AN ASSESSMENT OF A TWO-ECHELON INVENTORY SYSTEM AGAINST ALTERNATIVE SYSTEMS

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ABSTRACT

AN ASSESSMENT OF A TWO-ECHELON INVENTORY SYSTEM AGAINST ALTERNATIVE SYSTEMS

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In this study, we focus on a real life problem that involves a single item which is used in military operations. The items in use fail according to a Poisson process and lead times are deterministic. Four alternative inventory control models are developed. Among these models, a two-echelon system consisting of a depot in the upper and several bases in the lower echelon is operated currently. This system is compared to a single-echelon system that consists of several bases. The comparison reveals the importance of the holding cost incurred for the items in-transit between the depot and the base which is ignored in most of the studies in literature. Both the two and single-echelon models are also extended to have repair ability. A continuous-review base-stock policy is used for all models. Exact models are formulated. The results are obtained under various lead time, unit costs and demand parameters. Results of four different settings are compared and the findings are reported.

Keywords: Stochastic inventory control, single-echelon, multi-echelon, repair ability, exact model, continuous review

ÖZ

İKİ KADEMELİ ENVANTER SİSTEMİNİN ALTERNATİF SİSTEMLERE KARŞI DEĞERLENDİRİLMESİ

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Bu çalışmada, askeri operasyonlarda kullanılan bir malzemeyle ilgilenen gerçek hayat kaynaklı envanter problemine odaklanılmıştır. Kullanılan malzemeler Poisson sürecine uygun sıklıkta bozulmaktadırlar ve ön zamanlar değişken değildir. Dört farklı envanter kontrol modeli geliştirilmiştir. Bu modeller arasında, üst kademede depo alt kademede ise bir çok üs bulunan, iki kademeli sistem güncel olarak işletilmektedir. Bu sistem bir çok üsten oluşan tek kademeli bir sistemle karşılaştırılmıştır. Karşılaştırma sonucu; pek çok çalışmada göz ardı edilen depo ile üs arasında taşınmakta olan malzeme için harcanan maliyetin önemi ortaya çıkmıştır. Ayrıca, hem iki hem de tek kademeli sistemler, tamir yeteneği kazanacak şekilde geliştirilmişlerdir. Tüm modeller için, sürekli takip edilen, baz stok seviyesini öngören bir poliçe kullanılmıştır. Kesin sonuç veren bir model formulize edilmiştir. Sonuçlar, değişik ön zaman, birim maliyetler ve talep değerleri için elde edilmiştir. Dört farklı envanter modeli sonuçları karşılaştırılmış ve tespitler raporlanmıştır.

Anahtar Kelimeler: Rassal envanter kontrolü, tek kademe, çok kademe, tamir yeteneği, kesin model, sürekli takip

To My Family,

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CHAPTER 1

INTRODUCTION

In this study, a real life inventory control problem of a single item, which is being used in the military operations, is considered. The item has a critical importance in the operations and it is supplied by a single domestic supplier. In the current system, there is a two-echelon inventory setting which consists of a single stock point, a central depot, in the upper echelon and several stock points, bases, in the lower echelon. The users are the military units which are very close to the operation sites. The demand originates from the item failures in the users, which are regarded as the end customers, and it is stochastic. User demand occurs first in the lower echelon, and then it is reflected to the upper echelon (See Figure 1).

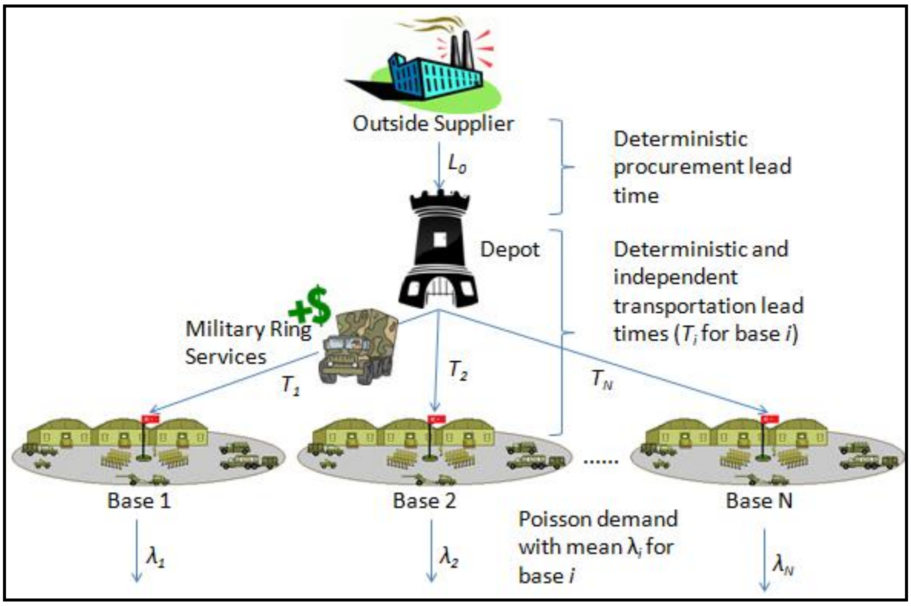

Figure 1-Two-echelon model

The depot replenishes the bases and orders from the outside supplier. The depot and the bases are connected via military ring services. The items are transported using these links incurring a holding cost for the items in-transit. Any unsatisfied demand is backordered at all stock points. The lead times are assumed to be deterministic. The goal is to operate the inventory control system at the minimum expected long run average cost. The stock levels, lead times, unit procurement, holding and backorder costs are the key drivers of this cost. For the rest of the study; the expected long run average cost is denoted by cost.

The headquarters are aware that the current system has room for improvement. For instance, although the items require high technology to be produced; since the outside supplier is a domestic firm, the repair ability, which would enable some portion of the failed items be repaired, can be acquired in the system. This repair ability at the stock points would replace the procurement lead times with relatively shorter repair lead time. Moreover the repair costs for each item repaired is expected to be lower than the procurement costs.

The current system is centralized due to the nature of the system and it is advantageous as the risk of backorder occurrences is reduced. This enables the headquarters to maintain an easy control on the system-wide inventory as the risks of the demand uncertainty are pooled. However, the ring services that are being used throughout the system are a bit risky, since the services have close proximity to the military operation sites, and the transportation lead times are long and the transportation process is very costly due to inventory holding cost of the items in-transit which is ignored in most studies in the literature. This makes the headquarters analyze the option of not operating the depot to reduce the risks and costs involved in transportation and to avoid the transportation lead time.

Considering these possible changes, we consider four different alternative inventory systems. The first one is the actual model which consists of two-echelons, including one depot in the upper and several bases in the lower one. In the two-echelon model with repair ability, only the depot is improved to have the repair ability. Another alternative considered by the headquarters is not to operate the depot, which corresponds to the single-echelon model with no repair ability. Finally, the bases in the single-echelon model are granted the repair ability which entails the analysis of single-echelon model with repair ability. The models considered in this study are presented in Table 1.

Table 1- Models in the study

Model	**Representation**	**Explanation**
Two-echelon model	TE model	Current model that is operated.
Two-echelon model with repair ability	TR model	Depot is granted with repair ability; the failed items can be repaired with probability ρ.
Single-echelon model	SE model	Depot is closed and the bases replenish from the outside supplier themselves.
Single-echelon model with repair ability	SR model	Depot is closed; the bases replenish from the outside supplier and they are also granted with repair ability.

As the items are very critical for military operations, if the considered stock point is granted with repair ability, a technician would always be available for each failed item due to criticality. Therefore, ample capacity assumption would be valid for repair facilities in the models. The ample capacity assumption also holds for the outside supplier; that is, any order placed to the outside supplier will be received exactly in the procurement lead time.

The main goal of the study is to ensure the minimum cost. Therefore the basic research questions are stated as follows:

i. What are the effects of operating a single-echelon system, instead of a two echelon system, on the performance of the inventory system?

ii. What are the effects of acquiring repair ability on the performance of both two and single-echelon models?

iii. What are the effects of the changes in cost, demand and lead time parameters on the system performance?

The study is organized as follows: In Chapter 2, literature survey on inventory control systems that are related to this study is presented and the problem is defined. In Chapter 3, the models are introduced and analyzed. Furthermore the algorithms that are used to optimize the models are introduced. In Chapter 4, we perform a detailed computational analysis and present our findings. Through the computational study the improvements of the models in consideration are assessed under different lead time, unit cost and demand settings. Finally, we conclude in Chapter 5 by summarizing the findings and providing future research directions.

CHAPTER 2

LITERATURE REVIEW & PROBLEM DEFINITION

In this study, we deal with four different inventory control systems, including both single and two-echelon systems with and without repair ability, and we restrict our attention to the related studies in the inventory management literature.

2.1. Literature Review

In Hadley and Whitin (1961), a single-echelon, multi-facility model for low demand items is developed, where, the demand follows a Poisson process and lead times are constant. Hadley and Whitin (1963) is the first to give the exact costs for a single echelon inventory system facing Poisson demand and constant lead times. Das (1977) optimizes a single-echelon base stock model in which customers are willing to wait for backorders for a fixed amount of time. It is stated that relaxation of the time limit does not necessarily reduce the costs. Smith (1977) obtains an approximate solution for a single echelon system that follows a base stock policy for a consumable item. The facilities may have emergency shipments and no backorders are allowed. In this model, how to evaluate and find optimal (S-$1,S$) policies for a system with zero replenishment costs and stochastic lead times is demonstrated. Schmidt and Moinzadeh (1991) consider a single facility that can have emergency shipments with a shorter lead time and higher costs than regular orders. In this paper both backorder and lost sale cases are considered. Schmidt and Nahmias (1985) deal with a model considering a single perishable item. The excess demand is lost. This model becomes similar to the model in Smith (1977) when the life time is assumed to be infinite. Moinzadeh

(1989) analyzes a model with Poisson demand and constant lead times. The excessive demand is either backordered or lost. This model is a generalization of Smith (1977) and Hadley and Whitin (1963) when lead times are assumed to be constant.

Among the papers that consider single-echelon models; some of them have repairable items. Phelps (1962) considers the decision when to repair versus when to procure in a single facility system with multiple items. Schrady (1967) determines the optimal procurement and repair quantities by examining a single-facility model, which allows item condemnation and has deterministic demands. In this model, repair process is considered as a separate model and repair orders are batched in two or more. Allen and D'Eposo (1967) consider a single facility model with *(r,Q)* policy. In this model the items are repairable and approximate steady state distributions are derived for stochastic demand and constant lead times. Simon (1968) derives the exact results for this model.

There are few papers that consider the comparison of two and single-echelon models. Muckstadt and Thomas (1980) compare these two systems for a multi-item situation where demands are Poisson and the lead times are stochastic. The lead times are represented with independent but not identically distributed random variables. They model the two-echelon model under budget constraint and a fill rate constraint is used for the single-echelon model. They conclude that the use of two-echelon model is a better alternative. Haussmann and Erkip (1994) consider a single-item inventory system with Poisson demand where the lead times are constant and all facilities follow *(S-1,S)* policy. It is a more relevant study to ours. It is stated that if all information about the system is available and reflected in the objective function, optimal results for two-echelon systems will always dominate the single-echelon systems; however, when managerial and organizational issues are considered, this may not be always true. The study explores the improvement of a two-echelon system against a single-echelon one. They obtain optimal results subject to a fill rate constraint and stated that single-echelon model can be 3-5% sub-optimal against the two-echelon.

Two-echelon models are the most common models that have been analyzed in the literature of inventory control. Axsater (1990) presents recursive procedures to determine exact costs and discusses the determination of optimal results for a two-echelon system having a single item with independent Poisson demand and constant lead time. The item flow is very similar to that of our two-echelon models. Unsatisfied demands are backordered. The study does not provide any information on steady state distributions. Compared to approximation models like Sherbrooke (1968) and Graves (1985) have presented, the model in Axsater (1990) requires a longer time to obtain an optimal solution. Svoronos and Zipkin (1991) consider a system that has more than two-echelons, with a central depot that procures from an outside supplier which has an ample capacity. Differently than many other models in the literature, this model considers the transportation lead times between the hierarchical facilities. They obtain approximate results for expected inventory and backorder levels and shows that transportation time variances significantly affect the system performance.

Nahmias (1981) state that there are two key components in the design of a multi-echelon system with repairable items: Those are: (i) characterization of the service performance for a given stock level and (ii) searching systematically over possible stock levels and finding the best. In this study, we follow such a procedure to find the optimal base stock levels and policy parameters.

Graves (1996) considers a two-echelon system with Poisson demand and stochastic lead times. He develops a new model with a new allocation scheme called virtual allocation that increases tractability for a two-echelon system with stochastic demand and deterministic lead times. It is stated in the study that having a depot benefit the system in two ways: One is the *joint order effect,* which pools the risk of replenishment lead times and the other is the *depot effect*, which balances the inventory levels at the bases and pools risks over backorder occurrences. One of the findings in the study is that although both the depot and the bases should hold safety stock, most of it should be in the bases. Moinzadeh and Aggarwal (1997) consider a two-echelon model that has a consumable item with Poisson demand and deterministic lead times. In this study, although the unsatisfied demands are backordered; the depot can place emergency orders and uses information of remaining lead time of the inventories on order to determine whether to place an expensive but immediate emergency order or not. Axsater (2001) provides a different cost structure that would be used in decentralized control in two-echelon systems with a depot in the upper echelon and several bases in the lower echelon. In this cost structure the depot also pays a penalty for the backorder occurrences.

There are many studies that consider two-echelon models with repair ability. Among them, METRIC can be regarded as a milestone study. Sherbrooke (1968) presents METRIC for a two-echelon model with repair ability. The system is a conservative one, i.e. all items are repaired and none procured, with compound Poisson demand and stochastic lead times. In this study, he derives the approximate net inventory distribution at each site. The objective of the study is to minimize the backorder levels at the bases subject to a limited budget. METRIC is a pioneering study for multi-echelon inventory systems; especially for the military applications. Simon (1971) considers a model whose environment is similar to METRIC except that lead times are assumed to be deterministic and demand is Poisson. He provides exact analysis to derive the distribution of the backorder levels in each facility in the system. Richards (1976) derives exact expressions for the inventory levels and depot backorder level in an environment similar to METRIC where demands are Poisson, lead times are stochastic and not all of the items are repairable, i.e. item condemnation is included. The results of this study are used by Graves (1985) and more accurate results with respect to METRIC are obtained via two parameter approximation. Graves (1985) considers a two-echelon system in which the depot acts like a central repair facility. The demands are Poisson but the conditional distribution of the outstanding orders is binomial due to first come first serve policy. In the study, exact and approximate methods are presented to determine steady state inventory level distributions at the bases and the depot. The use of binomial disaggregation for specifying the backorder levels in the bases in Graves (1985), encourages us to use the same method for specifying the backorder levels in the bases. This method is also used in the analysis of the two-echelon models in Axsater (2000).

Axsater (1993) reviews and compares his previous model in Axsater (1990) with the model in Graves (1985) and METRIC. Instead of minimizing inventory subject to a service constraint; he considers finding the optimal base stock levels that minimize the holding and backorder costs. Recursive procedures are used to evaluate the cost function and to find the optimal base stock levels.

Moinzadeh and Lee (1986) analyze a multi-echelon system that has a single repairable item and they develop a decision rule to select between the base stock, i.e. (S-1,S), and (r,Q) policies. It is stated that when the demand rate is high, (r,Q) policy is better; however, when the lead times are long and/or there are many bases in multi-echelon system, then (S-1,S) policy is better than the (r,Q) policy. Lee (1987) considers a two-echelon model with repairable items. The demands are Poisson and lead times are constant. The bases replenishes from the depot. Unsatisfied orders are backordered. In case of any stock out at the base, the base can place and emergency order to another base laterally.

Wang et. al. (2000) compares the METRIC model and the model in Svoronos and Zipkin (1991) when the lead times are stochastic and represented by independent but not necessarily identically distributed random variables due to dependency of the lead times on the depot. As a result, he proves that shorter lead times results in shorter delays and states that independent and identically distributed lead time assumption results in greater errors when demand levels are low. Regarding this result, we develop a model that allows non-identical lead times.

Caglar et.al (2004) deal with a two-echelon inventory system with repairable items. In the model, the depot acts like a repair facility as in Graves (1985) and they develop a heuristic to get an approximate solution. The steady state distributions are similarly specified to our two-echelon model with repair ability. Although we deal with an exact model to obtain the optimal policy parameters, the algorithms that we develop in our study are similar to the ones used in Caglar et.al (2004). They first solve the single depot problem for a given base stock level of the depot, and obtain the optimal base stock level that give the minimum cost for each of the bases. This procedure is called H1. Then, according to results obtained, H1 is repeated for the newly given base stock level of the depot to obtain better cost values. The algorithm is iteratively computed until maximum iteration number is reached. In our study, we run a procedure similar to H1 and repeat it until an intuitively given upper-bound is reached; thus we obtain the optimal policy parameters.

The papers that are most relevant to our study are presented in Table 2.

2.2. Problem Definition

The item that is considered in this study has a critical importance in military use and it is very expensive. There is a single domestic supplier. The demand originates from the item failures in the users, which are the military units. The military units are regarded as the end-customers. Currently, a two-echelon inventory system is operated to support the users. There is a depot in the upper echelon and several bases in the lower echelon. User demand occurs at the bases. The bases replenish the items from the depot and the depot replenishes from the outside supplier.

The items are transported via military ring services between the depot and the bases. Any unsatisfied demand is backordered at all stock points. The lead times are assumed to be deterministic. The performance measure of the system is long run average cost. In this study, to reduce the cost of the system, we consider several modifications on the current system. For instance, although the items require high technology, the repair ability, which would enable some portion of the failed items be repaired, can be acquired in the system since the outside supplier is a domestic firm. This repair ability at the stock points would replace the procurement lead times with relatively shorter repair lead time. Moreover the repair costs for each item repaired is presumably lower than the procurement costs. Therefore, the models with repair ability are worth analyzing.

The advantage of having a two-echelon system is that it enables an easy and reliable control over system-wide inventory and reduces the risk of backorder occurrences as the risks of the demand uncertainty are pooled. However, the ring services being used throughout the system are very close to the operation sites, therefore there are noticeable risks for the items and that situation incurs high holding costs for the items in-transit. Moreover, there are many other items, which are not considered in this study, transported via these services; therefore the transportation lead times can be unexpectedly long. This makes the headquarters analyze the option of not operating the depot, along with the ring services, to reduce the risks involved in transportation. This options requires that each stock point in the lower echelon be responsible for its own inventory and procure items from the outside supplier on its own. When this option is considered, the transportation risks are entitled to the outside supplier.

Table 2- Classification of Reviewed Studies

Nu.	Author	Year	Echelon	Item	Repair	Unsatisfied Demand	Lead time	Ordering Policy	Exact Model	Objective	Constraint
1	Sherbrooke	1968	Two	Multi	Yes*	Backordered	Stochastic	(S-1,S)	No	Minimize Backorder	Budget
2	Allen & D'eposo	1968	Single	Single	Yes	Backordered	Deterministic	(r,Q)	No	Derivation of Cost	-
3	Simon	1971	Two	Single	Yes	Backordered	Deterministic	(S-1,S)*	Yes	Steady State Characteristics	-
4	Richards	1976	Single	Single	Yes	Backordered	Stochastic	(r,Q)	Yes	Steady State Characteristics	-
5	Das	1977	Single	Single	No	Partially Lost	Stochastic	(S-1,S)	No	Minimize Cost	Delay Time
6	Smith	1977	Single	Single	No	Lost	Deterministic	(S-1,S)	No	Minimize Cost	-
7	Muckstadt & Thomas	1980	Two*	Multi	No	Backordered*	Stochastic**	(S-1,S)	Yes	Minimize Cost	Fill Rate
8	Graves	1985	Two	Single	Yes	Backordered	Stochastic*	(S-1,S)	No	Steady State Characteristics	-
9	Schmidt & Nahmias	1985	Single*	Single	No	Lost	Deterministic	(S-1,S)	No	Steady State Characteristics	-
10	Moinzadeh & Lee	1986	Two	Single	Yes	Backordered	Deterministic	(r,Q)	No	Minimize Cost	-
11	Lee	1987	Two	Single	Yes	Backordered	Deterministic	(S-1,S)	No	Minimize Cost	Fill Rate
12	Moinzadeh	1989	Single*	Single	No	Partially Lost	Deterministic	(S-1,S)	No	Steady State Characteristics	-

Table 2- Classification of Reviewed Studies (Continued)

Nu.	Author	Year	Echelon	Item	Repair	Unsatisfied Demand	Lead time	Ordering Policy	Exact Model	Objective	Constraint
13	Axsater	1990	Two	Single	No	Backordered	Stochastic*	(S-1,S)	Yes	Minimize Cost	-
14	Svoronos & Zipkin	1991	Two	Single	No	Backordered	Stochastic	(S-1,S)	No	Steady State Characteristics	-
15	Schmidt& Moinzadeh	1991	Single*	Single	No	Partially Lost	Deterministic	(S-1,S)	No	Steady State Characteristics	-
16	Hausmann & Erkip	1994	Two*	Single	No	Backordered*	Deterministic	(S-1,S)	Yes	Minimize Cost	Budget
17	Graves	1996	Two	Single	No	Backordered	Deterministic	(s, S)	No	Minimize Cost	-
18	Moinzadeh & Aggarwal	1997	Two	Single	No	Backordered	Deterministic	(S-1,S)	No	Minimize Cost	-
19	Wang et.al.	2000	Two	Single	Yes	Backordered	Stochastic**	(S-1,S)	Yes	Steady State Characteristics	Fill Rate
20	Axsater	2001	Two**	Single	No	Backordered	Deterministic	(S-1,S)	No	Minimize Cost	-
21	Caglar et.al	2004	Two	Multi	Yes	Backordered	Stochastic	(S-1,S)	No	Minimize Cost	Waiting Time
22	*This Study*	2011	Two*	Single	Yes	Backordered	Deterministic	(S-1,S)	Yes	Minimize Cost	-

Single* - Single facility model ; Two* - Compares two-echelon models with single-echelon ones ; Two** - Decentralized Control.
Yes* - The system is conservative, i.e. all of the failed items are repairable.
Backordered* - Each backordered item requires emegency order.
Stochastic* - Lead time of the depot is deterministic ; Stochastic** - Lead times are independent but not identically distributed.
(S-1,S)* - Depot follows (s,S) policy ;

Therefore, four different alternative inventory control models are built to be analyzed. The first one is the current system which consists of two-echelons, including one depot in the upper and several bases in the lower one. The first alternative is the two-echelon model with repair ability. In this model, the depot acquires the repair ability and this enables the depot repair some portion of the failed items.

The other major modification that should be considered is cancelling the depot out, which would also cancel the use of ring services for the considered item. Therefore, single-echelon model is created. The stock points in this model do not have repair ability and they are entitled to control their own inventory individually which includes supplying the items from the outside supplier. In this model, the responsibility to deliver the items to the stock points is given to the outside supplier. Finally, to see the effect of repair ability acquisition in the bases for single-echelon model, single-echelon model with repair ability is considered.

In each of these models the main goal is to ensure the minimum system wide cost through determining the optimal base stock levels for each stock point. Therefore the basic research questions that will be answered throughout the study can be stated as follows:

i. What are the effects of operating a single-echelon system, instead of a two echelon system, on the performance of the inventory system?

ii. What are the effects of acquiring repair ability on the performance of both two and single-echelon models?

iii. What are the effects of the changes in cost, demand and lead time parameters on the system performance?

CHAPTER 3

MODELS

As expressed in the previous chapters, four alternative inventory systems are considered throughout the study (recall Table 1 in Chapter 1). In both the TE and SE models, failed items are discarded; and a replenishment order is placed to the outside supplier by the depot in the TE model and by the bases individually in the SE model. However in the TR and SR models, the depot and the bases have the ability to repair some of the failed items, respectively. In these models, the demand is satisfied either by repaired or procured items. Among these systems, TE model is in operation currently.

In all models, a single item is considered and inventory is reviewed continuously. A continuous review base stock policy (S-$1,S$) is followed by all facilities; that is, the items are not batched for procurement or repair requests, but they individually create an order as soon as demand occurs. The demand originates from the item failures in the users, which are regarded as the end-customers. When an item fails, the user immediately returns the item to the base; thus user demand occurs at the bases and they are reflected to the depot in two-echelon systems. The demand across bases is independent of each other. Lateral shipments between the bases are not allowed and the unsatisfied demand at any stock point is backordered. The demand that base i faces is a pure Poisson process with rate λ_i. As all demands that occur at the bases are reflected to the depot in two-echelon systems, the demand process of the depot is the superposition of the demand processes of the bases; therefore the depot demand also follows a Poisson process. The outside supplier supplies the bases or the depot in the systems with a single or two-echelon structure, respectively. If the system has repair ability, there is a certain probability that a return is repairable. The inspection for detecting return's reparability is performed by the bases. The inspection is assumed to require no lead time; therefore, in the SR model, each base immediately places an order to the outside supplier or starts the repair process after a demand occurrence. In the TR model, if the item is repairable, it is sent to the depot by the bases to be repaired; and the depot immediately starts the repair process. Both the outside supplier and the repair facilities are assumed to have ample capacity; therefore no queues occur in replenishment or repair processes.

There are three kinds of lead times: (i) the procurement lead time, which is the time required for a facility to procure an item from the outside supplier; (ii) the transportation lead time, which is the time required for items to be transported from the depot to the bases or vice versa (iii) the repair lead time which is the time required for an item to be repaired. The lead times of different bases are not necessarily identical and all lead times are assumed to be deterministic. However, the time required for a base to replenish its stocks from the depot can turn out to be stochastic due to waiting time caused by the backorder occurrences at the depot.

As the decision criterion, a long run average cost that includes the following is considered:

- Inventory holding cost per unit per unit time
    - ✓ for physical stock at all stock points
    - ✓ for in-transit stock between the depot and the bases in two-echelon systems
    - ✓ for items in repair process for the systems where the repair ability exists

- Backorder cost at the bases, which is incurred per unit per unit time

- Procurement cost that is incurred for each item procured from outside supplier.

- Repair cost that is incurred for each item repaired in the systems with repair ability.

The notation used in the thesis context is presented in Table 3:

Table 3- Notation

i	Index for stock points
$I=\{0,1,\ldots, N\}$	Set of stock points, $i=0$ represents the depot in two-echelon systems.
S_i	Base stock level of the stock point $i \in I$
$C_i\,(S_i)$	Expected long-run average cost at stock point i for a given $S_i > 0$ in the SE model.
$C_i^r\,(S_i)$	Expected long-run average cost at stock point i for a given $S_i > 0$ in the SR model.
$C_i\,(S_i\,, S_0)$	Expected long-run average cost at base $i \in I\text{-}\{0\}$ for a given $S_i > 0$ and $S_0 > 0$ in the TE model.
$C_i^r(S_i\,, S_0)$	Expected long-run average cost at base $i \in I\text{-}\{0\}$ for a given $S_i > 0$ and $S_0 > 0$ in the TR model.
$\{D_i(t),\ t\geq 0\}$	Poisson process governing demand at base $i \in I\text{-}\{0\}$.
$\{D_0(t),\ t\geq 0\}$	Poisson process governing demand at depot in two-echelon systems
$D_i(t_1,t_2]$	Demand for base i in the interval $(t_1,t_2]$.
λ_i	Demand rate for the base i.
L_i	Procurement lead time for base i.
R_i	Repair lead time for base i.

Table 3- Notation (Continued)

T_i	Transportation lead time for base $i \in I\text{-}\{0\}$.
$NI_i(t)$	Net inventory for base i at time t.
$NI_i^+(t)$	Positive net inventory; stands for the inventory on hand for stock point $i \in I$ at time t.
$NI_i^-(t)$	Negative net inventory, stands for the outstanding backorders for stock point $i \in I$ at time t.
$IO_i(t)$	Inventory on order of base i at time t.
$IP_i(t)$	Inventory position of base i at time t. $IP_i(t) = NI_i(t) + IO_i(t)$.
c_p	Unit procurement cost.
c_r	Unit repair cost.
h_i	Unit inventory holding cost for base i.
h_t	Unit inventory holding cost for in-transit inventory.
b_i	Unit backorder cost for base i.
ρ	Repair ratio. That is the probability that a failed item is repaired (for both TR and SR models).

Note that $D_i(t_1,t_2]$ is Poisson with rate $\lambda_i\ (t_2\text{-}t_1)$, since $\{D_i(t),\ t\geq 0\}$ is a Poisson process. Although the current system is the TE model; for the sake of simplicity, we start with the simpler single-echelon models. The rest of this chapter is organized as follows: In Section 3.1, the SE model is considered and analyzed. In this model, the depot is not operated anymore. In Section 3.2; the SR model, which is an extension of the model in Section 3.1, where the bases have partial repair ability, is considered. In Section 3.3, the analysis of the TE model is presented. Finally, in Section 3.4, the TR model, which is the extension of the actual model, where only the depot has partial repair ability is considered and analyzed.

3.1. Single Echelon Model

The first model is the SE model with N-bases each of which independently replenishes from an outside supplier. The system can be illustrated as shown in Figure 2. Note that the $(S\text{-}1,S)$ policy is a special case of the (r,Q) policy where $r=S\text{-}1$ and $Q=S$; therefore there are many examples of the SE model developed in this study. In Axsater (2000), models similar to ours can be found with given optimization methods. Moreover the derivations in this model is based on the expected average inventory levels; therefore they are similar to those in Das (1977), Smith (1977) except that they considered a limited waiting time for backorder occurrences and emergency order, respectively.

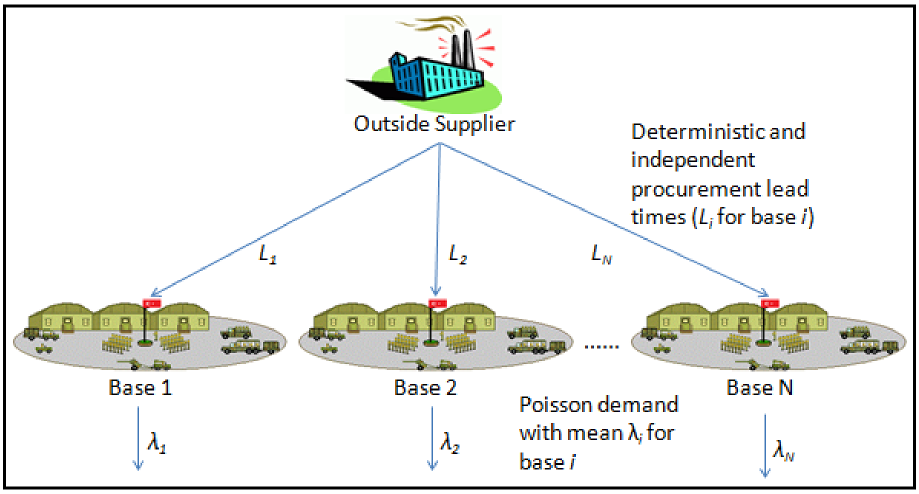

Figure 2- Single-echelon model

When a demand occurs at base i, if there are any items available, the base immediately meets the demand. In the case of a shortage, the base backorders the demand and incurs a unit backorder cost, b_i, per unit time. As demand occurs, the base immediately places an order to the outside supplier and receives it in procurement lead time, L_i, and incurs a unit procurement cost; c_p. Base i incurs a unit holding cost, h_i, for inventory on-hand per unit time. Events triggered by a demand (failure) occurrence are illustrated in Figure 3.

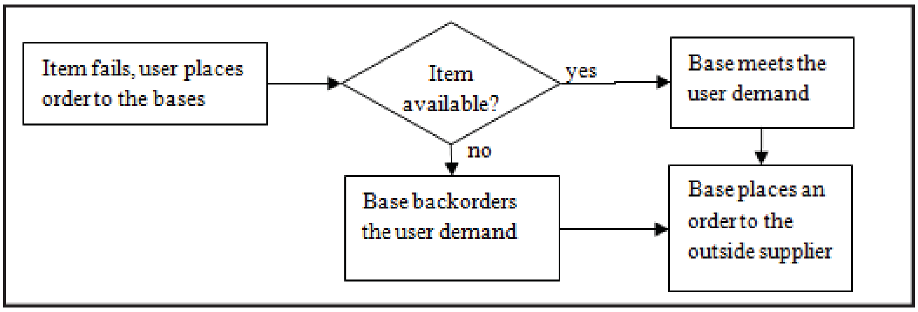

Figure 3- Events for single-echelon model with no repair ability

Total cost of the system is the sum of the costs of individual bases, since bases operate independently. The average cost of base *i*, $C_i(S_i)$ consists of average holding, backordering and procurement costs over time. Therefore, the problem of finding the optimal order-up-to levels can be formulated as:

$$\text{Minimize} \quad \sum_{i=1}^{N} C_i(S_i)$$

s.t. $S_i \in \{0,1,2,...\}$, $\quad \forall i \in \{1,...,N\}$

where; $C_i(S_i) = h_i E[NI_i^+] + b_i E[NI_i^-] + c_p \lambda_i$ (3.1)

The objective function consists of holding, backorder and procurement costs of all bases. The terms $E[NI_i^+]$ and $E[NI_i^-]$ in (3.1) stand for the expected inventory on-hand and expected backorders per unit time in the long-run, respectively. For deriving the long run average cost; we first need to observe the limiting behavior of the net inventory, NI_i.

Recall that inventory position, *IP(t)*, is, by definition, equal to the sum of net inventory, *NI(t)*, and inventory on order, *IO(t)* at time *t*, i.e. *IP(t) = NI(t) + IO(t)*. Since base stock policy, (*S-1,S*) is applied; the inventory position at time *t*, *IP(t)*, of any stock point always equals to its base stock level. Besides, the inventory on order at time t, *IO(t)*, is the orders that are placed but not received by time *t*. Let *L* denote the lead time of a stock point to replenish an item. The orders that are placed before *t-L* will be ready at the base by time *t*; whereas the orders placed in the interval of (*t-L,t]* are in-transit to the base. Therefore, the inventory on order, *IO(t)* is the orders placed in the interval of (*t-L,t]*; that is *IO(t)=D(t)-D(t-L)=D(L)*. As all of the orders are triggered immediately after a demand occurrence, the inventory on order is equal to the demand during the lead time, which is Poisson with rate *λL,* i.e., *D(L)~Poisson (λL).* Hence, the limiting distribution of *IO(t)* is also Poisson with rate *λL*. From now on, we let $D_i(L_i)$ denote the limiting behavior of the inventory on order for base *i*,

Noting that, $IP_i(t) = S_i \;\forall t$, and *IP(t) = NI(t) + IO(t);* we have

$S_i = NI_i + D_i(L_i)$, which results in;

$NI_i = S_i - D_i(L_i)$.

Then, we have,

$$E[NI_i^+] = E[(S_i - D_i(L_i))^+] = \sum_{j=0}^{S_i} (S_i - j) P(D_i(L_i) = j), \qquad (3.2)$$

$$E[NI_i^-] = E[(D_i(L_i) - S_i)^+] = \sum_{j=S_i+1}^{\infty} (j - S_i) P(D_i(L_i) = j), \qquad (3.3)$$

where,

$$P\{D_i(L_i) = j\} = \frac{e^{-\lambda_i L_i}(\lambda_i L_i)^j}{j!}.$$

Therefore; using (3.2) and (3.3) in (3.1), the cost of a single base *i* becomes:

$$C_i(S_i) = h_i \sum_{j=0}^{S_i} (S_i - j) P(D_i(L_i) = j) + b_i \sum_{j=S_i+1}^{\infty} (j - S_i) P(D_i(L_i) = j) + c_p \lambda_i \, .$$

As the bases are independent of each other, the problem is separable to N independent sub-problems, each of which corresponds to a single base problem. To ensure that a local optimal is the global optimal in a single base problem; it would be enough to show convexity of the objective cost function of a single base. As the cost function, $C_i(S_i)$, is not continuous; we use the difference functions to show the convexity of $C_i(S_i)$ with respect to the decision variable S_i.

The 1st order difference function of $C_i(S_i)$ can be stated as follows:

$$\Delta C_i(S_i) = C_i(S_i+1) - C_i(S_i) = \begin{pmatrix} h_i \sum_{j=0}^{S_i+1} (S_i + 1 - j) P(D_i(L_i) = i) + b_i \sum_{j=S_i+2}^{\infty} (j - S_i - 1) P(D_i(L_i) = j) + c_p \lambda_i \\ -h_i \sum_{j=0}^{S_i} (S_i - j) P(D_i(L_i) = j) - b_i \sum_{j=S_i}^{\infty} (j - S_i) P(D_i(L_i) = j) + c_p \lambda_i \end{pmatrix}$$

$$= \begin{bmatrix} h_i \left(\sum_{j=0}^{S_i+1} (S_i + 1 - j) P(D_i(L_i) = i) - \sum_{j=0}^{S_i} (S_i - j) P(D_i(L_i) = j) \right) \\ +b_i \left(\sum_{j=S_i+2}^{\infty} (j - S_i - 1) P(D_i(L_i) = j) - \sum_{j=S_i}^{\infty} (j - S_i) P(D_i(L_i) = j) \right) \end{bmatrix}$$

$$= \begin{bmatrix} h_i \begin{pmatrix} (S_i+1)P(D_i(L_i)=0) + S_i P(D_i(L_i)=1) + \ldots + 1P(D_i(L_i)=S_i) + 0P(D_i(L_i)=S_i+1) \\ -S_i P(D_i(L_i)=0) - (S_i-1)P(D_i(L_i)=1) - \ldots - 0P(D_i(L_i)=S_i) \end{pmatrix} \\ +b_i \begin{pmatrix} 1P(D_i(L_i)=S_i+2) + 2P(D_i(L_i)=S_i+3) + 3P(D_i(L_i)=S_i+4) + 4P(D_i(L_i)=S_i+5) + \ldots \\ -1P(D_i(L_i)=S_i+1) - 2P(D_i(L_i)=S_i+2) - 3P(D_i(L_i)=S_i+3) - 4P(D_i(L_i)=S_i+4) + \ldots \end{pmatrix} \end{bmatrix}$$

$$= \left[h_i \left(\sum_{j=0}^{S_i+1} P(D_i(L_i) = j) \right) + b_i \left(-\sum_{j=S_i+1}^{\infty} P(D_i(L_i) = j) \right) \right] . \qquad (3.4)$$

Similarly;

$$\Delta C_i(S_i+1) = C_i(S_i+2) - C_i(S_i+1) =$$

$$\left[h_i \left(\sum_{j=0}^{S_i+2} P(D_i(L_i)=j) \right) + b_i \left(-\sum_{j=S_i+2}^{\infty} P(D_i(L_i)=j) \right) \right]. \qquad (3.5)$$

Using (3.4) and (3.5), one can show that the 2nd order difference function of $C_i(S_i)$ is:

$$\Delta^2 C_i(S_i) = \Delta C_i(S_i+1) - \Delta C_i(S_i) = \left[\begin{array}{l} \left(h_i \sum_{j=0}^{S_i+2} \frac{(\lambda_i L_i)^j e^{-\lambda_i L_i}}{j!} - b_i \sum_{j=S_i+2}^{\infty} \frac{(\lambda_i L_i)^j e^{-\lambda_i L_i}}{j!} \right) \\ -\left(h_i \sum_{i=0}^{S_j+1} \frac{(\lambda_i L_i)^j e^{-\lambda_i L_i}}{j!} - b_i \sum_{i=S_j+1}^{\infty} \frac{(\lambda_i L_i)^j e^{-\lambda_i L_i}}{j!} \right) \end{array} \right]$$

$$= \left[\begin{array}{l} h_i \left(\begin{array}{l} \frac{(\lambda_i L_i)^0 e^{-\lambda_i L_i}}{0!} + \frac{(\lambda_i L_i)^1 e^{-\lambda_i L_i}}{1!} + \ldots + \frac{(\lambda_i L_i)^{S_i+1} e^{-\lambda_i L_i}}{(S_i+1)!} + \frac{(\lambda_i L_i)^{S_i+2} e^{-\lambda_i L_i}}{(S_i+2)!} \\ -\frac{(\lambda_i L_i)^0 e^{-\lambda_i L_i}}{0!} - \frac{(\lambda_i L_i)^1 e^{-\lambda_i L_i}}{1!} - \ldots - \frac{(\lambda_i L_i)^{S_i+1} e^{-\lambda_i L_i}}{(S_i+1)!} \end{array} \right) \\ -b_i \left(\begin{array}{l} \frac{(\lambda_i L_i)^{S_i+2} e^{-\lambda_i L_i}}{(S_i+2)!} + \frac{(\lambda_i L_i)^{S_i+3} e^{-\lambda_i L_i}}{(S_i+3)!} + \frac{(\lambda_i L_i)^{S_i+4} e^{-\lambda_i L_i}}{(S_i+4)!} + \ldots \\ -\frac{(\lambda_i L_i)^{S_i+1} e^{-\lambda_i L_i}}{(S_i+1)!} - \frac{(\lambda_i L_i)^{S_i+2} e^{-\lambda_i L_i}}{(S_i+2)!} - \frac{(\lambda_i L_i)^{S_i+3} e^{-\lambda_i L_i}}{(S_i+3)!} - \ldots \end{array} \right) \end{array} \right]$$

$$= \left[h_i \left(\frac{(\lambda_i L_i)^{S_i+2} e^{-\lambda_i L_i}}{(S_i+2)!} \right) + b_i \left(\frac{(\lambda_i L_i)^{S_i+1} e^{-\lambda_i L_i}}{(S_i+1)!} \right) \right] > 0 \qquad (3.6)$$

Therefore, we can conclude that $C_i(S_i)$ is convex in $S_i \geq 0$ and the optimal base stock level for base i, S_i^*, can be found from the first order optimality condition as follows:

$$S_i^* = min\{Si \in \{0,1,2,\ldots\} \mid \Delta C_i(S_i) \geq 0\} \qquad (3.7)$$

The optimal base stock levels are found by a local search procedure in which S_i levels are successively increased until optimality condition given in (3.7) is satisfied. As the bases are independent of each other, the sum of the minimum costs for each base ensures the minimum system wide costs.

3.2. Single Echelon Model with Repair Ability

In this case, we again have a single-echelon system consisting of N independent bases; but this time, the bases have a limited ability to repair the returned items. The system is illustrated in Figure 4. In this model, the environment is very similar to the previous model, except the demand distribution due to repair ability. The demand distribution in this model is inspired by the Simon (1971).

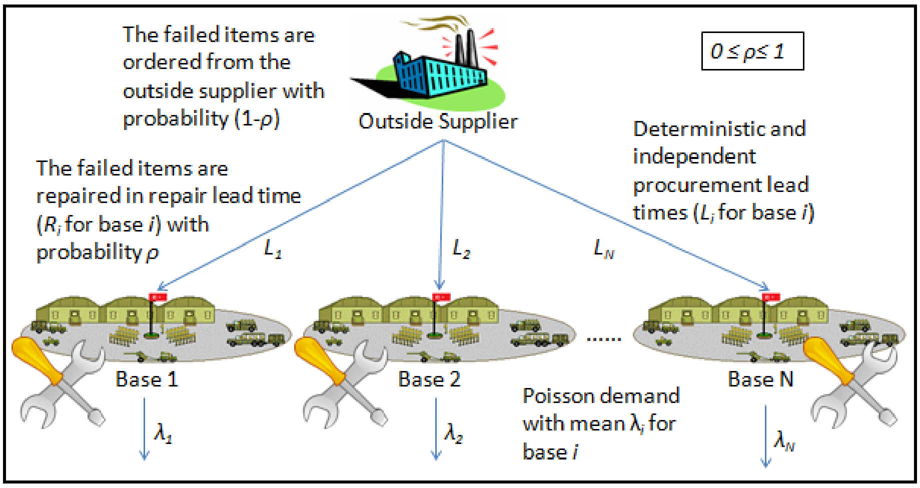

Figure 4- Single-echelon model with repair ability

Each base incurs a holding cost, h_i, for inventory on-hand. When a demand occurs at base i, if there are any available items, the base immediately meets the demand. The inspection, which is necessary to determine whether the failed item can be repaired or not, is performed in bases. With probability ρ, the item is repaired in repair lead time, R_i, incurring a unit cost of repair, c_r. Note that, items in repair incur inventory holding cost as well. If the item is not repairable, then an order is placed to the outside supplier and the order is received in procurement lead time, L_i, incurring a unit cost of procurement, c_p. However, if there is no item available when the demand occurs, the base backorders it incurring a unit backorder cost, b_i, per unit time. Events triggered by a demand (failure) occurrence are illustrated as in Figure 5.

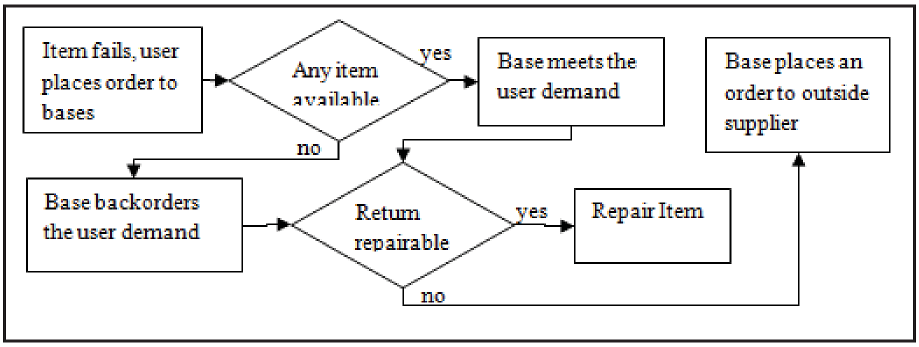

Figure 5- Events for single-echelon model with repair ability

Since the bases are independent, total cost of the system is total of the costs of each base. Therefore, the problem of finding the optimal order-up-to levels can be formulated as:

$$Minimize \ \sum_{i=1}^{N} C_i^r(S_i)$$

$$\text{s.t. } S_i \in \{0,1,2,...\}, \quad \forall i \in \{1,...,N\}$$

$$\text{where; } C_i^r(S_i) = h_i\left[E[NI_i^+] + \rho\lambda_i R_i\right] + b_i E[NI_i^-] + (1-\rho)c_p\lambda_i + \rho c_r\lambda_i \qquad (3.8)$$

The objective function consists of holding costs of items in both stock and repair process; backorder, procurement and repair costs for all bases. Similar to the previous section, we need to observe the limiting behavior of the net inventory levels of the bases for deriving the total average cost. The net inventory at time t,

$NI(t)$, is equal to the difference between the inventory position, $IP(t)$ and the inventory on order, $IO(t)$, i.e. $NI(t) = IP(t) - IO(t)$.

In this setting, it should be noted that the demand process, thus the inventory on order, $IO(t)$ splits into two independent Poisson processes: failed items that are repairable arrive at base i according to a Poisson process with rate $\rho\lambda_i$, and they are repaired in repair lead time, R_i; whereas non-repairable returns at base i are Poisson with rate $(1-\rho)\lambda_i$ and as they are discarded, new ones are procured in procurement lead time, L_i.

The inventory on order for base i equals, $IO_i(t)=D_i(t-R_i,t)+D_i(t-L_i,t)$, where $D_i(t-R_i,t)$ denote the repair orders outstanding at time t and $D_i(t-L,t)$ denote the procurement orders outstanding at time t.

Noting that, $IP_i(t) = S_i \ \forall t$, and $IP(t) = NI(t) + IO(t)$; we have

$S_i = NI_i + D_i(R_i)+D_i(L_i)$

Let, D_i^r be the random variable denoting the inventory on-order in base i for this setting. Then $E[D_i^r] = E[D_i(R_i)] + E[D_i(L_i)]$ for base i. The demand during lead time in steady state for repair and procurement processes, i.e. $E[D_i(R_i)]$ and $E[D_i(L_i)]$ are Poisson with rates $\rho\lambda_i R_i$ and $(1-\rho)\lambda_i L_i$, respectively. Both processes are independent and therefore, inventory on-order for the SR model is the superposition of these two Poisson processes. Therefore $E[D_i^r] = \lambda_i[(1-\rho)L_i+\rho R_i]$, Using the analysis in Section 3.1; for the expected values of both the positive and negative net inventory levels, i.e. $E[NI_i^+]$ and $E[NI_i^-]$ in (3.8), respectively, we have;

$$E[NI_i] = E[IP_i - D_i^r] = E[S_i - D_i^r].$$

Therefore; both the positive and negative expected net inventory can be stated as:

$$E[NI_i^+] = E[(S_i - D_i^r)^+] = \sum_{j=0}^{S_i} (S_i - j) P(D_i^r = j) \qquad (3.9),$$

$$E[NI_i^-] = E[(D_i^r - S_i)^+] = \sum_{j=S_i+1}^{\infty} (j - S_i) P(D_i^r = j) \qquad (3.10),$$

where $P(D_i^r = j) = \dfrac{e^{-\lambda_i(\rho R_i + (1-\rho)L_i)} \left[\lambda_i(\rho R_i + (1-\rho)L_i)\right]^j}{j!}$.

Inserting (3.9) and (3.10) to (3.8); the cost of a single base, say base *i*, becomes:

$$C_i^r(S_i) = \left(h_i \left[\left(\sum_{j=0}^{S_i} (S_i - i) P(D_i^r = j) \right) + \rho R_i \lambda_i \right] + b_i \sum_{j=S_i+1}^{\infty} (j - S_i) P(D_i^r = j) \right) + (1-\rho) c_p \lambda_i + \rho c_r \lambda_i$$

Each base operates independently of the other bases in the system; therefore we have *N* independent sub-problems, as we do in the SE model. Therefore, to ensure that a local optimal is the global optimal in a single base problem; it would be enough to show convexity of the cost function of a single base.

The 1st order difference function of $C_i^r(S_i)$ can be calculated as:

$$\Delta C_i^r(S_i) = C_i^r(S_i + 1) - C_i^r(S_i) =$$

$$\begin{bmatrix} h_i\left[\left(\sum_{j=0}^{S_i}(S_i+1-i)P(D_i^r=j)\right)+\rho R_i\lambda_i\right]+b_i\sum_{j=S_i+1}^{\infty}(j-S_i-1)P(D_i^r=j)+(1-\rho)c_p\lambda_i+\rho c_r\lambda_i \\ -\left(h_i\left[\left(\sum_{j=0}^{S_i}(S_i-i)P(D_i^r=j)\right)+\rho R_i\lambda_i\right]+b_i\sum_{j=S_i+1}^{\infty}(j-S_i)P(D_i^r=j)+(1-\rho)c_p\lambda_i+\rho c_r\lambda_i\right) \end{bmatrix}$$

$$=\begin{bmatrix} h_i\left(\sum_{j=0}^{S_i+1}(S_i+1-j)P(D_i^r=i)-\sum_{j=0}^{S_i}(S_i-j)P(D_i^r=j)\right) \\ +b_i\left(\sum_{j=S_i+2}^{\infty}(j-S_i-1)P(D_i^r=j)-\sum_{j=s_i}^{\infty}(j-S_i)P(D_i^r=j)\right) \end{bmatrix}$$

$$=\begin{bmatrix} h_i\begin{pmatrix}(S_i+1)P(D_i^r=0)+(S_i)P(D_i^r=1)+\ldots+1P(D_i^r=S_i)+0P(D_i^r=S_i+1) \\ -(S_i)P(D_i^r=0)-(S_i-1)P(D_i^r=1)-\ldots-0P(D_i^r=S_i)\end{pmatrix} \\ +b_i\begin{pmatrix}1P(D_i^r=S_i+2)+2P(D_i^r=S_i+3)+3P(D_i^r=S_i+4)+4P(D_i^r=S_i+5)+\ldots \\ -1P(D_i^r=S_i+1)-2P(D_i^r=S_i+2)-3P(D_i^r=S_i+3)-4P(D_i^r=S_i+4)+\ldots\end{pmatrix} \end{bmatrix}$$

$$=\left[h_i\left(\sum_{j=0}^{S_i+1}P(D_i^r=j)\right)+b_i\left(-\sum_{j=S_i+1}^{\infty}P(D_i^r=j)\right)\right]. \qquad (3.11)$$

Similarly;

$$\Delta C_i^r(S_i+1)=C_i^r(S_i+2)-C_i^r(S_i+1)=\left[h_i\left(\sum_{j=0}^{s_i+2}P(D_i^r=j)\right)+b_i\left(-\sum_{j=s_i+2}^{\infty}P(D_i^r=j)\right)\right]$$

(3.12) .

Using (3.11) and (3.12), one can calculate the 2nd order difference as:

$$\Delta^2C_i^r(S_i)=\Delta C_i^r(S_i+1)-\Delta C_i^r(S_i)=\begin{bmatrix}\left(h_i\sum_{j=0}^{S_i+2}\frac{(E[D_i^r])^j e^{-E[D_i^r]}}{j!}-b_i\sum_{j=S_i+2}^{\infty}\frac{(E[D_i^r])^j e^{-E[D_i^r]}}{j!}\right) \\ -\left(h_i\sum_{j=0}^{S_j+1}\frac{(E[D_i^r])^j e^{-E[D_i^r]}}{j!}-b_i\sum_{i=S_j+1}^{\infty}\frac{(E[D_i^r])^j e^{-E[D_i^r]}}{j!}\right)\end{bmatrix}$$

$$
= \left[\begin{array}{l} h_i \left(\begin{array}{l} \frac{(E[D_i^r])^0 e^{-E[D_i^r]}}{0!} + \frac{(E[D_i^r])^1 e^{-E[D_i^r]}}{1!} + \ldots + \frac{(E[D_i^r]]^{S_i+1} e^{-E[D_i^r]}}{(S_i+1)!} + \frac{(E[D_i^r])^{S_i+2} e^{-E[D_i^r]}}{(S_i+2)!} \\ - \frac{(E[D_i^r])^0 e^{-E[D_i^r]}}{0!} - \frac{(E[D_i^r])^1 e^{-E[D_i^r]}}{1!} - \ldots - \frac{(E[D_i^r])^{S_i+1} e^{-E[D_i^r]}}{(S_i+1)!} \end{array} \right) \\ -b_i \left(\begin{array}{l} \frac{(E[D_i^r])^{S_i+2} e^{-E[D_i^r]}}{(S_i+2)!} + \frac{(E[D_i^r])^{S_i+3} e^{-E[D_i^r]}}{(S_i+3)!} + \frac{(E[D_i^r])^{S_i+4} e^{-E[D_i^r]}}{(S_i+4)!} + \ldots \\ - \frac{(E[D_i^r])^{S_i+1} e^{-E[D_i^r]}}{(S_i+1)!} - \frac{(E[D_i^r])^{S_i+2} e^{-E[D_i^r]}}{(S_i+2)!} - \frac{(E[D_i^r])^{S_i+3} e^{-E[D_i^r]}}{(S_i+3)!} - \ldots \end{array} \right) \end{array} \right]
$$

$$
= \left[h_i \left(\frac{(E[D_i^r])^{S_i+2} e^{-E[D_i^r]}}{(S_i+2)!} \right) + b_i \left(\frac{(E[D_i^r])^{S_i+1} e^{-E[D_i^r]}}{(S_i+1)!} \right) \right] > 0. \qquad (3.13)
$$

Therefore, we can conclude that the function $C_i^r(S_i)$ is convex in $S_i \geq 0$ and the optimal base stock level for base i, S_i^*, can be found from the first order optimality condition as follows

$$
S_i^* = min\{S_i \in \{0,1,2,\ldots\} \mid \Delta C_i^r(S_i) \geq 0\}. \qquad (3.14)
$$

The optimal base stock levels are found by local search by increasing S_i values successively for each base until optimality condition given in (3.14) is satisfied. As bases are independent of each other, the sum of the minimum costs of each base ensures the minimum system wide costs.

In the previous model, i.e. when there is no repair ability, a procurement order is given for each demand. However, when repair ability is acquired, some of the items will be replenished by repair process in a shorter lead time than it could be via procurement process. Therefore the risk of backorder occurrence would decrease when the repair orders are in consideration. Thus, one can see that the expected order up to levels, i.e. base stock levels, S_i, in a model with repair ability is expected to be lower than the model without repair ability when the repair lead time, R_0 is lower than the procurement lead time, L_0; and, similarly the unit repair cost, c_p, is lower than the unit procurement cost, c_p.

3.3. Two Echelon Model

In this section, the TE model, which consists of a central depot and *N*-bases is considered. The model is illustrated as in Figure 1 in Chapter 1.

The environment considered in this model is similar to the one considered Moinzadeh and Aggarwal (1997). In Axsater (1990) and Axsater (2000), models very similar to our TE model are presented; however in these models the holding cost for the items in-transit is assumed to be negligible.

The demand occurs in the bases due to item failures in the users. When a demand occurs, the base immediately meets the demand if there are any available items. If there are no items available, then the demand is backordered and unit backorder cost, b_i, is incurred per unit time. As soon as a demand occurs at base *i*, it places an order to the depot. If there are any items available at the depot, demand of the base is immediately met and the item is transported to the base in transportation lead time, T_i. In-transit items in between the depot and the bases incur a unit holding cost, h_t, per unit time by the depot which is ignored by most of the papers in literature like Muckstadt and Thomas (1980), Moinzadeh and Aggarwal (1997), and Caglar et.al. (2004).

Since each demand occurrence creates an immediate order due to (S-$1,S$) policy, the demand faced by the depot is the superposition of the Poisson demand faced by the bases. Therefore the sum of the base demands forms the depot demand which is also governed by a Poisson process with rate $\lambda_0 = \sum_{i=1}^{N} \lambda_i$. The depot and bases incur a unit holding cost for inventory on-hand, h_0 and h_i for base i, respectively. In case of a shortage, the depot backorders the base demand.

As soon as a demand from a base is received, regardless of the inventory level, the depot places an order to outside supplier and receives it in procurement lead time, L_0, and incurs a unit procurement cost, c_p. When the item arrives at the depot, if there is any outstanding order, it is immediately sent to base; otherwise the item is kept in stock. As Axsater (2000, p.168) states; there is no backorder cost for the depot. It is assumed that the costs incurred by the backorders at the depot is reflected to the backorder costs in the bases, as base backorders are caused by the delays at the depot due to backorder occurrences. Replenishment of the items in each system element is conducted according to first in first out policy. Events triggered by a demand occurrence at a base are illustrated as in Figure 6.

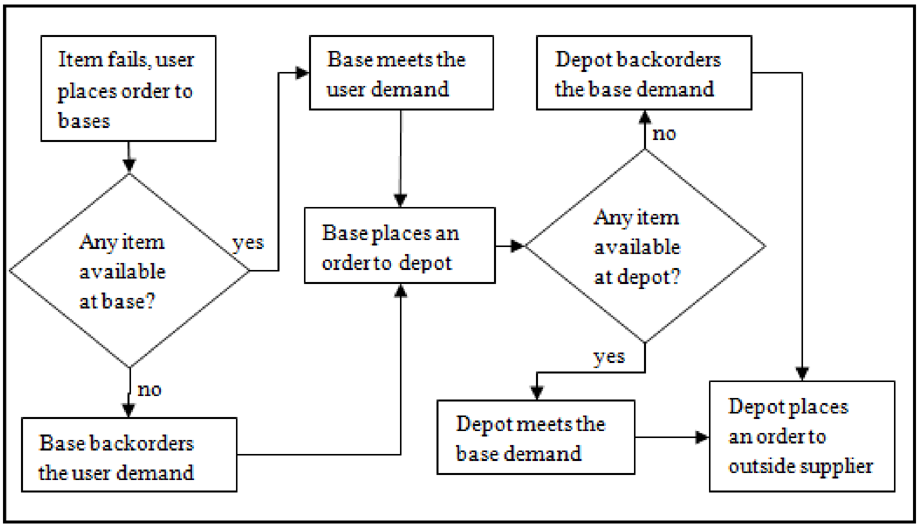

Figure 6- Events for two-echelon model with no repair ability

In this setting, we have procurement process at the depot and the transportation process for the items to reach from the depot to the base. The associated lead time for the depot is the procurement lead time, L_0; whereas the associated lead time for base i is the transportation lead time, T_i. Thus, the demand during lead time in steady state for procurement processes at the depot, $D_0(L_0)$, and transportation process for base i, $D_i(T_i)$ are Poisson with rates $\lambda_0 L_0$ and $\lambda_i T_i$, respectively.

Total cost of this setting is the sum of the costs at the depot and the bases. The cost of the depot, $C_0(S_0)$, consists of holding costs of the items in stock and items in-transit to the bases, and procurement cost. The cost of base i, $C_i(S_0,S_i)$, consists of holding cost of the items in stock and backorder costs. Our goal is to minimize total cost by specifying the optimal base stock levels of the depot and the bases. The problem can be modeled as:

$$Minimize\ C = C_0(S_0) + \sum_{i=1}^{N} C_i(S_0, S_i) \qquad (3.15)$$

s.t. $S_i \in \{0,1,2,...\}$

$S_0 > 0$, $\qquad \forall i \in \{0,1,...,N\}$

where;

$$C_0(S_0) = h_0 E[NI_0^{\,+}] + c_p \lambda_0 + h_t \sum_{i=1}^{N} \lambda_i T_i \qquad (3.16)$$

and

$$C_i(S_0, S_i) = h_i E[NI_i^{\,+}] - b_i E[NI_i^{\,-}]. \qquad (3.17)$$

We start the analysis of the model by determining the limiting behavior of NI_0 in (3.16), which stands for the net inventory at the depot. Similar to the analysis in Section 3.1, the positive net inventory for the depot becomes

$$E[NI_0^+] = E[S_0 - D_0(L_0)] = \sum_{m=0}^{S_0} (S_0 - m) P(D_0(L_0) = m) \qquad (3.18)$$

where $P(D_0(L_0)=m)=\dfrac{(\lambda_0 L_0)^m e^{-\lambda_0 L_0}}{m!}$

Plugging (3.18) to (3.16) the cost of the depot can be stated as

$$C_0(S_0)=h_0\sum_{m=0}^{S_0}(S_0-m)\frac{(\lambda_0 L_0)^m e^{-\lambda_0 L_0}}{m!}+c_p\lambda_0+h_t\sum_{i=1}^{N}\lambda_i T_i \qquad (3.19).$$

The cost at the depot, (3.19), is a function of depot's own ordering policy. Therefore, it is only dependent on the base stock level of the depot; i.e., S_0.

We next consider the limiting behavior of net inventory at the base i, NI_i. It should be noted that the analysis differs from the one in Section 3.1 since the replenishments of the bases are dependent also on the backorders at the depot. Recall that we have $NI_i(t) = IP_i(t) - IO_i(t)$, and $IP_i(t) = S_i$ for all t. However, $IO_i(t)$ is not equal to the demand during the transportation lead time since the depot may not ship an item immediately when the base places an order. Hence, $IO_i(t)$, is the sum of the number of backordered items at the depot that belongs to base i at time $t-T_i$ and the demand that occurred in the base in interval of $(t-T_i, t]$. Let's define $B_i(t-T_i)$ as the random variable that denotes the backorders at the depot at time $t-T_i$ that belongs to base i; where $\sum_{i=1}^{N} B_i(t-T_i)=NI_0^-(t-T_i)$. Then, it is appropriate to state the inventory on order of base i as;

$IO_i(t)= B_i(t-T_i)+ D_i(t-T_i,t)$.

Hence, the net inventory for base i becomes,

$NI_i(t) = IP_i(t)- B_i(t-T_i) - D_i(T_i)$.

Thus, the limiting behavior of the net inventory of base i can be observed as;

$NI_i = S_i - B_i - D_i(T_i)$.

To obtain the $B_i(t\text{-}T_i)$ values, binomial disaggregation Graves (1985) is used. In binomial distribution, the new backorder occurrences at the depot are caused by base *i* with probability λ_i/λ_0 due to Poisson demand, as the demands are filled according to first-in-first-out policy; hence, as it is suggested by Axsater (2000, p.164), the conditional distribution of the backorders at the bases is binomial, given the total number of backorders at the depot. Then, we have

$$P(B_i = k) = \sum_{a=k}^{\infty} P(NI_0^- = a)\binom{a}{k}\left(\frac{\lambda_i}{\lambda_0}\right)^k\left(1-\frac{\lambda_i}{\lambda_0}\right)^{a-k}, \qquad (3.20) \text{ for k > 0}$$

and

$$P(B_i = 0) = 1 - \sum_{k=1}^{\infty} P(B_i = k).$$

The probability mass function of negative net inventory at the depot can be expressed as;

$$P(NI_0^- = a) = P(NI_0 = -a) = P(D_0(L_0) = S_0 + a) = \frac{(\lambda_0 L_0)^{S_0+a} e^{-\lambda_0 L_0}}{(S_0+a)!}. \qquad (3.21)$$

Hence, plugging (3.21) into (3.20), we have

$$P(B_i = k) = \sum_{a=k}^{\infty} \frac{(\lambda_0 L_0)^{S_0+a} e^{-\lambda_0 L_0}}{(S_0+a)!}\binom{a}{k}\left(\frac{\lambda_i}{\lambda_0}\right)^k\left(1-\frac{\lambda_i}{\lambda_0}\right)^{a-k}. \qquad (3.22)$$

The expected on-hand inventory and backorder levels for the bases are

$$E[NI_i^+] = \sum_{j=0}^{S_i} jP(NI_i = j) \qquad (3.23),$$

and

$$E[NI_i^-] = \sum_{j=-\infty}^{-1} jP(NI_i = j) \quad (3.24),$$

respectively.

In (3.23) and (3.24), $P(NI_i = j)$ denotes the probability mass function of the net inventory level at base *i* in the steady state. Recalling that net inventory level can be stated as $NI_i = S_i - B_i - D_i(T_i)$, we condition on B_i to obtain $P(NI_i = j)$:

$P(NI_i = j) = P(S_i - B_i - D_i(T_i) = j) = P(D_i(T_i) = S_i - j - k)$, and

$$P(D_i(T_i) = S_i - j - k) = \sum_{k=0}^{S_i - j} P(B_i = k) \frac{(\lambda_i T_i)^{S_i - j - k} e^{-\lambda_i T_i}}{(S_i - j - k)!}. \qquad (3.25)$$

Plugging (3.22) in (3.25) we have

$$P(NI_i = j) = \sum_{k=0}^{S_i - j} \sum_{a=k}^{\infty} \frac{(\lambda_0 L_0)^{S_0 + a} e^{-\lambda_0 L_0}}{(S_0 + a)!} \binom{a}{k} \left(\frac{\lambda_i}{\lambda_0}\right)^k \left(1 - \frac{\lambda_i}{\lambda_0}\right)^{a-k} \frac{(\lambda_i T_i)^{S_i - j - k} e^{-\lambda_i T_i}}{(S_i - j - k)!} \quad . (3.26)$$

By plugging (3.26) to (3.23) and (3.24), the expected net inventory levels of the bases can be expressed as

$$E[NI_i^+] = \sum_{j=0}^{S_i} j \sum_{k=0}^{S_i - j} \frac{(\lambda_i T_i)^{S_i - j - k} e^{-\lambda_i T_i}}{(S_i - j - k)!} \sum_{a=k}^{\infty} \frac{(\lambda_0 L_0)^{S_0 + a} e^{-\lambda_0 L_0}}{(S_0 + a)!} \binom{a}{k} \left(\frac{\lambda_i}{\lambda_0}\right)^k \left(1 - \frac{\lambda_i}{\lambda_0}\right)^{a-k} \quad (3.27)$$

and

$$E[NI_i^-] = \sum_{j=-\infty}^{-1} j \sum_{k=0}^{S_i - j} \frac{(\lambda_i T_i)^{S_i - j - k} e^{-\lambda_i T_i}}{(S_i - j - k)!} \sum_{a=k}^{\infty} \frac{(\lambda_0 L_0)^{S_0 + a} e^{-\lambda_0 L_0}}{(S_0 + a)!} \binom{a}{k} \left(\frac{\lambda_i}{\lambda_0}\right)^k \left(1 - \frac{\lambda_i}{\lambda_0}\right)^{a-k} \quad (3.28),$$

respectively.

Finally, plugging (3.27) and (3.28) to (3.17) the cost of the base i can be expressed as

$$C_i(S_0,S_i)=\begin{pmatrix} h_i\sum_{j=0}^{s_i} j\sum_{k=0}^{s_i-j}\frac{(\lambda_i T_i)^{s_i-j-k}e^{-\lambda_i T_i}}{(s_i-j-k)!}\sum_{a=k}^{\infty}\frac{(\lambda_0 L_0)^{S_0+a}e^{-\lambda_0 L_0}}{(S_0+a)!}\binom{a}{k}\left(\frac{\lambda_i}{\lambda_0}\right)^k\left(1-\frac{\lambda_i}{\lambda_0}\right)^{a-k} \\ -b_i\sum_{j=-\infty}^{-1} j\sum_{k=0}^{s_i-j}\frac{(\lambda_i T_i)^{s_i-j-k}*e^{-\lambda_i T_i}}{(s_i-j-k)!}\sum_{a=k}^{\infty}\frac{(\lambda_0 L_0)^{S_0+a}e^{-\lambda_0 L_0}}{(S_0+a)!}\binom{a}{k}\left(\frac{\lambda_i}{\lambda_0}\right)^k\left(1-\frac{\lambda_i}{\lambda_0}\right)^{a-k}\end{pmatrix}$$

(3.29).

We now describe how to obtain the optimal base stock levels for the depot and the bases in order to minimize the system-wide costs. We first check whether the cost function is convex or not. First, we show that the cost function (3.15) is convex in S_i for a given value of S_0 is shown. For this purpose, we define:

$$X_k=\sum_{a=k}^{\infty}\frac{(\lambda_0 L_0)^{S_0+a}e^{-\lambda_0 L_0}}{(S_0+a)!}\binom{a}{k}\left(\frac{\lambda_i}{\lambda_0}\right)^k\left(1-\frac{\lambda_i}{\lambda_0}\right)^{a-k} \qquad (3.30)$$

$$Y_{S_i-j-k}=\frac{(\lambda_i T_i)^{S_i-j-k}e^{-\lambda_i T_i}}{(S_i-j-k)!} \qquad (3.31)$$

When (3.30) and (3.31) are plugged into (3.29), the cost of the base i becomes

$$C_i(S_0,S_i)=\left(h_i\sum_{j=0}^{S_i} j\sum_{k=0}^{S_i-j}Y_{S_i-j-k}X_k-b_i\sum_{j=-\infty}^{-1} j\sum_{k=0}^{S_i-j}Y_{S_i-j-k}X_k\right) \qquad (3.32).$$

To investigate convexity of (3.32), we need 1st order difference function, which is

$$\Delta C_i(S_0,S_i)=C_i(S_0,S_i+1)-C_i(S_0,S_i)$$

$$=\left[\begin{array}{l}\left(h_i\sum_{j=0}^{S_i+1} j\sum_{k=0}^{S_i+1-j}Y_{S_i+1-j-k}X_k-b_i\sum_{j=-\infty}^{-1} j\sum_{k=0}^{S_i+1-j}Y_{S_i+1-j-k}X_k\right)\\ -\left(h_i\sum_{j=0}^{S_i} j\sum_{k=0}^{S_i-j}Y_{S_i-j-k}X_k-b_i\sum_{j=-\infty}^{-1} j\sum_{k=0}^{S_i-j}Y_{S_i-j-k}X_k\right)\end{array}\right]$$

$$
=\left[\begin{array}{l}
h_i\left(\begin{array}{l}
0\sum_{k=0}^{S_i+1}Y_{S_i+1-k}X_k+1\sum_{k=0}^{S_i}Y_{S_i-k}X_k+\ldots+S_i\sum_{k=0}^{1}Y_{1-k}X_k+(S_i+1)\sum_{k=0}^{0}Y_{0-k}X_k\\
-0\sum_{k=0}^{S_i}Y_{S_i-k}X_k-1\sum_{k=0}^{S_i-1}Y_{S_i-1-k}X_k-\ldots-(S_i-1)\sum_{k=0}^{1}Y_{1-k}X_k-S_i\sum_{k=0}^{0}Y_{0-k}X_k
\end{array}\right)\\
-b_i\left(\begin{array}{l}
-1\sum_{k=0}^{S_i+2}Y_{S_i+2-k}X_k-2\sum_{k=0}^{S_i+3}Y_{S_i+3-k}X_k+3\sum_{k=0}^{S_i+4}Y_{S_i+4-k}X_k+\ldots\\
-\left(-1\sum_{k=0}^{S_i+1}Y_{S_i+1-k}X_k-2\sum_{k=0}^{S_i+2}Y_{S_i+2-k}X_k-3\sum_{k=0}^{S_i+3}Y_{S_i+3-k}X_k+\ldots\right)
\end{array}\right)
\end{array}\right]
$$

$$
=\left(h_i\sum_{j=0}^{S_i+1}\sum_{k=0}^{S_i+1-j}Y_{S_i+1-j-k}X_k-b_i\sum_{j=1}^{\infty}\sum_{k=0}^{S_i+j}Y_{S_i+j-k}X_k\right)
$$

Similarly:

$$
\Delta C_i(S_0,S_i+1)=C_i(S_0,S_i+2)-C_i(S_0,S_i+1)
$$

$$
=\left(h_i\sum_{j=0}^{S_i+2}\sum_{k=0}^{S_i+2-j}Y_{S_i+2-j-k}X_k-b_i\sum_{j=1}^{\infty}\sum_{k=0}^{S_i+1+j}Y_{S_i+1+j-k}X_k\right)
$$

Then, the 2nd order difference is:

$$
\Delta^2C_i(S_0,S_i)=\Delta C_i(S_0,S_i+1)-\Delta C_i(S_0,S_i)
$$

$$
=\left(\begin{array}{l}
h_i\sum_{j=0}^{S_i+2}\sum_{k=0}^{S_i+2-j}Y_{S_i+2-j-k}X_k-b_i\sum_{j=1}^{\infty}\sum_{k=0}^{S_i+1+j}Y_{S_i+1+j-k}X_k\\
-\left(h_i\sum_{j=0}^{S_i+1}\sum_{k=0}^{S_i+1-j}Y_{S_i+1-j-k}X_k-b_i\sum_{j=1}^{\infty}\sum_{k=0}^{S_i+j}Y_{S_i+j-k}X_k\right)
\end{array}\right)
$$

$$
=\left[\begin{array}{l}
h_i\left(\begin{array}{l}
\sum_{k=0}^{S_i+2}Y_{S_i+2-k}X_k+\sum_{k=0}^{S_i+1}Y_{S_i+1-k}X_k+\sum_{k=0}^{S_i}Y_{S_i-k}X_k+\ldots+\sum_{k=0}^{1}Y_{1-k}X_k+\sum_{k=0}^{0}Y_{0-k}X_k\\
-\sum_{k=0}^{S_i+1}Y_{S_i+1-k}X_k-\sum_{k=0}^{S_i}Y_{S_i-k}X_k-\ldots-\sum_{k=0}^{1}Y_{1-k}X_k-\sum_{k=0}^{0}Y_{0-k}X_k
\end{array}\right)\\
-b_i\left(\begin{array}{l}
\sum_{k=0}^{S_i+2}Y_{S_i+2-k}X_k+\sum_{k=0}^{S_i+3}Y_{S_i+3-k}X_k+\sum_{k=0}^{S_i+4}Y_{S_i+4-k}X_k+\ldots\\
-\sum_{k=0}^{S_i+1}Y_{S_i+1-k}X_k-\sum_{k=0}^{S_i+2}Y_{S_i+2-k}X_k-\sum_{k=0}^{S_i+3}Y_{S_i+3-k}X_k+\ldots
\end{array}\right)
\end{array}\right]
$$

$$= \left(h_i \sum_{k=0}^{S_i+2} Y_{S_i+2-k} X_k + b_i \sum_{k=0}^{S_i+1} Y_{S_i+1-k} X_k \right)$$

As Y_{S_i+2-k} , Y_{S_i+1-k} and X_k are non-negative, and we have positive cost parameters; value of $\Delta^2 C_i(S_0, S_i)$ is also non-negative and therefore $C_i(S_0, S_i)$ is convex with respect to $S_i \geq 0$, for a given $S_0 \geq 0$ value.

However, Axsater (2000, p.267) states that the total cost, (3.15), is not necessarily convex with respect to order-up-to level of the depot. This is shown with a counter-example. In this counter-example, as we are searching whether the total cost function is convex in S_0, for given values of S_i's; the parameters are set as given in Table 4:

Table 4- Parameters for the TE model in the counter-example.

Parameters	
Base Stocks (S)	$S_1 = 5$, $S_2 = 5$
Demand (λ)	$\lambda_0 = 16$, $\lambda_1 = 8$, $\lambda_2 = 8$
Lead Time (L)	$L_0 = 2$, $L_1 = 2$, $L_2 = 2$
Costs	$h_0 = 1$, $h_1 = 1$, $h_2 = 1$, $h_t = 1$, $b_1 = 5$, $b_2 = 5$, $c_p = 3$

In Figure 7. $C_0(S_0)+ \Sigma C_i(S_0,S_i)$ values for different S_0 values are plotted. As it can be seen in the figure $C_0(S_0)+ \Sigma C_i(S_0,S_i)$ is not convex in S_0 for given S_i values.

The plotted graph means that the local optimum is not necessarily the global optimum. In this model, among the costs components that form the cost, only holding and backorder costs are dependent on the base stock level of the depot, S_0.

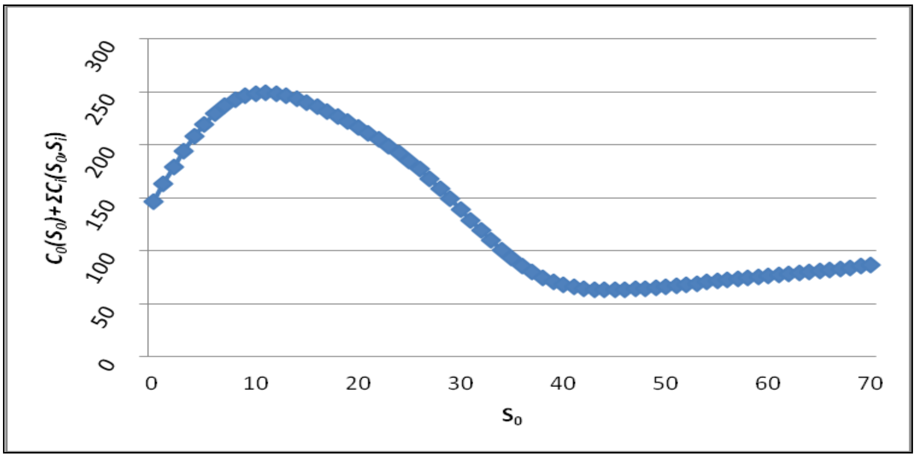

Figure 7- Counter example for convexity of "$C_0(S_0) + \Sigma C_i(S_0,S_i)$"

The procurement and in-transit inventory holding costs are related to demand solely. As the base stock level of depot, S_0, increases, the probability of a backorder occurrence decreases and becomes zero for sufficiently large S_0; this results in no backorder cost for the bases in the long run as the base stock level of the depot increases. However, as S_0 increases, eventually the inventory on hand increases; therefore it is expected that as S_0 increases, the holding cost components in total cost also increases. Regarding this, it can be said that as base stock infinitely increases the total cost also increases to infinity; therefore using that fact we can be sure that there is an upper-bound beyond which we can find a global optimum for S_0.

At this point, the problem is divided into two parts. One is the base problem which finds the optimal base stock levels for the bases, S_i^*; and the other deals with finding the optimal base stock level of the depot, S_0^*. An iterative procedure is used to deliver the optimal results for the problem. This procedure involves in solving the base problem for a given value of S_0. As the objective function is convex in $S_i > 0$; the optimal base stock level for base i, S_i^*, can again be found from the first order optimality condition, i.e., $S_i^* = \mathit{min}\ \{S_i \in \{0,1,2,...\}|$

$\Delta C_i(S_i) \geq 0\}$ for a given value of S_0. This procedure is repeated for all S_0 $\in \{0,1,2,...\}$ until an intuitively assigned upper-bound and the optimal order-up-to level of the depot, S_0^*, which ensures the minimum objective cost function value, is found by evaluation of all order-up-to levels, S_0, with a local search procedure. During the analysis, if the assigned upper-bound for S_0 ever becomes binding, the upper-bound is extended to a larger value and the local search procedure is restarted.

3.4. Two Echelon Model with Repair Ability

The last case is a two-echelon system consisting of a depot and *N* independent bases, but this time the depot has a limited ability to repair the returned items. The system is illustrated as shown in Figure 8. The model is an extension of the model presented in Section 3.3 where the depot acquires the repair ability and becomes a central repair facility. The main difference between the two models is the demand distributions. The demand distribution used in this model is similar to that is used in Simon (1971). Furthermore, depot's being the central repair facility is very similar to the papers like Graves (1985), Wang et.al.(2000) and Caglar et.al. (2004).

In this case, the events and the cost incurrence are almost the same as the TE model. The difference is that some of the items are repairable with probability ρ. The inspection, which is necessary to determine whether a failed item can be repaired or not, is performed in the bases. If the item is repairable, then base *i* returns the failed item to the depot in transportation lead time, T_i; incurring a holding cost for items in-transit. The failed item is repaired at the depot in repair lead time, R_0, incurring a unit cost of repair, c_r. The items in repair also incur a holding cost of h_0 at the depot. After the item is repaired, it is returned to the base that it belongs in transportation lead time T_i. This time the in-transit inventory holding cost, h_t, is incurred by the depot. Note that, the time required for satisfying a repair order from a base, is the sum of repair and transportation lead times, i.e. R_0+T_i; whereas it is L_0 for a procurement order. The events are illustrated in Figure 9.

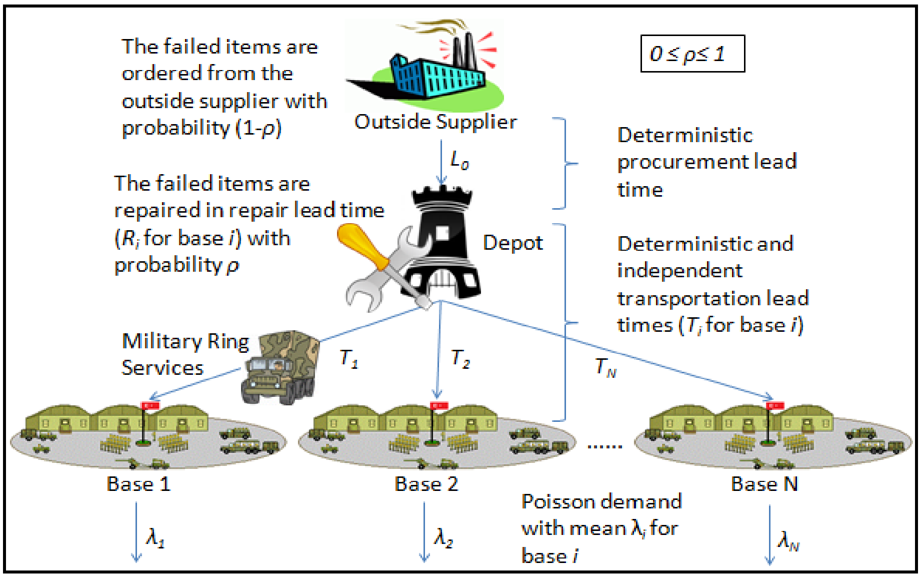

Figure 8- Two-echelon model with repair ability

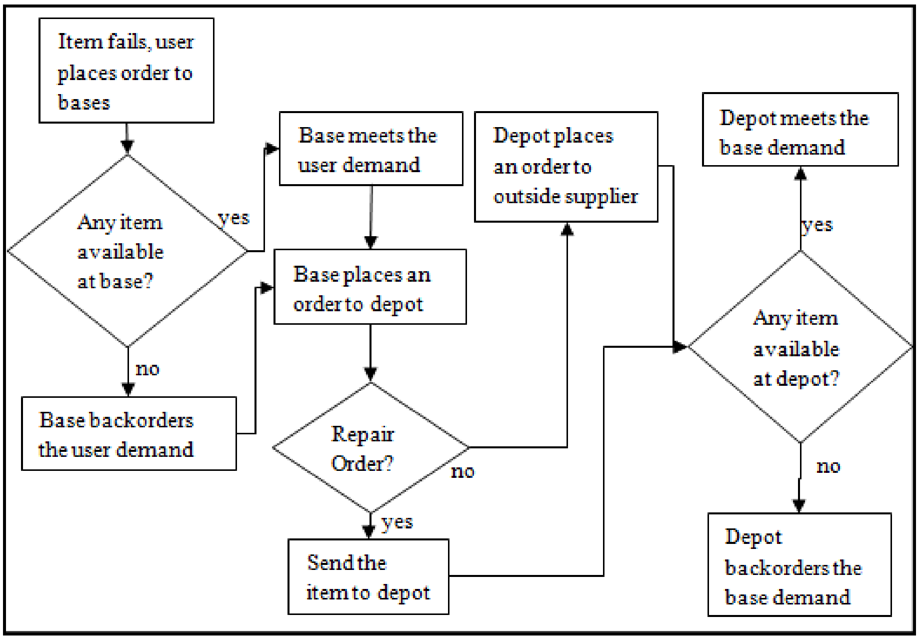

Figure 9- Events for two-echelon model with repair ability

The failed items can be repaired at the depot with probability ρ. The cost of the depot, $C_0^r(S_0)$ consists of procurement cost, repair cost and holding costs of both the items in stock, repaired or being repaired and items in-transit; whereas the cost of a single base *i*, $C_i^r(S_0, S_i)$ consists of backorder costs and holding cost of the items both in stock and in-transit. Our goal is to minimize total cost by specifying the optimal policy for base stock levels of both the depot and the bases; therefore the problem can be modeled as:

$$\textit{Minimize } C_r = C_0^{\,r}(S_0) + \sum_{i=1}^{N} C_i^{\,r}(S_0, S_i) \qquad (3.33)$$

s.t. $S_i \in \{0,1,2,...\}$

$S_0 > 0$, $\qquad \forall i \in \{0,1,...,N\}$

Where;

$$C_0^{\,r}(S_0) = h_0 E[NI_0^{\,+}] + c_p \lambda_0 (1-\rho) + c_r \lambda_0 \rho + h_t \sum_{i=1}^{N} \lambda_i T_i + h_0 \sum_{i=1}^{N} \rho \lambda_i R_0 \quad (3.34)$$

And $\quad C_i^{\,r}(S_0, S_i) = h_i E[NI_i^{\,+}] - b_i E[NI_i^{\,-}] + h_t \rho \lambda_i L_i \qquad (3.35)$

In the system, the depot demand is again the superposition of the base demands as it is in Section 3.3; therefore it is governed by a Poisson process with rate λ_0. However, it can be stated that as some of the items can be repaired in the depot with probability ρ, the Poisson demand of the depot is split into two kinds of demands; that is, a demand that occurs at the depot is either a repair or a procurement demand which are also Poisson. The rate of the repair demand is $\rho\lambda_0$ and the rate of the consumable demand is $(1-\rho)\lambda_0$.

Let the inventory on order for the depot equals, $IO_0(t)=D_0(t\text{-}R_0,t)+D_0(t\text{-}L_0,t)$, where $D_0(t\text{-}R_0,t)$ denote the repair orders outstanding at time t and $D_0(t\text{-}L_0,t)$ denote the procurement orders outstanding at time t.

Noting that, $IP_0(t) = S_0 \; \forall t$, and $IP_0(t) = NI_0(t) + IO_0(t)$; we have

$S_0 = E[NI_0] + E[D_0(R_0)+D_0(L_0)]$.

Let D_0^r be the random variable denoting the limiting behavior of the inventory on-order in depot for this setting, then $E[D_0^r] = E[D_0(L_0)] + E[D_0(R_0)]$. The expected demand during lead time at steady state for procurement process, $E[D_0(L_0)]$, and the repair process at the depot, $E[D_0(R_0)]$, is $(1\text{-}\rho)\lambda_0 L_0$ and $\rho\lambda_0(R_0+T_i)$, respectively; Both procurement and repair in the depot are independent Poisson processes. Therefore, the inventory on-order for the depot is superposition of these two Poisson processes. Therefore;

$E[D_0^r] = \lambda_0[(1\text{-}\rho)L_0+\rho(R_0+T_i)]$ for the depot.

The net inventory level for the depot, $E[NI_0^+]$, is derived regarding analysis in Section 3.1.

$$E[NI_0^+] = \sum_{m=0}^{S_0}(S_0 - m)P(D_0^r = m) \qquad (3.36)$$

where $P(D_0^r = m) = \dfrac{(\lambda_0(\rho(R_0+T_i)+(1-\rho)L_0))^m e^{-\lambda_0(\rho(R_0+T_i)+(1-\rho)L_0)}}{m!}$.

Therefore, plugging (3.36) into (3.34), the cost of depot can be stated as:

$$C_0^{\,r}(S_0) = h_0 \sum_{m=0}^{S_0} (S_0 - m) \frac{(\lambda_0(\rho(R_0 + T_i) + (1-\rho)L_0))^m e^{-\lambda_0(\rho(R_0+T_i)+(1-\rho)L_0)}}{m!} + C_p \lambda_0 + h_0 \sum_{i=1}^{N} \lambda_i T_i . \tag{3.37}$$

Next, the limiting behavior of the net inventory level of base *i* is considered. The analysis is the same as in Section 3.3. Note that the only difference is the inventory on-order levels of the depot. The expected inventory on-order for depot in this setting is $E[D_0^r] = \lambda_0[(1-\rho)L_0+\rho(R_0+T_i)]$; whereas it is $E[D_0(L_0)] = \lambda_0 L_0$ in the TE model. This difference has an effect only on the limiting behavior, as we condition the optimal base stock level of a base to the base stock level of the depot. Therefore, using the analysis in Section 3.3, the expected positive and negative net inventory levels of base *i* can be stated as:

$$E[NI_i^+] = \sum_{j=0}^{S_i} j \sum_{k=0}^{S_i-j} \frac{(\lambda_i T_i)^{S_i-j-k} e^{-\lambda_i T_i}}{(S_i - j - k)!} \sum_{a=k}^{\infty} \frac{(E[D_0^r])^{S_0+a} e^{-E[D_0^r]}}{(S_0+a)!} \binom{a}{k} \left(\frac{\lambda_i}{\lambda_0}\right)^k \left(1 - \frac{\lambda_i}{\lambda_0}\right)^{a-k}$$

(3.38) and

$$E[NI_i^-] = \sum_{j=-\infty}^{-1} j \sum_{k=0}^{S_i-j} \frac{(\lambda_i T_i)^{S_i-j-k} e^{-\lambda_i T_i}}{(S_i - j - k)!} \sum_{a=k}^{\infty} \frac{(E[D_0^r])^{S_0+a} e^{-E[D_0^r]}}{(S_0+a)!} \binom{a}{k} \left(\frac{\lambda_i}{\lambda_0}\right)^k \left(1 - \frac{\lambda_i}{\lambda_0}\right)^{a-k}$$

(3.39), respectively.

Finally, plugging (3.38) and (3.39) into (3.35), the cost function of base *i* becomes

$$C_i(S_0, S_i) = \begin{pmatrix} h_i \displaystyle\sum_{j=0}^{S_i} j \sum_{k=0}^{S_i-j} \frac{(\lambda_i T_i)^{S_i-j-k} e^{-\lambda_i T_i}}{(S_i - j - k)!} \sum_{a=k}^{\infty} \frac{(E[D_0^r])^{S_0+a} e^{-E[D_0^r]}}{(S_0+a)!} \binom{a}{k} \left(\frac{\lambda_i}{\lambda_0}\right)^k \left(1 - \frac{\lambda_i}{\lambda_0}\right)^{a-k} \\ -b_i \displaystyle\sum_{j=-\infty}^{-1} j \sum_{k=0}^{S_i-j} \frac{(\lambda_i T_i)^{S_i-j-k} e^{-\lambda_i T_i}}{(S_i - j - k)!} \sum_{a=k}^{\infty} \frac{(E[D_0^r])^{S_0+a} e^{-E[D_0^r]}}{(S_0+a)!} \binom{a}{k} \left(\frac{\lambda_i}{\lambda_0}\right)^k \left(1 - \frac{\lambda_i}{\lambda_0}\right)^{a-k} \end{pmatrix}$$

(3.40)

In the model, optimal base stock levels for both depot and the bases which ensure the minimum system wide costs are searched. The function must be convex to ensure that the local optimum is the global optimum. To show the convexity, difference functions are used as the cost functions are discrete.

Here, as it is done in section 3.3. the convexity of the cost function is checked. First, the convexity in S_i, for a given value of S_0 is shown. For this purpose, we define:

$$X_k^r = \sum_{a=k}^{\infty} \frac{(E[D_0^r])^{S_0+a} e^{-E[D_0^r]}}{(S_0+a)!} \binom{a}{k} \left(\frac{\lambda_i}{\lambda_0}\right)^k \left(1-\frac{\lambda_i}{\lambda_0}\right)^{a-k} \qquad (3.41)$$

$$Y_{S_i-j-k}^r = \frac{(\lambda_i T_i)^{S_i-j-k} e^{-\lambda_i T_i}}{(S_i-j-k)!} \qquad (3.42)$$

When (4.41) and (4.42) are applied to (4.40), the total cost of the base becomes:

$$C_i^r(S_0,S_i) = \left(h_i \sum_{j=0}^{S_i} j \sum_{k=0}^{S_i-j} Y_{S_i-j-k}^r X_k^r - b_i \sum_{j=-\infty}^{-1} j \sum_{k=0}^{S_i-j} Y_{S_i-j-k}^r X_k^r \right) \qquad (3.43)$$

To investigate convexity of (3.43), we need 1st order difference function, which is:

$$\Delta C_i^r(S_0,S_i) = C_i^r(S_0,S_i+1) - C_i^r(S_0,S_i)$$

$$= \left[\begin{array}{l} \left(h_i \sum_{j=0}^{S_i+1} j \sum_{k=0}^{S_i+1-j} Y_{S_i+1-j-k}^r X_k^r - b_i \sum_{j=-\infty}^{-1} j \sum_{k=0}^{S_i+1-j} Y_{S_i+1-j-k}^r X_k^r \right) \\ -\left(h_i \sum_{j=0}^{S_i} j \sum_{k=0}^{S_i-j} Y_{S_i-j-k}^r X_k^r - b_i \sum_{j=-\infty}^{-1} j \sum_{k=0}^{S_i-j} Y_{S_i-j-k}^r X_k^r \right) \end{array} \right]$$

$$
=\left[\begin{array}{l}
h_i\left(\begin{array}{l}
0\sum_{k=0}^{S_i+1}Y_{S_i+1-k}X_k^r+1\sum_{k=0}^{S_i}Y_{S_i-k}X_k^r+\ldots+S_i\sum_{k=0}^{1}Y_{1-k}X_k^r+(S_i+1)\sum_{k=0}^{0}Y_{0-k}X_k^r\\
-0\sum_{k=0}^{S_i}Y_{S_i-k}X_k^r-1\sum_{k=0}^{S_i-1}Y_{S_i-1-k}X_k^r-\ldots-(S_i-1)\sum_{k=0}^{1}Y_{1-k}X_k^r-S_i\sum_{k=0}^{0}Y_{0-k}X_k^r
\end{array}\right)\\
-b_i\left(\begin{array}{l}
-1\sum_{k=0}^{S_i+2}Y_{S_i+2-k}X_k^r-2\sum_{k=0}^{S_i+3}Y_{S_i+3-k}X_k^r+3\sum_{k=0}^{S_i+4}Y_{S_i+4-k}X_k^r+\ldots\\
-\left(-1\sum_{k=0}^{S_i+1}Y_{S_i+1-k}X_k^r-2\sum_{k=0}^{S_i+2}Y_{S_i+2-k}X_k^r-3\sum_{k=0}^{S_i+3}Y_{S_i+3-k}X_k^r+\ldots\right)
\end{array}\right)
\end{array}\right]
$$

$$
=\left(h_i\sum_{j=0}^{S_i+1}\sum_{k=0}^{S_i+1-j}Y_{S_i+1-j-k}^rX_k^r-b_i\sum_{j=1}^{\infty}\sum_{k=0}^{S_i+j}Y_{S_i-j-k}^rX_k^r\right)
$$

Similarly:

$$
\Delta C_i^r(S_0,S_i+1)=C_i^r(S_0,S_i+2)-C_i^r(S_0,S_i+1)
$$

$$
=\left(h_i\sum_{j=0}^{S_i+2}\sum_{k=0}^{S_i+2-j}Y_{S_i+2-j-k}^rX_k^r-b_i\sum_{j=1}^{\infty}\sum_{k=0}^{S_i+1+j}Y_{S_i+1+j-k}^rX_k^r\right)
$$

Then, the 2nd order difference is:

$$
\Delta^2C_i^r(S_0,S_i)=\Delta C_i^r(S_0,S_i+1)-\Delta C_i^r(S_0,S_i)
$$

$$
=\left(\begin{array}{l}
h_i\sum_{j=0}^{S_i+2}\sum_{k=0}^{S_i+2-j}Y_{S_i+2-j-k}^rX_k^r-b_i\sum_{j=1}^{\infty}\sum_{k=0}^{S_i+1+j}Y_{S_i+1+j-k}^rX_k^r\\
-\left(h_i\sum_{j=0}^{S_i+1}\sum_{k=0}^{S_i+1-j}Y_{S_i+1-j-k}^rX_k^r-b_i\sum_{j=1}^{\infty}\sum_{k=0}^{S_i+j}Y_{S_i+j-k}^rX_k^r\right)
\end{array}\right)
$$

$$
=\left[\begin{array}{l}
h_i\left(\begin{array}{l}
\sum_{k=0}^{S_i+2}Y_{S_i+2-k}X_k+\sum_{k=0}^{S_i+1}Y_{S_i+1-k}X_k+\sum_{k=0}^{S_i}Y_{S_i-k}X_k+\ldots+\sum_{k=0}^{1}Y_{1-k}X_k+\sum_{k=0}^{0}Y_{0-k}X_k\\
-\sum_{k=0}^{S_i+1}Y_{S_i+1-k}X_k-\sum_{k=0}^{S_i}Y_{S_i-k}X_k-\ldots-\sum_{k=0}^{1}Y_{1-k}X_k-\sum_{k=0}^{0}Y_{0-k}X_k
\end{array}\right)\\
-b_i\left(\begin{array}{l}
\sum_{k=0}^{S_i+2}Y_{S_i+2-k}X_k+\sum_{k=0}^{S_i+3}Y_{S_i+3-k}X_k+\sum_{k=0}^{S_i+4}Y_{S_i+4-k}X_k+\ldots\\
-\sum_{k=0}^{S_i+1}Y_{S_i+1-k}X_k-\sum_{k=0}^{S_i+2}Y_{S_i+2-k}X_k-\sum_{k=0}^{S_i+3}Y_{S_i+3-k}X_k+\ldots
\end{array}\right)
\end{array}\right]
$$

$$= \left(h_i \sum_{k=0}^{S_i+2} Y^r_{S_i+2-k} X^r_k + b_i \sum_{k=0}^{S_i+1} Y^r_{S_i+1-k} X^r_k \right)$$

As $Y^r_{S_i+2-k}$, $Y^r_{S_i+1-k}$ and X^r_k are non-negative, and we have positive cost parameters; value of $\Delta^2 C^r_i(S_0, S_i)$ is also non-negative and therefore $C^r_i(S_0, S_i)$ is convex with respect to $S_i \geq 0$, for a given $S_0 \geq 0$ value. However, similar to the previous model, the total cost, (3.33), is not necessarily convex in S_0. We show that with a counter-example. In this counter-example, the parameters are set as given in Table 5:

Table 5- Parameters for the TR model in the counter-example.

Parameters	
Base Stocks (*S*)	$S_1 = 5$, $S_2 = 5$
Demand (λ)	$\lambda_0 = 16$, $\lambda_1 = 8$, $\lambda_2 = 8$
Lead Time (*L*)	$L_0 = 1$, $L_1 = 1$, $L_2 = 1$, $R_0 = 1$
Costs	$h_0 = 1$, $h_1 = 1$, $h_2 = 1$, $h_t = 1$, $b_1 = 5$, $b_2 = 5$, $c_p = 3$
Repair ratio (ρ)	$\rho = 0.2$

In Figure 10 $C_0^r(S_0) + \Sigma C_i^r(S_0,S_i)$ values for different S_0 values are plotted. As it can be seen in the figure $C_0^r(S_0) + \Sigma C_i^r(S_0,S_i)$ is not convex in S_0 for given S_i values.

The plotted graph in Figure 10 means that the local optimum is not necessarily the global optimum. But we can still optimize the model. For optimizing the model, an upper-bound for base stock levels of the depot, S_0, is intuitively set and the problem is divided into two parts as it is done in Section 3.3. One is the base problem which deals with finding the optimal base stock levels for the bases, S_i^*; and the other deals with finding the optimal base stock level of the depot, S_0^*. An iterative procedure is used to deliver the optimal results for the problem. This procedure involves in solving *N* independent base problem for a given value of S_0.

As the objective function is convex in $S_i > 0 \ \forall i$; the optimal base stock level for base i, S_i^*, can again be found from the first order optimality condition given in (3.7), for a given value of S_0. This procedure is repeated for all $S_0 \in \{0,1,2,...\}$ until an intuitively assigned upper-bound.

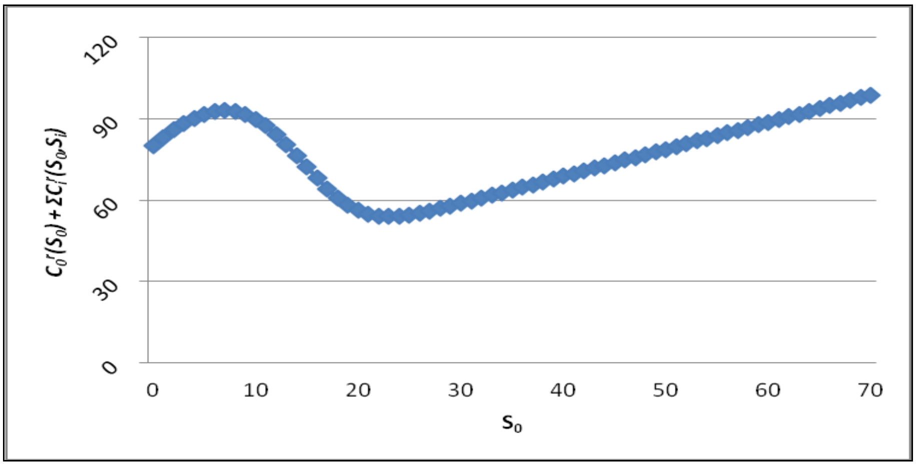

Figure 10- Counter example for convexity of "$C_0^r(S_0) + \Sigma C_i^r(S_0,S_i)$"

The optimal order-up-to level of the depot, S_0^*, which ensures the minimum objective cost function value, is found by evaluation of all order-up-to levels, S_0, with a local search procedure. During the analysis, if the assigned upper-bound for S_0 ever becomes binding, the upper-bound is extended to a larger value and the local search procedure is restarted.

CHAPTER 4

COMPUTATIONAL STUDY

In this chapter, the computational study on the comparison of the alternative models and observations about the research questions are presented. As analyzed in Chapter 3; there are four alternative models to be considered. Among these four models the two-echelon model is the one that is being operated currently. Due to risks and costs in transportation process between the depot and the bases, the management considers shutting down the depot and operating the single-echelon model. In the base case parameter setting, the procurement lead time of a base in single-echelon model is assumed to be equal to the replenishment lead time of a base in the two-echelon model which is the sum of the procurement lead time of the depot, L_0, and transportation lead time of base i, T_i. Regarding the transportation lead times and in-transit inventory holding costs, this assumption could result in the single-echelon model outperforming the two-echelon model. Recall that the first research question considers the effects of operating a SE model, instead of a TE model. Therefore in Section 4.1, the improvement of the system after such a change is considered.

Moreover, the stock points may acquire the ability to repair the failed items. The procurement lead times in both two and single-echelon models will then be replaced by the repair lead time which is presumably shorter than the procurement lead times. Furthermore, the procurement cost will be replaced by the repair cost which is expected to be lower. As, the second research question is about the effect of the repair ability acquisition; the improvement caused by the repair ability in both TE and SE models are considered in Section 4.2.

The third research question seeks the effects of the changes in cost, demand and lead time parameters on the system performance in all of the models. This may be regarded as a sensitivity analysis for each of the models considered in this study. Therefore, the computational study is conducted for the models under various parameter settings and the findings are presented in both Sections 4.1 and 4.2.

In the base case scenario, there are 3 bases and 1 depot in two-echelon models; and 3 bases in single-echelon models. The parameter values that are estimated in accordance with the real life case are presented in Table 6:

Table 6- Parameters of the Base Case Scenario

Parameters	
Demand (λ)	$\lambda_0 = 9, \lambda_1 = 3, \lambda_2 = 3, \lambda_3 = 3$
Lead Time (L)	$L_0 = 3$; $T_1 = 1$, $T_2 = 1$, $T_3 = 1$; $L_i = 4$; $R_0 = 2$; $R_i = 2$
Costs (10.000$)	$c_p = 4, h_i = 0.02, b_i = 60, c_r = 1$
Repair Ratio (ρ)	$\rho = 0.4$

The parameters are set in accordance with the actual data obtained from real life case. Cost parameters consist of holding, backorder, unit procurement and repair costs and they are assumed to be the same for all stock points in all models along with the demand parameters for the bases. The unit time is a week and the mean of the Poisson demand is 3 items per week. In the current system, procurement lead time of the depot is 3 weeks; whereas the transportation lead time between the depot and the bases is a week. For single-echelon systems, the procurement lead time is 4 weeks. The repair lead time is assumed to be 2 weeks for all stock points that have repair ability.

The holding costs, h_0 for the depot and h_i for base i, depend on the procurement cost, c_p, via an interest rate. We assume that the holding cost per item per week in base i, h_i, is 0.5% of the procurement cost. The in-transit inventory holding cost, h_t, is assumed to be the same. The backorder cost per unit per week in base i, b_i, is set to be 15 times of the procurement cost.

Generally, it is assumed that the procurement lead time, L_0, may change due to bureaucracy in the auction processes; whereas, distance is the key driver in transportation lead time, T_i, change. The change in repair lead time, R_0, depends on the availability of the spare parts which are not considered in this thesis study. Demand, λ, may increase due to obsolescence or decrease by more careful use in the military units. The procurement cost, c_p, depends on the prices and may change depending on the market. Similarly the repair cost, c_r, is dependent on the spare parts; and therefore, the market. Holding costs, h_0 and h_i, changes according to the interest rates; whereas the holding cost for the items in-transit is mainly dependent on the safety and insurance expenses.

The optimization algorithms presented in Chapter 3 are implemented in Microsoft Visual C# and the computational study is conducted on Intel(R) Core(TM)2 Duo CPU P7350 2GHz processor and 3 GB of RAMs computer. The input parameters and the logical structure are represented correctly and the code is checked and verified to cover every step in the algorithm. The program is the accurate representation of the real life case; as all the data are based on the real life case and a Chi-square goodness of fit test is conducted for the Poisson demand (See Appendix). Therefore, the model is valid.

4.1. Two-Echelon Model versus Single-Echelon Model

As it is stated in Muckstadt and Thomas (1980) and Hausmann and Erkip (1994), a two-echelon model outperforms the single-echelon model, especially when it is provided that the demand distribution and total replenishment lead time are the same along with the cost parameters. One of the main factors on this fact is the risk pooling effect of the depot which is stated in Graves (1996) as the uncertainty in demand can be controlled by the depot centrally. But in our two-echelon model, a holding cost is incurred for each item being transported from the depot to the base; therefore the improvement of the TE model over the SE model could be hindered. In this section the effects of the changes in lead time, demand and cost parameters are analyzed and the parameters presented in Table 6 are used as the base case scenario, unless otherwise mentioned. Recall that two and single-echelon models are referred to as TE and SE models, respectively, throughout the study for the sake of simplicity.

The procurement cost, which constitutes over 95% of the total cost in most cases, is only dependent on total demand and it does not have any effect on optimization. As total demand is the same for both two and single-echelon models; the procurement cost is also the same under both cases. Therefore the procurement cost is excluded in the comparison of the performances of single and two-echelon models for a more effective assessment.

We start our analysis by considering the effects of the lead time. We first focus on the effects of an increase in the total replenishment lead time. As replenishment lead time increases, it is expected that both models get worse in terms of cost. This is due to the increasing base stock levels as demand during lead time increases. The replenishment lead time is the procurement lead time for the SE model whereas it has two components for the TE model; procurement lead time of the depot and transportation lead times of the bases. Therefore, it should be noted whether the increase in replenishment lead time is caused by the procurement or transportation lead time. Figure 11 and Figure 12 illustrate the effects of the increase in lead times on costs of the TE and SE models, respectively.

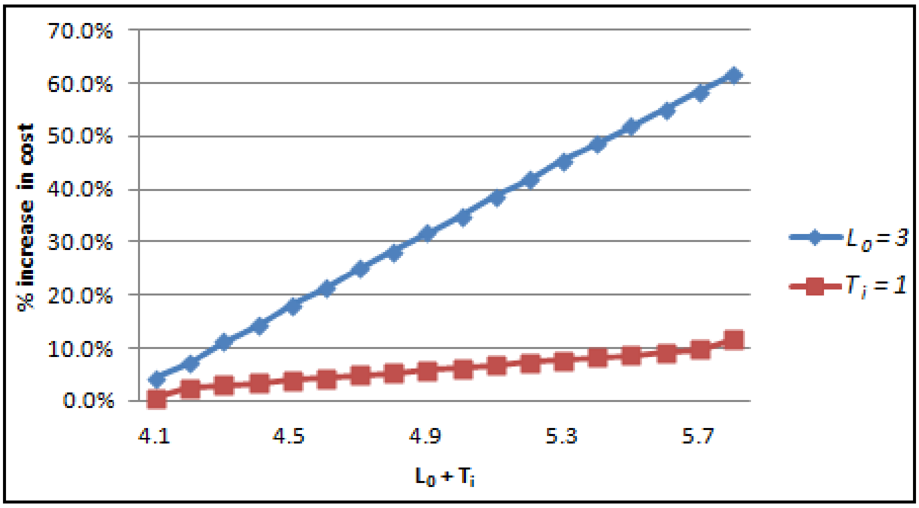

Figure 11- Cost increase in lead time for TE model vs. L_0+T_i

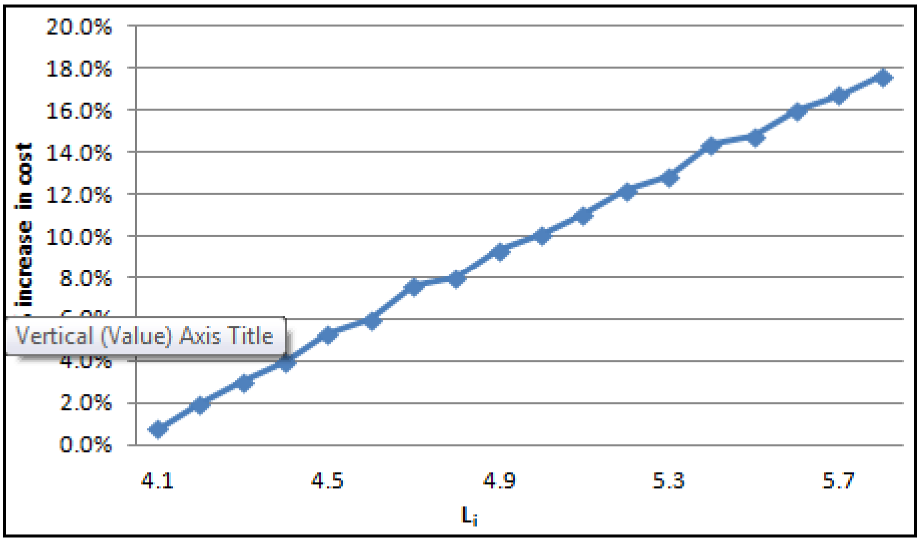

Figure 12- Cost increase in lead time for SE model vs. L_i

For both models, it can be said that if replenishment lead time increases the costs increase. This is due to increasing base stock levels and uncertainty in demand. As it can be seen in Figure 11 if the increase in replenishment lead time is caused by the transportation lead time, the effect is more dramatic because of the increasing in-transit inventory holding cost. To see this effect more clearly, optimal results of both TE and SE models are obtained and compared for different h_t values. The improvement of the TE model over SE model is assessed and plotted in Figure 13 for different h_t values. The improvement is formulated as;

$$\frac{Cost_{SE} - Cost_{TE}}{Cost_{SE} - c_p \sum_{i=1}^{N} \lambda_i} * 100 \qquad (4.1),$$

where; $Cost_{SE}$ and $Cost_{TE}$ denote the costs of SE and TE model, respectively. In the assessment of the improvement of TE model over SE model; the procurement cost is excluded in(4.1) since it is the same for both models under the same demand rates.

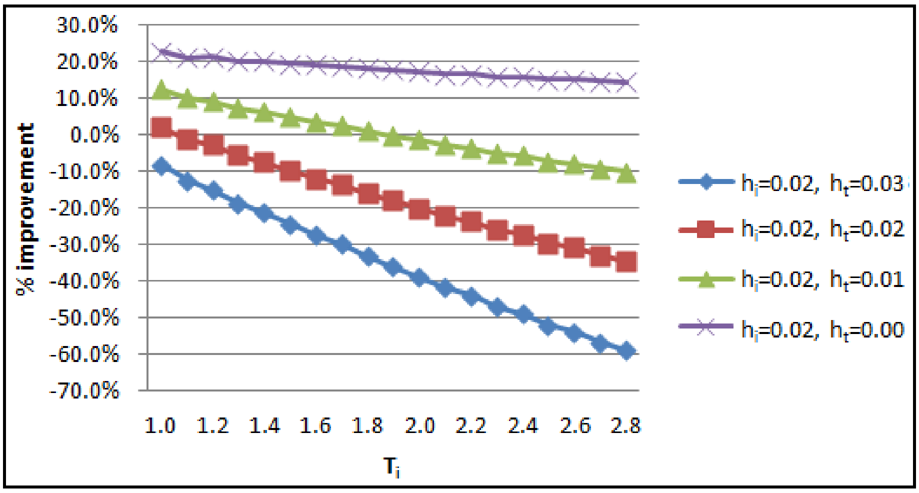

Figure 13- Improvement of TE over SE model vs. T_i (L_0=3, L_i=L_0+T_i)

In Figure 13, the replenishment lead times for both single and two-echelon systems are increasing. In the TE model, the procurement lead time for the depot,

L_0, is constant and set as 3 weeks. The increase in replenishment lead time is due to the transportation lead time, T_i. Although, expected costs of both models increase in lead times, the effect is more pronounced for TE model as T_i increase. It can be seen in Figure 13 that as the transportation lead time increases the improvement of the TE model over the SE model deteriorates, which means the cost is increasing more rapidly in transportation lead time for TE. Moreover; as long as holding cost is incurred for items in-transit; the single echelon model outperforms the two-echelon model after some specific increase in T_i. For example; in the base case, if T_i increases more than 6%, the SE model outperforms the TE model When h_t is 0.01, an 86% increase in transportation lead time is necessary for TE model to be outperformed by SE model (See Figure 13). However, in these assessments, h_t is assumed to be decreasing while holding cost rates, h_i, for each base i, remain constant. If h_i for each base i also changes in accordance with h_t; then the % improvement curves shift upwards for decreasing h_i (See Figure 14). This is also an effect of in-transit holding cost.

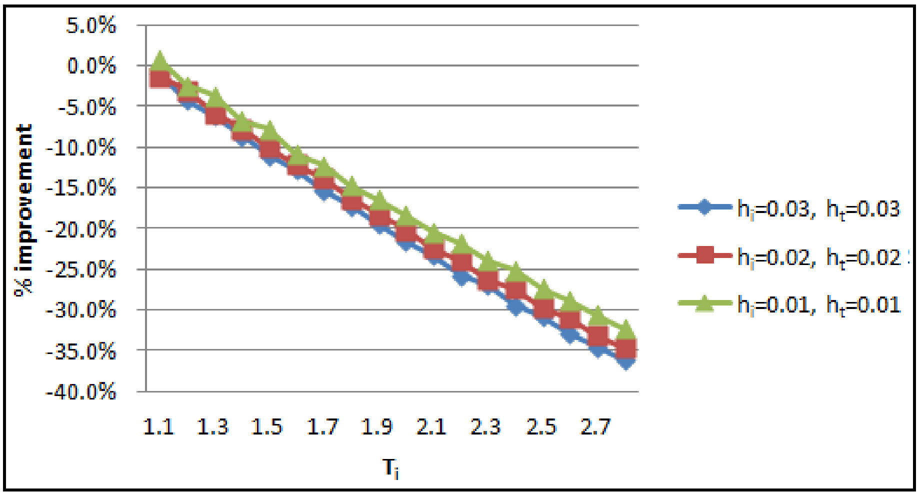

Figure 14- Improvement of TE over SE model vs. T_i (L_0=3, L_i=L_0+T_i)

As procurement lead time, L_0, gets smaller; i.e, set as 2 weeks or a week, while T_i & L_i increase; the % improvement curves presented in Figure 14 shift downwards. In such a case, the effects of inventory holding cost for the items in-transit in the two echelon system become more pronounced and the single-echelon model becomes more attractive. If inventory holding cost is not incurred for in-transit inventory, the two-echelon model would outperform the single-echelon model, no matter how large the procurement or in-transit inventory holding cost rates are, due to risk pooling effects of the depot.

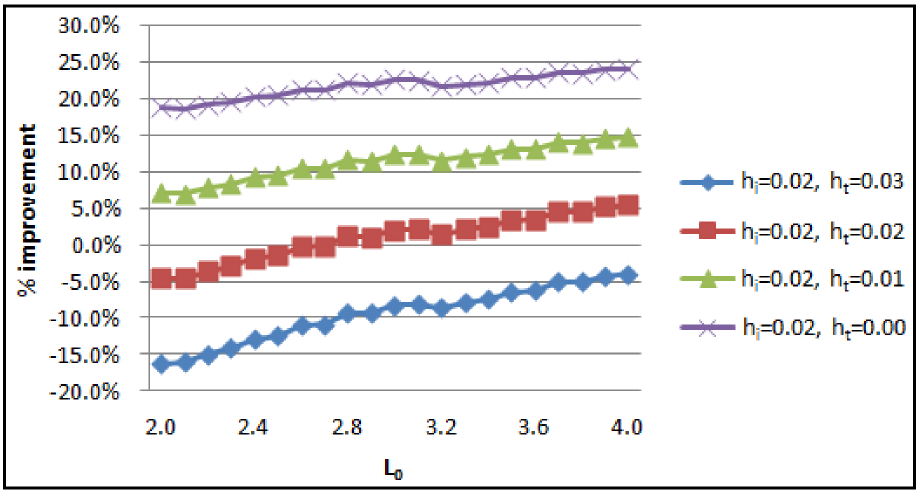

Figure 15- Improvement of TE over SE model vs. L_0 $(T_i=1, L_i=L_0+T_i)$

In Figure 15 the replenishment lead times are increasing for both SE and TE models. For the TE model, L_0 is increasing whereas transportation lead time for base i, T_i, is constant and set as 1 week. Although, the expected costs are increasing for both models, the increase in expected cost of TE model is less than the increase in SE model. This is because of the increasing joint order and depot effects presented by Graves (1996) as L_0 increases. In Figure 15, it is observed that as L_0 increases the improvement of TE model over SE model also increases. But, if L_0 decreases almost 10%, the SE model becomes more attractive against TE model for the base case. However, if holding cost rates for items in-transit, h_t, decreases; this effect neutralizes (see Figure 15).

Up to this point, the expected costs of TE and SE models and the improvements of TE model over SE model are compared as the lead time changes. It has been observed that the improvement of TE over SE model is more sensitive to T_i change. Moreover, if the increase in replenishment lead time is due to L_0 increase; although, the expected cost values increase for both models, the improvement of TE model over SE model also increases. In all these comparisons and assessments, the replenishment lead times of TE and SE models are consistent; i.e, the same through the analysis. Hence, the individual effects of the changes in lead times are not considered. Therefore, in this part of the analysis, the effects of the lead times changes; such as, change in L_0 and T_i are considered, individually.

If the procurement lead time of the depot, L_0, increases while all of the other parameters remains as they are in the base case; the optimal system wide base stock levels and the expected cost increases as expected (See Figure 16). As, the procurement cost does not change in L_0, it is excluded from the cost in Figure 16, to see the effect more clearly.

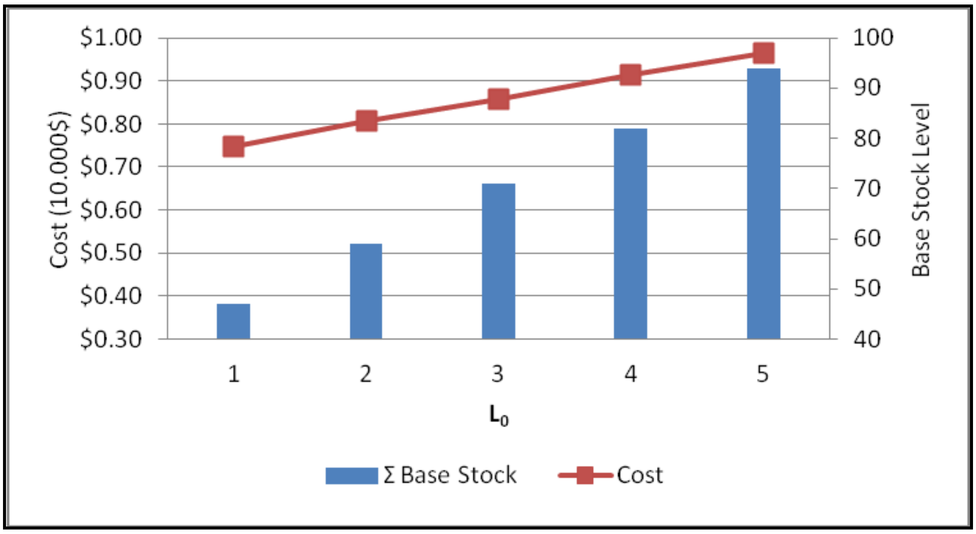

Figure 16- Cost and base-stock levels vs. L_0 (T_i=1)

Furthermore, as L_0 increases while all other parameters constant, the improvement of the two-echelon model over single-echelon one decreases. In the base case, after a 6% or more increase in L_0 the TE model is outperformed by the SE model. However, as h_t decreases, more than 6 % increase in L_0 is necessary for TE model to lose its attractiveness to SE model. For example; when h_t is assumed to be 0.01, the TE model loses attractiveness against SE model after a 70% increase in L_0. The graph that reflects the effect of procurement lead time increase can be seen in Figure 17.

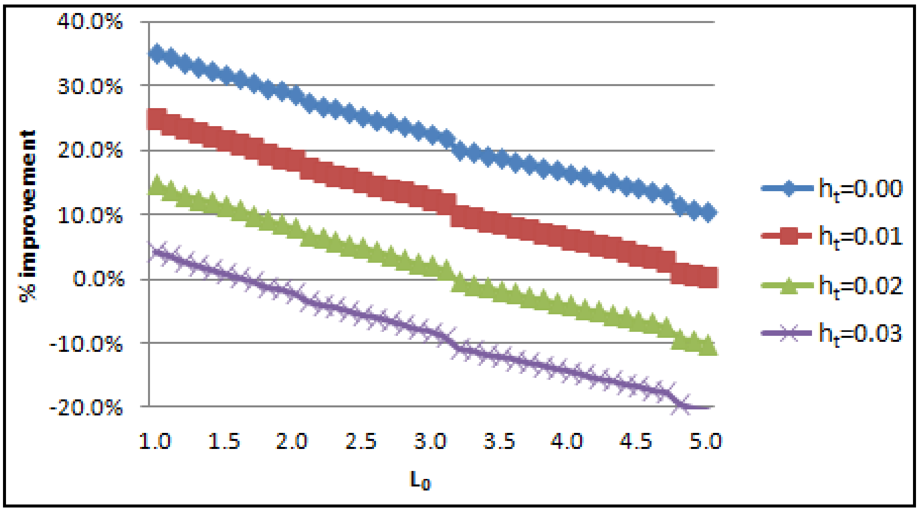

Figure 17- Improvement of TE over SE model vs. L_0 (T_i=1, L_i=4)

If transportation lead time for base *i*, T_i, increases while all other parameters remain constant; the effects are more apparent, although they are very similar with the effects of L_0 increase. The effect of T_i increase is more noticeable than the effect of L_0 increase due to holding cost incurred for in-transit inventory. For the base case, a 5% increase in transportation lead time results in, SE model outperforming the TE model. However the increase in T_i, that is necessary for TE model to be outperformed by SE model increases to 50% when h_t=0.01. The related graph is plotted in Figure 18.

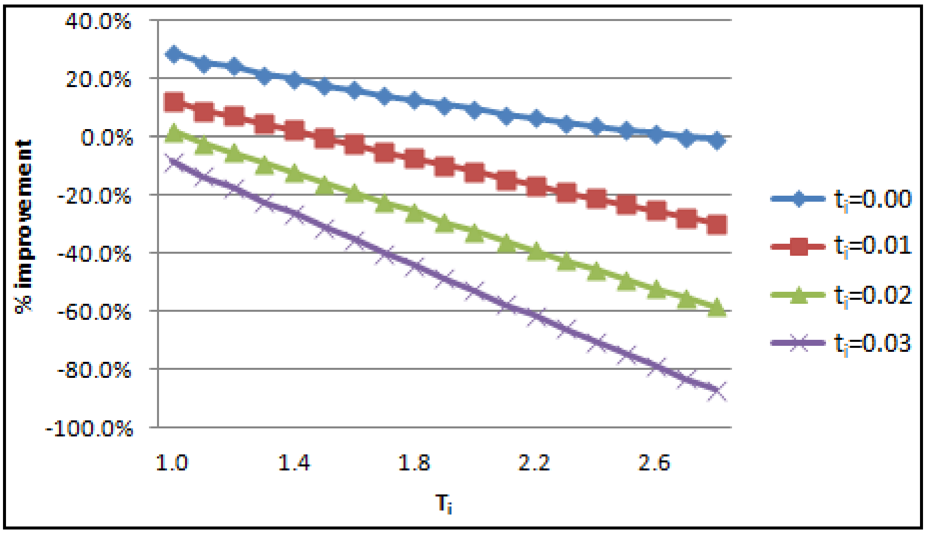

Figure 18- Improvement of TE over SE model vs. T_i (L_0=3, L_i=4)

If demand increases, the costs increase for both models. The main factor in this is the procurement costs. However, for increasing demand, holding costs both for inventory in-stock and in-transit also increases. The increase in expected cost is more pronounced for the TE model and that is expected. As demand increases, the items in-transit also increases and this incurs extra holding cost for those items. In Figure 19, %improvement curves are plotted in demand for various lead time settings. It is observed in the graph that, as demand increases in the base case, SE model becomes more attractive than the TE model. For increasing L_0, the %improvement curve shifts upwards; however if T_i increases, the %improvement curve shifts downwards as expressed in the previous cases. It can also be observed in Figure 19 that the worst % improvement of TE model over SE model is observed when the share of the T_i in replenishment lead time is the most. This is mainly due to in-transit inventory holding cost. When Figure 19 is compared to Figure 20, the undeniable effect of the h_t can be observed.

To see the effects of the in-transit inventory holding cost, TE and SE models are considered where no holding cost is incurred for items in-transit (see Figure 20).

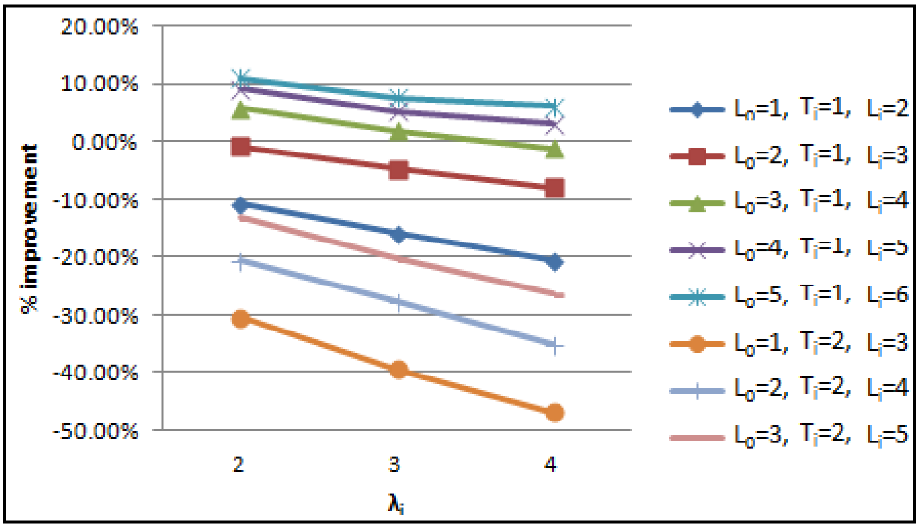

Figure 19- Improvement of TE over SE model vs. λ (h_t=0.02)

Although the expected cost values increase in demand; it is observed that if no in-transit inventory cost is incurred, then the improvement of TE over SE model increases for most of the cases due to increasing risk pooling effects of the depot.

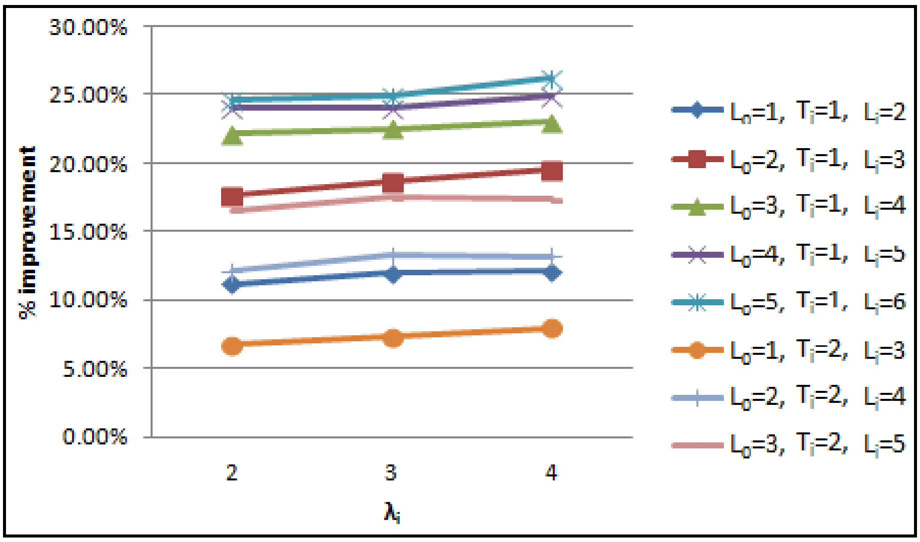

Figure 20- Improvement of TE over SE model vs. λ (h_t=0)

As holding cost rate, h_i, increases; the expected cost values increase due to increasing holding costs for items in-stock; although the system wide optimal base stock levels decreases in the base case scenario (See Figure 21). If h_t increases in accordance with h_i, then the increase in expected cost becomes a bit more significant. However, although the holding costs increase as holding cost rates, h_i, increase in the TE model; the improvement of TE over SE model seems not to be affected much by $\underline{h_i}$ change when no holding cost is incurred for items in-transit, i.e. $h_t=0$ (See Figure 22). This indifference to h_i change may be due to high backorder cost rate, b_i.

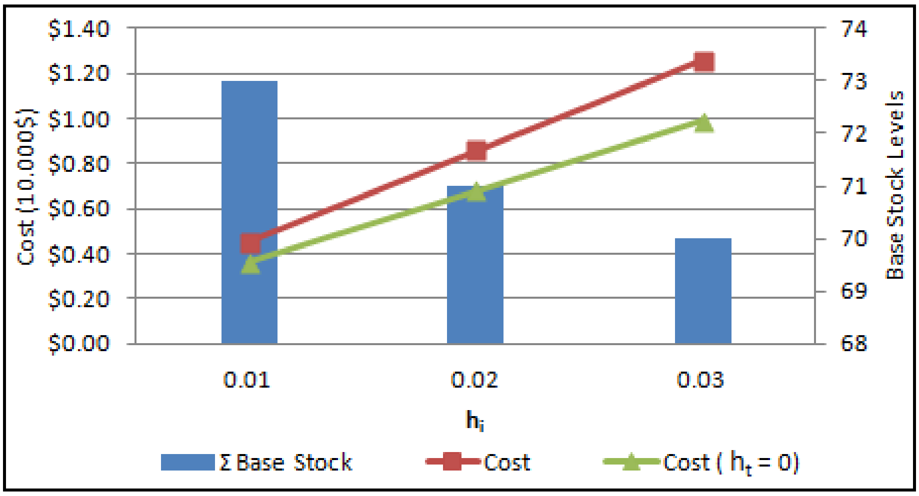

Figure 21- Cost and total base stock level vs. h_i

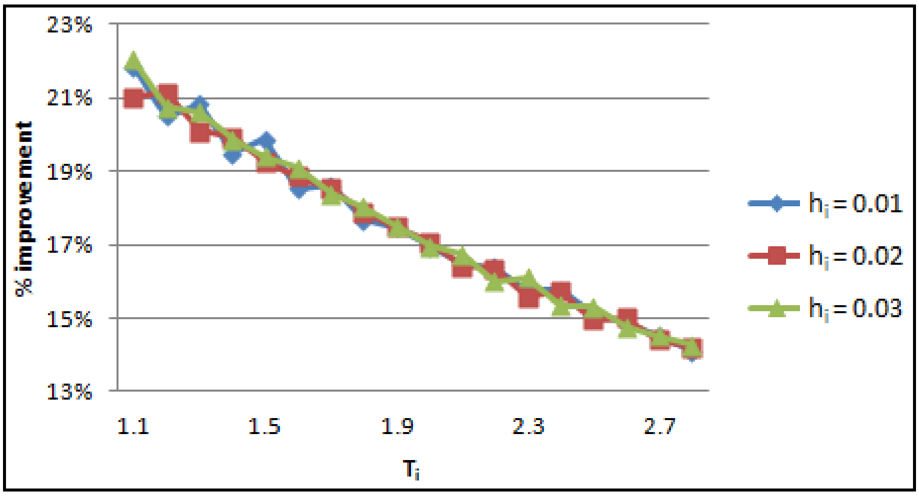

Figure 22- Improvement of TE over SE model vs. T_i ($L_0=3$, $L_i=L_0+T_i$, $h_t=0$)

Due to criticality of the items in military use; lack of a single item would result in fatal problems; therefore, b_i is assumed to be very high according to h_i. Therefore, the computations are conducted with lower b_i values such as, 5 and 0.4. It is observed that as backorder cost rate decreases, the system tends to hold fewer items in stock and the system wide optimal base stock level decrease; dependently, the holding costs decrease (See Figure 23).

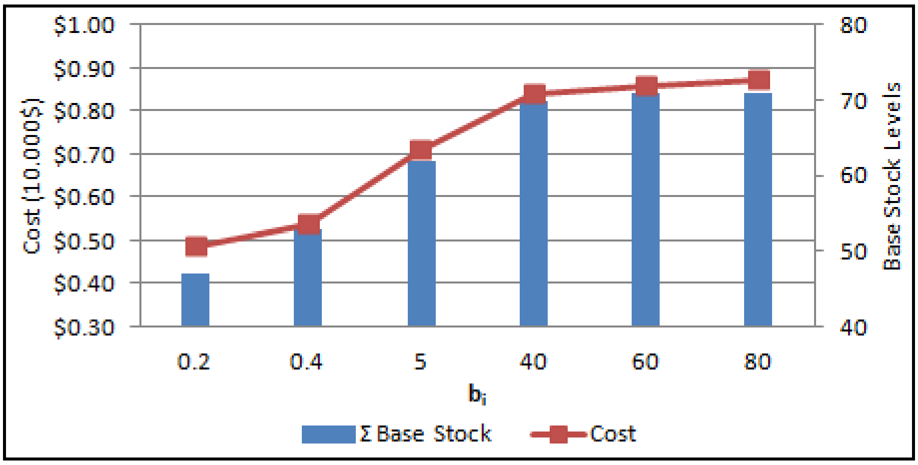

Figure 23- Cost and total base stock level vs. b_i

The system wide cost decreases as b_i decreases, and the in-transit inventory holding cost increases relatively. Therefore, for significantly decreasing b_i, the improvement of the two-echelon model also decreases (See Figure 24). It can be observed that the improvement of TE model over SE model gets better as L_0 increases. This is mainly caused by the increasing risk pooling effects of the depot in lead time. Furthermore, it can also be observed in Figure 24 that as T_i increases, the improvement of TE model deteriorates significantly for decreasing b_i. That is because of relatively increasing holding costs of the items in-transit.

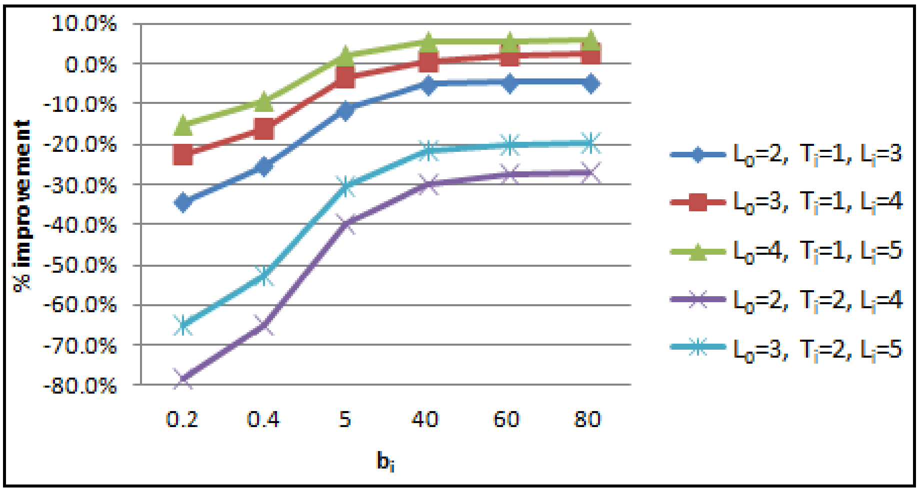

Figure 24- Improvement of TE over SE model vs. b_i (h_t=0,02)

However, if no holding cost is incurred for the items in-transit; i.e, h_t=0, the improvement of the TE model over SE shows no significant patterns as b_i changes (See Figure 25); although the expected costs are increasing in b_i. When, Figure 24 and 25 are compared, the importance and the effect of the in-transit inventory holding cost can easily be noticed.

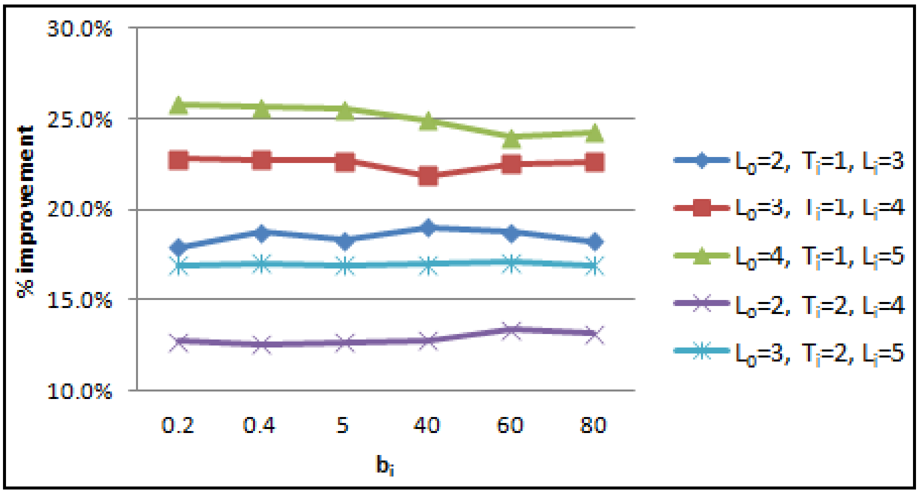

Figure 25- Improvement of TE over SE model vs. b_i (h_t=0)

Besides the changes in parameter settings, the environment of the problem can also differ. In this part of the study, the case where TE model has 4 bases instead of 3 bases is considered. The parameter setting for the system with 4 bases is presented in Table-7.

Table 7- Parameters of the Base Case Scenario with 4 Bases

Parameters	
Demand (λ)	$\lambda_0 = 9, \lambda_1 = 2.25, \lambda_2 = 2.25, \lambda_3 = 2.25, \lambda_4 = 2.25$
Lead Time (L)	$L_0 = 3$; $T_i = 1$; $L_i = 3{,}5$; $R_0 = 2$; $R_i = 2$
Costs (10.000$)	$c_p = 4, h_i = 0.02, b_i = 60, c_r = 1$

When an additional base is being operated in the base case, the expected optimal costs increases along with the system wide base stock levels (Figure 26). Although, system wide demand is not changed; demand faced by each base is decreased. Hence, the optimal base stock levels per base decrease. However, a new base can be regarded as a risk factor for potential backorder occurrences in the system. Therefore, the system wide optimal base stock levels increase with the new setting. As the items in stock increases, the holding cost incurred for the items in stock also increases and eventually, the system with 4 bases incurs higher expected cost than the current system (See Figure 26).

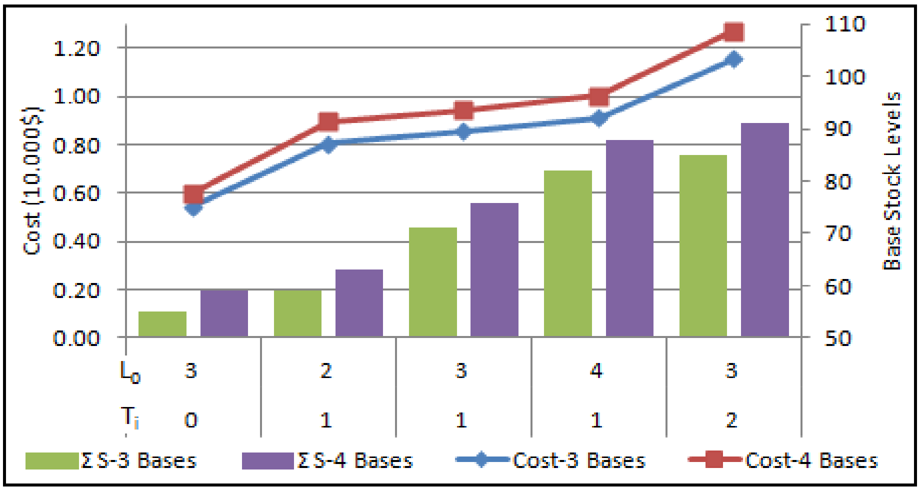

Figure 26- Cost and total base stock level vs. L_0 and T_i

When, the improvements of the TE models with 4 bases and 3 bases over the SE models are compared, it is observed that the TE model with 4 bases is more tolerant to T_i increase compared to the current model (See Figure 27). This may be due to increasing base stock levels; regarding that, the holding cost incurred for items in stock increases in base stock levels and the increase in holding cost relatively decreases the in-transit inventory holding cost which reduces the negative effects of the holding cost incurred for items in-transit.

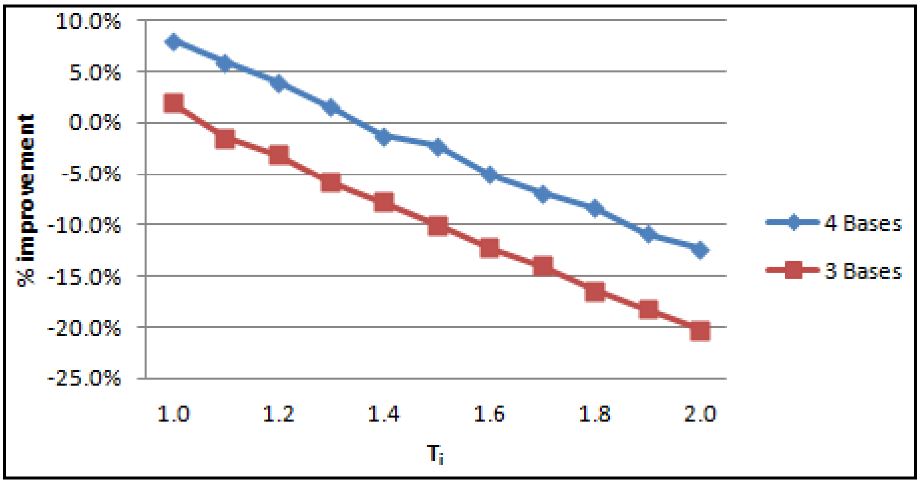

Figure 27- Improvement of TE over SE models vs T_i (3 Bases vs. 4 Bases)

It is observed in Figure 28 that the TE model with 4 bases presents better improvement against the SE models not only in T_i, but also in L_0. This is mainly due to increasing the risk pooling effects of the depot in L_0.

With 4 bases instead of 3 bases in the lower echelon, a TE model is more robust against being outperformed by a SE model due to change in parameters. Because, although the expected holding and backorder costs increase as an additional base is operated; the increase in costs of TE model is less than the increase in SE. This is mainly due to more noticeable risk pooling effects of the depot in TE model with 4 bases.

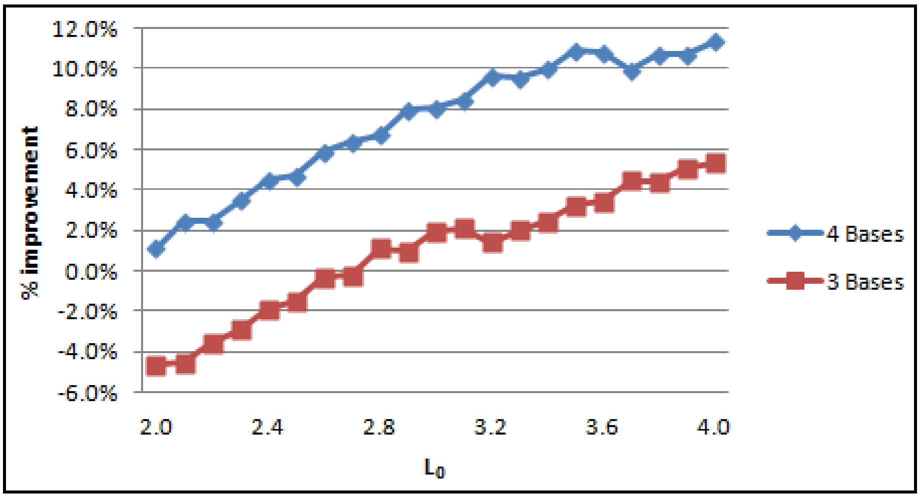

Figure 28- Improvement of TE over SE models vs L_0 (3 Bases vs. 4 Bases

4.2. Models with No Repair Ability versus Models with Repair Ability

The second research question requires the comparison between the systems with and without repair ability. In the two-echelon model the repair ability is acquired only by the depot whereas it is acquired by all bases in the single-echelon model. After the repair ability is acquired, there are two types of orders: One is procurement orders and the other is repair orders. Before starting the analysis, note that the repair ability acquisition creates two main changes in the model. One of them is the new cost parameters such as repair cost, repair holding cost for items in repair and extra in-transit inventory holding costs for items being sent to the depot for repair. The other change is in the lead times as for each repair order the procurement lead time is replaced by a presumably lower repair lead time.

In the analysis of the effects of repair ability acquisition, the two and single-echelon models are considered separately. Recall that the TE model with repair ability is represented as, TR; whereas the SE model with repair ability is referred to as SR for the sake of simplicity and convenience. First, the computational results related to TR are presented.

Two Echelon Models

When the systems with repair ability are considered, the key factor in the models with repair ability is the ratio of repairable items in total demand; i.e. repair ratio, ρ. It is observed in Figure 29 that the % improvement of TR model over TE model increases in ρ. This is because the procurement cost, c_p and lead time, L_0 are exchanged by presumably lower repair costs, c_r and lead times, R_0, for each repair order. The % improvement is formulated as;

$$\frac{Cost_{TE} - Cost_{TR}}{Cost_{TE}} * 100 \qquad (4.2),$$

where; $Cost_{TE}$ and $Cost_{TR}$ denote the costs of TE and TR model, respectively. In the assessment of the improvement of TR model over TE model; the procurement costs cannot be excluded in (4.2) because they are not the same for both models anymore.

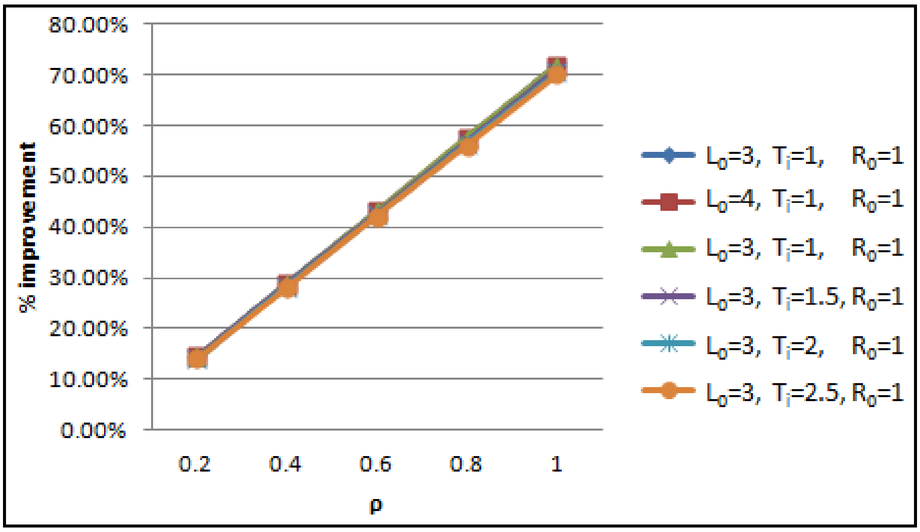

Figure 29- Improvement of TR over TE model vs. ρ

Although, the improvement of the system with repair ability over the system with no repair ability increases as repair ratio increases, the % improvements seem to be indifferent both for L_0 and T_i. However, with a closer look in the base case, it can be seen that there is an increase in the % improvement in L_0, and a decrease in the % improvement in T_i (See Figure 30 and 31, respectively).

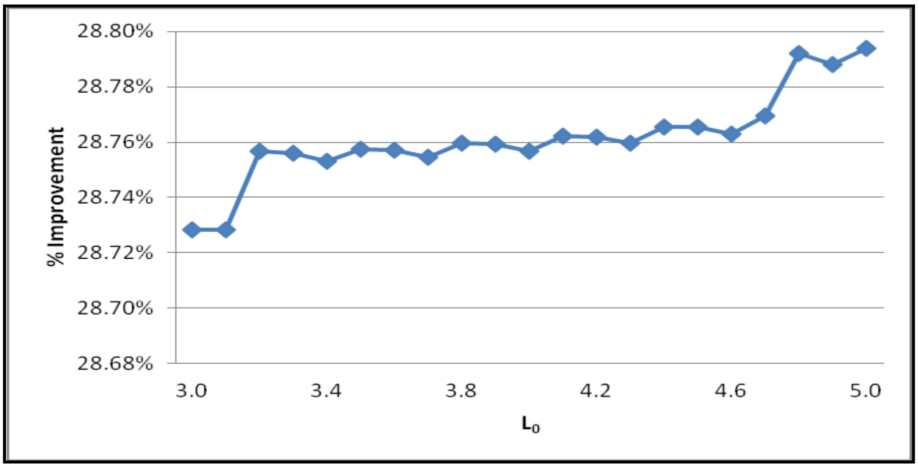

Figure 30- Improvement of TR over TE vs. L_0

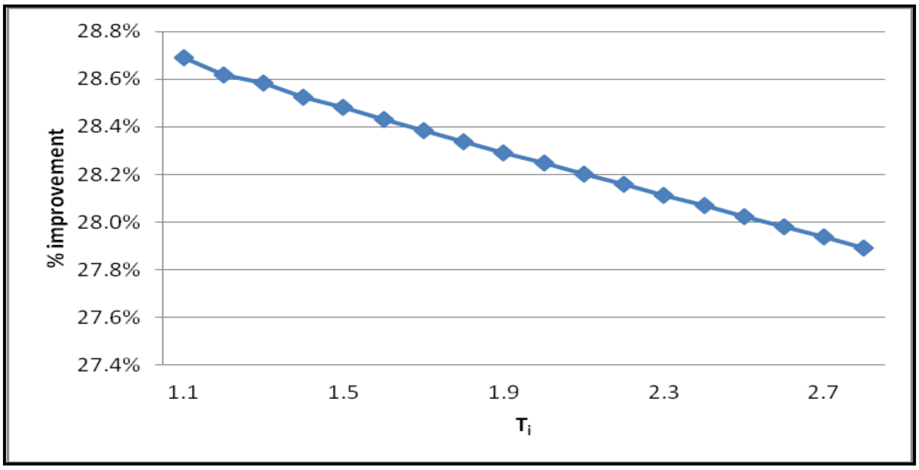

Figure 31- Improvement of TR over TE vs. T_i

The increase in the % improvement in L_0 is a result of both the decreasing effect of in-transit inventory holding cost and relative decrease in repair lead time as procurement lead time increases. However, the decrease in the % improvement in T_i is caused by increasing effect of the holding costs for items in-transit which is additionally incurred for the items sent from base to depot to be repaired. It can be seen in both graphs in Figure 30 and 31 that the %improvements are changing very slightly. This is not surprising; because, on the contrary to the previous section, the procurement cost cannot be excluded in this analysis.

It is previously expressed that for increasing T_i, SE model can become a better alternative instead of TE. However In Figure 32, it is observed that for increasing T_i, *TR* model outperforms both models even with a repair ratio of 20%; i.e, ρ=0.2.

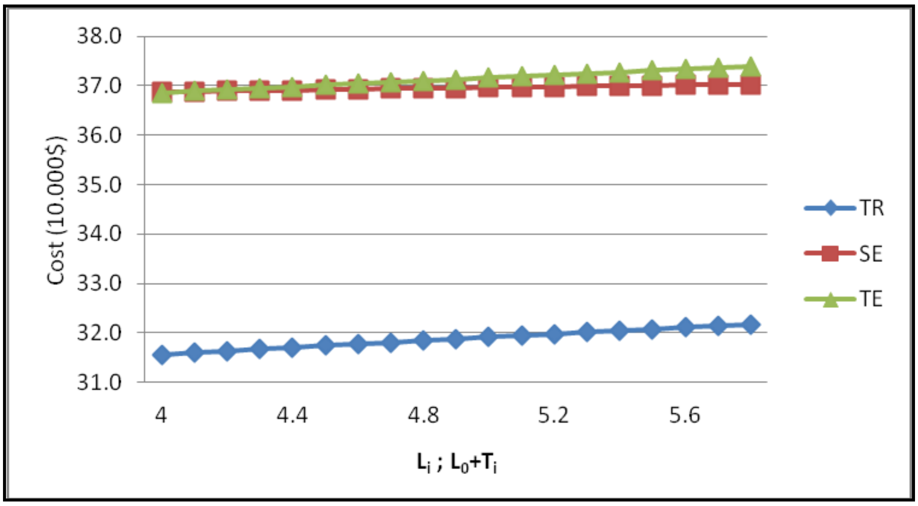

Figure 32- Cost for TR, SE and TE model vs. L_i and L_0+L_i (L_0=3, ρ=0.2)

Besides the repair ratio, ρ, the effect of the repair lead time, R_0, change is also considered. It can be seen in Figure 33 that the % improvement decreases in R_0; because it takes longer time for an item to be repaired and this increases the holding cost for item repaired or being repaired. In L_0, % improvement curves shift up very slightly; that is because, for increasing L_0, R_0 decreases relatively. However, as T_i increases, the % improvement curves noticably shift downwards. The reason is that as T_i increases, the holding costs incurred for items in transit increases, when it is recalled that this cost is incurred both for items being sent from depot to base and for items that are sent from base to depot for repair.

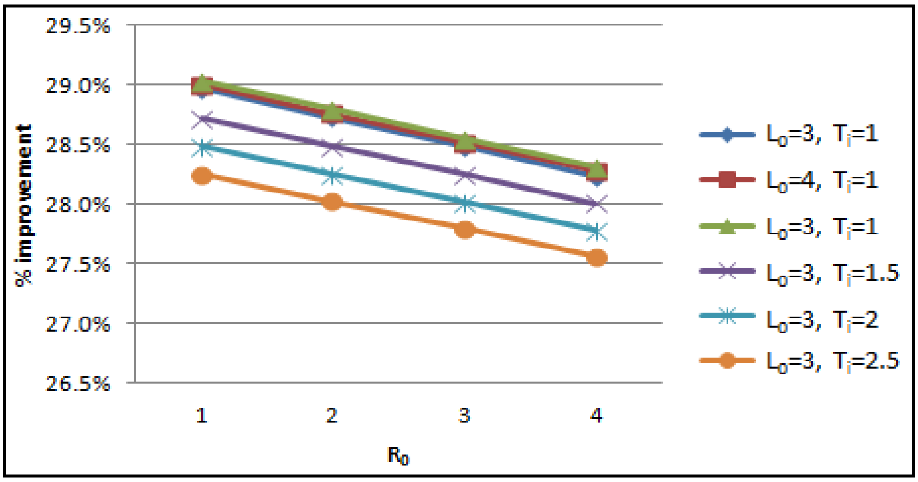

Figure 33- Improvement of TR over TE model vs. R_0

Although the % improvement decreases in R_0, the TR model is more attractive than the TE model and the increase in R_0 does not affect the preference of the TR model against TE model. In Figure 34, the % improvements seems indifferent in R_0 increase. However, the repairable model is extremely sensitive to an increase in c_r.

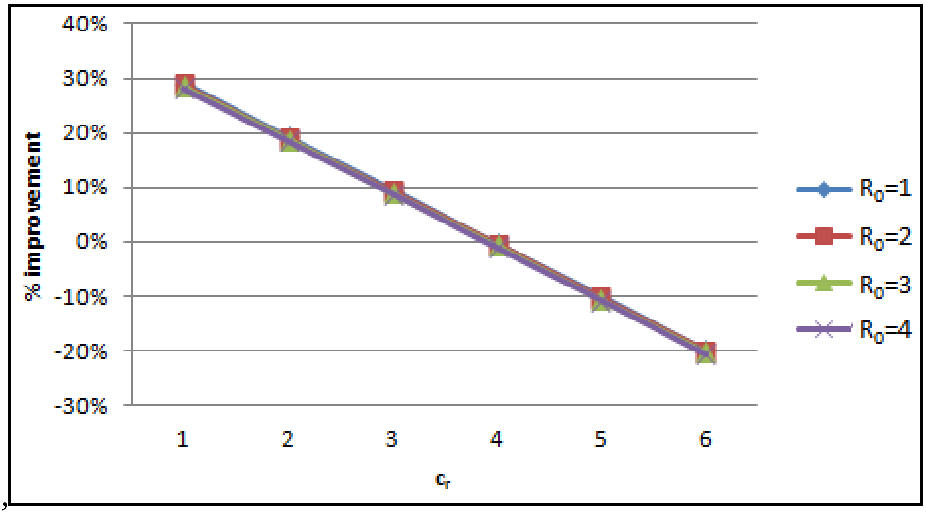

Figure 34- Improvement of TR over TE model vs. c_r (c_p=4)

If the repair cost per item, c_r increases while procurement cost per item, c_p remains constant the % improvement of the TR model over the TE model significantly decreases and when $c_r = c_p$; TR model is outperformed by the TE model.

Single-Echelon Models

Similar to the comparison of TE and TR models, when repair ratio increases, the % improvement of SR model over SE model, increases significantly. In Figure 35, the % improvement curves in L_i shift upwards as ρ increases. The % improvement is formulated as;

$$\frac{Cost_{SE} - Cost_{SR}}{Cost_{SE}} * 100 \qquad (4.3),$$

where; $Cost_{SE}$ and $Cost_{SR}$ denote the costs of SE and SR model, respectively. In the assessment of the improvement of SR model over SE model; the procurement cost cannot be excluded in (4.3) because they are not the same for both models anymore..

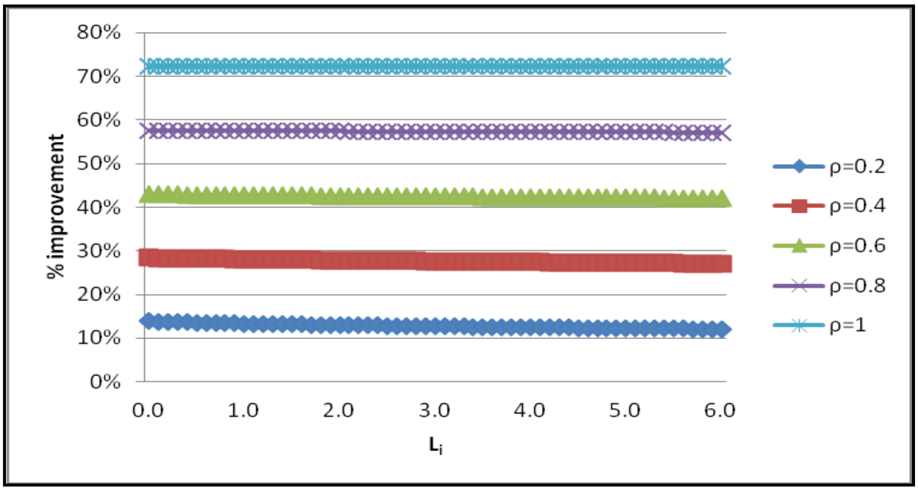

Figure 35- Improvement of SR over SE model vs. L_i

Although the % improvement seems to be steady in L_i, with a closer look in the base case parameters, it can be seen in Figure 36 that it decreases as lead time increases. This is expected, because; as L_i increases, R_i decreases relatively which reduces the difference in the expected costs of SR and SE model. Therefore the improvement of the system with repair ability is hindered.

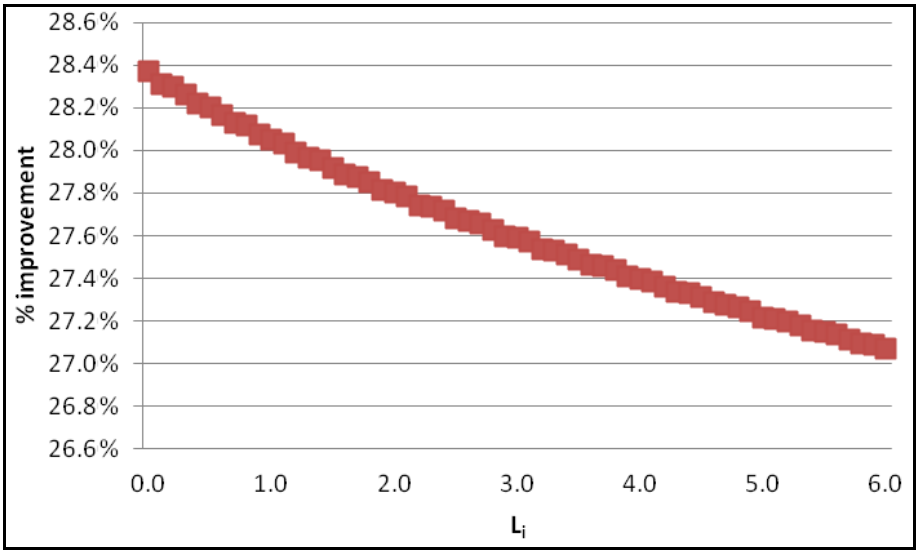

Figure 36- Improvement of SR over SE model vs. L_i (ρ=0.2)

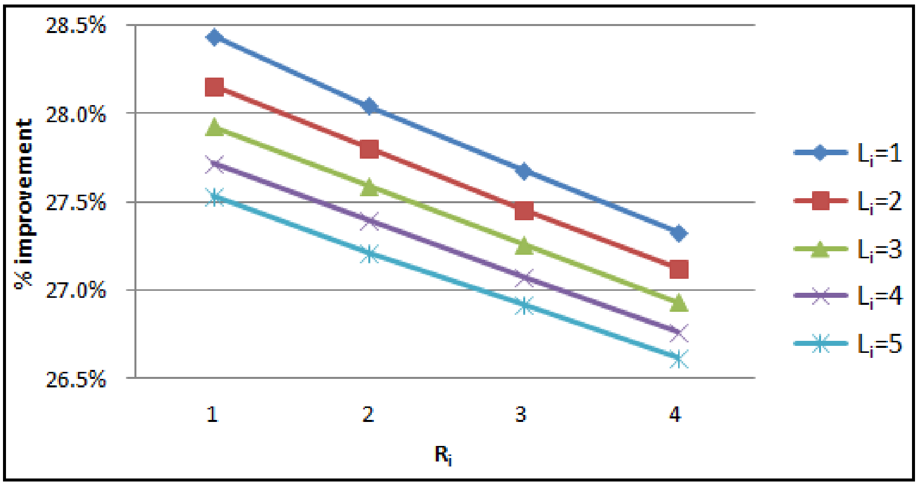

Figure 37- Improvement of SR over SE model vs. R_i

As repair lead time, R_i, increases; the decrease in % improvement of SR over SE shifts downwards. as it is seen in Figure 37. This is also expected as it is in two-echelon case; becasuse shorter repair lead times than the procurement lead time is one of the main advantages of the repair ability acquisition.

As R_i increases and becomes closer to L_i, the advantage of SR model decreases. However, due to lower repair cost per item, c_r, than the procurement cost per item, c_p; the SR system remains the better alternative against the SE model. But, as c_r increases and get close to c_p; the SR model loses advantage against the SE model. In Figure 38, when $c_r = c_p$, the SE model becomes the better alternative against the SR model no matter what R_i

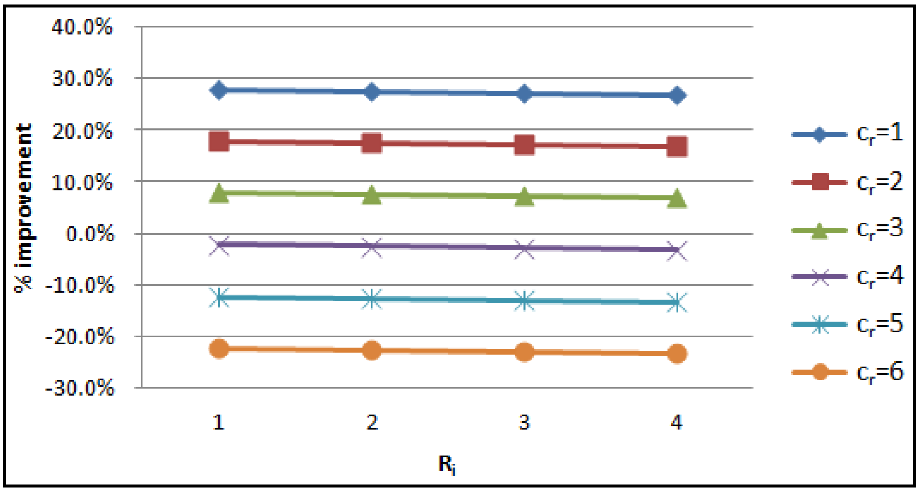

Figure 38- Improvement of SR over SE model vs. R_i

CHAPTER 5

CONCLUSION & FUTURE WORK

In this study, we deal with a real life problem that is encountered in the inventory control system being operated in military. A single item, which has a critical importance in military use, is considered. Currently, a two-echelon system, which consists of a depot in the upper and several bases in the lower echelon, is operated. The demand is stochastic and Poisson; and the lead times are assumed to be constant. Three alternative exact models are developed and presented along with the current system in Chapter 3. One of the alternative models considered is a single-echelon model that has several bases but a depot. The other alternatives are two and single-echelon models with repair ability. All of the models are optimized. The optimal results are compared and the models are experimented with parameter changes such as lead times, demand and cost rates in Chapter 4. The focus is on determining the conditions under which the current two-echelon model. The analyses are conducted accordingly and the findings are reported. The main contribution of the study is that two-echelon models are benchmarked with single-echelon models in an environment where the in-transit inventory holding cost cannot be ignored. Moreover, the importance of the repair ability for inventory systems is assessed.

Throughout the analysis, it is observed that in all of the models the system wide optimal base stock levels increase along with the total cost; as the lead time, backorder cost rate or demand increase.

In two-echelon models, the most critical observation is that as long as a holding cost is incurred for the items in-transit; the two-echelon models could lose its preference over the single-echelon model; especially when transportation lead time increase or procurement lead time decreases.

Although the increase in cost rates increases the costs for both models, they do not affect the improvement of the two-echelon models over single-echelon models. However, as long as the holding cost is incurred for items in-transit the changes in other cost rates become significant. For example, when backorder cost rate decreases or holding cost rate for items in stock increases, the improvement of the two-echelon model over the single-echelon model deteriorates. Furthermore, increasing demand effects the two-echelon model negatively as long as holding cost is incurred for items in-transit although the contrary is valid when there is no holding cost for the items in-transit.

It is observed that if the holding cost rate for items in-transit increases, while the holding cost rate for items in-stock remains the same; the performance of the two-echelon model deteriorates against the single-echelon model. But if the holding cost rate for items in-stock changes in accordance with the holding cost rate for items in-transit, the effect of the holding cost rate for items in-transit vanishes.

If four bases are operated instead of three bases, the expected long run average cost increases in both models. Moreover; the improvement of the two-echelon model over the single-echelon model increases and the two-echelon model becomes more robust against being outperformed by the single-echelon models due to changes in parameters.

In both two and single-echelon models, the acquisition of repair ability is preferable as long as unit repair cost is lower than unit procurement cost and repair lead time is lower than the procurement lead time. However, if repair cost is lower than the unit procurement cost, then the effect of the changes in other parameters on the improvement of the repairable models over no-repair models is very slight. Therefore unit repair cost's being lower than the unit procurement cost would be more than enough for repairable model to be preferable.

As future research direction, the bases can be allowed to ship laterally among themselves as Lee (1987) presented and finally the depot could be allowed to place emergency orders like in Moinzadeh and Schmidt (1991) and Moinzadeh and Aggarwal (1997). Besides, stochastic lead times can be allowed in all of the models.

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APPENDIX A

GOODNESS OF FIT TEST FOR DEMAND

In this thesis study, a real life inventory control problem of a single item is considered. Demand is originated from the item failures in the bases and it is assumed to be Poisson. To ensure that this assumption is valid; a goodness of fit test is conducted on the data obtained from the current system. We use the chi-square, χ^2, distribution for testing the goodness of fit of the data to Poisson distribution. The unit time in study context is a week, therefore to obtain the demand data, we classify the demand occurred in the current system weekly. The demand data of a base in the past 3 years (156 weeks) is checked and presented in Table 7. Among 156 weeks there are 39 weeks in which 3 items are demanded.

Table 8- Demand Data of the Current Model

Demand/ Week (k)	**Frequency observed (f_o)**	**Demand/ Week (k)**	**Frequency observed (f_o)**
0	11	**5**	12
1	25	**6**	7
2	36	**7**	1
3	39	**8**	2
4	23		

To determine whether the observed data fits to Poisson distribution or not, we first hypothesized the data as follows:

H_0 : The demand follows a Poisson distribution

H_1 : The demand does not follow a Poisson distribution

To test the hypothesis, we need the theoretical frequency. Since we, assumed that the demand is Poisson with mean 3 items/week, we test the hypothesis with λ=3. Recall that the Poisson probability mass function of Poisson distribution is:

$$P\{D = j\} = \frac{e^{-\lambda}(\lambda)^j}{j!} \qquad (1),$$

where, *D* is a random variable denoting the demand. In Table 8, the theoretical frequency is calculated using (1); whereas, theoretical frequency is the product of probability of distribution and total number of weeks observed, which is 156.

Table 9- Actual and Theoretical Frequencies

Demand/ Week (k)	**Frequency observed (f_o)**	**Probability of Distribution (λ=3)**	**Theoretical Frequency (f_t)**
0	11	0.0497	7.7667
1	25	0.1493	23.3003
2	36	0.2240	34.9505
3	39	0.2240	34.9505
4	23	0.1680	26.2129
5	12	0.1000	15.7277
6	7	0.0504	7.8638
7	1	0.0216	3.3702
8	2	0.0081	1.2638

In the computation of chi-square test to validate the Poisson assumption, equation in (2) is used.

$$\chi^2_{k-p-1} = \sum_k \frac{f_0 - f_t}{f_t} \qquad (2),$$

where *k–p–1* denotes the degrees of freedom and *p* denotes number of the parameters estimated from the data. Since the only parameter denoting both mean and the variance in Poisson distribution is λ and it is not estimated using the sample data, *p=0*. Therefore, the degrees of freedom are:

$$k - p - 1 = 9 - 0 - 1 = 8.$$

Using the standart table of chi-square distribution, with 0.05 level of significance, the critical value of χ^2 for 8 degrees of freedom is 15.51. Therefore, the decision criteria will be;

Reject H_0 if $\chi^2 > 15.51$; otherwise do not reject H_0.

Using the data in Table 8 and (2); for the current model, $\chi^2 = 4.3763 < 15.51$. Thus, we fail to reject H_0 and cannot conclude that the demand does not fit a Poisson distribution.