}_{X_3}\\
& + (\nabla_d R)\,\delta\Gamma^d_{ab}\wedge\delta g^{ab} + 2R_{ed}\ \nabla^e\delta\Gamma^d_{ab}\wedge\delta g^{ab} + 2(\nabla_d R^c{}_a)\,\delta\Gamma^d_{bc}\wedge\delta g^{ab}\\
& + 2g^{ce}(\nabla_b R_{da})\,\delta\Gamma^d_{ce}\wedge\delta g^{ab} + 2(\nabla_b R^c{}_d)\,\delta\Gamma^d_{ca}\wedge\delta g^{ab},\end{aligned}$$

with

$$X_1 = 4R_a{}^c\ \delta R_{cb}\wedge\delta g^{ab} \qquad X_2 = -2(\nabla^c R_{db})\ \delta\Gamma^d_{ac}\wedge\delta g^{ab} \qquad X_3 = 0.$$

It becomes

$$\delta B_{ab}\wedge\delta g^{ab} = \underbrace{R_c{}^f R_{ef}\ \delta g^{ce}\wedge\delta\ln|g|}_{B} + \underbrace{R^{cd}\ \delta R_{cd}\wedge\delta\ln|g|}_{H} - \underbrace{\frac{1}{2}(\nabla_e\nabla_c R)\,\delta g^{ce}\wedge\delta\ln|g|}_{Y}$$
$$-\underbrace{\frac{1}{2}\Box\delta R\wedge\delta\ln|g|}_{I} + \underbrace{\frac{1}{2}g^{ce}(\nabla_d R)\ \delta\Gamma^d_{ce}\wedge\delta\ln|g|}_{G} + \underbrace{\Box\delta R_{ab}\wedge\delta g^{ab}}_{J}$$
$$+\underbrace{(\nabla_c\nabla_d R_{ab})\ \delta g^{cd}\wedge\delta g^{ab}}_{K} - \underbrace{g^{cd}(\nabla_e R_{ab})\ \delta\Gamma^e_{dc}\wedge\delta g^{ab}}_{L} - \underbrace{2(\nabla_c\nabla_b R_{ea})\ \delta g^{ce}\wedge\delta g^{ab}}_{M}$$
$$-\underbrace{2\nabla^e\nabla_b\delta R_{ea}\wedge\delta g^{ab}}_{N} + \underbrace{(\nabla_d R)\ \delta\Gamma^d_{ab}\wedge\delta g^{ab}}_{D} + \underbrace{2R_{ed}\ \nabla^e\delta\Gamma^d_{ab}\wedge\delta g^{ab}}_{P}$$
$$+\underbrace{2(\nabla_d R^c{}_a)\ \delta\Gamma^d_{bc}\wedge\delta g^{ab}}_{R} + \underbrace{2g^{ce}(\nabla_b R_{da})\ \delta\Gamma^d_{ce}\wedge\delta g^{ab}}_{S} + \underbrace{2(\nabla_b R^c{}_d)\ \delta\Gamma^d_{ca}\wedge\delta g^{ab}}_{T}$$
$$+\underbrace{4R_a{}^c\ \delta R_{cb}\wedge\delta g^{ab}}_{V} - \underbrace{2(\nabla^c R_{db})\ \delta\Gamma^d_{ac}\wedge\delta g^{ab}}_{U}.$$

We also need the variation of the trace of the field equations. Starting from

$$\begin{aligned}\mathcal{B} = g^{ab}\mathcal{B}_{ab} &= -\frac{D}{2}R^{cd}R_{cd} + 2R_{ab}R^{ab} + \frac{D}{2}\Box R + \Box R - \underbrace{2\nabla_c\nabla_b R^c{}_a}_{=\Box R}\\ &= (2-\frac{D}{2})R^{cd}R_{cd} + 2R_{ab}R^{ab} + \frac{D}{2}\Box R,\end{aligned}$$

and using

$$\delta(R^{bc}R_{bc}) = 2R^{bc}\delta R_{bc} + 2R^d{}_c R_{de}\delta g^{ce},$$

$$\delta(g^{bc}\nabla_b\nabla_c R) = (\nabla_b\nabla_c R)\delta g^{bc} + \Box\delta R - g^{bc}(\nabla_d R)\delta\Gamma^d_{bc},$$

gives

$$\delta\mathcal{B} = (4-D)[R^{bc}\delta R_{bc} + 2R^d{}_c R_{de}\delta g^{ce}] + \frac{D}{2}[\nabla_b\nabla_c R)\delta g^{bc} + \Box\delta R - g^{bc}(\nabla_d R)\delta\Gamma^d_{bc}],$$

which finally yields the relavant term as

$$\frac{1}{2}\delta\mathcal{B}\wedge\delta\ln g = (2-\frac{D}{2})[\underbrace{R^{bc}\delta R_{bc}}_{A} + \underbrace{2R^d{}_c R_{de}\delta g^{ce}}_{B}]\wedge\delta\ln|g| + \frac{D}{4}[\underbrace{(\nabla_b\nabla_c R)\delta g^{bc}}_{C} + \underbrace{\Box\delta R}_{D} - \underbrace{g^{bc}(\nabla_d R)\delta\Gamma^d_{bc}}_{E}]\delta\ln|g|.$$

We will show that $\chi = \nabla_a J^a_\beta - \bigstar = 0$. All the terms are denoted with capital letters vanish except.

$$\begin{aligned}G &= -\frac{1}{2}g^{ea}(\nabla_b R)\ \delta\Gamma^b_{ea}\wedge\delta\ln|g|,\\ V &= 4R_d{}^b\delta g^{ad}\wedge\nabla_a\delta\Gamma^c_{bc} - 4R_d{}^b\delta g^{cd}\wedge\nabla_a\delta\Gamma^a_{bc} - 4R_a{}^c\delta R_{cd}\wedge\delta g^{ab} = 0.\end{aligned}$$

Then,

$$\chi = -(\nabla_a \nabla_c R^a{}_b)\,\delta \ln|g| \wedge \delta g^{bc} + \frac{1}{2}(\Box R_{bc})\,\delta \ln|g| \wedge \delta g^{bc} - \underbrace{2g^{cd} g^{be}\ \nabla_a \delta R_{de} \wedge \delta\Gamma^a_{bc}}_{X_1}$$

$$+ 2g^{be}\ \nabla^d \delta R_{de} \wedge \delta\Gamma^c_{bc} - \nabla^a \delta R \wedge \delta\Gamma^b_{ab} - 2g^{ea} R_b{}^d\ \delta\Gamma^b_{ea} \wedge \delta\Gamma^c_{dc}$$

$$- 2g^{db} R^a{}_c\ \delta\Gamma^c_{ad} \wedge \delta\Gamma^d_{be} + \underbrace{2g^{ad} g^{be}\ \nabla_c \delta R_{db} \wedge \delta\Gamma^c_{ae}}_{X_1} + \underbrace{2g^{ad}\ \nabla^e \delta R_{db} \wedge \delta\Gamma^b_{ae}}_{X_2}$$

$$+ 2(\nabla_c R_d{}^e)\ \delta g^{dc} \wedge \delta\Gamma^a_{ea} - \underbrace{2g^{eb}\ \nabla^a \delta R_{bc} \wedge \delta\Gamma^c_{ae}}_{X_2} - \frac{1}{2} g^{ea} (\nabla_b R)\ \delta\Gamma^b_{ae} \wedge \delta \ln|g|$$

$$- \frac{1}{2} g^{ab}\, \Box \delta R_{ab} \wedge \delta \ln|g|,$$

where the terms X_1 and X_2 also vanish. In order to show that the remaning terms are zero, we first consider

$$\delta(2g^{dc} g^{ef} \nabla_c R_{df}) = \delta(\nabla^e R) \wedge \delta\Gamma^a_{ea} = (\nabla_d R)\delta g^{ed} + \nabla^e \delta R$$

$$= 2(\nabla_c R_d{}^e)\delta g^{dc} + (\nabla_f R)\delta g^{ef} + 2g^{de} g^{ef} \delta(\nabla_c R_{df}),$$

which gives

$$0 = -\nabla^e \delta R \wedge \delta\Gamma^a_{ea} + 2(\nabla_c R_d{}^e)\ \delta g^{dc} \wedge \delta\Gamma^a_{ea} + 2g^{ef}\ \nabla^d \delta R_{df} \wedge \delta\Gamma^a_{ea} - 2g^{dc} R_b{}^e\ \delta\Gamma^b_{dc} \wedge \delta\Gamma^a_{ea}$$

$$- 2g^{ef} R^c{}_b\ \delta\Gamma^b_{cf} \wedge \delta\Gamma^a_{ea}.$$

Note that are the terms with $\delta\Gamma$ in χ. For the terms with $\delta \ln|g|$, we take

$$\delta(g^{ac} g^{bd} \nabla_c \nabla_d R_{ab}) = \delta(\nabla^a \nabla^b R_{ab}) = \frac{1}{2}\delta(g^{ac} \nabla_c R) = \frac{1}{2}(\nabla_a \nabla_c R)\delta g^{ac} + \frac{1}{2} g^{ac} \delta(\nabla_a \nabla_c R)$$

$$= \frac{1}{2}(\nabla_a \nabla_c R)\delta g^{ac} + \frac{1}{2}\Box \delta R - \frac{1}{2} g^{ac} (\nabla_d R)\delta\Gamma^d_{ca}$$

$$= \frac{1}{2}(\nabla_a \nabla_c R)\delta g^{ac} + (\nabla^a \nabla_d R_{ab})\delta g^{bd} + g^{ac} g^{be} \delta(\nabla_c \nabla_e R_{ab})$$

where

$$g^{ac} g^{be} \delta(\nabla_c \nabla_e R_{ab}) = g^{ac} g^{be} \Big[\nabla_c \nabla_e \delta R_{ab} - (\nabla_c R_{db})\delta\Gamma^d_{ae} - R_{db} \nabla_c \Gamma^d_{ae} - R_{ad} \nabla_c \Gamma^d_{be} - (\nabla_f R_{ab})\delta\Gamma^f_{ce}$$

$$- (\nabla_c R_{fb})\delta\Gamma^f_{ca} - (\nabla_e R_{af})\delta\Gamma^f_{cb}\Big]$$

$$= \nabla^a \nabla^b \delta R_{ab} - (\nabla^a R_d{}^e)\delta\Gamma^d_{ea} - g^{ac} R_d{}^e \nabla_c \Gamma^d_{ae} - \frac{1}{2} g^{be} (\nabla_d R)\delta\Gamma^d_{be} - g^{be} R^c{}_d \nabla_c \Gamma^d_{be}$$

$$- (\nabla_f R^{ce})\delta\Gamma^f_{ce} - \frac{1}{2}(\nabla_f R) g^{ca} \delta\Gamma^f_{ca} - (\nabla^b R^c{}_f)\delta\Gamma^f_{cb},$$

and

$$\delta(g^{ac} g^{bd} \nabla_c \nabla_d R_{ab}) = \frac{1}{2}(\nabla_a \nabla_c R)\delta g^{ac} + \frac{1}{2} g^{ab} \Box \delta R_{ab} + \frac{1}{2}\Box R_{ab} \delta g^{ab} + \nabla_c R_{ab} \nabla^c \delta g^{ab}$$

$$+ \frac{1}{2} R_{ab} \Box \delta g^{ab} - \frac{1}{2} g^{ac} (\nabla_d R)\delta\Gamma^d_{ac}$$

$$= \frac{1}{2}(\nabla_a \nabla_c R)\delta g^{ac} + (\nabla^a \nabla_d R_{ab})\delta g^{bd} + \nabla^a \nabla^b \delta R_{ab} - 2(\nabla^a R_d{}^e)\delta\Gamma^d_{ae} - R_d{}^e \nabla^a \delta\Gamma^d_{ae}$$

$$- \frac{1}{2} g^{be} (\nabla_d R)\delta\Gamma^d_{be} - g^{be} R^c{}_d \nabla_c \delta\Gamma^d_{be} - (\nabla_f R^{ce})\delta\Gamma^f_{ce} - \frac{1}{2} g^{ca} (\nabla_f R)\delta\Gamma^f_{ca}.$$

Using

$$(\nabla^a R_d{}^e)\delta\Gamma^d_{ae} = -\frac{1}{2}(\nabla^c R_{ab})\nabla_c \delta g^{ab} \quad , \qquad R^e{}_d \nabla^a \delta\Gamma^d_{ae} = -\frac{1}{2} R_{ab} \Box \delta g^{ab}$$

shows that the terms with $\delta \ln |g|$ are also zero.

APPENDIX E

EXTERIOR DERIVATIVE OF THE SYMPLECTIC TWO-FORM

We show that $\delta J^a = -\frac{1}{2} J^a \wedge \delta \ln|g|$ and therefore $\delta\omega = 0$. Again, a general proof for theories arising from a local gravity action was already given.

E.1 R^2 Term

The symplectic current is

$$\begin{aligned} J^a_\alpha \;=\; & -Rg^{bc}\,\delta\ln|g| \wedge \delta\Gamma^a_{bc} - 2g^{bc}\,\delta R \wedge \delta\Gamma^a_{bc} - 2R\,\delta g^{bc} \wedge \delta\Gamma^a_{bc} \\ & + g^{ab}R\,\delta\ln|g| \wedge \delta\Gamma^c_{bc} + 2g^{ab}\,\delta R \wedge \delta\Gamma^c_{bc} + 2R\,\delta g^{ab} \wedge \delta\Gamma^c_{bc} \\ & - (\nabla_b R)\,\delta\ln|g| \wedge \delta g^{ab} - 2\nabla_b \delta R \wedge \delta g^{ab} - 2(\nabla_b R)\,\delta g^{ab} \wedge \delta\ln|g| \\ & -2\nabla^a \delta R \wedge \delta\ln|g|, \end{aligned}$$

whose exterior derivative is given by

$$\begin{aligned} \delta J^a_\alpha \;=\; & -R\,\delta g^{bc} \wedge \delta\ln|g| \wedge \delta\Gamma^a_{bc} - g^{bc}\,\delta R\delta\ln|g| \wedge \delta\Gamma^a_{bc} - \underbrace{2\,\delta g^{bc} \wedge \delta R \wedge \delta\Gamma^a_{bc}}_{X^a_1} - \underbrace{2\,\delta R \wedge \delta g^{bc} \wedge \delta\Gamma^a_{bc}}_{X^a_1} \\ & + R\,\delta g^{ab} \wedge \delta\ln|g| \wedge \delta\Gamma^c_{bc} + g^{ab}\delta R \wedge \delta\ln|g| \wedge \delta\Gamma^c_{bc} + \underbrace{2\delta g^{ab} \wedge \delta R \wedge \delta\Gamma^c_{bc}}_{X^a_2} + \underbrace{2\delta R \wedge \delta g^{ab} \wedge \delta\Gamma^c_{bc}}_{X^a_2} \\ & - \underbrace{(\nabla_b\delta R) \wedge \delta\ln|g| \wedge \delta g^{ab}}_{X^a_3} - \underbrace{2\,\nabla_b\delta R \wedge \delta g^{ab} \wedge \delta\ln|g|}_{X^a_3} - \underbrace{2\,\delta g^{ab} \wedge \nabla_b\delta R \wedge \delta\ln|g|}_{X^a_3}\,. \end{aligned}$$

Terms X^a_1 and X^a_2 give no contribution and $X_3 = -(\nabla_b\delta R) \wedge \delta\ln|g| \wedge \delta g^{ab}$. Therefore, we have

$$\begin{aligned} \delta J^a_\alpha \;=\; & -\underbrace{R\,\delta g^{bc} \wedge \delta\ln|g| \wedge \delta\Gamma^a_{bc}}_{A^a} - \underbrace{g^{bc}\,\delta R\delta\ln|g| \wedge \delta\Gamma^a_{bc}}_{B^a} + \underbrace{R\,\delta g^{ab} \wedge \delta\ln|g| \wedge \delta\Gamma^c_{bc}}_{C^a} + \underbrace{g^{ab}\delta R \wedge \delta\ln|g| \wedge \delta\Gamma^c_{bc}}_{D^a} \\ & - \underbrace{(\nabla_b\delta R) \wedge \delta\ln|g| \wedge \delta g^{ab}}_{E^a}, \end{aligned}$$

which is equal to

$$
\begin{aligned}
-\frac{1}{2}\,J^a_\alpha\wedge\delta\ln|g| \;&=\; \underbrace{g^{bc}\,\delta R\wedge\delta\Gamma^a_{bc}\wedge\delta\ln|g|}_{B^a}+\underbrace{R\,\delta g^{bc}\wedge\delta\Gamma^a_{bc}\wedge\delta\ln|g|}_{A^a}-\underbrace{g^{ab}\,\delta R\wedge\delta\Gamma^c_{bc}\wedge\delta\ln|g|}_{D^a}\\
&\quad-\underbrace{R\,\delta g^{ab}\wedge\delta\Gamma^c_{bc}\wedge\delta\ln|g|}_{C^a}+\underbrace{\nabla_b\delta R\wedge\delta g^{ab}\wedge\delta\ln|g|}_{E^a}.
\end{aligned}
$$

E.2 R^2_{ab} Term

The symplectic current is

$$
\begin{aligned}
J^a_\beta \;&=\; -R^{bc}\,\delta\ln|g|\wedge\delta\Gamma^a_{bc}+R^{ab}\,\delta\ln|g|\wedge\delta\Gamma^c_{bc}-(\nabla_b R^a_c)\,\delta\ln|g|\wedge\delta g^{bc}+\frac{1}{2}(\nabla^a R_{bc})\,\delta\ln|g|\wedge\delta g^{bc}\\
&\quad-4R^c_d\,\delta g^{bd}\wedge\delta\Gamma^a_{bc}-2g^{bd}g^{ce}\,\delta R_{de}\wedge\delta\Gamma^a_{bc}+2R_d^{\;b}\,\delta g^{ad}\wedge\delta\Gamma^c_{bc}+2R^a_e\,\delta g^{be}\wedge\delta\Gamma^c_{bc}\\
&\quad+2g^{ad}g^{be}\,\delta R_{de}\wedge\delta\Gamma^c_{bc}+\frac{1}{2}(\nabla_b R)\,\delta\ln|g|\wedge\delta g^{ab}+\underbrace{\frac{1}{2}\,\delta\ln|g|\wedge\nabla^a\delta R}_{=\frac{1}{2}g^{ab}\,\delta\ln|g|\wedge\nabla_b\delta R}-2(\nabla_c R_{db})\,\delta g^{ad}\wedge\delta g^{bc}\\
&\quad-2g^{ad}\,\nabla_c\delta R_{db}\wedge\delta g^{bc}+2R^a_e\,\delta\Gamma^e_{bc}\wedge\delta g^{bc}+\nabla_d R_{bc}\,\delta g^{ad}\wedge\delta g^{bc}+\nabla^a\delta R_{bc}\wedge\delta g^{bc}.
\end{aligned}
$$

Its exterior derivative is

$$
\begin{aligned}
\delta J^a_\beta \;&=\; -\underbrace{\delta R^{bc}\wedge\delta\ln|g|\wedge\delta\Gamma^a_{bc}}_{A^a}+\underbrace{\delta R^{ab}\wedge\delta\ln|g|\wedge\delta\Gamma^c_{bc}}_{B^a}-\underbrace{\delta(\nabla_b R^a_c)\wedge\delta\ln|g|\wedge\delta g^{bc}}_{C^a}\\
&\quad+\underbrace{\frac{1}{2}\,\delta(\nabla^a R_{bc})\wedge\delta\ln|g|\wedge\delta g^{bc}}_{D^a}-\underbrace{4\,\delta R^c_d\wedge\delta g^{bd}\wedge\delta\Gamma^a_{bc}}_{E^a}-2g^{ce}\,\delta g^{bd}\wedge\delta R_{de}\wedge\delta\Gamma^a_{bc}\\
&\quad-2g^{bd}\,\delta g^{ce}\wedge\delta R_{de}\wedge\delta\Gamma^a_{bc}+\underbrace{2\delta R^b_d\wedge\delta g^{ad}\wedge\delta\Gamma^c_{bc}}_{F^a}+\underbrace{2\,\delta R^a_e\wedge\delta g^{be}\wedge\delta\Gamma^c_{bc}}_{G^a}\\
&\quad+2g^{be}\,\delta g^{ad}\wedge\delta R_{de}\wedge\delta\Gamma^c_{bc}+2g^{ad}\,\delta g^{be}\,\delta R_{de}\wedge\delta\Gamma^c_{bc}+\frac{1}{2}\,(\nabla_b\delta R)\wedge\delta\ln|g|\wedge\delta g^{ab}\\
&\quad+\frac{1}{2}\,\delta g^{ab}\wedge\delta\ln|g|\wedge\nabla_b\delta R-\underbrace{2\,\delta(\nabla_c R_{db})\wedge\delta g^{ad}\wedge\delta g^{bc}}_{H^A}-2\,\delta g^{ad}\wedge\nabla_c\delta R_{db}\wedge\delta g^{bc}\\
&\quad-\underbrace{2g^{ad}\,\delta(\nabla_c\delta R_{db})\wedge\delta g^{bc}}_{I^a}+\underbrace{2\,\delta R^a_e\wedge\delta\Gamma^e_{bc}\wedge\delta g^{bc}}_{K^a}+\underbrace{\delta(\nabla_d R_{bc})\wedge\delta g^{ad}\wedge\delta g^{bc}}_{L^a}\\
&\quad+\underbrace{\delta(\nabla^a\delta R_{bc})\wedge\delta g^{bc}}_{M^a},
\end{aligned}
$$

We need to expand some terms in this expression.

$$
\begin{aligned}
A^a \;&=\; -(g^{bd}g^{ce}R_{de})\wedge\delta\ln|g|\wedge\delta\Gamma^a_{bc}\\
&=\; -R^c_d\,\delta g^{bd}\wedge\delta\ln|g|\wedge\delta\Gamma^a_{bc}-R^b_e\,\delta g^{ce}\wedge\delta\ln|g|\wedge\delta\Gamma^a_{bc}-g^{bd}g^{ce}\,\delta R_{de}\wedge\delta\ln|g|\wedge\delta\Gamma^a_{bc}\\
&=\; -2R^c_d\,\delta g^{bd}\wedge\delta\ln|g|\wedge\delta\Gamma^a_{bc}-g^{bd}g^{ce}\,\delta R_{de}\wedge\delta\ln|g|\wedge\delta\Gamma^a_{bc},
\end{aligned}
$$

$$
\begin{aligned}
B^a &= \delta(g^{ac}g^{bd}R_{cd}) \wedge \delta \ln|g| \wedge \delta\Gamma^e_{\ be} \\
&= R_c^{\ b}\,\delta g^{ac} \wedge \delta \ln|g| \wedge \delta\Gamma^e_{\ be} + R^a_{\ d}\,\delta g^{bd} \wedge \delta \ln|g| \wedge \delta\Gamma^e_{\ be} + g^{ac}g^{bd}\,\delta R_{cd} \wedge \delta \ln|g| \wedge \delta\Gamma^e_{\ be}.
\end{aligned}
$$

Using

$$
\begin{aligned}
\delta(\nabla_b R^a_{\ c}) &= \delta(g^{ae}\nabla_b R_{ec}) = \nabla_b R_{ec}\,\delta g^{ae} + g^{ae}\,\delta(\nabla_b R_{ec}) \\
&= \nabla_b R_{ec}\,\delta g^{ae} + g^{ae}\,\nabla_b \delta R_{ec} - g^{ae}R_{cf}\,\delta\Gamma^f_{\ eb} - R^a_{\ f}\,\delta\Gamma^f_{\ cb},
\end{aligned}
$$

C^a can be written as

$$
\begin{aligned}
C^a &= -\nabla_b R_{ec}\,\delta g^{ae} \wedge \delta \ln|g| \wedge \delta g^{bc} - g^{ae}\,\nabla_b \delta R_{ec} \wedge \delta \ln|g| \wedge \delta g^{bc} + g^{ae}R_{cd}\,\delta\Gamma^d_{\ eb} \wedge \delta \ln|g| \wedge \delta g^{bc} \\
&\quad + R^a_{\ d}\,\delta\Gamma^d_{\ cb} \wedge \delta \ln|g| \wedge \delta g^{bc}.
\end{aligned}
$$

From

$$
\begin{aligned}
\delta(\nabla^a R_{bc}) &= \delta(g^{ad}\nabla_d R_{bc}) = \nabla_d R_{bc}\,\delta g^{ad} + g^{ad}\,\delta(\nabla_d R_{bc}) \\
&= \nabla_d R_{bc}\,\delta g^{ad} + g^{ad}(\nabla_d \delta R_{bc} - R_{ec}\,\delta\Gamma^e_{\ bd} - R_{eb}\,\delta\Gamma^e_{\ cd}) \\
&= \nabla_d R_{bc}\,\delta g^{ad} + \nabla^a \delta R_{bc} - g^{ad}R_{ec}\,\delta\Gamma^e_{\ bd} - g^{ad}R_{eb}\,\delta\Gamma^e_{\ cd},
\end{aligned}
$$

we have

$$
\begin{aligned}
D^a &= \frac{1}{2}\nabla_d R_{bc}\,\delta g^{ad} \wedge \delta \ln|g| \wedge \delta g^{bc} + \frac{1}{2}\nabla^a \delta R_{bc} \wedge \delta \ln|g| \wedge \delta g^{bc} - \frac{1}{2}g^{ad}R_{ec}\,\delta\Gamma^e_{\ bd} \wedge \delta \ln|g| \wedge \delta g^{bc} \\
&\quad - \frac{1}{2}g^{ad}R_{eb}\,\delta\Gamma^e_{\ cd} \wedge \delta \ln|g| \wedge \delta g^{bc} \\
&= \frac{1}{2}\nabla_d R_{bc}\,\delta g^{ad} \wedge \delta \ln|g| \wedge \delta g^{bc} + \frac{1}{2}\nabla^a \delta R_{bc} \wedge \delta \ln|g| \wedge \delta g^{bc} - g^{ad}R_{eb}\,\delta\Gamma^e_{\ cd} \wedge \delta \ln|g| \wedge \delta g^{bc}.
\end{aligned}
$$

$$
E^a = -\underbrace{4R_{ed}\,\delta g^{ce} \wedge \delta g^{be} \wedge \delta\Gamma^a_{\ bc}}_{=0} - 4g^{ce}\,\delta R_{ed} \wedge \delta g^{bd} \wedge \delta\Gamma^a_{\ bc},
$$

$$
F^a = 2R_{de}\,\delta g^{eb} \wedge \delta g^{ab} \wedge \delta\Gamma^c_{\ bc} + 2g^{eb}\,\delta R_{de} \wedge \delta g^{ab} \wedge \delta\Gamma^c_{\ bc},
$$

$$
G^a = 2R_{de}\,\delta g^{ad} \wedge \delta g^{be} \wedge \delta\Gamma^c_{\ bc} + 2g^{ad}\,\delta R_{de} \wedge \delta g^{be} \wedge \delta\Gamma^c_{\ bc},
$$

$$
H^a = -2\nabla_c \delta R_{db} \wedge \delta g^{ad} \wedge \delta g^{bc} + 2R_{eb}\,\delta\Gamma^e_{\ dc} \wedge \delta g^{ad} \wedge \delta g^{bc} + 2R_{ed}\,\delta\Gamma^e_{\ bc} \wedge \delta g^{ad} \wedge \delta g^{bc}.
$$

Using

$$
\delta(\nabla_c \delta R_{db}) = \delta(\partial_c \delta R_{db} - \Gamma^e_{\ dc}\,\delta R_{eb} - \Gamma^e_{\ bc}\,\delta R_{de}) = -\delta\Gamma^e_{\ dc} \wedge \delta R_{eb} - \delta\Gamma^e_{\ bc} \wedge \delta R_{de},
$$

I^a becomes

$$
I^a = 2g^{ad}\,\delta\Gamma^e_{\ dc} \wedge \delta R_{eb} \wedge \delta g^{bc} + 2g^{ad}\,\delta\Gamma^e_{\ bc} \wedge \delta R_{de} \wedge \delta g^{bc}.
$$

$$K^a = 2R_{de}\,\delta g^{ad}\wedge\delta\Gamma^e_{\ bc}\wedge\delta g^{bc} + 2g^{ad}\,\delta R_{de}\wedge\delta\Gamma^e_{\ bc}\wedge\delta g^{bc},$$

$$\begin{aligned}
L^a &= \nabla_d\delta R_{bc}\wedge\delta g^{ad}\wedge\delta g^{bc} - R_{be}\,\delta\Gamma^e_{\ cd}\wedge\delta g^{ad}\wedge\delta g^{bc} - R_{ec}\,\delta\Gamma^e_{\ bd}\wedge\delta g^{ad}\wedge\delta g^{bc}\\
&= \nabla_d\delta R_{bc}\wedge\delta g^{ad}\wedge\delta g^{bc} - 2R_{be}\,\delta\Gamma^e_{\ cd}\wedge\delta g^{ad}\wedge\delta g^{bc}.
\end{aligned}$$

With the help of

$$\begin{aligned}
\delta(g^{ad}\nabla_d\delta R_{bc}) &= \delta g^{ad}\wedge\nabla_d\delta R_{bc} + g^{ad}\delta(\partial_d\delta R_{bc} - \Gamma^e_{\ bd}\,\delta R_{ec} - \Gamma^e_{\ cd}\,\delta R_{be})\\
&= \delta g^{ad}\wedge\nabla_d\delta R_{bc} - g^{ad}\,\delta\Gamma^e_{\ bd}\wedge\delta R_{ec} - g^{ad}\,\delta\Gamma^e_{\ cd}\wedge\delta R_{be},
\end{aligned}$$

we get

$$\begin{aligned}
M^a &= \delta g^{ad}\wedge\nabla_d\delta R_{bc}\wedge\delta g^{bc} - g^{ad}\,\delta\Gamma^e_{\ bd}\wedge\delta R_{ec}\wedge\delta g^{bc} - g^{ad}\,\delta\Gamma^e_{\ cd}\wedge\delta R_{be}\wedge\delta g^{bc}\\
&= \delta g^{ad}\wedge\nabla_d\delta R_{bc}\wedge\delta g^{bc} - 2g^{ad}\,\delta\Gamma^e_{\ bd}\wedge\delta R_{ec}\wedge\delta g^{bc}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\delta J^a_\beta = &-\underbrace{2g^{ce}\,\delta g^{bd}\wedge\delta R_{de}\wedge\delta\Gamma^a_{\ bc}}_{X^a_1} - \underbrace{2g^{bd}\,\delta g^{ce}\wedge\delta R_{de}\wedge\delta\Gamma^a_{\ bc}}_{X^a_1} + \underbrace{2g^{be}\,\delta g^{ad}\wedge\delta R_{de}\wedge\delta\Gamma^c_{\ bc}}_{X^a_2}\\
&+ \underbrace{2g^{ad}\,\delta g^{be}\,\delta R_{de}\wedge\delta\Gamma^c_{\ bc}}_{X^a_3} + \underbrace{\frac{1}{2}(\nabla_b\delta R)\wedge\delta\ln|g|\wedge\delta g^{ab}}_{X^a_4} + \underbrace{\frac{1}{2}\delta g^{ab}\wedge\delta\ln|g|\wedge\nabla_b\delta R}_{X^a_4}\\
&- \underbrace{2\,\delta g^{ad}\wedge\nabla_c\delta R_{db}\wedge\delta g^{bc}}_{X^a_5} - 2R^c_{\ d}\,\delta g^{bd}\wedge\delta\ln|g|\wedge\delta\Gamma^a_{\ bc} - g^{bd}g^{ce}\,\delta R_{de}\wedge\delta\ln|g|\wedge\delta\Gamma^a_{\ bc}\\
&+ R_c^{\ b}\,\delta g^{ac}\wedge\delta\ln|g|\wedge\delta\Gamma^e_{\ be} + R^a_{\ d}\,\delta g^{bd}\wedge\delta\ln|g|\wedge\delta\Gamma^e_{\ be} + g^{ac}g^{bd}\,\delta R_{cd}\wedge\delta\ln|g|\wedge\delta\Gamma^e_{\ be}\\
&- \nabla_b R_{ec}\,\delta g^{ae}\wedge\delta\ln|g|\wedge\delta g^{bc} + g^{ae}\,\nabla_b\delta R_{ec}\wedge\delta\ln|g|\wedge\delta g^{bc} + \underbrace{g^{ae}R_{cd}\,\delta\Gamma^d_{\ eb}\wedge\delta\ln|g|\wedge\delta g^{bc}}_{X^a_{12}}\\
&+ R^a_{\ d}\,\delta\Gamma^d_{\ cb}\wedge\delta\ln|g|\wedge\delta g^{bc} + \frac{1}{2}\nabla_d R_{bc}\,\delta g^{ad}\wedge\delta\ln|g|\wedge\delta g^{bc} + \frac{1}{2}\nabla^a\delta R_{bc}\wedge\delta\ln|g|\wedge\delta g^{bc}\\
&- \underbrace{g^{ad}R_{eb}\,\delta\Gamma^e_{\ cd}\wedge\delta\ln|g|\wedge\delta g^{bc}}_{X^a_{12}} - \underbrace{4g^{ce}\,\delta R_{ed}\wedge\delta g^{bd}\wedge\delta\Gamma^a_{\ bc}}_{X^a_1} + \underbrace{2R_{de}\,\delta g^{eb}\wedge\delta g^{ab}\wedge\delta\Gamma^c_{\ bc}}_{X^a_6}\\
&+ \underbrace{2g^{eb}\,\delta R_{de}\wedge\delta g^{ab}\wedge\delta\Gamma^c_{\ bc}}_{X^a_2} + \underbrace{2R_{de}\,\delta g^{ad}\wedge\delta g^{be}\wedge\delta\Gamma^c_{\ bc}}_{X^a_6} + \underbrace{2g^{ad}\,\delta R_{de}\wedge\delta g^{be}\wedge\delta\Gamma^c_{\ bc}}_{X^a_3}\\
&- \underbrace{2\nabla_c\delta R_{db}\wedge\delta g^{ad}\wedge\delta g^{bc}}_{X^a_5} + \underbrace{2R_{eb}\,\delta\Gamma^e_{\ dc}\wedge\delta g^{ad}\wedge\delta g^{bc}}_{X^a_7} + \underbrace{2R_{ed}\,\delta\Gamma^e_{\ bc}\wedge\delta g^{ad}\wedge\delta g^{bc}}_{X^a_8}\\
&+ \underbrace{2g^{ad}\,\delta\Gamma^e_{\ dc}\wedge\delta R_{eb}\wedge\delta g^{bc}}_{X^a_9} + \underbrace{2g^{ad}\,\delta\Gamma^e_{\ bc}\wedge\delta R_{de}\wedge\delta g^{bc}}_{X^a_{10}} + \underbrace{2R_{de}\,\delta g^{ad}\wedge\delta\Gamma^e_{\ bc}\wedge\delta g^{bc}}_{X^a_8}\\
&+ \underbrace{2g^{ad}\,\delta R_{de}\wedge\delta\Gamma^e_{\ bc}\wedge\delta g^{bc}}_{X^a_{10}} + \underbrace{\nabla_d\delta R_{bc}\wedge\delta g^{ad}\wedge\delta g^{bc}}_{X^a_{11}} - \underbrace{2R_{be}\,\delta\Gamma^e_{\ cd}\wedge\delta g^{ad}\wedge\delta g^{bc}}_{X^a_7}\\
&+ \underbrace{\delta g^{ad}\wedge\nabla_d\delta R_{bc}\wedge\delta g^{bc}}_{X^a_{11}} - \underbrace{2g^{ad}\,\delta\Gamma^e_{\ bd}\wedge\delta R_{ec}\wedge\delta g^{bc}}_{X^a_9}.
\end{aligned}$$

All the terms from X_1^a to X_{12}^a vanish and we have

$$\begin{aligned}
\delta J_\beta^a \quad = \quad & -\underbrace{2R^c_{\ d}\,\delta g^{bd}\wedge\delta\ln|g|\wedge\delta\Gamma^a_{\ bc}}_{A^a}-\underbrace{g^{bd}g^{ce}\,\delta R_{de}\wedge\delta\ln|g|\wedge\delta\Gamma^a_{\ bc}}_{B^a}+\underbrace{R_c^{\ b}\delta g^{ac}\wedge\delta\ln|g|\wedge\delta\Gamma^e_{\ be}}_{C^a}\\
&+\underbrace{R^a_{\ d}\,\delta g^{bd}\wedge\delta\ln|g|\wedge\delta\Gamma^e_{\ be}}_{D^a}+\underbrace{g^{ac}g^{bd}\,\delta R_{cd}\wedge\delta\ln|g|\wedge\delta\Gamma^e_{\ be}}_{E^a}-\underbrace{\nabla_b R_{ec}\,\delta g^{ae}\wedge\delta\ln|g|\wedge\delta g^{bc}}_{F^a}\\
&+\underbrace{g^{ae}\,\nabla_b\delta R_{ec}\wedge\delta\ln|g|\wedge\delta g^{bc}}_{G^a}+\underbrace{R^a_{\ d}\,\delta\Gamma^d_{\ cb}\wedge\delta\ln|g|\wedge\delta g^{bc}}_{H^a}+\underbrace{\frac{1}{2}\nabla_d R_{bc}\,\delta g^{ad}\wedge\delta\ln|g|\wedge\delta g^{bc}}_{I^a}\\
&+\underbrace{\frac{1}{2}\nabla^a\delta R_{bc}\wedge\delta\ln|g|\wedge\delta g^{bc}}_{K^a},
\end{aligned}$$

which is the same as

$$\begin{aligned}
-\frac{1}{2}\,\delta J_\beta^a\wedge\delta\ln|g| \quad = \quad & \underbrace{2R^c_{\ d}\,\delta g^{bd}\wedge\delta\Gamma^a_{\ bc}\wedge\delta\ln|g|}_{A^a}+\underbrace{g^{bd}g^{ce}\,\delta R_{de}\wedge\delta\Gamma^a_{\ bc}\wedge\delta\ln|g|}_{B^a}-\underbrace{R^b_{\ d}\,\delta g^{ad}\wedge\delta\Gamma^c_{\ dc}\wedge\delta\ln|g|}_{C^a}\\
&-\underbrace{R^a_{\ e}\,\delta g^{be}\wedge\delta\Gamma^c_{\ bc}\wedge\delta\ln|g|}_{D^a}-\underbrace{g^{ad}g^{be}\,\delta R_{de}\wedge\delta\Gamma^c_{\ bc}\wedge\delta\ln|g|}_{E^a}\\
&+\underbrace{(\nabla_c R_{db})\,\delta g^{ad}\wedge\delta g^{bc}\wedge\delta\ln|g|}_{F^a}+\underbrace{g^{ad}\,\nabla_c\delta R_{db}\wedge\delta g^{bc}\wedge\delta\ln|g|}_{G^a}\\
&-\underbrace{R^a_{\ e}\,\delta\Gamma^e_{\ bc}\wedge\delta g^{bc}\wedge\delta\ln|g|}_{H^a}-\underbrace{\frac{1}{2}\nabla_d R_{bc}\,\delta g^{ad}\wedge\delta g^{bc}\wedge\delta\ln|g|}_{I^a}-\underbrace{\frac{1}{2}\,\nabla^a\delta R_{bc}\wedge\delta g^{bc}\wedge\delta\ln|g|}_{K^a}.
\end{aligned}$$

APPENDIX F

GAUGE INVARIANCE OF THE SYMPLECTIC TWO-FORM

For the calculation of the conserved charges, we need to find out how the symplectic current changes under the gauge transformations. We will write the change in the symplectic current as the covariant derivative of an antisymmetric tensor $\nabla_a \mathcal{F}^{ab}$ and some additional terms that vanish on-shell.

F.1 R^2 Term

The relevant terms transform as

$$\begin{aligned} &\delta\Gamma^a_{bc} \to \delta\Gamma^a_{bc} + \xi^e R_{ec}{}^a{}_b + \nabla_c\nabla_b\xi^a && \delta R \to \delta R + (\nabla_e R)\xi^e, \\ &\delta\Gamma^c_{bc} \to \delta\Gamma^c_{bc} - R_{ec}{}^a{}_b\xi^e + \nabla_c\nabla_b\xi^c && \nabla^a(\delta R) \to \nabla^a\delta R + (\nabla^a\nabla_e R)\xi^e + (\nabla_e R)\nabla^a\xi^e, \\ &\delta\ln|g| \to \delta\ln|g| + 2\nabla_d\xi^d && \nabla_b(\delta R) \to \nabla_b(\delta R) + (\nabla_b\nabla_d R)\xi^d + (\nabla_d R)\nabla_b\xi^d. \end{aligned}$$

Therefore, the symplectic current transforms as

$$\begin{aligned} J^a \to &-\frac{1}{2}g^{bc}R\,(\delta\ln|g| + 2\nabla_e\xi^e) \wedge (\delta\Gamma^a_{bc} + R^a{}_{bec}\xi^e + \nabla_c\nabla_b\xi^a) \\ &- g^{bc}\,[\delta R + (\nabla_e R)\xi^e] \wedge [\delta\Gamma^a_{bc} + R^a{}_{bec}\xi^e + \nabla_c\nabla_b\xi^a] \\ &- R\,(\delta g^{bc} - \nabla^b\xi^c - \nabla^c\xi^b) \wedge [\delta\Gamma^a_{bc} + R^a{}_{bec}\xi^e + \nabla_c\nabla_b\xi^a] \\ &+ \frac{1}{2}R(\delta\ln|g| + 2\nabla_d\xi^d) \wedge [\delta\Gamma^c_{bc} - R_{eb}\xi^e + \nabla_c\nabla_b\xi^e] \\ &+ g^{ab}\,[\delta R + (\nabla_e R)\xi^e] \wedge [\delta\Gamma^a_{bc} + R^a{}_{bec}\xi^e + \nabla_c\nabla_b\xi^a] \\ &+ R\,(\delta g^{ab} - \nabla^a\xi^b - \nabla^b\xi^a) \wedge [\delta\Gamma^c_{bc} - R_{eb}\xi^e + \nabla_c\nabla_b\xi^a] \\ &- \frac{1}{2}(\nabla_b R)g^{ab}(\delta\ln|g| + 2\nabla_d\xi^d) \wedge (\delta g^{ab} - \nabla^a\xi^b - \nabla^b\xi^a) \\ &+ (\delta g^{ab} - \nabla^a\xi^b - \nabla^b\xi^a) \wedge [\nabla_b\delta R + (\nabla_b\nabla_d R)\xi^d + (\nabla_d R)(\nabla_b\xi^d)] \\ &- (\nabla_b R)\,(\delta g^{ab} - \nabla^a\xi^b - \nabla^b\xi^a) \wedge (\delta\ln|g| + 2\nabla_d\xi^d) \\ &+ (\delta\ln g + 2\nabla_d\xi^d) \wedge [\nabla^a\delta R + (\nabla^a\nabla_e R)\xi^e + (\nabla_e R)\nabla^a\xi^e], \end{aligned}$$

from which the change in the symplectic current can be written as

$$
\begin{aligned}
\triangle J^a_\alpha = & -\underbrace{\frac{1}{2}RR^a{}_e\,\delta\ln g\wedge\xi^e}_{X^a_2} - \frac{1}{2}R\,\delta\ln|g|\wedge\Box\xi^e - Rg^{bc}\ \nabla_e\xi^e\wedge\Gamma^a_{bc} - \underbrace{R^a{}_b\,\delta R\wedge\xi^b}_{X^a_3} \\
& - \delta R\wedge\Box\xi^a - g^{bc}(\nabla_e R)\,\xi^e\wedge\delta\Gamma^a_{bc} - RR^a{}_{bec}\,\delta g^{bc}\wedge\xi^e - R\,\delta g^{bc}\wedge\nabla_b\nabla_c\xi^a \\
& + 2R\,\nabla^b\xi^c\wedge\delta\Gamma^a_{bc} - \underbrace{\frac{1}{2}RR^a{}_b\,\delta\ln|g|\wedge\xi^b}_{X^a_2} + \frac{1}{2}R\,\delta\ln|g|\wedge\nabla_c\nabla^a\xi^c + g^{ab}R\,\nabla_d\xi^d\wedge\delta\Gamma^c_{bc} \\
& - \underbrace{R^a{}_b\,\delta R\wedge\xi^b}_{X^a_3} + \delta R\wedge\nabla_b\nabla^a\xi^b + g^{ab}(\nabla_d R)\,\xi^d\wedge\delta\Gamma^c_{bc} - RR_{db}\,\delta g^{ab}\wedge\xi^d \\
& + R\,\delta g^{ab}\wedge\nabla_d\nabla^b\xi^d + R\,\delta\Gamma^c_{bc}\wedge(\nabla^a\xi^b+\nabla^b\xi^a) + \underbrace{\frac{1}{2}(\nabla_b R)\,\delta\ln|g|\wedge(\nabla^a\xi^b+\nabla^b\xi^a)}_{X^a_4} \\
& + \underbrace{(\nabla_b R)\,\delta g^{ab}\wedge\nabla_d\xi^d}_{X^a_1} + (\nabla_b\nabla_d R)\,\delta g^{ab}\wedge\xi^d + (\nabla_d R)\,\delta g^{ab}\wedge\nabla_b\xi^d \\
& + \nabla_d\delta R\wedge(\nabla^a\xi^b+\nabla^b\xi^a) - \underbrace{2(\nabla_b R)\,\delta g^{ab}\wedge\nabla_d\xi^d}_{X^a_1} + \underbrace{(\nabla_b R)\,(\nabla^a\xi^b+\nabla^b\xi^a)\wedge\delta\ln|g|}_{X^a_4} \\
& + (\nabla^a\nabla_d R)\,\delta\ln|g|\wedge\xi^d + (\nabla_d R)\,\delta\ln|g|\wedge\nabla^a\xi^d + 2\,\nabla_d\xi^d\wedge\nabla^a\delta R
\end{aligned}
$$

where

$$
\begin{aligned}
X^a_1 &= -(\nabla_b R)\,\delta g^{ab}\wedge\nabla_d\xi^d \qquad & X^a_2 &= -RR^a{}_b\,\delta\ln|g|\wedge\xi^b, \\
X^a_3 &= -2R^a{}_b\,\delta R\wedge\xi^b \qquad & X^a_4 &= -\frac{1}{2}(\nabla_b R)\,\delta\ln|g|\wedge(\nabla^a\xi^b+\nabla^b\xi^a).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\triangle J^a_\alpha = &\underbrace{\frac{1}{2}R\,\delta\ln|g|\wedge\Box\xi^a}_{A^a_1} - \underbrace{g^{bc}R\,\nabla_d\xi^d\wedge\delta\Gamma^a_{bc}}_{A^a_8} - \underbrace{\delta R\wedge\Box\xi^a}_{A^a_{18}} - \underbrace{g^{bc}(\nabla_d R)\,\xi^d\wedge\delta\Gamma^a_{bc}}_{A^a_9}\\
&- \underbrace{RR^a_{\ bdc}\,\delta g^{bc}\wedge\xi^d}_{A^a_{14}} - \underbrace{R\,\delta g^{bc}\wedge\nabla_b\nabla_c\xi^a}_{A^a_6} + \underbrace{2R\,\nabla^b\xi^c\wedge\delta\Gamma^a_{bc}}_{A^a_{13}} + \underbrace{\frac{1}{2}R\,\delta\ln|g|\wedge\nabla_c\nabla^a\xi^c}_{A^a_2}\\
&+ \underbrace{g^{ab}R\,\nabla_d\xi^d\wedge\delta\Gamma^c_{bc}}_{A^a_5} + \underbrace{\delta R\wedge\nabla_b\nabla^a\xi^b}_{A^a_{17}} + \underbrace{g^{ab}(\nabla_d R)\,\xi^d\wedge\delta\Gamma^c_{bc}}_{A^a_{10}} - \underbrace{RR_{db}\,\delta g^{ab}\wedge\xi^d}_{A^a_{15}}\\
&+ \underbrace{R\,\delta g^{ab}\wedge\nabla_d\nabla_b\xi^d}_{A^a_3} + \underbrace{R\,\delta\Gamma^c_{bc}\wedge(\nabla^b\xi^a+\nabla^a\xi^b)}_{A^a_4} + \underbrace{(\nabla_b\nabla_c R)\,\delta g^{ab}\wedge\xi^c}_{A^a_{21}}\\
&+ \underbrace{(\nabla_c R)\,\delta g^{ab}\wedge\nabla_b\xi^c}_{A^a_{12}} + \underbrace{\nabla_b\delta R\wedge(\nabla^b\xi^a+\nabla^a\xi^b)}_{A^a_{19}} + \underbrace{(\nabla^a\nabla_b R)\,\delta\ln|g|\wedge\xi^b}_{A^a_{22}}\\
&+ \underbrace{(\nabla_b R)\,\delta\ln|g|\wedge\nabla^a\xi^b}_{A^a_{11}} + \underbrace{2\,\nabla_b\xi^b\wedge\nabla^a\delta R}_{A^a_{20}} - \underbrace{(\nabla_b R)\,\delta g^{ab}\wedge\nabla_c\xi^c}_{A^a_{16}} - \underbrace{RR^a_{\ b}\,\delta\ln|g|\wedge\xi^b}_{A^a_{23}}\\
&- \underbrace{2R^a_{\ b}\,\delta R\wedge\xi^b}_{A^a_{24}} - \underbrace{\frac{1}{2}(\nabla_b R)\,\delta\ln|g|\wedge(\nabla^a\xi^b+\nabla^b\xi^a)}_{A^a_7}\,.
\end{aligned}
$$

We will manipulate all the terms to construct the antisymmetric part. Then, we will show that all the remaning terms vanish on-shell. We start with

$$A^a_1 = \nabla_b(-\frac{1}{2}R\ \delta\ln|g|\wedge\nabla^b\xi^a) + \frac{1}{2}(\nabla_b R)\ \delta\ln|g|\wedge\nabla^b\xi^a + \frac{1}{2}R\ \delta\ln|g|\wedge\nabla^b\xi^a,$$

$$A^a_2 = \nabla_b(\frac{1}{2}R\ \delta\ln|g|\wedge\nabla^a\xi^b) - \frac{1}{2}(\nabla_b R)\ \delta\ln|g|\wedge\nabla^a\xi^b - \frac{1}{2}R\ \nabla_b\delta\ln|g|\wedge\nabla^a\xi^b,$$

which give

$$\sum_{i=1}^{4} A^a_i = \nabla_b(\underbrace{R\,\delta\ln|g|\wedge\nabla^{[a}\xi^{b]}}_{Q^{ab}_1}) - (\nabla_b R)\ \delta\ln|g|\wedge\nabla^a\xi^b + 2R\ \delta\Gamma^c_{bc}\wedge\nabla^b\xi^a.$$

We proceed with

$$
\begin{aligned}
A^a_5 &= \nabla_b(Rg^{ad}\ \xi^b\wedge\delta\Gamma^c_{dc}) - g^{ad}(\nabla_b R)\ \xi^b\wedge\delta\Gamma^c_{dc} - g^{ad}R\ \xi^b\wedge\nabla_b\delta\Gamma^c_{dc}\\
&= \nabla_b(2Rg^{ad}\ \xi^b\wedge\delta\Gamma^c_{dc}) - 2g^{ad}(\nabla_b R)\ \xi^b\wedge\delta\Gamma^c_{dc} - 2g^{ad}R\ \xi^b\wedge\nabla_b\delta\Gamma^c_{dc} - Rg^{ad}\ \nabla_b\xi^b\wedge\delta\Gamma^c_{dc}.
\end{aligned}
$$

We first write

$$
\begin{aligned}
\sum_{i=1}^{4} A^a_i = &\nabla_b Q^{ab}_1 - (\nabla_b R)\ \delta\ln|g|\wedge\nabla^a\xi^b + \nabla_b(2Rg^{bd}\ \delta\Gamma^c_{dc}\wedge\xi^a) - 2(\nabla^b R)\ \delta\Gamma^c_{bc}\wedge\nabla^b\xi^a\\
&- 2R\nabla^b\delta\Gamma^c_{bc}\wedge\xi^a,
\end{aligned}
$$

and then add the contribution of A_5^a

$$\sum_{i=1}^{5} A_i^a = \nabla_b Q_1^{ab} + \nabla_b(\underbrace{R\nabla^{[b}\ \delta \ln|g| \wedge \xi^{a]}}_{Q_2^{ab}}) - (\nabla_b R)\ \delta \ln|g| \wedge \nabla^a \xi^b - 2(\nabla^b R)\ \delta\Gamma^c_{bc} \wedge \xi^a$$
$$- 2R\nabla^b \delta\Gamma^c_{bc} \wedge \xi^a - 2g^{ad}(\nabla_b R)\ \xi^b \wedge \delta\Gamma^c_{dc} - 2g^{ad} R\ \xi^b \wedge \nabla_b \delta\Gamma^c_{dc} - R g^{ad}\ \nabla_b \xi^b \wedge \delta\Gamma^c_{dc}.$$

Taking

$$A_6^a = -R\,\delta g^{bc} \wedge \nabla_b \nabla_c \xi^a = -\nabla_b(R\,\delta g^{bc} \wedge \nabla_c \xi^a) + (\nabla_b R)\,\delta g^{bc} \wedge \nabla_c \xi^a + R\,\nabla_b \delta g^{bc} \wedge \nabla_c \xi^a,$$
$$A_7^a = R\,\delta g^{ab} \wedge \nabla_d \nabla_b \xi^b = \nabla_b(R\,\delta g^{ac} \wedge \nabla_c \xi^b) - (\nabla_b R)\,\delta g^{ac} \wedge \nabla_c \xi^b - R\,\nabla_b \delta g^{ac} \wedge \nabla_c \xi^b,$$

gives

$$A_6^a + A_7^a = \nabla_b(\underbrace{2R\,\delta g^{c[a} \wedge \nabla_c \xi^{b]}}_{Q_3^{ab}}) - (\nabla_b R)\,\delta g^{ac} \wedge \nabla_c \xi^b - R\ \nabla_b \delta g^{ac} \wedge \nabla_c \xi^b + (\nabla_b R)\,\delta g^{bc} \wedge \nabla_c \xi^a$$
$$+ \underbrace{R\,\nabla_b \delta g^{bc} \wedge \nabla_c \xi^a}_{A^a},$$

where we write the last term for further simplification as

$$\begin{aligned}
A^a &= \nabla_b(R\,\nabla_c \delta g^{bc} \wedge \xi^a) - (\nabla_b R)\,\nabla_c \delta g^{bc} \wedge \xi^a - R\,\nabla_b \nabla_c \delta g^{bc} \wedge \xi^a \\
&= \nabla_b(R\,\nabla_c \delta g^{bc} \wedge \xi^a) + g^{de}(\nabla_b R)\,\delta\Gamma^b_{de} \wedge \xi^a + (\nabla_b R) g^{db}\ \delta\Gamma^e_{de} \wedge \xi^a \\
&+ R\,\nabla_b(g^{de}\delta\Gamma^b_{de} + g^{db}\delta\Gamma^e_{de}) \wedge \xi^a \\
&= \nabla_b(2R\ \nabla_c \delta g^{c[b} \wedge \xi^{a]}) + \nabla_b(R\nabla_c \delta g^{ac} \wedge \xi^b) \\
&+ (\nabla_b R) g^{de}\ \delta\Gamma^b_{de} \wedge \xi^a + (\nabla_b R) g^{db}\ \delta\Gamma^e_{de} \wedge \xi^a + R\nabla_b(g^{de}\delta\Gamma^b_{de} + g^{db}\delta\Gamma^e_{de}) \wedge \xi^a \\
&= \nabla_b Q_4^{ab} + \underbrace{(\nabla_b R)\ \xi^b \wedge \delta\Gamma^a_{de}}_{\text{cancels with } A_9^a} + \underbrace{g^{de} R\ \nabla_b \xi^b \xi^b \wedge \delta\Gamma^a_{de}}_{\text{cancels with } A_5^a} + g^{de} R\ \xi^b \wedge \nabla_b \delta\Gamma^a_{de} + g^{ad}(\nabla_b R)\ \xi^b \wedge \delta\Gamma^e_{de} \\
&+ \underbrace{g^{ad} R\ \nabla_b \xi^b \wedge \delta\Gamma^e_{de}}_{\text{cancels with } A_{10}^a} + g^{ad} R\ \xi^b \wedge \nabla_b \delta\Gamma^e_{de} + (\nabla_b R) g^{de}\ \delta\Gamma^b_{de} \wedge \xi^a + (\nabla^d R)\ \delta\Gamma^e_{de} \wedge \xi^a \\
&+ g^{de} R\ \nabla_b \delta\Gamma^b_{de} \wedge \xi^a + R\ \nabla^d \delta\Gamma^e_{de} \wedge \xi^a,
\end{aligned}$$

from which we get

$$\sum_{i=1}^{12} A_i^a = \sum_{i=1}^{4} \nabla_b Q_i^{ab} + \underbrace{2(\nabla^b R)\,\xi^a \wedge \delta\Gamma^c_{bc}}_{X_8^a} + \underbrace{2R\,\xi^a \wedge \nabla^b \delta\Gamma^c_{bc}}_{X_6^a} - \underbrace{(\nabla_b R)\,\delta \ln|g| \wedge \nabla^a \xi^b}_{X_4^a}$$
$$- \underbrace{2g^{ad}(\nabla_b R)\,\xi^b \wedge \delta\Gamma^c_{dc}}_{X_3^a} - \underbrace{2g^{ad} R\,\xi^b \wedge \nabla_b \delta\Gamma^c_{dc}}_{X_7^a} - \underbrace{(\nabla_b R)\,\delta g^{ac} \wedge \nabla_c \xi^b}_{X_5^a}$$
$$- R\,\nabla_b \delta g^{ac} \wedge \nabla_c \xi^b + (\nabla_b R)\,\delta g^{bc} \wedge \nabla_c \xi^a + \underbrace{g^{de}(\nabla_b R)\,\xi^b \wedge \delta\Gamma^a_{de}}_{X_2^a} + \underbrace{g^{de} R\,\nabla_b \xi^b \wedge \delta\Gamma^a_{de}}_{X_1^a}$$
$$+ g^{de} R \xi^b \wedge \nabla_b \delta\Gamma^a_{de} + \underbrace{g^{ad}(\nabla_b R)\,\xi^b \wedge \delta\Gamma^e_{de}}_{X_3^a} + g^{ad} R\,\nabla_b \xi^b \wedge \delta\Gamma^e_{de}$$
$$+ \underbrace{g^{ad} R\ \xi^b \wedge \nabla_b \delta\Gamma^e_{de}}_{X_7^a} + g^{de}(\nabla_b R)\ \delta\Gamma^b_{de} \wedge \xi^a + \underbrace{2(\nabla^d R)\,\delta\Gamma^e_{de} \wedge \delta\Gamma^e_{de}}_{X_8^a}$$
$$+ g^{de} R\,\nabla_b \delta\Gamma^b_{de} \wedge \xi^a + \underbrace{R\,\nabla^d \delta\Gamma^e_{de} \wedge \xi^a}_{X_6^a} - \underbrace{g^{bc} R\ \nabla_d \xi^d \wedge \delta\Gamma^a_{bc}}_{X_1^a} - \underbrace{g^{bc}(\nabla_d R)\,\xi^d \wedge \delta\Gamma^a_{bc}}_{X_2^a}$$
$$+ \underbrace{g^{ab}(\nabla_d R)\ \xi^d \wedge \delta\Gamma^c_{dc}}_{X_3^a} + \underbrace{(\nabla_b R)\delta \ln|g| \wedge \nabla^a \xi^b}_{X_4^a} + \underbrace{(\nabla_c R)\ \delta g^{ab} \wedge \nabla_b \xi^c}_{X_5^a}.$$

All terms except the followings vanish

$$X_6^a = R\,\xi^a \wedge \nabla^c \delta\Gamma^b_{cb} \qquad X_7^a = -g^{ad} R\,\xi^b \wedge \nabla_b \delta\Gamma^c_{dc} \qquad X_8^a = (\nabla^b R)\,\xi^a \wedge \delta\Gamma^c_{bc}$$

and we have

$$\sum_{i=1}^{12} A_i^a = \sum_{i=1}^{4} \nabla_b Q_i^{ab} - \underbrace{R\,\nabla_b \delta g^{ac} \wedge \nabla_c \xi^b}_{Y^a} + (\nabla_b R)\ \delta g^{bc} \wedge \nabla_c \xi^a - g^{de} R\,\xi^b \wedge \delta\Gamma^a_{de}$$
$$+ g^{de}(\nabla_b R)\,\delta\Gamma^b_{de} \wedge \xi^a + g^{de} R\,\nabla_b \delta\Gamma^b_{de} \wedge \xi^a + R\ \xi^a \wedge \nabla^c \delta\Gamma^b_{cb} - g^{ad} R\,\xi^b \wedge \nabla_b \delta\Gamma^c_{dc}$$
$$+ (\nabla^b R)\,\xi^a \wedge \delta\Gamma^c_{bc}.$$

We write A^a_{13} as

$$A^a_{13} = \nabla_b(\underbrace{2R\,\xi_c \wedge \nabla^{[a} \delta g^{b]c}}_{Q_5^{ab}}) + (\nabla_b R)\,\xi_c \wedge \nabla^b \delta g^{ac} + R\,\xi_c \wedge \Box \delta g^{ac} - (\nabla_b R)\,\xi_c \wedge \nabla^a \delta g^{bc}$$
$$- R\,\xi_c \wedge \nabla_b \nabla^a \delta g^{bc} - \underbrace{R\,\nabla_b \xi^c \wedge \nabla_c \delta g^{ab}}_{Y^a},$$

and then add it to $\sum_{i=1}^{12} A_i^a$ (terms Y^a gives zero)

$$\sum_{i=1}^{13} A_i^a = \sum_{i=1}^{5} \nabla_b Q_i^{ab} + (\nabla_b R)\,\delta g^{bc} \wedge \nabla_c \xi^a + \underbrace{g^{de} R\,\xi^b \wedge \nabla_b \delta\Gamma^a_{de}}_{E^a} + g^{de}(\nabla_b R)\,\Gamma^b_{de} \wedge \xi^a$$
$$+ \underbrace{g^{de} R\ \nabla_b \delta\Gamma^b_{de} \wedge \xi^a}_{A^a} + \underbrace{R\ \xi^a \wedge \nabla^c \delta\Gamma^b_{cb}}_{B^a} - \underbrace{g^{ad} R\ \xi^b \wedge \nabla_b \delta\Gamma^c_{dc}}_{F^a} + (\nabla^b R)\,\xi^a \wedge \delta\Gamma^c_{bc}$$
$$+ (\nabla^b R)\,\xi^a \wedge \delta\Gamma^c_{bc} + (\nabla^b R)\,\xi_c \wedge \nabla_b \delta g^{ac} + \underbrace{R\ \xi_c \wedge \Box\delta g^{ac}}_{D^a} - (\nabla_b R)\,\xi_c \wedge \nabla^a \delta g^{bc}$$
$$- \underbrace{R\,\xi_c \wedge \nabla_b \nabla^a \delta g^{bc}}_{C^a}.$$

Here

$$A^a + B^a = R g^{bc}\ \delta R_{bc} \wedge \xi^a \qquad E^a = -R\ \xi^b \wedge \nabla_b \nabla_c \delta g^{ac} - \frac{1}{2} R \xi^b \wedge \nabla_b \nabla^a \delta \ln g,$$
$$F^a = -\frac{1}{2} R\ \xi^b \wedge \nabla_b \nabla^a \delta \ln g \qquad E^a + F^a = -R\ \xi^b \wedge \nabla_b \nabla_c \delta g^{ac} - R \xi^b \wedge \nabla_b \nabla^a \delta \ln g,$$

and we have

$$\sum_{i=1}^{16} A_i^a = \sum_{i=1}^{6} \nabla_b Q_i^{ac} + g^{de}(\nabla_b R)\ \delta\Gamma^b_{de} \wedge \xi^a + (\nabla^b R)\ \xi^a \wedge \delta\Gamma^d_{bd} + g^{bc} R\ \delta R_{bc} \wedge \xi^a - 2 g^{ab} R\ \delta R_{bc} \wedge \xi^c$$
$$- 2 R R_{bc}\ \delta g^{ac} \wedge \xi^b + (\nabla_b \nabla_c R)\ \delta g^{ac} \wedge \xi^b - (\nabla_c \nabla_b R)\ \delta g^{bc} \wedge \xi^a + g^{bd}(\nabla_c R)\ \delta\Gamma^c_{bd} \wedge \xi^a$$
$$+ (\nabla^d R)\ \delta\Gamma^e_{de} \wedge \xi^a + 2 g^{ac}(\nabla_d R)\ \xi^b \wedge \delta\Gamma^d_{bc},$$

with $Q_5^{ab} = 2(\nabla_c R)\ \delta g^{c[b} \wedge \xi^{a]}$. Writing

$$A^a_{17} = \nabla_b(\delta R \wedge \nabla^a \xi^b) - \nabla_b \delta R \wedge \nabla^a \xi^b \qquad A^a_{18} = -\nabla_b(\delta R \wedge \nabla^b \xi^a) + \nabla_b \delta R \wedge \nabla^b \xi^a,$$

gives

$$A^a_{17} + A^a_{18} + A^a_{19} = \nabla_b(\underbrace{2\,\delta R \wedge \nabla^{[a} \xi^{b]}}_{Q_7^{ab}}) + 2\ \nabla_b \delta R \wedge \nabla^b \xi^a.$$

Together with A^a_{20}

$$A^a_{20} = 2\,\nabla_b \xi^b \wedge \nabla^a \delta R = \nabla_b(2\,\xi^b \wedge \nabla^a \delta R) - 2\ \xi^b \wedge \nabla_b \nabla^a \delta R,$$

they read

$$\sum_{i=17}^{20} A_i^a = \nabla_b Q_7^{ab} + \nabla_b(\underbrace{2\,\nabla^b \delta R \wedge \xi^a - 2\,\nabla^a \delta R \wedge \xi^b}_{Q_8^{ab}}) - 2\,\Box\delta R \wedge \xi^a - 2\,\xi^b \wedge \nabla_b \nabla^a \delta R,$$

and we finally have

$$\sum_{i=1}^{24} A_i^a = \sum_{i=1}^{8} \nabla_b Q_i^{ab} + \underbrace{g^{de}(\nabla_b R)\,\delta\Gamma^b_{de} \wedge \xi^a}_{Z_2^a} + \underbrace{(\nabla^b R)\,\xi^a \wedge \delta\Gamma^d_{bd}}_{Z_1^a} + \underbrace{g^{bc} R\,\delta R_{bc} \wedge \xi^a}_{X_5^a} - \underbrace{2g^{ab} R\,\delta R_{bc} \wedge \xi^c}_{X_7^a}$$
$$- \underbrace{2RR_{bc}\,\delta g^{ac} \wedge \xi^b}_{X_3^a} + \underbrace{2(\nabla_b\nabla_c R)\,\delta g^{ac} \wedge \xi^b}_{X_4^a} - \underbrace{(\nabla_c\nabla_b R)\ \delta g^{bc} \wedge \xi^a}_{X_{10}^a} + \underbrace{g^{bd}(\nabla_c R)\,\xi^a \wedge \delta\Gamma^c_{bd}}_{Z_2^a}$$
$$+ \underbrace{(\nabla^d R)\,\delta\Gamma^e_{de} \wedge \xi^a}_{Z_1^a} - \underbrace{2\ \Box\delta R \wedge \xi^a}_{X_9^a} - \underbrace{2\ \xi^b \wedge \nabla_b\nabla^a \delta R}_{X_8^a} + \underbrace{(\nabla_b\nabla_c R)\ \delta g^{ab} \wedge \xi^c}_{X_4^a}$$
$$+ \underbrace{(\nabla^a\nabla_b R)\ \ln g \wedge \xi^b}_{X_7^a} - \underbrace{RR^a{}_b\,\delta\ln g \wedge \xi^b}_{X_1^a} - \underbrace{2R^a{}_b\,\delta R \wedge \xi^b}_{X_6^a} + \underbrace{2g^{de}(\nabla_b R)\,\delta\Gamma^b_{de} \wedge \xi^a}_{X_{11}^a}$$
$$+ \underbrace{2g^{ac}(\nabla_d R)\,\xi^b \wedge \delta\Gamma^d_{bc}}_{X_{12}^a}$$

Terms Z_1^a and Z_2^a vanish. We need the following expression to show that the remaining terms are zero.

$$\bigstar^a = \underbrace{\frac{1}{2} g^{bc}\ \delta\mathcal{A}_{bc} \wedge \xi^a}_{C^a} + \underbrace{\mathcal{A}_{bc}\ \xi^c \wedge \delta g^{ab}}_{B^a} + \underbrace{g^{ac}\ \xi^b \wedge \delta\mathcal{A}_{bc}}_{D^a} + \underbrace{\frac{1}{2}\mathcal{A}^{ab}\ \xi_b \wedge \delta\ln g}_{A^a} - \delta\mathcal{A} \wedge \xi^a$$

where

$$\mathcal{A}_{ab} = 2RR_{ab} - \frac{1}{2} g_{ab} R^2 + 2g_{ab}\Box R - 2\nabla_a\nabla_b R,$$

$$\delta\mathcal{A}_{bc} = 2R_{bc}\delta R + 2R\delta R_{bc} - \frac{1}{2}R^2\delta g_{bc} - g_{bc}R\delta R + 2(\Box R)\delta g_{bc} + 2g_{bc}(\nabla_d\nabla_e R)\delta g^{de}$$
$$+ 2g_{bc}\Box\delta R - 2Rg_{bc}g^{de}(\nabla_f R)\delta\Gamma^f_{de} - 2\nabla_b\nabla_c\delta R + 2(\nabla_d R)\Gamma^d_{bc},$$

and

$$\delta\mathcal{A} = \Big(2 - \frac{D}{2}\Big)R\delta R + (D-1)\Big[(\nabla_a\nabla_b R)\delta g^{ab} + \Box\delta R - g^{ab}(\nabla_c R)\delta\Gamma^c_{ab}\Big].$$

Now we can calculate

$$A^a = RR^{ab}\ \xi_b \wedge \delta\ln g - \underbrace{\frac{1}{4}R^2\ \xi^a \wedge \delta\ln g}_{K_1^a} + \underbrace{(\Box R)\ \xi^a \wedge \delta\ln g}_{K_2^a} - 2(\nabla^a\nabla^b R)\ \xi_b \wedge \delta\ln g,$$

$$B^a = 2RR_{bc}\ \xi^c \wedge \delta g^{ab} - \underbrace{\frac{1}{2}R^2\ \xi_b \wedge \delta g^{ab}}_{K_4^a} + \underbrace{2(\Box R)\ \xi_b \wedge \delta g^{ab}}_{K_3^a} - 2(\nabla_b\nabla_c R)\ \xi^c \wedge \delta g^{ab},$$

$$
C^a = \underbrace{R\ \delta R \wedge \xi^a}_{K_5^a} + g^{bc} R\ \delta R_{bc} \wedge \xi^a - \underbrace{\frac{1}{4} R^2\ \delta \ln g \wedge \xi^a}_{K_1^a} - \underbrace{\frac{1}{2}(DR)\ \delta R \wedge \xi^a}_{K_5^a} + (\Box R)\ \delta \ln g \wedge \xi^a
$$

$$
+ \underbrace{D(\nabla_b \nabla_c R)\ \delta g^{bc} \wedge \xi^a}_{K_6^a} + \underbrace{D\ (\Box \delta R) \wedge \xi^a}_{K_7^a} - D g^{bc} (\nabla_d R)\ \delta \Gamma^d_{bc} \wedge \xi^a - \underbrace{(\Box \delta R) \wedge \xi^a}_{K_7^a}
$$

$$
+ \underbrace{g^{bc} (\nabla_d R)\ \delta \Gamma^d_{bc} \wedge \xi^a}_{K_8^a},
$$

$$
D^a = 2R^a{}_b\ \xi^b \wedge \delta R + 2g^{ac} R\ \xi^b \wedge \delta R_{bc} + \underbrace{\frac{1}{2} R^2\ \xi_b \wedge \delta g^{ab}}_{K_4^a} - \underbrace{R\ \xi^a \wedge \delta R}_{K_5^a}
$$

$$
- \underbrace{2(\Box R)\ \xi_b \wedge \delta g^{ab}}_{K_3^a} + \underbrace{2(\nabla_b \nabla_c R)\ \xi^a \wedge \delta g^{bc}}_{K_6^a} + \underbrace{2\ \xi^a \wedge \Box \delta R}_{K_7^a}
$$

$$
- 2g^{bc} (\nabla_d R)\ \xi^a \wedge \delta \Gamma^d_{bc} - 2\ \xi^b \wedge \nabla_b \nabla^a \delta R + 2g^{ac} (\nabla_d R)\ \xi^b \wedge \delta \Gamma^d_{bc}.
$$

When we calculate $A^a + B^a + C^a + D^a$ all the K terms except the followings vanish

$$
K_5^a = (4 - D) R\ \delta R \wedge \xi^a \qquad K_6^a = 2(D-2)(\nabla_b \nabla_c R)\ \delta g^{bc} \wedge \xi^a,
$$

$$
K_7^a = (2D - 6)\ \Box \delta R \wedge R \xi^a \qquad K_8^a = (2D-6) g^{bc} (\nabla_d R)\ \xi^a \wedge \delta \Gamma^d_{bc},
$$

and the result is

$$
A^a + B^a + C^a + D^a = \underbrace{R R^{ab}\ \xi_b \wedge \delta \ln g}_{X_1^a} - \underbrace{(\nabla^a \nabla^b R)\ \xi_b \wedge \delta \ln g}_{X_2^a} + \underbrace{2 R R_{bc}\ \xi^c \wedge \delta g^{ab}}_{X_3^a}
$$

$$
- \underbrace{2(\nabla_b \nabla_c R)\ \xi^c \wedge \delta g^{ab}}_{X_4^a} + \underbrace{g^{bc} R\ \delta R_{bc} \wedge \xi^a}_{X_5^a} + \underbrace{2R^a{}_b\ \xi^b \wedge \delta R}_{X_6^a} + \underbrace{2g^{ac} R\ \xi^b \wedge \delta R_{bc}}_{X_7^a}
$$

$$
- \underbrace{2\ \xi^b \wedge \nabla_b \nabla^a \delta R}_{X_8^a} + \underbrace{2g^{ac} (\nabla_d R)\ \xi^b \wedge \delta \Gamma^d_{bc}}_{X_{12}^a}
$$

$$
+ \Big[\Big(2 - \frac{D}{2}\Big) R\ \delta R \wedge \xi^a + (D-1)(\nabla_b \nabla_c R)\ \delta g^{bc} \wedge \xi^a + (D-1)\ \Box \delta R \wedge \xi^a
$$

$$
+ (D-1) g^{bc} (\nabla_d R)\ \xi^a \wedge \delta \Gamma^d_{bc} \Big] - \underbrace{2\ \Box \delta R \wedge \xi^a}_{A_9^a} - \underbrace{2 g^{bc} (\nabla_d R)\ \xi^a \wedge \delta \Gamma^d_{bc}}_{A_{11}^a}
$$

$$
- \underbrace{(\nabla_b \nabla_c R)\ \delta g^{bc} \wedge \xi^a}_{A_{10}^a}.
$$

The term in square brackets is equal to $\delta \mathcal{A} \wedge \xi^a$. The other terms are equal to the extra terms in $\sum_{i=1}^{24} A_i^a$.

F.2 R_{ab}^2 Term

The symplectic current transforms as

$$
\begin{aligned}
J^a_\beta \to & -R^{cb}(\delta \ln|g| + 2\nabla_d\xi^d) \wedge (\delta\Gamma^a_{cb} + R_{ec}{}^a{}_b\xi^e + \nabla_c\nabla_b\xi^a) \\
& + R^{ab}(\delta \ln|g| + 2\nabla_d\xi^d) \wedge (\delta\Gamma^c_{bc} - R_{eb}\xi^e + \nabla_c\nabla_b\xi^e) \\
& - (\nabla_c R^a{}_b)(\delta \ln|g| + 2\nabla_d\xi^d) \wedge (\delta g^{bc} - \nabla^b\xi^c - \nabla^c\xi^b) \\
& + \frac{1}{2}(\nabla^a R_{cb})(\delta \ln|g| + 2\nabla_d\xi^d) \wedge (\delta g^{bc} - \nabla^b\xi^c - \nabla^c\xi^b) \\
& - 4R_d{}^b(\delta g^{cd} - \nabla^c\xi^d - \nabla^d\xi^c) \wedge (\delta\Gamma^a_{bc} + R_{ec}{}^a{}_b\xi^e + \nabla_c\nabla_b\xi^a) \\
& - 2g^{cd}g^{be}[\delta R_{de} + (\nabla_f R_{de})\xi^f + R_{df}\nabla_e\xi^f + R_{ef}\nabla_d\xi^f] \wedge (\delta\Gamma^a_{bc} + R_{fc}{}^a{}_b\xi^f + \nabla_c\nabla_b\xi^a) \\
& + 2R_d{}^b(\delta g^{ad} - \nabla^a\xi^d - \nabla^d\xi^a) \wedge (\delta\Gamma^c_{bc} - R_{eb}\xi^e + \nabla_c\nabla_b\xi^c) \\
& + 2R_a{}^e(\delta g^{be} - \nabla^b\xi^e - \nabla^e\xi^b) \wedge (\delta\Gamma^c_{bc} - R_{db}\xi^b + \nabla_c\nabla_b\xi^c) \\
& + 2g^{ad}g^{be}[\delta R_{de} + (\nabla_f R_{de})\xi^f + R_{fe}\nabla_d\xi^f + R_{fd}\nabla_e\xi^f] \wedge (\delta\Gamma^c_{bc} - R_{cb}\xi^c + \nabla_c\nabla_b\xi^c) \\
& - \frac{1}{2}(\nabla_b R)(\delta g^{ab} - \nabla^a\xi^b - \nabla^b\xi^a) \wedge (\delta \ln|g| + 2\nabla_d\xi^d) \\
& - \frac{1}{2}[\nabla^a\delta R + (\nabla^a\nabla_e R)\xi^e + (\nabla_e R)\nabla^a\xi^e] \wedge (\delta \ln|g| + 2\nabla_d\xi^d) \\
& - 2(\nabla_c R_{db})(\delta g^{ad} - \nabla^a\xi^d - \nabla^d\xi^a) \wedge (\delta g^{cb} - \nabla^c\xi^b - \nabla^b\xi^c) \\
& - \Big\{2g^{ad}[\nabla_c\delta R_{db} + (\nabla_c\nabla_f R_{db})\xi^f + (\nabla_f R_{db})\nabla_c\xi^f + (\nabla_c R_{df})\nabla_b\xi^f \\
& + R_{df}\nabla_c\nabla_b\xi^f + (\nabla_c R_{bf})\nabla_d\xi^f + R_{bf}\nabla_c\nabla_d\xi^f] \wedge (\delta g^{cb} - \nabla^c\xi^b - \nabla^b\xi^c)\Big\} \\
& + 2R^a{}_e(\delta\Gamma^e_{cb} - R_{dc}{}^e{}_b\xi^d + \nabla_c\nabla_b\xi^e) \wedge (\delta g^{cb} - \nabla^c\xi^b - \nabla^b\xi^c) \\
& + (\nabla_d R_{cb})(\delta g^{ad} - \nabla^a\xi^d - \nabla^d\xi^a) \wedge (\delta g^{bc} - \nabla^b\xi^c - \nabla^c\xi^b) \\
& + \Big\{[\nabla^a\delta R_{cb} + (\nabla^a\nabla_f R_{cb})\xi^f + (\nabla_f R_{cb})\nabla^a\xi^f + (\nabla^a R_{cf})\nabla_b\xi^f \\
& + R_{cf}\nabla^a\nabla_b\xi^f + (\nabla^a R_{bf})\nabla_c\xi^f + R_{bf}\nabla^a\nabla_c\xi^f] \wedge (\delta g^{cb} - \nabla^c\xi^b - \nabla^b\xi^c)\Big\},
\end{aligned}
$$

which gives the change in the symplectic current as

$$
\begin{aligned}
\triangle J^a_\beta = &- \underbrace{R^{cb} R_{ec}{}^{a}{}_{b}\ \delta \ln|g| \wedge \xi^e}_{E^a_1} - \underbrace{R^{cb}\ \delta \ln|g| \wedge \nabla_c \nabla_b \xi^a}_{B^a_1} - \underbrace{2R^{cb}\ \nabla_d \xi^d \wedge \delta\Gamma^a_{cb}}_{B^a_{18}} \\
&- \underbrace{4R_d{}^b R_{ec}{}^{a}{}_{b}\ \delta g^{cd} \wedge \xi^e}_{E^a_2} - \underbrace{4R_d{}^b\ \delta g^{cd} \wedge \nabla_c \nabla_b \xi^a}_{B^a_{16}} + \underbrace{4R_d{}^b\ \nabla^c \xi^d \wedge \delta\Gamma^a_{bc}}_{L^a_3} + \underbrace{4R_d{}^b\ \nabla^d \xi^c \wedge \delta\Gamma^a_{bc}}_{b_{15}} \\
&- \underbrace{2R_f{}^{dae}\ \delta R_{de} \wedge \xi^f}_{E^a_3} - \underbrace{2\ \delta R_{de} \wedge \nabla^d \nabla^e \xi^a}_{B^a_8} - \underbrace{2(\nabla_f R^{bc})\ \xi^f \wedge \delta\Gamma^a_{bc}}_{B^a_{17}} - \underbrace{2R^c{}_f\ (\nabla^b \xi^f) \wedge \delta\Gamma^a_{bc}}_{L^a_3} \\
&- \underbrace{2R^c{}_f\ (\nabla^b \xi^f) \wedge \delta\Gamma^a_{bc}}_{L^a_3} - \underbrace{R^{ab} R_{eb}\ \delta \ln|g| \wedge \xi^e}_{E^a_4} + \underbrace{R^{ab}\ \delta \ln|g| \wedge \nabla_c \nabla_b \xi^c}_{B^a_2} + \underbrace{2R^{ab}\ \nabla_d \xi^d \wedge \delta\Gamma^c_{bc}}_{B^a_{20}} \\
&- \underbrace{2R_d{}^b R_{eb}\ \delta g^{ad} \wedge \xi^e}_{E^a_5} + \underbrace{2R_d{}^b\ \delta g^{ad} \wedge \nabla_c \nabla_b \xi^c}_{B^a_{12}} - \underbrace{2R_d{}^b\ (\nabla^a \xi^d) \wedge \delta\Gamma^c_{bc}}_{L^a_6} - \underbrace{2R_d{}^b\ (\nabla^d \xi^a) \wedge \delta\Gamma^c_{bc}}_{B^a_4} \\
&- \underbrace{2R^a{}_e R_{bd}\ \delta g^{be} \wedge \xi^d}_{E^a_6} + \underbrace{2R^a{}_e\ \delta g^{be} \wedge \nabla_c \nabla_b \xi^c}_{B^a_{13}} - \underbrace{2R^a{}_e\ (\nabla^b \xi^e) \wedge \delta\Gamma^c_{bc}}_{L^a_6} - \underbrace{2R^a{}_e\ (\nabla^e \xi^b) \wedge \delta\Gamma^c_{bc}}_{B^a_6} \\
&- \underbrace{2g^{ad} R^e{}_c\ \delta R_{de} \wedge \xi^c}_{E^a_7} + \underbrace{2g^{ad}\ \delta R_{de} \wedge \nabla_c \nabla^e \xi^c}_{B^a_7} + \underbrace{2(\nabla_f R^{ab})\ \xi^f \wedge \delta\Gamma^c_{bc}}_{B^a_{19}} + \underbrace{2R^a{}_f\ (\nabla^b \xi^f) \wedge \delta\Gamma^c_{bc}}_{L^a_4} \\
&+ \underbrace{2R^b{}_f\ (\nabla^a \xi^f) \wedge \delta\Gamma^c_{bc}}_{L^a} - \underbrace{\nabla_b R\ \delta g^{ab} \wedge \nabla_d \xi^d}_{B^a_{33}} + \frac{1}{2}(\nabla_b R)[\underbrace{\nabla^a \xi^b}_{L^a_5} + \underbrace{\nabla^b \xi^a}_{B^a_3}] \wedge \delta \ln|g| - \underbrace{\nabla^a \delta R \wedge \nabla_d \xi^d}_{B^a_{27}} \\
&- \underbrace{\frac{1}{2}(\nabla^a \nabla_b R)\ \xi^b \wedge \delta \ln g}_{E^a_8} - \underbrace{\frac{1}{2}(\nabla_b R)\ (\nabla^a \xi^b) \wedge \delta \ln g}_{L^a_5} + (\nabla_c R^a{}_b)\ \delta \ln|g| \wedge [\underbrace{\nabla^c \xi^b}_{B^a_{10}} + \underbrace{\nabla^b \xi^c}_{B^a_5}] \\
&- \underbrace{2(\nabla_c R^a{}_b)\ (\nabla_d \xi^d) \wedge \delta g^{bc}}_{B^a_{29}} - \underbrace{(\nabla^a R_{cb})\ \delta \ln|g| \wedge \nabla^b \xi^c}_{B^a_{11}} + \underbrace{(\nabla^a R_{cb})\ \nabla_d \xi^d \wedge \delta g^{bc}}_{B^a_{30}} \\
&- \underbrace{2(\nabla_d a R_{cb})\ \delta g^{ab} \wedge \nabla^c \xi^b}_{B^a_{23}} + (\nabla_d R_{cb})\ \delta g^{cb} \wedge [\underbrace{(\nabla^a \xi^d)}_{L^a_1} + \underbrace{(\nabla^d \xi^a)}_{B^a_{28}}] - \underbrace{2g^{ab}\ \delta(\nabla_d R_{bc}) \wedge \nabla^b \xi^c}_{B^a_{25}} \\
&+ \underbrace{(\nabla_e R_{cb})\ (\nabla^a \xi^e) \wedge \delta g^{bc}}_{L^a_1} + \underbrace{(\nabla^a \nabla_e R_{bc})\ \xi^e \wedge \delta g^{bc}}_{E^a_9} + \underbrace{2(\nabla^a R_{ec})\ \nabla_b \xi^e \wedge \delta g^{bc}}_{B^a_{22}} + \underbrace{2R_e{}^a{}_{cf} R_b{}^f\ \xi^e \wedge \delta g^{bc}}_{E^a_{10}} \\
&+ 2(\nabla_c R_{db})\ \delta g^{ad} \wedge [\underbrace{(\nabla^b \xi^c)}_{B^a_{32}} + \underbrace{(\nabla^c \xi^b)}_{B^a_{21}}] + 2(\nabla_c R_{db})\ [\underbrace{(\nabla^d \xi^a)}_{B^a_{14}} + \underbrace{(\nabla^a \xi^d)}_{L^a_2}] \wedge \delta g^{bc} \\
&+ 2g^{ab}\ \delta(\nabla_c R_{db}) \wedge [\underbrace{(\nabla^b \xi^c)}_{} + \underbrace{(\nabla^c \xi^b)}_{}] - \underbrace{2(\nabla_e R_b{}^a)\ (\nabla_c \xi^e) \wedge \delta g^{bc}}_{} - \underbrace{2(\nabla_c \nabla_e R_b{}^a)\ \xi^e \wedge \delta g^{bc}}_{} \\
&- \underbrace{2(\nabla_c R_b{}^e)\ (\nabla^a \xi_e) \wedge \delta g^{bc}}_{} - \underbrace{2(\nabla_c R^{ae})\ (\nabla_b \xi_e) \wedge \delta g^{bc}}_{B^a_{24}} - \underbrace{2R_{ecbf} R^{af}\ \xi^e \wedge \delta g^{bc}}_{E^a_{12}} - \underbrace{2R_{ec}{}^a{}_f R_b{}^f\ \xi^e \wedge \delta g^{bc}}_{E^a_{13}} .
\end{aligned}
$$

We proceed as we did for the R^2 term. We will rewrite some terms to obtain an expression, which is the covariant derivative of an antisymmetric tensor.

$$B_1^a = \nabla_c[-R^{cb}\ \delta\ln|g| \wedge \nabla_b\xi^a] + \frac{1}{2}(\nabla^b R)\ \delta\ln|g| \wedge \nabla_b\xi^a + R^{cb}\ \nabla_c\delta\ln|g| \wedge \nabla_b\xi^a$$
$$= \nabla_c[-R^{cb}\ \delta\ln g \wedge \nabla_b\xi^a] + \underbrace{\frac{1}{2}(\nabla^b R)\ \delta\ln|g| \wedge \nabla_b\xi^a}_{B_3^a} + \underbrace{2R^{cb}\ \delta\Gamma^d_{cd} \wedge \nabla_b\xi^a}_{B_4^a},$$

$$B_2^a = \nabla_c[R^{ab}\ \delta\ln g \wedge \nabla_b\xi^c] - (\nabla_c R^{ab})\ \delta\ln|g| \wedge \nabla_b\xi^c - R^{ab}\ \nabla_c\delta\ln|g| \wedge \nabla_b\xi^c$$
$$= \nabla_c[R^{ab}\ \delta\ln g \wedge \nabla_b\xi^c] - \underbrace{(\nabla_c R^{ab})\ \delta\ln|g| \wedge \nabla_b\xi^c}_{B_5^a} - \underbrace{2R^{ab}\ \delta\Gamma^d_{cd} \wedge \nabla_b\xi^c}_{B_6^a}.$$

Then,

$$\sum_{i=1}^{6} B_i^a = \nabla_c Q_1^{ac} + 4R^{cb}\ \Gamma^d_{cd} \wedge \nabla_b\xi^a \quad \text{where} \quad Q_1^{ac} = 2R^{b[a}\ \delta\ln|g| \wedge \nabla_b\xi^{c]}.$$

For B_7^a and B_8^a we have

$$B_7^a = \nabla_c[2g^{ad}\ \delta R_{de} \wedge \nabla^e\xi^c] - 2g^{ad}\ \nabla_c\delta R_{de} \wedge \nabla^e\xi^c,$$

$$B_8^a = \nabla_c[-2g^{cb}\ \delta R_{be} \wedge \nabla^e\xi^a] + 2\ \nabla^b\delta R_{be} \wedge \nabla^e\xi^a,$$

which gives

$$B_7^a + B_8^a = \nabla_c Q_2^{ac} - 2g^{ad}\ \nabla_c\delta R_{de} \wedge \nabla^e\xi^c + 2\ \nabla^b\delta R_{be} \wedge \nabla^e\xi^a \quad \text{where} \quad Q_2^{ac} = 4g^{d[a}\ \delta R_{de} \wedge \nabla^e\xi^{c]}.$$

With B_9^a

$$B_9^a = 2g^{ad}\ \delta(\nabla_c R_{db}) \wedge \nabla^b\xi^c = 2g^{ad}\ \nabla_c\delta R_{db}\ \wedge \nabla^b\xi^c - 2g^{ad}[R_{eb}\ \delta\Gamma^e_{cd} + R_{de}\ \delta\Gamma^e_{bc}] \wedge \nabla^b\xi^c$$
$$= 2g^{ad}\ \nabla_c\delta R_{db} \wedge \nabla^b\xi^c - 2g^{ad}R_{eb}\ \delta\Gamma^e_{cd} \wedge \nabla^b\xi^c - 2R^a{}_e\ \delta\Gamma^e_{bc} \wedge \nabla^b\xi^c,$$

they yield

$$B_7^a + B_8^a + B_9^a = \nabla_c Q_2^{ac} + 2\ \nabla^b\delta R_{be} \wedge \nabla^e\xi^a - 2g^{ad}R_{eb}\ \delta\Gamma^e_{cd} \wedge \nabla^b\xi^c - 2R^a{}_e\ \delta\Gamma^e_{bc} \wedge \nabla^b\xi^c.$$

Adding B_{10}^a and B_{11}^a

$$B_{10}^a = \nabla_c[\nabla^c R^a{}_b\ \delta\ln|g| \wedge \xi^b] - \Box R^a{}_b\ \ln|g| \wedge \xi^b - 2\nabla^c R^a{}_b\ \delta\Gamma^d_{cd} \wedge \xi^b,$$

$$B_{11}^a = \nabla_c[-\nabla^a R_b{}^c\ \delta\ln|g| \wedge \xi^b] + \nabla_c\nabla^a R_b{}^c\ \delta\ln|g| \wedge \xi^b + 2\nabla^a R_b{}^c\ \Gamma^d_{cd} \wedge \xi^b,$$

gives

$$B_{10}^a + B_{11}^a = \nabla_c Q_3^{ac} - \Box R^a{}_b\ \ln|g| \wedge \xi^b - 2\nabla^c R^a{}_b\ \delta\Gamma^d_{cd} \wedge \xi^b + \nabla_c\nabla^a R_b{}^c\ \delta\ln|g| \wedge \xi^b + 2\nabla^a R_b{}^c\ \Gamma^d_{cd} \wedge \xi^b$$

where $Q_3^{ac} = 2\nabla^{[c}R^{a]}{}_b\ \delta \ln g \wedge \xi^b$. We write B_{12}^a as

$$B_{12}^a = \nabla_c[2R_d{}^b\ \delta g^{ad} \wedge \nabla_b\xi^c] - 2\nabla_c R_d{}^b\ \delta g^{ad} \wedge \nabla_b\xi^c - 2R_d{}^b\ \nabla_c\delta g^{ad} \wedge \nabla_b\xi^c$$
$$= \nabla_c Q_4^{ac} - 2\nabla_c R_d{}^b\ \delta g^{ad} \wedge \nabla_b\xi^c - 2R_d{}^b\ \nabla_c\delta g^{ad} \wedge \nabla_b\xi^c + 2\nabla_c R_d{}^b\ \delta g^{cd} \wedge \nabla_b\xi^a$$
$$+ 2R_d{}^b\ \nabla_c\delta g^{cd} \wedge \nabla_b\xi^a + 2R_d{}^b\ \delta g^{cd} \wedge \nabla_c\nabla_b\xi^a,$$

where $Q_4^{ac} = 4R_d{}^b\ \delta g^{d[a} \wedge \nabla_b\xi^{c]}$, and B_{13}^a as

$$B_{13}^a = \nabla_c[2R^a{}_e\ \delta g^{be} \wedge \nabla_b\xi^c] - 2\nabla_c R^a{}_e\ \delta g^{be} \wedge \nabla_b\xi^c - 2R^a{}_e\ \nabla_c\delta g^{be} \wedge \nabla_b\xi^c$$
$$= \nabla_c Q_5^{ac} - 2\nabla_c R^a{}_e\ \delta g^{be} \wedge \nabla_b\xi^c - 2R^a{}_e\ \nabla_c\delta g^{be} \wedge \nabla_b\xi^c + 2\nabla_c R^c{}_e\ \delta g^{be} \wedge \nabla_b\xi^a$$
$$+ 2R^c{}_e\ \nabla_c\delta g^{be} \wedge \nabla_b\xi^a + 2R^c{}_e\ \delta g^{be} \wedge \nabla_c\nabla_b\xi^a$$

where $Q_5^{ac} = 4R_e{}^{[a}\ \delta g^{jbej} \wedge \nabla_b\xi^{c]}$. Their sum is

$$B_{12}^a + B_{13}^a = \nabla_c[Q_4^{ac} + Q_5^{ac}] - 2\nabla_c R_d{}^b\ \delta g^{ad} \wedge \nabla_b\xi^c + 2R_d{}^b g^{ea}\ \delta\Gamma^d_{ec} \wedge \nabla_b\xi^c + \underbrace{2R^{eb}\ \delta\Gamma^a_{ec} \wedge \nabla_b\xi^c}_{B_{15}^a}$$
$$+ \underbrace{2\nabla_c R_d{}^b\ \delta g^{cd} \wedge \nabla_b\xi^a.}_{B_{14}^a}$$

Writing B_{17}^a as

$$B_{17}^a = -\nabla_c[2R^{bd}\ \xi^c \wedge \delta\Gamma^a_{bd}] + 2R^{bd}\ \nabla_c\xi^c \wedge \delta\Gamma^a_{bd} + 2R^{bd}\ \xi^c \wedge \nabla_c\delta\Gamma^a_{bd},$$

gives

$$B_{17}^a + B_{18}^a = \nabla_c Q_6^{ac} + 2R^{bd}\ \xi^c \wedge \nabla_c\delta\Gamma^a_{bd} - 2(\nabla_c R^{bd})\ \xi^a \wedge \delta\Gamma^c_{bd} - 2R^{bd}\ \nabla_c\xi^a \wedge \delta\Gamma^c_{bd} - 2R^{bd}\ \xi^a \wedge \nabla_c\delta\Gamma^c_{bd}$$

where $Q_6^{ac} = 4R^{bd}\ \xi^{[a} \wedge \delta\Gamma^{c]}_{bd}$. B_{19}^a and B_{20}^a gives

$$B_{19}^a = \nabla_c[2R^{ab}\ \xi^c \wedge \delta\Gamma^d_{bd}] - 2R^{ab}\ \nabla_c\xi^c \wedge \delta\Gamma^d_{bd} - 2R^{ab}\ \xi^c \wedge \nabla_c\delta\Gamma^d_{bd},$$

$$B_{19}^a + B_{20}^a = \nabla_c Q_7^{ac} - 2R^{ab}\ \xi^c \wedge \nabla_c\delta\Gamma^d_{bd} + (\nabla^b R)\ \xi^a \wedge \delta\Gamma^d_{bd} + 2R^{cb}\ \nabla_c\xi^a \wedge \delta\Gamma^d_{bd} + 2R^{cb}\ \xi^a \wedge \nabla_c\delta\Gamma^d_{bd},$$

where $Q_7^{ac} = 4R^{b[a}\ \xi^{c]} \wedge \Gamma^d_{bd}$. We also have

$$B_{21}^a = \nabla_c[2(\nabla^c R_{db})\ \delta g^{ad} \wedge \xi^b] - 2\Box R_{db}\ \delta g^{ad} \wedge \xi^b - 2(\nabla^c R_{db})\ \nabla_c\delta g^{ad} \wedge \xi^b,$$

$$B_{22}^a = \nabla_c[2(\nabla^a R_{eb})\ \xi^e \wedge \delta g^{bc}] - 2\nabla_c\nabla^a R_{eb}\ \xi^e \wedge \delta g^{bc} - 2(\nabla^a R_{eb})\ \xi^e \wedge \nabla_c\delta g^{bc},$$

$$B_{21}^a + B_{22}^a = \nabla_c Q_8^{ac} - 2\Box R_{db}\ \delta g^{ad} \wedge \xi^b + 2(\nabla^c R_{db})g^{ea}\ \delta\Gamma^d_{ec} \wedge \xi^b + 2(\nabla^c R^e{}_b)\ \delta\Gamma^a_{ec} \wedge \xi^b$$
$$- 2\nabla_c\nabla^a R_{eb}\ \xi^e \wedge \delta g^{bc} + 2(\nabla^a R_e{}^d)\ \xi^e \wedge \delta\Gamma^c_{cd} + 2(\nabla^a R^{eb})g^{cd}\ \xi^e \wedge \delta\Gamma^b_{cd}$$

where $Q_8^{ac} = 4(\nabla^{[c} R_{db})\ \delta g^{a]d} \wedge \xi^b$ and

$$B_{23}^a = \nabla_c[-2(\nabla_d R^c{}_b)\ \delta g^{ad} \wedge \xi^b] + 2(\nabla_c \nabla_d R^c{}_b)\ \delta g^{ad} \wedge \xi^b + 2(\nabla_d R^c{}_b)\ \nabla_c \delta g^{ad} \wedge \xi^b,$$

$$B_{24}^a = \nabla_c[2(\nabla_b R^{ae})\ \delta g^{cb} \wedge \xi_e] - 2(\nabla_c \nabla_d R^{ae})\ \delta g^{cb} \wedge \xi_e - 2(\nabla_b R^{ae}) \nabla_c \delta g^{cb} \wedge \xi_e,$$

$$\begin{aligned} B_{23}^a + B_{24}^a = {} & \nabla_c Q_9^{ac} + 2(\nabla_c \nabla_d R^c{}_b)\ \delta g^{ad} \wedge \xi^b - 2(\nabla_d R^c{}_b) g^{ea}\ \delta\Gamma^d_{ec} \wedge \xi^b - 2(\nabla^e R^c{}_b) g^{ea}\ \delta\Gamma^a_{ec} \wedge \xi^b \\ & - 2(\nabla_c \nabla_d R^{ae})\ \delta g^{cb} \wedge \xi_e + 2(\nabla^d R^{ae})\ \delta\Gamma^c_{cd} \wedge \xi_e + 2(\nabla_b R^{ae}) g^{cd}\ \Gamma^b_{cd} \wedge \xi_e, \end{aligned}$$

where $Q_9^{ac} = 4\nabla_b R^{e[a}\ \delta g^{c]b} \wedge \xi_e$. Two more contribution comes from

$$B_{25}^a = \nabla_c[-2g^{ad} g^{ce} \delta(\nabla_d R_{be} \wedge \xi^b)] + 2g^{ad} g^{ce} \nabla^c \delta(\nabla_d R_{bc} \wedge \xi^b),$$

$$B_{26}^a = \nabla_c[2g^{ae} g^{cd} \delta(\nabla_d R_{eb} \wedge \xi^b)] - 2g^{ad} \nabla^e \delta(\nabla_e R_{db} \wedge \xi^b),$$

$$B_{25}^a + B_{26}^a = \nabla_c Q_{10}^{ac} + 2g^{ad} \nabla^c \delta(\nabla_d R_{bc} \wedge \xi^b) - 2g^{ad} \nabla^e \delta(\nabla_e R_{db} \wedge \xi^b)$$

where $Q_{10}^{ac} = 4g^{d[a} g^{c]e}\ \delta(\nabla_e R_{db} \wedge \xi^b)$, and

$$\begin{aligned} B_{27}^a &= \nabla_c[-\nabla^a \delta R \wedge \xi^c] + \nabla_c \nabla^a \delta R \wedge \xi^c \\ &= \nabla_c Q_{11}^{ac} + \nabla_c \nabla^a \delta R \wedge \xi^c - \Box R \wedge \xi^a - \nabla^c \delta R \wedge \nabla_c \xi^a, \end{aligned}$$

where $Q_{11}^{ac} = 2\nabla^{[c} \delta R \wedge \xi^{a]}$. When B_{28}^a is added to it,

$$\begin{aligned} B_{27}^a + B_{28}^a &= \nabla_c Q_{11}^{ac} + \nabla_c \nabla^a \delta R \wedge \xi^c - \Box R \wedge \xi^a + 2R^e{}_d\ \delta\Gamma^d_{ec} \wedge \nabla^c \xi^a + R_b{}^e\ \delta\Gamma^b_{ec} \wedge \nabla^c \xi^a \\ & \quad - (\nabla^c R_{bd})\ \delta g^{bd} \wedge \nabla_c \xi^a - g^{bd}\ \nabla_c \delta R_{bd} \wedge \nabla_c \xi^a + (\nabla_d R_{cb})\ \delta g^{cb} \wedge \nabla^d \xi^a \\ &= \nabla_c Q_{11}^{ac} + \nabla_c \nabla^a \delta R \wedge \xi^c - \Box R \wedge \xi^a + 2R^e{}_d\ \delta\Gamma^d_{ec} \wedge \nabla^c \xi^a - g^{bd}\ \nabla_c \delta R_{bd}) \wedge \nabla_c \xi^a. \end{aligned}$$

Let us write $B_{29}^a + B_{30}^a$ in a different way which was derived earlier.

$$B_{29}^a + B_{30}^a = 2g^{ae}\ \nabla_d \xi^d \wedge \nabla^b \delta R_{be} - 2g^{bc} R^a{}_e\ \nabla_d \xi^d \wedge \delta\Gamma^e_{cb} - g^{bc}\ \nabla_d \xi^d \wedge \nabla^a \delta R_{bc}.$$

Using B_{33}^a and C^a

$$B_{33}^a = \nabla_c[-\nabla_b R\ \delta g^{ab} \wedge \xi^c] + \nabla_c \nabla_b R\ \delta g^{ab} \wedge \xi^c + \nabla_b R\ \nabla_c \delta g^{ab} \wedge \xi^c,$$

$$C^a = \nabla_c[\nabla_b R\ \delta g^{cb} \wedge \xi^a] - \nabla_c \nabla_b R\ \delta g^{cb} \wedge \xi^a - \nabla_b R\ \nabla_c \delta g^{cb} \wedge \xi^a,$$

gives

$$\begin{aligned} B_{33}^a + C^a = {} & \nabla_c Q_{12}^{ac} + \nabla_c \nabla_b R\ \delta g^{ab} \wedge \xi^c - \nabla^e R\ \delta\Gamma^a_{ec} \wedge \xi^c - \nabla^b R g^{ae}\ \delta\Gamma^b_{ec} \wedge \xi^c \\ & - \nabla_c \nabla_b R\ \delta g^{cb} \wedge \xi^a + \nabla^d R\ \delta\Gamma^c_{cd} \wedge \xi^a + \nabla^b R g^{cd}\ \delta\Gamma^b_{cd} \wedge \xi^a, \end{aligned}$$

where $Q^{ac}_{12} = 2\nabla_b R\ \delta g^{b[c} \wedge \xi^{a]}$. Hence,

$$\sum_{i=1}^{12} B^a_i = \sum_{i=1}^{b=12} \nabla_c Q^{ac}_i + \underbrace{4R^{cb}\ \Gamma^d_{cd} \wedge \nabla_b \xi^a}_{P^a_3} + \underbrace{2\ \nabla^b \delta R_{be} \wedge \nabla^e \xi^a}_{P^a_{13}} - \underbrace{2g^{ad} R_{eb}\ \delta\Gamma^e_{cd} \wedge \nabla^b \xi^c}_{P^a_4} - \underbrace{2R^a{}_e\ \delta\Gamma^e_{bc} \wedge \nabla^b \xi^c}_{P^a_5}$$

$$- \Box R^a{}_b\ \delta \ln g \wedge \xi^b - \underbrace{2\nabla^c R^a{}_b\ \delta\Gamma^d_{cd} \wedge \xi^b}_{P^a_6} + \nabla_c \nabla^a R_b{}^c\ \delta \ln g \wedge \xi^b + \underbrace{2\nabla^a R_b{}^c\ \Gamma^d_{cd} \wedge \xi^b}_{P^a_7}$$

$$- \underbrace{2\nabla_c R_d{}^b\ \delta g^{ad} \wedge \nabla_b \xi^c}_{P^a_2} + \underbrace{2R_d{}^b g^{ea}\ \delta\Gamma^d_{ec} \wedge \nabla_b \xi^c}_{P^a_4} - \underbrace{2R^{eb}\ \delta\Gamma^c_{ce} \wedge \nabla_b \xi^a}_{P^a_3} - \underbrace{2R_a{}^b g^{ce}\ \delta\Gamma^d_{ec} \wedge \nabla^b \xi^a}_{P^a_{15}}$$

$$- \underbrace{2\nabla_c R^a_e\ \delta g^{be} \wedge \nabla_b \xi^c}_{P^a_1} + \underbrace{2R^a{}_e g^{db}\ \delta\Gamma^e_{cd} \wedge \nabla_b \xi^c}_{P^a_5} + \underbrace{2R^{ad}\ \delta\Gamma^b_{cd} \wedge \nabla_b \xi^c}_{P^a_{17}} - \underbrace{2R^c{}_e\ \delta\Gamma^e_{cd} \wedge \nabla^d \xi^a}_{P^a_9}$$

$$- \underbrace{2R^{cd}\ \delta\Gamma^b_{cd} \wedge \nabla_b \xi^a}_{P^a_8} + \underbrace{2R_d{}^b\ \nabla^d \xi^c \wedge \delta\Gamma^a_{bc}}_{P^a_{16}} + 2R_e{}^c R_{cb}{}^a{}_d \delta g^{be} \wedge \xi^d 2R^{bd}\ \xi^c \wedge \nabla_c \delta\Gamma^a_{bd}$$

$$- 2(\nabla_c R^{bd})\ \xi^a \wedge \delta\Gamma^c_{bd} - \underbrace{2R^{bd}\ \nabla_c \xi^a \wedge \delta\Gamma^c_{bd}}_{P^a_8} - 2R^{bd}\ \xi^a \wedge \nabla_c \delta\Gamma^c_{bd} - 2R^{ab}\ \xi^c \wedge \nabla_c \delta\Gamma^d_{bd}$$

$$+ (\nabla^b R)\ \xi^a \wedge \delta\Gamma^d_{bd} + \underbrace{2R^{cb}\ \nabla_c \xi^a \wedge \delta\Gamma^d_{bd}}_{P^a_3} + 2R^{cb}\ \xi^a \wedge \nabla_c \delta\Gamma^d_{bd} - 2\Box R_{db}\ \delta g^{ad} \wedge \xi^b$$

$$+ 2(\nabla^c R_{db}) g^{ea}\ \delta\Gamma^d_{ec} \wedge \xi^b + 2(\nabla^c R^e{}_b)\ \delta\Gamma^a_{ec} \wedge \xi^b - 2\nabla_c \nabla^a R_{eb}\ \xi^e \wedge \delta g^{bc} + \underbrace{2(\nabla^a R_e{}^d)\ \xi^e \wedge \delta\Gamma^c_{cd}}_{P^a_7}$$

$$+ 2(\nabla^a R^{eb}) g^{cd}\ \xi^e \wedge \delta\Gamma^b_{cd} + 2(\nabla_c \nabla_d R^c{}_b)\ \delta g^{ad} \wedge \xi^b - 2(\nabla_d R^c{}_b) g^{ea}\ \delta\Gamma^d_{ec} \wedge \xi^b$$

$$- 2(\nabla^e R^c{}_b) g^{ea}\ \delta\Gamma^a_{ec} \wedge \xi^b - 2(\nabla_c \nabla_d R^{ae})\ \delta g^{cb} \wedge \xi_e + \underbrace{2(\nabla^d R^{ae})\ \delta\Gamma^c_{cd} \wedge \xi_e}_{P^a_6} + 2(\nabla_b R^{ae}) g^{cd}\ \Gamma^b_{cd} \wedge \xi_e$$

$$+ 2g^{ad} \nabla^c \delta(\nabla_d R_{bc} \wedge \xi^b) - 2g^{ad} \nabla^e \delta(\nabla_e R_{db} \wedge \xi^b) + \nabla_c \nabla^a \delta R \wedge \xi^c - \Box R \wedge \xi^a + \underbrace{2R^e{}_d\ \delta\Gamma^d_{ec} \wedge \nabla^c \xi^a}_{P^a_9}$$

$$- \underbrace{g^{bd}\ \nabla_c \delta R_{bd}) \wedge \nabla_c \xi^a}_{P^a_{10}} + \underbrace{2g^{ae}\ \nabla_d \xi^d \wedge \nabla^b \delta R_{be}}_{P^a_{12}} - \underbrace{2g^{bc} R^a{}_e\ \nabla_d \xi^d \wedge \delta\Gamma^e_{cb}}_{P^a_{14}} - \underbrace{g^{bc}\ \nabla_d \xi^d \wedge \nabla^a \delta R_{bc}}_{P^a_{11}}$$

$$+ \nabla_c \nabla_b R\ \delta g^{ab} \wedge \xi^c - \nabla^e R\ \delta\Gamma^a_{ec} \wedge \xi^c - \nabla^b R g^{ae}\ \delta\Gamma^b_{ec} \wedge \xi^c - \nabla_c \nabla_b R\ \delta g^{cb} \wedge \xi^a$$

$$+ \nabla^d R\ \delta\Gamma^c_{cd} \wedge \xi^a + \nabla^b R g^{cd}\ \delta\Gamma^b_{cd} \wedge \xi^a - \underbrace{2(\nabla_e R_b{}^a)\ (\nabla_c \xi^e) \wedge \delta g^{bc}}_{P^a_1} + \underbrace{2(\nabla_c R_{db})\ \delta g^{ad} \wedge \nabla^d \xi^c}_{P^a_2}.$$

where

$$P^a_{10} = \nabla_c[-g^{bd}\ \nabla^c \delta R_{bd} \wedge \xi^a] + g^{bd}\ \Box \delta R_{bd} \wedge \xi^a,$$

$$P^a_{11} = \nabla_c[g^{bd}\ \nabla^a \delta R_{bd} \wedge \xi^c] - g^{bd}\ \nabla_c \nabla^a \delta R_{bd} \wedge \xi^c,$$

$$P^a_{10} + P^a_{11} = \nabla_c Q^{ac}_{13} + g^{bd}\ \Box \delta R_{bd} \wedge \xi^a - g^{bd}\ \nabla_c \nabla^a \delta R_{bd} \wedge \xi^c \ \ where\ \ Q^{ac}_{13} = 2g^{bd}\ \nabla^{[a} \delta R_{bd} \wedge \xi^{c]},$$

$$P^a_{12} = \nabla_c[2g^{ae}\ \xi^c \wedge \nabla^b \delta R_{be}] - 2g^{ae}\ \xi^c \wedge \nabla_c \nabla^b \delta R_{be},$$

$$P^a_{13} = \nabla_c[2g^{ce}\ \nabla^b \delta R_{be} \wedge \xi^a] - 2g^{ce}\ \nabla_c \nabla^b \delta R_{be} \wedge \xi^a,$$

$$P^a_{12} + P^a_{13} = \nabla_c Q^{ac}_{14} - 2g^{ae}\ \xi^c \wedge \nabla_c \nabla^b \delta R_{be} - 2\ \nabla^e \nabla^b \delta R_{be} \wedge \xi^a \ \ where\ \ Q^{ac}_{14} = 4g^{e[a}\ \xi^{c]} \wedge \nabla^b \delta R_{be},$$

$$P^a_{14} = \nabla_c[-2g^{bd} R^a{}_e\ \xi^c \wedge \delta\Gamma^e_{db}] + 2g^{bd}(\nabla_c R^a{}_e)\ \xi^c \wedge \delta\Gamma^e_{db} + 2g^{bd} R^a{}_e\ \xi^c \wedge \nabla_c \delta\Gamma^e_{db},$$

$$P^a_{15} = \nabla_c[-2g^{be} R_d{}^c\ \delta\Gamma^d_{be} \wedge \xi^a] + 2g^{be}(\nabla_c R_d{}^c)\ \delta\Gamma^d_{be} \wedge \xi^a + 2g^{be} R_d{}^c\ \nabla_c \delta\Gamma^d_{be} \wedge \xi^a,$$

$$P^a_{14} + P^a_{15} = \nabla_c Q^{ac}_{15} + 2g^{bd}(\nabla_c R^a{}_e)\ \xi^c \wedge \delta\Gamma^e_{db} + 2g^{bd} R^a{}_e\ \xi^c \wedge \nabla_c \delta\Gamma^e_{db} + g^{be}(\nabla_c R_d{}^c)\ \delta\Gamma^d_{be} \wedge \xi^a + 2g^{be} R_d{}^c\ \nabla_c \delta\Gamma^d_{be} \wedge \xi^a$$

where $Q^{ac}_{15} = 4g^{be} R_d{}^{[c}\ \xi^{a]} \wedge \delta\Gamma^d_{be}$,

$$P^a_{16} = \nabla_c[2R^{cb}\ \xi^d \wedge \delta\Gamma^a_{bd}] - (\nabla^b R)\ \xi^d \wedge \delta\Gamma^a_{bc} - 2R^{cb}\ \xi^d \wedge \nabla_c \delta\Gamma^a_{bd},$$

$$P^a_{17} = \nabla_c[2R^{ad}\ \delta\Gamma^c_{bd}] - 2(\nabla_c R^{ad})\ \delta\Gamma^c_{bd} \wedge \xi^b - 2R^{ad}\ \nabla_c \delta\Gamma^c_{bd} \wedge \xi^b,$$

$$P^a_{16} + P^a_{17} = \nabla_c Q^{ac}_{16} - (\nabla^b R)\ \xi^d \wedge \delta\Gamma^a_{bc} - 2R^{cb}\ \xi^d \wedge \nabla_c \delta\Gamma^a_{bd} - 2(\nabla_c R^{ad})\ \delta\Gamma^c_{bd} \wedge \xi^b - 2R^{ad}\ \nabla_c \delta\Gamma^c_{bd} \wedge \xi^b$$

where $Q^{ac}_{16} = 4R^{d[a}\ \delta\Gamma^{c]}_{bd} \wedge \xi^b$. Then,

$$
\begin{aligned}
\sum_{i=1}^{b=16} B_i^a = {} & \sum_{i=1}^{b=16} \nabla_c Q_i^{ac} - \Box R^a{}_b\ \delta\ln g\wedge\xi^b + \nabla_c\nabla^a R_b{}^c\ \delta\ln g\wedge\xi^b + \underbrace{2R_e{}^c R_{cb}{}^a{}_d\delta g^{be}\wedge\xi^d}_{A^a} \\
& + \underbrace{2R^{bd}\ \xi^c\wedge\nabla_c\delta\Gamma^a_{bd}}_{K^a} - 2(\nabla_c R^{bd})\ \xi^a\wedge\delta\Gamma^c_{bd} - \underbrace{2R^{bd}\ \xi^a\wedge\nabla_c\delta\Gamma^c_{bd}}_{C^a} - \underbrace{2R^{ab}\ \xi^c\wedge\nabla_c\delta\Gamma^d_{bd}}_{D^a} \\
& + \underbrace{(\nabla^b R)\ \xi^a\wedge\delta\Gamma^d_{bd}}_{E^a} + \underbrace{2R^{cb}\ \xi^a\wedge\nabla_c\delta\Gamma^d_{bd}}_{C^a} - 2\Box R_{db}\ \delta g^{ad}\wedge\xi^b + 2(\nabla^c R_{db})g^{ea}\ \delta\Gamma^d_{ec}\wedge\xi^b \\
& + \underbrace{2(\nabla^c R^e{}_b)\ \delta\Gamma^a_{ec}\wedge\xi^b}_{G^a} - 2\nabla_c\nabla^a R_{eb}\ \xi^e\wedge\delta g^{bc} + 2(\nabla^a R_{eb})g^{cd}\ \xi^e\wedge\delta\Gamma^b_{cd} + 2(\nabla_c\nabla_d R^c{}_b)\ \delta g^{ad}\wedge\xi^b \\
& - 2(\nabla_d R^c{}_b)g^{ea}\ \delta\Gamma^d_{ec}\wedge\xi^b - \underbrace{2(\nabla^e R^c{}_b)g^{ea}\ \delta\Gamma^a_{ec}\wedge\xi^b}_{G^a} - 2(\nabla_c\nabla_d R^{ae})\ \delta g^{cb}\wedge\xi_e + 2(\nabla_b R^{ae})g^{cd}\ \Gamma^b_{cd}\wedge\xi_e \\
& + 2g^{ad}\nabla^c\delta(\nabla_d R_{bc}\wedge\xi^b) - 2g^{ad}\nabla^e\delta(\nabla_e R_{db}\wedge\xi^b) + \underbrace{\nabla_c\nabla^a\delta R\wedge\xi^c}_{J^a} - \underbrace{\Box\delta R\wedge\xi^a}_{I^a} \\
& + \nabla_c\nabla_b R\ \delta g^{ab}\wedge\xi^c - \underbrace{\nabla^e R\ \delta\Gamma^a_{ec}\wedge\xi^c}_{H^a} - \nabla^b R g^{ae}\ \delta\Gamma^b_{ec}\wedge\xi^c - \nabla_c\nabla_b R\ \delta g^{cb}\wedge\xi^a \\
& + \underbrace{\nabla^d R\ \delta\Gamma^c_{cd}\wedge\xi^a}_{E^a} + \underbrace{\nabla_b R g^{cd}\ \delta\Gamma^b_{cd}\wedge\xi^a}_{F^a} + g^{bd}\ \Box\delta R_{bd}\wedge\xi^a - g^{bd}\ \nabla_c\nabla^a\delta R_{bd}\wedge\xi^c \\
& - 2g^{ae}\ \xi^c\wedge\nabla_c\nabla^b\delta R_{be} - 2\ \nabla^e\nabla^b\delta R_{be}\wedge\xi^a \\
& + 2g^{bd}(\nabla_c R^a{}_e)\ \xi^c\wedge\delta\Gamma^e_{db} + \underbrace{2g^{bd}R^a{}_e\ \xi^c\wedge\nabla_c\delta\Gamma^e_{db}}_{M^a} + \underbrace{g^{be}(\nabla_c R_d{}^c)\ \delta\Gamma^d_{be}\wedge\xi^a}_{F^a} + 2g^{be}R_d{}^c\ \nabla_c\delta\Gamma^d_{be}\wedge\xi^a \\
& - (\nabla^b R)\ \xi^d\wedge\delta\Gamma^a_{bc} - \underbrace{2R^{cb}\ \xi^d\wedge\nabla_c\delta\Gamma^a_{bd}}_{H^a} - \underbrace{2(\nabla_c R^{ad})\ \delta\Gamma^c_{bd}\wedge\xi^b}_{L^a} - \underbrace{2R^{ad}\ \nabla_c\delta\Gamma^c_{bd}\wedge\xi^b}_{D^a} \\
& - R^{cb}R_{ec}{}^a{}_b\ \delta\ln g\wedge\xi^e - \underbrace{4R_d{}^b R_{ec}{}^a{}_b\ \delta g^{cd}\wedge\xi^e}_{B^a} - 2R_f{}^{dae}\ \delta R_{de}\wedge\xi^f - R^{ab}R_{eb}\ \ln g\wedge\xi^e \\
& - 2R_d{}^b R_{eb}\ \delta g^{ad}\wedge\xi^e - 2R^a{}_e R_{bd}\ \delta g^{be}\wedge\xi^d - 2g^{ad}R^e{}_c\ \delta R_{de}\wedge\xi^c - \frac{1}{2}(\nabla^a\nabla_b R)\ \xi^b\wedge\delta\ln g \\
& + (\nabla^a\nabla_e R_{bc})\ \xi^e\wedge\delta g^{bc} + \underbrace{2R_e{}^a{}_{cf}R_b{}^f\ \xi^e\wedge\delta g^{bc}}_{A^a} \\
& - 2(\nabla_c\nabla_e R_b{}^a)\ \xi^e\wedge\delta g^{bc} - 2R_{ecbf}R^{af}\ \xi^e\wedge\delta g^{bc} - \underbrace{2R_{ec}{}^a{}_f R_b{}^f\ \xi^e\wedge\delta g^{bc}}_{B^a}.
\end{aligned}
$$

Now, we need to use

$$-\Box\delta R\wedge\xi^a=\xi^a\wedge\Box\delta R=\xi^a\wedge\nabla_c\nabla^c\delta(g^{bd}R_{bd})\qquad;\nabla_c\delta g^{bd}=-g^{be}\delta\Gamma^a_{ce}-g^{ae}\delta\Gamma^b_{ce}$$

$$=\xi^a\wedge\nabla_c\nabla^c[\delta g^{bd}R_{bd}+g^{bd}\delta R_{bd}]$$

$$=\xi^a\wedge\nabla_c[(\nabla^c\delta g^{bd})R_{bd}+\delta g^{bd}\nabla_cR_{bd}+g^{bd}\nabla_c\delta R_{bd}]$$

$$=R_{bd}\ \xi^a\wedge\Box\delta g^{bd}+2(\nabla_cR_{bd})\ \xi^a\wedge\nabla^c\delta g^{bd}+\Box R_{bd}\ \xi^a\wedge\delta g^{bd}+g^{bd}\ \xi^a\wedge\Box\delta R_{bd}$$

$$=R_{bd}\ \xi^a\wedge\Box\delta g^{bd}-2(\nabla^cR^e{}_d)\ \xi^a\wedge\delta\Gamma^d_{ce}-2(\nabla^cR_b{}^e)\ \xi^a\wedge\delta\Gamma^b_{ce}+\Box R_{bd}\ \xi^a\wedge\delta g^{bd}+g^{bd}\ \xi^a\wedge\Box\delta R_{bd}$$

$$=R_{bd}\ \xi^a\wedge\Box\delta g^{bd}-4(\nabla^cR^e{}_d)\ \xi^a\wedge\delta\Gamma^d_{ce}+\Box R_{bd}\ \xi^a\wedge\delta g^{bd}+g^{bd}\ \xi^a\wedge\Box\delta R_{bd},$$

which gives

$$I^a=2R_b{}^e\ \nabla^c\delta\Gamma^b_{ce}\wedge\xi^a-4(\nabla^cR^e{}_d)\ \xi^a\wedge\delta\Gamma^d_{ce}+\Box R_{bd}\ \xi^a\wedge\delta g^{bd}+g^{bd}\ \xi^a\wedge\Box\delta R_{bd}.$$

The other terms are

$$K^a=R_{bd}\ \xi^c\wedge\nabla_c\nabla^a\delta g^{bd}-2R_b{}^d\ \xi^c\wedge\nabla_c\nabla_d\delta g^{ba},$$

$$L^a=-2R^{bc}\ \xi^d\wedge\nabla_c\delta\Gamma^a_{bd}$$

$$=-2R^{bc}\ \xi^d\wedge\nabla_c\delta\Gamma^a_{bd}+2R_{ed}g^{ac}\ \xi^b\wedge\nabla^d\delta\Gamma^e_{bc}-2R_{ed}g^{ac}\ \xi^b\wedge\nabla^d\delta\Gamma^e_{bc}$$

$$=2R_{ed}\ \xi_b\wedge\nabla^d\nabla^b\delta g^{ae}+2R_{ed}g^{ac}\ \xi^b\wedge\nabla^d\delta\Gamma^e_{bc}$$

$$=2R_{ed}\ \xi_b\wedge[\nabla^d,\nabla^b]\delta g^{ae}+2R_{ed}\ \xi_b\wedge\nabla^b\nabla^d\delta g^{ae}+2R_{ed}g^{ac}\ \xi^b\wedge\nabla^d\delta\Gamma^e_{bc}$$

$$=2R_{ed}R^{dba}{}_c\ \xi_b\wedge\delta g^{ce}+2R_{ed}R^{dbe}{}_c\ \xi_b\wedge\delta g^{ac}+2R_{ed}\ \xi_b\wedge\nabla^b\nabla^d\delta g^{ae}+2R_{ed}g^{ac}\ \xi^b\wedge\nabla^d\delta\Gamma^e_{bc},$$

$$J^a=\nabla_c\nabla^a\delta R\wedge\xi^c=\nabla_c\nabla^a[\delta(g^{bd}R_bd)]\wedge\xi^c=\nabla_c\nabla^a[\delta g^{bd}R_bd+g^{bd}\delta R_{bd}]\wedge\xi^c$$

$$=\nabla_c[(\nabla^a\delta g^{bd})R_bd+\delta g^{bd}\nabla^aR_{bd}+g^{bd}\nabla^a\delta R_{bd}]\wedge\xi^c$$

$$=\nabla_c[(\nabla^a\delta g^{bd})R_bd+\delta g^{bd}\nabla^aR_{bd}+g^{bd}\nabla^a\delta R_{bd}]\wedge\xi^c$$

$$=\Big[(\nabla_c\nabla^a\delta g^{bd})R_{bd}+\nabla^a\delta g^{bd}(\nabla_cR_{bd})+(\nabla_c\delta g^{bd})(\nabla^aR_{bd})+\delta g^{bd}(\nabla_c\nabla^aR_{bd})+g^{bd}(\nabla_c\nabla^a\delta R_{bd})\Big]\wedge\xi^c$$

$$=(\nabla_c\nabla^a\delta g^{bd})R_{bd}\wedge\xi^c-2(\nabla_fR^e{}_d)g^{ac}\ \delta\Gamma^d_{ce}\wedge\xi^f-2(\nabla^aR^e{}_d)\ \delta\Gamma^d_{ce}\wedge\xi^c+(\nabla_c\nabla^aR_{bd})\ \delta g^{bd}\wedge\xi^c$$

$$+\,g^{bd}\ (\nabla_c\nabla^a\delta R_{bd})\wedge\xi^c.$$

With the following identities

$$g^{bd}R^a{}_e\ \xi^c\wedge\nabla_c\delta\Gamma^e_{db}=-2R^a{}_e\ \xi^c\wedge\nabla_c\nabla_d\delta g^{de}-R^{ae}\ \xi^c\wedge\nabla_c\nabla_e\delta\ln|g|$$

$$=2R^{ae}\ \xi^c\wedge\nabla_c\nabla^d\delta g_{de}-R^{ae}\ \xi^c\wedge\nabla_c\nabla_e\delta\ln|g|$$

$$=2R^{ae}\ \xi^c\wedge[\nabla_c,\nabla^d]\delta g_{de}+2R^{ae}\ \xi^c\wedge\nabla^d\nabla_c\delta g_{de}-R^{ae}\ \xi^c\wedge\nabla_c\nabla_e\delta\ln|g|,$$

$$2R_e{}^a\ \xi^b\wedge\nabla^d\delta\Gamma^e_{bd}=-R_e{}^a\ \xi^c\wedge\nabla_d\nabla_c\delta g^{de}-R_e{}^a\ \xi^c\wedge\Box\delta g^{ce}+R_e{}^a\ \xi_c\wedge\nabla_d\nabla^e\delta g^{cd}$$

$$=R^{ea}\ \xi^c\wedge\nabla^d\nabla_c\delta g_{ed}+R^{ea}\ \xi^c\wedge\Box\delta g_{ce}-R^{ea}\ \xi^c\wedge\nabla^d\nabla_e\delta g_{cd},$$

We obtain,

$$
\begin{aligned}
&2R^{ae}\ \xi^c \wedge [\nabla_c, \nabla^d]\delta g_{de} + R^{ae}\ \xi^c \wedge \nabla^d\nabla_c \delta g_{de} + R^{ea}\ \xi^c \wedge \nabla^d\nabla_e \delta g_{cd} - R^{ea}\ \xi^c \wedge \Box\delta g_{ce} \\
&- R^{ae}\ \xi^c \wedge \nabla_c\nabla_e \delta \ln g \\
&= 2R^{ae}R_c{}^d{}_d{}^b\ \xi^c \wedge \delta g_{be} + 2R^{ae}R_c{}^d{}_e{}^b\ \xi^c \wedge \delta g_{db} \\
&+ R^{ae}\ \xi^c \wedge \left\{\nabla^d\nabla_c \delta g_{de} + \nabla^d\nabla_e \delta g_{cd} - \Box\delta g_{ce} - \nabla_c\nabla_e \delta \ln g\right\} \\
&= -2R^{ae}R_c{}^b\ \xi^c \wedge \delta g_{be} + 2R^{ae}R_c{}^d{}_e{}^b\ \xi^c \wedge \delta g_{db} + 2R^{ae}\ \xi^c \wedge \delta R_{ce},
\end{aligned}
$$

which gives

$$
\begin{aligned}
M^a &= -2R^{ae}R_c{}^b\ \xi^c \wedge \delta g_{be} + 2R^{ae}R_c{}^d{}_e{}^b\ \xi^c \wedge \delta g_{db} + 2R^{ae}\ \xi^c \wedge \delta R_{ce} + 2R_e{}^a\ \xi^b \wedge \nabla^d \delta\Gamma^e_{bd} \\
&= 2R^a{}_e R_{cb}\ \xi^c \wedge \delta g^{be} - 2R^{ae}R_{cdeb}\ \xi^c \wedge \delta g^{db} + 2R^{ae}\ \xi^c \wedge \delta R_{ce} + 2R_e{}^a\ \xi^b \wedge \nabla^d \delta\Gamma^e_{bd}.
\end{aligned}
$$

As a result, we can write

$$
\begin{aligned}
\sum_{i=1}^{b=16} = \sum_{i=1}^{b=16} \nabla_c Q_i^{ac} &- \underbrace{\Box R^a{}_b\ \delta \ln g \wedge \xi^b}_{L_8^a} + \underbrace{\nabla_c \nabla^a R_b{}^c\ \delta \ln g \wedge \xi^b}_{L_9^a} - \underbrace{2(\nabla_c R^{bd})\ \xi^a \wedge \delta\Gamma^c_{bd}}_{L_{10}^a} \\
&- \underbrace{2\Box R_{db}\ \delta g^{ad} \wedge \xi^b}_{L_{11}^a} + \underbrace{2(\nabla^c R_{db}) g^{ea}\ \delta\Gamma^d_{ec} \wedge \xi^b}_{L_{12}^a} - \underbrace{2\nabla_c \nabla^a R_{eb}\ \xi^e \wedge \delta g^{bc}}_{L_{13}^a} + \underbrace{2(\nabla^a R_{eb}) g^{cd}\ \xi^e \wedge \delta\Gamma^b_{cd}}_{L_{14}^a} \\
&+ \underbrace{2(\nabla_c \nabla_d R^c{}_b)\ \delta g^{ad} \wedge \xi^b}_{L_{15}^a} - \underbrace{2(\nabla_d R^c{}_b) g^{ea}\ \delta\Gamma^d_{ec} \wedge \xi^b}_{L_{16}^a} - \underbrace{2(\nabla_c \nabla_d R^{ae})\ \delta g^{cb} \wedge \xi_e}_{L_{17}^a} \\
&+ \underbrace{2(\nabla_b R^{ae}) g^{cd}\ \Gamma^b_{cd} \wedge \xi_e}_{L_{18}^a} + \underbrace{2g^{ad} \nabla^c \delta(\nabla_d R_{bc} \wedge \xi^b)}_{L_{19}^a} - \underbrace{2g^{ad} \nabla^e \delta(\nabla_e R_{db} \wedge \xi^b)}_{L_{20}^a} + \underbrace{\nabla_c \nabla_b R\ \delta g^{ab} \wedge \xi^c}_{L_{21}^a} \\
&- \underbrace{\nabla^b R g^{ae}\ \delta\Gamma^b_{ec} \wedge \xi^c}_{L_{22}^a} - \underbrace{\nabla_c \nabla_b R\ \delta g^{cb} \wedge \xi^a}_{L_{23}^a} + \underbrace{g^{bd}\ \Box \delta R_{bd} \wedge \xi^a}_{L_3^a} - \underbrace{g^{bd}\ \nabla_c \nabla^a \delta R_{bd} \wedge \xi^c}_{L_4^a} \\
&- \underbrace{2g^{ae}\ \xi^c \wedge \nabla_c \nabla^b \delta R_{be}}_{L_{24}^a} - \underbrace{2\ \nabla^e \nabla^b \delta R_{be} \wedge \xi^a}_{L_{25}^a} + \underbrace{2g^{bd}(\nabla_c R^a{}_e)\ \xi^c \wedge \delta\Gamma^e_{db}}_{L_{26}^a} + \underbrace{2g^{be} R_d{}^c\ \nabla_c \delta\Gamma^d_{be} \wedge \xi^a}_{L_{27}^a} \\
&- \underbrace{2(\nabla_c R^{ad})\ \delta\Gamma^c_{bd} \wedge \xi^b}_{L_{28}^a} - \underbrace{R^{cb} R_{ec}{}^a{}_b\ \delta \ln g \wedge \xi^e}_{L_{29}^a} - \underbrace{2R_f{}^{dae}\ \delta R_{de} \wedge \xi^f}_{L_{30}^a} - \underbrace{R^{ab} R_{eb}\ \ln g \wedge \xi^e}_{L_{31}^a} \\
&- \underbrace{2R_d{}^b R_{eb}\ \delta g^{ad} \wedge \xi^e}_{L_{45}^a} - \underbrace{2R^a{}_e R_{bd}\ \delta g^{be} \wedge \xi^d}_{L_6^a} - \underbrace{2g^{ad} R^e{}_c\ \delta R_{de} \wedge \xi^c}_{L_{32}^a} - \underbrace{\frac{1}{2}(\nabla^a \nabla_b R)\ \xi^b \wedge \delta \ln g}_{L_{33}^a} \\
&+ \underbrace{(\nabla^a \nabla_e R_{bc})\ \xi^e \wedge \delta g^{bc}}_{(extra-1)^a} - \underbrace{2(\nabla_c \nabla_e R_b{}^a)\ \xi^e \wedge \delta g^{bc}}_{L_{34}^a} - \underbrace{2R_{ecbf} R^{af}\ \xi^e \wedge \delta g^{bc}}_{L_7^a} \\
&+ \underbrace{2R_d{}^b R_{ec}{}^a{}_b\ \xi^e \wedge \delta g^{cd}}_{extra-3} - \underbrace{2R^{bd}\ \xi^a \delta R_{bd}}_{L_{35}^a} + \underbrace{2R^{ab}\ \xi^c \wedge \delta R_{bc}}_{L_5^a} + \underbrace{2(\nabla_b R) g^{cd}\ \delta\Gamma^b_{cd} \wedge \xi^a}_{L_{36}^a} \\
&+ \underbrace{2R_b{}^e\ \nabla^c \delta\Gamma^b_{ce} \wedge \xi^a}_{L_{37}^a} - \underbrace{4(\nabla^c R^e{}_d)\ \xi^a \wedge \delta\Gamma^d_{ce}}_{L_{38}^a} + \underbrace{\Box R_{bd}\ \xi^a \wedge \delta g^{bd}}_{L_{39}^a} + \underbrace{g^{bd}\ \xi^a \wedge \Box \delta R_{bd}}_{L_3^a} + \underbrace{R_{bd}\ (\nabla_c \nabla^a \delta g^{bd}) \wedge \xi^c}_{L_1^a} \\
&- \underbrace{2(\nabla_f R^e{}_d) g^{ac}\ \delta\Gamma^d_{ce} \wedge \xi^f}_{L_{40}^a} - \underbrace{2(\nabla^a R^e{}_d)\ \delta\Gamma^d_{ce} \wedge \xi^c}_{L_{41}^a} + \underbrace{(\nabla_c \nabla^a R_{bd})\ \delta g^{bd} \wedge \xi^c}_{(extra-2)^a} + \underbrace{g^{bd}\ (\nabla_c \nabla^a \delta R_{bd}) \wedge \xi^c}_{L_4^a} \\
&+ \underbrace{R_{bd}\ \xi^c \wedge \nabla_c \nabla^a \delta g^{bd}}_{L_1^a} - \underbrace{2R_b{}^d\ \xi^c \wedge \nabla_c \nabla_d \delta g^{ba}}_{L_2^a} + \underbrace{2R_{ed} R^{dba}{}_c\ \xi_b \wedge \delta g^{ce}}_{(extra-4)^a} + \underbrace{2R_{ed} R^{dbe}{}_c\ \xi_b \wedge \delta g^{ac}}_{L_{42}^a} \\
&+ \underbrace{2R_{ed}\ \xi_b \wedge \nabla^b \nabla^d \delta g^{ae}}_{L_2^a} + \underbrace{2R_{ed} g^{ac}\ \xi^b \wedge \nabla^d \delta\Gamma^e_{bc}}_{L_{43}^a} + \underbrace{2R^a{}_e R_{cb}\ \xi^c \wedge \delta g^{be}}_{L_6^a} - \underbrace{2R^{ae} R_{cdeb}\ \xi^c \wedge \delta g^{bd}}_{L_7^a} \\
&+ \underbrace{2R^{ae}\ \xi^c \wedge \delta R_{ce}}_{L_5^a} + \underbrace{2R_e{}^a\ \xi^b \wedge \nabla^d \delta\Gamma^e_{bd}}_{L_{44}^a},
\end{aligned}
$$

with

$$L_5^a = 4R^{ab}\ \xi^c \wedge \delta R_{bc},$$

$$L_6^a = 4R^a{}_e R_{cb}\ \xi^c \wedge \delta g^{be},$$

$$L_7^a = [-2R^{ae}R_{cdeb} - 2R_{cdbe}R^{ae}]\ \xi^c \wedge \delta g^{bd} = 0,$$

$$\begin{aligned}(extra-1)^a + (extra-2)^a &= [\nabla^a, \nabla_e]R_{bc}\ \xi^e \wedge \delta g^{bc} \\ &= R^a{}_{ebd}R^d{}_c\ \xi^e \wedge \delta g^{bc} + R^a{}_{ecd}R_b{}^d\ \xi^e \wedge \delta g^{bc} \\ &= 2R^a{}_{ebd}R^d{}_c\ \xi^e \wedge \delta g^{bc} \\ &= 2R^a{}_{ebc}R^c{}_d\ \xi^e \wedge \delta g^{bd},\end{aligned}$$

$$(extra-3)^a = 2R_d{}^b R_{ec}{}^a{}_b R^c{}_d\ \xi^e \wedge \delta g^{cd} = 2R_d{}^c R_{eb}{}^a{}_c R^c{}_d\ \xi^e \wedge \delta g^{bd},$$

$$(extra-4)^a = 2R_{ed}R_{ce}{}^a{}_c\ \xi_b \wedge \delta g^{ce} = 2R_{bc}R^{cea}{}_d\ \xi_e \wedge \delta g^{db} = 2R_{dc}R^{cea}{}_b\ \xi_e \wedge \delta g^{db},$$

$$\begin{aligned}\sum_{i=1}^{4}(extra-i)^a &= 2R_{dc}\ \xi^e \wedge \delta g^{bd}[R^a{}_{eb}{}^c + R_{eb}{}^{ac} + R^c{}_e{}^a{}_b] \\ &= 2R_{dc}\ \xi^e \wedge \delta g^{bd}[R^a{}_{eb}{}^c + R^{ac}{}_{eb} + R^a{}_b{}^c{}_e] \\ &= 0.\end{aligned}$$

Therefore, extra part vanishes.

$$\begin{aligned}Q_\beta^{ac} = {} & 2R^{b[a}\ \delta \ln g \wedge \nabla_b \xi^{c]} + 4g^{d[a}\ \delta R_{de} \wedge \nabla^e \xi^{c]} + 2\nabla^{[c}R^{a]}{}_b\ \delta \ln g \wedge \xi^b + 4R_d{}^b\ \delta g^{d[a} \wedge \nabla_b \xi^{c]} \\ & + 4R_e{}^a\ \delta g^{jbej} \wedge \nabla_b \xi^{c]} + 4R^{bd}\ \xi^{[a} \wedge \delta\Gamma^{c]}_{bd} + 4R^{b[a}\ \xi^{c]} \wedge \delta\Gamma^d_{bd} + 4(\nabla^{[c}R_{db})\ g^{a]d} \wedge \xi^b \\ & + 4\nabla_b R^{e[a}\ \delta^{c]b} \wedge \xi_e + 4g^{d[a}g^{c]e}\ \delta(\nabla_e R_{db}) \wedge \xi^b + 2\nabla^{[c}\ \delta R \wedge \xi^{a]} + 2\nabla_b R\ \delta g^{b[c} \wedge \xi^{a]} \\ & + 2g^{bd}\nabla^{[a}\ \delta R_{bd} \wedge \xi^{c]} + 4g^{e[a}\ \xi^{c]} \wedge \nabla^b \delta R_{be} + 4g^{be}R_d{}^{[c}\ \xi^{a]} \wedge \Gamma^d_{be} + 4R^{d[a}\ \delta\Gamma^{c]}_{bd} \wedge \xi^b.\end{aligned}$$