A MULTILEVEL STRUCTURAL MODEL OF MATHEMATICAL THINKING IN DERIVATIVE CONCEPT

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JANUARY 2012

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A THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES
OF
MIDDLE EAST TECHNICAL UNIVERSITY

BY

UTKUN ÖZDİL

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY
IN
SECONDARY SCIENCE AND MATHEMATICS EDUCATION

JANUARY 2012

Approval of the thesis:

A MULTILEVEL STRUCTURAL MODEL OF MATHEMATICAL THINKING IN DERIVATIVE CONCEPT

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ABSTRACT

A MULTILEVEL STRUCTURAL MODEL OF MATHEMATICAL THINKING IN DERIVATIVE CONCEPT

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Ph. D., Department of Secondary Science and Mathematics Education
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January 2012, 307 pages

The purpose of the study was threefold: (1) to determine the factor structure of mathematical thinking at the within-classroom and at the between-classroom level; (2) to investigate the extent of variation in the relationships among different mathematical thinking constructs at the within- and between-classroom levels; and (3) to examine the cross-level interactions among different types of mathematical thinking. Previous research was extended by investigating the factor structure of mathematical thinking in derivative at the within- and between-classroom levels, and further examining the direct, indirect, and cross-level relations among different types of mathematical thinking. Multilevel analyses of a cross-sectional dataset containing two independent samples of undergraduate students nested within classrooms showed that the within-structure of mathematical thinking includes enactive, iconic, algorithmic, algebraic, formal, and axiomatic thinking, whereas the between-structure contains formal-axiomatic, proceptual-symbolic, and conceptual-embodied thinking. Major findings from the two-level mathematical thinking model revealed that: (1) enactive, iconic, algebraic, and axiomatic thinking varied primarily as a function of formal and algorithmic thinking; (2) the strongest direct effect of formal-axiomatic thinking was on proceptual-symbolic thinking; (3) the nature of the relationships was cyclic at the between-classroom level; (4) the within-classroom mathematical thinking constructs significantly moderate the relationships among conceptual-embodied, proceptual-symbolic, and formal-axiomatic thinking; and (5) the between-classroom mathematical thinking constructs moderate the relationships among enactive, iconic, algorithmic, algebraic, formal, and axiomatic thinking. The challenges when using multilevel exploratory factor analysis, multilevel confirmatory factor analysis, and multilevel structural equation modeling with categorical variables are emphasized. Methodological and educational implications of findings are discussed.

Keywords: Multilevel Exploratory Factor Analysis, Multilevel Confirmatory Factor Analysis, Multilevel Structural Equation Modeling, Mathematical Thinking, Derivative

ÖZ

TÜREV KAVRAMINDAKİ MATEMATİKSEL DÜŞÜNMENİN ÇOK AŞAMALI YAPISAL MODELİ

Özdil, Utkun
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Ocak 2012, 307 sayfa

Bu çalışmanın üç amacı vardır: (1) matematiksel düşünmenin sınıf-içi ve sınıflar-arası aşamada faktör yapısını belirlemek; (2) farklı matematiksel düşünme tipleri arasındaki ilişkilerin sınıf-içi ve sınıflar-arası aşamalardaki değişimini araştırmak; ve (3) farklı matematiksel düşünme tipleri arasındaki karşı-aşama ilişkilerini incelemek. Önceki araştırmalar türev kavramında matematiksel düşünmenin, sınıf-içi ve sınıflar-arası faktör yapısı araştırılarak ve farklı matematiksel düşünme tipleri arasındaki direkt, indirekt, ve karşı-aşama ilişkileri incelenerek genişletilmiştir. Birbirinden bağımsız iki örneklemde sınıflar içine geçmiş lisans öğrencilerini içeren kesitsel veri setinin çok aşamalı analizleri matematiksel düşünmenin sınıf-içi aşamada eylemsel, görüntüsel, algoritmik, cebirsel, biçimsel, ve belitsel düşünme tiplerini içerdiğini göstermekte iken sınıflar-arası aşamada ise biçimsel-belitsel, yöntemsel-sembolik, ve kavramsal-şekilsel düşünme tiplerini kapsadığını göstermiştir. İki-aşamalı matematiksel düşünme modelinin ana bulguları:

(1) eylemsel, görüntüsel, algoritmik, cebirsel, biçimsel, ve belitsel düşünme temelde biçimsel ve algoritmik düşünmenin işlevi ile değişimektedir; (2) biçimsel-belitsel düşünmenin en güçlü direkt etkisi yöntemsel-sembolik düşünme üzerindedir; (3) sınıflar-arası aşamada matematiksel düşünme ilişkileri döngüsel bir yapıya sahiptir; (4) sınıf-içi düşünme yapıları biçimsel-belitsel, yöntemsel-sembolik, ve kavramsal-şekilsel düşünme tipleri arasındaki ilişkilere aracılık etmektedir; ve (5) sınıflar-arası düşünme yapıları eylemsel, görüntüsel, algoritmik, cebirsel, biçimsel, ve belitsel düşünme tipleri arasındaki ilişkilere aracılık etmektedir. Kategorik değişkenlerle çok aşamalı açımlayıcı faktör analizi, çok aşamalı doğrulayıcı faktör analizi, ve çok aşamalı yapısal denklem modelleme kullanımında karşılaşılabilecek sorunlar belirtilmiştir. Bulguların yöntembilimsel ve eğitimsel uygulamaları tartışılmıştır.

Anahtar Kelimeler: Çok Aşamalı Açımlayıcı Faktör Analizi, Çok Aşamalı Doğrulayıcı Faktör Analizi, Çok Aşamalı Yapısal Denklem Modelleme, Matematiksel Düşünme, Türev

To My Grandmother…

Fadime KIVRAK

ACKNOWLEDGEMENTS

I would like to express my gratitude to my supervisor Prof. Dr. Behiye UBUZ for her valuable support and gentle guidance. Her expertise and sincere encouragement provided me support throughout this study.

I would like to thank my committee members Assoc. Prof. Dr. Esen UZUNTİRYAKİ, Assoc. Prof. Dr. Nihat BOZ, Assist. Prof. Dr. Yeşim ÇAPA AYDIN, and Assist. Prof. Dr. Çiğdem HASER for their feedback and insight.

Thanks are owed to Prof. Dr. Giray BERBEROĞLU and Prof. Dr. Ertan İBİKLİ; their encouraging efforts, constructive criticism and suggestions during this research. I am grateful to the technical support of Prof. Dr. Linda Muthén, Prof. Dr. Bengt Muthén, and Prof. Dr. Levent Dümenci through out my practices with the Mplus software. I am very much thankful to Prof. Dr. Hamza ŞAHİN, Prof. Dr. Mustafa SAFRAN, Assoc. Prof. Dr. Yezdan BOZ, Instructor Dr. Muhiddin UĞUZ, Assoc. Prof. Dr. Naim UZUN, Assist. Prof. Dr. Burçin ACAR ŞEŞEN, Resch. Asst. Bahattin ŞAHİN who helped me at the very beginning of the development of the instrument and during its administration. For the main study I am grateful to Prof. Dr. Gül Asiye AÇIK, Prof. Dr. Mustafa DİLEK, Assoc. Prof. Dr. Yeşim ZORAL, Assoc. Prof. Dr. İzzet GÖRGEN, Assoc. Prof. Dr. Sabahattin DENİZ, Assoc. Prof. Dr. Mustafa GÜLSU, Assoc. Prof. Dr. Sertap KAVASOĞLU, Assist. Prof. Dr. Niyazi ŞAHİN, Assist. Prof. Dr. Ayten KİRİŞ, Assist. Prof. Dr. Pınar Seda SARIER ÇETİN, Assist. Prof. Dr. Hülya OLMUŞ, Assist. Prof. Dr. Gülcan ÇETİN, and Resch. Asst. Elçin GÖKMEN.

I am indebted to my family Mukadder, Ömer, Tuğba and Orkun AYDIN; and Esin and Cihan ÖZDİL who were my source of strength and inspiration in every step of the way. There are no words to describe my appreciation and gratitude for their support and close lovingly encouragement.

I would like to extend thanks to my husband Sinan ÖZDİL, for his continuous encouragement, great morale support, motivation, and confidence. I am especially appreciative of the patience he put into this long and difficult process.

And last but not least I am grateful to my friends Ebru, Gökay, and Duru UYAR, Duygu and Erkan VURAL, Ebru KAYA, Fulya KULA, Beril YILMAZ SENEM, Hanife Can ŞEN, Mesture and Mirkan ALTAY, Ayşegül ERYILMAZ ÇEVİRGEN, Gökçe, Gürdal, and Çağla GÖKERİ, İlham ÜÇOK, Koray KUTLU, Sena and Hakan ERDOĞAN, and Özlem and Serhan MOĞOL. Thank you for sharing your love, time and interest. Your enormous morale support contributed greatly throughout the stress and difficulties.

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ENACTHK: Enactive Thinking

ICONTHK: Iconic Thinking

ALGOTHK: Algorithmic Thinking

ALGETHK: Algebraic Thinking

FORMTHK: Formal Thinking

AXIOTHK: Axiomatic Thinking

CONCPTHK: Conceptual-Embodied Thinking

PROCPTHK: Proceptual-Symbolic Thinking

FORMAXTHK: Formal-Axiomatic Thinking

TDT: Thinking-in-Derivative Test

SEM: Structural Equation Modeling

CFA: Confirmatory Factor Analysis

MLM: Multilevel Modeling

MEFA: Multilevel Exploratory Factor Analysis

MCFA: Multilevel Confirmatory Factor Analysis

MSEM: Multilevel Structural Equation Modeling

ITEM1-ITEM30: Items of Thinking-in-Derivative Test

CHAPTER 1

INTRODUCTION

University mathematics posits an ambitious set of outcome goals for student learning in calculus. These outcome goals all point to the importance of students' developing sophisticated and interconnected understandings of calculus concepts, procedures, and principles, not simply an ability to state definitions, memorize formulas, and apply algorithms. Increased emphasis is being placed not only on students' capacity to understand the substance of calculus but also on their capacity to "think in calculus". In recent years, mathematics educators have convincingly argued that full understanding of calculus consists of more than knowledge of mathematical definitions, symbols, facts, and procedures (e.g., Kaput, 1994; Robert & Speer, 2001; Tall, 1994). Complete understanding, they argue, includes the ability to employ the processes of mathematical thinking, in essence: framing definitions, examining symbols, solving problems, making conjectures, abstracting theorems, and so on. Students should not view calculus as a bounded system of advanced concepts and procedures to be absorbed but, rather, as a dynamic process of reasoning logically (Dubinsky, 1994; Tall, 1992) to endorse more valid inferences.

Mathematical thinking in general and that of calculus in particular are matters of great of importance for contemporary education at the university level. Over the years, mathematics educators have established studies in order to incorporate mathematical thinking into undergraduate instruction with the clear intention of encouraging both formal and informal vision of thinking in the field (e.g., Epp, 1994; Raman, 2002; Tall, 2008). This intense process of mathematical thinking has helped researchers to recognize the need to make clear definitions in the specific field of calculus, even though there is little aggreement on a common definition for "mathematical thinking". The notion of mathematical thinking is being recognized as an increasingly complex phonemenon compared with the simplicity that characterized our conceptions in past decades. Formerly, our conceptions of mathematical thinking consisted primarily of the product of abstract mathematical thought. Thus, at the most general level, characteristics of mathematical thinking that were put forth by the information-processing theory were the ability to apprehend information, induce relationships, and apply those relationships (Spearman, 1923). On the part of cognitive psychology, however, the abilities relevant to mathematical thinking posit three elements: analytical thought, practical thought, and creativity (Sternberg, 1988). Information-processing theory largely focused on the amount of conscious mental manipulation required by mathematical tasks, whereas cognitive psychology attended to what mental processes and representations are relevant to mathematical thinking. For any theory, both mathematical thinking and its counterpart, abstract thought, are critically important for extending the quality of students' sequential reasoning.

In the same vein, the concerns about the term collectively characterized it as a certain kind of abstraction in collegiate level mathematics (Edwards, Dubinsky, & McDonald, 2005; Tall, 1991). Studies by Fischbein (1983; 1993) raised our awareness that mathematical thinking involves much more than abstract mathematical thought rather it encompasses the formal, algorithmic, and intuitive processes. Hughes-Hallett's (1991) notion of mathematical representations made explicit what concerned many mathematics education researchers, namely, that mathematical thinking grounds on the relationships among symbolic, numeric, graphical, and verbal representations of mathematics. Tall's (2004) transformation of Fischbein and Hughes-Hallett's perspectives on mathematical thinking put a cognitively guided mathematical interface on their more general notions as Tall focused on thinking through three worlds of mathematics that we recognize today as influencing our conception of mathematical thinking interacting within a developmental conceptual-embodied, proceptual-symbolic, and formal-axiomatic worlds of mathematics. Selden and Selden (2005) concluded that mathematical thinking draws its meaning from a synthesis of processes including formalizing, algorithmatizing, visualizing, generalizing, and proving.

A considerable body of research documented that students possess limited grasp in mathematical thinking while defining the derivative concept or describing differentiation symbols [Formal Thinking] (Habre & Abboud, 2006; Orton, 1983; Zandieh, 1997; Zandieh & Knapp, 2006); proving differentiation theorems [Axiomatic Thinking] (Davis, 1993; Dreyfus, 1999; Epp, 2003; Selden & Selden, 1995; Tall, 1989; Viholainen, 2007); solving routine differentiation problems [Algorithmic Thinking] (Ali & Tall, 1996; Borgen & Manu, 2000; Kendal & Stacey, 2000, 2003; Tsamir, Raslan, & Dreyfus, 2006); modeling real-life phenomena by differentiation symbolism [Enactive Thinking] (Klymchuk, Zverkova, Gruenwald, & Sauerbier, 2010; Malaspina & Font, 2010; Maull & Berry, 2000; Ubuz & Ersoy, 1997; Villegas, Castro, & Gutiérrez, 2009; White & Mitchelmore, 1996; Yoon, Dreyfus, & Thomas, 2010); interpreting graphs [Iconic Thinking] (Asiala, Cottrill, Dubinsky, & Schwingendorf, 1997; Berry & Nyman, 2003; Baker, Cooley, & Triugeoros, 2000; Haciomeroglu, Aspinwall, & Presmeg, 2010; Hahkiöniemi, 2004; Ubuz, 2001; 2007); and unfolding a network of theoretical hypotheses that can be applied to differentiation algorithms [Algebraic Thinking] (Hahkiöniemi, 2006; Meel, 1998; Selden, Selden, Hauk, & Mason, 1999).

Although the existent literature has mirrored a general relational pattern, whether the pattern exists for specific types of thinking has not been thoroughly explored. Currently, there is no study that directly addresses the relationships among different types of mathematical thinking. However, research that concentrates on mathematical thinking has provided with two sets of studies for relational approach to thinking: (a) studies that include tasks triggering relationships among two types of thinking (Clark et al., 1997; Engelbrecht et al., 2005; Habre & Abboud, 2006; Martin, 2000; Selden et al., 1999; Ubuz & Ersoy, 1997); and (b) studies that involve tasks prompting relationships among a combination of three (e.g., Infante, 2007; Muhundan, 2005) or four (e.g., Bingolbali & Monaghan, 2008; Viholainen, 2008) types of thinking. Taken together, studies highlight a number of difficulties that spring from students' weak mathematical thinking processes while solving derivative tasks. These difficulties illustrate the drawbacks of overreliance on a particular type of mathematical thinking in the absence of another thinking type, thus led us to assume that critical relationships do exist among formal, axiomatic, algorithmic, algebraic, iconic, and enactive thinking.

To document students' difficulties with solving derivative tasks, researchers attempted to take students' written solutions as an approximate snapshot of the construction of and the relationships among different types of mathematical thinking, frequently using task-based interviews (e.g., Habre & Abboud, 2006; Hahkiöniemi, 2006; Klymchuk, Zverkova, Gruenwald, & Sauerbier, 2010; Maull & Berry, 2000; Orton, 1983) and to a lesser extent quantitative methods, particularly experimental designs (e.g., Meel, 1998; Ubuz & Ersoy, 1997; Ubuz, 2007). Empirical findings provide a sizable body of evidence that there are unique and joint effects of mathematical thinking constructs on one another.

Viewed together, previous research has contributed valuably to the literature on mathematical thinking in derivative. Yet, research on types of mathematical thinking, as well as the possible linkages among them, (a) is inferred from empirical findings that provide substantial evidence in favor of the positive unilateral interrelations among different types of mathematical thinking, (b) is informed by insights from qualitative data, (c) has failed to provide a specific structural model that describes the detailed relationships among six types of mathematical thinking. And most importantly, previous research has failed to incorporate a multilevel perspective, thus almost completely neglected the critical differentiation between student-level relationships and classroom-level relationships. Many of the afore-cited studies fit within a single level framework of analysis, in which researchers present an explicit lens for designating the relationships among different types of mathematical thinking at the within-classroom level. However, these studies do not inform us about the extent to which classrooms differ with respect to the relationships among different types of mathematical thinking. Finding relationships only at the student level does not, allow one to reflect on the following questions: (a) Are there reliable differences in mathematical thinking across classrooms? and (b) Do most classrooms have a similar mix of students with a complete mastery of mathematical thinking? Furthermore, the inferences based on a single level of analysis cannot highlight the stability of relationships among different types of mathematical thinking at within- and between-classroom levels and further cannot explain to what extent these types influence one another, directly or indirectly at both levels of analysis. Since students perform as members of a classroom rather than as individual students (O'Connell & McCoach, 2008), ignoring the influences from any level of the hierarchical structure may cause statistical problems and bias in the interpretation of results (de Leeuw & Meijer, 2008). In this vein, a multilevel approach is required whenever the relationships among different types of mathematical thinking are examined (Raudenbush & Bryk, 2002).

The present study takes these foundations of mathematical thinking and extends on this phenemenon by interpenetrating the term within the blend of six distinct but interrelated types as: (a) enactive thinking, (b) iconic thinking, (c) algorithmic thinking, (d) algebraic thinking, (e) formal thinking, and (f) axiomatic thinking.

An important background development in the process of mathematical thinking has been the emergence of perspectives in the calculus reform movements. These perspectives employed Schoenfeld's (1992) account of "learning to think mathematically" that is characterized through the development of a mathematical point of view. Researchers asserted that this mathematical point of view grounds on refined ideas, to some extent, relative to its both lower-and-higher order types and is dependent, at least in part, on the translations between these types. It is surprising to note, however, that few studies have empirically examined undergraduate students' enactive, iconic, algorithmic, algebraic, formal, and axiomatic thinking (Stewart & Thomas, 2007; Tall, 2008) and the interrelations among them (Christou, Pitta-Pantazi, Souyoul, & Zachariades, 2005). Although, there is a growing number of scholars question students' enactive, iconic, algorithmic, algebraic, formal, and axiomatic thinking in concepts such as functions, limits, and/or continuity (Dorier, 1995; Leron, Hazzan, & Zazkis, 1995; Lima & Tall, 2008; Przenioslo, 2004, 2007; Selden & Selden, 1995; Semadeni, 2008; Tall, 2005, 2008) it is only recently that these efforts were employed to investigate the concept of derivative (Hahkiöniemi, 2006; Ubuz, 2001, 2007; Viholainen, 2005, 2008; Zandieh, 2000). This is troubling in the light of studies reflecting difficulties that students encounter with the meaning of derivative, proofs of differentiation theorems, graph sketches of the derivative, graphical interpretations of the derivative, and optimization problems.

Clearly, grounded on functions and fueled by the limits, the concept of derivative capitilizes on mathematical thinking as a multifaceted concept (Zandieh, 2000) to incorporate different types of mathematical thinking that were previously reserved largely for studies with secondary students (e.g., Johanning, 2004; Lesh & Doerr, 2002; Lima & Tall, 2008; Mitchelmore & White, 2000; Stacey & McGregor, 2000; Stylianides, 2007) and to a lesser extent studies with undergraduate students (e.g., Alcock & Simpson, 2004; Ali & Tall, 1996; Ubuz & Ersoy, 1997). From a theoretical perspective, studies with undergraduate students portray that the classroom environment captures the array of students as a whole and reflects the general cognitive body in students' mathematical thinking. Accordingly, the mathematical thinking nestled in the classroom environment emerges from the relationships among individual students' mathematical thinking. It is therefore necessary to delineate a global factor structure at the between-classroom level – a composite of students' mathematical thinking at the within-classroom level –. That is, individual students' enactive, iconic, algorithmic, algebraic, formal, and axiomatic thinking may cover the broad spectrum of classroom's mathematical thinking in terms of conceptual-embodied, proceptual-symbolic, and formal-axiomatic thinking (Christou, Pitta-Pantazi, Souyoul, & Zacharides, 2005; Tall, 2004). Given that overview, three compilations emerge. First, conceptual-embodied thinking covers enactive and iconic thinking as it is a blend of visuo-spatial processes. Second, proceptual-symbolic thinking sets on algorithmic and algebraic thinking as it is an amalgam of application and manipulation processes. Third, formal-axiomatic thinking unravels formal and axiomatic thinking as it is a composite of descriptive and expository processes. Viewed together, it is revealing to consider that the structure of mathematical thinking at the between-classroom level casts a compact vision of three factors.

It is also important to note that, in one of the very few published articles taking a single-level approach to mathematical thinking, Christou et al. (2005) examined two single-level models for specifying the nature of the developmental trend in freshmen university students' mathematical thinking of the function concept. The authors used students' written responses to open-ended tasks in the context of function concept reflecting the conceptual-embodied, proceptual-symbolic, and formal-axiomatic thinking. Two main research questions were addressed. First, do different tasks in the context of functions can be categorized as conceptual-embodied, proceptual-symbolic, or formal-axiomatic thinking? Second, what are the relationships among conceptual-embodied, proceptual-symbolic, or formal-axiomatic thinking? Christou et al. (2005) found that each of the tasks employed in the study can represent three distinct types of thinking. Furthermore, as predicted by the authors, there were statistically significant and meaningful effects of different thinking types on each other. The results reaffirmed the developmental trend (Watson, Spirou, & Tall, 2003) as the effect of proceptual-symbolic thinking on formal-axiomatic thinking was largely mediated by conceptual-embodied thinking.

While no specific model of mathematical thinking has emerged, there are two lines of literature that pertain to the hypothesized two-level models of mathematical thinking in derivative. These include (a) findings from research portraying students' difficulties that may allude to a number of linkages among different types of mathematical thinking, and (b) findings from research providing a set of structured patterns associated with a model of task performance in different types of mathematical thinking.

On the basis of this empirical evidence, investigating the relationships among undergraduate students' different types of mathematical thinking at the within-classroom and between-classroom level simultaneously is a noteworthy issue for further inquiry. A basic premise of this research is that mathematics education research focusing on students' mathematical thinking can complement and inform innovative efforts initiated by calculus sensitive to different types of mathematical thinking in the concept of derivative. Broadly speaking, this study nestles a traditional research approach that seeks to investigate students' difficulties with the concept of derivative (e.g, Hirst, 1972; Orton, 1983; Thurston, 1972, 1994) and it reflects the increased recognition of the importance of mathematical thinking in the learning of derivative (e.g, Tall, 1991; Viholainen, 2008) within remarkable connections to the relationships at two levels. The present study intends to serve as a backdrop for multilevel research on mathematical thinking that seeks for the factor structure of mathematical thinking at the within- and between-classroom levels, widening the perspective by the exploration of within- and between-classroom relationships of undergraduate students' mathematical thinking in the derivative concept. Thus this research would illuminate mathematics education researchers to take into account the multilevel structure of mathematical thinking in their attempts to investigate students' progress in several subjects (e.g., numerical analysis, linear algebra) as well as various concepts (e.g., integral, vectors). Using MSEM as a method, researchers can obtain greater insight into multilevel and mediational mechanisms when the within- and between-classroom relationships are simultaneously estimated. In particular, this study is one of the few attempts for employing MSEM techniques with categorical data (e.g., Grilli & Rampichini, 2007). Thus, the analysis steps taken in the present study would guide researchers who are interested in using multilevel approaches with categorical variables. Accordingly, the multilevel models estimated in the present study would shed light into the teaching perspectives of mathematics instructors by giving prompt to the relationships among different types of mathematical thinking at the within- and between-classroom levels.

1.1 PURPOSE OF THE STUDY

The purpose of the study was threefold: (a) to determine the factor structure of mathematical thinking at the within-classroom and at the between-classroom level; (b) to investigate the extent of variation in the relationships among different types of mathematical thinking at the within-classroom and at the between-classroom level; and (c) to examine the cross-level interactions among different types of mathematical thinking. More specifically, it was hypothesized that the application of multilevel exploratory and confirmatory factor analyses would yield a distinct latent factor structure at the within- and between-classroom levels (Hypothesis 1). In regard to this distinction and preliminary evidence from the pattern of results in mathematical thinking research it was hypothesized that there would be statistically significant and positive relationships among different types of mathematical thinking at the within- and between-classroom levels. Accordingly, it was expected to find statistically significant variation in and across classrooms (Hypothesis 2). In addition, it was hypothesized that there would be statistically significant cross-level interactions among different types of mathematical thinking (Hypothesis 3). The main research questions were (a) What is the factor structure of mathematical thinking at the within- and between-classroom levels?; (b) What are the relationships among different types of mathematical thinking at the within- and between-classroom levels?; and (c) What are the cross-level interactions among different types of mathematical thinking?

Students' scores on the researcher-developed Thinking-in-Derivative Test were used to run multilevel exploratory factor analysis, multilevel confirmatory factor analysis, and multilevel structural equation modeling analysis. The items on the test were grouped to identify the latent variables at the within- and between-classroom levels. The latent variables included at the within-classroom level were conceptualized as: enactive thinking, iconic thinking, algorithmic thinking, algebraic thinking, formal thinking, and axiomatic thinking. The latent variables introduced at the between-classroom level were conceptual-embodied thinking, proceptual-symbolic thinking, and formal-axiomatic thinking. Two-level models were tested for the hypotheses given above. The main two-level structural model that was hypothesized to frame the multilevel relationships among different types of mathematical thinking is presented in Figure 1.1.

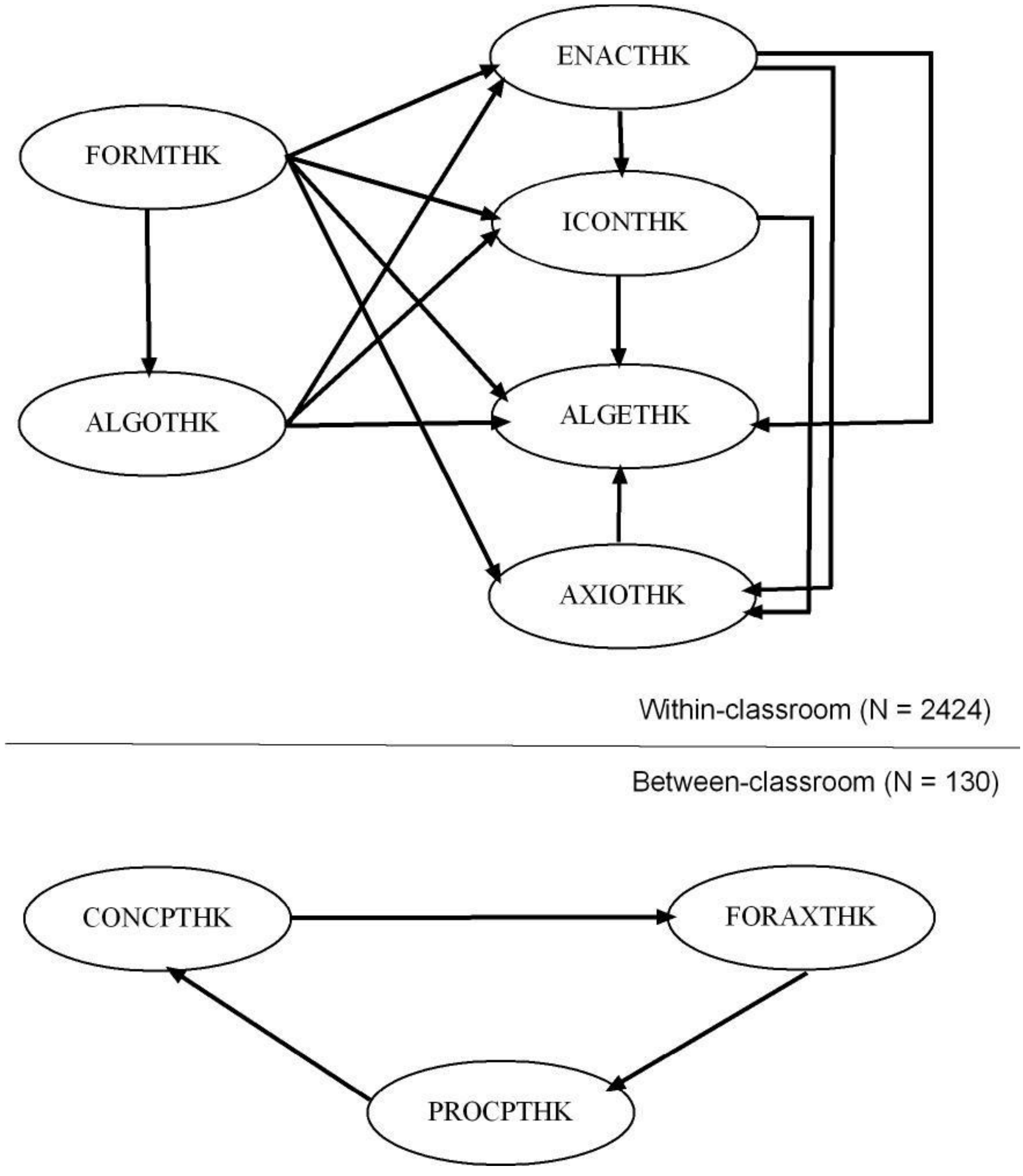

Figure 1.1 The Hypothesized Multilevel Model of Mathematical Thinking in Derivative

1.2 DEFINITION OF TERMS

The definition of the latent variables included in the hypothesized two-level structural models is given below:

1. Enactive Thinking (ENACTHK)

Enactive thinking employs students to think about the mathematical situations around them in the physical world as an inclusive conception of the modeling of real-life phenomena by mathematical symbolism. Thus, it not only encompasses the mental perceptions of real-world objects but also internal conceptions of visual and spatial imagery (Tall, 2004). The set of enactive views includes knowledge of the real-world applicability of mathematical ideas and interpretations of mathematics to make sense of situations.

2. Iconic Thinking (ICONTHK)

Iconic thinking unravels mathematical visualization as the use of physical senses and actions. It includes processes that students employ to make sense on graphical representations in a way that accounts for how visualization of mathematics occurs and what visual mathematics may be learned through embodiments of physical objects (Bruner, 1966). Henceforth, effective progress in iconic thinking requires making graph interpretations or graph constructions, thus developing visual manipulations within relevant images.

3. Algorithmic Thinking (ALGOTHK)

Algorithmic thinking is conveyed to the use of routine mathematical procedures and techniques. It nestles dynamics (Sirotic & Zazkis, 2007) that is concerned about the application of rules, procedures, and algorithms. It can be defined as processes in which algorithms emerge as a product of students' own thinking (Rasmussen, Zandieh, King, & Teppo, 2005). Effective progress in algorithmic thinking requires identifying the situation to which procedure applies, the correct order of algorithms, the correct completion of steps, and finally recognizing the correctly completed procedure.

4. Algebraic Thinking (ALGETHK)

Algebraic thinking involves the interplay between the understanding of a theoretical syntax within relevant symbols and the unpacking of theoretical situations into algorithms. It includes processes that unravel symbolic language from the emerging syntax of theorems and meaning of algorithms. It refers to the use of any of a variety of representations that handle quantitative situations in a relational way (Bednarz, Kieran, & Lee, 1996). Driscoll (1999) defined the term as "the capacity to represent quantitative situations so that relations among variables become apparent" (p. 1). Derry, Wilsman, and Hackbarth (2007) intensively emphasized that effective progress in algebraic thinking posits the interplay among the understanding of a theoretical syntax within relevant symbols, representing theoretical situations within procedures, and analyzing theoretical structures within changes in various algorithms.

5. Formal Thinking (FORMTHK)

Formal thinking grounds on mathematical definitions, symbols, and facts directed towards the recognition of mathematical terminology. Thus, it draws on the symbolic language of mathematics as well as the logic of its exposition. Based on definitions, axioms, and symbols, this form of thinking gives prompt to an active reasoning process (Fischbein, 1983). Effective progress in formal thinking requires stating concept definitions and recalling relevant symbols, facts and theorems.

6. Axiomatic Thinking (AXIOTHK)

Axiomatic thinking nestles proofs and proving, which are the bearers of verification, justification, and refutation. In essence, it is accepted to be at the core of mathematical thinking by its very abstract nature (Stylianides & Stylianides, 2008). Effective progress in axiomatic thinking requires the justification of empirical arguments as methods for validating theoretical generalizations.

7. Conceptual-embodied Thinking (CONCPTHK)

Conceptual-embodied thinking forms the basis for all mathematical activities that begin with students' perceptions and actions on the real-life phenomena (Watson & Tall, 2002). Effective progress on conceptual-embodied thinking involves reflection on the visual aspects of mathematical objects together with the embodiment of the real-life phenomena.

8. Proceptual-symbolic Thinking (PROCPTHK)

Proceptual-symbolic thinking is grounded on both mathematical calculations and symbolic manipulations (Gray & Tall, 2001; Tall & Thomas, 1991). The joint engagements in effective proceptual-symbolic thinking result in using automatized techniques based on symbolism that resonate dually as both algorithmic processes and algebraic manipulations.

9. Formal-axiomatic Thinking (FORMAXTHK)

Formal-axiomatic thinking is based on formal definitions and proof (Tall, 1989, 1994). Effective progress in formal-axiomatic thinking provides supportive structures for the fundamental definitions, symbols, facts. Additionally, it mediates processes through an array of a logical synthesis in proving theorems.

1.3 THE HYPOTHESIZED TWO-LEVEL MODEL OF MATHEMATICAL THINKING

On the basis of the theory of mathematical thinking and multilevel structural equation modeling the hypothesized Mathematical Thinking Model (see Figure 1.1) was developed. The set of related hypotheses predicting the two-level structural relations was basically split along two lines. The first line of hypotheses is derived from the review of related literature. The common threads of this line included (a) findings from research portraying students' difficulties that may allude to a number of linkages among different types of mathematical thinking, and (b) findings from research providing a set of structured patterns associated with a model of task performance in different types of mathematical thinking. Although this first line of hypotheses has yielded significant findings, with each of the previously mentioned threads considered a distinct facet of the two-level relationships among different types of mathematical thinking, another important line of hypotheses was derived from the review of methodological issues related to multilevel structural equation modeling. As with the first line of review, this line of inquiry has added to my understanding of the latent factor structure of mathematical thinking at the within-classroom level and at the between-classroom level. Because both lines of hypotheses are concerned with theoretical and methodological issues, though varying in terms of the focus of the research, it seems reasonable to assume that theoretical and methodological research that examines student- and classroom-related relations lead to a more complete understanding of two-level relationships among different types of mathematical thinking.

1.4 SIGNIFICANCE OF THE STUDY

Over the past decade or so, concerns in various fields with methodological issues in conducting research with hierarchical (clustered or nested) data have led to the development of multilevel modeling techniques. The multilevel theory specifies whether the variables belong to the within- or between-group level and which direct, indirect, and/or total effects as well as cross-level interaction effects should be expected. Apart from direct, indirect, and total effects, cross-level interaction effects between the individual level (e.g., student) and group level (e.g., classroom) which require the specification of processes within individuals that ground these individuals to be differentially influenced by the certain aspects of the within- and between-group effects can be put forth in multilevel models (Hox, 2002).

Despite the existence of hierarchical data structures in social sciences, previous research addressed either one of the Structural Equation Modeling (SEM) or the Multilevel Modeling (MLM) methods (Raudenbush & Byrk, 2002). Generally, SEM methods permit researchers to build and test models including both endogenous and exogenous latent variables simultaneously (the measurement model and, the structural model). MLM methods, on the other hand, allow for the variance attributable to the between-group level (group level) to be portioned from the variance associated with the within-group level (individual level), permitting the estimation of more accurate standard errors and more reliable information about between-and within-group effects (Raudenbush & Byrk, 2002). However, application of either methodology alone to the hierarchical data would produce several analytical difficulties and misspecifications about the complex relations that exist within and between groups. In a way, MLM represents a blind spot on the fact that variables are themselves related directly or indirectly to desired outcomes disregarding endogenous outcomes may be simultaneously related to each other (Kaplan & Elliot, 1997). Use of SEM alone would ignore the clustered sampling that is often used to design educational data and would produce biased results in the estimation of structural regression coefficients (Muthén, 1989a, 1989b). To resolve these difficulties, attempts have been made to integrate MLM with SEM for studying complex sample data. More recently, multilevel structural equation modeling (MSEM) has become a vigorous line of methodological research. Similar to the applications of the hierarchical linear model to regression in the context of the multilevel model, MSEM is a direct generalization of SEM in the context of the multilevel model (Cheung & Au, 2005; Raudenbush & Sampson, 1999) which allows the specification of separate structural models with direct, indirect, total effects as well as cross-level interaction effects within and between groups (Heck, 2001).

Applications of MSEM to educational research are still rare (see, e.g., Heck & Thomas, 2008). Thus, the present study would be an attractive approach in that it allows the incorporation of a substantive theory about the relations among mathematical thinking constructs within a nested structure (e.g., students nested within classrooms) by giving access to the investigation of the direct, indirect, total, and cross-level effects among different types of mathematical thinking.

The mathematical thinking constructs were estimated using the indicators via SEM (by use of confirmatory factor analysis). This analysis in SEM was then expanded in the MSEM (by use of multilevel exploratory and confirmatory factor analyses) to determine the latent factor structure of mathematical thinking at the within-classroom level and at the between-classroom level. While no specific model of mathematical thinking has emerged, the present study would serve as a critical first step in designating the factor structure of mathematical thinking in general, and that in part mathematical thinking in derivative.

The relationship among students' conceptual-embodied thinking, proceptual-symbolic thinking, and formal-axiomatic thinking has long been acknowledged as a critical factor in calculus success (Tall, 2004). The interrelation among different mathematical thinking types was particularly investigated in the domains of functions (Christou et al., 2005; Tall, 2004, 2005, 2008), eigenvalues and eigenvectors (Lapp, Nyman, & Berry, 2010; Stewart, 2008; Stewart & Thomas, 2009), linear dependency/independency (Ertekin, Solak, & Yazici, 2010; Stewart & Thomas, 2007, 2009), limits (Pinto & Tall, 2002; Tall, 2001; Weber & Alcock, 2004), and derivative (Hahkiöniemi, 2006) but derivative is, perhaps, one of the concepts that has attracted the least research attention in the framework for mathematical thinking based on the theory of mathematical thinking thus far. In addressing the relationship, the present study would illuminate the interrelations among different types of mathematical thinking specific to the derivative which is one of the fundamental concepts of calculus.

Researchers assessed the conceptual-embodied thinking through tasks that involve modeling real-life applications (enactive thinking) and visualizing applications in the context of graph constructions/interpretations (iconic thinking).

The proceptual-symbolic thinking was assessed through tasks that involve procedural computations (algorithmic thinking) and interpretations of generalized expressions and functional relationships (algebraic thinking). Finally, the formal-axiomatic thinking was assessed through tasks that involve "what" and "which" type of questions in the context of primary level of definitions and symbols (formal thinking) and/or that involve "prove that" or "show that" questions in the context of abstract level of if-then statements (axiomatic thinking). These three mathematical thinking constructs, however, were not classified as enactive, iconic, algorithmic, algebraic, formal, and axiomatic thinking rather generally simplified as embodied, symbolic, and formal thinking. Henceforth, the present study would provide a distinct facet of mathematical thinking by differentiating among these types at the within- and between-classroom levels.

In regard to the measures of performance, the stream of research on mathematical thinking put forth researchers' tendency to rely on self-developed open-ended questions while few of them was found to utilize multiple-choice questions in examining students' mathematical thinking. And those that are available were restricted to matching or true/false questions (Engelbrecht, Harding, & Potgieter, 2005; Goerdt, 2007; Stewart, 2008). Accordingly, the multiple-choice test developed in the present study would provide a different educational measurement perspective in addressing the relationships among different types of mathematical thinking.

Student adoption of mathematical thinking and the use of mathematical thinking engagement are primary factors when trying to understand and predict student achievement in calculus. The major contribution of the present study is that its findings support the relationships among different types of mathematical thinking in the range of factor structure known to delineate these relationships at the within- and between-classroom levels. The multilevel analyses shed considerable light on both the multilevel factor structure of mathematical thinking and the multilevel relationships among different types of mathematical thinking. Put succinctly, these two-level models demonstrate the importance of mathematical thinking in higher education. Evidence from no less an important multiple-choice measure as the Thinking-in-Derivative Test provided strong support for the claim that, classrooms, and the differences between them, matter. The two-level modeling efforts in the current study made clear that more simply, students and classrooms matter when promoting the relationships among different types of mathematical thinking.

CHAPTER 2

REVIEW OF RELATED LITERATURE

This chapter involves the review of related literature concerning the derivative concept, mathematical thinking contexts in the derivative concept, and the interrelations among these contexts.

2.1 DERIVATIVE

The teaching and learning of calculus has long been acknowledged as a critical theme in much research and is still considered to be an extremely important component of mathematics education from upper secondary school to university. Numerous studies have examined students' understanding of calculus in secondary (e.g., Hahkiöniemi, 2006; Lima & Tall, 2008; Presmeg & Balderas-Canas, 2001; Radford, 2000) and undergraduate level (e.g., Moore, 1994; Viholainen, 2008a, 2008b; Ubuz & Ersoy, 1997; Ubuz, 2007). Recent reviews of this literature suggest that a more substantial progress has been made in investigating undergraduate students' understandings of and difficulties with calculus (Bingolbali & Monaghan, 2008; Carlson, Oerhtman, & Engelke, 2010; Crouch & Haines, 2004; Lima & Tall, 2008; Martin & Harel, 1989; Newman-Ford, Lloyd, & Thomas, 2009; Orton, 1983; Raman, 2002; Selden & Selden, 1995; Ubuz & Kırkpınar, 2000; Ubuz, 2007, 2011). Researchers studied students' understanding of particular calculus concepts such as functions (Even, 1998; Leinhardt, Zaslavsky, & Stein, 1990), continuity (Bezuidenhout, 2001; Tall & Vinner, 1981); convergence (Alcock & Simpson, 2004; 2005), limits (Bergé, 2006; Szydlik, 2000; Williams, 2001), derivative (Orton, 1983; Zandieh & Knapp, 2006), and integral (Ferrini-Mundi & Graham, 1991; Lithner, 2003; Tall, 1996).

Specifically, derivative is, perhaps, one of the concepts that has attracted the least research attention thus far. Prominent researchers recently concluded that mapping out students' understandings of and difficulties with the derivative concept can be an important part of undergraduate mathematics education research that seeks to refine and build on students' mathematical thinking and learning at undergraduate level (Artigue, Batanero, & Kent, 2007).

A number of studies generally provide thick information about the detailed nature of students' conceptions or misconceptions across differentiation subjects. Such information is of fundamental importance in developing an understanding of the extent to which students are able to perform in solving differentiation problems and, therefore, in interpreting results about the difficulties they encounter with the derivative concept. In a 1990 study, Amit and Vinner reported on both correct and incorrect ideas present in an undergraduate student's interpretations of a graph of a function with a tangent line drawn at a point. Researchers indicated that the participant was able to read off the value of the function at the point of tangency and use the slope of the tangent line to determine the value of the derivative at the point of tangency. However, he was likely to assume that the derivative is the tangent line itself instead of the slope of that line. An early study of students' graphical understanding of derivative was conducted by Asiala et al. (1997). In particular, students who were given a graph of a function and a tangent line were required to specify the point at which a tangent line is drawn to a curve. Also they were required to calculate the gradient of the tangent line. The results revealed varying degrees of difficulties positing students' lack in working with local graphical data and interpreting the derivative as the slope of a tangent line at a point. Later studies by Ubuz (2001, 2007) updated the results and expanded the analyses to four common misconceptions that students held in the graphical understanding of the derivative with mainly similar findings. The researcher pointed to these misconceptions as follows: (a) derivative at a point gives the function of a derivative, (b) tangent equation is the derivative function, (c) derivative at a point is the value of the tangent equation at that point, and (d) derivative at a point is the value of the tangent equation at that point. As Baker, Cooley, and Trigueros (2000), Berry and Nyman (2003), and others have noted, when students were asked to sketch the graph of a function given its analytic properties (first and second derivatives, the value of limits, and continuity) they failed to coordinate the properties of the graph of a function and/or make connections between the graphs of a derived function and the function itself. Students encountered particular difficulties in interpreting the cusp at a point, the vertical tangent at a point or the removal of continuity.

Since the concept of derivative is closely aligned with the concept of function and the concept of limit, students must consider rate and slope in terms of the covariation of the derivative function and the original function. However, many undergraduate students do not have a fine-grained knowledge about the details of this covariation (Crocker, 1991; Thompson, 1994).

As Vinner (1982), Tall (1986), and Ellison (1993) have noted, students tend to describe derivative implicitly as the slope of the function or the slope of the curve at a point. Two explanations that account for this deficiency have been proposed. Firstly, researchers suggested that although students had intuitive notions of rate as speed and slope as steepness, few students describe the derivative as the slope of the tangent line to the curve. That is, deficiencies in defining the concept of derivative may stem, in part, from the fact that students superficially mention on the tangent line and/or the secant lines approaching to a tangent line. In addition, researchers proposed that these difficulties may be related to the breadth of the meaning of the definition of the derivative. They argue that the definition of derivative by describing the limit of the slope of secant lines approaching a tangent is a complex phenomenon, particularly because it requires students to describe the gradient at a point on the curve as the slope of the line connecting to nearby points. In some ways, this second explanation is a subset of the first, because it proposes that students initially think of the derivative as the slope of one secant line very close to the tangent (Heid, 1984). Both of these explanations are apparently plausible and may account for some of the difficulties that students have in understanding the concept of derivative. However, it is not clear whether students are able to distinguish between the tangent line and its slope which would yield in a complete understanding of derivative. Viholainen (2006) acknowledged that students' criterion for differentiability was generally the existence of an unambiguous tangent line. Their intuitive conception about the tangent misleadingly fulfill students' two misconceptions as follows: (a) a curve and its tangent have one and only one common point, and (b) a tangent keeps the whole curve in the same semi-plane (Biza, Christou, & Zachariades, 2008; Biza, Nardi, & Zachariades, 2009). Specifically, the majority of the students did not reflect the essence of tangent enough and that their efforts result in ambiguous conclusions.

A third explanation, which has not been offered much in the previous literature, focuses on whether students are able to respond to the question "What is a derivative?". Importantly, the mathematical nature of this question nicely explains apparent deficiencies in students understanding of the derivative concept. According to the literature, students are reasonably accurate when referring to the graphical interpretation of the derivative. For example, Vinner (1992) found that some students tend to define derivative as the slope of the tangent line to a curve whereas some others evoked the concept something to be calculated and related the derivative to the procedural methods of obtaining it. In another study, the way that students defined the derivative captured the notions of ratio, limit, and function and that students tend to use the difference quotient in the limiting process by analyzing a sequence of average rates of change as the difference in the denominator of the ratios goes to zero (Zandieh, 2000). Interestingly, although students know that there is a ratio and a limit involved, they guess that the derivative is a function because of the symbolic differentiation formula as taking the derivative means that decreasing the exponent for the function. Moreover, students were not able to think about the symbolic difference quotient, the derivative at a point, or the derivative as a function simultaneously. This clearly articulated that students generally avoid using definitions (Pinto, 1998; Viholainen, 2007), and rather tend to use procedural methods in those definitions (Juter, 2005).

On the other hand, studies that investigated students' competencies in solving routine differentiation problems show that they perform quite fast and adequately implement differentiation techniques. For example, Viholainen (2007) found that without any problems students performed a short calculation based on the definition of the derivative which led them to determine whether a function is differentiable or not at a given point. In accordance with previous research, Viholainen (2006, 2008a) pointed that students mainly direct their attention to carrying out the procedure rather than thinking about the prerequisites of the differentiating method they applied. Replicating her earlier studies Viholainen (2011) concluded that students are aware of the observed critical features of differentiation, however they fail to emphasize connections and simultaneously use these connections while interpreting the used expressions and drawings in differentiation.

Overall, research literature on the learning of the concept of derivative has been structured around several primary themes that are concerned about the difficulties students' encounter. Fundamental to these difficulties are students' lack in the meaning of the derivative concept and the related symbolic notations (Orton, 1983; Viholainen, 2008a; Zandieh & Knapp, 2006), the graphical interpretations of the derivative (Asiala, Cottrill, Dubinsky, & Schwingendorf, 1997; Kendal & Stacey, 1999), modeling real-life situations within the applications of differentiation (Asiala et al., 1997; Aspinwall & Miller, 2001), and applying procedures that utilize the differentiation rules (Aspinwall & Miller, 2001; Kendal & Stacey, 1999) or the differentiation theorems (Clark, Cordero, Cottrill, Czarnocha et al., 1997).

Students' deficiencies in contemplating advanced ideas and strategies by filtering them through their existing conceptions about the concept of derivative have provided an extensive outline for the relationships among the enactive, iconic, algorithmic, algebraic, formal, and axiomatic thinking. Although most of the research cited in these works involved the notable efforts on the nature of advanced mathematical thinking, it nevertheless can provide a lens for considering research on the derivative concept. In that respect, research examining the learning of derivative tends to utilize qualitative methods, particularly task-based interviews (Asiala, Cottrill, Dubinsky, & Schwingendorf, 1997; Borgen & Manu, 2002; Cooley, 2002; Doerr & Tripp, 1999; Gainsburg, 2006; Habre & Abboud, 2006; Haciomeroglu, Aspinwall, & Presmeg, 2010; Hahkiöniemi, 2006; Maull & Berry, 2000; Orton, 1983; Tsamir, Rasslan, & Dreyfus, 2006; Zandieh, 1997; Zazkis & Liljedahl, 2002) and to a lesser extent quantitative methods, particularly experimental designs (Aspinwall & Miller, 2001; Chappell & Killpatrick, 2003; Crocker, 1991; Meel, 1998; Tall, 2001; Ubuz & Ersoy, 1997; Ubuz, 2001, 2007).

At the outset the bulk of these reviews put forth the fact that there is a move in the higher education research to focus on the advanced mathematical thinking genres (Mamona-Downs & Downs, 2002, 2008) and in partial on students' difficulties with grasping the sense of advanced mathematics (Selden & Selden, 2005). The parallel concern has been with the concept of derivative (Aspinwall & Miller, 2001; Baker, Cooley, & Trigueros, 2000; Orton, 1983; Zandieh, 2000) indicating that while there has been attention to the role of mathematical thinking in mathematics education for some time (Dreyfus, 1990, 1991; English, 2002; Tall, 1991; Selden & Selden, 2005) it is only recently that an awareness of the significance of the mathematical thinking for the learning of derivative are emerging (Hahkiöniemi, 2006; Viholainen, 2005, 2006, 2007, 2008a, 2008b). These background developments raise a number of issues concerning mathematical thinking and the function of their relationships within the learning of the derivative concept itself.

2.2 MATHEMATICAL THINKING IN DERIVATIVE

Thinking in general refers to the means used by individuals to improve their understanding of, and exert some control over, their environment (Burton, 1984, p. 36). Associated with mathematical thinking, these means involve mastering concepts and procedures, inducing relationships, and applying those relationships (Pimm, 1995; Schoenfeld, 1991; Sternberg, 1996). A number of studies have defined mathematical thinking as the type of thinking process used in doing mathematics (Chapman, 2011). In that research, several different approaches to the conceptualization of mathematical thinking are represented, from multiple ways to multiple perspectives about the nature of mathematical thinking. Differences in conceptualization of mathematical thinking mirror an intense debate during the last decade about what dimensions should be included as associates of mathematical thinking. The issue of mathematical thinking was linked to mathematical processes (Mason, Burton, & Stacey, 1982), relational understanding (Skemp, 1976), conceptual knowledge (Hiebert & Lefevre, 1986), reflective abstraction (Dubinsky, 1991), mathematical sense-making (Schoenfeld, 1994), a dynamic process (Mason, Burton, & Stacey, 1982), mental activity (Watson & Mason, 1998), and advanced thinking (Tall, 1991). In the present study, I decided to take the much cited conceptualization of Mason, Burton, and Stacey (1982) as a point of departure. Based on a thorough review of the literature, Mason, Burton, and Stacey (1982) argued that mathematical thinking should be restricted to dimensions concerning the nature of thinking and the process of thinking, and that each dimension could be expressed as a mental activity or method used in learning mathematics.

Within this framework, O'Daffer and Thornquist (1993) stressed that mathematical thinking involves mental activities that provide the understanding of ideas, discovering the relationships among the ideas, drawing conclusions about the ideas, supporting the relationships among the ideas, and solving problems involving the ideas. Both quantitative and qualitative studies have, at least in part, confirmed this conceptualization. Studies based on slightly different theoretical frameworks have resulted in slightly same conceptualizations, probably reflecting that instruments tend to confirm the dimensionality in the theory of mathematical thinking underlying the construction of the questions to some extent. Viewed together, these questions indicated that mathematical thinking may be prompted within enactive, iconic, algorithmic, algebraic, formal, and axiomatic thinking, which was also the conclusion of a review of various empirical studies by David Tall. Tall (2004) suggested that mathematical thinking may be captured within a model with different types, distinguishing between conceptual-embodied, proceptual-symbolic, and formal-axiomatic thinking. He proposed that conceptual-embodied thinking nestles mental perceptions of real-world and internal conceptions of visuo-spatial imagery. In his study proceptual-symbolic thinking was represented by algorithmic actions and algebraic processes, whereas formal-axiomatic thinking was presented in a theoretical stance that underpins definitions, facts, symbols, and proofs. These distinctions indicate that it might be fruitful to explore mathematical thinking under six dimensions as enactive, iconic, algorithmic, algebraic, formal, and axiomatic thinking. Enactive thinking employs students to think about the situations around them in the physical world as an inclusive conception of the modeling of real-life phenomena by mathematical symbolism; iconic thinking unravels mathematical visualization as the use of physical senses and actions; algorithmic thinking is conveyed to the use of routine mathematical procedures and techniques; algebraic thinking involves the interplay between the understanding of a theoretical syntax within relevant symbols and the unpacking of theoretical situations into algorithms; formal thinking grounds on mathematical definitions, symbols, and facts directed towards the recognition of mathematical terminology; and axiomatic thinking nestles proofs and proving which are the bearers of verification, justification, and refutation (Artigue, Batanero, & Kent, 2007; Battista, 2007; Harel & Sowder, 2007; Kieran, 1992; Lesh & Zawojewski, 2007; Tall, 1992).

Students focus on different aspects of a particular task in which they develop six distinct but interrelated mathematical thinking. It is impossible to untie the merging of enactive, iconic, algorithmic, algebraic, formal, and axiomatic thinking in that every mathematical task calls out some paths of embodiment, symbolism, and formalism. A mathematical task links to the aspects that accompany a specific type of thinking and yet displays the aspects of all other thinking types. Accordingly, different types of mathematical thinking is used when tackling in appropriate tasks in any context area such as derivative, although relevant questions of a mathematical nature might more readily expose a variety of thinking types in differentiation at the same time. A derivative task is appropriate to specific types of thinking when it provokes or responds to the use of the components exemplified in the following lines. When students are solving an optimization task, their progress is energized by a blend of mathematical thinking. In its most general form such a task echoes the aspects of enactive thinking that employs students in modeling a real-life application of derivative. As a formal thinking, students attempt to disentangle the real-life phenomenon to recognize the definition of derivative along with differentiation symbols and rules. At this primary level, they are expected to understand the given conditions, identify the given quantities, and determine the unknowns. Students might advance these explorations into iconic thinking by drawing a diagram to identify the given and required quantities on the diagram and they might draw back to formal thinking to assign a symbol to the quantity that is to be maximized or minimized, also select symbols for other quantities and label the diagram with these symbols. They, then step out to enactive thinking by developing a mathematical model (e.g., an equation). In order to come up with a manageable model, algebraic thinking is inevitable in that they have to find equations relating variables simultaneously; expressing some in terms of others until they are left with just one equation connecting two variables. At the abstract level, students might broaden these relations to axiomatic thinking of Fermat's Theorem to justify that "If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$." Finally, to find the maximum or minimum value students access algorithmic thinking by applying the fundamental differentiation formula algorithms (e.g., the product rule).

2.2.1 ENACTIVE THINKING

The term "enactive thinking" includes processes that individuals employ to think about the situations around them in the physical world. Thus, it not only encompasses the mental perceptions of real-world objects but also internal conceptions of visual and spatial imagery (Tall, 2004). The set of enactive views includes knowledge of the real-world, applicability of mathematical ideas, and interpretations of mathematics to make sense of situations. Gravemeijer (1999) argued that in enactive thinking mathematics comes to the fore as a natural extension of students' experiences with the real-life applications. When given a mathematical situation to explore students have the opportunity to demonstrate the information about real-world in order to make some predictions for a reasonable solution. Many researchers (Blum & Niss, 1991; Greer, 1997; Lesh & Harel, 2003; Mousoulides, Christou, & Sriraman, 2008; Toumasis, 2004; Zbiek & Conner, 2006) described enactive thinking processes as an inclusive conception of models and modeling to accommodate appropriate mathematical ideas and techniques which, in general terms, referred to the modeling of real-life phenomena by mathematical symbolism (Watson, Spirou, & Tall, 2003). The set of models and modeling views coined the term *"model"* as any mathematical representation of certain aspects of real-life phenomena which is created using mathematical concepts such as functions, equations (Edwards & Hamson, 1990). The term *"modeling"*, on the other hand, was linked to both the development and the application of mathematical skills necessary to get efficient answers for the real-life phenomena. The accounts of these descriptions collectively identified that enactive ideas nestle elements of both a treated-as-real world and a mathematics world, processes of which involve the interactions among elements such as the real-world situation and the mathematical solution. A step toward discerning the similarities and distinctions between modeling and enactive thinking processes is the fact that enactive thinking requires a certain form of looking at a real-life phenomenon; namely, it requires an enactive point of view. However, to adopt a particular enactive point of view is to make use of models. This conception of enactive thinking acknowledges the connected undertakings of practical situations and numerical integrations (Huber & Lunday, 2006) where students are required to describe the problem, manipulate the problem to develop a model, and build links between the model and the real-life problem (Lesh & Doerr, 2002). To conceive of enactive thinking as a consequence of a more general model development denotes that mathematics is not to be seen as something untouchable. On the contrary, it is a set of ideas to be retrieved and used to describe certain parts of the real-life that are the result of a coordination of enactive thinking and of a search for the mathematics based on enactive thinking (Lakoff & Nunez, 1997). The enactive thinking processes thus involve structuring, interpreting, and solving real-life problems (Blum, 2002). My focus on enactive thinking in the concept of derivative mandates applications of differentiation to reflect within the little empirical work that has been done towards the better understanding of enactive thinking involved in modeling real-life applications of the derivative (Asiala, Cottrill, Dubinsky, & Schwingendorf, 1997; Gainsburg, 2006; Presmeg & Balderas-Canas, 2001; Ubuz & Ersoy, 1997). Associated with any enactive thinking process are a mathematizable real-world situation, a mathematical solution, a question that prompted the enactive thinking, and the relationships among them. To reflect this within enactive thinking in derivative and my interest in mathematical thinking in general, I assume the enactive thinking context to be any mathematical insight from optimization problems (Klymchuk, Zverkova, Gruenwald, & Sauerbier, 2010; Malaspina & Font, 2010). Such kind of applications of derivative holds unique qualities that make them capable of triggering the production of enactive thinking. Accordingly, the ways in which students perceive the required application of the derivative and impose their ideas in their solutions in real-life situations reflect their enactive thinking. In this sense, enactive thinking might serve as a venue for mathematical thinking in the derivative where processes focus on students' assumptions about and awareness of pertinent mathematics available in finding the largest value or the smallest value of a function that a function can take on an interval and/or looking for the largest or smallest value of a function subject to some kind of constraint.

2.2.2 ICONIC THINKING

Bruner (1966) coined the term "iconic representation" to unravel mathematical visualization to conceptualize the use of physical senses and actions in a way that accounts for how embodiment of mathematics occurs and what visual mathematics may be learned through embodiments of physical objects. Drawing on existing descriptions, iconic thinking is involved in processes generated from building a mental scheme to depict visual information (Presmeg, 1985, 1986, 1997) that is used interchangeably with the notion of "visualization" characterized both as a "noun"- the product, the visual image- and as a "verb"- the process, the activity- (Bishop, 1989, p. 7). Nemirovsky and Nobre (1997) suggested that iconic thinking presents students a form of seeing the unseen that scan through these different roles. This form fits with the view of iconic thinking in Friel, Curcio, and Bright's (2001) work. According to these researchers "iconic sense" is a construct for presenting certain forms of visualization that can be transmitted to others whereas "iconic interpretation" is the ability to read a graph and gain meaning from it. Taken literally, Costa and Rocha (2005) articulated that the iconic form of thinking results from the mental manipulation of images, intellectual operations related to the transformation of images, and mental construction of relationships among images. In a more figurative sense, it provides learners to illustrate embodied notions that help transcend the mathematical actions including physical manifestations (Zazkis, Dubinsky, & Dauterman, 1996). The embodiment of mathematical actions, via step-by-step graph sketches, then as graph interpretions, and as visual objects in their own right is well-represented in the literature at the secondary level (Mitchelmore & White, 2000; Noss, Healy, & Hoyles, 1997; Presmeg, 1985; Presmeg & Balderas-Canas, 2001; Stylianou, 2002) and at the undergraduate level (Alcock & Simpson, 2004, 2005; Asiala et al., 1997; Aspinwall et al., 1997, Aspinwall & Miller, 2001; Berry & Nyman, 2003; Kendal & Stacey, 1999; Lima & Tall, 2008; Pegg & Tall, 2005; Przenioslo, 2004, 2005; Roorda, Vos, & Goedhart, 2007; Stewart & Thomas, 2007; Ubuz, 2007). The pervasiveness of undergraduate students' iconic thinking has been affirmed to utilize learning in convergence of sequences (Alcock & Simpson, 2004; 2005), limits (Pinto & Tall, 2002; Przenioslo, 2004), convergence of sequences (Przenioslo, 2005), derivative (Asiala et al., 1997; Aspinwall et al., 1997, 2001; Berry & Nyman, 2003; Ferrini-Mundy, 1987; Kendal & Stacey, 1999; Orton, 1983; Roorda, Vos, & Goedhart, 2007; Zandieh, 2000), and logic (Carreiras & Santamaria, 1997) emerging a consensus that students' broad interpretations of graphs related to advanced problem solving is enhanced by engaging in iconic thinking processes. The findings further shown that the meaning of calculations on the given tasks were ensured by iconic transformations that usually attempted to transform the original graph into an algorithmic form for which a solving procedure was already known.

The concept of derivative nestles versatile iconic thinking processes (e.g., slope of a tangent line) that include both a procedural and conceptual fluency in translation within and between graphical representations (Viholainen, 2008a, 2008b). Thus, the iconic stance in the concept of derivative implies that students can benefit from constructing visual ideas underpinning derivative concepts (e.g., extrema points) by performing actions that have graphical manifestations, condensing these to processes and encapsulating these as interpretations in the visual world. For instance, interpreting the image of a curved graph of a derivative function intersecting the x-axis may be linked with the extremas of the corresponding function or sketching the graph of a function f(x) may be linked to the graph of its derivative $f'(x)$.

2.2.3 ALGORITHMIC THINKING

Several researchers (Fischbein, 1983; 1994; Gray & Tall, 1994; Lima & Tall, 2008; Marrongelle, 2007; Sirotic & Zazkis, 2007; Stewart & Thomas, 2007; Tall, 2002; Tirosh, Fischbein, Graeber, & Wilson, 1998) described algorithmic thinking simply as the capability to apply procedures. Henceforth, most of them used mathematical procedures and techniques to convey algorithmic thinking as a process. Their viewpoints jointly underscored the importance of algorithmic thinking in operating on with mathematical properties and extending these properties to wider computations. In other words, algorithmic thinking was accepted to be stored in mathematical competencies including routine practices that are technically justified and which have to be actively trained. In this sense, algorithmic thinking nestles dynamics (Sirotic & Zazkis, 2007) that is concerned about the practice and mastery of a set of computational procedures in which algorithms emerge as a product (Rasmussen, Zandieh, King, & Teppo, 2005).

There are currently increased efforts to highlight the algorithmatization process in reducing mathematical conditions to procedural forms throughout secondary grades (Cooper, 2003; Johanning, 2004; Lima & Tall, 2008; Stacey & MacGregor, 2000) and university (Ali & Tall, 1996; Clark et al., 1997; Orton, 1983; Roorda, Vos, & Goedhart, 2007; Williams, 2001). Researchers mainly argued that the ready-made procedures presented with little connection to reasoning may encourage limitations in students' mathematical thinking (Gravemeijer & van Galen, 2003). In terms of the contextualization of operations algorithmically in a stepwise flaw of procedures it is widely accepted that, when engaged in algorithmic thinking, students may exhibit an automatized progress, thus perform more spontoneously on the given task. Marrongelle (2007) underlined algorithmic thinking processes as a controversial part of mathematics in the sense of the efficient mechanisms it provides for communicating generalized procedures on the one hand and the limitations it endengers by preventing students to develop mathematical insights on the other.

It has been extensively documented that algorithmic thinking in solving derivative tasks often trigger responses aligned with procedures (Ali & Tall, 1996; Aspinwall & Miller, 2001; Clark et al., 1997; Kendal & Stacey, 1999; Orton, 1983; Viholainen, 2005; Zandieh, 2000). Fischbein's (1983) analysis of algorithmic thinking facilitates our understanding of the building blocks proposed in the concept of derivative. His analysis incorporates two dimensions: (a) the ability to describe and use the rules along with relevant procedures; and (b) the ability to algorithmatize theoretical justifications. During the first dimension, students integrate the rules to apply procedures (e.g., knowing that the derivative rule for a constant function is "if $f(x) = c$, then $f'(x) =$ 0." and applying this rule to find the derivative of $f(x) = 3$.). Next, these steps are organized into algorithms, which reflect the explanation of why certain procedures hold for this problem situation (e.g., $f'(3) = 0$ because from the definition of derivative:

$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{x-x}{h} = \lim_{h\to 0} 0 = 0;$ accordingly $f'(3) = \lim_{h\to 0} \frac{f(3+h)-f(3)}{h} = \lim_{h\to 0} \frac{3-3}{h} = \lim_{h\to 0} 0 = 0$). Thus, algorithmic thinking requires identifying the situation to which procedure applies, the correct order of algorithms, the correct completion of steps, and finally recognizing the correctly completed procedure.

2.2.4 ALGEBRAIC THINKING

The constitution of a symbolic language and the concomitant rise of symbolic thinking contain the germ of algebraic thinking that shed lights to the emerging syntax of theorems and meaning of algorithms. Algebraic thinking about the syntax of theorems and about the meaning of algorithms to restructure thinking in adaptive response to changing the theoretical situation demands may be particularly critical in testing the hypotheses of a theorem. Thus, the itemization into algebraic thinking by Mason (1996) brought the process further as an activity. Unwrapping theorems and pushing symbols, he viewed algebraic thinking simply in 'algorithm seeking'. Zazkis and Liljedahl (2002) supported that the very formation of this algorithm in the mind of the student, in whatever form it is envisioned, is algebraic thinking (p. 399). In characterizing algebraic thinking, Derry, Wilsman, and Hackbarth (2007) intensively emphasized the interplay among the understanding of a theoretical syntax within relevant symbols, representing theoretical situations within procedures, and analyzing theoretical structures within changes in various algorithms. Algebraic thinking, hence, involve the algorithmizations students use to endow them with meaning in their encounter with theorems (Radford, 2000). Even and Tirosh (2002) postulated that the ability to think algebraically unfolds a network of connections to enrich the theoretical justifications of respective procedures. Researchers put forward students' difficulties with developing such algebraic techniques to acquire and make sense of theorems (Gainsburg, 2006; Radford, 2000; Radford & Puig, 2007). The transition from the understanding of the distinctive manner in which simple procedures and compounded algorithms stand for the theorems they represent to the grasping of the sense of procedures carried out on those theorems was taken as a shift that nestles complexities for students. These complexities were evidenced to significantly hinder students' ability to succeed with more advanced mathematics (Ladson-Billings, 1998). To better understand the algebraic thinking in the concept of derivative, it may be helpful to notice that, the theorems has to be represented through symbols, for it is the initial point of the advanced mathematical thinking process. The symbols then express the algorithms as general procedures in the theorem, the ultimate goal of being algebraic representation. For instance, the theoretical text of the Mean Value Theorem unfolds as algorithmic calculations from this representation of the theorem to derive inequalities such as $|\tan b - \tan a| \leq |b-a|$, where $-\pi/2 < a,b < \pi/2$. Moreover, Mean Value Theorem figures out the connection between the slope at a point (local picture of derivative) and the average slope across an interval (global picture of derivative) where students make algebraic manipulations with functions that are continuous in the closed interval [a, b] and differentiable in the open interval (a, b).

Followed by a move towards algebraic thinking and algebraic symbolism in mathematical abstractions, some research which was set out to trace the development of students' use of algebraic thinking concerned secondary school students (Cooper, 2003; Irwin & Britt, 2005; Johanning, 2004; Radford, 2000; Stacey & MacGregor, 2000; Steele & Johanning, 2004; Sutherland, 1989; Tall & Thomas, 1991; Zazkis & Liljedahl, 2002), and to a lesser extent, undergraduate students (Borko et al., 2005; Derry, Wilsman, & Hackborth, 2007). Researchers reported that algebraic thinking is necessary for mathematics throughout several mathematical subjects including division (Cooper, 2003), patterns (Radford, 2000; Zazkis & Liljedahl, 2002), functions (Derry, Wilsman, & Hackborth, 2007), numbers (Irwin & Britt, 2005; Zazkis & Liljedahl, 2002), equations (Kramarski & Hirsch, 2003), and word problems

(Stacey & MacGregor, 2000; Steele & Johanning, 2004). Collectively, findings echoed students' difficulties with making the transition from arithmetic to algebra that requires them to reason about a theorem and to express appropriate algorithms (Kieran, 1989; Kramarski & Hirsch, 2003; Sutherland, 1989; Tall & Thomas, 1991). Henceforth, researchers suggested that for meaningful algebraic thinking to occur it is not sufficient to see the general theorem in the particular algorithms, indeed it must be expressed algebraically.

2.2.5 FORMAL THINKING

Formal thinking draws on the symbolic language of mathematics as well as the logic of its exposition. Based on definitions, symbols, and theorems, this form of thinking gives prompt to an active reasoning process (Fischbein, 1983). The standpoint of Fischbein, here, parallels that formal thinking is constructed through the ability to connect mathematical symbols and notations with relevant definitions (Bergsten, 2004). This cognitive process is facilitated by the knowledge about how the mathematical realm works (Tsamir & Tirosh, 2008) in a declarative manner. Tsamir and Bazzini (2000) argued that a wider perspective of the mathematical realm encompasses the knowledge of how to validate concepts in mathematical context by their definitions or notations. Formal thinking, then, involves the capacity to reason propositionally and the ability to make deductions beginning from formal definitions (Fischbein, 1994) followed by their relevant notations, and further pertinent theorems. Sirotic and Zazkis (2007) suggested that in order to understand the depth of the formal thinking it is necessary to examine students' ability to recall the symbols relevant to a concept, state the meanings of these symbols in their own words, and sequentially implement these meanings in a broader mathematical context of facts and rules. Researchers generally accept that formal thinking processes can be carried out by rote (Semadeni, 2008; Tirosh, Fischbein, Graeber, & Wilson, 1998), and thus become declarative and in part procedural. However, the importance of formal thinking cannot be overemphasized that it has a powerful complementary role in two aspects: (a) it helps students to exert meanings from visual images related to definitions and theorems (Orton, 1983); and (b) it provides students the opportunity to apply appropriate procedures relevant to a concept in conjunction with its definition and symbolic use (Zandieh & Knapp, 2006). Probably, formal thinking enables students to systematically take turns between fundamental concept definitions, notations, and rules. In a deeper sense, these aspects articulated that students' access to formal thinking indicate the successful discrimination of the concept definition (e.g., rate of change) denoted by a symbol (e.g., $\frac{dy}{dx}$). Accordingly, formal thinking evokes in the concept definition and its notation incorporated into the facts and rules (e.g., f(x) = sinx $\Rightarrow f'$(x) = $\frac{df}{dx}$ = cosx.

Recent attention to address student learning in calculus has generated renewed efforts to understand the essential role of formal thinking at the secondary level (Gray & Tall, 2007; Lima & Tall, 2008) and at the undergraduate level (Alcock & Simpson, 2004, 2005; Orton, 1983; Przenioslo, 2004, 2005; Szydlik, 2000; White & Mitchelmore, 1996; Williams, 1998; Williams, 2001). A number of researchers have commented on the fundamentality of formal thinking on the basis of undergraduate mathematics in learning convergence of sequences (Alcock & Simpson, 2004; 2005), limits (Przenioslo, 2004, 2005), functions (Williams, 1998), equivalence relations (Chin & Tall, 2000, 2001), and derivative (Orton, 1983; Viholainen, 2008; White & Mitchelmore, 1996; Zandieh, 1998, 2000) contending that students' understanding of fundamental formal concepts is underdeveloped. Collectively, they supported that the acceleration of formal thinking slights further subtle processes in solving more sophisticated calculus problems.

The chief characteristic of formal thinking in the concept of derivative is the fact that it is framed by a world of concrete concepts (e.g., definitions, symbols, notations) and higher evolved forms of abstractions from fundamental mathematical concepts that offer students an array of lines of conceptual development. From this point of view, the concept of derivative can be regarded to grasp the essence of the functions and limits from which it is abstracted. This abstraction also addresses the meaning of symbolization of derivative as dy/dx which is then used in differentiation rules and further in the statements of theorems under some circumstances (Thurston, 1972, 1994).

2.2.6 AXIOMATIC THINKING

In terms of axiomatic thinking, what Stylianides and Stylianides (2008) assert is that the process of proving and the concept of proof are the bearers of the cognitive activity of axiomatising in which the properties are deduced solely from the theorems. This was initially discussed at some length in *Advanced mathematical thinking* (Tall, 1991), which was in essence an overview of axiomatic thinking as a combination of verification, justification, or refutation, and in particular of its emphasis on rigorous formal mathematics. Taking a comprehensive and subjective stance Harel and Sowder (2007) supported the term 'proof' to include both formal and informal arguments (justification, verification, and explanation) in an attempt to imbue mathematical thinking with a spirit of axiomatic enquiry. Then, through a closer examination of axiomatic thinking in practice, it came to the further conclusion that proof is the central idea of modern mathematics and proving is the style of presentation of that mathematics (Epp, 2003; Tall, 2001). In his further study, Tall (2002) used the hyphenated term *theorem-proving* to describe the overarching view of axiomatic thinking that encompasses the following major processes that are frequently involved in the process of making sense of and establishing mathematical proof: (a) predicting and experimenting in the real world, (b) using statements that are available for justification, (c) employing forms of valid mathematical argumentation, (d) connecting a sequence of assertions for or against a mathematical claim, and (e) communicating with appropriate forms of mathematical expressions. The choice of a hyphenated term to encompass these five processes reflected Tall's intention to view the processes in an integral way. Also, given that the term "proving" has been associated with many different aspects of mathematical processes that are related to axiomatic thinking (e.g., inductive reasoning, deductive reasoning), the hyphenated term theorem-proving clarified that the focus is on aspects of axiomatic thinking related to the proofs of theorems. In this context, proving and proof, Stylianides and Stylianides (2008) suggested, are formally constituted building blocks of axiomatic thinking that empower students to reflect on the set of accepted statements (e.g, definitions, axioms, theorems), interpret the modes of argumentation (e.g., logical rules of inference), and make transformations among the modes of argument representation (e.g., verbal, graphical, symbolic) in a certain mathematical way – much in the same manner that mathematicians do in using *true* statements, *valid* arguments, and *appropriate* representations.

A great number of educational researchers have focused on how axiomatic thinking is dealt with by elementary school students (Stylianides, 2007; Stylianides & Stylianides, 2008), secondary school students (Hanna, 2000; Hanna & Jahnke, 1996; Hoyles & Küchemann, 2002; Miyakawa & Winslow, 2009; Yackel & Hanna, 2003), undergraduate students (Bills & Tall, 1998; Chin & Tall, 2002; Furinghetti & Morselli, 2009; Gibson, 1998; Martin & Harel, 1989; Weber, 2006), and teachers (Corleis, Schwarz, Kaiser, & Leung, 2008; Hatzikiriakou & Metallidou, 2008; Schwarz, Leung, Buchholtz, Kaiser, Stillman, Brown, & Vale, 2008). They recommended that axiomatic thinking is central to doing mathematics throughout several mathematical subjects including integral (Bills & Tall, 1998), continuity and convergence (Gibson, 1998), number theory (Furinghetti & Morselli, 2009), equivalence relations (Chin & Tall, 2002), divisibility (Martin & Harel, 1989), group homomorphisms (Weber, 2006), group isomorphisms (Leron, Hazzan, & Zazkis, 1995), logic (Inglis & Simpson, 2008), derivative (Davis, 1993; Mills & Tall, 1988), and geometry (Schwarz et al., 2008; Weiss, Herbst, & Chen, 2009). There is a body of research reporting that students lack rich axiomatic thinking to derive the truth of a statement (Goetting, 1995; Martin & Harel, 1989), distinguish between inductive and deductive arguments (Morris, 2002), reason on proof settings (Sowder & Harel, 2003), validate the given statement (Selden & Selden, 2003), focus on the proof methods (Sowder, 2004; Weber, 2001; Weber & Alcock, 2004). As the focus of attention drew on the understanding that mathematics is founded on axiomatic thinking and is not a set of arbitrary rules to mechanically apply, researchers point to the involvement of proof-based textbooks in the curriculum (Stacey & Vincent, 2009; Stylianides, 2009) and the integration of dynamic geometry environments to mathematics classes (Hoyles & Healy, 1999; Olivero & Robutti, 2001) that provide students the opportunity to make sense on the essence of axiomatic thinking.

When students are introduced the axiomatic thinking, a new focus of attention occurs within the transition from using symbols and computations to give answers to selecting certain properties as definitions and theorems to build up the other properties by logical inferences (Gray, Pinto, Pitta, & Tall, 1999; Mariotti, 2006). Consequently, the empirical evidence for the growing emphasis on axiomatic thinking showed that the mathematical processes it mediates are not something transparent for the students (Harel & Sowder, 2007) as they step into the rigid forms of mathematics.

Axiomatic thinking deposited in the concept of derivative, the rigorous system that they form, and the advanced mathematical thinking processes that they mediate offer students certain lines of theoretical development, malleable directions of logical growth that the students can pursue and transform in accordance to the proving activities they engage with. For instance, providing a justification for the truth of the given function f(x) has exactly one root by reference to the Mean Value Theorem is formally cast by axiomatic thinking.

In a broader sense mathematical thinking processes that are provoked when solving different tasks may be analyzed from the perspective of "Three Worlds of Mathematics" (Tall, 2004), which proposes the existence of conceptual-embodied, proceptual-symbolic, and formal-axiomatic thinking.

2.2.7 CONCEPTUAL-EMBODIED THINKING

Most studies on students' conceptual-embodied thinking fall in one of two categories, that is, the positive effect approach with a focus on students' visual and spatial imagery lead to great leaps of insight in mathematics, and the negative effect approach that calls for blind alleys of error in the embodiment of mathematics. Obviously, there are many arguments both for and against each one of the two approaches. For example, an advantage of the positive effect approach is that conceptual-embodied thinking is involved in processes generated from a mental scheme to depict visual information (Presmeg, 1986, 2006; Zimmerman & Cunningham, 1991). At the same time, this approach points that conceptual-embodied thinking is used interchangeably with "visualization" characterized both as a "noun"-the product (e.g., visual image) and as a "verb"-the process (e.g., activity) (Bishop, 1989, p.7). Yet to deny the importance of conceptual-embodied thinking, however, is to deny the roots of many of the most profound mathematical ideas relevant to functions, continuity, and differentiation (Tall, 1991). Nemirovsky and Nobre (1997) expressed a concern about diverting attention from important mathematical subjects, and this concern of importance assigned to seeing the unseen gave rise to conceptual-embodied thinking to be a fundamental source of presenting certain forms of visualization. Research into these converse approaches showed that students lack the ability to represent, transform, and generalize visual representations of the derivative that trigger the harmonization of conceptual-embodied thinking (Asiala et al., 1997; Aspinwall et al., 1997, 2001).

Researchers have long reported the difficulties in conceptual-embodied thinking encountered by students referring to basic calculus concepts as the derivative (Asiala, Cottrill, Dubinsky, & Schwingendorf, 1997; Gainsburg, 2006; Presmeg & Balderas-Canas, 2001; Ubuz & Ersoy, 1997). Many of the main difficulties has been subject to their lack in the ability to communicate, document, and reflect on visual information depicted in applications of derivative (Orton, 1983; Yoon, Dreyfus, & Thomas, 2010). The origins of such difficulties may be analyzed in light of various perspectives: interpreting a graph and constructing its derivative graph (e.g., Ubuz, 2007) and/or solving optimization problems (e.g., Villegas, Castro, & Guttiérrez, 2009).

Students generally have very weak conceptual-embodied thinking skills in the calculus (Epp, 1987; Selden, Selden, & Mason, 1994), which in turn leads to lack of conceptualizations of the graphical implications of the first derivative, second derivative, and continuity together with the coordination of these elements to sketch the graph of the relevant function (Baker, Cooley, & Trigueros, 2000). Many students are proficient at, for example, differentiating a function and finding its critical values and inflection points (Lima & Tall, 2008; Pegg & Tall, 2005). However, they fail to conceptualize these actions and work with these actions if they are presented in graphical form rather than in equation form. Furthermore, when the intervals of increasing/decreasing values and/or concavity are presented to the students they fail to visualize the graphical implications of these differentiation features (Roorda, Gos, & Goedhart, 2007). Concomitantly, while students were attempting to solve a conceptual-embodied thinking task they demonstrated difficulties with coordinating information in two major areas of concern (Baker, Cooley, & Trigueros, 2000). First, they failed to coordinate overlapping information across contiguous intervals on the domain. Second, they failed to coordinate differentiation features that were explicitly stated rather than derived from a differentiation formula. The main obstacles that students encountered were identified as the cusp point, the vertical tangent, removal of continuity, and the second derivative.

Many of the problems in conceptual-embodied thinking displayed by students involved confusions by the conditions that a graph is increasing and concave down and/or graphical implications of functions with cusps (Aspinwall et al., 1997; Slavit, 1995; Vinner, 1989). Researchers reported that in their access to conceptual-embodied thinking students showed a strong tendency to use the first derivative to gain most of their information from the graph (Asiala et al., 1996). Apparently, students displayed a reasonable conceptual-embodied thinking in analyzing the functions with the first derivative and building the relationship between the slope of the tangent and the derivative. However, students were lack of conception about recognizing the first and the second derivative as a function and notably they could not interpret the relationship of the second derivative to the first (Thompson, 1994). Eventually, students should transmit conceptual-embodied thinking to consider the first derivative as itself a function in order to make sense on the second derivative.

Beginning with graph constructions and/or interpretations students' difficulties in conceptual-embodied thinking led in the direction of their obstacles in modeling optimization problems. For example, in area optimization a characterization of students made evident that there is a strong relationship between understanding the structure of the real-life situation (e.g., optimization) and exposing conceptual-embodied thinking in the construction, use, and articulation of differentiation (Villegas, Castro, & Guttiérrez, 2009). The constituent parts of this body of relationship may relate to any of the difficulties in conceptual-embodied thinking mentioned earlier. As students mobilize opportunities to engage in conceptual-embodied thinking the following obstacles arise: visualization of the optimization scenario (e.g., drawing a diagram of the area), labeling the variables in the scenario, deciding on the variables to be minimized or maximized, writing this as a function of the variables in terms of just one variable, solving for the variable in the equation $f' = 0$ (Campos & Estrada, 1999; Tall, 2005). In cultivating conceptual-embodied thinking in their encounter with real-life applications of the derivative students were not able to find the values of variable for which f' does not exist or check each zero of the derivative to see whether this zero of f' corresponds to a relative minimum or a relative maximum (Porzio, 1999).

2.2.8 PROCEPTUAL-SYMBOLIC THINKING

The nature of proceptual-symbolic thinking has been a source of discussion and debate for many years (see, e.g., Fischbein, 1983; Lima & Tall, 2008). Proceptual-symbolic thinking occupies a privileged position in calculus, being a core thinking type for all students. This is largely due to its perceived usefulness in triggering responses aligned with procedures in especially solving differentiation problems (Zandieh, 2000). The value of proceptual-symbolic thinking lies in its potential to prompt students to identify the situation to which procedure applies, the correct order of algorithms, the correct completion of procedural steps, and finally to recognize the correctly completed procedure. Often there is a conflict between this emphasis on usefulness and those who see proceptual-symbolic thinking as a form of mere integration of rules to apply procedures, emphasizing technical implementations and focused routines inwards (Ali & Tall, 1996).

Proceptual-symbolic thinking in derivative was singled out from three components: a process of algorithms that produces differentiation and a symbol that represents either the process or the concept of derivative (dy/dx). In terms of this contextualization researchers argued that ready-made procedures encourage students to operate in a stepwise flaw of algorithms (Gravemeijer & van Galen, 2003). While algorithmic approaches to differentiation may help students to formalize procedures, they may actually hamper understanding unless proper techniques are available for the student to fold back to. Accordingly, students face many obstacles in activating proceptual-symbolic thinking in derivative. Among the most objective ones are procedural obstacles identified by Orton (1983). On the one hand, they are related to both algorithmic and algebraic methods in conjunction with various unconscious schemes of thinking; on the other hand, they are related to the concept of derivative and related proceptual terms (algorithm, procedure). It has been widely documented that while differentiating a function students could not apply the proper algorithmic techniques and obtain the correct solution (Marongelle, 2007). A perennial problem in students' proceptual-symbolic thinking is thus how to apply differentiation procedures. The end product for students is often an algorithm step which focuses on the computing, and less so on the broader differentiation perspective. These problems are often reinforced by typical differentiation questions that ask students to "solve, find, evaluate, determine, differentiate", etc. (see Ferrini-Mundy & Guether-Graham, 1991). Approaching proceptual-symbolic thinking from a practical point of view, researchers accepted that students exhibit an automatized progress, thus perform more spontaneously on the given differentiation task (Orton, 1983; Roorda, Vos, & Goedhart, 2007; Viholainen, 2005). Clearly, this leads students lack the ability to solve the derivative of a function at a given point and evaluate the correctness of their solutions (Tsamir, Rasslan, & Dreyfus, 2006).

Another difficulty in students' proceptual-symbolic thinking in derivative stems from its dual nature. Indeed, proceptual-symbolic thinking in derivative can be understood in two essentially different ways: algebraically- as a production of symbolic language and algorithmically- as a computational process. Those two ways of understanding the proceptual-symbolic thinking in derivative, although apparently ruling out one another, should, however, complete each other and constitute a coherent unity in understanding a theoretical syntax within relevant differentiation symbols, representing theoretical situations in differentiation within procedures and analyzing theoretical structures within changes in various differentiation algorithms (Derry, Wilsman, & Hackbarth, 2007). Although students knew how to apply an algebraic algorithm, they did not understand the mathematical underpinnings involved in the differentiation theorems, and that had mixed ideas regarding the algorithms involved (Zandieh & Knapp, 2006).

Undoubtedly, a deep analysis of research on proceptual-symbolic thinking goes far beyond the implementation of simple algorithms in derivative. The functioning of these implementations is also to a great extent dependent on the quality of students' theoretical justifications of respective differentiation procedures that go with these implementations. With the availability of effective proceptual-symbolic thinking, there is the opportunity to enrich algebraic manipulations and algorithmic calculations by supporting students to unfold a network in which compounded procedures and algorithms that stand for the theorems (Gainsburg, 2006; Radford & Puig, 2007).

2.2.9 FORMAL-AXIOMATIC THINKING

The analysis of formal-axiomatic thinking in derivative produced by students has been a recurring theme in the mathematics education literature. Generally these types of analysis are of two kinds: those that concentrate on the content and structure of derivative (e.g., the meaning of derivative, the symbol of derivative) and those that concentrate on the theory of differentiation (e.g., proofs and proving of differentiation theorems). Reports on students' difficulties with formal-axiomatic thinking in derivative thus fit into these two categories.

This research has given rise for analyzing student understanding of the concept of derivative (Zandieh, 1997, 2002), which describes formal-axiomatic thinking that is important for students to make sense of concrete differentiation terminology and be able to use this terminology in order to manage successfully differentiation issues that come up in their mathematical practice. In an effort to explore a scheme for describing students' difficulties with formal-axiomatic thinking in differentiation, researchers demonstrated that these difficulties were inextricably linked to the extent to which a student may be qualified in elaborating the meaning of derivative (Hirst, 1972; Orton, 1980, 1983; Viholainen, 2008; White & Mitchelmore, 1996). Results articulated that students lack the ability to find and describe a coherent structure for the derivative based on a context standing for the whole derivative concept. More specifically, students did not have a very complete understanding of derivative in terms of representing the derivative graphically as the slope of the tangent line to a curve at a point, verbally as the instantaneous rate of change, physically as speed or velocity, and symbolically as the limit of the difference quotient (Zandieh & Knapp, 2006). Students' lack in formal-axiomatic thinking in rate of change and tangent lines added on their difficulties with the symbolic language of derivative comprising deficiencies in the meaning of $\frac{dy}{dx}$, $\frac{\Delta y}{\Delta x}, \frac{\partial y}{\partial x}, \frac{\delta y}{\delta x}$ (Ubuz, 1994, 1996). Thus, it was obvious that mathematically, students were not able to develop formal-axiomatic thinking in their interpretations indicating that the derivative concept involves a ratio, a limit, and a function.

Studies on formal-axiomatic thinking in derivative have showed, as well, a range of difficulties that students encounter in tackling with proofs and proving in differentiation (Hahkiöniemi, 2006; Tall, 1986, 2004). The authenticity in proofs and proving requires students to be able to identify differentiation situations in which proof is called for, recognize important theoretical differences among these situations, and stage appropriate opportunities for proving to engage in formal-axiomatic thinking (Przenioslo, 2004, 2005). Given the central role that formal-axiomatic thinking can play in the proving activity that takes place in differentiation (Zandieh, 2000), however, students have a scant acceleration in proving whether a function is differentiable at a given point (Zandieh, 2007) and proving differentiation theorems (Zandieh, 1997). Moreover, with regard to the relationship between differentiability and continuity, under the assumption of certain axioms students failed to demonstrate whether a statement is necessarily true (Zandieh, 1997).

Formal-axiomatic thinking is thus a complex mathematical progress combining processes of concise differentiation terminology and the heuristic processes of producing a conjecture relevant to differentiation (Gray, Pinto, Pitta, & Tall, 1999) and that the outcome of this progress reflects an important part of the differentiating process.

When we focus explicitly on the literature in mathematical thinking, it becomes apparent that many researchers have emphasized the value of the relationships among enactive, iconic, algorithmic, algebraic, formal, and axiomatic thinking in the teaching and learning of calculus (e.g., Edwards, Dubinsky, & McDonald, 2005; Mitchelmore & White, 2000, Przenioslo, 2005) in general, and the concept of derivative (e.g., Orton, 1983; Viholainen, 2008) in partial. The following section provides an indepth understanding for the relationship among enactive, iconic, algorithmic, algebraic, formal, and axiomatic thinking together with the relationship among conceptual-embodied, proceptual-symbolic, and formal-axiomatic thinking that delineate the paths in regard to the within- and between-classroom associations, respectively.

2.3 WITHIN- AND BETWEEN-CLASSROOM RELATIONS OF MATHEMATICAL THINKING IN DERIVATIVE

The stream of research on the relationships among enactive, iconic, algorithmic, algebraic, formal, and axiomatic thinking that reported the unique and joint effects of thinking types on one another converge to promote students' performance in derivative (e.g., Borgen & Manu, 2002; Hahkiöniemi, 2006; Maull & Berry, 2000; Orton, 1983; Selden, Selden, Hauk, & Mason, 1999; Zandieh, 2000). This implies that the relative contribution of each thinking type to performance in derivative may serve as a target or source beyond the relative contributions of other thinking constructs. By its very mathematical nature, students' thinking in concept definitions, symbols, and rules directs their attention towards the relevant features of the variables and conditions in the problem context. The arrangement of this thinking leads them to operate on formulas and extend these formulas to algorithms. The dynamics nestled in the practice and mastery of procedures provides students the ability to read a graph, gain meaning from it, and make visual interpretations. It is then, in a more figurative sense, students illustrate embodied notions that help transcend graph constructions. Taken literally, these iconic incorporations facilitate the coordination of enactive ideas in manipulating these ideas to develop a model and linking the model to real-life applications. The formation of such processes prompts students to think algebraically in unfolding the network of theoretical justifications. Algebraic ideas, hence, involve the algorithmatizations that students use to endow them with meaning in their encounter with hypothesis testing. These building blocks further empower students to reflect on a set of theorems or "if-then" statements and make inferences. The evidence in this causal picture led us to predict that two-level relationships among enactive, iconic, algorithmic, algebraic, formal, and axiomatic thinking.

The relationship among students' conceptual-embodied thinking, proceptual-symbolic thinking, and formal-axiomatic thinking has long been acknowledged as a critical factor in calculus success (Tall, 2004). The interrelation among different thinking types was particularly investigated in the domains of functions (Christou, Pitta-Pantazi, Souyoul, & Zachariades, 2005; Tall, 2004, 2005, 2008), eigenvalues and eigenvectors (Lapp, Nyman, & Berry, 2010; Stewart, 2008; Stewart & Thomas, 2009), linear dependency/independency (Ertekin, Solak, & Yazici, 2010; Stewart & Thomas, 2007, 2009), limits (Pinto & Tall, 2002; Tall, 2001; Weber & Alcock, 2004), and derivative (Hahkiöniemi, 2006) but derivative is, perhaps, one of the concepts that has attracted the least research attention in the framework for mathematical thinking based on the theory of three worlds of mathematics thus far. In addressing the relationship, most researchers reported that types of thinking describe a hierarchy that grows in sophistication and that these interconnected developments are available to, and are used by students further through their progress in mathematics (Tall, 2008).

Researchers assessed the conceptual-embodied thinking through tasks that involve modeling real-life applications (enactive thinking) and visualizing applications in the context of graph constructions/interpretations (iconic thinking). The proceptual-symbolic thinking was assessed through tasks that involve procedural computations (algorithmic thinking) and interpretations of generalized expressions and functional relationships (algebraic thinking). Finally, the formal-axiomatic thinking was assessed through tasks that involve "what" and "which" type of questions in the context of primary level of definitions and symbols (formal thinking) and/or that involve "prove that" or "show that" questions in the context of abstract level of if-then statements (axiomatic thinking). These three mathematical thinking constructs, however, were not classified as enactive, iconic, algorithmic, algebraic, formal, and axiomatic thinking rather generally simplified as embodied, symbolic, and formal thinking. Whether researchers are speaking of embodied, symbolic, and formal thinking they hold to the same premise that any of these types of thinking involve enactive, iconic, algorithmic, algebraic, formal, and axiomatic thinking (Bruner, 1966; Fischbein, 1983; Hughes-Hallett, 1991; Tall, 2004).

In the case of derivative concept, these research findings brought about six noteworthy points to delineate the context of mathematical thinking tasks. First, the context of enactive thinking tasks were mainly reserved for rate of change and population growth, and to a lesser extent for optimization. Second, the context of iconic thinking tasks mostly involved either one of graph constructions or graph interpretations of a function and its derivative but ignored the assessment of both processes simultaneously. Third, the context of algorithmic tasks was limited to the applications of simple differentiation formulas such as power rule or the product rule thereby excluding the computations of other differentiation algorithms such as evaluating the derivative of a function via limit of a difference quotient. Fourth, the context of algebraic thinking tasks required students to make explanations for why a certain differentiation rule holds at a specific point or whether a solution exists for a given function at a specific point, however, did not provoke them to test the hypotheses of fundamental differentiation theorems such as the Mean Value Theorem or Rolle's Theorem. Fifth, the context of formal thinking tasks only included the meaning of the derivative concept or the meaning of the symbol dy/dx, thereby not going further to comprise other essential differentiation concepts, symbols, and facts. Lastly, the context of axiomatic thinking tasks were based on the justification of conditional inferences relevant to basic differentiation rules. While a substantial research literature was available to document the context of iconic, algorithmic, and formal thinking tasks, far less research has served us to outline the context of enactive, algebraic, and axiomatic thinking tasks in derivative concept. For example, in the case of axiomatic thinking tasks much research effort has gone into assessing students within the context of number theory (Alcock & Inglis, 2008; Furinghetti & Morselli, 2009; Martin & Harel, 1989; Morris, 2002; Stylianides & Stylianides, 2009), set theory (Moore, 1994; Stewart, 2008), matrix algebra (Sowder & Harel, 2003), and/or logic (Inglis & Simpson, 2008; Nelson & Hannan, 2002; Stylianides, Stylianides, & Philippou, 2004). The bulk of assessment contexts reviewed to outline the tasks that reflect different types of mathematical thinking further put forth researchers' tendency to rely on self-developed open-ended questions while few of them was found to utilize multiple-choice questions in examining students' mathematical thinking. And those that are available were restricted to matching or true/false questions (Engelbrecht, Harding, & Potgieter, 2005; Goerdt, 2007; Stewart, 2008).

An important development in the search for the most conventional setting for mathematical representation of real-life phenomena has been the emergence of enactive thinking perspectives where students are required to model applications of mathematics to the given situation. These views result in the interpretation of mathematical models by which students explore mathematical demands paying attention to the potential enactive thinking encapsulated in the use of the model. Thus, a growing number of researchers point to a very powerful enactive chain in the concept of derivative that can be traced from the way students think about the real-life phenomena sustained by a model in general and the modeling of area/volume optimization (Carreira, 2001; Davis, 2007; Doerr & Tripp, 1999; Gravemeijer, 1999; Liu & Niess, 2006; Mousoulides, Christou, & Sriraman, 2008; Ubuz & Ersoy, 1997). They acknowledged that the enactive thinking highlights the net of ideas strongly combined with iconic aspects and properties of the model that also extends some aspects of formal thinking which incorporated and rearranged pieces of algorithmic and algebraic thinking embedded in the model. The results documented that students struggle to make the leap from the real-life phenomena to the differentiation. Students' conceptualizations about the enactive thinking were dominated by modeling into a function, differentiating the function, and setting the function to zero. Bagni and Menghini (2005) supported that their tendency to model the real-life situations into formulae was one of the main obstacles to the appreciation of finding the optimal way of doing something, which further led them to reluctance on retaining the iconic setting.

Studies mainly focused on the effect of computer technology in helping students to create models through data collection, experimentation, and representation as they employ enactive thinking by modeling the real life situations into the problems of optimization (Drijvers, 2000; Klymchuk et al., 2010; Malaspina & Font, 2010), rate of change (Confrey & Smith, 1994; Doerr & Zangor, 2000; Gravemeijer & Doorman, 1999) and maximum/minimum (Ubuz & Ersoy, 1997). The findings provided evidence about the paths of enactive thinking to gear forms of describing and organizing real-life phenomena, to project iconic thinking (Orton, 1983; Ubuz & Ersoy, 1997) onto enactive thinking, and to rebuild formal (Gravemeijer & Doorman, 1999; Orton, 1983) and algorithmic (Gravemeijer & Doorman, 1999; Orton, 1983) thinking. These paths further affirmed that the production of enactive thinking in tackling with a model into applied derivative problems also indicates axiomatic thinking (Zandieh, 2000) can become algorithms as algorithms become part of the theorization. Yackel, Rasmussen, and King (2000) stated that the mathematics explored within the real-life situation is given some sensible meaning from its relation with tangible links to different types of mathematical thinking. It is widely acknowledged that when students are offered real-life problems that involve the use of models into derivative, they are challenged and compelled to unpack enactive thinking underlying applications of derivative (e.g., population growth). According to Drijvers (2000), once the construction of enactive scaffolds takes place, that is, once the enactive senses start to be appropriated, the activity of optimizing (e.g., the area of a rectangle) becomes valid. Gravemeijer and Doorman (1999) supported the mediating nature of enactive thinking in the way students engaged in the process of enchaining definitions, algorithms, and graphs. They stated that students were regularly drawing on their formal thinking repertoire to understand and make sense of the the real-life problem to exert the meaning of the derivative as "rate of change" together within algorithms. In accord with the findings of Zandieh and Knapp (2006), many students' enactive thinking flowed as an extension of their formal and iconic thinking that indicated a bodily basis for rate of change as an approximation for the derivative. Through its iconic thinking potentiality, enactive thinking opens up the scope of algorithmic (Drijvers, 2000), algebraic (Kutzler, 2000), formal (Doerr & Zangor, 2000), and axiomatic (Zandieh & Knapp, 2006) thinking in the concept of derivative. Kutzler (2000) emphasized that enactive ideas provide a double anchoring for the production of algebraic thinking offering duplicated references by the links between the definition of derivative as "the slope of a tangent line at a given point" embedded in the optimization context and referential theorems. He further indicated that enactive thinking links to the representations of algorithms and theorems compensating each other in the language of algebraic thinking and is even used as a form of visualization within iconic thinking to state the effects of different definitions of the derivative such as rate of change and the slope of a tangent line.

The presence, role, extent, and constraints of iconic thinking in conjunction with visual-graphical states in the problem-solving processes of university students as they solved derivative problems challenged research approaches because of the difficulties in apprehending the construction and use of visualization. Iconic thinking used in solving derivative problems is accepted to be related with not only formal (Berry & Nyman, 2003; Orton, 1983; Viholainen, 2005), algorithmic (Hahkiöniemi, 2006; Orton, 1983; Tall, 2002), algebraic (Kutzler, 2000), and axiomatic (Viholainen, 2008) ideas, but often laden with insights of enactive thinking (Presmeg & Balderas-Canas, 2001; Zandieh & Knapp, 2006). Researchers persistently acknowledge that the graphical representations of the derivative concept influence students' generating formal meanings and making algorithmic inventions. Tall (1994) argued that for the advanced mathematical thinking ways of formalization, symbolization, and algorithmatization to emerge in students' activities; accompanying visual mathematics should be experienced. Following Tall's remarks, Berry and Nyman (2003) take this line of argument one step further, by emphasizing a dialectical relation between images-in-use and formal-algorithmic sense making. According to this point of view, it is in the process of visualizing those formalizations and algorithmatizations emerge and develop their meaning. In this process, students' links between the graph of a function and its derivative shape the very formal thinking of the function itself from which they emerge, while at the same time, formal thinking shape the iconic manipulations that emerge. It is also possible to have the students demonstrate a fully algorithmic thinking, the interaction between procedural experiences of the differentiation rules and graphical interpretations of the extremas Aspinwall, Shaw, and Presmeg (1997) presupposed, that grow in concert with visual thinking. According to Viholainen (2008), it is these aspects of iconic thinking that may enable axiomatic solution processes of students. He pointed on students' engagements in iconic thinking through skilful drive along the formal definition of differentiability to its iconic aspects relevant to continuity as they prove whether the given function is differentiable at a given point. In contrast, Delos Santos and Thomas (2001) documented that iconic thinking is attempted for the purpose of sense making in the use of formal symbol dy/dx both as a derivative of y with respect to x and as a slope of the tangent line; however these attempts did not seem to cast light on students' iconic interpretations of dy/dx= k sufficiently to enable any further progress. Students' reactions to the derivative problems in iconic thinking context with regard to their attempts made to introduce symbols, usually in the form of memorized formulas, were reminiscent of the iconic thinking of students in Asiala et al.'s (1997) research. They persistently seek for an expression representing the function to differentiate rather than interpreting the derivative as a slope of the tangent line. Tall (2002) nevertheless inferred the presence of iconic thinking genesis in students' solving applications of the derivative that went straight to the sublimation of enactive thinking and its experiential foundations of iconic thinking.

Recent research underpinning the way derivative is learnt stress the importance of algorithmic thinking focused on its interconnectedness to formal (Orton, 1983; Viholainen, 2008), algebraic (Kendal & Stacey, 1999), axiomatic (Selden & Selden, 1995; Tall, 1989, 1998), enactive (Carreira, 2001), and iconic (Zandieh, 2000) ideas. The value of algorithmic thinking lies in its potential to reduce complex problems to simple procedures. However, as researchers have witnessed, algorithmic processes aligned only procedurally often emerge from automatical actions without an ability to understand advanced foundations of derivative's practical value. Viholainen (2008) calls on the part of algorithmic thinking which is often deeply ingrained, confound the development of formal and iconic thinking in representing problem situations, thus impel students to be reluctant on drawing inferences based on essential aspects of definitions or graphs of derivative (Weber & Alcock, 2004). In a similar vein, Eisenberg (1994) underscored "avoidance to visualize" and concurred with several researchers (Eisenberg & Dreyfus, 1991; Tall & Vinner, 1981; Vinner, 1989; Vinner & Dreyfus, 1989) who documented students' predominant reliance on routine algorithmic applications of the derivative concept in creating graphic solutions. This implies that students' procedural approaches can easily dominate their coming to grips with iconic thinking by developing and testing hypotheses (Marrongelle, 2007). However, the algorithmic solutions of students which were to some extent inspired by representations of iconic thinking formed the starting point for a series of discussions, in which a model-based representation of enactive thinking in derivative problems is emerged. Students' challenges of algorithmic techniques contributed to their solving conflicts in their modeling everyday life applications of the derivative (Orton, 1983; Ubuz & Ersoy, 1997; Viholainen, 2005). Stylianou (2002) suggested that the ability to present the derivative concept within realistic contexts and exhibit flexible transitions between procedures and organization of advanced enactive thinking insights strengthen the management of derivative tasks.

Although algorithmic thinking processes were understood as a collection of routine procedures, Clark et al. (1997) and Orton (1983) have argued that these routines can make advanced mathematical ideas explicit in solving theorem-based derivative problems that nestle axiomatic thinking. The claim is that algorithmic thinking is important in all facets of the concept of derivative including theorems. The argument for including algorithmic ideas as part of theorems is that students go through algebraic essences in the theorem and extract relevant numeric information by procedural insights. This parallels Gravemeijer and van Galen (2003) who presented algorithmic thinking in which algorithms emerge as the product of students' both algebraic and axiomatic thinking generalizations. Algebraically, to apply the Mean Value Theorem, is to be able to carry out the inductions involved through algorithms, not merely to be able to state its underlying certain facts. Thus, what a student thinks through algorithmic techniques, in addition to stateable theoretical knowledge, includes operational inventions (Delos Santos & Thomas, 2001) and further visual translations (Zandieh, 1998). Despite its controversial nature, differentiating algorithms which work well serve as a building block for the implementation of differentiation methods used in exploring differentiability (Viholainen, 2006). Indeed, they nestle links within the differentiation context that reflects standard closed form solutions fulfilled with algorithmatizing activities.

Algebraic thinking has been considered to be an essential characteristic of mathematics in terms of views of its role and its essence. A more recent tendency among researchers is to investigate the forms to which it adheres. Nathan and Koedinger (2000) sketch this adherence by an initial emphasis on theorems that almost exclusively involve formal thinking linkages of definitions and symbols. This particular way, in which students formally acted underpins the emergence of their algebraic thinking was regulated by a conceptually established process where formal thinking plays a central role. The immersing and initiating students into the particularities of algebraic meanings in which the applications of algorithms are grounded delineate the operational character of algebraic thinking. According to Sfard and Linchevski (1994), students' emergent algebraic thinking, as required in the application of algorithms, appeared hence as the orientation of a highly specialized kind of procedural praxis requiring a critical use of algorithms to achieve specific expressions of algebraic thinking. Iconically, from the point of view of the students, every iconic representation is a legitimate component of a theorem that it can convey algebraic thinking insights to obviate the need for rigor in the algebraic thinking acquired through visualization (Francis, 1996). Derry et al. (2007) argued visualization through algebraic thinking makes it possible not only to transform statements in the theorem, alter definitions, and manipulate symbols but also to examine algorithms in theorems that are inaccessible without algebraic thinking. They appeared to think of students' algebraic thinking development largely within the iconic thinking precedence view. However, analyses of Bagni and Menghini (2005) suggested an enactive thinking direction for the development of algebraic thinking that circumvents many of the misunderstandings caused by iconic thinking. In this enactive thinking direction, translating a theorem into an equation demands the contraction of the theoretical statements. That is, the equation narrates the theorem to allow for manipulations of algebraic thinking. With these views in mind, Radford and Puig (2006) framed students' algebraic thinking processes to be more effective when they are employed in the form of both enactive and iconic thinking. However, Furinghetti and Morselli (2009) documented that embedding the models or visual properties at issue in a form of representation suitable to an algebraic thinking is one of the main difficulties of the required axiomatic statements. In justifying axiomatic statements, it is this difficulty which prevents students to adopt increasingly sophisticated modes of algebraic thinking. Selden and Selden (2005) cited students' weak ability in translating back-and-forth between algebraic and axiomatic thinking practices while unpacking algebraic statements into the axiomatic language of proofs. This adds to the deficiencies in students to trivialize theorems and relevant algorithms in moving from informal reasons to proper axiomatic arguments. It is important to explicitly stress a closing remark that fundamental to laying a foundation for meaningful algebraic thinking is to employ versatile thinking of complemented overall grasp of the advanced mathematics (Tall & Thomas, 1991).

The sequence of research leading to the development of formal thinking for pervading sophisticated insights has provided fertile ground to investigate the links to enactive (Carreira, 2001), iconic (Pinto & Tall, 2001), algorithmic (Bagni & Menghini, 2005), algebraic (Derry, Wilsman, & Hackbarth, 2007), and axiomatic (Epp, 2003; Hanna, 2000) thinking. The analyses presented in these studies reflected the potentiality of formal thinking in gearing advanced mathematics throughout secondary level (Gray & Tall, 2007; Lima & Tall, 2008; Sajka, 2003) and undergraduate level (Alcock & Simpson, 2004, 2005; Przenioslo, 2004, 2005; Szydlik, 2000; Zandieh, 2000). The emergent relations delineating formal thinking at university can be viewed in limits (Przenioslo, 2004, 2005), functions (Williams, 1998), and derivative (Hahkiöniemi, 2006; Orton, 1983; Viholainen, 2005; White & Mitchelmore, 1996; Zandieh, 2000).

There is a body of research suggesting that students lack rich understandings of the formal definition of the derivative along with relevant rules and symbols that prevent them from correctly applying algorithms (Asiala et al., 1997; Clark et al., 1997; Viholainen, 2005) and explicitly interpreting graphs of a function and its derivative (Berry & Nyman, 2003; Orton, 1983). For example, Orton (1983) reported that students are often unable to give the correct definition of the derivative for clearly stating the symbols of differentiation as well as the approaches to differentiation. There were clear indications from his study that students' formal thinking processes are basically concerned with ratio and proportion. Moreover, they do not regularly see the connections between the meaning of rate of change and its graphical representations. The deficits in stating the rate of change formula prohibit students from carrying out the appropriate procedures in derivative and further adopting its definition to model real-life situations in the given task. Although it seems that students' difficulties with the application of derivative rules are typically altered throughout their formal thinking in translation and interpretation of the symbols (Chin & Tall, 2000; Clark et al., 1997; Roorda, Vos, & Goedhart, 2007), it is equally clear that students' interpretations of those symbols are in place before they begin their illustrations on graphs (Delos Santos & Thomas, 2001). Much that is accepted as a sign that students are in possession of algorithmic and iconic thinking operations of the derivative consists in their being able to state certain facts, rules, and theorems. Their appropriation of formal ideas re-emerge to aid in customizing maximum-minimum viewpoints in optimization problems (Ubuz & Ersoy, 1997; Ubuz,

2007) or theorem-based applications of differentiability (Viholainen, 2005) and further in exploiting links between the graph of a function and its derivative (Berry & Nyman, 2003). Students' subsequent codings of the definition of derivative and relevant notations into axiomatic forms postulate that formal methods can harmonize the essence of differentiation (Zandieh, 2000). Moreover, formal thinking reflects the evolution of the derivative concept from an algebraic thinking application of the differentiability theorem to an algorithmic thinking manipulation grasping the relationships between continuity and differentiability (Bagni & Menghini, 2005) characteristic of the capacity to algorithmatize the theorem by recalling the theorem statement when desired. Tall (2004), citing the shift in students' thinking about embodied structures to more formal notions, highlighted that, for students, formal thinking can function as a gateway in building from definitions and marginalizing fundamental symbols, notations, facts, and rules.

The notion of axiomatic thinking in the derivative has traditionally been associated, almost exclusively, with university mathematics. In recent years, many researchers have recommended that axiomatic thinking become central to all students' experiences with derivative throughout the execution of proofs in increasing/decreasing functions (Epp, 2003), differentiability (Mills & Tall, 1988), and differentiation rules (Tall, 1989; Viholainen, 2007). Researchers have found that, in general, students' difficulties with generating axiomatic thinking are related to the following factors: (a) understanding the nature of proofs, (b) formulating statements, and (c) applying procedures to deduce the truth of statements. They have identified students' abrupt introduction to proof in university as a possible explanation for these difficulties that students face with proof (Harel & Sowder, 1998; Martin & Harel, 1989; Selden & Selden, 2003), thereby proposing that students engage with axiomatic thinking in a coherent duplication of different types of mathematical thinking. Henceforth, a richly differentiated mathematical thinking of axiomatic statements (Bell, 1976; de Villiers, 1990, 1999; Hanna & Jahnke, 1996; Hanna, 2000) arise as

(a) demonstrating formal definitions and symbols, (b) yielding algorithms within relevant procedures, (c) incorporating a well-known fact into an axiomatic framework from an enactive perspective, (d) communicating the algebraic transmission of theorems within the meaning of a definition, consequences of an assumption, and systematization of an algorithm, and (e) constructing iconic representations to verify or explain the statement. Barwise and Etchemendy (1991) documented that the content of axiomatic thinking in the derivative directs to the manipulation of the aforementioned processes rather than solely to the interpretation of formal structure of sentences. They indicated that axiomatic thinking in the derivative proceeds on the basis of explicit rules of differentiation that taken as a whole apply to different types of mathematical thinking. Given this compression, successful engagement with axiomatic thinking requires several advanced thinking processes by students. One such process is to understand the information in the proofs (Gibson, 1998). Indicative of the start off for an axiomatic thinking attempt is recognizing the definitions and symbols of all the elements in the statement to generate hypotheses and draw conclusions. The amalgam of formal thinking, an output of definitions and symbols helps students to become entangled with the completeness of the axiomatic thinking. This power of formal definitions and symbols to evoke axiomatic thinking caused Epp (2003) to suggest that it is important for students to learn to express the definitions of axiomatic-based concepts such as not, or, if-then, if-and only if to consolidate facts about negations of axiomatic statements. For instance, they should insert the meaning of “if-then” to understand that a conditional statement is false if, and only if, its hypothesis is true and its conclusion is false. In axiomatic statements, formal thinking starts as simple structures of definitions and grows in interiority with the symbols. However, many students lack respect for formal thinking such as using words like if-then, if-and only if or symbols like $\forall$, $\exists$ that make a crucial difference to the interpretation of these axiomatic statements (Inglis & Simpson, 2008). Pointing out the axiomatic thinking was further emphasized in having students charmed to encounter algorithmic implications whose command of axiomatic thinking is transitive (Epp, 2003; Hanna, 2000). Another important ability for successful engagement with axiomatic thinking is to discover algebraic thinking for showing the conclusion of a theorem is true whenever its hypothesis is true. This algebraic thinking argument arises from a sequence of connected ideas and connections between these ideas to form an axiomatic thinking argument (Gibson, 1998). According to Giaquinto (1994), such axiomatic thinking arguments can in turn lead students to grasp the structure of algebraic thinking arguments and their ramifications that yield an explicit understanding of every link within the explorations precisely formulated for tentative interpretations of a theorem. An important ability for successful performance in this process comes to the scene that presents the role of enactive thinking. When the complexity and subtlety of axiomatic ideas become conceived as a zigzag path (Stylianides & Stylianides, 2008) between enactive thinking attempts to generate valid arguments and criticisms of these attempts, students become able to sensitize the importance of enactive thinking that leads them to unambiguous and meaningful conclusions. Epp (2003) suggested that students should be challenged to convey very clearly axiomatic thinking the interplay of experimentation and its relation to the real world within enactive-supported explorations. Finally, one other important ability for engagement with axiomatic thinking is to use iconic thinking. The studies that examined the use of iconic thinking representations within axiomatic statements, and in particular their potential contribution to exhibit axiomatic thinking, acknowledged that iconic thinking plays a heuristic role in employing axiomatic thinking viewpoints (Barwise & Etchemendy, 1991; Epp, 2003; Gibson, 1998; Hanna, 2000; Harel & Sowder, 2007; Zimmerman & Cunningham, 1991). Such considerations have gained in scope and status in part because iconic thinking representations have increased the possibilities of visual aspects to promote both exploration and experimentation. Although, there is strong evidence for the impact of iconic thinking on exerting axiomatic thinking aspects, some researchers also realize that misleading visualizations abound (Brown, 1999). However, iconic thinking is widely intensified to have promise in providing evidence for not only an axiomatic statement but also for its justification. Iconic thinking aids have been welcomed as visual accompaniments to axiomatic statements, where they can inspire both the theorem to be proved and approaches to the proof itself. Gibson (1998) postulated that students found axiomatic statements more understandable when they think about them in iconic thinking terms and that iconic thinking reduces the burden proving has placed in their minds. In the same vein, Hanna (2000) reported that using iconic thinking representations help students organize their axiomatic thinking and give them concrete mathematical objects to hang onto while they deal with the abstraction of axiomatic statements.

The focal point of the most well-known debate relating to conceptual-embodied thinking concentrates on the fact that conceptual-embodied thinking is applicable and useful for analyzing and finding solutions for very diverse practices in differentiation beginning with proceptual-symbolic thinking (Campos & Estrada, 1999; Klymchuk et al., 2010) and ending with formal-axiomatic thinking (Malaspina & Font, 2010; Porzio, 1999). While most analysis of conceptual-embodied thinking was focused on perspectives of visualization, researchers aimed to explain, discuss, and exemplify how processes which are prior to accompanying conceptual-embodied thinking influence proceptual-symbolic and formal-axiomatic thinking. They were mostly interested in the pre-proceptual-symbolic thinking processes, or as Tall (2002) describes, the part of routine applications which is on the periphery of conceptual-embodied thinking. Taken together, conceptual-embodied thinking is largely concerned with proceptual-symbolic thinking and to some degree concerned with formal-axiomatic thinking (Christou et al., 2005).

Concomitantly, proceptual-symbolic thinking in derivative can be coded and recoded in different conceptual-embodied thinking implementations (Tall, 2002). Moreover, it can be transferrable over formal-axiomatic thinking situations and conditions (Stewart & Thomas, 2009). Thus, proceptual-symbolic thinking is particularly important for upholding the formal-axiomatic thinking innovations that are deemed essential for conceptual-embodied thinking. This connection can be interpreted as differentiation requirements, emphasizing effective use of procedures and mastery of algorithms in operational form.

From this interactional point of view, formal-axiomatic thinking in derivative is conceived of as a process that can be described as development of derivative theorems, with the help of basic differentiation terminology (Tall, 1999). This approach can be referred to as the integration of formal-axiomatic thinking into conceptual-embodied and proceptual-symbolic thinking. Within this incorporation, formal-axiomatic thinking can be seen as a process of visual change of actions in derivative graphs (conceptual-embodied thinking) that may take place when students engage in meaningful guidance of differentiation rules (Watson & Tall, 2002). This process may also receive its content and structure through the use of differentiation techniques (Gray & Tall, 2007).

Taken as a whole, the underlying thread of the relationships among different types of mathematical thinking in derivative is an advanced blend of thinking processes supplemented with specialized mathematical meanings at the within- and between-classroom levels. It comes equipped with an extensive range of concepts and subconcepts of the derivative, including definitions, symbols, notations, algorithms, theorems, graphs, diagrams, and model applications for presenting the derivative at an advanced level. The considerations on this interrelationship collectively embedded the understanding of derivative as the object of investigation in mathematics education (Habre & Abboud, 2006; Hahkiöniemi, 2004, 2005a, 2005b, 2006; Sanchez-Matamoros, Garcia, & Llinares, 2008; Ubuz, 2007; Zandieh, 2000). This is at variance with Harel, Selden, and Selden's (2006) emphasis on the importance of an inclusive conception of derivative to accommodate a set of significant ideas necessary to perform different mathematical thinking structures around the relationships among the conceptual views, procedural applications, and visual constructions of the derivative. Drawing on Dubinsky, Cordero, Hillel, and Zazkis (1998) who suggested researchers to design frameworks that take into account the interrelation of advanced mathematical concepts, procedures, and representations, the present study intends to orient the function of enactive, iconic, algorithmic, algebraic, formal, and axiomatic thinking in derivative with reference to their both within- and between-classroom transitions.

2.4 THE PRESENT STUDY

Having established these facts mentioned above, the present study aimed to test the interrelationships among mathematical thinking constructs (enactive, iconic, algorithmic, algebraic, formal, and axiomatic thinking). We estimated two-level structural models to investigate the hypothesized effects of the selected thinking constructs on each other. The structural relationships among these constructs were interpreted as indices of effects of one construct on the other at both within- and between-levels. Three main research questions were addressed.

First, what is the factor structure of mathematical thinking at the within- and between-classroom levels? More specifically, we hypothesized that the application of multilevel exploratory and confirmatory factor analyses would demonstrate a distinct latent factor structure at the within-classroom and at the between-classroom level (Hypothesis 1).

Second, to what extent do the relationships among different types of mathematical thinking vary at the within- and between-classroom levels? Based on preliminary evidence from the pattern of results in mathematical thinking research it was expected to find statistically significant variation in the relationships among enactive, iconic, algorithmic, algebraic, formal, and axiomatic thinking at the within-classroom level; and in the relationships among conceptual-embodied, proceptual-symbolic, and formal-axiomatic thinking at the between-classroom level (Hypothesis 2).

Finally, what are the cross-level interactions among different types of mathematical thinking at the within- and between-classroom levels? Based on preliminary evidence from the pattern of results in mathematical thinking research it was expected to find statistically significant cross-level interactions at the within- and between-classroom levels. More specifically, in regard to testing lower level mediation models, it was hypothesized that enactive, iconic, algorithmic, algebraic, formal, and axiomatic thinking will moderate the relationships among conceptual-embodied, proceptual-symbolic, and formal-axiomatic thinking; whereas in regard to testing upper level mediation models, it was hypothesized that conceptual-embodied, proceptual-symbolic, and formal-axiomatic thinking will moderate the relationships among enactive, iconic, algorithmic, algebraic, formal, and axiomatic thinking (Hypothesis 3).

From a theoretical point of view mathematical thinking is framed by enactive, iconic, algorithmic, algebraic, formal, and axiomatic thinking. Students attached to the effective use of models, graphical representations, procedures, hypotheses, definitions, and proofs are more apt to align their progress with a sophisticated blend of mathematical thinking. Many students predominantly emphasize one type of thinking when doing mathematics, while using the other type of thinking infrequently. However, the essence of mathematical thinking most likely requires developing competence with both informal mathematics (e.g., iconic thinking and algorithmic thinking) and formal mathematics (e.g., algebraic thinking and axiomatic thinking). Similar issues of concern point in the same direction: towards the hypothesis that students should espouse the blend of mathematical thinking in other subject areas such as linear algebra, real analysis, numerical analysis etc. In this sense, mathematical thinking model in the present study presents two-level relations specific for calculus as well as provides insightful directions that can be adapted to measures in other subject areas at the university level.

From a structural point of view using a multilevel framework we were able to establish two-level associations among different types of mathematical thinking. Such associations hold the promise to better integrate within- and between-classroom models mathematical thinking. In this vein, the mathematical thinking model in the present study offers researchers a concentrated focus on the interrelations of mathematical thinking to further interrogate and/or understand their affects within other cognitive constructs such as mathematical reasoning (e.g., inductive and deductive reasoning) (Simon, 1996), mathematical understanding (e.g., instrumental and relational understanding) (Skemp, 1978, 1987), and mathematical knowledge (e.g., declarative, conditional, and procedural knowledge) (Aydin & Ubuz, 2010).

CHAPTER 3

METHODOLOGY

This chapter involves the methodology of the study comprising population and sample, the development of the instrument together with their validity and reliability, procedure, data collection and analysis including the multilevel structural equation modeling techniques.

A nonexperimental design with hierarchical data was used. Students were considered to be nested within classrooms. Information was obtained for each level of nesting in the sample, thus the focus was on the within-classroom (students) and the between-classroom (classrooms) level. No control group of students who had experienced a nonmathematical vision existed since undergraduate students in state universities that had adopted, in wholesale vision, a calculus course at their first and/or second years at the university. The lack of experimental manipulation indicates that the results of the present study support inferences about the relationships among different types of mathematical thinking together with the magnitude of these relationships. However, as a consequence of the cross-sectional designs the results of the present study do not generally support strong causal inferences.

3.1 POPULATION AND SAMPLE

The target population of this study consists of all undergraduate students in Turkey who enrolled in calculus courses during the 2009-2010 and 2010-2011 academic years. The accessible population is all undergraduate students attending to state universities in eight different cities in Turkey. Total number of university students in Turkey attending to the state and foundation universities is 3.529.334. Of these students 924.536 (424.330 females and 500.206 males) major in a four year undergraduate program at state universities. For the aim of the present study two independent samples of undergraduate students who were enrolled in Calculus courses at their first or second year at the university. The total number of the students in these two different samples was 3523.

Sample 1 included 1099 undergraduates (548 females and 551 males) within 72 classrooms from 8 state universities. Participants were freshmen (n = 415), sophomores (n = 154), juniors (n = 317), and seniors (n = 213) with an age range of 17 to 25, comprising 348 students from Faculty of Education, 364 students from Faculty of Arts and Sciences, and 387 students from Faculty of Engineering. Sample 2 involved 2424 undergraduates (1161 females and 1263 males) within 134 classrooms from 9 state universities in Turkey. Participants were sophomores (n = 924), juniors (n = 911), and seniors (n = 589) with an age range from 17 to 25, including 835 students from Faculty of Education, 817 students from Faculty of Arts and Sciences, and 772 students from Faculty of Engineering. The mean number of students per classroom for Sample 1 and Sample 2 was M= 15.25 and 18.64, respectively. State universities are institutions of higher education that accept students on the basis of their scores on the University Entrance Examination (YGS) conducted by the Student Selection and Placement Center (OSYM). This assessment measures high school students' general educational development and their capability to complete university-level work with 240 multiple-choice questions equally distributed under two domains: Common General Knowledge Courses (Turkish Literature, History, General Geography, Philosophy, Mathematics, Geometry, Physics, Chemistry, and Biology) and Content Courses (Turkish Literature, Geography of Turkey, Geography of Countries, Psychology, History, Sociology, Logic, Mathematics, Geometry, Physics, Chemistry, and Biology). The first domain assesses students cumulatively on the ninth and tenth grade curriculum, whereas the latter requests them to answer questions generally related to the eleventh and twelfth grade curriculum. In this sense, although the subject names are the same, their contents are different in terms of secondary school curriculum. It takes students 195 minutes to complete this test. To be accepted to state universities students are required to take the adequate score on these two domains for making preferences of departments and grant an academic degree. In every semester, students attending to state universities have to pay a certain fee to the Institute of Higher Education (YOK) depending on the departments they are accepted.

For the purpose of the present study, as well as the methodological issues (e.g., complex sampling designs) concerning multilevel modeling techniques (students within classrooms) cluster sampling was used. In regard to the *multistage* (Stapleton, 2006) sampling procedures, the first level of selection (i.e., primary sampling unit) included random selection of classrooms in a faculty and then the sampling of all students in the selected classrooms. Characteristics of students indicated that the Sample 1 and Sample 2 mirrored the target population in terms of gender distribution, age, high school performance, and socioeconomic status. The selection here mainly revolves around the question of whether differences in mathematical thinking are more a reflection of state universities which were included in the present study. Couched in how students from diverse graduate degrees and different mathematical backgrounds would reflect on the items contextualized in different types of mathematical thinking, the participants have the potential to represent the general student body in state universities. Furthermore, both samples were identified as appropriate on the basis of the departments that students were majoring. That is, the participants have the potential to represent the general student body in Faculty of Education, Faculty of Arts and Sciences, and Faculty of Engineering included in the present study.

3.2 INSTRUMENT

3.2.1 THINKING-IN-DERIVATIVE TEST (TDT)

A multiple-choice test covering the content of derivative was developed to measure undergraduate students' mathematical thinking (see Appendix A). The development of the multiple-choice test items related to the six theory-driven thinking constructs: enactive thinking, iconic thinking, algorithmic thinking, algebraic thinking, formal thinking, and axiomatic thinking to be included in the Thinking-in-Derivative Test (TDT) are presented in the following lines. Furthermore, the validation of the six-factor structure of the TDT by use of the confirmatory factor analysis and further validation of this structure followed by the examination of subgroup validity coefficients, and the estimation of the reliability of students' scores on the TDT is also addressed.

At the beginning of the test development process, it was primarily asserted whether the content of the TDT is intended to be unidimensional or multidimensional. Accordingly, for the general content domain of the test a name was provided (i.e., Thinking-In-Derivative), and in regard to the review of the related literature on the derivative concept and mathematical thinking, six subdomains (enactive thinking, iconic thinking, algorithmic thinking, algebraic thinking, formal thinking, and axiomatic thinking) were identified. Since it was believed subdomains were useful in covering the concept of derivative, a multidimensional interpretation was considered to be appropriate.

Raymond and Neustel (2006) recommended a practice analysis as a means to establish the criticality and frequency of use of knowledge and thinking skills in a content domain. They indicated that it is important to define the construct and to link the definition to the test via a survey of the content. Regarding the context of the derivative concept in a standard calculus course content, a practice analysis was conducted to identify the content domain (i.e., the derivative concept and applications of derivative) and its subdomains (i.e., basic concepts, rules, and facts, graph sketches and/or interpretations, maximum and minimum problems). The design of the TDT outlined by the practice analysis was intended to speak to the idea that the test is multidimensional with well-defined categories of content subdomains. In this sense, the test development activities proceeded with the idea that each item reflects one of these different subdomain content categories.

After the practice analysis was completed, four alignment criteria (Webb, 2006) were utilized to frame the content of the TDT: (a) the content topics of derivative to be included in the TDT, (b) the complexity of the test items, (c) the range of derivative content to be covered, and (d) the degree of emphasis to be given to specific derivative content expectations. To respond correctly to TDT items, students not only need to be familiar with the differentiation content being assessed, but also they need to exhibit a range of cognitive behaviors to energize their mathematical thinking in the differentiation topic. More specifically, mathematical thinking in the differentiation topic consists of having the ability to proceed effectively in numerous behaviors like recalling, recognizing, computing, retrieving, selecting, representing, modeling, solving, analyzing, generalizing, synthesizing, integrating, and justifying. Thus, each mathematical thinking domain included items developed to address specific cognitive behaviors. Ultimately, this process led to the development of the general test specifications that identify the behaviors that students should activate when engaging in different mathematical thinking subdomains (see Table 3.1). Item development was accordingly based on these test specifications, and that each item has a content identifier that uniquely places it in one and only one subdomain.

This process consequently included an indepth investigation of the calculus textbooks (e.g., Adams, 1999; Balcı & Aral, 2003; Balcı, 1997; Bittinger, 2004; Goldstein, 2007; LaTorre, Kenelly, Fetta Reed, Harris, & Carpenter, 2005; Stewart, 2003; Swokowski, Olinick, & Pence, 1994), calculus books (e.g., Boyer, 1949; Chartrand, Polimeni, & Zhang, 2008; Harcharras & Mitrea, 2007; Nelsen, 1993, 2000; Solow, 2001; Thompson & Gardner, 1999), calculus course materials (e.g., the handout materials of M119 calculus course offered in Middle East Technical University and the handout materials of M109 calculus course offered in Ankara University), calculus lecture notes (e.g., Akkoç, 1982), journal articles (e.g., Modica, 2010; Orton, 1983, Zandieh & Knapp, 2006), doctoral dissertations (e.g., Hahkiöniemi, 2006; Ubuz, 1996; Viholainen, 2008) and university entrance examination questions (e.g., 1981-2010). The typical questions in the existing instruments tended to be open-ended to which students were asked to provide their responses and explanations. This approach was shifted to the multiple-choice format for this study. In explication, multiple-choice items are used to tap more adequately all the aspects of a construct (Haladyna, Downing, & Rodriguez, 2002) and the reliance on this format contributes to overemphasize on testing student learning at the expense of the more difficult-to-measure cognitive abilities (Frederiksen, 1984).

Despite the dominant role that open-ended questions have played in assessing mathematical thinking of students, drawing a direct link from the open-ended question solutions to student misconceptions and errors facilitated the development of multiple-choice items. The indepth analysis of the results of previous research that utilized open-ended assessment context served as an important lever for developing the distractors.

Guided by the practice analysis and general test specifications, a question pool was constructed including 183 multiple-choice items contextualized in enactive, iconic, algebraic, algorithmic, formal, and axiomatic thinking. Multiple-choice items were then re-evaluated to eliminate those which appeared redundant or ambiguous and those which appeared irrelevant to a specific subdomain. Items were chosen on the basis of their membership in the content subdomains denoted in the test specifications.

Table 3.1 The Cognitive Behaviors for the Mathematical Thinking Subdomains

Mathematical Thinking Subdomain	Cognitive Behaviors
Enactive Thinking	Model the differentiation problem in the context of a real-life phenomenon. Generate an appropriate model such as an equation to translate the differentiation problem into a function to be maximized or minimized.
Iconic Thinking	Retrieve information from the graph of a function and/or derivative function to interpret a set of graphical information. Construct graphical representations for a given differentiation relationship.
Algorithmic Thinking	Compute derivatives via an appropriate differentiation method. Evaluate derivatives to carry out algorithmic differentiation procedures. Solve routine differentiation problems.
Algebraic Thinking	Integrate linkages between a differentiation theorem and its hypotheses. Synthesize the differentiation theorem to apply its hypotheses. Justify the truth or falsity of a differentiation theorem statement by reference to its hypotheses.
Formal Thinking	Recall the definitions, notations, and conventions relevant to differentiation concepts. Recognize the differentiation facts, rules, and terminology.
Axiomatic Thinking	Analyze differentiation theorems to determine, describe, and use relationships between mathematical facts and situations. Generalize the justifications to which the inferences are applicable by restating them in more widely terms of differentiation.

Following that it was aimed to assess the content validity of the subdomains and to identify any particular items which may still be ambiguous or irrelevant. In order to achieve these goals, initially, this first version of the TDT was submitted to the advisor was submitted a table presenting the item number, the context of the item to the relevant thinking dimension, the objective of the item, the solution of the item, and the answer key of the item. To rank how well the items fit with construct definitions, a definition of enactive, iconic, algorithmic, algebraic, formal, and axiomatic thinking was also provided. The advisor was requested to analyze the questions in terms of their contribution to the relevant thinking context, comprehensiveness, and appropriateness to the content of differentiation in calculus course. With regard to her comments and recommendations 123 questions were dropped in terms of the overlaps in the objectives of the questions. Furthermore, the structure, syntax, and distractors of the questions were revised. For the context of enactive and algorithmic thinking questions it was suggested to reconsider the overlaps in the objectives. In terms of algorithmic questions overlaps in procedures on which students are expected to progress were taken into consideration, while for the context of enactive questions overlaps in equations for which a model will be generated were examined. For the context of the iconic thinking questions it was suggested to equal the number of questions on graph construction and graph interpretation within considering the integration of more unfamiliar functions. For the context of formal thinking questions it was suggested to reexamine the distractors within considerations of students' common misconceptions and errors in the definitions, facts, and notations of the concept of the derivative. For the context of algebraic thinking questions it was suggested to focus on the more sophisticated linkages between a differentiation theorem and its hypotheses rather than simple syntheses of a differentiation theorem to apply its hypotheses. For the context of axiomatic thinking questions it was suggested to eliminate the hints in the distractors.

The second version of the TDT with 60 multiple-choice test items was submitted to a professor in the Department of Mathematics at Ankara University along with the definitions of thinking contexts and objectives of the questions to gather trustworthy judgements and suggestions. Regarding the professor's comments the theoretical definitions of the differentiation theorems were integrated into the syntax of the axiomatic thinking questions. Furthermore, in terms of algebraic thinking questions, some of the functions on which the hypotheses of the differentiation theorems to be applied were revised. The revised questions were consulted with the advisor. Regarding this crosscheck the advisor suggested to shift the syntax of algebraic thinking questions to a more plausible sense that would trigger students to focus on the hypotheses of differentiation theorems for the given functions to justify whether they apply the given theorem. The suggestions on the formal thinking questions in this second version mainly focused on the revisions for the distractors in which the term being defined as part of its relevant misconceptions and errors. This process retained 32 multiple-choice items in the TDT.

The third version of TDT including the revised 32 questions was reconsulted with the advisor. Regarding her comments two questions in the enactive thinking context were dropped mainly in response to the particular overlaps in their objectives. In this process the duration of the test and the requirement for at least four indicators per latent variable (Jöreskog & Sörbom, 1993) were also taken into consideration.

The final version of TDT including 30 multiple-choice items was administered to three third grade university students from the Department of Elementary Mathematics Education in Middle East Technical University, two research assistants from the Department of Secondary Science and Mathematics Education in Middle East Technical University, and two students attending to Master of Science Without Thesis program at the Department of Secondary Science and Mathematics Education in Gazi University. One of the students attending to Master of Science Without Thesis program at the Department of

Secondary Science and Mathematics Education in Gazi University was requested to solve each item with the researcher using a think-aloud procedure. These processes enabled to determine the time that the items take to solve and to ensure the clarity and intelligibility of the items. Upon the completion of the TDT students were consulted about the overall design of the test. All students confirmed that the items were clear and understandable. Taking into account their suggestions, only some wordings and the placement of items were revised.

The revised final version of the TDT was reevaluated by the advisor and no more further revisions were made on the test. These 30 multiple-choice test included six items in formal thinking context, five items in axiomatic thinking context, four items in algebraic thinking context, five items in iconic thinking context, five items in algorithmic thinking context, and five items in enactive thinking context which were confirmed to be understandable and appropriate for undergraduate students, thus content valid to administer. The sequence of items with their respective mathematical thinking contexts is presented in Table 3.2. The table of specifications of the 30 multiple-choice test items is presented in Table 3.3.

Table 3.2 Items Contextualized in Mathematical Thinking Contexts

Item	Mathematical Thinking Context
1 2 3 4 5 6	Formal Thinking
7 8 9 10 11	Axiomatic Thinking
12 13 14 15	Algebraic Thinking
16 17 18 19 20	Iconic Thinking
21 22 24 25 26	Algorithmic Thinking
23 27 28 29 30	Enactive Thinking

Table 3.3 Table of Specifications for the Items of the TDT

Items	Objectives
1	Recall the definition of the derivative
2	Recall the definition of the inflection point
3	Recall the definitions of the increasing function and decreasing function
4	Recall the definitions of the local maximum and local minimum
5	Recognize the product rule for differentiation
6	Recognize the notations relevant to differentiation
7	Analyze Rolle's Theorem to make valid inferences from given information
8	Analyze the Mean Value Theorem to make valid inferences from given information
9	Analyze Fermat's Theorem to make valid inferences from given information
10	Analyze the Intermediate Value Theorem to make valid inferences from given information
11	Analyze the differentiability/continuity theorem to make valid inferences from given information
12	Synthesize the hypotheses of the Mean Value Theorem to establish results on a given interval
13	Justify that the given partial function implies the hypotheses of Rolle's Theorem, Fermat's Theorem, and the Intermediate Value Theorem
14	Justify that the given polynomial function implies the hypotheses of Rolle's Theorem and Fermat's Theorem
15	Synthesize the Mean Value Theorem to determine whether the given functions imply its hypotheses on a given interval

Table 3.3 (continued)

Items	Objectives
16	Retrieve information from the graph of a function to compute the derivative of another function
17	Retrieve information from the graph of a derivative function to determine the inflection points
18	Construct the derivative graph of a function with reference to the graph of the original function
19	Construct the graph of a function with reference to the graph of the derivative function
20	Retrieve information from the graph of a function to compute the derivative of another function
21	Compute the derivative of a given function
22	Evaluate the values on an interval where the given function is always decreasing
23	Model a rectangle to maximize the area
24	Evaluate the derivative of a given function via using the limit of a difference quotient
25	Compute the product of the unknowns in a function via using its local extremum point and the inflection point
26	Evaluate whether a given partial function is differentiable at a given point
27	Model a parabola to maximize the side length of a triangle
28	Model a quarter-circle to maximize the area of a rectangle
29	Model a cylinder to minimize the height of an oil can
30	Model a square to maximize the volume of a rectangular prism box

The scoring criteria representing the six thinking constructs were specified for correct and incorrect responses. Based on this, the highest score 1 was awarded for correct responses, while the lowest score of 0 was reserved for incorrect responses. The possible scores on the TDT ranged from 0 to 30.

The pilot study of the TDT was conducted during the spring semester of 2009-2010 academic year. At this stage four phases were involved: test administration, confirmatory factor analysis, subgroup validity analysis, and reliability analysis.

3.2.1.1 Test Administration

The TDT including 30 multiple-choice items was administered to 766 undergraduate students (352 female and 414 male). Cross-sectional data for freshmen, sophomores, juniors, and seniors attending to the Faculty of Education (N = 253, 33%), Faculty of Arts and Sciences (N = 284, 37%), and Faculty of Engineering (N = 229, 29%) from nine universities residing in seven different cities of Turkey were collected. Of these students 10.4% were from the department of elementary mathematics teacher education, 6.5% from secondary mathematics teacher education, 11.6% from secondary chemistry teacher education, 4.4% from secondary biology teacher education, 21.8% from mathematics, 7.7% from physics, 7.6% from statistics, 0.5% from geological engineering, 12.7% from civil engineering, and 16.7% from food engineering. The sample for the pilot study had an age range from 17 to 24.

Students were from different sections of either one of the Fundamentals of Mathematics, General Mathematics, or Calculus classes offered by the Department of Mathematics and were allocated to these sections according to their major department at the first grade. They were also requested demographic data including their university, department, gender, grade level, cumulative grade point average, high school mathematics achievement, and type of the graduated high school.

The TDT was administered to classes during regular course hours by either the researcher or the lecturers. The 60-min study was administered in lectures selected by the chair of the departments of the participating faculties based on availability of testing time. In classes where the researcher was not available, lecturers were present during the study administration. On the occasions that they were present, lecturers were given information about the study; they did not communicate with their students during the testing session and did not assist in the data collection.

The test administration process guided through the validation of the six-factor structure of the TDT together with the reliability of the test scores.

3.2.1.2 Confirmatory Factor Analysis

A confirmatory factor analysis (CFA) was conducted on students' scores on the 30 TDT items to provide supportive evidence to the six-factor structure of the TDT. As any multiple-choice achievement test is likely to have some degree of multidimensionality (Tate, 2002, 2004). CFA provides the evidence for asserting that a set of item responses is sufficiently multidimensional when a construct definition (i.e., mathematical thinking) posits several dimensions (i.e., enactive thinking, iconic thinking) in the face of theory (Kline, 2005; Thompson & Daniel, 1996). Furthermore, with an initial theory CFA can be utilized without conducting an exploratory factor analysis (Brown, 2006).

The analyses employed the LISREL 8.7 (Jöreskog & Sörbom, 1993) statistical software package in calculating weighted least squares estimates. An alternative to the Pearson product-moment correlation coefficient is the tetrachoric correlation. A tetrachoric correlation is used when observed variables are categorical which are assumed to represent underlying bivariate normal distributions. In other words, it measures the linear relationship between two observed, categorical variables that are manifestations of latent, normal continuous variables. Thus, a tetrachoric correlation is a more appropriate measure of the relationship between two multiple-choice items on a test than the

Pearson correlation (Olsson, 1979). It follows that the matrix of tetrachoric correlations is more appropriate than the matrix of Pearson correlations for confirmatory factor analysis when binary (0/1) multiple-choice test items serve as observed variables (McLeod, Swygert, & Thissen, 2001).

Because a set of multiple-choice items does not have a multivariate normal distribution, the maximum likelihood (ML) estimation that is typically employed in confirmatory factor analysis is not the best method of estimation for testing the factor structure of such variables (Bollen, 1989). Instead, it is suggested to apply weighted least square estimation (WLS) to the matrix of tetrachoric correlations, where the weight matrix is defined in terms of the asymptotic covariance matrix among all tetrachoric correlations (Schumacker & Lomax, 2004). Thus, all analyses were conducted on the asymptotic covariance matrix of the estimated tetrachoric correlations. Thompson (2004) argued that "correlated factors are usually expected and almost always provide a better fit to the data" (p. 118). Thus, the evaluation of the correlations among the factors in these models was an effort to maximize model-to-data fit (Kieffer & Reese, 2009). It should also be stressed that the factor loading of the first item at each latent factor was fixed to 1 in order to set the metric of the mathematical thinking constructs automatically.

Lance and Vandenberg (2002) suggested that evidence for discriminant validity is provided when other theoretically substantial factor models are demonstrated to fit worse to the target model. Discriminant validity of the scores on the TDT was tested in a way that the superiority of the theoretical model as compared to three other substantial models was investigated. Three alternative factor analytic models were tested: a common factor model, a three-factor model, and a null model. The common factor model was specified such that all items loaded on a single general factor as mathematical thinking proposing that enactive, iconic, algorithmic, algebraic, formal, and axiomatic thinking are not conceptually or statistically distinct. The three-factor model was specified such that items loaded on three factors proposing thinking dimensions as conceptual-embodied, proceptual-symbolic, and formal-axiomatic thinking. Tall (2002) distinguished mathematical thinking types as conceptual-embodied, proceptual-symbolic, and formal-axiomatic thinking. However, it is impossible to untie the merging of enactive, iconic, algorithmic, algebraic, formal, and axiomatic thinking in that every mathematical task calls out some paths of embodiment, symbolism, and formalism. The null model implied all the items are uncorrelated proposing that each item on the TDT is a single factor.

Chi-square difference testing was used to make model comparisons (Lee & Song, 2001). Furthermore, multiple criteria including the ratio of chi-square to the degrees of freedom (χ^2 / df), the root mean square residual (RMR), goodness-of-fit index (GFI), adjusted-goodness-of-fit index (AGFI), root mean square error of approximation (RMSEA), and comparative fit index (CFI) were used to test model-data-fit. It is suggested substantively interpretive models with chi-square ratios of three or less, a RMR below .05, a GFI above .90, an AGFI above .90, a RMSEA from .06 to .08, and a CFI above .95 as good fitting (Schreiber, Stage, King, Nora, & Barlow, 2006; Schreiber, 2008).

3.2.1.3 Further Validation

As further validation evidence, subgroup validity was demonstrated. It is appropriate to present subgroup validity evidence when groups whose scores are expected to differ on a test do so in the hypothesized direction (Hinkin, 1995). In the current study, gender and faculty affiliation were expected to differentiate students on the six dimensions of the TDT. Gender was coded with 1 = female and 2 = male; and faculty affiliation was coded as 1 = Faculty of Education, 2 = Faculty of Arts and Sciences, and 3 = Faculty of Engineering. The dependent variables were six dimensions of the TDT. The predictions were based on two lines of review of the literature. First, gender effect in mathematical thinking has long been subject to mathematics education research at elementary, middle, and secondary levels (e.g., Fennema & Sherman, 1977; Friedman, 1989;

Grootenboer & Hemmings, 2007; Rothman & McMillan, 2003; Tartre & Fennema, 1995), while there is by now a sparse literature at the university level (Edge & Friedberg, 1984; Stage & Kloosterman, 1995; Ubuz & Kırkpınar, 2000; Ubuz, 2011). Many of these studies are couched in the differences between males and females in favor of males (Ercikan, McCreith, & Lapointe, 2005; Grootenboer & Hemmings, 2007; Leder, 1992; Oakes, 1990). However, meta-analyses of the causes, correlates, and effects of gender tend to reflect diverse results and provide an empirical support for either small or no significant differences (Friedman, 1989; Hyde, Fennema, & Lamon, 1990). Second, faculty affiliation merits special attention for revealing how institutional factors facilitate for some students, even while inhibiting for others, the accumulation and exchange of various kinds of mathematical thinking (Maull & Berry, 2000; Praslon, 1999; Ubuz, 2011; Ubuz & Kırkpınar, 2000; Ubuz & Ersoy, 1997). Although many studies appeared to assume that faculty affiliation has no bearing in students' mathematical thinking (Asiala et al., 1997; Bezuidenhout, 1998), there is considerable evidence in support of the fact that faculty affiliation is essential to predict students' progress in different types of mathematical thinking (Bingolbali & Ozmantar, 2009; Bingolbali & Monaghan, 2008; Ubuz & Kirkpinar, 2000; Ubuz & Ersoy, 1997).

Multivariate analysis of variance was generated to check these issues.

3.2.1.4 Reliability Analysis

The reliability analysis was undertaken to produce a revised test made up of items that are significantly intercorrelated. Because the focus of the development of TDT was in a multidimensional sense positing that it covers six specific contexts of mathematical thinking, the reliability analysis was conducted separately for the scores on each dimension using Kuder-Richardson Formula 20 (KR-20) reliability coefficient as a measure of internal consistency. The items were also investigated in terms of two criteria (Pallant, 2005): An item was deleted if (a) it had a corrected item-total correlation lower than .30, and (b) deletion of the item resulted in a substantial increase in the reliability coefficient of the mathematical thinking dimension.

3.3 PROCEDURE

In the fall semester of 2009-2010 academic year extensive and detailed information was obtained about advanced mathematical thinking, the teaching and learning of the derivative concept at the university level, the teaching and learning of calculus, and multilevel structural equation modeling through review of related literature. In the spring semester of 2009-2010 TDT was developed. Prior to the pilot study and the main study permission was taken from the METU Human Research Ethics Committee in January. Subsequently, permission was taken from the President's Office of the state universities that participated in the present study. Accordingly, the TDT was piloted. In the summer break, data analyses of the TDT was conducted. The 6-month rental edition of the TESTFACT 4 program was purchased for the item analyses of the TDT. Once the program was practiced via exercises and the manual that were electronically available, the reliability analyses of the students' scores were conducted.

During September-December 2010 the main study was conducted in terms of the administration of TDT as part of calculus or appropriate classes. After the data were collected Mplus 6.1 software program was purchased (STBML60005110) with regard to its availability and flexibility in multilevel structural equation modeling with categorical variables. Until the program was shipped to the researcher, multilevel exploratory factor analysis, multilevel confirmatory factor analysis, and multilevel structural equation modeling techniques were practiced via the demo version of Mplus on the Mplus website (www.statmodel.com). Mplus course videos, handouts, and electronic version of the User's Guide were further followed from the official website.

3.4 DATA COLLECTION

The testing was conducted during September-December in the fall semester of 2010-2011 academic year. The 60-min study was administered in lectures selected by the chair of the departments of the participating faculties based on availability of testing time. In classes where the researcher was not available, lecturers were present during the study administration. On the occasions that they were present, lecturers were given information about the study; they did not communicate with their students during the testing session and did not assist in the data collection.

3.5 DATA ANALYSES

The preliminary data analysis began by the preparation of the data files. The data gathered through the TDT were obtained in PASW Statistics 18 files separately for the Sample 1 and Sample 2. Descriptive analyses were run to get indepth information about the data. The descriptive statistics are reported in Appendix B and Appendix C for Sample 1 and Sample 2, respectively.

In addition, single-level structural equation modeling analyses were run to validate the instrument and to determine the latent mathematical thinking constructs that would be introduced to the two-level structural model in the subsequent multilevel analyses, whereas multilevel structural equation modeling analyses were run to determine the factor structure of mathematical thinking at the within- and between-classroom levels and to investigate the relationships among different types of mathematical thinking at the within- and between-classroom levels. PASW Statistics 18 (SPSS Inc., 2010) was used to run the descriptive analyses, LISREL 8.7 (Jöreskog & Sörbom, 1993) was used to run single-level structural equation modeling analyses, TESTFACT 4 (Bock, Gibbons, Schilling, Muraki, Wilson, & Wood, 2003) was used to run the reliability analyses, and Mplus version 6.1 (Muthén & Muthén, 1998-2010) was used to run the multilevel analyses.

3.5.1 MISSING DATA ANALYSES

Most of the data analysis procedures in educational research are designed for complete samples. The underlying idea is that the data are generated randomly from the population of interest. The invalidation of this assumption that the data are drawn randomly, missing data may cause a number of problems: (a) bias in parameter estimates; (b) inflation of Type I and Type II error rates; (c) degrade in the performance of confidence intervals; and (d) reduction in statistical power (Collins, Schafer, & Kim, 2001).

Missing data is a potentially serious methodological problem in nonexperimental research with crosssectional design (Trautwein & Lüdtke, 2009). In most educational tests item nonresponse relates to omitted items, not reached items, or multiple response items. The response to an item is classified as *omitted* if it is missing, and at least one of the following items has a presented correct, incorrect, or multiple response. The response to an item is classified as *nonreached* each response from a block of items follows a contiguous sequence of missing responses to items which contain the last item the block. The response to an item is classified as *multiple response* if the respondent marks more than one choices. In general, students are assumed to narrow down the range of possible correct answers. The selected response is then classified as either correct or incorrect if the selection contains both the correct and the incorrect answer. However, if each marked response is incorrect this selection is inappropriate. In the present study, the multiple responses were treated as missing as they would not provide an authentic determinacy for students' actual mathematical thinking performance on the relevant item.

A total of 1200 undergraduates were in Sample 1 with 45 students having no response in either one of the items but an index of descriptive information (e.g., gender, university, faculty, and department). These students were automatically dropped (3.75% loss) and 1155 students were remained for the further missing data analyses.

For the items considered in Sample 1, the average percentage of missing data was 2.15% ranging from 1.7% to 4.1%. The lowest missing rate was for the formal thinking item, Item 6 that requested students to state the mathematical symbol standing for the derivative concept. The highest missing rate was for the algebraic thinking item, Item 15. Given a number of functions students were requested to select the one that satisfies the hypotheses of the Mean Value Theorem. There are two possible explanations for the relatively higher missing rate at this item. First, students who are mostly familiar with items that require factual knowledge of derivative theorems might be less likely to answer this item which requires an inquiry-based mathematical thinking. Second, some students might not tend to make sense on the fundamentals of Mean Value Theorem. Thus, missing rate for this item was ignorable and that; there were only unintentionally missing data. The nonreached items toward the end of the TDT had a nonresponse rate ranging from 1.4% to 2.3%.

A total of 2576 undergraduates were in Sample 2 with 76 students having no response in either one of the instruments but an index of descriptive information (e.g., gender, university, faculty, and department). These students were automatically dropped (2.95% loss) and 2500 students were remained for the further missing data analyses.

For the items considered in the Sample 2, the average percentage of missing data was 5.04% ranging from 0% to 12%. The lowest missing rate was for the formal thinking item, Item 5 that requested students to state the product rule. The highest missing rate was for the axiomatic thinking item, Item 9. Given Fermat's Theorem students were requested to select the correct conjecture. There are two possible explanations for the relatively higher missing rate at this item. First, students who are mostly familiar with items that require routine procedures and technical algorithms might be less likely to answer this item which requires a sophisticated and a nonroutine manner of mathematical thinking. Second, some students might not tend to make sense on Fermat's Theorem. Thus, missing rate for this item was ignorable and that; there were only unintentionally missing data. The nonreached items toward the end of the TDT had a nonresponse rate ranging from 2.4% to 3.3%.

For both samples the nonreached responses on the TDT signaled that some students were unable to complete the whole multiple-choice test within the given duration. On the other hand, the omitted responses did not follow a long contiguous set of items. It was likely that students did not attend to certain items. When an answer to an item was missing the preceding and following items were not. This might possibly be a consequence of not knowing. If there was a long sequence of missing responses to consecutive items, it would be concluded that the first few responses may be due to the fact that not knowing and the rest due to the fact that not attending. This indicated that some students were unable to complete the whole multiple-choice test within the given duration.

Following that the type of missingness for item nonresponse in the data set was investigated separately for Sample 1 and Sample 2 by conducting Little's MCAR test (Little, 1988) on the PASW Statistics 18. Both samples were classified according to the values of the multiple-choice items that are subject to missing data due to nonresponse. The classification of missing data mechanisms depends on whether the probability of the items' missing data depends on the state of the item and/or other independent variables (Rubin, 1976).

For the Sample 1, results demonstrated that the data were missing completely at random (MCAR) ($p= .318$, $p> .05$). Similarly, for the Sample 2, results revealed that the data were missing completely at random (MCAR) ($p= .456$, $p> .05$). Given the low percentage of overall nonresponse rate (2.15% and 5.04%, respectively) of the data and that, the data are MCAR; it was opted for the listwise deletion procedure that would give unbiased estimates (Acock, 2005; Cohen, Cohen, West, & Aiken, 2003). In a special case of MCAR, the probability of missing data of the items depends neither on the item itself nor on other independent variables. When data are MCAR the missing data mechanism can be regarded as ignorable (Little & Rubin, 2002) in the sense that inference does not depend on that mechanism (Chen & Astebro, 2003).

Taken together, the overall levels of loss at 30 multiple-choice items included in the present study was considered low enough to support the feasibility of further multilevel exploratory factor analyses with 1099 students and, multilevel confirmatory analysis and multilevel structural equation modeling with 2424 students.

3.5.2 POWER ANALYSIS

Power analysis is an important aspect of multilevel structural equation modeling studies. The purpose of the power analysis is to provide the researcher with information needed to address the research questions in a precise fashion. Statistical inference is drawn upon the relationships among four variables: alpha, beta, power, effect size, and sample size (Tabachnick & Fidell, 2007). For any statistical model, the relationships among the four are such that each is a function of the other three. Thus, in research planning, it has been widely acknowledged to know the sample size necessary to attain the desired power for the specified alpha and hypothesized effect size (Cohen, 1988). The power of the study should be large enough to conduct a research of value. Similarly, the effect size of the study should be large enough to make meaningful inferences and so that the results are statistically and practically significant.

Power is the probability of rejecting a false null hypothesis and the specification for power is .80 which is a convention for general use (Cohen et al., 2003). On the other hand, an effect size measure is the size of the relationship among variables (Weinfurt, 1995) which is expressed as the proportion of explained variance in the dependent variables. In other words, effect size is an indicator of the association that exists between two or more variables. Multiple regression analysis provides a better understanding for the multilevel structural equation modeling. The multiple correlation indices are multiple correlation (R), squared multiple correlation (R^2), and adjusted squared multiple correlation ($R_{adj}{}^2$). Squared multiple correlation is a measure of the strength of the linear relationship. The measure of effect size is equivalent to the R^2 used in multiple regression. The squared multiple correlation indicates the amount of variance explained by the set of independent variables (Raykov & Penev, 2010). It is used as a model fit criterion in multiple regression analysis (Schumacler & Lomax, 2004). Cohen (1988) suggested a classification of effect sizes, which were measured in terms of R^2. This classification indicated for effect sizes: 0.01 is small, 0.09 is medium and 0.25 or greater is large. In social studies, small to medium effect sizes emerge (Weinfurt, 1995). Furthermore, multilevel structural equation modeling techniques require large sample sizes at each level of analysis (Rabe-Hesketh, Skrondal, & Zheng, 2007; Schreiber & Griffin, 2004).

In regard to the aforementioned assumptions, the alpha, beta, power, and effect size were fixed to carry out the multilevel analyses. Setting these indices a priori to the research enabled to determine the minimum sample size required to estimate the two-level structural models.

Prior to the study, the alpha which is the probability of making a Type I Error, was set at .01. The power was set at .95 as it is a large enough value indicating that the relationships are worth investigating. Based on the power, the beta was fixed at .05 (power = 1 – β). Taken with the conventional value of .25 as a large effect, the effect size for the population was fixed at .40. In the present study the number of the within-classroom variables was 6 and the number of the between-classroom variables was 3. However, at the beginning of the study the number of variables introduced to the two-level models was set at 100 which is the maximum number to obtain the index L. Consequently, Cohen's sample size table was used to determine the sufficient sample size of this multilevel study.

The sample size of the study was computed as follows:

$\alpha = .01$, $\beta = .05$, power = .95, effect size = .40

$$f^2 = \frac{1}{1-R^2} = \frac{.40}{1-.40} = \frac{2}{3}$$

In regard to Cohen's table with α = .01, k_b =100, power = .95, L equals to 70.37.

$n = \frac{L}{f^2} + k_b + 1$ = 70.37/.66 + 100 + 1 = 207.621 ≅ 208

Thus, the adequate sample size for the present study was computed as 207.621 indicating that the sample size should be at least 208, approximately. The two independent sample sizes were 1099 and 2424 at the within-classroom level. From a methodological standpoint, these values were far above the necessary sample size set for the beginning of the study. Furthermore, the sample size at the between-classroom level was 72 and 130 for Sample 1 and Sample 2, respectively. These values were also large enough to test two-level structural models.

3.6 OVERVIEW MULTILEVEL STRUCTURAL EQUATION MODELING

Over the past decade or so, concerns in various fields with methodological issues in conducting research with hierarchical (clustered or multilevel) data have led to the development of multilevel modeling techniques. The theory of multilevel modeling grounds on sociology underlying that the effects of the social context on individuals must be mediated by intervening cognitive or psychological processes dependent on the characteristics of the social context (Chan, 1998; Erbring & Young, 1979; Stinchcombe, 1968). Therefore, multilevel theory specifies whether the variables belong to the within- or between-classroom level and which direct and/or indirect effects as well as cross-level interaction effects should be expected. More specifically, cross-level interaction effects between the individual and group level which require the specification of processes within individuals that ground these individuals to be differentially influenced by the certain aspects of the within- and between-classroom effects can be put forth in multilevel models (Hox, 2002).

Despite the existence of hierarchical data structures in social sciences, previous research addressed either one of the Structural Equation Modeling (SEM) or the Multilevel Modeling (MLM) methods (Raudenbush & Byrk, 2002). Generally, SEM methods permit researchers to build and test models including both endogenous and exogenous latent variables simultaneously (the measurement model and the structural model). MLM methods, on the other hand, allow for the variance attributable to the between-classroom level (group level) to be portioned from the variance associated with the within-classroom level (individual level), permitting the estimation of more accurate standard errors and more reliable information about between-and within-classroom effects (Raudenbush & Byrk, 2002). However, application of either methodology alone to the hierarchical data would produce several analytical difficulties and mispecifications about the complex relations that exist within and between groups. In a way, MLM represents a blind spot on the fact that variables are themselves related directly or indirectly to desired outcomes disregarding endogenous outcomes may be simultaneously related to each other (Hoffman, 1997; Kaplan & Elliot, 1997b). Use of SEM alone would ignore the clustered sampling that is often used to design educational data and would produce biased results in the estimation of structural regression coefficients (Muthén, 1989a, 1989b).

To resolve these difficulties, attempts have been made to integrate MLM with SEM for studying complex sample data. More recently, multilevel structural equation modeling (MSEM) has become a vigorous line of methodological research. Similar to the applications of the hierarchical linear model to regression in the context of the multilevel model, MSEM is a direct generalization of SEM in the context of the multilevel model (Cheung & Au, 2005; Raudenbush & Sampson, 1999) which allows the specification of separate structural models with direct and indirect effects within and between groups (Heck, 2001). The most critical aspect of MSEM is that the magnitude of the regression coefficients (i.e., factor loadings) are standardized separately at the within- and between-group levels. Thus, there is no relation in the proportion of variance accounted for in one level versus the other level (Byrne, 2012, p. 365). As such, there is no correspondence between interpretations of relationships at the within- versus between-group levels.

Several types of multilevel models can be investigated with SEM techniques including two-level measurement models that define latent constructs through their observed indicators, path models that investigate two-level relationships among observed variables, and two-level structural models that focus on the relationships among latent and observed variables. Applications of MSEM to educational research are still rare (Heck & Thomas, 2008). However, it is an attractive approach because it allows the researcher to incorporate a substantive theory about the relations among variables within a nested structure (e.g., students nested within schools) as well as relations at a group level (in this case schools) by giving access to the investigation of one of the aforementioned models.

To summarize, the tradition that prompted an interest in multilevel structural models is the practice of examining nested educational structure. Multilevel structural models have a natural appeal to educational researchers because the structure of education is often hierarchical. Examples for multilevel educational structure are plentiful. In schools, students are nested within classrooms, and classrooms are nested within teachers. Guo and Zhao (2000) pointed that multilevel structural models offer a number of the following advantages. First, they provide a convenient framework for investigating hierarchical data. Second, such models correct for the biases in parameter estimates that result from the clustering effect. Third, multilevel structural models provide unbiased standard errors and thus correct confidence intervals and significance tests. Finally, in line with the first advantage, estimates of the variances and covariances of random effects at the within and between levels enable researchers to decompose the total variance in the outcome variable into portions associated with the within level and the between level.

3.6.1 ASSUMPTIONS OF MULTILEVEL STRUCTURAL EQUATION MODELING

Inferential statistical methods are based on assumptions. Provided certain assumptions are met, methods that researchers use will generally function as intended. However, if the assumptions are violated it is likely that certain procedures will not produce the desired results at least under some data-analytic conditions.

In educational research, regarding the assumptions, practical issues are usually of concern especially when applying a sophisticated statistical method, such as MSEM. Since multilevel structural models assume large sample sizes, the minimum sample sizes required at the within- and between-classroom levels is one the fundamental issues. However, MSEM approaches are relatively new and that only a few studies have investigated the sample size requirements of these approaches. Although researchers agreed that problems would occur for small and inadequate between-classroom samples, they showed a controversy in that between-classroom sample size should be at least 100 (Hox & Maas, 2001) and/or at least 30 to 50 (Stapleton, 2006) for good performance of the estimator being used. In support of this controversy, illustrated examples using MSEM in the literature vary a lot in the sample sizes employed. Some studies satisfy the between-classroom sample size larger than 100 criterion (Duncan, Alpert, & Duncan, 1998; Kaplan & Elliott, 1997a, 1997b; Muthén, 1994) while some studies fulfill the between–classroom sample size should be between 30 to 50 criterion (Heck, 2001; Hox, 1998; Maas & Hox, 2005).

The issue of small between-classroom sample size becomes even more critical when applying MSEM to educational research because the classroom-level sample size is always small, whereas the student-level sample size is comparatively large. The multilevel analyses on comparing the student and classroom factor structures conducted by several researchers shed light into this issue (e.g., Burstein, 1980; D'Haanens, Damme, & Onghena, 2010; Goldstein & Bonnet, & Rocher, 2007; Goldstein & Rasbash, 1996; Nasser & Hagtvet, 2006;

Muthén, 1991; Webster & Fischer, 2000). It is easy to observe that the typical between-classroom sample size is around 30 to 200, whereas the within-classroom sample size can be as large as 300 to 3500 in educational research.

Although the between-classroom sample size is relatively small in educational research, it is unclear whether the large within-classroom sample size is beneficial to the overall performance of MSEM procedures, especially at the between-classroom level. Some researchers suggested a trade-off between sample sizes at different levels of analysis, partially for multilevel regression (Mok, 1995; Snijders & Bosker, 1992; Tabachnick & Fidell, 2007). They indicated that increasing within-classroom sample size may reduce the between-classroom sample size requirement. That is, increasing the within-classroom sample size would subsequently increase the precision on the parameter estimates at the between-classroom level. Although most of the previous research suggested that increasing the between-classroom sample size is more beneficial than increasing the within-classroom sample size (e.g., Snijders & Bosker, 1992), it is not clear whether the large within-classroom sample size in educational research will help the between-classroom model fit or parameter estimates in MSEM. Since most of these suggestions are based on simulation studies with artificial data, the effects of sample sizes on real data sets remains unknown. Considerations derived from the sample size assumption are likely to be generalizable to those with suggestions in educational research indicating that sufficient sample sizes for MSEM is at least 200 at the within-classroom level (e.g., Heck & Thomas, 2008) and at least 30 at the between-classroom level (e.g., Stapleton, 2006).

Provided that the sample sizes are large enough at both within- and between-classroom levels, multilevel structural equation modeling is appropriate when data are collected at multiple levels simultaneously. As mentioned before, the term "levels" refers to how data are organized and more important statistically, to whether observations are dependent or independent. One of the important characteristics of such data is the within-classroom observations are not independent. Students in a classroom share whatever characteristics the classroom has, and the overall performance of the classroom has in common the characteristics of the student. This lack of independence means that traditional multiple regression analysis or conventional single-level SEM analysis in which within-classroom observations are treated as independent observations cannot be used because such ordinary least-squares techniques violate the fundamental assumption: the independence of observations. In a multilevel data structure, units of observations (i.e., students) are randomly sampled from populations at different levels simultaneously. For example, in a study of mathematical thinking, individual students are sampled to provide a basis for making inferences about classrooms. At the same time, the classrooms are meant to provide a basis for making inferences about each student. Thus, the error associated with sampling at each level of analysis should be estimated. That is, the within-classroom relationships found in a sample of between-classroom would probably be similar to, but not exactly the same as, the relationships found in another classroom. MSEM takes into account simultaneously the sampling error at each level of analysis to estimate the factor structure of educational constructs at the within-classroom and between-classroom levels.

In this vein, the assumptions underlying the multilevel structural model are similar to the assumptions in ordinary single-level multiple regression analysis: linear relationships, homoscedasticity, and normal distribution of the residuals (Maas & Hox, 2004). Specifically, the assumptions of multilevel structural equation models emphasize multivariate normal distributions for all residuals, independence of residuals for different levels, and independence of residuals for different units in the same level (Curran, 2003). In the case of latent variable, the local independence implies that the residual covariance matrix is diagonal (Raudenbush & Byrk, 2002). If the multilevel model fits the data well then assumptions relevant to uncorrelated residuals are assumed to be not violated. In multilevel structural equation modeling, it is known that violations of these assumptions lead to highly inaccurate parameter estimates and standard errors (Tabachnick & Fidell, 2007). Prominent among these violations is that the correlated errors among the students within a classroom violate the independent observations assumption, resulting in downwardly biased standard error estimates, overly large test statistics, and inflated Type I error rates (Snijders & Bosker, 1992). However, provided that the sample size is large, MSEM can be regarded as a robust analysis method. In response to the challenge of appropriately analyzing hierarchical data, it preserves the original data structure while explicitly modeling the within-classroom homogeneity of errors by allowing the estimation of error terms for both the student and the classroom (Krull & MacKinnon, 2001). In the case of severe violations, for example when the variance of the error terms differ across observations, this method has the advantage that heterocedasticity can be modeled directly (Goldstein, 1995).

Distributional assumptions are made about the errors at the within-classroom and at the between-classroom level in the multilevel structural model. The within-classroom level errors as well as the between-classroom level errors are both assumed to be independently and normally distributed. For multilevel structural models with categorical outcomes the normality assumption is not realistic (Dedrick et al., 2009). Associated with revealing the robustness of multilevel structural equation modeling with categorical variables, the normality conditions investigated in the present study were indicated by skewness and kurtosis values for each observed variable. Lei and Lomax (2005) categorized the absolute values of skewness and kurtosis less than 1.0 as slight nonnormality, the values between 1.0 and 2.3 as moderate nonnormality, and the values above 2.3 as severe nonnormality. In contrast, Fabrigar et al. (1999) accepted that nonnormality is severe if skewness is above 2 and kurtosis is above 7. With categorical variables, the values of skewness and kurtosis are generally not a problem for model assumptions as it is for continuous variables where normality is violated (Muthén, 1989a). Additionally, categorical variable methodology takes into account the floor and ceiling effects. Ceiling effect occurs when students' scores on a test pile up at the high end, whereas floor effect occurs when students' scores on a test pile up at the low end. The percentage of students scoring at the highest and the lowest levels reflect whether a test is too easy or too difficult to cause an undesirable measurement outcome that would possibly inflate the multivariate normality assumption (Croacker & Algina, 1986). The percentage of responses at the lowest or the highest level response option at 25% or more can be taken as a large floor or ceiling effect (Haladyna, 1994).

Also implicit in the assumptions about the mediation analyses in the present study is that multilevel mediation models (i.e., cross-level mediator models) make all of the standard assumptions of the multilevel structural models (i.e., linearity, normality, homogeneity of error variance, and independence of errors).

3.6.2 CONCEPTUAL BACKGROUND FOR MULTILEVEL STRUCTURAL EQUATION MODELS

The estimation procedures in multilevel models with categorical observed variables have been rendered possible the widespread availability of a class of statistical models (Finney & DiStefano, 2006; Kaplan, 2009). The distinguishing feature of such models lies beneath whether they assume a logistic distribution or a normal distribution. Logit or probit link functions therefore introduce an additional source of classification of models which is based on tethrachoric correlations. In general logit and probit link functions provide similar fits and conclusions (Goldstein, Bonnet, & Rocher, 2007) it is suggested the link function choice should be based primarily on ease of interpretation and computational demands (Hedeker, 2008).

In the context of latent variable modeling with categorical variables, the detailed descriptions of the single-level latent variable model, multilevel regression model, multilevel logit model, and multilevel probit model are presented below.

3.6.2.1 The Single-Level Latent Variable Model

To provide a familiar starting point, I begin with a review of the single-level latent variable model. My review focuses on a specific single-level model with categorical variables (1 = correct; 0 = incorrect) that educational researchers are likely to estimate. A simple latent variable model of this type relates the responses on a set of items to one or more underlying latent factors. A basic version can be expressed as follows.

For a student (*i*) who responds to item (*r*), the probability (π_{ri}) of a correct response can be given by

$$g(\pi_{ri}) = \beta_{0r} + \lambda_r \theta_i$$
$$\theta_i \sim N(0, \sigma_\theta{}^2)$$
$$y_{ri} \sim binomial(1, \pi_{ri})$$

where *g* is a link function, the logit or the probit, and the response, y_{ri} , is 1 if the item is correctly responded and 0 if the item is incorrectly responded and the y_{ri} are mutually independent. This is just a categorical factor model with a single factor (θ) and a set of factor loadings, λ_r. The term β_{0r} refers to the "facility" for Item *r*, as it belongs to the fixed part of the model.

An important assumption in this model is that the responses y_{ri} are conditionally independent. Because some of the items involve responses to the similar context (e.g., graph interpretation of the derivation function, modeling real life applications of the derivative), it is possible that this assumption will be violated, as the conditional probability of a correct response to an item may depend on the outcome with respect to a similar item responded previously. When the factor loadings λ_r are constrained to be equal this refers to a model with a logit link. However, the two link functions are, in fact, very similar and that the estimated probabilities are very close. Goldstein, Bonnet, and Rocher (2007) suggested that the probit link is more advantageous than the logit link in terms of computational issues and the interpretation with regard to the underlying normal propensity distribution for the responses. Here it is wise to note that latent variable models with categorical variables are often motivated and described by using the "threshold" concept. Such models assume that a continuous latent variable underlies the observed categorical response. Then, a threshold *X* determines if the dichotomous response *Y* equals 0 ($y_{ri} \leq X$) or 1 ($y_{ri} > X$). Thus, for a categorical response there is an underlying continuous response for an item with a threshold value (*X*) such that responses above that threshold value are correct and those below that value are in correct.

3.6.2.2 The Multilevel Regression Model

For a description of the multilevel regression model in its most general form a simple two-level model can be expressed as follows.

Assume that we have data from *j* classrooms, with a different number of students n_j in each classroom. On the within-classroom level, we have the outcome variable Y_{ij}. Furthermore, we have one explanatory variable X_{ij} on the within-classroom level, and one between-classroom explanatory variable Z_j. To model these data, we essentially have a separate regression model at the within-classroom level and at the between-classroom level as follows:

$$Y_{ij} = \beta_{0j} + \beta_{1j}X_{ij} + e_{ij}. \tag{1}$$

The variation of the regression coefficients β_j can be modeled by a between-classroom regression model as follows:

$$\beta_{0j} = \gamma_{00} + \gamma_{01}Z_j + u_{0j} \tag{2}$$

and

$$\beta_{1j} = \gamma_{10} + \gamma_{11}Z_j + u_{1j}. \tag{3}$$

In equation (2), γ_{00} is the average intercept and γ_{01} is the coefficient estimating the association between Y_{ij} and Z_j. In equation (3), γ_{10} is the average slope between Y_{ij} and Z_j. Accordingly, γ_{11} represents how γ_{10} changes for differences in Z_j. Both u_{0j} and u_{1j} are error terms at the between-classroom level, where u_{0j} is the deviation from the overall mean (or average intercept) for classroom j and u_{1j} is the deviation from the overall mean regression coefficient, γ_{10} , for classroom j.

Substituting equations (2) and (3) into (1) and rearranging terms yields the following single equation:

$$Y_{ij} = \gamma_{00} + \gamma_{10}X_{ij} + \gamma_{01}Z_j + \gamma_{11}X_{ij}Z_j + u_{1j}X_{ij} + u_{0j} + e_{ij}. \qquad (4)$$

The segment ($\gamma_{00} + \gamma_{10}X_{ij} + \gamma_{01}Z_j + \gamma_{11}X_{ij}Z_j$) in equation (4) contains all the fixed coefficients, that is it refers to the fixed part of the model. The segment ($u_{1j}X_{ij} + u_{0j} + e_{ij}$) in this combined model contains all the random error terms and is referred to as the random part of the model. The term $X_{ij}Z_j$ is an interaction term that appears in the model because the varying regression slope β_{1j} of the within-classroom level variable X_{ij} with the between-classroom variable Z_j is modeled.

The multilevel structural models for categorical outcomes can be derived through the aforementioned multilevel latent variable model conceptualization. It is assumed that there exists a latent continuous variable $Y_{ij}{}^*$ underlying Y_{ij}. In this case only the categorical within-classroom variable Y_{ij} is observed directly, but $Y_{ij}{}^*$ is not. However, it is known that $Y_{ij}{}^* > 0$ if $Y_{ij} = 1$ and $Y_{ij}{}^* \leq 0$ if $Y_{ij} = 0$. A multilevel model for $Y_{ij}{}^*$ equivalent to (4) can be written as

$$Y_{ij}{}^* = \beta_0 + \beta_1X_{ij} + u_j + e_{ij}. \qquad (5)$$

Conditional on the random effect u_j at the between-classroom level, either a logit multilevel model or a probit multilevel model can be derived from equation (5) depending on the assumption that whether e_{ij} in (5) has a standard logistic distribution or a normal distribution.

3.6.2.3 The Multilevel Logit Model

At the most basic level, the multilevel logit model is conceptually equivalent to equation (5), a cross-level direct effects model. Similarly, suppose that we observe Y_{ij}, a categorical response for student i in classroom j and X_{ij}, an explanatory variable at the within-classroom level. The probability of the response equal to 1 can be defined as $p_{ij} = \Pr(Y_{ij} = 1)$ and p_{ij} can be modeled using a logit link function. In this case the standard assumption is that Y_{ij} has a Bernoulli distribution. Then the two-level model is as follows.

$$\log\left[\frac{p_{ij}}{1-p_{ij}}\right] = \beta_0 + \beta_1 X_{ij} + u_j \tag{6}$$

where u_j is the random effect at the between-classroom level. It should be noted that without u_j this combined model (6) would be an ordinary logistic regression model. As in the case of multilevel regression models, u_j is assumed to be normally distributed and further conditional on u_j, Y_{ij}s are assumed to be independent. More specifically, the within-classroom logit model is as follows

$$\log\left[\frac{p_{ij}}{1-p_{ij}}\right] = \beta_0 + \beta_1 X_{ij} + u_j \tag{7}$$

and the between-classroom logit model can be expressed as

$$\beta_{0j} = \beta_0 + u_j \ . \tag{8}$$

Relative to equations (7) and (8), equation (6) is the so-called combined logit model. Collectively, because the error terms are assumed to follow a logistic distribution and the random effects are assumed to follow a normal distribution, these models are referred to as multilevel logit models.

3.6.2.4 The Multilevel Probit Model

A cross-level direct effects model is called a multilevel probit model when both the error terms and the random effects are assumed to follow a normal distribution. In the resulting multilevel probit model from these assumptions, the error terms have mean 0 and variance 1, as is the case in the variance of the standard normal distribution.

From a multilevel latent variable modeling perspective, denoted by

$Y_i = (Y_{ij} : j = 1, 2, \dots, n_i)^T$ is the vector of categorical measurements on student *i* (*i* = 1, 2,…, N). It is assumed that each Y_{ij} takes on a value of 0 or 1. Conditionally on a set of random effects b_i, the model for the probability that a correct response be observed on the *j*th student in the *i*th classroom is specified as

$$\varphi^{-1}(P[Y_{ij} = 1|b_i]) = X_{ij}{}^T\beta + Z_{ij}{}^T b_i, \tag{9}$$

where X_{ij} is a vector of covariates having fixed effects β and Z_{ij} stands for a vector of covariates that possibly overlaps with X_{ij}, having *q*-dimension random effects $b_i \sim N\,(0, \Sigma_b)$.

The existence of a latent variable $Y_{ij}{}^*$ indicates that it is continuously distributed and related to the actual response through a certain threshold. In the context of independent categorical data this approach motivates the standard probit model. In such models, it is assumed that the observed categorical response is actually obtained by dichotomizing an observed continuous latent variable. Similar to the logit model, it is further assumed that there is a need for a certain threshold value (e.g., a cut-off value). Provided that an intercept term is included in the model this cut-off value can be 0. Likewise, this threshold can be chosen as (Y_{ij} = 1) if $Y_{ij}{}^* > 0$ for a correct response and for an incorrect response (Y_{ij} = 0) if $Y_{ij}{}^* \leq 0$. Accordingly, if it is assumed that $Y_{ij}{}^*$ is normally distributed, then the random-effects regression model can be expressed as follows.

$$Y_{ij}{}^* = X_{ij}{}^T\beta + Z_{ij}{}^T b_i + \varepsilon_{ij}{}^* \tag{10}$$

where the error terms $\varepsilon_{ij}{}^*$ are assumed to be normally distributed with mean 0 and variance 1. When the variance parameter is fixed to 1 the derived model for the categorical within-classroom variable Y_{ij} is exactly equation (9) that poists the multilevel probit model.

3.6.2.5 The Cross-Level Moderator Model

The basic cross-level moderator model involves a three-factor system in which an initial independent latent variable affects a mediational variable, which, in turn, affects a dependent latent variable. This system aims to determine whether the relation between the independent and the dependent variable due, wholly or partially, to the meditational variable. Given that the independent variables can reside at both within- and between-classroom levels cross-level models may take several forms as shown in Figure 3.1 and Figure 3.2 (Krull & MacKinnon, 2001). In essence, for instance, *upper level mediation* exists when the effect of a between-classroom independent variable on a within-classroom dependent variable is mediated by another between-classroom independent variable (2→2→1 mediation).

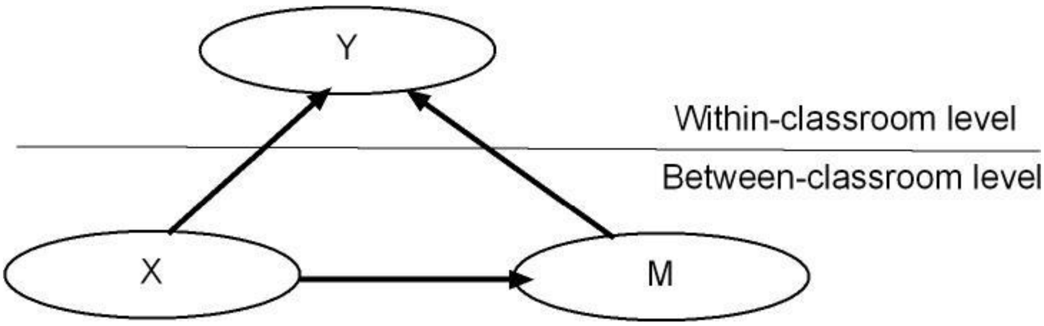

Figure 3.1 Upper level mediation in a two-level model (2→2→1)

Lower level mediation exists when the meditational variable is a within-classroom variable (e.g., 2→1→1 mediation).

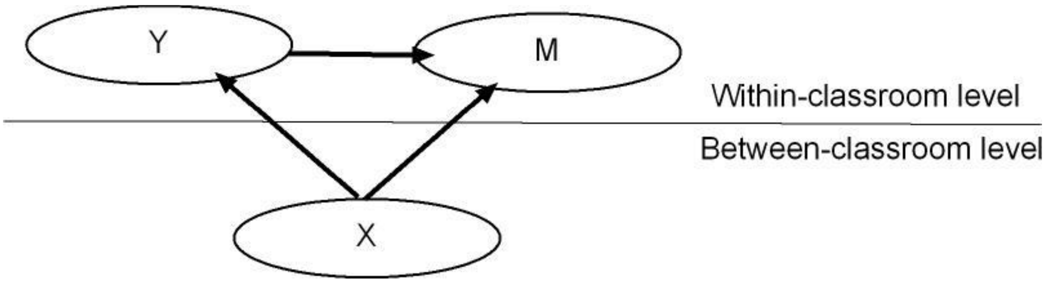

Figure 3.2 Lower level mediation in a two-level model (2→1→1)

Baron and Kenny (1986) suggested three preconditions to support a mediation hypothesis. First, there should be a significant relationship between the independent variable and the dependent variable. Second, there should be a significant relationship between the independent variable and the meditational variable. Third, there should be a significant relationship between the meditational variable and the dependent variable.

The statistical model for testing a lower level, 2→1→1 mediation can be expressed at the within-classroom level and at the between-classroom level as follows:

$$Y_{ij} = \beta_{0j} + \beta_{1j} M_{ij} + e_{ij}. \tag{11}$$

The variation of the regression coefficients β_j can be modeled by a between-classroom regression model as follows:

$$\beta_{0j} = \gamma_{00} + \gamma_{01} Z_j + u_{0j} \tag{12}$$

and

$$\beta_{1j} = \gamma_{10} + \gamma_{11} Z_j + u_{1j} \tag{13}$$

In equation (11), Y_{ij} refers to the within-classroom dependent variable, β_{0j} refers to within-classroom intercept, β_{1j} refers to slope of the mediator M_{ij}, and e_{ij} refers to overall error term. In equation (12), γ_{00} refers to intercept of between-classroom regression predicting β_{0j}, γ_{01} refers to slope of between-classroom regression (independent between-classroom variable Z_j) predicting β_{0j}, and u_{0j} refers to error term for within-classroom intercept β_{0j}. In equation (13), γ_{10} refers to intercept of between-classroom regression predicting β_{1j}, γ_{11} refers to refers to slope of between-classroom regression (Z_j) predicting β_{1j}, and u_{1j} refers to error term for within-classroom slope β_{1j}. Then the meditational equation for *i*th student in classroom *j* at the within-classroom level can be depicted as

$$M_{ij} = \beta_{0j} + e_{ij} \tag{14}$$

and at the between-classroom level it can be expressed as

$$\beta_{0j} = \gamma_{00} + \gamma_{01} Z_j + u_{0j} \,. \tag{15}$$

Of interest several mediation hypotheses including 1→2→1 and 2→1→2 designs can be tested with similar cross-level moderator models (Preacher, Zyphur, & Zhang, 2010).

3.6.3 ESTIMATION OF MULTILEVEL STRUCTURAL EQUATION MODELS

Model estimation involves finding parameter values such as factor loadings, factor variances and covariances to evaluate the difference between the observed and reproduced covariance matrices (Long, 1997). Early use of multilevel modeling was limited to cases of perfectly balanced sampling designs (i.e., equal group sizes) yielding closedform mathematical formulas available to estimate the variance and covariance components (Raudenbush & Byrk, 2002). When sampling designs were unbalanced, problems occurred in using iterative estimation procedures to obtain efficient estimates (Harville, 1977). Determining the extent to which the clustering is present is the critical first step to decide whether MSEM will dominate over single-level techniques (Longford, 1993).

Where variability due to the clustering is present across within- and between-classrooms, statistical concerns about MSEM under different sampling conditions and emerging solutions have drawn the attention of researchers (Hox & Maas, 2001; Muthén & Satorra, 1995). In general, with *balanced* group sizes and continuous observed variables, full information maximum likelihood (FIML) estimation can be used (Muthén, 1984). FIML estimation depends on large sample sizes preferably at both within- and between-classroom levels for the estimates to have desirable asymptotic properties (Muthén, 1990). However, FIML is computationally demanding to use practically, when applied to unbalanced groups. It produces incorrect chi-square values, fit indexes and standard errors (Kaplan, 2009). To offset this, with *unbalanced* group sizes, Muthén's quasi-likelihood (MULM) estimator is probably the most widely used procedure in MSEM (Kaplan & Elliott, 1997). Although it gives less information than FIML, empirical findings reveal that the MUML estimation gives similar results as FIML estimation with rough approximations to the chi-square test statistics and standard errors of parameter estimates (Hox & Maas, 2001; Muthén, 1991). Specifically, it is appropriate to use weighted least squares estimation with mean (WLSM) estimator or the maximum likelihood estimation with robust standard errors (MLR) with unbalanced group sizes and categorical variables. WLSM provides weighted least square parameter estimates using a diagonal weight matrix with standard errors and mean-adjusted chi-square test statistic that use a full weight matrix (Muthén & Satorra, 1995). The use of WLSM is suggested when there is large number of categorical variables (i.e., > 15) included in the model estimation process (Muthén, du Toit, & Spicic, 1997). On the other hand, MLR provides maximum likelihood parameter estimates with standard errors and a chi-square test statistic which requires a numerical integration algorithm.

MLR uses full information estimation with logit and/or probit links (default is the logit in Mplus), whereas WLSM uses limited information estimation with only probit links. However, MLR uses the same model as WLSM when assuming normal factors and using probit links. Both estimators are robust to nonnormality of data and nonindependence of observations for unbalanced group sizes (Hox & Maas, 2004). Thus, the WLSM estimator to illustrate the overall procedures of MSEM was used in the present study. However, MLR estimator using a numerical integration algorithm was also used when the estimation of multilevel models with categorical variables took considerable time to converge on a solution.

3.6.4 STEPS IN MULTILEVEL STRUCTURAL EQUATION MODELING WITH CATEGORICAL VARIABLES

A four-step procedure is followed in preceding MSEM with categorical variables (Muthén & Muthén, 1998-2010). In a preliminary analysis step the proportions of within- and between-classroom variance (intraclass correlation coefficients) of the study variables are computed. This enables to determine the percentage of total variation attributable to between-classroom variation and whether it is of value to use multilevel modeling approaches. Subsequently, three steps are taken into consideration: multilevel exploratory factor analysis (MEFA), multilevel confirmatory factor analysis (MCFA), and multilevel structural equation modeling. The conceptual background for each of these steps is given in the following lines.

It is important to note that researchers who are interested in MSEM with continuous variables should follow Muthén's (1989a, 1989b) five-step procedure: conventional structural equation modeling of the total sample covariance matrix, estimation of between-classroom variation, estimation of pooled within-classroom structure, estimation of between-classroom structure, and estimation of the within- and between-classroom structure simultaneously.

3.6.4.1 Step 1 Intraclass Correlation Coefficient (ICC)

The computation of ICCs is a crucial preliminary analysis which displays whether multilevel analyses are required: if the proportion of variance attributable to the classrooms is large enough (i.e., ρ≥ .10), further multilevel analyses can be dispensed with (Bickel, 2007). If ICCs for the items are not large (i.e., close to zero), it is worthwhile to conduct conventional single-level SEM analyses to obtain unbiased chi-square model/data fit statistic, parameter estimates, and standard errors (Julian, 2001).

Muthén (1994) suggested computing the estimated intraclass correlation coefficients (ICC) for each item to get a rough indication of the amount of between-classroom variation. For continuous variables, ICC is defined as

$$\rho = \frac{\sigma_B^2}{\sigma_B^2 + \sigma_W^2}$$

where σ_B^2 and σ_W^2 are the between- and within-classroom variances, respectively.

For categorical variables with logit link function takes the form of

$$\rho = \frac{\sigma_B^2}{\sigma_B^2 + \frac{\pi^2}{3}}$$

whereas, for probit link function it is formulated as

$$\rho = \frac{\sigma_B^2}{\sigma_B^2 + 1} .$$

3.6.4.2 Step 2 Multilevel Exploratory Factor Analysis (MEFA)

Traditionally, exploratory factor analysis has been conducted on data from cross-sectional designs using students as the unit of analysis (i.e., a single-level exploratory factor analysis). However, data from such designs have shown that characteristics of educational constructs may not be as stable as previously hypothesized (Goldstein, 1987; Muthén, 1994; O'Connell & McCoach, 2008). Using single timepoint measures, then, inherently confound student and classroom variation on the given occasion (e.g., multiple-choice test, attitude scale) that the characteristic of an educational construct is measured. Simply collecting cross-sectional data where single timepoint observations are nested within classrooms provides a method to tease apart within- and between-classroom variation for variables of interest and the factors that may underlie them.

Technically, MEFA is used to determine the number of continuous latent variables that are needed to explain the correlations among a set of categorical observed variables at the within- and between-classroom levels. The continuous latent variables are referred to as factors, and the observed categorical variables are referred to as factor indicators. This way, two separate exploratory factor analysis models are obtained: one that accounts for the structure of the items at within-classroom level and the other that accounts for the structure of the items at between-classroom level.

MEFA can be conducted in two ways. First, the within-classroom structure of an educational construct might be previously determined by a pilot study using single-level factor analysis procedures. As the factor structure at the between-classroom level is unknown, the number of factors at the between-classroom level can be specified with regard to the decions made upon the theory and the review of related literature holding the within-classroom model constant. Second, if both within- and between-classroom structures are not known then maximum number of factors that can be extracted can be determined by the formula given below:

$a = pm + m(m+1)/2 + p - m^2$ and $b = p\ (p + 1)/2$; where p is the number of observed variables and m is the number of factors. The $a \leq b$ criterion; where a is the number of parameters to be estimated and b is the number of variances/covariances should be supplied. Along with Fabrigar, Wegener, MacCallum, and Strahan (1999) specifying too few factors (i.e., underfactoring) and/or too many factors (i.e., overfactoring) at both levels should be attentively taken into consideration. It should also be noted that it is typically suggested to include fewer factors at the between-classroom level than at the within-classroom level (Heck & Thomas, 2008; Hox, 2002).

Once the number of factors to be extracted at the within- and between-classroom levels is determined and MEFA is conducted, suggestions such as (a) eigenvalue-greater-than-one rule (Kaiser, 1960), (b) pattern matrix coefficients have to exceed .30 on at least one factor (Hair, Black, Babin, Anderson, & Tatham, 2006), and (c) at least three significant coefficients is required to identify a factor (Zwick & Velicer, 1986) can be followed to guide the analytical decisions. Based on the results, D'Haenens, Damme, and Onghena (2010) suggested that different factor solutions can be compared regarding (a) the interpretation of factors (i.e., dropping the factors that are poorly measured), and (b) the number of items included in each factor (i.e., dropping items that poorly measure factors).

3.6.4.3 Step 3 Multilevel Confirmatory Factor Analysis (MCFA)

MCFA is a theory-based technique that is used to determine if the number of factors and the factor loadings of the observed variables on them conform to what is hypothesized. It serves as a multilevel measurement model in which latent variables are prescribed and thus applied to cross-validate the factor structure at the within- and between-classroom levels that emerged from MEFA. The measurement model for MCFA is a multivariate regression model that describes the relationships between a set of observed dependent variables (i.e., categorical factor indicators) and a set of continuous latent variables (i.e., factors). The relationships are described by a set of linear regression equations for categorical factor indicators. When WLSM estimator is used these equations are probit regression equations whereas they are logistic regression equations for MLR estimator (Muthén & Muthén, 1998-2010). From a methodological point of view, in MCFA researchers posit an a priori MEFA structure and test the ability of a solution based on this structure to fit the data at each level of analysis (e.g., within-classroom and between-classroom) within three considerations: (a) the factor solution is well-defined and it evinces interpretable and distinct factors (Nezlek, 2008); (b) parameter estimates are consistent with theory and in the line of a priori predictions (McDonald & Marsh, 1990); and (c) the goodness-of-fit statistics are at a reasonable size (Dedrick, Ferron, Hess, Hogarty, Kromrey, Lang et al., 2009).

3.6.4.4 Step 4 Multilevel Structural Equation Modeling (MSEM)

MSEM is a regression-based technique that examines both the direct and indirect relationships among latent variables along with the total effects within a nested structure. In this sense, the multilevel structural model is an indication of the extent which the hypothesized relationships at the within- and between-classroom levels are supported by the data (Stapleton, 2006). Multilevel constructs within a given methodological network may be combined in a variety of different ways to estimate models differing in structure. Two broad classes of models involve cross-level and homologous multilevel models.

Cross-level models demonstrate the relationships at different levels of analysis. The most common cross-level models are the *cross-level direct effects model* and the *cross-level moderator model*. Klein and Kozlowski (2000) distinguished between these two types of models in response to the need to estimate a particular model to test a direct effect or a mediation hypothesis in a specific design. In a cross-level direct effects model it is hypothesized that a predictor variable at one level of analysis influences an outcome variable at a different level of analysis. Such a model typically suggests that a predictor variable at the between-classroom level influences an outcome variable at the within-classroom level. A cross-level mediator model on the other hand describes the interaction effects suggesting that variables at two different levels of analysis interact to predict an outcome at the within-classroom level and/or between-classroom level. Thus, for example, a within-classroom variable might moderate the effects of a between-classroom variable on a within-classroom outcome (2-1-1 model). Alternatively, a between-classroom variable might moderate the effects of a within-classroom on a within-classroom outcome (1-2-1 model). Mathematically, these two-cross level models are equivalent (Bauer, Preacher, & Gil, 2006). This line of reasoning suggests several mediation models can be created based on the cross-level mediation hypotheses whereby lead to the specification of upper level mediation (e.g., 2-2-1 and 1-2-2 designs) as well as lower level mediation (e.g., 1-1-1, 1-1-2, and 2-1-2 designs) (Kenny, Korchmaros, & Bolger, 2003; Pituch & Stapleton, 2008; Preacher, Zyphur, & Zhang, 2010).

The second class of multilevel structural models – the homologous multilevel models- specifies that relationships among variables hold at multiple levels of analysis. Researchers might propose for example that the association between two variables is not a cross-level relationship but a multilevel, homologous relationship. That is, at both the within- and between-classroom levels the same relationship is hypothesized. Models of this type have the primary value of generalizing both latent constructs and functional relationships linking the latent constructs across within- and between-classroom levels (Klein & Kozlowski, 2000).

Ideally, whether we are considering a cross-level or a homologous multilevel model the strategies that compare the alternative models should be emphasized. Rather than merely making a series of model trimming or model building attempts in the proposed model through examining modification indices (Heck & Thomas, 2008), alternative models should be tested in terms of these modifications which are made sparingly with regard to the theory and statistical power. Subsequently, Chi-square Difference Testing should be conducted in order to determine which model (i.e., the proposed model or the alternative model) fits the data well. For nested and nonnested models different procedures are applied as well as in response to the estimator being used (Bentler & Bonett, 1980).

3.6.5 MULTILEVEL STRUCTURAL EQUATION MODELING WITH MPLUS

Hierarchical and nested data structures are common in education (Morris, 1995). However, the use of multilevel models is a promising but still underutilized approach (Dedrick et al., 2009), and that these models are more complicated than conventional single-level models. Much of the methodological work in MSEM is continuing at present with a tendency in hard thinking about hierarchical data, iterative model checking, and developing better software. The pervasiveness of hierarchical data has led to a proliferation of statistical software, referred to under a number of names including HLM (Raudenbush & Byrk, 2002), LISREL (Jöreskog & Sörbom, 1993), EQS (Bentler, 1995), Mplus (Muthén & Muthén, 1998), and MLwiN (Rasbash, Charlton, Browne, Healy, & Cameron, 2009). Although the MSEM approach and corresponding computer software, HLM have been widely accepted in the analysis of crosscultural multivariate data, the techniques have not been widely applied to the analysis of multilevel educational data structures with Mplus (Hox, 1995; Muthén, 1994). Mplus 6.1 (Muthén & Muthén, 1998-2010) program is attractive for multilevel modeling. It provides a flexible framework that makes possible the specification and testing of a wide variety of theoretical models. A defining feature of the software program is its providing the opportunity to handle numerous types of models with categorical observed and latent variables. Mplus can be used for single-level regression and path analysis, exploratory factor analysis, confirmatory factor analysis, structural equation modeling, mixture modeling, multilevel exploratory factor analysis, multilevel confirmatory factor analysis, multilevel structural equation modeling, multilevel mixture modeling, and further missing data modeling within Bayesian analysis and Monte Carlo simulation studies. The user-friendly interface of the program in analyzing multilevel data considerably enhance the analytic possibilities with MSEM for research designs where individuals are nested within groups. Furthermore, licensed users have the opportunity to register the Mplus Discussion Board and request help from Mplus Support Service when needed.

The Mplus multilevel modeling framework draws on the unifying theme of latent variable modeling and the unique use of both observed and latent variables. Continuous latent variables that are formed by categorical observed variables are used to represent random effects corresponding to the variation in coefficients across groups in the nested data structure. The goal of the multilevel analysis is then to decompose the variation in a set of relationships into variance components associated with each level of a hierarchical data structure and explain the variation present at each level simultaneously designating the direct, indirect, and total effects (Bollen, 1987).

After Mplus is installed, it can be run from the Mplus Editor for Windows including a language generator and a graphics module. Apart from other software programs Mplus does not provide a pictorial diagram of the multilevel models. The user language for Mplus consists of a set of ten commands each of which has several options. However, for most analyses only a small subset of commands is amounted. The ten commands include TITLE, DATA, VARIABLE, DEFINE, ANALYSIS, MODEL, OUTPUT, SAVEDATA, PLOT, and MONTECARLO. For simplicity, only the commands that were used in the present study will be detailed in the following lines.

The TITLE command is used to provide a title for the analysis. The DATA command is used to provide information about the data set to be analyzed. In terms of this command, the location of the data set to be analyzed is specified. The data must be numeric except for certain missing value options and must reside in an external ASCII file. Researchers can save SPSS files as FIXED ASCII format and specify the number of observations and number of groups in the data set. The FILE option is a required option under this command that is used to specify the name and the location of the ASCII file. The VARIABLE command is used to provide information about the naming and description of the variables in the data set. Variable names are generated as a list under the

NAMES option. The USEVARIABLES option is used to select variables for a specific analysis. NAMES option includes all the variables in the data set whereas the USEVARIABLES option generates only the variables that are to be used from the NAMES statement of the VARIABLE command. New variables can be created by the DEFINE command. The CATEGORICAL option is used to specify which dependent variables are treated as binary or ordered categorical whereas the NOMINAL option is used to specify which dependent variables are treated as unordered categorical in the model and its estimation. There are two options specific to two-level models: WITHIN and BETWEEN. The WITHIN/BETWEEN options are used with the analysis TYPE = TWOLEVEL to identify the variables in the within/between-level and modeled only on the within/between level of analysis. The CLUSTER option specifies the variable that contains the clustering information (i.e., group, class, school).

The ANALYSIS command is used to describe the technical details of the analysis that involves type of analysis, the statistical estimator, the parameterization of the model, and specifics of the computational algorithms. In order to capture the four step procedure of the MSEM techniques the ANALYSIS command presents the options as TYPE = TWOLEVEL BASIC (Step 1), TYPE = TWOLEVEL EFA # # UW # # UB (Step 2), and TYPE = TWOLEVEL (Step 3 and Step 4). Depending on the analysis type there are several ESTIMATOR and ALGORITHM options.

The MODEL command is used to provide a description of the model to be estimated. This command has variations for use with models with indirect and cross-level effects. In order to be able to specify a model, researchers need to make four important distinctions about the variables in the VARIABLES command. The distinctions are whether the variables are observed or latent, whether variables are dependent or independent, whether variables are at the within-level or at the between-level, and the scale of the observed dependent variables.

The underlying framework of Mplus comprises the measurement model including the indicators of the latent variables and the structural model involving the relationships among the observed and latent variables. Accordingly, a model may consist of only a measurement model as in the multilevel confirmatory factor, only a structural model as in a multilevel path analysis, or both the measurement and structural model as in the multilevel structural analysis. In a TYPE = TWOLEVEL analysis there are three major options: BY, ON, and WITH. BY defines latent variables, ON defines regression relationships, WITH defines correlational relationships, @ fixes a parameter at a default value or a specific value, | names and defines random effect variables, XWITH defines interactions between variables. Furthermore, MODEL INDIRECT: describes the relationships for which indirect and total effects are requested. Under this option IND describes a specific indirect effect or a set of indirect effects and VIA describes a set of indirect effects that includes specific mediators. The most important variations of the MODEL command are %WITHIN% which describes the within part and %BETWEEN% which describes the between part of a two-level model.

The OUTPUT command is used to request additional output such as sample statistics (SAMPSTAT), standardized and unstandardized parameter estimates (STANDARDIZED), and/or modification indices (MODINDICES).

The Mplus output initially provides the input setup which contains the restatement of the input file. The restatement of the input instructions arranges a record of which input file produced the results in the output. How Mplus interpreted the input instructions and read the data are shown in the summary of the analysis specifications. Researchers should check the number of observations, read the warnings and error messages generated by the program to appropriately understand and modify the analysis. The summary of analysis results provides fit statistics (e.g., chi-square test statistic, degrees of freedom, and p-value for the analysis model), parameter estimates, and standard errors. The type of regression coefficient (logit or probit) produced during model estimation is determined by the scale of the dependent variable (continuous or categorical) and the estimator being used (ML, MLR, WLS, WLSM, etc.). The parameter estimates, standard errors, and the value of the parameter estimate divided by the standard error are also provided in the output. The value of the parameter estimate divided by the standard error (Est./S.E.) is a statistical test of significance (z-test). The critical value for a two-tailed test at the .05 level refers to an absolute value greater than 1.96.

SAMPSTAT option is used to request the sample statistics such as sample means, variances, covariances, and correlations. For categorical variables using WLSM estimation these include sample thresholds, sample tetrachoric correlations, and sample probit regression coefficients. The STANDARDIZED option is used to request standardized parameter estimates and their standard errors. Mplus output provides three types of standardizations as the default: STDYX, STDY, and STD. For standardization, STDYX uses the variances of the continuous latent variables and the outcome variables. The standardized coefficient is interpreted as the change in y in y standard deviation units for a standard deviation change in x. Since such a standard deviation change of a binary variable is not meaningful, STDY should be used for categorical covariates. The standardized coefficient is interpreted as the change in y in y standard deviation units when x changes from zero to 1. Finally, STD uses only the variances of the continuous latent variables.

The MODINDICES option is used to request modification indices and expected parameter change indices. To request modification indices for all matrices MODINDICES (ALL) is specified. However, for simplicity it is suggested to specify MODINDICES (3.84). The modification indices are the amount chi-square will drop the parameter is estimated as part of the model. The chi-square value of 3.84 refers to the value that should be exceeded for one degree of freedom at the .05 level, and that the desired modification index should be at least 3.84 and large enough to free the parameter.

3.6.5.1 The Goodness-of-Fit Criteria for Multilevel Structural Equation Modeling

In the present study Mplus Version 6.1 (Muthén & Muthén, 1998-2010) was used to formulate and estimate the two-level models. For the evaluation of the adequacy of the multilevel models in explaining the data a set of goodness-of-fit indices provided by the Mplus program. Model-data-fit indices recommended by Schreiber, Stage, King, Nora, and Barlow (2006) were the chi-square goodness-of-fit statistic (χ^2), the root mean square error of approximation (RMSEA), the standardized root mean square residual (SRMR), the comparative fit index (CFI), and the Tucker-Lewis index (TLI). For lack of any standard cut-off criteria for MSEM with categorical variables, a small (relative to its degrees of freedom) non-significant chi-square, a RMSEA from .06 to .08, a SRMR close to .08, a CFI above .90, and a TLI above .90 were displayed as good fitting (Hu & Bentler, 1999; Marsh, Hau, & Bentler, 2004; Yu, 2002). Hox (2002) noted that goodness-of-fit indices are more sensitive to the evaluation of within-model misspecifications (i.e., models at the within-classroom level) and less sensitive to the evaluation of between-model misspecifications (i.e., models at the between-classroom level). Interpretations of goodness-of-fit criteria are given in detail below:

1. *Chi-Square Statistic (χ^2)*

Chi-square is a measure of overall fit of the model to the data (Jöreskog & Sörbom, 1993). A nonsignificant chi-square value implies that the model fits the data. However, chi-square statistic has been widely criticized because of its sensitivity to sample size (Bollen, 1989; 1993; Kelloway, 1998; Kline, 2005). It tends to indicate a significant probability level when the sample size increases (Schumacker & Lomax, 2004). In MSEM lower values of chi-square are intended.

2. Root Mean Squared Error of Approximation (RMSEA)

The RMSEA is a measure of discrepancy per degree of freedom. It is suggested that values from .07 to .10 indicate a moderate fit, the values from .04 to .06 indicate a good fit, and the values from .00 to .03 indicate an excellent fit to the data (Hu & Bentler, 1999).

3. Standardized Root Mean Square Residual (SRMR)

The SRMR reflects the average magnitude of the residuals. It has a lower bound of 0 and upper bound of 1 (Hu & Bentler, 1999). In Mplus, SRMR values are available for within-classroom level and between-classroom level models. The values less than .05 indicate a good fit to the data (Ryu & West, 2009), however values close to .08 are also acceptable (Marsh, Hau, & Bentler, 2004).

4. Comparative Fit Index (CFI)

The CFI provides an indication of how much better the current model is than a baseline model. It varies along a 0 to 1 continuum in which values at or greater than .90 and .95 indicate an acceptable and excellent fits to the data respectively (McDonald & Marsh, 1990).

5. Tucker Lewis Index (TLI)

The TLI which is known as the Non-normed Fit Index (NNFI) refers to the number of degrees of freedom in a model. Similar to CFI, it varies along a 0 to 1 continuum in which values at or greater than .90 and .95 indicate an acceptable and excellent fits to the data respectively (McDonald & Marsh, 1990; Ryu, 2008).

3.6.6 MULTILEVEL STRUCTURAL VARIABLES OF THE STUDY

3.6.6.1 Within-Classroom Level Variables

Six within-classroom level variables were introduced to the multilevel structural model of mathematical thinking on the basis of the results of the confirmatory factor analysis of students’ scores on TDT at the pilot study. These variables included *enactive thinking*, *iconic thinking*, *algorithmic thinking*, *algebraic thinking*, *formal thinking*, and *algorithmic thinking*.

3.6.6.2 Between-Classroom Level Variables

Three within-classroom level variables were introduced to the multilevel structural model of mathematical thinking on the basis of the results of the MEFA and MCFA. These variables included *conceptual-embodied thinking*, *proceptual-symbolic thinking*, and *formal-axiomatic thinking*.

CHAPTER 4

RESULTS

Results of the present study are given in two main sections as preliminary analysis and multilevel structural equation modeling. Preliminary analysis includes the confirmatory factor analysis, reliability analysis, and the further validation analysis of the TDT. Multilevel structural equation modeling involves the validation of multilevel structural equation modeling assumptions, computation of intraclass correlation coefficients, multilevel exploratory factor analysis, multilevel confirmatory factor analysis, and two-level structural equation modeling.

4.1 PRELIMINARY ANALYSIS

Evidence accumulated through a cross-sectional data set and summarized with preliminary analysis substantiated that the TDT is a valid and reliable instrument to measure undergraduate students' mathematical thinking in derivative. This data set also demonstrated the existence of six different types of mathematical thinking: enactive, iconic, algorithmic, algebraic, formal, and axiomatic thinking. It is important to note that multiple-choice items presents a risk of guessing. However, the inspection of students' written responses on the test papers provided evidence for controlling the guessing factor.

4.1.1 RESULTS OF CONFIRMATORY FACTOR ANALYSIS FOR THE THINKING-IN-DERIVATIVE TEST

Confirmatory factor analysis (CFA) was conducted to specify the observed variables that indicate the latent factors of the Thinking-in-Derivative Test (TDT).

The CFA supported the six-factor solution that emerged from the aforementioned hypotheses. The weighted least square estimations appeared between .39 and .97; demonstrating that the factor coefficients of each item on the related context were at a reasonable size to define enactive thinking (ENACTHK), iconic thinking (ICONTHK), algorithmic thinking (ALGOTHK), algebraic thinking (ALGETHK), formal thinking (FORMTHK), and axiomatic thinking (AXIOTHK). More specifically, Items 1, 2, 3, 4, 5, and 6 comprised FORMTHK, whereas items 7, 8, 9, 10, and 11 constituted AXIOTHK. Items 12, 13, 14, and 15 were amount to ALGETHK while ICONTHK involved items 16, 17, 18, 19, and 20. Items 21, 22, 24, 25, and 26 reflected ALGOTHK whereas items 23, 27, 28, 29, and 30 mirrored ENACTHK.

Fit statistics of squared multiple correlation (R^2) were calculated for each observed variable (Item 1 – Item 30) that represent the latent variables. The values of R^2 that equal to the proportion of explained variance indicated the reliability of the items that specify ENACTHK, ICONTHK, ALGOTHK, ALGETHK, FORMTHK, and AXIOTHK were substantial in size ranging from .70 to .90.

Table 4.1 presents the standardized estimates and reliability coefficients for the items in the six dimensions of the TDT generated in this phase.

Table 4.1 Standardized estimates and reliability coefficients of items in TDT

Items	ENACTHK	ICONTHK	ALGOTHK	R^2
24	1.00			.70
29	.69			.83
28	.76			.79
30	.60			.89
27	.65			.85
20		1.00		.74
19		.87		.87
16		.85		.74
18		.94		.81
17		.67		.77
23			1.00	.77
25			.92	.78
22			.79	.70
21			.81	.73
26			.65	.83

Table 4.1 (continued)

Items	ALGETHK	FORMTHK	AXIOTHK	R^2
15	1.00			.90
14	.88			.88
12	.77			.80
13	.87			.86
4		1.00		.86
5		.85		.84
3		.73		.84
2		.74		.83
6		.93		.88
1		.90		.88
9			1.00	.80
10			.77	.87
8			.67	.77
11			.90	.84
7			.89	.80

4.1.2 RESULTS OF FURTHER VALIDATION ANALYSIS FOR THE THINKING-IN-DERIVATIVE TEST

4.1.2.1 Results Of Discriminant Validity Analysis

The model comparisons were presented in Table 4.2. Testing of four different models revealed that the target model fit the data better than the common, three-factor, and null models across fit indices. Furthermore, the chi-square difference tests indicated the superiority of the target model as compared to these alternate theoretically plausible models. In addition, the difference in practical fit between the four models was substantial. Collectively, these results offered supplementary evidence for the six theory-driven dimensions of the TDT, especially at the within-classroom level. The LISREL input files for the CFA models are given in Appendices. Specifically, the LISREL input files for the common factor model, the three-factor model, the null model, and the theoretical target model are presented in Appendix D, Appendix E, Appendix F, and Appendix G, respectively.

Table 4.2 Goodness-of-fit statistics for tests of discriminant validity

Model	Statistics		
	χ^2	Df	NC (χ^2/df)
Target	789.77	372	2.1
Common factor	1284.83	350	3.6
Three-factor	989.79	341	2.8
Null	1580.14	390	4.5
	RMSEA	RMR	GFI
Target	.03	.07	.98
Common factor	.05	.08	.92
Three-factor	.05	.08	.94
Null	.06	.09	.91
	AGFI		CFI
Target	.95		.98
Common factor	.91		.95
Three-factor	.93		.96
Null	.89		.95
	$\Delta\chi^2$	Δdf	ΔCFI
Target	-	-	-
Common factor	495.06	1.5	.03
Three-factor	199.02	.7	.02
Null	790.37	2.4	.03

The more parsimonious, common factor model showed a poorer fit to the data than the target model. The comparison of the target model to the common factor model across goodness of fit indices revealed the target model fits the data better. In addition, the chi-square difference test indicated the superiority of the target model as compared to the common factor model ($\Delta\chi^2 = 495.06$, $\Delta df = -22$, $p < .001$). The significance of the chi-square supported that it is unlikely to take the common factor model as a correct alternate. In addition, the difference in practical fit between the two models ($\Delta CFI = .03$) was substantial, which was greater than the cutoff point ($\Delta CFI = .01$) suggested by Cheung and Rensvold (2002). Taken together, these findings provided additional support for the subscale dimensionality of the TDT.

As is revealed in Table 4.2, the target model fit the data better than the three factor model across fit indices. The chi-square difference test indicated the superiority of the target model as compared to the null model ($\Delta\chi^2 = 199.02$, $\Delta df = -31$, $p < .001$). The significance of the chi-square was in favor of the six-factor model. Additionally, the difference in practical fit between the two models was significant ($\Delta CFI = .02$).

It is widely acknowledged that a null model is expected to have a poorer fit to the data than a target model. However, a null model can establish discriminant validity if it is shown to fit significantly worse than the target model. As is demonstrated in Table 4.2, the target model again had a better fit to the data. The chi-square difference test indicated the superiority of the target model as compared to the null model ($\Delta\chi^2 = 790.37$, $\Delta df = 18$, $p < .001$). The difference in practical fit between the two models was considerable in size ($\Delta CFI = .03$) and that these results offered supplementary evidence of the existence of the six a priori dimensions of the TDT.

4.1.2.2 Results Of Subgroup Validity Analysis

Results of the multivariate analysis of variance revealed that the six dimensions of the TDT differentiate between females and males. Preliminary assumption testing on multivariate normality and homogeneity of variance-covariance matrices was generated and no violations were detected. In terms of testing multivariate normality with categorical variables, the floor and ceiling effects were investigated. Turning first to the frequencies on the 30 items demonstrated percentages of responses on the correct choice (from 15.5% to 19.3%) and on the incorrect choice (from 10.8% to 23%). The percentage of responses at the lowest or the highest level response option was well below the cut-off 25% value (Haladyna, Downing, & Rodriguez, 2002; Haladyna, 1994). Thus, the data did not violate the assumption of the multivariate normality. This further indicated that there were no substantial multivariate outliers in the data.

To test whether the data violate the assumption of homogeneity of variance-covariance matrices, Box's Test of Equality of Covariance Matrices was checked. The significance value was .028 for gender. Since this value was larger than the significance value of .001 (Tabachnick & Fidell, 2007) the data did not violate the assumption of homogeneity of variance-covariance matrices. Consistent with the predictions, results of the multivariate analysis revealed a significant main effect for female/male difference (Wilk's Lambda = .98, $F (6, 759) = 2.06$, $p < .001$, $\eta^2 = .016$) suggesting that the female and male students differed on a linear combination of the six dimensions of the TDT. The partial eta squared of .016 would be interpreted as a small effect (Cohen, 1988). The follow-up univariate analyses revealed that there was a significant difference between females and males on enactive thinking, $F (1, 764) = 3.28$, $p < .001$, $\eta^2 = .004$. Males ($M = 2.89$, $SD = 1.45$) outperformed females ($M = 2.70$, $SD = 1.42$) in modeling real life applications of the derivative within optimization problems. On iconic, algorithmic, and algebraic thinking males again scored higher than females; however they did not appear significant.

Further validation evidence indicated that the six dimensions of the TDT differentiate among students majoring in Faculty of Education, Faculty of Arts and Sciences, and Faculty of Engineering. To test whether the data violate the assumption of homogeneity of variance-covariance matrices, Box's Test of Equality of Covariance Matrices was checked. The significance value was .044 for faculty affiliation. Since this value was larger than the significance value of .001 (Tabachnick & Fidell, 2007) the data did not violate the assumption of homogeneity of variance-covariance matrices. In line with the predictions, results of the multivariate analysis revealed a significant main effect for faculty affiliation difference (Wilk's Lambda = .74, $F(12, 1516) = 19.99$, $p < .001$, $\eta^2 = .137$) suggesting that students from Faculty of Education, Faculty of Arts and Sciences, and Faculty of Engineering differed on a linear combination of the six dimensions of the TDT. The partial eta squared of .137 would be interpreted as a large effect (Cohen, 1988). The follow-up univariate analyses revealed that there was a significant difference among students attending different faculties in all six dimensions of the TDT ($p < .001$). In addition students majoring in Faculty of Arts and Sciences outperformed their counterparts who are majoring in Faculty of Education and Faculty of Engineering on all six dimensions of the TDT. This finding demonstrated that scientific students were more apt to energize their enactive, iconic, algorithmic, algebraic, formal, and axiomatic thinking than teaching and /or engineering students.

4.1.3 RESULTS OF RELIABILITY ANALYSIS

The necessary steps taken with TESTFACT 4 program for estimating the reliability coefficients indicated that the reliability analysis resulted in the acceptance of all the items for further analyses. The TESTFACT 4 input file is presented in Appendix H.

Results documented that none of the items violated the aforementioned criteria. The KR-20 reliability coefficient for the scores on enactive thinking was .78, iconic thinking was .97, algorithmic thinking was .96, algebraic thinking was .91, formal thinking was .93, and axiomatic thinking was .87. These findings reflected high reliability of the six dimensions of TDT.

4.2 RESULTS OF TESTING MULTILEVEL STRUCTURAL EQUATION MODELING ASSUMPTIONS

The present study used MSEM as a useful quantitative method in specifying, estimating, and testing the hypothesized theoretical models that describe the relationships among mathematical thinking at the within- and between-classroom levels. Accordingly, it was focused on the validity of models and the directed effects among model parameters. Because multivariate normality of observed data is a typical assumption for multivariate regression analysis, it is certainly a crucial assumption for MSEM analysis when parameter estimation methods such as WLSM and MLR are used (Schumacker & Lomax, 2004). Any violation of the normality and independence assumptions can produce inaccurate parameter estimates, which will lead to improper interpretation of the results. Therefore, the present study initially sought to know the robustness of MSEM analysis results under normality, sample size, and estimation methods.

In Sample 1, the inspection of the absolute values of skewness and kurtosis for the observed variables (item1 through item30) revealed that the values for skewness ranged from .016 to 1.43 and the values for kurtosis ranged from .068 to 2.00. Similarly, in Sample 2, the inspection of the absolute values of skewness and kurtosis for the observed variables (item1 through item30) revealed that the values for skewness ranged from .008 to 1.77 and the values for kurtosis ranged from .858 to 2.00. For both samples, investigation of the values of skewness and kurtosis for each observed variable were acceptable and that there is only slight nonnormality. That is, the findings concluded that nonnormality has negligible effects on parameter estimates. The values of skewness and kurtosis values of each item in TDT for Sample 1 and Sample 2 are presented in Appendix I and Appendix J, respectively.

Students' scores on each item on the TDT, total scores on each mathematical thinking dimension, and total scores on the overall test were investigated in order to determine whether floor or ceiling effects exist. Descriptive statistics were generated to characterize the TDT score distributions, including percentage floor and ceiling effects of each item on the TDT, total scores on each mathematical thinking dimension, and total scores on the overall test for Sample 1 and Sample 2, respectively. The results of these analyses are presented in Appendix K, Appendix L, and Appendix M for Sample 1 and in Appendix N, Appendix O, and Appendix P for Sample 2, respectively.

As evidenced by the skewness and kurtosis values of Sample 1 and Sample 2, there were no floor and ceiling effects in students' scores on each item on the TDT, total scores on each mathematical thinking dimension, and total scores on the overall test.

Turning first to Sample 1, the frequencies on the 30 items demonstrated percentages of responses on the correct choice (from 35.5% to 79.2%) and on the incorrect choice (from 20.8% to 53%). Results revealed that the formal thinking question, Item 5 (79.2%) showed a ceiling effect whereas enactive thinking question, Item 29 showed a floor effect (53%). The remaining items showed almost equal percentages of responses on the correct and incorrect choices indicating that there were no floor and ceiling effects. None of the sub-total scores of the TDT showed floor or ceiling effects. The percentage of responses at the lowest (total = 0) and highest response (total = 6) ranged from .07% to 15.7% for formal thinking. The percentage of responses at the lowest (total = 0) and highest response (total = 4) ranged from 8.2% to 15.8% for algebraic thinking. The percentage of responses at the lowest (total = 0) and highest response (total = 5) ranged from 3.8% to 17.3%, from 6.4% to 23.4%, from 5.2% to 21.8%, and from 4.6% to 17.4% for enactive, iconic, algorithmic, and axiomatic thinking, respectively. In the same vein, the floor and ceiling effects were not pronounced for students' total scores on the TDT. Scores ranged from 3 to 30 with only 116 students obtaining top scores on the test. The percentage of responses at the lowest response (total = 0) and highest response (total = 30) ranged from .0% to 10.6%. Given the large sample size (N = 1099), results indicated no floor or ceiling effects.

Moving to Sample 2, the frequencies on the 30 items demonstrated percentages of responses on the correct choice (from 42.5% to 83.1%) and on the incorrect choice (from 33.6% to 57.5%). As with Sample 1, formal thinking question, Item 5 (83.1%) evidenced a ceiling effect. On the other hand, iconic thinking question, Item 29 showed a floor effect (33.6%). The remaining items showed almost equal percentages of responses on the correct and incorrect choices indicating that there were no floor and ceiling effects. None of the sub-total scores of the TDT showed floor or ceiling effects. The percentage of responses at the lowest (total = 0) and highest response (total = 6) ranged from 1.5% to 9.4% for formal thinking. The percentage of responses at the lowest (total = 0) and highest response (total = 4) ranged from 11.2% to 12.9% for algebraic thinking. The percentage of responses at the lowest (total = 0) and highest response (total = 5) ranged from 4.9% to 15.6%, from 5.7% to 15.9%, from 5.7% to 16.8%, and from 4.5% to 14.2% for enactive, iconic, algorithmic, and axiomatic thinking, respectively. In the same vein, the floor and ceiling effects were not pronounced for students' total scores on the TDT. Scores ranged from 3 to 30 with only 101 students obtaining top scores on the test. The percentage of responses at the lowest response (total = 0) and highest response (total = 30) ranged from .0% to 4.2%. Given the large sample size (N = 2424), results indicated no floor or ceiling effects.

Collectively, floor and ceiling effects demonstrated by the percentage of responses at the lowest and highest response in students' scores on each item on the TDT, total scores on each mathematical thinking dimension, and total scores on the overall were negligible and minimal. In essence, the modest floor and ceiling effects meant that the TDT was sufficiently challenging but not overtly difficult. This strength of the TDT provided support for the multivariate normality assumption.

In the present study WLSM and MLR estimators were used to test the theoretical models. The weighted least squares and maximum likelihood estimation methods used commonly in MSEM are asymptotic, which translates to the assumption that the sample size is large (Maas & Hox, 2004). The accuracy of estimation methods with adequate sample sizes has been markedly criticized in relation to the assumptions of MSEM (Hox & Maas, 2001). This concerns especially the between-classroom level, because the sample size at the highest level (the sample of classrooms) is always smaller than the sample size at the lowest level (the sample of students). Along with the suggestions of several researchers (Muthén, 1994; Raudenbush & Byrk, 2002; Snijders & Bosker, 1999; Stapleton, 2006), the present had the adequate sample size at the within-classroom level (N = 2424) and at the between-classroom-level (N = 130).

The assumptions relevant to local dependence in consonance with the multivariate normal distributions for all residuals, independence of residuals for different levels, and independence of residuals for different units in the same level were also met by the use of WLSM and MLR estimators. Along with Raudenbush and Bryk (2002) the estimation methods used in the present study imply that the residual covariance matrix is diagonal. Since all the multilevel models fit the data well assumptions relevant to uncorrelated residuals were assumed to be not violated.

Taken together, the assumptions of normality and independence of observations were met via the use of accurate estimators (WLSM and MLR) for models with categorical variables which was available in Mplus program.

4.3 MULTILEVEL STRUCTURAL EQUATION MODELING

According to the preliminary analysis (e.g., confirmatory factor analysis) with students' scores on the TDT, the observed variables that represent the six different types of mathematical thinking constructs were determined and then introduced to the multilevel analyses. Two independent samples were used for multilevel analyses. Sample 1 was used for multilevel exploratory factor analysis, whereas Sample 2 was used for multilevel confirmatory analysis and multilevel structural equation modeling All multilevel analyses were conducted via Mplus 6.1 software program using either one of the weighted least squares with mean and maximum likelihood with robust standard errors estimators.

The results of the multilevel structural equation modeling that offer the relationships among students' different types of mathematical thinking at the within-classroom and between-classroom level are presented partially. The results follow a sequence of four steps. In *Step 1*, initial descriptive analyses were run including the calculation of means and standard deviations of the variables. Intraclass correlation coefficients (ICCs) were then computed to determine the proportions of within- and between-classroom variance for all the study variables. In *Step 2*, multilevel exploratory factor analysis (MEFA) was conducted on the TDT scores of Sample 1 to determine the underlying a priori structure of the items in TDT by defining a set of common latent factors. In *Step 3*, multilevel confirmatory factor analysis (MCFA) was conducted on the TDT scores of Sample 2 to cross-validate the factor structure that emerged from Step 2. Finally, in *Step 4*, multilevel structural equation modeling (MSEM) was conducted on the TDT scores of Sample 2 to estimate the relationships among six mathematical thinking constructs. This step includes two stages. At the first stage, a theoretical two-level model was tested to highlight the essence of the interrelations among six different types of mathematical thinking at the within- and between-classroom levels. This cross-level direct effects model was tested to provide an estimate of how much of the variance in student and classroom levels could be explained in terms of the relationships among mathematical thinking constructs, and thus a grasp of the relationships that exhibit differences at each level. In regard to the relationships specified at this first stage, at the second stage, additional cross-level moderator models were tested to investigate the within-classroom relations of mathematical thinking as a function of between-classroom factors as well as the between-classroom relations of mathematical thinking as a function of within-classroom factors.

4.3.1 RESULTS OF DESCRIPTIVE ANALYSIS AND INTRACLASS CORRELATION COEFFICIENTS

The means, standard deviations, and intraclass correlation coefficients of the main variables included in the study are presented in Table 4.3 for Sample 1 and Sample 2, separately. The Mplus input file for Step 1 is given in Appendix Q.

Table 4.3 Means, standard deviations, and intraclass correlations

Variable	Sample 1 (N = 1099)			Sample 2 (N = 2424)		
	M	**SD**	**ICC**	**M**	**SD**	**ICC**
FORMTHK						
Item 1	.48	.50	.26	.49	.50	.45
Item 2	.56	.49	.34	.53	.49	.29
Item 3	.59	.49	.22	.49	.50	.27
Item 4	.61	.48	.17	.57	.49	.19
Item 5	.79	.40	.56	.83	.37	.49
Item 6	.56	.49	.23	.52	.49	.35
AXIOTHK						
Item 7	.60	.49	.39	.64	.47	.19
Item 8	.49	.50	.31	.48	.49	.21
Item 9	.68	.46	.44	.73	.44	.24
Item 10	.46	.49	.33	.42	.49	.30
Item 11	.59	.49	.43	.55	.49	.32
ALGETHK						
Item 12	.48	.50	.28	.45	.49	.15
Item13	.48	.50	.22	.45	.49	.13
Item14	.57	.49	.20	.55	.49	.14
Item 15	.56	.49	.22	.51	.49	.13

Table 4.3 (continued)

Variable	Sample 1 (N = 1099)			Sample 2 (N = 2424)		
	M	**SD**	**ICC**	**M**	**SD**	**ICC**
ICONTHK						
Item 16	.59	.49	.30	.57	.49	.21
Item 17	.50	.50	.20	.43	.49	.17
Item 18	.62	.48	.37	.57	.49	.35
Item 19	.65	.47	.32	.66	.47	.23
Item 20	.64	.48	.37	.60	.48	.38
ALGOTHK						
Item 21	.63	.48	.43	.61	.48	.31
Item 22	.64	.48	.33	.61	.48	.24
Item 24	.59	.49	.27	.58	.49	.31
Item 25	.56	.49	.39	.55	.49	.32
Item 26	.54	.49	.28	.48	.49	.22
ENACTHK						
Item 23	.65	.47	.40	.65	.47	.26
Item 27	.49	.50	.28	.50	.50	.15
Item 28	.59	.49	.37	.48	.48	.30
Item 29	.47	.49	.22	.49	.49	.17
Item 30	.62	.48	.29	.49	.49	.24

With means ranging from .46 to .79 and .42 to .83, students in Sample 1 and Sample 2 progressed in moderate to high levels of mathematical thinking, and there was considerable variation across the six mathematical thinking constructs examined. The difference between the mathematical thinking types with the highest reported score on Item 5 (formal thinking) and the lowest reported score on Item 10 (axiomatic thinking) was approximately more than half a standard deviation. Students' achievement levels on formal thinking items were greater than those on the axiomatic thinking items. This finding builds on and extends the understanding of mathematical thinking in three significant ways. First, as it is the case at secondary level, the impact of declarative editions of mathematical thinking comes fore to the ground at the undergraduate level. Second, faculty-wide adoptions of calculus curricula are used as part of a formal thinking routine rather than more sophisticated lines of axiomatic thinking. This provides a more accurate overall picture of students' mathematical thinking specifically in derivative concept and therefore may not be typical of all concepts. However, the low mean and standard deviation of Item 1 was .48 and .49 for Sample 1 and Sample 2, respectively. This finding added a brushstroke to the emerging picture of mathematical thinking in university classrooms indicating that students failed in the construction of meaning for the derivative. For many students it appeared hard to structure the derivative concept as an instantaneous rate of change or the slope of a tangent.

As expected the emerging body of the results of descriptive analysis concluded that students were more successful in more familiar items. For instance, in both samples, students performed slightly higher in algorithmic thinking items with regard to facility with both arithmetic and symbolic manipulation procedures mixed over differentiation algorithms.

Multilevel analysis was employed to account for the wide variability between classrooms and the interdependency of students within the same classroom. Preliminary analysis showed that there were meaningful between-classroom differences in students' mathematical thinking. The ICCs indicated the proportion of total variance that is located between classrooms; given the same total variance, the higher the ICC, the more similar the mathematical thinking of students in the same classrooms and the more different the mathematical thinking of students in different classrooms. Theoretically, the values of ICC range between 0 and 1.

Results of the analysis with Sample 1 yielded that the ICCs were somewhat higher for axiomatic thinking (ranging from .31 in items to .44) than for formal thinking (ranging from .17 in items to .56), algebraic thinking (ranging from .20 in items to .28), iconic thinking (ranging from .20 in items to .37), algorithmic thinking (ranging from .27 in items to .43), and enactive thinking (ranging from .22 in items to .40). With Sample 2, The ICCs were somewhat higher for formal thinking (ranging from .19 in items to .45) than for axiomatic thinking (ranging from .19 in items to .32), algebraic thinking (ranging from .13 in items to .15), iconic thinking (ranging from .17 in items to .20), algorithmic thinking (ranging from .22 in items to .32), and enactive thinking (ranging from .15 in items to .31).

Taken as a whole, ICC values of around .17 to .56 ascertained that the nested structure of the data set should be modeled using multilevel analyses (Muthén, 1994; Snijders & Bosker, 1999).

4.3.2 RESULTS OF MULTILEVEL EXPLORATORY FACTOR ANALYSIS

The Mplus input file for Step 2 is given in Appendix R. In line with Hypothesis 1, a series of MEFAs was conducted to determine the factor structure for the mathematical thinking constructs at both the within-classroom and the between-classroom levels with Sample 1. Geomin rotation was used for all models that varied in the number of factors specified at each level of the nested data structure (from 1 to 6 factors). Primarily two recommendations were followed in preceding the MEFA. First, if latent factors are indicated by categorical variables it is suggested to include at most four factors at the within- or between-classroom levels (Muthén & Muthén, 1998-2010). However, overfactoring introduces much less error to factor loading estimates than underfactoring (Fabrigar, Wegener, MacCallum, & Strahan, 1999). Along with these recommendations, the confirmatory factor analysis results of the pilot TDT data which supported the six-factor solution (χ^2 (789.77, N = 766) = 2.12, RMSEA = .03, RMR= .07, GFI = .96, AGFI = .95, and CFI= .98) were pursued and six factors were specified at the within-classroom level of the data structure. This progress resulted in models that could be estimated (i.e., the model estimation converged). Second, it is typically suggested to include fewer factors at the between-classroom level than at the within-classroom level (Heck, 2001). As the factor structure at the between-classroom level was unknown 1, 2, and 3 factors at the between-classroom level, holding the within-classroom six-factor model constant. In line with Christou et al. (2005), broader factors were found at the between-classroom level.

As shown in Table 4.4, models including a minimum of three factors at both the within- and between-classroom levels, respectively, fit reasonably well according to the fit indices. In particular, the models that specified (a) 6 Within – 2 Between factors and (b) 6 Within – 3 Between factors fit best according to the more rigorous cutoff values for the RMSEA and SRMR. These two models were compared by means of a chi-square difference test. Since WLSM estimation was used, the difference between two scaled chi-square values for nested models was not distributed as a chi-square. Therefore, chi-square values were corrected before performing a difference test (Muthén & Muthén, 1998-2010). Using chi-square difference testing of the WLSM chi-square values (Satorra & Bentler, 2001) for the two nested models, the 6 Within - 3 Between factors model fit significantly better, $\Delta\chi^2$ (df = 28) = 122.04, $p < .01$.

The pattern matrix coefficients for the 6 Within – 3 Between factors model are presented in Table 4.5. These coefficients are factor loadings that represent the unique contribution of each item to the factor, thus accounting for the interfactor correlations. Along with Hair et al. (2006), all items had pattern matrix coefficients greater than .30 indicating that they correspond precisely to the hypothesized factors at within and between levels. At the within-classroom level of the MEFA, items 7, 8, 9, 10, and 11 loaded on the first factor; this factor is referred to as axiomatic thinking. Items 1, 2, 3, 4, 5, and 6 loaded on

Table 4.4 Overall model fit for multilevel exploratory factor analysis

Model (Factors)	χ^2 (df)	RMSEA	CFA/TLI	SRMR (Within/Between)
1 Within-1Between	3183.98(810)	.058	.73/.71	.079/.131
2 Within-1 Between	3046.97(781)	.051	.79/.77	.072/.131
3 Within-1 Between	2723.03 (753)	.049	.82/.79	.066/.131
4 Within-1Between	2481.08 (726)	.047	.84/.81	.061/.131
5 Within-1 Between	2334.73 (700)	.046	.85/.82	.056/.131
6 Within-1 Between	2203.70 (675)	.045	.86/.82	.052/.131
1 Within-2 Between	3477.69 (781)	.056	.76/.73	.079/.067
2 Within-2 Between	2676.69 (752)	.048	.82/.80	.072/.067
3 Within-2 Between	2328.93 (724)	.045	.85/.82	.066/.067
4 Within-2 Between	2062.98 (697)	.042	.87/.84	.061/.067
5 Within-2 Between	1893.84 (671)	.041	.89/.85	.056/.067
6 Within-2 Between	1735.68 (646)	.039	.90/.87	.052/.067
1 Within-3 Between	3353.82 (753)	.056	.76/.73	.079/.053
2 Within-3 Between	2568.00 (724)	.048	.83/.80	.072/.053
3 Within-3 Between	2225.14 (696)	.045	.86/83	.066/.053
4 Within-3 Between	1962.20 (669)	.042	.88/.95	.061/.053
5 Within-3 Between	1793.89 (643)	.040	.90/.96	.052/.053
6 Within-3 Between	1636.032 (618)	.039	.92/.97	.042/.053

the second factor; this factor is referred to as formal thinking. Items 16, 17, 18, 19, 20 loaded on the third factor while items 21, 22, 24, 25, and 26 loaded on the fourth factor. These factors were referred to as iconic thinking and algorithmic thinking, respectively. Items 12, 13, 14, and 15 loaded on the fifth factor whereas items 23, 27, 28, 29, and 30 loaded on the sixth factor. These factors constituted the algebraic thinking and enactive thinking, respectively. At the between-classroom level the Factor 1 was reserved for items 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, and 11 and referred to as formal-axiomatic thinking (FORMAXTHK). Items 12, 13, 14, 15, 21, 22, 24, 25, and 26 constituted Factor 2 and referred to as proceptual-symbolic thinking (PROCPTHK). Finally, items 16, 17, 18, 19, 20, 23, 27, 28, 29, and 30 revolved around conceptual-embodied thinking (CONCPTHK) and aggregate under Factor 3. Overall, the factors emerged at the between-classroom level were composites of the factors at the within-classroom level. In line with theory (Tall, 2008), items on FORMAXTHK -a combination of items on FORMTHK and AXIOTHK- were based on formal definitions and proof. On the other hand, items on PROCPTHK grew out of items on ALGOTHK and ALGETHK underscoring algorithms and hypotheses that function both as procedures to implement and theorems to manipulate. Likewise, items on CONCPTHK appeared as a blend of items on ENACTHK and ICONTHK which were based on perception of and reflection on properties of graphical representations or assets of models that are initially visualized but then symbolized.

Table 4.5 Pattern matrix coefficients for the 6 within – 3 between factors model

	Factor					
Mathematical thinking	1	2	3	4	5	6
Within-classroom variables						
Item 9	.81					
Item11	.72					
Item7	.68					
Item8	.65					
Item 10	.57					
Item 3		.80				
Item 4		.60				
Item 6		.54				
Item 2		.50				
Item 5		.44				
Item 1		.43				
Item 20			.59			
Item 16			.53			
Item 19			.50			
Item 18			.48			
Item 17			.46			

Table 4.5 (continued)

	Factor					
Mathematical thinking	1	2	3	4	5	6
Within-classroom variables						
Item 23				.83		
Item 22				.69		
Item 26				.65		
Item 25				.63		
Item 21				.43		
Item 14					.94	
Item 13					.88	
Item 12					.69	
Item 15					.53	
Item 29						.76
Item 24						.61
Item 28						.57
Item 27						.52
Item 30						.51

Table 4.5 (continued)

	Factor		
Mathematical thinking	1	2	3
Between-classroom variables			
Item 5	.96		
Item 10	.95		
Item 2	.95		
Item 3	.92		
Item 9	.88		
Item 8	.81		
Item 4	.78		
Item 11	.76		
Item 7	.66		
Item 1	.58		
Item 6	.45		
Item 13		.97	
Item 22		.90	
Item 21		.88	
Item 25		.87	
Item 23		.78	
Item 14		.74	
Item 15		.73	
Item 26		.61	
Item 12		.44	
Item 18			.97
Item 16			.96
Item 24			.94
Item 28			.94
Item 20			.90
Item 30			.89
Item 29			.88
Item 27			.64
Item 19			.63
Item 17			.57

Interfactor correlations among the 6 within-classroom factors and the 3 between-classroom factors, separately, are presented in Table 4.6. At the within-classroom level, statistically significant and positive correlations were found between (a) axiomatic thinking factor and the remaining five factors, (b) formal thinking factor and, algorithmic and enactive thinking factors, (c) iconic thinking and, algorithmic, algebraic, and enactive thinking factors, and (d) algorithmic thinking and, algebraic and enactive thinking factors. Likewise, at the between-classroom level results revealed statistically significant and positive correlations among formal-axiomatic, proceptual-symbolic, and conceptual-embodied thinking factors. The correlations among the between-classroom factors were generally larger than those found among within-classroom factors. The highest correlation was between formal-axiomatic thinking and proceptual-symbolic thinking factors ($r = .45$, $p < .01$), whereas the lowest correlation appeared between formal-axiomatic thinking and conceptual- thinking factors ($r = .14$, $p < .01$).

Table 4.6 Interfactor correlations at the within- and between-classroom levels

Mathematical thinking	1	2	3	4	5	6
Within-classroom level						
1. Axiomatic thinking	-					
2. Formal thinking	.26*	-				
3. Iconic thinking	.10*	.07	-			
4. Algorithmic thinking	.27*	.28*	.29*	-		
5. Algebraic thinking	.11*	.02	.18*	.37*	-	
6. Enactive thinking	.25*	.19*	.43*	.30*	.09	-
Betwen-classroom level						
1. Formal-axiomatic thinking	-					
2. Proceptual-symbolic thinking	.45*	-				
3. Conceptual-embodied thinking	.14*	.32*	-			

* $p < .001$

4.3.3 RESULTS OF MULTILEVEL CONFIRMATORY FACTOR ANALYSIS

The Mplus input file for Step 3 is given in Appendix S. In regard to the expectations along with Hypothesis 1, the MCFA on students' scores on the 30 TDT items provided supportive evidence to the 6 Within – 3 Between structure. Results of this fully unconditional model showed a fairly good fit relative to the assessment criteria. The relation yielded (χ^2 (792) = 1399.98; p= .001; RMSEA= .03 SRMR= (Within/Between) = .02/.04 CFI= .94; TLI= .93). The standardized coefficients for the items in the 6 Within – 3 Between factor structure were demonstrated in Table 4.7. The completely standardized solution was displayed, which refers to a standardization based on the variances of both mathematical thinking constructs and the items (i.e., StdYX). It should also be stressed that the factor loading of the first item at each latent factor was fixed to 1 in order to set the metric of the mathematical thinking constructs automatically. Accordingly, the within-level WLSM estimates appeared between .32 and .92 while the between-level WLSM estimates ranged from .36 to .86. These values were all significant at $p< .001$; demonstrating that the factor coefficients of each item on the related mathematical thinking dimension were at a reasonable size to define ENACTHK, ICONTHK, ALGOTHK, ALGETHK, FORMTHK, and AXIOTHK at the within-classroom level and FORMAXTHK, PROCPTHK, and CONCPTHK at the between-classroom level.

Table 4.7 Standardized estimates of the multilevel confirmatory factor analysis at the within- and between-classroom levels*

	Factor		
Mathematical thinking	AXIOTHK	FORMTHK	ICONTHK
Within-classroom variables			
Item 9	1.00 (.00)		
Item11	.42 (.05)		
Item7	.56 (.07)		
Item8	.42 (.06)		
Item 10	.67 (.08)		
Item 3		1.00 (.00)	
Item 4		.92 (.09)	
Item 6		.52 (.10)	
Item 2		.47 (.10)	
Item 5		.85 (.12)	
Item 1		.62 (.12)	
Item 20			1.00 (.00)
Item 16			.66 (.06)
Item 19			.56 (.04)
Item 18			.83 (.07)
Item 17			.49 (.04)

* Standardized errors are given in parantheses.

Table 4.7 (continued)*

	Factor		
Mathematical thinking	ALGOTHK	ALGETHK	ENACTHK
Within-classroom variables			
Item 23	1.00 (.00)		
Item 22	.84 (.05)		
Item 26	.69 (.05)		
Item 25	.89 (.07)		
Item 21	.70 (.05)		
Item 14		1.00 (.00)	
Item 13		.69 (.07)	
Item 12		.46 (.07)	
Item 15		.31 (.03)	
Item 29			1.00 (.00)
Item 24			.87 (.18)
Item 28			.32 (.10)
Item 27			.64 (.14)
Item 30			.89 (.10)

* Standardized errors are given in parantheses.

Table 4.7 (continued)*

	Factor		
Mathematical thinking	FORMAXTHK	PROCPTHK	CONCPTHK
Between-classroom variables			
Item 5	1.00 (.00)		
Item 10	.56 (.12)		
Item 2	.86 (.14)		
Item 3	.81 (.14)		
Item 9	.82 (.14)		
Item 8	.43 (.10)		
Item 4	.62 (.10)		
Item 11	.87 (.16)		
Item 7	.59 (.10)		
Item 1	.79 (.16)		
Item 6	.73 (.15)		
Item 13		1.00 (.00)	
Item 22		.36 (.08)	
Item 21		.41 (.03)	
Item 25		.87 (.04)	
Item 24		.81 (.03)	
Item 14		.80 (.03)	
Item 15		.62 (.06)	
Item 26		.37 (.04)	
Item 12		.43 (.07)	
Item 18			1.00 (.00)
Item 16			.74 (.08)
Item 23			.83 (.09)
Item 28			.82 (.08)
Item 20			.41 (.12)
Item 30			.74 (.08)
Item 29			.38 (.07)
Item 27			.44 (.07)
Item 19			.54 (.07)
Item 17			.44 (.07)

* Standardized errors are given in parantheses.

In addition, with regard to the 6 Within – 3 Between model, the amount of explained variance were computed for each mathematical thinking construct at the within- and between-classroom levels. The percentage variance extracted (R^2) reflects the overall amount of variance in the items accounted for by each mathematical thinking construct. Guidelines suggest that R^2 values above .50 indicate a better representation of the latent construct by the items (Tabachnick & Fidell, 2007). At the within-classroom level, R^2s for ENACTHK, ICONTHK, ALGOTHK, ALGETHK, FORMTHK, and AXIOTHK were .55, .57, .58, .63, .53, and .63, respectively. At the between-classroom level, R^2s for FORMAXTHK, PROCPTHK, and CONCPTHK were .53, .53, .54, respectively. Collectively, the R^2s, in general, suggested a fairly good representation of the mathematical thinking constructs by the items with higher percentages for ALGETHK and AXIOTHK.

4.3.4 RESULTS OF MULTILEVEL STRUCTURAL EQUATION MODELING

4.3.4.1 Results of Cross-Level Direct Effects Model

The Mplus input for Step 4 is given in Appendix T. The multilevel model of the relationships among different types of mathematical thinking is presented in Figure 4.1. As hypothesized (Hypothesis 2) results revealed statistically significant and positive relationships among enactive, iconic, algorithmic, algebraic, formal, and axiomatic thinking at the within-classroom level and formal-axiomatic, proceptual-symbolic, and conceptual-embodied thinking at the between-classroom level.

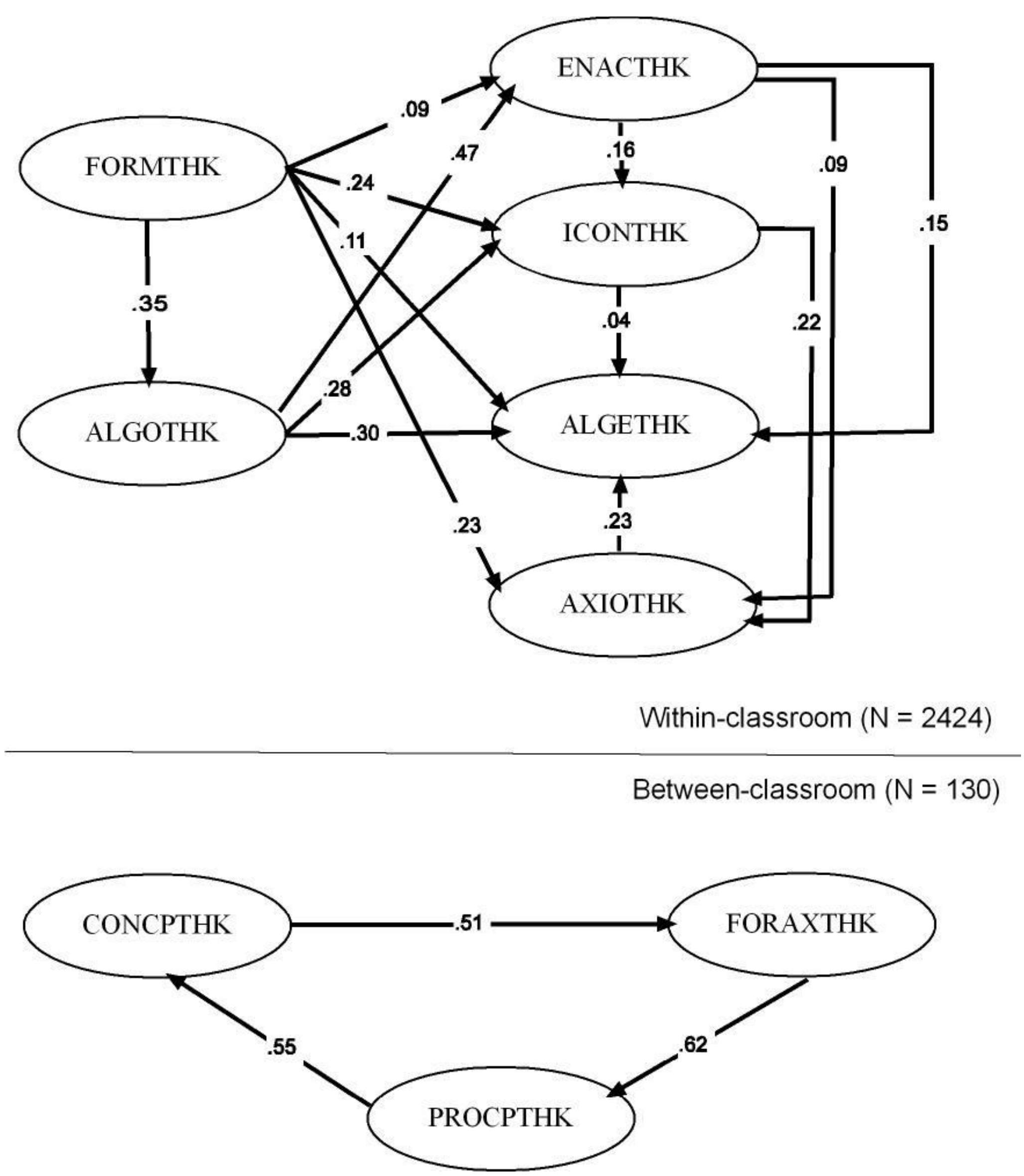

Figure 4.1 Multilevel model of mathematical thinking in derivative*
* All the paths are significant at $p < .001$

Primarily the intercorrelations among these constructs that are demonstrated in Table 4.8 were investigated. Overall, the correlations were in the expected direction, and a similar pattern of associations was found for each mathematical thinking construct. At both within- and between-classroom levels, higher attainment in a particular type of mathematical thinking was associated with a higher level of competency in other types. For most mathematical thinking types, the size of the correlations was considerably higher at the between-classroom level than at the within-classroom level-this pattern is well known from student achievement studies in educational research (O'Connell & McCoach, 2008). Furthermore, the ICCs for mathematical thinking constructs signified substantial between-classroom variation with values exceeding .10 (Bickel, 2007). The ICC values for enactive, iconic, algorithmic, algebraic, formal, and axiomatic thinking were .31, .33, .41, .28, .48, and .39, respectively. The ICCs were somewhat higher for the algorithmic and formal thinking than for algebraic thinking. Most of the variance for enactive, iconic, algebraic, and axiomatic thinking resides at the within-classroom level. Meaningful between-classroom differences were also found for formal-axiomatic, proceptual-symbolic, and conceptual-embodied thinking with values of .53, .41, and .39, respectively.

Table 4.8 Intercorrelations among factors at the within- and between-classroom levels

Mathematical thinking	1	2	3	4	5	6
Within-classroom level						
1. Axiomatic thinking	-					
2. Formal thinking	.38	-				
3. Iconic thinking	.50	.46	-			
4. Algorithmic thinking	.27	.24	.22	-		
5. Algebraic thinking	.26	.39	.35	.26	-	
6. Enactive thinking	.24	.34	.24	.33	.34	-
Betwen-classroom level						
1. Formal-axiomatic thinking	-					
2. Proceptual-symbolic thinking	.76	-				
3. Conceptual-embodied thinking	.72	.74	-			

Results indicated that the variances of proceptual-symbolic and conceptual-embodied thinking lay quite predominantly at the between-classroom level whereas the variance of formal-axiomatic thinking settles roughly equally within- and between-classroom levels. Taken together, these findings highlight the need to consider the within and between-classroom levels separately when probing for the relationships among different types of mathematical thinking.

The hypothesized theoretical model attained a good fit to the data, χ^2 (18) = 303.11; p= .001; RMSEA= .02 SRMR= (Within/Between) = .01/.02 CFI= .98; TLI= .97. As shown in Figure 4.1 direct, indirect, and total effects were examined for significance at the .001 level. A direct effect can be interpreted as a causal effect denoted by the directional relation among the latent constructs. An indirect effect represents the portion of the relationship between two variables that is mediated by one or more variables. The total effect is equal to the sum of the direct and indirect effects.

Turning first to the within-classroom level, formal thinking had statistically significant and positive direct effects on enactive (β = .09), iconic (β = .24), algorithmic (β = .35), algebraic (β = .11), and axiomatic (β = .23), thinking. Students who had adequate knowledge of definitions, symbols, and facts relevant to the derivative concept were able to integrate modeling situations into optimization procedures, retrieve information from the graph of a derivative function, solve routine differentiation problems, reason on the algebraic changes within theoretical differentiation structures, and synthesize the steps in differentiation proofs. Formal thinking also had indirect effects on enactive (γ = .16), iconic (γ = .10), and algebraic (γ = .01) thinking through influencing algorithmic thinking. Similarly, mediated by enactive thinking it had indirect effects on iconic (γ = .01), algebraic (γ = .01), and axiomatic (γ = .09) thinking. Formal thinking further had significant indirect effects on algebraic (γ = .01) and axiomatic (γ = .05) thinking moderated by iconic thinking. Taken as a whole, these findings implied that students who were more prone to recall the fundamental differentiation terminology tend to excel in diverse types of thinking as pertained to differentiation techniques, optimization models, and/or rough differentiation sketches.

The three direct effects of algorithmic thinking were on enactive (β = .47), iconic (β = .28), and algebraic (β = .30) thinking. Its strongest impact was on enactive thinking, while to a lesser extent on algebraic and iconic thinking. One explanation might be that there is very much of differentiation procedures which can be used to perform a task that is "enactive" in nature. The execution of such a task radically differs from others because students are required to cultivate a series of technical steps in order to find a maximum or minimum value (e.g., assign a symbol to the quantity that is to be maximized/minimized, select a symbol for other unknown quantities, and express a function of more than one variable). Through the influence of enactive thinking, substantial indirect effects appeared among algorithmic thinking and, iconic (γ = .07), algebraic (γ = .06), and axiomatic (γ = .04), thinking. Mediated by iconic thinking, algorithmic thinking had indirect effects on algebraic (γ = .01) and axiomatic (γ = .06) thinking. These indirect effects of algorithmic thinking through enactive and iconic thinking made up an important segment in classifying students' performance into a set of structured thinking patterns associated with different components of the derivative concept. Algebraic thinking was specified dependent of enactive (β = .15), iconic (β = .04), and axiomatic (β = .23) thinking; and this specification was not disproved by the data. The stronger association between algebraic and axiomatic thinking was expected due to the connected contextualization of the tasks aiming to expose students' repertoire of differentiation theorems to build relationships among the hypotheses relative to these theorems. Enactive thinking had the strongest direct effect on iconic thinking (β = .16) and to a lesser extent on axiomatic thinking (β = .09). This finding was expected in that, students who were able to draw a diagram to model the optimization phenomenon would typically be more likely to draw the graph of a derivative function and/or given the derivative function sketch the graph of the original function. Interestingly, the direct effect of iconic thinking on axiomatic thinking (β = .22) was somewhat higher than expected. One likely explanation for the unexpected result is due to the intrinsic potential of graphical representations for making interpretations about the transformation between differentiation theory and visual practice.

Concerning the relationships at the between-classroom level, the two-level model depicted a cyclic interrelation among formal-axiomatic, proceptual-symbolic, and conceptual-embodied thinking. Formal-axiomatic thinking had a significant and positive direct effect on proceptual-symbolic thinking (β = .62), whereas proceptual-symbolic thinking had a significant direct effect on conceptual-embodied thinking (β = .55), and in turn conceptual-embodied thinking had a significant direct effect on formal-thinking (β = .51). The strongest relationship between formal-axiomatic and proceptual-symbolic thinking signified that classrooms equipped with more rigorous aspects of differentiation tend to enforce their theoretical progress in the more flexible grounds of differentiation techniques. Though being automatized such flexibility provides classrooms to further incorporate their procedural knowledge of differentiation into more visual forms and that to become fairly more proficient in driving an algorithm within their model and/or graph constructions. This cyclic nature of the associations was also evident in the indirect effects among between-classroom variables. The effect of formal-axiomatic thinking on conceptual-embodied thinking was mediated by proceptual-thinking (γ = .34), while the effect proceptual-symbolic thinking on formal-axiomatic thinking was mediated by conceptual-embodied thinking (γ = .28). In turn, conceptual-embodied thinking had a significant indirect effect on proceptual-symbolic thinking (γ = .31) through influencing formal-axiomatic thinking. Collectively, these indirect effects reaffirmed the important role of mediation among different types of mathematical thinking in the classroom context. Along with the statistically significant and positive direct and indirect effects, it can be concluded that the total effects of the mathematical thinking constructs at the within- and between-classroom levels were all substantial in size.

The estimation of the mathematical thinking model enabled to further evaluate the extent of variation at the within- classroom level and at the between-classroom level for each relevant mathematical thinking construct. At the within-classroom level, results revealed that the amount of variance explained was .26, .29, .12, .16, .15, and .18 for enactive, iconic, algorithmic, algebraic, formal, and axiomatic thinking, respectively. Moving to the between-classroom level the amount of explained variance increased to .48, .56, and .51 for formal-axiomatic, proceptual-symbolic, and conceptual-embodied thinking, respectively. Taken together, the R^2s calculated as the change in the variance components for the within- and between-classroom levels as reported by Mplus indicated that all mathematical thinking constructs introduced to the two-level model were practically significant contributors for one another.

4.3.4.2 Results of Cross-Level Moderator Models

The within-classroom relations as a function of between-classroom factors were tested with a total of 42 cross-level moderator models. On the other hand, the between-classroom relations as a function of within-classroom factors were tested with a total of 18 cross-level moderator models.

The Mplus input files for testing each cross-level moderator model are presented for the within-classroom moderations and between-classroom moderations in Appendix U and Appendix V, respectively.

As hypothesized (Hypothesis 3), results revealed that enactive, iconic, algorithmic, algebraic, formal, and axiomatic thinking significantly moderated the relationships among conceptual-embodied, proceptual-symbolic, and formal-axiomatic thinking. In that respect, conceptual-embodied, proceptual-symbolic, and formal-axiomatic thinking significantly moderated the relationships among enactive, iconic, algorithmic, algebraic, formal, and axiomatic thinking.

4.3.4.2.1 Results of the Within-Classroom Relations as a Function of Between-Classroom Factors

Results documented that the between-classroom factors, conceptual-embodied, proceptual-symbolic, formal-axiomatic thinking, significantly moderated the relationships among the within-classroom factors –enactive, iconic, algorithmic, algebraic, formal, and axiomatic thinking- specified in the cross-level direct effects model. Statistically, the upper level moderations reflected that the between-classroom factors that are observed to be significantly related to the random coefficients can be termed cross-level interactions and simply denote that a between-classroom factor influences a within-classroom slope. The gamma coefficients (γ) for the within-classroom relations as a function of between-classroom factors are presented in Table 4.9.

To examine whether between-classroom factors mediated the formal thinking effects on algebraic thinking, formal-axiomatic, conceptual-embodied, and proceptual-symbolic thinking were introduced as predictor variables separately. The regression coefficients for formal-axiomatic (γ = .84), conceptual-embodied (γ = .09), proceptual-symbolic (γ = .34) thinking were significant and of considerable size. To examine whether between-classroom factors mediated the formal thinking effects on algorithmic thinking, formal-axiomatic, conceptual-embodied, and proceptual-symbolic thinking were introduced as predictor variables separately. The regression coefficients for formal-axiomatic (γ = .22), conceptual-embodied (γ = .25), proceptual-symbolic (γ = .61) thinking were significant and of considerable size. To examine whether between-classroom factors mediated the formal thinking effects on axiomatic thinking, formal-axiomatic, conceptual-embodied, and proceptual-symbolic thinking were introduced as predictor variables separately.

The regression coefficients for formal-axiomatic (γ = .30), conceptual-embodied (γ = .10), proceptual-symbolic (γ = .16) thinking were significant and of considerable size. To examine whether between-classroom factors mediated the formal thinking effects on enactive thinking, formal-axiomatic, conceptual-embodied, and proceptual-symbolic thinking were introduced as predictor variables separately. The regression coefficients for formal-axiomatic (γ = .46), conceptual-embodied (γ = .51), proceptual-symbolic (γ = .45) thinking were significant and of considerable size. To examine whether between-classroom factors mediated the formal thinking effects on iconic thinking, formal-axiomatic, conceptual-embodied, and proceptual-symbolic thinking were introduced as predictor variables separately. The regression coefficients for formal-axiomatic (γ = .40), conceptual-embodied (γ = .20), proceptual-symbolic (γ = .44) thinking were significant and of considerable size.

Collectively, these results implied that students' formal thinking is significantly associated with their algebraic, algorithmic, axiomatic, enactive, and iconic thinking in classrooms with higher formal-axiomatic, conceptual-embodied, and proceptual-symbolic thinking. The strongest mediation effect was that of formal-axiomatic thinking on the relationship between formal and algebraic thinking. This signaled that when students' strong knowledge of derivative, indicative of their hypothetical knowledge of differentiation theorems, is coupled with the classroom's knowledge of fundamental differentiation rules and theorems, this results in stronger cross-level formal-algebraic relationships. The lowest effect was of conceptual-embodied thinking that mediates the relationship between formal and algebraic thinking. This result was expected in that students in classrooms where higher levels of real-life applications of the derivative are possessed by visualization might not tend to use the embodied and/or visual aspects of the derivative concept to build links between the definition of derivative and the algebraic manipulation of derivative theorems. Furthermore, the lack of significance in the mediation effect of formal-axiomatic thinking on the effects of formal thinking on enactive and iconic thinking supported this line of reasoning, suggesting that the formal thinking effects on enactive and iconic thinking stem from students' patterning of proceptual-symbolic and/or conceptual-symbolic thinking rather than from their theorizing formal-axiomatic thinking. This finding signaled that the combination of more rigorous knowledge of differentiation symbols, facts, and axioms in a classroom do not offer support for students' linking the definition of derivative to the identification of the maximum/minimum value of a function or the construction of the graph of a derivative function.

To examine whether between-classroom factors mediated the iconic thinking effects on algebraic thinking, formal-axiomatic, conceptual-embodied, and proceptual-symbolic thinking were introduced as predictor variables separately. The regression coefficients for formal-axiomatic (γ = .03), conceptual-embodied (γ = .03), proceptual-symbolic (γ = .04) thinking were significant and of considerable size. To examine whether between-classroom factors mediated the iconic thinking effects on axiomatic thinking, formal-axiomatic, conceptual-embodied, and proceptual-symbolic thinking were introduced as predictor variables separately. The regression coefficients for formal-axiomatic (γ = .61), conceptual-embodied (γ = .02), proceptual-symbolic (γ = .15) thinking were significant and of considerable size.

Taken together, these results implied that students' iconic thinking is significantly associated with their algebraic and axiomatic thinking in classrooms with higher formal-axiomatic, conceptual-embodied, and proceptual-symbolic thinking. The strongest mediation effect was that of formal-axiomatic thinking on the relationship between iconic and axiomatic thinking. The mediation might present an important aspect of classroom's theoretical structure of differentiation has an influence on students' functioning in finding the local maxima or minima of a differentiable function in regard to the Fermat's Theorem applied to investigating the local maxima/minima of a function on a graph. Somewhat unexpectedly, however, the mediation of conceptual-embodied thinking on that relationship was nonsignificant. The lowest and in part almost equal mediation effects were that of formal-axiomatic and conceptual-embodied thinking on the relationship between iconic and algebraic thinking.

To examine whether between-classroom factors mediated the enactive thinking effects on algebraic thinking, formal-axiomatic, conceptual-embodied, and proceptual-symbolic thinking were introduced as predictor variables separately. The regression coefficients for formal-axiomatic (γ = .70), conceptual-embodied (γ = .06), proceptual-symbolic (γ = .04) thinking were significant and of considerable size. To examine whether between-classroom factors mediated the enactive thinking effects on axiomatic thinking, formal-axiomatic, conceptual-embodied, and proceptual-symbolic thinking were introduced as predictor variables separately. The regression coefficients for formal-axiomatic (γ = .11), conceptual-embodied (γ = .14), proceptual-symbolic (γ = .20) thinking were significant and of considerable size. To examine whether between-classroom factors mediated the enactive thinking effects on iconic thinking, formal-axiomatic, conceptual-embodied, and proceptual-symbolic thinking were introduced as predictor variables separately. The regression coefficients for formal-axiomatic (γ = .58), conceptual-embodied (γ = .24), proceptual-symbolic (γ = .53) thinking were significant and of considerable size.

The findings mirrored that students' enactive thinking is significantly associated with their algebraic, axiomatic, and iconic thinking in classrooms with higher formal-axiomatic, conceptual-embodied, and proceptual-symbolic thinking. The strongest mediation of formal-axiomatic thinking was again brought on the scene for the relationship between enactive and algebraic thinking. Although the mediation of formal-axiomatic thinking on the relationship between enactive and iconic thinking was strong it did not appear to be significant. This finding clearly supported the fact that classroom's rigid forms of logical arguments in differentiation do not function in students' relating the differentiation processes applicable in a graph to each other and to relevant real-life situations.

To examine whether between-classroom factors mediated the algorithmic thinking effects on algebraic thinking, formal-axiomatic, conceptual-embodied, and proceptual-symbolic thinking were introduced as predictor variables separately. The regression coefficients for formal-axiomatic (γ = .73), conceptual-embodied (γ = .09), proceptual-symbolic (γ = .05) thinking were significant and of considerable size. To examine whether between-classroom factors mediated the algorithmic thinking effects on enactive thinking, formal-axiomatic, conceptual-embodied, and proceptual-symbolic thinking were introduced as predictor variables separately. The regression coefficients for formal-axiomatic (γ = .36), conceptual-embodied (γ = .44), proceptual-symbolic (γ = .29) thinking were significant and of considerable size. To examine whether between-classroom factors mediated the algorithmic thinking effects on iconic thinking, formal-axiomatic, conceptual-embodied, and proceptual-symbolic thinking were introduced as predictor variables separately. The regression coefficients for formal-axiomatic (γ = .49), conceptual-embodied (γ = .22), proceptual-symbolic (γ = .11) thinking were significant and of considerable size.

Results demonstrated that students' algorithmic thinking is significantly associated with their algebraic, axiomatic, and iconic thinking in classrooms with higher formal-axiomatic, conceptual-embodied, and proceptual-symbolic thinking. In contrast to the aforementioned statistically significant and strong effects, classroom's formal-axiomatic thinking did not have a role in the paths of algebraic and enactive thinking associated with algorithmic thinking. However, classroom's portrait of formal-axiomatic thinking was still a significant and strong mediator students' calculations from graphical forms (e.g., the gradient of a tangent at a point) or using differentiation procedures to obtain numerical results from graphs (e.g., interpolation for approximating an average rate of change).

To examine whether between-classroom factors mediated the axiomatic thinking effects on algebraic thinking, formal-axiomatic, conceptual-embodied, and proceptual-symbolic thinking were introduced as predictor variables separately. The regression coefficients for formal-axiomatic (γ = .68), conceptual-embodied (γ = .09), proceptual-symbolic (γ = .02) thinking were significant and of considerable size.

Taken as a whole, results confirmed that students' axiomatic thinking is significantly associated with their algebraic thinking in classrooms with higher formal-axiomatic, conceptual-embodied, and proceptual-symbolic thinking. The strongest mediation effect was that of formal-axiomatic thinking on the relationship between axiomatic and algebraic thinking. The mediation might mirror once again the important role of classroom's body of formal differentiation theory in students' having sufficiently well-formed concept-oriented knowledge to move fluently between theoretical perspectives and algebraic manifestations. It is likely for students to link between a theorem and its hypotheses in classrooms where appropriate theoretical and conceptual perspectives are chosen to solve a differentiation problem. The lowest mediation effect was that of conceptual-embodied thinking. However, this finding supported that the strength of visual and enactive interpretations is extremely sensitive to the accurate conception of what constitutes a differentiation proof and the ability to derive algebraic inferences from that logical conceptions. As expected, the mediation effect of proceptual-symbolic thinking on the relationship between axiomatic and algebraic thinking was statistically nonsignificant. This implied that the classroom's tendency in practicing rote application of algorithms and procedures at the expense of hypotheses of differentiation theorems did not mediate students' understanding the nature of a differentiation theorem to exclusively involve in the algebraic manipulations relevant to its hypotheses.

Table 4.9 Gamma coefficients for the within-classroom relations as a function of between-classroom factors

Within-Classroom Level Relationship	Between-Classroom Level Factor	Coefficient γ
FORMTHK→ALGETHK	FORMAXTHK	.84*
	CONCPTHK	.09*
	PROCPTHK	.34*
FORMTHK→ALGOTHK	FORMAXTHK	.22*
	CONCPTHK	.25*
	PROCPTHK	.61*
FORMTHK→AXIOTHK	FORMAXTHK	.30*
	CONCPTHK	.10*
	PROCPTHK	.16*
FORMTHK→ENACTHK	FORMAXTHK	.46
	CONCPTHK	.51*
	PROCPTHK	.45*
FORMTHK→ICONTHK	FORMAXTHK	.40
	CONCPTHK	.20*
	PROCPTHK	.44*
ICONTHK→ALGETHK	FORMAXTHK	.03*
	CONCPTHK	.03*
	PROCPTHK	.04*
ICONTHK→AXIOTHK	FORMAXTHK	.61*
	CONCPTHK	.02
	PROCPTHK	.15*
ENACTHK→ALGETHK	FORMAXTHK	.70*
	CONCPTHK	.06*
	PROCPTHK	.04*
ENACTHK→AXIOTHK	FORMAXTHK	.11*
	CONCPTHK	.14*
	PROCPTHK	.20*

* Significant values at $p < .001$

Table 4.9 (continued)

Within-Classroom Level Relationship	Between-Classroom Level Factor	Coefficient γ
ENACTHK→ICONTHK	FORMAXTHK	.58
	CONCPTHK	.24*
	PROCPTHK	.53*
ALGOTHK→ALGETHK	FORMAXTHK	.73
	CONCPTHK	.09*
	PROCPTHK	.05*
ALGOTHK→ENACTHK	FORMAXTHK	.36
	CONCPTHK	.44*
	PROCPTHK	.29*
ALGOTHK→ICONTHK	FORMAXTHK	.49*
	CONCPTHK	.22*
	PROCPTHK	.11*
AXIOTHK→ALGETHK	FORMAXTHK	.68*
	CONCPTHK	.09*
	PROCPTHK	.02

* Significant values at $p < .001$

4.3.4.2.2 Results of the Between-Classroom Relations as a Function of Within-Classroom Factors

Results documented that the within-classroom factors -enactive, iconic, algorithmic, algebraic, formal, and axiomatic thinking- significantly moderated the relationships among the between-classroom factors, conceptual-embodied, proceptual-symbolic, formal-axiomatic thinking, specified in the cross-level direct effects model. Statistically, the lower level moderations reflected that the within-classroom factors that are observed to be significantly related to the random coefficients can be termed cross-level interactions and simply denote that a within-classroom factor influences a between-classroom slope.

To examine whether within-classroom factors mediated the formal-axiomatic thinking effects on proceptual-symbolic thinking, enactive, iconic, algorithmic, algebraic, formal, and axiomatic thinking were introduced as predictor variables separately. The regression coefficients for formal (γ = .09), axiomatic (γ = .05), iconic (γ = .07), enactive (γ = .06), algebraic (γ = .08), and algorithmic (γ = .10) thinking were significant and of considerable size.

To examine whether within-classroom factors mediated the proceptual-symbolic thinking effects on conceptual-symbolic thinking, enactive, iconic, algorithmic, algebraic, formal, and axiomatic thinking were introduced as predictor variables separately. The regression coefficients for formal (γ = .03), axiomatic (γ = .03), iconic (γ = .04), enactive (γ = .05), algebraic (γ = .03), and algorithmic (γ = .00) thinking were significant and of considerable size.

To examine whether within-classroom factors mediated the conceptual-embodied thinking effects on formal-axiomatic thinking, enactive, iconic, algorithmic, algebraic, formal, and axiomatic thinking were introduced as predictor variables separately. The regression coefficients for formal (γ = .14), axiomatic (γ = .13), iconic (γ = .05), enactive (γ = .08), algebraic (γ = .07), and algorithmic (γ = .11) thinking were significant and of considerable size.

Collectively, results demonstrated that for most within-classroom factors, the mediation effects were considerable in size and statistically significant. Notably, the nonsignificant effects appeared only in the relationship between proceptual-symbolic and conceptual-embodied thinking mediated by formal, algorithmic, and algebraic thinking. The strongest mediation effect was that of formal thinking on the relationship between conceptual-embodied and formal-axiomatic thinking. This finding obviously indicated that in a classroom the links between visual and theoretical conceptualizations of the derivative concept are primarily formed through or facilitated by students' basic knowledge of the definitions, symbols, rules, and facts relevant to differentiation. To a lesser extent, the strong mediation effect of axiomatic thinking on the relationship between conceptual-embodied and formal-axiomatic thinking was also reasonable. Similarly, students' individual norms of thinking about theorems may moderate the classroom's line of reasoning in graphs of a derivative function registered in theoretical aspects of the original function or vice versa. In line with the predictions, the lowest mediation effect was that of axiomatic thinking on the relationship between proceptual-symbolic and conceptual-embodied thinking. One possible reason might be that in a classroom when technical knowledge of differentiation procedures, indicative of interpreting derivative graphs as representing functions, is coupled with students' operating on theorems, this may not result in too strong essence in making procedural and visual connections between corresponding differentiation processes.

Table 4.10 Gamma coefficients for the between-classroom relations as a function of within-classroom factors

Within-Classroom Level Relationship	Between-Classroom Level Factor	Coefficient γ
FORMAXTHK→PROCPTHK	ENACTHK	.06*
	ICONTHK	.07*
	ALGOTHK	.10*
	ALGETHK	.08*
	FORMTHK	.09*
	AXIOTHK	.05*
PROCPTHK→CONCPTHK	ENACTHK	.04*
	ICONTHK	.05*
	ALGOTHK	.03
	ALGETHK	.00
	FORMTHK	.03
	AXIOTHK	.03*
CONCPTHK→FORMAXTHK	ENACTHK	.05*
	ICONTHK	.08*
	ALGOTHK	.07*
	ALGETHK	.11*
	FORMTHK	.14*
	AXIOTHK	.13*

* Significant values at $p < .001$

CHAPTER 5

DISCUSSION, CONCLUSION AND IMPLICATIONS

This chapter presents the discussion and conclusion of the results, educational implications, limitations, and recommendations for future research.

5.1 DISCUSSION

The purpose of the study was threefold: (a) to determine the factor structure of mathematical thinking at the within-classroom and at the between-classroom level; (b) to investigate the extent of variation in mathematical thinking at the within-classroom and at the between-classroom level; and (c) to examine the cross-level interactions among different types of mathematical thinking.

So first, when a researcher wants to test whether the factor structure of mathematical thinking differs across levels of analysis (i.e., within-classroom and between-classroom), by means of multilevel exploratory and confirmatory factor analyses does it matter whether he/she controls for the nesting of students within classrooms?

The results of the multilevel exploratory factor analysis showed that the 30 multiple-choice items of the Thinking-in-Derivative Test could be divided into a six-factor structure (enactive, iconic, algorithmic, algebraic, formal, and axiomatic thinking) at the within-classroom level and a three-factor structure (formal-axiomatic, proceptual-symbolic, and conceptual-embodied thinking) at the between-classroom level. All three factors at the between-classroom level were clear combinations of several factors at the within-classroom level, albeit with a different order of the items. The distinct factor structure was in line with previous research (Schwartz, 1992, 1994) documenting that the same items can cluster differently at the within versus between levels of analysis. This differentially clustering of items happened with mathematical thinking constructs as well. The within-classroom factors formal thinking and axiomatic thinking generated the formal-axiomatic thinking factor, whereas algorithmic thinking and algebraic thinking factors aggregated under proceptual-symbolic thinking factor at the between-classroom level. Additionally, at the within-classroom level enactive thinking and iconic thinking factors emerged, whereas at the between-classroom level a blend of these two factors appeared as conceptual-embodied thinking factor. Although hard to identify straightforwardly, this conceptual meaning of between-classroom factors are consistent with the conceptualizations of Tall (2004) stating that three different types of mathematical thinking (e.g., formal-axiomatic thinking, proceptual-symbolic thinking, and conceptual-embodied thinking) are provoked when solving advanced mathematics problems. In accordance with this line of conceptualization, Tall further indicated that a mathematical task links to the aspects that accompany a specific type of thinking (e.g., formal-axiomatic thinking) and yet displays the aspects of all other types (formal thinking and axiomatic thinking).

As regards the size of the factor loadings it was noticed much higher loadings at the between-classroom level (from .35 to .97) than at the within-classroom level (from .32 to .67). Here, all factor loadings were significant. The more inclusive between-classroom factors correlated moderately, while among the larger group of interfactor correlations at the within-classroom level, at least some correlations were low. This finding supported Shieh and Fouladi (2003) suggesting that when using these constructs in further multilevel analysis to investigate relationships low correlations are to be preferred. Identical conclusions concerning both levels of analysis were solutions with a simple structure and low to moderate interfactor correlations. However, compared to the factor loadings at the within-classroom level higher factor loadings at the between-classroom level were obtained.

The 6 Within – 3 Between factor structure that emerged in the multilevel exploratory factor analysis was cross-validated in the multilevel confirmatory factor analysis with an independent sample of undergraduate students. Results of the multilevel confirmatory factor analysis were marginally acceptable proved by high goodness-of-fit indices indicating that the model fits the data well, thus confirming that students' mathematical thinking in derivative can be designated by 6 factors at the within-classroom level and more global 3 factors at the between-classroom level. Furthermore, high percentages of variance extracted indicate an adequate representation of the mathematical thinking constructs specified at the within- and between-classroom levels.

Concerning the factor structure, exactly the same factors for conventional confirmatory factor analysis and multilevel confirmatory factor analysis were found at the within-classroom level. Most items appeared in a different order when comparing confirmatory factor analysis with multilevel confirmatory factor analysis at the within-classroom level. Thus, items that were most crucial for an underlying thinking dimension within confirmatory factor analysis were not the most crucial for that same underlying dimension within multilevel confirmatory factor analysis. Similarly, a clear distinction was related to the size of the factor loadings of the items. Regarding multilevel confirmatory factor analysis at the within-classroom level, most items had a slightly lower factor loading. The same phenomenon was observed when using multilevel exploratory factor analysis. This might be due to the downward-correction of the tetrachoric correlation matrix when controlling for the dependency of the students' responses (Muthén & Satorra, 1995).

Obviously, these findings implied that the constructed factors of both multilevel exploratory factor analysis and multilevel confirmatory factor analysis were reliable measures of students' mathematical thinking, and thus provided a step forward to estimate a two-level model of mathematical thinking.

Moving to the second goal of the present study the relationships among different types of mathematical thinking derived at the within- and between-classroom levels were investigated. The amount of variance identified at each level of the nested data structure provides an indication of whether or not mathematical thinking constructs are accessed rigidly across classrooms or varying relative to a student's own potential of progress in mathematical thinking. The results of the present study are somewhat disappointing in that the amount of variance explained was not large, particularly at the within-classroom level where most of the variability resides for mathematical thinking constructs (e.g., enactive thinking, iconic thinking, and algebraic thinking). Part of the problem may be due to the categorical nature of the data, which served to constrain the relationships being modeled (Snijders & Boskers, 1999). The proportion of variance suggests there is considerable amount of variability in undergraduate students' different types of mathematical thinking between classrooms and that this aspect of mathematical thinking may be indicative of students' overall intellectual abilities to bring in their diverse thinking processes. Further inspection of the ICCs across two levels for each of the mathematical thinking constructs indicates that algebraic thinking had the lowest and formal-axiomatic thinking had the highest amount of between-classroom variance, 28% and 53%, respectively. From a methodological perspective, these findings suggest that researchers should pay careful attention to the clustering of students nested within classrooms when examining the relationships among different types of mathematical thinking in cross-sectional studies at educational settings (O'Connell & McCoach, 2008).

The multilevel model tested in the current study suggests an interesting conceptualization of the relationships among different types of mathematical thinking at both within- and between-classroom levels and sheds light on the pattern (direction and magnitude) of these relations. Thus, the present study contributes to the existing body of literature on students' mathematical thinking by introducing a more sensible representation of the complexity of the relationships among the factors underlying students' mathematical thinking. The multilevel structural equation modeling analysis provided to capture this complexity while taking into account the hierarchical nature of the data. Accordingly, with respect to the multilevel structural model estimating the relationships among different types of mathematical thinking, interesting findings emerged at both within- and between-classroom levels.

At the within-classroom level, formal thinking had the strongest predictive power on algorithmic thinking; that as students become talented in defining the derivative concept, they exhibit a higher progress in applying differentiation algorithms. This finding supports the findings of previous studies (Habre & Abboud, 2006; Orton, 1983; Roorda, Vos, & Goedhardt, 2007), which provided evidence that students who were able to define and/or symbolize the derivative concept would be more apt to manage the algorithms and procedures within relevant derivative problems. To a lesser extent, formal thinking also had statistically significant and positive direct effects on students' gains in enactive thinking, iconic thinking, algebraic thinking, and axiomatic thinking. this finding supports previous research that underlined students' core basis of differentiation terminology students' understanding of the core differentiation terminology positively contributes to their successful utilization of max/min problems (Ubuz & Ersoy, 1997), graph constructions/interpretations of the derivative function (Ubuz, 2007), transformations between corollaries and differentiation procedures (Viholainen, 2006), and proofs in relation to differentiability (Tall, 1998). The strongest direct effect of algorithmic thinking on enactive thinking provided support to the results of previous studies (Thompson, 1994; Ubuz & Ersoy, 1997; Villegas, Castro, & Gutiérrez, 2009) indicating that students' capability to use appropriate differentiation techniques would lead them to master the steps to be taken in modeling optimization problems. In accordance with the findings of previous researchers (Aspinwall et al., 1997; Gray, Loud, & Sokolowski, 2009; Marrongelle, 2007), algorithmic thinking had a significant and positive direct effect on iconic thinking and algebraic thinking. That is, students who are equipped with the procedural knowledge of the derivative would be able to effectively visualize the derivative as a slope of the tangent line to the curve and algebraically unfold a network of connections in the Mean Value Theorem. The significantly positive direct effect of enactive thinking on iconic thinking, algebraic thinking, and axiomatic thinking presented in the Mathematical Thinking Model could throw some light on the relationships raised by several researchers (Derry, Wilsman, & Hackbarth, 2007; Ferrini-Mundy, 1987; Tall, 1989; Ubuz & Ersoy, 1997; Sowder & Harel, 2003) where students who had the ability to determine the max/min value for a given function integrate this appropriately into their identification of the absolute max/min points on the graph a function, exploration of conditions under which a differentiation theorem can be true, and justification of statements in one theorem with regards to the conditions satisfied in another theorem. For instance, a student who is more apt to model a max/min situation would be more inclined to confirm that in the statement of Rolle's Theorem, f(x) is a continuous function on the closed interval [a,b], henceforth by the Intermediate Value Theorem it achieves a maximum and a minimum on [a,b]. The significant effect of iconic thinking on algebraic thinking could also be documented for axiomatic thinking. Our study was in agreement with those studies (Asiala et al., 1997; Bingolbali & Monaghan, 2008) on the account that students' inclusive manner of engagement in graphical representations of the derivative affects their testing necessary hypotheses to determine whether a function is differentiable or translating empirical arguments to derivative proofs. The sense of this relationship clearly articulated the ways to harmonize the visual demonstration of a derivative theorem within the more rigid fashion of formal mathematics and thereby expedite students' actions on "proof without words". Another central result that was framed in the within-classroom level of the Mathematical Thinking Model is that algebraic thinking was positively associated with axiomatic thinking. This finding was consistent with previous research (Thompson, 1994; Zandieh & Knapp, 2006), which reported that students' repertoire of differentiation theory enriched by relevant axioms and conjectures would likely to influence their understandings of the theoretical syntax that is to be represented by various algorithms. The pervasive effects of formal thinking on enactive thinking and iconic thinking and to a lesser extent algebraic thinking held algorithmic thinking as an essential mediator. In contrast, the indirect effect of formal thinking on iconic thinking, algebraic thinking, and axiomatic thinking through influencing enactive thinking was negligible. Similarly, formal thinking had minor indirect effects on algebraic thinking and axiomatic thinking mediated by iconic thinking. These findings elucidated that to some extent students' knowledge of derivative concept draws upon their practical knowledge of algorithms, models, and graphs relevant to differentiation and follows a continuum of theoretical differentiation strategies. Similar issues of concern were evident in the relatively small indirect effects of algorithmic thinking on algebraic thinking and axiomatic thinking through influencing enactive thinking and iconic thinking. Furthermore, mediated by enactive thinking, students' algorithmic thinking had an indirect effect on iconic thinking. Obviously these results affirmed that students' proper use of visualization skills would serve as a link between their procedural and logical reasoning while solving derivative problems.

Turning to the relationships emerged at the between-classroom level provided support in favor of the fact that the effects of one type of mathematical thinking on the other represents important features of the link between formal-axiomatic thinking, proceptual-symbolic thinking, and conceptual-embodied thinking (Stewart & Thomas, 2007, 2009). Our study has the new perspective of a link among other studies investigating the interrelationships among different types of mathematical thinking. Coupled with the cyclic associations, it holds the strongest promise in unfolding the relationship where formal-axiomatic thinking influenced the gains in proceptual-symbolic thinking as the gains in proceptual-symbolic thinking influenced conceptual-embodied thinking and in turn conceptual-embodied thinking influenced formal-axiomatic thinking. Taken as a whole, this shows that a classroom's profile of mathematical thinking in derivative cannot be drawn without designating each individual student's knowledge of definitions and theorems, knowledge of algorithms and algebraic representations and, knowledge of models and graphical representations relevant to the derivative concept. From this perspective, the strongest direct effect was from formal-axiomatic thinking to proceptual-symbolic thinking followed by proceptual-symbolic thinking to conceptual-embodied thinking, and to a lesser extent conceptual-embodied thinking to formal-axiomatic thinking. In line with previous research (Christou et al., 2005; Stewart & Thomas, 2007; Stewart, 2008; Tall, 2004) these relationships imply that classrooms' mathematical thinking in derivative builds from patterns of declarative and strategic knowledge: recalling the definition of derivative; proving differentiation formulas. Once this type of thinking is practiced and becomes routine, it can be symbolized as algorithms and used dually as hypotheses on which the procedures can be performed: solving a derivative problem; elaborating the conditions of a derivative theorem in order to determine whether a given function satisfies these conditions. As the focus of attention switches from theory to the manipulation of algorithms, classrooms' mathematical thinking shifts to the embodiment of physical conceptions and actions: modeling a real-life phenomenon; constructing the graph of a derivative function. Throughout classrooms' mathematical thinking processes in derivative, embodiment and visualization give specific meanings in varied contexts of differentiation while symbolism in algorithms and algebraic manipulations requires students to have computational power. Along with Tall (2008), these relationships give a cyclic parsimony to the between-classroom framework of mathematical thinking in derivative by accompanying an approach that interlinks embodiment, symbolism, and formalism. Another central result that framed this approach is that formal-axiomatic thinking, proceptual-symbolic thinking, and conceptual-embodied thinking, and the indirect relations among the three mediate the links relative to the derivative concept and differentiation processes. Researchers' (Cavallaro et al., 2007) considerations for each type of mathematical thinking to become predominant on the others via mediation resonate with our results. By becoming reflectively engaged in each type of mathematical thinking, i.e., by becoming involved in justifications and applications of the derivative, visualization-mediated practices offer students a wide array of mathematical thinking in derivative. For instance, in relation to the items (2, 19, 25) provided in the Thinking-in-Derivative Test, in a classroom where students appropriately define the inflection point would be more successful in finding the product of the unknown variables in a function when the local extreme and the inflection points are given. Such classrooms would thereby more effectively interpret the graph of a second derivative function in order to identify its inflection points.

The relationships among different types of mathematical thinking at the between-classroom level offers an integrated understanding that pointed knowledge of concepts and theorems more strongly forms the grounds for the acquisition of procedures and fosters the algebraic manipulations (Christou et al., 2005). This result might be partly attributable to the focus of university teaching of mathematics on formal-axiomatic and proceptual-symbolic thinking. Therefore, classrooms are better equipped to deal with those types of mathematical thinking in terms of linking within particular definitions (e.g., derivative), symbols (e.g., dy/dx), theorems (e.g., proof of Rolle's Theorem), computations (e.g., derivative operations), and algebraic manipulations (e.g., syntax of Rolle's Theorem).

Finally, the third goal of the present study was the exploration of so-called cross-level effects, which are present when a within-classroom mathematical thinking construct proves to impact the relationship among two mathematical thinking constructs at the between-classroom level as well as when a between-classroom level mathematical thinking construct proves to impact the relationship among two mathematical thinking constructs at the within-classroom level (Baron & Kenny, 1986; James & Brett, 1984). All the cross-level effects in the present study implied that the magnitude of the relation between two mathematical thinking constructs at one level of analysis varies as a function of another mathematical thinking construct at the other level.

The nested series of cross-level moderator models suggest that both lower-level and upper-level models fit the data reasonably well. Thus, it is appropriate to further decompose or probe the cross-level effects to better understand the structure of the relations at the within- and between-classroom levels (Aiken & West, 1991).

The connotations of these cross-level effects are twofold. First, it is suggested that mathematical thinking needs to be studied in its educational context, which is characterized by relational phenomena that cannot be understood in terms of students independently. For undergraduate students attending different faculties, perhaps the most prominent educational context is their immediate classroom (Artigue, 2001). Although mathematical thinking has its origin at the student level of analysis, it may form a shared cognition or a collective climate at the classroom level. This bottom-up process of mathematical thinking posits to several theories that provide the theoretical underpinnings for the emergence of mathematical thinking as a classroom-level construct as well (Kozlowski & Klein, 2000a, 2000b). Sociocultural theory (Cobb, 1994; Yackel, Rasmussen, & King, 2000) argues that students use knowledge gathered from others in their direct educational contexts to form processes in mathematical thinking. Given that students of the same classroom are exposed to the same instruction, they will possess shared enactive, iconic, algorithmic, algebraic, formal, and axiomatic thinking and form common processes regarding the general conceptual-embodied, proceptual-symbolic, and formal-axiomatic thinking practices in the classroom. Similarly, research on constructivist theory reveals that students learn from each other, via interactions with existing students in the classroom, the procedures dictating how mathematical thinking is generally carried out and how students' mathematical thinking is generally activated in the same classroom (von Glasersfeld, 1987). This type of thinking exchange takes place most frequently among students in the same classroom, thereby fostering the formation of relatively homogeneous mathematical thinking in the classroom. Second, students' mathematical thinking is influenced not only by his or her own enactive, iconic, algorithmic, algebraic, formal, and axiomatic thinking but also by the conceptual-embodied, proceptual-symbolic, and formal-axiomatic thinking of the classroom. As a result, conceptual-embodied, proceptual-symbolic, and formal-axiomatic thinking, once formed as part of a between-classroom context, will have top-down influences on students' enactive, iconic, algorithmic, algebraic, formal, and axiomatic thinking. In sum, in line with the above theories, the significant cross-level effects obtained in the present study support the use of classroom as an appropriate level to examine the existence of mathematical thinking as shared processes among students.

5.2 CONCLUSION

Beyond the well-researched phenomenon of mathematics achievement, mathematical thinking has received little attention within mathematics education research. Findings of the present study showed that mathematical thinking has a distinct factor structure at the within- and between-classroom levels, and that significant relationships among different types of mathematical thinking are present at both levels.

The application of multilevel structural equation modeling techniques to data from the multiple-choice Thinking-in-Derivative Test was revealing. Advocates of educational measurement and assessment have, at times, attacked multiple-choice tests as biased, discriminatory, or inadequate especially in testing higher-order mathematical thinking skills. The analyses reported in the present study provide little support and comfort to those critics in the field.

The central point emerging from the analyses is that classroom matters when promoting the relationships among different types of mathematical thinking. Mathematical thinking was simultaneously examined as a composite of the relationships at the within- and between-classroom levels. Classrooms are complex systems, and mathematical thinking should be explored as a system in educational settings. Along with the direct and indirect effects, the cross-level interactions among different types of mathematical thinking showed that enactive, iconic, algorithmic, algebraic, formal, and axiomatic thinking in relation to the concept of derivative are differentially affected by facets of the classrooms' conceptual-embodied, proceptual-symbolic, and formal-axiomatic thinking as activated by students. This demonstrated that classrooms systematically differ from one another in their conceptual-embodied, proceptual-symbolic, and formal-axiomatic thinking, and that this variation can to a large extent be explained by students' enactive, iconic, algorithmic, algebraic, formal, and axiomatic thinking. The data also support the assumption that students' individual enactive, iconic, algorithmic, algebraic, formal, and axiomatic thinking is not only influenced by how he or she accesses to a particular type of mathematical thinking, but also by the composition of the classroom in terms of conceptual-embodied, proceptual-symbolic, and formal-axiomatic thinking. This information is of particular value for instructors and mathematicians, for whom the cognitive well-being of their students should be a desired educational outcome in itself.

At the very least this study can serve to animate research in education to move beyond individual differences among students and beyond the conventional experimental and correlational research on student learning. The intent is for these multilevel structural equation modeling approaches to become more widely used and to further studies that investigate variance within- and between-classrooms. The vision is that these multilevel structural equation modeling approaches will provide a means for unifying methodologies to better understand the interactions of students as learners in a classroom, mathematical thinking relations as educational treatments, and classrooms as the contexts in which they occur.

As mathematical thinking is a cognitive-intensive and contentious aspect of university learning, more research is clearly warranted into how mathematical thinking types affect students' performance – and into how mathematical thinking experiences in classroom settings influence learning in higher education.

5.3 IMPLICATIONS

Throughout this study two-level structural equation models were used to explore the factor structure of mathematical thinking at student and classroom levels and to investigate the relationships among different types of mathematical thinking in and across classrooms. Although complex, these two-level models, nevertheless, are revealing and point to a number of directions for future research on mathematical thinking, efforts that go beyond investigating individual differences among students or group differences among classrooms for that matter.

Most importantly, the findings of the present study argue for methodological focus not only on individual student but also on collective classroom level. While the mechanisms of mathematical thinking must be investigated more thoroughly, particularly in terms of how mathematical thinking is structured, significant differences across relationships indicate that the factor structure of mathematical thinking does matter. Thus, the results of the multilevel exploratory and confirmatory factor analyses offer some preliminary implications for mathematics education researchers. First, when a researcher is interested in the relevant dimensions of cognitive constructs at the classroom level, in order to compare classrooms with regard to their positions on the measures for classroom process variables (e.g., thinking, reasoning, and understanding) multilevel exploratory factor analysis should be considered. Second, researchers should use multilevel confirmatory factor analysis to test whether the structure of any cognitive construct differs in and across levels of analysis. Indeed, the same items can cluster differently at the within- versus between- levels of analysis. By statistical means of the multilevel confirmatory factor analysis researchers might allow their theory to dictate the two-level analyses they opt to perform, and not their statistical analyses determining their theory. Thus, this differentially clustering of items provides a methodology for systematical investigation of the robustness of measures at the within- and between-classroom levels.

The results, with regard to the relationships among different types of mathematical thinking at the within- and between-classroom levels attest to the power of mathematical thinking at the university, call claims of teaching and learning effects at the undergraduate level, and emphasize the importance of different mathematical constructs both as outcome and mediator variables. In a sense, the two-level theoretical model identifies basic building blocks of mathematical thinking in calculus. The mathematical thinking model has the potential to inform mathematicians about the development of a thinking-prompted calculus curricula by clarifying and providing working relationships. Evidence from the within-classroom level relationships, is the emergent trend, that overall, formal and algorithmic thinking gear other mathematical thinking types. This gives mathematicians the opportunity to understand how to introduce undergraduate students to factual information and computational techniques in order to comprehensively teach all the facets of the six types of mathematical thinking in derivative. The mathematical thinking model also helps mathematicians to identify the purpose of the differentiation activities that are incorporated into the calculus curriculum from a connected perspective. This model gives mathematicians access to a viable range of in practice linkages across six types of mathematical thinking (e.g., the relationship between formal and algorithmic thinking), some of which had previously been accessible only in principle in calculus courses (e.g., the relationship between axiomatic and algebraic thinking). On the basis of the significant relationship between iconic and axiomatic thinking, for instance, a mathematician may design an instruction which facilitates proving (AXIOTHK) with effective utilization of visualizing (ICONTHK). Such an instruction may be designed to direct the attention of students to the visual content of a theorem statement, rather than solely to the theoretical structure of its proof. Thus, in their teaching, mathematicians may put forth logical essences and proof construction by manipulating both visual (ICONTHK) and theoretical (AXIOTHK) processes in an integrated manner. With this instruction, axiomatic thinking goes well beyond simple inspection of conjectures, principles, and/or postulates. Rather, it derives from iconic thinking with the incorporation of diagrams, tables, and/or graphs into the proving process. In the same vein, the significant relationship between iconic and algebraic thinking provides mathematicians with an instructional perspective that highlights algebraic thinking proceeds on the basis of explicit assumptions of theorems that, taken as a whole, apply to both hypothetical (ALGETHK) and visual (ICONTHK) processes. As the between-classroom relations in this study clearly show, conceptual-embodied, proceptual-symbolic, and formal-axiomatic thinking can be used to enhance the role of embodiment, symbolism, and formalism in the classroom. The use of a cyclic approach may pose two questions that in essence a single issue: How can this approach best be integrated into the calculus curriculum as a whole, and how can it best be used to promote mathematical thinking? The attractive and engaging techniques offered by mathematicians may raise the profile of mathematical thinking at the between-classroom level. In this regard, the curriculum designs and instructional methods can primarily take on from tasks that energize conceptual-embodied thinking, follow through tasks that mobilize formal-axiomatic thinking, and end up with tasks that prompt proceptual-symbolic thinking. As is evident in the between-classroom relations mathematical thinking is not characterized by the replacement of one type of thinking by another that supposedly is “higher” or “more abstract”; rather it is characterized by the development and interlinking of different types of thinking that can develop alongside and in combination with one another. On the other hand, the cross-level interactions among mathematical thinking types presented in the context of students nested within classrooms can direct mathematicians attention to reconsider each individual students’ mathematical thinking can carry a wealth of information about the broad spectrum of classroom’s mathematical thinking or vice versa. The results of the upper-level mediation models suggest that formal-axiomatic thinking is a key mediator of the within-classroom relations. As such, measures of formal-axiomatic thinking may potentially be useful for identifying students who are likely to struggle in calculus and may require a specific intervention. Such intervention may need to include an amalgam that improves defining and/or proving processes (FORMAXTHK), even during secondary school. Because of the strong mediation of formal-axiomatic thinking in within-classroom relations, interventions may need to focus on ways to make calculus more dependent on basic mathematical terminology and abstract mathematical theory. Furthermore, results of the lower-level mediation models demonstrate that some within-classroom factors minimally mediate the between-classroom relations, whereas some have strong mediator effects. For instance, algorithmic thinking is the strongest mediator of the relationship between formal-axiomatic and proceptual-symbolic thinking. This mediation may affect both students and classrooms. It may be helpful, therefore, to teach formalism and symbolism in conjunction with computational techniques that target the mobilization of algorithmic thinking within the linkages between formal-axiomatic and proceptual-symbolic thinking. For instance, a useful teaching strategy may be to instruct students initially to access formal-axiomatic thinking to energize proceptual-symbolic thinking for formulating a solution plan that grounds on the basic concepts, theory, hypotheses, and algorithms and then, penetrate algorithmic thinking into this association by implementing technical procedures. Mathematicians and mathematics educators may also integrate this cross-level relationship into a measurement context to develop useful diagnostic tools for determining why students have deficits in connecting formal-axiomatic and proceptual-symbolic thinking, and thus for determining what type of mathematical thinking (in this case algorithmic thinking that mobilizes computational fluency) is needed to address these deficits. In this vein, to some extent, the norms of enhancing authentic mathematical tasks are not well developed in both secondary school and university classrooms. One of the challenges that face mathematics education is to develop worthwhile tasks that promote students' mathematical thinking and thus foster classroom's mathematical thinking. Drawing on this connection, mathematics education researchers may develop multiple-choice and/or open-ended question pools to construct item banks and/or testlets including tasks that challenge students to demonstrate fruitful mathematical thinking.

Collectively, the harmony of the two-level relationships as well as the the interactions mainly provide an important avenue for the mathematicians and mathematics educators whose instructional focus is on the teaching and learning of advanced mathematics. However, this harmony further orchestrates with the educational aims of secondary school mathematics teachers whose teaching is directed towards precalculus concepts and procedures. On the side of secondary education, the findings again support the idea that mathematical thinking types lie on a continuum and influence one another and hence elucidate the role that individual student's mathematical thinking in a specific task plays as an underlying mechanism in classroom's mathematical thinking. Thus, the findings of this research have two main implications for secondary mathematics education. First, developing educational approaches and tasks that allow and trigger the interaction among different types of mathematical thinking can enhance, factually, conceptually and procedurally, the learning outcomes of young students' problem solving in precalculus subjects. Given the traditionally and internationally highlighted need to support students to construct a well-connected web of mathematical thinking, tasks such as the ones used in this research that enable students to apply more than one type of mathematical thinking and possess into the connections among these different types of mathematical thinking can be highly beneficial in helping secondary students develop their understanding of the derivative concept and the differentiation process. A second implication of this study is that educational activities that offer secondary students the opportunity to work on and elaborate different types of mathematical thinking that brings overt thinking aspects of a task into interaction, hence enhancing students' mathematical thinking development. As a set, in addition to extending the research literature on students' thinking in derivative, this research may enhance information available to curriculum designers. Specifically, the two-level relationships may enable students' mathematical thinking to be described and delineated in a coherent and systematic manner. Taking into consideration that mathematical thinking types do improve one another (i.e., two-level relationships, cross-level interactions), this model offers mathematics teachers and instructors a framework of students' mathematical thinking while solving various formats of derivative tasks. This model can thus be used as a tool in mathematics teachers' instruction for organizing instruction and building mathematical tasks, in general, and constructing derivative tasks, in partial. Taken together, these implications further encourage reflection on both educational and instructional approaches in the transition from secondary school to university.

The constructivist theory of mathematical thinking (Edwards, Dubinsky, & McDonald, 2005) suggests that high-quality instruction fosters the development of students' higher-order enactive, iconic, algorithmic, algebraic, formal, and axiomatic thinking and thus enhances the classroom's experience of conceptual-embodied, proceptual-symbolic, and formal-axiomatic thinking. The present results supports this hypothesis in the derivative context, emphasizing that – in order to trigger sophisticated rather than naive mathematical thinking in mathematics– instructors need to set effective tasks that are well integrated into lectures, that reinforce classroom thinking. The distinction among different types of mathematical thinking at the within- and between-classroom levels illuminates alternative ways in which useful instructional designs for calculus courses could be implemented. One of the challenges that face higher education is to develop worthwhile tasks that require students to effectively employ heuristics in order to maximize the chances that they arrive at the best possible solution. Fruitful learning areas might be created by the inclusion of thinking-specific tasks that primarily activates students' thinking about definitions/symbols/facts (formal thinking), continues through about procedures/algorithms (algorithmic thinking), thinking about real-life phenomena (enactive thinking), and thinking about graphical representations (iconic thinking), and finally outlines thinking about algebraic manipulations (algebraic thinking) together with thinking about proofs/proving (axiomatic thinking). An issue that needs to be followed up in classroom contexts concerns the specific sequence of instructional tactics to be presented to undergraduate students. The cyclic interrelationships at the between-classroom level established in the present study underline that none of the thinking types are sufficient on their own. For instance, the leap to fundamental definitions and theorems (formal-axiomatic thinking) can only be achieved when algorithmic and/or algebraic manipulations (proceptual-symbolic thinking) are enhanced with visual constructions and interpretations (conceptual-embodied thinking) or vice versa. Henceforth, the combination of thinking types at the within- and between-classroom levels casts light on the synthesis of taking into account the thinking capabilities of the student as an individual and as a member of the classroom, and in turn brings the engagement in sophisticated ways of mathematical thinking in calculus. From an assessment perspective, the two-level model appears to be valuable in providing mathematicians with useful background on students' mathematical thinking and in enabling them to monitor general growth in a classroom's mathematical thinking. The viability of using this model for informing mathematical thinking in regular classroom situations provides opportunities for fine-tuning the relationships among different types of mathematical thinking and making these relationships more effective for generating instructional programs that build on the mathematical thinking portrait of students and classrooms.

Whilst a variety of relationships at the within- and between-classroom levels were identified, at the between-classroom level, these are largely relationships beyond the control of the classroom (namely the universities and/or faculties). Nonetheless, these relationships need to be considered by policy-makers. Furthermore, identifying these relationships helps to explain the overall mathematical thinking and to alert those in authority as to the effect of different types of mathematical thinking on students' achievement in mathematics at the university. Furthermore, with reform in calculus classes with a concentrated focus on the development of mathematical thinking, educational practioners and department chairs can become deeply concerned with instructional methods appropriate for both students and classrooms. The two-level relationships in the present study require a serious commitment to restructuring calculus classes that different types of mathematical thinking should be provoked to ensure success in advanced mathematics courses. More specifically, issues such as faculty affiliation truly may be more salient in some classrooms than in others. Bingolbali and Ozmantar (2009) point out, is the fact that faculty affiliation has, by and large, certain connotations that send signals to the structure of calculus courses as to what mathematics is useful or appropriate for students. Lecturers might make deliberate amendments to their instructions and put more emphasis on certain aspects of the same concepts in regard to the students of different faculties. For instance, while teaching the rules of differentiation, a lecturer might put more emphasis on proving the chain rule at Faculty of Arts and Sciences whilst at Faculty of Engineering on technically applying the procedure beneath the chain rule to evaluate the derivative. If university personnel can do more to create equal opportunities for students, then students might tend to experience higher levels of mathematical thinking. That is, for instance some students from Faculty of Education who do not feel supported in those environments where mathematical thinking is likely to be less fostered may experience additional cognitive rejection and achievement problems.

Finally, the present research is description of some sort as it provides alternative accounts of enactive, iconic, algorithmic, algebraic, formal, and axiomatic thinking. More than focusing on the well-known types such as conceptual-embodied, proceptual-symbolic, and formal-axiomatic thinking, it underlines the non-transparency of embodiment, symbolism, and formalism; it draws attention back to the roots of mathematical thinking (e.g., Bruner, 1966; Fischbein, 1983) and that, it tends to look through in search of a comprehensible framework of mathematical thinking. Henceforth, this study provides a more detailed way to look at how descriptions of mathematical thinking types are constructed and deployed to specify certain characteristics of each thinking type. Accordingly, mathematics education researchers can look for and examine in any particular mathematics domain to see what these descriptions make possible and what alternative descriptions of mathematical thinking can be provided. In the guidance of this study, researchers may ask themselves: (a) How have the mathematical thinking types in this study been constructed? Could alternative types have been constructed? (b) What inferences about mathematical thinking do these descriptions make possible?

(c) What kinds of descriptions appear in the categorization of mathematical thinking types? Where do this categorization come from? What alternative forms of categorization could be used? What difference would these alternative categorizations make? (d) How do these questions combine to construct a different multilevel model of mathematical thinking? As previously mentioned, the above-highlighted questions are likely to be familiar to mathematics education researchers. They are not, however, the only people interested in different types of mathematical thinking; they are also of interest to mathematics teachers, mathematics lecturers, government advisors or textbook writers. Advanced mathematics curricula, for example, often include descriptive lists of higher-order thinking skills that students should acquire, thereby describing mathematical thinking as something that can be distinguished and categorized into different types to which teachers and/or lecturers can be held accountable. In this case, this study offers a starting point for how such a description and categorization might be put into educational practice.

5.4 LIMITATIONS

With its large sample size, hierarchical dataset, and subject specificity, the present study provides well-founded insights into the factor structure of mathematical thinking at the within- and between-classroom levels and the relationships among different types of mathematical thinking at these levels. However, there are some limitations that should be addressed in future studies.

It is important to note that even though the present study examined certain types of mathematical thinking as dependent variables and some others as predictors, the cross-sectional nature of the data does not permit to infer cause-effect relationships. Furthermore, the relationships among different types of mathematical thinking was investigated in the calculus classroom only and specifically on the derivative concept. It will be of importance to examine these hypotheses in other subject domains (e.g., numerical analysis) and concepts (e.g., interpolation). On the other hand, a 30 item multiple-choice test was the sole source of information about students' mathematical thinking. Although multiple-choice tests provide us easy-to-measure student learning and content (Haladyna, 1994), their use presents a risk of guessing. That is, students may choose the correct answer even if they do not access to the mathematical thinking needed to reach a solution. In view of presenting students enough number of test items (n = 30) and including sufficient number of distracters for these items (options = 4), however, it was considered the chance of a student guessing the right answer to be decreased.

Although this investigation was based on two very strong data sets including independent samples, the data used from the Thinking-in-Derivative Test was limited to university students in particular regions attending. Similar studies should be conducted in various provinces to shed light on the relationships among different types of mathematical thinking at the within- and between-classroom levels.

Mathematical thinking in classrooms is a complex phenomenon, and the nature of mathematical thinking and, in all likelihood, its stimulation across grade levels might differ markedly. The present study used data from second, third, and fourth graders. More studies examining the relationships among different types of mathematical thinking should involve first year university students. This would provide an indepth understanding of what characteristics of mathematical thinking exert at which grade levels, and in what forms of relationships. Moreover, the proposed models were tested for the data from students at the undergraduate level. There is always the possibility that the data from elementary and/or secondary students might affect the pattern of results at the within-classroom level as well as at the between-classroom level.

In the present study only two levels of analysis were used to explore the relationships among different types of mathematical thinking. From a theoretical point of view, however, one would probably argue that in educational settings classrooms are the natural frame of reference for students' mathematical thinking (e.g., O'Connell & McCoach, 2008; Hill & Goldstein, 1998). From a methodological point of view, it is preferable to include all substantial levels of analysis in multilevel modeling (e.g., Raudenbush & Byrk, 2002). Thus, the present study would have been even stronger if grade level and/or faculty have been used as additional units in the hierarchical sampling process. However, this three-level modeling approach would make the study considerably more complex given that applications of three-level modeling are fairly recent in educational research and software programs providing the use of this technique are still developing.

5.5 RECOMMENDATIONS FOR FUTURE RESEARCH

Mathematical thinking in higher education is an issue of tremendous importance for students, classrooms, and lecturers. The present study significantly extends previous mathematical thinking research by focusing on the two-level structural relationships among different types of mathematical thinking. As a consequence of the pattern of results, the following recommendations could be inferred. The present study shows that it is possible to separate within- and between-classroom variance in responses to mathematical thinking items. Identifying the university as a source of variance (e.g., students nested within classrooms nested within universities), future research may incorporate more than two levels of nesting. Moreover, multilevel techniques are not limited to cross-sectional data but can be used in longitudinal data where cluster sampling has been employed. This study encountered some theoretical issues that lend support to the use of multilevel modeling techniques and specification of the direct and indirect effects of the constructs at both levels. Therefore, future research requires the cross-validation and/or replication of this present study with any multilevel analysis in combination with multiple indicators-multiple causes (MIMIC) and multiple-group approaches. Meanwhile, qualitative studies are important to better understand students' different types of mathematical thinking in educational settings. For example, interviews could be conducted with students to better understand the nature and function of mathematical thinking in these settings, as their progress in different types of mathematical thinking may be influenced by classroom differences relating to significant others' mathematical thinking. Analogously, a sense of understanding may be gathered through conducting classroom observations to detect individual and collective development in different types of mathematical thinking.

With within-classroom and between-classroom effects simultaneously accounted for, certain factors at the student and classroom levels might significantly contribute to the explanation of variance in different types of mathematical thinking. Identifying the predictors of mathematical thinking at the within- and between-classroom levels is important for future research to better address the pathways of mathematical thinking in educational settings. In addition to the call for investigating the influence of affective factors (e.g, metacognition, self-efficacy, self-regulation) on mathematical thinking (O'Connell & McCoach, 2008) personal variables such as gender, socioeconomic status or institutional factors such as faculty affiliation and grade level can be introduced to the multilevel models of mathematical thinking. Future research should examine these factors more specifically as possible predictors of mathematical thinking that may help target within- and between-classroom interventions more effectively. It is clear that factors at within- and between-classroom levels should be assessed when examining different aspects of mathematical thinking and developing initiatives to enhance both students' and classrooms' mathematical thinking.

In line with the factor structure of mathematical thinking at the within- and between-classroom levels, the present study employed a cross-level multilevel model. However, it is important to realize that the application of multilevel exploratory factor analysis is partly subjective in nature. Due to the exploratory decisions that researchers make in order to select the most appropriate factor solution, independent researchers might come to different results at the within- and between-classroom levels. Therefore, future researchers may choose to estimate a homologous multilevel model in which relationships among variables hold at both levels of analysis. For instance, researchers may postulate the existence of shared mathematical thinking constructs and thus use the aggregated measures of within-classroom constructs at the between-classroom level (e.g., formal thinking at the within-classroom level and average formal thinking at the between-classroom level). Such model specifications are valuable in that they allow the researchers to generalize the mathematical thinking constructs and functional interrelations linking these constructs across different levels of the educational system.

The student and the classroom become face-to-face in an instructional encounter mathematical learning can be characterized as participating in mathematical thinking practices (Rasmussen, Zandieh, King, & Teppo, 2005). Exhaustively, mathematical thinking types at the within-classroom level and at the between-classroom level are tendered as important examples of such practices. Closely related to the direct/indirect/cross-level effects of mathematical thinking constructs, the present study can guide research in the overutilization of mathematical thinking types as centerpieces around which to craft different item formats (e.g., open response, true/false).

Furthermore, future studies can provide further insights into the development of mathematical thinking and pinpoint practical implications for high-quality mathematical thinking by examining which characteristics of mathematical tasks enhance the experience of mathematical thinking and which elicit ineffective mathematical thinking. Research on general instructional quality in mathematics has shown cognitively activating mathematical tasks to be positively linked to achievement (Henningsen & Stein, 1997; Stein, Grover, & Henningsen, 1996). Similar patterns of results might be expected to emerge between mathematical thinking constructs and mathematical tasks.

Another avenue on which future researchers might consider embarking is the mathematics education community has now a new, well-validated measure of mathematical thinking. With its very multidimensional nature, the Thinking-in-Derivative Test measures enactive, iconic, algorithmic, algebraic, formal, and axiomatic thinking. Future research should consider adapting this measure to test whether the interrelationships differs in other cognitive domains such as mathematical reasoning, understanding, and knowledge.

In conclusion, this research represents the first known attempt at taking a cross-level, student-classroom approach to the study of mathematical thinking. The data provide a promising pattern of results that will hopefully be explored by researchers integrating longitudinal and/or growth models. It would be important to conduct longitudinal studies that follow cohorts of students to examine how they progress in different types of mathematical thinking over time, and how the relationships among different types of mathematical thinking may be influenced by a broad spectrum of variables, from exogenous factors to students and to classrooms at the within- and between-classroom levels. Henceforth, experimental studies are needed to better highlight the issue of causation, thereby complementing longitudinal studies. Given the robustness of mathematical thinking on highly relevant outcomes at the university (Tall, 2004), this research continues to have direct implications on how students and classrooms interact in terms of the relationships among different types of mathematical thinking.

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APPENDIX A

THINKING-IN-DERIVATIVE TEST

Üniversiteniz:
Bölümünüz:
Sınıfınız:
Ağırlıklı Not Ortalamanız:
Lise Matematik Başarınız: **1 ( ) 2 ( ) 3 ( ) 4 ( ) 5( )**
Mezun Olduğunuz Lise Türü: **Devlet ( ) Özel ( ) Anadolu ( )**
Fen ( ) Anadolu Öğretmen ( )
Cinsiyet: **Kız ( )** **Erkek ( )**

Bu test türev kavramı ile ilgili 30 sorudan oluşmaktadır. Her sorunun tek doğru cevabı vardır. Doğru olduğunu düşündüğünüz cevabınızı işaretleyiniz. Lütfen her soruyu cevaplamaya dikkat ediniz. Teşekkürler! ☺

SPECIMEN ITEM FOR FORMAL THINKING

Türev sembolü aşağıdakilerden hangisidir?

A) $\frac{\delta y}{\delta x}$ B) $\frac{\Delta y}{\Delta x}$ C) $\frac{d}{dx}$ D) $\frac{\partial y}{\partial x}$ E) $\frac{dy}{dx}$

SPECIMEN ITEM FOR ALGEBRAIC THINKING

$f:[6,15]\rightarrow R$ şeklinde tanımlanan f fonksiyonu için f(6)= -2 ve $f'(x)\leq 10$ dur. Bu fonksiyon [6,15] aralığında **Ortalama Değer Teoremi**'nin koşullarını sağladığına göre aşağıdakilerden hangisi doğrudur?

I. f(15) in en büyük değeri 88 dir.
II. f fonksiyonunun [6, 15] aralığındaki ortalama değeri 45 dir.
III. f fonksiyonu [6, 15] aralığında 45 değerini en az bir kez alır.

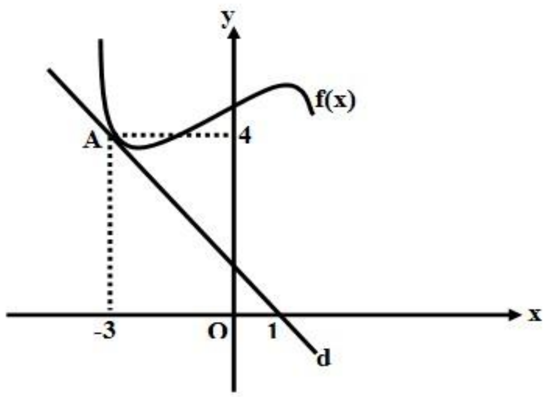
A) Yalnız I B) Yalnız II C) I ve III D) II ve III E) I, II, ve III

SPECIMEN ITEM FOR AXIOMATIC THINKING

"**Rolle Teoremi:** f: [a,b]→R fonksiyonu sürekli ve her $x \in (a,b)$ noktasında türevlenebilir olsun. Eğer $f(a) = f(b)$ ise (a,b) aralığında, $f'(c) = 0$ olacak şekilde en az bir c noktası vardır." Teoremi ile ilgili olarak aşağıdaki çıkarımlardan hangisi doğrudur?

A) f fonksiyonunun (a,b) aralığında birden çok kökü vardır.
B) f fonksiyonunun (a,b) aralığında en az bir kritik noktası vardır.
C) f fonksiyonuna (c, f(c)) noktasında çizilen teğet x eksenine diktir.
D) f fonksiyonunun (a,b) aralığında birinci türevi her zaman pozitif veya her zaman negatiftir.
E) f fonksiyonuna (a,b) aralığında çizilen kiriş doğrusunun eğimi ile teğet doğrusunun eğimi birbirinden farklıdır.

SPECIMEN ITEM FOR ICONIC THINKING



Yukarıdaki şekilde, d doğrusu f fonksiyonunun grafiğine A noktasında teğettir. h(x)= x. f(x) olduğuna göre, $h'(-3)$ aşağıdakilerden hangisidir?

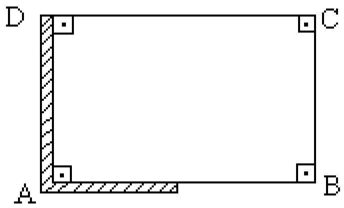
A) -4 B) -2 C) 0 D) 2 E) 7

SPECIMEN ITEM FOR ALGORITHMIC THINKING

$f(x) = [1 + (x + x^2)^3]^4$ olduğuna göre, $f'(x)$ türev fonksiyonunun x= 1 için değeri aşağıdakilerden hangisidir?

A) $2^3.3^5$ B) $2^3.3^7$ C) $2^4.3^6$ D) $2^4.3^8$ E) $2^5.3^{10}$

SPECIMEN ITEM FOR ENACTIVE THINKING



Yukarıdaki şekilde dikdörtgen biçimindeki bir bahçenin [AD] kenarının tümü ile [AB] kenarının yarısına taş duvar örülmüş, kenarlarının geriye kalan kısmına bir sıra tel çekilmiştir. Kullanılan telin uzunluğu 120 m olduğuna göre, bahçenin alanı en fazla kaç m^2 olabilir?

A) 1200 B) 1250 C) 2300 D) 2350 E) 2400

APPENDIX B

DESCRIPTIVE STATISTICS OF SAMPLE 1

Table B.1 Descriptive statistics of Sample1

	Cumulative Grade Point Average				
Grade Level	1	2	3	4	Total
1	28	201	159	27	415
2	3	70	68	13	154
3	6	123	175	13	317
4	0	58	133	22	213
5	-	-	-	-	-
Total	37	452	535	75	1099
Prior Mathematics Achievement					
1	4	7	2	0	13
2	3	61	73	5	142
3	10	121	89	9	229
4	14	124	135	19	292
5	6	139	236	42	423
Total	37	452	535	75	1099

APPENDIX C

DESCRIPTIVE STATISTICS OF SAMPLE 2

Table C.1 Descriptive statistics of Sample2

	Cumulative Grade Point Average				
Grade Level	1	2	3	4	Total
1	-	-	-	-	-
2	30	450	389	55	924
3	26	428	407	50	911
4	9	239	300	41	589
5	-	-	-	-	-
Total	65	1117	1096	146	2424
Prior Mathematics Achievement					
1	4	10	10	4	28
2	6	89	73	6	174
3	14	196	147	17	374
4	23	404	389	43	859
5	18	418	477	76	989
Total	65	1117	1096	146	2424

APPENDIX D

THE LISREL INPUT FILE FOR THE COMMON-FACTOR MODEL OF THE TDT

Real Data Set

Observed Variables
QUES1 QUES2 QUES3 QUES4 QUES5 QUES6 QUES7 QUES8 QUES9
QUES10 QUES11 QUES12 QUES13 QUES14 QUES15 QUES16 QUES17
QUES18
QUES19 QUES20 QUES21 QUES22 QUES23 QUES24 QUES25 QUES26
QUES27 QUES28 QUES29 QUES30
Correlation matrix from File: tdt.cor
Asymptotic Covariance Matrix from File: tdt.acm
Sample Size = 766
Latent Variables
Think
Relationships
QUES2 QUES3 QUES4 QUES6 QUES7 QUES8 QUES9 QUES10 QUES11 =
Think
QUES12 QUES13 QUES14 QUES15 QUES21 QUES22 QUES23 QUES25
QUES26 = Think
QUES16 QUES17 QUES18 QUES19 QUES20 QUES24 QUES27 QUES28
QUES30 = Think
QUES1 = 1*Think
Path Diagram
Admissibility Check = 1000
Iterations = 5000
Method of Estimation: Diagonally Weighted Least Squares
End of problem

APPENDIX E

THE LISREL INPUT FILE FOR THE THREE-FACTOR MODEL OF THE TDT

Real Data Set

Observed Variables
QUES1 QUES2 QUES3 QUES4 QUES5 QUES6 QUES7 QUES8 QUES9
QUES10 QUES11 QUES12 QUES13 QUES14 QUES15 QUES16 QUES17
QUES18
QUES19 QUES20 QUES21 QUES22 QUES23 QUES24 QUES25 QUES26
QUES27 QUES28 QUES29 QUES30

Correlation matrix from File: tdt.cor
Asymptotic Covariance Matrix from File: tdt.acm
Sample Size = 766
Latent Variables
Formaxthink Procsymthink Concembthink
Relationships
QUES2 QUES3 QUES4 QUES6 QUES7 QUES8 QUES9 QUES10 QUES11 =
Formaxthink
QUES13 QUES14 QUES15 QUES21 QUES22 QUES23 QUES25 QUES26 =
Procsymthink
QUES17 QUES18 QUES19 QUES20 QUES24 QUES27 QUES28 QUES30
=Concembthink

QUES1 = 1*Formaxthink
QUES12 = 1*Procsymthink
QUES16 = 1*Concembthink

Set Error Covariance Between QUES6 and QUES2 Free
Set Error Covariance Between QUES6 and QUES1 Free
Set Error Covariance Between QUES2 and QUES1 Free

```
Set Error Covariance Between QUES30 and QUES15 Free
Set Error Covariance Between QUES10 and QUES4 Free
Set Error Covariance Between QUES8 and QUES1 Free
Path Diagram
Admissibility Check = 1000
Iterations = 5000
Method of Estimation: Diagonally Weighted Least Squares
End of problem
```

APPENDIX F

THE LISREL INPUT FILE FOR THE NULL MODEL OF THE TDT

```
Real Data Set
Observed Variables
QUES1 QUES2 QUES3 QUES4 QUES5 QUES6 QUES7 QUES8 QUES9
QUES10 QUES11 QUES12 QUES13 QUES14 QUES15 QUES16 QUES17 QUES18
QUES19 QUES20 QUES21 QUES22 QUES23 QUES24 QUES25 QUES26 QUES27
QUES28 QUES29 QUES30
Correlation matrix from File: tdt.cor
Asymptotic Covariance Matrix from File: tdt.acm
Sample Size = 766
Latent Variables
Enacthink Iconthink Algethink Algothink Formthink Axiothink
Relationships
QUES2 QUES3 QUES4 QUES5 QUES6 = Formthink
QUES8 QUES9 QUES10 QUES11 = Axiothink
QUES13 QUES14 QUES15 = Algethink
QUES17 QUES18 QUES19 QUES20 = Iconthink
QUES22 QUES23 QUES25 QUES26 = Algothink
QUES27 QUES28 QUES29 QUES30 = Enacthink
QUES1 = 1* Formthink
QUES7 = 1* Axiothink
QUES12 = 1* Algethink
QUES16 = 1* Iconthink
QUES21 = 1* Algothink
QUES24 = 1* Enacthink
Let Correlation Between Latent Variables Equal to 0.
Let Correlation Between Observed Variables Equal to 0.
Let Correlation Between Latent Variables and Observed Variables Equal to 0.
Path Diagram
Admissibility Check = 1000
Iterations = 5000
Method of Estimation: Diagonally Weighted Least Squares
End of problem
```

APPENDIX G

THE LISREL INPUT FILE FOR THE TARGET MODEL OF THE TDT

```
Real Data Set
Observed Variables
QUES1-QUES30
Correlation matrix from File: tdt.cor
Asymptotic Covariance Matrix from File: tdt.acm
Sample Size = 766
Latent Variables
Enacthink Iconthink Algethink Algothink Formthink Axiothink
Relationships
QUES2 QUES3 QUES4 QUES5 QUES6 = Formthink
QUES8 QUES9 QUES10 QUES11 = Axiothink
QUES13 QUES14 QUES15 = Algethink
QUES17 QUES18 QUES19 QUES20 = Iconthink
QUES22 QUES23 QUES25 QUES26 = Algothink
QUES27 QUES28 QUES29 QUES30 = Enacthink
QUES1 = 1*Formthink
QUES7 = 1*Axiothink
QUES12 = 1*Algethink
QUES16 = 1*Iconthink
QUES21 = 1*Algothink
QUES24 = 1*Enacthink
Set Error Covariance Between QUES26 and QUES2 Free
Set Error Covariance Between QUES21 and QUES10 Free
Set Error Covariance Between QUES6 and QUES5 Free
Set Error Covariance Between QUES12 and QUES5 Free
Set Error Covariance Between QUES15 and QUES1 Free
Set Error Covariance Between QUES5 and QUES4 Free
Set Error Covariance Between QUES25 and QUES6 Free
Set Error Covariance Between QUES30 and QUES27 Free
Set Error Covariance Between QUES23 and QUES2 Free
Set Error Covariance Between QUES23 and QUES5 Free
Set Error Covariance Between QUES4 and QUES1 Free
```

Set Error Covariance Between QUES13 and QUES1 Free
Set Error Covariance Between QUES18 and QUES2
Path Diagram
Method of Estimation: Diagonally Weighted Least Squares
End of problem

APPENDIX H

THE TESTFACT4 INPUT FILE FOR RELIABILITY ANALYSIS OF THE TDT

```
>TITLE
THINKING IN DERIVATIVE TEST
            ITEM AND TEST STATISTICS
>PROBLEM  NITEMS=30, RESPONSE=5;
>NAMES     QUES1, QUES2, QUES3, QUES4, QUES5,
          QUES6, QUES7, QUES8, QUES9, QUES10,
          QUES11, QUES12, QUES13, QUES14, QUES15,
          QUES16, QUES17, QUES18, QUES19, QUES20
          QUES21, QUES22, QUES23, QUES24, QUES25,
          QUES26, QUES27, QUES28, QUES29, QUES30;
>RESPONSE   'A','B','C','D','E';
>KEY       EBADCEBDEBBACACBAEAEDCDEADABEC;
>RELIABIITY KR20;
>PLOT      PBISERIAL, CRITERION, FACILITY;
>TETRACHORIC NDEC=3, LIST;
>SAVE CORRELAT, FSCORES
>INPUT     NIDCHAR=7, SCORES, FILE='MLT.DAT';
(7A1,T11,30A1)
>STOP
```

APPENDIX I

DESCRIPTIVE STATISTICS OF THE ITEMS OF TDT FOR SAMPLE1

Table I.1 Descriptive statistics of the items of TDT for Sample1

Item	Min	Max	Mean	SD	Skewness	Kurtosis
QUES1	0	1	.48	.50	.06	1.99
QUES2	0	1	.56	.49	-.25	-1.94
QUES3	0	1	.59	.49	-.34	-1.88
QUES4	0	1	.61	.48	-.43	-1.81
QUES5	0	1	.79	.40	-1.43	.06
QUES6	0	1	.56	.49	-.25	-1.93
QUES7	0	1	.60	.49	-.41	-1.82
QUES8	0	1	.49	.50	.04	-2.00
QUES9	0	1	.68	.46	-.77	-1.40
QUES10	0	1	.46	.49	.17	-1.97
QUES11	0	1	.59	.49	-.36	-1.87
QUES12	0	1	.48	.50	.09	-1.99
QUES13	0	1	.48	.50	.09	-1.99
QUES14	0	1	.57	.49	-.27	-1.92
QUES15	0	1	.56	.49	-.22	-1.95

Table I. 1 (continued)

Item	Min	Max	Mean	SD	Skewness	Kurtosis
QUES16	0	1	.59	.49	-.38	-1.85
QUES17	0	1	.50	.50	-.01	-2.00
QUES18	0	1	.62	.48	-.48	-1.76
QUES19	0	1	.65	.47	-.61	-1.62
QUES20	0	1	.64	.48	-.57	-1.66
QUES21	0	1	.63	.48	-.54	-1.70
QUES22	0	1	.64	.48	-.58	-1.66
QUES23	0	1	.65	.47	-.60	-1.63
QUES24	0	1	.59	.49	-.35	-1.87
QUES25	0	1	.56	.49	-.24	-1.94
QUES26	0	1	.54	.49	-.16	-1.97
QUES27	0	1	.49	.50	.02	-2.00
QUES28	0	1	.59	.49	-.37	-1.86
QUES29	0	1	.47	.49	.12	-1.98
QUES30	0	1	.62	.48	-.47	-1.77

APPENDIX J

DESCRIPTIVE STATISTICS OF THE ITEMS OF TDT FOR SAMPLE2

Table J.1 Descriptive statistics of the items of TDT for Sample2

Item	Min	Max	Mean	SD	Skewness	Kurtosis
QUES1	0	1	.49	.50	.01	-2.01
QUES2	0	1	.53	.49	-.12	-1.98
QUES3	0	1	.49	.50	.01	-2.00
QUES4	0	1	.57	.49	-.29	-1.91
QUES5	0	1	.83	.37	-1.77	1.13
QUES6	0	1	.52	.49	-.09	-1.99
QUES7	0	1	.64	.47	.60	-1.63
QUES8	0	1	.48	.49	.06	-1.99
QUES9	0	1	.73	.44	-1.06	-.85
QUES10	0	1	.42	.49	.30	-1.91
QUES11	0	1	.55	.49	-.21	-1.95
QUES12	0	1	.45	.49	.19	-1.96
QUES13	0	1	.45	.49	.19	-1.95
QUES14	0	1	.55	.49	-.22	-2.00
QUES15	0	1	.51	.49	-.04	-1.91

Table J.1 (continued)

Item	Min	Max	Mean	SD	Skewness	Kurtosis
QUES16	0	1	.57	.49	-.29	-1.93
QUES17	0	1	.43	.49	.26	-1.90
QUES18	0	1	.57	.49	-.30	-1.51
QUES19	0	1	.66	.47	-.69	-1.81
QUES20	0	1	.60	.48	-.43	-1.77
QUES21	0	1	.61	.48	-.47	-1.77
QUES22	0	1	.61	.48	-.47	-1.77
QUES23	0	1	.65	.47	-.65	-1.56
QUES24	0	1	.58	.49	-.36	-1.87
QUES25	0	1	.55	.49	-.21	-1.95
QUES26	0	1	.48	.49	.05	-1.99
QUES27	0	1	.49	.50	-.00	-2.00
QUES28	0	1	.60	.48	-.41	-1.82
QUES29	0	1	.46	.49	.15	-1.97
QUES30	0	1	.57	.49	-.28	-1.92

APPENDIX K

PERCENTAGE FLOOR AND CEILING EFFECTS OF THE ITEMS OF TDT FOR SAMPLE 1

Table K.1 Percentage floor and ceiling effects of items of TDT for Sample 1

Item	Min	Max	Mean	SD	%incorrect	%correct
QUES1	0	1	.48	.50	51.7	48.3
QUES2	0	1	.56	.49	43.8	56.2
QUES3	0	1	.59	.49	41.5	58.5
QUES4	0	1	.61	.48	39.3	60.7
QUES5	0	1	.79	.40	20.8	79.2
QUES6	0	1	.56	.49	43.6	56.4
QUES7	0	1	.60	.49	39.8	60.2
QUES8	0	1	.49	.50	51.0	49.0
QUES9	0	1	.68	.46	32.0	68.0
QUES10	0	1	.46	.49	54.2	45.8
QUES11	0	1	.59	.49	41.1	58.9
QUES12	0	1	.48	.50	52.4	47.6
QUES13	0	1	.48	.50	52.4	47.6
QUES14	0	1	.57	.49	43.2	56.8
QUES15	0	1	.56	.49	44.5	55.5
QUES16	0	1	.59	.49	40.7	59.3
QUES17	0	1	.50	.50	49.6	50.4
QUES18	0	1	.62	.48	38.2	61.8
QUES19	0	1	.65	.47	35.3	64.7
QUES20	0	1	.64	.48	36.1	63.9
QUES21	0	1	.63	.48	36.9	63.1
QUES22	0	1	.64	.48	36.0	64.0
QUES23	0	1	.65	.47	35.5	64.5
QUES24	0	1	.59	.49	41.3	58.7
QUES25	0	1	.56	.49	43.9	56.1
QUES26	0	1	.54	.49	45.8	54.2
QUES27	0	1	.49	.50	50.5	49.5
QUES28	0	1	.59	.49	40.8	59.2
QUES29	0	1	.47	.49	53.0	47.0
QUES30	0	1	.62	.48	38.5	61.5

APPENDIX L

DESCRIPTIVE STATISTICS FOR SCORES ON THE SUBDOMAINS OF TDT FOR SAMPLE 1

Table L.1 Descriptive statistics for scores on the subdomains of TDT for Sample 1

Subdomain	Min	Max	Mean	SD	Skewness	Kurtosis
ENACTHK	0	5	2.81	1.42	.04	-.93
ICONTHK	0	5	3.00	1.53	-.22	-.99
ALGOTHK	0	5	2.96	1.51	-.17	-1.03
ALGETHK	0	4	2.07	1.19	.12	-.95
FORMTHK	0	6	3.59	1.52	.05	-.85
AXIOTHK	0	5	2.81	1.44	-.00	-.93

APPENDIX M

DESCRIPTIVE STATISTICS FOR TOTAL SCORES ON TDT FOR SAMPLE 1

Table M.1 Descriptive statistics for total scores on TDT for Sample 1

Total Score	Min	Max	Mean	SD	Skewness	Kurtosis
MATTHK	3	30	17.27	6.53	.50	-.56

APPENDIX N

PERCENTAGE FLOOR AND CEILING EFFECTS OF THE ITEMS OF TDT FOR SAMPLE 2

Table N.1 Percentage floor and ceiling effects of items of TDT for Sample 2

Item	Min	Max	Mean	SD	%incorrect	%correct
QUES1	0	1	.49	.50	50.5	49.5
QUES2	0	1	.53	.49	46.8	53.2
QUES3	0	1	.49	.50	50.2	49.8
QUES4	0	1	.57	.49	42.8	57.2
QUES5	0	1	.83	.37	16.9	83.1
QUES6	0	1	.52	.49	47.6	52.4
QUES7	0	1	.64	.47	35.6	64.4
QUES8	0	1	.48	.49	51.5	48.5
QUES9	0	1	.73	.44	26.4	73.6
QUES10	0	1	.42	.49	57.5	42.5
QUES11	0	1	.55	.49	44.8	55.2
QUES12	0	1	.45	.49	54.9	45.1
QUES13	0	1	.45	.49	54.8	45.2
QUES14	0	1	.55	.49	44.3	55.7
QUES15	0	1	.51	.49	49.0	51.0
QUES16	0	1	.57	.49	42.8	57.2
QUES17	0	1	.43	.49	56.5	43.5
QUES18	0	1	.57	.49	42.5	57.5
QUES19	0	1	.66	.47	33.6	66.4
QUES20	0	1	.60	.48	39.3	60.7
QUES21	0	1	.61	.48	38.5	61.5
QUES22	0	1	.61	.48	38.4	61.6
QUES23	0	1	.65	.47	34.4	65.6
QUES24	0	1	.58	.49	41.1	58.9
QUES25	0	1	.55	.49	44.7	55.3
QUES26	0	1	.48	.49	51.3	48.7
QUES27	0	1	.49	.50	50.2	49.8
QUES28	0	1	.60	.48	39.9	60.1
QUES29	0	1	.46	.49	53.8	46.2
QUES30	0	1	.57	.49	43.0	57.0

APPENDIX O

DESCRIPTIVE STATISTICS FOR SCORES ON THE SUBDOMAINS OF TDT FOR SAMPLE 2

Table O.1 Descriptive statistics for scores on the subdomains of TDT for Sample 2

Subdomain	Min	Max	Mean	SD	Skewness	Kurtosis
ENACTHK	0	5	2.78	1.43	-.02	-.91
ICONTHK	0	5	2.85	1.43	-.13	-.85
ALGOTHK	0	5	2.86	1.47	-.15	-.95
ALGETHK	0	4	1.96	1.20	.11	-.93
FORMTHK	0	6	3.45	1.44	-.03	-.61
AXIOTHK	0	5	2.84	1.38	-.11	-.87

APPENDIX P

DESCRIPTIVE STATISTICS FOR TOTAL SCORES ON TDT FOR SAMPLE 2

Table P.1 Descriptive statistics for total scores on TDT for Sample 2

Total Score	Min	Max	Mean	SD	Skewness	Kurtosis
MATTHK	3	30	16.76	5.57	.36	-.11

APPENDIX Q

THE MPLUS INPUT FILE FOR THE STEP 1

```
TITLE:     Intraclass Correlation Coefficients MSEM
DATA:      FILE IS msemave.dat;
           VARIANCES = NOCHECK;
           VARIABLE:  NAMES ARE class ques1-ques30;
                      USEVARIABLES ARE ques1-ques30;
                      CATEGORICAL ARE ques1-ques30;
                      CLUSTER IS class;
ANALYSIS:  TYPE IS TWOLEVEL BASIC;
           ESTIMATOR IS WLSM;
OUTPUT:    SAMPSTAT STANDARDIZED;
```

APPENDIX R

THE MPLUS INPUT FILE FOR THE STEP 2

```
TITLE:     Two-Level Exploratory Factor Analysis
DATA:      FILE IS mefa.dat;
           VARIANCES = NOCHECK;
VARIABLE:  NAMES ARE class ques1-ques30;
           USEVARIABLES ARE class ques1-ques30;
           CATEGORICAL ARE ques1-ques30;
           CLUSTER IS class;
ANALYSIS:  TYPE = TWOLEVEL EFA 1 6 UW 1 3 UB;
           ESTIMATOR IS WLSM;
           ROTATION = GEOMIN;
```

APPENDIX S

THE MPLUS INPUT FILE FOR THE STEP 3

```
TITLE: Two-Level Confirmatory Factor Analysis
DATA: FILE IS mcfa.dat;
        VARIANCES = NOCHECK;
VARIABLE: NAMES ARE class ques1-ques30;
            USEVARIABLES ARE class ques1-ques30;
            CATEGORICAL ARE ques1-ques30;
            CLUSTER IS class;
ANALYSIS: TYPE = TWOLEVEL;
            ESTIMATOR IS WLSM;
MODEL:

      %WITHIN%
      formthkw BY ques3@1 ques4 ques6 ques2 ques5 ques1;
      axiothkw BY ques9@1 ques11 ques7 ques8 ques10;
      algethkw BY ques14@1 ques13 ques12 ques15;
      iconthkw BY ques20@1 ques16 ques19 ques18 ques17;
      algothkw BY ques24@1 ques22 ques26 ques25 ques21;
      enacthkw BY ques29@1 ques23 ques28 ques27 ques30;

      %BETWEEN%
      formaxthk BY ques5@1 ques10 ques2 ques3 ques9 ques8
      ques4 ques11 ques7 ques1 ques6;
      procepthk BY ques13@1 ques22 ques21 ques25
      ques24 ques14 ques15 ques26 ques12;
      concpthk BY ques18@1 ques16 ques23 ques28 ques20
      ques30 ques29 ques27 ques19 ques17;
 OUTPUT: SAMPSTAT STANDARDIZED;
```

APPENDIX T

THE MPLUS INPUT FILE FOR THE STEP 4

```
TITLE:    Two-Level Structural Equation Modeling
DATA:     FILE IS msem.dat;
          VARIANCES = NOCHECK;
VARIABLE: NAMES ARE class ques1-ques30;
          USEVARIABLES ARE class ques1-ques30;
          CATEGORICAL ARE ques1-ques30;
          CLUSTER IS class;
ANALYSIS:
          TYPE = TWOLEVEL;
          ESTIMATOR = MLM;
MODEL:
      %WITHIN%
      formthk BY ques3@1 ques4 ques6 ques2 ques5 ques1;
      axiothk BY ques9@1 ques11 ques7 ques8 ques10;
      algethk BY ques14@1 ques13 ques12 ques15;
      iconthk BY ques20@1 ques16 ques19 ques18 ques17;
      algothk BY ques24@1 ques22 ques26 ques25 ques21;
      enacthk BY ques29@1 ques23 ques28 ques27 ques30;
      iconthk ON formthk;
      algothk ON formthk;
      enacthk ON formthk;
      algethk ON formthk;
      axiothk ON formthk;
      enacthk ON algothk;
      iconthk ON algothk;
      algethk ON algothk;
      iconthk ON enacthk;
      algethk ON enacthk;
      axiothk ON enacthk;
      algethk ON iconthk;
      axiothk ON iconthk;
      algethk ON axiothk;
      %BETWEEN%
      formaxthk BY ques5@1 ques10 ques2 ques3 ques9 ques8
      ques4 ques11 ques7 ques1 ques6;
      procepthk BY ques13@1 ques22 ques21 ques25
      ques24 ques14 ques15 ques26 ques12;
      concpthk BY ques18@1 ques16 ques23 ques28 ques20
      ques30 ques29 ques27 ques19 ques17;
```

```
                formaxthk ON concpthk;
                procpthk ON formaxthk;
                concpthk ON procpthk;
    MODEL INDIRECT:
                enacthk IND algothk formthk;
                iconthk IND algothk formthk;
                algethk IND algothk formthk;
                iconthk IND enacthk formthk;
                algethk IND enacthk formthk;
                axiothk IND enacthk formthk;
                algethk IND iconthk formthk;
                axiothk IND iconthk formthk;
                iconthk IND enacthk algothk;
                algethk IND enacthk algothk;
                axiothk IND enacthk algothk;
                algethk IND iconthk algothk;
                axiothk IND iconthk algothk;
OUTPUT: SAMPSTAT STANDARDIZED MODINDICES (3.84);
```

APPENDIX U

THE MPLUS INPUT FILES FOR THE STEP 5

CROSS-LEVEL MODELS AT THE WITHIN-CLASSROOM LEVEL

MODEL 1

```
TITLE:    Cross-Level Effects
DATA:     FILE IS msemagg.dat;
          VARIANCES = NOCHECK;
VARIABLE:  NAMES ARE class ques1-ques30;
          USEVARIABLES ARE class ques18 ques16 ques23 ques28
          ques20 form alge;
          WITHIN = form alge;
          CATEGORICAL ARE ques18 ques16 ques23 ques28 ques20;
          CLUSTER IS class;
ANALYSIS:
         TYPE = TWOLEVEL RANDOM;
         ALGORITHM = INTEGRATION;
MODEL:
        %WITHIN%
        concptw BY ques18 ques16 ques23 ques28 ques20;
        form BY ques3@1 ques4 ques6 ques2 ques5 ques1;
        alge BY ques14@1 ques13 ques12 ques15;
        int | concptw XWITH form;
        alge ON form int;
        %BETWEEN%
        concpt BY ques18 ques16 ques23 ques28 ques20;
 OUTPUT: SAMPSTAT;
```

MODEL 2

```
TITLE:    Cross-Level Effects
DATA:     FILE IS msemagg.dat;
          VARIANCES = NOCHECK;
VARIABLE:  NAMES ARE class ques1-ques30;
          USEVARIABLES ARE class ques5 ques10 ques2 ques3 ques9
          form alge;
          WITHIN = form alge;
          CATEGORICAL ARE ques5 ques10 ques2 ques3 ques9;
```

```
        CLUSTER IS class;
ANALYSIS:
        TYPE = TWOLEVEL RANDOM;
        ALGORITHM = INTEGRATION;
MODEL:
        %WITHIN%
        formaxw BY ques5 ques10 ques2 ques3 ques9;
        form BY ques3@1 ques4 ques6 ques2 ques5 ques1;
        alge BY ques14@1 ques13 ques12 ques15;
        int | formaxw XWITH form;
        alge ON form int;
        %BETWEEN%
        formax BY ques5 ques10 ques2 ques3 ques9;
OUTPUT: SAMPSTAT;
```

MODEL 3

```
TITLE:    Cross-Level Effects
DATA:     FILE IS msemagg.dat;
        VARIANCES = NOCHECK;
VARIABLE:  NAMES ARE class ques1-ques30;
        USEVARIABLES ARE class ques13 ques22 ques21 ques25
        ques24 form alge;
        WITHIN = form alge;
        CATEGORICAL ARE ques13 ques22 ques21 ques25 ques24;
        CLUSTER IS class;
ANALYSIS:
        TYPE = TWOLEVEL RANDOM;
        ALGORITHM = INTEGRATION;
MODEL:
        %WITHIN%
        procptw BY ques13 ques22 ques21 ques25 ques24;
        form BY ques3@1 ques4 ques6 ques2 ques5 ques1;
        alge BY ques14@1 ques13 ques12 ques15;
        int | procptw XWITH form;
        alge ON form int;
        %BETWEEN%
        procpt BY ques13 ques22 ques21 ques25 ques24;
OUTPUT: SAMPSTAT;
```

MODEL 4

```
TITLE:    Cross-Level Effects
DATA:     FILE IS msemagg.dat;
        VARIANCES = NOCHECK;
VARIABLE:  NAMES ARE class ques1-ques30;
        USEVARIABLES ARE class ques18 ques16 ques23 ques28
        ques20 form algo;
        WITHIN = form algo;
        CATEGORICAL ARE ques18 ques16 ques23 ques28 ques20;
```

```
CLUSTER IS class;
ANALYSIS:
      TYPE = TWOLEVEL RANDOM;
      ALGORITHM = INTEGRATION;
MODEL:
      %WITHIN%
        concptw BY ques18 ques16 ques23 ques28 ques20;
        form BY ques3@1 ques4 ques6 ques2 ques5 ques1;
        algo BY ques24@1 ques22 ques26 ques25 ques21;
      int | concptw XWITH form;
      algo ON form int;
      %BETWEEN%
      concpt BY ques18 ques16 ques23 ques28 ques20;
OUTPUT: SAMPSTAT;
```

MODEL 5

```
TITLE:    Cross-Level Effects
DATA:     FILE IS msemagg.dat;
      VARIANCES = NOCHECK;
VARIABLE:  NAMES ARE class ques1-ques30;
      USEVARIABLES ARE class ques5 ques2 ques10 ques3 ques9
      form algo;
      WITHIN = form algo;
      CATEGORICAL ARE ques5 ques2 ques10 ques3 ques9;
      CLUSTER IS class;
ANALYSIS:
      TYPE = TWOLEVEL RANDOM;
      ALGORITHM = INTEGRATION;
MODEL:
      %WITHIN%
      formaxw BY ques5 ques10 ques2 ques3 ques9;
      form BY ques3@1 ques4 ques6 ques2 ques5 ques1;
      algo BY ques24@1 ques22 ques26 ques25 ques21;
      int | formaxw XWITH form;
      algo ON form int;
      %BETWEEN%
      formax BY ques5 ques2 ques10 ques3 ques9;
OUTPUT: SAMPSTAT;
```

MODEL 6

```
TITLE:    Cross-Level Effects
DATA:     FILE IS msemagg.dat;
      VARIANCES = NOCHECK;
VARIABLE:  NAMES ARE class ques1-ques30;
      USEVARIABLES ARE class ques13 ques22 ques21 ques25
      ques24 form algo;
      WITHIN = form algo;
      CATEGORICAL ARE ques13 ques22 ques21 ques25 ques24;
```

```
CLUSTER IS class;
ANALYSIS:
    TYPE = TWOLEVEL RANDOM;
    ALGORITHM = INTEGRATION;
MODEL:
    %WITHIN%
    procptw BY ques13@1 ques22 ques21 ques25 ques24;
    form BY ques3@1 ques4 ques6 ques2 ques5 ques1;
    algo BY ques24@1 ques22 ques26 ques25 ques21;
    int | procptw XWITH form;
    algo ON form int;
    %BETWEEN%
    procpt BY ques13@1 ques22 ques21 ques25 ques24;
OUTPUT: SAMPSTAT
```

MODEL 7

```
TITLE:   Cross-Level Effects
DATA:    FILE IS msemagg.dat;
    VARIANCES = NOCHECK;
VARIABLE:  NAMES ARE class ques1-ques30;
    USEVARIABLES ARE class ques18 ques16 ques23 ques28
    ques20
    form axio;
    WITHIN = form axio;
    CATEGORICAL ARE ques18 ques16 ques23 ques28 ques20;
    CLUSTER IS class;
ANALYSIS:
    TYPE = TWOLEVEL RANDOM;
    ALGORITHM = INTEGRATION;
MODEL:
    %WITHIN%
    concptw BY ques18 ques16 ques23 ques28 ques20;
    int | concptw XWITH form;
    axio ON form int;
    %BETWEEN%
    concpt BY ques18 ques16 ques23 ques28 ques20;
OUTPUT: SAMPSTAT;
```

MODEL 8

```
TITLE:   Cross-Level Effects
DATA:    FILE IS msemagg.dat;
    VARIANCES = NOCHECK;
VARIABLE:  NAMES ARE class ques1-ques30;
    USEVARIABLES ARE class ques5 ques10 ques2 ques3 ques9
    form axio;
    WITHIN = form axio;
    CATEGORICAL ARE ques5 ques10 ques2 ques3 ques9;
    CLUSTER IS class;
```

```
ANALYSIS:
        TYPE = TWOLEVEL RANDOM;
        ALGORITHM = INTEGRATION;
MODEL:
       %WITHIN%
       formaxw BY ques5 ques10 ques2 ques3 ques9;
       form BY ques3@1 ques4 ques6 ques2 ques5 ques1;
       axio BY ques9@1 ques11 ques7 ques8 ques10;
       int | formaxw XWITH form;
       axio ON form int;
       %BETWEEN%
       formax BY ques5 ques10 ques2 ques3 ques9;
 OUTPUT: SAMPSTAT;
```

MODEL 9

```
TITLE:    Cross-Level Effects
DATA:      FILE IS msemagg.dat;
         VARIANCES = NOCHECK;
VARIABLE:  NAMES ARE class ques1-ques30;
         USEVARIABLES ARE class ques13 ques22 ques21 ques25
         ques24
         form axio;
         WITHIN = form axio;
         CATEGORICAL ARE ques13 ques22 ques21 ques25 ques24;
         CLUSTER IS class;
ANALYSIS:
        TYPE = TWOLEVEL RANDOM;
        ALGORITHM = INTEGRATION;
MODEL:
       %WITHIN%
       procptw BY ques13 ques22 ques21 ques25 ques24;
       form BY ques3@1 ques4 ques6 ques2 ques5 ques1;
       axio BY ques9@1 ques11 ques7 ques8 ques10;
       int | procptw XWITH form;
       axio ON form int;
       %BETWEEN%
       procpt BY ques13 ques22 ques21 ques25 ques24;
 OUTPUT: SAMPSTAT;
```

MODEL 10

```
TITLE:    Cross-Level Effects
DATA:      FILE IS msemagg.dat;
         VARIANCES = NOCHECK;
VARIABLE:  NAMES ARE class ques1-ques30;
         USEVARIABLES ARE class ques18 ques16 ques23 ques28
         ques20
         form enac;
         WITHIN = form enac;
```

```
        CATEGORICAL ARE ques18 ques16 ques23 ques28 ques20;
        CLUSTER IS class;
ANALYSIS:
        TYPE = TWOLEVEL RANDOM;
        ALGORITHM = INTEGRATION;
MODEL:
      %WITHIN%
      concptw BY ques18 ques16 ques23 ques28 ques20;
        form BY ques3@1 ques4 ques6 ques2 ques5 ques1;
        enac BY ques29@1 ques23 ques28 ques27 ques30;
      int | concptw XWITH form;
      enac ON form int;
      %BETWEEN%
      concpt BY ques18 ques16 ques23 ques28 ques20;
 OUTPUT: SAMPSTAT;
```

MODEL 11

```
TITLE:    Cross-Level Effects
DATA:     FILE IS msemagg.dat;
        VARIANCES = NOCHECK;
VARIABLE:  NAMES ARE class ques1-ques30;
        USEVARIABLES ARE class ques5 ques10 ques2 ques3 ques9
        form enac;
        WITHIN = form enac;
        CATEGORICAL ARE ques5 ques10 ques2 ques3 ques9;
        CLUSTER IS class;
ANALYSIS:
        TYPE = TWOLEVEL RANDOM;
        ALGORITHM = INTEGRATION;
MODEL:
      %WITHIN%
      formaxw BY ques5 ques10 ques2 ques3 ques9;
      form BY ques3@1 ques4 ques6 ques2 ques5 ques1;
      enac BY ques29@1 ques23 ques28 ques27 ques30;
      int | formaxw XWITH form;
      enac ON form int;
      %BETWEEN%
      formax BY ques5 ques10 ques2 ques3 ques9;
 OUTPUT: SAMPSTAT;
```

MODEL 12

```
TITLE:    Cross-Level Effects
DATA:     FILE IS msemagg.dat;
        VARIANCES = NOCHECK;
VARIABLE:  NAMES ARE class ques1-ques30;
        USEVARIABLES ARE class ques13 ques22 ques21 ques25
        ques24
        form enac;
```

```
WITHIN = form enac;
CATEGORICAL ARE ques13 ques22 ques21 ques25 ques24;
CLUSTER IS class;
ANALYSIS:
TYPE = TWOLEVEL RANDOM;
ALGORITHM = INTEGRATION;
MODEL:
%WITHIN%
procptw BY ques13 ques22 ques21 ques25 ques24;
form BY ques3@1 ques4 ques6 ques2 ques5 ques1;
enac BY ques29@1 ques23 ques28 ques27 ques30;
int | procptw XWITH form;
enac ON form int;
%BETWEEN%
procpt BY ques13 ques22 ques21 ques25 ques24;
OUTPUT: SAMPSTAT;
```

MODEL 13

```
TITLE: Cross-Level Effects
DATA: FILE IS msemagg.dat;
VARIANCES = NOCHECK;
VARIABLE: NAMES ARE class ques1-ques30;
USEVARIABLES ARE class ques18 ques16 ques23 ques28
ques20
form icon;
WITHIN = form icon;
CATEGORICAL ARE ques18 ques16 ques23 ques28 ques20;
CLUSTER IS class;
ANALYSIS:
TYPE = TWOLEVEL RANDOM;
ALGORITHM = INTEGRATION;
MODEL:
%WITHIN%
concptw BY ques18 ques16 ques23 ques28 ques20;
int | concptw XWITH form;
icon ON form int;
%BETWEEN%
concpt BY ques18 ques16 ques23 ques28 ques20;
OUTPUT: SAMPSTAT;
```

MODEL 14

```
TITLE: Cross-Level Effects
DATA: FILE IS msemagg.dat;
VARIANCES = NOCHECK;
VARIABLE: NAMES ARE class ques1-ques30;
USEVARIABLES ARE class ques5 ques10 ques2 ques3 ques9
form icon;
WITHIN = form icon;
```

```
        CATEGORICAL ARE ques5 ques10 ques2 ques3 ques9;
        CLUSTER IS class;
ANALYSIS:
        TYPE = TWOLEVEL RANDOM;
        ALGORITHM = INTEGRATION;
MODEL:
       %WITHIN%
       formaxw BY ques5 ques10 ques2 ques3 ques9;
       form BY ques3@1 ques4 ques6 ques2 ques5 ques1;
       icon BY ques20@1 ques16 ques19 ques18 ques17;
       int | formaxw XWITH form;
       icon ON form int;
       %BETWEEN%
       formax BY ques5 ques10 ques2 ques3 ques9;
OUTPUT: SAMPSTAT;
```

MODEL 15

```
TITLE:    Cross-Level Effects
DATA:     FILE IS msemagg.dat;
        VARIANCES = NOCHECK;
VARIABLE:  NAMES ARE class ques1-ques30;
        USEVARIABLES ARE class ques13 ques22 ques21 ques25
        ques24
        form icon;
        WITHIN = form icon;
        CATEGORICAL ARE ques13 ques22 ques21 ques25 ques24;
        CLUSTER IS class;
ANALYSIS:
        TYPE = TWOLEVEL RANDOM;
        ALGORITHM = INTEGRATION;
MODEL:
       %WITHIN%
       procptw BY ques13 ques22 ques21 ques25 ques24;
       form BY ques3@1 ques4 ques6 ques2 ques5 ques1;
       icon BY ques20@1 ques16 ques19 ques18 ques17;
       int | procptw XWITH form;
       icon ON form int;
       %BETWEEN%
       procpt BY ques13 ques22 ques21 ques25 ques24;
OUTPUT: SAMPSTAT;
```

MODEL 16

```
TITLE:    Cross-Level Effects
DATA:     FILE IS msemagg.dat;
        VARIANCES = NOCHECK;
VARIABLE:  NAMES ARE class ques1-ques30;
        USEVARIABLES ARE class ques13 ques22 ques21 ques25
       ques24
```

```
          icon alge;
          WITHIN = icon alge;
          CATEGORICAL ARE ques13 ques22 ques21 ques25 ques24;
          CLUSTER IS class;
ANALYSIS:
         TYPE = TWOLEVEL RANDOM;
         ALGORITHM = INTEGRATION;
         MITERATIONS = 1000000;
MODEL:
        %WITHIN%
        procptw BY ques13 ques22 ques21 ques25 ques24;
        icon BY ques20@1 ques16 ques19 ques18 ques17;
        alge BY ques14@1 ques13 ques12 ques15;
        int | procptw XWITH icon;
        alge ON icon int;
        %BETWEEN%
        procpt BY ques13 ques22 ques21 ques25 ques24;
 OUTPUT: SAMPSTAT;
```

MODEL 17

```
TITLE:    Cross-Level Effects
DATA:     FILE IS msemagg.dat;
          VARIANCES = NOCHECK;
VARIABLE:  NAMES ARE class ques1-ques30;
          USEVARIABLES ARE class ques13 ques22 ques21 ques25
         ques24
          icon alge;
          WITHIN = icon alge;
          CATEGORICAL ARE ques13 ques22 ques21 ques25 ques24;
          CLUSTER IS class;
ANALYSIS:
         TYPE = TWOLEVEL RANDOM;
         ALGORITHM = INTEGRATION;
         MITERATIONS = 1000000;
MODEL:
        %WITHIN%
        formaxw BY ques5 ques10 ques2 ques3 ques9;
        icon BY ques20@1 ques16 ques19 ques18 ques17;
        alge BY ques14@1 ques13 ques12 ques15;
        int | formaxw XWITH icon;
        alge ON icon int;
        %BETWEEN%
        formax BY ques5 ques10 ques2 ques3 ques9;
 OUTPUT: SAMPSTAT;
```

MODEL 18

```
TITLE:    Cross-Level Effects
DATA:     FILE IS msemagg.dat;
```

```
VARIANCES = NOCHECK;
VARIABLE:  NAMES ARE class ques1-ques30;
        USEVARIABLES ARE class ques13 ques22 ques21 ques25
        ques24
        icon alge;
        WITHIN = icon alge;
        CATEGORICAL ARE ques13 ques22 ques21 ques25 ques24;
        CLUSTER IS class;
ANALYSIS:
       TYPE = TWOLEVEL RANDOM;
       ALGORITHM = INTEGRATION;
       MITERATIONS = 1000000;
MODEL:
      %WITHIN%
      concptw BY ques18 ques16 ques23 ques28 ques20;
      icon BY ques20@1 ques16 ques19 ques18 ques17;
      alge BY ques14@1 ques13 ques12 ques15;
      int | concptw XWITH icon;
      alge ON icon int;
      %BETWEEN%
      concpt BY ques18 ques16 ques23 ques28 ques20;
OUTPUT: SAMPSTAT;
```

MODEL 19

```
TITLE:    Cross-Level Effects
DATA:     FILE IS msemagg.dat;
        VARIANCES = NOCHECK;
VARIABLE:  NAMES ARE class ques1-ques30;
        USEVARIABLES ARE class ques18 ques16 ques23 ques28
        ques20
        icon axio;
        WITHIN = icon axio;
        CATEGORICAL ARE ques18 ques16 ques23 ques28 ques20;
        CLUSTER IS class;
ANALYSIS:
        TYPE = TWOLEVEL RANDOM;
        ALGORITHM = INTEGRATION;
MODEL:
      %WITHIN%
      concptw BY ques18 ques16 ques23 ques28 ques20;
      icon BY ques20@1 ques16 ques19 ques18 ques17;
      axio BY ques9@1 ques11 ques7 ques8 ques10;
      int | concptw XWITH icon;
      axio ON icon int;
      %BETWEEN%
      concpt BY ques18 ques16 ques23 ques28 ques20;
OUTPUT: SAMPSTAT;
```

MODEL 20

```
TITLE:    Cross-Level Effects
DATA:     FILE IS msemagg.dat;
          VARIANCES = NOCHECK;
VARIABLE:  NAMES ARE class ques1-ques30;
          USEVARIABLES ARE class ques5 ques10 ques2 ques3 ques9
          icon axio;
          WITHIN = icon axio;
          CATEGORICAL ARE ques5 ques10 ques2 ques3 ques9;
          CLUSTER IS class;
ANALYSIS:
         TYPE = TWOLEVEL RANDOM;
         ALGORITHM = INTEGRATION;
MODEL:
        %WITHIN%
        formaxw BY ques5 ques10 ques2 ques3 ques9;
        icon BY ques20@1 ques16 ques19 ques18 ques17;
        axio BY ques9@1 ques11 ques7 ques8 ques10;
        int | formaxw XWITH icon;
        axio ON icon int;
        %BETWEEN%
        formax BY ques5 ques10 ques2 ques3 ques9;
 OUTPUT: SAMPSTAT;
```

MODEL 21

```
TITLE:    Cross-Level Effects
DATA:     FILE IS msemagg.dat;
          VARIANCES = NOCHECK;
VARIABLE:  NAMES ARE class ques1-ques30;
          USEVARIABLES ARE class ques13 ques22 ques21 ques25
         ques24
          icon axio;
          WITHIN = icon axio;
          CATEGORICAL ARE ques13 ques22 ques21 ques25 ques24;
          CLUSTER IS class;
ANALYSIS:
         TYPE = TWOLEVEL RANDOM;
         ALGORITHM = INTEGRATION;
MODEL:
        %WITHIN%
        procptw BY ques13 ques22 ques21 ques25 ques24;
        icon BY ques20@1 ques16 ques19 ques18 ques17;
        axio BY ques9@1 ques11 ques7 ques8 ques10;
        int | procptw XWITH icon;
        axio ON icon int;
        %BETWEEN%
        procpt BY ques13 ques22 ques21 ques25 ques24;
 OUTPUT: SAMPSTAT;
```

MODEL 22

```
TITLE:    Cross-Level Effects
DATA:     FILE IS msemagg.dat;
        VARIANCES = NOCHECK;
VARIABLE:  NAMES ARE class ques1-ques30;
        USEVARIABLES ARE class ques18 ques16 ques23 ques28
        ques20
        enac alge;
        WITHIN = enac alge;
        CATEGORICAL ARE ques18 ques16 ques23 ques28 ques20;
        CLUSTER IS class;
ANALYSIS:
        TYPE = TWOLEVEL RANDOM;
        ALGORITHM = INTEGRATION;
MODEL:
       %WITHIN%
       concptw BY ques18 ques16 ques23 ques28 ques20;
       int | concptw XWITH enac;
       alge ON enac int;
       %BETWEEN%
       concpt BY ques18 ques16 ques23 ques28 ques20;
 OUTPUT: SAMPSTAT;
```

MODEL 23

```
TITLE:    Cross-Level Effects
DATA:     FILE IS msemagg.dat;
        VARIANCES = NOCHECK;
VARIABLE:  NAMES ARE class ques1-ques30;
        USEVARIABLES ARE class ques5 ques10 ques2 ques3 ques9
        enac alge;
        WITHIN = enac alge;
        CATEGORICAL ARE ques5 ques10 ques2 ques3 ques9;
        CLUSTER IS class;
ANALYSIS:
        TYPE = TWOLEVEL RANDOM;
        ALGORITHM = INTEGRATION;
MODEL:
       %WITHIN%
       formaxw BY ques5 ques10 ques2 ques3 ques9;
       alge BY ques14@1 ques13 ques12 ques15;
       enac BY ques29@1 ques23 ques28 ques27 ques30;
       int | formaxw XWITH enac;
       alge ON enac int;
       %BETWEEN%
       formax BY ques5 ques10 ques2 ques3 ques9;
 OUTPUT: SAMPSTAT;
```

MODEL 24

```
TITLE:    Cross-Level Effects
DATA:     FILE IS msemagg.dat;
          VARIANCES = NOCHECK;
VARIABLE:  NAMES ARE class ques1-ques30;
          USEVARIABLES ARE class ques13 ques22 ques21 ques25
          ques24
          enac alge;
          WITHIN = enac alge;
          CATEGORICAL ARE ques13 ques22 ques21 ques25 ques24;
          CLUSTER IS class;
ANALYSIS:
         TYPE = TWOLEVEL RANDOM;
         ALGORITHM = INTEGRATION;
MODEL:
        %WITHIN%
        procptw BY ques13 ques22 ques21 ques25 ques24;
        alge BY ques14@1 ques13 ques12 ques15;
        enac BY ques29@1 ques23 ques28 ques27 ques30;
        int | procptw XWITH enac;
        alge ON enac int;
        %BETWEEN%
        procpt BY ques13 ques22 ques21 ques25 ques24;
 OUTPUT: SAMPSTAT;
```

MODEL 25

```
TITLE:    Cross-Level Effects
DATA:     FILE IS msemagg.dat;
          VARIANCES = NOCHECK;
VARIABLE:  NAMES ARE class ques1-ques30;
          USEVARIABLES ARE class ques18 ques16 ques23 ques28
          ques20
          enac axio;
          WITHIN = enac axio;
          CATEGORICAL ARE ques18 ques16 ques23 ques28 ques20;
          CLUSTER IS class;
ANALYSIS:
         TYPE = TWOLEVEL RANDOM;
         ALGORITHM = INTEGRATION;
MODEL:
        %WITHIN%
        concptw BY ques18 ques16 ques23 ques28 ques20;
        enac BY ques29@1 ques23 ques28 ques27 ques30;
        axio BY ques9@1 ques11 ques7 ques8 ques10;
        int | concptw XWITH enac;
        axio ON enac int;
        %BETWEEN%
        concpt BY ques18 ques16 ques23 ques28 ques20;
```

```
OUTPUT: SAMPSTAT;
```

MODEL 26

```
TITLE:    Cross-Level Effects
DATA:     FILE IS msemagg.dat;
          VARIANCES = NOCHECK;
VARIABLE:  NAMES ARE class ques1-ques30;
          USEVARIABLES ARE class ques5 ques10 ques2 ques3 ques9
          enac axio;
          WITHIN = enac axio;
          CATEGORICAL ARE ques5 ques10 ques2 ques3 ques9;
          CLUSTER IS class;
ANALYSIS:
         TYPE = TWOLEVEL RANDOM;
         ALGORITHM = INTEGRATION;
MODEL:
        %WITHIN%
        formaxw BY ques5 ques10 ques2 ques3 ques9;
        enac BY ques29@1 ques23 ques28 ques27 ques30;
        axio BY ques9@1 ques11 ques7 ques8 ques10;
        int | formaxw XWITH enac;
        axio ON enac int;
        %BETWEEN%
        formax BY ques5 ques10 ques2 ques3 ques9;
 OUTPUT: SAMPSTAT;
```

MODEL 27

```
TITLE:    Cross-Level Effects
DATA:     FILE IS msemagg.dat;
          VARIANCES = NOCHECK;
VARIABLE:  NAMES ARE class ques1-ques30;
          USEVARIABLES ARE class ques13 ques22 ques21 ques25
          ques24
          enac axio;
          WITHIN = enac axio;
          CATEGORICAL ARE ques13 ques22 ques21 ques25 ques24;
          CLUSTER IS class;
ANALYSIS:
         TYPE = TWOLEVEL RANDOM;
         ALGORITHM = INTEGRATION;
MODEL:
        %WITHIN%
        procptw BY ques13 ques22 ques21 ques25 ques24;
        enac BY ques29@1 ques23 ques28 ques27 ques30;
        axio BY ques9@1 ques11 ques7 ques8 ques10;
        int | procptw XWITH enac;
        axio ON enac int;
        %BETWEEN%
```

```
procpt BY ques13 ques22 ques21 ques25 ques24;
OUTPUT: SAMPSTAT;
```

MODEL 28

```
TITLE:    Cross-Level Effects
DATA:     FILE IS msemagg.dat;
          VARIANCES = NOCHECK;
VARIABLE:  NAMES ARE class ques1-ques30 form axio icon algo
          alge enac
          agg1-agg30 mathk formagg axioagg iconagg algoagg
          algeagg enacagg formax procpt concpt formaxag
          procptag concptag;
          USEVARIABLES ARE class ques18 ques16 ques23 ques28
          ques20
          enac icon;
          WITHIN = enac icon;
          CATEGORICAL ARE ques18 ques16 ques23 ques28 ques20;
          CLUSTER IS class;
ANALYSIS:
         TYPE = TWOLEVEL RANDOM;
         ALGORITHM = INTEGRATION;
MODEL:
        %WITHIN%
        concptw BY ques18 ques16 ques23 ques28 ques20;
        alge BY ques14@1 ques13 ques12 ques15;
        enac BY ques29@1 ques23 ques28 ques27 ques30;
        int | concptw XWITH enac;
        icon ON enac int;
        %BETWEEN%
        concpt BY ques18 ques16 ques23 ques28 ques20;
 OUTPUT: SAMPSTAT;
```

MODEL 29

```
TITLE:    Cross-Level Effects
DATA:     FILE IS msemagg.dat;
          VARIANCES = NOCHECK;
VARIABLE:  NAMES ARE class ques1-ques30;
          USEVARIABLES ARE class ques5 ques10 ques2 ques3 ques9
          enac icon;
          WITHIN = enac icon;
          CATEGORICAL ARE ques5 ques10 ques2 ques3 ques9;
          CLUSTER IS class;
ANALYSIS:
         TYPE = TWOLEVEL RANDOM;
         ALGORITHM = INTEGRATION;
MODEL:
        %WITHIN%
        formaxw BY ques5 ques10 ques2 ques3 ques9;
```

```
        enac BY ques29@1 ques23 ques28 ques27 ques30;
        icon BY ques20@1 ques16 ques19 ques18 ques17;
        int | formaxw XWITH enac;
        icon ON enac int;
        %BETWEEN%
        formax BY ques5 ques10 ques2 ques3 ques9;
 OUTPUT: SAMPSTAT;
```

MODEL 30

```
TITLE:    Cross-Level Effects
DATA:     FILE IS msemagg.dat;
          VARIANCES = NOCHECK;
VARIABLE:  NAMES ARE class ques1-ques30;
          USEVARIABLES ARE class ques13 ques22 ques21 ques25
          ques24
          enac icon;
          WITHIN = enac icon;
          CATEGORICAL ARE ques13 ques22 ques21 ques25 ques24;
          CLUSTER IS class;
ANALYSIS:
         TYPE = TWOLEVEL RANDOM;
         ALGORITHM = INTEGRATION;
MODEL:
        %WITHIN%
        procptw BY ques13 ques22 ques21 ques25 ques24;
        enac BY ques29@1 ques23 ques28 ques27 ques30;
        icon BY ques20@1 ques16 ques19 ques18 ques17;
        int | procptw XWITH enac;
        icon ON enac int;
        %BETWEEN%
        procpt BY ques13 ques22 ques21 ques25 ques24;
 OUTPUT: SAMPSTAT;
```

MODEL 31

```
TITLE:    Cross-Level Effects
DATA:     FILE IS msemagg.dat;
          VARIANCES = NOCHECK;
VARIABLE:  NAMES ARE class ques1-ques30;
          USEVARIABLES ARE class ques18 ques16 ques23 ques28
          ques20
          algo alge;
          WITHIN = algo alge;
          CATEGORICAL ARE ques18 ques16 ques23 ques28 ques20;
          CLUSTER IS class;
ANALYSIS:
         TYPE = TWOLEVEL RANDOM;
         ALGORITHM = INTEGRATION;
MODEL:
```

```
    %WITHIN%
    concptw BY ques18 ques16 ques23 ques28 ques20;
    algo BY ques24@1 ques22 ques26 ques25 ques21;
    alge BY ques14@1 ques13 ques12 ques15;
    int | concptw XWITH algo;
    alge ON algo int;
    %BETWEEN%
    concpt BY ques18 ques16 ques23 ques28 ques20;
OUTPUT: SAMPSTAT;
```

MODEL 32

```
TITLE:    Cross-Level Effects
DATA:     FILE IS msemagg.dat;
          VARIANCES = NOCHECK;
VARIABLE:  NAMES ARE class ques1-ques30;
          USEVARIABLES ARE class ques5 ques10 ques2 ques3 ques9
          algo alge;
          WITHIN = algo alge;
          CATEGORICAL ARE ques5 ques10 ques2 ques3 ques9;
          CLUSTER IS class;
ANALYSIS:
         TYPE = TWOLEVEL RANDOM;
         ALGORITHM = INTEGRATION;
MODEL:
    %WITHIN%
    formaxw BY ques5 ques10 ques2 ques3 ques9;
    algo BY ques24@1 ques22 ques26 ques25 ques21;
    alge BY ques14@1 ques13 ques12 ques15;
    int | formaxw XWITH algo;
    alge ON algo int;
    %BETWEEN%
    formax BY ques5 ques10 ques2 ques3 ques9;
OUTPUT: SAMPSTAT;
```

MODEL 33

```
TITLE:    Cross-Level Effects
DATA:     FILE IS msemagg.dat;
          VARIANCES = NOCHECK;
VARIABLE:  NAMES ARE class ques1-ques30;
          USEVARIABLES ARE class ques13 ques22 ques21 ques25
          ques24
          algo alge;
          WITHIN = algo alge;
          CATEGORICAL ARE ques13 ques22 ques21 ques25 ques24;
          CLUSTER IS class;
ANALYSIS:
         TYPE = TWOLEVEL RANDOM;
         ALGORITHM = INTEGRATION;
```

```
MODEL:
      %WITHIN%
      procptw BY ques13 ques22 ques21 ques25 ques24;
      algo BY ques24@1 ques22 ques26 ques25 ques21;
      alge BY ques14@1 ques13 ques12 ques15;
      int | procptw XWITH algo;
      alge ON algo int;
      %BETWEEN%
      procpt BY ques13 ques22 ques21 ques25 ques24;
 OUTPUT: SAMPSTAT;
```

MODEL 34

```
TITLE:    Cross-Level Effects
DATA:     FILE IS msemagg.dat;
          VARIANCES = NOCHECK;
VARIABLE: NAMES ARE class ques1-ques30;
          USEVARIABLES ARE class ques18 ques16 ques23
          ques28 ques20
          algo enac;
          WITHIN = algo enac;
          CATEGORICAL ARE ques18 ques16 ques23 ques28
          ques20;
          CLUSTER IS class;
ANALYSIS:
         TYPE = TWOLEVEL RANDOM;
         ALGORITHM = INTEGRATION;
MODEL:
        %WITHIN%
        concptw BY ques18 ques16 ques23 ques28 ques20;
        algo BY ques24@1 ques22 ques26 ques25 ques21;
        enac BY ques29@1 ques23 ques28 ques27 ques30;
        int | concptw XWITH algo;
        enac ON algo int;
        %BETWEEN%
        concpt BY ques18 ques16 ques23 ques28 ques20;
 OUTPUT: SAMPSTAT;
```

MODEL 35

```
TITLE:    Cross-Level Effects
DATA:     FILE IS msemagg.dat;
        VARIANCES = NOCHECK;
VARIABLE: NAMES ARE class ques1-ques30;
        USEVARIABLES ARE class ques5 ques10 ques2 ques3 ques9
        algo enac;
        WITHIN = algo enac;
        CATEGORICAL ARE ques5 ques10 ques2 ques3 ques9;
```

```
        CLUSTER IS class;
ANALYSIS:
       TYPE = TWOLEVEL RANDOM;
       ALGORITHM = INTEGRATION;
MODEL:
      %WITHIN%
      formaxw BY ques5 ques10 ques2 ques3 ques9;
      algo BY ques24@1 ques22 ques26 ques25 ques21;
      enac BY ques29@1 ques23 ques28 ques27 ques30;
      int | formaxw XWITH algo;
      enac ON algo int;
      %BETWEEN%
      formax BY ques5 ques10 ques2 ques3 ques9;
 OUTPUT: SAMPSTAT;
```

MODEL 36

```
TITLE:   Cross-Level Effects
DATA:    FILE IS msemagg.dat;
        VARIANCES = NOCHECK;
VARIABLE:  NAMES ARE class ques1-ques30;
        USEVARIABLES ARE class ques13 ques22 ques21
        ques25 ques24 algo enac;
        WITHIN = algo enac;
        CATEGORICAL ARE ques13 ques22 ques21 ques25 ques24;
        CLUSTER IS class;
ANALYSIS:
       TYPE = TWOLEVEL RANDOM;
       ALGORITHM = INTEGRATION;
MODEL:
      %WITHIN%
      procptw BY ques13 ques22 ques21 ques25 ques24;
      algo BY ques24@1 ques22 ques26 ques25 ques21;
      enac BY ques29@1 ques23 ques28 ques27 ques30;
      int | procptw XWITH algo;
      enac ON algo int;
      %BETWEEN%
      procpt BY ques13 ques22 ques21 ques25 ques24;
 OUTPUT: SAMPSTAT;
```

MODEL 37

```
TITLE:   Cross-Level Effects
DATA:    FILE IS msemagg.dat;
        VARIANCES = NOCHECK;
VARIABLE:  NAMES ARE class ques1-ques30;
        USEVARIABLES ARE class ques18 ques16 ques23 ques28
        ques20
        algo icon;
        WITHIN = algo icon;
```

```
        CATEGORICAL ARE ques18 ques16 ques23 ques28 ques20;
        CLUSTER IS class;
ANALYSIS:
        TYPE = TWOLEVEL RANDOM;
        ALGORITHM = INTEGRATION;
MODEL:
       %WITHIN%
       concptw BY ques18 ques16 ques23 ques28 ques20;
       icon BY ques20@1 ques16 ques19 ques18 ques17;
       algo BY ques24@1 ques22 ques26 ques25 ques21;
       int | concptw XWITH algo;
       icon ON algo int;
       %BETWEEN%
       concpt BY ques18 ques16 ques23 ques28 ques20;
 OUTPUT: SAMPSTAT;
```

MODEL 38

```
TITLE:    Cross-Level Effects
DATA:     FILE IS msemagg.dat;
        VARIANCES = NOCHECK;
VARIABLE:  NAMES ARE class ques1-ques30;
        USEVARIABLES ARE class ques5 ques10 ques2 ques3 ques9
        algo icon;
        WITHIN = algo icon;
        CATEGORICAL ARE ques5 ques10 ques2 ques3 ques9;
        CLUSTER IS class;
ANALYSIS:
        TYPE = TWOLEVEL RANDOM;
        ALGORITHM = INTEGRATION;
MODEL:
       %WITHIN%
       formaxw BY ques5 ques10 ques2 ques3 ques9;
       icon BY ques20@1 ques16 ques19 ques18 ques17;
       algo BY ques24@1 ques22 ques26 ques25 ques21;
       int | formaxw XWITH algo;
       icon ON algo int;
       %BETWEEN%
       formax BY ques5 ques10 ques2 ques3 ques9;
 OUTPUT: SAMPSTAT;
```

MODEL 39

```
TITLE:    Cross-Level Effects
DATA:     FILE IS msemagg.dat;
        VARIANCES = NOCHECK;
VARIABLE:  NAMES ARE class ques1-ques30;
        USEVARIABLES ARE class ques13 ques22 ques21
        ques25 ques24 algo icon;
        WITHIN = algo icon;
```

```
    CATEGORICAL ARE ques13 ques22 ques21 ques25 ques24;
    CLUSTER IS class;
ANALYSIS:
    TYPE = TWOLEVEL RANDOM;
    ALGORITHM = INTEGRATION;
MODEL:
    %WITHIN%
    procptw BY ques13 ques22 ques21 ques25 ques24;
    icon BY ques20@1 ques16 ques19 ques18 ques17;
    algo BY ques24@1 ques22 ques26 ques25 ques21;
    int | procptw XWITH algo;
    icon ON algo int;
    %BETWEEN%
    procpt BY ques13 ques22 ques21 ques25 ques24;
OUTPUT: SAMPSTAT;
```

MODEL 40

```
TITLE:    Cross-Level Effects
DATA:     FILE IS msemagg.dat;
    VARIANCES = NOCHECK;
VARIABLE:  NAMES ARE class ques1-ques30;
    USEVARIABLES ARE class ques18 ques16 ques23
    ques28 ques20
    axio alge;
    WITHIN = axio alge;
    CATEGORICAL ARE ques18 ques16 ques23 ques28
    ques20;
    CLUSTER IS class;
ANALYSIS:
    TYPE = TWOLEVEL RANDOM;
    ALGORITHM = INTEGRATION;
MODEL:
    %WITHIN%
    concptw BY ques18 ques16 ques23 ques28 ques20;
    alge BY ques14@1 ques13 ques12 ques15;
    axio BY ques9@1 ques11 ques7 ques8 ques10;
    int | concptw XWITH axio;
    alge ON axio int;
    %BETWEEN%
    concpt BY ques18 ques16 ques23 ques28 ques20;
OUTPUT: SAMPSTAT;
```

MODEL 41

```
TITLE:    Cross-Level Effects
DATA:     FILE IS msemagg.dat;
```

```
        VARIANCES = NOCHECK;
VARIABLE:  NAMES ARE class ques1-ques30;
        USEVARIABLES ARE class ques5 ques10 ques2 ques3
        ques9
        axio alge;
        WITHIN = axio alge;
        CATEGORICAL ARE ques5 ques10 ques2 ques3 ques9;
        CLUSTER IS class;
ANALYSIS:
        TYPE = TWOLEVEL RANDOM;
        ALGORITHM = INTEGRATION;
MODEL:
      %WITHIN%
      formaxw BY ques5 ques10 ques2 ques3 ques9;
      alge BY ques14@1 ques13 ques12 ques15;
      axio BY ques9@1 ques11 ques7 ques8 ques10;
      int | formaxw XWITH axio;
      alge ON axio int;
      %BETWEEN%
      formax BY ques5 ques10 ques2 ques3 ques9;
 OUTPUT: SAMPSTAT;
```

MODEL 42

```
 TITLE:    Cross-Level Effects
DATA:     FILE IS msemagg.dat;
        VARIANCES = NOCHECK;
VARIABLE:  NAMES ARE class ques1-ques30;
        USEVARIABLES ARE class ques13 ques22 ques21
        ques25 ques24 axio alge;
        WITHIN = axio alge;
        CATEGORICAL ARE ques13 ques22 ques21 ques25
        ques24;
        CLUSTER IS class;
ANALYSIS:
        TYPE = TWOLEVEL RANDOM;
        ALGORITHM = INTEGRATION;
MODEL:
      %WITHIN%
      procptw BY ques13 ques22 ques21 ques25 ques24;
        alge BY ques14@1 ques13 ques12 ques15;
        axio BY ques9@1 ques11 ques7 ques8 ques10;
      int | procptw XWITH axio;
      alge ON axio int;
      %BETWEEN%
      procpt BY ques13 ques22 ques21 ques25 ques24;
 OUTPUT: SAMPSTAT;
```

APPENDIX V

THE MPLUS INPUT FILES FOR THE STEP 5

CROSS-LEVEL MODELS AT THE BETWEEN-CLASSROOM LEVEL

MODEL 1

```
TITLE:    Cross-Level Effects
DATA:     FILE IS msemagg.dat;
          VARIANCES = NOCHECK;
VARIABLE:  NAMES ARE class ques1-ques30;
          USEVARIABLES ARE class  ques16-ques20 formax concpt;
          BETWEEN = formax concpt;
          CATEGORICAL ARE ques16-ques20;
          CLUSTER IS class;
ANALYSIS:
          TYPE = TWOLEVEL RANDOM;
          ALGORITHM = INTEGRATION;
MODEL:
        %WITHIN%
        iconthk BY ques16-ques20;
        %BETWEEN%
        iconthkb BY ques16-ques20;
        concpt BY ques18@1 ques16 ques23 ques28 ques20
        ques30 ques29 ques27 ques19 ques17;
        formax BY ques5@1 ques10 ques2 ques3 ques9 ques8
        ques4 ques11 ques7 ques1 ques6;
        int | iconthkb XWITH concpt;
        formax ON concpt int;
OUTPUT: SAMPSTAT;
```

MODEL 2

```
TITLE:    Cross-Level Effects
DATA:     FILE IS msemagg.dat;
          VARIANCES = NOCHECK;
VARIABLE:  NAMES ARE class ques1-ques30;
          USEVARIABLES ARE class  ques12-ques15 formax concpt;
          BETWEEN = formax concpt;
          CATEGORICAL ARE ques12-ques15;
```

```
        CLUSTER IS class;
ANALYSIS:
        TYPE = TWOLEVEL RANDOM;
        ALGORITHM = INTEGRATION;
MODEL:
        %WITHIN%
        algethk BY ques12-ques15;
        %BETWEEN%
        algethkb BY ques12-ques15;
        concpt BY ques18@1 ques16 ques23 ques28 ques20
        ques30 ques29 ques27 ques19 ques17;
        formax BY ques5@1 ques10 ques2 ques3 ques9 ques8
        ques4 ques11 ques7 ques1 ques6;
        int | algethkb XWITH concpt;
        formax ON concpt int;
OUTPUT: SAMPSTAT;
```

MODEL 3

```
TITLE:    Cross-Level Effects
DATA:     FILE IS msemagg.dat;
          VARIANCES = NOCHECK;
VARIABLE:  NAMES ARE class ques1-ques30;
          USEVARIABLES ARE class ques21 ques22 ques24 ques25
          ques26
          formax concpt;
          BETWEEN = formax concpt;
          CATEGORICAL ARE ques21 ques22 ques24 ques25 ques26;
          CLUSTER IS class;
ANALYSIS:
        TYPE = TWOLEVEL RANDOM;
        ALGORITHM = INTEGRATION;
MODEL:
        %WITHIN%
        algothk BY ques21 ques22 ques24 ques25 ques26;
        %BETWEEN%
        algothkb BY ques21 ques22 ques24 ques25 ques26;
        concpt BY ques18@1 ques16 ques23 ques28 ques20
        ques30 ques29 ques27 ques19 ques17;
        formax BY ques5@1 ques10 ques2 ques3 ques9 ques8
        ques4 ques11 ques7 ques1 ques6;
        int | algothkb XWITH concpt;
        formax ON concpt int;
OUTPUT: SAMPSTAT;
```

MODEL 4

```
TITLE:    Cross-Level Effects
DATA:     FILE IS msemagg.dat;
          VARIANCES = NOCHECK;
```

```
VARIABLE:  NAMES ARE class ques1-ques30;
           USEVARIABLES ARE class  ques7-ques11 formax concpt;
           BETWEEN = formax concpt;
           CATEGORICAL ARE ques7-ques11;
           CLUSTER IS class;
ANALYSIS:
          TYPE = TWOLEVEL RANDOM;
          ALGORITHM = INTEGRATION;
MODEL:
        %WITHIN%
        axiothk BY ques7-ques11;
        %BETWEEN%
        axiothkb BY ques7-ques11;
        concpt BY ques18@1 ques16 ques23 ques28 ques20
        ques30 ques29 ques27 ques19 ques17;
        formax BY ques5@1 ques10 ques2 ques3 ques9 ques8
        ques4 ques11 ques7 ques1 ques6;
        int | axiothkb XWITH concpt;
        formax ON concpt int;
 OUTPUT: SAMPSTAT;
```

MODEL 5

```
TITLE:    Cross-Level Effects
DATA:     FILE IS msemagg.dat;
          VARIANCES = NOCHECK;
VARIABLE:  NAMES ARE class ques1-ques30;
           USEVARIABLES ARE class  ques23 ques27 ques28 ques29
          ques30  formax concpt;
           BETWEEN = formax concpt;
           CATEGORICAL ARE ques23 ques27 ques28 ques29 ques30;
           CLUSTER IS class;
ANALYSIS:
          TYPE = TWOLEVEL RANDOM;
          ALGORITHM = INTEGRATION;
MODEL:
        %WITHIN%
        enacthk BY ques23 ques27 ques28 ques29 ques30;
        %BETWEEN%
        enacthkb BY ques23 ques27 ques28 ques29 ques30;
        concpt BY ques18@1 ques16 ques23 ques28 ques20
        ques30 ques29 ques27 ques19 ques17;
        formax BY ques5@1 ques10 ques2 ques3 ques9 ques8
        ques4 ques11 ques7 ques1 ques6;
        int | enacthkb XWITH concpt;
        formax ON concpt int;
 OUTPUT: SAMPSTAT;
```

MODEL 6

```
TITLE:    Cross-Level Effects concpt formax
DATA:     FILE IS msemagg.dat;
          VARIANCES = NOCHECK;
VARIABLE: NAMES ARE class ques1-ques30;
          USEVARIABLES ARE class  ques1-ques6 formax concpt;
          BETWEEN = formax concpt;
          CATEGORICAL ARE ques1-ques6;
          CLUSTER IS class;
ANALYSIS:
         TYPE = TWOLEVEL RANDOM;
MODEL:
        %WITHIN%
        formthk BY ques1-ques6;
        %BETWEEN%
        formthkb BY ques1-ques6;
        concpt BY ques18@1 ques16 ques23 ques28 ques20
        ques30 ques29 ques27 ques19 ques17;
        formax BY ques5@1 ques10 ques2 ques3 ques9 ques8
        ques4 ques11 ques7 ques1 ques6;
        int | formthkb XWITH concpt;
        formax ON concpt int;
 OUTPUT: SAMPSTAT;
```

MODEL 7

```
TITLE:    Cross-Level Effects
DATA:     FILE IS msemagg.dat;
          VARIANCES = NOCHECK;
VARIABLE: NAMES ARE class ques1-ques30;
          USEVARIABLES ARE class  ques12-ques15 formax procpt;
          BETWEEN = formax procpt;
          CATEGORICAL ARE ques12-ques15;
          CLUSTER IS class;
ANALYSIS:
         TYPE = TWOLEVEL RANDOM;
         ALGORITHM = INTEGRATION;
MODEL:
        %WITHIN%
        algethk BY ques12-ques15;
        %BETWEEN%
        algethkb BY ques12-ques15;
        formax BY ques5@1 ques10 ques2 ques3 ques9 ques8
        ques4 ques11 ques7 ques1 ques6;
        procpt BY ques13@1 ques22 ques21 ques25
       ques24 ques14 ques15 ques26 ques12;
        int | algethkb XWITH formax;
        procpt ON formax int;
 OUTPUT: SAMPSTAT;
```

MODEL 8

```
TITLE:    Cross-Level Effects
DATA:     FILE IS msemagg.dat;
        VARIANCES = NOCHECK;
VARIABLE:  NAMES ARE class ques1-ques30;
        USEVARIABLES ARE class  ques21 ques20 ques22 ques24
        ques25
        ques26 formax procpt;
        BETWEEN = formax procpt;
        CATEGORICAL ARE ques21 ques20 ques22 ques24 ques25
        ques26;
        CLUSTER IS class;
ANALYSIS:
        TYPE = TWOLEVEL RANDOM;
        ALGORITHM = INTEGRATION;
MODEL:
       %WITHIN%
       algothk BY ques21 ques20 ques22 ques24 ques25 ques26;
       %BETWEEN%
       algothkb BY ques21 ques20 ques22 ques24 ques25 ques26;
       formax BY ques5@1 ques10 ques2 ques3 ques9 ques8
       ques4 ques11 ques7 ques1 ques6;
       procpt BY ques13@1 ques22 ques21 ques25
       ques24 ques14 ques15 ques26 ques12;
       int | algothkb XWITH formax;
       procpt ON formax int;
OUTPUT: SAMPSTAT;
```

MODEL 9

```
TITLE:    Cross-Level Effects
DATA:     FILE IS msemagg.dat;
        VARIANCES = NOCHECK;
VARIABLE:  NAMES ARE class ques1-ques30;
        USEVARIABLES ARE class  ques7-ques11 formax procpt;
        BETWEEN = formax procpt;
        CATEGORICAL ARE ques7-ques11;
        CLUSTER IS class;
ANALYSIS:
        TYPE = TWOLEVEL RANDOM;
        ALGORITHM = INTEGRATION;
MODEL:
       %WITHIN%
       axiothk BY ques7-ques11;
       %BETWEEN%
       axiothkb BY ques7-ques11;
       formax BY ques5@1 ques10 ques2 ques3 ques9 ques8
       ques4 ques11 ques7 ques1 ques6;
       procpt BY ques13@1 ques22 ques21 ques25
```

```
ques24 ques14 ques15 ques26 ques12;
 int | axiothkb XWITH formax;
 procpt ON formax int;
OUTPUT: SAMPSTAT;
```

MODEL 10

```
TITLE:    Cross-Level Effects
DATA:     FILE IS msemagg.dat;
          VARIANCES = NOCHECK;
VARIABLE: NAMES ARE class ques1-ques30;
          USEVARIABLES ARE class  ques23 ques27 ques28 ques29
         ques30  formax procpt;
          BETWEEN = formax procpt;
          CATEGORICAL ARE ques23 ques27 ques28 ques29 ques30;
          CLUSTER IS class;
ANALYSIS:
         TYPE = TWOLEVEL RANDOM;
         ALGORITHM = INTEGRATION;
MODEL:
        %WITHIN%
        enacthk BY ques23 ques27 ques28 ques29 ques30;
        %BETWEEN%
        enacthkb BY ques23 ques27 ques28 ques29 ques30;
        formax BY ques5@1 ques10 ques2 ques3 ques9 ques8
        ques4 ques11 ques7 ques1 ques6;
        procpt BY ques13@1 ques22 ques21 ques25
       ques24 ques14 ques15 ques26 ques12;
       int | enacthkb XWITH formax;
       procpt ON formax int;
 OUTPUT: SAMPSTAT;
```

MODEL 11

```
TITLE:    Cross-Level Effects
DATA:     FILE IS msemagg.dat;
          VARIANCES = NOCHECK;
          VARIABLE:  NAMES ARE class ques1-ques30;
          USEVARIABLES ARE class  ques1-ques6 formax procpt;
          BETWEEN = formax procpt;
          CATEGORICAL ARE ques1-ques6;
          CLUSTER IS class;
ANALYSIS:
         TYPE = TWOLEVEL RANDOM;
         ALGORITHM = INTEGRATION;
MODEL:
        %WITHIN%
        formthk BY ques1-ques6;
        %BETWEEN%
        formthkb BY ques1-ques6;
```

```
      formax BY ques5@1 ques10 ques2 ques3 ques9 ques8
       ques4 ques11 ques7 ques1 ques6;
       procpt BY ques13@1 ques22 ques21 ques25
      ques24 ques14 ques15 ques26 ques12;
      int | formthkb XWITH formax;
      procpt ON formax int;
OUTPUT: SAMPSTAT;
```

MODEL 12

```
TITLE:    Cross-Level Effects
DATA:     FILE IS msemagg.dat;
          VARIANCES = NOCHECK;
VARIABLE: NAMES ARE class ques1-ques30;
          USEVARIABLES ARE class  ques16-ques20 formax procpt;
          BETWEEN = formax procpt;
          CATEGORICAL ARE ques16-ques20;
          CLUSTER IS class;
ANALYSIS:
         TYPE = TWOLEVEL RANDOM;
         ALGORITHM = INTEGRATION;
MODEL:
        %WITHIN%
        iconthk BY ques16-ques20;
        %BETWEEN%
        iconthkb BY ques16-ques20;
        formax BY ques5@1 ques10 ques2 ques3 ques9 ques8
        ques4 ques11 ques7 ques1 ques6;
        procpt BY ques13@1 ques22 ques21 ques25
       ques24 ques14 ques15 ques26 ques12;
        int | iconthkb XWITH formax;
        procpt ON formax int;
 OUTPUT: SAMPSTAT;
```

MODEL 13

```
 TITLE:    Cross-Level Effects
DATA:     FILE IS msemagg.dat;
          VARIANCES = NOCHECK;
VARIABLE: NAMES ARE class ques1-ques30 form axio icon algo alge
          enac
          agg1-agg30 mathk formagg axioagg iconagg algoagg
          algeagg enacagg formax procpt concpt formax
          procpt concpt;
          USEVARIABLES ARE class  ques12-ques15 procpt concpt;
          BETWEEN = procpt concpt;
          CATEGORICAL ARE ques12-ques15;
          CLUSTER IS class;
ANALYSIS:
         TYPE = TWOLEVEL RANDOM;
```

```
 ALGORITHM = INTEGRATION;
MODEL:
  %WITHIN%
  algethk BY ques12-ques15;
  %BETWEEN%
  algethkb BY ques12-ques15;
  procpt BY ques13@1 ques22 ques21 ques25
  ques24 ques14 ques15 ques26 ques12;
  concpt BY ques18@1 ques16 ques23 ques28 ques20
  ques30 ques29 ques27 ques19 ques17;
  int | algethkb XWITH procpt;
  concpt ON procpt int;
OUTPUT: SAMPSTAT;
```

MODEL 14

```
TITLE:   Cross-Level Effects
DATA:    FILE IS msemagg.dat;
   VARIANCES = NOCHECK;
VARIABLE:  NAMES ARE class ques1-ques30;
   USEVARIABLES ARE class  ques21 ques20 ques22 ques24
   ques25
   ques26 procpt concpt;
   BETWEEN = procpt concpt;
   CATEGORICAL ARE ques21 ques20 ques22 ques24 ques25
   ques26;
   CLUSTER IS class;
ANALYSIS:
   TYPE = TWOLEVEL RANDOM;
   ALGORITHM = INTEGRATION;
MODEL:
  %WITHIN%
  algothk BY ques21 ques20 ques22 ques24 ques25 ques26;
  %BETWEEN%
  algothkb BY ques21 ques20 ques22 ques24 ques25 ques26;
  procpt BY ques13@1 ques22 ques21 ques25
  ques24 ques14 ques15 ques26 ques12;
  concpt BY ques18@1 ques16 ques23 ques28 ques20
  ques30 ques29 ques27 ques19 ques17;
  int | algothkb XWITH procpt;
  concpt ON procpt int;
OUTPUT: SAMPSTAT;
```

MODEL 15

```
TITLE:   Cross-Level Effects
DATA:    FILE IS msemagg.dat;
   VARIANCES = NOCHECK;
VARIABLE:  NAMES ARE class ques1-ques30;
   USEVARIABLES ARE class  ques7-ques11 procpt concpt;
```

```
BETWEEN = procpt concpt;
CATEGORICAL ARE ques7-ques11;
CLUSTER IS class;
ANALYSIS:
TYPE = TWOLEVEL RANDOM;
ALGORITHM = INTEGRATION;
MODEL:
%WITHIN%
axiothk BY ques7-ques11;
%BETWEEN%
axiothkb BY ques7-ques11;
procpt BY ques13@1 ques22 ques21 ques25
ques24 ques14 ques15 ques26 ques12;
concpt BY ques18@1 ques16 ques23 ques28 ques20
ques30 ques29 ques27 ques19 ques17;
int | axiothkb XWITH procpt;
concpt ON procpt int;
OUTPUT: SAMPSTAT;
```

MODEL 16

```
TITLE:    Cross-Level Effects
DATA:     FILE IS msemagg.dat;
          VARIANCES = NOCHECK;
VARIABLE:  NAMES ARE class ques1-ques30;
          USEVARIABLES ARE class  ques23 ques27 ques28 ques29
          ques30  procpt concpt;
          BETWEEN = procpt concpt;
          CATEGORICAL ARE ques23 ques27 ques28 ques29 ques30;
          CLUSTER IS class;
ANALYSIS:
          TYPE = TWOLEVEL RANDOM;
          ALGORITHM = INTEGRATION;
MODEL:
        %WITHIN%
        enacthk BY ques23 ques27 ques28 ques29 ques30;
        %BETWEEN%
        enacthkb BY ques23 ques27 ques28 ques29 ques30;
        procpt BY ques13@1 ques22 ques21 ques25
        ques24 ques14 ques15 ques26 ques12;
        concpt BY ques18@1 ques16 ques23 ques28 ques20
        ques30 ques29 ques27 ques19 ques17;
        int | enacthkb XWITH procpt;
        concpt ON procpt int;
 OUTPUT: SAMPSTAT;
```

MODEL 17

```
TITLE:    Cross-Level Effects formax procpt
DATA:     FILE IS msemagg.dat;
          VARIANCES = NOCHECK;
VARIABLE:  NAMES ARE class ques1-ques30;
          USEVARIABLES ARE class  ques1-ques6 procpt concpt;
          BETWEEN = procpt concpt;
          CATEGORICAL ARE ques1-ques6;
          CLUSTER IS class;
ANALYSIS:
         TYPE = TWOLEVEL RANDOM;
         ALGORITHM = INTEGRATION;
MODEL:
        %WITHIN%
        formthk BY ques1-ques6;
        %BETWEEN%
        formthkb BY ques1-ques6;
       procpt BY ques13@1 ques22 ques21 ques25
       ques24 ques14 ques15 ques26 ques12;
      concpt BY ques18@1 ques16 ques23 ques28 ques20
      ques30 ques29 ques27 ques19 ques17;
        int | formthkb XWITH procpt;
        concpt ON procpt int;
 OUTPUT: SAMPSTAT;
```

MODEL 18

```
TITLE:    Cross-Level Effects
DATA:     FILE IS msemagg.dat;
          VARIANCES = NOCHECK;
VARIABLE:  NAMES ARE class ques1-ques30;
          USEVARIABLES ARE class  ques16-ques20 procpt concpt;
          BETWEEN = procpt concpt;
          CATEGORICAL ARE ques16-ques20;
          CLUSTER IS class;
ANALYSIS:
         TYPE = TWOLEVEL RANDOM;
         ALGORITHM = INTEGRATION;
MODEL:
        %WITHIN%
        iconthk BY ques16-ques20;
        %BETWEEN%
        iconthkb BY ques16-ques20;
        procpt BY ques13@1 ques22 ques21 ques25
        ques24 ques14 ques15 ques26 ques12;
        concpt BY ques18@1 ques16 ques23 ques28 ques20
        ques30 ques29 ques27 ques19 ques17;
        int | iconthkb XWITH procpt;
        concpt ON procpt int;
 OUTPUT: SAMPSTAT;
```

CURRICULUM VITAE

PERSONAL INFORMATION

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EDUCATION

Degree	**Institution**	**Year of Graduation**
High School	TED Zonguldak College	1997
BS	Ankara University Mathematics	2002
Non-thesis MS	Mugla University Secondary Science and Mathematics Education	2004
MS	Middle East Technical University Secondary Science and Mathematics Education	2007

RESEARCH INTEREST

Structural Equation Modeling, Multilevel Modeling, Knowledge of Mathematics

FOREIGN LANGUAGES

Advanced English