OPTION PRICING WITH FRACTIONAL BROWNIAN MOTION

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ABSTRACT

OPTION PRICING WITH FRACTIONAL BROWNIAN MOTION

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Traditional financial modeling is based on semimartingale processes with stationary and independent increments. However, empirical investigations on financial data does not always support these assumptions. This contradiction showed that there is a need for new stochastic models. *Fractional* Brownian motion (*fBm*) was proposed as one of these models by Benoit Mandelbrot. *FBm* is the only continuous Gaussian process with dependent increments. Correlation between increments of a *fBm* changes according to its self-similarity parameter H. This property of *fBm* helps to capture the correlation dynamics of the data and consequently obtain better forecast results. But for values of H different than $1/2$, *fBm* is not a semimartingale and classical Itô formula does not in that case. This gives rise to need for using the white noise theory to construct integrals with respect to *fBm* and obtain fractional Itô formulas. In this thesis, the representation of *fBm* and its fundamental properties are examined. Construction of Wick-Itô-Skorohod (*WIS*) and fractional *WIS* integrals are investigated. An Itô type formula and Girsanov type theorems are stated. The financial applications of *fBm* are mentioned and the Black&Scholes price of a European call option on an asset which is assumed to follow a geometric *fBm* is derived. The statistical aspects of *fBm* are investigated. Estimators for the self-similarity parameter H and simulation methods of *fBm* are summarized. Using the *R/S* methodology of Hurst, the estimations of the parameter H are obtained and these values are used to evaluate the fractional Black&Scholes prices of a European call option with different maturities. Afterwards, these values are compared to Black&Scholes price of the same option to demonstrate the effect of long-range dependence on the option prices. Also, estimations of H at different time scales are obtained to investigate the multiscaling in financial data. An outlook of the future work is given.

Keywords: fractional Brownian motion, long-range dependence, option pricing, Hurst parameter, self-similarity

ÖZ

KESİRLİ BROWN HAREKETİ İLE OPSİYON FİYATLAMASI

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Geleneksel finansal modelleme bağımsız durağan semimartingale süreçleri üzerine kurulmuştur. Ancak, finansal veri üzerinde yapılan çalışmalar bu önkabulleri her zaman desteklememiştir. Bu durum, yeni finansal modellere olan ihtiyacı ortaya koymuştur. Kesirli Brown hareketi (*kBh*) Benoit Mandelbrot tarafından bu yeni modellerden biri olarak önerilmiştir. *KBh*'nin artımları arasındaki korelasyon, kendine benzerlik parametresi *H*'nin değerine göre değişir. *KBh*, *H*'nin $1/2$'den büyük değerleri için uzun-dönemli bağlılık gösterir. Bu özellik verinin korelasyon dinamiklerinin yakalanmasında kullanılabilmekte ve böylece daha iyi öngörü sonuçları elde edilmektedir. Bu tezde, *kBh*'nin temsili ve temel özellikleri incelenmiştir. *KBh*'ye göre Wick-Itô-Skorohod (*WIS*) ve kesirli *WIS* integrallerinin yapılandırılması araştırılmıştır. Bu integraller için Itô tarzı formüller ve Girsanov tarzı teoremler ifade edilmiştir. Finansal uygulamalarda *fBm* kullanımı özetlenmiş ve Avrupa tipi alım opsiyonu için kesirli Black&Scholes fiyatı elde edilmiştir. *KBh*'nin istatistiksel özellikleri incelenmiştir. Kendine-benzerlik parametresi *H* için tahmin yöntemleri ve *kBh* için simülasyon yöntemleri özetlenmiştir. *R/S* yöntemi uygulanmış ve elde edilen *H* tahmin değerleri Avrupa tipi alım opsiyonunun kesirli Black&Scholes fiyatının elde edilmesinde kullanılmıştır. Daha sonra elde edilen bu fiyatlar Black&Scholes fiyatları ile karşılaştırılarak opsiyon fiyatlarında uzun dönem bağlılık etkisi gösterilmiştir.

Ayrıca farklı zaman ölçekleri için H parametresi tahminleri elde edilmiş ve finansal verinin çoklu ölçeklenme ihtimaline değinilmiştir. Gelecekteki çalışmalar için bir bakış açısı verilmiştir.

Anahtar Kelimeler: kesirli Brown hareketi, uzun dönem bağlılık, opsiyon fiyatlama, Hurst parametresi, kendine benzerlik

To my family

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CHAPTER 1

Introduction

Applying methods of physics to financial markets is a well-established paradigm since the work of Bachelier [1] and the proof of Einstein [16] on the distribution of Brownian motion. For more than a hundred years, whenever there is a financial concept that is really hard to explain or model, a very sophisticated method of physics is chosen and adopted to financial modeling. Recently, quantum theory and chaos theory have drawn the attention of financial analysts. Turbulence phenomena has been used to model price dynamics, and hence the fractals has been shown to exist in the financial markets [27]. A fractal is “a rough or fragmented geometric shape that can be split into parts, each of which is (at least approximately) a reduced-size copy of the whole”, by Benoit Mandelbrot’s words [28]. Fractals are best known with their self-similarity property. When you look at a fractal, at different scales, you see smaller shapes that is similar to what you saw at a larger scale. This is called self-similarity. After the fractals widely known, many natural phenomena has been shown to exhibit self-similarity. Trees, galaxies, lungs, brain, etc. all shown to have a self-similarity index which characterizes the behavior of the process. When statistically investigated, self-similarity shows itself in the sense of finite dimensional distributions. Most of the stochastic processes used in financial modeling are self-similar in the sense that they generate variance self-similarly, for different time scales. For example, the most widely known stochastic process, Brownian motion is $\frac{1}{2}$ self-similar, that is, generating a variance proportional to square root of the time it has been observed. Sharing the same property with fractals, therefore, stochastic processes can be regarded as fractals, as shown, again, by Mandelbrot [32].

On the other hand, a hydrogeologist named Edwin Hurst, showed in his study of the Nile River [25] that the cycle of floods and dryness exhibits a specific type of behavior which has not been considered then. His work suggested a different covariance structure than that of Brownian motion. After this work, Mandelbrot tried to use this behavior to model financial data, and the *fractional* Brownian motion (*fBm*) has been defined [29]. The self-similarity parameter was called the name of the hydrologist Hurst. The most important property of *fBm* is the variability of its self-similarity parameter, H. According to the value of this parameter, *fBm* exhibits positive or negative correlation between increments, corresponding to 'persistent' and 'antipersistent' cases respectively. This parameter is used to capture the correlation dynamics of the process to be modeled. If $H = \frac{1}{2}$, *fBm* becomes the standard Brownian motion. When persistent, *fBm* is said to exhibit long-range dependence.

It is a well known empirical fact in financial markets that the correlation between increments do not decay at a rate of a Markov process. Some analysts argued that the noise in the data generated this correlation structure and filtering may help to overcome this difficulty. But filtering may also cause some information of the data to be ignored, so *fBm* would be used to model this characteristic of the market data. However, using *fBm* as a tool is not as easy as the standard Brownian motion. Especially, for financial economics, it was not considered as a proper tool because it is not a semimartingale when $H \neq \frac{1}{2}$, therefore it is possible to generate arbitrage in a fractional Brownian market. Classical Itô type formulas does not work because of its interesting variational properties. It is stationary but its increments are not independent and therefore it is not a Lèvy process, but it is Gaussian and its distribution is characterized only by its first and second moments. This makes *fBm* the only continuous Gaussian process with long-range dependence and the only alternative to properly model the dependence between observations.

After almost a decade, *fBm* had drawn the attention of the time series analysts. Granger [20] and Hosking [23] defined the discrete analog of the *fBm* as the fractionally integrated ARMA, ARFIMA or FARIMA processes. ARFIMA processes captures the slow decay rate of correlations between increments with only one additional parameter: d. The fractionally integrated process was constructed upon the integrated processes, ARIMA, defined by Box and Jenkins [5]. In their setting, d, an integer, is the number of differencing needed to obtain a stationary process from a non-stationary process. For fractionally integrated series, d can be noninteger, and this is termed as the fractional differencing. ARFIMA models gave better forecast results especially in the long-term. Many researchers found the evidence of long-range dependence in foreign exchange markets [9], commodity prices [27], and electricity prices [43].

In order to use *fBm* in financial modeling, one must define an Itô type formula and a risk-neutral measure, as done in the Brownian motion case. These contributions are made mainly by Øksendal and Hu in [34] and [24], Elliott and Van der Hoek [17] and Norros and Valkeila [32]. In this work, we mainly follow their approximations to price an option in fractional Brownian markets. To mathematically define *fBm*, some difficult mathematical concepts such as the Gaussian white noise theory, fractional calculus are needed. Malliavin calculus is used to obtain further results. But in this thesis, we will not use Malliavin calculus, so the definitions and formulas are results of fractional calculus used in the Gaussian white noise theory [22]. Using the fractional white noise calculus, the price of a European call option is presented.

From a statistical point of view, the long-range dependence property is not easy to estimate, especially in the time domain, because it is defined as an asymptotic behavior and one must find a cutoff point in estimation procedure. Long computational time needed for precision. An intuitive estimator of long-range dependence, and also the first one, is the *Rescaled Range* statistic of Hurst. Hurst's discovery of the long-range dependence is still used as a basic tool to obtain a first idea about the long-range dependence characteristic of a process. Another useful tool is spectral analysis. In the spectral domain, the long-range dependence can be detected by investigating the behavior of the spectral density for the zero frequency components. This is the main reason why fast Fourier transform is widely used for estimating H and simulating *fBm* paths.

In Chapter 2, we give the definition of *fBm* and its properties which are basically the long-range dependence, self-similarity, and path differentiability.

In Chapter 3, basic tools of white noise analysis that is used in construction of *fBm* and the definition of integrals with respect to *fBm* as defined by Øksendal and Biagini in [34] summarized. We present the Itô type formula for the Wick-Ito-Skorohod (*WIS*) integral. Then we will mention briefly about the fractional white noise theory and Skorohod integrability as used in the definition of fractional *WIS* integrals in [24]. Then the pathwise integrals with respect to *fBm* is defined and Itô formula for forward pathwise integral are given. The question of arbitrage is addressed and examples of arbitrage portfolios are presented.

In Chapter 4, some of the financial applications using *fBm* are mentioned. A modification of the Black&Scholes formula to the *fBm* case is given where the main difference is the assumption that the underlying asset follows a geometric fractional Brownian motion, that is:

$$dS(t) = \mu S(t)dt + \sigma S(t)dB^H(t),$$

where $B^H(t)$ is a *fBm*.

In Chapter 5, the statistical characteristics of *fBm* and ARFIMA processes, the discrete analog of *fBm*, are given and its characteristics are presented. Also, how these characteristics can be used in estimation and simulation are summarized. Two of the estimation procedures, periodogram and R/S analysis are applied to Dow Jones Industrial Average Index and exchange rate series and results are used to demonstrate the effect of long-range dependence on the option prices.

In Chapter 6, we give a brief conclusion and outlook of our study.

CHAPTER 2

Fractional Brownian motion and its properties

2.1 Definition and Properties

We begin with the definition and basic properties of the fractional Brownian motion (*fBm*). Some of these properties differ from the properties of the standard Brownian motion and this situation makes it harder to use it as a tool in financial mathematics. The stochastic integral (moving average) representation is first provided by Mandelbrot and Van Ness [29].

Definition 2.1.1 *Let H be a constant belonging to $(0,1)$. A fractional Brownian motion (*fBm*) $B^H = (B^H(t))_{t\geqslant 0}$ with Hurst index H is a centered Gaussian process with covariance function*

$$E[B^{(H)}(t)B^{(H)}(s)] = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H})E[(B^H(1))^2] \tag{2.1}$$

For $H = \frac{1}{2}$, the *fBm* is a standard Brownian motion as can be seen by letting $H = \frac{1}{2}$

$$E[B^{1/2}(s)B^{1/2}(t)] = \frac{1}{2}(t + s - |t-s|) = \min(t,s).$$

A standard *fBm* B^H has the following properties:

1. $B^H(0) = 0$ and $E[B^{(H)}(t)] = 0$ a.s. for all $t \geqslant 0$.
2. B^H has homogeneous increments, i.e., $B^{(H)}(t+s) - B^{(H)}(s)$ has the same law of $B^H(t)$ for $s, t \geqslant 0$.
3. B^H is Gaussian with $E[(B^H(t))^2] = t^{2H}$ for all $H \in (0,1)$ and $t \geq 0$.
4. B^H has continuous trajectories.

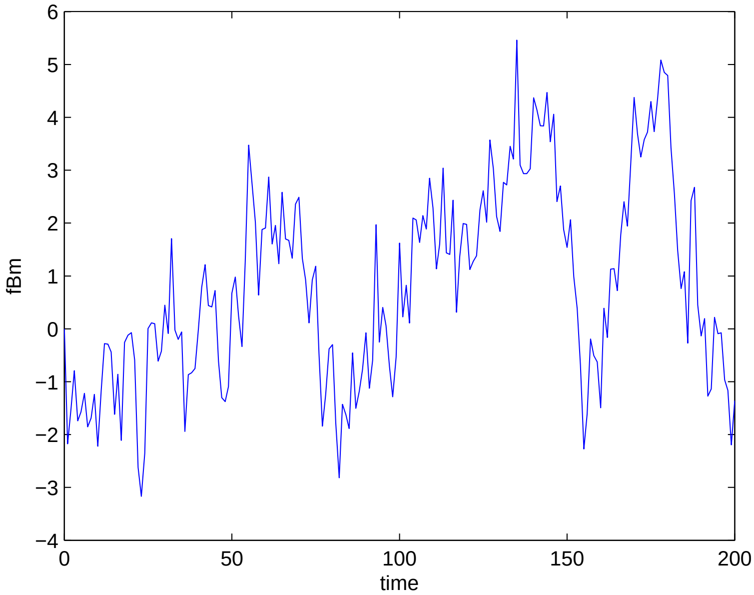

Figure 2.1: Simulated path of *fBm* with $H = 0.3$

As known, the distribution of a Gaussian process is determined by its mean and covariance structure. Since we know that *fBm* is Gaussian, its mean and the specific covariance structure determines a unique Gaussian process.

The dependence between the increments of a *fBm* may bring to one's mind the questions about stationarity of *fBm*. In order to see the stationarity, it is enough to carry out a simple calculation:

$$\begin{aligned}
E[((B^H(t+h) - B^H(h))(B^H(s+h) - B^H(h)))] &= E[B^H(t+h)B^H(s+h)] - E[B^H(t+h)B^H(h)] \\
&\quad -E[B^H(s+h)B^H(s+h)] + E[(B^H(h))^2] \\
&= \frac{1}{2}[((t+h)^{2H} + (s+h)^{2H} - |t-s|^{2H}) - \\
&\quad ((t+h)^{2H} + h^{2H} - t^{2H}) \\
&\quad -((s+h)^{2H} + h^{2H} - s^{2H}) + 2h^{2H}]E[(B^H(1))] \\
&= \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H})E[(B^H(1))] \\
&= E[B^H(t)B^H(s)],
\end{aligned}$$

with $s, t, h > 0$. Hence we obtain that $[B^H(t+h) - B^H(h)] = [B^H(t)]$ in distribution. The *fBm* has stationary increments which are not independent. According to Cont and Tankov, *fBm* is a self-similar Gaussian process but not a Lèvy process because it does not have independent increments [11].

2.1.1 Stochastic integral representation

This representation of *fBm* gives it the name *fractional* because of the notion of fractional calculus used in the stochastic integral. In [29] it is proved that $B^H(t)$ defined as follows is a *fBm* with Hurst index $H \in (0,1)$:

$$\begin{aligned} B^H(t) &= C(H)\int_{\mathbb{R}}\Big((t-s)_+^{H-\frac{1}{2}} - (-s)_+^{H-\frac{1}{2}}\Big)dB(s) \quad (2.2)\\ &= C(H)\Big(\int_{-\infty}^{0}((t-s)_+^{H-\frac{1}{2}} - (-s)_+^{\frac{1}{2}})\Big)dB(s) + \int_0^{+\infty}\Big((t-s)_+^{H-\frac{1}{2}} - (-s)_+^{H-\frac{1}{2}}\Big)dB(s)\\ &= C(H)\Big(\int_{-\infty}^{0}((t-s)^{H-\frac{1}{2}} - (-s)^{H-\frac{1}{2}})dB(s) + \int_0^{t}(t-s)^{H-\frac{1}{2}}dB(s)\Big), \quad (2.3)\end{aligned}$$

where

$$C(H) = E[(B^H(1))^2]^{1/2}[\int_{-\infty}^{0}((t-s)^{H-1/2} - (-s)^{H-1/2})^2 ds + \frac{1}{2H}]^{-1/2}$$

and $(x)_+ = \max\{x, 0\}$. According to this definition, we can obtain the variance of *fBm*, using the Itô isometry :

$$\begin{aligned} E[(B^H(t))^2] &= C(H)^2 E[(\int_{\mathbb{R}}\Big((t-s)_+^{H-\frac{1}{2}} - (-s)_+^{H-\frac{1}{2}}\Big)dB(s))^2]\\ &= C(H)^2 E[\int_{\mathbb{R}}\Big((t-s)_+^{H-\frac{1}{2}} - (-s)_+^{H-\frac{1}{2}}\Big)^2 ds]\\ &= C(H)^2\int_{\mathbb{R}}[(t-s)_+^{H-\frac{1}{2}} - (-s)_+^{H-\frac{1}{2}}]^2 ds.\end{aligned}$$

Hence, by the change variables $s = tu$, we obtain

$$\begin{aligned} C(H)^2\int_{\mathbb{R}}[t^{H-\frac{1}{2}}\Big((1-u)_+^{H-\frac{1}{2}} - (-u)_+^{H-\frac{1}{2}}]^2 tdu &= C(H)^2\frac{t^{2H}}{t}\int_{\mathbb{R}}[(1-u)_+^{H-\frac{1}{2}} - (-u)_+^{H-\frac{1}{2}}]^2 tdu\\ &= t^{2H}E[(B^H(1))^2].\end{aligned}$$

The kernel in 2.2 is obtained by using fractional calculus. For preliminary information on fractional calculus, we refer to Appendix. Using the stochastic integral representation, one can show that the variance of *fBm* satisfies (2.1):

$$\begin{aligned} E[|B^H(t) - B^H(s)|^2] &= C(H)^2 E\Big[\int_{\mathbb{R}}[(t-u)_+^{H-\frac{1}{2}} - (-u)_+^{H-\frac{1}{2}} - (s-u)_+^{H-\frac{1}{2}} - (-u)_+^{H-\frac{1}{2}}]dB(u)\Big]^2\\ &= C(H)^2 E\Big[\int_{\mathbb{R}}[(t-u)_+^{H-\frac{1}{2}} - (s-u)_+^{H-\frac{1}{2}}]^2 du\Big],\end{aligned}$$

by the change of variables $u = u^{'} + s$, we obtain

$$= C(H)^2 \int_{\mathbb{R}} [(t-s-u^{'})_+^{H-\frac{1}{2}} - (-u)_+^{H-\frac{1}{2}}]^2 du^{'}$$

Then by change of variables $|t-s|v = u^{'}$,

$$\begin{aligned} C(H)^2 \frac{|t-s|^{2H}}{|t-s|} \int_{\mathbb{R}} [(1-v)_+^{H-\frac{1}{2}} - (-v)_+^{H-\frac{1}{2}}]^2 dv |t-s| &= |t-s|^{2H} \int_{\mathbb{R}} [(1-v)_+^{H-\frac{1}{2}} - (-v)_+^{H-\frac{1}{2}})] dv \\ &= |t-s|^{2H} E[(B^H(1))^2]. \end{aligned}$$

In the beginning the term $E[(B^H(1))^2]$ seems a bit unnecessary. From self-similarity property, one may think that $E[(B^H(1))] = 1^{2H} = 1$. However, the self-similarity does not imply anything on the variance of the process. Let us consider that we constructed a *fBm* $\hat{B}^H(t)$ with a variance of $\sigma^2_{\hat{B}^H(1)} = 10$. This only magnifies the variance of the process, but now the variance of $B^H(1)$ will not be equal to 1, but instead it will equal to 10. From self-similarity we know $B^H(t) = t^H B^H(1)$ so the variance of the process will always depend on the variance of $B^H(1)$. Keeping this in mind, under the assumption $B^H(1) = 1$, one obtains the following covariance function for a *standard fBm*:

$$E[B^H(t)B^H(s)] = \frac{1}{2}[t^{2H} + s^{2H} - |t-s|^{2H}]$$

Although the representation (2.2) is the most used one, it is not unique. It is shown in [40] that

$$\int_{-\infty}^{\infty} [a((t-x)_+^{H+1/2} - (-x)_+^{H-1/2}) + b((t-x)_-^{H+1/2} - (-x)_-^{H-1/2})] dB(x)$$

is a *fBm* up to a constant.

2.1.2 The representation of fBm over a finite interval

The stochastic integral, or moving average, representation of *fBm* is based on the integration over the whole real line. By the following approach in [32], a *fBm* can be represented over a finite interval using the kernel $K_H(t,s)$ in:

$$B^H(t) := \int_0^t K_H(t,s) dB(s), \qquad t \geqslant 0. \tag{2.4}$$

Here,

1. For $H > 1/2$:

   - $K_H(t,s) = C_H s^{1/2-H} \int_s^t |u-s|^{H-3/2} u^{H-1/2} du$
     where $c_H = [H(2H-1)/\beta(2-2H, H-1/2)]^{1/2}$ and $t > s$.

2. For $H < 1/2$:

   - $K_{t,s} = b_H[(\frac{t}{s})^{H-1/2}(t-s)^{H-1/2} - \left(H - \frac{1}{2}\right)s^{1/2-H} \int_s^t (u-s)^{H-1/2} u^{H-3/2} du\right)]$
     with $b_H = [2H/((1-2H)\beta(1-2H, H+1/2))]^{1/2}$ and $t > s$,

where $\beta(\cdot,\cdot)$ denotes the Beta function. For the proof, see [32]. Again, the effect of the parameter H can be seen on the kernels.

2.1.3 Long-range dependence

The most important property of *fBm* is the ability to change its covariance structure depending on the parameter H. For $H = 1/2, B^H(t)$ is a standard Brownian motion , which is a process with independent increments. But for $H \neq 1/2$, the increments of *fBm* are not independent. In order to see this, we compute the covariance between $B^H(t+h) - B^H(t)$ and $B^H(s+h) - B^H(s)$ with $s + h \leq t$ and $t - s = nh$ is

$$
\begin{aligned}
E[(B^H(t+h) - B^H(t))(B^H(s+h) - B^H(s))] &= \frac{1}{2}[(t-s+h)^{2H} + (t-s-h)^{2H} - 2(t-s)^{2H}] \\
&= \frac{1}{2}[((n+1)h)^{2H} + ((n-1)h)^{2H} - 2(nh)^{2H}],
\end{aligned}
$$

so we have

$$
E[(B^H(t+h) - B^H(t))(B^H(s+h) - B^H(s))] = \frac{h^{2H}}{2}[(n+1)^{2H} + (n-1)^{2H} - 2n^{2H}].
$$

As can be seen, *fBm* has positively correlated increments when $H > 1/2$ and negatively correlated increments when $H < 1/2$. These correspond to the cases known as the ‘persistence’ and ‘antipersistence’ cases respectively. Many natural phenomena are shown to exhibit ’persistence’, that is, having positive correlation between increments. This property can be interpreted as some kind of feedback occuring in the process.

Long-range dependence can be very useful in empirical studies. This property can be used to capture the long-term behavior of stock markets. Long-range dependence best realized in the correlation structure and the spectral density of the process. In this section , we present the definition based on correlations, in [3].

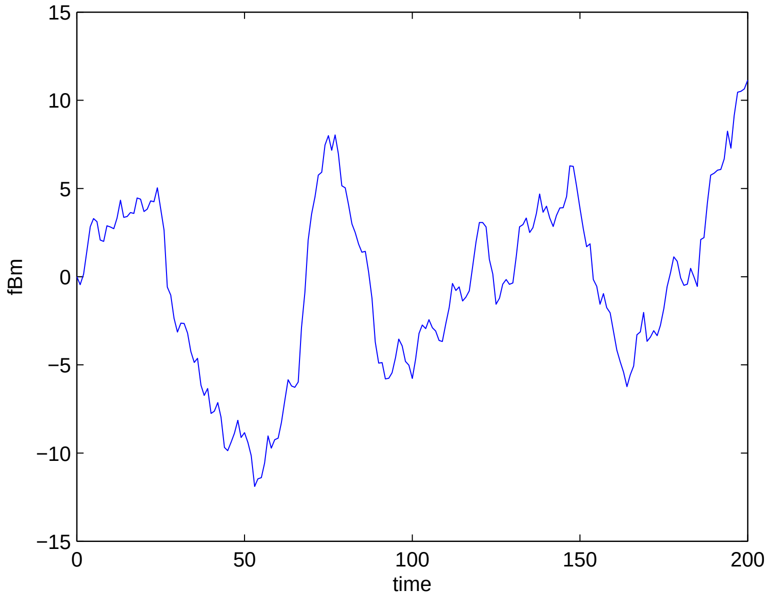

Figure 2.2: Simulated path of *fBm* with $H = 0.7$

Definition 2.1.2 *A stationary process X_n exhibits long-range dependence (or long memory) if the autocovariance function $\rho(n) := cov(X_k, X_{k+n})$ satisfies*

$$lim_{n\to\infty} \frac{\rho(n)}{cn^{-\alpha}} = 1 \tag{2.5}$$

for some constant c and $\alpha \in (0, 1)$.

In the presence of long-range dependence, the dependence between X_{k+n} and X_k decays slowly as n increases and

$$\sum_{n=1}^{\infty} \rho(n) = \infty, \tag{2.6}$$

We can show that the *fBm* exhibits long-range dependence using its covariance function. The covariance between increments is,

$$\begin{aligned} \rho(n) &:= E[(B^H(t+h) - B^H(t))(B^H(s+h) - B^H(s))] \\ &= \frac{h^{2H}}{2}[(n+1)^{2H} + (n-1)^{2H} - 2n^{2H}]. \end{aligned}$$

Using the expansions

$$\begin{aligned} (1+x)^{\alpha} &= 1 + \alpha x + \frac{\alpha(\alpha-1)x^2}{2} + \ldots \\ (1-x)^{\alpha} &= 1 - \alpha x + \frac{\alpha(\alpha-1)x^2}{2} - \ldots \end{aligned}$$

we obtain

$$\rho(n) = 1 + 2H\frac{1}{n} + \frac{2H(2H-1)}{2}\frac{1}{n^2} + 1 + -2H\frac{1}{n} + \frac{2H(2H-1)}{2}\frac{1}{n^2} + ...$$
$$\cong \frac{n^{2H}}{2}(2H(2H-1))\frac{1}{n^2} = n^{2H-2}(H(2H-1)).$$

So, the definition of long-range dependence holds for *fBm* with $c = H(2H-1)$ and $\alpha = 2-2H$. We obtain

1. For $H > 1/2$, $\sum_{n=1}^{\infty} \rho_H(n) = \infty$.

2. For $H < 1/2$, $\sum_{n=1}^{\infty} |\rho_H(n)| < \infty$.

The finiteness property for the sum of correlations does not explain the correlation function of *fBm*. When $H < 1/2$, the correlations alternate in sign and in the limit they sum up to a constant. This situation leads to the mean-reverting property of *fBm* when $H < 1/2$.

2.1.4 Self-similarity

The term 'self-similar' was basically used for defining a fractal. Until recently, this term was not used in statistics. But later, fractal processes have been used to model many natural phenomena such as the organization of cells, metal surfaces, lightening strikes, etc. and so 'self-similarity' has become a well known term. For stochastic processes, it is defined in the finite dimensional distributions sense. There is a set of self-similar processes used in modeling. In fact, in statistical terms, a process must be self-similar to be used easily in modeling. Self-similar processes are invariant under scaling of time and space. Another widely used set of processes consists α-stable processes. α-stable processes also can exhibit both persistence and 'antipersistence'. When a stationary process is self-similar with index H and is α-stable, then the existence of moments limits the values of α and H. For $0 < \alpha \leq 2$, $H \in (0, 1/\alpha)$ if $\alpha < 1$ and $H \in (0, 1]$ if $\alpha \geq 1$. For a given H, there is a single Gaussian H self-similar process and that is *fBm*. Standard Brownian motion is the case with $\alpha = 2$, *see*[40]. There is also an important relation between the statistical fractal dimension of a process and its self-similarity parameter.

Definition 2.1.3 *The real-valued process $X(t), t \geq 0$ is self-similar with index $H > 0$ if for all $a > 0$ the finite-dimensional distributions of $X(at), t \geq 0$ are identical to the finite-dimensional* *distributions of* $a^H X(t), t \geq 0$*; i.e., if for any* $d \geq 1, t_1, t_2, ..., t_d \geq 0$ *and any* $a > 0$*,*

$$(X(at_1), X(at_2), \dots, X(at_d)) =^d (a^H X(t_1), a^H X(t_2), \dots, a^H X(t_d)). \tag{2.7}$$

The quantity $D = 1/H$ is called the *statistical fractal dimension of* X, which is one of the basic connections between fractals and statistics. Standard Brownian motion is $H = \frac{1}{2}$ self-similar. Using the stochastic integral representation of *fBm* it can be shown that *fBm* is H self-similar. Denoting the kernel in (2.2) as $\int_{\mathbb{R}} ((t-s)_+^{H-1/2} - (-u)_+^{H-1/2}) = l_H(t,u)$, we can write

$$B^H(at) := \int_{\mathbb{R}} l_H(at,u) dB(u) = a^{H-1/2} \int_{\mathbb{R}} l_H(t, a^{-1}u) dB(u),$$

using the substitution $v = ua^{-1}$, we obtain

$$\begin{aligned} a^{H-1/2} \int_{\mathbb{R}} l_H(t,v) dB(av) &= a^{H-1/2} a^{1/2} \int_{\mathbb{R}} l_H(t,v) dB(v) \\ &= a^H B^H(t), \end{aligned}$$

where the last equality follows from the $\frac{1}{2}$ self-similarity of Brownian motion.

It is also possible to see the self-similarity by using the covariance function of *fBm*:

$$\begin{aligned} \mathbb{E}[B^H(at_1) B^H(at_2)] &= \frac{1}{2}[(at_1)^{2H} + (at_2)^{2H} - |at_1 - at^2|^{2H}] \\ &= \frac{a^{2H}}{2}[(t_1)^{2H} + (t_2)^{2H} - |t_1 - t^2|^{2H}] \\ &= a^{2H} \mathbb{E}[B^H(t_1) B^H(t_2)]. \end{aligned}$$

In distribution function, self-similarity can also be seen:

$$\begin{aligned} F_t(x) &= \mathbb{P}(B^H(t) \leq x) \\ &= \mathbb{P}(t^H B^H(1) \leq x) \\ &= F_1(\frac{x}{t^H}). \end{aligned}$$

Therefore, once we know the distribution of a self-similar process over the unit interval, it is possible to obtain the distribution of the process over the whole real line.

There is a strong relation between the self-similarity parameter H and the Hölder exponent. It was Mandelbrot's work that tied these two exponents together. Hölder continuity of *fBm* is needed to have a continuous version of it by the Kolmogorov-Chentsov theorem. The following theorem in [14] gives the Hölder continuity property of *fBm*:

Theorem 2.1.4 *Let $H \in (0,1)$. The* fBm B^H *admits a version whose sample paths are almost surely Hölder continuous of order strictly less than H.*

The Hölder continuity of *fBm* follows from

$$E[|B^H(t) - B^H(s)|^\alpha] = C_\alpha |t-s|^{H\alpha},$$

where $\alpha > 0$ and C_α is a constant. For the proof of theorem we refer to [14].

2.1.5 Path differentiability

One needs stochastic calculus to define an integral with respect to *fBm* and this is because the *fBm* sample path is not differentiable. The following lemma in [29] states this property.

Lemma 2.1.5 *Let $H \in (0,1)$. The* fBm *sample path $B^H(\cdot)$ is not differentiable. For every $t_0 \in [0,\infty)$ $\limsup_{t\to 0} |\frac{B^H(t)-B^H(t_0)}{t-t_0}| = \infty$ with probability one.*

Proof. We basically follow the proof in [29]. Let us assume that $B^H(0) = 0$. Then define the random variable

$$R_{t,t_0} := \frac{B^H(t) - B^H(t_0)}{t - t_0}$$

that represents the incremental ratio of B^H. Using the self-similarity property of *fBm*, we can see that

$$R_{t,t_0} \triangleq \frac{(t-t_0)^H(B^H(1))}{(t-t_0)} =^d (t-t_0)^{H-1}(B^H(1)),$$

where $\triangleq$ denotes the equality in distribution. If we define the event

$$A(t,\omega) := \sup_{0\le s\le t} \frac{|B^H(s)|}{s} > d,$$

then, for sequence $(t_n)_{n\in\mathbb{N}}$ decreasing to zero, we have $A(t_n,\omega) \supseteq A(t_{n+1},\omega)$ and

$$A(t_n,\omega) \supseteq (|\frac{B^H(t_n)}{t_n}| > d) \triangleq (|B^H(1)| > t_n^{1-H} d)$$

from the self-similarity of *fBm*. Consider the sequence $t_n = t - t_0 > 0$ going to zero as $t_0 \to t$. And since the probability of the last term tends to 1 as $n \to \infty$, $B^H(t)$ does not have differentiable sample paths. ∎

2.1.6 *FBm* with $H \neq \frac{1}{2}$ is not a semimartingale

In financial mathematics, the most important property which is required for a process in order to use it to model the price process of a financial asset is that it *should not* generate arbitrage. Arbitrage in *fBm* markets will be defined in chapters that follow. In general, the term *free lunch* , that is, a portfolio of assets that has no intrinsic value at the beginning and has positive value at a distinct future time with positive probability, is used to define arbitrage. It is known that if the underlying price process is modeled by using a semimartingale, there is no arbitrage opportunities in the market. But it is shown in [38] that *fBm* is not a semimartingale. In order to see it, we begin with the definition of a semimartingale. By Theorem 9 in [37], every semi-martingale is a decomposable process such that

$$S(t) = S(0) + M(t) + A(t),$$

where $M(0) = A(0) = 0$, M is a locally square integrable martingale, and A is a right-continuous process with left limits (cádlág) with paths of finite variation. The fact that *fBm* is not a semi-martingale unless $H = 1/2$ is proved by using the p-variation of B^H.

Let $(X(t))_{t\in[0,T]}$ be a stochastic process and consider a partition $\pi = 0 = t_0 < t_1 < ... < t_n = T$. Put

$$S_p(X,\pi) := \sum_{i=1}^{n} |X(t_k) - X_{t_{k-1}}|^p.$$

The p-variation of X over the interval [0,T] is defined as

$$V_p(X,[0,T]) := \sup_{\pi} S_p(X,\pi)$$

where π is the partition defined above. The index of p-variation of a process is defined as $I(X,[0,T]) := \inf\{p > 0; V_p(X,[0,T]) < \infty\}$.

Now let us define, for $p > 0$,

$$Y_{n,p} = n^{pH-1} \sum_{i=1}^{n} |B^H(\frac{i}{n}) - B^H(\frac{i-1}{n})|^p.$$

Using the self-similarity property we obtain,

$$Y_{n,p} \triangleq \tilde{Y}_{n,p} = n^{-1} \sum_{i=1}^{n} |B^H(i) - B^H(i-1)|^p$$

and by Ergodic theorem (see [38] for references),

$$\tilde{Y}_{n,p} = n^{-1} \sum_{i=1}^{n} |B^H(i) - B^H(i-1)|^p$$

converges to $\mathbb{E}[|B^H(1)|^p]$ in L^1 as n tends to infinity. Using these equalities, it follows that

$$\begin{aligned} V_{n,p} &= \sum_{i=1}^{n} |B^H(\frac{i}{n}) - B^H(\frac{i-1}{n})|^p \\ &\triangleq \frac{\sum_{i=1} n|B^H(i) - B(i-1)|^p}{n^{pH}}. \end{aligned}$$

So, when $pH > 1$, $V_{n,p} \to 0$ in probability and

when $pH < 1$, $V_{n,p} \to \infty$ in probability. Therefore, we can see that:

$$I(B^H, [0,T]) = \frac{1}{H}$$

If $B^H(t)$ were a semimartingale, it would have a Doob-Meyer decomposition. The problem is that, this decomposition consists of a continuous local martingale and a finite variation process, but the *fBm* has zero variation if $H > 1/2$ and infinite variation if $H < 1/2$. This contradicts the usual assumptions of a semimartingale and therefore *fBm* is not a semi-martingale. This non-semimartingale property of *fBm* makes it difficult to define an integral with respect to it. Classical Itô type integration is well-defined for semi-martingales. As a consequence, we need different approaches for the construction of stochastic integrals with respect to *fBm*. In the next chapter, we will summarize some of these integral definitions.

CHAPTER 3

Integration with respect to *fBm*

As we saw, *fBm* is not a semimartingale and it has zero quadratic variation when $H > 1/2$ and infinite quadratic variation when $H < 1/2$. So the definition of stochastic integrals with respect to *fBm* can not be defined with the quadratic variation of the Itô integral for Brownian motion. In general, we will review two approximations: The first approximation defines the integral in the *white noise space* and the second in a pathwise sense. Although they have different results, both approximations give better results for $H > 1/2$. We begin with some preliminary knowledge of the *white noise space* and its specific tools.

3.1 *WIS* integral with respect to *fBm* with $0 < H < 1$

The Gaussian white noise theory was first introduced by Hida [22]. *WIS* (Wick-Itô-Skorohod) integral is defined in the white noise analysis framework and using some advanced mathematical concepts such as the *Wick calculus*, it is possible to obtain an Itô formula and Girsanov theorem for *fBm*. We follow the approximation for the definition of WIS integral with respect to *fBm* in [34].

3.1.1 The White noise probability measure

The White noise theory is a very useful tool for the analysis of Gaussian random variables. The Gaussian property of *fBm* made it attractive to analyze *fBm* in this setting. The special case Brownian motion is defined in a natural way using the White noise probability measure. The definition of this measure is as follows:

Definition 3.1.1 *Let $S(\mathbb{R})$ denote the Schwartz space of rapidly decreasing smooth functions on $\mathbb{R}$, and let $\Omega := S'(\mathbb{R})$ be its dual, usually called the* space of tempered distributions. *Let $\mathbb{P}$ be the probability measure on the Borel σ-algebra $\mathcal{F} := B(S'(\mathbb{R}))$ defined by the property that*

$$\int_{\mathbb{R}} \exp(i < \omega, f >) d\mathbb{P}(\omega) = \exp\Big(-\frac{1}{2} \|f\|^2_{L^2(\mathbb{R})} \Big), \tag{3.1}$$

where $i = \sqrt{-1}$ and $< \omega, f >= \omega(f)$ is the action of $\omega \in \Omega = S'(\mathbb{R})$ on $f \in S(\mathbb{R})$.

The measure $\mathbb{P}$ is called the *white noise probability measure.* Its existence follows from the Bochner-Minlos theorem. In (3.1), by expanding both sides to Taylor series, and using the properties of the characteristic function $\varphi_{<\omega, f>}$ of $< \omega, f >$, which is φ admits the Taylor expansion:

$$\begin{aligned} \varphi(< \omega, f >) &= \sum_{k=0}^{n} \frac{(it)^k}{k!} E(< \omega, f >^k) + o(|t|^n) \\ &= 1 + (it)E(< \omega, f >) - t^2 \frac{1}{2} E(< \omega, f >^2) + \dots, \end{aligned}$$

with $t = 1$ we obtain

$$E[< \omega, f >] = 0 \qquad \forall f \in S(\mathbb{R}).$$

As can be seen in from the Taylor expansion of the characteristic function, there is a useful isometry property of the *white noise probability measure*:

$$E[< \omega, f >^2] = \|f\|^2_{L^2(\mathbb{R})} \qquad \forall f \in S(\mathbb{R}).$$

The expectation of a function F with respect to this measure is defined by

$$E[F(\omega)] = \int_{\Omega} F(\omega) d\mathbb{P}(\omega).$$

Based on these definitions, the random variable $< \omega, f >$ is defined for arbitrary $f \in L^2(\mathbb{R})$ as a limit in $L^2(\mathbb{R})$:

$$< \omega, f >= \lim_{n \to \infty} < \omega, f >, \qquad \text{limit in} \quad L^2(\mathbb{R}),$$

where $f_n \in S(\mathbb{R})$ is a sequence converging to $f \in L^2(\mathbb{R})$. Hence, one can extend $< \omega, f >$ for $\omega \in S'(\mathbb{R})$ $f \in L^2(\mathbb{R})$. We define $\tilde{B}(t)$ by taking $f = I_{[0,t]}(\cdot)$ and the idea that any function $f \in L^2(\mathbb{R})$ can be approximated using step functions. Indeed,

$$\tilde{B}(t) := \tilde{B}(t, \omega) :=< \omega, I_{[0,t]}(\cdot) >$$

is well-defined as an element of $L^2(\mathbb{P})$ for all $t \in \mathbb{R}$, where

$$I_{[0,t]}(s) = \begin{cases} 1, & \text{if} \quad 0 \le s \le t, \\ -1, & \text{if} \quad t \le s \le 0, \\ 0, & \text{otherwise.} \end{cases}$$

This definition shows us that $\tilde{B}(t)$ is Gaussian with $E[< \omega, I_{[0,t]} >] = 0$ and $E[< \omega, I_{[0,t]} >^2] = 1$, i.e, a standard normal variable. In order to obtain a continuous version of $\tilde{B}(t)$, we need the well-known theorem of Kolmogorov and Chentsov:

Theorem 3.1.2 *Suppose that a process* $X = X(t); 0 \le t \le T$ *on a probability space* $(\Omega, F, \mathbb{P})$ *satisfies the condition*

$$E|X(t) - X(s)|^{\alpha} \le C|t - s|^{1+\beta}, \quad 0 \le s, t \le T,$$

for some positive constants α,β *and* C*. Then there exists a continuous modification* $\tilde{X} = \{\tilde{X(t)}; 0 \le t \le T\}$ *of* X*, which is locally Hölder continuous with exponent* γ*, for every* $\gamma \in (0, \beta/\alpha)$*, i.e.,*

$$\mathbb{P}[\omega : \sup_{0<t-s<h(\omega)} \frac{|\tilde{X}(t,\omega) - \tilde{X}(s,\omega)|}{|t-s|^{\gamma}} \le \delta] = 1, \quad s, t \in [0, T].$$

where $h(\omega)$ *is an a.s. positive random variable and* δ *is an appropriate constant.*

So by Kolmogorov-Chantsov theorem, the process $\tilde{B}(t)$ has a continuous version, which will be denoted by $B(t)$. Brownian motion $B(t)$ is defined as a natural element of the Gaussian white noise space. Its covariance function can be computed by

$$E[B(t_1)B(t_2)] = \int_{\mathbb{R}} I_{[0,t_1]} I_{[0,t_2]}(s) ds = \begin{cases} \min\{|t_1|, |t_2|\}, & \text{if } t_1, t_2 > 0, \\ 0, & \text{otherwise.} \end{cases}$$

In this context, integral of an arbitrary $f \in L^2(\mathbb{R})$ with respect to Brownian motion can be defined as follows:

$$< \omega, f >= \int_{\mathbb{R}} f(t) dB(t), \quad \text{for all deterministic } f \in L^2(\mathbb{R}),$$

with $E[< \omega, f >] = 0$ and $E[< \omega, f >^2] = ||f||^2_{L^2(\mathbb{R})}$ is the well-known Itô isometry.

As we mentioned his name, let us give two theorems that is of fundamental necessity for defining the integral with respect to *fBm*. The following theorem is known as the *Wiener-Itô chaos expansion theorem I*. First we give the definition of the *iterated Itô integral* of a symmetric function.

Let $\hat{L}^2(\mathbb{R}^n)$ be the set of all symmetric deterministic functions $f \in L^2(\mathbb{R}^n)$. If $f \in \hat{L}^2(\mathbb{R}^n)$, the *iterated Itô integral of f* is defined by

$$\begin{aligned} I_n(f) &:= \int_{\mathbb{R}^n} f(t) dB^{\bigotimes n}(t) \\ &:= n! \int_{\mathbb{R}} [\int_{-\infty}^{t_n} \dots [\int_{-\infty}^{t_2} f(t_1, \dots, t_n) dB(t_1)] dB(t_2) \dots dB(t_n)]. \end{aligned}$$

Theorem 3.1.3 *Let $F \in L^2(\mathbb{P})$. Then there exists a unique sequence $\{f_n\}_{n=0}^{\infty}$ of functions $f_n \in \hat{L}^2(\mathbb{R}^n)$ such that*

$$F(\omega) = \sum_{n=0}^{\infty} I_n(f_n),$$

where the convergence is in $L^2(\mathbb{P})$ and $I_0(f_0) := E[F]$. Moreover, there is the following isometry

$$E[F^2] = \sum_{n=0}^{\infty} n! \|f_n\|^2_{L^2(\mathbb{R}^n)}.$$

Example 3.1.4 *Now let us find the chaos expansion of Brownian motion $B(t)$. We know that $B(t) \in L^2(\mathbb{P})$, and it has the representation*

$$\begin{aligned} B(t) &= \int_0^T I[0, t](s) dB(s) \\ &= I_1(f_1). \end{aligned}$$

Therefore, we have $f_1 = I_{[0,t]}$ and $f_n = 0$ for $n > 1$.

In addition to Theorem 3.1.3, the *Wiener-Itô chaos expansion theorem II* is of fundamental importance for defining the *Skorohod integral* of a random variable in the *white noise space*.

In order to be able to give the second chaos expansion theorem, we have to define some of the most basic elements used. For detailed proofs and information, we refer to [34].

Let $\{\xi_k\}_{k=1}^{\infty}$ be the Hermite functions defined as

$$\xi_n(x) = \pi^{-1/4}((n-1)!)^{-1/2} h_{n-1}(\sqrt{2}x) e^{-x^2/2} \quad \text{for} n = 1, 2, ...$$

where

$$h_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n}(e^{x^2/2}) \quad \text{for} n = 1, 2, \dots,$$

are the Hermite polynomials. Using the definition, it is easy to compute the Hermite polynomials

$h_0(x) = 1, \quad h_1(x) = x, \quad h_2 = x^2 - 1,$

$h_3(x) = x^3 - 3x, \quad h_4(x) = x^4 - 6x^2 + 3, \ldots.$

The generating function of Hermite polynomials is given by

$$\exp(tx - \frac{x^2}{2}) = \sum_{n=0}^{\infty} \frac{t^n}{n!} h_n(x), \quad \forall t, x \in \mathbb{R}.$$

Then $\xi_k \in S(\mathbb{R})$ and for their upper bound, the following relation is known. There exist constants C and ϑ such that

$$|\xi_n(x)| \leq \begin{cases} Cn^{-1/12} & \text{if } |x| \leq 2\sqrt{n} \\ Ce^{\vartheta x^2} & \text{if } |x| > 2\sqrt{n} \end{cases}$$

The $\{\xi_n\}_{n=1}^{\infty}$ constitutes an orthonormal basis for $L^2(\mathbb{R})$, see [34] for references. Instead of the *iterated Itô integrals*, the second chaos expansion theorem is based on the *Hermite functions* and their products. The order of the *Hermite functions* to be used is stated by multi-indices $\alpha = (\alpha_1, \alpha_2, \ldots)$ of finite length; it has finite non-zero elements α_i where $\alpha_i \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$ for all i. Let $\mathcal{J}$ be the set of all multi-indices $\alpha = (\alpha_1, \alpha_2, \ldots)$ of finite length $l(\alpha) = \max\{i; \alpha_i \neq 0\}$. With $\alpha! = \alpha_1!\alpha_2! \ldots \alpha_n!$ and $|\alpha| = \alpha_1 + \ldots + \alpha_n$, $H_\alpha(\omega)$ is defined by

$$H_\alpha(\omega) = h_{\alpha_1}(< \omega, \xi_1 >) h_{\alpha_2}(< \omega, \xi_2 >) \ldots h_{\alpha_n}(< \omega, \xi_n >).$$

Thus, as an example

$$\begin{aligned} H_{(2,0,3,1)}(\omega) &= h_2(< \omega, \xi_1 >) h_0(< \omega, \xi_2 >) h_3(< \omega, \xi_3 >) h_1(< \omega, \xi_4 >) \\ &= (< \omega, \xi_1 >^2 - 1)(< \omega, \xi_3 >^3 - 3 < \omega, \xi_3 >) < \omega, \xi_4 >, \end{aligned}$$

since $h_0(x) = 1, \quad h_1(x) = x, \quad h_2 = x^2 - 1, \quad h_3(x) = x^3 - 3x$. If we denote the unit vectors of $L^2(\mathbb{R})$ by $\varepsilon^{(k)} = (0, 0, \ldots, 0, 1)$ with 1 on the kth entry, and 0 otherwise, where $k = 1, 2, \ldots$. Using these unit vectors, we obtain a useful equality for our computations:

$$\begin{aligned} H_{\varepsilon^{(k)}}(\omega) &= h_1(< \omega, \xi_k >) \\ &= < \omega, \xi_k > = \int_{\mathbb{R}} \xi_k(t) dB(t), \end{aligned}$$

which corresponds to the *chaos expansion* of $< \omega, \xi_k >$ in terms of multiple Itô integrals. The main result obtained by Itô gives the *chaos expansion* of $H_\alpha(\omega)$:

$$H_\alpha(\omega) = \int_{\mathbb{R}^{|\alpha|}} \xi^{\hat{\otimes}\alpha}(x) dB^{\otimes |\alpha|}(x),$$

where $\hat{\otimes}$ denotes symmetrized tensor product. See [34] and [24] for details.

Theorem 3.1.5 Let $F \in L^2(\mathbb{P})$. Then there exists a unique family $c_\alpha, \alpha \in \mathcal{J}$ of constants $c_\alpha \in \mathbb{R}$ such that

$$F(\omega) = \sum_{\alpha \in J} c = \alpha H_\alpha(\omega) \qquad \textit{with convergence in} \quad L^2(\mathbb{P});$$

furthermore, there is the following isometry

$$E[F^2] = \sum_{\alpha \in J} c_\alpha^2 \alpha!.$$

Let us find the chaos expansion of Brownian motion $B(t_0)$ for some $t_0 \in \mathbb{R}$ in the sense of *iterated Itô integrals*. We can write

$$B(t_0) = \int_{\mathbb{R}} I_{[0,t_0]}(s) dB(s).$$

We know that ξ_k's constitute a basis for $L^2(\mathbb{R})$. When we expand in Fourier series we obtain

$$\int_{\mathbb{R}} \sum_{k=1}^{\infty} \langle I_{[0,t_0]}, \xi_k \rangle_{L^2(\mathbb{R})} \xi_k(s) dW(s),$$

where the inner product of $I_{[0,t_0]}$ and ξ_k is

$$\begin{aligned} \langle I_{[0,t_0]}, \xi_k \rangle_{L^2(\mathbb{R})} &= \int_{\mathbb{R}} I_{[0,t_0]}(u) \xi_k(u) du \\ &= \int_0^{t_0} \xi_k(u) du. \end{aligned}$$

So we see that the expansion of $B(t_0)$ is

$$B(t_0) = \sum_{k=1}^{\infty} (\int_0^{t_0} \xi_k(u) du) \int_{\mathbb{R}} \xi_k(s) dW(s),$$

where $c_\alpha = \int_0^{t_0} \xi_k(u) du$ and $H_{\varepsilon^k}(\omega) = \xi_k(s) dW(s)$. We see that

$$B(t) = \sum_{k=1}^{\infty} c_\alpha H_{\varepsilon^{(k)}}(\omega)$$

holds for 1-dimensional Brownian motion. Another way to see this is considering $B(t)$ as $< \omega, I_{[0,t]} >$, the action of ω on f:

$$\begin{aligned} B(t) &= < \omega, I_{[0,t]}(.) >=< \omega, \sum_{k=1}^{\infty} \langle I_{0,t}, \xi_k \rangle_{L^2(\mathbb{R})} \xi_k(.) > \\ &= \sum_{k=1}^{\infty} \int_0^t \xi_k(s) ds < \omega, \xi_k > \\ &= \sum_{k=1}^{\infty} \int_0^t \xi_k(s) ds H_{\varepsilon^{(k)}}. \end{aligned}$$

This expansion shows us that, when regarded as a map, $B(\cdot) : \mathbb{R} \to (S)^*$, $B(t)$ is differentiable with respect to t and the resulting process is called the *white noise*, which, indeed, gives its name to this theoretical framework. The white noise process, denoted $W(t)$ has the expansion

$$W(t) = \frac{d}{dt}B(t) = \sum_{k=1}^{\infty} \xi_k(t) H_{\varepsilon^{(k)}}. \tag{3.2}$$

The white noise process plays an important role in the definition of the Skorohod integral. After giving the definition of this integral, we will see how it is possible to obtain an Itô type formula using the Wick-Itô-Skorohod integral with respect to *fBm*. Skorohod integral, can be seen as an extension of the Itô integral to the integrands that may not be adapted. But to define this integral, there is another necessary concept of the white noise theory: the *Wick product*. After we define the *Wick product*, we will be able to compute stochastic integrals using the chaos expansion of functions.

Definition 3.1.6 *(*Wick Product*)*
If $F_i(\omega) = \sum_{\alpha\in\mathcal{J}} c_\alpha^{(i)} H_\alpha(\omega); \quad i = 1, 2$ *are two elements of* $(S)^*$ *we define their* Wick product $(F_1 \diamond F_2)(\omega)$ *by*

$$(F_1 \diamond F_2)(\omega) = \sum_{\alpha,\beta\in J} c_\alpha^{(1)} c_\beta^{(2)} H_{\alpha+\beta}(\omega) = \sum_{\gamma\in\mathcal{J}} (\sum_{\alpha+\beta=\gamma} c_\alpha^{(1)} c_\beta^{(2)}) H_\gamma(\omega). \tag{3.3}$$

For calculations of the *Wick product* of two functions, we remark some useful properties of $H_\alpha(\omega)$ which are needed:

$$H_{\varepsilon^{(i)}+\varepsilon^{(j)}}(\omega) = \begin{cases} H_{\varepsilon^{(i)}} H_{\varepsilon^{(j)}}(\omega), & \text{if } i \neq j \\ H^2_{\varepsilon^{(i)}}(\omega) - 1, & \text{if } i = j \end{cases}$$

There are some advantages of using the *Wick calculus* when dealing with square integrable random variables ([17],[34]). We summarize some of these advantages as follows:

1. If F is deterministic, then $F \diamond G = F \cdot G$.

2. If $f \in L^2(\mathbb{R})$ is deterministic, then

$$\begin{aligned} \int_{\mathbb{R}} f(t)dB(t) &= <\omega, f>= \sum_{k=1}^{\infty} \langle f, \xi_k \rangle_{L^2(\mathbb{R})} <\omega, \xi_k> \\ &= \sum_{k=1}^{\infty} \langle f, \xi_k \rangle_{L^2(\mathbb{R})} H_{\varepsilon^{(k)}}(\omega). \end{aligned}$$

3. If g(t) $\in L^2(\mathbb{R})$ is deterministic, then

$$
\begin{aligned}
[\int_{\mathbb{R}} f(t)dB(t)] \diamond [\int_{\mathbb{R}} g(t)dB(t)] &= \sum_{i,j=1}^{\infty} \langle f, \xi_i \rangle_{L^{\mathbb{R}}} \langle g, \xi_i \rangle_{L^{\mathbb{R}}} H_{\varepsilon^{(i)}+\varepsilon^{(j)}}(\omega) \\
&= [\int_{\mathbb{R}} f(t)dB(t)] \cdot [\int_{\mathbb{R}} g(t)dB(t)] - \langle f, g \rangle_{L^{\mathbb{R}}}.
\end{aligned}
$$

Another property of *Wick product* which is used to compute the values of stochastic integrals is that when $\|f\|_2 = 1$, $< \omega, f >^{\diamond n} = h_n(< \omega, f >)$

As we will see, *Skorohod* integral with respect to *fBm* makes it possible to compute the values of stochastic integrals with respect to *fBm*. Let us give the definition of *Skorohod* integral.

Definition 3.1.7 *Let $g(t, \omega)$, $\quad \omega \in \Omega$, $\quad t \in [0, T]$, be a stochastic process that is assumed to be (t,ω)-measurable, that is, $g(t, \omega)$ is F-measurable for all $t \in [0, T]$ and $E[g^2(t, \omega)] < \infty$ $\forall t \in [0, T]$. Then we can find the chaos expansion of the random variable $\omega \to g(t, \omega)$ and obtain the functions $f_{n,t}(t_1, t_2, \ldots, t_n)$ such that*

$$g(t, \omega) = \sum_{k=1}^{\infty} I_n(f_{n,t}(\cdot)).$$

These functions depend only on the parameter t, then we write

$$f_{n,t}(t_1, t_2, \ldots, t_n) = f_n(t_1, t_2, \ldots, t_n, t)$$

now the symmetrization of f_n, denoted $\tilde{f}_n$ is a function of $n+1$ variables $t_1, t_2, \ldots, t_n, t$ is given by, with $t_{n+1} = t$

$$
\begin{aligned}
\tilde{f}_{n,t}(t_1, t_2, \ldots, t_n) &= \frac{1}{n+1}[f_n(t_1, t_2, \ldots, t_n) + f_n(t_1, t_2, \ldots, t_n, t) + \ldots \\
&\quad f_{n,t}(t_1, t_2, \ldots, t_{n+1}, t_i) + \ldots,
\end{aligned}
$$

where the sum is over the permutations σ of the indices $(1, \ldots, n+1)$ which interchange the last component with one of the others and leave the rest in place. Now the Skorohod integral of g can be defined by

$$\delta(g) := \int_{\mathbb{R}} g(t, \omega) \delta B(t) := \sum_{n=0}^{\infty} I_n(\tilde{f}_n(., t)).$$

The relation between the Skorohod integral and the *Wick product* is given by the following equality for Skorohod integrable functions $Y(t, \omega)$:

$$\int_{\mathbb{R}} Y(t, \omega) dB(t) = \int_R Y(t, \omega) \diamond W(t) dt.$$

Using this relation, let us compute the following stochastic integral as an example. Earlier in this chapter, we obtained expansions of both $B(t)$ and $W(t)$. Using these chaos expansions, we will demonstrate the computation of the *Wick product* of $B(t)$ and $W(t)$ [17]:

Example 3.1.8

$$\begin{aligned} B(t) \diamond W(t) &= \sum_{i,j=1}^{\infty} [\xi_i(t) \int_0^t \xi_j(u)du] H_{\varepsilon^{(i)}\varepsilon^{(j)}}(\omega) \\ &= \sum_{i,j=1}^{\infty} [\xi_k(t) H_{\varepsilon^{(i)}}(\omega) \int_0^t \xi_j(u)du H_{\varepsilon^{(i)}}(\omega) - \xi_k(t) \int_0^t \xi_j(u)du] \\ &= B(t)W(t) - \sum_{i=1}^{\infty} \xi_k(t) \int_0^t \xi_j(u)du. \end{aligned}$$

As we mentioned before, it is possible obtain the results of stochastic integrals of the type $\int_0^T B(t)dB(t)$ *as follows:*

$$\begin{aligned} \int_0^T B(t)dB(t) &= \int_0^T B(t) \diamond W(t)dt \\ &= \sum_{i,j=1}^{\infty} [\int_0^T \xi_i(t) \int_0^t \xi_j(u)dudt] H_{\varepsilon^{(i)}+\varepsilon^{(j)}} \\ &= \frac{1}{2} \sum_{i,j=1}^{\infty} [\int_0^T \xi_i(u)du \int_0^T \xi_i(u)du] H_{\varepsilon^{(i)}+\varepsilon^{(j)}} \\ &= \frac{1}{2} [\sum_{i,j=1}^{\infty} (\int_0^T \xi_i(u)du) H_{\varepsilon^{(i)}}]^2 - \frac{1}{2} \sum_{i=1}^{\infty} [\int_0^T \xi_i(u)du]^2 \\ &= \frac{1}{2} B(T)^2 - \frac{1}{2} T, \end{aligned}$$

where we used the Perseval's identity

$$\sum_i \langle g, \xi_i \rangle^2_{L^2(\mathbb{R})} = ||g||^2_{L^2(\mathbb{R})}$$

to obtain $\sum_{i=1}^{\infty} [\int_0^T \xi_i(u)du]^2 = T$.

3.1.2 Operator M

The name *fractional* comes from the notion of fractional calculus used in the definition of *fBm*. Intuitively, *fBm* can be thought as fractionally integrated Brownian motion. Fractional differentiation and integration is widely used in physics, especially in turbulence phenomena. At first sight, there no direct connection between fractional integration and long-range dependence or the specific covariance structure of *fBm*, but as we will see, when used together, notions from fractional calculus and white noise theory can provide a way to build this connection. Now we will see how this connection has been established via the operator M. In fractional calculus notions, M operator is the fractional integral operator of order $\alpha = H - \frac{1}{2}$, so it is the main reason of the name 'fractional' Brownian motion.

Definition 3.1.9 *Let* $0 < H < 1$. *The operator* $M = M_H$ *is defined on functions* $f \in S(\mathbb{R})$ *by*

$$\widehat{Mf}(y) = |y|^{1/2-H}\widehat{f}(y), \tag{3.4}$$

where $y \in \mathbb{R}$ *and*

$$\widehat{f}(y) := \int_{\mathbb{R}} e^{ixy} f(x)dx \tag{3.5}$$

denotes the Fourier transform of g.

Equivalently, for every $0 < H < 1$ the operator M can be defined as

$$Mf(x) = -\frac{d}{dx}\frac{C_H}{(H-1/2)}\int_{\mathbb{R}}(t-x)|t-x|^{H-3/2}f(t)dt, \tag{3.6}$$

where $f \in S(\mathbb{R})$ and

$$C_H = 2\Gamma(H-\frac{1}{2})\cos[\frac{\Pi}{2}(H-\frac{1}{2})]^{-1}[\Gamma(2H+1)\sin(\Pi H)]^{1/2}$$

with $\Gamma(\cdot)$ denoting the classical Gamma function. For $0 < H < 1/2$ we have

$$Mf(x) = C_H \int_{\mathbb{R}} \frac{f(x-t) - f(x)}{|t|^{3/2-H}}dt.$$

For $H = 1/2$

$$Mf(x) = f(x)$$

For $1/2 < H < 1$ we have

$$Mf(x) = C_H \int_{\mathbb{R}} \frac{f(t)}{|t-x|^{3/2}}dt.$$

The operator M extends from the *Schwartz space* $S(\mathbb{R})$ to the space

$$\begin{aligned} L_H^2 &:= \{f : \mathbb{R} \to \mathbb{R}(deterministic) : |y|^{1/2-H}\hat{f}(y) \in L^2(\mathbb{R})\} \\ &=: \{f : \mathbb{R} \to \mathbb{R} : Mf(x) \in L^2(\mathbb{R})\} \\ &=: \{f : \mathbb{R} \to \mathbb{R} : ||f||_H < \infty\}, \end{aligned}$$

where

$$||f||_H := ||Mf||_{L^2(\mathbb{R})}.$$

Then we are in a new space L^2_H, defined by M operator operated on $f \in L^2(\mathbb{R})$. The elements of the space L^2_H are fractionally integrated functions of the *Schwartz space*. The inner product and norm in this space are also defined using the M operator. The inner product in this space is $\langle f, g\rangle_H = \langle Mf, Mg\rangle_{L^2(\mathbb{R})}$. One of the problems is that $L^2_H(\mathbb{R})$ is not closed with respect to the inner product (see [34] for references). In particular, the indicator function $I_{[0,t]}(\cdot)$ belongs to $L^2_H(\mathbb{R})$ for fixed $t \in \mathbb{R}$. We write

$$MI_{[0,t]}(x) := M[0,t](x),$$

and if $f, g \in L^2(\mathbb{R}) \cap L^2_H\mathbb{R}$, then

$$\begin{aligned}
\langle f, Mg\rangle_{L^2(\mathbb{R})} &= \langle \widehat{f, Mg}\rangle_{L^2(\mathbb{R})}\rangle_{L^2(\mathbb{R})} \\
&= \int_{\mathbb{R}} |y|^{1/2-H} \hat{f}(y)\hat{g}(y)dy = \langle \widehat{Mf, g}\rangle_{L^2(\mathbb{R})} \\
&= \langle Mf, g\rangle_{L^2(\mathbb{R})}.
\end{aligned}$$

Using the properties of M operator there are several ways of computing $MI_{[a,b]}$, which is of practical importance in defining the *fBm*:

$$M[a,b](x) = -\frac{d}{dx}\int_a^b (t-x)|t-x|^{H-3/2}dt; \tag{3.7}$$

using the change of variables $\lambda = t - x$, we get

$$\begin{aligned}
M[a,b](x) &= -\frac{d}{dx}\frac{C_H}{(H-1/2)}\int_{a-x}^{b-x} \lambda|\lambda|^{H-3/2}d\lambda \\
&= \frac{C_H}{(H-1/2)} - [(x-b)|b-x|^{H-3/2} - (x-a)|a-x|^{H-3/2}] \\
&= \frac{[\Gamma(2H+1)\sin(\pi H)]^{1/2}}{2\Gamma(H+1/2)\cos[\pi/2(H+1/2)]}\Big[\frac{b-x}{|b-x|^{3/2-H}} - \frac{a-x}{|a-x|^{3/2-H}}\Big].
\end{aligned}$$

and $Mf \in L^2(\mathbb{R})$ for this choice of f. By using 3.4 and Perceval's Theorem, we have, for $0 < H < 1$,

$$\begin{aligned}
\int_{\mathbb{R}} [M[a,b](x)]^2 dx &= \frac{1}{2\Pi}\int_{\mathbb{R}} [\widehat{M[a,b]}(s)]^2 ds \\
&= \frac{1}{2\Pi}\int_{\mathbb{R}} |s|^{1-2H}\frac{|e^{-ibs} - e^{-ias}|^2}{|s|^2}ds \\
&= (b-a)^{2H},
\end{aligned} \tag{3.8}$$

where the following relation is used

$$\widehat{I_{[a,b]}}(s) = \frac{[e^{-ibs} - e^{-ias}]}{-is}.$$

We refer to [34] and [17] for the proof and details.

Since $M[s,t] = M[0,t] - M[0,s]$ for $s < t$, and using the following equation,

$$M[0,t]M[0,s] = \frac{(M[0,s])^2 + (M[0,t])^2 - (M[s,t])^2}{2},$$

we have

$$\int_{\mathbb{R}} M[0,t](x)M[0,s](x)dx = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}), \tag{3.9}$$

which holds for arbitrary $s, t \in \mathbb{R}$ since $\int_{\mathbb{R}}[M[a,b](x)]dx = (b-a)^{2H}$. So we see that the operator M is defined in such a way to obtain the specific covariance structure of *fBm*. When indicator function is used instead of M, one would obtain the covariance of a standard Brownian motion. But using the M operator, for a specific value of H, one can obtain the dependence between increments of the resulting process; *fBm*. In the white noise space, *fBm* is defined in an analog way of Brownian motion. We can see the operator M is the only difference between these definitions. For $\forall t \in \mathbb{R}$, define

$$B^H(t) := B^H(t,\omega) :=< \omega, M[0,t](\cdot) >,$$

where $< \omega, f >= \omega(f)$ is the action of $\omega \in \Omega = S^{'}(\mathbb{R})$ on $f \in S(\mathbb{R})$. Since the measure used in definition of *fBm* is the white noise measure, $\tilde{B}^H(t)$ is Gaussian, $\tilde{B}^H(0) = \mathbb{E}[\tilde{B}^H(t)] = 0$ a.s. for all $t \in \mathbb{R}$, and by using (3.9) we see that

$$\begin{aligned} E[\tilde{B}^H(t)\tilde{B}^H(s)] &= \int_{\mathbb{R}} M[0,t](x)M[0,s](x)dx \\ &= \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}). \end{aligned}$$

Therefore, the continuous version of $\tilde{B}^H(t)$, $B^H(t)$ is a *fBm*, as defined in Chapter 1.

Integral with respect to *fBm* in the white noise space was defined by following an approximation similar to that of standard Brownian motion. As usual, we begin with the step functions. Let $f(x) = \Sigma_j a_j I_{[t_j, t_{j+1}]}$ be a step function, then by the definitions above and linearity, we can write

$$
\begin{aligned}
<\omega, Mf> \;&:=\; \sum_j a_j <\omega, M[t_j, t_{j+1}]> &(3.10)\\
&=\; \sum_j a_j (B^H(t_{j+1}) - B^H(t_j))\\
&=: \int_{\mathbb{R}} f(t) dB^H(t). &(3.11)
\end{aligned}
$$

A Gaussian random variable is characterized by its first two moments. To obtain this characterization for $\int_{\mathbb{R}} f(t)dB^H(t)$, the Itô isometry for this random variable must be defined. This can be done by using the Itô isometry for Brownian motion case and the definition of the norm in $L^2_H(\mathbb{R})$:

$$|| <\omega, Mf> ||_{L^2(\mathbb{P})} = ||Mf||_{L^2(\mathbb{R})} = ||f||_H.$$

Comparing to the definition of Itô integral in the white noise space:

$$<\omega, f>= \int_{\mathbb{R}} f(t)dB(t) \qquad \text{for all deterministic} f \in \mathbb{R},$$

we see that

$$<\omega, Mf>= \int_{\mathbb{R}} f(t)dB^H = \int_{\mathbb{R}} Mf(t)dB(t), \quad f \in L^2_H(\mathbb{R}).$$

It is surprising to see the integral at the beginning forming into the last term, an Itô integral with respect to Brownian motion $B(t)$! The amazing properties of the operator M makes it possible to compute the integral of a function with respect to a *fBm* with an arbitrary H by only operating the operator M on the function under consideration. We will use this result in the following subsections.

Since $Mf \in L^2(\mathbb{R})$ for all $f \in S(\mathbb{R})$, using the isometry, $M : S^{'}(\mathbb{R}) \to S^{'}(\mathbb{R})$ satisfies the following relation:

$$< M\omega, f >=< \omega, Mf >, \quad f \in S(\mathbb{R}) \quad \text{for} \quad \mathbb{P}-a.e. \quad \omega \in \Omega = S^{'}(\mathbb{R}).$$

We saw that the operator M is defined based on $f \in S(\mathbb{R})$, then this last equality is hard to give meaning at first sight because there is no explicit interpretation of the term $M\omega$, where $\omega \in \Omega = S^{'}(\mathbb{R})$. The isometry property states that these two random variables are equal in the mean-square sense. We know that the M operator is in fact the fractional integration operator of order $\alpha = H - \frac{1}{2}$ and this gives us the idea of the notion of *fractionally integrated* white noise, or as an analog to the *fBm* case, the *fractional* white noise. This interpretation is true indeed, and to obtain a fractional Black& Scholes formula, the *fractional* white noise calculus is used. But the term *fractional* white noise is used in the sense of a probability measure, not in the sense of the process *fractional* white noise defined in the next subsection.

Now let us define

$$e_k(x) = M^{-1}\xi_k(x), \quad k = 1, 2, \ldots, \tag{3.12}$$

then $e_{k}{}_{k=1}^{\infty}$ are orthonormal in $L^2_H(\mathbb{R})$ and the closed linear span of $\{e_k\}_{k=1}^{\infty}$ contains $L^2_H(\mathbb{R})$ (see [34]). And keep in mind that $Me_k(x) = \xi_k(x)$.

3.1.3 WIS integral

So far, we presented the definitions of Brownian motion and *fBm* in the white noise space. We proceed with the definition of Skorohod integral. To decide if a function is Skorohod integrable, the following function spaces, the *Hida space (S) of stochastic test functions* and *Hida space* $(S)^*$ *of stochastic distributions* are used. We briefly give the definitions of these spaces. For details, see [22].

Definition 3.1.10 1. *The* Hida space (S) of stochastic test functions *is defined to be all* $\psi \in L^2(\mathbb{P})$, *whose expansion*

$$\psi(\omega) = \sum_{\alpha\in\mathcal{J}} a_\alpha H_\alpha,$$

satisfies

$$||\psi||_k^2 := \sum_{\alpha\in\mathcal{J}} a_\alpha^2 \alpha!(2\mathbb{N})^{k\alpha} < \infty \quad \forall k = 1, 2, \ldots,$$

where $(2\mathbb{N})^\gamma = (2.1)^{\gamma_1}(2.2)^{\gamma_2}\ldots(2m)^{\gamma_m}$, *if* $\gamma = (\gamma_1, \ldots, \gamma_m) \in \mathcal{J}$.

2. *The* Hida space $(S)^*$ of stochastic distributions *is defined to be the set of formal expansions*

$$G(\omega) = \sum_{\alpha\in\mathcal{J}} b_\alpha H_\alpha(\omega),$$

such that

$$||G||_q^2 := \sum_{\alpha\in\mathcal{J}} b_\alpha^2 \alpha!(2\mathbb{N})^{-q\alpha} < \infty \quad \textit{for some} \quad q < \infty.$$

Definition 3.1.11 *Suppose that* $Z : \mathbb{R} \rightarrow (S)^*$*, is a given function with the property that*

$$\langle Z(t), \psi \rangle \in L^1(\mathbb{R}, dt), \qquad \forall \psi \in (S),$$

then $\int_{\mathbb{R}} Z(t)dt$ *is defined to be the unique element of* $(S)^*$ *such that*

$$\langle \int_{\mathbb{R}} Z(t)dt, \psi \rangle = \int_{\mathbb{R}} \langle Z(t), \psi \rangle dt \quad \forall \psi \in (S).$$

If the last equality holds, then $Z(t)$ *is said to be* $dt-$ *integrable in* $(S)^*$.

If a function is $dt-$ *integrable* in $(S)^*$, we can define Skorohod integral of it with respect to an element of $(S)^*$. Since $B^H(t)$ is an element of $(S)^*$ [34], we the Skorohod integral of a function with respect to *fBm* is given as follows:

Definition 3.1.12 *(*WIS*) integral*
If a function $Y(t, \omega)$ *is* $dt-$ *integrable in* $(S^*$*, then we say that* Y *is Wick Itô Skorohod (*WIS*) integrable and define its* WIS *integral with respect to* $B^H(t)$ *by*

$$\int_{\mathbb{R}} Y(t, \omega)dB^H(t) := \int_{\mathbb{R}} Y(t) \diamond W^H(t)dt,$$

The fractional analog of the white noise process, *fractional white noise* process is defined in these spaces using the operator M. These definitions are in the sense of finite dimensional distributions and in this sense, *fBm* $B^H(t)$ is differentiable with respect to t.

Example 3.1.13 *Let* $H_\alpha(\omega)$ *be as defined before and* $\varepsilon^{(k)}$ *are the unit vectors denoted by*

$$\varepsilon^{(k)} = (0, \ldots, 1, 0, 0)$$

with only the kth entry being 1 *and all the others are 0,* $\quad k = 0, 1, \ldots$*. Now let us find the chaos expansion of* $B^H(t)$ *in a similar approach that we have followed when we computed the chaos expansion of* $B(t)$*. As we expect, the only difference is the operator* M*, operating on the* *indicator function:*

$$\begin{aligned}
B^H(t) &= < \omega, M[0,t](.) >=< M\omega, I_{[0,t](.)} > \\
&= < M\omega, \sum_{k=1}^{\infty} \langle I_{[0,t]}, e_k \rangle_H e_k(.) > \\
&= < M\omega, \sum_{k=1}^{\infty} \langle M_{[0,t]}, Me_k \rangle_{L^2(\mathbb{R})} e_k(.) > \\
&= \sum_{k=1}^{\infty} \langle M_{[0,t]}, \xi_k \rangle_{L^2(\mathbb{R})} < M\omega, e_k > \\
&= \sum_{k=1}^{\infty} \langle I_{[0,t]}, M\xi_k \rangle_{L^2(\mathbb{R})} < \omega, Me_k > \\
&= \sum_{k=1}^{\infty} \int_0^t M\xi_k(s) ds H_{\varepsilon^{(k)}}(\omega).
\end{aligned}$$

The key element in definition of *WIS* integral, as seen, is the *fractional white noise* $W^H(t)$. Again in a similar approach that took us to the expansion of white noise $W(t)$, we use the expansion of $B^H(t)$ and take its derivative with respect to t in $(S)^*$:

$$W^H(t) = \sum_{k=1}^{\infty} M\xi_k H_{\varepsilon^{(k)}}(\omega)$$

since

$$\frac{dB^H(t)}{dt} = W^H(t), \qquad \text{in} \quad (S)^*.$$

As we mentioned before, in the construction of *fractional* white noise in this setting, M has operated on $\xi_k(s) \in S(\mathbb{R})$, not on $\omega \in \Omega$. Since the expansion of *fractional* white noise is known, it is possible to compute the WIS integral of a function $f \in L_H^2(\mathbb{R})$, with respect to *fBm* $B^H(t)$. As done in the standard Brownian motion case, *Wick calculus* is used in this computation:

$$\begin{aligned}
\int_{\mathbb{R}} f(t) \diamond W^H(t) dt &= \sum_{k=1}^{\infty} [\int_{\mathbb{R}} f(t) M\xi_k(t) dt] H_{\varepsilon^{(k)}}(\omega) \\
&= \sum_{k=1}^{\infty} \langle f, M\xi_k \rangle_{L^2} H_{\varepsilon^{(k)}(\omega)} \\
&= \sum_{k=1}^{\infty} \langle Mf, \xi_k \rangle_{L^2} H_{\varepsilon^{(k)}(\omega)} = \int_{\mathbb{R}} Mf \diamond W(t) dt \\
&= \int_{\mathbb{R}} Mf(t) dB(t).
\end{aligned}$$

When it comes to the definition of this stochastic integral over a finite time interval, indicator function, as expected, appears to be solving the problem:

$$\int_0^T f(t,\omega)dB^H(t) := \int_{\mathbb{R}} f(t)I_{[0,T]} \diamond W^H(t)dt.$$

Using the *Wick* calculus in $(S)^*$, we can compute the Skorohod integral $\int_0^T B^H(t)dB^H(t)$ as follows:

$$\begin{aligned}
\int_0^T B^H(t)dB^H(t) &= \int_0^T B^H(t) \diamond W^H(t)dt = \int 0^T B^H(t) \diamond \frac{dB^H(t)}{dt}dt \\
&= \frac{1}{2}[(B^H(t))^{\diamond 2}]_0^T = \frac{1}{2}(B^H(T))^{\diamond 2} = \frac{1}{2}(<\omega, M[0,T]>)^{\diamond 2} \\
&= \frac{1}{2}[(<\omega, M[0,T]>)^2 - \langle M[0,T], M[0,T]\rangle_{L^2(\mathbb{R})}] \\
&= \frac{1}{2}(B^H(T))^2 - \frac{1}{2}||M[0,T]||^2_{L^2(\mathbb{R})} = \frac{1}{2}(B^H(T))^2 - \frac{1}{2}T^{2H}.
\end{aligned}$$

This result shows us that the stochastic integral has expectation zero, as in the case of Itô integral for standard Brownian motion. This property is coined with the notion of martingale for stochastic integrals. Then, in the white noise universe, this stochastic integral behaves like a martingale. Generally speaking, when we use *Wick* calculus in $(S)^*$, we can deal with *fBm* in a similar fashion to that is used in the standard Brownian motion case.

As we mentioned, *Skorohod* integral is an extension of the Itô integral. Now let us give the definition of the Itô exponential to *WIS* integral.

Example 3.1.14 *The* WIS *(*Wick*) exponential*

The Wick *exponential is defined as*

$$\exp^{\diamond} F = \sum_{n=0}^{\infty} \frac{1}{n!}F^{\diamond n}.$$

In general we have ([24]),

$$\exp^{\diamond}[<\omega, Mf>] = \exp\Big(<\omega, Mf> - \frac{1}{2}||Mf||^2_{L^2(\mathbb{R})}\Big). \tag{3.13}$$

When we take the expectation of the *Wick* exponential, we see that

$$\begin{aligned}
E[\exp\Big(<\omega, Mf> - \frac{1}{2}||Mf||^2_{L^2(\mathbb{R})}\Big)] &= E[\exp\Big(<\omega, Mf> \Big)]\exp\Big(-\frac{1}{2}||Mf||^2_{L^2(\mathbb{R})}\Big) \\
&= \exp\Big(\frac{1}{2}||Mf||^2_{L^2(\mathbb{R})}\Big)\exp\Big(-\frac{1}{2}||Mf||^2_{L^2(\mathbb{R})}\Big) \\
&= 1,
\end{aligned}$$

which is an analogous property to stochastic exponential process; the *Wick* exponential has constant expectation.

The *WIS* analog of the stochastic exponential is the *WIS* exponential and using its properties and *Wick calculus* in $(S)^*$, it is possible to solve fractional SDEs. Let us consider the fractional stochastic differential equation

$$dX(t) = \alpha(t)X(t)dt + \beta(t)X(t)dB^H(t), \qquad t \geq 0, \tag{3.14}$$

which is the differential form of

$$X(t) = X(0) + \int_0^t \alpha(s)X(s)ds + \int_0^t \beta(s)X(s)dB^H(s),$$

where $\alpha(\cdot), \beta(\cdot)$ are locally bounded deterministic functions.Using the definition of *fractional white noise* in $(S)^*$, we can write this equation as a differential equation in $(S)^*$:

$$\begin{aligned} \frac{dX(t)}{dt} &= \alpha(t)X(t) + \beta(t)X(t) \diamond W^H(t) \\ &= X(t) \diamond [\alpha(t) + \beta(t)W^H(t)], \end{aligned}$$

When all the products are considered to be in the *Wick* sense, this equation is the differential equation for the exponential. The solution of this equation is obtained using the *Wick calculus* [34]:

$$X(t) = X(0) \diamond \exp^{\diamond}(\int_0^t \alpha(s)ds + \int_0^t \beta(s)dB^H(s)), \tag{3.15}$$

where

$$\int_0^t \beta(s)dB^H(s) = \int_{\mathbb{R}} \beta(s)I_{[0,t]}(s)dB^H(s).$$

Using the *Wick* exponential, our solution can be written as:

$$X(t) = X(0) \diamond \exp(\int_0^t \beta(s)dB^H(s) + \int_0^t \alpha(s)ds - \frac{1}{2}\int_{\mathbb{R}}(M_s(\beta(s)I_{[0,t]}(s)))^2 ds), \tag{3.16}$$

where M_s is the operator M acting on the variable s. If $X(0) = x$ is deterministic, the solution of the fractional SDE becomes

$$X(t) = x\exp(\int_0^t \beta(s)dB^H(s) + \int_0^t \alpha(s)ds - \frac{1}{2}\int_{\mathbb{R}}(M_s(\beta(s)I_[0,t](s)))^2 ds).$$

Furthermore if $\beta(s) = \beta$, $\alpha(s) = \alpha$ are constants, we obtain

$$X(t) = x\exp(\beta B^H(t) + \alpha t - \frac{1}{2}\beta^2 t^{2H}).$$

Now we present the Itô type formula for *fBm*. The *WIS* exponential, plays an important part in obtaining this formula.

Theorem 3.1.15 *Let $H \in (0,1)$. Assume that $f(s,x) : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ belongs to $C^{1,2}(\mathbb{R}\times\mathbb{R})$, and assume that the random variables $f(t, B^H(t))$, $\int_0^t \frac{\partial f}{\partial s}(s, B^H(s))ds$ and $\int_0^t \frac{\partial^2 f}{\partial x^2}(s, B^H(s))s^{2H-1}ds$ all belong to $L^2(\mathbb{P})$. Then*

$$f(t, B^H(t)) = f(0,0)+\int_0^t \frac{\partial f}{\partial s}(s, B^{H(s)})ds+\int_0^t \frac{\partial f}{\partial x}(s, B^{H(s)})dB^H(s)+H\int_0^t \frac{\partial^2}{\partial x^2}(s, B^H(s))s^{2H-1}ds.$$

Proof. We follow the approximation in [34] for the proof of Itô formula.

Let $\alpha \in \mathbb{R}$ be a constant, and let $\beta : \mathbb{R} \to \mathbb{R}$ be a deterministic function. Define

$$g(t,x) = \exp(\alpha x + \beta(t)),$$

and put

$$Y(t) = g(t,x).$$

Using the *WIS* exponential, we can write $Y(t)$ as:

$$\begin{aligned} Y(t) &= \exp(\alpha B^H(t)) \exp(\beta(t)) \\ &= \exp\diamond(\alpha B^H(t) + \frac{1}{2}\alpha^2 t^{2H}) \exp(\beta(t)). \end{aligned}$$

Using the *Wick* calculus in $(S)^*$, we have the following equality for the term $\frac{d}{dt}Y(t)$:

$$\begin{aligned} \frac{d}{dt}Y(t) &= \exp\diamond(\alpha B^H + \frac{1}{2}\alpha^2 t^{2H}) \diamond (\alpha W^H(t) + H\alpha^2 t^{2H-1}) \exp(\beta(t)) \\ &\quad + \exp^{\diamond}(\alpha B^H(t) + \frac{1}{2}\alpha^2 t^{2H}) \exp(\beta(t))\beta^{'}(t) \\ &= Y(t)\beta^{'} + Y(t) \diamond (\alpha W^H(t)) + Y(t)H\alpha^2 t^{2H-1}. \end{aligned}$$

Hence,

$$Y(t) = Y(0) + \int_0^t Y(s)\beta^{'}(s)ds + \int_0^t Y(s)\alpha dB^H(s) + H\int_0^t Y(s)\alpha^2 s^{2H-1}ds.$$

If we write this last equations in terms of $g(t,x)$, we see,

$$g(t, B^H(t)) = g(0,0)+\int_0^t \frac{\partial g}{\partial s}(s, B^H(s))ds+\int_0^t \frac{\partial g}{\partial x}(s, B^H(s)dB^H(s)+H\int_0^t \frac{\partial^2 g}{\partial s^2}(s, B^H(s))s^{2H-1}ds,$$

which is the fractional Itô formula. ∎

For a function in the form of f, there exists a sequence $f_n(t,x)$ of linear combinations of $g(t,x)$, since it is shown in [34] that the linear combinations of $g(t,x)$ is dense in $(S)^*$. Therefore we can write

$$f_n(t,x) \to f(t,x), \quad \frac{\partial f_n}{\partial t}(t,x) \to \frac{\partial f}{\partial t}(t,x), \quad \frac{\partial f_n}{\partial x}(t,x) \to \frac{\partial f}{\partial x}(t,x)$$

and $\partial^2 f_n(t,x)/\partial x \to \partial^2 f(t,x)/\partial x^2$, pointwise dominatedly as $n \to \infty$. Using these arguments, we define

$$\begin{aligned} f_n(t, B^H(t)) &= f_n(0,0) + \int_0^t \frac{\partial f_n}{\partial s}(s, B^H(s))ds + \int_0^t \frac{\partial f_n}{\partial x}(s, B^H(s))dB^H(s) \\ &\quad + H \int_0^t \frac{\partial^2 f_n}{\partial x^2}(s, B^H(s)) s^{2H-1} ds. \end{aligned}$$

Taking the limit in $L^2(\mathbb{P})$ (and also in $(S)^*$), we obtain

$$\begin{aligned} f(t, B^H(t)) &= f(0,0) + \int_0^t \frac{\partial f}{\partial s}(s, B^H(s))ds + \lim_{n\to\infty} \int_0^t \frac{\partial f_n}{\partial x}(s, B^H(s))dB^H(s) \\ &\quad + H \int_0^t \frac{\partial^2 f_n}{\partial x^2}(s, B^H(s)) s^{2H-1} ds. \end{aligned}$$

Using the continuity of $s \to \partial f_n(s, B^H(s))/\partial x$ in $(S)^*$, we can write

$$\int_0^t \frac{\partial f_n}{\partial x}(s, B^H(s))dB^H(s) = \int_0^t \frac{\partial f_n}{\partial x}(s, B^H(s)) \diamond W^H(s)ds,$$

which converges to

$$\int_0^t \frac{\partial f}{\partial x}(s, B^H(s)) \diamond W^H(s)ds$$

in $(S)^*$ as $n \to \infty$. Comparing these limit arguments we obtain the fractional Itô formula. This proof is in the Hida distribution space $(S)^*$. The only difference between the *fractional* Itô formula and the Itô formula is the form of the quadratic variation term. Other than this, the general proof of Itô formula holds. To obtain this term, let us recall the relation between the $L^2(\mathbb{R})$ and $L^2_H(\mathbb{R})$, i.e., $\|f\|_H = \|Mf\|_{L^2(\mathbb{R})}$. Let us use this equality to compute the quadratic variation of the process

$$Z(t) = \int_0^t \mu ds + \int_0^t \sigma dB^H(s),$$

where μ and σ are constants, for convenience. Using the operator M we can obtain the quadratic variation of $Z(t)$ as

$$\begin{aligned} d\langle Z\rangle(s) &= d(M_v \sigma I_{[0,s](v)})^2 \\ &= d(\sigma^2 s^{2H}) = 2H\sigma^2 s^{2H-1} ds, \end{aligned}$$

which is two times the quadratic variation term in the *fractional* Itô formula.

3.2 Fractional White noise calculus for *fBm* with $H > 1/2$

In this section, we will summarize how the *white noise theory* is modified to obtain the proper tools for pricing an option whose price dynamics is modeled by a geometric *fractional* Brownian motion. The idea of a *fractional white noise* probability measure was mentioned before. Now we will see how this idea is modeled using the *white noise analysis* mainly in [24] and [34]. The definition of the stochastic integral is again in the sense of a limit of the Riemann sums.

We saw that one way to construct an integral with respect to *fBm* is using fractional calculus by the operator M and then defining the integral in the *WIS* sense. Now we will see that there is another way of defining an integral with respect to *fBm*. A new kernel $\phi(s,t)$ was introduced for this purpose.

Proof. Let us define, for fixed $1/2 < H < 1$,

$$\phi(s,t) = H(2H-1)|s-t|^{2H-2}, \qquad s,t \in \mathbb{R}.$$

The function $\phi(\cdot,\cdot)$ is defined in a specific manner that

$$\int_0^t \int_0^t \phi(u,v)dudv = t^{2H}, \tag{3.17}$$

and for $s,t > 0$,

$$\int_0^t \int_0^s \phi(u,v)dudv = \frac{1}{2}(s^{2H} + t^{2H} - |t-s|^{2H}), \tag{3.18}$$

Let us give a proof of these equalities which are essential for the definition of the *fractional white noise*. There are three cases to consider: $i)s < t$, $ii)s = t$, $iii)s < t$. Suppose that $s \neq t$. Then

$$\int_0^t \int_0^s \phi(u,v)dudv = H(2H-1)\int_0^t \int_0^s |u-v|^{2H-2}dudv,$$

where

$$\int_0^t \int_0^s |u-v|^{2H-2}dudv = \int_0^s \int_0^s |u-v|^{2H-2}dudv.$$

If $s = t = r$, then

$$\begin{aligned}\int_0^t \int_0^s |u-v|^{2H-2}dudv &= \int_0^r \int_0^r |u-v|^{2H-2}dudv \\ &= 2\int_0^r \int_v^r |u-v|^{2H-2}dudv,\end{aligned}$$

where we can write the last equality since $\phi(u, v)$ is unchanged under the transformation $(u, v) \to (v, u)$. In using the change of variables $w = u - v \Rightarrow dw = du$, we obtain

$$\begin{aligned}
\int_0^r \int_v^r |u-v|^{2H-2} du dv &= \int_0^r \int_v^r (u-v)^{2H-2} du dv \\
&= \int_0^r \int_0^{r-v} w^{2H-2} dw dv = \frac{1}{2H-1} \int_0^r (r-v)^{2H-1} dv,
\end{aligned}$$

changing variables as $\lambda = r - v \Rightarrow d\lambda = -dv$, we have

$$\frac{1}{2H-1} \int_0^r \lambda^{2H-1} d\lambda = \frac{1}{2H(2H-1)} r^{2H}.$$

Furthermore, for the second part of the integral

$$\begin{aligned}
\int_0^t \int_0^s |u-v|^{2H-2} du dv &= \int_0^s \int_s^t |u-v|^{2H-2} du dv, \\
&= \int_0^s \int_s^t (u-v)^{2H-2} du dv, \quad \text{if} \quad t > s. \\
\text{Let} \quad \eta = u - v, \quad \text{then} & \\
&= \int_0^s \int_{s-u}^{t-u} \eta^{2H-2} d\eta du \\
&= \frac{1}{2H-1} \int_0^s [(t-u)^{2H-1} - (s-u)^{2H-1}] du \\
&= \frac{1}{2H-1} [\int_0^s (t-u)^{2H-1} du - \int_0^s (s-u)^{2H-1} du] \\
&= \frac{1}{2H(2H-1)} [t^{2H} - (t-s)^{2H} - s^{2H}].
\end{aligned}$$

Hence, when $t \geq s$,

$$\int_0^t \int_0^s \phi(u, v) du dv = \frac{1}{2}[s^{2H} + t^{2H} - (t-s)^{2H}].$$

Since $\phi(u, v)$ is unchanged under the map $(u, v) \mapsto (v, u)$,

$$\int_0^t \int_0^s \phi(u, v) du dv = \frac{1}{2}[s^{2H} + t^{2H} - |t-s|^{2H}],$$

for any $s, t > 0$. ■

The function $\phi(s, t)$ takes the place of the operator M in this setting. Using $\phi(s, t)$, an isometry and a new function space is defined as follows: Let $S(\mathbb{R})$ denote the *Schwartz space* as before. If the following holds for $f \in S(\mathbb{R})$

$$|f|_\phi^2 := \int_{\mathbb{R}} \int_{\mathbb{R}} f(s) f(t) \phi(s, t) ds dt < \infty,$$

then it is said that $f \in L_\phi^2(\mathbb{R})$. If $f, g \in L_\phi^2(\mathbb{R})$, the inner product in this space is defined by

$$\langle f, g \rangle_\phi := \int_{\mathbb{R}} \int_{\mathbb{R}} f(s) g(t) \phi(s, t) ds dt,$$

then $L^2_\phi(\mathbb{R})$, the completion of $S(\mathbb{R})$, becomes a separable Hilbert space. Let $L_H(\mathbb{R})$ denote the subspace of deterministic functions in $L^2_\phi(\mathbb{R})$. An isometry from $L^2_\phi(\mathbb{R})$ to $L^2(\mathbb{R})$ is defined by the following lemma which is proved in [14]:

Lemma 3.2.1 *Let*

$$I_-^{H-1/2} f(u) = c_H \int_u^\infty (t-u)^{H-3/2} f(t)dt, \tag{3.19}$$

where

$$c_H = \sqrt{\frac{H(2H-1)\Gamma(\frac{3}{2}-H)}{\Gamma(H-\frac{1}{2})\Gamma(2-2H)}},$$

and Γ *denotes the gamma function. Then* $I_-^{H-1/2}$ *is an isometry from* $L^2_\phi(\mathbb{R})$ *to* $L^2(\mathbb{R})$.

The fractional white noise measure, as we will see, is the fractional analog of the *white noise measure*. It is defined by using the tools and definitions of the white noise calculus. Some of the basic definitions are as follows. For detailed information and proofs, we refer to [14]. In this setting, the usual construction of the *white noise space* was used. So, $S(\mathbb{R})$ denotes the *Schwartz space* of rapidly decreasing functions on $\mathbb{R}$ and $\Omega = S'(\mathbb{R})$, the space of tempered distributions, is the dual of $S(\mathbb{R})$. Now the map $f \to \exp(-\frac{1}{2}|f|^2_\phi)$ is positive definite on $S(\mathbb{R})$. By the Bochner-Minlos theorem, there exists a probability measure μ_ϕ on Ω such that

$$\int_\Omega \exp(i\langle\omega, f\rangle)d\mu_\phi(\omega) = \exp(-\frac{1}{2}|f|^2_\phi),$$

for $f \in S(\mathbb{R})$. $\langle\omega, f\rangle$ is a Gaussian random variable with the first two moments given as follows

$$E_{\mu_\phi}[\langle\cdot, f\rangle] = 0,$$

and

$$E_{\mu_\phi}[\langle\cdot, f\rangle^2] = |f|^2_\phi.$$

Now, under this measure, we can define the *fBm* in a more natural way by

$$\tilde{B}^H(t) = \tilde{B}^H(t,\omega) = \langle\omega, I_{[0,t]}(\cdot)\rangle,$$

where the indicator function is the same that we defined in the white noise section. As we see, this measure is the *fractional* analog of the white noise probability measure. As long as we use this measure, we do not need any operator operating on the indicator function $I_{[0,t]}(\cdot)$. Again, by Kolmogorov-Chentsov theorem, $\tilde{B}^H(t)$ has a continuous version denoted by $B^H(t)$, which is a standard *fBm* under the measure μ_ϕ. We denote the natural filtration of $B^H(t)$ by $\mathcal{F}_t^H$ and endow Ω with this filtration.

For $f \in L^2_H(\mathbb{R})$, the integral with respect to *fBm* can be defined in the usual way by considering simple integrands first. Let us define

$$f_m(t) = \sum_i a_i^m I_{[t_i, t_{i+1})}(t),$$

and set

$$\int_{\mathbb{R}} f_m dB^H(t) = \sum_i a_i^{(m)} (B^H(t_{i+1}) - B^H_{t_i}).$$

Defining the integral, as in the classical case, as the limit of these sums with

$$\lim_{m\to\infty} f_m = f \tag{3.20}$$

in $L^2_\phi(\mathbb{R})$, we have

$$\int_{\mathbb{R}} f(t) dB^H(t) = \lim_{m\to\infty} \int_{\mathbb{R}} f_m dB^H(t).$$

The limit exists in $L^2(\mu_\phi)$ because of the isometry

$$E(\int_R f_m(t) dB^H(t))^2 = ||f_m||^2_\phi.$$

Now we can, by approximating f with step functions, write

$$\langle \omega, f \rangle = \int_{\mathbb{R}} f(t) dB^H(t, \omega). \tag{3.21}$$

The exponential function is of practical use in this setting as in the classical Itô integral approximation. Let $L^p(\mu_\phi) = L^p$ denote the space of all random variables $F : \Omega \to \mathbb{R}$ such that

$$||F||_{L^p(\mu_\phi)} = E[|F|^p]^{1/p} < \infty.$$

The exponential functional $\varepsilon : L^2_H(\mathbb{R}) \to L^1(\mu_\phi)$ is defined as

$$\begin{aligned} \varepsilon(f) : &= \exp\big(\int_{\mathbb{R}} f(t) dB^H(t) - \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} f(s) f(t) \phi(s,t) ds dt\big) \\ &= \exp\big(\int_{\mathbb{R}} f(t) dB^H(t) - \frac{1}{2} ||f||^2_H\big). \end{aligned}$$

If $f \in L^2_H(\mathbb{R})$, the $\varepsilon(f) \in L^p(\mu_\phi)$ for each $p \geq 1$. If we denote the linear span of the exponentials by χ, that is,

$$\chi = \{ \sum_{k=1}^n a_k \varepsilon(f_k) : n \in \mathbb{N}, a_k \in \mathbb{R}, f_k \in L^2_\phi(\mathbb{R}) \quad \text{for} \quad k \in \{1, \ldots, n\}\}.$$

is a dense set of $L^2(\mu_\phi)$, for the proof we refer to [24] and [31]. In order to define the *fractional chaos expansion theorems*, one needs an orthonormal basis of $L_\phi(\mathbb{R})$. This basis is defined in [24] by the following equation:

$$e_n(u) = (I_-^{H-1/2})^{-1}(\xi_n)(u),$$

where the $\xi_n(x)$ are the Hermite functions. The *fractional Wiener-Itô chaos expansion theorem* is stated in terms of the orthonormal basis $\{e_n\}_{n=1}^\infty$ of $L^2_\phi(\mathbb{R})$. Let $\mathcal{J} = (\mathbb{N}_{\not{0}})_s$ denote the set of all finite multi-indices $\alpha = (\alpha_1, \dots, \alpha)$, where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. If $\alpha = (\alpha_1, \dots, \alpha) \in \mathcal{J}$, we write

$$\tilde{H}_\alpha(\omega) := h_{\alpha_1}(\langle \omega, e_1 \rangle) \dots h_{\alpha_m}(\langle \omega, e_m \rangle).$$

In particular, if $\varepsilon^{(i)} := (0, \dots, 0, 1, 0, \dots, 0)$ denotes the ith unit vector, we get

$$\tilde{H}_{\varepsilon^{(i)}}(\omega) = h_1(\langle \omega, e_i \rangle) = \langle \omega, e_i \rangle$$

Now let us give the *fractional Wiener-Itô chaos expansion theorem*,

Theorem 3.2.2 *Let $F \in L^2(\mu_\phi)$. Then there exist constants $c_\alpha \in \mathbb{R}$, $\alpha \in \mathcal{J}$, such that*

$$F(\omega) = \sum_{\alpha \in \mathcal{J}} c_\alpha \tilde{H}_\alpha(\omega),$$

where the convergence holds in $L^2(\mu_\phi)$. Moreover,

$$\|F\|^2_{\mu_\phi} = \sum_{\alpha \in \mathcal{J}} \alpha! c_\alpha^2,$$

where $\alpha! = \alpha_1! \dots \alpha_m!$ if $\alpha = (\alpha_1, \dots, \alpha) \in \mathcal{J}$.

When computing the chaos expansion in terms of the Hermite functions, the inner product $\langle f, e_i \rangle_H$ gives us the coefficients of the chaos expansion. Using the definition of the inner product in $L^2_\phi(\mathbb{R})$, one can see

$$\langle f, e_i \rangle = \int_{\mathbb{R}} \int_{\mathbb{R}} f(u) e_i(v) \phi(u, v) du dv.$$

Therefore, we have the expansion

$$\int_{\mathbb{R}} f(s) dB^H(s) = \sum_{i=1}^{\infty} \langle f, e_i \rangle_H \tilde{H}_{\varepsilon^{(i)}}(\omega), \qquad f \in L^2_H(\mathbb{R}).$$

Taking $f = I_{[0,t]}$, we obtain *fBm*,

$$B^H(t) = \sum_{i=1}^{\infty} [\int_0^t (\int_{-\infty}^{\infty} e_i(v) \phi(u, v) dv)] \tilde{H}_{\varepsilon^{(i)}}(\omega)$$

The *fractional white noise* process $W^H(t)$ is obtained by using the differentiability of $B^H(t)$ in the fractional Hida test function and distribution spaces defined by the following:

Definition 3.2.3 *1. The fractional Hida test function space: Define $(S)_H$ to be the set of all $\psi(\omega) = \sum_{\alpha\in\mathcal{J}} a_\alpha \tilde{H}(\omega) \in L^2(\mu_\phi)$ such that*

$$||\psi||^2_{H,k} := \sum_{\alpha\in\mathcal{J}} \alpha! a_\alpha^2 (2\mathbb{N})^{k\alpha} < \infty \qquad k \in \mathbb{N},$$

where

$$(2\mathbb{N})^\gamma = \prod_j (2j)^{\gamma j} \qquad \textit{if} \qquad \gamma = (\gamma_1, \ldots, \gamma_m) \in \mathcal{J}.$$

2. The fractional Hida distribution space: Define $(S)^*_H$ to be the set of all formal expansions

$$G(\omega) = \sum_{\beta\in\mathcal{J}} b_\beta \tilde{H}_\beta(\omega),$$

such that

$$||G||^2_{H,-q} := \sum_{\beta\in\mathcal{J}} \beta! b_\beta^2 (2\mathbb{N})^{-q\beta} < \infty \qquad \textit{for some} \qquad q \in \mathbb{N}.$$

For the proofs, details on these spaces and the topologies, we refer to [24].

We saw that the white noise process plays a fundamental role in the definition of *WIS* integral. This holds also for the fractional *WIS* integral. In the fractional Hida distribution space, the fractional white noise at time t is defined by:

$$W^H(t) = \sum_{i=1}^{\infty} [\int_{\mathbb{R}} e_i(v)\phi(t,v)dv] \tilde{H}_{\varepsilon^{(i)}}(\omega).$$

The *fBm* is differentiable with respect to t in $(S)^*_H$ and $W^H(t)$ is integrable in $(S)^*_H$. In fact, for $0 \leq s \leq t$, we have,

$$\int_0^t W^H(s)ds = \sum_{i=1}^{\infty} \{\int_0^t [\int_{\mathbb{R}} e_i(v)\phi(u,v)dv]du\} \tilde{H}_{\varepsilon^{(i)}}(\omega) = B^H(t).$$

Therefore we can write

$$\frac{d}{dt} B^H(t) = W^H(t) \qquad \text{in} \qquad (S)^*_H.$$

Using the *fractional white noise process*, the *fractional WIS integral* can be defined using the *Wick product*. Suppose we can approximate the random variable $Z(t)$ with the step functions F_i using the partition $0 = t_1 < t_2 < \ldots < t_{n+1} = t$:

$$Z(t) = \sum_{i=1}^{n} F_i(\omega) I_{[t_i,t_{i+1})}(t), \quad \text{where} \qquad F_i \in (S)^*_H$$

then we can approximate the stochastic integral with respect to *fBm* as

$$\begin{aligned}\int_0^t Z(s)dB^H(s) &= \sum_{i=1}^{n} F_i(\omega) \diamond (B^H(t_{i+1} - B^H(t_i))) \\ &= \sum_{i=1}^{n} F_i(\omega) \diamond (\int_{t_i}^{t_{i+1}} W^H(s)ds) \\ &= \sum_{i=1}^{n} [F_i(\omega) \diamond (\int_{t_i}^{t_{i+1}} \sum_{j=1}^{\infty} \int_{\mathbb{R}} e_j(v)\phi(s,v)dv)\tilde{H}_{\varepsilon^{(j)}}]ds \\ &= \sum_{i=1}^{n} [\int_{t_i}^{t_{i+1}} [F_i(\omega) \diamond \sum_{j=1}^{\infty} \int_{\mathbb{R}} e_j(v)\phi(s,v)dv)\tilde{H}_{\varepsilon^{(j)}}]ds \\ &= \sum_{i=1}^{n} \int_{t_i}^{t_{i+1}} F_i(\omega) \diamond W^H(s)ds.\end{aligned}$$

Furthermore, when the mesh size of the partition goes to zero, the last term converges to $\int_0^t Z(s)dB^H(s)$ in $(S)_H^*$.

As we have seen so far, the *fractional WIS* and *WIS* integrals are constructed in a very similar manner. Now let us look at the solution of a geometric fractional Brownian motion in this setting:

$$dX(t) = \mu X(t)dt + \sigma X(t)dB^H(t)$$

with $X(0) = x > 0$, μ and σ are constants. As before, this equation can be written in $(S)_H^*$ as

$$\frac{dX(t)}{dt} = \mu X(t) + \sigma X(t) \diamond W^H(t) = (\mu + \sigma W^H(t)) \diamond X(t).$$

Using *Wick calculus*, the solution can be shown to be

$$X(t) = x \exp^{\diamond} (\mu t + \sigma \int_0^t W^H(s)ds),$$

which is the *Wick exponential*.

It is shown in [34] that

$$\exp^{\diamond}(\langle \omega, f \rangle) = \varepsilon(f) \qquad \text{for all} \qquad f \in L_H^2(\mathbb{R}).$$

Using this definition, we can write

$$x \exp^{\diamond}(\mu t + \sigma B^H(t))X(t) = x \exp (\sigma B^H(t) + \mu t - \frac{1}{2}\sigma^2 t^{2H}).$$

Under the *fractional white noise measure* μ_ϕ we have

$$\begin{aligned} E_{\mu_\phi}[X(t)] &= x\exp\big(\mu t - \frac{1}{2}\sigma^2 t^{2H}\big) E_{\mu_\phi}[\exp(\sigma^2 B^H(t))] \\ &= x\exp\big(\mu t - \frac{1}{2}\sigma^2 t^{2H}\big)\exp\big(\frac{1}{2}\sigma^2 t^{2H}\big) \\ &= x\exp(\mu t), \end{aligned}$$

where we used the definition of the *fractional white noise measure* to compute the expectation on the right-hand side.

The main advantage of defining *fBm* in the fractional white noise space is the *quasi*-martingale property of *fBm* in this setting. In order to show this important property, first, the fractional version of iterated Itô integrals and then *fractional(or quasi)-conditional* expectation are defined in the following pair of spaces. For details, see [24] and [31]:

Definition 3.2.4 *1. Let $k \in \mathbb{N}$. Consider the function*

$$\psi(\omega) = \sum_{n=0}^{\infty} \int_{\mathbb{R}^n} f_n d(B^H)^{\otimes},$$

with $f_n \in \hat{L}^2_\phi(\mathbb{R}^n)$, where $\hat{L}^2_\phi(\mathbb{R}^n)$ is the set of functions $f(x_1, \ldots, x_n)$ which are symmetric with respect to its n variables and satisfies $\|f\|^2_{L^2_\phi(\mathbb{R}^n)} = \langle f, f\rangle_{L^2_\phi(\mathbb{R}^n)} < \infty$. It is said that ψ belongs to the space $(G)_k = (G)_k(\mu_\phi)$, if

$$\|\psi\|^2_{(G)_k} := \sum_{n=0}^{\infty} n!\|f_n\|^2_{L^\phi(\mathbb{R}^n)} e^{2kn} < \infty,$$

then (G) is defined as

$$(G) = (G)(\mu_\phi) = \bigcap_{k=1}^{\infty} (G)_k(\mu_\phi).$$

2. Let $q \in \mathbb{N}$. The function F with the formal expansion

$$F = \sum_{n=0}^{\infty} \int_{\mathbb{R}^n} f_n d(B^H)^{\otimes}(t),$$

where $f_n \in \hat{L}^2_\phi(\mathbb{R})$. It is said that F belongs to the space $(G)_{-q} = (G)_{-q}(\mu_\phi)$ if

$$\|G\|^2_{(G)_{-q}} = \sum_{n=0}^{\infty} n!\|f_n\|^2_{L^2_\phi}(\mathbb{R}^n) e^{-2qn} < \infty.$$

Then $(G)^*$ is defined by

$$(G)^* = (G)^*(\mu_\phi) = \bigcup_{q\in\mathbb{N}} (G)_{-q}(\mu_\phi),$$

Equipped with the proper topologies, $(G)^*$ is the dual of (G).

The *fractional(or quasi)*-conditional expectation of a function is defined on these spaces.

Definition 3.2.5 *Let* $F = \sum_{n=0}^{\infty} \int_{\mathbb{R}^n} f_n(s) d(B^H)^{\otimes}$. *The* quasi-conditional expectation *of F with respect to* $\mathcal{F}_t^H = \mathcal{B}(B^H(s), s \leq t)$ *is defined by*

$$\tilde{E}_t[G] := \tilde{E}[G|F_t^H] := \sum_{n=0}^{\infty} \int_{\mathbb{R}} g_n(s) I_{[0\leq s\leq t]}(s) dB^H(s).$$

The following lemma in [31] gives the properties of Wick product under *quasi*-expectation.

Lemma 3.2.6

1. *Let* $F \in (G)^*$, *then we have that* $\tilde{E}_t[F] \in (G)^*$.

2. *Let* $F, G \in (G)^*$, *then we have that* $\tilde{E}_t[F \diamond G] = \tilde{E}_t[F] \diamond \tilde{E}_t[G]$.

3. *Let* $F \in L^2(\mu_\phi)$, *then we have* $\tilde{E}_t[F] = F \Leftrightarrow F$ *is* F_t^H *- measurable.*

In the first chapter, we saw that *fBm* is not a semimartingale. This property makes the use of *fBm* in financial modeling and defining integrals with respect to it difficult. But, the fractional white noise probability measure μ_ϕ is constructed in such a way that WIS integrals with respect to *fBm* are *quasi*-martingales under this measure. The definition of *quasi*-martingale is as follows

Definition 3.2.7 *An* $\mathcal{F}_t^H$ *- adapted stochastic process* $M(t, \omega)$ *is a* quasi-martingale *if* $M(t) \in (G)^*$ *for all t and* $\tilde{E}_s[M(t)] = M(s)$ *for all* $t \geq s$.

The following lemma in [24] states that, when considered as an element of the *fractional white noise space*, *fBm* is a *quasi*-martingale. This property is of fundamental importance for financial applications of *fBm*.

Lemma 3.2.8

1. $B^H(t)$ *is a quasi-martingale*

2. *Let* $f \in L_\phi^2(\mathbb{R})$, *then the* Wick exponential

$$\zeta(t) = \exp^{\diamond}(\langle \omega, I_{[0,t]} f \rangle) = \exp(\int_0^t f(s) dB^H(s) - \frac{1}{2} ||f||_{L_H^2(\mathbb{R})})$$

is a quasi-martingale.

3. *Let* $f \in L_{\phi}^{1,2}$ *and* $M(t) := \int_0^t f(s,\omega)dB^H(s)$. *Then* $M(t)$ *is a quasi-martingale.*

Using this lemma, the following theorem states the *quasi*-martingale property of a functional of *fBm*. Basically, this theorem is of fundamental importance to price an option in fractional Brownian markets since the price of an option is defined as a function of *fBm* at a distant stopping time T.

Theorem 3.2.9 *For every* $0 < t < T$ *and* $\lambda \in C$ *we have* $\tilde{E}_t[e^{\lambda B^H(T)}] = e^{\lambda B^H(t) + \frac{\lambda^2}{2}(T^{2H} - t^{2H})}$.

Proof. We know from Lemma 3.2.8 that the *Wick exponential* is a *quasi*-martingale. The *Wick exponential* is the solution of

$$dX(t) = \lambda X(t)dB^H(t), \qquad X(0) = 1.$$

Using the *quasi*-martingale property of $X(t)$, we can write

$$\tilde{E}_t[X(T)] = X(t),$$

where $X(t) = \exp \lambda B^H(t) - \frac{\lambda^2}{2}t^{2H}$. Therefore we have

$$\tilde{E}_t[\exp \lambda B^H(T) - \frac{\lambda^2}{2}T^{2H}] = \exp \lambda B^H(t) - \frac{\lambda^2}{2}t^{2H},$$

where we see that

$$\tilde{E}_t[\exp \lambda B^H(T)] = \exp \lambda B^H(t) + \frac{\lambda^2}{2}(T^{2H} - t^{2H}).$$

■

The increments of *fBm* are Gaussian and the next theorem in [31] gives the distribution of the increment $B^H(T) - B^H(t)$.

Theorem 3.2.10 *Let* f *be a function that* $E[f(B^H(T))] < \infty$. *Then, for every* $t \leq T$ *the following holds*

$$\tilde{E}_t[f(B^H(T))] = \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi(T^{2H} - t^{2H})}} \exp\Big(-\frac{(x - B^H(t))^2}{2(T^{2H} - t^{2H})}\Big) f(x)dx. \tag{3.22}$$

Proof. Let $\hat{f}$ denote the Fourier transform of f:

$$\hat{f}(\eta) = \int_{\mathbb{R}} e^{-ix\eta} f(x)dx$$

Then f is the inverse Fourier transform of $\hat{f}$

$$f(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{ix\eta}\hat{f}(\eta)d\eta.$$

Then we have

$$f\Big(B^H(T)\Big) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{iB^H(T)\eta}\hat{f}(\eta)d\eta.$$

Now, if we take the *quasi*-conditional expectation of $f(B^H(T))$, we obtain

$$\begin{aligned}
\tilde{E}_t[f(B^H(T))] &= \tilde{E}_t\Big[\frac{1}{2\pi}\int_{\mathbb{R}} e^{i\eta B^H(T)}\hat{f}(\eta)d\eta\Big] \\
&= \frac{1}{2\pi}\int_{\mathbb{R}} \tilde{E}_t\Big[e^{i\eta B^H(t)}\Big]\hat{f}(\eta)d\eta \\
&= \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\eta B^H(t)-\frac{\eta^2}{2}(T^{2H}-t^{2H})}\hat{f}(\eta)d\eta \\
&= g(B^H(t))
\end{aligned}$$

and the last function, g, is the inverse Fourier transform of the product between $e^{-\frac{\xi^2}{2}(T^{2H}-t^{2H})}$ and $\hat{f}$. One of these functions, $e^{-\frac{\xi^2}{2}(T^{2H}-t^{2H})}$, looks familiar. In fact, it is the Fourier transform, or the characteristic function, of Gaussian distribution with mean zero and variance $(T^{2H} - t^{2H})$. If we denote the density of this function as

$$n_{T,t} = \frac{1}{\sqrt{2\pi(T^{2H}-t^{2H})}}\exp\Big(-\frac{x^2}{2(T^{2H}-t^{2H})}\Big)$$

and, the Fourier transform of this function as $\hat{n}_{t,T}(\eta)$, we see the following holds:

$$g(B^H(t)) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\eta B^H(t)}\hat{n}_{t,T}(\eta)\hat{f}(\eta)d\eta.$$

Using the fact that the Fourier transform of a convolution is the product of the Fourier transform of the two functions, and the function g being the inverse Fourier transform of the product between two Fourier transforms, it follows that g is the convolution of $n_{t,T}$ and f, i.e.,

$$\begin{aligned}
g(B^H(t)) &= \int_{\mathbb{R}} n_{t,T}(B^H(t)-y)f(y)dy \\
&= \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi(T^{2H}-t^{2H})}}\exp\Big(-\frac{(B^H(t)-y)^2}{2(T^{2H}-t^{2H})}\Big)f(y)dy \\
&= \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi(T^{2H}-t^{2H})}}\exp\Big(-\frac{(y-B^H(t))^2}{2(T^{2H}-t^{2H})}\Big)f(y)dy,
\end{aligned}$$

where the last line of the equation comes from the symmetry property of Gaussian distribution and completes the proof. Let us give the result of this theorem when $f = I_A$. Let $A \in \mathcal{B}(\mathbb{R})$. Then

$$\tilde{E}_t[I_A(B^H(T))] = \int_A \frac{1}{\sqrt{2\pi(T^{2H}-t^{2H})}}\exp\Big(-\frac{(x-B^H(t))^2}{2(T^{2H}-t^{2H})}\Big)dx.$$

■ As we saw, these useful results can be obtained for *fBm* when regarded as an element of the white noise space. But when one defines pathwise integrals with respect to *fBm*, its quadratic variational property is lost. We will see some of these pathwise integrals and corresponding Itô formula in the next section.

3.3 Pathwise integrals with respect to *fBm* with $H > 1/2$

As for the standard Brownian motion case, pathwise integrals with respect to *fBm* is defined by taking the limit of Riemann sums of the type:

$$\sum_{i=1}^{n} f(t_i)[B^H(t_{i+1}) - B^H(t_i)],$$

where $0 = t_1 < t_2 < \ldots < t_n = T$ is a partition of $[0, T]$. Using this approximation to the stochastic integral, three types of pathwise integrals has been defined in [33] and [39]. The following definition in [34] summarizes these definitions:

Definition 3.3.1 *Let $H \in (0, 1)$ and $(y_t)_{t\in[0,T]}$ be a process with integrable trajectories. Then*

1. *The symmetric integral of y with respect to B^H is defined as*

$$\lim_{\epsilon\to 0} \frac{1}{2\epsilon} \int_0^T y(s)[B^H(s+\epsilon) - B^H(s-\epsilon)]ds.$$

2. *The forward integral of y with respect to B^H is defined as*

$$\lim_{\epsilon\to 0} \frac{1}{\epsilon} \int_0^T y(s)[B^H(s+\epsilon) - B^H(s)]ds.$$

3. *The backward integral is defined as*

$$\lim_{\epsilon\to 0} \frac{1}{\epsilon} \int_0^T y(s)[B^H(s-\epsilon) - B^H(s)]ds.$$

whenever the limit exists in probability.

When $H = \frac{1}{2}$, the symmetric integral is a generalization of the Stratonovich integral for the standard Brownian motion, and the forward integral extends the Itô integral. So we mainly deal with the forward integral with respect to *fBm*. Using the definition of the symmetric integral, one can obtain

$$
\begin{aligned}
& \lim_{\epsilon \to 0} \int_0^T B^H(s) \frac{B^H(s+\epsilon) - B^H(s-\epsilon)}{2\epsilon} ds = \\
= & \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \left(\int_0^T B^H(s) B^H(s+\epsilon) ds - \int_{-\epsilon}^{T-\epsilon} B^H(s) B^H(s+\epsilon) ds \right) \\
= & \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \left(\int_0^T B^H(s) B^H(s+\epsilon) ds - \int_{T-\epsilon}^{T} B^H(s) B^H(s+\epsilon) ds \right) \\
= & \frac{1}{2}(B^H(T))^2 - \frac{1}{2}(B^H(0))^2,
\end{aligned}
$$

which does not have constant expectation, and therefore, is not a martingale. When $H < 1/2$, the infinite quadratic variation of *fBm* makes it difficult to define these integrals. For the rest of the section, we assume $H > 1/2$.

Using the forward integral definition, a simple calculation yields:

$$
\begin{aligned}
E[\sum_{i=1}^{N} B^H(t_i)(B^H(t_{i+1}) - B^H(t_i))] &= \sum_{i=1}^{N} [E(B^H(t_i)B^H(t_{i+1})) - E((B^H(t_i))^2)] \\
&= \sum_{i=1}^{N} [\frac{1}{2}[(t_{i+1})^{2H} - (t_i)^{2H}]] \\
&= \frac{1}{2} t^{2H},
\end{aligned}
$$

which is not constant, shows that the forward integral is not a martingale.

Using the forward integral, several authors have obtained Itô type formulas. We will mention three of them. The first formula is obtained in [19]. The well known *Taylor* series is used in order to obtain the formula.

Theorem 3.3.2 *Let*

$$dX(t) = \mu(t)dt + \sigma dB^H(t)$$

and $X(0) = x$ *be the fractional forward process. Suppose* $f \in C^2(\mathbb{R})$ *and put* $Y(t) = f(t, X(t))$. *The following holds*

$$dY(t) = \frac{\partial f}{\partial t}(t, X(t))dt + \frac{\partial f}{\partial x}(t, X(t))dX(t).$$

Proof. Approximating the pathwise integral by the limit of the sums on the partition $0 = t_0 < t_1 < \ldots < t_N = t$,

$$
\begin{aligned}
Y(t) - Y(0) &= \sum_j [Y(t_{j+1}) - Y(t_j)] \\
&= \sum_j \frac{\partial f}{\partial t}(t_j, X(t_j))\triangle t_j + \sum_j \frac{\partial f}{\partial x}(t_j, X(t_j))\triangle X(t_j) \\
&\quad + \frac{1}{2}\sum_j \frac{\partial^2 f}{\partial x^2}(t_j, X(t_j))\triangle(X(t_j))^2 + \sum_j o((\triangle t_j)^2) + o((\triangle(X(t_j)))^2) \\
&= \sum_j \frac{\partial f}{\partial t}(t_j, X(t_j))\triangle t_j + \sum_j \int_{t_j}^{t_{j+1}} \frac{\partial f}{\partial x}(t_j, X(t_j))\triangle X(t_j) \\
&\quad + \frac{1}{2}\sum_j \frac{\partial^2 f}{\partial x^2}(t_j, X(t_j))\triangle(X(t_j))^2 + \sum_j o((\triangle t_j)^2) + o((\triangle(X(t_j)))^2),
\end{aligned}
$$

■ taking the limit as the mesh of the partition goes to zero, and using the fact that *fBm* has zero quadratic variation when $H > 1/2$, we obtain the Itô formula for the pathwise forward integral.

In [41], Shiryaev, using another version of Taylor's expansion and the fact that the *fBm* has zero quadratic variation when $H > 1/2$, obtains the same formula with a similar apporach. He uses Taylor's theorem with remainder.

Theorem 3.3.3 *If the* $(n+1)$*th derivative of* f *is continuous on an interval containing* c *and* x*, and if* $P_n(x)$ *is the* Taylor *polynomial of degree* n *for about the point* $x = c$*, then the remainder* $R_n(x) = f(x) - P_n(x)$ *in* Taylor's *formula can be given by*

$$R_n(x) = \frac{1}{n!}\int_c^x (x-t)^n f^{(n+1)}(t)dt,$$

where $f^{(n+1)}$ *denotes the* $(n+1)$*th derivative of* f.

Writing explicitly, the theorem states that,

$$F(x) = F(y) + f(y)(x-y) + \int_0^x f^{'}(u)(x-u)du.$$

Since the quadratic variation of *fBm* vanishes when $H > 1/2$, a function of *fBm* acts like a deterministic function in this formulation. Let us consider a sequence $T^n = t^{(n)}(m), m \geq 1$,

$n \geq 1$, for $t^n(m) \quad (0 = t^{(n)}(1) \leq t^{(n)}(2), \ldots)$, we have

$$\begin{aligned} F(B^H(t)) - F(B^H(0)) &= \sum_m [F(B^H(t \wedge t^{(n)}(m+1))) - F(B^H(t \wedge t^{(n)}(m)))] \\ &= \sum_m f(B^H(t \wedge t^{(n)}(m+1)))(B^H(t \wedge t^{(n)}(m+1)) - B^H(t \wedge t^{(n)}(m))) \\ &\quad + R^{(n)}(t), \end{aligned}$$

where

$$R^n(t) = \sum_m \int_{B^H(t \wedge t^{(n)}(m))}^{B^H(t \wedge t^{(n)}(m+1))} f^{'}(B^H(t \wedge t^{(n)}(m+1)) - u) du$$

and f is the first derivative of F. Then we expand the function F to *Taylor's* formula with $x = B^H_{t \wedge t^{(n)}(m+1)}$ and $y = B^H(t \wedge t^{(n)}(m+1))$, and obtain

$$\begin{aligned} F(B^H(t \wedge t^{(n)}(m+1))) - F(B^H(t \wedge t^{(n)}(m))) &= f(B^H(t \wedge t^{(n)}(m)))(B^H(t \wedge t^{(n)}(m+1)) \\ &\quad - B^H(t \wedge t^{(n)}(m))) + \\ &\quad \int_{B^H(t \wedge t^{(n)}(m))}^{B^H(t \wedge t^{(n)}(m+1))} f^{'}(u)(B^H(t \wedge t^{(n)}(m+1)) - u) du \end{aligned}$$

with $\mathbb{P}(\sup_{0 \leq u \leq t} |f^{'}(B^H(u))| < \infty) = 1$ from the bounded quadratic variation property of $f^{'}(B^H)$, and for $H \in (\frac{1}{2}, 1)$

$$\mathbb{P} - \lim_{n \to \infty} \sum_m |B^H(t \wedge t^{(n)}(m+1)) - B^H(t \wedge t^{(n)}(m))|^2 = 0,$$

to obtain an upper bound for $R^{(n)}(t)$. If we take the *supremum* of the derivative form of the remainder, we can use the limit above to obtain

$$|R^{(n)}(t)| \leq \frac{1}{2} \sup_{0 \leq u \leq t} |f^{'}(B^H(u))| \sum_m |B^H(t \wedge t^{(n)}(m+1)) - B^H(t \wedge t^{(n)}(m))|^2 \to^{\mathbb{P}} 0.$$

This is the same formula derived before, using the *Taylor's* series, taking the limit in probability of the pathwise integral of a function of *fBm*, we see that it has no quadratic variation part to complicate things. Taking the limit as the mash of the partition T^n going to zero, summing both sizes on m, and with $\mathbb{P} - \lim_n \sum_m f(B^H(t \wedge t^{(n)}(m)))(B^H(t \wedge t^{(n)}(m+1)) - B^H(t \wedge t^{(n)}(m)))$ existing, we obtain the Itô formula, for $H > \frac{1}{2}$,

$$\begin{aligned} & \mathbb{P} - \lim_n \sum_m F(B^H(t \wedge t^{(n)}(m+1))) - F(B^H(t \wedge t^{(n)}(m))) = \\ & \mathbb{P} - \lim_n \sum_m f(B^H(t \wedge t^{(n)}(m)))(B^H(t \wedge t^{(n)}(m+1)) - B^H(t \wedge t^{(n)}(m))) \\ & + \int_{B^H(t \wedge t^{(n)}(m))}^{B^H(t \wedge t^{(n)}(m+1))} f^{'}(u)(B^H(t \wedge t^{(n)}(m+1)) - u) du \\ = \quad & F(B^H(t)) - F(B^H(0)) = \int_0^t f(B^H(u)) dB^H(u). \end{aligned}$$

The stochastic integral is well-defined at least for functions $g = (g(u))$ of the type $g(u) = g(B^H(u))$, with $\int_0^t g(u)dS_\mu(u) = \int_0^t g(u)\mu du + \int_0^t g(u)dB^H(u)$. We can write this Itô type formula in the differential form as

$$dF(t, B^H(t)) = \partial_1 F(t, B^H(t))dt + \partial_2 F(t, B^H(t))dB^H(t).$$

Example 3.3.4 *Let us consider the function $Y(t) = e^{\mu t + B^H(t)}$ and apply the Itô formula for pathwise integration model.*

$$\begin{aligned} dY(t) &= d(e^{\mu t + B^H(t)}) = \mu e^{\mu t + B^H(t)}dt + e^{\mu t + B^H(t)}dB^H(t) \\ &= (\mu dt + dB^H(t))Y(t). \end{aligned}$$

Using this market model and Itô type formula, it can be shown that it is possible to generate arbitrage in a market model with *fBm*

Another Itô type formula was obtained by Dai and Hayde in [12]. Their formula permits the integrands to be stochastic processes but requires a number of restrictions on the integrands. For detailed information see [12]. Under these restrictions, they proved that for the following type of processes, their Itô type formula holds. Consider the process

$$X(t) = X(0) + \int_0^t a(s,\omega)ds + \int_0^t b(s,\omega)dB^H(s),$$

then the following holds;

$$Y(t) = Y(0) + \int_0^t [\frac{\partial U}{\partial t}(s, X(s)) + a(s,\omega)\frac{\partial U}{\partial x}(s, X(s))]ds + \int_0^t b(s,\omega)\frac{\partial U}{\partial x}(s, X(s))dB^H(s)$$

or in differential form

$$dY(t) = [\frac{\partial U}{\partial t}(t, X(t)) + a(s,\omega)\frac{\partial U}{\partial x}(t, X(t))]dt + b(t,\omega)\frac{\partial U}{\partial x}(t, X(t))dB^H(t),$$

where $U(t, x) : [0, T] \times \mathbb{R} \to \mathbb{R}$ is a two variable function with uniformly continuous partial derivatives $\frac{\partial U}{\partial t}$, $\frac{\partial U}{\partial x}$ and $\frac{\partial^2 U}{\partial x^2}$ and all partial derivatives are in $L^2(\mathbb{P})$. As we have seen before in other formulas, there is no quadratic variation term in their formula too.

For a stochastic differential equation driven by a *fBm* to have a unique solution, there are some conditions as in the standard Brownian motion case. The following theorem in [12] states these conditions:

Theorem 3.3.5 *Let $f(s, x)$ and g(s) be Borel functions such that*

1. $g : [0, \infty) \to \mathbb{R}$ *is bounded,*

2. $|f(s, x)| \leq K|x| + K$,

3. $|f(s, x) - f(s, y)| \leq K|x - y|$,

where K is a positive constant. Then the stochastic differential equation

$$\begin{aligned} dX(t) &= f(t, X(t))dt + g(s)dB^H(s) \\ X(t_0) &= A(\omega) \end{aligned}$$

has a unique solution whose paths are continuous. And if $f(s, X(t)) = \mu X(t)$ *and* $g(s) = \sigma X(t)$,*where* σ *and* μ *are constants, the solution is*

$$X(t) = A \exp\left(\mu(t - t_0) + \sigma(B^H(t) - B^H(t_0))\right),$$

where $t > t_0$ *and* $A(\omega)$ *is a positive random variable such that* $E|A(\omega)|^2 < \infty$.

In order to see this, we apply the Itô formula to SDE and define $S(t) = \exp(X(t))$. Then we have

$$\begin{aligned} dS(t) &= \mu \exp(X(t))dt + \sigma \exp(X(t))dB^H(t) \\ &= \mu S(t)dt + \sigma S(t)dB^H(t). \end{aligned}$$

Their proof is based on the same arguments as the other two Itô type formulas are based on. As we have seen, using the pathwise integration approximation, one can obtain an Itô type formula with no quadratic variation term. But for pricing an option, the Itô formula is not alone enough. In the following section, we will obtain the pricing formula for an option whose price process is assumed to follow a fractional SDE.

CHAPTER 4

Option pricing using *fBm*

Stochastic processes are used in financial modeling for more than a hundred years. The work of Bachelier [1] was the first attempt to use Brownian motion for financial modeling. But after 1973, when Black&Scholes published their work on option pricing [6] and derived their famous formula, using probability theory in finance has moved to a new level. In this thesis, we focus on the derivation of a Black&Scholes formula for the price of a European option where the underlying asset is assumed to follow a geometric fractional Brownian motion.

An option gives its holder the right, but not the obligation, to buy or sell a certain amount of a financial asset, by a certain date, for a certain strike price. There are two sides of this transaction. One party is the buyer of the option and the second party is the writer of the option. An option is specified by the following quantities:

- the type of the option: call option is the option to buy and the put option is the option to sell;
- the underlying asset: a stock, a bond, a currency, etc.;
- the amount of the underlying asset to be purchased or sold;
- the expiration date: an *American* option can be exercised at any time until maturity, while a *European* option can only be exercised at maturity;
- the exercise price which is the price for the transaction if the option is exercised

The price of the option is called the *premium*. Therefore, to price an option means to compute the *premium*. Let us have a look at the case of a European call option on a stock with maturity date T and strike price K. If we denote the price of the stock at T by $S(T)$, the buyer of the option makes a profit of $(S(T) - K)$ by exercising the option since he/she buys the stock for K which indeed has a price $S(T)$. Therefore, the value of the call at maturity is given by

$$\max((S(T) - K), 0).$$

How much value does this option have at $t = 0$? This question was answered by Black&Scholes, where $S(t)$ was assumed to follow a geometric Brownian motion. In the following section, we will see the computation of this value when the underlying asset is assumed to follow a geometric fractional Brownian motion.

4.1 Financial applications of *fBm*

Using a process which is not a semi-martingale for financial modeling causes some problems to be solved. The specific construction of *fBm* in the white noise space makes the real world financial interpretations of the integrals with respect to it difficult. Another difficulty is the possibility of arbitrage in the fractional markets. These problems are not solved at the moment but there are theoretical results of using *fBm* in finance. In the next subsection we will show how the fractional Black&Scholes price of a European call option was obtained and give arbitrage examples in the fractional markets.

4.1.1 The fractional Black&Scholes formula

We mainly follow [17], [24] and [31] for the derivation of the fractional Black&Scholes formula. We begin with the definition of the risk-neutral measure under which the discounted asset prices are martingales which means that the stock price process generates a riskless return equal to bank deposit rate r.

In the case of *fBm*, defining a risk-neutral measure is not trivial. The following *fractional* version of the Girsanov theorem in [24] gives the definition of the Radon-Nikodym derivative process in the *fractional white noise space*.

4.1.1.1 Fractional Girsanov theorem

Theorem 4.1.1 *Let $T > 0$ and γ be a continuous function with* $\mathrm{supp}\gamma \subset [0,T]$. *Let K be a function with* $\mathrm{supp}K \subset [0,T]$ *and such that,*

$$\langle K, g\rangle_\phi = \langle \gamma, f\rangle_{L^2(\mathbb{R})}, \qquad \forall f \in S(\mathbb{R}), \quad supp f \subset [0,T],$$

, i.e.,

$$\int_{\mathbb{R}} K(s)\phi(s,t)ds = \gamma(t), 0 \le t \le T.$$

Define a probability measure $\mu_{\phi,\mu}$ on the σ-algebra $\mathcal{F}_T^H$ generated by $B^H(s); 0 \le s \le T$ by

$$\frac{d\mu_{\phi,\mu}}{d\mu_\phi} = \exp^\diamond\left(-\langle\omega, K\rangle\right) = \exp(-\langle\omega, K\rangle + \frac{1}{2}\|K\|_H),$$

then $\tilde{B}^H(t) = B^H(t) + \int_0^t \gamma_s ds, \quad 0 \le t \le T$ is a fractional Brownian motion under $\mu_{\phi,\mu}$.

As we saw, the fractional analog of the geometric Brownian motion, the geometric fractional Brownian motion is obtained by using the white noise analysis concepts. Also, a fractional Itô formula is derived and a fractional Radon-Nikodym derivative process is defined. There are the basic tools one needs to compute the Black&Scholes price of an option. Following basically the approaches in [31] and [24], we give the price of a European call option. We remark that when $H > 1/2$, the *fractional WIS* integral and the *WIS* integral definitions coincide and the Itô formula for the *WIS* integral holds. Furthermore, the space $L_H^2(\mathbb{R})$ of deterministic functions also coincide [34].

Let $B^H(t)$, $0 \le t \le T$, be a *fBm* on a probability space $(\Omega, \mathcal{F}^H, \mathbb{P})$, and let $\mathcal{F}^H(t)$ be a filtration for this *fBm*. Let us consider a fractional SDE which is the stock price process

$$dS(t) = \mu S(t)dt + \sigma S(t)dB^H(t), \qquad 0 \le t \le T. \tag{4.1}$$

where the differential is in the *Wick* sense. We know that the solution of this equation can be obtained by applying the Itô formula for WIS integrals. We first change measure to risk-neutral measure to be able to compute the price of the option. Using the fractional Radon-Nikodym derivative process:

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = \exp\left(\int_{\mathbb{R}} \theta(s)dB^H(s) - \frac{1}{2}\|\theta\|^2_{L^2_H(\mathbb{R})}\right),$$

with

$$\int_{\mathbb{R}} \theta(s)\phi(s,t)ds = \frac{\mu - r}{\sigma},$$

then, under $\tilde{\mathbb{P}}$, the new process

$$\tilde{B}^H(t) := \frac{\mu - r}{\sigma} t + B^H(t),$$

is a *fBm*. We use this new *fBm* to write the price process under the risk-neutral measure $\tilde{\mathbb{P}}$:

$$dS(t) = rS(t)dt + \sigma S(t)d\tilde{B}^H(t), \qquad 0 \le t \le T,$$

where r is the risk-free rate and we are now in the risk-neutral world. Let us apply the Itô formula to solve this equation. Taking $f(S(t)) = \ln(S(t))$, which is a function of only $S(t)$ and applying the Itô formula yields

$$\begin{aligned} \ln(S(t)) &= \ln(S(0)) + \int_0^t \frac{1}{S(s)} dS(s) - H \int_0^t \frac{1}{S^2(s)} d[S,S](s) \\ &= \ln(S(0)) + \int_0^t (rds + \sigma d\tilde{B}^H(s)) - H \int_0^t \frac{1}{S^2(s)} \sigma^2 S^2(s) s^{2H-1} ds \\ &= \ln(S(0)) + (rt - \frac{1}{2}\sigma^2 t^{2H} + \sigma \tilde{B}^H(t)). \end{aligned}$$

Therefore, we can write

$$S(t) = S(0) \exp(rt - \frac{1}{2}\sigma^2 t^{2H} + \sigma \tilde{B}^H(t)).$$

Furthermore, we can apply Itô formula to obtain

$$S(T) = S(t) \exp(r(T-t) - \frac{1}{2}\sigma^2 (T^{2H} - t^{2H}) + \sigma(\tilde{B}^H(T) - \tilde{B}^H(t))), \tag{4.2}$$

which we will use to compute the price of a European option at t.

Now we present the *fractional Black&Scholes formula* as given in [17] , [31]. The formula looks the same as the classical Black&Scholes formula but there is a slight difference in the borders of the integration of the standard normal distribution denoted by d_1 and d_2.

Theorem 4.1.2 *The price of a European call option with strike price K and maturity T at time t is equal to:*

$$C(t, S(t)) = S(t)N(d_1) - Ke^{-r(T-t)}N(d_2), \tag{4.3}$$

where r is the risk-free interest rate and d_1 and d_2 are given by

$$d_1 = \frac{\ln(S(t)/K) + r(T-t) + \frac{1}{2}\sigma^2(T^{2H} - t^{2H})}{\sigma\sqrt{T^{2H} - t^{2H}}}$$

and

$$d_2 = d_1 - \sigma\sqrt{T^{2H} - t^{2H}} = \frac{\ln(S(t)/K) + r(T-t) - \frac{1}{2}\sigma^2(T^{2H} - t^{2H})}{\sigma\sqrt{T^{2H} - t^{2H}}}$$

Proof. The fundamental theorems of asset pricing state that, under the risk-neutral measure, discounted asset prices are martingales [42]). As we know, *fBm* is not a martingale but a *quasi*-martingale and we will use this property to obtain the price of an option. We can write,

$$\begin{aligned} C(t, S(t)) &= \tilde{E}_t[e^{-r(T-t)} \max((S(T) - K), 0)] \\ &= \tilde{E}_t[e^{-r(T-t)} S(T) I_{\{S(T)>K\}}] - K e^{-r(T-t)} \tilde{E}_t[I_{\{S(T)>K\}}]. \end{aligned}$$

For this expectation to have a value greater than 0, the following should hold

$$S(T) > K \Rightarrow \ln S + rT - \frac{\sigma^2}{2} T^{2H} + \tilde{B}^H(T) > \ln K$$

so let us denote

$$d_2^* = \frac{\ln(K/S) - rT + \frac{1}{2}\sigma^2 T^{2H}}{\sigma}.$$

Then, we see that the following holds:

$$\tilde{B}^H(T) > d_2^*.$$

Using (3.24) we obtain

$$\begin{aligned} \tilde{E}_t[I_{\{S(T)>K\}}] &= \tilde{E}_t[I_{\{B^H(T)>d_2^*\}}] \\ &= \int_{d_2^*}^{\infty} \frac{1}{\sqrt{2\pi(T^{2H} - t^{2H})}} \exp(-\frac{(x - \tilde{B}^H(t))^2}{2(T^{2H} - t^{2H})}) \\ &= \int_{\frac{d_2^* - \tilde{B}^H(t)}{\sqrt{T^{2H} - t^{2H}}}}^{\infty} \frac{1}{\sqrt{2\pi}} \exp(-\frac{z^2}{2}) dz \\ &= \int_{-\infty}^{\frac{\tilde{B}^H(t) - d_2^*}{\sqrt{T^{2H} - t^{2H}}}} \frac{1}{\sqrt{2\pi}} \exp(-\frac{z^2}{2}) dz \\ &= N(d_2), \end{aligned}$$

where $\frac{\tilde{B}^H(t) - d_2^*}{\sqrt{T^{2H} - t^{2H}}}$ is a standard normal variable and we see that the last line holds when we write the equations that define d^* and d_2 explicitly:

$$\begin{aligned} \frac{\tilde{B}^H(t) - d_2^*}{\sqrt{T^{2H} - t^{2H}}} &= \frac{\tilde{B}^H(t) - (\frac{\ln(S/K) - rT + \frac{\sigma^2}{2} T^{2H}}{\sigma})}{\sqrt{T^{2H} - t^{2H}}} \\ &= \frac{\sigma \tilde{B}^H(t) - \ln(K/S) + rT - \frac{\sigma^2 T^{2H}}{2}}{\sigma \sqrt{T^{2H} - t^{2H}}} = d_2 = \frac{\ln(S(t)/K) + r(T - t) - \frac{\sigma^2 (T^{2H} - t^{2H})}{2}}{\sigma \sqrt{T^{2H} - t^{2H}}}, \end{aligned}$$

where we obtain the following equality for $S(t)$

$$\ln(S(t)) = \ln S + rt - \frac{\sigma^2}{2} t^{2H} + \sigma \tilde{B}^H(t),$$

which is true for $S(t)$ and therefore the equation holds.

Now we can move to the second part to compute $\tilde{E}_t[S(T)I_{\{S(T)>K\}}]$. If we write

$$\tilde{E}_t[e^{-r(T-t)}S(t)e^{r(T-t)-\frac{\sigma^2}{2}(T^{2H}-t^{2H})+\sigma\tilde{B}^H(T-t)}I_{\{S(T)>K\}}],$$

we can use this to write this conditional expectation in the integral form since $S(t)$ is $\mathcal{F}^H(t)$ measurable and we know the distribution of $B^H(T-t)$. We have

$$S(t)\int_{-\infty}^{\infty} e^{-\frac{\sigma^2}{2}(T^{2H}-t^{2H})-\sigma\sqrt{T^{2H}-t^{2H}}z}\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz = S(t)\int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2}dy,$$

where

$$z = -\frac{\tilde{B}^H(T)-\tilde{B}^H(t)}{\sqrt{T^{2H}-t^{2H}}},$$

is a standard normal random variable and

$$y = z + \sigma\sqrt{T^{2H}-t^{2H}}.$$

Now let us investigate the borders of the integral. $(S(T)-K)_+$ is positive if and only if

$$S(T) > K \Leftrightarrow \frac{\ln(S(t)/K) + r(T-t) - \frac{\sigma^2}{2}(T^{2H}-t^{2H})}{\sigma\sqrt{T^{2H}-t^{2H}}} > z,$$

since $d_1 = d_2 + \sigma\sqrt{T^{2H}-t^{2H}}$, and $y = z + \sigma\sqrt{T^{2H}-t^{2H}}$, we have

$$S(t)\int_{-\infty}^{d_1} e^{-\frac{1}{2}y^2}dy = S(t)N(d_1),$$

which gives us the price of a European call option at t. ■

The key element in computing the fractional Black&Scholes price of an option is estimating the Hurst exponent H. We will summarize most widely used procedures for estimating H in the following chapter.

4.1.2 WIS portfolios

Using *Wick* calculus in finance has been questioned and it is shown that it is possible to build portfolios that can generate arbitrage. The approximation of Øksendal and Hu in [34] is not easy to apply nor interpret. They adopt the perspective of quantum mechanics, where the 'eye of the beholder' can effect the observations, to financial markets. One of the difficulties of their approximation comes into sight when one tried to build a numerical algorithm on it. The *test functions*, a key element in their modeling, are in the form of Radon-Nikodym derivative process, but to obtain the price of the asset, they use *Wick calculus*. We assume a market with one risk-free asset and a risky asset $S_0(t)$ and $S_1(t)$, respectively. The price dynamics of these assets are given as

$$dS_0(t) = rS_0(t)dt, S_0(0) = 1, \qquad (bankaccount)$$

and

$$dS(t) = \mu S(t)dt + \sigma S(t)dB^H(t), \quad S(0) = x > 0, \quad (stock)$$

where $r, \mu, \sigma \neq 0$ and $x > 0$ are constants and the stochastic integral with respect to B^H is in the *WIS* sense. We know that the solution of this SDE is a geometric fractional Brownian motion, namely,

$$S(t) = x \exp(\sigma B^H(t) + \mu t - \frac{1}{2}\sigma^2 t^{2H}), \qquad t \geq 0.$$

In their adaptation of quantum mechanical point of view, $S(t)$ does not represent the observed stock price at time t. Instead, it represents the total firm value that is not, and probably can not, be observed directly, but changes according to different market observers. I think this point of view also holds for the real world. If there were really one stock price that has been agreed upon, then which motive would drive the price processes and create price dynamics? In my opinion, if it was the case with the financial markets, then there would be no financial markets at all. In the *WIS* sense, this idea is modeled as follows:

First, we begin with regarding $S(t, \omega)$ as a stochastic distribution in ω, as an element of $(S)^*$. Then, the observed stock price, denoted as $\bar{S}(t)$ is obtained applying $S(t, .) \in (S)^*$ to a stochastic test function $\psi \in (S)$. Here, $S(t)$ denotes the generalized stock price. Following this interpretation, we see

$$\bar{S}(t) := \langle S(t, .), \psi(.) \rangle = \langle S(t), \psi \rangle,$$

where $\langle S(t), \psi \rangle$ denotes the action of a stochastic distribution $S(t, \cdot) \in (S)^*$ on a stochastic test function, $\psi(\cdot) \in (S)$. In this setting, stochastic test functions of the type of ψ are called the *market observers*. They are assumed to be in the form of a *Wick* exponential:

$$\begin{aligned} \psi(\omega) &= \exp^{\diamond}(\int_{\mathbb{R}} h(t)dB^H(t)) \\ &= \exp(\int_{\mathbb{R}} h(t)dB^H(t) - \frac{1}{2}\|h\|_H^2) \qquad \text{for some} \quad h \in L_H^2(\mathbb{R}). \end{aligned}$$

The set of all linear combinations of such ψ is dense in both (S) and $(S)^*$ and moreover they have a very specific property: they are normalized in the sense

$$E[\exp^{\diamond}(\int_{\mathbb{R}} h(t)dB^H(t))] = 1, \quad \forall h \in L^2_H(\mathbb{R}).$$

The definition of the portfolio value process is also defined by using *Wick* calculus. We are still in the *stochastic test functions space (S)* and *stochastic distribution space* $(S)^*$, the portfolio process is defined as in the case of the stock price process. So, if we can not directly observe the prices of a stock at time t directly, then we can not observe the value of a portfolio directly, neither. We need the definition of a *generalized portfolio*. A *generalized portfolio* is defined to be the adapted process

$$\theta(t) = \theta(t,\omega) = (\theta_0(t,\omega), \theta_1(t,\omega)), \qquad (t,\omega) \in [0,T] \times \Omega,$$

such that $\theta(t,\omega)$ is measurable with respect to $\mathcal{B}[0,T] \otimes F^H$, where $\mathcal{B}[0,T]$ is the Borel σ-algebra generated by $\{B^H(s)\}_{s\geq 0}$, and $\theta_0(t)$ is the fraction of wealth invested in the bank account and $\theta(t)$ is the fraction of wealth invested in the stock. In this setting, portfolio process is defined to be a function of both time and ω, a random variable depends on not only the realized path but takes into consideration all of the states that can be generated by the probabilistic element ω.

The perspective that led from *generalized stock price* to *observed stock price* also applied to the *generalized portfolio* process. Holding in mind that we defined the actual *observed price* at time t as $\bar{S}(t) = \langle S(t,.), \psi(.)\rangle$, then the actual *observed* number of stocks held in our portfolio process is given by

$$\bar{\theta}(t) := \langle \theta(t,\cdot), \psi(\cdot)\rangle.$$

According to this definition, the actual *observed wealth* held in the risky asset at t, denoted $\bar{U}(t)$, is defined by

$$\bar{U}(t) = \bar{\theta}(t)\bar{S}(t) = \langle \theta(t,\cdot), \psi(\cdot)\rangle\langle S(t), \psi\rangle.$$

The *generalized total wealth process* $U(t)$ is also defined by using the *Wick* product. The *Wick* product reduces to ordinary product for deterministic functions, so we have

$$U(t,\cdot) = \theta(t,\cdot) \diamond S(t,\cdot) = \theta_0(t)S_0(t) + \theta(t) \diamond S(t).$$

If we consider a partition of an interval of time $[0, T]$, t_k, and consider a discrete time market model then the *generalized portfolio wealth* at t_k will be

$$\theta(t) = \theta(t_k, \omega), \quad \text{where} \quad t_k \leq t < t_{k+1},$$

and the change in the *generalized wealth process* between t_k and t_{k+1} is

$$\triangle U(t_k) = \theta(t_k) \diamond \triangle S(t_k).$$

If we take the sum of the two sides over k and take the limit as the mash of the partition goes to zero, we obtain the continuous time equation of the *generalized wealth process*, which is

$$U(T) = U(0) + \int_0^T \theta(t) \diamond dS(t) = U(0) + \int_0^T \theta(t) dS(t).$$

Writing the equations for $dS(t)$ and $dS_0(t)$ the self-financing property is obtained

$$U(T) = U(0) + \int_0^T r\theta_0(t) S_t dt + \int_0^T \mu\theta(t) \diamond S(t) dt + \int_0^T \sigma\theta(t) \diamond S(t) dB^H(t).$$

Definition 4.1.3 *A* generalized portfolio $\theta(t)$ *in the* WIS *model is called* WIS self-financing *if*

$$dU^\theta(t) = \theta(t) dS(t),$$

or, explicitly,

$$U^\theta(t) = U^\theta(0) + \int_0^t \theta_0(s) dS_0(s) + \int_0^t \theta(s) dS(s),$$

where the integral with respect to the stock prices process $S(t)$ is a WIS integral, under the assumption that the two integrals exist. A generalized portfolio is called WIS admissible if it is WIS self-financing and $\theta(s) \diamond S(s)$ is Skorohod integrable.

In order to see the self-financing property, we will use the risk-neutral measure which is defined by the Girsanov theorem for *fBm*. There are several approaches to obtain a Girsanov type formula for *fBm*. This one is obtained for $0 < H < 1$ in [17] as an adaptation of the classical Girsanov theorem.

Let us consider the SDE of a risky asset S(t),

$$dS(t) = \mu S(t) dt + \sigma S(t) dB^H(t),$$

where the drift term μ is interpreted as the riskless rate of return on this asset and is different from the riskless deposit rate r. From a financial mathematical point of view, the classical

Girsanov theorem provides a useful tool to find equivalent measures to obtain a specific process which neutralizes the risk of an underlying asset generating a return that is different than the riskless deposit rate r. Theorem uses the Radon-Nikodym derivative process to change the measure in the following way:

Let $\tilde{\mathbb{P}}$ and $\mathbb{P}$ be two equivalent measures in the sense that they have the same negligible sets, then Radon-Nikodym density is defined by

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = \exp\Big(\int_{\mathbb{R}} \psi(s)dB(s) - \frac{1}{2}||\psi||^2_{L^2(\mathbb{R})}\Big),$$

then, the Girsanov theorem states that the process $\tilde{B}(t)$ defined by

$$\tilde{B}(t) := B(t) - \int_0^t \psi(s)ds,$$

is a standard Brownian motion under $\tilde{\mathbb{P}}$. Now we remember the definition of *fBm* in terms of Brownian motion and try to obtain a standard *fBm* under this new measure $\tilde{\mathbb{P}}$. If we define the process $\tilde{B}^H$

$$\tilde{B}^H(t) := \int_{\mathbb{R}} M[0,t](s)d\tilde{B}(s),$$

then it is a standard *fBm* under $\tilde{\mathbb{P}}$. We can see that

$$\tilde{B}^H(t) = B^H(t) - \int_{\mathbb{R}} M[0,t](s)\psi(s)ds.$$

In order to obtain a specific drift value, the riskless rate of return r in our case, we must solve the following equation:

$$\begin{aligned} S(t) &= S(0) + \int_0^t \mu S(s)ds + \int_0^t \sigma S(s)dB^H(s) \\ &= S(0) + \int_0^t rS(s)ds + \int_0^t \sigma S(s)d\tilde{B}^H(s), \end{aligned}$$

where we obtain

$$\int_0^t \mu S(s)ds = \int_0^t rS(s)ds + \sigma \int_0^t S(s)M\psi(s)ds.$$

If we write this equation in the differential form

$$\mu S(t)dt = rS(t)dt + \sigma S(t)M\psi(t)dt,$$

finally we see that

$$M\psi(t) = \frac{\mu - r}{\sigma}t.$$

It is shown in [17], using the following property of the operator M,

$$M^{-1}[0,t] = M_{1-H}[0,t],$$

it is possible to see the explicit form of $\psi(t)$:

$$\psi(t) = \frac{(\mu - r)[(T - t)^{1/2-H} + t^{1/2-H}])}{2\sigma\Gamma(3/2 - H)\cos[\pi/2(1/2 - H)]},$$

then, $\tilde{B}^H(t)$ defined by

$$\tilde{B}^H(t) := \frac{\mu - r}{\sigma}t + B^H(t),$$

is a standard fractional Brownian motion under $\tilde{\mathbb{P}}^H$. When we write the self-financing condition under the new measure $\tilde{\mathbb{P}}^H$ by using the new *fBm* $\tilde{B}^H(t)$ instead of $B^H(t)$, as done in the standard Brownian motion case, our SDE becomes

$$U^\theta(t) = U^\theta(0) + \int_0^t r\theta_0(s)S_0(s)ds + \int_0^t r\theta(s) \diamond S(s)ds + \int_0^t \sigma\theta(s) \diamond S(s)dB^H(s)$$

and, in differential form,

$$dU^\theta(t) = r\theta_0(t)S_0(t)dt + r\theta(t) \diamond S(t)dt + \sigma\theta(t) \diamond S(t)dB^H(t).$$

Now we can compute the discounted wealth process as follows:

$$\begin{aligned} d(e^{-rt}U^\theta(t)) &= -re^{-rt}U^\theta(t)dt + e^{-rt}dV^\theta(t) \\ &= -re^{-rt}(\theta(t) \diamond S(t))dt + e^{-rt}(r\theta_0(t)S_0(t)dt + r\theta(t) \diamond \S(t)dt + \sigma\theta(t) \diamond S(t)d\tilde{B}^H(t)) \\ &= e^{-rt}\sigma\theta(t) \diamond S(t)d\tilde{B}^H(t). \end{aligned}$$

Finally, we give the definition of strong arbitrage in the *WIS* market:

Definition 4.1.4 *A* WIS *admissible portfolio $\theta(t)$ is called a* strong arbitrage *if the* generalized total wealth process $U^\theta(t)$ *satisfies*

$$\begin{aligned} &U^\theta(0) = 0, \\ &U^\theta(T) \in L^2(\tilde{\mathbb{P}}^H) \qquad \textit{and}, \\ &U^\theta(T) \geq 0 \qquad \textit{a.s.}, \\ &\tilde{\mathbb{P}}^H(U^\theta(T) > 0) > 0, \end{aligned}$$

Taking the expectation with respect to risk-neutral measure $\tilde{\mathbb{P}}^H$ we see that

$$e^{-rT}\tilde{E}[U^\theta(T)] = U^\theta(0),$$

so we see that there is no strong arbitrage.

Although in [34], it is stated that the *WIS* model being free of *strong arbitrage* is not in conflict with *fBm* not being a martingale, according to [4], arbitrage is possible in fractional markets because the *Wick* product is used in the definition of the portfolio value process. But, there is something that leads to the opposition of using the *Wick* product for the definition of a portfolio value process. After the *Wick* product being applied, the *generalized total wealth process* becomes a function only of time; because the *Wick* product takes all the possible realizations to consideration,that is, it can not be calculated pathwisely. As a consequence, one must take into consideration all the states of nature in order to be able to compute the value of *generalized total wealth process* at a specific time *t*. This leads to some unwanted situations such as a portfolio consisting of a positive amount of stocks having a negative value, as shown in [4].

4.1.3 Arbitrage in *fBm* models

One of the most important concepts in financial economics is arbitrage. In real world terms, it means that there are more than one price for the same financial asset. This is not interesting for practitioners since this is the basic motivation for the financial corporations to be founded. But in financial mathematics, the existence of arbitrage in a model makes it impossible to define a so-called equilibrium price of the asset. *fBm* is one of these models that causes arbitrage and therefore it is forbidden until there is a solution to this problem. Bjork and Hult investigates the definitions of self-financing and arbitrage portfolios in detail in [4]. They argued that the self-financing condition is a fundamental concept in financial economics that depends on the stochastic integral concept used to construct it. It is stressed that replacing the Itô integral with Wick integral is not a proper way of defining a new self-financing condition, and may result in ‘nonsense’.

It is also shown that when replacing the Itô self-financing condition with Wick self-financing condition, may cause some serious problems, even breaking trading laws. To see this they consider a market consisting of

$$dS(t) = S(t) \diamond dW^H(t)$$

and a bank account with zero interest rate,i.e., R(t)=1. If we consider a portfolio value process

V associated with the portfolio $h = (h_0, h_1)$, then we have

$$dV^h(t) = h_0(t)R(t) + h_1 S(t) = h_0(t) + h_1(t)S(t)$$

with ordinary products and let us recall the *Wick* self-financing condition

$$dV^h(t) = h_0(t)dR(t) + h_1(t)S(t) \diamond dW^H(t)$$

since in this setting $R(t) = 1$, we see that

$$dV^h(t) = h_1(t)S(t) \diamond dW^H(t).$$

Let us consider two stopping times $t_0 \leq t_1$. The portfolio that is constructed with ordinary product satisfies

$$V(t_1) - V(t_0) = h(t_0)(S(t_1) - S(t_0)).$$

and the Wick based portfolio satisfies

$$V(t_1) - V(t_0) = \int_{t_0}^{t_1} h(t_0)S(u) \diamond dW^H(u).$$

As we mentioned before, Wick product is a product between random variables, therefore

$$\int_{t_0}^{t_1} h(t_0)S(u) \diamond dW^H(u)$$

does not in general coincide with

$$h(t_0) \int_{t_0}^{t_1} S(u) \diamond dW^H(u)$$

causing the Wick self-financing condition to differ from the standard self-financing condition. They also construct simple a portfolio strategy which is self-financing in the standard sense but not in the Wick sense. A portfolio strategy with initial capital $c > 0$. Putting all the money in the bank account at $t = 0$ and holding it there until $t = 1$ where the rate is equal to zero, resulting in no change in our wealth. Then, at $t = 1$, buy $\frac{c}{S(1)}$ shares with all your money,where $S(1)$ is the price of the risky asset at $t = 1$, and hold this position until $t = 2$. Then the value of this portfolio at $t = 2$ is

$$V(2) = \frac{c}{S(1)} S(2).$$

Since, by construction, this portfolio is self-financing, i.e. no capital added or withdrawn between $t = 0$ and $t = 2$, the definition of Wick self-financing must include this strategy. Then it is shown that

$$c\frac{S(2)}{S(1)} \neq c + \int_0^2 h^1(u)S(u) \diamond dW^H(u),$$

where $h^0(t) = I_{(0,1]}(t)$ and $h^1(t) = \frac{c}{S(1)} I_{(1,2]}$. Taking expectation, we see that

$$E(x + \int_0^t h^1(u) S(u) \diamond dW^H(u)) = c,$$

whereas

$$\begin{aligned} E(\frac{c}{S(1)} S(2)) &= xE(\exp(W^H(2) - \frac{2^{2H}}{2}) \exp(-W^H(1) + \frac{1^{2H}}{2})) \\ &= c \exp(-\frac{1}{2}(2^{2H} - 1)) E(\exp(W^H(2) - W^H(1))) \\ &= c \exp(-\frac{1}{2}(2^{2H} - 1)) x \exp(\frac{1}{2}|2 - 1|^{2H}) \\ &= c \exp(1 - 2^{2H-1}) \neq c, \end{aligned}$$

when $H \neq \frac{1}{2}$. As we see, the expectation is equal to c only when $H = \frac{1}{2}$, the standard Brownian motion case. Although these results are discouraging, there are some important points that should be taken into consideration before deciding whether the *fBm* is a suitable model for finance or not. One of them is the transaction costs. In real world, there are transaction costs unlike the theoretical world. In [21], it is shown that geometric *fBm* model is free of arbitrage under transaction costs, of any magnitude. The other point is transaction time. Although high-frequency trading is becoming more popular as time goes on, continuous trading is still impossible. In [8], arbitrage is excluded by introducing a minimal amount of time $h > 0$ that must lie between two consecutive transactions.

Since these work show that it is possible to exclude arbitrage from *fBm* models by some realistic assumptions, it looks possible to build a proper model in financial mathematics using *fBm*. Future work may solve these problems and *fBm* may then replace the standard Brownian motion in financial mathematics.

CHAPTER 5

Estimation and Simulation

5.1 Statistical aspects of the *fBm*

The main motivation behind using *fBm* in modeling is to use its flexible covariance structure to capture the covariance structure of the data. This can only be done by estimating the Hurst exponent H. The Hurst exponent can be estimated using the statistical properties that are specific to *fBm*. Since *fBm* is a special case among Gaussain random variables, we can use the modification of the basic results to obtain an estimate of the self-similarity parameter H. Let us begin with the usual assumptions and results used in the statistical theory. One of them is : *The variance of the sample mean is equal to the variance of one observation divided by the sample size*, that is,

$$\text{var}(\overline{X}) = \sigma^2 n^{-1}, \tag{5.1}$$

where $X_1, \ldots, X_n$ are observations with common mean $\mu = E(X_i)$, σ^2 =var$[X_i] = E[(X_i - \mu)^2]$ and $\overline{X} = n^{-1} \sum_{i=1}^{n} X_i$. This result can only be obtained under assumptions numbered below:

1. The population mean $\mu = E[X_i]$ and the population variance $\sigma^2 = E[X_i]$ exists and finite.

2. $X_1, \ldots, X_n$ are uncorrelated, that is

$$\rho(i, j) = 0$$

for $i \neq j$, where

$$\rho(i, j) = \frac{\gamma(i, j)}{\sigma^2}$$

is the autocorrelation between X_i and X_j, and

$$\gamma(i, j) = E[(X_i - \mu)(X_j - \mu)]$$

is the autocovariance between X_i and X_j. The first assumption seems to be required for being able to use Gaussian distributions in statistics and depend on the distribution function F. But the second assumption, in the case of *fBm*, does not hold because of its long-memory property, although it is a Gaussian process with finite first two moments. One of the questions arose is what happens when the second assumption does not hold? Will the other assumptions be affected? As we will see, when this assumption does not hold, the decay rate of the variance of the sample mean changes.

For $\overline{X}$ to be meaningful, $E[X_i] = \mu$ is assumed to be constant. The variance of $\overline{X} = n^{-1} \sum_{i=1}^{n} X_i$ is equal to

$$\text{var}[\overline{X}] = n^{-2} \sum_{i,j=1}^{n} \gamma(i, j) = n^{-2}\sigma^2 \sum_{i,j=1}^{n} \rho(i, j)$$

and if the correlations for $i \neq j$ sum up to zero, then

$$\sum_{i,j=1}^{n} \gamma(i, j) = \sum_{i,j=1}^{n} \gamma(i, i) = \sum_{i=1}^{n} \sigma^2,$$

and (0.1) holds since $\rho(i, i) = 1$. But if

$$\sum_{i=1}^{n} \rho(i, j) \neq 0,$$

then we have

$$\begin{aligned} var(\overline{X}) &= n^{-2}\sigma^2 \sum_{i,j=1}^{n} \rho(i, j) = n^{-2}\sigma^2(n + \sum_{i \neq j} \rho(i, j)) \\ &= \sigma^2 n^{-1}(1 + \nu_n(\rho)), \end{aligned}$$

where $\nu_n(\rho)$ is the non-zero correction term and the first assumption does not hold [3]. But as the number of observations n goes to infinity,

$$\nu(\rho) = \lim_{n\to\infty} \nu_n(\rho) = \lim_{n\to\infty} n^{-1} \sum_{i \neq j} \rho(i, j)$$

exists and greater than -1. Then we asymptotically have

$$var(\overline{X}) \approx \sigma^2 n^{-1}[1 + \nu(\rho)] = \sigma^2 n^{-1} c(\rho).$$

where $c(\rho)$ is constant. This relation is first realized by Edwin Hurst in [25] and then several estimation techniques has been developed based on this relation.

5.2 Estimation of H

As we have seen, long term behavior of a process can be modeled by *fBm* . There are different estimation techniques that can be used to estimate the parameter H. We begin with the estimation of H under the assumption that the data under consideration is a sample path of a *fBm*.

5.2.1 The *R/S* statistic

The Nile River has been a great inspiration for people since the early ages, especially its floods. It has a characteristic long-term behavior which we know define to be ‘persistent’. Long periods of floods were followed by long periods of drought. One interesting thing is that there is reasonably reliable historical data ranging from 622 A.D. to 1281 A.D. There were long periods of which the maximal level tended to stay high and on the other hand long periods of low levels, but the overall series look stationary. These characteristics has drawn the attention of Edwin Hurst, who was a hydrologist, when he was trying to find a way to regularize the flow of the Nile River. He has done a statistical discovery by empirically generating a biased sequence of random draws, using playcards. Suppose we want to calculate the capacity of a reservoir of the ideal capacity for the time interval $(t, t+k)$, assuming the time is discrete and there is no storage losses. The ideal capacity is defined as follows: the outflow is uniform, at time $t+k$ the reservoir is as full as it was at time t and the reservoir never overflows. When X_i denote the inflow at time i and $Y_j = \sum_{i=1}^{j} X_i$ is the cumulative inflow up to time j, the ideal capacity can be shown to be equal to

$$\begin{aligned} R(t,k) &= \max_{0\le i\le k}[Y_{t+i} - Y_t - \frac{i}{k}(Y_{t+k} - Y_t)] \\ &\quad - \min_{0\le i\le k}[Y_{t+i} - Y_t - \frac{i}{k}(Y_{t+k} - Y_t)], \end{aligned}$$

$R(t,k)$ is called the adjusted range. It is standardized to be able to study the properties which are independent of the scale. The scale coefficient $S(t,k)$ is given by

$$S(t,k) = \sqrt{k^{-1} \sum_{i=t+1}^{t+k} (X_i - \overline{X}_{t,k})},$$

where $X_{t,k} = k^{-1} \sum_{i=t+1}^{t+k} X_i$, $S^2(t,k)$ is equal to $\frac{k-1}{k}$ times the sample variance of $X_{t+1}, \ldots, X_{t+k}$. Then, the ratio

$$R/S = \frac{R(t,k)}{S(t,k)}$$

is the *rescaled adjusted range* or *R/S* statistic. When Hurst plotted the logarithm of this statistic against several values of k, he observed that for large values of k, $\log R/S$ was scattered around a straight line with slope that is grater than $\frac{1}{2}$. In probabilistic terminology,

$$\log E[R/S] \approx a + H \log k, \qquad \text{with} \qquad H > \frac{1}{2}.$$

The slope exceeding $\frac{1}{2}$ was in contradiction with the assumption that the underlying process was Markov with independent increments. Since the R/S statistic is based on cumulative sums and sample variance, it should satisfy the basic statistical results used in modeling, which we argued in the beginning of this subsection, and H would be close to $\frac{1}{2}$. At first, this looked like a special case for the Nile River, but after Hurst's discovery, many natural records has been shown to act in a similar way to the Nile River, with R/S statistic for some $H > \frac{1}{2}$, and this situation has began to be known as the *Hurst* effect. *fBm* was built by Mandelbrot in order to model the *Hurst* effect, or the Noah effect in Mandelbrot's terms. For *fBm*, it is known as the 'persistence' case. Mandelbrot applied the R/S analysis to financial time series such as interest rates, commodity prices and stock market data and found evidence of persistence, or long-memory in series. He thought this was the evidence of arbitrage in the market, as now is the case for *fBm*. Let us denote $Q = Q(t,k) = R(t,k)/S(t,k)$ then the following theorem in [30] can explain why $Q(t,k)$ is useful in terms of detecting the long-range dependence:

Theorem 5.2.1 *Let X_t be such that X_t^2 is ergodic and $t^{-\frac{1}{2}} \sum_{s=1}^{t} X_s$ converges weakly to Brownian motion as t tends to infinity. Then, as $k \to \infty$,*

$$k^{-\frac{1}{2}} Q \to_d \zeta,$$

where ζ is a nondegenerate random variable.

Let us assume the central limit theorem holds for the process X_t. Then it can be seen that since the R/S statistic Q is obtained by subtracting the weighted average from the cumulative sum and then scaling with the sample variance times a constant, $k^{\frac{1}{2}} Q$ should converge to a well-defined random variable, a Gaussian for instance. For long-memory processes, the slope of the plot of $\log(R/S)$ versus $\log k$, expected to be $H > 1/2$, for sufficiently large lags k.

Mandelbrot states that the asymptotic behavior of the R/S remains unaffected even in the case of α-stable processes with infinite variance. The algorithm of the R/S method can be summarized as follows:

1. Calculate Q for a sufficient number of different values of t and k.
2. Plot $\log Q$ against $\log k$.
3. Estimate the regression coefficients. Then, the estimate of H is the slope coefficient of the regression.

In terms of application, there arise some difficulties like: How to decide the value of k that the asymptotic behavior of the process starts? Is there a bias in the estimate of H? Is linear regression the proper tool for this estimation? These problems make it difficult to use and interpret the R/S statistic. However, the R/S statistic is useful in getting a first idea about the dependence structure of the data.

5.2.2 The Correlogram

The correlogram is a standard method in time series analysis. It is based on plotting the correlations against the lag k, where

$$\hat{\rho}(k) = \frac{\hat{\gamma(k)}}{\hat{\gamma}(0)}$$

is the sample correlations. When one draws two horizontal lines at the levels $\pm 2/\sqrt{n}$, correlations outside this lines are considered significant at 0.05 level, since this is the confidence interval for sample correlations as a limit case [36]. But this is the case when the sample under consideration is uncorrelated. If this is not the case, the significance level would differ from the uncorrelated level. A more suitable method can be obtained by takin the logarithm of both sides to linearize the relationship between k and $\rho(k)$ as the limit case. For a long-memory process, we have seen that the decay rate of correlations is k^{2H-1} (k^{2d-1}) as $k \to \infty$. This property is used to estimate H by plotting $\log|\rho(k)|$ against $\log k$ and using, again, linear regression. When the asymptotic decay of the correlations is hyperbolic, the slope coefficient is close to $2H-2$. The correlogram method does not give meaningful results when k is small or when H is close to $\frac{1}{2}$. One can also use the asymptotic decay rate of partial correlations,$k^{-H-\frac{1}{2}}$ to estimate H and this method would also have the same difficulties in application terms.

5.2.3 Variance Plot method

The third term we will mention is again based on the logarithmic plots, but uses another property of long-range dependent processes. As we have seen, one of the implications of the long-memory can be seen by looking at the variance of the sample mean. From theorem 2.2 in [3], we have

$$\mathrm{var}(\overline{X}_n) \approx cn^{2H-2},$$

where $c > 0$ is a constant. Using this relation, the following method is defined:

1. Let k be an integer with $2 \leq k \leq n/2$ and a sufficient number of sub-samples l_k of length k, calculate the sub-sample means $X_1(k), X_2(k), \ldots, X_{l_k}(k)$ and the overall mean
$$\overline{X}(k) = l_k^{-1} \sum_{j=1}^{l_k} \overline{X}_j(k).$$
2. For each k, calculate the sub-sample variances $s^2(k)$:
$$s^2(k) = (l_k - 1)^{-1} \sum_{k=1}^{l_k} (\overline{X}_j(k) - \overline{X}(k)).$$
3. Plot $s^2(k)$ against $\log k$.

This method also has the same difficulties and drawbacks as the first two methods mentioned. Again, we use the slope of the plot to obtain an estimation of H.

5.2.4 Absolute moments method

This method is a generalization of the variance plot method. The basic idea is,again, using the self-similarity property of *fBm*. See [15] for references. The quantity

$$ABs_m = \frac{1}{M} \sum_{i=0}^{M-1} |X_i^{(m)} - \overline{X^{(m)}}|^n$$

for $n = 1, 2, \ldots$. From the asymptotic behavior of the variance, the following relation can be shown to be satisfied for *fBm*

$$\begin{aligned} E|X_i^m - E[X_i^m]|^n &\sim cM^{H-1} \\ &= m^{n(H-1)} cM^{H-1} \end{aligned}$$

for big m as $M \to \infty$, and c is an appropriate constant. These relations are not fully proven to hold. But it is known that $E[ABs_m]$ is proportional to $m^{n(H-1)}$ [15]. This method is generally used with $n = 1$, and when $n = 2$, it is the same method with variance plot method.

5.2.5 Variance of the regression residuals

This method has been proposed by Peng in [35]. Method based on dividing the sample into blocks of size m and then linearly regress the series on a line $\alpha^k + \beta^k i$. After the residuals

$$\epsilon_i^k = \sum_{j=km}^{km+i-1} X_j - \alpha^k + \beta^k i,$$

are obtained, their variance is computed for each block. The average of this sample variance over all blocks are plotted versus m, again, on a log-log scale. When linear regression is used, the slope coefficient is equal to $2H$.

5.2.6 Periodogram method

This method is based on the idea of detecting long-range dependence in the frequency domain. The estimation procedure begin with computing the periodogram of the sample. It is shown in [18] that the periodogram is an unbiased estimator of the spectral density. Periodogram is defined by

$$I(\lambda) = \sum_{-(N-1)}^{N-1} \hat{\gamma}(j) \exp(ij\lambda),$$

where $\hat{\gamma}(j)$ is the sample autocovariance computed as

$$\hat{\gamma}(j) = \frac{1}{N} \sum_{k=0}^{N-|j|-1} (X_k - \bar{X})(X_{k+|j|} - \bar{X}).$$

It is shown in [22] that

$$I(\lambda) = \frac{1}{N} \Big| \sum_{k=0}^{N-1} (X_k - \bar{X}) \exp(ik\lambda) \Big|^2$$

The periodogram is symmetric around zero, as the spectral density is. The periodogram is asymptotically an unbiased estimator of the spectral density $s(\lambda)$ [5], that is,

$$\lim_{N \to \infty} E[I(\lambda)] = s(\lambda)$$

We compute the periodogram of the sample to investigate the behavior of the spectral density near the origin. Then we can use the relation:

$$s(\lambda) \sim c_s|\lambda|^{1-2H} \qquad (|\lambda \to 0),$$

which can be written as

$$\log s(\lambda) \sim \log c_s + (1-2H)\log|\lambda|.$$

The $I(\lambda)$ is usually calculated at the Fourier frequencies

$$\lambda_{k,n} = \frac{2\pi k}{n}, \qquad k = 1, \ldots, n^*,$$

where n^* is the integer part of $(n-1)/2$, to obtain an estimate of H. For processes that do not exhibit long-range dependence, the periodogram ordinates at Fourier frequencies are exponentially independent random variables with means $s(\lambda_1), \ldots, s(\lambda_k)$ [3]. Then the following relation holds approximately

$$\log I(\lambda_{k,n}) \approx \log c_s + (1-2H)\log\lambda_{k,n} + \log\eta_k,$$

where η_k are independent standard exponential random variables, with $E[\log\eta] = -C = -0.577215\ldots$, where C is the Euler constant [3]. If we define

$$y_k = \log I(\lambda_{k,n}), \quad x_k = \log\lambda_{k,n}, \quad \beta_0 = \log c_s - C, \quad \beta_1 = 1-2H$$

and the error terms as

$$e_k = \log\eta + C,$$

then we can write

$$y_k = \beta_0 + \beta_1 x_k + e_k.$$

Geweke-Porter-Hudak suggested applying a least squares regression procedure to estimate H in [18]. Their method is based on the following relation

$$\hat{H} = \frac{1-\hat{\beta}_1}{2}.$$

Although this method is computationally effective, there are some drawbacks of the method. First of all, the desired behavior of the spectral density, proportionality to λ^{1-2H} can be detected only around a small neighborhood of frequency zero and this makes it more difficult for the method to capture the asymptotically defined notion of long-memory. For details, see 3. Another way for estimating H is to use the discrete time analog of *fBm*: Fractionally integrated ARMA models. Using these processes, it is possible to construct maximum likelihood estimation procedures to estimate H. We will give the definition and basic properties of these models in the following subsection.

5.3 Fractionally Integrated ARMA Models

The classical ARMA models are based on the stationarity assumption of the series. In [5], integrated time series models are introduced to model nonstationary time series. This models later used to construct fractionally integrated time series models, which is the discrete time analog of *fBm*. In the continuous setting, the derivative of the *fBm*, the fractional white noise process,$W^H(t)$ is defined to be the $(\frac{1}{2} - H)$th fractional derivative of the white noise $W(t)$. Using this definition, the discrete time analogue of $B^H(t)$ and $W^H(t)$ is defined in the following way. First we remark that the discrete-time analogue of Brownian motion is the random walk process $\{y_t\}$, defined by

$$\nabla y_t = (1 - B)y_t = a_t,$$

where B is the backshift operator defined by $By_t = y_{t-1}$ and $\{a_t\}$ are independent identically distributed random variables with zero mean and unit variance. The first difference of $\{y_t\}$ is the discrete time white noise process $\{a_t\}$. In time-series modeling, the process $\{y_t\}$ is said to be an integrated process of order d. The operator $(1 - B)$ is called the difference operator. Since integrated processes is used to construct the discrete-time analog of Brownian motion and white noise processes, we can proceed by using fractional integration to obtain a discrete analogue of *fBm*. Based on the definition of $W^H(t)$, discrete time analog of this process is defined to be the $(\frac{1}{2}-H)$th fractional difference of the discrete-time white noise. The fractional difference operator ∇^d defined by the binomial series:

$$\nabla^d = (1 - B)^d = \sum_{k=0}^{\infty} \binom{n}{k} (-B)^k = 1 - dB - \frac{1}{2}d(1 - d)B^2 - \frac{1}{6}d(1 - d)(2 - d)B^3 - \ldots,$$

When we write $d = H - \frac{1}{2}$, the discrete-time analogue of $W^H(t)$ is the process $y_t = \nabla^{-d}a_t$, fractionally integrated (summed) discrete-time white noise or $\nabla^d y_t = a_t$ fractionally differenced discrete-time Brownian motion. $\{x_t\}$ is called an ARIMA(0,d,0) process, an extension of ARIMA processes to noninteger d.

5.3.1 The fractional ARIMA(0,*d*,0) process

This process is formally defined in [23] as the discrete-time stochastic process $\{y_t\}$ which may be represented as

$$\nabla^d y_t = a_t,$$

where the $\{a_t\}$ is i.i.d. with mean zero and $\sigma_a = 1$ for convenience. The following theorem gives the basic properties of this process

Theorem 5.3.1 *1. When $d < 1/2$ ($H < 1$), $\{y_t\}$ is a stationary process and has the infinite moving-average representation*

$$y_t = \theta(B)a_t = \sum_{k=0}^{\infty} \theta_k a_{t-k},$$

where

$$\theta_k = \frac{d(1+d)\ldots(k-1+d)}{k!} = \frac{(k+d-1)!}{k!(d-1)!}$$

as $k \to \infty$, $\theta_k \sim k^{d-1}/(d-1)!$.

2. *When $d > -1/2$ ($H > 0$), y_t is invertible and has the infinite autoregressive representation*

$$\pi(B)y_t = \sum_{k=0}^{\infty} \pi_k y_{t-k} = a_t,$$

where

$$\pi_k = \frac{-d(1-d)\ldots(k-1-d)}{k!} = \frac{(k-d-1)!}{k!(-d-1)!}$$

as $k \to \infty$, $\pi_k \sim k^{-d-1}/(-d-1)!$.

3. *The spectral density of y_t is*

$$s(\lambda) = (2\sin\frac{1}{2}\lambda)^{-2d} \quad \textit{for} \quad 0 < \omega \leq \pi \quad \textit{and} \quad s(\lambda) \sim \lambda^{-2d} \textit{as} f \to 0.$$

4. *The covariance function of $\{y_t\}$ is*

$$\gamma_k = E(y_t y_{t-k}) = \frac{(-1)^k(-2d)!}{(k-d)!(-k-d)!}$$

and the correlation function of $\{y_t\}$ is

$$\rho_k = \gamma_k/\gamma_0 = \frac{(-d)!(k+d-1)!}{(d-1)!(k-d)!}, \qquad \textit{for} \quad k = 0, \pm 1, \ldots,$$

$$\rho_k = \frac{d(d+1)\ldots(k-1+d)}{(1-d)(2-d)\ldots(k-d)}, \qquad \textit{for} \quad k = 1, 2, \ldots,$$

The first autocovariance and autocorrelation are $\gamma_0 = (-2d)!/[(-d)!]^2$ and $\rho_1 = d/(1-d)$. And as $k \to \infty$

$$\rho_k \sim \frac{(-d)!}{(d-1)!} k^{2d-1}.$$

5. The partial correlations of y_t are

$$\phi_{kk} = d/(k - d).$$

The θ_k coefficients from the infinite moving-average representation can easily be obtained by using the difference operator $(1 - B)^{-d}$ instead of $\theta(L)$, i.e., $y_t = (1 - L)^{-d} a_t$ and using the binomial expansion of the fractional difference operator $\sum_{k=0}^{\infty} \binom{-d}{k}$ and we obtain θ_k. In this case, with $-d$ instead of d, the difference operator becomes an integration operator and it shows us that the process $\{y_t\}$ is indeed fractionally integrated white noise. Analogously, to obtain θ_k coefficients, we write the infinite autoregressive representation of $\{y_t\}$ as

$$\pi(L)y_t = (1 - L)^d y_t = a_t \tag{5.2}$$

and see that when the ARIMA(0,d,0) process $\{y_t\}$ is fractionally differenced, the resulting process is white noise process $\{a_t\}$. According to Theorem 5.2.2, the ARIMA(0,d,0) process $\{y_t\}$ is both stationary and invertible when $-\frac{1}{2} < d < \frac{1}{2}$. If d is in this interval we see that $0 < H < 1$, and the ARIMA(0,d,0) model coincides with the *fBm*. Now let us investigate the relation between two parameters $d$and H. We know that $d = H - \frac{1}{2}$, so d is positive when $H > 1/2$, and the difference operator $(1-L)^d$ indeed works as a difference operator. However, when $H < 1/2$, d is negative and the difference operator becomes an integration operator, and to obtain the white noise process $\{a_t\}$, we fractionally integrate the ARIMA(0,d,0) process $\{y_t\}$. One of the things that draws our attention is the hyperbolic decay rate of θ_k and π_k being different from the exponential decay rate of an ARMA process. As the theorem stated, it is also possible to identify long-range dependence in the frequency domain via the behavior of the spectral density of $\{y_t\}$ at low frequencies. If $0 < d < 1/2$, $\{y_t\}$ is a process with long-range dependence as we expected it to be since $1/2 < H < 1$.

In the frequency domain, it is also possible to detect long-range dependence by looking at the spectral density of the process at low frequencies. The spectral density of $\{y_t\}$ is concentrated at low frequencies, is a decreasing function of frequency and goes to infinity as the frequency goes to zero, but still integrable at zero. This fact may help in understanding why the word ‘persistence’ is used sometimes instead of long-range dependence. If $1/2 < H < 1$, the persistent behavior of *fBm* can be seen by the process being dominated by the low-frequency components, therefore causing the process to seem more deterministic and like consisting of waves with greater periods. Recall that, in this case, the quadratic variation of the *fBm* goes to zero, may be seen as another reason for its persistent behavior. The ARIMA(0,d,0) process has a different behavior when $-\frac{1}{2} < d < 0$ ($0 < H < 1/2$). It exhibits short-memory, or 'antipersistence', in Mandelbrot's terminology. The correlations and partial correlations are all negative, except, of course, $\rho_0 = 1$, and decay hyperbolically to zero as it is the general case for ARIMA(0,d,0) processes. Again, this difference in behavior reflects on the behavior of the spectral density of the process. Unlike the case of $d > 0$, the spectral density is dominated by high-frequency components, the spectral density $s(\lambda)$ is an increasing function of frequency λ and vanishes at $\lambda = 0$. As we see, the result of the domination of high-frequency component is the mean-reverting behavior of the ARIMA(0,d,0) process, and the *fBm*. This situation is usually interpreted as acting 'more chaotic', or 'wild' but the definition of 'more chaotic' is not explicit enough. Nonetheless, one can see that the statistical fractal dimension of *fBm* is inversely related with the parameter H and this can be a way of interpreting this chaotic behavior. As mentioned, one of the characteristic of a fractionally integrated process is the hyperbolic decay rate of the correlations. The correlations of an ARMA(p, q) process decays exponentially. But the hyperbolic decay rate does not give us a clue about the value of the parameter d because the decay rate of the partial correlations is k^{-1} which is independent of d. But the behavior of the partial linear regression coefficients are used to distinguish between different values of d. The Durbin-Levinson method [5] can be used to obtain these coefficients. Let us consider a fractionally integrated model is approximated by the first two lags of the series. Then the best linear prediction of the next observations, in the mean square sense, is

$$\hat{y}_3 = \pi_1 y_1 + \pi_2 y_2.$$

If we consider this model as an AR(2) model fitted to an AR(3) model, then we can obtain the partial correlation coefficients recursively using the Durbin-Levinson method since we know the correlations $\rho(k)$. The partial linear regression coefficients ϕ_{kj} for $1 \leq j \leq k$ is obtained by this method is

$$\phi_{kj} = -\binom{k}{j}\frac{(j-d-1)!(k-d-j)!}{(-d-1)!(k-d)!}$$

as $j, k \to \infty$ with $j/k \to 0$ we have

$$\phi_{jk} \sim -j^{-d-1}/(-d-1)!.$$

5.3.2 The fractional ARIMA(p, d, q) process

The ARIMA(0,d,0) process may be helpful in modeling of long-range dependence but time-series generally have more a few more characteristics to be considered when one try to model. In order to preserve the hyperbolic decay of the correlation function, fractional difference operator is applied to ARMA processes. The resulting process does not have too many parameters and seem to have enough parameters to model the ARMA models with slowly decaying correlations. Let us recall the definition of an ARMA(p, q) process $\{x_t\}$:

Definition 5.3.2 *The ARMA(p,q) process* $\{x_t\}$ *is defined as*

$$\phi(B)x_t = \theta(B)a_t, \tag{5.3}$$

where

$$\phi(B) = 1 - \phi_1 B - \phi_2 B^2 - \ldots - \phi_p B^p,$$
$$\theta(B) = 1 - \theta_1 B - \theta_2 B^2 - \ldots - \theta_p B^p.$$

Then the ARFIMA(p, d, q) process is formally defined in [23] as the stochastic process $\{y_t\}$ that can be represented as

$$\phi(L) \triangledown^d y_t = \theta(B)a_t.$$

Using this process, it is possible to model both short-range and long-range correlation structures. The ARMA part of the process can be used to model short-range structure, while the parameter d is chosen so that the long-range behavior of a series can be captured. One of the difficulties of working with ARFIMA(p,d,q) models is that the AR and MA representation weights are complicated functions of the hypergeometric function. In 10, an alternative method for calculation of these autocorrelations were provided as:

$$\gamma_k = \sum_{k=-q}^{q} b_j \sum_{n=1}^{p} \theta_n C(k - p - j; \psi_n),$$

where

$$C(k - p - j; \psi_k) = \psi_k^{2p} \sum_{m=0}^{\infty} \psi_k^m \gamma^*_{k-p-j-m} + \sum_{n=1}^{\infty} \psi_k^n \gamma^*_{k-p-j+n}$$

and γ^* is the autocovariance at lag k of an ARFIMA(0,d,0) process. The b_k and ψ_j are given by:

$$b_k = [\psi_k \prod_{1}^{p}(1 - \psi_i - \psi_k) \prod_{m \neq k}^{p}(\psi_k - \psi_m)], \quad \text{for} k = 1, 2, \ldots, p$$

$$\psi_j = \sum_{i=0}^{q-|j|} \theta_i \theta_{i+|j|}.$$

The equation that defines the ARFIMA(p, d, q) process can be interpreted in several ways. For instance, let us write

$$(1 - L)^d y_t = \tilde{y}_t,$$

where $\tilde{y}_t$ is an ARMA process defined by

$$\tilde{y}_t = \phi(L)^{-1}\theta(L)a_t,$$

which can be interpreted as, after passing $\{y_t\}$ through the fractional difference operator (infinite linear filter) $(1 - L)^d$, one obtains an ARMA process. And we can write,

$$y_t = \phi(L)^{-1}\theta(B)y_t^*,$$

where y_t^* is an ARIMA(0,d,0) process defined by

$$y_t^* = (1 - L)^{-d} a_t.$$

The effect of parameter d, can be seen directly in behavior of the spectral density function of an ARFIMA(p, d, q) process. To see this effect, we must compute the spectral density function of $\{y_t\}$, by using the spectral density function of an ARMA(p, q) process. Let us denote the spectral density of the ARMA(p, q) process $\{\tilde{y}_t\}$ by

$$s_{\tilde{y}}(\lambda) = \frac{\sigma_a |\theta(e^{i\lambda})|^2}{2\pi |\phi(e^{i\lambda})|^2}.$$

Since the process $\{\tilde{y}_t\}$ is obtained from the process $\{y_t\}$ by applying the linear filter $(1 - B)^d$, we can use this relation in [22] to compute the spectral density of the process $\{y_t\}$. In the representation

$$(1 - L)^{-d}\tilde{y}_t = y_t,$$

we can write $e^{i\lambda}$ instead of L and use the result of Priestley in [36] to obtain the spectral density of $\{y_t\}$:

$$s_y(\lambda) = |1 - e^{i\lambda}|^{-2d} s_{\tilde{y}_t}.$$

Using the equality $|1 - e^{i\lambda}| = 2\sin(\frac{1}{2}\lambda)$ [3], we take the limit as frequency goes to zero to obtain the behavior of the spectral density around zero. Because $\lim_{\lambda\to 0} \lambda^{-1}(2\sin(\frac{1}{2}\lambda)) = 1$, this behavior is can be seen in the following equality:

$$s_y(\lambda) \sim \frac{\sigma^2|\theta(1)|^2}{2\pi|\phi(1)|^2}|\lambda|^{-2d} = s_{\tilde{y}}(0)|\lambda|^{-2d}.$$

Thus, when $d > 0$, the spectral density goes to infinity around zero frequency. This corresponds to the case $H > 1/2$ for the *fBm*. So the long-range dependence can be seen in the behavior of the spectral density dominated by low frequency components. As for the *fBm*, the ARFIMA(p, d, q) model is easier to work with when long-range dependence exists, i.e., when $d > 0$. The following is an alternative definition of long-range dependence based on the spectral densities:

Definition 5.3.3 *Let $\{y_t\}$ be a stationary process for which the following holds: There exists a real number $\beta \in (0, 1)$ and a constant $c_s > 0$ such that*

$$\lim_{\lambda\to 0} s(\lambda)/[c_s|\lambda|^{-\beta}] = 1.$$

Then $\{y_t\}$ is called a stationary process with long-memory or long-range dependence.

The following theorem in [3] gives an equivalent definition of long-range dependence for stationary processes:

Theorem 5.3.4 1. *Suppose the following holds holds with $0 < \alpha = 2-2H < 1$ and $c_n > 0$,*

$$\lim_{n\to\infty} \frac{\rho(n)}{c_\rho n^{-\alpha}} = 1,$$

then the spectral density s exists and

$$\lim_{\lambda\to 0} s(\lambda)/[c_s(H)|\lambda|^{1-2H}] = 1,$$

where

$$c_s = \sigma^2\pi^{-1}c_\rho\Gamma(2H - 1)\sin(\pi - \pi H),$$

and $\sigma^2 = var(y_t)$.

2. *Suppose the long-range dependence property holds for $\{y_t\}$ with $0 < \beta = 2H - 1 < 1$. Then*

$$\lim_{k\to\infty} \rho(k)/[c_\rho k^{2H-2}] = 1$$

where $c_\rho = \frac{c_\gamma}{\sigma^2}$, *and*

$$c_\gamma = 2c_s\Gamma(2-2H)\sin(\pi H - \frac{1}{2}\pi).$$

One may use these asymptotic equalities to obtain explicit formulas for the covariances and correlations. As $|k| \to \infty$,

$$\gamma(k) \sim \frac{\sigma^2|\theta(1)|^2}{\pi|\phi(1)|^2}\Gamma(1-2d)\sin(\pi d)$$

and for the correlations

$$\rho(k) \sim \frac{\frac{\sigma^2|\theta(1)|^2}{\pi|\phi(1)|^2}\Gamma(1-2d)\sin(\pi d)|k|^{2d-1}}{\int_{-\pi}^{\pi} s(\lambda)d\lambda}.$$

One of the problems that may arise is that the data to be modeled by an ARFIMA process has a greater order of integration, i.e., $d > 1$. In this case, as we have seen, the ARFIMA process is not stationary. This problem may be solved easily by differencing series until we obtain $d < 1/2$.

The reason for the fractional integration and the long-range dependence has been widely questioned. One of the explanations made is the aggregation of independent AR(1) processes causing the fractional behavior by Granger in [20].

5.3.3 Maximum-likelihood method

Maximum likelihood method is probably the most reliable estimation procedure, but it has drawbacks. The maximum likelihood function of a fractionally integrated process is difficult to derive and requires long computational time. Since *fBm* is Gaussian, the joint distribution of a sample $X = (X_1, X_2, \ldots, X_n)'$ can be shown to be equal to

$$l(x;\theta^o) = (2\pi)^{-\frac{n}{2}}|\Sigma(\theta^o)|^{-\frac{1}{2}}e^{-\frac{1}{2}x'\Sigma^{-1}(\theta^o)}x,$$

where $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ [3]. Then the log-likelihood function is

$$\begin{aligned} L_n(x;\theta^o) &= \log h(x;\theta^o) \\ &= \frac{n}{2}\log 2\pi - \frac{1}{2}\log|\Sigma(\theta^o)| - \frac{1}{2}x'\Sigma^{-1}x. \end{aligned}$$

The MLE of θ^o is obtained by maximizing the log-likelihood function with respect to parameter vector θ. Under mild regularity conditions, this maximization can be shown to be equivalent to solving the following equations

$$L_n^{'}(x;\hat{\theta}) = 0.$$

The asymptotic distribution of $\hat{\theta}$ then can be derived using the Taylor expansion of $L_n^{'}$. For details, see[3]. Sowell derives the exact formulation of the maximum likelihood function in [44]. Another approach would be using the conditional distribution of x_n given $x_{n-1}, x_{n-2}, \ldots, x_1$ to decompose the likelihood function as

$$l(x) = l_1(x_1)l_2(x_2|x_1)\ldots l_n(x_n|x_1,\ldots,x_{n-1}),$$

where $l_j(x_j|x_1,\ldots,x_{j-1})$ are one dimensional Gaussian densities which are fully characterized by its mean and variance. Since we use the time series modeling point of view, the mean μ_j can be obtained by the best linear prediction of x_j given $x_1,\ldots,x_{j-1}$,

$$\mu_j = E[X_j|X_1,\ldots,X_{j-1}] = \hat{X}_j = \sum_{s=1}^{j-1} \beta_{j-1,s} X_{j-s},$$

where the coefficients $\beta_{j-1,s}$ are the partial linear regression coefficients and can be obtained by the Durbin-Levinson algorithm [5]. The variance of l_j is equal to the expected mean square error of $\hat{X}_j$

$$\sigma_j^2 = E[(\hat{X}_j - \mu_j)^2|X_1,\ldots,X_{j-1}].$$

MLE methods are reliable but difficult to implement in terms of computational time.

5.3.4 Whittle's approximate maximum likelihood function

using the spectral properties of the process, an approximate MLE method has been proposed by Whittle in [46]. In general, the estimation methods which used to estimate d, are based on Fourier transform techniques as shown in [44]. This is because the effect of the fractional differencing parameter can be seen on the behavior of the spectral density of the process and there is correspondence between the spectral density and the covariance matrix of the process. Whittle's estimator uses this correspondence to approximate the covariance matrix and its inverse. The idea is using the equality,

$$\lim_{n\to\infty} \frac{1}{n} \log|\Sigma_n(\theta)| = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log s(\lambda;\theta)d\lambda,$$

to obtain the approximation $\log|\Sigma_n(\theta)| \approx n(2\pi)^{-1}\int_{-\pi}^{pi} \log s(\lambda;\theta)d\lambda$. It is shown in [6] that the following approximation holds. Let us define the $n \times n$ matrix A by

$$A(\theta) = [\alpha(j-l)]_{j,l=1,\ldots,n},$$

where its elements are defined by

$$\alpha(j-l) = (2\pi)^{-2} \int_{-\pi}^{\pi} \frac{1}{s(\lambda;\theta)} e^{i(j-l)\lambda} d\lambda,$$

then the matrix A is asymptotically inverse of the covariance matrix Σ_n as shown in [3]. The Whittle estimator is easier to implement when discretized. The elements $\alpha(j-l)$ are approximated by

$$\hat{\alpha}(k) = 2\frac{1}{(2\pi)^2} \sum_{j=1}^{m} \frac{1}{f(\lambda_{j,m})} e^{ik\lambda_{j,m}} \frac{2\pi}{m},$$

where

$$\lambda_{j,m} = \frac{2\pi j}{m} \quad (j = 1, \ldots, m^*),$$

and m^* denotes the integer part of $(m-1)/2$. Then the estimation of parameters can be obtained by minimizing

$$\hat{L}_W(\theta) = 2\frac{1}{2\pi}[\sum_{j=1}^{m^*} \log(\lambda_{j,m};\theta)\frac{2\pi}{m} \sum_{j=1}^{m^*} \frac{I(\lambda_{j,m})}{f(\lambda_{j,m};\theta)} \frac{2\pi}{m}].$$

If, in addition, one represents the parameter vector as $\theta = (\theta_1, \eta)$ such that

$$s(\lambda;\theta) = \theta_1 s(\lambda;\theta^*)$$

with the following condition holds

$$\int_{-\pi}^{\pi} \log s(\lambda;\theta^*) d\lambda = 0,$$

where $\theta^* = (1,\eta)$, then minimizing the Whittle likelihood amounts to minimizing $I(\lambda_{j,m})s^{-1}(\lambda_{j,m};\theta^*)$ with respect to η. For details, see [6].

5.4 Simulation of *fBm*

5.4.1 Durbin-Levinson Method

This method is also known as the Hosking method and is an algorithm to simulate stationary Gaussian processes in general. The idea behind this algorithm is to obtain Yule-Walker estimates of a AR($p+1$) process using the parameters of AR(p) process fitted to the same time series. The partial autocorrelation function can therefore be approximated. Assume we have $X_0, X_1, \ldots, X_n$, the first n observations of a process, Durbin-Levinson method generates X_{n+1}

using these past observations. Let γ denote the autocovariance function of a zero-mean series, i.e.

$$\gamma(k) = E[X_n X_{n+k}], \quad \text{for} \quad k, n = 0, 1, \ldots.$$

Since we can magnify the variance of the process as we like, we assume, for now, it has unit variance; $\gamma(0) = \sigma^2 = 1$. Let $\Gamma(n) = [\gamma(i-j)]_{i,j=0,\ldots,n}$ be the covariance matrix and define the $(n+1)$-column vector as $g(n)_k = \gamma(k+1), \quad k = 0, \ldots, n$. The $(n+1)\times(n+1)$-matrix $T(n) = [I_{i=n-j}]_{i,j=0,\ldots,n}$ is defined in such a way that premultiplying this matrix with a column vector or postmultiplying this matrix with a row vector transposes the vector. Using these we can write

$$\Gamma(n+1) = \begin{pmatrix} 1 & g(n)' \\ g(n) & \Gamma(n) \end{pmatrix} = \begin{pmatrix} \Gamma(n) & F(n)g(n) \\ g(n)'F(n) & 1 \end{pmatrix}$$

where the prime denotes the vector transpose. The mean is $\mu_n = c(n)'\Gamma(n)^{-1}(X_n \ldots X_1 X_0)'$ and the variance is $\sigma_n^2 = 1 - c(n)'\Gamma(n)^{-1}c(n)$. The method generates the next estimates of the variance and the mean recursively using the recursion

$$\sigma_{n+1}^2 = \sigma^2 - \frac{(\gamma(n+2) - \tau_n)^2}{\sigma_n^2}$$

with $\tau_n = d(n)'F(n)c(n) = c(n)'F(n)d(n)$ and $d(n) = \Gamma(n)^{-1}c(n)$. A recursion for $d(n+1)$ is also obtained

$$d(n+1) = \begin{pmatrix} d(n) & \phi_n F(n)d(n) \\ & \phi_n \end{pmatrix}$$

with $\phi_n = \frac{\gamma(n+2)-\tau_n}{\sigma_n^2}$. Recursion starts with $\mu_0 = \gamma(1)X_0$, $\sigma_0^2 = 1 - \gamma(1)^2$ and $\tau_0 = \gamma(1)^2$. A sample of *fBm* is obtained by computing the cumulative sums.

5.4.2 Cholesky method

As expected, Cholesky method is based on the Cholesky decomposition of the covariance matrix. When a matrix is symmetric positive definite, the Cholesky method can be used. First we write

$$\Gamma(n) = M(n)M(n),$$

where $M(n)$ is an $(n+1)\times(n+1)$ lower triangular matrix, that is, the (i, j) element of $M(n)$ is zero for $j > i$. The elements of $M(n)$ can be computed since the element (i, j) of $M(n)M(n)'$ and $\Gamma(n)$ should be equal for $j \leq i$ because $M(n)$ is lower triangular, but then they will be equal because of the symmetry of $\Gamma(n)$, and this means

$$\gamma(i-j) = \sum_{k=0}^{j} m_{ik} m_{jk}, \qquad j \leq i.$$

If $i = j = 0$, we obtain the variance as $\gamma(0) = m_{00}^2$. If $i = 1$, we have the two equations

$$\gamma(1) = m_{10} m_{00}, \qquad \gamma(0) = m_{10}^2 + m_{11}^2$$

to determine m_{10} and m_{11}. Since we can compute $M(n+1)$ from $M(n)$ by adding a row and enough zeros to keep it lower triangular we can determine the row that we will add by

$$\begin{aligned}
m_{n+1,0} &= \frac{\gamma(n+1)}{m_{00}}, \\
m_{n+1,j} &= \frac{1}{m_{jj}}(\gamma(n+1-j) - \sum_{k=0}^{j-l} m_{n+1,k} m_{jk}) \qquad (0 < j \leq n), \\
m_{n+1,n+1}^2 &= \gamma(0) - \sum_{k=0}^{n} m_{n+1,k}^2.
\end{aligned}$$

The Cholesky method requires the positive definiteness of $\Gamma(n+1)$ to obtain a real matrix $M(n+1)$. If we denote by $S(n) = (S_i)_{i=0,\ldots,n}$ an $(n+1)$-column vector of i.i.d. standard normal variables. The idea behind the method is simulating $X(n) = M(n)S(n)$ recursively. Since for every $n \geq 0$ $X(n)$ has the covariance matrix

$$\mathrm{Cov}(M(n)S(n)) = M(n)\mathrm{Cov}(S(n))M(n)' = M(n)M(n)' = \Gamma(n)$$

and zero mean, the simulated process has the characteristics we want. If $L(n+1)$ is computed, X_{n+1} can be computed by

$$X_{n+1} = \sum_{k=0}^{n+1} m_{n+1,k} V_k.$$

The Cholesky method is slower than the Durbin-Levinson method, although, in principle, they both compute the matrix $M(n)$.

5.4.3 Davies and Harte method

The method is proposed by Davies and Harte in [13] to simulate a stationary Gaussian time series of length n with autocovariances $\gamma(0), \gamma(1) \ldots, \gamma(n-1)$. It is described as follows:

1. Define

$$\lambda_k = \frac{2\pi(k-1)}{2n-2}$$

for $k = 1, \ldots, 2n-2$, and the finite Fourier transform f_k of the sequence $\gamma(0), \gamma(1), \ldots, \gamma(n-2), \gamma(n-1), \gamma(n-2), \ldots, \gamma(1)$.

$$f_k = \sum_{j=1}^{n-1} \gamma(j-1) e^{i(j-1)\lambda_k} + \sum_{j=n}^{2n-2} \gamma(2n-j-1) e^{i(j-1)\lambda_k}$$

for $k = 1, \ldots, 2n-2$. To move to next step, it must be verified that $f_k > 0$.

2. Simulate two independent series of zero mean normal random variables $d_i, \quad i = 1, \ldots, n$ and $h_j, \quad j = 2, \ldots, n-1$, such that

$$\mathrm{var}(d_1) = \mathrm{var}(d_n) = 2,$$

and for $k \neq 1, n$,

$$\mathrm{var}(d_k) = \mathrm{var}(h_k) = 1$$

and $h_1 = h_n = 0$.

3. Define the complex random variables z_k as

$$z_k = d_k + ih_k, \qquad (k = 1, \ldots, n),$$

and

$$z_k = d_{2n-k} - ih_{2n-2}, \qquad (k = n+1, \ldots, 2n-2).$$

4. $t = 1, \ldots, n$,

5. Define

$$X_t = \frac{1}{2\sqrt{n-1}} \sum_{k=1}^{2n-2} \sqrt{f_k} e^{i(t-1)\lambda_k} z_k.$$

In order to obtain a specific Gaussian process, for instance *fBm*, one must use the covariance function of *fBm*. There are also simulation methods for *fBm* which are based on its representations. For detailed information on simulation methods, see [15].

5.5 Application

We applied R/S analysis and periodogram methods to estimate the Hurst parameter of Dow Jones Industrial Average (DJIA) index, Turkish Lira/Dollar, Euro/Dollar, and Turkish Lira/Euro exchange rate data. The data is obtained from the FOREX platform of MIG bank. Although

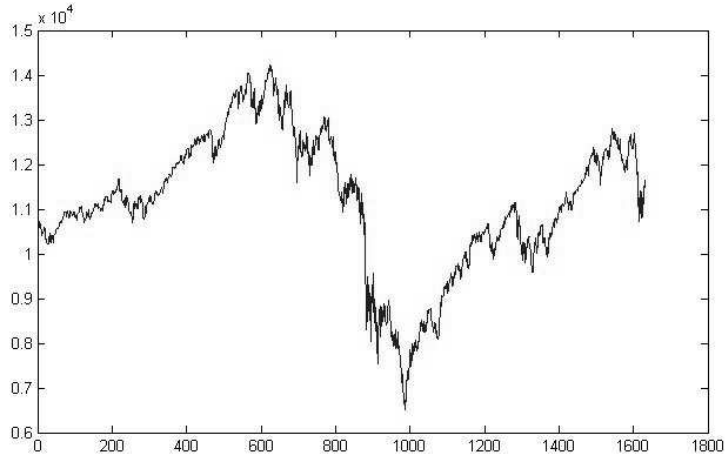

Figure 5.1: DJIA index between 07.09.2005-01.09.2011

periodogram method gave values very close to 0.5, the R/S analysis estimated higher values of H.

In the *WIS* model, one of the differences from the classical Itô integral setting is the variance of the stock return replaced by the variance of the firm value process. But this replacement is not easy to do in real world terms since the firm value process can not be observed for small time intervals. There are different methods to compute the value of a firm and this difference may also lead to different prices. Leaving the solutions of these problems to future work, we compute the price of a European call option on DJIA using the variance of the daily return series in our computation to see the difference between the fractional Black&Scholes and classical Black&Scholes prices.

A statistic for testing the long-memory hypothesis was proposed by Lo in [26]. This test is called the *modified R/S*. For details on this test, see [26] and [45]. The estimated values of *H* and *modified R/S* values are given below:

The mean of the daily return series is very close to zero. There is no apparent trend in the series and this in favour of our statistical modelling methodology. In the presence of trends, estimators of long-memory may be biased towards accepting the long-memory hypothesis.

The estimated H value of the daily return series of DJIA is greater than $1/2$, so we can use the theoretical results of *WIS* and fractional *WIS* models and evaluate the price of an option written on DJIA.

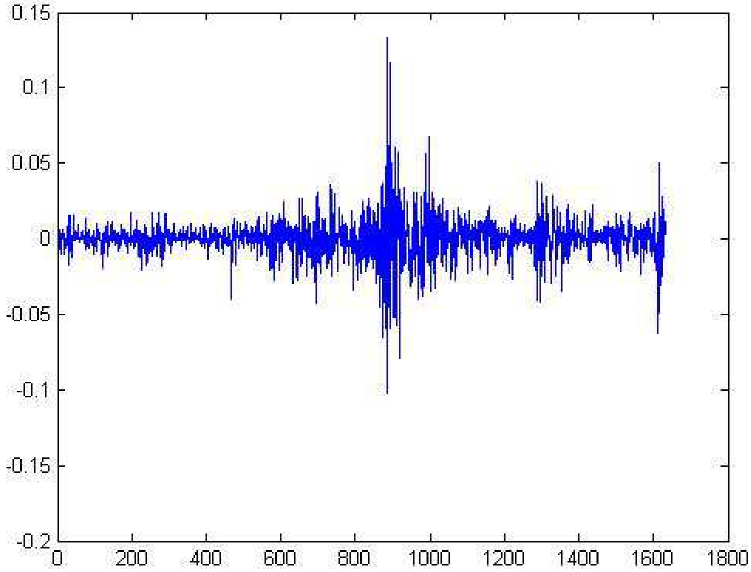

Figure 5.2: Daily return series of DJIA

Mean	Median	Std. Deviation	Estimated H
0.00005386	0.0003857	0.2134	0.59247

Figure 5.3: Descriptive statistics of DJIA daily return series

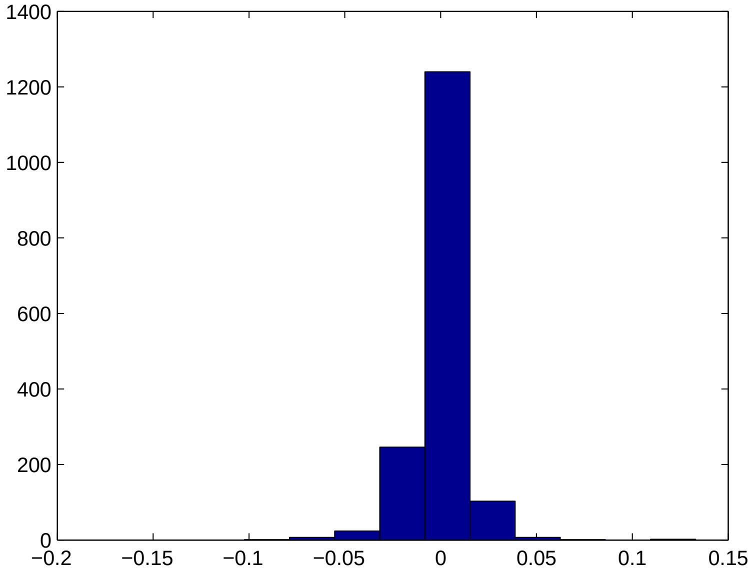

Figure 5.4: Histogram of the daily return series

	1 Hour	4 Hours	Daily
USD/TRY	0.58241 1.0734	0.5763 1.4394	0.58135 1.2038
EUR/TRY	0.56129 1.4077	0.60888 1.2548	0.58393 0.9835
EUR/USD	0.59202 1.0860	0.59899 1.133	0.6038 1.0837

Figure 5.5: Estimated H values and modified R/S

	T=0.7	T=1.5	T=2.3
fBs Price	16.1574	22.1823	24.6968
Bs Price	16.4656	21.6710	24.5639

Figure 5.6: Comparison of prices

The *modified R/S* statistic of Lo [26] rejects the long-memory hypothesis for all cases but this does not surprise us because it is shown in [45] that this statistic is biased towards rejecting the long-memory hypotheses. But in some cases, it gives values that is near the accepting value and this gives some idea about the existence of long-memory in our data.

Using these estimated values of H, we computed the fractional Black&Scholes price to compare to the Black&Scholes price for the same option. The results are summarized in the following table. We followed the methodology in [48].

As we can see, the difference between fBS and BS prices are very small. This is because the estimated values of H are close to 0.5. But even this small difference may be of importance in the case of multiple transactions. The effect of H on the option prices may be used to decide whether an option is overpriced or underpriced. But in order to properly price an option, one needs better and unbiased estimators of H. One of the possible future works may be on this topic.

CHAPTER 6

Conclusion and Outlook

In this thesis, *fractional* Brownian motion as a model in finance has been considered. We began with the definition and properties of *fBm*. We basically explain the self-similarity, stationarity and long-range dependence properties of *fBm* and conclude the first chapter with the non-semimartingale property of *fBm* with $H \neq 1/2$. This property makes defining an integral with respect to *fBm* more difficult than the standard Brownian motion case. Then, in the second chapter, we summarized different approximations to defining the integral with respect to *fBm*. In the *WIS* setting, we showed how an Itô type formula and the fractional analog of Girsanov theorem is derived and then we use this Itô formula to obtain the price of a European call option using the Black&Scholes approach to option pricing. Disadvantages and question of arbitrage when using *fBm* in financial applications is addressed and arbitrage examples are given. We continued with the time series models that has the long-range dependence property of *fBm*, ARFIMA models. Some estimation methods for *fBm* and ARFIMA models are presented. The most widely used estimation procedures such as Hurst's R/S analysis, correlogram and linear regression in the frequency domain have been reviewed. Some of the simulation methods to generate a sample of *fBm* are briefly mentioned. To illustrate the effect of long memory on the option prices, we applied R/S analysis to DJIA index and exchange rate series at different time scales to see the self-similarity characteristics of the series. Then, using the estimated value we compared the standard Black&Scholes prices to fractional Black&Scholes prices for different maturities. When we estimated the Hurst parameter H, we saw that it varies over different time scales. This characteristic of the financial data has been considered by Mandelbrot and his students Calvet and Fisher and it led to a multifractal model for financial data [7]. The future work may be on relating the multifractal model and white noise analysis notions to obtain a proper tool for pricing an option in the fractional markets.

There are also other possible ways to characterize the long-range dependence property of *fBm* such as implementing information theoretical concepts and notions of financial economics for modeling the financial markets using *fBm* in a more realistic manner. In addition to these, consistent and robust estimators of H must be developed. The asymptotic definition of long-range dependence makes it difficult to estimate H in a finite sample and unless this problem is solved, *fBm* can not be used more actively. A Bayesian approach to 'multifractal white noise model' may be used to solve these problems.

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APPENDIX A

Basic fractional calculus notions

As we mentioned in the introduction, *WIS* and fractional *WIS* integrals with respect to *fBm* are defined by using fractional calculus in the white noise theory. The operator M is indeed the fractional integration operator and the isometry from $L^2_\phi(\mathbb{R})$ to $L^2(\mathbb{R})$ is again defined by the fractional integration operator $I_-^{H-1/2}$. In this appendix, we briefly summarize the basic notions of fractional calculus by following [47].

A.1 Fractional calculus on a finite interval

Definition A.1.1 *Let f be a deterministic real-valued function that belongs to $L^1(a,b)$, where (a,b) is a finite interval of $\mathbb{R}$. The* fractional Riemann Liouville integrals of order $\alpha > 0$ *are determined at almost every $x \in (a,b)$ and defined as the*

1. *Left-sided version:*
$$I^\alpha_{a+} f(x) := \frac{1}{\Gamma(\alpha)} \int_a^x (x-y)^{\alpha-1} f(y)dy.$$
2. *Right-handed version:*
$$I^\alpha_{b-} f(x) := \frac{1}{\Gamma(\alpha)} \int_x^b (y-x)^{\alpha-1} f(y)dy$$
where $\Gamma(\cdot)$ denotes the Gamma function. As we see from this definition, $I_-^{H-1/2}$ is the fractional Riemann Liouville integral *of order $H-\frac{1}{2}$. For $\alpha = n \in \mathbb{N}$ one obtains the $n-$order integrals*
$$I^n_{a+} f(x) = \int_a^x \int_a^{x_{n-1}} \dots \int_a^{x_2} f(x_1)dx_1dx_2\dots dx_n$$
and
$$I^n_{b-} f(x) = \int_x^b \int_{x_{n-1}}^b \dots \int_{x_2}^b f(x_1)dx_1dx_2\dots dx_n.$$

Definition A.1.2 *For $\alpha < 1$, the* fractional Liouville derivative *is defined as*

$$D^{\alpha}_{a+} f := \frac{d}{dx} I^{1-\alpha}_{a+} f$$

and

$$D^{\alpha}_{b-} f := \frac{d}{dx} I^{1-\alpha}_{b-} f,$$

if the right-hand sides are well-defined (or determined).

For any $f \in L^1(a, b)$ one obtains

$$D^{\alpha}_{a+} I^{\alpha}_{a+} f = f, \qquad D^{\alpha}_{b-} I^{\alpha}_{b-} f = f.$$

For details, see [47].

A.2 Fractional calculus on the whole real line

The left- and right-sided fractional integral and derivative operators on $\mathbb{R}$ for $\alpha \in (0, 1)$ are defined as follows (see [34] for references)

Definition A.2.1 *Let $\alpha \in (0, 1)$. The fractional integrals I^{α}_{+} and I^{α}_{-} of a function ϕ on the whole real line are defined, respectively, by*

$$I^{\alpha}_{+} f(x) := \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{x} (x - y)^{\alpha - 1} f(y) dy, \quad x \in \mathbb{R},$$

and

$$I^{\alpha}_{-} f(x) := \frac{1}{\Gamma(\alpha)} \int_{x}^{\infty} (x - y)^{\alpha - 1} f(y) dy, \quad x \in \mathbb{R}.$$