MASSIVE HIGHER DERIVATIVE GRAVITY THEORIES

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ABSTRACT

MASSIVE HIGHER DERIVATIVE GRAVITY THEORIES

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In this thesis massive higher derivative gravity theories are analyzed in some detail. One-particle scattering amplitude between two covariantly conserved sources mediated by a graviton exchange is found at tree-level in D dimensional (Anti)-de Sitter and flat spacetimes for the most general quadratic curvature theory augmented with the Pauli-Fierz mass term. From the amplitude expression, the Newtonian potential energies are calculated for various cases. Also, from this amplitude and the propagator structure, a three dimensional unitary theory is identified. In the second part of the thesis, the found three dimensional unitary theory is studied in more detail from a canonical point of view. The general higher order action is written in terms of gauge-invariant functions both in flat and de Sitter backgrounds. The analysis is extended by adding static sources, spinning masses and the gravitational Chern-Simons term separately to the theory in the case of flat spacetime. For all cases the microscopic spectrum and the masses are found. In the discussion of curved spacetime, the masses are found in the relativistic and non-relativistic limits. In the Appendix, some useful calculations that are frequently used in the bulk of the thesis are given.

Keywords: Higher derivative gravity, Massive spin-2 fields, Unitarity in gravity.

ÖZ

KÜTLELİ YÜKSEK TÜREVLİ KÜTLEÇEKİM KURAMLARI

Güllü, İbrahim
Doktora, Fizik Bölümü
Tez Yöneticisi : Prof. Dr. Bayram Tekin

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Bu tez çalışmasında, kütleli yüksek türevli kütleçekim kuramları ayrıntılı olarak analiz edildi. Kovariant olarak korunan iki kaynak arasında gerçekleşen graviton değiş-tokuşu aracılığıyla tek parçacıklı saçılma genliği, D boyutlu (Anti)-de Sitter ve düz uzay-zamanlarda, Pauli-Fierz kütle terimi eklenmiş en genel ikinci dereceden eğrilikli kütleçekim kuramı için hesaplanmıştır. Bu genlik ifadesi kullanılarak, Newton potansiyel enerjisi birçok durum için hesaplanmıştır. Ayrıca, bu genlik ve yürütücü yapısından üç boyutlu üniter bir kuram bulunmuştur. Tezin ikinci bölümünde, bulunan bu üç boyutlu kuram, kanonik açıdan, ayrıntılı bir şekilde incelenmiştir. Genel yüksek mertebeli eylem, hem düz hem de de Sitter uzay-zamanı için ayar dönüşm̃leri altında değişmez fonksiyonlar cinsinden yazılmıştır. Bu analiz, kurama durgun kaynaklar, spinli kütleler ve kütleçekimsel Chern-Simons terimleri eklenerek düz uzay-zaman durumunda geniş-letildi. Bütün durumlar için mikroskobik spektrum ve kütleler bulundu. Tartışmanın eğri uzay-zaman kısmında kütleler göreli ve göreli olmayan limitler için bulundu. Ek bölümünde, tezin içinde sıkça kullanılan bazı yararlı hesaplar verilmiştir.

Anahtar Kelimeler: Yüksek türevli kütleçekim, Kütleli spin-2 alanlar, Kütleçekimde üniterlik.

To my “honey beetle”

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INTRODUCTION

1.1 Introduction

General Relativity (GR) [1] can be thought as a modification (albeit a major one) of Newtonian gravity. The history of gravity starts with Newton's $\frac{1}{r^2}$ force law in 1687. The precise form of this law is

$$F = -\frac{Gm_1m_2}{r^2}, \tag{1.1}$$

where G is the universal Newton's constant and m_i are the masses of the interacting particles. The minus sign indicates that the force is attractive. The Celestial mechanics gave successful verifications of this law. A well-known example of the success of Newton's theory is the prediction of a new planet. After the discovery of the orbit of Uranus there were attempts to calculate this orbit theoretically. However, these attempts did not work exactly since there were deviations between the calculated and observed orbits [2, 3]. Using Newton's law (1.1) the location of Neptune was predicted and just after this calculation it was observed exactly at that location [2, 3]. Hence an "outer" mass was responsible for the perturbations of the orbit of Uranus. Newton's law, with the help of "dark matter" (Neptune in this case) worked well for the case of Uranus. After this resolution, there were studies of all planets. When Mercury was studied, another problem was found: The calculations of precession of Mercury's perihelion was in contradiction with the observed value. The calculations gave larger value than the observed ones. The first attempt to solve this problem was again to invoke the "dark matter" idea. It was thought that there must be a planet, Vulcan, in the solar system [3, 4]. However, this did not solve the problem since Vulcan was never found. The solution came with modifying gravity: replacing Newtonian gravity with the Einsteinian one [3].

In GR the spacetime geometry is determined by the matter, energy, pressure etc. and the dynamics of matter or light is determined by the geometry of spacetime. GR is constructed with the idea of equivalence principle and the general coordinate invariance (and with the added assumption that the equations are wave-type equations with second derivatives on the basic fields). The Einstein equation is

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa T_{\mu\nu}, \tag{1.2}$$

where the left hand side is the geometry part and it is known as the Einstein tensor and the right hand side is the matter part. In (1.2), $R_{\mu\nu}$ and R are the Ricci tensor and Ricci scalar, respectively. These quantities can be thought as a measure of the curvature of spacetime. They depend on the spacetime metric $g_{\mu\nu}$ and its derivatives. The connection is taken to be metric compatible ($\nabla g = 0$). In the right hand side, there is a coupling constant related to the Newton's constant which in four dimensions is $\kappa = 8\pi G$, and $T_{\mu\nu}$ is the energy-momentum tensor coming from the matter sector.

Einstein's equation, right after was introduced, was immediately applied to solve the then more-than fifty year old problem of Mercury's orbit. After the Schwarzschild solution was found, which is the unique static spherically symmetric time-independent solution to (1.2), the perihelion precession of Mercury was calculated and the answer was found to be compatible with the observed value [3, 5, 6, 7]. There were also other predictions that came out from GR such as the bending of light passing by the sun and the gravitational redshift of light which were also supported by observations. Inspite of these successful solutions and predictions there are some observations that GR in its basic form as given in (1.2) cannot explain. The data taken from supernova explosions [8, 9, 10] show that the universe has an accelerated expansion. GR cannot explain this phenomenon in its pure form[1].

If GR is taken as the correct theory, there must exist a dark energy component in the matter-energy budget of the universe which can be represented as a constant term, the cosmological constant, Λ. The density of the dark energy is estimated to be 75% of the total energy density of the universe and numerically it reads

$$\rho_\Lambda = 4 \times 10^{-6} \frac{GeV}{cm^3}, \tag{1.3}$$

which is very small compared to the vacuum energy density that comes from the energy

[1] $T_{\mu\nu}$ is taken as the standard energy-momentum tensor of the matter.

density of the vacuum of quantum field theory (QFT);

$$\rho_\Lambda \sim 10^{-123}\rho_{QFT}, \tag{1.4}$$

[10, 11, 12]. This mismatch of the "experimental" and "theoretical" values of the energy density of the vacuum is known as the cosmological constant problem. Therefore, it can be thought that GR is not the full theory and it must be modified in the infrared (IR), namely at ultra-large distances or ultra-weak interacting regimes, such that without a need of dark energy the accelerating expansion of the universe happens. [Admittedly, this alone will not solve the question of why QFT vacuum is almost empty.] One of the suitable candidates for this modification is to give a tiny mass to the graviton in the theory, in such a way that massive GR still passes the solar system tests.

From the perspective of QFT, at low energies, GR can be thought as a weakly interacting massless spin-2 field. The story starts with the classification of particles with respect to their spins and masses, as degrees of freedom [10, 13]. The representations of these degrees of freedom are fields. For spin-0 mode of massless particles the representations is a scalar field, and for spin-1 it is a vector field. For spin-2 mode the symmetry is the general coordinate invariance when the interactions are taken into account, and around flat and maximally symmetric spaces, one has a massless helicity-2 particle. This points to GR [10, 14, 15, 16, 17, 18, 19, 20] with the action

$$I = \frac{1}{\kappa}\int d^4x\,\sqrt{-g}R\,, \tag{1.5}$$

where g is the determinant of the spacetime metric and the rest are defined in section 1.5.2. Therefore, it is also natural to combine the quantum theory with GR.

There are other motivations to try to modify GR and perhaps construct a quantum gravity theory. The Schwarzschild solution of (1.2) is a black hole with a curvature singularity, to explain or resolve this type of singularities a quantum gravity theory is needed. Also, to understand the very early universe when both the quantum effects and gravitational effects are dominant at the same time, a unification of gravity and quantum mechanics is necessary. There are attempts to construct quantum gravity theories. String theory is one such candidate. Higher curvature gravity theories also are candidates. With this motivation it is also possible to add higher curvature terms to Einstein gravity (1.5). The higher curvature terms are negligible at low energies, but they dominate at high energy domains. Another reason to introduce the higher curvature terms is that the Einstein-Hilbert action (1.5) is not renormalizable. Higher curvature terms make the theory renormalizable but ruin the unitarity, namely yields negative norm states in the scattering matrix [21, 22].

To summarize the argument, we can say that GR needs modification at both the IR and ultraviolet (UV) regimes. Therefore, in this thesis Einstein gravity will be modified both by adding mass and higher curvature terms. Hence we modify the theory at both the UV and IR regimes.

The outline of the thesis is as follows: In the next sections of this chapter the massive gravity and higher derivative gravity models are discussed separately. In the massive gravity model mainly the Pauli-Fierz [23, 24] model is studied and also the nature of the van Dam-Veltman-Zakharov [25, 26] discontinuity between the strictly massless theory and the arbitrarily small mass theory is discussed. Then some specific three dimensional massive theories are studied. After that, some technical details will be given about the relevant spacetimes, the cosmological constant and the higher dimensional gravity models. At the end of this Chapter the higher derivative Pais-Uhlenbeck oscillators are briefly reviewed. In the second chapter the general higher curvature massive gravity theory is analyzed. Its unitarity structure is studied by computing the tree level scattering amplitude in generic D dimensions. Also, the Newtonian potentials are calculated. The third chapter is devoted to the analysis of the three dimensional unitary theory. These two chapters are based on the papers "*Massive Higher Derivative Gravity in D-dimensional Anti-de Sitter Spacetimes*" [27] and "*Canonical Structure of Higher Derivative Gravity in 3D* " [28] respectively. Then, the conclusion part comes. Also, an appendix part is added so that some of the calculations can be followed easily.

1.2 Modifying Gravity

In this part the modification of the Einstein theory is discussed. By modifying gravity we mean that we change the degrees of freedom of the theory in some way. First the mass term is added to the theory which changes the degrees of freedom from two to five in $3+1$ dimensions. Also, adding higher derivative terms change the degrees of freedom. The massive gravity theory is discussed first. Then the higher derivative gravity theory is considered. At the end of this part, two 3-dimensional massive higher derivative gravity theories are introduced.

1.2.1 Modifying Gravity with Mass Terms:

There are two ways to give mass to a gravity theory. One is to add directly the mass to the action. The other way is to introduce scalar fields. When these fields are evaluated at the background the general covariance is broken and the graviton gets mass. Both ways end up in the same class of theories. In this discussion we will follow the first way. The mass is added to the Einstein-Hilbert theory in a Lorentz invariant way as follows

$$I = \int d^4x \sqrt{-g} \left[\frac{1}{\kappa} R - \frac{1}{2} m^2 h^{\mu\nu} \left(h_{\mu\nu} - \eta_{\mu\nu} h \right) \right], \tag{1.6}$$

which is known as the Pauli-Fierz (PF) action, the second part is the PF mass term [23, 24] and m^2 is the mass parameter. We first consider this theory in a flat background. In (1.6) $\eta_{\mu\nu}$ is the flat spacetime metric and $h_{\mu\nu}$ is the linear part of the metric perturbation, $g_{\mu\nu} = \eta_{\mu\nu} - h_{\mu\nu}$, and $h = \eta_{\mu\nu} h^{\mu\nu}$ is trace of this metric perturbation. To see more explicitly how the added PF part gives mass to the graviton, we should linearize the theory around the background spacetime. From both (1.2) and (1.5), the linearized Einstein tensor can be written in the absence of sources as

$$\mathcal{G}^L_{\mu\nu} = \frac{1}{2} \left(\partial^\sigma \partial_\mu h_{\nu\sigma} + \partial^\sigma \partial_\nu h_{\mu\sigma} - \partial^2 h_{\mu\nu} - \partial_\mu \partial_\nu h \right) - \frac{1}{2} \eta_{\mu\nu} \left(-\partial^2 h + \partial^\sigma \partial^\rho h_{\sigma\rho} \right) = 0, \tag{1.7}$$

where $\partial^2 \equiv \partial_\mu \partial^\mu$ and the linear parts of the Ricci tensor and Ricci scalar are

$$R^L_{\mu\nu} = \frac{1}{2} \left(\partial^\sigma \partial_\mu h_{\nu\sigma} + \partial^\sigma \partial_\nu h_{\mu\sigma} - \partial^2 h_{\mu\nu} - \partial_\mu \partial_\nu h \right), \; R^L = -\partial^2 h + \partial^\sigma \partial^\mu h_{\sigma\mu}. \tag{1.8}$$

If the metric perturbation is constrained to be transverse and traceless, that are $\partial^\mu h_{\mu\nu} = 0$ and $h = 0$, then (1.7) yields

$$\partial^2 \left(h_{\mu\nu} - \eta_{\mu\nu} h \right) = 0, \tag{1.9}$$

and for massive particles it is known that the Klein-Gordon equation must hold that is

$$\left(\partial^2 - m^2 \right) \phi = 0, \tag{1.10}$$

for the mostly plus signature of the metric. To write (1.9) as a massive field equation, the PF term must be added:

$$\left(\partial^2 - m^2 \right) \left(h_{\mu\nu} - \eta_{\mu\nu} h \right) = 0. \tag{1.11}$$

Since this term cannot be obtained from the curvature terms, it is added to the action (1.5) by brute force. Later on, it will be seen that modifying GR with higher curvature terms will generate this mass term automatically when the coupling constants of the higher curvature terms have a special combination. From the transverse and traceless conditions it can be seen easily that the PF theory describes a massive spin-2 field. The metric perturbation is a symmetric rank-2 tensor, so that it has ten independent degrees of freedom in 3+1 dimensions. However, from the transversality condition there are four constraints which eliminate four degrees of freedom leaving six. The tracelessness condition also eliminates one degree of freedom and the theory has five degrees of freedom. We must also note that this mass term is the unique ghost and tachyon-free Lorentz-invariant combination. When the sign is changed in the middle of the PF mass term, then it produces tachyons in its propagator structure [10].

The massless limit of the PF theory must tend to Einstein gravity as the usual continuity arguments in physics dictate. However, the $m^2 \to 0$ limit does not give the results of the $m^2 = 0$ theory. For two point sources the interaction potential, in the Newtonian limit of GR, in four dimensions is

$$U = -\frac{Gm_1 m_2}{r}. \tag{1.12}$$

However, the massive theory gives

$$U = -\frac{4}{3}\frac{Gm_1 m_2}{r}, \tag{1.13}$$

where the Newtonian potential is greater than the usual one. This effect is known as the van Dam-Veltman-Zakharov (vDVZ) discontinuity[2] [25, 26]. The linearized PF theory is different from the linearized GR in the massless limit. This discontinuity appears in flat backgrounds [29, 30, 31]. However, in curved spacetime this discontinuity does not appear. From the general amplitude equation (2.30) it can be seen more explicitly. There is a $\frac{M^2}{\Lambda}$ fraction by which the flat spacetime limit and massless limit do not commute. Going to flat spacetime limit first the vDVZ discontinuity appears in the massless limit, but taking the massless limit first the discontinuity disappears in four dimensions [27, 30]. Hence, the introduction of a cosmological constant to the theory solves this problem since the smallness of the mass can be compared with another measurable quantity. [One could argue that discontinuity has been mainly replaced by the non-commutativity of the limits, which is a valid objection.] From the pole structure of the general amplitude equation (2.30), it can be seen that for de Sitter spacetime ($\Lambda > 0$) a pole produces a tachyon and it is absent for anti-de Sitter ($\Lambda < 0$) spacetime[3].

[2] Redefinition of the Newton's constant does not solve the problem, since then the deflection of light changes by 25%.

[3] We should also note that Vainshtein claimed that [32] the discontinuity disappears at the nonlinear level once the finite Schwarzschild radius of one of the scattering particles is taken into account.

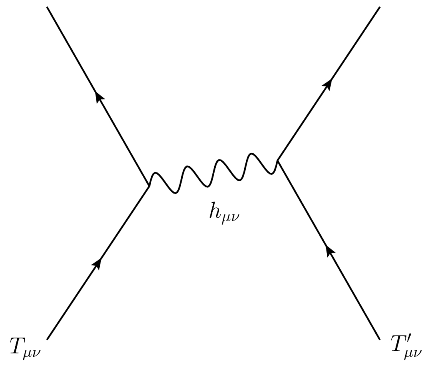

Figure 1.1: The tree-level one-particle scattering amplitude between two background covariantly conserved sources mediated by a graviton exchange.

1.2.2 Modifying Gravity with Higher Curvature Terms:

To make GR perturbatively renormalizable, quadratic curvature terms $\alpha R^2 + \beta R_{\mu\nu} R^{\mu\nu} + \gamma R_{\mu\sigma\nu\rho} \times R^{\mu\sigma\nu\rho}$ must be added to the Einstein-Hilbert action (1.5) [21]. But adding these terms makes the theory nonunitary because of the non-decoupling ghost introduced by the middle term $\beta R_{\mu\nu} R^{\mu\nu}$ [22]. When this term is excluded the unitarity of the theory is regained but the theory becomes nonrenormalizable. To have a consistent quantum gravity, both of these properties must be provided.

Before discussing further, the notion of unitarity must be explained. With unitarity, it is meant that the theory must be both ghost and tachyon free. Ghost is a particle that has a negative kinetic energy and tachyon is a particle that has negative mass-squared (in flat space). In a more technical language the signs of the propagators must be correct that is (for the $(-,+,+,+)$ signature) the (scalar propagator)

$$D(p) \sim \frac{1}{p^2 + m^2} \tag{1.14}$$

where p is the four-momentum. In the canonical form, the Lagrangian should be

$$\mathcal{L} = \frac{1}{2} \psi(t, \vec{x}) \left(\Box - m^2 \right) \psi(t, \vec{x}), \tag{1.15}$$

the kinetic term is the d'Alembertian operator and it must be positive. The mass-squared must again be greater than zero. Note that only the tree-level unitarity is considered here. The Feynman diagram of the tree-level scattering amplitude is shown in the figure .

1.2.3 Topologically Massive Gravity

In the discussion to follow, we will need a specific 3D theory that has been studied a lot in the literature. This is the topologically massive gravity (TMG) theory introduced in [33, 34] with the action

$$I = \int d^3x \sqrt{-g} \left[\frac{1}{\kappa} R - \frac{1}{2\mu} \epsilon^{\lambda\mu\nu} \Gamma^{\rho}{}_{\lambda\sigma} \left(\partial_\mu \Gamma^{\sigma}{}_{\rho\nu} + \frac{2}{3} \Gamma^{\sigma}{}_{\mu\beta} \Gamma^{\beta}{}_{\nu\rho} \right) \right]. \tag{1.16}$$

Here μ is the coupling constant of the gravitational Chern-Simons term. The sign of μ is not fixed. $\epsilon^{\lambda\mu\nu}$ is the three dimensional anti-symmetric tensor which is connected to the Levi-Civita symbol as $\epsilon^{\lambda\mu\nu} \equiv -\frac{1}{\sqrt{-g}} \tilde{\epsilon}^{\lambda\mu\nu}$. The pure Einstein-Hilbert action does not propagate any degrees of freedom in 3D. After the gravitational Chern-Simons term is introduced, TMG propagates a single massive particle with helicity $+2$ or -2 (not both since the theory is not parity-invariant). The action for this theory can be written in terms of the gauge-invariant functions[4] (3.45) as

$$I = \frac{1}{2} \int d^3x \left[\frac{1}{\kappa} \left(\phi q + \sigma^2 \right) + \frac{1}{\mu} \sigma \left(q + \Box \phi \right) \right]. \tag{1.17}$$

Taking the variation with respect to the q field one of the other fields can be eliminated. This variation yields

$$\delta q : \ \frac{1}{\kappa} \phi + \frac{1}{\mu} \sigma = 0, \tag{1.18}$$

from which $\phi = -\frac{\kappa}{\mu} \sigma$ follows. Putting this result back into (1.17) yields the action

$$I = -\frac{\kappa}{2\mu^2} \int d^3x \left[\sigma \Box \sigma - \frac{\mu^2}{\kappa^2} \sigma^2 \right]. \tag{1.19}$$

From (1.19) it can be seen that this theory propagates only a single massive degree of freedom with mass $m^2 = \frac{\mu^2}{\kappa^2}$. Not to have a negative kinetic energy term, κ must be chosen negative which yields the "wrong-sign" Einstein-Hilbert term. In fact this model propagates a single massive scalar spin-2 mode [33]. Since the spin of the particle depends on the sign of μ, this model is a parity violating theory, as noted above. The detailed discussion and calculations for this model is given in Chapter 3.

[4] Note that the gauge-invariant functions will be discussed later in the third chapter.

1.2.4 New Massive Gravity and Critical Gravity

Another interesting higher derivative massive gravity theory is the recently introduced "New Massive Gravity" [35, 36]. The action for this model is

$$I = -\frac{1}{\kappa}\int d^3x\,\sqrt{-g}\left\{-R + \frac{1}{m^2}\left(R_{\mu\nu}^2 - \frac{3}{8}R^2\right)\right\}, \tag{1.20}$$

where there is a relation between the coupling constants of the higher curvature terms as $8\alpha + 3\beta = 0$ and the mass reads in terms of the coupling constants as $m^2 = -\frac{1}{\kappa\beta}$. For the unitarity of the theory, κ must be negative, and $\beta > 0$. This model also implies the wrong-sign Einstein-Hilbert term.

This model is not a higher derivative theory at the linearized level. The canonical form of (1.20) is (see more details in section 3.2)

$$I = \int d^3x\left\{-\frac{1}{2\kappa}\left(\phi\Box\phi + \frac{1}{\kappa\beta}\phi^2\right) + \frac{\beta}{2}\left(\sigma\Box\sigma + \frac{1}{\kappa\beta}\sigma^2\right)\right\}, \tag{1.21}$$

where it can be seen that this model has two modes propagating with the same masses. Therefore, this theory is parity invariant. Also, in this form the necessity of negative κ can be seen explicitly.

After this theory was introduced [35] a research activity started in massive 3D gravity theories. Especially in Anti-de Sitter (AdS) backgrounds some classical solutions and conserved charges of this theory was given [27, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46]. The question of how the theory gains mass was answered in [47, 48]. In this work Higgs mechanism of the general quadratic theory in three dimensions was given by writing the theory in a Weyl invariant form. Breaking the Weyl invariance in the background, graviton gains mass, scalar field remains massless and the Weyl gauge field gains mass or can remain massless depending on the parameters.

The four and higher dimensional extensions of this theory, called "the critical gravity", was also investigated in [49, 50, 51]. For these cases the special points are again set to zero $3\alpha + \beta = 0$ and $4\alpha(D-1) + D\beta = 0$ for four and the generic D dimensional cases respectively, for which the massive scalar mode vanishes [27]. In the critical gravity, the mass of spin-2 is set to zero and the energy of black holes become zero for the AdS vacuum. It is also found that the energy of the massless spin-2 modes vanishes on-shell. There remains only spin-2 logarithmic modes with positive energy.

1.3 Constant Curvature Spacetimes

In our discussion we will need the maximally symmetric backgrounds which are the flat, Anti-de Sitter (AdS) and de Sitter (dS) geometries. Maximally symmetric means that the curvature of the spacetime is constant everywhere and in any direction. Therefore, if the curvature is known in one point it is enough to know the curvature at every point. There are only three maximally symmetric spacetimes that are Minkowski, which is flat so that the curvature is zero everywhere, dS and AdS. These spacetimes are classified with respect to the curvature scalar R, the dimensionality of spacetime and the signature of the metric. De Sitter space is a $(D-1)$-dimensional surface embedded into a D dimensional flat spacetime for which the signature of the metric reads $(-,+,+,...,+)$. For the AdS spacetime the signature of the embedding metric reads $(-,+,+,...,-)$. For AdS spacetime there is no horizon but for dS there is a horizon [52, 53, 54]. Examples of the maximally symmetric flat spaces are planes with Euclidean signature $(+,+,+)$ and its higher dimensional generalizations. Spheres $\left(S^D\right)$ are dS or positively curved spacetimes, and hyperboloids $\left(H^D\right)$ are negatively curved or AdS spacetimes.

The Riemann tensor of a maximally symmetric D-dimensional spacetime is

$$R_{\mu\sigma\nu\rho} = \frac{R}{D\left(D-1\right)}\left(g_{\mu\nu}g_{\sigma\rho} - g_{\mu\rho}g_{\sigma\nu}\right), \tag{1.22}$$

where the curvature scalar R is constant and D is the dimensionality of the spacetime [7]. For AdS or dS the curvature scalar is $R = \frac{2D\Lambda}{(D-2)}$. Here, Λ is the cosmological constant and for $\Lambda < 0$ the spacetime becomes AdS, negatively curved spacetime, and for $\Lambda > 0$ it is dS, positively curved spacetime. If $\Lambda = 0$ then the spacetime is flat.

The cosmological constant is related to the vacuum energy density in $D = 4$ as

$$\rho_{\text{vac}} = \frac{\Lambda}{8\pi G}. \tag{1.23}$$

Here ρ is the energy density of the vacuum which is proportional to the pressure of a perfect fluid $T_{\mu\nu} = (\rho + p)\, U_\mu U_\nu + p g_{\mu\nu}$. To see this proportionality the energy-momentum tensor in the Einstein equation is divided into matter and vacuum pieces,

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G\left(T^{\text{M}}_{\mu\nu} + T^{\text{vac}}_{\mu\nu}\right), \tag{1.24}$$

where the vacuum energy-momentum tensor is $T^{\text{vac}}_{\mu\nu} = -\rho_{\text{vac}}g_{\mu\nu}$. The vacuum energy-momentum tensor becomes a perfect if $-\rho_{\text{vac}} = p_{\text{vac}}$. The Einstein equation can be written with a cosmological constant term as

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}. \tag{1.25}$$

The relation between the vacuum energy density and the cosmological constant can be seen from these equations [7]. The maximally symmetric solutions of this equation are dS and AdS (and also Minkowski) spacetimes, for the pure vacuum case.

In the first part of the analysis the higher curvature massive gravity is studied in a D dimensional (A)dS spacetime (AdS_D). In this way the effects of higher dimensions can be seen. Also, with this general approach a unitary theory may be found in $D \geq 5$.

1.4 Canonical Analysis

In this part some technical details of the canonical analysis are given. To put a Lagrangian into the canonical form means to write it in terms of scalar fields that are in the form of harmonic oscillator Lagrangians. The field equations of these Lagrangians are Klein-Gordon equation. So that, KG equation can be seen as simple harmonic oscillators. From these oscillators the particle spectrum can be studied. Also, for investigation of the unitarity of a theory, this form is useful. Therefore, first the KG equation for free particle is reviewed. Then the generalization to higher derivative of this equation known as Pais-Uhlenbeck oscillator is discussed.

1.4.1 The Klein-Gordon Equation

The story starts from the quantization of the non-relativistic energy equation for a free particle. The energy of a free particle is

$$E = \frac{p^2}{2m}, \tag{1.26}$$

where E is the energy and p is the momentum of the particle. To quantize this, the variables are promoted to being operators by taking $E \to H = i\hbar\partial_t$ and $\vec{p} \to -i\hbar\vec{\nabla}$, where $\hbar$ is the reduced Planck constant, and $\frac{\partial}{\partial t} \equiv \partial_t = \partial_0$, $\nabla \equiv \frac{\partial}{\partial x}$ for one dimensional space[5]. With these

[5] Here the derivative is $\partial_\mu = \left(\frac{\partial}{\partial t}, \nabla\right)$ and the four momentum is $P^\mu = (E, \vec{p}) = i\partial^\mu$ in the natural units.

substitutions (1.26) turns to the well known Schrödinger equation

$$i\hbar\partial_t\psi(t,\vec{x}) = -\frac{\hbar^2}{2m}\nabla^2\psi(t,\vec{x}), \tag{1.27}$$

where $\psi(t,\vec{x})$ is a complex function. A relativistic version of this equation can be obtained by upgrading the relativistic dispersion relation[6] [55]

$$E^2 = p^2c^2 + m^2c^4, \tag{1.28}$$

to an operator equation

$$\left(\frac{1}{c^2}\partial_t^2 - \nabla^2 + \frac{m^2c^2}{\hbar^2}\right)\psi(t,\vec{x}) = 0, \tag{1.29}$$

where c is the speed of light. Going to natural units $c = 1$, $\hbar = 1$ and defining the d'Alembertian operator $\Box \equiv \partial_\mu\partial^\mu$, where the components of the partial derivative is $\partial^\mu = \left(\partial^0, -\vec{\nabla}\right)$ (in Minkowski background with the signature $(-,+,+,+)$) yields[7]

$$\left(\Box - m^2\right)\psi(t,\vec{x}) = 0, \tag{1.30}$$

which is the classical relativistic wave equation of a massive particle. Taking the Fourier transform of $\psi(t,\vec{x}) = \int \frac{d^3x}{2\pi}e^{i\vec{p}.\vec{x}}\psi(t,\vec{p})$ and putting it into (1.29), the equation of motion for a harmonic oscillator is found with the frequency $\omega_p = \sqrt{|p|^2+m^2}$, that is [56, 57]

$$\left(\partial_t^2 + \omega_p^2\right)\psi(t,\vec{p}) = 0. \tag{1.31}$$

In the field theoretical approach the same result can be found by taking the real spin-0 particle Lagrangian density with mass m

$$\mathcal{L}(t,x) = -\frac{1}{2}\left(\partial_\mu\psi\partial^\mu\psi + m^2\psi^2\right), \tag{1.32}$$

where the field depends both on time and space $\psi = \psi(t,x)$. The Euler-Lagrange equations $\frac{\partial\mathcal{L}}{\partial\psi} = \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial^\mu\psi)}$ of (1.32) gives (1.30) [56, 57].

After getting some basic informations about the relativistic wave equation, the higher derivative terms can be added to the Lagrangian to generalize this discussion. The field equations of the higher derivative Lagrangians are known as the Pais-Uhlenbeck oscillators [58].

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[6] If the equation is not squared the operators will be inside the square root $E = \sqrt{p^2c^2+m^2c^4} \rightarrow i\hbar\partial_t\psi = \sqrt{(-i\hbar\nabla)^2c^2+m^2c^4}\psi$. This form has some problems in evaluating the operators. Also, it is nonlocal.

[7] Note that for the signature $(+,-,-,-)$ the KG equation reads $\left(\Box + m^2\right)\psi = 0$.

1.4.2 Pais-Uhlenbeck Oscillators

The Lagrangian density of the higher derivative real scalar field is [59]

$$\mathcal{L} = -\frac{1}{2}\psi\left(\prod_{i=1}^{N}\left(\Box + m_i^2\right)\right)\psi. \tag{1.33}$$

For simplicity N is set to two, i. e. $N = 2$, and the examination is done in the nonrelativistic limit, that is all the space derivatives are dropped. For these conditions the Lagrangian density becomes

$$\mathcal{L} = -\frac{1}{2}\left\{\ddot{q}^2 - \left(\omega_1^2 + \omega_2^2\right)\dot{q}^2 + \omega_1^2\omega_2^2 q^2\right\}, \tag{1.34}$$

where ω_i is used instead of m_i. Also, note that the $\omega_1 = \omega_2$ case will be different from the $\omega_1 \neq \omega_2$ case, where the first case is the degenerate case and the Lagrangian density becomes purely quadratic

$$\mathcal{L} = -\frac{1}{2}\left\{\ddot{q} + \omega^2 q\right\}^2. \tag{1.35}$$

The Euler-Lagrange equation for (1.34) is

$$\frac{d^2}{dt^2}\left(\frac{\partial\mathcal{L}}{\partial\ddot{q}}\right) - \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}}\right) + \frac{\partial\mathcal{L}}{\partial q} = 0, \tag{1.36}$$

which can be obtained by taking the variation of (1.34) with respect to the variable $q = q(t)$. For n-th order variational problem (1.36) takes the form

$$\sum_{n=0}^{N}(-1)^n\frac{d^n}{dt^n}\left(\frac{\partial\mathcal{L}}{\partial q^{(n)}}\right) = 0 \tag{1.37}$$

in the non-relativistic limit. This equation (1.37) can be obtained by taking the variation of the general Lagrangian $L = L\left(q, \dot{q}, \ddot{q}, \ldots, q^{(n)}\right)$, which is [60, 61]

$$\delta S = \int_{t_i}^{t_f} dt\delta q\left[-\sum_{n=0}^{N}\left(-\frac{d}{dt}\right)^n\frac{\partial L}{\partial q^{(n)}}\right] + \left[\sum_{n=0}^{N-1}P_{q^{(n)}}\delta q^{(n)}\right]_{t_i}^{t_f}, \tag{1.38}$$

where $P_{q^{(n)}}$ is defined as

$$P_{q^{(n)}} \equiv \sum_{k=n+1}^{N}\left(-\frac{d}{dt}\right)^{k-n-1}\frac{\partial L}{\partial q^{(k)}}. \tag{1.39}$$

The Hamiltonian for the generic case is written [60, 61] as

$$H = \sum_{n=0}^{N-1}P_{q^{(n)}}q^{(n+1)} - L. \tag{1.40}$$

From (1.37) the equations of motion can be found for $N = 2$ as

$$q^{(4)} + \left(\omega_1^2 + \omega_2^2\right)\ddot{q} + \omega_1^2\omega_2^2 q = 0, \tag{1.41}$$

where $q^{(4)}$ is the fourth time derivative of q. The Hamiltonian (1.40) and the momentum (1.39) of the system for $N = 2$ reads

$$H = -\frac{1}{2}\left\{\ddot{q}^2 - 2q^{(3)}\dot{q} - \left(\omega_1^2 + \omega_2^2\right)\dot{q}^2 - \omega_1^2\omega_2^2 q^2\right\}, \tag{1.42}$$

and

$$P_{q^{(0)}} = \left(\omega_1^2 + \omega_2^2\right)\dot{q} + q^{(3)} \quad \text{and} \quad P_{q^{(1)}} = -\ddot{q}, \tag{1.43}$$

respectively. For the non-degenerate case $\omega_1 \neq \omega_2$ with the Pais-Uhlenbeck variables $Q_1 \equiv q + \frac{\ddot{q}}{\omega_2^2}$ and $Q_2 \equiv q + \frac{\ddot{q}}{\omega_1^2}$ can be diagonalized as

$$H = \frac{\omega_1^4}{2\left(\omega_2^2 - \omega_1^2\right)}\left(\dot{Q}_2^2 + \omega_2^2\dot{Q}_2^2\right) - \frac{\omega_2^4}{2\left(\omega_2^2 - \omega_1^2\right)}\left(\dot{Q}_1^2 + \omega_1^2\dot{Q}_1^2\right), \tag{1.44}$$

where the second term is a ghost term. However, for the degenerate case there can be a unitary region for proper constants [59].

1.5 Conventions

In what follows we will use the following conventions:

1.5.1 Flat Spacetime:

The flat spacetime metric has the mostly positive signature $\mathrm{diag}\left(\eta_{\mu\nu}\right) = (-, +, +, +, \dots)$ in its diagonal components. Since the spacetime is flat, the curvature tensors and scalars are zero $R^{\sigma}{}_{\mu\rho\nu} = R_{\mu\nu} = R = 0$. The partial derivative reads $\partial^\mu = \left(\partial^t, -\partial^i\right) = \left(\partial^0, -\vec{\nabla}\right)$ and $\partial_\mu = (-\partial_t, -\partial_i) = \left(-\partial_0, -\vec{\nabla}\right)$. The d'Alembertian operator becomes $\Box = \partial^2 = \partial_\mu\partial^\mu = -\partial_0^2 + \partial_i^2 = -\partial_0^2 + \nabla^2$. The Greek indices run as $\mu = 0, 1, 2, \dots$ that counts all the coordinates of the spacetime. The Latin indices run as $i = 1, 2, 3, \dots$ and they denote the spatial coordinates.

1.5.2 The Curved Spacetime:

The metric has the signature $(-, +, +, +, \dots)$ and the Christoffel connection reads

$$\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2}g^{\sigma\lambda}\left(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu}\right), \tag{1.45}$$

and the commutation relation of the covariant derivatives are

$$\left[\nabla_\mu, \nabla_\nu\right] V^\sigma = R_{\mu\nu}{}^{\sigma}{}_{\lambda} V^\lambda, \tag{1.46}$$

where

$$R_{\mu\nu}{}^{\sigma}{}_{\lambda} = \partial_\mu \Gamma^\sigma_{\nu\lambda} - \partial_\nu \Gamma^\sigma_{\mu\lambda} + \Gamma^\sigma_{\mu\rho}\Gamma^\rho_{\nu\lambda} - \Gamma^\sigma_{\nu\rho}\Gamma^\rho_{\mu\lambda} \tag{1.47}$$

is Riemann tensor. The covariant derivative acts on the covariant and contravariant vectors as

$$\nabla_\mu \omega_\nu = \partial_\mu \omega_\nu - \Gamma^\lambda_{\mu\nu}\omega_\lambda, \quad \nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda. \tag{1.48}$$

The Ricci tensor and Ricci scalar are defined as

$$R_{\mu\nu} = g^{\rho\sigma} R_{\rho\mu\sigma\nu} \quad R = g^{\mu\nu} R_{\mu\nu}, \tag{1.49}$$

respectively. The d'Alembertian is $\Box = \nabla^\mu \nabla_\mu$. The Greek and Latin indices have the same meaning as in the flat spacetime case.

CHAPTER 2

MASSIVE HIGHER DERIVATIVE GRAVITY IN D-DIMENSIONAL ANTI-DE SITTER SPACETIMES

2.1 Introduction

In gravity there is no theory which is unitary and renormalizable at the same time. To get a renormalizable theory in four dimensions, higher derivative terms, $\alpha R^2 + \beta R_{\mu\nu}^2$, are added to Einstein-Hilbert action [21]. In doing so one can gain renormalizability but loses unitary. The coupling constant of the square of the Ricci tensor introduces a non-decoupling ghost term. However, if we omit this term then we gain unitary yet lose renormalizability. If a theory is non-unitary it shows itself as a repulsive force in the Newtonian limit between static sources. Therefore the theory has a better UV behaviour because of this repulsive force. In field theory this is what usually happens: To have a better behaved theory, ghosts are introduced during the renormalization process but they decouple at the end if the theory is unitary. Therefore, bartering unitarity with renormalizability can not be accepted.

In three dimensions there is a perturbatively renormalizable and tree-level unitary theory [35, 36] in flat background for a special choice of coupling constants of higher derivative terms that is $8\alpha + 3\beta = 0$ and for a reversed sign of Einstein-Hilbert term [62, 63, 64, 65] and this theory is named as “New Massive Gravity” (NMG), that is a parity-preserving spin-2 theory. However, we do not know that if this special ratio between α and β will survive renormalization at a given loop level. Another interesting thing about this theory is that at the linearized level the theory has a massive graviton with helicities ± 2 in its spectrum. Therefore, the theory has a non-linear extension to the Pauli-Fierz (PF) mass term. For this reason this theory solves an old problem of finding a non-linear extension of PF mass term. The equivalence of NMG and

PF was shown in [35][1]. The physical meaning of this equivalence must be well understood since NMG theory is background diffeomorphism invariant not only at non-linear level but also at the linear level. However, PF theory is invariant under Killing symmetries of the $2 + 1$ dimensional Minkowski spacetime. To have a better understanding about these symmetries a quite interesting approach was put forward in [65]: In the absence of the Einstein-Hilbert term, the Weyl invariant form of the linearized NMG is written. Therefore, introducing the Einstein-Hilbert term at the linearized level breaks this symmetry and produces the mass of the graviton. As a result one can think that Einstein-Hilbert term provides the mass and the higher derivative terms gives the kinetic energy. This perspective explains the sign change of Einstein-Hilbert term. Looking back, this result is expected since pure Einstein gravity is non-dynamical and does not give any propagation in three dimensions. It is like the mass term in a scalar field theory at the linearized level. When a kinetic energy is introduced to the theory then it plays a role in the dynamics. In the case of NMG the kinetic energy comes from the higher derivative terms. Consequently, the mass of the graviton comes from Einstein-Hilbert term by breaking Weyl invariance. This point may be important for constructing massive gravity theories in other dimensions.

In this chapter, the most general quadratic curvature gravity theory will be considered in a D dimensional (anti)-de Sitter background. We also augment this theory with a PF mass term. We will study the propagator structure of this theory by finding its one-particle exchange amplitude between two covariantly conserved sources. For this reason we first linearize the action

$$\begin{aligned} I &= \int d^D x\, \sqrt{-g} \left\{ \frac{1}{\kappa} R - \frac{2\Lambda_0}{\kappa} + \alpha R^2 + \beta R_{\mu\nu}^2 + \gamma \left(R^2_{\mu\nu\sigma\rho} - 4R^2_{\mu\nu} + R^2 \right) \right\} \\ &\quad + \int d^D x\, \sqrt{-g} \left\{ -\frac{M^2}{4\kappa} \left(h^2_{\mu\nu} - h^2 \right) + \mathcal{L}_{\text{matter}} \right\}, \end{aligned} \tag{2.1}$$

where Λ_0 is the bare cosmological constant. κ is related to the D-dimensional Newton's constant as $\kappa \equiv 2\Omega_{D-2} G_D$ where Ω_{D-2} is the $D-2$ dimensional solid angle. The other parameters are the coupling constants α, β, γ and M^2 is the mass parameter. In total we have a seven parameter theory which is the most general quadratic model including the dimensions and cosmological constants[2]. Because of these parameters the theory potentially has various in-

[1] Note that, beside this parity-preserving theory, there is also the parity violating Topologically Massive Gravity (TMG) [33, 34].

[2] In the absence of the source terms and at the linearized level, one can reduce the number of parameters in

teresting limits and discontinuities. At this level, there are no constraints on the parameters. They may be positive, negative or zero. After computing the tree-level amplitude we will constrain the parameters not to have any ghosts or tachyons in the theory. Also, some terms do not contribute to the equations of motion for some specific dimensions. For example, for $D = 4$ the Gauss-Bonnet term becomes a surface term and for $D = 3$ it vanishes identically. Therefore, in three dimensions all the information is carried by the Ricci tensor and the Riemann tensor has no more information. In three dimensions the model can be extended by adding the Chern-Simons term $\mu\left(\Gamma\partial\Gamma + \frac{2}{3}\Gamma^3\right)$ to (2.1). But here we will stick to (2.1). In the next chapter we study this case in detail[3]. The theory reduces to R^2 model with the PF term for two dimensions. In this chapter we consider $D \geq 3$. The theory that we consider has general covariance except for the PF mass term.

Various limits of the spin-2 model which is defined by the linearization of (2.1) have been studied in the literature. Though, for some certain limits of the above action there may still be some interesting new models. One of them is NMG which is the case for $D = 3$ and in flat background without the PF mass term and $8\alpha + 3\beta = 0$. Apart from finding such new models we will also explore discontinuities of the full seven parameter theory. The discontinuities come out while the order of limits are changed when some of the parameters approach zero. One such discontinuity is the so called van vDVZ discontinuity. The resolution of this discontinuity comes from the introduction of the cosmological constant. Then taking $\frac{M^2}{\Lambda} \to 0$ limit GR results are recovered at tree-level [29, 30, 31, 32, 69][4]. However, the discontinuity reappears when the quantum corrections are taken into account [70]. Up to now the linear theory has been considered. Once, we consider the non-linear theory a ghost arises in massive gravity and this effect is known as the Boulware-Deser instability [71].

The ingredients of this chapter is as follows: In the second section we will analyze the linear equations of motion which is obtained around an AdS background. We also discuss certain special limits. In the next section we calculate the one-particle scattering amplitude and the following sections will be devoted to the discussions of some limits and discontinuities. At the last section the conclusions and discussions will be given.

the action [66, 67], but here for the sake of generality we shall work with (2.1).

[3] See [59, 68] for this case, without the higher curvature terms.

[4] Another resolution of the discontinuity may follow even in flat space if the Schwarzschild radius of the scattering objects is taken as a second mass scale in the theory [32].

2.2 Linearized Equations

The field equations can be found by taking the variation of the action (2.1) with respect to the metric. After variation we get the field equations as follows:

$$\begin{aligned}&\frac{1}{\kappa}\left(R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R+\Lambda_0 g_{\mu\nu}\right)+2\alpha R\left(R_{\mu\nu}-\frac{1}{4}g_{\mu\nu}R\right)+(2\alpha+\beta)\left(g_{\mu\nu}\Box-\nabla_\mu\nabla_\nu\right)R\\&+2\gamma\left[RR_{\mu\nu}-2R_{\mu\sigma\nu\rho}R^{\sigma\rho}+R_{\mu\sigma\rho\tau}R_\nu{}^{\sigma\rho\tau}-2R_{\mu\sigma}R_\nu{}^\sigma-\frac{1}{4}g_{\mu\nu}\left(R^2_{\tau\lambda\sigma\rho}-4R^2_{\sigma\rho}+R^2\right)\right]\\&+\beta\Box\left(R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R\right)+2\beta\left(R_{\mu\sigma\nu\rho}-\frac{1}{4}g_{\mu\nu}R_{\sigma\rho}\right)R^{\sigma\rho}+\frac{M^2}{2\kappa}\left(h_{\mu\nu}-\bar{g}_{\mu\nu}h\right)=\tau_{\mu\nu}.\end{aligned}\tag{2.2}$$

Here $\tau_{\mu\nu}$ is energy-momentum tensor coming from the source terms, etc... . The background metric $\bar{g}_{\mu\nu}$ namely the vacuum, is a non-singular solution to the field equations in the absence of the matter terms. The vacuum is the (anti)-de Sitter space which is the maximally symmetric vacuum with the Riemann, Ricci tensors and Ricci scalar curvature given respectively as

$$\bar{R}_{\mu\rho\nu\sigma}=\frac{2\Lambda}{(D-1)(D-2)}\left(\bar{g}_{\mu\nu}\bar{g}_{\rho\sigma}-\bar{g}_{\mu\sigma}\bar{g}_{\nu\rho}\right),\quad \bar{R}_{\mu\nu}=\frac{2\Lambda}{D-2}\bar{g}_{\mu\nu},\quad \bar{R}=\frac{2D\Lambda}{D-2}.\tag{2.3}$$

We made all the contractions with respect to the background metric. If we put the background metric (2.3) in the field equations (2.2), the effective cosmological constant Λ can be found in terms of α,β,γ, and Λ_0:

$$\frac{\Lambda-\Lambda_0}{2\kappa}+f(\alpha,\beta,\gamma)\Lambda^2=0,\tag{2.4}$$

where $f(\alpha,\beta,\gamma)\equiv(\alpha D+\beta)\frac{(D-4)}{(D-2)^2}+\gamma\frac{(D-3)(D-4)}{(D-1)(D-2)}$. This is a second order equation with respect to Λ. Therefore it has two solutions which are

$$\Lambda=-\frac{1}{4\kappa f}\left[1\pm\sqrt{1+8\kappa f\Lambda_0}\right],\tag{2.5}$$

and for reality of the vacuum $8\kappa\Lambda_0 f\geq-1$. Hence the spacetime can be both dS or AdS. One of these solutions vanishes in the absence of Λ_0. In this case the non-vanishing root becomes $\Lambda=-\frac{1}{2\kappa f}$. There are some exceptional points of this equation of motion for which the effective cosmological constant becomes equal to the bare cosmological constant. In four dimensions $\Lambda=\Lambda_0$ since $f(\alpha,\beta,\gamma)$ becomes zero. Also, in three dimensions if we set $3\alpha+\beta=0$ then we again have $f(\alpha,\beta,\gamma)=0$ and $\Lambda=\Lambda_0$. Moreover, if we set $\gamma=0$ and $\alpha D+\beta=0$, which gives $f(\alpha,\beta,\gamma) = 0$, the theory again has $\Lambda_0 = \Lambda$. In three dimensions without setting $3\alpha + \beta = 0$, we have $\Lambda_\pm = \frac{1 \pm \sqrt{1-8\kappa(3\alpha+\beta)\Lambda_0}}{4\kappa(3\alpha+\beta)}$ and for Λ to be real $1 \geq 8\kappa\,(3\alpha+\beta)\,\Lambda_0$.

The next step is to find the linear equation of motion of (2.2) around the constant curvature background, $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$. By using (2.5) we eliminate the bare cosmological constant and after linearization the equation of motion becomes [72]

$$\begin{aligned} T_{\mu\nu}(h) &= a\,\mathcal{G}^L_{\mu\nu} + (2\alpha+\beta)\left(\bar{g}_{\mu\nu}\bar{\Box} - \bar{\nabla}_\mu\bar{\nabla}_\nu + \frac{2\Lambda}{D-2}\bar{g}_{\mu\nu}\right)R^L \\ &+ \beta\left(\bar{\Box}\mathcal{G}^L_{\mu\nu} - \frac{2\Lambda}{D-1}\bar{g}_{\mu\nu}R^L\right) + \frac{M^2}{2\kappa}\left(h_{\mu\nu} - \bar{g}_{\mu\nu}h\right), \end{aligned} \tag{2.6}$$

Note that the PF term is already linear. Here we define a new constant in terms of the other coupling constants as

$$a \equiv \frac{1}{\kappa} + \frac{4\Lambda D}{D-2}\alpha + \frac{4\Lambda}{D-1}\beta + \frac{4\Lambda(D-3)(D-4)}{(D-1)(D-2)}\gamma. \tag{2.7}$$

$T_{\mu\nu}(h)$ is the energy-momentum tensor which contains all higher order terms and the source $\tau_{\mu\nu}$. $\mathcal{G}^L_{\mu\nu}$ is the linear cosmological Einstein tensor

$$\mathcal{G}^L_{\mu\nu} = R^L_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}R^L - \frac{2\Lambda}{D-2}h_{\mu\nu}. \tag{2.8}$$

$R^L_{\mu\nu}$ and R^L are the linearized Ricci tensor and the linearized scalar curvature $R^L \equiv \left(g^{\mu\nu}R_{\mu\nu}\right)^L$, respectively. They read

$$R^L_{\mu\nu} = \frac{1}{2}\left(\bar{\nabla}^\sigma\bar{\nabla}_\mu h_{\nu\sigma} + \bar{\nabla}^\sigma\bar{\nabla}_\nu h_{\mu\sigma} - \bar{\Box}h_{\mu\nu} - \bar{\nabla}_\mu\bar{\nabla}_\nu h\right),\ R^L = -\bar{\Box}h + \bar{\nabla}^\sigma\bar{\nabla}^\mu h_{\sigma\mu} - \frac{2\Lambda}{D-2}h. \tag{2.9}$$

To calculate the scattering amplitude we need the trace of (2.6), that is $T = \bar{g}_{\mu\nu}T^{\mu\nu}$ and it reads

$$\left[(4\alpha(D-1) + D\beta)\,\bar{\Box} - (D-2)\left(\frac{1}{\kappa} + 4f\Lambda\right)\right]R^L - \frac{M^2}{\kappa}(D-1)h = 2T. \tag{2.10}$$

This equation is a wave equation, and for $4\alpha(D-1) + D\beta = 0$ something special happens, since the dynamical part of this equation vanishes. Actually, for $D = 3$ this choice will lead us to the NMG. After computing the tree level scattering amplitude between two sources, and constraining the theory to have no ghosts or tachyons will give us this special point. Before moving on we can still analyze some limits of the theory at the linearized level. For this purpose we drop the sources. Since there are no sources we can not get the unitarity regions but we can capture the ranges of the parameters in which tachyons drop out. The discussion bifurcates whether M^2 is zero or not. First we will consider $M^2 \neq 0$.

2.2.1 Massive case:

We take the divergence and double divergence of (2.6) to find the constraints on the deviation part of the metric, $h_{\mu\nu}$. Since, (2.6) is divergence free without the mass term, the constraints come from the divergence of the mass term as

$$\bar{\nabla}^{\mu} h_{\mu\nu} - \bar{\nabla}_{\nu} h = 0, \qquad \bar{\nabla}^{\mu}\bar{\nabla}^{\nu} h_{\mu\nu} - \bar{\Box} h = 0. \tag{2.11}$$

Replacing the covariant derivatives that appear in the background Ricci scalar in (2.9) which is

$$\begin{aligned}
\bar{\nabla}^{\sigma}\bar{\nabla}^{\mu} h_{\sigma\mu} &= \left[\bar{\nabla}^{\sigma}, \bar{\nabla}^{\mu}\right] h_{\sigma\mu} + \bar{\nabla}^{\sigma}\bar{\nabla}^{\mu} h_{\sigma\mu}, \\
&= \bar{R}^{\sigma\mu}{}_{\sigma}{}^{\lambda} h_{\lambda\mu} + \bar{R}^{\sigma\mu}{}_{\mu}{}^{\lambda} h_{\sigma\lambda} + \bar{\nabla}^{\sigma}\bar{\nabla}_{\sigma} h, \\
&= \frac{2\Lambda}{(D-2)} \bar{g}^{\mu\lambda} h_{\lambda\mu} - \frac{2\Lambda}{(D-2)} \bar{g}^{\sigma\lambda} h_{\sigma\lambda} + \bar{\nabla}^{\sigma}\bar{\nabla}_{\sigma} h, \\
&= \bar{\nabla}^{\sigma}\bar{\nabla}_{\sigma} h,
\end{aligned}$$

and using the first equation in (2.11) lead to $R^L = -\frac{2\Lambda}{D-2} h$. For the flat space case we have $R^L = 0$. This choice forces the field to be traceless (2.10), and from the first equation of (2.11) it becomes transverse. The field $h_{\mu\nu}$ has $\frac{D(D+1)}{2}$ independent components. The transverse-traceless condition gives $D+1$ equations that eliminates $D+1$ of the independent components. Thus we are left with the remaining $(D+1)(D-2)/2$ independent components. The linear Einstein tensor becomes $\mathcal{G}^L_{\mu\nu} = R^L_{\mu\nu} = -\partial^2 h_{\mu\nu}$, where we have changed the places of the covariant derivatives in (2.9). With these constraints the field equation for the remaining independent components of the field becomes

$$\left(\beta\partial^4 + \frac{1}{\kappa}\partial^2 - \frac{M^2}{\kappa}\right) h_{\mu\nu} = 0, \tag{2.12}$$

which is a quadratic equation that describes two massive excitations with masses

$$m^2_{\pm} = -\frac{1}{2\kappa\beta} \pm \frac{1}{2|\kappa\beta|}\sqrt{1 + 4\kappa\beta M^2}. \tag{2.13}$$

To have real masses $M^2 \geq -\frac{1}{4\beta\kappa}$, and at the saturation point the two masses become equal. These masses become non-tachyonic when the parameters are chosen properly. If κ and β have the same sign, one of the excitation becomes negative and produces a tachyon. If they have opposite signs, the term inside the square root must be $0 < 4 \mid \kappa\beta \mid M^2 < 1$. In this case both excitations become non-tachyonic. However, this theory is not unitary as we will see.

For the generic Λ case the trace equation reads

$$\left[(4\alpha(D-1) + D\beta)\Box - (D-2)\left(\frac{1}{\kappa} + 4\Lambda f\right) + \frac{M^2}{2\kappa\Lambda}(D-1)(D-2)\right] h = 0. \tag{2.14}$$

In this equation we can see that h becomes a dynamical scalar field. The dynamical part can be eliminated by setting $4\alpha(D-1) + D\beta = 0$. After choosing this special point we still have two options to satisfy (2.14). We may set $h = 0$ and the field becomes again transverse and traceless, and the field equations read

$$\left[\left(a + \beta\bar{\Box}\right)\left(\frac{2\Lambda}{(D-1)(D-2)} - \frac{1}{2}\bar{\Box}\right) + \frac{M^2}{2\kappa}\right] h_{\mu\nu} = 0, \tag{2.15}$$

and it has generically two excitations

$$m_{1,2} = \frac{2\Lambda}{(D-1)(D-2)} - \frac{a}{2\beta} \pm \sqrt{\left(\frac{2\Lambda}{(D-1)(D-2)} + \frac{a}{2\beta}\right)^2 + \frac{M^2}{\kappa}}. \tag{2.16}$$

After computing the one-particle exchange amplitude we will again discuss this theory and see that for special points, this theory will give us a ghost and tachyon free theory. The other option to satisfy (2.14) is tuning the mass as

$$M^2 = \frac{2\Lambda\kappa}{D-1}\left(\frac{1}{\kappa} - \frac{\Lambda\beta(D-4)}{D-1} + 4\Lambda\gamma\frac{(D-3)(D-4)}{(D-1)(D-2)}\right), \tag{2.17}$$

which is the partially massless point, vanishes for flat background and for curved background a higher derivative gauge invariance appears [73]. Therefore, the field has one less degree of freedom (DOF) compared to the massive one. Moreover, the mass is allowed to be negative in AdS as long as it satisfies the Breitenlohner-Freedman type bound. In four dimensions higher derivative terms vanish and the mass only depends on the cosmological constant, $M^2 = \frac{2\Lambda}{3}$. In higher dimensions the partially massless theory depends on the higher derivative terms.

2.2.2 Massless case:

Without the Pauli-Fierz mass term the theory becomes invariant under background diffeomorphisms $\delta_\xi h_{\mu\nu} = \bar{\nabla}_\mu \xi_\nu + \bar{\nabla}_\nu \xi_\mu$, since $\delta_\xi \mathcal{G}^L_{\mu\nu} = 0$ and $\delta_\xi R^L = 0$. As we mentioned above the divergence and double divergence of (2.6) do not give any constraint on $h_{\mu\nu}$. Therefore, R^L becomes a dynamical variable for $T = 0$. If the coefficient in front of the d'Alembertian operator is fixed to zero than the dynamics of R^L vanishes. The equation (2.10) is satisfied for two conditions: R^L can be zero or the coefficient, $\frac{1}{\kappa} + 4\Lambda f$, can be set to zero in which case R^L need not vanish. The later one is not acceptable because the field equations leave a gauge invariant object undetermined.

2.3 Tree-level Amplitude

To find the Newton potentials and to analyze the particle spectrum of (2.1), we are going to find the tree-level scattering amplitude between two covariantly conserved sources which is defined as

$$A \equiv \frac{1}{4} \int d^D x \, \sqrt{-\bar{g}} T'_{\mu\nu}(x) \, h^{\mu\nu}(x) \, , \tag{2.18}$$

by considering the full theory (2.6). In this equation $T'_{\mu\nu}$ is one of the sources and $h^{\mu\nu}$ is the deviation which is produced by the unprimed source. The factor $\frac{1}{4}$ is put to get the correct Newton's constant. Since the field has independent components we need to decompose $h^{\mu\nu}$ to eliminate the unphysical parts of it. Therefore, the decomposition must be done in such a way that the physical components of $h_{\mu\nu}$ will be determined by $T^{\mu\nu}$. The usual choice is to define

$$h_{\mu\nu} \equiv h^{TT}_{\mu\nu} + \bar{\nabla}_{(\mu} V_{\nu)} + \bar{\nabla}_\mu \bar{\nabla}_\nu \phi + \bar{g}_{\mu\nu} \psi, \tag{2.19}$$

where $h^{TT}_{\mu\nu}$ is the transverse and traceless part of the deviation. V_μ is the vector part with a symmetrization that is defined with a $\frac{1}{2}$ factor and it is divergence free. ϕ and ψ are scalar functions. With this definition (2.19), the amplitude equation becomes

$$A = \frac{1}{4} \int d^D x \, \sqrt{-\bar{g}} \left(T'_{\mu\nu} h^{TT\mu\nu} + T' \psi \right), \tag{2.20}$$

where the terms in the middle of (2.19) vanish since we have covariantly conserved sources and the total derivative vanishes at the boundary by use of Stoke's theorem. In (2.20) the tensorial quantities and ψ must be written in terms of the trace of energy momentum tensor. In order to get this we take the trace, divergence and double divergence of (2.19)

$$h = \bar{\Box}\phi + D\psi, \qquad \bar{\Box} h = \bar{\Box}^2 \phi + \frac{2\Lambda}{(D-2)} \bar{\Box}\phi + \bar{\Box}\psi, \tag{2.21}$$

where we used $\bar{\nabla}^\nu \bar{\nabla}^\mu h_{\mu\nu} = \bar{\Box} h$. This condition comes from the nonzero mass term and it is not a gauge condition. From these two equations we can write $\bar{\Box}\phi$ in terms of $\bar{\Box}\psi$. We hit the first equation with the $\bar{\Box}$ operator and subtract it from the second one which gives us

$$\bar{\Box}\phi = \frac{(D-1)(D-2)}{2\Lambda}\bar{\Box}\psi, \tag{2.22}$$

which can be put into the first equation in (2.21) yielding

$$h = \left(\frac{(D-1)(D-2)}{2\Lambda}\bar{\Box} + D\right)\psi. \tag{2.23}$$

This equation can be inserted into (2.10) so that ψ can be written in terms of the trace of the energy momentum tensor as follows

$$\psi = \left\{\frac{\Lambda}{\kappa} + 4\Lambda f - c\Lambda\bar{\Box} - \frac{M^2}{2\kappa}(D-1)\right\}^{-1}\left(\frac{(D-1)(D-2)}{2\Lambda}\bar{\Box} + D\right)^{-1} T, \tag{2.24}$$

where $c \equiv \frac{4(D-1)\alpha}{D-2} + \frac{D\beta}{D-2}$. Now we will write $h^{TT}_{\mu\nu}$ such that it is determined by the trace of the energy momentum tensor. First of all, we will find the transverse and traceless part of the field equations by using the Lichnerowicz operator $\triangle^{(2)}_L$ acting on spin-2 symmetric tensors

$$\triangle^{(2)}_L h_{\mu\nu} = -\bar{\Box}h_{\mu\nu} - 2\bar{R}_{\mu\rho\nu\sigma}h^{\rho\sigma} + 2\bar{R}^{\rho}{}_{(\mu}h_{\nu)\rho}. \tag{2.25}$$

Some properties of this operator (that we need) were collected in [30]

$$\begin{aligned}
\triangle^{(2)}_L \nabla_{(\mu}V_{\nu)} &= \nabla_{(\mu}\triangle^{(1)}_L V_{\nu)}, \quad \triangle^{(1)}_L V_\mu = (-\Box + \Lambda)V_\mu, \quad \nabla^\mu \triangle^{(2)}_L h_{\mu\nu} = \triangle^{(1)}_L \nabla^\mu h_{\mu\nu},\\
\triangle^{(2)}_L g_{\mu\nu}\phi &= g_{\mu\nu}\triangle^{(0)}_L \phi, \quad \triangle^{(0)}_L \phi = -\Box\phi, \quad \nabla^\mu \triangle^{(1)}_L V_\mu = \triangle^{(0)}_L \nabla^\mu V_\mu.
\end{aligned} \tag{2.26}$$

Using these properties we have

$$\mathcal{G}^{LTT}_{\mu\nu} = \frac{1}{2}\triangle^{(2)}_L h^{TT}_{\mu\nu} - \frac{2\Lambda}{(D-2)}h^{TT}_{\mu\nu}. \tag{2.27}$$

With this equation the transverse traceless part of the deviation can be written in terms of the transverse traceless part of the energy momentum tensor as

$$h^{TT}_{\mu\nu} = 2\left\{(\beta\bar{\Box} + a)(\triangle^{(2)}_L - \frac{4\Lambda}{D-2}) + \frac{M^2}{\kappa}\right\}^{-1} T^{TT}_{\mu\nu}. \tag{2.28}$$

The only thing that is left is to decompose the energy momentum tensor so that we can connect $h^{TT}_{\mu\nu}$ with the energy momentum tensor. To find this we first decompose the energy momentum tensor as (2.19). Then we take the double divergence of this equation, keeping in mind that we have covariantly conserved sources. Also taking the trace, one can find

$$\begin{aligned}
T^{TT}_{\mu\nu} = T_{\mu\nu} - \frac{\bar{g}_{\mu\nu}}{D-1}T \quad &+ \quad \frac{1}{D-1}\left(\bar{\nabla}_\mu\bar{\nabla}_\nu + \frac{2\Lambda\bar{g}_{\mu\nu}}{(D-1)(D-2)}\right)\\
&\times\left(\bar{\Box} + \frac{2\Lambda D}{(D-1)(D-2)}\right)^{-1} T.
\end{aligned} \tag{2.29}$$

We are ready to compute the scattering amplitude by putting (2.24, 2.28, 2.29) into (2.20). After doing manipulations by using (2.26) and suppressing the integral sign the amplitude equation becomes

$$\begin{aligned} 4A &= 2T'_{\mu\nu}\left\{(\beta\bar{\Box}+a)(\triangle_L^{(2)}-\frac{4\Lambda}{D-2})+\frac{M^2}{\kappa}\right\}^{-1}T^{\mu\nu} \\ &+ \frac{2}{D-1}T'\left\{(\beta\bar{\Box}+a)(\bar{\Box}+\frac{4\Lambda}{D-2})-\frac{M^2}{\kappa}\right\}^{-1}T \\ &- \frac{4\Lambda}{(D-2)(D-1)^2}T'\left\{(\beta\bar{\Box}+a)(\bar{\Box}+\frac{4\Lambda}{D-2})-\frac{M^2}{\kappa}\right\}^{-1}\left\{\bar{\Box}+\frac{2\Lambda D}{(D-2)(D-1)}\right\}^{-1}T \\ &+ \frac{2}{(D-2)(D-1)}T'\left\{\frac{1}{\kappa}+4\Lambda f-c\bar{\Box}-\frac{M^2}{2\kappa\Lambda}(D-1)\right\}^{-1}\left\{\bar{\Box}+\frac{2\Lambda D}{(D-2)(D-1)}\right\}^{-1}T. \end{aligned} \tag{2.30}$$

From this main result we can consider various limits. Solving this integral for non zero cosmological constant is highly nontrivial. However, the particle spectrum of the theory can be considered by analyzing the pole structure of the amplitude. For the general case there are four poles which read as

$$\begin{aligned} \bar{\Box}_1 &= -\frac{2\Lambda D}{(D-1)(D-2)}, \\ \bar{\Box}_{2,3} &= \frac{1}{\beta}\left\{-\left(\frac{a}{2}+\frac{2\Lambda\beta}{(D-2)}\right)\pm\sqrt{\left(\frac{a}{2}+\frac{2\Lambda\beta}{(D-2)}\right)^2-\beta\left(\frac{4\Lambda a}{(D-2)}-\frac{M^2}{\kappa}\right)}\right\}, \\ \bar{\Box}_4 &= \frac{1}{c}\left(\kappa^{-1}+4\Lambda f-\frac{M^2}{2\kappa\Lambda}(D-1)\right). \end{aligned} \tag{2.31}$$

To have tachyon free theory these poles must be positive. By choosing the constants properly such a theory can be found. However, this is not the only restriction on the parameters that appear in the general quadratic theory. With these poles the residues can be found. These residues must be negative in order not to have any ghost terms. This is the other restriction that we use while finding a ghost and tachyon free model. However, in the most general form the residues are cumbersome. Therefore, we will restrict the theory and compute the poles and residues for some interesting limits. Also, we will compute the Newtonian potentials for these limits.

2.4 Massive theory in flat spacetime:

Looking at (2.30), we can see that the massless limit and the flat space limit do not commute because of the $\frac{M^2}{\Lambda}$ term. Therefore, if we want to get the usual Newtonian potentials in flat space and without mass term, the flat space limit must be taken before the massless limit. Otherwise, we encounter the well known vDVZ discontinuity. Let us first look at the limits which produce the vDVZ discontinuity that means we first take the flat space limit $\Lambda \to 0$ and go to massless limit $M^2 \to 0$. With the flat space limit (2.30) becomes;

$$4A = -2T'_{\mu\nu}\left\{\beta\partial^4 + \frac{1}{\kappa}\partial^2 - \frac{M^2}{\kappa}\right\}^{-1} T^{\mu\nu} + \frac{2}{D-1}T'\left\{\beta\partial^4 + \frac{1}{\kappa}\partial^2 - \frac{M^2}{\kappa}\right\}^{-1} T, \tag{2.32}$$

here the Lichnerowicz operator goes to $\triangle_L^{(2)} = -\partial^4$ and this can be seen from (2.25). The spectrum of (2.32) has two massive excitations which can be found by solving the second order equation for ∂^2 and the masses are the same as (2.13). Modifying (2.32) with the mass terms give

$$4A = -\frac{2}{\beta(m_-^2 - m_+^2)}\left\{T'_{\mu\nu}\left(\frac{1}{\partial^2 - m_+^2} - \frac{1}{\partial^2 - m_-^2}\right)T^{\mu\nu} - \frac{1}{(D-1)}T'\left(\frac{1}{\partial^2 - m_+^2} - \frac{1}{\partial^2 - m_-^2}\right)T\right\}. \tag{2.33}$$

For $\beta < 0$ and $\beta > 0$, (2.33) produces a massive ghost. Therefore, β must be set to zero to avoid ghost terms. The Newtonian potential energy (U) can be calculated by use of Green's function technique. For simplicity we take the sources as $T'_{00} \equiv m_1\delta(x-x_1)$, $T_{00} \equiv m_2\delta(x'-x_2)$, where m_1 and m_2 are masses of the sources, and the other terms of the energy-momentum tensor are taken to be zero. Moreover, we calculate these potentials in four and three dimensions. The Green's function for modified Helmholtz equation $\nabla^2 - k^2$ reads $\frac{1}{2\pi}K_0\left(k|r_1 - r_2|\right)$ and $\frac{e^{-k|r_1-r_2|}}{4\pi|r_1-r_2|}$ [74] for two and three space dimensions, respectively. Then the potential energies read

$$U = \begin{cases} \frac{1}{2\beta(m_+^2 - m_-^2)}\frac{m_1 m_2}{4\pi}[K_0(m_- r) - K_0(m_+ r)] & D = 3, \\ \frac{m_1 m_2}{3\beta(m_+^2 - m_-^2)}\frac{1}{4\pi r}[e^{-m_- r} - e^{-m_+ r}] & D = 4, \end{cases} \tag{2.34}$$

where $r \equiv |\vec{x_1} - \vec{x_2}|$ and $m_\pm^2$ are defined in (2.13). If we take the $\beta \to 0$ limit to take care of the ghost term, (2.34) becomes

$$U = \begin{cases} -\frac{\kappa}{8\pi}m_1 m_2 K_0(Mr) & D = 3, \\ -\frac{4}{3}\frac{Gm_1 m_2}{r}e^{-Mr} & D = 4. \end{cases} \tag{2.35}$$

The first equation of (2.35) is the Newtonian limit of massive gravity in three dimensions and it gives attractive force when κ is positive. For this case the $M \to 0$ limit does not exist. As $x \to 0$, $K_0(x) \to -\ln(x/2) + \gamma_E$, which gives the expected $\frac{1}{r}$ force for small separation of the sources. The second equation of (2.35) is the Newtonian limit of massive gravity in four dimensions. Unlike the three dimensional case, $M \to 0$ limit exists. However, the $\frac{4}{3}$ term indicates the famous vDVZ discontinuity. G is the Newton constant and it is taken as $\frac{\kappa}{16\pi}$. Therefore, in three dimensions massive gravity gives the correct Newtonian limit despite that in four dimensions it does not give the expected limit.

2.5 Massless theory in flat spacetime:

In the above discussion we see that first taking the flat space limit and then going to massless limit does not give us the expected Newtonian limit. In this section we first take the massless limit and then go to the flat space limit. When we set $M^2 = 0$ and $\Lambda \to 0$ in (2.30), we get

$$\begin{aligned} 4A = \; & -2T'_{\mu\nu}\left\{\beta\partial^4 + \frac{1}{\kappa}\partial^2\right\}^{-1} T^{\mu\nu} + \frac{2}{(D-1)}T'\left\{\beta\partial^4 + \frac{1}{\kappa}\partial^2\right\}^{-1} T \\ & -\frac{2}{(D-1)(D-2)}T'\left\{c\partial^4 - \frac{1}{\kappa}\partial^2\right\}^{-1} T. \end{aligned} \tag{2.36}$$

Unlike (2.32), (2.36) has three poles that are

$$\partial_1^2 = 0, \qquad \partial_2^2 = -\frac{1}{\kappa\beta}, \qquad \partial_3^2 = \frac{1}{\kappa c}. \tag{2.37}$$

To make a full analysis we need the residues of these poles which read

$$Res(\partial_1^2) = \frac{2\kappa(3-D)}{(D-2)} \qquad Res(\partial_2^2) = \frac{2\kappa(D-2)}{(D-1)} \qquad Res\left(\partial_3^2\right) = -\frac{2\kappa}{(D-1)(D-2)}.$$

From the second pole we see that $\kappa\beta < 0$ for not having tachyon and form the residue of this pole $\kappa < 0$ not to have a ghost. For negative κ the residue of the massless pole fixes the dimension to three, since for $D > 3$ makes the residue positive in which case it produces a massless ghost. The residue of the third pole becomes positive for negative κ. To eliminate this residue one has to set $c = 0$. For these conditions the theory becomes unitary and has its special name as New Massive Gravity (NMG). If we look to the Newtonian potential which is written as

$$U = \frac{\kappa}{8\pi}m_1 m_2\left(K_0(m_g r) - K_0(m_0 r)\right) \qquad\qquad D = 3, \tag{2.38}$$

where $m_g^2 \equiv -\frac{1}{\kappa\beta}$ and $m_0^2 \equiv \frac{1}{\kappa(8\alpha+3\beta)}$. For negative κ, the second term gives a massive ghost which decouples in the case of $8\alpha + 3\beta = 0$. In this case only the first term lives and gives an attractive force. Also, if we choose the Pauli-Fierz mass term as $M = m_g$ the NMG has the same Newtonian limit as the usual massive gravity. For $D > 3$, a massive ghost comes out and does not decouple unless $\beta = 0$. For example we can look at $D = 4$:

$$U = -\frac{Gm_1m_2}{r}\left(1 - \frac{4}{3}e^{-m_g r} + \frac{1}{3}e^{-m_a r}\right), \tag{2.39}$$

where $m_a^2 \equiv \frac{1}{2\kappa(3\alpha+\beta)}$. The term in the middle is the ghost term [21]. When we take $\beta \to 0$ the ghost term vanishes and taking $m_a \to \infty$ the last term drops out and we are left with the usual Newtonian potential in which case the vDVZ discontinuity does not appear. Therefore, vDVZ discontinuity does not appear in a curved background, since we can compare the smallness of the mass of graviton with another scale that is the cosmological constant. We can also generalize the Newtonian potentials to D dimensions. For this we need the general Green's function for the operators $\frac{1}{\partial^2}$ and $\frac{1}{\partial^2 - m_i^2}$. They are

$$G\left(x, x'\right) = \frac{1}{(2\pi)^{\frac{D-1}{2}}} \frac{1}{r^{D-3}} \left[2^{\frac{(D-5)}{2}} \Gamma\left(\frac{D-3}{2}\right)\right]$$

and

$$G\left(x, x'\right) = \frac{1}{(2\pi)^{\frac{D-1}{2}}} \frac{1}{r^{\frac{D-3}{2}}} \left[\left(\frac{1}{m_i^2}\right)^{\frac{3-D}{4}} K_{\frac{D-3}{2}}\left(r\, m_i^2\right)\right],$$

respectively. Γ is the gamma function and K_ν is the modified Bessel function. We first write the amplitude equation (2.36) as

$$\begin{aligned} 2A = {} & \frac{(D-2)\,\kappa\, m_1\, m_2}{(D-1)} \delta(x-x_1) \frac{1}{\partial^2 - m_g^2} \delta(x-x_2) \\ & - \frac{\left(D^2 - 4D + 3\right)\kappa\, m_1 m_2}{(D-1)\,(D-2)} \delta(x-x_1) \frac{1}{\partial^2} \delta(x-x_2) \\ & - \frac{\kappa\, m_1\, m_2}{(D-1)(D-2)} \delta(x-x_1) \frac{1}{\partial^2 - m_a^2} \delta(x-x_2). \end{aligned} \tag{2.40}$$

Taking again static sources, we find the potential as

$$\begin{aligned} U = {} & \frac{(D-2)\,\kappa\, m_1\, m_2}{(D-1)} \frac{1}{(2\pi)^{\frac{D-1}{2}}} \frac{1}{r^{\frac{D-3}{2}}} \left[\left(\frac{1}{m_g^2}\right)^{\frac{3-D}{4}} K_{\frac{D-3}{2}}\left(r\, m_g^2\right)\right] \\ & - \frac{\kappa\, m_1\, m_2}{(D-1)(D-2)} \frac{1}{(2\pi)^{\frac{D-1}{2}}} \frac{1}{r^{\frac{D-3}{2}}} \left[\left(\frac{1}{m_a^2}\right)^{\frac{3-D}{4}} K_{\frac{D-3}{2}}\left(r\, m_a^2\right)\right] \\ & - \frac{\left(D^2 - 4D + 3\right)\kappa\, m_1 m_2}{(D-1)\,(D-2)} \frac{1}{(2\pi)^{\frac{D-1}{2}}} \frac{1}{r^{D-3}} \left[2^{\frac{(D-5)}{2}} \Gamma\left(\frac{D-3}{2}\right)\right]. \end{aligned} \tag{2.41}$$

For any desired dimension this equation gives the Newtonian potential in flat spacetime and without any mass term. For $\kappa > 0$ the first term signals the ghost problem for all dimensions and the last two terms give attractive forces. We can confirm [21] that in any dimension the quadratic curvature gravity theory is not unitary except for the NMG point.

For curved backgrounds the calculation of the Newtonian potential is complicated because of the Lichnerowicz operator. Nonetheless, we can say that in curved backgrounds there will be an additional term to the Newtonian potentials. Apart from the term that contains Lichnerowicz operator, other terms can be calculated as above.

2.6 Conclusion and Discussion for Chapter-2

The tree level scattering amplitude of the general quadratic gravity theory with a Pauli-Fierz mass term in D dimensions is found and various limits of this amplitude equation is studied. For these limits the Newtonian potentials for three and four dimensions are calculated. First the flat space limit is taken and then the massless limit. Looking at the potential energies it is seen that the theory is not unitary. For taking care of the ghost term the $\beta \to 0$ limit is taken. In this limit the theory becomes unitary but the theory suffers from the vDVZ discontinuity in four dimensions. Massive gravity gives the correct Newtonian potentials in three dimensions since the $M^2 \to 0$ limit does not exist. The second limit that is taken is the massless limit and then the flat space limit. Analyzing the pole structure and the residues of the amplitude and calculating the Newtonian potential energies it is seen that the theory is unitary for a special condition for the coupling constants of the theory that is $8\alpha + 3\beta = 0$ in three dimensions. Also, the coupling constant of Hilbert-Einstein term becomes negative. Apart from three dimensions the general quadratic theory is non-unitary. To make the theory unitary again the $\beta \to 0$ limit is taken and for this case it becomes unitary. The usual Newtonian potential is obtained in four dimensions by taking all coupling constants of higher derivative terms to zero. Also, the Newtonian potentials for generic dimensions in flat space time are calculated.

CHAPTER 3

CANONICAL STRUCTURE OF HIGHER DERIVATIVE GRAVITY IN 3D

3.1 Introduction

In the previous chapter we saw that the general quadratic gravity theory has a special case in three dimensions. This special case is $8\alpha + 3\beta = 0$ and $\kappa < 0$ for the Lagrangian $\kappa^{-1}R + \alpha R^2 + \beta R_{\mu\nu}^2$ and named as New Massive Gravity (NMG) [35, 36]. Also, the parity violating extension of NMG is found [35, 36] by adding a gravitational Chern-Simons term to the theory. These theories have massive ghost-free spin-2 particles in their free spectrum around both flat and (anti)-de Sitter (A)dS. The most interesting property of NMG is that it gives a non linear extension to the Pauli-Fierz mass term for spin-2 particles. It is the only known example which provides this nonlinear extension. Thus, NMG is a suitable candidate to be a perturbatively well defined quantum gravity theory in three dimensions if it is unitary beyond tree level.

The theory is studied in many directions. Its ghost-freedom and tree level unitarity [17, 27, 35, 36, 62] and Newtonian limits [27] have been studied out. Furthermore, its classical solutions and issues related to the classical solutions were studied [35, 36, 37, 38, 40, 41, 42]. Also, its supergravity extension were given in [75].

In this chapter we study the canonical structure of the general quadratic curvature theory and give an explicitly gauge invariant analysis of it in three dimensions. The analysis is done for both flat and de Sitter (dS) spacetimes. In the flat space analysis the gravitational Chern-Simons term is added to the theory. By writing the canonical form of the general quadratic action we can easily see how NMG is singled out among all other higher curvature theories as a regular "harmonic oscillator" which can be thought as massive free field. These type oscillators do not have the Ostragradskian instability which spoil every higher time derivative theory [76][1]. Except of NMG, at the linearized level, all the other quadratic theories are ghost-ridden Pais-Uhlenbeck oscillators. Moreover, the discussion is extended by discussing the Newtonian limits, weak fields, the scattering of particles with mass and spin at the tree level.

The content of this chapter is as follows: In the first section the flat spacetime analysis is done by writing the canonical structure for both quadratic curvature theory with and without the gravitational Chern-Simons term. Also, the effects of static sources are analyzed in detail. Moreover, the weak field solutions are obtained in circularly symmetric case. In the second section the discussion is extended to curved spacetimes namely to de Sitter spacetime. Some of the useful calculations are given in the Appendix.

3.2 Higher-derivative spin-2 in flat spacetime

We start the analysis with the flat space case since it is simpler than the curved space background which is discussed in the next section. Without any constraints on the parameters the general action is

$$I = \int d^3x \, \sqrt{-g} \left(\frac{1}{\kappa} R + \alpha R^2 + \beta R_{\mu\nu}^2 \right). \tag{3.1}$$

To get the desired spin-2 model the action must be expanded around flat background $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where $\eta_{\mu\nu}$ is the usual flat spacetime metric with signature $(-,+,+)$ and $h_{\mu\nu}$ is the spin-2 field which actually is a symmetric rank-2 tensor and without any constraints has not only spin-2 component but also spin-1 and spin-0 components. In this section we do not consider the gravitational Chern-Simons term. However, in the next section we add it to (3.1). To get the action for $h_{\mu\nu}$ we first linearize the field equations that comes from the variation of (3.1) and then integrate the linear field equations. While integrating them, one must pay attention to the overall sign factor since the signs will become important in the discussion of unitarity. Up to boundary terms (3.1) becomes

$$I = -\frac{1}{2} \int d^3x \, h_{\mu\nu} \left[\frac{1}{\kappa} \mathcal{G}_L^{\mu\nu} + (2\alpha + \beta) \left(\eta^{\mu\nu} \Box - \partial^\mu \partial^\nu \right) R_L + \beta \Box \mathcal{G}_L^{\mu\nu} \right]. \tag{3.2}$$

[1] It was claimed that adding interactions might yield stable higher-time derivative theories [77].

Here, $\mathcal{G}_L^{\mu\nu}$ is the linearized Einstein tensor and R_L is the linearized Ricci scalar which is defined as $R_L \equiv \left(g_{\mu\nu}R^{\mu\nu}\right)_L$. The linearized Einstein and Ricci tensors, and linearized curvature scalar are as follows

$$\mathcal{G}_L^{\mu\nu} = R_L^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}R_L, \qquad R_L = \partial_\alpha\partial_\beta h^{\alpha\beta} - \Box h,$$
$$R_L^{\mu\nu} = \frac{1}{2}\left(\partial_\sigma\partial^\mu h^{\nu\sigma} + \partial_\sigma\partial^\nu h^{\mu\sigma} - \Box h^{\mu\nu} - \partial^\mu\partial^\nu h\right), \qquad h = \eta^{\mu\nu}h_{\mu\nu}, \tag{3.3}$$

where $\Box$ is the D'Alembertian operator that is $\Box = \partial_\mu\partial^\mu = -\partial_0^2 + \nabla^2$. Since the metric is perturbed around flat background, we raise and lower the indices with the flat metric $\eta_{\mu\nu}$. To analyze the canonical structure and explore the free fields $h_{\mu\nu}$ must be decomposed. This is done as follows:

$$h_{ij} \equiv \left(\delta_{ij} + \hat{\partial}_i\hat{\partial}_j\right)\phi - \hat{\partial}_i\hat{\partial}_j\chi + \left(\epsilon_{ik}\hat{\partial}_k\hat{\partial}_j + \epsilon_{jk}\hat{\partial}_k\hat{\partial}_i\right)\xi,$$
$$h_{0i} \equiv -\epsilon_{ij}\partial_j\eta + \partial_i N_L, \qquad h_{00} \equiv N, \tag{3.4}$$

where ϕ, χ, ξ, η, N_L, N are free functions of time and space $(t,\ \vec{x})$ and $\hat{\partial}_i \equiv \partial_i/\sqrt{-\nabla^2}$.

After decomposing the spin-2 field, the components of the linearized Einstein tensor must be written in terms of these free functions. Let us give some details of these calculations.

We first write the trace of the spin-2 field:

$$\eta^{\mu\nu}h_{\mu\nu} = h = \eta^{00}h_{00} + \eta^{ij}h_{ij}. \tag{3.5}$$

Putting (3.4) in (3.5) gives us

$$h = -N + h_i^i = -N + \phi + \chi,$$

where we have used $\hat{\partial}_i\hat{\partial}^i = -1$ and note that $h_i^i = \phi + \chi$. The linear Einstein tensor has three components that are $\mathcal{G}_{00}^L$, $\mathcal{G}_{0i}^L$ and $\mathcal{G}_{ij}^L$. Using (3.3) and summing the repeated indices these components can be written as

$$\mathcal{G}_{00}^L = \frac{1}{2}\left(\partial^i\partial^j h_{ij} - \partial^i\partial_i h_k^{\ k}\right),$$

$$\mathcal{G}_{0i}^L = \frac{1}{2}\left(\partial_j\partial_0 h_i^{\ j} + \partial_j\partial_i h_0^{\ j} - \partial_j\partial^j h_{0i} - \partial_0\partial_i h_k^{\ k}\right),$$

$$\begin{aligned}\mathcal{G}_{ij}^L =& \frac{1}{2}\left(\partial_0\partial_i h_j^{\ 0} + \partial_0\partial_j h_i^{\ 0} - \partial_0\partial^0 h_{ij} - \partial_k\partial^k h_{ij} + \partial_i\partial_j h_0^{\ 0} - \partial_i\partial_j h_k^{\ k}\right)\\ &- \frac{1}{2}\delta_{ij}\left(\partial^0\partial^0 h_{00} + 2\partial^0\partial^k h_{0k} + \partial^l\partial^k h_{lk} + \partial_0\partial^0 h_{00} - \delta^{kl}\partial_0\partial^0 h_{kl} + \partial_k\partial^k h_{00} - \delta^{kl}\partial_n\partial^n h_{kl}\right).\end{aligned}$$

After using (3.4) these components can be written in terms of gauge invariant objects as follows

$$\mathcal{G}^L_{00} = -\frac{1}{2}\nabla^2\phi, \qquad \mathcal{G}^L_{0i} = -\frac{1}{2}\left(\epsilon_{ik}\partial_k\sigma + \partial_i\dot{\phi}\right),$$
$$\mathcal{G}^L_{ij} = -\frac{1}{2}\left[\left(\delta_{ij} + \hat{\partial}_i\hat{\partial}_j\right)q - \hat{\partial}_i\hat{\partial}_j\ddot{\phi} - \left(\epsilon_{ik}\hat{\partial}_k\hat{\partial}_j + \epsilon_{jk}\hat{\partial}_k\hat{\partial}_i\right)\dot{\sigma}\right], \tag{3.6}$$

where "$\cdot$" means differentiation with respect to time t and q, σ, and ϕ are defined as

$$q \equiv \nabla^2 N - 2\nabla^2\dot{N}_L + \ddot{\chi}, \qquad \sigma \equiv \dot{\xi} - \nabla^2\eta, \tag{3.7}$$

which are gauge invariant under the transformation $\delta_\zeta h_{\mu\nu} = \partial_\mu\zeta_\nu + \partial_\nu\zeta_\mu$ and the Laplacian operator is $\nabla^2 = \partial_i^2 = \partial_i\partial^i = \partial_i\partial_i$. Note that, ϕ is already a gauge invariant component of $h_{\mu\nu}$.

Let us show this invariance of these objects: First we define the three vector as

$$\xi_\mu = \left(\Lambda_0,\, \epsilon_{ij}\partial_j + \partial_i K\right), \tag{3.8}$$

and then we transform the components of the spin-2 fields. From the transformations of h_{00} and h_{0i} we get

$$\eta' = \eta - \dot{\Lambda}, \qquad N'_L = N_L + \dot{K} + \Lambda_0, \qquad N' = N + 2\dot{\Lambda}_0. \tag{3.9}$$

From the last component we get three more equations by multiplying $h'_{ij} = h_{ij} + \partial_i\xi_j + \partial_j\xi_i$ with $\partial_j\partial_i$, $\epsilon_{il}\partial_l$ and δ_{ij} that are

$$\chi' = \chi + 2\nabla^2 K, \qquad \xi' = \xi - \nabla^2\Lambda, \qquad \phi' = \phi, \tag{3.10}$$

respectively. From here we can see explicitly the gauge invariance of ϕ. Using (3.9) and (3.10), one can get (3.7). The linearized Ricci scalar can be written in terms of the gauge invariant functions:

$$\begin{aligned} R^L &= -\left(\partial_\alpha\partial^\alpha\eta^{\mu\nu} - \partial^\mu\partial^\nu\right)h_{\mu\nu} \\ &= \partial_0^2 h_i^{\ i} + \partial_i^2\left(h_{00} - h_i^{\ i}\right) - 2\partial_0\partial_i h_{0i} + \partial_i\partial_j h_{ij}, \end{aligned}$$

again using (3.4) the scalar curvature becomes

$$R^L = \ddot{\phi} + \ddot{\chi} - \nabla^2\phi + \nabla^2 N - 2\nabla^2 N_L.$$

Putting the first equation of (3.7) finally we get

$$R_L = q - \Box\phi. \tag{3.11}$$

Although we have six free functions, the Bianchi identity $\partial_\mu \mathcal{G}_L^{\mu\nu} = 0$ reduces the number of arbitrary functions to three. Therefore, there are two sets of functions that can be used which are ϕ, σ, q and ϕ, σ, R_L. We can write the total action in terms of these gauge invariant free functions. Let us start with the Einstein-Hilbert part of the action that is

$$I_{EH} = -\frac{1}{2\kappa}\int d^3x\, h_{\mu\nu}\mathcal{G}_L^{\mu\nu},$$

then doing the summation about the repeated indices and using (3.4), (3.6) and (3.7) we get

$$I_{EH} = -\frac{1}{2\kappa}\int d^3x\, h_{\mu\nu}\mathcal{G}_L^{\mu\nu} = \frac{1}{2\kappa}\int d^3x\left(\phi q + \sigma^2\right). \tag{3.12}$$

Here we can see that the pure Einstein theory does not have any propagating degrees of freedom since there is not a dynamical part which shows itself as a D'Alembertian operator. With the same route we can find the other two parts of the action. Instead of using this technique one can also use the self-adjointness of the operators. In both ways one can write the quadratic parts of the action in explicitly gauge invariant forms. They are

$$I_{2\alpha+\beta} = -\frac{2\alpha+\beta}{2}\int d^3x\, h_{\mu\nu}\left(\eta^{\mu\nu}\Box - \partial^\mu\partial^\nu\right)R_L = \frac{2\alpha+\beta}{2}\int d^3x\, R_L^2, \tag{3.13}$$

$$I_\beta = -\frac{\beta}{2}\int d^3x\, h_{\mu\nu}\Box\mathcal{G}_L^{\mu\nu} = -\frac{\beta}{2}\int d^3x\left(-2\mathcal{G}^L_{\mu\nu}\mathcal{G}_L^{\mu\nu} + \frac{1}{2}R_L^2\right) = \frac{\beta}{2}\int d^3x\left(q\Box\phi + \sigma\Box\sigma\right). \tag{3.14}$$

In the first action we move all the derivatives on $h_{\mu\nu}$ and that combination gives us again the linearized Ricci scalar. In the second action I_β, first the derivatives are moved on the spin-2 field and then (3.3) is used. Finally using the Bianchi identity one can achieve (3.14). Adding (3.12), (3.13) and (3.14) the total action comes out in gauge invariant combinations as

$$I = \frac{1}{2}\int d^3x\left[\frac{1}{\kappa}\phi q + (2\alpha+\beta)(q-\Box\phi)^2 + \beta q\Box\phi\right] + \frac{\beta}{2}\int d^3x\left(\sigma\Box\sigma + \frac{1}{\kappa\beta}\sigma^2\right). \tag{3.15}$$

From this equation, it can be seen immediately that σ shows itself as a single massive scalar field with mass $m_g^2 = -\frac{1}{\kappa\beta}$. Not to have negative mass $\kappa\beta$ must be negative and not to have ghost β must be positive for the σ field. For these signs κ must be chosen negative. For the remaining part of the action the discussion bifurcates whether $2\alpha + \beta = 0$, or not. Let us discuss these cases separately:

3.2.1 $2\alpha + \beta \neq 0$ case:

In this case the q field can be eliminated by taking the variation with respect to it. Doing so gives

$$q = -\frac{1}{2(2\alpha+\beta)}\left(\frac{\phi}{\kappa} + \beta\Box\phi\right) + \Box\phi, \tag{3.16}$$

and inserting the q field into (3.15), the action for the ϕ field becomes

$$I_\phi = \frac{1}{2}\int d^3x \left[\frac{\beta(8\alpha+3\beta)}{4(2\alpha+\beta)}(\Box\phi)^2 + \frac{(4\alpha+\beta)}{2\kappa(2\alpha+\beta)}\phi\Box\phi - \frac{1}{4\kappa^2(2\alpha+\beta)}\phi^2\right]. \tag{3.17}$$

In this action there are some special limits. The first limit is the NMG limit $8\alpha + 3\beta = 0$. For this limit the higher derivative term drops out and the ϕ field describes a single massive degree of freedom

$$I_{NMG,\phi} = -\frac{1}{2\kappa}\int d^3x \left(\phi\Box\phi + \frac{1}{\kappa\beta}\phi^2\right), \tag{3.18}$$

which has the same mass as the σ field. Since both degrees of freedom have the same masses in the NMG limit, the theory is parity-invariant. Note that, not to have a ghost κ must again be negative. Also, NMG is not a higher derivative theory at the linearized level. The theory does not have the Ostragradski instability which appears in theories that have higher order derivatives.

Another interesting limit is $4\alpha + \beta = 0$. The middle term in (3.17) drops out and the action (3.17) becomes

$$I_\phi = \frac{1}{4\kappa}\int d^3x \left(\kappa\beta(\Box\phi)^2 - \frac{1}{\kappa\beta}\phi^2\right). \tag{3.19}$$

This theory is tachyonic for $\kappa < 0$ and gives a ghost term for $\kappa > 0$, keeping in mind that $\kappa\beta < 0$.

Taking $\beta = 0$ also drops the higher derivative term. For this case the action reads

$$I_\phi = \frac{1}{2\kappa}\int d^3x \left(\phi\Box\phi - \frac{1}{8\kappa\alpha}\phi^2\right). \tag{3.20}$$

In this case, to avoid ghost term $\kappa > 0$ and not to have tachyon $\alpha > 0$, where the mass can be defined as $m^2 = \frac{1}{8\kappa\alpha}$.

For the general coupling constants, the above action (3.17) is a higher derivative Pais-Uhlenbeck oscillator. In order to decouple the fields we define new fields that are linear in terms of old fields. By inspection the fields can be defined as follows

$$\varphi_1 \equiv \phi - \frac{\Box\phi}{m_g^2}, \qquad \varphi_2 \equiv \phi - \frac{\Box\phi}{m_s^2}, \tag{3.21}$$

and with these definitions the uncoupled Lagrangian must be as follows

$$K_1\varphi_1\left(\Box - m_s^2\right)\varphi_1 + K_2\varphi_2\left(\Box - m_g^2\right)\varphi_2.$$

Here K_1 and K_2 are unknowns. To find these unknown terms we put (3.21) into above Lagrangian and compare that result with (3.17). After getting the unknown factors the action can be written in terms of simple oscillators and (3.17) becomes

$$I_\phi = \frac{1}{64\kappa\left(2\alpha+\beta\right)^2}\int d^3x\left[\left(8\alpha+3\beta\right)^2\varphi_1\left(\Box - m_s^2\right)\varphi_1 - \beta^2\varphi_2\left(\Box - m_g^2\right)\varphi_2\right], \tag{3.22}$$

and the two fields φ_1, φ_2 have masses m_s and m_g respectively. m_g is defined above and m_s is defined as

$$m_s^2 = \frac{1}{\kappa\left(8\alpha+3\beta\right)}.$$

Note that for the general case the only restriction on the coupling constants is $2\alpha+\beta \neq 0$. For $\kappa < 0$ the φ_1 field gives ghost and for $\kappa > 0$ the φ_2 has negative kinetic energy. Also, not to have negative mass term $8\alpha + 3\beta$ must be positive for positive κ and negative for negative κ.

3.2.2 $2\alpha + \beta = 0$ case:

In this case we can follow two routes: we either start from (3.15) or from (3.22). These two actions give different results and as we have seen above in the second action this limit provides a singular theory, while in the first action it does not.

We start with (3.22) and take $\epsilon \equiv 2\alpha+\beta \to 0$ limit. By this limit $m_s = \frac{m_g^2}{\left(1-\frac{4\epsilon}{\beta}\right)} \approx m_g^2\left(1+\frac{4\epsilon}{\beta}\right)$ where in the last step we have taken the Taylor expansion around small ϵ. Note that m_g^2 does not depend on ϵ. With the same limit we can write the decoupled fields in terms of ϵ as $\varphi_2 = \phi - \frac{\Box\phi}{m_g^2\left(1+\frac{4\epsilon}{\beta}\right)} \approx \varphi_1 + \frac{4\epsilon}{\beta m_g^2}$ and again φ_1 does not depend on ϵ. With these expansions, (3.22) gives us up to second order

$$I_\phi = \frac{1}{8\kappa\epsilon}\int d^3x\left\{\frac{\beta}{m_g^2}\left[\left(\Box - m_g^2\right)\phi\right]^2 - 4\epsilon\phi\left(\Box - m_g^2\right)\phi + O\left(\epsilon^2\right)\right\}. \tag{3.23}$$

This action is a degenerate Pais-Uhlenbeck oscillator which is known as ghost free for some parameter ranges. However, if we start from (3.15) and set $2\alpha + \beta = 0$, we get

$$I_\phi = \frac{\beta}{2} \int d^3x \left(q\Box\phi - m_g^2 q\phi \right). \tag{3.24}$$

Taking variation with respect to q or ϕ gives us a massive wave equation. However, these equations do not say anything about the ghost structure or tachyonic behaviour of the theory. So that, we must again separate the fields by redefining the fields such that $q \equiv m_g^2 (\Psi_1 + \Psi_2)$ and $\phi \equiv \Psi_1 - \Psi_2$. With these definitions (3.24) turns into

$$I = \frac{m_g^2 \beta}{2} \int d^3x \left[\left(\Psi_1 \Box \Psi_1 - m_g^2 \Psi_1^2 \right) - \left(\Psi_2 \Box \Psi_2 - m_g^2 \Psi_2^2 \right) \right]. \tag{3.25}$$

From the above discussion we know that β must be positive not to have ghost term, but in this case if it is taken positive, Ψ_2 becomes a ghost excitation. As we discussed in the previous chapter the Newtonian limit of this theory is interesting [27]. The Newtonian potential of this theory becomes zero when two static sources are taken into account, since the ghost excitation that is the repulsive component cancels the spin-2 part which is the attractive component. This is the same situation that happens in the pure Einstein gravity.

3.2.3 Adding static and spinning sources

Up to now the analysis depended on theories without any interaction. However, when interaction enters into the picture it may change the particle spectrum of the theory. So that, we turn our analysis to source dependent higher derivative gravity. First we add static sources to our analysis and then we generalize this analysis by adding spinning masses in flat spacetime.

3.2.3.1 Static Sources:

The matter can be added to the theory by the usual gravity-matter coupling that is

$$I_{\text{source}} = \frac{1}{2} \int d^3x \, h_{\mu\nu} T^{\mu\nu}. \tag{3.26}$$

For the static source case we take $T^{00} = \rho(\vec{x})$, $T^{0i} = 0$, $T^{ij} = 0$, and the above action becomes

$$I_{\text{source}} = \frac{1}{2} \int d^3x \, N \rho(\vec{x}) = \frac{1}{2} \int d^3x \left(\frac{1}{\nabla^2} q + 2\dot{N}_L - \frac{1}{\nabla^2} \ddot{\chi} \right) \rho(\vec{x}), \tag{3.27}$$

where we used $q \equiv \nabla^2 N - 2\nabla^2 \dot{N}_L + \ddot{\chi} \Rightarrow N = \frac{1}{\nabla^2} q + 2\dot{N}_L - \frac{1}{\nabla^2}\ddot{\chi}$. After dropping the boundary terms and taking the source terms static, that is $\dot{\rho}(\vec{x}) = 0$, and using the symmetry property of the Green's function in the first term of (3.27), the source action becomes

$$I_{\text{source}} = \frac{1}{2}\int d^3x\, q \frac{1}{\nabla^2}\rho.$$

We add this action to the general total action (3.15)

$$I = \frac{1}{2}\int d^3x \left[\frac{1}{\kappa}\phi q + (2\alpha + \beta)(q - \Box\phi)^2 + \beta q \Box\phi + q\frac{1}{\nabla^2}\rho\right] + \frac{\beta}{2} I_\sigma, \tag{3.28}$$

where I_σ is the action that is only constructed from the σ field and is already in the simple harmonic oscillator form. To eliminate the last term in the parenthesis of (3.28) we define new fields so that we can decouple the fields easily. The last and the first terms can be combined and redefined as a new field $\varphi \equiv \phi + \kappa\frac{1}{\nabla^2}\rho$ and by inspection the q field can be redefined as $\tilde{q} \equiv q + \kappa\rho$. Putting these new fields into (3.28) and taking $\Box = -\partial_t^2 + \nabla^2$, we get the total action as

$$\begin{aligned} I = \frac{1}{2}\int d^3x \Big[&\frac{1}{\kappa}\left(\varphi\tilde{q} - \kappa\varphi\rho + \sigma^2\right) + (2\alpha + \beta)(\tilde{q} - \Box\varphi)^2 \\ &+ \beta\left(\tilde{q}\Box\varphi - \kappa\rho\Box\varphi - \kappa\tilde{q}\rho + \kappa^2\rho^2 + \sigma\Box\sigma\right)\Big]. \end{aligned} \tag{3.29}$$

Let us concentrate on the NMG case, that is $8\alpha + 3\beta = 0$. For this case $2\alpha + \beta = \frac{\beta}{4}$. Taking variation with respect to $\tilde{q}$ field, the following equation comes out

$$\frac{1}{\kappa}\varphi + \frac{\beta}{2}(\tilde{q} - \Box\varphi) + \beta(\Box\varphi - \kappa\rho) = 0.$$

From this equation $\tilde{q}$ can be defined in terms of the other fields as

$$\tilde{q} = 2\kappa\rho - \Box\varphi - \frac{2}{\kappa\beta}\varphi. \tag{3.30}$$

Putting (3.30) into (3.29) the full action becomes

$$I = \frac{1}{2}\int d^3x \left[\beta\left(\sigma\Box\sigma - m_g^2\sigma^2\right) - \frac{1}{\kappa}\left(\varphi\Box\varphi - m_g^2\varphi^2\right) + \varphi\rho\right]. \tag{3.31}$$

The last term in the action is the interaction part and it gives an attractive potential energy for negative κ. In order to find the potential we take the variation of (3.31) with respect to φ field that gives

$$\frac{2}{\kappa}\left(\Box - m_g^2\right)\varphi = \rho \Rightarrow \frac{2}{\kappa}\left(\Box - m_g^2\right)\varphi = \frac{\kappa}{2}\frac{1}{\Box - m_g^2}\rho. \tag{3.32}$$

Putting the φ into the interaction part of (3.31) the potential energy reads

$$U=\frac{\kappa}{4}\int d^2x\,\rho_1\frac{1}{\nabla^2-m_g^2}\rho_2=\frac{\kappa}{8\pi}m_1m_2K_0\left(m_gr\right). \tag{3.33}$$

From this equation it can be seen easily that for negative κ the interaction part gives an attractive potential energy, that means in NMG case adding static sources does not spoil the unitarity structure or the tachyonic behaviour. In (3.33) the sources are defined as $\rho_1(\vec{x})=m_1\delta^{(2)}(\vec{x}-\vec{x}_1)$, $\rho_2(\vec{x})=m_2\delta^{(2)}(\vec{x}-\vec{x}_2)$, and K_0 is the modified Bessel function of the second kind. This result also matches with the result that we found in the previous chapter [27].

3.2.3.2 Spinning masses:

For this case the energy-momentum tensor must be written as

$$T_{00}=m\delta^{(2)}(\vec{r}-\vec{r}_1), \qquad T^i_{\ 0}=\frac{1}{2}j\,\epsilon^{ij}\partial_j\delta^{(2)}(\vec{r}-\vec{r}_1), \qquad T_{ij}=0,$$

where m is the mass and j is the spin of the point-like source. The general amplitude and Newtonian potential was calculated for static sources at flat spacetime in D-dimensions in the previous chapter [27]. They become

$$\begin{aligned}4A \;=\; & \int d^3x\left\{-2T'_{\mu\nu}\left[\beta\partial^4+\frac{1}{\kappa}\partial^2\right]^{-1}T^{\mu\nu}+T'\left[\beta\partial^4+\frac{1}{\kappa}\partial^2\right]^{-1}T\right.\\ & \left.-T'\left[(8\alpha+3\beta)\partial^4-\frac{1}{\kappa}\partial^2\right]^{-1}T\right\},\end{aligned}$$

and

$$U=\frac{\kappa\,m_1\,m_2}{2}\frac{1}{(2\pi)}\left[K_0\left(r\,m_g\right)-K_0\left(r\,m_a\right)\right]$$

in three dimensions. The only added part from the spin will come from the T_{0i} components of the energy-momentum tensor. It will read as

$$-4T'_{i0}\left(\beta\partial^4+\frac{1}{\kappa}\partial^2\right)^{-1}T^{i0}=-\frac{j_1j_2}{\beta m_g^2}\partial_i\delta^{(2)}(\vec{r}-\vec{r}_1)\left(\frac{1}{\partial^2}-\frac{1}{\partial^2-m_g^2}\right)\partial_i\delta^{(2)}(\vec{r}-\vec{r}_2).$$

The Newtonian potential of these operators was calculated in the previous chapter. Using that result the potential energy reads

$$\begin{aligned}
-4T'_{i0}\left\{\beta\partial^4+\frac{1}{\kappa}\partial^2\right\}^{-1}T^{i0} &= -\frac{4}{\beta}T'_{i0}\frac{1}{\partial^2(\partial^2+\frac{1}{\beta\kappa})}T_{i0} \\
&= \frac{4}{\beta m_g^2}T'_{i0}\frac{1}{\partial^2}T_{i0}-\frac{4}{\beta m_g^2}T'_{i0}\frac{1}{\partial^2-m_g^2}T_{i0} \\
&= +\frac{j_1 j_2}{\beta m_g^2}\epsilon^{ij}\partial_j\delta^2(\vec{r}-\vec{r}_1)\frac{1}{\partial^2}\epsilon^{ik}\partial_k\delta^2(\vec{r}-\vec{r}_2) \\
&\quad -\frac{j_1 j_2}{\beta m_g^2}\epsilon^{ij}\partial_j\delta^2(\vec{r}-\vec{r}_1)\frac{1}{\partial^2-m_g^2}\epsilon^{ik}\partial_k\delta^2(\vec{r}-\vec{r}_2).
\end{aligned}$$

Using $\epsilon^{ij}\epsilon^{ik}=\delta^{jk}$ the spinning part of the amplitude becomes

$$\begin{aligned}
-4T'_{i0}\left\{\beta\partial^4+\frac{1}{\kappa}\partial^2\right\}^{-1}T^{i0} &= \frac{j_1 j_2}{\beta m_g^2}\partial_i\delta^2(\vec{r}-\vec{r}_1)\frac{1}{\partial^2}\partial_i\delta^2(\vec{r}-\vec{r}_2) \\
&\quad -\frac{j_1 j_2}{\beta m_g^2}\partial_i\delta^2(\vec{r}-\vec{r}_1)\frac{1}{\partial^2-m_g^2}\partial_i\delta^2(\vec{r}-\vec{r}_2).
\end{aligned}$$

Using the Green's function of these operators and taking the differentiations in front of the functions and carrying the space integrations, we get

$$\begin{aligned}
-4T'_{i0}\left\{\beta\partial^4+\frac{1}{\kappa}\partial^2\right\}^{-1}T^{i0} &= -\frac{j_1 j_2}{2\pi\beta m_g^2}\vec{\nabla}^2 ln|\vec{r}_1-\vec{r}_2| \\
&\quad -\frac{j_1 j_2}{2\pi\beta m_g^2}\vec{\nabla}^2 K_0\left(m_g|\vec{r}_1-\vec{r}_2|\right).
\end{aligned}$$

Note that $\vec{\nabla}^2 ln|\vec{r}_1-\vec{r}_2|=-2\pi\delta^2\left(\vec{r}_1-\vec{r}_2\right)$ and if we assume that the sources are in separate positions, that is $\vec{r}_1\neq\vec{r}_2$ the first term in the above equation vanishes and the second term becomes $\left(\vec{\nabla}^2-m_g^2\right)K_0\left(m_g|\vec{r}_1-\vec{r}_2|\right)=2\pi\delta^2\left(\vec{r}_1-\vec{r}_2\right)=0\Rightarrow\vec{\nabla}^2K_0\left(m_g|\vec{r}_1-\vec{r}_2|\right)=m_g^2K_0\left(m_g|\vec{r}_1-\vec{r}_2|\right)$ and the amplitude takes the form

$$-4T'_{i0}\left(\beta\partial^4+\frac{1}{\kappa}\partial^2\right)^{-1}T^{i0}=-\frac{j_1 j_2}{2\pi\beta}K_0\left(m_g\left|\vec{r}_1-\vec{r}_2\right|\right). \tag{3.34}$$

With this spinning part, the total Newtonian potential energy, U = Amplitude/time, can be written as

$$U=\frac{\kappa}{8\pi}\left(m_1m_2+4m_g^2j_1j_2\right)K_0\left(m_g\left|\vec{r}_1-\vec{r}_2\right|\right)-\frac{\kappa}{8\pi}m_1m_2K_0\left(m_s\left|\vec{r}_1-\vec{r}_2\right|\right). \tag{3.35}$$

The signs of j_1 and j_2 are not fixed so that they can be both positive or negative. Therefore, potential energy coming from this part could be both attractive or repulsive. In the NMG case this situation does not change since for this case only the last term in (3.35) drops out.

3.2.4 Weak field approximation

Up to this point we have found some results from the linearized theory. In this section we will try to get some of the above results from the nonlinear theory (3.1). We start with the ansatz

$$ds^2 = -f(r)\,dt^2 + \frac{b^2(r)}{f(r)}dr^2 + r^2 d\theta^2, \tag{3.36}$$

and insert it to (3.1). Here $f(r)$ and $b(r)$ are functions and the action will be varied with respect to these functions[2]. We will consider only the NMG theory. To find an approximate solution the functions can be defined as follows: $f(r) = 1 + \int^r dr\, a(r)$, $b(r) = 1 + \int^r dr\, v(r)$. Here, $v(r)$ and $a(r)$ are small functions. Putting the ansatz into (3.1) with the defined functions we get

$$\frac{4}{\kappa}v + 2\beta v'' + 2\beta a'' + r\beta a''' = 0, \tag{3.37}$$

$$\beta r^2 a'' + \frac{2}{\kappa}r^2 a + 2r\beta v' - 2\beta v = 0, \tag{3.38}$$

up to first order in $v(r)$ and $a(r)$. Here, $'$ denotes differentiation with respect to r. From (3.37) and (3.38), $v(r)$ can be written in terms of $a(r)$ as $v(r) = a(r) + \frac{r}{2}a'(r)$. To have an ordinary differential equation we put $v(r)$ into (3.38) and we get

$$r^2 a'' + ra' - a\left(m_g^2 r^2 + 1\right) = 0. \tag{3.39}$$

By solving this equation we can find $a(r)$ as $a(r) = c_1 I_1\left(m_g r\right) + c_2 K_1\left(m_g r\right)$, where c_1 and c_2 are constants. From this equation also $v(r)$ can be determined using $v(r) = a(r) + \frac{r}{2}a'(r)$. From the above ansatz we know that $g_{00} \approx -1 - \int^r dr\, a(r)$. For g_{rr} we first take the square of $b(r)$ and drop the second order term. Then we expand $\frac{1}{f(r)}$ up to first order that is $\frac{1}{1+\epsilon} \approx 1 - \epsilon$. Combining these approximations we get $g_{rr} \approx 1 + \int^r dr\, [2v(r) - a(r)]$. We know that for decaying fields that is for $r \to \infty$, $a(r) \to 0$. Since I_1 diverges to infinity for decaying fields c_1 must vanish, and the metric components become

$$g_{00} \approx -1 + cK_0\left(m_g r\right), \qquad g_{rr} \approx 1 + dK_1\left(m_g r\right). \tag{3.40}$$

Here c and d are constants. When we compare these components with Schwarzschild solution we see that the constants must be related to the mass of the sources. This is consistent with our earlier result (3.33).

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[2] See the details of this Weyl trick in [78].

3.2.5 Higher-derivative gravity plus a Chern-Simons term

In this part the above discussion will be extended by adding the gravitational Chern-Simons term to the quadratic gravity theory in flat space. We will follow the same route; first we will decompose the linear form of the action that comes from the gravitational Chern-Simons term. Then the coupled field will be decomposed by taking the Fourier transform of the Lagrangian and writing it in a matrix form and diagonalizing this matrix. The general quadratic action with gravitational Chern-Simons term is [33, 34]

$$I = \int d^3x\, \sqrt{-g}\left[\frac{1}{\kappa}R + \alpha R^2 + \beta R_{\mu\nu}^2 - \frac{1}{2\mu}\epsilon^{\lambda\mu\nu}\Gamma^{\rho}{}_{\lambda\sigma}\left(\partial_\mu \Gamma^{\sigma}{}_{\rho\nu} + \frac{2}{3}\Gamma^{\sigma}{}_{\mu\beta}\Gamma^{\beta}{}_{\nu\rho}\right)\right], \tag{3.41}$$

where $\epsilon_{012} = 1$, and μ is the Chern-Simons coupling with an arbitrary sign[3].

The linearization of the gravitational Chern-Simons bit yields

$$I_{CS} = -\frac{1}{2\mu}\int d^3x\, \epsilon_{\mu\alpha\beta}\mathcal{G}_L^{\alpha\nu}\partial^\mu h^\beta{}_\nu + O\left(h^3\right).$$

First we write all the summations that is

$$\begin{aligned} I_{CS} &= -\frac{1}{2\mu}\int d^3x\, \epsilon_{\mu\alpha\beta}\mathcal{G}_L^{\alpha\nu}\partial^\mu h^\beta{}_\nu = -\frac{1}{2\mu}\int d^3x\, \eta^{\rho\nu}\epsilon^{\mu\alpha\beta}\mathcal{G}^L_{\alpha\rho}\partial_\mu h_{\beta\nu} \\ &= \frac{1}{2\mu}\int d^3x\, \epsilon_{ij}\left(-\mathcal{G}^L_{i0}\partial_0 h_{j0} + \mathcal{G}^L_{00}\partial_i h_{j0} - \mathcal{G}^L_{j0}\partial_i h_{00} - \mathcal{G}^L_{jk}\partial_0 h_{ik} + \mathcal{G}^L_{0k}\partial_j h_{ik} - \mathcal{G}^L_{ik}\partial_j h_{0k}\right), \end{aligned} \tag{3.42}$$

where $\epsilon_{ij} \equiv \epsilon_{0ij}$. Using (3.4) and (3.6) the terms that appear in the last line in the above equation are

$$\begin{aligned} &-\epsilon_{ij}\mathcal{G}^L_{0i}\partial_0 h_{0j} = \frac{1}{2}\left(-\sigma\nabla^2\dot{N}_L + \ddot{\phi}\nabla^2\eta\right), \qquad \epsilon_{ij}\mathcal{G}^L_{00}\partial_i h_{0j} = -\frac{1}{2}\nabla^2\phi\nabla^2\eta, \\ &-\epsilon_{ij}\mathcal{G}^L_{0j}\partial_i h_{00} = \frac{1}{2}\sigma\nabla^2 N, \qquad -\epsilon_{ij}\mathcal{G}^L_{jk}\partial_0 h_{ik} = \frac{1}{2}\left(q\dot{\xi} - \ddot{\phi}\dot{\xi} + \sigma\ddot{\chi} - \sigma\ddot{\phi}\right), \\ &\epsilon_{ij}\mathcal{G}^L_{0k}\partial_j h_{ik} = \frac{1}{2}\left(\sigma\nabla^2\phi + \dot{\xi}\nabla^2\phi\right), \qquad -\epsilon_{ij}\mathcal{G}^L_{ik}\partial_j h_{0k} = \frac{1}{2}\left(-q\nabla^2\eta - \sigma\nabla^2\dot{N}_L\right), \end{aligned} \tag{3.43}$$

where we have taken out total derivatives in the needed steps and dropped the boundary terms. With these equations the action of the gravitational Chern-Simons term takes the form

$$I_{CS} = \frac{1}{2\mu}\int d^3x\, \left[\sigma\left(q + \Box\phi\right)\right], \tag{3.44}$$

[3] Without the α, β terms, but with a Pauli-Fierz mass term, canonical analysis was carried out in [59, 68].

where we have defined the q and σ terms in the previous section. With these, action (3.41) becomes

$$I=\frac{1}{2}\int d^3x\left[\frac{1}{\kappa}\left(\phi q+\sigma^2\right)+(2\alpha+\beta)(q-\Box\phi)^2+\beta(q\Box\phi+\sigma\Box\sigma)+\frac{1}{\mu}\sigma(q+\Box\phi)\right], \tag{3.45}$$

where all the fields are gauge invariant. For the general case, that is $2\alpha+\beta\neq 0$ and both $\alpha\neq 0$, $\beta\neq 0$, we eliminate the q field by taking the variation of (3.45) with respect to the q field. From this equation of motion we get

$$0=\frac{1}{\kappa}\phi+2(2\alpha+\beta)(q-\Box\phi)+\beta\Box\phi+\frac{1}{\mu}\sigma, \tag{3.46}$$

and

$$q=(4\alpha+\beta)\Box\phi-\frac{1}{\kappa}\phi-\frac{1}{\mu}\sigma. \tag{3.47}$$

Putting (3.47) in (3.45), the general action takes the following form

$$\begin{aligned}I=\frac{1}{2}\int d^3x&\left\{\beta\left[\sigma\Box\sigma+\left(\frac{1}{\kappa\beta}-\frac{1}{4\mu^2\beta(2\alpha+\beta)}\right)\sigma^2\right]+\left[\frac{1}{\mu}+\frac{(4\alpha+\beta)}{2\mu(2\alpha+\beta)}\right]\sigma\Box\phi\right.\\&\left.-\frac{1}{2\kappa\mu(2\alpha+\beta)}\sigma\phi+\frac{1}{\kappa}\left[\frac{\beta\kappa(8\alpha+3\beta)}{4(2\alpha+\beta)}(\Box\phi)^2+\frac{(4\alpha+\beta)}{2(2\alpha+\beta)}\phi\Box\phi-\frac{1}{4\kappa(2\alpha+\beta)}\phi^2\right]\right\}.\end{aligned} \tag{3.48}$$

For a proper analysis we must decouple the σ and the ϕ fields. This decoupling procedure can be done mathematically for generic α and β but since we are interested in the NMG case we take the $8\alpha+3\beta=0$ point. For this case the action reads

$$I_{NMG-CS}=\frac{\beta}{2}\int d^3x\left\{\left[\sigma\Box\sigma-\left(m_g^2+\frac{1}{\mu^2\beta^2}\right)\sigma^2\right]+\frac{2m_g^2}{\beta\mu}\sigma\phi+m_g^2\left(\phi\Box\phi-m_g^2\phi^2\right)\right\}.$$

To decouple these fields we will follow a different route: First we take the Fourier transform of the fields and put the Lagrangian in a matrix form then we diagonalize this matrix.

Taking the Fourier transform of the Lagrangian and putting in the matrix form yields

$$L_{FT}=\begin{pmatrix}\tilde{\sigma} & m_g\tilde{\phi}\end{pmatrix}\begin{bmatrix}-\left(k^2+m_g^2+\frac{1}{\mu^2\beta^2}\right) & \frac{m_g}{\beta\mu}\\ \frac{m_g}{\beta\mu} & -\left(k^2+m_g^2\right)\end{bmatrix}\begin{pmatrix}\tilde{\sigma}\\ m_g\tilde{\phi}\end{pmatrix}.$$

Using the eigenvalue equation $det(\lambda I-A)=0$ the eigenvalues of the A matrix can be found where I is 2×2 identity matrix, A is the above matrix and λ is the eigenvalues of the A matrix. Solving the eigenvalue equation for λ we get

$$\lambda^2+\left[2\left(k^2+m_g^2\right)+\frac{1}{\mu^2\beta^2}\right]\lambda+\left(k^2+m_g^2\right)^2+\frac{k^2}{\mu^2\beta^2}=0,$$

with roots

$$\lambda_{\pm} = -k^2 - m_g^2 - \frac{1}{2\mu^2\beta^2} \pm \frac{1}{\mu\beta}\sqrt{m_g^2 + \frac{1}{4\mu^2\beta^2}}. \tag{3.49}$$

Now, let's find eigenvectors. For $\lambda_{\pm}$;

$$\begin{bmatrix} -\frac{1}{2\mu^2\beta^2} \mp \frac{1}{\mu\beta}\sqrt{m_g^2 + \frac{1}{4\mu^2\beta^2}} & \frac{m_g}{\beta\mu} \\ \frac{m_g}{\beta\mu} & \frac{1}{2\mu^2\beta^2} \mp \frac{1}{\mu\beta}\sqrt{m_g^2 + \frac{1}{4\mu^2\beta^2}} \end{bmatrix} \begin{bmatrix} u_1^{\pm} \\ u_2^{\pm} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{3.50}$$

$$\begin{bmatrix} u_1^{\pm} \\ u_2^{\pm} \end{bmatrix} = N_{\pm} \begin{bmatrix} 1 \\ \frac{1}{2\mu\beta m_g} \pm \frac{1}{m_g}\sqrt{m_g^2 + \frac{1}{4\mu^2\beta^2}} \end{bmatrix}, \; N_{\pm} = \left[1 + \left(\frac{1}{2\mu\beta m_g} \pm \frac{1}{m_g}\sqrt{m_g^2 + \frac{1}{4\mu^2\beta^2}}\right)^2\right]^{-\frac{1}{2}} \tag{3.51}$$

where $N_{\pm}$ are the normalization factors. Then we construct a modal matrix from the components of the eigenvectors such that

$$P = \begin{bmatrix} N_+ & N_- \\ N_+\left(\frac{1}{2\mu\beta m_g} + \frac{1}{m_g}\sqrt{m_g^2 + \frac{1}{4\mu^2\beta^2}}\right) & N_-\left(\frac{1}{2\mu\beta m_g} - \frac{1}{m_g}\sqrt{m_g^2 + \frac{1}{4\mu^2\beta^2}}\right) \end{bmatrix}, \tag{3.52}$$

and since the eigenvectors are orthogonal to each other the inverse of the modal matrix is equal to its transpose. With the inverse of this modal matrix we can define new fields such that $\Psi = P^{-1}X$, where X is the original field,

$$\Psi = \begin{pmatrix} \tilde{\Psi}_+ \\ \tilde{\Psi}_- \end{pmatrix} = \begin{bmatrix} N_+ & N_+\left(\frac{1}{2\mu\beta m_g} + \frac{1}{m_g}\sqrt{m_g^2 + \frac{1}{4\mu^2\beta^2}}\right) \\ N_- & N_-\left(\frac{1}{2\mu\beta m_g} - \frac{1}{m_g}\sqrt{m_g^2 + \frac{1}{4\mu^2\beta^2}}\right) \end{bmatrix} \begin{pmatrix} \tilde{\sigma} \\ m_g\tilde{\phi} \end{pmatrix}. \tag{3.53}$$

Then, the diagonalized A matrix can be constructed as $D = P^{-1}AP$ which is just $D = \begin{bmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{bmatrix}$, from which the transformed Lagrangian reads $L_{FT} = \Psi D \Psi$. Then, the action in the decoupled form becomes

$$\begin{aligned} I_{NMG-CS} = \frac{\beta}{2}\int d^3x \Bigg[&\Psi_+\Box\Psi_+ - \left(m_g^2 + \frac{1}{2\mu^2\beta^2} - \frac{1}{\mu\beta}\sqrt{m_g^2 + \frac{1}{4\mu^2\beta^2}}\right)\Psi_+^2 \\ &\Psi_-\Box\Psi_- - \left(m_g^2 + \frac{1}{2\mu^2\beta^2} + \frac{1}{\mu\beta}\sqrt{m_g^2 + \frac{1}{4\mu^2\beta^2}}\right)\Psi_-^2\Bigg], \end{aligned} \tag{3.54}$$

or in a more concrete form

$$I_{NMG-CS} = \frac{\beta}{2}\int d^3x \left(\Psi_+\Box\Psi_+ - m_+^2\Psi_+^2 + \Psi_-\Box\Psi_- - m_-^2\Psi_-^2\right), \tag{3.55}$$

where the masses read

$$m_{\pm}^2 = m_g^2 + \frac{1}{2\mu^2\beta^2} \pm \frac{1}{\mu\beta}\sqrt{m_g^2 + \frac{1}{4\mu^2\beta^2}}, \tag{3.56}$$

and the fields are inverse Fourier transformed. The masses are same with those of [35, 36, 75]. Since masses of the helicity modes are different this theory is a parity violating theory. To check the results of topologically massive gravity we take $\beta \to 0$ limit. In this limit m_+ diverges and drops out, therefore we are left with a single degree of freedom that has a mass $m_- = -|\mu|/\kappa$ [33, 34]. This result can be seen by Taylor expanding the square root part of (3.56) up to third order.

3.3 Higher-derivative spin-2 in a de Sitter background

Up to now our analysis was based on the flat space time. Now, we will change the background to a constant curvature background. Specifically we will study the canonical structure of higher derivative gravity whose action is defined as

$$I = \int d^3x \sqrt{-g}\left[\frac{1}{\kappa}(R-2\Lambda_0) + \alpha R^2 + \beta R_{\mu\nu}^2\right], \tag{3.57}$$

in an (anti)-de Sitter background. Here Λ_0 is the bare cosmological constant. The linearization of (3.57) yields

$$I = -\frac{1}{2}\int d^3x \sqrt{-\bar{g}}\, h_{\mu\nu}\left[a\mathcal{G}_L^{\mu\nu} + (2\alpha+\beta)\left(\bar{g}^{\mu\nu}\Box - \nabla^\mu\nabla^\nu + \frac{2}{\ell^2}\bar{g}^{\mu\nu}\right)R_L + \beta\left(\Box\mathcal{G}_L^{\mu\nu} - \frac{1}{\ell^2}\bar{g}^{\mu\nu}R_L\right)\right], \tag{3.58}$$

where $1/\ell^2$ is the cosmological constant and $a \equiv \frac{1}{\kappa} + \frac{12}{\ell^2}\alpha + \frac{2}{\ell^2}\beta$. The cosmological constant can be related to the coupling constants α, β, κ and the bare cosmological constant as $\frac{1}{\ell^2} = \frac{1}{4\kappa(3\alpha+\beta)}\left[1 \pm \sqrt{1-8\kappa\Lambda_0(3\alpha+\beta)}\right]$ [27, 72, 79]. We will carry the analysis in a de Sitter (dS) background since it is simpler than an anti-de Sitter (AdS) background. Nevertheless, from these expressions one can find the results in AdS spacetime by taking $\ell \to i\ell$ transformation. This transformation can be taken since the results that come in the dS background are analytic in ℓ[4]. For dS, we define the background metric $\bar{g}_{\mu\nu}$ in the Poincaré form as

$$ds^2 = \frac{\ell^2}{t^2}\left(-dt^2 + dx^2 + dy^2\right), \tag{3.59}$$

by which all the raising and lowering operations and covariant derivatives will be made. The perturbation can be defined as

[4] To keep the signature intact, one also needs to Wick rotate a space coordinate.

$$g_{\mu\nu} = \frac{\ell^2}{t^2}\eta_{\mu\nu} + h_{\mu\nu}, \tag{3.60}$$

where $\frac{\ell^2}{t^2}\eta_{\mu\nu} = \bar{g}_{\mu\nu}$ and $\eta_{\mu\nu}$ is the flat spacetime metric. With this perturbation the linearized forms of Einstein and Ricci tensors and scalar curvature can be written as

$$\mathcal{G}^L_{\mu\nu} = R^L_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}R_L - \frac{2}{\ell^2}h_{\mu\nu},$$
$$R^L_{\mu\nu} = \frac{1}{2}\left(\nabla^\sigma\nabla_\mu h_{\nu\sigma} + \nabla^\sigma\nabla_\nu h_{\mu\sigma} - \Box h_{\mu\nu} - \nabla_\mu\nabla_\nu h\right), \qquad R_L = \nabla_\alpha\nabla_\beta h^{\alpha\beta} - \Box h - \frac{2}{\ell^2}h, \tag{3.61}$$

where D'Alembertian operator defined as $\Box \equiv \nabla_\mu\nabla^\mu = \frac{t^2}{\ell^2}\eta^{\mu\nu}\nabla_\mu\nabla_\nu$. The deviation part of the metric $h_{\mu\nu}$ can be decomposed into its "spatial" tensor h_{ij}, vector h_{0i} and "scalar" h_{00} parts as follows

$$\begin{aligned}
h_{ij} &\equiv \frac{\ell^2}{t^2}\left[\left(\delta_{ij} + \hat{\nabla}_i\hat{\nabla}_j\right)\phi - \hat{\nabla}_i\hat{\nabla}_j\chi + \left(\tilde{\epsilon}_i{}^k\hat{\nabla}_k\hat{\nabla}_j + \tilde{\epsilon}_j{}^k\hat{\nabla}_k\hat{\nabla}_i\right)\xi\right] \\
&= \frac{\ell^2}{t^2}\left[\left(\delta_{ij} + \hat{\nabla}_i\hat{\nabla}_j\right)\phi - \hat{\nabla}_i\hat{\nabla}_j\chi + \frac{t^2}{\ell^2}\left(\tilde{\epsilon}_{ik}\hat{\nabla}_k\hat{\nabla}_j + \tilde{\epsilon}_{jk}\hat{\nabla}_k\hat{\nabla}_i\right)\xi\right], \\
h_{0i} &\equiv \frac{\ell^2}{t^2}\left(-\tilde{\epsilon}_i{}^k\nabla_k\eta + \partial_i N_L\right) = \frac{\ell^2}{t^2}\left(-\frac{t^2}{\ell^2}\tilde{\epsilon}_{ij}\nabla_j\eta + \partial_i N_L\right), \\
h_{00} &\equiv \frac{\ell^2}{t^2}N,
\end{aligned} \tag{3.62}$$

where $\hat{\nabla}_i \equiv \nabla_i/\sqrt{-\nabla_k^2}$ and the covariant derivative is defined for the two-dimensional space metric $\gamma_{ij} = \frac{\ell^2}{t^2}\delta_{ij}$. The Latin indices are $i = 1,2$ for space dimensions. Since the components of the two dimensional metric has no space dependence and is flat, the covariant derivative reduces to partial derivative, $\nabla_i \to \partial_i$, and $\hat{\partial}_i \equiv \partial_i/\sqrt{-\partial_k^2}$. $\tilde{\epsilon}_{ik}$ is the Levi-Civita tensor which is related to the corresponding tensor density as

$$\tilde{\epsilon}_{ik} = \sqrt{\gamma}\epsilon_{ik} \quad \Rightarrow \quad \tilde{\epsilon}_{ik} = \frac{\ell^2}{t^2}\epsilon_{ik}. \tag{3.63}$$

The convention for ϵ_{ik} is $\epsilon_{12} = 1$ (the convention for Levi-Civita tensor density for the upper indices is $\epsilon^{12} = 1$ naturally with the induced metric). Therefore, the final result of the above decomposition becomes

$$h_{ij} = \frac{\ell^2}{t^2}\left[\left(\delta_{ij} + \hat{\partial}_i\hat{\partial}_j\right)\phi - \hat{\partial}_i\hat{\partial}_j\chi + \left(\epsilon_{ik}\hat{\partial}_k\hat{\partial}_j + \epsilon_{jk}\hat{\partial}_k\hat{\partial}_i\right)\xi\right],$$
$$h_{0i} = \frac{\ell^2}{t^2}\left(-\epsilon_{ij}\partial_j\eta + \partial_i N_L\right), \qquad h_{00} = \frac{\ell^2}{t^2}N. \tag{3.64}$$

The raising and lowering operations for spatial indices are done with δ_{ij}. Note that, the specific choice of decomposition involves $\frac{\ell^2}{t^2}$ coefficients and these coefficients help us to check the flat space limit, $\ell \to \infty$, $\frac{\ell}{t} \to 1$ at every step in our calculations.

At this stage we will find the gauge invariant combinations that will be constructed from the six scalar functions. In flat space case ϕ is gauge invariant. However, for curved background it is not gauge invariant anymore. The components of $h_{\mu\nu}$ transforms under the gauge transformations $\delta_\zeta h_{\mu\nu} = \nabla_\mu \zeta_\nu + \nabla_\nu \zeta_\mu$ as

$$\delta_\zeta \phi = 2\frac{t}{\ell^2}\zeta_0, \qquad \delta_\zeta \chi = 2\frac{t^2}{\ell^2}\left(\partial_i^2 \kappa + \frac{1}{t}\zeta_0\right), \qquad \delta_\zeta \xi = \frac{t^2}{\ell^2}\partial_i^2 \zeta,$$
$$\delta_\zeta \eta = \frac{t^2}{\ell^2}\left(\dot{\zeta} + \frac{2}{t}\zeta\right), \qquad \delta_\zeta N_L = \frac{t^2}{\ell^2}\left(\dot{\kappa} + \zeta_0 + \frac{2}{t}\kappa\right), \qquad \delta_\zeta N = 2\frac{t^2}{\ell^2}\left(\dot{\zeta}_0 + \frac{1}{t}\zeta_0\right), \tag{3.65}$$

where the components of ζ_μ are defined as $\zeta_\mu \equiv \left(\zeta_0, -\epsilon_{ij}\partial_j \zeta + \partial_i \kappa\right)$. Looking at the linearized Bianchi identity, $\nabla_\mu \mathcal{G}^{\mu\nu} = 0$, there should be at least three independent gauge invariant combinations which are constructed out of the six scalar fields and their derivatives. These combinations can be found by inspections. However, looking at the independent components of the gauge-invariant tensor $\mathcal{G}_L^{\mu\nu}$ the gauge-invariant combinations can be found easily. Following this route four gauge-invariant functions can be found:

$$f \equiv \frac{\ell}{t}\left[\phi - \frac{2}{t}N_L + \frac{1}{t}\frac{1}{\nabla^2}\left(\dot{\phi} + \dot{\chi} - \frac{2}{t}N\right)\right], \qquad p \equiv \frac{\ell}{t}\left(\dot{\phi} - \frac{1}{t}N\right),$$
$$q \equiv \frac{\ell}{t}\left[\nabla^2 N + \ddot{\chi} - 2\nabla^2 \dot{N}_L - \frac{1}{t}\left(\dot{N} - 2\nabla^2 N_L + \dot{\chi}\right) + \frac{2}{t^2}N\right], \qquad \sigma \equiv \frac{\ell}{t}\left(\dot{\xi} - \nabla^2 \eta\right), \tag{3.66}$$

and from the Bianchi identity we get a relation between these functions as

$$t\nabla^2\left(\dot{f} - p + \frac{f}{t}\right) - \dot{p} - q = 0. \tag{3.67}$$

We can find the components of the linearized Einstein tensor in terms of these gauge invariant fields, that are

$$\mathcal{G}^L_{00} = -\frac{t}{2\ell}\nabla^2 f, \qquad \mathcal{G}^L_{0i} = -\frac{t}{2\ell}\left(\partial_i p + \epsilon_{ik}\partial_k \sigma\right),$$
$$\mathcal{G}^L_{ij} = -\frac{t}{2\ell}\left[\left(\delta_{ij} + \hat{\partial}_i\hat{\partial}_j\right)q - \hat{\partial}_i\hat{\partial}_j \dot{p} - \left(\epsilon_{ik}\hat{\partial}_k\hat{\partial}_j + \epsilon_{jk}\hat{\partial}_k\hat{\partial}_i\right)\dot{\sigma}\right]. \tag{3.68}$$

We can also write the linearized curvature scalar in terms of the gauge-invariant fields

$$R_L = \frac{t^3}{\ell^3}\left(q - \nabla^2 f + \dot{p}\right) = \frac{t^4}{\ell^3}\nabla^2\left(\dot{f} - p\right). \tag{3.69}$$

In the second equality we used the Bianchi identity (3.67).

Using the above relations (3.58) can be written in terms of the gauge-invariant functions. The Einstein-Hilbert part of the action takes the following form:

$$I_{EH} = -\frac{a}{2}\int d^3x\,\sqrt{-\bar{g}}\,h_{\mu\nu}\mathcal{G}_L^{\mu\nu} = \frac{a}{2}\int d^3x\left[\frac{\ell^2}{t^2}fR_L + \frac{t}{\ell}\left(f\nabla^2 f + p^2 + \sigma^2\right)\right]. \tag{3.70}$$

For the $2\alpha+\beta$ part of the action, using the self-adjointness of the involved operators, $h_{\mu\nu}$ can be replaced by some gauge-invariant combinations. Also, doing so makes the computations simpler as in the flat space case. With this trick, the $2\alpha+\beta$ part of the action becomes

$$I_{2\alpha+\beta} = -\frac{(2\alpha+\beta)}{2}\int d^3x\,\sqrt{-\bar{g}}h_{\mu\nu}\left(\bar{g}^{\mu\nu}\Box - \nabla^\mu\nabla^\nu + \frac{2}{\ell^2}\bar{g}^{\mu\nu}\right)R_L = \frac{(2\alpha+\beta)}{2}\int d^3x\,\sqrt{-\bar{g}}R_L^2. \tag{3.71}$$

For the β part of the general action, we have

$$I_\beta = -\frac{\beta}{2}\int d^3x\,\sqrt{-\bar{g}}h_{\mu\nu}\left(\Box\mathcal{G}_L^{\mu\nu} - \frac{1}{\ell^2}\bar{g}^{\mu\nu}R_L\right) = -\frac{\beta}{2}\int d^3x\,\sqrt{-\bar{g}}\left[\left(\Box h_{\mu\nu}\right)\mathcal{G}_L^{\mu\nu} - \frac{1}{\ell^2}hR_L\right].$$

First, writing $R^L_{\mu\nu}$ (3.61) such that the indices of covariant derivatives μ and ν stay at the left, and then using the Bianchi identity, the following action can be found for the β part

$$I_\beta = -\frac{\beta}{2}\int d^3x\,\sqrt{-\bar{g}}\left(-2\mathcal{G}^L_{\mu\nu}\mathcal{G}_L^{\mu\nu} + \frac{1}{2}R_L^2 + \frac{2}{\ell^2}h_{\mu\nu}\mathcal{G}_L^{\mu\nu}\right).$$

If we had not used this method and computed $h_{\mu\nu}\Box\mathcal{G}_L^{\mu\nu}$ directly, to put the result into an explicitly gauge-invariant form would have been more difficult and taken more time. Finally, the general action can be written in terms of the gauge-invariant form by collecting all the parts that are computed above as follows

$$\begin{aligned} I = \frac{1}{2}\int d^3x\,\Bigg\{&\left(a + \frac{2\beta}{\ell^2}\right)\left[\frac{\ell^2}{t^2}fR_L + \frac{t}{\ell}\left(f\nabla^2 f + p^2 + \sigma^2\right)\right] + (2\alpha+\beta)\frac{\ell^3}{t^3}R_L^2 \\ &+\beta\frac{t^3}{\ell^3}\Bigg[\dot{\sigma}^2 + \sigma\nabla^2\sigma + \dot{p}^2 + p\nabla^2 p + \left(\nabla^2 f\right)^2 \\ &+\frac{\ell^3}{t^3}R_L\nabla^2 f - \frac{\ell^3}{t^3}R_L\dot{p} - \dot{p}\nabla^2 f\Bigg]\Bigg\}. \end{aligned} \tag{3.72}$$

The above action gives (3.15) when the flat space limit is taken. Also, the fields that appear in (3.72) are not independent. However, defining new field such that $\varphi \equiv \nabla^2 f$, and after using the Bianchi identity (3.67), the above action (3.72) takes a rather simple form in which the σ field decouples

$$\begin{aligned} I = \frac{1}{2}\int d^3x\,\Bigg\{&\left(a + \frac{2\beta}{\ell^2}\right)\frac{t}{\ell}\left(-tp\varphi + p^2\right) + (2\alpha+\beta)\frac{t^5}{\ell^3}\left(\dot{\varphi} - \nabla^2 p\right)^2 \\ &+\beta\frac{t^3}{\ell^3}\left(\dot{p}^2 - p\nabla^2 p - \varphi^2 - t\varphi\nabla^2 p - t\dot{p}\dot{\varphi} - \varphi\dot{p}\right)\Bigg\} + I_\sigma, \end{aligned} \tag{3.73}$$

where the σ action is

$$I_\sigma = \frac{1}{2}\int d^3x \left[\beta\frac{t^3}{\ell^3}\left(\dot{\sigma}^2 + \sigma\nabla^2\sigma\right) + \left(a + \frac{2\beta}{\ell^2}\right)\frac{t}{\ell}\sigma^2\right]. \tag{3.74}$$

In flat spacetime, cosmological Einstein theory does not have any propagating degrees of freedom for vanishing α and β in $D = 3$. Just like the flat spacetime, without higher curvature terms the total action does not have any propagating degrees of freedom. For non-vanishing α and β, (3.73) has three degrees of freedom. From the decoupled field we can read the mass if (3.74) is put in a correct canonical form. For a minimally coupled scalar field the correct canonical form is

$$I = -\frac{1}{2}\int d^3x\,\sqrt{-g}\left(\partial_\mu\Phi\partial^\mu\Phi + m^2\Phi^2\right) = -\frac{1}{2}\int d^3x\left\{\frac{\ell}{t}\left[-\dot{\Phi}^2 + (\partial_i\Phi)^2\right] + \frac{\ell^3}{t^3}m^2\Phi^2\right\}. \tag{3.75}$$

To get the correct canonical dimension the σ field is rescaled as $\sigma \to \frac{\ell^2}{t^2}\sigma$. After this rescaling, (3.74) becomes

$$I_\sigma = -\frac{\beta}{2}\int d^3x\left[\frac{t}{\ell}\left(-\dot{\sigma}^2 + (\nabla\sigma)^2\right) - \frac{\ell^3}{t^3}\left(\frac{a}{\beta} + \frac{2}{\ell^2}\right)\sigma^2\right], \tag{3.76}$$

and for the σ field the mass reads

$$m_g^2 = -\frac{a}{\beta} - \frac{2}{\ell^2} = -\frac{1}{\kappa\beta} - \frac{12\alpha}{\ell^2\beta} - \frac{4}{\ell^2} = -\frac{1}{\kappa\beta} - \frac{4\,(3\alpha - \beta)}{\beta\ell^2}. \tag{3.77}$$

Unlike the flat space case, diagonalizing the φ, p action is complicated for the general coupling constants. Since we want to see the oscillators of this theory we can use other methods. One of the methods is to Fourier transform the fields in the $\vec{x}$ space and then computing the zero-momentum limit, that is dropping the Laplacians in the action. The action does not change the number of degrees of freedom by doing this manipulation, since, ∇^2 (field) is not the lowest order term anymore. Another method is to directly work on the equations of motion. Both methods will be studied separately below.

3.3.1 Masses from the nonrelativistic limit

In this part we take the nonrelativistic limit of the general action (3.73) by dropping the ∇^2 terms. Since the σ part of the action is already in an oscillator form, it is not taken into account. Then the action becomes

$$I = \frac{1}{2}\int d^3x\left[\left(a + \frac{2\beta}{\ell^2}\right)\frac{t}{\ell}\left(-tp\varphi + p^2\right) + (2\alpha + \beta)\frac{t^5}{\ell^3}\dot{\varphi}^2 + \beta\frac{t^3}{\ell^3}\left(\dot{p}^2 - \varphi^2 - t\dot{p}\dot{\varphi} - \varphi\dot{p}\right)\right]. \tag{3.78}$$

At this step we must decouple the φ and p fields. For this reason we rescale the φ field as $\varphi \to \frac{1}{t}\varphi$. Also, we separate the constant $2\alpha + \beta = \frac{\beta}{4} + \frac{8\alpha+3\beta}{4}$ to see clearly the NMG limit. With these (3.78) becomes

$$I = \frac{1}{2}\int d^3x \left[\left(a + \frac{2\beta}{\ell^2}\right)\frac{t}{\ell}\left(-p\varphi + p^2\right) + \frac{\beta}{4}\frac{t^3}{\ell^3}\left(\dot{\varphi}^2 - \frac{\varphi^2}{t^2} + 4\dot{p}^2 - 4\dot{p}\dot{\varphi}\right)\right.$$
$$\left. + \frac{(8\alpha+3\beta)}{4}\frac{t^3}{\ell^3}\left(\dot{\varphi}^2 + \frac{3\varphi^2}{t^2}\right)\right]. \tag{3.79}$$

However, this rescaling is not enough to separate the fields. Therefore, we define a new field $\Phi \equiv \varphi - 2p$. With this definition the φ and Φ fields decouple. The Φ field action can be read as

$$I_\Phi = \frac{\beta}{8}\int d^3x \left[\frac{t^3}{\ell^3}\dot{\Phi}^2 + \frac{t}{\ell}\left(\frac{a}{\beta} + \frac{2}{\ell^2}\right)\Phi^2\right], \tag{3.80}$$

and gives the same mass as the σ field (3.77). Therefore, the Φ field becomes the spin-2 helicity partner of the σ field. The action for the φ field reads

$$I_\varphi = \frac{(8\alpha+3\beta)}{8}\int d^3x \left[\frac{t^3}{\ell^3}\dot{\varphi}^2 - \frac{1}{(8\alpha+3\beta)}\frac{t}{\ell}\left(a - \frac{24\alpha}{\ell^2} - \frac{6\beta}{\ell^2}\right)\varphi^2\right], \tag{3.81}$$

which is the spin-0 mode. To read the mass of this mode, we must put it into the canonical form. For this reason we again rescale the φ field $\varphi \to \frac{\ell^2}{t^2}\varphi$. After this rescaling the mass can be read as

$$m_s^2 = \frac{1}{\kappa(8\alpha+3\beta)} - \frac{4}{\ell^2}\left(\frac{3\alpha+\beta}{8\alpha+3\beta}\right). \tag{3.82}$$

For the NMG case this mode drops out and we are left with the m_g^2 and this result matches with [35]. Note that in [35] the analysis was carried out by introducing auxiliary fields, and these fields can be eliminated in such a way that the action gives spin-2 field with a Pauli-Fierz mass. However, we reach the same results by using canonical analysis. For general coupling constants, to make the same analysis as in [35] we must introduce two auxiliary fields and rewrite the action (3.1) in terms of these fields:

$$\mathcal{L} = \frac{1}{\kappa}\sqrt{-g}\mathcal{G}^L_{\mu\nu}\left[R - f^{\mu\nu}G_{\mu\nu} - \phi R + \frac{m_1^2}{2}\phi^2 + \frac{m_2^2}{4}\left(f^{\mu\nu}f_{\mu\nu} - f^2\right)\right], \tag{3.83}$$

where ϕ and $f_{\mu\nu}$ are auxiliary fields and the masses are $m_1^2 = -\frac{4}{\kappa(8\alpha+3\beta)}$ and $m_2^2 = -\frac{1}{\kappa\beta}$. Then we linearize (3.83) around flat background, that is

$$\kappa\mathcal{L}_{linearized} = -\left(\frac{1}{2}h^{\mu\nu} + f^{\mu\nu}\right)\mathcal{G}^L_{\mu\nu} - \phi R_L - \frac{2}{\kappa(8\alpha+3\beta)}\phi^2 - \frac{1}{4\kappa\beta}\left(f^{\mu\nu}f_{\mu\nu} - f^2\right). \tag{3.84}$$

Here we can see explicitly that for the NMG limit the ϕ field drops out and $f_{\mu\nu}$ can be chosen as $f_{\mu\nu} = -h_{\mu\nu}$. However, for the generic case it is not clear how the fields ϕ, $f_{\mu\nu}$ and $h_{\mu\nu}$ decouples. One possible way is to rescale the $h_{\mu\nu}$ field [80].

In this section we have discussed the canonical structure of the generic action in the non-relativistic limit. It would also be interesting to get the same results from the analysis of the relativistic equations. The next section is devoted to the discussion of the relativistic equations for the NMG limit.

3.3.2 Equations of motions in the NMG case

The action (3.73) for the $8\alpha + 3\beta = 0$ case becomes

$$I = \frac{\beta}{2}\int d^3x \left\{ m_g^2 \frac{t}{\ell}\left(tp\varphi - p^2\right) + \frac{t^5}{4\ell^3}\left(\dot{\varphi} - \nabla^2 p\right)^2 \right. \\ \left. + \frac{t^3}{\ell^3}\left(\dot{p}^2 - p\nabla^2 p - \varphi^2 - t\varphi\nabla^2 p - t\dot{p}\dot{\varphi} - \varphi\dot{p}\right)\right\}, \tag{3.85}$$

where we dropped the σ field. From this action one can claim that there are two degrees of freedom, which is in conflict with our earlier results [35, 36]. However, if we look at the Hessian matrix $\mathcal{H} = \frac{\partial^2 \mathcal{L}}{\partial \dot{q}_i \partial \dot{q}_j}$, that is

$$\mathcal{H} = \frac{\beta t^3}{4\ell^3}\begin{pmatrix} t^2 & -2t \\ -2t & 4 \end{pmatrix}, \tag{3.86}$$

and compute its determinant, $\det \mathcal{H} = 0$, we see that there should be a constraint in this model. Therefore the time derivatives of the fields cannot be separated in terms of the canonical momenta

$$\Pi_\varphi \equiv \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = \frac{\beta t^5}{4\ell^3}\left(\dot{\varphi} - \nabla^2 p - \frac{2}{t}\dot{p}\right), \qquad \Pi_p \equiv \frac{\partial \mathcal{L}}{\partial \dot{p}} = \frac{\beta t^3}{2\ell^3}\left(2\dot{p} - t\dot{\varphi} - \varphi\right).$$

Since the equations of motion are needed we take the variations of (3.85) with respect to φ and p fields which yield

$$\delta\varphi: \quad \frac{m_g^2 t^2}{\ell}p - \frac{t^3}{\ell^3}\left(2\varphi + t\nabla^2 p + \dot{p}\right) - \frac{1}{2\ell^3}\partial_0\left[t^5\left(\dot{\varphi} - \nabla^2 p\right) - 2t^4\dot{p}\right] = 0, \tag{3.87}$$

and

$$\delta p: \quad \frac{m_g^2 t}{\ell}\left(t\varphi - 2p\right) - \frac{t^5}{2\ell^3}\left(\dot{\varphi} - \nabla^2 p + \frac{4}{t^2}p + \frac{2}{t}\varphi\right) - \frac{1}{\ell^3}\partial_0\left[t^3\left(2\dot{p} - t\dot{\varphi} - \varphi\right)\right] = 0. \tag{3.88}$$

To decouple the fields, we define $\dot{\varphi} = \nabla^2 p$ with a hint that this choice makes $R_L = 0$. With this definition, the other equation becomes

$$\frac{\ell}{t}\left(-\ddot{\varphi} - \frac{1}{t}\dot{\varphi} + \nabla^2\varphi\right) - \frac{\ell^3}{t^3}\left(m_g^2 - \frac{1}{\ell^2}\right)\varphi = 0. \tag{3.89}$$

This equation is not in the canonical wave equation form in dS. We again rescale the field $\varphi \to \varphi/t$ to have canonical wave equation and this rescaling yields

$$\frac{\ell}{t}\left(-\ddot{\varphi} + \frac{1}{t}\dot{\varphi} + \nabla^2\varphi\right) - \frac{\ell^3}{t^3}m_g^2\varphi = 0, \Rightarrow \left(\Box - m_g^2\right)\varphi = 0, \tag{3.90}$$

which is the same as the σ field. From the relativistic equations, we again see that for NMG limit the quadratic gravity theory is parity preserving theory in three dimensions.

3.4 Conclusions and Discussion for Chapter-3

In this chapter the canonical structure of the linearized quadratic gravity theories have been studied. The analysis is done in an explicitly gauge-invariant way. Moreover, the analysis is made both for flat and dS backgrounds in three dimensions. In flat background case, the general quadratic action has been written in canonical wave equation form. After the fields are decoupled they generate three harmonic oscillators. The coupling constants κ, α, β are chosen to have a ghost-free and non-tachyonic theory. When the coupling constants are fixed in this way, the NMG theory is singled out as the unique unitary higher derivative massive gravity (but not a higher-time derivative theory). Apart from this theory, all other higher derivative gravity models are higher-derivative Pais-Uhlenbeck oscillators which have ghost modes.

The analysis is also extended by adding static sources and spinning masses to the theory. The effects of the sources are described by computing the Newtonian potentials for both cases. In addition, the weak field limit of the theory is computed at the nonlinear level by using the circularly symmetric ansatz. Another extension that is done for the flat spacetime is adding the gravitational Chern-Simons term to the general quadratic theory. With this addition the NMG theory is investigated and in this limit it is found that the oscillators decouple with different masses. Therefore, this model is parity violating theory as expected.

In dS spacetime case, the general action is written in terms of three gauge-invariant functions. These functions are constructed from the derivatives of the components of metric perturbation. The fields are decoupled by two different ways. The first way to decouple fields is to go to the nonrelativistic limit by dropping out the two dimensional Laplacians in the action. The second decoupling way is to get the field equations in the relativistic form.

These gauge-invariant actions that are constructed in this chapter may be useful when nonlinearities and interactions are introduced to the theory. Apart from this usefulness there are other interesting points about the theories that we discussed in this chapter. Tuned values of the parameters give rise to uncommon phenomena, for instance partial masslessness or chiral gravity, which especially arise in (anti)-de Sitter spacetimes. These matters are open to study.

CHAPTER 4

CONCLUSION

In this dissertation, the most general quadratic curvature massive gravity theory is analyzed to find a specific unitary theory. For this aim the one-particle scattering amplitude is calculated. Also, to understand the found unitary theory, that is NMG, the most quadratic curvature theory is written in canonical form in three dimensions for both flat and constant curvature spacetimes.

In the second chapter we have studied the D dimensional quadratic gravity theory augmented with a Pauli-Fierz mass term. First the linear theory without sources is studied for both massive and massless limits. For the massive case, a wave equation is found with two massive excitations in flat spacetime. In curved spacetime the trace of the metric perturbation becomes a dynamical scalar field. When the dynamical part is dropped by fixing the coupling constants one solution of this equation is the partially massless point which comes out only in curved backgrounds. For the massless limit both the linearized Einstein tensor and the linearized Ricci scalar are background diffeomorphism invariant. Also, the scalar curvature must be zero in flat spacetime, but need not be zero in curved background when the dynamical part of the equation is eliminated by fixing the constants.

After the analysis in the linearized level, we have moved on to compute the one-particle scattering amplitude between two covariantly conserved sources. To find the exchange amplitude the perturbation part must be expressed in terms of the energy-momentum tensor. However, the components of the perturbation part of the metric are not independent. Hence, it is decomposed into its parts to obtain the independent components. After computing the necessary elements, the scattering amplitude is calculated. From this general amplitude equation, the poles and residues can be calculated to have an idea about the particle spectrum of the most general theory. Moreover, the Newtonian potentials can be computed from this amplitude equation. Nevertheless, the equations are very complicated for the most general case. The calculations are done only for some interesting limits. First the massive theory is considered in flat spacetime. For this case it is seen that the theory suffers from ghosts, and to get rid of the ghost term the coupling constant of the square of Ricci tensor must be taken to zero. Also, the Newtonian potentials are calculated for static point sources in four and three dimensions. From the four dimensional Newtonian potential, the vDVZ discontinuity is obtained. The massive gravity theory does not fit in with the pure Einstein theory. But in three dimensions it works well with the pure Einstein theory for small separations of the sources since the massless limit does not exist. The other interesting limit is the massless theory in flat spacetime. In this case first mass is set to zero and then the flat spacetime limit is taken. For this case the poles, corresponding residues and Newtonian potential energies are calculated. When the theory is constrained to ghost and tachyon freedom, from the poles, residues and Newtonian potentials, it is seen that the dimension must be set to three, the coupling constant of the Einstein-Hilbert term must taken to be negative and there must be a relation between the coupling constants of the higher curvature terms as $8\alpha + 3\beta = 0$. This theory is known as NMG. Therefore, this theory is obtained from a different perspective than that of [35, 36]. In three dimensions the Newtonian potential energy again has the same potential energy as the Einstein gravity at the NMG limit. The Newtonian potential energy in four dimensions has a ghost term and to get rid of this term $\beta \to 0$ limit must be taken. Also, for this limit the general Newtonian potential energy is calculated for D dimensions. From this equation it can be seen that for all dimensions there is a ghost term.

In the third chapter, the three dimensional quadratic curvature theory is studied in more detail. To understand how the NMG theory is singled out among other three dimensional theories, the most general action is written in canonical form in both flat and dS spacetimes. First the flat spacetime case is studied. To get the canonical form, the metric perturbation is decomposed in terms of six scalar functions. From these functions three gauge-invariant functions are constructed by the help of Bianchi identity. The components of linearized Einstein tensor is written in terms of these gauge-invariant combinations. With the help of these components and the linearized Ricci scalar, the most general quadratic action is written in the canonical form with these three gauge-invariant fields, one of which is automatically decoupled from the others. The decoupling of the other two fields depends on whether $2\alpha + \beta$ is zero or not.

These two cases are discussed separately. When this constant is set to a generic value apart from zero, one of the fields that has no dynamics in the action can be written in terms of the other dynamical field. After this manipulation, a decoupled action is computed. This action is analyzed for some special points and it is found that for NMG case, the higher-derivative term disappears and we are left with a parity-invariant theory. Except for the NMG case, this theory describes a ghost like excitation. When we set $2\alpha + \beta = 0$ at the action level and define a new field, a decoupled action can be written which has a ghost excitation. Then the discussion is extended by adding static sources and spinning masses to the action separately. For the static case an interaction part comes to the theory. The Newtonian potential energy of this interaction part becomes attractive for negative κ. By adding spinning masses, the spin-spin interaction part is also found. Unfortunately, this interaction can be repulsive or attractive depending on the signs of the spins. Moreover, some of the found results are also obtained from the nonlinear theory for the NMG limit. The last part for the flat spacetime discussion is to add the gravitational Chern-Simons term to the general quadratic action. For this case the masses of the excitations are found. Since, these masses are different, this is a parity-violating theory. Also, for the topologically massive gravity limit, the expected result is found.

In the second part of the third chapter, the discussion is extended to curved backgrounds, namely to dS backgrounds. In this part the calculations are repeated for this case and again three gauge invariant functions are found. With the help of these functions the generic action is written with one decoupled field and two coupled fields. For the decoupled field the mass is computed, and for the coupled fields the mass is obtained from both the nonrelativistic limit and the equations of motion.

In the Appendix A1-A5, we give some details of the computation which can be helpful to follow the discussion.

REFERENCES

[1] A. Einstein, Annalen Phys. **82**, 769-822, (1916).

[2] M. Grosser, *"The Discovery of Neptun",* Harvard Uni. Press, Cambridge, (1962).

[3] A. S. Goldhaber and M. M. Nieto, Rev. Mod. Phys. **82**, 939, (2010).

[4] U. J. J. le Verrier, Ann. Observ. Imp. Paris **V**, 1, (1859).

[5] S. Weinberg, "*Gravitation and Cosmology: Principles and Applications* of *The General Theory of Relativity*", Wiley, New York, (1972).

[6] R. d'Inverno, "*Introducing Einstein's Relativity*", Clarendon Press, Oxford, (1995).

[7] S. Carroll, *"Spacetime and Geometry",* Addison-Wesley, (2004).

[8] A. G. Risse *et al.*, Astron. J. **116** 1009, (1998).

[9] S. Perlmutter *et al.*, Astrophys. J. **517** 565, (1999).

[10] K. Hinterbichler, "*Theoretical Aspects of Massive Gravity*", arXiv: 1105.3735 [hep-th], (2011).

[11] V. A. Rubakov and P. G. Tinyakov, Phys.Usp. **51** 759, (2008).

[12] S. Weinberg, Rev. Mod. Phys. **61** 1, (1989).

[13] S. Weinberg, "*The Quantum Theory of Fields. Vol. 1: Foundations*". Cambridge, UK: Univ. Press, (1995).

[14] S. N. Gupta, Phys. Rev. **96** 1683, (1954).

[15] R. H. Kraichnan, Phys. Rev. **98** 1118, (1955).

[16] S. Weinberg, Phys. Rev. **138** B988, (1965).

[17] S. Deser, Gen. Rel. Grav **1** 9, (1970).

[18] D. G. Boulware and S. Deser, Ann. Phys. **89** 193, (1975).

[19] J. Fang and C. Fronsdal, J. Math. Phys. **20** 2264, (1979).

[20] R. M. Wald, Phys. Rev. **D33** 3613, (1986).

[21] K.S. Stelle, Phys. Rev. **D16**, 953, (1977).

[22] K.S. Stelle, Gen. Rel. Grav. **9** 353, (1978).

[23] W. Pauli and M. Fierz, Helv. Phys. Acta **12** 297, (1939).

[24] W. Pauli and M. Fierz, Proc. Roy. Soc. London **A 173** 211, (1939).

[25] H. van Dam and M. J. G. Veltman, Nucl. Phys. **B22** 397, (1970).

[26] V. I. Zakharov, JETP Lett. **12**, 312, (1970).

[27] İ. Gullu and B. Tekin, Phys. Rev. **D80** 064033, (2009).

[28] İ. Gullu, T. Ç. Şişman and B. Tekin, Phys. Rev. **D81** 104017, (2010).

[29] A. Higuchi, Nucl. Phys. B **282**, 397, (1987).

[30] M. Porrati, Phys. Lett. B **498**, 92, (2001).

[31] I. I. Kogan, S. Mouslopoulos and A. Papazoglou, Phys. Lett. B **503**, 173, (2001).

[32] A. Vainshtein, Surveys High Energ. Phys. **20**, 5, (2006).

[33] S. Deser, R. Jackiw and S. Templeton, Annals Phys. **140**, 372, (1982).

[34] S. Deser, R. Jackiw and S. Templeton Phys. Rev. Lett. **48**, 975, (1982).

[35] E. A. Bergshoeff, O. Hohm and P. K. Townsend, Phys. Rev. Lett. **102**, 201301, (2009).

[36] E. A. Bergshoeff, O. Hohm and P. K. Townsend, Phys. Rev. D **79**, 124042, (2009).

[37] G. Clement, Class. Quant. Grav. **26**, 105015, (2009).

[38] Y. Liu and Y. W. Sun, Phys. Rev. D **79**, 126001, (2009).

[39] Y. Liu and Y. W. Sun, JHEP **0905**, 039, (2009).

[40] E. Ayon-Beato, G. Giribet and M. Hassaine, JHEP **0905**, 029, (2009).

[41] J. Oliva, D. Tempo and R. Troncoso, JHEP **0907**, 011, (2009).

[42] M. Gurses, Class. Quant. Grav. **27**, 205018, (2010).

[43] H. Ahmedov and A. N. Aliev, Phys. Lett. B **694**, 143, (2010).

[44] H. Ahmedov and A. N. Aliev, Phys. Rev. Lett. **106**, 021301, (2011).

[45] H. Ahmedov and A. N. Aliev, Phys. Lett. D **83**, 084032, (2011).

[46] M. Gurses, T. Ç. Şişman and B. Tekin, "*Some Exact Solutions of All f(Ricci) Theories in Three Dimensions*", arXiv:1112.6346 [hep-th], (2011).

[47] S. Dengiz, B. Tekin, Phys. Rev. **D84** 024033, (2011).

[48] M. R. Tanhayi, S. Dengiz, B. Tekin, "*Unitarity of Weyl-Invariant New Massive Gravity and Generation of Graviton Mass via Symmetry Breaking*", arXiv: 1112.2338 [hep-th], (2011).

[49] H. Lu, C. N. Pope, Phys. Rev. Lett. **106** 181302 (2011).

[50] S. Deser, H. Liu, H. Lu, C. N. Pope, T. C. Sisman, B. Tekin, Phys. Rev. D **83** 061502, (2011).

[51] M. Porrati, M. M. Roberts, Phys. Rev. **D84** 024013, (2011).

[52] L. F. Abbott and S. Deser, Nucl. Phys. **B195**, 76, (1982).

[53] E. Witten, "*Quantum Gravity in de Sitter Space*", arXiv:0106109 [hep-th], (2001).

[54] A. Strominger, J. High Energy Phys. **10**, 034, (2001).

[55] D. Griffiths, "*Introduction to Elementary Particles*", WILEY-VCH, (2008).

[56] M. E. Peskin and D. V. Schroeder, "*An Introduction to Quantum Field Theory*", Addison-Wesley, (1995).

[57] G. Reinhardt, "*Field Quantization*", Springer, (1996).

[58] A. Pais and G. E. Uhlenbeck, Phys. Rev. **79**, 145, (1950).

[59] O. Sarioglu and B. Tekin, Class. Quant. Grav. **23**, 7541, (2006).

[60] M. Ostrogradski, Mem. Act. St. Petersbourg **VI 4**, 385, (1850).

[61] J. Z. Simon, Phys. Rev. **D41**, 3720, (1990).

[62] M. Nakasone and I. Oda, Prog. Theor. Phys. **121** 1389, (2009).

[63] I. Oda, JHEP **0905**, 064, (2009).

[64] M. Nakasone and I. Oda, Phys. Rev. **D 79**, 104012, (2009).

[65] S. Deser, Phys. Rev. Lett. **103**, 101302, (2009).

[66] M. A. Vasiliev, Nucl. Phys. B **616**, 106 (2001) [Erratum-ibid. B **652**, 407, (2003)].

[67] X. Bekaert, S. Cnockaert, C. Iazeolla and M. A. Vasiliev, "Nonlinear higher spin theories in various dimensions," arXiv: 0503128 [hep-th], (2005).

[68] S. Deser and B. Tekin, Class. Quant. Grav. **19**, L97, (2002).

[69] S. Deser and A. Waldron, Phys. Lett. B **501**, 134, (2001).

[70] F. A. Dilkes, M. J. Duff, J. T. Liu and H. Sati, Phys. Rev. Lett. **87**, 041301, (2001).

[71] D. G. Boulware and S. Deser, Phys. Rev. D **6**, 3368, (1972).

[72] S. Deser and B. Tekin, Phys. Rev. D **67**, 084009, (2003).

[73] S. Deser and A. Waldron, Nucl. Phys. B **607**, 577, (2001).

[74] G. B. Arfken and H. J. Weber, "*Mathematical Methods for Physicists*", Elsevier Academic Press, (2005).

[75] R. Andringa, E. A. Bergshoeff, M. de Roo, O. Hohm, E. Sezgin and P. K. Townsend, Class. Quant. Grav. **27**, 025010, (2010).

[76] R. P. Woodard, Lect. Notes Phys. **720**, 403, (2007).

[77] A. V. Smilga, Phys. Lett. B **632**, 433, (2006).

[78] S. Deser and B. Tekin, Class. Quant. Grav. **20**, 4877, (2003).

[79] S. Deser and B. Tekin, Phys. Rev. Lett. **89**, 101101, (2002).

[80] A. Hindawi, B. A. Ovrut and D. Waldram, Phys. Rev. D **53**, 5583, (1996).

APPENDIX A

GAUGE INVARIANCE OF THE GENERAL ACTION AND EQUATIONS OF MOTION

A.1 Introduction

In this chapter some of the calculations are given to follow the bulk of the dissertation. Here the gauge invariant action is calculated in dS spacetime. For this purpose, first the gauge invariance of the metric perturbation is written, then from the components of the Einstein tensor the gauge-invariant functions are defined. Then the action is written in terms of these gauge-invariant functions. In this part all the raising and lowering operations are made by the background metric and all covariant derivatives are defined with the background metric.

The metric is taken as

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}, \tag{A.1}$$

where $\bar{g}_{\mu\nu} = \frac{\ell^2}{t^2}\eta_{\mu\nu}$ is the background metric and $\eta_{\mu\nu}$ is the flat spacetime metric. The perturbation part of the metric $h_{\mu\nu}$ is decomposed as

$$\begin{aligned} h_{ij} &\equiv \left(\delta_{ij} + \hat{\partial}_i\hat{\partial}_j\right)\phi - \hat{\partial}_i\hat{\partial}_j\chi + \left(\epsilon_{ik}\hat{\partial}_k\hat{\partial}_j + \epsilon_{jk}\hat{\partial}_k\hat{\partial}_i\right)\xi, \\ h_{0i} &\equiv \left(-\epsilon_{ij}\partial_j\eta + \partial_i N_L\right), \qquad h_{00} \equiv N. \end{aligned} \tag{A.2}$$

Here ϵ_{ij} is the tensor density with the convention $\epsilon_{12} = 1$.

The trace of the perturbation metric is

$$\begin{aligned} h = \bar{g}^{\mu\nu}h_{\mu\nu} = \frac{t^2}{a^2}\eta^{\mu\nu}h_{\mu\nu} = \frac{t^2}{a^2}\left(-h_{00} + h_{ii}\right), \\ = \frac{t^2}{a^2}\left(-N + \phi + \chi\right). \end{aligned} \tag{A.3}$$

Some useful identities are as follows:

$$\begin{aligned}\partial_i h_{0i} &= \partial_i\left(-\epsilon_{ik}\partial_k\eta + \partial_i N_L\right) = \partial_i^2 N_L,\\ \partial_i\partial_j h_{ij} &= \partial_i\partial_j\left[\left(\delta_{ij} + \hat{\partial}_i\hat{\partial}_j\right)\phi - \hat{\partial}_i\hat{\partial}_j\chi + \left(\epsilon_{ik}\hat{\partial}_k\hat{\partial}_j + \epsilon_{jk}\hat{\partial}_k\hat{\partial}_i\right)\xi\right],\\ &= \partial_i\left[\left(\partial_i - \partial_i\right)\phi + \partial_i\chi - \epsilon_{ik}\partial_k\xi\right] = \partial_i^2\chi,\\ h_{ii} &= \left(\delta_{ii} + \hat{\partial}_i\hat{\partial}_i\right)\phi - \hat{\partial}_i\hat{\partial}_i\chi + \left(\epsilon_{ik}\hat{\partial}_k\hat{\partial}_i + \epsilon_{ik}\hat{\partial}_k\hat{\partial}_i\right)\xi = \phi + \chi.\end{aligned} \tag{A.4}$$

Later the six free functions are redefined as $\tilde{\psi} \equiv \frac{\ell^2}{t^2}\psi$, to get simple form of the results. With this redefinition, the metric (A.1) takes the form

$$g_{\mu\nu} = \frac{\ell^2}{t^2}\left(\eta_{\mu\nu} + h_{\mu\nu}\right). \tag{A.5}$$

In this appendix only the calculations of the dS part is given but the flat spacetime part is given at the needed steps by taking the limits $\ell \to \infty$, $\ell/t \to 1$.

A.2 The Gauge-Invariance of Metric Decomposition:

The gauge transformation is $\delta_\zeta h_{\mu\nu} = \nabla_\mu\zeta_\nu + \nabla_\nu\zeta_\mu$ where

$$\zeta_\mu = \left(\zeta_0,\ -\epsilon_{ij}\partial_j\zeta + \partial_i\kappa\right) \tag{A.6}$$

and ϵ_{ij} is the Levi-Civita tensor density with the convention $\epsilon_{12} = 1$. To calculate the gauge-invariance of the metric components the connection coefficients are needed. For (A.1) the coefficients can be calculated by use of

$$\Gamma^\mu_{\sigma\nu} = \frac{1}{2}\bar{g}^{\mu\lambda}\left(\partial_\sigma\bar{g}_{\nu\lambda} + \partial_\nu\bar{g}_{\sigma\lambda} - \partial_\lambda\bar{g}_{\nu\sigma}\right), \tag{A.7}$$

where for notational simplicity we omit the bar sign on $\Gamma^\mu_{\sigma\nu}$ and in what follows we shall also omit the bar on the covariant derivatives. There are only three non-trivial components of (A.7)

$$\Gamma^0_{00} = -\frac{1}{t},\ \Gamma^0_{ij} = -\frac{1}{t}\delta_{ij},\ \Gamma^i_{0j} = -\frac{1}{t}\delta_{ij}. \tag{A.8}$$

All other components of the connection coefficients are zero. For the full tensor the gauge invariance can be written as

$$h'_{\mu\nu} - h_{\mu\nu} = \partial_\mu\zeta_\nu + \partial_\nu\zeta_\mu - 2\Gamma^\lambda_{\mu\nu}\zeta_\lambda, \tag{A.9}$$

after decomposing the covariant derivative.

A.2.1 The h_{00} Component:

Let start with the h_{00} component. Using (A.9);

$$\begin{aligned} h'_{00} - h_{00} &= 2\partial_0\zeta_0 - 2\Gamma^0_{00}\zeta_0 - 2\Gamma^i_{00}\zeta_i \\ N' - N &= 2\left(\dot{\zeta}_0 + \frac{1}{t}\zeta_0\right), \end{aligned} \tag{A.10}$$

where in the second line, (A.8) is used. Taking $h_{\mu\nu} \to \frac{\ell^2}{t^2}h_{\mu\nu}$

$$\delta_\zeta N = 2\frac{t^2}{\ell^2}\left(\dot{\zeta}_0 + \frac{1}{t}\zeta_0\right). \tag{A.11}$$

A.2.2 The h_{i0} Component:

The h_{i0} component can be transformed again using (A.9) as

$$\begin{aligned} h'_{i0} - h_{i0} &= \partial_0\zeta_i + \partial_i\zeta_0 - 2\Gamma^j_{0i}\zeta_j - 2\Gamma^0_{0i}\zeta_0 \\ -\epsilon_{ij}\partial_j\eta' + \partial_i N'_L + \epsilon_{ij}\partial_j\eta - \partial_i N_L &= \partial_0\left(-\epsilon_{ij}\partial_j\zeta + \partial_i\kappa\right) + \partial_i\zeta_0 + \frac{2}{t}\delta_{ij}\left(-\epsilon_{kj}\partial_k\zeta + \partial_j\kappa\right) \\ -\epsilon_{ij}\partial_j\left(\eta' - \eta\right) + \partial_i\left(N'_L - N_L\right) &= -\partial_j\epsilon_{ij}\left(\dot{\zeta} - \frac{2}{t}\zeta\right) + \partial_i\left(\dot{\kappa} + \zeta_0 + \frac{2}{t}\kappa\right), \end{aligned} \tag{A.12}$$

where we have used (A.2) and (A.8) in the left and right hand sides of the second line, respectively. Equating both sides with respect to their coefficients, we found

$$\delta_\zeta N_L = \left(\dot{\kappa} + \zeta_0 + \frac{2}{t}\kappa\right), \quad \delta_\zeta\eta = \left(\dot{\zeta} - \frac{2}{t}\zeta\right), \tag{A.13}$$

and for the redefinition of the metric perturbation, (A.13) take the following form

$$\delta_\zeta N_L = \frac{t^2}{\ell^2}\left(\dot{\kappa} + \zeta_0 + \frac{2}{t}\kappa\right),\ \delta_\zeta\eta = \frac{t^2}{\ell^2}\left(\dot{\zeta} - \frac{2}{t}\zeta\right). \tag{A.14}$$

A.2.3 The h_{ij} Component:

From (A.9) the last term is can be written as

$$\begin{aligned} h'_{ij} - h_{ij} =& \partial_j\zeta_i + \partial_i\zeta_j - 2\Gamma^0_{ji}\zeta_0 - 2\Gamma^k_{ji}\zeta_k \\ =& \left(\delta_{ij} + \hat{\partial}_i\hat{\partial}_j\right)\phi' - \hat{\partial}_i\hat{\partial}_j\chi\prime + \left(\epsilon_{ik}\hat{\partial}_k\hat{\partial}_j + \epsilon_{jk}\hat{\partial}_k\hat{\partial}_i\right)\xi' \\ &- \left(\delta_{ij} + \hat{\partial}_i\hat{\partial}_j\right)\phi + \hat{\partial}_i\hat{\partial}_j\chi - \left(\epsilon_{ik}\hat{\partial}_k\hat{\partial}_j + \epsilon_{jk}\hat{\partial}_k\hat{\partial}_i\right)\xi \\ =& \partial_j\left(-\epsilon_{ik}\partial_k\zeta + \partial_i\kappa\right) + \partial_i\left(-\epsilon_{jk}\partial_k\zeta + \partial_j\kappa\right) + \frac{2}{t}\delta_{ij}\zeta_0, \end{aligned} \tag{A.15}$$

where we have again used (A.2),(A.6) and (A.8). The (A.15) is multiplied with ∂_j in order to eliminate ϕ field. Note that $\partial_j \hat{\partial}_i \hat{\partial}_j = -\partial_i$ and $\hat{\partial}_i \hat{\partial}_i = -1$. Then

$$\partial_i \chi\prime - \epsilon_{ik}\partial_k \xi' - \partial_i \chi + \epsilon_{ik}\partial_k \xi = -\epsilon_{ik}\partial_j^2 \partial_k \zeta + 2\partial_i \partial_j^2 \kappa + \frac{2}{t}\partial_i \zeta_0, \tag{A.16}$$

and equating both sides with respect to their coefficients, we get

$$\begin{aligned} \partial_i (\chi\prime - \chi) &= 2\partial_j^2 \kappa + \frac{2}{t}\zeta_0, \\ \delta_\zeta \chi &= 2\left(\partial_j^2 \kappa + \frac{1}{t}\zeta_0\right), \end{aligned} \tag{A.17}$$

and

$$\begin{aligned} -\epsilon_{ik}\partial_k (\xi' - \xi) &= -\epsilon_{ik}\partial_j^2 \partial_k \zeta \\ \delta_\zeta \xi &= \partial_j^2 \zeta. \end{aligned} \tag{A.18}$$

For the redefined metric, (A.17) and (A.18) becomes

$$\delta_\zeta \chi = 2\frac{t^2}{\ell^2}\left(\partial_j^2 \kappa + \frac{1}{t}\zeta_0\right). \tag{A.19}$$

and

$$\delta_\zeta \xi = \frac{t^2}{\ell^2}\partial_j^2 \zeta. \tag{A.20}$$

We can eliminate ξ field by multiplying (A.15) with δ_{ij} and using (A.17) we get

$$\phi' - \phi = \frac{2}{t}\zeta_0, \tag{A.21}$$

and for the redefined metric perturbation (A.21) reads

$$\delta_\zeta \phi = \frac{2t}{\ell^2}\zeta_0. \tag{A.22}$$

For the flat spacetime case the non-gauge-invariant functions become

$$\begin{aligned} &\delta_\zeta \phi = 0, \ \delta_\zeta \xi = \partial_j^2 \zeta, \ \delta_\zeta \chi = 2\partial_j^2 \kappa, \\ &\delta_\zeta N_L = \dot{\kappa} + \zeta_0, \ \delta_\zeta \eta = \dot{\zeta}, \delta_\zeta N = 2\dot{\zeta}_0. \end{aligned} \tag{A.23}$$

From (A.23) it can be seen that apart from ϕ function all other functions are not gauge-invariant.

In this part we see that the functions are not gauge-invariant. Therefore, the next section is devoted to find gauge invariant functions. For this purpose, we need to find the components of the linear Einstein tensor in terms of the free functions since it is known that linear Einstein tensor must satisfy the Bianchi identity. Therefore, its components must be gauge invariant.

In the next section we first find the components of Ricci tensor and scalar in terms of these six free functions. Then using these components, the components of the linear Einstein tensor are written. Then we define the gauge invariant functions and also check their invariance. Moreover, we see that Bianchi identity give us a relation between these gauge invariant functions.

A.3 Gauge-Invariant Combinations:

The Bianchi identity gives a clue to finding these gauge invariant combinations, constructed from the six free functions, since $\nabla_\mu \mathcal{G}^{\mu\nu}_L = 0$. Therefore, the components of the linear Einstein tensor must be composed of such combinations that are invariant under a gauge transformation. The components of the linear Einstein tensor are found one by one.

Before calculating the components of Einstein tensor the components of the linear Ricci tensor and the Scalar curvature must be calculated.

A.3.1 Ricci Tensor:

The Ricci tensor is

$$R^L_{\mu\nu} = \frac{1}{2}\left(\nabla^\sigma\nabla_\mu h_{\nu\sigma} + \nabla^\sigma\nabla_\nu h_{\mu\sigma} - \Box h_{\mu\nu} - \nabla_\mu\nabla_\nu h\right). \tag{A.24}$$

Extracting the covariant derivatives, the Ricci tensor becomes

$$\begin{aligned}
R^L_{\mu\nu} &= \frac{1}{2}\left(\nabla^\sigma\nabla_\mu h_{\nu\sigma} + \nabla^\sigma\nabla_\nu h_{\mu\sigma} - \Box h_{\mu\nu} - \nabla_\mu\nabla_\nu h\right),\\
&= \frac{1}{2}\left(\bar{g}^{\sigma\rho}\nabla_\rho\nabla_\mu h_{\nu\sigma} + \bar{g}^{\sigma\rho}\nabla_\rho\nabla_\nu h_{\mu\sigma} - \bar{g}^{\sigma\rho}\nabla_\rho\nabla_\sigma h_{\mu\nu}\right) - \frac{1}{2}\nabla_\mu\left(\partial_\nu h\right),\\
&= \frac{1}{2}\bar{g}^{\sigma\rho}\nabla_\rho\left(\nabla_\mu h_{\nu\sigma} + \nabla_\nu h_{\mu\sigma} - \nabla_\sigma h_{\mu\nu}\right) - \frac{1}{2}\left(\partial_\mu\partial_\nu h - \Gamma^\lambda_{\mu\nu}\partial_\lambda h\right),\\
&= \frac{1}{2}\bar{g}^{\sigma\rho}\nabla_\rho\Big[\left(\partial_\mu h_{\nu\sigma} - \Gamma^\lambda_{\mu\nu}h_{\lambda\sigma} - \Gamma^\lambda_{\mu\sigma}h_{\nu\lambda}\right) + \left(\partial_\nu h_{\mu\sigma} - \Gamma^\lambda_{\nu\mu}h_{\lambda\sigma} - \Gamma^\lambda_{\nu\sigma}h_{\mu\lambda}\right)\\
&\quad - \left(\partial_\sigma h_{\mu\nu} - \Gamma^\lambda_{\sigma\mu}h_{\lambda\nu} - \Gamma^\lambda_{\sigma\nu}h_{\mu\lambda}\right)\Big] - \frac{1}{2}\left(\partial_\mu\partial_\nu h - \Gamma^\lambda_{\mu\nu}\partial_\lambda h\right),
\end{aligned} \tag{A.25}$$

and

$$
\begin{aligned}
R^L_{\mu\nu} &= \frac{1}{2}\bar{g}^{\sigma\rho}\nabla_\rho\left[\partial_\mu h_{\nu\sigma} - 2\Gamma^\lambda_{\mu\nu}h_{\lambda\sigma} + \partial_\nu h_{\mu\sigma} - \partial_\sigma h_{\mu\nu}\right] - \frac{1}{2}\left(\partial_\mu\partial_\nu h - \Gamma^\lambda_{\mu\nu}\partial_\lambda h\right), \\
&= \frac{1}{2}\bar{g}^{\sigma\rho}\left\{\left[\partial_\rho\left(\partial_\mu h_{\nu\sigma}\right) - \Gamma^\alpha_{\rho\mu}\left(\partial_\alpha h_{\nu\sigma}\right) - \Gamma^\alpha_{\rho\nu}\left(\partial_\mu h_{\alpha\sigma}\right) - \Gamma^\alpha_{\rho\sigma}\left(\partial_\mu h_{\nu\alpha}\right)\right]\right. \\
&\qquad + \left[\partial_\rho\left(\partial_\nu h_{\mu\sigma}\right) - \Gamma^\alpha_{\rho\nu}\left(\partial_\alpha h_{\mu\sigma}\right) - \Gamma^\alpha_{\rho\mu}\left(\partial_\nu h_{\alpha\sigma}\right) - \Gamma^\alpha_{\rho\sigma}\left(\partial_\nu h_{\mu\alpha}\right)\right] \\
&\qquad \left. - \left[\partial_\rho\left(\partial_\sigma h_{\mu\nu}\right) - \Gamma^\alpha_{\rho\sigma}\left(\partial_\alpha h_{\mu\nu}\right) - \Gamma^\alpha_{\rho\mu}\left(\partial_\sigma h_{\alpha\nu}\right) - \Gamma^\alpha_{\rho\nu}\left(\partial_\sigma h_{\mu\alpha}\right)\right]\right\} \\
&\qquad - \bar{g}^{\sigma\rho}\left[\partial_\rho\left(\Gamma^\lambda_{\mu\nu}h_{\lambda\sigma}\right) - \Gamma^\alpha_{\rho\mu}\Gamma^\lambda_{\alpha\nu}h_{\lambda\sigma} - \Gamma^\alpha_{\rho\nu}\Gamma^\lambda_{\mu\alpha}h_{\lambda\sigma} - \Gamma^\alpha_{\rho\sigma}\Gamma^\lambda_{\mu\nu}h_{\lambda\alpha}\right] \\
&\quad - \frac{1}{2}\left(\partial_\mu\partial_\nu h - \Gamma^\lambda_{\mu\nu}\partial_\lambda h\right),
\end{aligned}
\tag{A.26}
$$

after collecting terms in parenthesis we obtain,

$$
\begin{aligned}
2R^L_{\mu\nu} &= \bar{g}^{\sigma\rho}\left\{\partial_\rho\left(\partial_\mu h_{\nu\sigma} + \partial_\nu h_{\mu\sigma} - \partial_\sigma h_{\mu\nu}\right) - \Gamma^\alpha_{\rho\mu}\left(\partial_\alpha h_{\nu\sigma} + \partial_\nu h_{\alpha\sigma} - \partial_\sigma h_{\alpha\nu}\right)\right. \\
&\qquad - \Gamma^\alpha_{\rho\nu}\left(\partial_\mu h_{\alpha\sigma} + \partial_\alpha h_{\mu\sigma} - \partial_\sigma h_{\mu\alpha}\right) - \Gamma^\alpha_{\rho\sigma}\left(\partial_\mu h_{\nu\alpha} + \partial_\nu h_{\mu\alpha} - \partial_\alpha h_{\mu\nu}\right) \\
&\qquad \left. -2\left[\partial_\rho\left(\Gamma^\lambda_{\mu\nu}h_{\lambda\sigma}\right) - \Gamma^\alpha_{\rho\mu}\Gamma^\lambda_{\alpha\nu}h_{\lambda\sigma} - \Gamma^\alpha_{\rho\nu}\Gamma^\lambda_{\mu\alpha}h_{\lambda\sigma} - \Gamma^\alpha_{\rho\sigma}\Gamma^\lambda_{\mu\nu}h_{\lambda\alpha}\right]\right\} \\
&\quad - \left(\partial_\mu\partial_\nu h - \Gamma^\lambda_{\mu\nu}\partial_\lambda h\right).
\end{aligned}
\tag{A.27}
$$

Putting $\bar{g}^{\sigma\rho} = \frac{t^2}{\ell^2}\eta^{\sigma\rho}$ and summing σ and ρ indices we get

$$
\begin{aligned}
R^L_{\mu\nu} &= -\frac{t^2}{2\ell^2}\left\{\partial_0\left(\partial_\mu h_{\nu 0} + \partial_\nu h_{\mu 0} - \partial_0 h_{\mu\nu}\right) - \Gamma^\alpha_{0\mu}\left(\partial_\alpha h_{\nu 0} + \partial_\nu h_{\alpha 0} - \partial_0 h_{\alpha\nu}\right)\right. \\
&\qquad - \Gamma^\alpha_{0\nu}\left(\partial_\mu h_{\alpha 0} + \partial_\alpha h_{\mu 0} - \partial_0 h_{\mu\alpha}\right) - \Gamma^\alpha_{00}\left(\partial_\mu h_{\nu\alpha} + \partial_\nu h_{\mu\alpha} - \partial_\alpha h_{\mu\nu}\right) \\
&\qquad \left. -2\left[\partial_0\left(\Gamma^\lambda_{\mu\nu}h_{\lambda 0}\right) - \Gamma^\alpha_{0\mu}\Gamma^\lambda_{\alpha\nu}h_{\lambda 0} - \Gamma^\alpha_{0\nu}\Gamma^\lambda_{\mu\alpha}h_{\lambda 0} - \Gamma^\alpha_{00}\Gamma^\lambda_{\mu\nu}h_{\lambda\alpha}\right]\right\} \\
&\quad + \frac{t^2}{2\ell^2}\left\{\partial_i\left(\partial_\mu h_{\nu i} + \partial_\nu h_{\mu i} - \partial_i h_{\mu\nu}\right) - \Gamma^\alpha_{i\mu}\left(\partial_\alpha h_{\nu i} + \partial_\nu h_{\alpha i} - \partial_i h_{\alpha\nu}\right)\right. \\
&\qquad - \Gamma^\alpha_{i\nu}\left(\partial_\mu h_{\alpha i} + \partial_\alpha h_{\mu i} - \partial_i h_{\mu\alpha}\right) - \Gamma^\alpha_{ii}\left(\partial_\mu h_{\nu\alpha} + \partial_\nu h_{\mu\alpha} - \partial_\alpha h_{\mu\nu}\right) \\
&\qquad \left. -2\left[\partial_i\left(\Gamma^\lambda_{\mu\nu}h_{\lambda i}\right) - \Gamma^\alpha_{i\mu}\Gamma^\lambda_{\alpha\nu}h_{\lambda i} - \Gamma^\alpha_{i\nu}\Gamma^\lambda_{\mu\alpha}h_{\lambda i} - \Gamma^\alpha_{ii}\Gamma^\lambda_{\mu\nu}h_{\lambda\alpha}\right]\right\} \\
&\quad - \frac{1}{2}\left(\partial_\mu\partial_\nu h - \Gamma^\lambda_{\mu\nu}\partial_\lambda h\right).
\end{aligned}
\tag{A.28}
$$

With the help of (A.8), (A.28) becomes

$$
\begin{aligned}
R^L_{\mu\nu} = -\frac{t^2}{2\ell^2}\Big\{ & \partial_0\left(\partial_\mu h_{\nu 0} + \partial_\nu h_{\mu 0} - \partial_0 h_{\mu\nu}\right) + \frac{2}{t}\left(\partial_\mu h_{\nu 0} + \partial_\nu h_{\mu 0} - \partial_0 h_{\mu\nu}\right) \\
& + \frac{1}{t}\left(\partial_\mu h_{\nu 0} + \partial_\nu h_{\mu 0} - \partial_0 h_{\mu\nu}\right) \\
& -2\left[\partial_0\left(\Gamma^\lambda_{\mu\nu} h_{\lambda 0}\right) + \frac{1}{t}\left(2\Gamma^\lambda_{\mu\nu} h_{\lambda 0} + \Gamma^\lambda_{\mu\nu} h_{\lambda 0}\right)\right]\Big\} \\
+ \frac{t^2}{2\ell^2}\Big\{ & \partial_i\left(\partial_\mu h_{\nu i} + \partial_\nu h_{\mu i} - \partial_i h_{\mu\nu}\right) - \Gamma^\alpha_{i\mu}\left(\partial_\alpha h_{\nu i} + \partial_\nu h_{\alpha i} - \partial_i h_{\alpha\nu}\right) \\
& - \Gamma^\alpha_{i\nu}\left(\partial_\mu h_{\alpha i} + \partial_\alpha h_{\mu i} - \partial_i h_{\mu\alpha}\right) + \frac{2}{t}\left(\partial_\mu h_{\nu 0} + \partial_\nu h_{\mu 0} - \partial_0 h_{\mu\nu}\right) \\
& -2\left[\partial_i\left(\Gamma^\lambda_{\mu\nu} h_{\lambda i}\right) - \Gamma^\alpha_{i\mu}\Gamma^\lambda_{\alpha\nu} h_{\lambda i} - \Gamma^\alpha_{i\nu}\Gamma^\lambda_{\mu\alpha} h_{\lambda i} + \frac{2}{t}\Gamma^\lambda_{\mu\nu} h_{\lambda 0}\right]\Big\} \\
- \frac{1}{2} & \left(\partial_\mu\partial_\nu h - \Gamma^\lambda_{\mu\nu}\partial_\lambda h\right),
\end{aligned}
\tag{A.29}
$$

finally, we end up with

$$
\begin{aligned}
R^L_{\mu\nu} = -\frac{t^2}{2\ell^2}\Big\{ & \partial_0\left(\partial_\mu h_{\nu 0} + \partial_\nu h_{\mu 0} - \partial_0 h_{\mu\nu}\right) - \partial_i\left(\partial_\mu h_{\nu i} + \partial_\nu h_{\mu i} - \partial_i h_{\mu\nu}\right) \\
& + \frac{1}{t}\left(\partial_\mu h_{\nu 0} + \partial_\nu h_{\mu 0} - \partial_0 h_{\mu\nu}\right) + \Gamma^\alpha_{i\mu}\left(\partial_\alpha h_{\nu i} + \partial_\nu h_{\alpha i} - \partial_i h_{\alpha\nu}\right) \\
& + \Gamma^\alpha_{i\nu}\left(\partial_\mu h_{\alpha i} + \partial_\alpha h_{\mu i} - \partial_i h_{\mu\alpha}\right) \\
& -2\left[\partial_0\left(\Gamma^\lambda_{\mu\nu} h_{\lambda 0}\right) - \partial_i\left(\Gamma^\lambda_{\mu\nu} h_{\lambda i}\right) + \frac{1}{t}\Gamma^\lambda_{\mu\nu} h_{\lambda 0} + \Gamma^\alpha_{i\mu}\Gamma^\lambda_{\alpha\nu} h_{\lambda i} + \Gamma^\alpha_{i\nu}\Gamma^\lambda_{\mu\alpha} h_{\lambda i}\right]\Big\} \\
- \frac{1}{2} & \left(\partial_\mu\partial_\nu h - \Gamma^\lambda_{\mu\nu}\partial_\lambda h\right).
\end{aligned}
\tag{A.30}
$$

With this result the components of the Ricci tensor can be calculated.

A.3.1.1 The R^L_{00} Component:

Using (A.30), R^L_{00} component becomes

$$
\begin{aligned}
R^L_{00} = -\frac{t^2}{2\ell^2}\{ & \partial_0\left(\partial_0 h_{00} + \partial_0 h_{00} - \partial_0 h_{00}\right) - \partial_i\left(\partial_0 h_{0i} + \partial_0 h_{0i} - \partial_i h_{00}\right) \\
& + \frac{1}{t}\left(\partial_0 h_{00} + \partial_0 h_{00} - \partial_0 h_{00}\right) \\
& + \Gamma^\alpha_{i0}\left(\partial_\alpha h_{0i} + \partial_0 h_{\alpha i} - \partial_i h_{\alpha 0}\right) + \Gamma^\alpha_{i0}\left(\partial_0 h_{\alpha i} + \partial_\alpha h_{0i} - \partial_i h_{0\alpha}\right) \\
& -2\left[\partial_0\left(\Gamma^\lambda_{00} h_{\lambda 0}\right) - \partial_i\left(\Gamma^\lambda_{00} h_{\lambda i}\right) + \frac{1}{t}\Gamma^\lambda_{00} h_{\lambda 0} + \Gamma^\alpha_{i0}\Gamma^\lambda_{\alpha 0} h_{\lambda i} + \Gamma^\alpha_{i0}\Gamma^\lambda_{0\alpha} h_{\lambda i}\right]\Big\} \\
- \frac{1}{2} & \left(\partial_0\partial_0 h - \Gamma^\lambda_{00}\partial_\lambda h\right),
\end{aligned}
\tag{A.31}
$$

rearranging and doing some cancellations yield

$$
\begin{aligned}
R^L_{00} = &-\frac{t^2}{2\ell^2}\left\{\partial_0^2 h_{00} - 2\partial_0\partial_i h_{0i} + \partial_i^2 h_{00} + \frac{1}{t}\partial_0 h_{00} + 2\Gamma^\alpha_{i0}\left(\partial_\alpha h_{0i} + \partial_0 h_{\alpha i} - \partial_i h_{\alpha 0}\right)\right.\\
&\left.-2\left[\partial_0\left(\Gamma^\lambda_{00}h_{\lambda 0}\right) - \partial_i\left(\Gamma^\lambda_{00}h_{\lambda i}\right) + \frac{1}{t}\Gamma^\lambda_{00}h_{\lambda 0} + 2\Gamma^\alpha_{i0}\Gamma^\lambda_{\alpha 0}h_{\lambda i}\right]\right\} \\
&-\frac{1}{2}\left(\partial_0\partial_0 h - \Gamma^\lambda_{00}\partial_\lambda h\right).
\end{aligned} \tag{A.32}
$$

Summing the repeated indices and using (A.8) gives

$$
\begin{aligned}
R^L_{00} = &-\frac{t^2}{2\ell^2}\left(\partial_0^2 h_{00} - 2\partial_0\partial_i h_{0i} + \partial_i^2 h_{00}\right) - \frac{t}{2\ell^2}\left(3\partial_0 h_{00} - 2\partial_0 h_{ii} - 2\partial_i h_{0i}\right)\\
&+\frac{2}{\ell^2}h_{ii} - \frac{1}{2}\left(\partial_0\partial_0 + \frac{1}{t}\partial_0\right)h.
\end{aligned} \tag{A.33}
$$

Putting (A.2) in (A.33) yields,

$$
\begin{aligned}
R^L_{00} = &-\frac{t^2}{2\ell^2}\left(\ddot{N} - 2\partial_i^2\dot{N}_L + \partial_i^2 N\right) - \frac{t}{2\ell^2}\left(3\dot{N} - 2\dot{\phi} - 2\dot{\chi} - 2\partial_i^2 N_L\right)\\
&+\frac{2}{\ell^2}\left(\phi+\chi\right) - \frac{1}{2}\left(\partial_0\partial_0 + \frac{1}{t}\partial_0\right)\left[\frac{t^2}{\ell^2}\left(-N+\phi+\chi\right)\right],\\
= &-\frac{t^2}{2\ell^2}\left(\ddot{N} + \partial_i^2 N - 2\partial_i^2\dot{N}_L\right) - \frac{t}{2\ell^2}\left(3\dot{N} - 2\dot{\phi} - 2\dot{\chi} - 2\partial_i^2 N_L\right) + \frac{2}{\ell^2}\left(\phi+\chi\right)\\
&+\frac{t^2}{2\ell^2}\left(\ddot{N} - \ddot{\phi} - \ddot{\chi}\right) + \frac{5t}{2\ell^2}\left(\dot{N} - \dot{\phi} - \dot{\chi}\right) + \frac{2}{\ell^2}\left(N - \phi - \chi\right),
\end{aligned} \tag{A.34}
$$

and after the cancellations the final answer of R^L_{00} component becomes

$$
R^L_{00} = -\frac{t^2}{2\ell^2}\left(\partial_i^2 N - 2\partial_i^2\dot{N}_L + \ddot{\phi} + \ddot{\chi}\right) + \frac{t}{2\ell^2}\left(2\dot{N} - 3\dot{\phi} - 3\dot{\chi} + 2\partial_i^2 N_L\right) + \frac{2}{\ell^2}N. \tag{A.35}
$$

Also, taking the $t/\ell \to 1$ and $\ell \to \infty$ limits the flat spacetime case of this component can be got

$$
R^L_{00} = -\frac{1}{2}\left(\partial_i^2 N - 2\partial_i^2\dot{N}_L + \ddot{\phi} + \ddot{\chi}\right). \tag{A.36}
$$

A.3.1.2 The R^L_{0i} Component:

From (A.30) R^L_{0j} term can be written as

$$
\begin{aligned}
R^L_{0j} = &-\frac{t^2}{2\ell^2}\left\{\partial_0\left(\partial_0 h_{j0} + \partial_j h_{00} - \partial_0 h_{0j}\right) - \partial_i\left(\partial_0 h_{ji} + \partial_j h_{0i} - \partial_i h_{0j}\right)\right.\\
&+\frac{1}{t}\left(\partial_0 h_{j0} + \partial_j h_{00} - \partial_0 h_{0j}\right)\\
&+\Gamma^\alpha_{i0}\left(\partial_\alpha h_{ji} + \partial_j h_{\alpha i} - \partial_i h_{\alpha j}\right) + \Gamma^\alpha_{ij}\left(\partial_0 h_{\alpha i} + \partial_\alpha h_{0i} - \partial_i h_{0\alpha}\right)\\
&\left.-2\left[\partial_0\left(\Gamma^\lambda_{0j}h_{\lambda 0}\right) - \partial_i\left(\Gamma^\lambda_{0j}h_{\lambda i}\right) + \frac{1}{t}\Gamma^\lambda_{0j}h_{\lambda 0} + \Gamma^\alpha_{i0}\Gamma^\lambda_{\alpha j}h_{\lambda i} + \Gamma^\alpha_{ij}\Gamma^\lambda_{0\alpha}h_{\lambda i}\right]\right\}\\
&-\frac{1}{2}\left(\partial_0\partial_j h - \Gamma^\lambda_{0j}\partial_\lambda h\right),
\end{aligned} \tag{A.37}
$$

rearranging and doing some cancellations we get

$$R^L_{0j} = -\frac{t^2}{2\ell^2}\Bigg\{\partial_0\partial_j h_{00} - \partial_0\partial_i h_{ij} - \partial_j\partial_i h_{0i} + \partial_i^2 h_{0j} + \frac{1}{t}\partial_j h_{00}$$
$$+ \Gamma^\alpha_{i0}\left(\partial_\alpha h_{ji} + \partial_j h_{\alpha i} - \partial_i h_{\alpha j}\right) + \Gamma^\alpha_{ij}\left(\partial_0 h_{\alpha i} + \partial_\alpha h_{0i} - \partial_i h_{0\alpha}\right)$$
$$-2\left[\partial_0\left(\Gamma^\lambda_{0j}h_{\lambda 0}\right) - \partial_i\left(\Gamma^\lambda_{0j}h_{\lambda i}\right) + \frac{1}{t}\Gamma^\lambda_{0j}h_{\lambda 0} + \Gamma^\alpha_{i0}\Gamma^\lambda_{\alpha j}h_{\lambda i} + \Gamma^\alpha_{ij}\Gamma^\lambda_{0\alpha}h_{\lambda i}\right]\Bigg\}$$
$$- \frac{1}{2}\left(\partial_0\partial_j h - \Gamma^\lambda_{0j}\partial_\lambda h\right). \tag{A.38}$$

Again, using (A.8) in (A.38) gives us

$$R^L_{0j} = -\frac{t^2}{2\ell^2}\Bigg\{\partial_0\partial_j h_{00} - \partial_0\partial_i h_{ij} - \partial_j\partial_i h_{0i} + \partial_i^2 h_{0j}$$
$$+ \frac{1}{t}\partial_j h_{00} - \frac{1}{t}\left(\partial_i h_{ji} + \partial_j h_{ii} - \partial_i h_{ij}\right) - \frac{1}{t}\left(\partial_0 h_{0j} + \partial_0 h_{0j} - \partial_j h_{00}\right)$$
$$-2\left[-\frac{1}{t}\partial_0 h_{j0} + \frac{1}{t^2}h_{j0} + \frac{1}{t}\partial_i h_{ji} - \frac{1}{t^2}h_{j0} + \frac{1}{t^2}h_{0j} + \frac{1}{t^2}h_{0j}\right]\Bigg\}$$
$$- \frac{1}{2}\left(\partial_0\partial_j h + \frac{1}{t}\partial_j h\right), \tag{A.39}$$

doing cancellations and summing the same terms yields

$$R^L_{0j} = -\frac{t^2}{2\ell^2}\left(\partial_0\partial_j h_{00} - \partial_0\partial_i h_{ij} - \partial_j\partial_i h_{0i} + \partial_i^2 h_{0j}\right) - \frac{t}{2\ell^2}\left(2\partial_j h_{00} - 2\partial_i h_{ij} - \partial_j h_{ii}\right)$$
$$+ \frac{2}{\ell^2}h_{0j} - \frac{1}{2}\left(\partial_0 + \frac{1}{t}\right)\partial_j h. \tag{A.40}$$

Putting (A.2) in (A.40) bring forth

$$R^L_{0j} = -\frac{t^2}{2\ell^2}\left(\partial_j\dot{N} - \partial_j\dot{\chi} + \epsilon_{ji}\partial_i\dot{\xi} - \partial_j\partial_i^2 N_L - \epsilon_{jk}\partial_k\partial_i^2\eta + \partial_j\partial_i^2 N_L\right)$$
$$- \frac{t}{2\ell^2}\left(2\partial_j N - 2\partial_j\chi + 2\epsilon_{ji}\partial_i\xi - \partial_j\phi - \partial_j\chi\right) + \frac{2}{\ell^2}\left(-\epsilon_{ji}\partial_i\eta + \partial_j N_L\right) \tag{A.41}$$
$$- \frac{1}{2}\left(\partial_0 + \frac{1}{t}\right)\left[\frac{t^2}{\ell^2}\partial_j\left(-N + \phi + \chi\right)\right].$$

Rearranging and opening the parenthesis in the last line in (A.41) gives us

$$R^L_{0j} = -\frac{t^2}{2\ell^2}\left[\partial_j\left(\dot{N} - \dot{\chi}\right) + \epsilon_{jk}\partial_k\left(\dot{\xi} - \partial_i^2\eta\right)\right] - \frac{t}{2\ell^2}\left[\partial_j\left(2N - 3\chi - \phi\right) + 2\epsilon_{ji}\partial_i\xi\right]$$
$$+ \frac{2}{\ell^2}\left(-\epsilon_{ji}\partial_i\eta + \partial_j N_L\right) + \frac{t^2}{2\ell^2}\partial_j\left(\dot{N} - \dot{\chi} - \dot{\phi}\right) + \frac{t}{2\ell^2}\partial_j\left(3N - 3\chi - 3\phi\right), \tag{A.42}$$

and the overall result comes out as

$$R^L_{0j} = -\frac{t^2}{2\ell^2}\left[\partial_j\dot{\phi} + \epsilon_{jk}\partial_k\left(\dot{\xi} - \partial_i^2\eta\right)\right] + \frac{t}{2\ell^2}\left[\partial_j\left(N - 2\phi\right) - 2\epsilon_{ji}\partial_i\xi\right]$$
$$+ \frac{2}{\ell^2}\left(-\epsilon_{ji}\partial_i\eta + \partial_j N_L\right). \tag{A.43}$$

The flat spacetime limit, $t/\ell \to 1$ and $\ell \to \infty$, of this expression is

$$R^L_{0j} = -\frac{1}{2}\left[\partial_j\dot{\phi} + \epsilon_{jk}\partial_k\left(\dot{\xi} - \partial_i^2\eta\right)\right]. \tag{A.44}$$

A.3.1.3 The R^L_{jk} Component:

The last term is R^L_{jk} and it can be written from (A.30) as

$$\begin{aligned}
R^L_{jk} = -\frac{t^2}{2\ell^2}\Big\{&\partial_0\left(\partial_j h_{k0}+\partial_k h_{j0}-\partial_0 h_{jk}\right)-\partial_i\left(\partial_j h_{ki}+\partial_k h_{ji}-\partial_i h_{jk}\right)\\
&+\frac{1}{t}\left(\partial_j h_{k0}+\partial_k h_{j0}-\partial_0 h_{jk}\right)\\
&+\Gamma^\alpha_{ij}\left(\partial_\alpha h_{ki}+\partial_k h_{\alpha i}-\partial_i h_{\alpha k}\right)+\Gamma^\alpha_{ik}\left(\partial_j h_{\alpha i}+\partial_\alpha h_{ji}-\partial_i h_{j\alpha}\right)\\
&-2\left[\partial_0\left(\Gamma^\lambda_{jk}h_{\lambda 0}\right)-\partial_i\left(\Gamma^\lambda_{jk}h_{\lambda i}\right)+\frac{1}{t}\Gamma^\lambda_{jk}h_{\lambda 0}+\Gamma^\alpha_{ij}\Gamma^\lambda_{\alpha k}h_{\lambda i}+\Gamma^\alpha_{ik}\Gamma^\lambda_{j\alpha}h_{\lambda i}\right]\Big\}\\
-\frac{1}{2}&\left(\partial_j\partial_k h-\Gamma^\lambda_{jk}\partial_\lambda h\right),
\end{aligned} \tag{A.45}$$

and using (A.8) yields

$$\begin{aligned}
R^L_{jk} = &-\frac{t^2}{2\ell^2}\left(2\partial_0\partial_j h_{k0}-2\partial_j\partial_i h_{ki}-\partial_0^2 h_{jk}+\partial_i^2 h_{jk}\right)\\
&-\frac{t}{2\ell^2}\left(2\partial_j h_{k0}-3\partial_0 h_{jk}+2\eta_{jk}\partial_0 h_{00}-2\eta_{jk}\partial_i h_{0i}\right)\\
&+\frac{2}{\ell^2}h_{jk}-\frac{1}{2}\left(\partial_j\partial_k h+\frac{1}{t}\eta_{jk}\partial_0 h\right).
\end{aligned} \tag{A.46}$$

Putting (A.2) in (A.46) gives us

$$\begin{aligned}
R^L_{jk} = \left(-\frac{t^2}{2\ell^2}\right)\Big\{&\partial_j\left[-2\epsilon_{kl}\partial_l\left(\dot\eta+\frac{1}{t}\eta\right)+\partial_k\left(2\dot N_L-N+\frac{2}{t}N_L\right)\right]\\
&-\left(\partial_0^2+\frac{3}{t}\partial_0+\frac{4}{t^2}\right)\left[\left(\delta_{jk}+\hat\partial_j\hat\partial_k\right)\phi-\hat\partial_j\hat\partial_k\chi+\left(\epsilon_{jl}\hat\partial_l\hat\partial_k+\epsilon_{kl}\hat\partial_l\hat\partial_j\right)\xi\right]\\
&+\left[\delta_{jk}\partial_i^2\phi-\left(\epsilon_{jl}\partial_l\partial_k-\epsilon_{kl}\partial_l\partial_j\right)\xi\right]\\
&+\frac{1}{t}\delta_{jk}\left[2\left(\dot N-\partial_i^2 N_L\right)+\left(-\dot N+\dot\phi+\dot\chi\right)+\frac{2}{t}\left(-N+\phi+\chi\right)\right]\Big\},
\end{aligned} \tag{A.47}$$

where $\left(\epsilon_{jl}\partial_l\partial_k-\epsilon_{kl}\partial_l\partial_j\right)\xi$ should be eliminated due to the fact that j and k are symmetric. Then the final form for R^L_{jk} becomes

$$\begin{aligned}
R^L_{jk} = \left(-\frac{t^2}{2\ell^2}\right)\Big\{&\partial_j\left[-2\epsilon_{kl}\partial_l\left(\dot\eta+\frac{1}{t}\eta\right)+\partial_k\left(2\dot N_L-N+\frac{2}{t}N_L\right)\right]\\
&-\left(\partial_0^2+\frac{3}{t}\partial_0+\frac{4}{t^2}\right)\left[\left(\delta_{jk}+\hat\partial_j\hat\partial_k\right)\phi-\hat\partial_j\hat\partial_k\chi+\left(\epsilon_{jl}\hat\partial_l\hat\partial_k+\epsilon_{kl}\hat\partial_l\hat\partial_j\right)\xi\right]\\
&+\delta_{jk}\partial_i^2\phi+\frac{1}{t}\delta_{jk}\left[2\left(\dot N-\partial_i^2 N_L\right)+\left(-\dot N+\dot\phi+\dot\chi\right)+\frac{2}{t}\left(-N+\phi+\chi\right)\right]\Big\}.
\end{aligned} \tag{A.48}$$

The flat spacetime result of this component is

$$R^L_{jk} = \left(-\frac{1}{2}\right)\Big\{\partial_j\left[-2\epsilon_{kl}\partial_l\left(\dot{\eta}+\frac{1}{t}\eta\right)+\partial_k\left(2\dot{N}_L - N + \frac{2}{t}N_L\right)\right]$$
$$-\left(\partial_0^2+\frac{3}{t}\partial_0+\frac{4}{t^2}\right)\left[\left(\delta_{jk}+\hat{\partial}_j\hat{\partial}_k\right)\phi - \hat{\partial}_j\hat{\partial}_k\chi + \left(\epsilon_{jl}\hat{\partial}_l\hat{\partial}_k+\epsilon_{kl}\hat{\partial}_l\hat{\partial}_j\right)\xi\right]$$
$$+\delta_{jk}\partial_i^2\phi + \frac{1}{t}\delta_{jk}\left[2\left(\dot{N}-\partial_i^2 N_L\right)+\left(-\dot{N}+\dot{\phi}+\dot{\chi}\right)+\frac{2}{t}\left(-N+\phi+\chi\right)\right]\Big\}, \quad \text{(A.49)}$$

where the $t/\ell \to 1$ and $\ell \to \infty$ limits are used.

In order to calculate the components of Einstein tensor the Ricci scalar is also needed. The next section is devoted to the calculation of Ricci scalar.

A.3.2 Ricci Scalar

Now, let us work on the R_L term: It is defined as

$$R_L \equiv \nabla_\mu\nabla_\nu h^{\mu\nu} - \Box h - 2\Lambda h,$$
$$= \frac{t^4}{\ell^4}\eta^{\rho\mu}\eta^{\sigma\nu}\nabla_\rho\nabla_\sigma h_{\mu\nu} - \frac{t^2}{\ell^2}\eta^{\mu\nu}\nabla_\mu\nabla_\nu h - \frac{2}{\ell^2}h, \quad \text{(A.50)}$$

where in the second line the upper indices are lowered. In (A.50) there are two terms that must be written in terms of the free functions. These are

$$\nabla_\mu\nabla_\nu h = \partial_\mu\partial_\nu h - \Gamma^\lambda_{\mu\nu}\partial_\lambda h, \quad \text{(A.51)}$$

and

$$\nabla_\rho\nabla_\sigma h_{\mu\nu} = \partial_\rho\left(\nabla_\sigma h_{\mu\nu}\right) - \Gamma^\alpha_{\rho\sigma}\left(\nabla_\alpha h_{\mu\nu}\right) - \Gamma^\alpha_{\rho\mu}\left(\nabla_\sigma h_{\alpha\nu}\right) - \Gamma^\alpha_{\rho\nu}\left(\nabla_\sigma h_{\mu\alpha}\right),$$
$$= \partial_\rho\left(\partial_\sigma h_{\mu\nu} - \Gamma^\lambda_{\sigma\mu}h_{\lambda\nu} - \Gamma^\lambda_{\sigma\nu}h_{\mu\lambda}\right) - \Gamma^\alpha_{\rho\sigma}\left(\partial_\alpha h_{\mu\nu} - \Gamma^\lambda_{\alpha\mu}h_{\lambda\nu} - \Gamma^\lambda_{\alpha\nu}h_{\mu\lambda}\right) \quad \text{(A.52)}$$
$$- \Gamma^\alpha_{\rho\mu}\left(\partial_\sigma h_{\alpha\nu} - \Gamma^\lambda_{\sigma\alpha}h_{\lambda\nu} - \Gamma^\lambda_{\sigma\nu}h_{\alpha\lambda}\right) - \Gamma^\alpha_{\rho\nu}\left(\partial_\sigma h_{\mu\alpha} - \Gamma^\lambda_{\sigma\mu}h_{\lambda\alpha} - \Gamma^\lambda_{\sigma\alpha}h_{\mu\lambda}\right).$$

After the application of contractions, using (A.8) and doing some simple manipulations these terms become

$$
\begin{aligned}
-\frac{t^2}{\ell^2}\eta^{\mu\nu}\nabla_\mu\nabla_\nu h &= -\frac{t^2}{\ell^2}\eta^{\mu\nu}\left(\partial_\mu\partial_\nu h - \Gamma^\lambda_{\mu\nu}\partial_\lambda h\right), \\
&= \frac{t^2}{\ell^2}\left[\left(\partial_0\partial_0 h - \Gamma^\lambda_{00}\partial_\lambda h\right) - \left(\partial_i\partial_i h - \Gamma^\lambda_{ii}\partial_\lambda h\right)\right], \\
&= \frac{t^2}{\ell^2}\left[\partial_0\partial_0 h - \partial_i\partial_i h + \frac{1}{t}\left(\partial_0 h - 2\partial_0 h\right)\right], \\
&= \frac{t^2}{\ell^2}\left[\partial_0\partial_0 h - \partial_i\partial_i h - \frac{1}{t}\partial_0 h\right], \\
&= \frac{t^2}{\ell^2}\left(\partial_0\partial_0 h - \partial_i\partial_i h\right) - \frac{t}{\ell^2}\partial_0 h,
\end{aligned}
\tag{A.53}
$$

and

$$
\begin{aligned}
\frac{t^4}{\ell^4}\eta^{\rho\mu}\eta^{\sigma\nu}\nabla_\rho\nabla_\sigma h_{\mu\nu} = &\frac{t^4}{\ell^4}\Big[\partial_0\left(\partial_0 h_{00} - \Gamma^\lambda_{00}h_{\lambda 0} - \Gamma^\lambda_{00}h_{0\lambda}\right) - \Gamma^\alpha_{00}\left(\partial_\alpha h_{00} - \Gamma^\lambda_{\alpha 0}h_{\lambda 0} - \Gamma^\lambda_{\alpha 0}h_{0\lambda}\right) \\
&-\Gamma^\alpha_{00}\left(\partial_0 h_{\alpha 0} - \Gamma^\lambda_{0\alpha}h_{\lambda 0} - \Gamma^\lambda_{00}h_{\alpha\lambda}\right) - \Gamma^\alpha_{00}\left(\partial_0 h_{0\alpha} - \Gamma^\lambda_{00}h_{\lambda\alpha} - \Gamma^\lambda_{0\alpha}h_{0\lambda}\right)\Big] \\
&- \frac{t^4}{\ell^4}\Big[\partial_0\left(\partial_i h_{0i} - \Gamma^\lambda_{i0}h_{\lambda i} - \Gamma^\lambda_{ii}h_{0\lambda}\right) - \Gamma^\alpha_{0i}\left(\partial_\alpha h_{0i} - \Gamma^\lambda_{\alpha 0}h_{\lambda i} - \Gamma^\lambda_{\alpha i}h_{0\lambda}\right) \\
&-\Gamma^\alpha_{00}\left(\partial_i h_{\alpha i} - \Gamma^\lambda_{i\alpha}h_{\lambda i} - \Gamma^\lambda_{ii}h_{\alpha\lambda}\right) - \Gamma^\alpha_{0i}\left(\partial_i h_{0\alpha} - \Gamma^\lambda_{i0}h_{\lambda\alpha} - \Gamma^\lambda_{i\alpha}h_{0\lambda}\right)\Big] \\
&- \frac{t^4}{\ell^4}\Big[\partial_i\left(\partial_0 h_{i0} - \Gamma^\lambda_{0i}h_{\lambda 0} - \Gamma^\lambda_{00}h_{i\lambda}\right) - \Gamma^\alpha_{i0}\left(\partial_\alpha h_{i0} - \Gamma^\lambda_{\alpha i}h_{\lambda 0} - \Gamma^\lambda_{\alpha 0}h_{i\lambda}\right) \\
&-\Gamma^\alpha_{ii}\left(\partial_0 h_{\alpha 0} - \Gamma^\lambda_{0\alpha}h_{\lambda 0} - \Gamma^\lambda_{00}h_{\alpha\lambda}\right) - \Gamma^\alpha_{i0}\left(\partial_0 h_{i\alpha} - \Gamma^\lambda_{0i}h_{\lambda\alpha} - \Gamma^\lambda_{0\alpha}h_{i\lambda}\right)\Big] \\
&+ \frac{t^4}{\ell^4}\Big[\partial_i\left(\partial_j h_{ij} - \Gamma^\lambda_{ji}h_{\lambda j} - \Gamma^\lambda_{jj}h_{i\lambda}\right) - \Gamma^\alpha_{ij}\left(\partial_\alpha h_{ij} - \Gamma^\lambda_{\alpha i}h_{\lambda j} - \Gamma^\lambda_{\alpha j}h_{i\lambda}\right) \\
&-\Gamma^\alpha_{ii}\left(\partial_j h_{\alpha j} - \Gamma^\lambda_{j\alpha}h_{\lambda j} - \Gamma^\lambda_{jj}h_{\alpha\lambda}\right) - \Gamma^\alpha_{ij}\left(\partial_j h_{i\alpha} - \Gamma^\lambda_{ji}h_{\lambda\alpha} - \Gamma^\lambda_{j\alpha}h_{i\lambda}\right)\Big].
\end{aligned}
\tag{A.54}
$$

Summing the repeated indices in (A.54) and again using (A.8) yields

$$
\nabla_\mu\nabla_\nu h^{\mu\nu} = \frac{t^4}{\ell^4}\left(\partial_0\partial_0 h_{00} - 2\partial_0\partial_i h_{0i} + \partial_i\partial_j h_{ij}\right) + \frac{t^3}{\ell^4}\left(\partial_0 h_{00} - \partial_0 h_{ii}\right). \tag{A.55}
$$

Putting (A.53) and (A.55) into (A.50) and using $\Lambda = 1/\ell^2$ gives us

$$
\begin{aligned}
R_L = &\left[\frac{t^4}{\ell^4}\left(\partial_0\partial_0 h_{00} - 2\partial_0\partial_i h_{0i} + \partial_i\partial_j h_{ij}\right) + \frac{t^3}{\ell^4}\left(\partial_0 h_{00} - \partial_0 h_{ii}\right)\right] \\
&+ \left[\frac{t^2}{\ell^2}\left(\partial_0\partial_0 - \partial_i\partial_i\right) - \frac{t}{\ell^2}\partial_0 - \frac{2}{\ell^2}\right]h.
\end{aligned}
\tag{A.56}
$$

Putting (A.2) in (A.50) yields

$$
\begin{aligned}
R_L =& \left[\frac{t^4}{\ell^4}\left(\ddot{N} - 2\partial_i^2\dot{N}_L + \partial_i^2\chi\right) + \frac{t^3}{\ell^4}\left(\dot{N} - \dot{\phi} - \dot{\chi}\right)\right] \\
&+ \left[\frac{t^2}{\ell^2}\left(\partial_0\partial_0 - \partial_i\partial_i\right) - \frac{t}{\ell^2}\partial_0 - \frac{2}{\ell^2}\right]\left[\frac{t^2}{\ell^2}\left(-N + \phi + \chi\right)\right], \\
=& \left[\frac{t^4}{\ell^4}\left(\ddot{N} - 2\partial_i^2\dot{N}_L + \partial_i^2\chi\right) + \frac{t^3}{\ell^4}\left(\dot{N} - \dot{\phi} - \dot{\chi}\right)\right] \\
&+ \left[\frac{t^4}{\ell^4}\left(\partial_0\partial_0 - \partial_i\partial_i\right) + \frac{4t^3}{\ell^4}\partial_0 + \frac{2t^2}{\ell^4} - \frac{t^3}{\ell^4}\partial_0 - \frac{2t^2}{\ell^4} - \frac{2t^2}{\ell^4}\right]\left(-N + \phi + \chi\right), \\
=& \left[\left(\frac{t^4}{\ell^4}\partial_0\partial_0 + \frac{t^3}{\ell^4}\partial_0\right)N + \left(\frac{t^4}{\ell^4}\partial_i^2 - \frac{t^3}{\ell^4}\partial_0\right)\chi - \frac{t^4}{\ell^4}2\partial_i^2\dot{N}_L - \frac{t^3}{\ell^4}\partial_0\phi\right] \\
&+ \left[\frac{t^4}{\ell^4}\left(\partial_0\partial_0 - \partial_i\partial_i\right) + \frac{3t^3}{\ell^4}\partial_0 - \frac{2t^2}{\ell^4}\right]\left(-N + \phi + \chi\right), \\
=& \left[\left(\frac{t^4}{\ell^4}\partial_i\partial_i - \frac{2t^3}{\ell^4}\partial_0 + \frac{2t^2}{\ell^4}\right)N + \left(\frac{t^4}{\ell^4}\partial_0^2 + \frac{2t^3}{\ell^4}\partial_0 - \frac{2t^2}{\ell^4}\right)\chi - \frac{t^4}{\ell^4}2\partial_i^2\dot{N}_L\right] \\
&+ \left[\frac{t^4}{\ell^4}\left(\partial_0\partial_0 - \partial_i\partial_i\right) + \frac{2t^3}{\ell^4}\partial_0 - \frac{2t^2}{\ell^4}\right]\phi,
\end{aligned}
\tag{A.57}
$$

and finally the scalar curvature read

$$
R_L = \frac{t^4}{\ell^4}\left(\partial_i^2 N + \ddot{\chi} - 2\partial_i^2\dot{N}_L + \ddot{\phi} - \partial_i^2\phi\right) + \frac{2t^3}{\ell^4}\left(-\dot{N} + \dot{\phi} + \dot{\chi}\right) - \frac{2t^2}{\ell^4}\left(-N + \phi + \chi\right). \tag{A.58}
$$

In the flat spacetime limit (A.58) becomes

$$
R_L = \partial_j^2 N + \ddot{\chi} - 2\partial_i^2\dot{N}_L + \left(\partial_0^2 - \partial_j^2\right)\phi, \tag{A.59}
$$

which is the flat spacetime result for the scalar curvature.

With the results that we found in the above calculations, (A.35), (A.43), (A.48) and (A.58), we are ready to find the components of the Einstein tensor in terms of the free functions of metric perturbation.

A.3.3 The Einstein Tensor

The linear Einstein tensor is defined as follows

$$
\mathcal{G}_{\mu\nu}^L = R_{\mu\nu}^L - \frac{1}{2}\bar{g}_{\mu\nu}R_L - 2\Lambda h_{\mu\nu}. \tag{A.60}
$$

The components of the Einstein tensor is investigated term by term.

A.3.3.1 The $\mathcal{G}^L_{00}$ Term:

The 00-component of linear Einstein tensor,$\mathcal{G}^L_{00}$, can be written by using (A.35), (A.58) and (A.2) as follows,

$$
\begin{aligned}
\mathcal{G}^L_{00} &= R^L_{00} - \frac{1}{2}\bar{g}_{00}R_L - 2\Lambda h_{00}, \\
&= \left[-\frac{t^2}{2\ell^2}\left(\partial_i^2 N - 2\partial_i^2\dot{N}_L + \ddot{\phi} + \ddot{\chi}\right) + \frac{t}{2\ell^2}\left(2\dot{N} - 3\dot{\phi} - 3\dot{\chi} + 2\partial_i^2 N_L\right) + \frac{2}{\ell^2}N\right] - \frac{2}{\ell^2}N \\
&\quad - \frac{1}{2}\frac{\ell^2}{t^2}\eta_{00}\left\{\frac{t^4}{\ell^4}\left[\partial_i^2 N + \ddot{\chi} - 2\partial_i^2\dot{N}_L + \left(\partial_0^2 - \partial_i^2\right)\phi\right]\right. \\
&\quad \left. + \frac{2t^3}{\ell^4}\left(-\dot{N} + \dot{\phi} + \dot{\chi}\right) - \frac{2t^2}{\ell^4}\left(-N + \phi + \chi\right)\right\}, \\
&= \left[-\frac{t^2}{2\ell^2}\left(\partial_i^2 N - 2\partial_i^2\dot{N}_L + \ddot{\phi} + \ddot{\chi}\right) + \frac{t}{2\ell^2}\left(2\dot{N} - 3\dot{\phi} - 3\dot{\chi} + 2\partial_i^2 N_L\right)\right] \\
&\quad + \left\{\frac{t^2}{2\ell^2}\left[\partial_i^2 N + \ddot{\chi} - 2\partial_i^2\dot{N}_L + \left(\partial_0^2 - \partial_i^2\right)\phi\right] + \frac{t}{\ell^2}\left(-\dot{N} + \dot{\phi} + \dot{\chi}\right) - \frac{1}{\ell^2}\left(-N + \phi + \chi\right)\right\},
\end{aligned}
\tag{A.61}
$$

and doing cancellations and summations we get

$$
\mathcal{G}^L_{00} = -\frac{t^2}{2\ell^2}\partial_i^2\phi - \frac{t}{2\ell^2}\left(\dot{\phi} + \dot{\chi} - 2\partial_i^2 N_L\right) - \frac{1}{\ell^2}\left(-N + \phi + \chi\right). \tag{A.62}
$$

To simplify (A.62), the metric perturbation functions are redefined as $\phi \to \frac{\ell^2}{t^2}\phi$ and the time derivative transforms as$\dot{\phi} \to -2\frac{\ell^2}{t^3}\phi + \frac{\ell^2}{t^2}\dot{\phi}$. With these redefinitions the 00-component of the Einstein tensor becomes

$$
\begin{aligned}
\mathcal{G}^L_{00} &= -\frac{1}{2}\partial_i^2\phi - \frac{t}{2\ell^2}\left(-2\frac{\ell^2}{t^3}\phi + \frac{\ell^2}{t^2}\dot{\phi} - 2\frac{\ell^2}{t^3}\chi + \frac{\ell^2}{t^2}\dot{\chi} - 2\frac{\ell^2}{t^2}\partial_i^2 N_L\right) - \frac{1}{t^2}\left(-N + \phi + \chi\right) \\
&= -\frac{1}{2}\partial_i^2\phi - \left(-\frac{1}{t^2}\phi + \frac{1}{2t}\dot{\phi} - \frac{1}{t^2}\chi + \frac{1}{2t}\dot{\chi} - \frac{1}{t}\partial_i^2 N_L\right) - \frac{1}{t^2}\left(-N + \phi + \chi\right) \\
&= -\frac{1}{2}\partial_i^2\phi - \left(\frac{1}{2t}\dot{\phi} + \frac{1}{2t}\dot{\chi} - \frac{1}{t^2}N\right) + \frac{1}{t}\partial_i^2 N_L \\
&= -\frac{1}{2}\left[\partial_i^2\phi + \frac{1}{t}\left(\dot{\phi} + \dot{\chi} - \frac{2}{t}N\right) - \frac{2}{t}\partial_i^2 N_L\right].
\end{aligned}
\tag{A.63}
$$

From the last line of (A.63) a gauge invariant function can be defined:

$$
f \equiv \frac{\ell}{t}\left[\phi + \frac{1}{t\nabla^2}\left(\dot{\phi} + \dot{\chi} - \frac{2}{t}N\right) - \frac{2}{t}N_L\right], \tag{A.64}
$$

then

$$
\mathcal{G}^L_{00} = -\frac{t}{2\ell}\nabla^2 f. \tag{A.65}
$$

Also, for flat spacetime this term becomes

$$
\mathcal{G}^L_{00} = -\frac{1}{2}\nabla^2\phi, \tag{A.66}
$$

where $f = \phi$ for the flat spacetime background.

A.3.3.2 The $\mathcal{G}^L_{0j}$ Term:

The $0j$-component of linear Einstein tensor,$\mathcal{G}^L_{0j} = R^L_{0j} - 2\Lambda h_{0j}$, can be written by using (A.43) and (A.2) as follows,

$$\begin{aligned}
\mathcal{G}^L_{0j} &= \left\{ -\frac{t^2}{2\ell^2}\left[\partial_j\dot{\phi} + \epsilon_{jk}\partial_k\left(\dot{\xi} - \partial_i^2\eta\right)\right] + \frac{t}{2\ell^2}\left[\partial_j\left(N - 2\phi\right) - 2\epsilon_{ji}\partial_i\xi\right] + \frac{2}{\ell^2}h_{0j}\right\} - \frac{2}{\ell^2}h_{0j}, \\
&= -\frac{t^2}{2\ell^2}\left[\partial_j\dot{\phi} + \epsilon_{jk}\partial_k\left(\dot{\xi} - \partial_i^2\eta\right)\right] + \frac{t}{2\ell^2}\left[\partial_j\left(N - 2\phi\right) - 2\epsilon_{ji}\partial_i\xi\right]. \qquad \text{(A.67)}
\end{aligned}$$

Again doing the same redefinitions, $\phi \to \frac{\ell^2}{t^2}\phi$ and $\dot{\phi} \to -2\frac{\ell^2}{t^3}\phi + \frac{\ell^2}{t^2}\dot{\phi}$, (A.67) becomes

$$\begin{aligned}
\mathcal{G}^L_{0j} &= -\frac{1}{2}\left[\partial_j\left(-\frac{2}{t}\phi + \dot{\phi}\right) + \epsilon_{jk}\partial_k\left(-\frac{2}{t}\xi + \dot{\xi} - \partial_i^2\eta\right)\right] + \frac{1}{2t}\left[\partial_j\left(N - 2\phi\right) - 2\epsilon_{ji}\partial_i\xi\right] \\
&= -\frac{1}{2}\left[\partial_j\dot{\phi} + \epsilon_{jk}\partial_k\left(\dot{\xi} - \partial_i^2\eta\right) - \frac{1}{t}\partial_j N\right]. \qquad \text{(A.68)}
\end{aligned}$$

From (A.68) two gauge-invariant combinations can be defined as

$$p \equiv \frac{\ell}{t}\left(\dot{\phi} - \frac{1}{t}N\right), \quad \sigma \equiv \frac{\ell}{t}\left(\dot{\xi} - \partial_i^2\eta\right). \qquad \text{(A.69)}$$

With these definitions, (A.69), the $0j$-component of the linear Einstein tensor reads

$$\mathcal{G}^L_{0j} = -\frac{t}{2\ell}\left[\partial_j p + \epsilon_{jk}\partial_k\sigma\right]. \qquad \text{(A.70)}$$

For the flat spacetime this component yields

$$\mathcal{G}^L_{0j} = -\frac{1}{2}\left[\partial_j\dot{\phi} + \epsilon_{jk}\partial_k\sigma\right], \qquad \text{(A.71)}$$

with the flat spacetime version of the gauge-invariant functions, that are

$$p \equiv \dot{\phi}, \quad \sigma \equiv \left(\dot{\xi} - \partial_i^2\eta\right). \qquad \text{(A.72)}$$

A.3.3.3 The $\mathcal{G}^L_{jk}$ Term:

The jk-component of linear Einstein tensor,$\mathcal{G}^L_{jk}$, can be written by putting (A.48), (A.58) and (A.2) into the following equation

$$\mathcal{G}^L_{jk} = R^L_{jk} - \frac{1}{2}\frac{\ell^2}{t^2}\delta_{jk}R_L - \frac{2}{\ell^2}h_{jk}, \qquad \text{(A.73)}$$

that yields

$$\begin{aligned}
\mathcal{G}^L_{jk} = \left(-\frac{t^2}{2\ell^2}\right)\Bigg\{\partial_j\left[-2\epsilon_{kl}\partial_l\left(\dot{\eta}+\frac{1}{t}\eta\right)+\partial_k\left(2\dot{N}_L-N+\frac{2}{t}N_L\right)\right] \\
-\left(\partial_0^2+\frac{3}{t}\partial_0+\frac{4}{t^2}\right)\left[\left(\delta_{jk}+\hat{\partial}_j\hat{\partial}_k\right)\phi-\hat{\partial}_j\hat{\partial}_k\chi+\left(\epsilon_{jl}\hat{\partial}_l\hat{\partial}_k+\epsilon_{kl}\hat{\partial}_l\hat{\partial}_j\right)\xi\right] \\
+\delta_{jk}\partial_i^2\phi+\frac{1}{t}\delta_{jk}\left[2\left(\dot{N}-\partial_i^2N_L\right)+\left(-\dot{N}+\dot{\phi}+\dot{\chi}\right)+\frac{2}{t}\left(-N+\phi+\chi\right)\right]\Bigg\} \\
-\frac{1}{2}\delta_{jk}\left\{\frac{t^2}{\ell^2}\left[\partial_i^2N+\ddot{\chi}-2\partial_i^2\dot{N}_L+\left(\partial_0^2-\partial_i^2\right)\phi\right]+\frac{2t}{\ell^2}\left(-\dot{N}+\dot{\phi}+\dot{\chi}\right)-\frac{2}{\ell^2}\left(-N+\phi+\chi\right)\right\} \\
-\frac{2}{\ell^2}h_{jk},
\end{aligned} \tag{A.74}$$

and with some cancellations it takes the following form

$$\begin{aligned}
\mathcal{G}^L_{jk} = \left(-\frac{t^2}{2\ell^2}\right)\Bigg\{\partial_j\left[-2\epsilon_{kl}\partial_l\left(\dot{\eta}+\frac{1}{t}\eta\right)+\partial_k\left(2\dot{N}_L-N+\frac{2}{t}N_L\right)\right] \\
-\left(\partial_0^2+\frac{3}{t}\partial_0\right)\left[\hat{\partial}_j\hat{\partial}_k\phi-\hat{\partial}_j\hat{\partial}_k\chi+\left(\epsilon_{jl}\hat{\partial}_l\hat{\partial}_k+\epsilon_{kl}\hat{\partial}_l\hat{\partial}_j\right)\xi\right]\Bigg\} \\
-\frac{1}{2}\delta_{jk}\left\{\frac{t^2}{\ell^2}\left(\partial_i^2N+\ddot{\chi}-2\partial_i^2\dot{N}_L\right)-\frac{t}{\ell^2}\left(\dot{N}+2\partial_i^2N_L\right)+\frac{3t}{\ell^2}\dot{\chi}\right\}.
\end{aligned} \tag{A.75}$$

After redefining all the functions as $\phi \to \frac{\ell^2}{t^2}\phi$ for which the first second derivatives are $\dot{\phi} \to -\frac{2\ell^2}{t^3}\phi+\frac{\ell^2}{t^2}\dot{\phi}$, and $\ddot{\phi} \to \frac{6\ell^2}{t^4}\phi-\frac{4\ell^2}{t^3}\dot{\phi}+\frac{\ell^2}{t^2}\ddot{\phi}$ respectively, (A.75) becomes

$$\begin{aligned}
-2\mathcal{G}^L_{jk} = 2\epsilon_{kl}\partial_j\partial_l\left(\frac{1}{t}\eta-\dot{\eta}\right)+\partial_j\partial_k\left(-\frac{2}{t}N_L+2\dot{N}_L-N\right) \\
+\hat{\partial}_j\hat{\partial}_k\left(\frac{1}{t}\dot{\phi}-\ddot{\phi}\right)-\hat{\partial}_j\hat{\partial}_k\left(\frac{1}{t}\dot{\chi}-\ddot{\chi}\right)+\left(\epsilon_{jl}\hat{\partial}_l\hat{\partial}_k+\epsilon_{kl}\hat{\partial}_l\hat{\partial}_j\right)\left(\frac{1}{t}\dot{\xi}-\ddot{\xi}\right) \\
+\delta_{jk}\left(\partial_i^2N-\frac{1}{t}\dot{\chi}+\ddot{\chi}-2\partial_i^2\dot{N}_L+\frac{2}{t}\partial_i^2N_L+\frac{2}{t^2}N-\frac{1}{t}\dot{N}\right).
\end{aligned} \tag{A.76}$$

Since the j and k indices are symmetric, the first term can be manipulated such that

$$\begin{aligned}
2\epsilon_{kl}\partial_j\partial_l\left(\frac{1}{t}\eta-\dot{\eta}\right) &= \left(\epsilon_{kl}\partial_j\partial_l+\epsilon_{jl}\partial_k\partial_l\right)\left(\frac{1}{t}\eta-\dot{\eta}\right) \\
&= -\left(\epsilon_{kl}\hat{\partial}_j\hat{\partial}_l+\epsilon_{jl}\hat{\partial}_k\hat{\partial}_l\right)\left(\frac{1}{t}\partial_i^2\eta-\partial_i^2\dot{\eta}\right)
\end{aligned} \tag{A.77}$$

where in the second line the unit derivatives are introduced. After this manipulation (A.76) becomes

$$\begin{aligned}
-2\mathcal{G}^L_{jk} = -\left(\epsilon_{kl}\hat{\partial}_j\hat{\partial}_l+\epsilon_{jl}\hat{\partial}_k\hat{\partial}_l\right)\left(\frac{1}{t}\partial_i^2\eta-\partial_i^2\dot{\eta}-\frac{1}{t}\dot{\xi}+\ddot{\xi}\right) \\
-\hat{\partial}_j\hat{\partial}_k\left(-\frac{1}{t}\dot{\phi}+\ddot{\phi}+\frac{2}{t^2}N-\frac{1}{t}\dot{N}\right) \\
+\left(\delta_{jk}+\hat{\partial}_j\hat{\partial}_k\right)\left(\partial_i^2N-\frac{1}{t}\dot{\chi}+\ddot{\chi}-2\partial_i^2\dot{N}_L+\frac{2}{t}\partial_i^2N_L+\frac{2}{t^2}N-\frac{1}{t}\dot{N}\right),
\end{aligned} \tag{A.78}$$

where the terms $\frac{2}{t^2}\hat{\partial}_k\hat{\partial}_j N$ and $\frac{1}{t}\hat{\partial}_k\hat{\partial}_j \dot{N}$ are added and subtracted. The second parenthesis in the first term is $\dot{\sigma}$ and the second term is $\dot{p}$. The last term can be defined as

$$q \equiv \frac{\ell}{t}\left(\partial_i^2 N - \frac{1}{t}\dot{\chi} + \ddot{\chi} - 2\partial_i^2 \dot{N}_L + \frac{2}{t}\partial_i^2 N_L + \frac{2}{t^2}N - \frac{1}{t}\dot{N}\right), \tag{A.79}$$

which is also gauge-invariant function. Finally, the jk-component of the linear Einstein tensor becomes

$$\mathcal{G}^L_{jk} = -\frac{t}{2\ell}\left\{-\left(\epsilon_{kl}\hat{\partial}_j\hat{\partial}_l + \epsilon_{jl}\hat{\partial}_k\hat{\partial}_l\right)\dot{\sigma} - \hat{\partial}_j\hat{\partial}_k\dot{p} + \left(\delta_{jk} + \hat{\partial}_j\hat{\partial}_k\right)q\right\}. \tag{A.80}$$

The flat spacetime version of this component is

$$\mathcal{G}^L_{jk} = -\frac{1}{2}\left\{-\left(\epsilon_{kl}\hat{\partial}_j\hat{\partial}_l + \epsilon_{jl}\hat{\partial}_k\hat{\partial}_l\right)\dot{\sigma} - \hat{\partial}_j\hat{\partial}_k\ddot{\phi} + \left(\delta_{jk} + \hat{\partial}_j\hat{\partial}_k\right)q\right\}, \tag{A.81}$$

where the function q reads

$$q = \partial_i^2 N - \frac{1}{t}\dot{\chi} + \ddot{\chi} - 2\partial_i^2 \dot{N}_L. \tag{A.82}$$

A.3.4 Gauge-Invariance of the Defined Functions

The functions can be checked whether they are gauge-invariant or not. Let us start with,

$$\delta_\zeta \sigma = \frac{\ell}{t}\left(\delta_\zeta \dot{\xi} - \nabla^2 \delta_\zeta \eta\right), \tag{A.83}$$

and using (A.11, A.14, A.19, A.20, A.22) and their derivatives (A.83) becomes

$$\delta_\zeta \sigma = \frac{\ell}{t}\left(\frac{2t}{\ell^2}\nabla^2\zeta + \frac{t^2}{\ell^2}\nabla^2\dot{\zeta} - \frac{t^2}{\ell^2}\nabla^2\dot{\zeta} - \frac{2t}{\ell^2}\nabla^2\zeta\right) = 0, \tag{A.84}$$

where we have used $\delta_\zeta\dot{\xi} = \frac{2t}{\ell^2}\partial_j^2\zeta + \frac{t^2}{\ell^2}\partial_j^2\dot{\zeta}$. Again using (A.11, A.14, A.19, A.20, A.22) the p function reads

$$\delta_\zeta p = \frac{\ell}{t}\left(\delta_\zeta\dot{\phi} - 2\frac{t}{\ell^2}\dot{\zeta}_0 - 2\frac{1}{\ell^2}\zeta_0\right), \tag{A.85}$$

$$\delta_\zeta p = \frac{\ell}{t}\left(\frac{2}{\ell^2}\zeta_0 + \frac{2t}{\ell^2}\dot{\zeta}_0 - 2\frac{t}{\ell^2}\dot{\zeta}_0 - 2\frac{1}{\ell^2}\zeta_0\right) = 0, \tag{A.86}$$

where we have again used the derivative of fields, $\delta_\zeta\dot{\phi} = \frac{2}{\ell^2}\zeta_0 + \frac{2t}{\ell^2}\dot{\zeta}_0$. The other two functions can be handled in the same way.

$$\delta_\zeta f = \frac{\ell}{t}\left[\delta_\zeta\phi + \frac{1}{t\nabla^2}\left(\delta_\zeta\dot{\phi} + \delta_\zeta\dot{\chi} - \frac{2}{t}\delta_\zeta N\right) - \frac{2}{t}\delta_\zeta N_L\right], \tag{A.87}$$

$$\begin{aligned}\frac{t}{\ell}\delta_\zeta f = \frac{1}{t\nabla^2}&\left[\frac{2}{\ell^2}\zeta_0 + \frac{2t}{\ell^2}\dot{\zeta}_0 + 4\frac{t}{\ell^2}\left(\nabla^2\kappa + \frac{1}{t}\zeta_0\right) + 2\frac{t^2}{\ell^2}\left(\nabla^2\dot{\kappa} + \frac{1}{t}\dot{\zeta}_0 - \frac{1}{t^2}\zeta_0\right) - 4\frac{t}{\ell^2}\left(\dot{\zeta}_0 + \frac{1}{t}\zeta_0\right)\right]\\ &+ \frac{2t}{\ell^2}\zeta_0 - \frac{2t}{\ell^2}\left(\dot{\kappa} + \zeta_0 + \frac{2}{t}\kappa\right) = 0,\end{aligned} \tag{A.88}$$

where we have used $\delta_\zeta \dot{\chi} = 4\frac{t}{\ell^2}\left(\partial_j^2\kappa + \frac{1}{t}\zeta_0\right) + 2\frac{t^2}{\ell^2}\left(\partial_j^2\dot{\kappa} + \frac{1}{t}\dot{\zeta}_0 - \frac{1}{t^2}\zeta_0\right)$. The q function is

$$\frac{t}{\ell}\delta_\zeta q = \nabla^2\delta_\zeta N - \frac{1}{t}\delta_\zeta\dot{\chi} + \delta_\zeta\ddot{\chi} - 2\partial_i^2\delta_\zeta\dot{N}_L + \frac{2}{t}\nabla^2\delta_\zeta N_L + \frac{2}{t^2}\delta_\zeta N - \frac{1}{t}\delta_\zeta\dot{N} \tag{A.89}$$

$$\begin{aligned}
\frac{t}{\ell}\delta_\zeta q = &\frac{2t^2}{\ell^2}\nabla^2\left(\dot{\zeta}_0 + \frac{1}{t}\zeta_0\right) - \frac{4}{\ell^2}\left(\nabla^2\kappa + \frac{1}{t}\zeta_0\right) - \frac{2t}{\ell^2}\left(\nabla^2\dot{\kappa} + \frac{1}{t}\dot{\zeta}_0 - \frac{1}{t^2}\zeta_0\right)\\
&+ \frac{4}{\ell^2}\left(\nabla^2\kappa + \frac{1}{t}\zeta_0\right) + \frac{4t}{\ell^2}\left(\nabla^2\dot{\kappa} + \frac{1}{t}\dot{\zeta}_0 - \frac{1}{t^2}\zeta_0\right)\\
&+ \frac{4t}{\ell^2}\left(\nabla^2\dot{\kappa} + \frac{1}{t}\dot{\zeta}_0 - \frac{1}{t^2}\zeta_0\right) + \frac{2t^2}{\ell^2}\left(\nabla^2\ddot{\kappa} + \frac{1}{t}\ddot{\zeta}_0 - \frac{1}{t^2}\dot{\zeta}_0 - \frac{1}{t^2}\dot{\zeta}_0 + \frac{2}{t^3}\zeta_0\right)\\
&- \frac{4t}{\ell^2}\nabla^2\left(\dot{\kappa} + \zeta_0 + \frac{2}{t}\kappa\right) - \frac{2t^2}{\ell^2}\nabla^2\left(\ddot{\kappa} + \dot{\zeta}_0 + \frac{2}{t}\dot{\kappa} - \frac{2}{t^2}\kappa\right)\\
&+ \frac{2t}{\ell^2}\nabla^2\left(\dot{\kappa} + \zeta_0 + \frac{2}{t}\kappa\right) + \frac{4}{\ell^2}\left(\dot{\zeta}_0 + \frac{1}{t}\zeta_0\right)\\
&- \frac{4}{\ell^2}\left(\dot{\zeta}_0 + \frac{1}{t}\zeta_0\right) - \frac{2t}{\ell^2}\left(\ddot{\zeta}_0 + \frac{1}{t}\dot{\zeta}_0 - \frac{1}{t^2}\zeta_0\right) = 0.
\end{aligned} \tag{A.90}$$

Therefore, these functions are indeed invariant under gauge transformations.

A.3.5 Ricci Scalar in Terms of Gauge Invariant functions

The Ricci scalar can also be written in terms of the gauge-invariant functions. We first write (A.58) with the modified functions, $\phi \to \frac{\ell^2}{t^2}\phi$. With these functions the Ricci scalar takes the form

$$R_L = \frac{t^2}{\ell^2}\left(\partial_i^2 N + \ddot{\chi} - 2\partial_i^2\dot{N}_L + \ddot{\phi} - \partial_i^2\phi - \frac{2}{t}\dot{\chi} + \frac{4}{t}\nabla^2 N_L - \frac{2}{t}\dot{\phi} + \frac{6}{t^2}N - \frac{2}{t}\dot{N}\right). \tag{A.91}$$

Using the definitions of the gauge-invariant functions, (A.64), (A.69) and (A.79), the Ricci scalar can be written as

$$R_L = \frac{t^3}{\ell^3}\left(q - \nabla^2 f + \dot{p}\right). \tag{A.92}$$

For the flat spacetime case

$$R_L = q - \nabla^2\phi + \ddot{\phi}, \tag{A.93}$$

and using $\Box = -\partial_0 + \nabla^2$ the Ricci scalar becomes

$$R_L = q - \Box\phi. \tag{A.94}$$

A.3.6 The Bianchi Identity

The Bianchi identity can also be written in terms of the gauge-invariant functions. First the identity is decomposed into its components by doing the summation in the repeated indices:

$$
\begin{aligned}
\nabla^{\mu}\mathcal{G}^{L}_{\mu\nu} &= \bar{g}^{\mu\sigma}\nabla_{\sigma}\mathcal{G}^{L}_{\mu\nu} = 0 &\text{(A.95)}\\
&= \bar{g}^{\mu\sigma}\left(\partial_{\sigma}\mathcal{G}^{L}_{\mu\nu} - \Gamma^{\lambda}_{\sigma\mu}\mathcal{G}^{L}_{\lambda\nu} - \Gamma^{\lambda}_{\sigma\nu}\mathcal{G}^{L}_{\lambda\mu}\right),\\
&= \bar{g}^{0\sigma}\left(\partial_{\sigma}\mathcal{G}^{L}_{0\nu} - \Gamma^{\lambda}_{\sigma 0}\mathcal{G}^{L}_{\lambda\nu} - \Gamma^{\lambda}_{\sigma\nu}\mathcal{G}^{L}_{\lambda 0}\right)\\
&\quad + \bar{g}^{i\sigma}\left(\partial_{\sigma}\mathcal{G}^{L}_{i\nu} - \Gamma^{\lambda}_{\sigma i}\mathcal{G}^{L}_{\lambda\nu} - \Gamma^{\lambda}_{\sigma\nu}\mathcal{G}^{L}_{\lambda i}\right),\\
0 &= -\frac{t^2}{\ell^2}\left(\partial_0\mathcal{G}^{L}_{0\nu} - \Gamma^{0}_{00}\mathcal{G}^{L}_{0\nu} - \Gamma^{0}_{0\nu}\mathcal{G}^{L}_{00} - \Gamma^{k}_{0\nu}\mathcal{G}^{L}_{k0}\right)\\
&\quad + \frac{t^2}{\ell^2}\delta^{ij}\left(\partial_j\mathcal{G}^{L}_{i\nu} - \Gamma^{0}_{ji}\mathcal{G}^{L}_{0\nu} - \Gamma^{0}_{j\nu}\mathcal{G}^{L}_{0i} - \Gamma^{k}_{j\nu}\mathcal{G}^{L}_{ki}\right). &\text{(A.96)}
\end{aligned}
$$

There is two equations depending on the ν index. For $\nu = 0$, (A.96) yields

$$
\begin{aligned}
0 &= -\frac{t^2}{\ell^2}\left(\partial_0\mathcal{G}^{L}_{00} - \Gamma^{0}_{00}\mathcal{G}^{L}_{00} - \Gamma^{0}_{00}\mathcal{G}^{L}_{00} - \Gamma^{k}_{00}\mathcal{G}^{L}_{k0}\right)\\
&\quad + \frac{t^2}{\ell^2}\delta^{ij}\left(\partial_j\mathcal{G}^{L}_{i0} - \Gamma^{0}_{ji}\mathcal{G}^{L}_{00} - \Gamma^{0}_{j0}\mathcal{G}^{L}_{0i} - \Gamma^{k}_{j0}\mathcal{G}^{L}_{ki}\right), &\text{(A.97)}
\end{aligned}
$$

and using (A.8),

$$
\begin{aligned}
0 &= -\frac{t^2}{\ell^2}\left(\partial_0\mathcal{G}^{L}_{00} + \frac{2}{t}\mathcal{G}^{L}_{00}\right)\\
&\quad + \frac{t^2}{\ell^2}\left(\partial_i\mathcal{G}^{L}_{i0} + \frac{2}{t}\mathcal{G}^{L}_{00} + \frac{1}{t}\mathcal{G}^{L}_{ii}\right),\\
0 &= -\frac{t^2}{\ell^2}\left(\partial_0\mathcal{G}^{L}_{00} - \partial_i\mathcal{G}^{L}_{i0} - \frac{1}{t}\mathcal{G}^{L}_{ii}\right). &\text{(A.98)}
\end{aligned}
$$

Using the gauge-invariant form of the components of the Einstein tensor and their derivatives, that are

$$
\mathcal{G}^{L}_{00} = -\frac{t}{2\ell}\nabla^2 f \Rightarrow \partial_0\mathcal{G}^{L}_{00} = -\frac{1}{2\ell}\nabla^2 f - \frac{t}{2\ell}\nabla^2\dot{f}, \tag{A.99}
$$

$$
\mathcal{G}^{L}_{i0} = -\frac{t}{2\ell}\left[\partial_i p + \epsilon_{ik}\partial_k\sigma\right] \Rightarrow \partial_i\mathcal{G}^{L}_{i0} = -\frac{t}{2\ell}\partial_i^2 p = -\frac{t}{2\ell}\nabla^2 p, \tag{A.100}
$$

and

$$
\begin{aligned}
\mathcal{G}^{L}_{ii} &= -\frac{t}{2\ell}\delta_{ij}\left\{-\left(\epsilon_{il}\hat{\partial}_j\hat{\partial}_l + \epsilon_{jl}\hat{\partial}_i\hat{\partial}_l\right)\dot{\sigma} - \hat{\partial}_j\hat{\partial}_i\dot{p} + \left(\delta_{ji} + \hat{\partial}_j\hat{\partial}_i\right)q\right\}\\
&= -\frac{t}{2\ell}\left(\dot{p} + q\right). &\text{(A.101)}
\end{aligned}
$$

With these equations, (A.99), (A.100) and (A.101), the Bianchi identity yields

$$0 = \left(-\frac{1}{2\ell}\nabla^2 f - \frac{t}{2\ell}\nabla^2 \dot{f} + \frac{t}{2\ell}\nabla^2 p + \frac{1}{2\ell}\left(\dot{p}+q\right)\right),$$

$$= \left(-\nabla^2 f - t\nabla^2 \dot{f} + t\nabla^2 p + \dot{p} + q\right), \tag{A.102}$$

and finally

$$t\nabla^2\left(\frac{1}{t}f + \dot{f} - p\right) - \dot{p} - q = 0. \tag{A.103}$$

Also, ν can be taken as $\nu = n$. For this choice (A.96) reads

$$\begin{aligned}
0 &= -\frac{t^2}{\ell^2}\left(\partial_0 \mathcal{G}^L_{0n} - \Gamma^0_{00}\mathcal{G}^L_{0n} - \Gamma^0_{0n}\mathcal{G}^L_{00} - \Gamma^k_{0n}\mathcal{G}^L_{k0}\right)\\
&\quad + \frac{t^2}{\ell^2}\delta^{ij}\left(\partial_j \mathcal{G}^L_{in} - \Gamma^0_{ji}\mathcal{G}^L_{0n} - \Gamma^0_{jn}\mathcal{G}^L_{0i} - \Gamma^k_{jn}\mathcal{G}^L_{ki}\right),\\
0 &= -\left(\partial_0 \mathcal{G}^L_{0n} + \frac{1}{t}\mathcal{G}^L_{0n} + \frac{1}{t}\mathcal{G}^L_{n0}\right)\\
&\quad + \left(\partial_i \mathcal{G}^L_{in} + \frac{2}{t}\mathcal{G}^L_{0n} + \frac{1}{t}\mathcal{G}^L_{0n}\right),\\
0 &= \left(\partial_0 \mathcal{G}^L_{0n} - \frac{1}{t}\mathcal{G}^L_{0n} - \partial_i \mathcal{G}^L_{in}\right).
\end{aligned} \tag{A.104}$$

Again using (A.100) and the derivative of (A.80), that is

$$\partial_i \mathcal{G}^L_{in} = -\frac{t}{2\ell}\left(-\epsilon_{nl}\partial_l\dot{\sigma} + \partial_n\dot{p}\right), \tag{A.105}$$

(A.104) yields

$$\begin{aligned}
0 &= -\frac{t}{2\ell}\left[\partial_n\dot{p} + \epsilon_{nk}\partial_k\dot{\sigma}\right] + \frac{1}{2\ell}\left[\partial_n p + \epsilon_{nk}\partial_k\sigma\right] + \frac{t}{2\ell}\left(-\epsilon_{nl}\hat{\partial}_i\hat{\partial}_l\dot{\sigma} + \partial_n\dot{p}\right),\\
0 &= -\partial_i\dot{p} - \epsilon_{ik}\partial_k\dot{\sigma} + \frac{1}{t}\left[\partial_i p + \epsilon_{ik}\partial_k\sigma\right] - \epsilon_{nl}\partial_l\dot{\sigma} + \partial_n\dot{p},\\
0 &= -2\epsilon_{nk}\partial_k\dot{\sigma} + \frac{1}{t}\left[\partial_n p + \epsilon_{nk}\partial_k\sigma\right],
\end{aligned} \tag{A.106}$$

and

$$0 = \epsilon_{nk}\partial_k\left(-2\dot{\sigma} + \frac{1}{t}\sigma\right) + \frac{1}{t}\partial_n p. \tag{A.107}$$

Using (A.103), (A.92) becomes

$$R_L = \frac{t^3}{\ell^3}\left(q - \nabla^2 f + \dot{p}\right) = \frac{t^4}{\ell^3}\nabla^2\left(\dot{f} - p\right). \tag{A.108}$$

Up to now, we write the components of linearized Ricci tensor and linear Einstein tensor in terms of gauge invariant functions. Moreover, the linear Ricci scalar is also written in a gauge invariant form. Now we can write (3.58) in terms of gauge invariant functions by using these identities. The next section is devoted to find a gauge invariant action.

A.4 Metric Decomposition and Gauge-Invariant Form Of The Higher Derivative Action

Linearized action is

$$I = -\frac{1}{2}\int d^3x\ \sqrt{\bar{g}}\, h_{\mu\nu}\left[a\mathcal{G}_L^{\mu\nu} + (2\alpha+\beta)\left(\bar{g}^{\mu\nu}\square - \nabla^\mu\nabla^\nu + 2\Lambda\bar{g}^{\mu\nu}\right)R_L + \beta\left(\square\mathcal{G}_L^{\mu\nu} - \Lambda\bar{g}^{\mu\nu}R_L\right)\right] \tag{A.109}$$

where background metric is

$$ds^2 = \frac{\ell^2}{t^2}\left[-dt^2 + dx^2 + dy^2\right] \quad \Rightarrow \quad \bar{g}_{\mu\nu} = \frac{\ell^2}{t^2}\eta_{\mu\nu}. \tag{A.110}$$

Note that the term in the parenthesis is the field equations of the quadratic curvature theory. The constant a in front of $\mathcal{G}_L^{\mu\nu}$ is [27]

$$a \equiv \frac{1}{\kappa} + 12\Lambda\alpha + 2\Lambda\beta. \tag{A.111}$$

Linearized form of Einstein and Ricci tensors, and Ricci scalar are given in (A.60, A.24, A.50) with the definition $\square \equiv \nabla_\mu\nabla^\mu = \frac{t^2}{\ell^2}\eta^{\mu\nu}\nabla_\mu\nabla_\nu$.

A.4.1 The Einstein-Hilbert Part

Let us work on the Einstein-Hilbert part first:

$$\begin{aligned} I_E &= -\frac{a}{2}\int d^3x\ \sqrt{\bar{g}}h_{\mu\nu}\mathcal{G}_L^{\mu\nu} = -\frac{a}{2}\int d^3x\, \frac{\ell^3}{t^3}\frac{t^4}{\ell^4}\eta^{\rho\mu}\eta^{\sigma\nu}h_{\rho\sigma}\mathcal{G}^L_{\mu\nu}, \\ &= -\frac{a}{2}\int d^3x\, \frac{t}{\ell}\eta^{\rho\mu}\eta^{\sigma\nu}h_{\rho\sigma}\mathcal{G}^L_{\mu\nu}. \end{aligned} \tag{A.112}$$

Writing Einstein-Hilbert part in terms of tensor components yields

$$I_E = -\frac{a}{2}\int d^3x\, \frac{t}{\ell}\left(h_{00}\mathcal{G}^L_{00} - 2h_{0i}\mathcal{G}^L_{0i} + h_{ij}\mathcal{G}^L_{ij}\right). \tag{A.113}$$

Let us calculate term by term this action:

A.4.1.1 Decomposition of $h_{00}\mathcal{G}^L_{00}$ term

Let's start with the first term, that is $h_{00}\mathcal{G}^L_{00}$ where

$$\mathcal{G}^L_{00} = R^L_{00} - \frac{1}{2}\bar{g}_{00}R_L - 2\Lambda h_{00}. \tag{A.114}$$

Using (A.62) and (A.2), (A.115) becomes

$$h_{00}\mathcal{G}^L_{00} = N\left[-\frac{t^2}{2\ell^2}\partial_i^2\phi - \frac{t}{2\ell^2}\left(\dot{\phi} + \dot{\chi} - 2\partial_i^2 N_L\right) - \frac{1}{\ell^2}\left(-N + \phi + \chi\right)\right]. \tag{A.115}$$

For the flat spacetime case (A.115) becomes

$$h_{00}\mathcal{G}^L_{00} = -\frac{1}{2}N\partial_i^2\phi. \tag{A.116}$$

A.4.1.2 Decomposition of $h_{0j}\mathcal{G}^L_{0j}$ term

Let us move to second term in Einstein-Hilbert action which is $-2h_{0j}\mathcal{G}^L_{0j}$. Using (A.67) and (A.2) $h_{0j}\mathcal{G}^L_{0j}$ becomes

$$\begin{aligned}
h_{0j}\mathcal{G}^L_{0j} &= \left(-2h_{0j}\left\{-\frac{t^2}{2\ell^2}\left[\partial_j\dot{\phi} + \epsilon_{jk}\partial_k\left(\dot{\xi} - \partial_i^2\eta\right)\right] + \frac{t}{2\ell^2}\left[\partial_j\left(N - 2\phi\right) - 2\epsilon_{jk}\partial_k\xi\right]\right\}\right),\\
&= \frac{t^2}{\ell^2}\left(h_{0j}\left\{\partial_j\left[\dot{\phi} - \frac{1}{t}\left(N - 2\phi\right)\right] + \epsilon_{jk}\partial_k\left[\left(\dot{\xi} - \partial_i^2\eta\right) + \frac{2}{t}\xi\right]\right\}\right),\\
&= \frac{t^2}{\ell^2}\left\{-\left(\partial_j h_{0j}\right)\left[\dot{\phi} - \frac{1}{t}\left(N - 2\phi\right)\right] - \left(\epsilon_{jk}\partial_k h_{0j}\right)\left[\left(\dot{\xi} - \partial_i^2\eta\right) + \frac{2}{t}\xi\right]\right\},
\end{aligned} \tag{A.117}$$

where the integral sign is suppressed and at the last line the boundary terms dropped. Also note that,

$$\partial_j h_{0j} = \partial_j^2 N_L, \tag{A.118}$$

and

$$\begin{aligned}
\epsilon_{jk}\partial_k h_{0j} &= \epsilon_{jk}\partial_k\left(-\epsilon_{ji}\partial_i\eta + \partial_j N_L\right) = -\epsilon_{ji}\epsilon_{jk}\partial_k\partial_i\eta + \epsilon_{jk}\partial_k\partial_j N_L,\\
&= -\delta_{ik}\partial_k\partial_i\eta = -\partial_i^2\eta.
\end{aligned} \tag{A.119}$$

Then (A.117) yields,

$$\begin{aligned}
\left(-2h_{0j}\mathcal{G}^L_{0j}\right) &= \frac{t^2}{\ell^2}\left\{\partial_i^2\eta\left[\left(\dot{\xi} - \partial_i^2\eta\right) + \frac{2}{t}\xi\right] - \partial_j^2 N_L\left[\dot{\phi} - \frac{1}{t}\left(N - 2\phi\right)\right]\right\},\\
&= \frac{t^2}{\ell^2}\left\{\left(\dot{\xi} - \partial_i^2\eta\right)\partial_i^2\eta - \dot{\phi}\left(\partial_j^2 N_L\right) + \frac{1}{t}\left[2\xi\partial_i^2\eta + \left(N - 2\phi\right)\left(\partial_j^2 N_L\right)\right]\right\}.
\end{aligned} \tag{A.120}$$

In $t/\ell \to 1$ and $\ell \to \infty$ limits (A.120) becomes

$$-2h_{0j}\mathcal{G}^L_{0j} = \left(\dot{\xi} - \partial_i^2\eta\right)\partial_i^2\eta - \dot{\phi}\left(\partial_j^2 N_L\right), \tag{A.121}$$

which is the result for the flat spacetime case.

A.4.1.3 Decomposition of $h_{jk}\mathcal{G}^L_{jk}$ term

Using (A.75) and the first equation in (A.2), $h_{jk}\mathcal{G}^L_{jk}$ takes the following form

$$h_{jk}\mathcal{G}^L_{jk} = \left(-\frac{t^2}{2\ell^2}\right)\left\{h_{jk}\partial_j\left[-2\epsilon_{kl}\partial_l\left(\dot{\eta}+\frac{1}{t}\eta\right)+\partial_k\left(2\dot{N}_L - N + \frac{2}{t}N_L\right)\right]\right.$$
$$\left. -h_{jk}\left(\partial_0^2+\frac{3}{t}\partial_0\right)\left[\hat{\partial}_j\hat{\partial}_k\phi - \hat{\partial}_j\hat{\partial}_k\chi + \left(\epsilon_{jl}\hat{\partial}_l\hat{\partial}_k + \epsilon_{kl}\hat{\partial}_l\hat{\partial}_j\right)\xi\right]\right\}$$
$$-\frac{1}{2}h_{jk}\delta_{jk}\left\{\frac{t^2}{\ell^2}\left(\partial_i^2 N + \ddot{\chi} - 2\partial_i^2\dot{N}_L\right) - \frac{t}{\ell^2}\left(\dot{N}+2\partial_i^2 N_L\right)+\frac{3t}{\ell^2}\dot{\chi}\right\}. \tag{A.122}$$

Note that $\partial_j h_{jk} = \partial_k\chi - \epsilon_{kl}\partial_l\xi$ and $\delta_{jk}h_{jk} = \phi + \chi$ and again the integral sign is suppressed. Taking the space derivative in front of the metric perturbation in the first term with dropping the boundary terms the equation becomes

$$h_{jk}\mathcal{G}^L_{jk} = \left(-\frac{t^2}{2a\ell^2}\right)\left\{-\left(\partial_k\chi - \epsilon_{kl}\partial_l\xi\right)\left[-2\epsilon_{kl}\partial_l\left(\dot{\eta}+\frac{1}{t}\eta\right)+\partial_k\left(2\dot{N}_L - N + \frac{2}{t}N_L\right)\right]\right.$$
$$\left. -h_{jk}\left(\partial_0^2+\frac{3}{t}\partial_0\right)\left[\hat{\partial}_j\hat{\partial}_k\phi - \hat{\partial}_j\hat{\partial}_k\chi + \left(\epsilon_{jl}\hat{\partial}_l\hat{\partial}_k + \epsilon_{kl}\hat{\partial}_l\hat{\partial}_j\right)\xi\right]\right\}$$
$$-\frac{1}{2}\left(\phi+\chi\right)\left\{\frac{t^2}{\ell^2}\left(\partial_i^2 N + \ddot{\chi} - 2\partial_i^2\dot{N}_L\right) - \frac{t}{\ell^2}\left(\dot{N}+2\partial_i^2 N_L\right)+\frac{3t}{\ell^2}\dot{\chi}\right\}. \tag{A.123}$$

The multiplication of the metric perturbation with the middle term is

$$\left[\left(\delta_{jk}+\hat{\partial}_k\hat{\partial}_j\right)\phi - \hat{\partial}_k\hat{\partial}_j\chi + \left(\epsilon_{kn}\hat{\partial}_n\hat{\partial}_j + \epsilon_{jn}\hat{\partial}_n\hat{\partial}_k\right)\xi\right]\times$$
$$\left[\hat{\partial}_j\hat{\partial}_k\phi^{(i)} - \hat{\partial}_j\hat{\partial}_k\chi^{(i)} + \left(\epsilon_{jl}\hat{\partial}_l\hat{\partial}_k + \epsilon_{kl}\hat{\partial}_l\hat{\partial}_j\right)\xi^{(i)}\right] = -\chi\phi^{(i)} + \chi\chi^{(i)} + 2\xi\xi^{(i)}, \tag{A.124}$$

where $^{(i)}$ denotes time derivative and $i = 1, 2$. Putting this equation with suitable derivative signs and doing the cancellations (A.123) turns to be

$$h_{jk}\mathcal{G}^L_{jk} = \left(-\frac{t^2}{2\ell^2}\right)\left\{\left(-2\phi\partial_i^2\dot{N}_L + 2\xi\partial_i^2\dot{\eta} + \phi\partial_i^2 N + \phi\ddot{\chi} + \chi\ddot{\phi} - 2\xi\ddot{\xi}\right)\right.$$
$$\left. +\frac{1}{t}\left(2\xi\partial_i^2\eta - 6\xi\dot{\xi} - 2\phi\partial_i^2 N_L - \phi\dot{N} - \chi\dot{N} + 3\phi\dot{\chi} + 3\chi\dot{\phi}\right)\right\}. \tag{A.125}$$

In the flat spacetime limit (A.125) becomes

$$h_{jk}\mathcal{G}^L_{jk} = -\frac{1}{2}\left(-2\phi\partial_i^2\dot{N}_L + 2\xi\partial_i^2\dot{\eta} + \phi\partial_i^2 N + \phi\ddot{\chi} + \chi\ddot{\phi} - 2\xi\ddot{\xi}\right),$$
$$= \phi\partial_i^2\dot{N}_L - \xi\partial_i^2\dot{\eta} - \frac{1}{2}\phi\partial_i^2 N - \frac{1}{2}\phi\ddot{\chi} - \frac{1}{2}\chi\ddot{\phi} + \xi\ddot{\xi}, \tag{A.126}$$

and with integration by parts on time differentiations (A.126) yields

$$h_{jk}\mathcal{G}^L_{jk} = -\dot{\phi}\partial_i^2 N_L + \dot{\xi}\partial_i^2\eta + \xi\ddot{\xi} - \phi\ddot{\chi} - \frac{1}{2}\phi\partial_i^2 N. \tag{A.127}$$

A.4.1.4 Decomposed form of Einstein-Hilbert Action

To get the final form of Einstein-Hilbert term, all the found results (A.115), (A.120), (A.125) are put in (A.113) to get

$$
\begin{aligned}
I_E =& -\frac{a}{2}\int d^3x\,\frac{t}{\ell}N\left[-\frac{t^2}{2\ell^2}\partial_i^2\phi - \frac{t}{2\ell^2}\left(\dot\phi + \dot\chi - 2\partial_i^2 N_L\right) - \frac{1}{\ell^2}\left(-N + \phi + \chi\right)\right] \\
&- \frac{a}{2}\int d^3x\,\frac{t^3}{\ell^3}\left\{\left(\dot\xi - \partial_i^2\eta\right)\partial_i^2\eta - \dot\phi\left(\partial_j^2 N_L\right) + \frac{1}{t}\left[2\xi\partial_i^2\eta + (N - 2\phi)\left(\partial_j^2 N_L\right)\right]\right\} \\
&- \frac{a}{2}\int d^3x\left(-\frac{t^3}{2\ell^3}\right)\Big\{\left(-2\phi\partial_i^2\dot N_L + 2\xi\partial_i^2\dot\eta + \phi\partial_i^2 N + \phi\ddot\chi + \chi\ddot\phi - 2\xi\ddot\xi\right) \\
&\qquad + \frac{1}{t}\left(2\xi\partial_i^2\eta - 6\xi\dot\xi - 2\phi\partial_i^2 N_L - \phi\dot N - \chi\dot N + 3\phi\dot\chi + 3\chi\dot\phi\right)\Big\}.
\end{aligned}
\tag{A.128}
$$

To see the terms that may cancel or sum after integration by parts operation, (A.128) is written in the following form

$$
\begin{aligned}
I_E = -\frac{a}{2}\int d^3x\left(-\frac{t^3}{2\ell^3}\right)\Big\{&\phi\ddot\chi + \chi\ddot\phi + 2\phi\partial_i^2 N + 2\dot\phi\partial_i^2 N_L - 2\phi\partial_i^2\dot N_L \\
&+ \left[2\left(\partial_i^2\eta\right)^2 - 2\dot\xi\partial_i^2\eta + 2\xi\partial_i^2\dot\eta - 2\xi\ddot\xi\right] \\
&+ \frac{1}{t}\left(3\phi\dot\chi + 3\chi\dot\phi + 2\phi\partial_j^2 N_L - 2\xi\partial_i^2\eta - 6\xi\dot\xi\right) \\
&+ \frac{1}{t}\left[N\left(\dot\phi + \dot\chi\right) - \dot N\left(\phi + \chi\right) - 4N\partial_i^2 N_L\right] \\
&+ \frac{2}{t^2}N\left(-N + \phi + \chi\right)\Big\}.
\end{aligned}
\tag{A.129}
$$

After doing integration by parts in the terms $\chi\ddot\phi$ and $\dot N(\phi + \chi)$, (A.129) becomes

$$
\begin{aligned}
I_E = -\frac{a}{2}\int d^3x\left(-\frac{t^3}{2\ell^3}\right)\Big\{&\phi\ddot\chi + \phi\left(\ddot\chi + \frac{6}{t}\dot\chi + \frac{6}{t^2}\chi\right) + 2\phi\partial_i^2 N \\
&- 2\phi\partial_i^2\dot N_L - \frac{6}{t}\phi\partial_i^2 N_L - 2\phi\partial_i^2\dot N_L \\
&+ \left[2\left(\partial_i^2\eta\right)^2 - 2\xi\partial_i^2\eta - 2\dot\xi\partial_i^2\eta - \frac{6}{t}\xi\partial_i^2\eta + 2\dot\xi^2 + \frac{6}{t}\xi\dot\xi\right] \\
&+ \frac{1}{t}\left(3\phi\dot\chi - 3\dot\chi\phi - \frac{6}{t}\chi\phi + 2\phi\partial_j^2 N_L - 2\xi\partial_i^2\eta - 6\xi\dot\xi\right) \\
&+ \frac{1}{t}\left[N\left(\dot\phi + \dot\chi\right) + N\left(\dot\phi + \dot\chi\right) + \frac{2}{t}N\left(\phi + \chi\right) - 4N\partial_i^2 N_L\right] \\
&+ \frac{2}{t^2}N\left(-N + \phi + \chi\right)\Big\},
\end{aligned}
\tag{A.130}
$$

and the final form of (A.130) in terms of metric perturbation functions is

$$I_E = -\frac{a}{2}\int d^3x\left(-\frac{t^3}{\ell^3}\right)\Big\{\phi\ddot{\chi} + \phi\partial_i^2\left(N - 2\dot{N}_L\right) + \left(-\partial_i^2\eta + \dot{\xi}\right)^2$$
$$+\frac{1}{t}\left(3\phi\dot{\chi} + N\left(\dot{\phi} + \dot{\chi}\right) - 2N\partial_i^2 N_L - 2\phi\partial_i^2 N_L - 4\xi\partial_i^2\eta\right)$$
$$+\frac{1}{t^2}\left(-N^2 + 2N\left(\phi + \chi\right)\right)\Big\}, \tag{A.131}$$

which is not gauge invariant yet. In flat spacetime limit (A.131) yields

$$I_E = \frac{a}{2}\int d^3x\left[\phi\ddot{\chi} + \phi\partial_i^2\left(N - 2\dot{N}_L\right) + \left(-\partial_i^2\eta + \dot{\xi}\right)^2\right]. \tag{A.132}$$

A.4.1.5 The Gauge-Invariant form of the Einstein-Hilbert term:

From (A.131) the gauge-invariant form of the Einstein-Hilbert action can be written. For this aim, first the functions are redefined as $\phi \to \frac{\ell^2}{t^2}\phi$. With this redefinition (A.131) becomes

$$I_E = -\frac{a}{2}\int d^3x\left(-\frac{\ell}{t}\right)\Big\{\phi\ddot{\chi} + \phi\partial_i^2\left(N - 2\dot{N}_L\right) + \left(-\partial_i^2\eta + \dot{\xi}\right)^2$$
$$+\frac{1}{t}\left(-\phi\dot{\chi} + N\left(\dot{\phi} + \dot{\chi}\right) - 2N\partial_i^2 N_L + 2\phi\partial_i^2 N_L\right)$$
$$-\frac{1}{t^2}N^2 + \frac{4}{t^2}\xi^2 - \frac{4}{t}\xi\dot{\xi}\Big\}, \tag{A.133}$$

and doing integration by parts in the last term

$$-\frac{4}{t^2}\xi\dot{\xi} = \partial_0\left(-\frac{4}{t^2}\xi\xi\right) - \frac{8}{t^3}\xi\xi + \frac{4}{t^2}\xi\dot{\xi}, \Rightarrow \xi\dot{\xi} = \frac{1}{t}\xi\xi, \tag{A.134}$$

where we have dropped the boundary term in the last equality. Then the Einstein-Hilbert action becomes

$$I_E = \left(-\frac{\ell}{t}\right)\left[\phi\ddot{\chi} + \phi\partial_i^2\left(N - 2\dot{N}_L\right) + \left(-\partial_i^2\eta + \dot{\xi}\right)^2 - \frac{1}{t^2}N^2\right]$$
$$-\frac{\ell}{t^2}\left[-\phi\dot{\chi} + N\left(\dot{\phi} + \dot{\chi}\right) - 2N\partial_i^2 N_L + 2\phi\partial_i^2 N_L\right], \tag{A.135}$$

where the integral sign and the overall coefficients are suppressed. Using (A.69) and adding and subtracting the terms $\frac{2}{t}\dot{\phi}N,\ \dot{\phi}^2,\ \frac{1}{t^2}N^2$ yields

$$I_E = \left(-\frac{\ell}{t}\right)\left[\frac{t^2}{\ell^2}\left(\sigma^2 + p^2\right) + \phi\ddot{\chi} + \phi\partial_i^2\left(N - 2\dot{N}_L\right) - \frac{2}{t^2}N^2\right]$$
$$-\frac{\ell}{t^2}\left[-\phi\dot{\chi} + N\left(3\dot{\phi} + \dot{\chi}\right) - 2N\partial_i^2 N_L + 2\phi\partial_i^2 N_L - \dot{\phi}^2\right]. \tag{A.136}$$

Again doing integration by parts with respect to time in the following terms terms $\frac{\ell}{t}\ddot{\chi}\phi$, $\frac{2\ell}{t}\phi\nabla^2\dot{N}_L$ and adding the term $t\phi\nabla\dot{\phi}$ freely, since it is a boundary term, to the action yields

$$I_E = \left(-\frac{\ell}{t}\right)\left[\frac{t^2}{\ell^2}\left(\sigma^2 + p^2\right) - \dot{\phi}\dot{\chi} + N\partial_i^2\phi + 2\dot{\phi}\partial_i^2 N_L - \frac{2}{t^2}N^2 - t\phi\nabla\dot{\phi}\right] - \frac{\ell}{t^2}\left[2N\dot{\phi} + N\dot{\phi} + N\dot{\chi} - 2N\partial_i^2 N_L - \dot{\phi}^2\right]. \tag{A.137}$$

With reordering the terms (A.137) can be written as follows

$$I_E = \left(-\frac{\ell}{t}\right)\left[\frac{t^2}{\ell^2}\left(\sigma^2 + p^2\right) - t\dot{\phi}\left(\frac{1}{t}\dot{\chi} - \frac{2}{t}\nabla^2 N_L - \frac{2}{t^2}N + \frac{1}{t}\dot{\phi} + \nabla^2\phi\right)\right] - \frac{\ell}{t}\left[N\left(\frac{1}{t}\dot{\chi} - \frac{2}{t}\nabla^2 N_L - \frac{2}{t^2}N + \frac{1}{t}\dot{\phi} + \nabla^2\phi\right)\right], \tag{A.138}$$

and putting (A.64) in (A.138) we obtain

$$I_E = \left(-\frac{\ell}{t}\right)\left[\frac{t^2}{\ell^2}\left(\sigma^2 + p^2\right) - \frac{t^2}{\ell}\dot{\phi}\nabla^2 f + \frac{t}{\ell}N\nabla^2 f\right]. \tag{A.139}$$

Finally, using (A.69) we get

$$I_E = -\left[\frac{t}{\ell}\left(\sigma^2 + p^2\right) - \frac{t^2}{\ell}\, p\nabla^2 f\right]. \tag{A.140}$$

We can also do some manipulations to write (A.140) in terms of the Ricci Scalar. When we look at the gauge-invariant Bianchi identity, we see that the term $t\,f\,\nabla^2\dot{f} + f\,\nabla^2 f$ must be added to the action (A.140). Since this term is boundary term it can be added freely. Therefore, the action becomes

$$I_E = \left(-\frac{t}{\ell}\right)\left[\left(\sigma^2 + p^2\right) - t\,f\nabla^2 p + t\,f\,\nabla^2\dot{f} + f\,\nabla^2 f\right], \tag{A.141}$$

and using Bianchi identity (A.141) becomes

$$I_E = \left(-\frac{t}{\ell}\right)\left[\left(\sigma^2 + p^2\right) + f\,(q + \dot{p})\right]. \tag{A.142}$$

Then, we add and subtract $f\,\nabla^2 f$ in (A.142) and using (A.92) we get

$$I_E = -\left[\frac{t}{\ell}\left(\sigma^2 + p^2 + f\,\nabla^2 f\right) + \frac{\ell^2}{t^2}f\,R_L\right], \tag{A.143}$$

and in the formal form we have

$$I_E = \frac{a}{2}\int\left[\frac{t}{\ell}\left(\sigma^2 + p^2 + f\,\nabla^2 f\right) + \frac{\ell^2}{t^2}f\,R_L\right]. \tag{A.144}$$

The flat spacetime limit of this action is

$$I_E = \frac{1}{2\kappa}\int\left[\sigma^2 + \dot{\phi}^2 + \phi\,\nabla^2\phi + \phi\,\left(q - \Box\phi\right)\right], \tag{A.145}$$

where the flat space version of the gauge-invariant form of the Ricci scalar (A.94) and $a = \frac{1}{\kappa}$ are used. Then

$$
\begin{aligned}
I_E &= \frac{1}{2\kappa} \int \left[\sigma^2 - \phi\ddot{\phi} + \phi\, \nabla^2\phi + \phi q - \phi\Box\phi\right] \\
&= \frac{1}{2\kappa} \int \left[\sigma^2 + \phi\left(-\partial_0^2 + \nabla^2\right)\phi + \phi q - \phi\Box\phi\right] \\
&= \frac{1}{2\kappa} \int \left(\sigma^2 + \phi q\right), \qquad \text{(A.146)}
\end{aligned}
$$

where in the first line integration by parts is done and in the second line the definition of the D'Alembertian $\Box = -\partial_0^2 + \nabla^2$ is used. Also, at the flat spacetime limit the coefficient a becomes $\frac{1}{\kappa}$, after taking $\Lambda = 0$ in (A.111).

A.4.2 The $2\alpha + \beta$ Part of The Higher Derivative Action

The $2\alpha + \beta$ of the action is

$$
I_{2\alpha+\beta} = -\frac{(2\alpha + \beta)}{2} \int d^3x \, \sqrt{\bar{g}} h_{\mu\nu} \left(\bar{g}^{\mu\nu}\Box - \nabla^\mu\nabla^\nu + 2\Lambda\bar{g}^{\mu\nu}\right) R_L, \qquad \text{(A.147)}
$$

and doing integration by parts and dropping the boundary terms the action becomes

$$
I_{2\alpha+\beta} = -\frac{(2\alpha + \beta)}{2} \int d^3x \, \sqrt{\bar{g}} R_L \left(\Box h - \nabla^\mu\nabla^\nu h_{\mu\nu} + 2\Lambda h\right), \qquad \text{(A.148)}
$$

where the term in the parenthesis is $-R_L$. Therefore, the action becomes,

$$
I_{2\alpha+\beta} = \frac{(2\alpha + \beta)}{2} \int d^3x \, \sqrt{\bar{g}} R_L^2, \qquad \text{(A.149)}
$$

since R_L is already gauge-invariant this part of the action is also gauge-invariant. The flat spacetime version of this action is

$$
I_{2\alpha+\beta} = \frac{(2\alpha + \beta)}{2} \int d^3x \, R_L^2, \qquad \text{(A.150)}
$$

where $\sqrt{-\bar{g}} = \frac{\ell^3}{\ell^3} = 1$ is taken.

A.4.3 The β Part of the Higher Derivative Action

Now, let us move to the β term of (A.109);

$$\begin{aligned} I_\beta &= -\frac{\beta}{2}\int d^3x\ \sqrt{\bar{g}}h_{\mu\nu}\left(\Box\mathcal{G}_L^{\mu\nu} - \Lambda\bar{g}^{\mu\nu}R_L\right), \\ &= -\frac{\beta}{2}\int d^3x\ \sqrt{\bar{g}}\left[\left(\Box h_{\mu\nu}\right)\mathcal{G}_L^{\mu\nu} - \Lambda h R_L\right], \end{aligned} \tag{A.151}$$

where in the second line the D'Alembertian is moved in front of the metric perturbation and the boundary terms were dropped. Observe that, the definition of $R^L_{\mu\nu}$, (A.24), involves $\Box h_{\mu\nu}$, therefore it can be written in terms of $\mathcal{G}^L_{\mu\nu}$. From (A.24)

$$\Box h_{\mu\nu} = -2R^L_{\mu\nu} + \nabla^\sigma\nabla_\mu h_{\nu\sigma} + \nabla^\sigma\nabla_\nu h_{\mu\sigma} - \nabla_\mu\nabla_\nu h, \tag{A.152}$$

and changing the order of the covariant derivatives yield

$$\begin{aligned} \Box h_{\mu\nu} = &-2R^L_{\mu\nu} + \nabla^\sigma\nabla_\mu h_{\nu\sigma} + \nabla^\sigma\nabla_\nu h_{\mu\sigma} - \nabla_\mu\nabla_\nu h \\ &+ 6\Lambda h_{\mu\nu} - 2\Lambda g_{\mu\nu}h, \end{aligned} \tag{A.153}$$

where we have used

$$\begin{aligned} \nabla^\sigma\nabla_\mu h_{\nu\sigma} &= \left[\nabla^\sigma, \nabla_\mu\right]h_{\nu\sigma} + \nabla_\mu\nabla^\sigma h_{\nu\sigma} \\ &= \bar{R}^\sigma{}_{\mu\nu}{}^\lambda h_{\lambda\sigma} + \bar{R}^\sigma{}_{\mu\sigma}{}^\lambda h_{\nu\lambda} + \nabla_\mu\nabla^\sigma h_{\nu\sigma} \\ &= 3\Lambda h_{\mu\nu} - g_{\mu\nu}\Lambda h + \nabla_\mu\nabla^\sigma h_{\nu\sigma}, \end{aligned} \tag{A.154}$$

and in the second line we have put the following identities

$$\bar{R}_{\mu\rho\nu\sigma} = \Lambda\left(\bar{g}_{\mu\nu}\bar{g}_{\rho\sigma} - \bar{g}_{\mu\sigma}\bar{g}_{\nu\rho}\right), \quad \bar{R}_{\mu\nu} = 2\Lambda\bar{g}_{\mu\nu}, \quad \bar{R} = 6\Lambda. \tag{A.155}$$

Then,

$$\begin{aligned} \Box h_{\mu\nu} &= -2R^L_{\mu\nu} + 6\Lambda h_{\mu\nu} - 2\Lambda g_{\mu\nu}h + \nabla_\mu\nabla^\sigma h_{\nu\sigma} + \nabla_\nu\nabla^\sigma h_{\mu\sigma} - \nabla_\mu\nabla_\nu h \\ &= -2\mathcal{G}^L_{\mu\nu} - \bar{g}_{\mu\nu}R_L + 2\Lambda h_{\mu\nu} - 2g_{\mu\nu}\Lambda h \\ &\quad + \nabla_\mu\nabla^\sigma h_{\nu\sigma} + \nabla_\nu\nabla^\sigma h_{\mu\sigma} - \nabla_\mu\nabla_\nu h, \end{aligned} \tag{A.156}$$

where in the second line we have used (A.60). Then the first term in (A.151) becomes,

$$\begin{aligned} \left(\Box h_{\mu\nu}\right)\mathcal{G}_L^{\mu\nu} = &-2\mathcal{G}^L_{\mu\nu}\mathcal{G}_L^{\mu\nu} - \bar{g}_{\mu\nu}\mathcal{G}_L^{\mu\nu}R_L + 2\Lambda h_{\mu\nu}\mathcal{G}_L^{\mu\nu} - 2\Lambda\bar{g}_{\mu\nu}\mathcal{G}_L^{\mu\nu}h \\ &+ \mathcal{G}_L^{\mu\nu}\nabla_\nu\nabla^\sigma h_{\mu\sigma} + \mathcal{G}_L^{\mu\nu}\nabla_\mu\nabla^\sigma h_{\nu\sigma} - \mathcal{G}_L^{\mu\nu}\nabla_\mu\nabla_\nu h. \end{aligned} \tag{A.157}$$

The last three terms in (A.157) becomes boundary term and drops out by the use of Bianchi identity, $\nabla_\mu \mathcal{G}_L^{\mu\nu} = 0$. Also, note that

$$\begin{aligned}\bar{g}_{\mu\nu}\mathcal{G}_L^{\mu\nu} &= \bar{g}_{\mu\nu}\left(R_L^{\mu\nu} - \frac{1}{2}\bar{g}^{\mu\nu}R_L - 2\Lambda h^{\mu\nu}\right)\\ &= \bar{g}_{\mu\nu}R_L^{\mu\nu} - \frac{3}{2}R_L - 2\Lambda h,\end{aligned} \tag{A.158}$$

and using

$$\begin{aligned}R_L \equiv \left(g^{\mu\nu}R_{\mu\nu}\right)_L &= \bar{g}^{\mu\nu}R^L_{\mu\nu} - h^{\mu\nu}\bar{R}_{\mu\nu}\\ &= \bar{g}^{\mu\nu}R^L_{\mu\nu} - h^{\mu\nu}\left(2\Lambda\bar{g}_{\mu\nu}\right)\\ &= \bar{g}^{\mu\nu}R^L_{\mu\nu} - 2\Lambda h,\end{aligned} \tag{A.159}$$

$$\bar{g}^{\mu\nu}R^L_{\mu\nu} = R_L + 2\Lambda h, \tag{A.160}$$

(A.158) becomes

$$\bar{g}_{\mu\nu}\mathcal{G}_L^{\mu\nu} = -\frac{1}{2}R_L. \tag{A.161}$$

Putting (A.161) in (A.157) yields

$$\left(\Box h_{\mu\nu}\right)\mathcal{G}_L^{\mu\nu} = -2\mathcal{G}^L_{\mu\nu}\mathcal{G}_L^{\mu\nu} + \frac{1}{2}R_L^2 + 2\Lambda h_{\mu\nu}\mathcal{G}_L^{\mu\nu} + \Lambda h R_L. \tag{A.162}$$

The total action for the β part of (A.109) becomes

$$I_\beta = -\frac{\beta}{2}\int d^3x\, \sqrt{\bar{g}}\left[-2\mathcal{G}^L_{\mu\nu}\mathcal{G}_L^{\mu\nu} + \frac{1}{2}R_L^2 + \frac{2}{\ell^2}h_{\mu\nu}\mathcal{G}_L^{\mu\nu}\right], \tag{A.163}$$

where $\Lambda \equiv \frac{1}{\ell^2}$ is used. The gauge-invariant form of the last term is known since it is the same as Einstein-Hilbert part. The middle term is already gauge-invariant. To have a total gauge-invariant action the first term must be written in that form.

A.4.3.1 The Gauge-Invariant form of $\mathcal{G}^L_{\mu\nu}\mathcal{G}_L^{\mu\nu}$:

Decomposing $\beta\int d^3x\, \sqrt{\bar{g}}\left(\mathcal{G}^L_{\mu\nu}\mathcal{G}_L^{\mu\nu}\right)$ gives us

$$\beta\int d^3x\, \sqrt{\bar{g}}\mathcal{G}^L_{\mu\nu}\mathcal{G}_L^{\mu\nu} = \beta\int d^3x\,\frac{t}{\ell}\left[\left(\mathcal{G}^L_{00}\right)^2 - 2\left(\mathcal{G}^L_{0i}\right)^2 + \left(\mathcal{G}^L_{ij}\right)^2\right]. \tag{A.164}$$

From (A.65, A.70, A.80), we have

$$\left(\mathcal{G}^L_{00}\right)^2 = \frac{t^2}{4\ell^2}\left(\nabla^2 f\right)^2, \quad \left(\mathcal{G}^L_{0j}\right)^2 = -\frac{t^2}{4\ell^2}\left(p\nabla^2 p + \sigma\nabla^2\sigma\right), \quad \left(\mathcal{G}^L_{jk}\right)^2 = \frac{t^2}{4\ell^2}\left[2\dot{\sigma}^2 + q^2 + \dot{p}^2\right], \tag{A.165}$$

and using (A.165) in the action (A.164) we get,

$$\sqrt{-\bar{g}}\mathcal{G}^L_{\mu\nu}\mathcal{G}^{\mu\nu}_L = \frac{t^3}{4\ell^3}\left(\nabla^2 f\right)^2 + \frac{t^3}{2\ell^3}\left(p\nabla^2 p + \sigma\nabla^2\sigma\right) + \frac{t^3}{4\ell^3}\left[2\dot{\sigma}^2 + q^2 + \dot{p}^2\right], \tag{A.166}$$

where $\sqrt{-\bar{g}} = \frac{\ell^3}{t^3}$ and note that we lowered the upper indices by using the background metric.

A.4.3.2 The Gauge-Invariant Form of the I_β Action:

Adding (A.166), square of (A.92) and (A.144) the action (A.163) becomes

$$\begin{aligned} I_\beta =& \frac{\beta}{2}\int d^3x \frac{t^3}{\ell^3}\left[\frac{1}{2}\left(\nabla^2 f\right)^2 + p\nabla^2 p + \sigma\nabla^2\sigma + \dot{\sigma}^2 + \frac{1}{2}q^2 + \frac{1}{2}\dot{p}^2 - \frac{1}{2}\left(q - \nabla^2 f + \dot{p}\right)^2\right] \\ &+ \frac{\beta}{2}\int d^3x \frac{2}{\ell^2}\left(\frac{t}{\ell}\left(\sigma^2 + p^2 + f\,\nabla^2 f\right) + \frac{\ell^2}{t^2} f\,R_L\right) \\ =& \frac{\beta}{2}\int d^3x \frac{t^3}{\ell^3}\left[p\nabla^2 p + \sigma\nabla^2\sigma + \dot{\sigma}^2 - q\dot{p} + q\nabla^2 f + \dot{p}\nabla^2 f\right] \\ &+ \frac{\beta}{2}\int d^3x \frac{2}{\ell^2}\left(\frac{t}{\ell}\left(\sigma^2 + p^2 + f\,\nabla^2 f\right) + \frac{\ell^2}{t^2} f\,R_L\right). \end{aligned} \tag{A.167}$$

In this action there is five gauge-invariant functions. By using the gauge-invariant form of R_L the number of functions can be reduced to four. Here, it is preferred to eliminate the q variable by using $R_L = \frac{t^3}{\ell^3}\left(q - \nabla^2 f + \dot{p}\right) \Rightarrow q = \frac{\ell^3}{t^3}R_L + \nabla^2 f - \dot{p}$,

$$\begin{aligned} I_\beta =& \frac{\beta}{2}\int d^3x \frac{t^3}{\ell^3}\left[p\nabla^2 p + \sigma\nabla^2\sigma + \dot{\sigma}^2 - \left(\frac{\ell^3}{t^3}R_L + \nabla^2 f - \dot{p}\right)\dot{p}\right. \\ &\left. + \left(\frac{\ell^3}{t^3}R_L + \nabla^2 f - \dot{p}\right)\nabla^2 f + \dot{p}\nabla^2 f\right] \\ &+ \frac{\beta}{2}\int d^3x \frac{2}{\ell^2}\left(\frac{t}{\ell}\left(\sigma^2 + p^2 + f\,\nabla^2 f\right) + \frac{\ell^2}{t^2} f\,R_L\right). \\ =& \frac{\beta}{2}\int d^3x \frac{t^3}{\ell^3}\left[p\nabla^2 p + \sigma\nabla^2\sigma + \dot{\sigma}^2 - \frac{\ell^3}{t^3}\dot{p}R_L - \dot{p}\nabla^2 f + \dot{p}^2 + \frac{\ell^3}{t^3}R_L\nabla^2 f + \left(\nabla^2 f\right)^2\right] \\ &+ \frac{\beta}{2}\int d^3x \frac{2}{\ell^2}\left(\frac{t}{\ell}\left(\sigma^2 + p^2 + f\,\nabla^2 f\right) + \frac{\ell^2}{t^2} f\,R_L\right). \end{aligned} \tag{A.168}$$

This action is the final result for the β part of the total action. Now everything is ready to write (A.109) in terms of the gauge-invariant functions.

Before going on, the flat spacetime limit of (A.168) can be written as

$$
\begin{aligned}
I_\beta &= \frac{\beta}{2}\int d^3x\left[\dot\phi\nabla^2\dot\phi+\sigma\nabla^2\sigma+\dot\sigma^2-\ddot\phi R_L-\ddot\phi\nabla^2\phi+\ddot\phi^2+R_L\nabla^2\phi+\left(\nabla^2\phi\right)^2\right]\\
&= \frac{\beta}{2}\int d^3x\left[-\ddot\phi\nabla^2\phi+\sigma\nabla^2\sigma-\sigma\ddot\sigma-\ddot\phi\left(q-\Box\phi\right)-\ddot\phi\nabla^2\phi+\ddot\phi^2+\left(q-\Box\phi\right)\nabla^2\phi+\left(\nabla^2\phi\right)^2\right]\\
&= \frac{\beta}{2}\int d^3x\left[-\ddot\phi\left(\nabla^2-\partial_0^2\right)\phi+\sigma\left(\nabla^2-\partial_0^2\right)\sigma+q\left(-\partial_0^2+\nabla^2\right)\phi\right.\\
&\qquad\left.-\Box\phi\left(-\partial_0^2+\nabla^2\right)\phi+\nabla^2\phi\left(-\partial_0^2+\nabla^2\right)\phi\right]\\
&= \frac{\beta}{2}\int d^3x\left[\Box\phi\left(-\partial_0^2+\nabla^2\right)\phi+\sigma\Box\sigma+q\Box\phi-\left(\Box\phi\right)^2\right],
\end{aligned}
\tag{A.169}
$$

where in the second line we have done integration by parts and used (A.94), in the third line some suitable combinations have done and in the last line the definition of the D'Alembertian operator ,$\Box=-\partial_0^2+\nabla^2$, have been used. Finally, for the flat spacetime limit (A.168) becomes

$$
I_\beta=\frac{\beta}{2}\int d^3x\left[\sigma\Box\sigma+q\Box\phi\right]. \tag{A.170}
$$

A.4.4 The Total Gauge-Invariant Action

Summing (A.144), (A.149) and (A.168) the total action can be written as

$$
\begin{aligned}
I &= \frac{1}{2}\int d^3x\left(a+\frac{2\beta}{\ell^2}\right)\left[\frac{t}{\ell}\left(\sigma^2+p^2+f\nabla^2 f\right)+\frac{\ell^2}{t^2}f R_L\right]+\left(2\alpha+\beta\right)\frac{\ell^3}{t^3}R_L^2\\
&+\frac{1}{2}\int d^3x\frac{t^3}{\ell^3}\beta\left[\dot\sigma^2+\sigma\nabla^2\sigma+\dot p^2+p\nabla^2 p+\left(\nabla^2 f\right)^2-\frac{\ell^3}{t^3}\dot p R_L+\frac{\ell^3}{t^3}R_L\nabla^2 f-\dot p\nabla^2 f\right].
\end{aligned}
\tag{A.171}
$$

This result is the gauge-invariant action for the general quadratic curvature theory in three dimensions. This action can also be simplified by defining $\varphi\equiv\nabla^2 f$ and using (A.92). For the first term

$$
\begin{aligned}
\frac{t}{\ell}\left(\sigma^2+p^2+f\nabla^2 f\right)+\frac{\ell^2}{t^2}f R_L &= \frac{t}{\ell}\left[\sigma^2+p^2+f\nabla^2 f+f\left(q-\nabla^2 f+\dot p\right)\right]\\
&= \frac{t}{\ell}\left[\sigma^2+p^2+f\left(q+\dot p\right)\right],
\end{aligned}
\tag{A.172}
$$

using the Bianchi identity (A.103)

$$
\frac{t}{\ell}\left(\sigma^2+p^2+f\nabla^2 f\right)+\frac{\ell^2}{t^2}f R_L=\frac{t}{\ell}\left[\sigma^2+p^2+t f\nabla^2\dot f-t f\nabla^2 p+f\nabla^2 f\right], \tag{A.173}
$$

and observe that $\frac{t^2}{\ell}f\nabla^2\dot f=\partial_0\left(\frac{t^2}{\ell}f\nabla^2 f\right)-\frac{2t}{\ell}f\nabla^2 f-\frac{t^2}{\ell}\dot f\nabla^2 f=-\frac{2t}{\ell}f\nabla^2 f-\frac{t^2}{\ell}f\nabla^2\dot f\Rightarrow$ $\frac{t^2}{\ell}f\nabla^2\dot f=-\frac{t}{\ell}f\nabla^2 f$ after dropping the boundary term. Then (A.173) becomes

$$
\frac{t}{\ell}\left(\sigma^2+p^2+f\nabla^2 f\right)+\frac{\ell^2}{t^2}f R_L=\frac{t}{\ell}\left[\sigma^2+p^2-t\varphi p\right]. \tag{A.174}
$$

The second term of (A.171) can be written as

$$\frac{\ell^3}{t^3} R_L^2 = \frac{t^5}{\ell^3}\left(\dot{\varphi} - \nabla^2 p\right)^2, \tag{A.175}$$

where it has been used $R_L = \frac{t^4}{\ell^3}\nabla^2\left(\dot{f} - p\right)$ which comes from using Bianchi identity (A.103) in the gauge-invariant form of Ricci scalar. The last term of (A.171) can be written without taking care of the σ field by using (A.108),

$$\begin{aligned}\frac{t^3}{\ell^3}\left[\dot{p}^2 + p\nabla^2 p + \left(\nabla^2 f\right)^2 - \frac{\ell^3}{t^3}\dot{p}R_L + \frac{\ell^3}{t^3}R_L\nabla^2 f - \dot{p}\nabla^2 f\right] &= \frac{t^3}{\ell^3}\left[p\nabla^2 p - \dot{p}q + (\dot{p}+q)\nabla^2 f\right]\\
&= \frac{t^3}{\ell^3}\left[p\nabla^2 p - \dot{p}q + t\nabla^2\left(\dot{f}-p\right)\nabla^2 f + \left(\nabla^2 f\right)^2\right]\\
&= \frac{t^3}{\ell^3}\left[p\nabla^2 p - \dot{p}q - t\nabla^2 p\nabla^2 f - \left(\nabla^2 f\right)^2\right],\end{aligned} \tag{A.176}$$

where in the last line the integration by parts is used for the term $\frac{t^4}{\ell^3}\nabla^2\dot{f}\nabla^2 f = -\frac{2t^3}{\ell^3}\left(\nabla^2 f\right)^2$. Again using the Bianchi identity (A.103) $q\dot{p} = t\nabla^2\left(\dot{f}-p\right)\dot{p} + \dot{p}\left(\nabla^2 f\right) - \dot{p}^2$. Then, (A.176) can be written as

$$\begin{aligned}&\frac{t^3}{\ell^3}\left[p\nabla^2 p - t\dot{p}\nabla^2\dot{f} - t\dot{p}\nabla^2 p - \dot{p}\left(\nabla^2 f\right) + \dot{p}^2 - t\nabla^2 p\nabla^2 f - \left(\nabla^2 f\right)^2\right] =\\
&\qquad\frac{t^3}{\ell^3}\left[p\nabla^2 p - t\dot{p}\dot{\varphi} - t\dot{p}\nabla^2 p - \dot{p}\varphi + \dot{p}^2 - t\varphi\nabla^2 p - \varphi^2\right] =\\
&\qquad\qquad\frac{t^3}{\ell^3}\left[-p\nabla^2 p - t\dot{p}\dot{\varphi} - \dot{p}\varphi + \dot{p}^2 - t\varphi\nabla^2 p - \varphi^2\right],\end{aligned} \tag{A.177}$$

where in the last line the integration by parts is used for $\frac{t^4}{\ell^3}\dot{p}\nabla^2 p = -\frac{2t^3}{\ell^3}p\nabla^2 p$. The final form of (A.171) becomes with (A.173), (A.175) and (A.177)

$$\begin{aligned}I = &\frac{1}{2}\int d^3x\left(a + \frac{2\beta}{\ell^2}\right)\left[\frac{t}{\ell}\left(p^2 - tp\varphi\right)\right] + \frac{1}{2}\int d^3x\,(2\alpha+\beta)\frac{t^5}{\ell^3}\left(\dot{\varphi} - \nabla^2 p\right)^2\\
&+ \frac{1}{2}\int d^3x\,\frac{t^3}{\ell^3}\beta\left[-p\nabla^2 p - t\dot{p}\dot{\varphi} - \dot{p}\varphi + \dot{p}^2 - t\varphi\nabla^2 p - \varphi^2\right] + I_\sigma,\end{aligned} \tag{A.178}$$

where we have defined

$$I_\sigma = \frac{1}{2}\int d^3x\left[\beta\frac{t^3}{\ell^3}\left(\dot{\sigma}^2 + \sigma\nabla^2\sigma\right) + \left(a + \frac{2\beta}{\ell^2}\right)\frac{t}{\ell}\sigma^2\right]. \tag{A.179}$$

In the flat spacetime limit, (A.178) becomes by summing (A.146), (A.150) and (A.170)

$$\begin{aligned}I = &\frac{1}{2}\int d^3x\left[\frac{1}{\kappa}\phi q + (2\alpha+\beta)(q - \Box\phi)^2 + \beta q\Box\phi\right]\\
&+ \frac{\beta}{2}\int d^3x\left[\sigma\Box\sigma + \frac{1}{\kappa\beta}\sigma^2\right].\end{aligned} \tag{A.180}$$

A.5 Gauge Invariance of The Field Equations

The field equations of the action can also be written in terms gauge-invariant functions. In the field equation only the $\Box\mathcal{G}^L_{\mu\nu}$ is unknown in terms of the gauge-invariant functions. Therefore in this part only this term is written in terms of gauge invariant functions. First the covariant derivatives are extracted

$$\begin{aligned}
\Box\mathcal{G}^L_{\mu\nu} &= \bar{g}^{\sigma\rho}\nabla_\rho\nabla_\sigma\mathcal{G}^L_{\mu\nu},\\
&= \bar{g}^{\sigma\rho}\nabla_\rho\left(\partial_\sigma\mathcal{G}^L_{\mu\nu} - \Gamma^\lambda_{\sigma\mu}\mathcal{G}^L_{\lambda\nu} - \Gamma^\lambda_{\sigma\nu}\mathcal{G}^L_{\mu\lambda}\right),\\
&= \frac{t^2}{\ell^2}\eta^{\sigma\rho}\left\{\partial_\rho\left(\partial_\sigma\mathcal{G}^L_{\mu\nu} - \Gamma^\lambda_{\sigma\mu}\mathcal{G}^L_{\lambda\nu} - \Gamma^\lambda_{\sigma\nu}\mathcal{G}^L_{\mu\lambda}\right) - \Gamma^\alpha_{\rho\sigma}\left(\partial_\alpha\mathcal{G}^L_{\mu\nu} - \Gamma^\lambda_{\alpha\mu}\mathcal{G}^L_{\lambda\nu} - \Gamma^\lambda_{\alpha\nu}\mathcal{G}^L_{\mu\lambda}\right)\right.\\
&\quad -\Gamma^\alpha_{\rho\mu}\left(\partial_\sigma\mathcal{G}^L_{\alpha\nu} - \Gamma^\lambda_{\sigma\alpha}\mathcal{G}^L_{\lambda\nu} - \Gamma^\lambda_{\sigma\nu}\mathcal{G}^L_{\alpha\lambda}\right) - \left.\Gamma^\alpha_{\rho\nu}\left(\partial_\sigma\mathcal{G}^L_{\mu\alpha} - \Gamma^\lambda_{\sigma\mu}\mathcal{G}^L_{\lambda\alpha} - \Gamma^\lambda_{\sigma\alpha}\mathcal{G}^L_{\mu\lambda}\right)\right\},\\
&= \frac{t^2}{\ell^2}\eta^{\sigma\rho}\left\{\left(\partial_\rho\partial_\sigma\mathcal{G}^L_{\mu\nu} - \mathcal{G}^L_{\lambda\nu}\partial_\rho\Gamma^\lambda_{\sigma\mu} - \Gamma^\lambda_{\sigma\mu}\partial_\rho\mathcal{G}^L_{\lambda\nu} - \mathcal{G}^L_{\mu\lambda}\partial_\rho\Gamma^\lambda_{\sigma\nu} - \Gamma^\lambda_{\sigma\nu}\partial_\rho\mathcal{G}^L_{\mu\lambda}\right)\right.\\
&\quad - \left(\Gamma^\alpha_{\rho\sigma}\partial_\alpha\mathcal{G}^L_{\mu\nu} - \Gamma^\alpha_{\rho\sigma}\Gamma^\lambda_{\alpha\mu}\mathcal{G}^L_{\lambda\nu} - \Gamma^\alpha_{\rho\sigma}\Gamma^\lambda_{\alpha\nu}\mathcal{G}^L_{\mu\lambda}\right)\\
&\quad - \left(\Gamma^\alpha_{\rho\mu}\partial_\sigma\mathcal{G}^L_{\alpha\nu} - \Gamma^\alpha_{\rho\mu}\Gamma^\lambda_{\sigma\alpha}\mathcal{G}^L_{\lambda\nu} - \Gamma^\alpha_{\rho\mu}\Gamma^\lambda_{\sigma\nu}\mathcal{G}^L_{\alpha\lambda}\right)\\
&\quad - \left.\left(\Gamma^\alpha_{\rho\nu}\partial_\sigma\mathcal{G}^L_{\mu\alpha} - \Gamma^\alpha_{\rho\nu}\Gamma^\lambda_{\sigma\mu}\mathcal{G}^L_{\lambda\alpha} - \Gamma^\alpha_{\rho\nu}\Gamma^\lambda_{\sigma\alpha}\mathcal{G}^L_{\mu\lambda}\right)\right\},
\end{aligned} \tag{A.181}$$

where we have used $\bar{g}_{\sigma\rho} = \frac{\ell^2}{t^2}\eta_{\sigma\rho}$ in the third line. Doing the summations in the repeated indices (A.181) becomes

$$\begin{aligned}
\Box\mathcal{G}^L_{\mu\nu} &= \frac{t^2}{\ell^2}\eta^{\sigma\rho}\left\{\partial_\rho\partial_\sigma\mathcal{G}^L_{\mu\nu} - \left(\mathcal{G}^L_{\lambda\nu}\partial_\rho\Gamma^\lambda_{\sigma\mu} + \mathcal{G}^L_{\mu\lambda}\partial_\rho\Gamma^\lambda_{\sigma\nu}\right)\right.\\
&\quad - \left(\Gamma^\lambda_{\sigma\mu}\partial_\rho\mathcal{G}^L_{\lambda\nu} + \Gamma^\lambda_{\sigma\nu}\partial_\rho\mathcal{G}^L_{\mu\lambda} + \Gamma^\alpha_{\rho\sigma}\partial_\alpha\mathcal{G}^L_{\mu\nu} + \Gamma^\alpha_{\rho\mu}\partial_\sigma\mathcal{G}^L_{\alpha\nu} + \Gamma^\alpha_{\rho\nu}\partial_\sigma\mathcal{G}^L_{\mu\alpha}\right)\\
&\quad +\Gamma^\alpha_{\rho\sigma}\left(\Gamma^\lambda_{\alpha\mu}\mathcal{G}^L_{\lambda\nu} + \Gamma^\lambda_{\alpha\nu}\mathcal{G}^L_{\mu\lambda}\right) + \left(\Gamma^\alpha_{\rho\mu}\Gamma^\lambda_{\sigma\alpha}\mathcal{G}^L_{\lambda\nu} + \Gamma^\alpha_{\rho\mu}\Gamma^\lambda_{\sigma\nu}\mathcal{G}^L_{\alpha\lambda}\right)\\
&\quad + \left.\left(\Gamma^\alpha_{\rho\nu}\Gamma^\lambda_{\sigma\mu}\mathcal{G}^L_{\lambda\alpha} + \Gamma^\alpha_{\rho\nu}\Gamma^\lambda_{\sigma\alpha}\mathcal{G}^L_{\mu\lambda}\right)\right\},\\
&= \frac{t^2}{\ell^2}\left\{\left(-\partial_0^2 + \partial_i^2\right)\mathcal{G}^L_{\mu\nu} - \left(-\mathcal{G}^L_{\lambda\nu}\partial_0\Gamma^\lambda_{0\mu} + \mathcal{G}^L_{\lambda\nu}\partial_i\Gamma^\lambda_{i\mu} - \mathcal{G}^L_{\mu\lambda}\partial_0\Gamma^\lambda_{0\nu} + \mathcal{G}^L_{\mu\lambda}\partial_i\Gamma^\lambda_{i\nu}\right)\right.\\
&\quad - \left(-\Gamma^\lambda_{0\mu}\partial_0\mathcal{G}^L_{\lambda\nu} + \Gamma^\lambda_{i\mu}\partial_i\mathcal{G}^L_{\lambda\nu} - \Gamma^\lambda_{0\nu}\partial_0\mathcal{G}^L_{\mu\lambda} + \Gamma^\lambda_{i\nu}\partial_i\mathcal{G}^L_{\mu\lambda} - \Gamma^\alpha_{00}\partial_\alpha\mathcal{G}^L_{\mu\nu} + \Gamma^\alpha_{ii}\partial_\alpha\mathcal{G}^L_{\mu\nu}\right)\\
&\quad - \left(-\Gamma^\alpha_{0\mu}\partial_0\mathcal{G}^L_{\alpha\nu} + \Gamma^\alpha_{i\mu}\partial_i\mathcal{G}^L_{\alpha\nu} - \Gamma^\alpha_{0\nu}\partial_0\mathcal{G}^L_{\mu\alpha} + \Gamma^\alpha_{i\nu}\partial_i\mathcal{G}^L_{\mu\alpha}\right)\\
&\quad + \left(-\Gamma^\alpha_{00} + \Gamma^\alpha_{ii}\right)\left(\Gamma^\lambda_{\alpha\mu}\mathcal{G}^L_{\lambda\nu} + \Gamma^\lambda_{\alpha\nu}\mathcal{G}^L_{\mu\lambda}\right)\\
&\quad + \left(-\Gamma^\alpha_{0\mu}\Gamma^\lambda_{0\alpha}\mathcal{G}^L_{\lambda\nu} + \Gamma^\alpha_{i\mu}\Gamma^\lambda_{i\alpha}\mathcal{G}^L_{\lambda\nu} - \Gamma^\alpha_{0\mu}\Gamma^\lambda_{0\nu}\mathcal{G}^L_{\alpha\lambda} + \Gamma^\alpha_{i\mu}\Gamma^\lambda_{i\nu}\mathcal{G}^L_{\alpha\lambda}\right)\\
&\quad + \left.\left(-\Gamma^\alpha_{0\nu}\Gamma^\lambda_{0\mu}\mathcal{G}^L_{\lambda\alpha} + \Gamma^\alpha_{i\nu}\Gamma^\lambda_{i\mu}\mathcal{G}^L_{\lambda\alpha} - \Gamma^\alpha_{0\nu}\Gamma^\lambda_{0\alpha}\mathcal{G}^L_{\mu\lambda} + \Gamma^\alpha_{i\nu}\Gamma^\lambda_{i\alpha}\mathcal{G}^L_{\mu\lambda}\right)\right\},
\end{aligned} \tag{A.182}$$

and using Christoffel connections (A.8), (A.182) yields

$$\Box\mathcal{G}^L_{\mu\nu} = \frac{t^2}{\ell^2}\left\{\left(-\partial_0^2+\partial_i^2\right)\mathcal{G}^L_{\mu\nu} - \left[2\mathcal{G}^L_{\mu\nu}\partial_0\left(\frac{1}{t}\right) + \mathcal{G}^L_{\lambda\nu}\partial_i\Gamma^\lambda_{i\mu} + \mathcal{G}^L_{\mu\lambda}\partial_i\Gamma^\lambda_{i\nu}\right]\right.$$
$$-\left(\frac{3}{t}\partial_0\mathcal{G}^L_{\mu\nu} + 2\Gamma^\lambda_{i\mu}\partial_i\mathcal{G}^L_{\lambda\nu} + 2\Gamma^\lambda_{i\nu}\partial_i\mathcal{G}^L_{\mu\lambda} - \frac{1}{t}\eta_{ii}\partial_0\mathcal{G}^L_{\mu\nu}\right) - \frac{2}{t}\partial_0\mathcal{G}^L_{\mu\nu}$$
$$\left.+\frac{2}{t^2}\mathcal{G}^L_{\mu\nu} + \left(-\frac{4}{t^2}\mathcal{G}^L_{\mu\nu} + \Gamma^\alpha_{i\mu}\Gamma^\lambda_{i\alpha}\mathcal{G}^L_{\lambda\nu} + \Gamma^\alpha_{i\nu}\Gamma^\lambda_{i\alpha}\mathcal{G}^L_{\mu\lambda} + 2\Gamma^\alpha_{i\mu}\Gamma^\lambda_{i\nu}\mathcal{G}^L_{\alpha\lambda}\right)\right\}. \tag{A.183}$$

Doing the essential cancellations (A.183) becomes

$$\Box\mathcal{G}^L_{\mu\nu} = \frac{t^2}{\ell^2}\left\{\left(-\partial_0^2 - \frac{3}{t}\partial_0 + \partial_i^2\right)\mathcal{G}^L_{\mu\nu} - \left(\mathcal{G}^L_{\lambda\nu}\partial_i\Gamma^\lambda_{i\mu} + \mathcal{G}^L_{\mu\lambda}\partial_i\Gamma^\lambda_{i\nu}\right)\right.$$
$$-\left(2\Gamma^\lambda_{i\mu}\partial_i\mathcal{G}^L_{\lambda\nu} + 2\Gamma^\lambda_{i\nu}\partial_i\mathcal{G}^L_{\mu\lambda}\right)$$
$$\left.+\Gamma^\alpha_{i\mu}\Gamma^\lambda_{i\alpha}\mathcal{G}^L_{\lambda\nu} + \Gamma^\alpha_{i\nu}\Gamma^\lambda_{i\alpha}\mathcal{G}^L_{\mu\lambda} + 2\Gamma^\alpha_{i\mu}\Gamma^\lambda_{i\nu}\mathcal{G}^L_{\alpha\lambda}\right\}. \tag{A.184}$$

Again doing summations in the repeated indices and dropping the derivatives of the Christoffel connections, $\partial_i\Gamma^\mu_{\nu\sigma} = 0$, (A.184) yields

$$\Box\mathcal{G}^L_{\mu\nu} = \frac{t^2}{\ell^2}\left\{\left(-\partial_0^2 - \frac{3}{t}\partial_0 + \partial_i^2\right)\mathcal{G}^L_{\mu\nu} - 2\left(\Gamma^0_{i\mu}\partial_i\mathcal{G}^L_{0\nu} + \Gamma^j_{i\mu}\partial_i\mathcal{G}^L_{j\nu} + \Gamma^0_{i\nu}\partial_i\mathcal{G}^L_{\mu 0} + \Gamma^j_{i\nu}\partial_i\mathcal{G}^L_{\mu j}\right)\right.$$
$$-\frac{1}{t}\Gamma^0_{i\mu}\mathcal{G}^L_{i\nu} - \frac{1}{t}\Gamma^i_{i\mu}\mathcal{G}^L_{0\nu} - \frac{1}{t}\Gamma^0_{i\nu}\mathcal{G}^L_{\mu i} - \frac{1}{t}\Gamma^i_{i\nu}\mathcal{G}^L_{\mu 0}$$
$$\left.+2\Gamma^0_{i\mu}\Gamma^0_{i\nu}\mathcal{G}^L_{00} + 2\Gamma^0_{i\mu}\Gamma^j_{i\nu}\mathcal{G}^L_{0j} + 2\Gamma^j_{i\mu}\Gamma^0_{i\nu}\mathcal{G}^L_{j0} + 2\Gamma^j_{i\mu}\Gamma^k_{i\nu}\mathcal{G}^L_{jk}\right\}. \tag{A.185}$$

By using (A.185) we can calculate the components term by term.

A.5.1 The $\Box\mathcal{G}^L_{00}$ Term

Setting $\mu = \nu = 0$ in (A.185) and using (A.8) we have

$$\Box\mathcal{G}^L_{00} = \frac{t^2}{\ell^2}\left\{\left(-\partial_0^2 - \frac{3}{t}\partial_0 + \partial_i^2\right)\mathcal{G}^L_{00} - 2\left(\Gamma^0_{i0}\partial_i\mathcal{G}^L_{00} + \Gamma^j_{i0}\partial_i\mathcal{G}^L_{j0} + \Gamma^0_{i0}\partial_i\mathcal{G}^L_{00} + \Gamma^j_{i0}\partial_i\mathcal{G}^L_{0j}\right)\right.$$
$$-\frac{1}{t}\Gamma^0_{i0}\mathcal{G}^L_{i0} - \frac{1}{t}\Gamma^i_{i0}\mathcal{G}^L_{00} - \frac{1}{t}\Gamma^0_{i0}\mathcal{G}^L_{0i} - \frac{1}{t}\Gamma^i_{i0}\mathcal{G}^L_{00} \tag{A.186}$$
$$\left.+2\Gamma^0_{i0}\Gamma^0_{i0}\mathcal{G}^L_{00} + 2\Gamma^0_{i0}\Gamma^j_{i0}\mathcal{G}^L_{0j} + 2\Gamma^j_{i0}\Gamma^0_{i0}\mathcal{G}^L_{j0} + 2\Gamma^j_{i0}\Gamma^k_{i0}\mathcal{G}^L_{jk}\right\},$$

and after simple manipulations (A.186) becomes

$$
\begin{aligned}
\Box \mathcal{G}^L_{00} &= \frac{t^2}{\ell^2}\left\{\left(-\partial_0^2 - \frac{3}{t}\partial_0 + \partial_i^2\right)\mathcal{G}^L_{00} - 2\left(\Gamma^j_{i0}\partial_i\mathcal{G}^L_{j0} + \Gamma^j_{i0}\partial_i\mathcal{G}^L_{0j}\right)\right. \\
&\qquad \left. -\frac{2}{t}\Gamma^i_{i0}\mathcal{G}^L_{00} + 2\Gamma^j_{i0}\Gamma^k_{i0}\mathcal{G}^L_{jk}\right\}, \\
&= \frac{t^2}{\ell^2}\left\{\left(-\partial_0^2 - \frac{3}{t}\partial_0 + \partial_i^2\right)\mathcal{G}^L_{00} - 2\left(-\frac{2}{t}\delta^j_i\partial_i\mathcal{G}^L_{j0}\right)\right. \\
&\qquad \left. +\frac{2}{t^2}\delta^i_i\mathcal{G}^L_{00} + \frac{2}{t^2}\delta^j_i\delta^k_i\mathcal{G}^L_{jk}\right\}, \\
&= \frac{t^2}{\ell^2}\left\{\left(-\partial_0^2 - \frac{3}{t}\partial_0 + \partial_i^2\right)\mathcal{G}^L_{00} + \frac{4}{t}\partial_i\mathcal{G}^L_{i0} + \frac{4}{t^2}\mathcal{G}^L_{00} + \frac{2}{t^2}\mathcal{G}^L_{ii}\right\}.
\end{aligned} \tag{A.187}
$$

Putting (A.65) and its derivatives, the derivative of (A.70) and the trace of (A.80), which are

$$
\begin{aligned}
&\partial_0\mathcal{G}^L_{00} = -\frac{1}{2\ell}\nabla^2 f - \frac{t}{2\ell}\nabla^2\dot{f}, \quad \partial_0^2\mathcal{G}^L_{00} = -\frac{1}{\ell}\nabla^2\dot{f} - \frac{t}{2\ell}\nabla^2\ddot{f}, \\
&\partial_i\mathcal{G}^L_{i0} = -\frac{t}{2\ell}\nabla^2 p, \quad \mathcal{G}^L_{ii} = -\frac{t}{2\ell}\left(\dot{p} + q\right),
\end{aligned} \tag{A.188}
$$

into (A.187) yields

$$
\begin{aligned}
\Box \mathcal{G}^L_{00} &= \frac{t^3}{2\ell^3}\left(\frac{2}{t}\nabla^2\dot{f} + \nabla^2\ddot{f} + \frac{3}{t^2}\nabla^2 f + \frac{3}{t}\nabla^2\dot{f} - \nabla^2\nabla^2 f\right) \\
&\quad + \frac{t^3}{2\ell^3}\left(-\frac{4}{t}\nabla^2 p - \frac{4}{t^2}\nabla^2 f - \frac{2}{t^2}\left(\dot{p} + q\right)\right), \\
&= \frac{t^3}{2\ell^3}\left(\frac{5}{t}\nabla^2\dot{f} + \nabla^2\ddot{f} - \frac{3}{t^2}\nabla^2 f - \nabla^2\nabla^2 f\right) \\
&\quad + \frac{t^3}{2\ell^3}\left(-\frac{4}{t}\nabla^2 p + \frac{2}{t^2}\nabla^2 f - \frac{2}{t^2}\left(\dot{p} + q\right)\right), \\
&= \frac{t^3}{2\ell^3}\left(\frac{5}{t}\nabla^2\dot{f} + \nabla^2\ddot{f} - \frac{3}{t^2}\nabla^2 f - \nabla^2\nabla^2 f\right) \\
&\quad + \frac{t^3}{2\ell^3}\left(-\frac{4}{t}\nabla^2 p - \frac{2}{t^2}\left[\left(\dot{p} + q\right) - \nabla^2 f\right]\right),
\end{aligned} \tag{A.189}
$$

and from Bianchi identity (A.103) the last term is $\frac{\ell^3}{t^3}R_L$ and the final result becomes

$$
\begin{aligned}
\Box \mathcal{G}^L_{00} &= \frac{t^3}{2\ell^3}\left(\nabla^2\ddot{f} + \frac{5}{t}\nabla^2\dot{f} - \nabla^2\nabla^2 f\right) \\
&\quad - \frac{t^3}{2\ell^3}\left(\frac{4}{t}\nabla^2 p + \frac{3}{t^2}\nabla^2 f + \frac{2\ell^3}{t^5}R_L\right).
\end{aligned} \tag{A.190}
$$

For the flat spacetime case (A.190) yields

$$
\Box \mathcal{G}^L_{00} = \frac{1}{2}\left(\nabla^2\ddot{f} - \nabla^2\nabla^2 f\right) \tag{A.191}
$$

A.5.2 The $\Box\mathcal{G}^L_{k0}$ Term

Let us continue with $\Box\mathcal{G}^L_{k0}$ component. Inserting $\mu = k$ and $\nu = 0$, the equation (A.185) becomes,

$$\begin{aligned}\Box\mathcal{G}^L_{k0} &= \frac{t^2}{\ell^2}\left\{\left(-\partial_0^2 - \frac{3}{t}\partial_0 + \partial_i^2\right)\mathcal{G}^L_{k0} - 2\left(\Gamma^0_{ik}\partial_i\mathcal{G}^L_{00} + \Gamma^j_{i0}\partial_i\mathcal{G}^L_{kj}\right)\right.\\ &\qquad \left. -\frac{1}{t}\Gamma^0_{ik}\mathcal{G}^L_{i0} - \frac{1}{t}\Gamma^i_{i0}\mathcal{G}^L_{k0} + 2\Gamma^0_{ik}\Gamma^j_{i0}\mathcal{G}^L_{0j}\right\},\\ &= \frac{t^2}{\ell^2}\left\{\left(-\partial_0^2 - \frac{3}{t}\partial_0 + \partial_i^2\right)\mathcal{G}^L_{k0} + \frac{2}{t}\left(\partial_k\mathcal{G}^L_{00} + \partial_i\mathcal{G}^L_{ki}\right) + \frac{5}{t^2}\mathcal{G}^L_{0k}\right\}.\end{aligned} \tag{A.192}$$

Using (A.70) and its derivatives, which are

$$\begin{aligned}\partial_0\mathcal{G}^L_{k0} &= -\frac{1}{2\ell}\left(\partial_k p + \epsilon_{kn}\partial_n\sigma\right) - \frac{t}{2\ell}\left(\partial_k\dot{p} + \epsilon_{kn}\partial_n\dot{\sigma}\right),\\ \partial_0^2\mathcal{G}^L_{k0} &= -\frac{1}{\ell}\left(\partial_k\dot{p} + \epsilon_{kn}\partial_n\dot{\sigma}\right) - \frac{t}{2\ell}\left(\partial_k\ddot{p} + \epsilon_{kn}\partial_n\ddot{\sigma}\right),\end{aligned} \tag{A.193}$$

the derivative of (A.80), that is

$$\partial_i\mathcal{G}^L_{ik} = -\frac{t}{2\ell}\left(\partial_k\dot{p} + \epsilon_{kn}\partial_n\dot{\sigma}\right), \tag{A.194}$$

and (A.8) into (A.192) yields

$$\begin{aligned}\frac{\ell^2}{t^2}\Box\mathcal{G}^L_{k0} &= \frac{1}{\ell}\left(\partial_k\dot{p} + \epsilon_{kn}\partial_n\dot{\sigma}\right) + \frac{t}{2\ell}\left(\partial_k\ddot{p} + \epsilon_{kn}\partial_n\ddot{\sigma}\right)\\ &\quad + \frac{3}{t}\frac{1}{2\ell}\left(\partial_k p + \epsilon_{kn}\partial_n\sigma\right) + \frac{3}{t}\frac{t}{2\ell}\left(\partial_k\dot{p} + \epsilon_{kn}\partial_n\dot{\sigma}\right)\\ &\quad - \frac{t}{2\ell}\left(\partial_k\partial_i^2 p + \epsilon_{kn}\partial_n\partial_i^2\sigma\right)\\ &\quad + \frac{2}{t}\left(-\frac{t}{2\ell}\partial_k\nabla^2 f - \frac{t}{2\ell}\left(\partial_k\dot{p} + \epsilon_{kn}\partial_n\dot{\sigma}\right)\right)\\ &\quad - \frac{5}{t^2}\frac{t}{2\ell}\left(\partial_k p + \epsilon_{kn}\partial_n\sigma\right),\end{aligned} \tag{A.195}$$

and

$$\begin{aligned}\Box\mathcal{G}^L_{k0} &= \frac{t^3}{2\ell^3}\partial_k\left(\ddot{p} + \frac{3}{t}\dot{p} - \nabla^2 p - \frac{2}{t^2}p - \frac{2}{t}\nabla^2 f\right)\\ &\quad + \frac{t^3}{2\ell^3}\epsilon_{kn}\partial_n\left(\ddot{\sigma} + \frac{3}{t}\dot{\sigma} - \nabla^2\sigma - \frac{2}{t^2}\sigma\right).\end{aligned} \tag{A.196}$$

The flat spacetime version of (A.196) is

$$\Box\mathcal{G}^L_{k0} = \frac{1}{2}\partial_k\left(\ddot{p} - \nabla^2 p\right) + \frac{1}{2}\epsilon_{kn}\partial_n\left(\ddot{\sigma} - \nabla^2\sigma\right). \tag{A.197}$$

A.5.3 The $\Box \mathcal{G}^L_{mn}$ Term

The last component can be calculated in the same way. The indices are renamed in (A.185) as $\mu = m$ and $\nu = n$. After using (A.8)

$$\frac{\ell^2}{t^2}\Box \mathcal{G}^L_{mn} = \left(-\partial_0^2 - \frac{3}{t}\partial_0 + \partial_i^2\right)\mathcal{G}^L_{mn} + \frac{2}{t}\left(\partial_m \mathcal{G}^L_{0n} + \partial_n \mathcal{G}^L_{m0}\right)$$
$$+ \frac{1}{t^2}\mathcal{G}^L_{mn} + \frac{1}{t^2}\mathcal{G}^L_{mn} - \frac{2}{t}\delta_{mn}\mathcal{G}^L_{00}. \tag{A.198}$$

After calculating the derivatives and doing suitable cancellations the final answer comes out as

$$\frac{2\ell^3}{t^3}\Box \mathcal{G}^L_{mn} = \left(\delta_{mn} + \hat{\partial}_m\hat{\partial}_n\right)\left(\ddot{q} + \frac{5}{t}\dot{q} + \frac{1}{t^2}q - \nabla^2 q - \frac{2}{t^2}\nabla^2 f\right)$$
$$- \hat{\partial}_m\hat{\partial}_n\left(\dddot{p} + \frac{5}{t}\ddot{p} + \frac{1}{t^2}\dot{p} - \nabla^2\dot{p} - \frac{4}{t}\nabla^2 p - \frac{2}{t^2}\nabla^2 f\right)$$
$$- \left(\epsilon_{mk}\hat{\partial}_k\hat{\partial}_n + \epsilon_{nk}\hat{\partial}_k\hat{\partial}_m\right)\left(\dddot{\sigma} + \frac{5}{t}\ddot{\sigma} + \frac{1}{t^2}\dot{\sigma} - \nabla^2\dot{\sigma} - \frac{2}{t}\nabla^2\sigma\right). \tag{A.199}$$

In the flat spacetime limit (A.199) becomes

$$2\Box \mathcal{G}^L_{mn} = \left(\delta_{mn} + \hat{\partial}_m\hat{\partial}_n\right)\left(\ddot{q} - \nabla^2 q\right) - \hat{\partial}_m\hat{\partial}_n\left(\dddot{p} - \nabla^2\dot{p}\right)$$
$$- \left(\epsilon_{mk}\hat{\partial}_k\hat{\partial}_n + \epsilon_{nk}\hat{\partial}_k\hat{\partial}_m\right)\left(\dddot{\sigma} - \nabla^2\dot{\sigma}\right) \tag{A.200}$$

With (A.190), (A.196), (A.199) and the results that are computed for $\mathcal{G}^L_{\mu\nu}$ and R_L are enough to write the gauge-invariant form of the equations of motion.

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- *"AdS Waves as Exact Solutions to Quadratic Gravity"*. Phys. Rev. D 83: 084015, (2011). İ. Güllü, M. Gürses, T. Ç. Şişman, B. Tekin.
- *"All Bulk and Boundary Unitary Cubic Curvature Theories in Three Dimensions"*. Phys. Rev. D 83: 024033, (2011). İ. Güllü, T. Ç. Şişman, B. Tekin.
- *"Unitary Analysis of General Born-Infeld Gravity Theories"*. Phys. Rev. D 82: 124023, (2010). İ. Güllü, T. Ç. Şişman, B. Tekin.
- *"c-functions in the Born-Infeld Extended New Massive Gravity"*. Phys. Rev. D 82: 024032, (2010). İ. Güllü, T. Ç. Şişman, B. Tekin.
- *"Born-Infeld-Horava Gravity"*. Phys. Rev. D 81: 104018, (2010). İ. Güllü, T. Ç. Şişman, B. Tekin.
- *"Born-Infeld Extension of New Massive Gravity"*. Class. Quant. Grav. 27: 162001, (2010). İ. Güllü, T. Ç. Şişman, B. Tekin.

- *"Canonical Structure of Higher Derivative Gravity in 3D"*. Phys. Rev. D 81: 104017, (2010). İ. Güllü, T. Ç. Şişman, B. Tekin.

- *"Massive Higher Derivative Gravity in D-dimensional Anti-de Sitter Spacetimes"*. Phys. Rev. D 80: 064033, (2009). İ. Güllü, B. Tekin.

- *"Charges in AdS spacetimes"*. Turk. J. Phys. 30: 419-436, (2006). İ. Güllü, B. Tekin.

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