CONTINUOUS TIME MEAN VARIANCE OPTIMAL PORTFOLIOS

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ABSTRACT

CONTINUOUS TIME MEAN VARIANCE OPTIMAL PORTFOLIOS

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The most popular and fundamental portfolio optimization problem is Markowitz's one period mean-variance portfolio selection problem. However, it is criticized because of its one period static nature. Further, the estimation of the stock price expected return is a particularly hard problem. For this purpose, there are a lot of studies solving the mean-variance portfolio optimization problem in continuous time. To solve the estimation problem of the stock price expected return, in 1992, Black and Litterman proposed the Bayesian asset allocation method in discrete time. Later on, Lindberg has introduced a new way of parameterizing the price dynamics in the standard Black-Scholes and solved the continuous time mean-variance portfolio optimization problem.

In this thesis, firstly we take up the Lindberg's approach, we generalize the results to a jump-diffusion market setting and we correct the proof of the main result. Further, we demonstrate the implications of the Lindberg parameterization for the stock price drift vector in different market settings, we analyze the dependence of the optimal portfolio from jump and diffusion risk, and we indicate how to use the method.

Secondly, we present the Lagrangian function approach of Korn and Trautmann and we de-

rive some new results for this approach, in particular explicit representations for the optimal portfolio process. In addition, we present the L^2-projection approach of Schweizer for the continuous time mean-variance portfolio optimization problem and derive the optimal portfolio and the optimal wealth processes for this approach. While, deriving these results as the underlying model, the market parameterization of Lindberg is chosen.

Lastly, we compare these three different optimization frameworks in detail and their attractive and not so attractive features are highlighted by numerical examples.

Keywords: Continuous-time mean-variance portfolio optimization problem, Lindberg parameterization, Lagrangian function approach, L^2-projection approach

ÖZ

SÜREKLİ ZAMAN BEKLENEN DEĞER VARYANS OPTİMAL PORTFÖYLERİ

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En popüler ve temel portföy optimizasyonu problemi Markowiz'in tek dönemli ortalama-varyans portföy seçim problemidir. Ancak tek dönemlik statik yapısı ve hisse senedi fiyatı beklenen getirisinin tahmininde yaşanılan problemler nedeniyle eleştirilmektedir. Bu amaçla, sürekli zamanda ortalama-varyans portföy problemini çözen bir çok çalışma bulunmaktadır. Hisse senedi beklenen getirisinde yaşanan problemi çözmek için 1992'de Black ve Litterman kesikli zaman Bayes yöntemini önermiştir. Daha sonra Lindberg, Black-Scholes standart fiyat dinamiklerini yeniden düzenleyerek sürekli zaman ortalama-varyans portföy problemini çözmüştür.

Bu tezde, öncelikle Lindberg'in yaklaşımı ele alınıp jump-difüzyon pazar dinamiği kullanılarak geliştirilmiş ve genel sonuçların ispatları düzeltilmiştir. Ayrıca, Lindberg'in fiyat parametrelendirmesinin etkileri, farklı hisse senedi fiyatı beklenen getirileri kullanılarak gösterilmiş, optimal portföyün jump ve difüzyon risklerine bağımlılığı analiz edilmiş ve yöntemin nasıl kullanılacağı gösterilmiştir.

İkinci olarak, Korn ve Trautman'ın Lagrange fonksiyonu yaklaşımı anlatılmış ve optimal portföy sürecinin tam olarak formunun ifade edilmesi ile ilgili çıkarılan yeni sonuçlar verilmiştir.

Ayrıca, Schweizer'in L^2- projeksiyon yönetimi kullanılarak optimal portföy ve optimal varlık süreçleri elde edilmiştir. Bu sonuçlar elde edilirken, piyasa modeli olarak Lindberg'in modeli kullanılmıştır.

Son olarak, bu üç farklı optimizasyon yönetimi ayrıntılı bir şekilde karşılaştırılmış, öne çıkan özellikleri örneklerle vurgulanmıştır.

Anahtar Kelimeler: Sürekli zaman ortalama-varyans portföy optimizasyon problemi, Lindberg parametrelendirmesi, Lagrange fonksiyonu yaklaşımı, L^2 projeksiyon yaklaşımı

To my family

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CHAPTER 1

INTRODUCTION

1.1 The Portfolio Problem: Introduction and Literature Review

One of the classical problems in financial mathematics is the portfolio optimization problem. An investor would like to have the highest possible return from an investment. However, this has to be counterbalanced by the amount of risk he is able to take. In doing so, he is faced with so-called portfolio optimization problem, i.e. he has to choose how to allocate the initial wealth to the different available assets to satisfy his requirements. Generally, the aim of the investor is to find the investment strategy that will generate the maximum possible return with a certain stability. This will be made more precise in the following.

Before the 1950's, asset allocation decisions were based on experts' opinions only, mainly without quantitative considerations. Later on with the study of Harry Markowitz in 1952 [45], modern portfolio theory based on analytic results has been started. In Markowitz's Mean-Variance (MV) portfolio optimization approach the problem is to minimize the level of risk modelled as the variance of the portfolio return for a given level of expected portfolio return. Here, the aim is to achieve a balancing between risk and return. In this approach the market is studied in a single period setting, that means the investor has to decide how to allocate the initial wealth at the beginning of the holding period and then can only wait until the end of the period. Although, criticized because of its static nature, the MV approach is still preferred by practitioners. It will be explained in detail in Section 1.2.

Markowitz's work was followed by Tobin [59] (1958) who extended the MV approach by adding the risk-free asset to the problem. In addition, in 1964 Sharpe [57] extended the MV optimization results and constructs the capital asset pricing model (CAPM) under conditions of systematic and non-systematic risk.

Instead of using variance as a risk measure there are a lot of papers in the literature that are modifying the MV approach by using new kind of risk measures.

In 1971, Jean [25] used three or more moments as a risk objective instead of just using variance. Shalit and Yitzhaki [56] (1984) presented a new approach based on stochastic dominance to construct optimal portfolios with Mean-Gini objective. Konno and Yamazaki [30] (1991) proposed absolute deviation as a risk measure for MV approach. Harlow [20] (1991) conducted optimal asset allocation with downside risk measure which is based on dispersion below a predefined target return. Young [66] (1998) proposed a new principle while choosing optimal portfolios which minimize maximum loss based on historical data. In 2000, Chekhlov et al. [11] (2000) proposed a new conditional drawdown at risk measure and solved the asset allocation problem.

Some of the authors preferred to use Value at Risk (VaR) as a risk measure. Among them are: Vorst [61] (2000), Wang [62] (2000), Campbell et al. [10] (2001), Gaivoronski and Pflug [18] (2005) and Benati and Rizzi [6] (2007).

One of the main drawbacks of the MV approach is its single period trading nature that the investor is just making decisions at the beginning. Indeed, multiperiod extensions of the MV approach are more practical. In discrete time multiperiod portfolio optimization, the holding period is divided into sequence of time spots and multiple investment decisions are taken.

In 1965, Tobin [60] derived optimal results for the discrete time multiperiod asset allocation problem. A multiperiod problem where the objective is to decide on optimal consumption and investment is then studied by Samuelson [51] in 1969.

Li and Ng [38] (2000) considered an analytical optimal solution to the multiperiod MV model. In 2004, Leippold [37] (2004) presented a geometric approach to the multiperod MV problem that simplifies its mathematical analysis.

In continuous time trading the investor can invest at any time during the holding period. Since the asset prices are frequently changing, modelling asset prices in continuous time is more realistic. However, continuous time modelling needs stochastic analysis and so, it is technically much more involved than studying at discrete time. Approaches to solve the continuous time portfolio optimization problem are mainly divided into two categories: Expected utility maximization problem and Continuous-time MV problem. Continuous-time portfolio optimization is dominated by expected utility maximization studies.

The starting point of continuous-time portfolio optimization is Merton's [46] (1969) study. He derived the optimal strategies for Geometric Brownian Motion (GBM) market setting by using stochastic optimal control approach. In his study, the optimal portfolio strategy and optimal consumption are found explicitly. Later on his study is extended. For instance, short selling constraint are added, the problem is solved in incomplete market setting and solved when the market parameters follow a Markov chain etc. [17], [36], [48] and [58].

After Merton's study and developments in financial mathematics a new method based on martingale representation theory is presented by Harrison and Kreps [21] (1979) and Harrison and Pliska [22] (1981). This martingale method is used and extended in expected utility maximization problems in some of the studies such as: Pliska [49] (1986), Karatzas et al. [27] (1987), Pages [47] (1987) and Cox and Huang [12] (1989).

In 1986, Pliska [49] used the martingale method to find an optimal strategy that maximizes the expected utility when the asset prices are semimartingales. In 1987, Karatzas et al. [27] directly used the martingale method when the market coefficients are constant. In 1987 Pages, [47] used the method when the market is incomplete, and in 1989 Cox and Huang [12] explicitly constructed optimal results when taking into account the non-negativity of consumption and wealth.

While using the martingale method for the expected utility maximization problems the market conditions are very important. When the market is complete the martingale method can directly be used, but when the market is incomplete or when there are short selling constraints in the problem, it is not possible to apply the martingale method directly. For such cases, duality methods that provide information about the main problem is needed. In 1991, Karatzas et al. [28] solved the expected utility maximization problem in an incomplete market setting by using the convex duality theory. In 1992, Xu and Shereve [64, 65] (1976) solved the problem when there are shortselling constraints.

All these continuous time expected utility maximization studies summarized above ignore the transaction costs which are very important in investment decisions. In reality, there are transaction costs and investors tries to avoid them. Magill and Constantinides's [44] study is one of the first studies that is taking into account transaction costs. Their study is followed by Davis and Norm [14] (1990), Korn [34] (1998), Akian et al. [1] (1995), Atkinson and Wilmott [3] (1995), Atkinson et al. [4] (1997), and Liu [43] (2002).

Using utility function gives an idea about investor's risk attitudes and is consistent with economic theory. However, in real world investors do not know their utility functions and defining a utility function is a difficult concept.

Modern portfolio theory seeks to find more realistic models. Although, the continuous time portfolio literature is dominated by expected utility maximization studies, the continuous time MV approach is studied just recently, in this thesis it is preferred to use continuous-time MV approach.

As mentioned before there are two mainly used techniques in expected utility maximization problem: stochastic optimal control method and martingale method. In the continuous-time MV optimization problems there are difficulties in using stochastic optimal control method. This method can not be directly used because of the variance term in the objective function. The reason for this is that the variance cannot be written as an expected utility due to the appearance of expectation $E(X)$ inside the variance $Var(X)$.

To solve this problem in 2000, Li and Ng [38] found an embedding technique and combine the variance term with the expected wealth term in the objective function of the problem. They found analytically optimal results for the discrete time multiperiod MV problem. Then Zhou and Li [67] (2000) extended the multiperiod embedding approach to the continuous time case. They formulated the problem as a bicriteria optimization problem by putting weight on the expected wealth and variance criteria and combine them in a single objective. Then the problem is embedded into a class of auxiliary stochastic linear quadratic (LQ) problem and can be solved analytically. In 2001, Li et al. [39] then solved the problem when shortselling is not allowed. In this study, efficient strategies and the efficient frontier is derived explicitly based on LQ control techniques. Lim [41] (2004) extended the solution to the case when the market is incomplete with random market coefficients. Hu and Zhou [23] (2005) solved the constrained stochastic LQ problem in continuous time MV problem when there is random appreciation, stochastic volatility and shortselling is not allowed.

In 2005, Xia [63] investigated the problem with bankruptcy prohibition for the incomplete market. Li and Zhou [40] (2006) studied the continuous time market when the investor specified the investment horizon and target terminal wealth. In 2009, Dai et al. [13] considered the problem by taking into account the transaction costs by using transformations that turned the problem into double obstacle problem.

In 2010, Basak and Chabakauri [5] also solved the continuous time MV problem with stochastic control method by using a different approach than embedding approach. They have derived a recursion formula for the mean-variance criteria accounting for its time inconsistency and analytically solved the problem.

The continuous MV problem is solved with the martingale method in 1989 by Richardson [50]. In 1995, Korn and Trautmann [31] solved the problem with positive terminal wealth constraints by extending the martingale method. Then in 2005 the martingale method is used by Bielecki [7] to solve the MV problem where all the market coefficients are random under bankruptcy prohibition.

Different approaches are also used in the continuous-time MV problem. For example Duffie [16] (1991) solved the MV hedging problem. His results can be used to solve the MV portfolio problem. Similarly, in 1994 Schweizer [53] solved the MV hedging problem when the asset prices are semimartingales by approximating random variables with stochastic integrals. As a special case the MV portfolio optimization problem is solved.

To solve a portfolio optimization problem all the market coefficients (such as the drift and diffusion coefficients of the stock price) have to be known. However, the estimation of the stock price drift is a particularly hard problem. As locally the quadratic variation dominates the mean of the log returns it takes a very long time until the drift coefficient can be estimated accurately. In 1998, Brennan [9] studied the effect of uncertainty about the mean rate of return of investment and concluded that it has a significant effect on portfolio decisions of the investors. In the discrete time case, Black and Litterman [8] found a way to estimate expected returns by combining equilibrium estimation and investors' views. To get around the problem of not having reliable estimates for the mean rates of return in a continuous-time framework, various authors considered the portfolio problem with incomplete information that included unobservable expected returns (see, e.g. [19, 15, 52].

A completely different approach has been given by Lindberg [42] (2009). He used a new way of parameterizing the price dynamics in the standard Black-Scholes setting. Its main idea is to decompose the mean rate of stock return into a sum made up of the risk-free return plus a term that is closely related to the volatility structure of the stock price processes.

1.1.1 Outline of the Thesis

In this chapter we give the background of portfolio optimization problem and Markowitz's [45] mean-variance approach.

In Chapter 2, the continuous time expected utility maximization portfolio problem and mostly used methods: the martingale method and the stochastic optimal control method are presented and explained with examples.

In Chapter 3, we firstly explain Lindberg's [42] continuous time mean-variance approach and his market parameterization. Then, we give our extensions by incorporation of Poisson jump process, and we show that the continuous-time Markowitz portfolio problem can still be solved explicitly. In addition, we give how to obtain a reliable estimate of the range of the drift parameters in Lindberg's parameterization. Lastly, we give numerical examples and a detailed analysis of the form of the optimal strategy in various market settings.

In Chapter 4, we first introduce the extended martingale approach given in Korn and Trautmann [31]. Then, the Lagrangian function approach to the continuous-time MV portfolio optimization, which has first been presented in Korn and Trautmann [31] and our new results for the explicit representation for the optimal portfolio process are given for Lindberg's [42] continuous time market model.

In Chapter 5, we first give the optimal solutions derived for the continuous-time MV portfolio optimization by using the L^2-projection technique of Schweizer [53] for Lindberg's [42] continuous time market model. Then, we give the attractive features of the three different approaches by numerical examples.

Chapter 6 concludes the thesis.

1.2 Markowitz's Mean-Variance Model

One of the most widely used portfolio optimization techniques is presented by Harry Markowitz [45] in 1952. He conducted the problem as a bicriteria optimization problem. In his approach, there are two criteria: the expected return and variance of the assets. There are different forms of this problem but the mostly used one is to minimize the portfolio return variance when the expected return is equal to a predefined level. In this model, variance is used as the measure of risk. Here, the problem is modelled as a one period discrete time optimization problem which means investors are making their investment decisions at the beginning of the period and can not take any actions until the end of the period.

In portfolio optimization, the utility maximization approach is consistent with the economic theory but as mentioned before determining the personal utility function is a hard concept. However, understanding the behavior of investors is a very important concept for portfolio optimization theory. In the MV approach it is assumed that all the investors are risk averse which means that given equal expected return investors prefer the investment opportunity with less risk. This behavior also leads to requiring the concavity of the utility function.

In the MV approach, the portfolio preference is based on expectation and variance of the asset returns. Under certain assumptions on the distribution of the asset return or about the form of the utility function, expectation and variance are sufficient to describe the portfolio preference and are then even consistent with the expected utility concept.

To see this let us perform the Taylor expansion of the utility function around the expectation of the terminal wealth and get the following expression given in (1.1) [24]:

$$u(W_T) = u(E[W_T]) + u'(E[W_T])(W_T - E[(W_T]) + \frac{1}{2}u''(E[W_T])(W_T - E[(W_T])^2 + O_3. \quad (1.1)$$

Here, O_3 is the term including higher than power second moments.

Assuming that the Taylor series converges and the expectation and summation is interchangeable [24],

$$E[u(W_T)] = u(E[W_T]) + \frac{1}{2}u''(E[W_T])\sigma^2[W_T] + E[O_3]. \quad (1.2)$$

If we do not have any idea about the utility function of the investor and if we assume that the asset returns are multivariate normally distributed then as the normal distribution is completely defined by expectation and variance, the third and higher order moments involved in $E[O_3]$ can be expressed as functions of mean and variance. As a result, risky asset returns can be sufficiently defined via mean and variance.

If we do not know the distribution of the asset returns and we assume that the investor's utility function is a quadratic one then as for the quadratic utility the 3^{rd} and higher order derivatives are zero so, $E[O_3] = 0$ again mean and variance are sufficient for our investment decisions.

The Markowitz approach relies on some assumptions such as [24]:

- Investors seek to maximize the expected return of the total wealth,
- All the investors are risk averse,
- In the market there are at least two assets,
- In the market unlimited shortselling is allowed,
- Assets are liquid and perfectly divisible and there are no transaction costs,
- Asset returns are linearly independent,
- Variance-covariance matrix is non-singular,

In the MV approach the problem can be formalized as;

$$\begin{aligned} &\min_{w}\{\frac{1}{2}w^T\Sigma w\}, \\ &\text{s.t.} \quad w^T\bar{r} = \bar{r}_p, \\ &w^T\mathbf{1} = 1. \end{aligned} \tag{1.3}$$

Here,

w_{nx1} is the weight vector,

$\bar{r}_{nx1}$ is the expected return of assets vector,

Σ_{nxn} is the variance-covariance matrix,

r_p is the target expected rates of return on portfolio,

$\mathbf{1}_{nx1}$ is the vector of ones,

n is the number of assets in the portfolio.

To solve this problem Lagrange multipliers method can be used. The Lagrangian function of this problem is,

$$L(w, \lambda_1, \lambda_2) = \frac{1}{2} w^T \Sigma w + \lambda_1 (r_p - w^T \bar{r}) + \lambda_2 (1 - w^T \mathbf{1}). \tag{1.4}$$

Taking the derivatives of (1.4) with respect to w, λ_1 and λ_2 the necessary first order conditions can be obtained as,

$$\frac{\partial L(w, \lambda_1, \lambda_2)}{\partial w} = \Sigma w - \lambda_1 \bar{r} - \lambda_2 \mathbf{1} = 0, \tag{1.5}$$

$$\frac{\partial L(w, \lambda_1, \lambda_2)}{\partial \lambda_1} = r_p - w^T \bar{r} = 0, \tag{1.6}$$

$$\frac{\partial L(w, \lambda_1, \lambda_2)}{\partial \lambda_2} = 1 - w^T \mathbf{1} = 0. \tag{1.7}$$

Since Σ is positive definite all the first order conditions are necessary and sufficient for a global optimum.

From (1.5), the equation for w is obtained as in (1.8):

$$w = \lambda_1 (\Sigma^{-1} \bar{r}) + \lambda_2 (\Sigma^{-1} \mathbf{1}). \tag{1.8}$$

When multiplying both sides in (1.8) with $\bar{r}^T$ and using (1.6), then we obtain

$$r_p = \lambda_1 (\bar{r}^T \Sigma^{-1} \bar{r}) + \lambda_2 (\bar{r}^T \Sigma^{-1} \mathbf{1}). \tag{1.9}$$

When multiplying both sides in (1.8) with $\mathbf{1}^T$ and using (1.7), then we get

$$\mathbf{1} = \lambda_1 (\mathbf{1}^T \Sigma^{-1} \bar{r}) + \lambda_2 (\mathbf{1}^T \Sigma^{-1} \mathbf{1}). \tag{1.10}$$

Solving (1.9) and (1.10) together when,

$$\begin{aligned} A &= \mathbf{1}^T \Sigma^{-1} \bar{r} = \bar{r}^T \Sigma^{-1} \mathbf{1}, \\ B &= \bar{r}^T \Sigma^{-1} \bar{r}, \\ C &= \mathbf{1}^T \Sigma^{-1} \mathbf{1}, \\ D &= BC - A^2. \end{aligned} \tag{1.11}$$

Then, the Lagrange multipliers are,

$$\begin{aligned} \lambda_1 &= \frac{Cr_p - A}{D}, \\ \lambda_2 &= \frac{B - Ar_p}{D}. \end{aligned} \tag{1.12}$$

Using these results the optimal weight vector becomes,

$$w* = r_p \left[\frac{C}{D} \Sigma^{-1} \bar{r} - \frac{A}{D} \Sigma^{-1} \mathbf{1} \right] + \left[\frac{B}{D} \Sigma^{-1} \mathbf{1} - \frac{A}{D} \Sigma^{-1} \bar{r} \right]. \tag{1.13}$$

The optimal portfolio weight vector represents the portfolios on the frontier. The variance of the frontier is,

$$w^T \Sigma w = \sigma_p^2 = \frac{1}{D} [Cr_p^2 - 2Ar_p + B], \tag{1.14}$$

and the standard deviation is

$$\sigma_p = \sqrt{\frac{1}{D} [Cr_p^2 - 2Ar_p + B]}. \tag{1.15}$$

Figure 1.1 shows the expected return and standard deviations of frontier portfolios. Here, the minimum variance portfolio is obtained by conducting the optimization without the expected

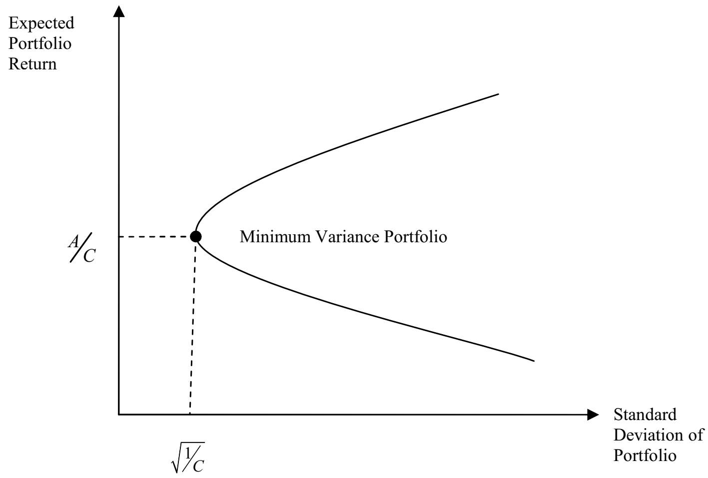

Figure 1.1: Portfolio frontier without risk-free asset

return constraint. It is the minimum variance frontier portfolio with expected return $\frac{A}{C}$. The frontier portfolios which have expected return higher than the minimum variance portfolio are called efficient portfolios. An efficient portfolio is the portfolio which has minimum variance for a given level of expected return or which has a maximum expected return for a given level of variance. All these portfolios form the MV efficient frontier.

In the original version of the MV model the risk-free asset is ignored. However, in the portfolio theory it is often assumed that there exists risk-free asset. The risk-free asset is the asset with (typically) low constant expected return. Thus, the variance of its return equals 0. When the risk-free asset is included the problem becomes,

$$\begin{aligned} &\min_{w}\{\frac{1}{2}w^T\Sigma w\} \\ \text{s.t.}\quad &w^T\bar{r} + (1 - w^T\mathbf{1})r_f = \bar{r}_p. \end{aligned} \tag{1.16}$$

Here, r_f is the risk-free asset return.

The Lagrangian function for this problem is

$$L(w, \lambda) = \frac{1}{2}w^T\Sigma w + \lambda(r_p - w^T\bar{r} - (1 - w^T\mathbf{1})r_f). \tag{1.17}$$

The necessary first order conditions are

$$\Sigma w - \lambda \bar{r} + \lambda \mathbf{1} r_f = 0, \tag{1.18}$$

$$r_p - w^T \bar{r} - (1 - w^T \mathbf{1}) r_f = 0. \tag{1.19}$$

When solving (the first order conditions together (1.20) and (1.21) are obtained when A, B, C and D are as in (1.11).

$$\lambda = \frac{r_p - r_f}{B - 2Ar_f + Cr_f^2} \tag{1.20}$$

$$w = \Sigma^{-1}(\bar{r} - r_f \mathbf{1}) \frac{(r_p - r_f)}{B - 2Ar_f + Cr_f^2} \tag{1.21}$$

Then the variance of the portfolio is

$$w^T \Sigma w = \sigma_p^2 = \frac{(r_p - r_f)^2}{B - 2Ar_f + Cr_f^2}. \tag{1.22}$$

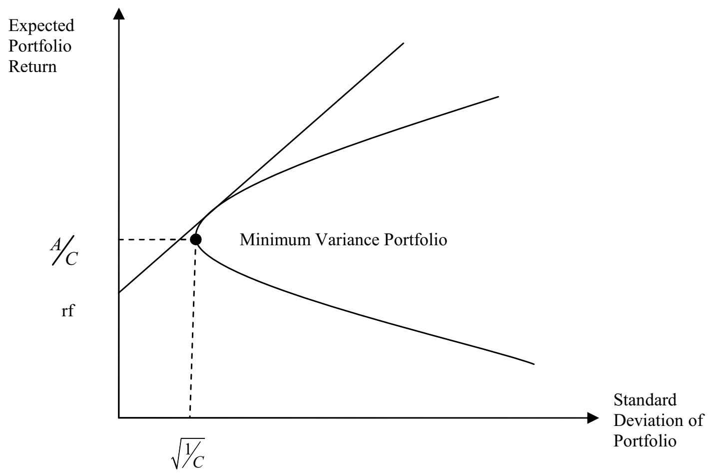

Figure 1.2: Portfolio frontier with risk-free asset

In Figure 1.2, we can see the efficient frontier when the risk-free asset is included into the portfolio. The efficient frontier without the risk-free asset is a curve but when it is added the frontier becomes a straight line. This efficient frontier line is called the capital market line. It starts with the risk-free asset return and is tangent to the efficient frontier. The tangency point represents the market portfolio which is the portfolio that includes all the risky assets in the market.

1.2.1 Criticisms of the Mean-Variance Approach

Markowitz MV approach is one of the most used portfolio optimization method as it is easy to use, but it is also often critized.

One criticism is that the portfolio variance is used as a risk measure. Variance is a symmetric risk measure which does not only limit the losses but also possible gains [33]. To solve this problem in the literature there are a lot of papers preferring different risk measures (see[6, **?**, 10, 11, 18, 20, 25, 30, 56, 61, 62, 66]).

The other main criticism is the effect of estimation errors. The optimal portfolio weights are highly sensitive to the estimation of the expected return of the different assets. To achieve good statistical estimates of expected return portfolio managers need large samples of historical data. Otherwise the MV approach yields unstable portfolio weights. However, it is not always possible to obtain such data. For this purpose, in 1992 Black and Litterman [8] proposed the Bayesian asset allocation approach to solve this problem.

The other criticism of the MV approach is its one period static nature [33]. As mentioned before, in the one period model the investor makes the asset allocation decision at the beginning and can not perform any action until the end of the period. However, in reality assets are highly volatile and the MV approach needs frequent rebalancing between time intervals. For this purpose, there are many discrete time multiperiod (see [37, 38]) and continuous time(see [7, 13, 23, 31, 39, 40, 41, 42, 50, 63, 67]) extensions.

CHAPTER 2

THE CONTINUOUS-TIME PORTFOLIO OPTIMIZATION PROBLEM: A GENERAL OVERVIEW

In this chapter, the continuous time expected utility maximization portfolio problem and the main two methods to solve this problem, the martingale method and the stochastic optimal control method, will be summarized.

2.1 The Financial Market Model

In this chapter, while explaining the continuous time portfolio optimization techniques, the standard Black Scholes market dynamics is used for the prices of n+1 assets, one of them is risk-free and the others are risky assets with the following price evolutions:

$$dS_0(t) = S_0(t)rdt, \;\; S_0(0) = 1, \tag{2.1}$$

$$dS_i(t) = S_i(t)(\mu_i dt + \sum_{j=1}^{n} \sigma_{i,j} dB_j(t)). \tag{2.2}$$

Here, r is the continuously compounded interest rate stated as the rate of the risk-free asset, $\mu = (\mu_1, \ldots, \mu_n)^T$ is the drift parameter vector, $\sigma := (\sigma_{i,j})_{i,j=1}^{n}$ is the volatility matrix. $B(t) = (B_1(t), \ldots, B_n(t))^T$ is an n-dimensional Brownian motion (consisting of components which are independent one-dimensional Brownian motions) which is defined on a complete probability space (Ω, F, P). μ_i, σ and r are all assumed to be F_t adapted, progressively measurable and uniformly bounded variables where F_t is the natural filtration. In addition, $\sigma^T\sigma$ is assumed to be positive definite (ω-wise). These properties of the market coefficients ensure the existence of a unique solutions of the stochastic differential equations for the price dynamics. The solutions are given by,

$$S_0(t) = \exp(rt), \tag{2.3}$$

and

$$S_i(t) = S_i(0) \exp\left(\left(\mu_i - \frac{1}{2}\sum_{j=1}^{n} \sigma_{i,j}^2\right) t + \sum_{j=1}^{n} \sigma_{i,j} B_j(t)\right). \tag{2.4}$$

Suppose the non-negative initial wealth of the investor is denoted by w and the portfolio process $\pi_i(t)$ is the proportion of wealth invested in i^{th} risky asset at time t. We assume that investors act in a self-financing way. This means that the investor's wealth only changes due to gains/losses from investment. The evolution of the investor's self-financing wealth process is given in Equation (2.5):

$$dW(t) = W(t)\left(\sum_{i=1}^{n} \pi_i(t)(\mu_i - r)dt + \sum_{i=1}^{n}\sum_{j=1}^{n} \pi_i(t)\sigma_{i,j} dB_j(t) + rdt\right). \tag{2.5}$$

The matrix notation of the self financing wealth process is

$$dW(t) = W(t)\{(1 - \pi_t^T \mathbf{1})rdt + \pi_t^T(\mu dt + \sigma dB(t)\}. \tag{2.6}$$

Here, $\pi_t \in A(w)$ is the admissible portfolio strategy vector defined in Definition 2.1.1 and $A(w)$ is the set of admissible strategies.

Definition 2.1.1 *The $\mathbb{R}^n$ valued, F_t progressively measurable stochastic portfolio process π_t is an admissible strategy if it is self-financing and the corresponding wealth process satisfies $W(t) \geq 0$ a.s. P when $w \geq 0$.*

For technical reasons, an admissible portfolio strategy should satisfy the integrability requirements. Since μ, r and σ are uniformly bounded,

$$\int_0^T \pi_t(t)^2 dt < \infty \ \ a.s.\ P, \tag{2.7}$$

ensures the existence and uniqueness of the solution of the wealth process.

2.2 The Continuous Time Portfolio Problem

For a given initial wealth $w \geq 0$ the aim of the investor is to maximize the expected utility of consumption over the trading period and/or expected utility of terminal wealth. We will not consider consumption in this thesis.

The aim of this thesis is to deal with the continuous time MV problem. For this reason, we introduce the methods that are mainly used in continuous time portfolio optimization.

First, we state the portfolio problem given as,

$$\max_{\{\pi_t \in A'(w)\}} E[U(W_T)], \tag{2.8}$$

where $U(\cdot)$ is the utility function defined in Definition 2.2.1 and $A'(w)$ is the set of all admissible strategies that additionally satisfy $E[U^(W_T)^-] < \infty$. When we define the problem for $\pi_t \in A(w)$ then we restrict the solution for the set of strategies which has finite expected utility and we would possibly exclude an optimal strategy. However, infinite possible utility is a desired property. Therefore, here the problem is defined on $A'(w)$ where in finite utility is allowed but only expectation over negative parts of the utility is finite.

Definition 2.2.1 *A function $U(\cdot) : (0, \infty) \to \mathbb{R}$, which is strictly concave and continuously differentiable, is called a utility function if*

$$U'(0) := \lim_{w \to 0} U'(w) = \infty \ \ and \ \ U'(\infty) := \lim_{w \to \infty} U'(w) = 0. \tag{2.9}$$

Examples of such utility functions are power utility $U(x) = \frac{1}{\alpha} x^\alpha, \quad$ where $\alpha \in (0, 1)$ and logarithmic utility function $U(x) = \ln(x)$ [33].

2.3 Continuous Time Portfolio Optimization Methods

In this section, the two basic methods used in continuous time portfolio optimization problem will be introduced. Here, for the definitions, theorems and proofs; studies of Korn (1997) [33], Karatzas and Shreve (1998)[29], Korn and Korn [35] and Akume (1995) [2] are used.

2.3.1 The Martingale Method

The martingale method is first presented by Harrison and Kreps [21] (1979) and Harrison and Pliska [22] (1981). Then it is used and extended in expected utility maximization problems in various papers such as Pliska [49] (1984), Karatzas et al. [27] (1987), Pages [47] (1987) and Cox and Huang [12] (1989).

The martingale method method consists of two steps. In the first step, the optimal terminal wealth is found by decomposing the dynamic portfolio optimization problem into a static portfolio optimization problem and following by a representation problem. This decomposition is based on the completeness of the market that will be verified and explained in detail in this section later on. The second step is the determination of the optimal strategy. This step is again depending on the completeness of the market and martingale representation theorem.

Before starting to introduce the steps of the martingale method some preliminary definitions and theorems are needed.

For the direct usage of martingale method completeness of the market is crucial. Completeness means that every (suitably integrable) F_T-measurable payment can be replicated by a wealth process corresponding to a suitable admissible portfolio process $\pi(t)$.

Completeness of the market especially implies the existence of a unique state price density given in (2.10).

$$H(t) := e^{-rt-\frac{\|\theta\|^2 t}{2}-\theta^T B(t)}, \tag{2.10}$$

where θ is the market price of risk,

$$\theta = \sigma^{-1}(\mu - r)\mathbf{1}. \tag{2.11}$$

There are two important theorems that built basis for the martingale method.

Theorem 2.3.1 *For every $\pi_t \in A(w)$ and corresponding wealth process $W(t)$, we have*

$$E[H(t)W(t)] \leq w. \tag{2.12}$$

This is the budget constraint which states that the expected discounted terminal wealth can not exceed the initial wealth.

Proof. Let $\pi_t \in A(w)$, by using the definitions of the state price density $H(t)$ and wealth process $W(t)$ and Ito's Lemma and product rule,

$$H(t)W(t) = w + \int_0^t H(s)dW_s + \int_0^t W(s)dH_s + < W, H >_t, \tag{2.13}$$

$$H(t)W(t) = w + \int_0^t H(s)W(s)(\pi_s^T \sigma - \theta^T)dB_s, \tag{2.14}$$

are obtained.

Since $\pi_t \in A(w)$, $w \geq 0$, $W(t) \geq$ and $H(t) \geq 0$, then the left side of the equation is non-negative. Here, the Ito integral part is a local martingale and Fatou's Lemma states that a non-negative local martingale is a super martingale. Thus,

$$E[H(t)W(t)] = E[w + \int_0^t H(s)W(s)(\pi_s^T \sigma - \theta^T)dB_s] \leq w, \tag{2.15}$$

completes the proof ■

Theorem 2.3.2 *Let $w \geq 0$ be given and ξ be a non-negative F_T measurable variable with*

$$w := E[H(T)\xi] < \infty. \tag{2.16}$$

Then there exists an admissible portfolio process $\pi_t \in A(w)$ such that $\xi = W(T)$ a.s. P

Proof. Let us define the non negative martingale,

$$M(t) := H(t)W(t) = E[H(T)\xi | F_t]. \tag{2.17}$$

Here, F_t is the Brownian filtration. Then, $M(t)$ is an F_t martingale with $M(0) = m$ $a.s.$ P. According to the martingale representation theorem there exists a progressively measurable $\mathbb{R}^n$ valued process $\psi(\cdot)$ satisfying;

$$\int_0^T \|\psi(s)\|^2 ds < \infty \; a.s. \; P \,, \tag{2.18}$$

and

$$M(t) = w + \int_0^t \psi^T(s) dB_s \; a.s. \; P. \tag{2.19}$$

In addition let us define,

$$W(t) := \frac{1}{H(t)} E[H(T)\xi | F_t]. \tag{2.20}$$

From the definition of $W(t)$ it is adapted to the filtration F_t, $W(t) \geq 0$ and as F_t is the Brownian filtration the conditional expectation of a random variable given F_0 $P - a.s.$ is constant that coincides with its unconditional expectation leads $W(0) = w$ $P - a.s.$ then $W(T) = \frac{1}{H(T)} E[H(T)\xi | F_T] = \xi$ $P - a.s.$

From the definitions of $M(t)$ and $W(t)$ we get

$$M(t) = H(t)W(t) = w + \int_0^t \psi^T(s) dB_s \; \text{ a.s P.} \tag{2.21}$$

From Ito's Lemma we obtain

$$H(t)W(t) = w + \int_0^t H(s)W(s)(\pi_t^T \sigma - \theta^T) dW_s. \tag{2.22}$$

Equalizing (2.21) and (2.22) yields

$$\pi_t = \sigma^{-1} \frac{\psi(t)}{W(t)H(t)} + \theta. \tag{2.23}$$

Since $W(t) \geq 0$ is continuous then π_t is an admissible strategy. Then $W(t)$ is the wealth process implying $\pi_t \in A(w)$. ■

2.3.1.1 Derivation of the Martingale Method

For every possible final wealth corresponding to an admissible strategy Theorem 2.3.1 yields that we have $E[H(T)W(T)] \leq w$. We thus consider the static optimization problem where we are only maximizing the expected utility over non-negative F_T measurable random variable ξ that satisfy this budget constraint. Indeed, from Theorem 2.3.2 we know that for the optimal ξ^* such random variable for the static problem, there exists an admissible strategy such that $\xi^* = W^*(T)$. As a result, we can consider the following static problem instead of our original portfolio problem (this will be verified rigorously later on):

$$\max_{\xi \in Z(w)} E[U(\xi)], \tag{2.24}$$
$$\text{where, } Z(w) = \{\xi \geq 0,\ \xi \text{ is } F_T \text{ measurable, } E[H(T)\xi] \leq w, \text{ and } E[U(\xi)^-] < \infty\}.$$

It will be verified later that the budget constraint is satisfied as equality for the optimal wealth.

As a result, the martingale method is the decomposition of the dynamic portfolio optimization problem into a static optimization problem and a representation problem that consists of finding optimal strategy.

Let us start with solving the static problem. It is a convex optimization problem with linear budget constraint so it can be solved - at least heuristically - with the Lagrange multiplier method. Its heuristic calculation is considered by defining the Lagrangian function,

$$L(\xi, y) = E[U(\xi)] - y(E[H(T)\xi] - w). \tag{2.25}$$

Here, $y > 0$ is the Lagrange multiplier and the (heuristic analogues to the) first order conditions are,

$$E[U'(\xi) - yH(T)] = 0 \ \text{ and} \tag{2.26}$$
$$E[H(T)\xi] - w = 0. \tag{2.27}$$

The first condition yields $\xi^* = (U')^{-1}(yH(T)) = I(yH(T))$. Here, $I(\cdot)$ is the inverse function which is strictly decreasing from $(0, \infty)$ into $(0, \infty)$. The second condition yields $E[H(T)\xi^*] =$

w. Then putting ξ^* into the second condition yields $X(y) := E[H(T)I(yH(T))] = w$ results in $y^* = X^{-1}(w)$ and $\xi^* = I(y^*H(T))$. This heuristic computation will be verified later on by a different method.

Firstly, let us start with the verification of optimal y. Assume that $X(y) < \infty$ for all $y > 0$. Then $X(y)$ satisfies the following conditions:

- $X(y)$ is strictly decreasing since $H(t) > 0$ and $I(\cdot)$ is strictly decreasing on $(0, \infty)$,
- $X(y)$ is continuous on $(0, \infty)$ from the dominated convergence theorem since $H(\cdot)$ and $I(\cdot)$ are continuous,
- $X(0) := \lim_{y\to 0} X(y) = \infty$ and $X(\infty) := \lim_{y\to\infty} X(y)) = 0$
  are results of the dominated convergence theorem, since $I(0) := \lim_{y\to 0} I(y) = \infty$ and $I(\infty) := \lim_{y\to\infty} I(y)) = 0$ and $I(\cdot)$ is strictly decreasing.

The properties of $X(y)$ imply the existence of $X^{-1}(w)$ on $(0, \infty)$, $X^{-1}(0) := \lim_{w\to 0} X^{-1}(w) = \infty$ and $X^{-1}(\infty) := \lim_{w\to\infty} X^{-1}(w) = 0$. Therefore, an optimal y can always be found from $X^{-1}(w) = y^*$.

The optimal solution of the static problem is $\xi^* = I(y^*H(T))$. From the definition of ξ^* and $X^{-1}(w)$ the budget constraint is satisfied as an equality. Positivity of $X^{-1}(w)$, $H(t)$ and $I(\cdot)$ results in the positivity of ξ^*. According to Theorem 2.3.2 existence of admissible optimal strategy is obvious. Thereafter, the optimal portfolio strategy can be found by using (2.23).

In the following we will rigorously verify the optimality of the portfolio strategy π_t^*. For this, we first derive a simply ingredient. By using the concavity of the utility function, we obtain the inequality (2.28):

$$U(I(y)) \geq U(w) + U'(y)(I(y) - w) = U(w) + y(I(y) - w). \tag{2.28}$$

Then, we obtain

$$U(\xi^*) \geq U(1) + X^{-1}(w)H(T)(\xi^* - 1). \tag{2.29}$$

From $a \geq b \Rightarrow a^- \leq b^- \leq |b|$ and using that $U(\cdot)$ is continuous and $E[H(T)]$ is bounded,

$$E[U(\xi^*)^-] \le |U(1)| + X^{-1}(w)(w + E[H(T)]) < \infty. \tag{2.30}$$

Thus, we have shown that the candidate for the optimal final wealth satisfies $E[U(\xi^*)^-] \le \infty$, and thus the corresponding portfolio process π_t^* is in $A'(w)$.

Lastly, the optimality of π_t^* and the corresponding final wealth $W^*(T)$ will be verified. Consider any admissible portfolio strategy $\pi_t \in A'(w)$. Due to (2.28), we get

$$U(\xi^*) \ge U(W(T)) + X^{-1}(w)H(T)(\xi^* - W(T)). \tag{2.31}$$

Taking expectation on both sides yields

$$\begin{aligned} E[U(\xi^*)] &\ge E[U(W(T)] + X^{-1}(w)E[H(T)(\xi^* - W(T))] && (2.32) \\ &\ge E[U(W(T)] + X^{-1}(w)(w - E[H(T)W(T)]). && (2.33) \end{aligned}$$

As a result from $(w - E[H(T)W(T)]) \ge 0$ and $X^{-1}(w) > 0$,

$$E[U(\xi^*)] \ge E[U(W(T))]. \tag{2.34}$$

Hence, we have rigorously proved the optimality of ξ^* and corresponding π_t^* without the need to use our initially made heuristic computations.

To understand the application of the martingale method even more, we will give an example.

Example 2.3.3

Suppose the asset prices and wealth processes are given as in (2.1), (2.2), (2.5) and the investor's utility function is a logarithmic utility $U(w) = \ln(w)$. Then the portfolio problem is

$$\max_{\pi_t \in A'(w)} E[\ln(W_T)]. \tag{2.35}$$

Then the static problem is

$$\max_{\xi \in Z(w)} E[\ln(\xi)], \tag{2.36}$$

where, $Z(w) = \{\xi > 0,\ \xi \text{ is } F_T \text{ measurable}, E[H(T)\xi] \leq w \text{ and } E[U(\xi)^-] < \infty\}$.

Since, $U'(w) = \frac{1}{w}$ and $I(y) = \frac{1}{y}$ then the optimal ξ is $\xi^* = I(y^* H(T)) = \frac{1}{y^* H(T)}$. For ξ^* to be optimal, y should be found as

$$X(y) = E[H(T)I(yH(T))] = E[H(T)\frac{1}{yH(T)}] = \frac{1}{y} = w, \tag{2.37}$$

$$X^{-1}(w) = y^* = \frac{1}{w}. \tag{2.38}$$

As a result, $\xi^* = \frac{w}{H(T)}$.

From (2.23), $\pi^* = \sigma^{-1}\theta$, since $M(t) = H(t)W^*(t) = E[H(T)\xi^*|F_t] = w$ and, so $\psi = 0$.

2.3.2 The Stochastic Optimal Control Method

The stochastic optimal control method is the other method typically used to solve the continuous time portfolio optimization. It was first applied by Merton [46] in 1969 to the expected utility maximization problem. This method depends on Bellman's principle of optimality rule and the derivation of the so-called Hamilton-Jacobi-Bellman (HJB) equation.

Let us start with defining a whole family of portfolio problems via the introduction of the value function

$$V(t,w) := \max_{\pi_t \in A'(t,w)} E_{t,w}[U(W_T)]. \tag{2.39}$$

Here, $E_{t,w}[U(W_T)] = E[U(W_T)|W(t) = w]$ is the expected utility from the terminal wealth given that the wealth process starts at time t with the initial value of w. The terminal value of the value function is obviously given by $V(T,w) = U(W(T))$.

For a given $(t,w) \in [0,T]\ (0,\infty)$ if we divide the time into $[t, t+\Delta t]$ and $[t+\Delta t, T]$ the Bellman principle of optimality states that an optimal strategy from t to T is also optimal from $t + \Delta t$

to T. In other words, it is to choose an optimal strategy on $[t, t + \Delta t]$ and behave optimally on $[t + \Delta t, T]$ using the fact that the optimal strategy on $[t, t + \Delta t]$ is at least as good as strategy on $[t, T]$.

When the problem is formalized according to the Bellman's principle the equality given in (2.40) is obtained.

$$V(t, w) = \max_{\pi_{t \in [t,t+\Delta t]} \in A(w)} E_{t,w}[V(t + \Delta t, W(t + \Delta t))]. \tag{2.40}$$

Formal application of Ito's Lemma (note that we have to assume sufficient differentiability of the value function) yields

$$\begin{aligned} V(t + \Delta t, W(t + \Delta t)) &= V(t, w) + \int_t^{\Delta t} V_t(s, W(s))ds \qquad (2.41) \\ &+ \int_t^{\Delta t} V_w(s, W(s))dWs + \frac{1}{2}\int_t^{\Delta t} V_{ww}(s, W(s))d < W, W > s \\ &= V(t, w) + \int_t^{\Delta t} (V_t(s, W(s)) + V_w(s, W(s))W(s)(r + \pi_t^T(\mu - r)) \\ &+ \frac{1}{2}V_{ww}(s, W(s))W^2(s)\pi_t^T \sigma\sigma^T \pi_t)ds + \int_t^{\Delta t} V_w(s, W(s))W(s)\pi_t^T \sigma dWs \end{aligned}$$

We further assume $E_{t,w}\left[\int_t^{\Delta t} V_w(s, W(s))W(s)\pi_t^T \sigma dWs\right] = 0$ which results in

$$\begin{aligned} E_{t,w}[V(t + \Delta t, W(t + \Delta t))] &= V(t, w) + E_{t,w}\bigg(\int_t^{\Delta t} (V_t(s, W(s)) \qquad (2.42) \\ &+ V_w(s, W(s))W(s)(r + \pi_t^T(\mu - r)) + \frac{1}{2}V_{ww}(s, W(s))W^2(s)\pi_t^T \sigma\sigma^T \pi_t)ds\bigg) \end{aligned}$$

After putting the result in (2.42) into (2.40), subtracting $V(t, w)$ from both sides, dividing both sides by Δt and taking the limit of $\Delta t \to 0$ formally yields the HJB equation given in (2.43):

$$\max_{\pi_t \in A(w)} \{V_t(t, w) + V_w(t, w)w(r + \pi_t^T(\mu - r)) + \frac{1}{2}V_{ww}(s, w)w^2\pi_t^T \sigma\sigma^T \pi_t)\} = 0. \tag{2.43}$$

To make all these heuristical derivations rigorous, we state a so-called verification theorem that says that a particular solution of the HJB-equation coincides with the value function of our problem. Further, it also gives a way to obtain a corresponding optimal trading strategy (for a proof of the verification theorem consider monographs on optimal portfolios or on stochastic control such as Korn and Korn (2001)):

Theorem 2.3.4 (Verification theorem for the HJB-equation) *Let the utility function of the portfolio problem be polynomially bounded. Then we have:*
a) If there exists a polynomially bounded in w solution $\tilde{V}(t,w)$ to the HJB-equation (2.43) with final condition $V(T,w) = U(w)$ then we have

$$\tilde{V}(t,w) \geq V(t,w) \; \forall (t,w) \in [0,T] \times (0,\infty). \tag{2.44}$$

b) If there exists a process π_t^* that attains the maximum in Equation (2.43) for all possible $(t,w) \in [0,T] \times (0,\infty)$ and that is an admissible portfolio process, then π_t^* is an optimal portfolio process for the portfolio problem.

The verification theorem now tells us exactly what we have to do to solve the HJB equation. We first solve the optimization problem in the HJB-equation formally. For this, we look at the it is needed to apply first order conditions to get,

$$V_w(t,w)w((\mu - r) + V_{ww}(t,w)w^2\pi_t^T\sigma\sigma^T = 0, \tag{2.45}$$

which yields

$$\pi_t^{opt} = -\frac{V_w(t,w)}{V_{ww}(t,w)w}(\sigma\sigma^T)^{-1}(\mu - r). \tag{2.46}$$

Note that we here assume that the first order conditions are sufficient to yield the maximum, a fact that we have to check later. Replacing this control in (2.43) gives us the following partial differential equation (PDE)

$$\{V_t(t,w) + V_w(t,w)w(r + (\pi_t^{opt})^T(\mu - r)) + \frac{1}{2}V_{ww}(s,w)w^2(\pi_t^{opt})^T\sigma\sigma^T\pi_t^{opt})\} = 0, \tag{2.47}$$
$$V(T, W(T)) = U(W(T)).$$

Solving the PDE given in (2.47) then yields the value function and the optimal portfolio strategy. However, note that we still have to check all the assumptions made in the calculations and the ones that are needed to apply the verification theorem.

To understand the application of the stochastic optimal control method better, we will give an example below.

Example 2.3.5

Suppose the asset prices and wealth processes are given in (2.1), (2.2), (2.5). When we want to solve the expected utility maximization problem for the investor with power utility function $U(w) = \frac{1}{1-\gamma} w^{1-\gamma}$, the portfolio problem is,

$$\begin{aligned} V(t, w) &= \max_{\pi_t \in A(w)} E\left[\frac{1}{1-\gamma} W(T)^{1-\gamma}\right] \\ V(T, w) &= \frac{1}{1-\gamma} w^{1-\gamma}. \end{aligned} \tag{2.48}$$

The HJB equation for the problem has the form of

$$\max_{\pi_t \in A(w)} \{V_t(t, w) + V_w(t, w) w(r + \pi_t^T(\mu - r)) + \frac{1}{2} V_{ww}(s, w) w^2 \pi_t^T \sigma \sigma^T \pi_t)\} = 0. \tag{2.49}$$

The first order condition yields

$$\pi_t^{opt} = -\frac{V_w(t, w)}{V_{ww}(t, w) w} (\sigma \sigma^T)^{-1} (\mu - r). \tag{2.50}$$

Inserting the (candidate for the) optimal strategy into the HJB-equation (2.49) gives the PDE given in (2.51):

$$\begin{aligned} V_t(t, w) + rw V_w(t, w) - \frac{1}{2} (\mu - r)^T (\sigma \sigma^T)^{-1} (\mu - r) \frac{V_w(t, w)^2}{V_{ww}(t, w)} &= 0, \\ V(T, w) &= \frac{1}{1-\gamma} w^{1-\gamma}. \end{aligned} \tag{2.51}$$

To solve these kind of PDEs, one usually tries a separation approach. Suppose $V(t, w) = \frac{1}{1-\gamma} w^{1-\gamma} f(t)$, where $f(T) = 1$, then we obtain

$$V_t(t,w) = \frac{1}{1-\gamma} w^{1-\gamma} f'(t), \tag{2.52}$$
$$V_w(t,w) = w^{-\gamma} f(t),$$
$$V_{ww}(t,w) = -\gamma w^{-\gamma-1} f(t).$$

With these relations the PDE given in (2.51) reduces to an ordinary differential equation (ODE) of the form

$$f'(t) + \left\{ (1-\gamma)(r + \frac{1}{2\gamma}(\mu - r)^T(\sigma\sigma^T)^{-1}(\mu - r)) \right\} f(t) = 0 \tag{2.53}$$
$$f(T) = 1.$$

If we define $K := (1-\gamma)(r + \frac{1}{2\gamma}(\mu - r)^T(\sigma\sigma^T)^{-1}(\mu - r))$ then the ODE reduces to

$$f'(t) + Kf(t) = 0 \tag{2.54}$$
$$f(T) = 1.$$

This simple ODE has the unique solution

$$f(t) = e^{K(T-t)}. \tag{2.55}$$

As a result we obtain

$$V(t,w) = \frac{1}{1-\gamma} w^{1-\gamma} f(t) = \frac{1}{1-\gamma} w^{1-\gamma} e^{K(T-t)}, \tag{2.56}$$
$$V_w(t,w) = w^{-\gamma} e^{K(T-t)},$$
$$V_{ww}(t,w) = -\gamma w^{-\gamma-1} e^{K(T-t)}.$$

$$\pi_t^{opt} = \frac{1}{\gamma}(\sigma\sigma^T)^{-1}(\mu - r). \tag{2.57}$$

It can now be directly verified that

- π_t^{opt} is an admissible strategy as it is constant,
- the value function is polynomially bounded in w and is concave. Thus, in particular the first order condition indeed is sufficient to deliver a maximum in the HJB-equation.

Consequently, all assumptions that are used to derive the solution to the HJB-equation and to apply the verification theorem are justified. Hence, the portfolio process π_t^{opt} is indeed an optimal portfolio process for our problem which is thus solved completely.

However, one has to admit that such explicit solutions to stochastic control problems are more the exception than the rule.

CHAPTER 3

THE DIRECT CONTINUOUS TIME MEAN-VARIANCE PORTFOLIO OPTIMIZATION APPROACH BY LINDBERG AND ITS EXTENSIONS

To solve a portfolio optimization problem all the market coefficients (such as the drift and diffusion coefficients of the stock price) have to be known. However, the estimation of the stock price drift is a particularly hard problem. As locally the quadratic variation dominates the mean of the log returns, it takes a very long time until the drift coefficient can be estimated accurately. In addition, when the expected return is estimated from historical data, portfolio weights are unstable over time. For this purpose Lindberg [42] (2009) solved the continuous time MV problem directly by using a new way of parameterizing the price dynamics in the standard Black-Scholes setting. In this chapter, we will firstly explain Lindberg's approach. Then, we will explain our extensions by incorporating Poisson jump process, and we will show that the continuous-time Markowitz portfolio problem can still be solved explicitly using Lindberg's method as in [42]. As a slight correction to [42] we will remark that the strategy, he claims to be optimal, is only optimal among the class of deterministic portfolio processes. In addition, we will show how to obtain a reliable estimate of the range of the drift parameters in Lindberg's parameterization. Lastly, we will give numerical examples and a detailed analysis of the form of the optimal strategy in various market settings. All these extensions, corrections and examples can be found in [54].

3.1 The Direct Continuous Time Mean-Variance Portfolio Optimization Approach by Lindberg

In this section, the continuous time approach of Lindberg will be explained. Since, the estimation of expected return is a difficult concept in portfolio optimization, in [42] the classical n asset Black-Scholes market model is modified with a new parameterization that the expected return can be expressed with the help of the volatility. Here, the market consists of $n+1$ assets: n risky assets and one risk-free asset. The i^{th} risky asset price process is denoted by S_i and has the following dynamics with constant model parameters:

$$dS_i(t) = S_i(t)(rdt + \sum_{j=1}^{n} \sigma_{i,j}[(\mu - r)dt + dB_j(t)]). \tag{3.1}$$

Here, r is the continuously compounded interest rate, $\mu > r$ is the common drift parameter, $\sigma := (\sigma_{i,j})_{i,j=1}^{n}$ is the volatility matrix. $B_j(t) \quad (j = 1, \dots, n)$ are the independent Brownian motions.

The risk-free asset has the usual price process dynamics of

$$dS_0(t) = S_0(t)rdt, \;\; S_0(0) = 1. \tag{3.2}$$

In the new parametrization, contrary to the classical Black-Scholes market the stock price processes are assumed to have equal positive drifts of $(\mu - r)$. Then there appears a relation between risk and return. This relation enables us to determine the expected return by means of the volatility matrix. This parametrization is suggested to overcome the estimation problems for the expected return. The aim of this parametrization and extensions will be explained in detail in Section 3.2.1.

The self-financing wealth process is defined as:

$$W(t) := w + \sum_{i=1}^{n} \int_0^t \frac{\pi_i(t)W(s)}{S_i(s)} dS_i(s) + \int_0^t \frac{(1 - \sum_{i=1}^{n} \pi_i(t))W(s)}{S_0(s)} dS_0(s). \tag{3.3}$$

Then the self-financing wealth process for this problem is given as

$$dW(t) = W(t)\left(\sum_{i=1}^{n}\sum_{j=1}^{n}\pi_i(t)\sigma_{ij}[(\mu - r)dt + dB_j(t)] + rdt\right). \tag{3.4}$$

Here, $\pi_i(t)$ is the admissible deterministic portfolio strategy which is the proportion of the wealth invested in ith stock at $t \in [0, T]$.

When we set $p_j := \sum_{i=1}^{n}\pi_i(t)\sigma_{ij}$ then the wealth process becomes,

$$dW(t) = W(t)\left(\sum_{j=1}^{n} p_j(t)[(\mu - r)dt + dB_j(t)] + rdt\right). \tag{3.5}$$

It turns out to be a standard Geometric Brownian motion process with explicit form of

$$W(t) = w\exp\left\{\sum_{j=1}^{n}\left[\int_0^t (p_j(s)(\mu - r) - \frac{1}{2}p_j^2(s))ds + \int_0^t p_j(s)dB_j(s)\right] + rt\right\}. \tag{3.6}$$

We consider the variant of Markowitz's Mean Variance problem where we minimize the variance in the presence of a lower bound on expected return. In this case in continuous time the problem is

$$\begin{aligned} &\min_{\pi_t \in A(w)} \operatorname{var}[W_T] \\ &E[W(T)] \geq w\exp\{\lambda T\}. \end{aligned} \tag{3.7}$$

Here, $A(w)$ is the set of all admissible strategies, w is the initial wealth and $\lambda > r$ is the continuously compounded required rate of return.

Here, it is assumed that $\lambda > r$ since when $\lambda \leq r$ the pure bond strategy satisfies the expectation constraint and so it will be the best strategy with zero variance.

In the case with $\lambda > r$ it is not possible to have an optimal zero variance strategy and thus investment in some risky asset is needed to satisfy the expectation constraint. Then, we obtain $(\mu - r)\displaystyle\sum_{j=1}^{n}\int_0^T p_j(t)dt > 0$ and from the law of total variance,

$$\operatorname{var}[W(T)] = E[\operatorname{var}[W(T)]|p] + \operatorname{var}[E[W(T))|p]] \tag{3.8}$$

we obtain

$$\operatorname{var}[W(T)] \geq E[\operatorname{var}[W(T)]|p] = w^2 E\left[\exp\left\{2\left((\mu-r)\sum_{j=1}^{n}\int_0^T p_j(t)dt + rT\right)\right\}\right.$$
$$\left.\left(\exp\left\{\sum_{j=1}^{n}\int_0^T p_j^2(t)\right\} - 1\right)\right]. \tag{3.9}$$

As a result, variance can be reduced by multiplying the strategy with a factor $\alpha \in (0,1)$ and still satisfies the expected return constraint. Hence, for the optimal strategy the expectation constraint has to be satisfied as an equality.

$$E[W^*(T)] = w\exp\{\lambda T\}. \tag{3.10}$$

The expected wealth and variance of the wealth are:

$$E[W(T)] = w\exp\left\{\sum_{j=1}^{n}\int_0^T (\mu-r)p_j(t)dt + rT\right\} \tag{3.11}$$

$$\operatorname{var}[W(T)] = w^2\exp\left\{2\left((\mu-r)\sum_{j=1}^{n}\int_0^T p_j(t)dt + rT\right)\right\}$$
$$\left(\exp\left\{\sum_{j=1}^{n}\int_0^T p_j^2(t)\right\} - 1\right). \tag{3.12}$$

When we set $I_p := \frac{1}{nT}\sum_{j=1}^{n}\int_0^T p_j(s)ds$ and using Jensen's and Cauchy Schwarz Inequalities we have

$$\operatorname{var}[W(T)] \geq w^2 E\left[\exp\{2((\mu-r)nI_p + r)T\}\left(\exp\{nTI_p^2\} - 1\right)\right] \tag{3.13}$$

When we set $X = \exp\{(\mu-r)nTI_p\}$ and rearrange the terms for the case $X \geq 1$ we will have

$$\operatorname{var}[W(T)] \geq w^2 E\left[\exp\{2rT\}X^2\left(\exp\left\{\frac{(\log(X)^2)}{(\mu-r)^2 nT}\right\} - 1\right)\right]. \tag{3.14}$$

Since the part inside the expectation at (3.14) is strictly convex in X and from (3.10) and (3.11) we have $E[X] = E[\exp\{(\mu - r)nTIp\}] = \exp\{(\lambda - r)T\}$, then we obtain

$$\mathrm{var}[W(T)] \geq w^2 \exp\{2\lambda T\} \left(\exp\left\{ \left(\frac{1}{n} \frac{\lambda - r}{\mu - r} \right)^2 nT \right\} - 1 \right). \tag{3.15}$$

As a result, the optimal strategy is obtained by solving

$$p_1^*(t) = \ldots = p_n^*(t) = \frac{1}{n} \frac{\lambda - r}{\mu - r}. \tag{3.16}$$

This strategy indeed attains the lower bound for the variance given in (3.15). Further, as the strategy is deterministic, we have $\mathrm{var}[E[W(T))|p]] = 0$ which shows that we are actually computing the variance and not just a lower bound for it.

When we use this result and the solution of the wealth process given in (3.6) the optimal wealth process has the following dynamics

$$W^*(t) = w \exp\left\{ \lambda t - \frac{t}{2n} \left(\frac{\lambda - r}{\mu - r} \right)^2 + \frac{1}{n} \frac{\lambda - r}{\mu - r} \sum_{j=1}^{n} B_j(t) \right\}. \tag{3.17}$$

3.2 The Direct Continuous Time Mean-Variance Portfolio Optimization Approach with Jump Extension

In this section, we will explain our extensions obtained by adding Poisson jump processes to the model and we will show that the continuous-time Markowitz portfolio problem can still be solved explicitly. The results can be found in [54].

3.2.1 The Market Model with Jumps

We again look at a financial market consisting of $n+1$ assets. One of the assets is assumed to be risk-free with a price denoted by $S_0(t)$ satisfying,

$$dS_0(t) = S_0(t) r dt, \quad S_0(0) = 1. \tag{3.18}$$

As always, r is the continuously compounded risk-free interest rate. The remaining n assets are risky assets with the following dynamics

$$dS_i(t) = S_i(t-)\left[(r + (\mu - r)(\sum_{j=1}^{n} \sigma_{i,j} + \sum_{j=1}^{n} \varphi_{i,j} \sqrt{\lambda_j})\ )dt \right.$$
$$\left. + \sum_{j=1}^{n} \sigma_{i,j} dB_j(t) + \sum_{j=1}^{n} \varphi_{i,j} dM_j(t)\right]. \tag{3.19}$$

Here, the process $B := \{(B_1(t), \ldots, B_n(t))^T \ : \ 0 \leq t \leq T)\}$ is an n-dimensional Brownion motion. The process,

$$M_j(t) := N_j(t) - \int_0^t \lambda_j ds, \tag{3.20}$$

is determined by the n-dimensional Poisson Process $N := \{(N_1(t), \ldots, N_n(t))^T \ : \ 0 \leq t \leq T)\}$ with intensities $(\lambda_1(t), \ldots, \lambda_n(t))^T$. Both these processes are defined on a probability space $(\Omega, F, \{F_t\}_{\{t \in [0,T]\}}, P)$ with $\{F_t\}_{\{t \in [0,T]\}}$ denoting the product filtration of the two canonical filtrations generated by the Brownian motion and the Poisson process.
$\mu > r$ is the common drift parameter. $\sigma := (\sigma_{i,j})_{1 \leq i,j \leq n}$ is the (constant) volatility matrix that is assumed to be non-singular, and $\varphi := (\varphi_i)_{1 \leq i \leq n}$ is the vector of jump height coefficients that expresses how the jump risk factors N_j affects stock i at a jump time where we assume $\varphi_i > -1$.

With the definition of $M_j(t)$ we can rewrite the stock price equation as,

$$dS_i(t) = S_i(t-)\left[(r + (\mu - r)(\sum_{j=1}^{n} \sigma_{i,j} + \sum_{j=1}^{n} \varphi_{i,j} \sqrt{\lambda_j}) \right.$$
$$\left. - \sum_{j=1}^{n} \varphi_{i,j} \lambda_j)dt + \sum_{j=1}^{n} \sigma_{i,j} dB_j(t) + \sum_{j=1}^{n} \varphi_{i,j} dN_j(t)\right]. \tag{3.21}$$

By applying the Ito formula it can be verified that we have

$$S_i(t) = S_i(0) \exp\left(\left(r + (\mu - r)(\sum_{j=1}^{n} \sigma_{i,j} + \sum_{j=1}^{n} \varphi_{i,j} \sqrt{\lambda_j}) - \sum_{j=1}^{n} \varphi_{i,j} \lambda_j - \right.\right.$$
$$\left.\left. -\frac{1}{2} \sum_{j=1}^{n} \sigma_{i,j}^2 \right) t + \sum_{j=1}^{n} \sigma_{i,j} B_j(t) + \sum_{j=1}^{n} \ln(1 + \varphi_{i,j}) N_j(t)\right). \tag{3.22}$$

This model is an extension of Lindberg's parameterization in [42] for the jump-diffusion framework. While the Brownian and the Poisson parts are in standard form, the choice of the drift parameter in (3.19) is an unconventional one. However, it can still be put into the standard Black-Scholes form by formally introducing

$$\hat{\mu}_i = r + (\mu - r)(\sum_{j=1}^{n} \sigma_{i,j} + \sum_{j=1}^{n} \varphi_{i,j} \sqrt{\lambda_j}), \tag{3.23}$$

which slightly resembles the representation including a market price of risk,

$$\hat{\mu}_i = r + (\hat{\mu}_i - r) = r + vol_i * \kappa_i.$$

with κ being the vector consisting of the different market prices of risk, vol_i is a measure for the volatility of stock i. What is, however, intriguing is the fact that μ is the same for all stocks. The parts of the stock drift that are actually special to stock i are all parts of the variance-covariance structure of the stock price returns. According to Lindberg [42], the interpretation of this representation is that the expected surplus return is determined by the exposure of the stocks to the different risk factors, i.e., the components of the Brownian motion and the Poisson processes. The rest of the mean rate of return is not special to each stock and therefore receives the common drift equal to the risk-free interest rate r.

There are two special consequences of this parameterization:

- Drift estimation will be performed via specification of the volatility structure of the stock price returns.
- Determination of the optimal investment strategy for the continuous-time mean-variance problem will be possible without the knowledge of the surplus drift rate when only the total amount of money that should be invested in the risky assets is known.

3.2.2 The Wealth Process Dynamics and the Portfolio Problem with Solution

In this section we will give the dynamics of the wealth process. In addition we will give the explicit solution to Markowitz's problem for the continuous time case in the jump-diffusion setting. We will mainly follow along the lines of [42]. To describe the action of a trader, we use the notion of a portfolio process $\pi(t) := (\pi_1(t), \ldots, \pi_n(t))^T$. It is an F_t-predictable strategy on (Ω, F, P) with

$$\int_0^T |\pi(t)^T(\hat{\mu}_i - r)|dt + \int_0^T \|\pi(t)^T\sigma\|^2 dt + \int_0^T \|\pi(t)^T\varphi\|^2 dt < \infty \qquad P - a.s.. \tag{3.24}$$

Let us consider an investor with initial wealth $w \geq 0$ who uses a portfolio process $\pi(t)$. By requiring that the investor acts in a self-financing way, the corresponding wealth process $W(t)$ has to satisfy:

$$W(t) = w + \int_0^t \sum_{i=1}^n \frac{\pi_i(s)W(s-)}{S_i(s-)} dS_i(s) + \int_0^t \frac{(1 - \sum_{i=1}^n \pi_i(s))W(s)}{S_0(s)} dS_0(s). \tag{3.25}$$

Note that we do not consider consumption on $[0, T]$. From Equations (3.18), (3.19), and (3.25), the wealth process dynamics are given by

$$\begin{aligned} dW(t) = W(t-)\Bigg[& \sum_{i=1}^n \pi_i(t)((\mu - r)(\sum_{j=1}^n \sigma_{i,j} + \sum_{j=1}^n \varphi_{i,j}\sqrt{\lambda_j}) - \sum_{j=1}^n \varphi_{i,j}\lambda_j)dt \\ & + \sum_{i=1}^n \pi_i(t)(\sum_{j=1}^n \sigma_{i,j} dB_j(t) + \sum_{j=1}^n \varphi_{i,j} dN_j(t)) + rdt \Bigg]. \end{aligned} \tag{3.26}$$

To simplify notation, the following abbreviations are used

$$p_j(t) = \sum_{i=1}^n \pi_i(t)\sigma_{i,j}, \quad q_j(t) = \sum_{i=1}^n \pi_i(t)\varphi_{i,j}. \tag{3.27}$$

Note that $p_j(t)$ and $q_j(t)$ are associated with the basic sources of risk, the jth component of the Brownian motion $W_j(t)$ and the j th jump component $N_j(t)$. This will allow an easier interpretation of the optimal strategy later on.

With this notation, the solution of equation (3.26) is given as

$$\begin{aligned} W(t) = w \exp\Bigg\{ & \sum_{j=1}^n \int_0^t ((\mu - r)(p_j(s) + q_j(s)\sqrt{\lambda_j}) - \lambda_j q_j(s) - \frac{1}{2} p_j(s)^2 + r)ds \\ & + \sum_{j=1}^n \int_0^t p_j(s) dB_j(s) + \sum_{j=1}^n \int_0^t \ln(1 + q_j(s)) dN_j(s) \Bigg\} \end{aligned} \tag{3.28}$$

The continuous-time Markowitz problem is formulated as

$$\begin{gathered} \min_{\pi \in D(x)} \operatorname{var}[W(T)] \\ E[W(T)] \geq w \exp(\gamma T) \end{gathered} \tag{3.29}$$

where $D(x)$ consists of all deterministic portfolio processes $\pi(\cdot)$ such that the corresponding wealth equation (3.26) (resp. Equation (3.28)) admits a unique non-negative solution $W(t)$ and γ is the continuously compounded required rate of return.

Theorem 3.2.1 *a) In the case of $\gamma > r$, the unique optimal deterministic portfolio process $\pi^*(\cdot)$ for the continuous-time Markowitz problem (3.29) is given as the unique solution of the following system of linear equations*

$$\pi_1(\sigma_{11} + \varphi_{11}\sqrt{\lambda_1}) + \ldots + \pi_n(\sigma_{n1} + \varphi_{n1}\sqrt{\lambda_1}) = \frac{1}{n}\frac{\gamma - r}{\mu - r},$$
$$\vdots$$
$$\pi_1(\sigma_{1n} + \varphi_{1n}\sqrt{\lambda_n}) + \ldots + \pi_n(\sigma_{nn} + \varphi_{nn}\sqrt{\lambda_n}) = \frac{1}{n}\frac{\gamma - r}{\mu - r}. \tag{3.30}$$

In particular, the optimal deterministic portfolio process is constant.

b) In the case of $\gamma \leq r$ the pure bond strategy, i.e.,

$$\pi^* = 0 \tag{3.31}$$

solves the continuous-time Markowitz problem (3.29).

Proof: In this part, we follow along the lines of the proof given in [42] for the pure diffusion setting (see the remark after the proof for the relation to [42]).

b) Under the assumption of $\gamma \leq r$ the pure bond strategy satisfies the expectation constraint as we then have

$$E[W(T)] = w \exp(rT) \geq w \exp(\gamma T). \tag{3.32}$$

As for this strategy the final wealth has zero variance, it is proved that in this case the pure bond strategy solves the continuous-time Markowitz problem.

a) For any deterministic portfolio process $\pi(\cdot)$, the expected terminal wealth is given as

$$E[W(T)] = w \exp\left\{\sum_{j=1}^{n}\int_0^T ((\mu - r)(p_j(s) + q_j(s)\sqrt{\lambda_j}))ds + rT\right\} \tag{3.33}$$

where we have used the description of $\pi(\cdot)$ via the processes $p(\cdot)$ and $q(\cdot)$ as defined in Equation (3.27). Note in particular that due to the assumptions of $\mu > r$ and of $\gamma > r$ we must have

$$\sum_{j=1}^{n}\int_0^T \left((\mu - r)(p_j(s) + q_j(s)\sqrt{\lambda_j})\right)ds > 0 \tag{3.34}$$

for every deterministic admissible portfolio process $\pi(\cdot)$. Further, in this case, the variance of the final wealth can be computed as

$$\begin{aligned}\mathrm{var}[W(T)] = w^2 \Bigg[\exp\left\{ 2\left(\sum_{j=1}^{n} \int_0^T ((\mu - r)(p_j(s) + q_j(s)\sqrt{\lambda_j}))ds + rT \right) \right\} \\ \left(\exp\left\{ \sum_{j=1}^{n} \int_0^T (p_j^2(s) + q_j^2(s)\lambda_j)ds \right\} - 1 \right) \Bigg]. \end{aligned} \tag{3.35}$$

Under the assumption of $\gamma > r$, the pure bond strategy is not admissible. Thus, it is necessary to invest in some risky asset to satisfy the expected terminal wealth constraint. As a result, the unconstrained minimum variance of zero cannot be attained. Note also that a portfolio process that satisfies the minimum expectation requirement in the optimization problem as a strict inequality cannot be optimal. To see this note that by the explicit form of the variance given in Equation (3.35) and the relation (3.34), one could reduce the variance by multiplying the strategy with a factor smaller than 1 and could still satisfy the expectation constraint for every strategy that originally produces an expected terminal wealth exceeding $\exp((\gamma - r)T)$. Hence, we can conclude that for the optimal portfolio process π^* the expectation constraint of the optimization problem must be satisfied as an equality. We can thus without loss of generality restrict ourselves to portfolio processes $\pi(\cdot)$ with a corresponding terminal wealth $W(T)$ satisfying

$$E[W(T)] = w \exp\{\gamma T\}. \tag{3.36}$$

By introducing the notation

$$s_j(t) = q_j(t)\sqrt{\lambda_j}, \tag{3.37}$$

we can rewrite Equation (3.35) as

$$\begin{aligned}\mathrm{var[W(T)]} = w^2 \Bigg[\exp\left\{ 2\left(\sum_{j=1}^{n} \int_0^T ((\mu - r)(p_j(s) + s_j(s)))ds + rT \right) \right\} \\ \left(\exp\left\{ \sum_{j=1}^{n} \int_0^T (p_j^2(s) + s_j^2(s))ds \right\} - 1 \right) \Bigg]. \end{aligned} \tag{3.38}$$

Using the Cauchy Schwarz inequality and $a^2 + b^2 \geq \frac{1}{2}(a+b)^2$ yields

$$\begin{aligned} \int_0^T p_j^2(s)ds + \int_0^T s_j^2(s)ds &\geq \frac{1}{T}\left[\left(\int_0^T p_j(s)ds \right)^2 + \left(\int_0^T s_j(s)ds \right)^2 \right] \\ &\geq \frac{1}{2T}\left[\left(\int_0^T \big(p_j(s) + s_j(s)\big)\, ds \right)^2 \right]. \end{aligned} \tag{3.39}$$

An application of Jensen's inequality yields

$$\frac{1}{2T}\left[\sum_{j=1}^{n}\left(\int_0^T \left(p_j(s)+s_j(s)\right)ds\right)^2\right] \geq \frac{1}{2nT}\left[\sum_{j=1}^{n}\int_0^T \left(p_j(s)+s_j(s)\right)ds\right]^2. \tag{3.40}$$

So, we finally arrive at

$$\sum_{j=1}^{n}\int_0^T \left(p_j(s)^2+s_j(s)^2\right)ds \geq \frac{1}{2nT}\left[\sum_{j=1}^{n}\int_0^T \left(p_j(s)+s_j(s)\right)ds\right]^2, \tag{3.41}$$

which directly implies

$$\begin{aligned}&\exp\left(\sum_{j=1}^{n}\int_0^T \left(p_j(s)^2+s_j(s)^2\right)ds\right)\\ &\qquad\geq \exp\left(\frac{1}{2nT}\left[\sum_{j=1}^{n}\int_0^T \left(p_j(s)+s_j(s)\right)ds\right]^2\right).\end{aligned} \tag{3.42}$$

With the help of notation

$$I := \frac{1}{2nT}\sum_{j=1}^{n}\int_0^T \left(p_j(s)+s_j(s)\right)ds, \tag{3.43}$$

relation (3.42), and the variance representation (3.38), we obtain

$$\mathrm{var}[W(T)] \geq w^2\left[\exp\{2((\mu-r)2nTI+rT)\}\left(\exp\left\{2nTI^2\right\}-1\right)\right]. \tag{3.44}$$

Using the new variable X

$$X := \exp\{(\mu-r)2nTI\}, \tag{3.45}$$

the variance inequality can be further simplified to get

$$\begin{aligned}&\mathrm{var}[W(T)] \geq\\ &\quad w^2\left[\exp\{2rT\}X^2\left(\exp\left\{2nT\left(\log\left(\exp\left\{\tfrac{(\mu-r)2nTI}{(\mu-r)2nT}\right\}\right)\right)^2\right\}-1\right)\right]\\ &\qquad = w^2\left[\exp\{2rT\}X^2\left(\exp\left\{\tfrac{log(X)^2}{(\mu-r)^2 2nT}\right\}-1\right)\right].\end{aligned} \tag{3.46}$$

The function in X in the last line above is strictly convex on $[1,\infty)$. As a result we have,

$$\mathrm{var}[W(T)] \quad \geq \quad w^2\exp\{2rT\}X^2\left(\exp\left\{\frac{log(X)^2}{(\mu-r)^2 2nT}\right\}-1\right) \tag{3.47}$$

From the definition of X and the fact that all candidates for an optimal portfolio process must satisfy the equality constraint (3.36) for the terminal wealth, we obtain

$$X = \exp\{(\mu-r)2nTI\} = \exp\{(\gamma-r)T\} \tag{3.48}$$

as X simply equals the expected wealth corresponding to the portfolio process characterized by $p(\cdot)$ and $q(\cdot)$. Eventually, we obtain the following lower bound for the variance of the terminal wealth of all possible candidates for an optimal deterministic portfolio process:

$$\operatorname{var}[W(T)] \geq w^2 \exp\{\gamma T\} \left(\exp\left\{ \left(\frac{1}{2n} \frac{\gamma - r}{\mu - r} \right)^2 2nT \right\} - 1 \right). \tag{3.49}$$

However, it is easy to check that the portfolio process π^* given by the solutions p^* and q^* of the system of linear equations

$$p_1^* + s_1^* + \ldots + p_n^* + s_n^* = \frac{1}{n} \frac{\gamma - r}{\mu - r} \tag{3.50}$$

has a variance that equals the lower bound in Equation (3.49). Consequently, $\pi^*(\cdot)$ is an optimal solution to the continuous-time Markowitz problem (3.29).

In contrast to the theorem in [42], we do not claim that we have solved the continuous-time Markowitz problem (3.29) for general portfolio processes. If one directly inspects Lindberg's proof, one will realize that certain independence assumptions of the portfolio process and the wealth process (respectively, the stock prices) have to be satisfied.

3.3 Examples and Results

In this section we will,

- analyze the form of the optimal portfolio process in different market settings,
- show how the range of the stock price drifts can be determined from the stock price volatilities and
- demonstrate how to use the parameterization of the drift process without actually estimating the drift parameter by using examples.

To highlight these aspects we will make use of Theorem 3.2.1 and the market setting introduced in Section 3.2.1. In this market the n risky assets are driven by an n-dimensional Brownian motions and an n-dimensional Poisson jump process, we are in an incomplete market setting. However, we will shortly see that the application of Theorem 3.2.1 is not effected by this incompleteness at all.

3.3.1 The Optimal Strategy

Let us recall that Theorem 3.2.1 yields the following equations for the optimal portfolio process that can uniquely be solved:

$$\pi_1(\sigma_{11} + \varphi_{11}\sqrt{\lambda_1}) + \ldots + \pi_n(\sigma_{n1} + \varphi_{n1}\sqrt{\lambda_n}) = \frac{1}{n}\frac{\gamma - r}{\mu - r} \tag{3.51}$$

$$\vdots \qquad = \qquad \vdots$$

$$\pi_1(\sigma_{1n} + \varphi_{1n}\sqrt{\lambda_1}) + \ldots + \pi_n(\sigma_{nn} + \varphi_{nn}\sqrt{\lambda_n}) = \frac{1}{n}\frac{\gamma - r}{\mu - r}. \tag{3.52}$$

As long as the n stocks are not linearly dependent with respect to their risks (which means that the coefficient matrix of the above linear system is regular), this simple system of linear equations in $(\pi_1, \ldots, \pi_n)$ has a unique solution. The bond component of the portfolio process π_0 is – as always – obtained as

$$\pi_0 = 1 - \pi_1 - \ldots - \pi_n. \tag{3.53}$$

We can discover the optimal strategy given in [42] by simply setting all φ-components equal to zero. We would also like to mention explicitly that due to the restriction to deterministic portfolios and the form of the optimality equations in Theorem 3.2.1, we do not need market completeness assumptions such as those in the model of [26]. There, the authors consider the general class of trading strategies but look at a complete market made up of a risk-free bond and n stocks driven by a d-dimensional Brownian motion and an $(n-d)$-dimensional Poisson process. Our model can recover the model in [26] by setting

$$\sigma_{ij} = 0 \text{ for } i = d+1, \ldots, n, \quad \varphi_{ij} = 0 \text{ for } i = 1, \ldots, d, \tag{3.54}$$

for $i = 1, \ldots, n$. Still we obtain a unique optimal deterministic portfolio process in this setting as before.

3.3.1.1 Interpretation of the Optimal Portfolio Process as $1/n$-strategy

Note that the equations in the linear system (3.51), (3.52) correspond to the fractions of wealth invested in the different risks (W_i, N_i), $(i = 1, \ldots, n)$, not the different assets. In [42], the optimal portfolio strategy is called a $1/n$-strategy. The motivation for this is the following: All the right-hand sides of the linear system (3.51), (3.52) are equal. Thus, if we define the left-hand sides as the fractions of wealth assigned to the ith risk component, which is generated by both the ith component of the Brownian motion and of the Poisson jump process (both weighted by their volatility and jump height parameters), this means that equal fractions of wealth are assigned to each such component risks. However, one has to be careful with the use of the term $1/n$-strategy. Even if all the right-hand side equal are equal to $1/n$ (which is the case for $\gamma = \mu$) this does still not mean that the investor assigns a fraction of $1/n$ of his wealth to each asset.

There are however situations when this can be the case. As an example, we look at the case of independent risky assets in a pure diffusion market with the choice of $\gamma = \mu$ and

$$\sigma_{ii} = 1, \ \sigma_{ij} = 0 \text{ for } i \neq j, \ \varphi_{ij} = 0 \ \forall i, j. \tag{3.55}$$

Then, the investor assigns the same fraction of wealth $1/n$ to each component of the Brownian motion which here also means that the investor assigns the same fraction of wealth to each risky asset. We then have a perfect $1/n$-strategy with no bond investment. However, in the general situation, the term $1/n$-strategy has to be understood in a more general sense as implicated above.

3.3.1.2 Impact of Jump Risk on the Form of the Optimal Strategy

Here, we will analyze the effect of the existence of jump risk to the optimal allocation of wealth to the different assets and risks, respectively. To illustrate this, we look at a simple example with just two stocks. To further simplify the notation, we assume

$$\gamma = \mu. \tag{3.56}$$

To demonstrate the effect of jump risk, we now have a look at two particular examples:

1. two independent stocks, each driven by a single Brownian motion,
2. two stocks, both driven by a Brownian motion and by a Poisson process.

We will choose the parameters that all stocks have the same expectation when divided by their current prices. In the first case, we obtain the optimal portfolio process of the form

$$\pi_1 = \frac{1}{2\sigma_{11}}, \ \ \pi_2 = \frac{1}{2\sigma_{22}}. \tag{3.57}$$

Equating the (relative) stock price means of the corresponding diffusion and jump-diffusion stock price leads to

$$\sigma_{ii} = \tilde{\sigma}_{ii} + \tilde{\varphi}_{ii}\sqrt{\tilde{\lambda}_i} \tag{3.58}$$

with the "tilde"-coefficients corresponding to the jump-diffusion setting. There is now a particular property of the stock price parameterization when jumps are included and the above equality of means is required: as long as both jump and diffusion parts are non-vanishing, then the variances of the (relative) stock prices differ as we have

$$\sigma_{ii}^2 = \left(\tilde{\sigma}_{ii} + \tilde{\varphi}_{ii}\sqrt{\tilde{\lambda}_i}\right)^2 \neq \tilde{\sigma}_{ii}^2 + \tilde{\varphi}_{ii}^2\tilde{\lambda}_i. \tag{3.59}$$

While the first equality is implied by the requirement (3.58), the second would be needed to ensure equality of the stock price variances. However, for non-vanishing diffusion and jump parts it can never be satisfied. So, note that the jump-diffusion stock price has a smaller variance than the diffusion price with the same mean if and only if we have

$$\tilde{\sigma}_{ii}\tilde{\varphi}_{ii} > 0. \tag{3.60}$$

This is also underlined by the form of the optimal strategies in the jump-diffusion setting

$$\pi_1 = \frac{1}{2}\frac{1}{\tilde{\sigma}_{11} + \tilde{\varphi}_{11}\sqrt{\tilde{\lambda}_1}}, \quad \pi_2 = \frac{1}{2}\frac{1}{\tilde{\sigma}_{22} + \tilde{\varphi}_{22}\sqrt{\tilde{\lambda}_2}}. \tag{3.61}$$

which exactly equal the ones of the diffusion setting given requirement (3.58). Thus, we obtain similar assertions for the portfolio variance as for the single stock variances above. Of course, one also has to consider the "mixed case" when the variance of the first stock in the diffusion setting is bigger than that of the jump-diffusion setting and the relation is turned around for the second stock.

3.3.1.3 The Expectation Constraint, Sign of the Drift, and Diversification

For simplicity, we now only look at a pure diffusion market with two independent stock price processes with the same volatility $\tilde{\sigma}^2$. We further assume

$$\sigma_{11} = \sqrt{\tilde{\sigma}^2} > 0. \tag{3.62}$$

We further set

$$\gamma = r + (\mu - r)\,\sigma_{11}. \tag{3.63}$$

In this case, pure investment in the first stock (i.e., a buy and hold strategy in the first stock only) would be sufficient to satisfy the expectation constraint. To demonstrate the diversification effect of the presence of a second stock, we look at two different cases,

$$Case\ 1: \ \sigma_{22} = \sigma_{11}, \quad Case\ 2: \ \sigma_{22} = -\sigma_{11}. \tag{3.64}$$

In Case 1 we obtain

$$\pi_1 = \pi_2 = \frac{1}{2} \quad (Case\ 1). \tag{3.65}$$

If we compare the variance var* of this optimal deterministic strategy with the one of the buy and hold strategy in the first stock, var^{bh}, then we obtain the relation

$$\frac{\text{var}^*}{\text{var}^{bh}} = \frac{\exp\left(0,5 * \sigma_{11}^2\right) - 1}{\exp\left(\sigma_{11}^2\right) - 1} \quad (Case\ 1). \tag{3.66}$$

which clearly shows the diversification effect expressed as a variance reduction while preserving the expectation constraint. In Case 2 we obtain

$$\pi_1 = -\pi_2 = \frac{1}{2} \quad (Case\ 2), \tag{3.67}$$

i.e., we make use of the bad performance of stock 2 by selling it short. However, we do not increase the relative position in stock 2, but put the gains from the short position in stock 1 into the bank account. This will even further reduce the minimal variance var$^-$ compared to the minimal one of Case 1, leading to

$$\frac{\text{var}^-}{\text{var}^*} = \exp\left(-2\left(\mu - r\right)\sigma_{11} T\right), \tag{3.68}$$

where we have implicitly assumed $\mu > r$ when talking about variance reduction.

3.3.2 Determination of the Stock Price Drift Parameters

It is a well-known problem when implementing continuous-time portfolio optimization methods that they strongly depend on an exact knowledge of the stock price drift parameters. However, they are the critical parameters to estimate. As the volatility dominates the short-term fluctuations of the stock price movements, it will take a long time to obtain a sufficiently small confidence interval for the drift parameters. Typically, the time horizon needed for such as small confidence interval makes the assumption of constant market coefficients over such a long time span is questionable.

On the other hand, in the diffusion setting, the variance-covariance matrix Σ of the log-returns of the stock prices can be estimated very accurately. From this one can then determine a volatility matrix σ as a square root, i.e., a matrix σ with $\Sigma = \sigma * \sigma^T$. This estimation of the variance-covariance matrix of the log-returns is more problematic in the jump-diffusion setting.

Below, we take a look at the various components of the market coefficients and their corresponding estimation.

3.3.2.1 Determination of the Common Drift μ

The common drift parameter μ that occurs in all stock price equations can be interpreted in (at least) two ways. First, we can view it as the common surplus for stock investment. It should then summarize the investor's view towards the stock market as a whole. μ could then be determined as the surplus part of the drift of a market portfolio or a suitable stock index.

As a second interpretation, μ can also represent the investor's risk aversity towards stock investment. To make this more clear, we assume that the investor has already decided about the total fraction of his wealth π_{sum} that he wants to invest into stocks. Then, μ can be obtained directly with the help of our main theorem. For this, we define the $(n+1)*(n+1)$-matrix ν via

$$\nu_{ij} = \begin{cases} \sigma_{ji} + \varphi_{ji}\sqrt{\lambda_i} & , \ 1 \le i, j \le n \\ -1 & , \ i = 1, \dots, n, \ j = n+1 \\ 1 & , \ i = n+1, \ j = 1, \dots, n \\ 0 & , \ i = j = n+1 \end{cases} . \tag{3.69}$$

We further introduce the two vectors

$$\tilde{\pi} = ((\pi_1, \dots, \pi_n), c)^T, \quad b = (0, \dots, 0, \pi_{sum})^T . \tag{3.70}$$

Then, by solving the linear system

$$\nu\tilde{\pi} = b \tag{3.71}$$

for $\tilde{\pi}$, Theorem 3.2.1 ensures that we obtain both the optimal portfolio process π and also the common drift parameter μ. For the latter, we additionally have to solve the equation

$$c = \frac{1}{n}\frac{\gamma - r}{\mu - r}. \tag{3.72}$$

Thus, no estimation of μ is necessary in this case. We obtain it implicitly from two decisions of the investor, the fixed total fraction of wealth invested in stocks and the use of the mean-variance criterion.

3.3.2.2 Drift Specification and Range of Possible Drifts

First of all, it is important to understand that the stock price parameterization of the Lindberg model still contains a lot of freedom. Although, (parts of) the stock price volatility enters the stock price drift, this does not mean that it can be estimated exactly from the data as is suggested in [42]. What we however can estimate accurately is the range of the stock price drift. We will highlight this below, first in the case of a pure diffusion market model.

Assume that μ is already specified and the variance-covariance matrix Σ of the log-returns with

$$\Sigma_{ij} = \frac{1}{t}\mathrm{Cov}\left(\ln\left(S_i(t)\right), \ln\left(S_j(t)\right)\right) = \sum_{k=1}^{n} \sigma_{ik}\sigma_{jk} \tag{3.73}$$

is already estimated. This can usually be done either statistically from time series of stock prices or implicitly from option prices. For us, this makes no difference. To specify the stock prices completely, we now have to choose an appropriate matrix σ that satisfies the Equation (3.73). However, there is a lot of freedom to do so according to the following lemma (see [42]).

Lemma 3.3.1 *Let the variance-covariance matrix Σ of the log-returns of the stock prices be given. Consider its Cholesky decomposition of*

$$\Sigma = LL^T. \tag{3.74}$$

Then, any matrix σ that satisfies Equation (3.73) can be written as

$$\sigma = LQ \tag{3.75}$$

with Q being a suitable orthogonal matrix.

We use this result to illustrate the determination of the range of possible drift parameters in the case of just two stocks in the diffusion setting. For this assume that Σ and L of Lemma

3.3.1 have the forms of

$$L = \begin{bmatrix} A & 0 \\ B & C \end{bmatrix}, \quad \Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12} & \Sigma_{22} \end{bmatrix} \tag{3.76}$$

In two dimensions, an orthogonal matrix Q can be represented as

$$Q = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \tag{3.77}$$

for some $\theta \in [0, 2\pi)$. Thus, the general form of σ is

$$\sigma = \begin{bmatrix} A\cos(\theta) & A\sin(\theta) \\ B\cos(\theta) - C\sin(\theta) & B\sin(\theta) + C\cos(\theta) \end{bmatrix}. \tag{3.78}$$

As from the Cholesky algorithm we know that we have

$$A = \sqrt{\Sigma_{11}} = \sqrt{\sigma_{11}^2 + \sigma_{12}^2}, \tag{3.79}$$

calculation of the extremes of the function

$$f(\theta) = \sin(\theta) + \cos(\theta) \tag{3.80}$$

results in

$$\mu_1 \in \left[r - (\mu - r)\sqrt{2\Sigma_{11}}, r + (\mu - r)\sqrt{2\Sigma_{11}}\right], \tag{3.81}$$

i.e. the range of the interval is determined by the common drift μ and the square-root of the variance of the log-return of stock 1. For stock 2 we obtain a slightly more complicated formula. However, this is only due to the labelling of our stocks. A relabelling would lead to exactly the analogous bounds for the drift μ_2 of stock 2. As the bounds have to be independent of the labelling, we can conclude that the volatility of the stock prices alone determine the stock specific part of the bounds. The entrances of the matrix σ are all correlated via θ, the stock price drifts are correlated, too. This will be demonstrated by Table 3.1 where we plot the surplus drift multipliers x_i, $(i = 1, 2)$ defined by

$$\mu_i = r + (\mu - r)\,x_i. \tag{3.82}$$

It is also worth highlighting a particular effect of our drift parameterizations in the case of independent stock prices with equal volatilities (i.e., $A = C$):

As independence implies $B = 0$, the additional drift parameters resulting from the volatility matrix without any further multiplication by an orthogonal matrix would be $x_1 = A$, $x_2 = C$

θ	0	$\pi/4$	$\pi/2$	$3\pi/4$	π	$5\pi/4$	$3\pi/2$	$7\pi/4$
x_1	A	$\sqrt{2}$A	A	0	-A	- $\sqrt{2}$A	-A	0
x_2	B+C	$\sqrt{2}$B	B-C	- $\sqrt{2}$C	-(B+C)	- $\sqrt{2}$B	-(B-C)	$\sqrt{2}$C

Table 3.1: Surplus drifts for two stocks

(see also Table 3.1). If, we look at the sum of the two surplus drift parameters then in the independent case we obtain

$$x_1 + x_2 = 2A\cos(\theta). \tag{3.83}$$

This in particular means that the maximum attainable surplus drift of $\sqrt{2}A$ for each of the single stocks cannot be attained simultaneously although both stocks are independent.

In the multi-asset diffusion case, one can still use Lemma 3.3.1. Then, an orthogonal matrix can be written as a product of Givens matrices which are of the form

$$G(i,k,\theta) = \begin{bmatrix} 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \cdots & cos(\theta) & \cdots & sin(\theta) & \cdots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \cdots & -sin(\theta) & \cdots & cos(\theta) & \cdots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix}. \tag{3.84}$$

An analysis for the possible range of the surplus drift parameters can then be done in a similar way to the case of just two stocks. However, it will be notationally more involved.

To deal with the drift specification in the jump diffusion part, we suggest to separate the jumps from the remaining stock price part. The reasons for this are twofold:

- In the diffusion setting the symmetry of the normal distribution of the Brownian motion increments did not allow a unique determination of the sign of the σ-parameters. For the Poisson jumps the sign of the jump height is always determined. The same is valid for the jump intensity parameters λ_i.

- The role of the φ-parameters as jump heights does not allow a second independent use as variance parameters as is the case for the σ-parameters.

3.3.3 Drift Choice and Its Impact on the Optimal Strategy

In this section, we will demonstrate the effect of the choice of the part of the stock price drifts that corresponds to the volatility parameters σ and φ.

In the pure diffusion setting, in [42] some examples for the choice of the σ-matrix are given and are discussed as a kind of different dependence modelling. We would like to take on another point of view towards the choice of σ: as every choice of σ in the way discussed above will not change the volatility matrix Σ, we interpret the choice of σ as that of the surplus parts of the stock drifts and do not relate its interpretation to volatility at all.

To illustrate this with some numbers, we look at the case of two risky assets and one bond in a jump-diffusion market. Instead of specifying the common drift μ, we demonstrate that if the investor has already decided about the fraction of wealth that should be invested into stocks, we can find the optimal strategy without knowing γ, r and μ. To do so, suppose we want to have $\pi_1 + \pi_2 = 1$. We further assume that the jump heights, jump intensities, and the Cholesky matrix L of the volatility matrix are given by

$$\lambda_1 = 1,\ \varphi_{11} = -0.2,\ \lambda_2 = \varphi_{22} = \varphi_{21} = \varphi_{12} = 0, \tag{3.85}$$

$$L = \begin{bmatrix} 0.4 & 0 \\ -0.4 & 0.6 \end{bmatrix}.$$

We then use the fact that the optimality equations for the optimal portfolio all have the same right hand sides and that we have already specified the desired fraction of wealth to be invested in the stock market. This will give us the following two equations if we choose $Q = I$ in Lemma 3.3.1, i.e., if we choose $\theta = 0$.

$$\begin{aligned} \pi_1(0.4 - 0.2 * 1) &= \pi_1(-0.4) + \pi_2 * 0.6, & (3.86) \\ \pi_1 + \pi_2 &= 1, & (3.87) \end{aligned}$$

leading to an optimal portfolio process of

$$\pi_1^{(0)} = \pi_2^{(0)} = 0.5. \tag{3.88}$$

Here, the upper index denotes the choice of θ.

If we use $\theta = 0.25 * \pi$ then this results in the new σ-matrix of

$$\sigma^{(0.25*\pi)} = \begin{bmatrix} 0.2828 & 0.2828 \\ -0.7071 & 0.1414 \end{bmatrix}, \tag{3.89}$$

which leads to a smaller surplus drift for stock 1. It is thus not surprising that investment in stock 2 gets more attractive leading to an optimal portfolio process of

$$\pi_1^{(0.25*\pi)} = -0.2180,\ \pi_2^{(0.25*\pi)} = 1.2180. \tag{3.90}$$

For $\theta = \pi$ we get the new σ-matrix of

$$\sigma^{(\pi)} = \begin{bmatrix} -0.4 & 0 \\ 0.4 & -0.6 \end{bmatrix} \tag{3.91}$$

and an optimal portfolio process of

$$\pi_1^{(\pi)} = 0.375,\ \pi_2^{(\pi)} = 0.625. \tag{3.92}$$

One could now try to give the choice of θ a geometric interpretation, but this intuition will become hard to transfer to the case of more than two stocks anyway.

The interpretations given in [42] are more related to the importance of the companies that are considered. Using a σ that is a symmetric matrix square root of the volatility matrix should correspond to the case when the companies influence each other in a symmetric way. On the other hand, the use of the Cholesky matrix as σ is advised if the first company can influence the second, but the second cannot influence the first one. This is interpreted by the fact that the first company only depends on one source of uncertainty (expressed by the zero of the second component in the first row of the σ-matrix) while the second company depends on both sources. Although this interpretation is a nice and intuitive one, it does not determine the sign of the additional drift parameters. To see this, simply note that the matrix square root is not unique. Thus, as already said, the new parameterization helps us in determining a possible range of the stock price drift. The decision of the exact values of this drift is still for a big part left to the investor. However, internal dependencies are respected as the simple examples above have shown.

CHAPTER 4

THE LAGRANGIAN FUNCTION APPROACH TO THE CONTINUOUS TIME MEAN-VARIANCE PORTFOLIO OPTIMIZATION AND GENERALIZATIONS

In this chapter, we will first introduce the extended martingale approach for a bigger class of utility functions defined in Definition 4.1.1 than the utility functions given in Definition 2.2.1. Then, the Lagrangian function approach to the continuous-time MV portfolio optimization with positive terminal wealth constraint, which has first been presented in Korn and Trautmann [31], and some new results particularly for the explicit representation for the optimal portfolio process will be given. Here, the innovative market parametrization given in (3.19) is used. The main results of the extended martingale method and the Lagrangian function approach is taken from [31] and [33] and the results of Lagrangian function approach for the market model given in (3.19) together with the new results for the optimal portfolio process are given in [55].

4.1 The Extended Martingale Approach

The martingale method for the class of utility functions defined in Definition 2.2.1 is given in Section 2.3.1.1. When the bigger class of utility functions given in Definition 4.1.1 are considered then some improvements are needed. These extensions are presented in detail by Korn [33] in 1997.

Definition 4.1.1 *A function $U(\cdot) : (0, \infty) \to R$ which is strictly concave, continuously differentiable and satisfies,*

$$U'(0) := \lim_{w \to 0} U'(w) > 0 \;\; and \;\; U'(z) = 0 \tag{4.1}$$

for unique value of $z \in [0, \infty]$ *will be called a utility function.*

This utility function definition is more general than the definition given in Definition 2.2.1 in the sense of including utility functions which are not strictly increasing and possibly having a finite slope at the origin such as quadratic utility functions.

As mentioned in Section 2.3.1.1 in the martingale method we need the inverse function $I(\cdot)$. Here, $U'(\cdot)$ is strictly decreasing on $[0, z]$ and this implies a strictly decreasing continuous inverse function $I^* : [0, U'(0)] \to [0, z]$ that can be extended to the whole positive real line by setting,

$$I(x) := \begin{cases} I^*(x), & x \in [0, U'(0)] \\ 0, & x \geq U'(0) \end{cases} \tag{4.2}$$

Here, since the utility functions are not necessarily strictly increasing the investor might not need his whole endowment w to gain the full optimal utility. Thus, our continuous time portfolio optimization problem becomes,

$$\max_{\pi_t \in A'(x),\, x \leq w} E[U(W^{x,\pi}(T))]. \tag{4.3}$$

Here, $A'(x) := \{\pi \in A(x) : E[U(W^{x,\pi}(T))^-] < \infty\}$ and $A(x)$ is the set of all admissible strategies with $x > 0$.

For the heuristic computation of the optimal terminal wealth process we need the function $X : (0, \infty) \to R$ defined in (4.4) as in Section 2.3.1.1.

$$X(y) := E[H(T)I(yH(T))] = w \;\; \forall\, y > 0. \tag{4.4}$$

We need to guarantee the existence of the inverse function $X^{-1}(w)$ for the optimal terminal wealth. The existence of $X^{-1}(w)$ is satisfied by Proposition 4.1.2.

Proposition 4.1.2 *$X(y) < \infty \ \ \forall \ y \in (0,\infty)$ and in the case of $U'(0) < \infty$ assume the market price of risk $\theta(t)$ $t \in [0,T]$ is deterministic with $\int_0^t \|\theta(s)\|^2 ds < \infty$. Then, $X(\cdot)$ is continuous on $(0,\infty)$ and strictly decreasing with,*

$$X(\infty) := \lim_{y\to\infty} X(y) = 0, \tag{4.5}$$

$$X(0) := \lim_{y\to 0} X(y) = \begin{cases} \infty, & \text{if } \lim_{z\to\infty} U'(z) = 0, \\ z_1 E[H(T)], & \text{else.} \end{cases},$$

where z_1 is attained from $U'(z_1) = 0$.

Proof.

i. Since $I(\cdot)$ is a monotone continuous function, the dominated convergence theorem leads $X(y)$ is also continuous.

ii. Since $I(\cdot)$ is strictly decreasing on $(0, U'(0))$ and $H(T)I(yH(T))$ is strictly decreasing in y on the set $yH(T) < U'(0)$ and zero on its complement.

If we can show that $P\left(yH(T) < U'(0)\right) > 0 \ \ for\ every\ y \in (0,\infty)$ then we can conclude that $X(y)$ is strictly decreasing in $y \in (0,\infty)$

In the case $U'(0) = \infty$, $P(yH(T) < U'(0)) > 0$ is obvious since $H(T)$ and $U'(0)$ are continuous.

In the case $U'(0) < \infty$, since $r(t)$ and $\theta(t)$ are uniformly bounded and $W(T)$ is the Brownian motion with normal distribution, we get $P(\ln(H(T)) < a) > 0$ for $a > 0$ which implies $P(yH(T) < U'(0)) > 0$.

For both cases, the strictly decreasing nature of $X(y)$ is proved.

iii. From the monotone convergence theorem we obtain $X(\infty) := \lim_{y\to\infty} X(y) = 0$ since, $I(\cdot)$ is monotone and $I(\infty) = 0$

iv. In the case of $\lim_{z\to\infty} U'(z) = 0$, because, $I(\cdot)$ is non-negative function, Fatou's Lemma implies,

$$\liminf_{y\to 0} X(y) \geq E[H(T) \liminf_{y\to 0} I(yH(T))] = \infty.$$

Conversely, when $\lim_{z\to\infty} U'(z) = 0$ is not satisfied then,

$\limsup_{y\to 0} X(y) \leq z_1 E[H(T)]$,
because, $I(\cdot) \leq z_1$ for every w. Then, Fatou's Lemma implies,

$\liminf_{y\to 0} X(y) \geq E[H(T) \liminf_{y\to 0} I(yH(T))] = z_1 E(H(T))$,

which completes the proof.

$\blacksquare$

Proposition 4.1.2 guarantees the existence of a continuous and strictly decreasing inverse function $X^{-1}(w) : [0, z^*] \to [0, \infty]$. Now we can find the optimal terminal wealth ξ^* and the optimal portfolio process π^* according to the following theorem.

Theorem 4.1.3 *Let $w > 0$. Then under the assumptions of $X(y) < \infty$, $U'(0) < \infty$ and $||\theta||^2 > 0$, the optimal terminal wealth process is given by*

$$\xi^* := \begin{cases} z_1 & \text{if } w \geq z_1 \\ I(y^* H(T)) & \text{else} \end{cases}, \tag{4.6}$$

and there exists an $w^* \in [0, w]$ and an optimal portfolio process $\pi_t^*,\ t \in [0, T]$ such that we have

$\pi_t^* \in A'(w^*)$, $W^{w^*,\pi^*} = \xi^*$ *a.s.*

In addition, π_t^* solves the problem given in (4.3).

Proof.

Case 1: $w \geq z_1$

Since $U(\cdot)$ attains its absolute maximum in $z_1 = I(0)$ we have

$U(z_1) = U(I(0)) \geq U(W^{w,\pi}(T))$ a.s.

for every $\pi \in A'(x)$ with $x \leq w$.

Then, $\xi^* = z_1 = I(0)$ is pathwise optimal and yields the optimal solution of the portfolio optimization problem.

In addition, there exists a portfolio process $\pi^*(t)$ with $\pi^* \in A(z^*)$ and we have $W^{z^*,\pi^*}(T) = \xi^*$ a.s. according to Theorem 2.3.2.

Since ξ^* is deterministic we have

$$E[W^{z^*,\pi^*}(T)^-] = U(z_1)^- < \infty,$$

which implies $\pi^* \in A'(z^*)$. That completes the proof for the first case.

Case 2: $w < z_1$

Since, U is concave, we have

$U(I(y)) \geq U(x) + U'(I(y))(I(y) - x) = U(x) + y(I(y) - x)\ \forall x \geq 0\ y \in (0, \infty)$

In the case of $y > U'(0)$, we have

$$U(I(y)) = U(0) \geq U(x) + U'(0)(0 - x) \geq U(x) + y(I(y) - x).$$

Therefore,

$$U(I(y)) \geq U(x) + y(I(y) - x).$$

When we use this result, we can see

$$U(\xi^*) \geq U(1) + X^{-1}(w)H(T)(B^* - 1).$$

Hence,

$$E[U(\xi^*)^-] \leq |U(1)| + X^{-1}(w)(x + E[H(T)]) < \infty. \tag{4.7}$$

Furthermore,

$$U(\xi^*) \geq U(W^{x,\pi}(T)) + X^{-1}(w)H(T)(\xi^* - W^{x,\pi}(T)),$$

imply

$E[U(\xi^*)] \geq J(x, \pi) + X^{-1}(w) E[H(T)(\xi^* - W^{x,\pi}(T))]$.

From the definition of ξ^* we have $E[H(T)\xi^*] = w$ and from Theorem 2.3.1 we obtain

$E[H(T)W^{x,\pi}(T)] \leq w$.

Thus, we arrive at

$E[H(T)\xi^*] - E[H(T)W^{x,\pi}(T))] \geq 0$ for all $x \leq w$.

Therefore,

$E[U(\xi^*)] \geq J(x, \pi)$ for every $\pi \in A'(x), \; x \leq w$.

As a result, according to Theorem 2.3.2, there exist a portfolio process $\pi^* \in A(x)$ and an optimal terminal wealth process $W^{x,\pi}(T) = \xi^*$ a.s.. As above we have concluded that $E[U(\xi^*)] < \infty$ in (4.7), then $\pi^* \in A'(x)$ completes the proof. ■

4.1.1 The Lagrangian Function Approach

The Lagrangian function approach to the continuous-time MV problem, which has first been presented in Korn and Trautmann [31], is based on the idea to reduce the solution of the mean-variance problem to solving a family of portfolio problems with suitable quadratic utility functions. Therefore, it can be solved with the extended martingale method.

In this section, we will give the solution of the continuous-time MV problem for Lindberg's market model given in (3.1) with the Lagrangian approach when there is a positive terminal wealth constraint. In addition, we will state our new results about the form of the optimal portfolio strategy.

The MV optimization problem reads as

$$\min_{\pi(\cdot) \in A'(x), \; x \leq w} \operatorname{var}\left[W^{y,\pi}(T)\right] \quad \text{s.t.} \quad E[W^{y,\pi}(T)] \geq w e^{\lambda T}. \tag{4.8}$$

Here, the set $A'(x)$ consists of the portfolio processes $\pi(\cdot)$ that are progressively measurable with respect to the underlying Brownian filtration, that satisfy

$$\int_0^T ||\pi(t)||^2 dt < \infty \quad \text{a.s.} \tag{4.9}$$

and that ensure a non-negative final wealth when starting with an initial wealth of x, i.e., that guarantee

$$W^{x,\pi}(T) \geq 0 \;\text{ a.s.,}\;\; W^{y,\pi}(0) = x. \tag{4.10}$$

Further, $\lambda > 0$ denotes the continuously compounded required mean rate of return.

Due to the special form of the variance as an objective function, the optimal portfolio process for our problem may not necessarily need the full initial investment of w. This is obvious if we are in the case of $\lambda < r$. Then, a suitable risk-free bond investment is sufficient to satisfy the constraint with a variance of 0. On top of this, due to $\lambda < r$ not the full initial capital w is needed.

From now on, we will therefore assume

$$\lambda > r. \tag{4.11}$$

In this case, the expected wealth constraint cannot be satisfied by following the pure bond strategy. Hence, the optimal strategy must include risky investment. As then a variance of zero can no longer be achieved, the optimal strategy must satisfy the expected terminal wealth constraint as an equality. Thus, the mean variance problem is equivalent to the following problem:

$$\max_{\pi(\cdot)\in A'(x),\; x\leq w} -\frac{1}{2}E[W^{x,\pi}(T) - we^{\lambda T}]^2 \;\text{ s.t. }\; E[W^{x,\pi}(T)] = we^{\lambda T}. \tag{4.12}$$

As described in Section 4.3 of [32], this problem can be solved by following a two-step procedure:

Step1: For arbitrary fixed $d \geq 0$ solve the unconstrained problem

$$\max_{\pi(\cdot)\in A'(x),\; x\leq w} -E\left[\frac{1}{2}(W^{x,\pi}(T) - (we^{\lambda T} + d))^2\right] + \frac{1}{2}d^2.$$

Step2: Minimize the solutions obtained in Step 1 over $d \in [0, \infty)$.

This two-step procedure is the analogon to the classical way of deterministic optimization to transform the solution of a constrained problem to solving a set of unconstrained problems and then figuring out the appropriate one. Here, the utility function in the unconstrained problems simply is the corresponding Lagrangian function. The part of the Lagrangian function that enters the maximization in Step 1 is a quadratic utility function, and we can concentrate on solving the problems

$$\max_{\pi(\cdot)\in A'(x),\ x\le w} E\left[-\frac{1}{2}(W^{x,\pi}(T) - (we^{\lambda T} + d))^2\right] \tag{4.13}$$

for (fixed) values $d \geq 0$. The optimal terminal wealth of the last problem equals:

$$\begin{aligned}
W^*(T) &= (we^{\lambda T} + d) \qquad\qquad \text{if } w \geq (we^{\lambda T} + d)E[H(T)] && (4.14)\\
W^*(T) &= I(Y(w,d)H(T)) \qquad \text{otherwise} \\
&= ((we^{\lambda T} + d) - Y(w,d)H(T))\mathbf{1}_{\{H(T)\le \frac{(we^{\lambda T}+d)}{Y(w,d)}\}} \\
&= ((we^{\lambda T} + d) - Y(w,d)H(T))^+ && (4.15)
\end{aligned}$$

In the second but last line, $I(\cdot)$ is the generalized inverse of the derivative of the utility function which in our case equals

$$I(x) = ((we^{\lambda T} + d) - x)^+.$$

Here, the positivity is due to the positivity constraint of the terminal wealth constraint.

$H(T)$ denotes the deflator function defined as

$$H(T) := e^{-rT - \frac{\|\theta\|^2 T}{2} - \theta^T B(T)}, \tag{4.16}$$

with $\theta = ((\mu - r), \ldots, (\mu - r))^T$ is the market price of risk vector of the stock prices as given in (3.1). $Y(w,d)$ is a constant that is determined by the requirement

$$w = E\left(H(T)((we^{\lambda T} + d) - Y(w,d)H(T))^+\right). \tag{4.17}$$

Note that due to our assumptions on λ, we are always in the situation of Equation (4.15) and it suffices to maximize over $\pi(\cdot) \in A'(w)$.

To have a completely specified solution of the quadratic problem (4.13), we still have to determine

- the constant $Y(w,d)$ and
- the trading strategy that yields the final wealth $W^*(T)$.

For the first issue we have to find $X(y)$ to find $Y(w,d)$. From Section 4.1we know

$$X(y) := E[H(T)I(yH(T))] = w. \tag{4.18}$$

From definition of $I(\cdot)$ we obtain,

$$X(y) = E[H(T)((we^{\lambda T} + d) - yH(T))^+] = (we^{\lambda T} + d)E\left[H(T)\mathbf{1}_{\left\{H(T)\leq \frac{(we^{\lambda T}+d)}{y}\right\}}\right] \tag{4.19}$$

$$-yE\left[H(T)^2\mathbf{1}_{\left\{H(T)\leq \frac{(we^{\lambda T}+d)}{y}\right\}}\right] \tag{4.20}$$

Since, $H(T) \leq \frac{(we^{\lambda T}+d)}{y}\}$,

When we set $a = \theta^T B(T)$ then $a \geq -\ln\left(\frac{(we^{\lambda T}+d)}{y}\right) - rT - \frac{||\theta||^2 T}{2}$,

and set $A_1 = -\ln\left(\frac{(we^{\lambda T}+d)}{y}\right) - rT - \frac{||\theta||^2 T}{2}$.

Then,

$$E\left[H(T)\mathbf{1}_{\left\{H(T)\leq \frac{(we^{\lambda T}+d)}{y}\right\}}\right] = \int_{A_1}^{\infty} e^{\left(-rT-\frac{||\theta||^2T}{2}-a\right)} e^{-\frac{a^2}{2||\theta||^2T}} \frac{1}{\sqrt{2\pi||\theta||^2T}} da$$

$$= e^{-rT}\int_{A_1}^{\infty} e^{-\frac{1}{2}\left(\frac{a+||\theta||^2T}{\sqrt{||\theta||^2T}}\right)^2} \frac{1}{\sqrt{2\pi||\theta||^2T}} da. \tag{4.21}$$

When we set $u = \left(\frac{a+||\theta||^2T}{\sqrt{||\theta||^2T}}\right)$ and $A_2 = \left(\frac{-\ln(\frac{(we^{\lambda T}+d)}{y}) + \frac{||\theta||^2T}{2} - rT}{\sqrt{||\theta||^2T}}\right)$ then $da = du\sqrt{||\theta||^2T}$

and

$$E\left[H(T)\mathbf{1}_{\left\{H(T)\leq \frac{(we^{\lambda T}+d)}{y}\right\}}\right] = e^{-rT}\int_{A_2}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\frac{-u^2}{2}} du = e^{-rT}\Phi(-A_2). \tag{4.22}$$

In addition,

$$E\left[H^2(T)\mathbf{1}_{\left\{H(T)\leq \frac{(we^{\lambda T}+d)}{y}\right\}}\right] = \int_{A_1}^{\infty} e^{\left(-2rT-||\theta||^2T-2a\right)} e^{-\frac{a^2}{2||\theta||^2T}} \frac{1}{\sqrt{2\pi||\theta||^2T}} da$$

$$= e^{-2rT+||\theta||^2T}\int_{A_1}^{\infty} e^{-\frac{1}{2}\left(\frac{a+2||\theta||^2T}{\sqrt{||\theta||^2T}}\right)^2} \frac{1}{\sqrt{2\pi||\theta||^2T}} da \tag{4.23}$$

When we set $v = \left(\frac{a+2||\theta||^2T}{\sqrt{||\theta||^2T}}\right)$ and $A_3 = \left(\frac{-\ln(\frac{(we^{\lambda T}+d)}{y}) + \frac{3||\theta||^2T}{2} - rT}{\sqrt{||\theta||^2T}}\right)$, then $da = dv\sqrt{||\theta||^2T}$

and

$$E\left[H(T)\mathbf{1}_{\left\{H(T)^2\leq \frac{(we^{\lambda T}+d)}{y}\right\}}\right] = e^{-2rT+||\theta||^2T}\int_{A_3}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\frac{-v^2}{2}} dv = e^{-2rT+||\theta||^2T}\Phi(-A_3). \tag{4.24}$$

As a result, the right side of Equation (4.17) can be computed as

$$X(y) := (we^{\lambda T} + d)e^{-rT}\Phi\left(\frac{\ln(\frac{(we^{\lambda T}+d)}{y}) - \frac{\|\theta\|^2 T}{2} + rT}{\sqrt{\|\theta\|^2 T}}\right) \\ -ye^{-2rT+\|\theta\|^2 T}\Phi\left(\frac{\ln(\frac{(we^{\lambda T}+d)}{y}) - \frac{3\|\theta\|^2 T}{2} + rT}{\sqrt{\|\theta\|^2 T}}\right) \tag{4.25}$$

with $\Phi(\cdot)$ being the distribution function of the standard Gaussian distribution.

Thus, to find $Y(w,d)$ we have to solve the non-linear equation $X(y) = w$ numerically. Having found this unique solution for a fixed d, we then have to numerically minimize the utility function of Step 1

$$-E\left[\frac{1}{2}\left(W^*(T) - \left(we^{\lambda T} + d\right)\right)^2\right] + \frac{1}{2}d^2 \tag{4.26}$$
$$= \frac{1}{2}\Big[d^2 - \left(we^{\lambda T} + d\right)^2 \Phi\left(f\left(Y(w,d),d\right)\right) \\ -Y(w,d)^2 e^{-2rT+\|\theta\|^2 T}\left(1 - \Phi\left(h\left(Y(w,d),d\right)\right)\right)\Big]$$

as a function of d over $[0, \infty\ )$ where we use the abbreviations

$$f(y,d) = \frac{\ln\left(\frac{y}{we^{\lambda T}+d}\right) - \left(r + \frac{1}{2}\|\theta\|^2\right)T}{\sqrt{\|\theta\|^2 T}}, \quad h(y,d) = f(y,d) + 2\sqrt{\|\theta\|^2 T}. \tag{4.27}$$

Having thus determined the optimal final wealth of

$$W^*(T) = [(we^{\lambda T} + d^*) - Y(w,d^*)H(T)]^+, \tag{4.28}$$

we can determine the corresponding portfolio process by a comparison argument. We summarize the result in the following theorem.

Theorem 4.1.4 *The optimal portfolio process $\pi^*(t)$ is given by*

$$\pi^*(t) = -\sigma^{-1}\theta\tilde{y}H(t)\frac{\tilde{f}_x(t,\tilde{y}H(t))}{\tilde{f}(t,\tilde{y}H(t))} \tag{4.29}$$
$$= \sigma^{-1}\theta\tilde{y}H(t)\frac{e^{\left(-2r+\|\theta\|^2\right)(T-t)}}{\tilde{f}(t,\tilde{y}H(t))}\Phi\left(\frac{\ln\left(\frac{\tilde{K}}{\tilde{y}H(t)}\right) + \left(r - \frac{3}{2}\|\theta\|^2\right)(T-t)}{\sqrt{\|\theta\|^2(T-t)}}\right)$$

with d^* and $Y(w,d^*)$ as defined above,

$$\tilde{K} := we^{\lambda T} + d^*, \;\; \tilde{y} := Y(w,d^*), \tag{4.30}$$

where $\tilde{f}_x(t,y)$ denotes the partial derivative of $\tilde{f}$ with respect to its second variable and $\tilde{f}$ is given by

$$\tilde{f}(t, yH(t)) = \tilde{K}e^{-r(T-t)}\Phi\left(\frac{\ln\left(\frac{\tilde{K}}{yH(t)}\right) + \left(r - \frac{1}{2}\|\theta\|^2\right)(T-t)}{\sqrt{\|\theta\|^2(T-t)}}\right) - \tag{4.31}$$
$$-yH(t)\, e^{-(2r-\|\theta\|^2)(T-t)}\Phi\left(\frac{\ln\left(\frac{\tilde{K}}{yH(t)}\right) + \left(r - \frac{3}{2}\|\theta\|^2\right)(T-t)}{\sqrt{\|\theta\|^2(T-t)}}\right).$$

.

Proof. Let us first point out that due to standard results of continuous-time finance (see e.g., Chapter 2 of [33]), the wealth process $W^*(t)$ that leads to the optimal final wealth of $W^*(T) = [(we^{\lambda T} + d^*) - Y(w,d^*)H(T)]^+$ satisfies

$$H(t)W^*(t) = E\left(H(T)W^*(T)\,|F_t\right). \tag{4.32}$$

This leads to

$$\begin{aligned} W^*(t) &= E\left(\frac{H(T)}{H(t)}W^*(T)\,|F_t\right) \\ &= \tilde{K}E\left(\frac{H(T)}{H(t)}1_{\tilde{y}H(T)>\tilde{K}}\,|F_t\right) - \tilde{y}E\left(\frac{H(T)}{H(t)}H(T)1_{\tilde{y}H(T)>\tilde{K}}\,|F_t\right) \\ &= \tilde{K}e^{-r(T-t)}\Phi\left(\frac{\ln\left(\frac{\tilde{K}}{\tilde{y}H(t)}\right) + \left(r - \frac{1}{2}\|\theta\|^2\right)(T-t)}{\sqrt{\|\theta\|^2(T-t)}}\right) - \\ &\quad -\tilde{y}H(t)\, e^{-(2r-\|\theta\|^2)(T-t)}\Phi\left(\frac{\ln\left(\frac{\tilde{K}}{\tilde{y}H(t)}\right) + \left(r - \frac{3}{2}\|\theta\|^2\right)(T-t)}{\sqrt{\|\theta\|^2(T-t)}}\right) \qquad (4.33) \\ &=: \tilde{f}(t, \tilde{y}H(t)). \end{aligned}$$

Here, the second but last equality is result of the explicit computation of the conditional expectations which are totally similar to the computations leading to Equation (4.25).

Applying Itô's rule to $W^*(t)$ yields

$$\begin{aligned} dW^*(t) &= d\tilde{f}(t, \tilde{y}H(t)) \\ &= \tilde{f}_t(t, \tilde{y}H(t))\,dt + \tilde{f}_x(t, \tilde{y}H(t))\,dx + \frac{1}{2}\tilde{f}_{xx}(t, \tilde{y}H(t))\,d < x, x > \\ &= \left(\tilde{f}_t(t, \tilde{y}H(t)) - \tilde{f}_x(t, \tilde{y}H(t))\,r\tilde{y}H(t)\right)dt \\ &\qquad -\tilde{f}_x(t, \tilde{y}H(t))\,\tilde{y}H(t)\,\theta^T dB(t). \qquad (4.34) \end{aligned}$$

Comparing this representation with the general form of the wealth equation

$$dW^*(t) = W^*(t)\left[\pi^*(t)^T\sigma(\mu - r)\,dt + \pi^*(t)^T\sigma dB(t)\right] \tag{4.35}$$

leads

$$\pi^*(t) = -\sigma^{-1}\theta\tilde{y}H(t)\,\frac{\tilde{f}_x(t,\tilde{y}H(t))}{\tilde{f}(t,\tilde{y}H(t))}. \tag{4.36}$$

$$\tag{4.37}$$

The second representation of (4.29) is then obtained by the explicit calculation of the partial derivative (this is similar to the calculation of the Delta in the Black-Scholes formula).

When we set $x = \tilde{y}H(t)$, then

$$\tilde{f}(t,x) = \tilde{K}e^{-r(T-t)}\Phi(d_1) - xe^{-(2r-\|\theta\|^2)(T-t)}\Phi(d_2),$$

where

$$\begin{aligned} d_1 &= \frac{\ln\left(\frac{\tilde{K}}{x}\right) + \left(r - \frac{1}{2}\|\theta\|^2\right)(T-t)}{\sqrt{\|\theta\|^2(T-t)}} \\ d_2 &= \frac{\ln\left(\frac{\tilde{K}}{x}\right) + \left(r - \frac{3}{2}\|\theta\|^2\right)(T-t)}{\sqrt{\|\theta\|^2(T-t)}}. \end{aligned} \tag{4.38}$$

The partial derivatives of d_1 and d_2 with respect to x are obtained as

$$\frac{\partial d_1}{\partial x} = \frac{\partial d_2}{\partial x} = -\frac{1}{x\sqrt{\|\theta\|^2(T-t)}}. \tag{4.39}$$

The partial derivative of $\tilde{f}(t,x)$ with respect to x are obtained as

$$\begin{aligned} \frac{\partial \tilde{f}(t,x)}{\partial x} = -\tilde{K}e^{-r(T-t)}\phi(d_1)\frac{1}{x\sqrt{\|\theta\|^2(T-t)}} + e^{-(2r-\|\theta\|^2)(T-t)}\phi(d_2)\frac{1}{\sqrt{\|\theta\|^2(T-t)}} \\ -e^{-(2r-\|\theta\|^2)(T-t)}\Phi(d_2), \end{aligned} \tag{4.40}$$

where ϕ is the normal density function.

From the definitions of d_1 and d_2 we can see that $d_2 = d_1 - \sqrt{\|\theta\|^2(T-t)}$. When we use this result, we can get the density function of d_2 in the form of d_1 as

$$\begin{aligned} \phi(d_2) = \frac{1}{\sqrt{2\pi}}e^{-\frac{d_2^2}{2}} &= \frac{1}{\sqrt{2\pi}}e^{-\frac{\left(d_1-\sqrt{\|\theta\|^2(T-t)}\right)^2}{2}} \\ &= \phi(d_1)e^{\left(-\frac{1}{2}d_1^2+\ln\left(\frac{\tilde{K}}{x}\right)+\left(r-\|\theta\|^2\right)(T-t)\right)} \\ &= \phi(d_1)\frac{\tilde{K}}{x}e^{\left(\left(r-\|\theta\|^2\right)(T-t)\right)}. \end{aligned} \tag{4.41}$$

When we put this result to (4.40), we will have

$$\frac{\partial \tilde{f}(t,x)}{\partial x} = -e^{-(2r-\|\theta\|^2)(T-t)}\Phi(d_2). \tag{4.42}$$

As a result the optimal portfolio strategy formula is

$$\pi^*(t) = \sigma^{-1}\theta\tilde{y}H(t)\frac{e^{\left(-2r+\|\theta\|^2\right)(T-t)}}{\tilde{f}(t,\tilde{y}H(t))}\Phi\left(\frac{\ln\left(\frac{\tilde{K}}{\tilde{y}H(t)}\right)+\left(r-\frac{3}{2}\|\theta\|^2\right)(T-t)}{\sqrt{\|\theta\|^2(T-t)}}\right). \tag{4.43}$$

■

CHAPTER 5

CONTINUOUS-TIME MEAN-VARIANCE OPTIMAL PORTFOLIO APPROACHES: A COMPARISON

In this chapter, we will first give the solution of the continuous time MV problem with the L^2 projection method of Schweizer [53]. Then, we will perform a detailed comparison of the three continuous time MV portfolio optimization approaches considered in this thesis:

- the Lagrangian function approach,
- the L^2-projection approach, and
- the direct approach by Lindberg.

The comparison results are illustrated by numerical examples. The solution of the continuous time MV problem and the comparison results are taken from [55].

5.1 The L^2 - Projection Approach

In 1995, Schweizer [53] solved quadratic optimization problems in finance by using L^2 projection techniques on the space of stochastic integrals with special conditions. As a general case, he aims to solve the Mean-Variance hedging problems. However, his results can be used for the MV portfolio optimization problem as well. If one gives up the non-negative final wealth requirement of the problem given in (4.8), then the solution of the MV problem can be solved by using L^2-projection techniques as given in Schweizer [53].

The MV portfolio optimization problem then reads as

$$\min_{\pi(\cdot)\in A''(y),\ y\leq w} \operatorname{var}[W^{y,\pi}(T)] \ \text{ s.t. } \ E[W^{y,\pi}(T)] \geq we^{\lambda T} \tag{5.1}$$

where the set $A''(y)$ consists of the portfolio processes $\pi(\cdot)$ that are progressively measurable with respect to the underlying Brownian filtration and that satisfy

$$\int_0^T \|\pi(t)\|^2 dt < \infty \text{ a.s. } . \tag{5.2}$$

As indicated in [32] this problem can be treated by using the same steps as in the Lagrangian function approach, i.e. by solving the following quadratic problems

$$\max_{\pi(\cdot)\in A''(w)} -E\left[\frac{1}{2}(W^{w,\pi}(T) - (we^{\lambda T} + d))^2\right] \tag{5.3}$$

where we have again used Assumption (4.11) and can thus restrict ourselves to consider only strategies from $A''(w)$.

Rewriting the unconstrained problem in Step 1 in the form of the problem in [53] with discounted asset prices and discounted wealth process

$$\tilde{S}_i(t) = S_i(t)e^{-rt}, \quad \tilde{W}(t) = W(t)e^{-rt}, \tag{5.4}$$

we look at the minimization problem

$$\min_{\pi(\cdot)\epsilon A(w)} \frac{1}{2}E\left[(\tilde{W}(T) - (we^{\lambda T} + d)e^{-rT})^2\right] . \tag{5.5}$$

Theorem 5.1.1 *The optimal final wealth $W^*(T,d)$ for the minimization problem (5.5) is given by*

$$W^*(T,d) = (we^{\lambda T} + d) - ((we^{\lambda T} + d) - we^{rT})e^{-\frac{3}{2}\|\theta\|^2 T - \theta^T B(T)}. \tag{5.6}$$

The corresponding portfolio process to the final wealth of $W^*(T,d)$ has the form of

$$\pi_i^{(d)}(t) = \frac{\varrho_i}{W^*(t,d)}\Big(((we^{\lambda T} + d) - we^{rT})e^{-r(T-t)-\frac{3}{2}\|\theta\|^2 t - \theta^T B(t)}\Big). \tag{5.7}$$

with the notation of $\varrho = (\sigma^T)^{-1}\theta$.

Proof. If we set,

$$c = 0, \ \ \varrho = (\sigma^T)^{-1}\theta, \ \ \xi^H(t) = 0, \ \ \xi^c(t) = \pi_t \frac{W^*(t)}{S_i(t)},$$

and use the notation of

$$H = (we^{\lambda T} + d)e^{-rT},\ G_T(\xi) = \tilde{W}(T),$$
$$V^H(t) = E_t[H|F_t] = (we^{\lambda T} + d)e^{-rT},$$

then according to the main theorem of [53] given in Appendix A.2 part of the thesis the optimal number of assets are found as,

$$\xi^c(t) = \tilde{\lambda}_i(t)[(we^{\lambda T} + d)e^{-rT} - \tilde{W}^*(t,d)], \tag{5.8}$$

where $\tilde{\lambda}_i(t) = \frac{\varrho_i}{\tilde{S}_i(t)}$.

Then, since $d\tilde{W}^*(t,d) = \sum_{i=1}^{n} \xi^c(t) d\tilde{S}_i(t)$, the optimal wealth process has the following stochastic differential equation:

$$d\tilde{W}^*(t,d) = ((we^{\lambda T} + d)e^{-rT} - \tilde{W}^*(t,d))(||\theta||^2 dt + \theta^T dB_t), \tag{5.9}$$

with solution given in (5.10):

$$\tilde{W}^*(t,d) = (we^{\lambda T} + d)e^{-rT} - ((we^{\lambda T} + d)e^{-rT} - w)e^{-\frac{3}{2}||\theta||^2 t - \theta^T B(t)}. \tag{5.10}$$

We know that $\tilde{W}^*(t,d) = W^*(t,d)e^{-rt}$. So, as a result,

$$W^*(t,d) = (we^{\lambda T} + d)e^{-r(T-t)} - ((we^{\lambda T} + d)e^{-r(T-t)} - we^{rt})e^{-\frac{3}{2}||\theta||^2 t - \theta^T B(t)}, \tag{5.11}$$

and the optimal terminal wealth process becomes

$$W^*(T,d) = (we^{\lambda T} + d) - ((we^{\lambda T} + d) - we^{rT})e^{-\frac{3}{2}||\theta||^2 T - \theta^T B(T)}. \tag{5.12}$$

When we use the result obtained in (5.10) and the fact that $\tilde{S}_i(t) = S_i(t)e^{-rt}$, then the optimal $\xi^c(t)$ becomes

$$\xi^c(t) = \frac{\varrho_i}{S_i(t)} \left((we^{\lambda T} + d)e^{-r(T-t)} - we^{rt} \right) e^{-\frac{3}{2}||\theta||^2 T - \theta^T B(T)}. \tag{5.13}$$

In Schweizer's paper the optimal strategy is given as the number of assets in the portfolio, but in our case we are dealing with the weights of assets in the portfolio so we have to multiply the $\xi^c(t)$ with $\frac{S_i(t)}{W^*(t)}$ to represent the strategy as the portfolio weights. Finally, we can obtain the optimal portfolio strategy as

$$\pi_i^{(d)}(t) = \frac{\varrho_i}{W^*(t,d)} \left(((we^{\lambda T} + d) - we^{rT})e^{-r(T-t)-\frac{3}{2}||\theta||^2 t - \theta^T B(t)} \right). \tag{5.14}$$

■

We can now use this explicit form of $W^*(T,d)$ to solve the mean-variance problem

$$\begin{aligned} &\min_{d \geq 0} \min_{\pi(\cdot) \in A''(w)} \frac{1}{2} \left[d^2 - E\left[(W^*(T,d) - (we^{\lambda T} + d))^2 \right] \right] \\ &= \min_{d \geq 0} \frac{1}{2} \left[d^2 - ((we^{\lambda T} + d) - we^{rT})^2 E\left(e^{-3||\theta||^2 T - 2\theta^T B(T)} \right) \right] \end{aligned} \tag{5.15}$$

The minimum of this deterministic function in d is attained for

$$d^* = \frac{(we^{\lambda T} - we^{rT})e^{-||\theta||^2 T}}{1 - e^{-||\theta||^2 T}}. \tag{5.16}$$

Hence, we have completely solved the mean-variance problem in the setting without non-negative wealth constraints:

Theorem 5.1.2 *Under Assumption (4.11) the solution of the mean-variance problem is given by the optimal final wealth*

$$W^*(T) = \frac{w}{1 - e^{-||\theta||^2 T}} \left[e^{\lambda T} - e^{(r - ||\theta||^2)T} - (e^{\lambda T} - e^{rT})e^{(r - ||\theta||^2)T} H(T) \right] \tag{5.17}$$

and the corresponding portfolio process of

$$\pi_i^*(t) = \varrho_i \frac{(e^{\lambda T} - e^{rT})e^{(r-\|\theta\|^2)t}H(t)}{e^{\lambda T} - e^{(r-\|\theta\|^2)T} - (e^{\lambda T} - e^{rT})e^{(r-\|\theta\|^2)t}H(t)} \,. \tag{5.18}$$

Proof. The proof consists of putting the explicit form (5.16) of d^* into the representations of $W^*(T, d^*)$ and $\pi^{(d)}(t)$ given in Theorem 5.1.1. ∎

5.2 Comparison of the Continuous-Time Mean-Variance Approaches

5.2.1 The Form of the Optimal Terminal Wealth

The first - and maybe most important - characteristic to compare among the different mean-variance approaches is the final wealth. There are two remarkable aspects. First, as in the L^2-projection approach, we have the weakest requirements on the portfolio process (followed by those of the Lagrangian function approach), we have the obvious ranking in terms of the smallest variance between the different methods:

$$\operatorname{var}\left(W_{LP}^*(T)\right) < \operatorname{var}\left(W_{LF}^*(T)\right) < \operatorname{var}\left(W_{DL}^*(T)\right). \tag{5.19}$$

Here, $W_{LP}^*(T)$ is the optimal terminal wealth of L^2-projection approach, $W_{LF}^*(T)$ is the optimal terminal wealth of Lagrangian function approach and $W_{DL}^*(T)$ is the optimal terminal wealth of the direct approach by Lindberg.

To see why we have even the strict inequalities above, note that the optimal wealth processes are unique in all three problems and have different behavior with respect to the sign of the final wealth: While the Lindberg wealth is strictly positive, the optimal final wealth in the Lagrangian function approach is non-negative with a positive probability to be zero. The optimal final wealth in the projection method attains positive and negative values with a positive probability.

To illustrate the different characteristics for the three optimal final wealths, we plot them as a function of the underlying Brownian motion $B(T)$ in Figure 5.1 (where we have used the following set of parameters: $n = 1, r = 0.05, \mu = 0.1, \sigma = 0.4, \lambda = 0.075, w = 1, T = 1$). Note that both the L^2 projection and the Lagrangian function approach nearly perfectly coincide inside the center area $[-2, 2]$ of the distribution of $B(T)$. However, although hard to view, the L^2 projection method is slightly closer to the lower bound for the expected final wealth than the Lagrangian function method. As expected, the L^2 projection method leads to negative values for the final wealth for values of $B(T) < -2$, while the Lagrangian function method gets 0 values.

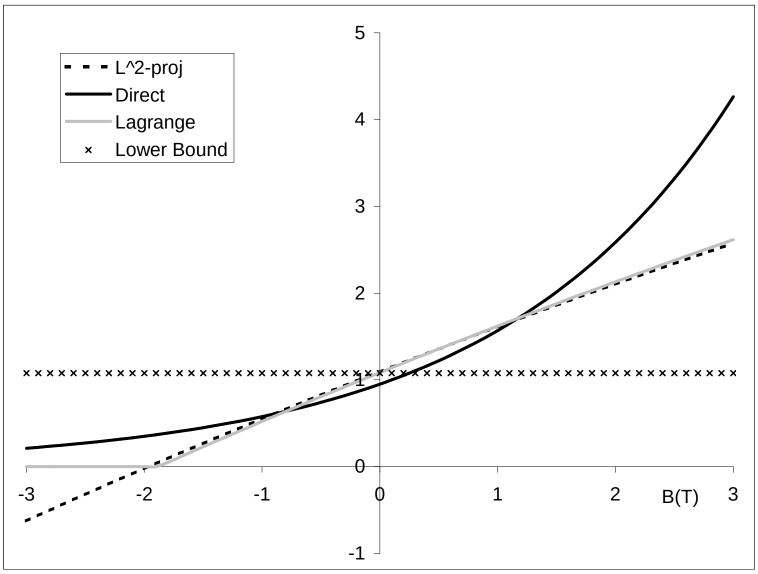

Figure 5.1: Comparison of final wealth as function of $B(T)$.

One is tempted to say that the final wealth of the direct method is the most attractive one. However, this is a bit misleading as it mainly performs well in the non-central part of the distribution of $B(T)$. In the center, the two other methods perform better. Note also that the final wealth of the direct method is an exponential function of $B(T)$, while the other two are more close to a linear such function (at least approximately).

In addition to Figure 5.1, to show the change in the characteristic behavior of the optimal weights for an increase in the risk-free rate r and in the required rate of return λ we plot the optimal weights as a function of $B(T)$ in Figure 5.2 and 5.3 by changing those two rates accordingly. In Figure 5.2 all the parameters of Figure 5.1 are kept constant and the risk-free rate is increased to 0.065. In Figure 5.3 all the parameters of Figure 5.1 are kept constant but the required rate of return is increased to 0.085.

In Figure 5.2 the optimal wealths of the Lagrangian function method and the L^2 projection approach nearly perfectly coincide in all values of $B(T)$. As we have increased the risk-free rate the difference between the risk-free rate and the required rate of risky asset then is very small. Therefore, the risky investment will be decreased and as can be seen from the figure, the optimal wealth of the L^2 projection approach does not attain negative values in the likely area of the final wealth. The direct approach maintains it's behavior in the central and non-central part of the distribution of $B(T)$. However, the difference in optimal wealths of the direct method and the Lagrangian function and L^2 projection method has decreased in all values of $B(T)$.

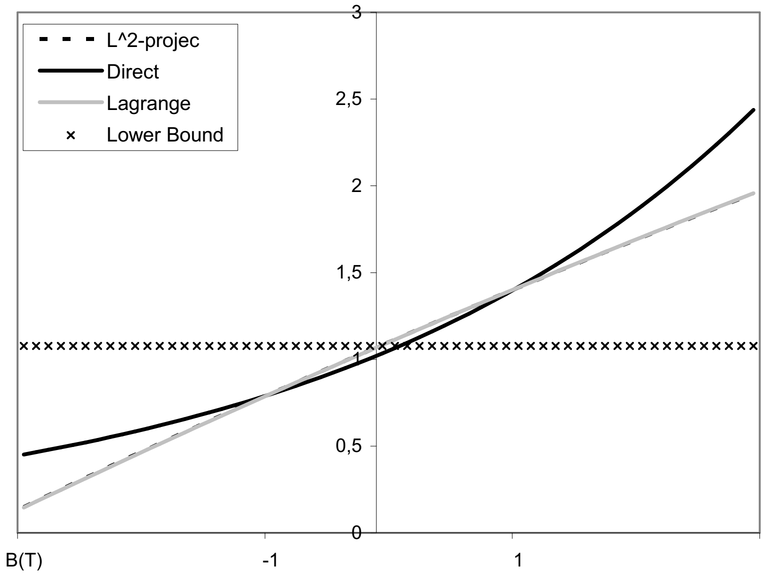

Figure 5.2: Comparison of final wealth as function of $B(T)$ when r=0.65

In Figure 5.3 inside the interval of [-1.5,-0.5] the distribution of $B(T)$ L^2 projection method performs better than the Lagrangian function approach. For $B(T)$>0.5 the Lagrangian function approach method performs better than the L^2 projection method. The L^2 projection approach is more risky than the Lagrangian function approach and when we increased the required rate of risky asset the L^2 projection approach leads to negative optimal terminal wealths for $B(T)$<-1,.5. Thus, the probability of getting negative values is higher than the case of Figure 5.1. The direct approach maintains it's behavior in the central and non-central part of the distribution of $B(T)$.

Note that in the L^2-projection approach the variance minimizing activities can lead to a negative wealth which indeed is very undesirable. While the Lagrangian function approach performs best by obtaining a non-negative wealth process with minimal variance, its main draw-

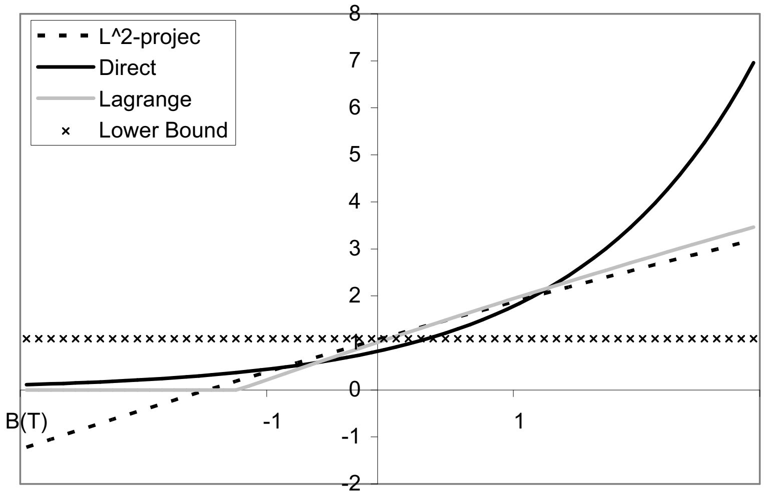

Figure 5.3: Comparison of final wealth as function of $B(T)$ when $\lambda = 0.085$

back are the complicated computations. Here, the fully explicit results of the direct approach and the L^2 projection approach are easier to obtain.

5.2.2 The Form of the Optimal Portfolio Process

The other characteristic to compare is the portfolio processes. Again, the ranking of simplicity is lead by the direct method as the optimal portfolio process for the direct method is just a constant portfolio process, given as the unique solution of the linear system (3.30).

Also, the optimal portfolio processes of the other two methods can be computed explicitly . The L^2-projection method has an optimal portfolio process of

$$\pi_i^*(t) = \frac{\varrho_i}{W_t^*}\left((we^{\lambda T} + d^*) - we^{rT})e^{\{-rt-\frac{3}{2}\|\theta\|^2 t-\theta^T B(t)\}}\right). \tag{5.20}$$

Note that $1/S_i(t)$ is hidden in $\hat{\lambda}_i$. Thus, the portfolio process is far from constant and the size of its moves depend heavily on the relation between $H(t)$ (the market portfolio) and the i. stock.

Finally, we have an explicit formula for the portfolio process in the Lagrangian function approach as given by Equation (4.29),

$$\pi^*(t) = \sigma^{-1}\theta\tilde{y}H(t)\frac{e^{\left(-2r+\|\theta\|^2\right)(T-t)}}{\tilde{f}(t,\tilde{y}H(t))}\Phi\left(\frac{\ln\left(\frac{\tilde{K}}{\tilde{y}H(t)}\right)+\left(r-\frac{3}{2}\|\theta\|^2\right)(T-t)}{\sqrt{\|\theta\|^2(T-t)}}\right) \tag{5.21}$$

To compare their performance, we look at the setting of our example used above to illustrate the behavior of the optimal final wealths. In the first example given in Figure 5.4 the market performs well (indicated by a high value of $B(T)$) and thus the L^2 projection and the Lagrangian function methods do not need a big stock investment to deal with the expectation constraint.

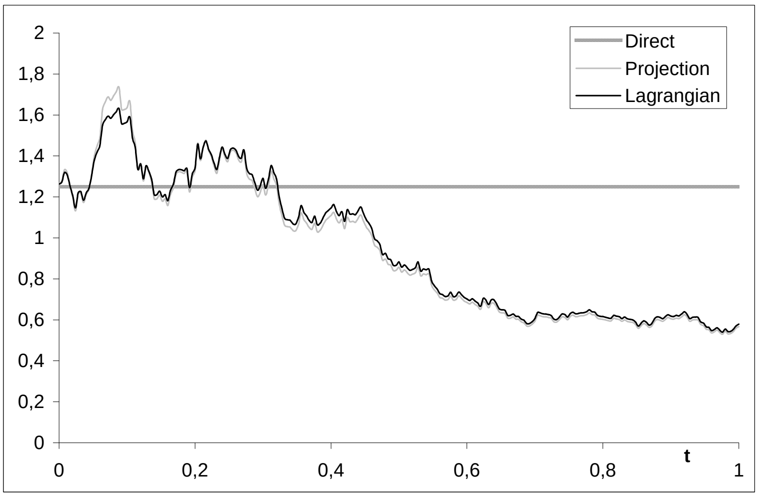

Figure 5.4: Performance of different portfolio strategies - good market performance

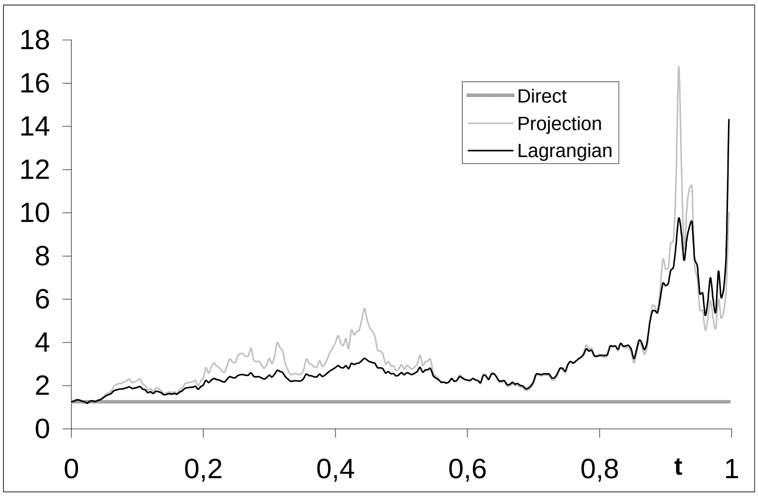

Figure 5.5: Performance of different portfolio strategies - bad market performance

In our second example given in Figure 5.5, the market performs badly (indicated by a low value of $B(T)$) and both the Lagrangian function and the L^2 projection method lead to a final wealth which is positive, but close to zero. Thus, both methods try to compensate for this by a stock investment which is financed by a short position in the bond. Note that the L^2 projection method typically is more risky than the Lagrangian function method for nearly all the time. Only close to the time horizon the Lagrangian function portfolio it exceeds the one of the L^2 projection method slightly.

CHAPTER 6

CONCLUSION

One of the classical problems in financial mathematics is the portfolio optimization problem. The most widely used portfolio optimization techniques is presented by Harry Markowitz [45] in 1952. Although his one period MV portfolio optimization approach is used wide-spread in application, its main problem is its static nature. In addition, to solve a portfolio optimization problem all the market coefficients (such as the drift and diffusion coefficients of the stock price) have to be known. However, the estimation of the stock price drift is a particularly hard problem.

For this purposes, in this thesis we have studied continuous time extension of MV portfolio optimization problem. To solve the expected return estimation problem, we picked up ideas of Lindberg presented in [42] intending to

- solve the continuous-time mean-variance problem,
- help estimating stock price drift coefficients in an intuitive way,
- and to interpret the optimal strategy as a $1/n$-strategy.

We have extended these ideas and the setting of [42] to a jump-diffusion framework. By doing so we modified the optimality statement given in [42] in showing that the assertion there is only valid for a restricted class of portfolio strategies, which is essentially equal to the deterministic portfolio processes. Further, we examined the scope and the interpretation of this new method by demonstrating effects such as

- the determination of the range of the drift parameters,

- the exact interpretation of the form of the optimal strategy (which is only a $1/n$-strategy in a very wide sense),
- the interpretation of the choice of the σ-matrix as a choice of surplus drift and not as a choice of the volatility structure (which indeed remains unchanged under such a choice).

Then, we have used the Lagrangian function approach to the continuous-time MV portfolio optimization, which has first been presented in Korn and Trautmann [31]. We have used the innovative market parametrization of Lindberg [42] and derived new results in particular explicit representation for the optimal portfolio process. In addition, we have derived the optimal solution for the continuous-time MV portfolio optimization with Lindberg's market parametrization by using the L^2-projection techniques as given in Schweizer [53].

Lastly, we have compared the three different approaches to the continuous-time mean-variance problem in the market setting suggested recently by Lindberg in [42]. All three approaches have their attractive features:

- The Lindberg approach is structurally very easy as its optimal portfolio process is a constant one. Further, it has an interpretation of a $1/n$-strategy (as dividing money equally to the different sources of risk) which is appealing.
- The L^2 projection approach leads to an explicit form for both the optimal final wealth and the optimal portfolio process. Further, it has the biggest range of application, up to a general semi-martingale setting as presented in [53] and many follow-up papers. Also, it provides the lowest variance among the three methods. However, it cannot guarantee a non-negative final wealth which is an undesirable feature for its application in real life.
- The Lagrangian function approach in its current form is restricted to the complete diffusion market setting. To obtain its optimal final wealth, one has to use an iteration procedure that also involves solving a non-linear equation numerically. On the positive side, we have derived an explicit representation for the optimal portfolio process that allows an implementation of the method. And on top of this, the performance of the Lagrangian function method is very close to that of the L^2 projection method, but it avoids the possibility for a negative final wealth.

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APPENDIX A

SCHWEIZER'S L^2 PROJECTION APPROACH

A.1 Preliminary Definitions

Let (Ω, F, P) be a probability space with a filtration $\mathbb{F} = (F_t)_{0\leq t\leq T}$, $T > 0$ be a fixed finite time horizon. Let $X = (X_t)_{0\leq t\leq T}$ be an $\mathbb{R}^d$ valued semimartingale in S^2_{loc} for the canonical decomposition

$$X = X_0 + M + A \tag{A.1}$$

of X, this means $M \in \mathcal{M}^2_{0,loc}$ and $|A^i|$ is the predictable finite variation part such that A^i of X^i is locally square integrable for each i. Let us choose versions of M and A such that M^i and A^i are right continuous with left limits (RCLL) for each i. We shall assume,

$$A^i << \langle M^i \rangle \tag{A.2}$$

with predictable density $\alpha^i = (\alpha^i_t)_{0\leq t\leq T}$ where $\langle M^i \rangle$ is the process associated to M^i.

Then fix a predictable increasing RCLL process $B = (B_t)_{0\leq t\leq T}$ null at 0 such that $\langle M^i \rangle << B$ for each i, for instance, choose $B = \sum_{i=1}^{d} \langle M^i \rangle$ then this implies $\langle M^i, M^j \rangle << B$ for all $i, j = 1, \ldots, d$ and define the predictable matrix-valued process $\sigma = (\sigma_t)_{0\leq t\leq T}$ by,

$$\sigma^{i,j}_t := \frac{d\langle M^i, M^j \rangle}{dB_t} \quad for \;\; i, j = 1, \ldots, d. \tag{A.3}$$

Therefore, each $\sigma^{i,j}_t$ is symmetric non-negative definite matrix.

If a predictable $\mathbb{R}^d$ valued process $\gamma = (\gamma_t)_{0\leq t\leq T}$ is defined by

$$\gamma_t^i := \alpha_t^i \sigma_t^{i,i} \quad for \;\; i = 1, \ldots, d. \tag{A.4}$$

Then, (A.2) and (A.3) imply that

$$A_t^i = \int_0^t \gamma_s^i dB_s \quad for \;\; i = 1, \ldots, d. \tag{A.5}$$

Definition A.1.1 *The space $L^2_{(loc)}(M)$ consists of all predictable $\mathbb{R}^d$ valued process $\upsilon = (\upsilon_t)_{0 \le t \le T}$ such that*

$$\left(\int_0^t \upsilon_s^T \sigma_s \upsilon_s dB_s \right)_{0 \le t \le T} \tag{A.6}$$

is locally integrable.

The space $L^2_{(loc)}(A)$ consists of all predictable $\mathbb{R}^d$ valued process $\upsilon = (\upsilon_t)_{0 \le t \le T}$ such that

$$\left(\int_0^t |\upsilon_s^T \gamma_s| dB_s \right)_{0 \le t \le T} \tag{A.7}$$

is locally square integrable.

Let us define the space $\Theta := L^2(M) \cap L^2(A)$.

If $\upsilon \in L^2_{(loc)}(M)$ the stochastic integral $\int \upsilon dM$ is well defined in $\mathcal{M}^2_{0,loc}$ and

$$\left\langle \int \upsilon dM, \int \psi dM \right\rangle_t = \int_0^t \upsilon_s^T \sigma_s \psi_s dB_s \quad \text{for} \;\; \upsilon, \psi \in L^2_{(loc)}(M) \tag{A.8}$$

If $\upsilon \in L^2_{(loc)}(A)$ the process $\int_0^t \upsilon_s^T dA_s := \sum_{i=1}^d \int_0^t \upsilon_s^i dA_s^i = \int_0^t \upsilon_s^T \gamma_s dB_s$ is well defined as a Riemann-Stilges integral and has locally square integrabe variation $\left| \int \upsilon^T dA \right| = \int |\upsilon^T \gamma| dB.$

Then, for any $\upsilon \in \Theta$, the stochastic integral process

$$G_t(\upsilon) := \int_0^t \upsilon_s dX_s, \tag{A.9}$$

is well defined and a semimartingale in $\mathcal{S}^2$ with canonical decomposition,

$$G(\upsilon) := \int \upsilon dM + \int \upsilon^T dA, \tag{A.10}$$

Let us define the predictable matrix valued process $J = (J_t)_{0\le t\le T}$ by setting

$$J_t^{i,j} := \sum_{0<s\le t} \Delta A_s^i \Delta A_s^j \quad for \quad i,j = 1,\ldots,d. \tag{A.11}$$

By (A.5) J can be written as

$$J_t^{i,j} := \int_0^t \kappa_s^{i,j} dB_s, \tag{A.12}$$

where the predictable matrix valued process $\kappa = (\kappa_t)_{0\le t\le T}$ is defined as $\kappa_t^{i,j} := \gamma_t^i \gamma_t^j \Delta B_t \quad for$ $i,j = 1,\ldots,d$.

Definition A.1.2 *X satisfies the extended structure condition (ESC) if there exists a predictable $\mathbb{R}^d$ valued process $\tilde{\lambda} = (\tilde{\lambda}_t)_{0\le t\le T}$ such that*

$$(\sigma_t + \kappa_t)\tilde{\lambda}_t = \gamma_t \quad P-a.s. \quad for\ all \quad t \in [0,T] \tag{A.13}$$

and

$$\tilde{K}_t := \int_0^t \tilde{\lambda}_s^T \gamma_s dB_s < \infty \quad P-a.s. \quad for\ all \quad t \in [0,T]. \tag{A.14}$$

When the RCLL version of $\tilde{K}_t$ is choosed then it is called the extended mean-variance tradeoff (EMVT) process of X.

Definition A.1.3 *A random variable $H \in L^2$ admits a strong FS decomposition if H can be written as*

$$H = H_0 + \int_0^T \xi_s^H dX_s + L_T^H \quad P-a.s., \tag{A.15}$$

where $H_0 \in \mathbb{R}$ is a constant, $\xi_s^H \in \Theta$ is a strategy and $L^H = (L_t^H)_{0\le t\le T}$ is in $\mathcal{M}^2$ with $E[L_0^H] = 0$ and strongly orthogonal to $\int \upsilon\, dM$ for every $\upsilon \in L^2(M)$.

A.2 The Main Theorem

Theorem A.2.1 *Suppose X satisfies the ESC and EMVT process $\tilde{K}$ is deterministic. If $H \in L^2$ admits a strong FS decomposition then the solution of the following quadratic optimization problem*

$$\min_{\upsilon \in \Theta} E[(H - c - G_T(\upsilon))^2] \tag{A.16}$$

for any $c \in \mathbb{R}$ is

$$\xi_t^{(c)} = \xi_t^H + \tilde{\lambda}_t(V_{t-}^H - c - G_{t-}(\xi^{(c)})) \tag{A.17}$$

where $V_t^H := H_0 + \int_0^t \xi_s^H dX_s + L_t^H$.

The problem given in (A.16) has a very natural interpretation in financial mathematics in option hedging. When we suppose X_t as the discounted asset price at time t, υ as the dynamic portfolio strategy that describes the number of shares of assets to be held, $c \in \mathbb{R}$ as the initial capital at time 0 and H as the contingent claim then the problem is called mean-variance hedging problem which is the minimization of expected square of net loss.

The solution of the quadratic problem given in (A.16) is the application of Hilbert's projection theorem. $G_T(\Theta)$ is a linear subspace of the Hilbert space L^2, then the projection theorem implies that a strategy $\xi \in \Theta$ solves the quadratic problem if and only if

$$E[(H - c - G_T(\xi))G_T(\upsilon)] = 0 \text{ for all } \upsilon \in \Theta \tag{A.18}$$

Since H admits a strong FS decomposition $H = V_T^H \quad P - a.s.$. Fix $\xi, \upsilon \in \theta$ and define a function $f : [0, T] \to \mathbb{R}$ by

$$f(t) := E[(V_t^H - c - G_t(\xi))G_t(\upsilon)]. \tag{A.19}$$

If we can show that $f(T) = 0$ for $\xi = \xi^{(c)}$ and arbitrary υ then the theorem will be proved. The product rule gives

$$f(t) = E[\int_0^t \upsilon_s^T \left((\sigma_s + \kappa_s)(\xi_s^H - \xi_s) + \gamma_s(V_{s-}^H - c - G_{s-}(\xi))\right) dB_s] +$$
$$E[\int_0^t \upsilon_s^T(\xi_s^H - \xi_s)G_{s-}(\upsilon)dB_s]. \qquad \text{(A.20)}$$

Then inserting $\xi = \xi^c$ and using (A.13), (A.13) and (A.17) yields,

$$f(t) = -E[\int_0^t (V_{s-}^H - c - G_{s-}(\xi^{(c)}))G_{s-}(\upsilon)d\tilde{K}_s] = -\int_0^t f(s-)d\tilde{K}_s \qquad \text{(A.21)}$$

since $\tilde{K}$ is deterministic then $f \equiv 0$ if $\xi = \xi^{(c)}$ so $\xi^{(c)}$ solves the quadratic optimization problem. For detailed proof of the theorem and construction of $\xi^{(c)}$ see [53].

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PUBLICATIONS

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