VIBRATION ANALYSIS OF CRACKED BEAMS ON ELASTIC FOUNDATION USING TIMOSHENKO BEAM THEORY

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ABSTRACT

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In this thesis, transverse vibration of a cracked beam on an elastic foundation and the effect of crack and foundation parameters on transverse vibration natural frequencies are studied. Analytical formulations are derived for a beam with rectangular cross section. The crack is an open type edge crack placed in the medium of the beam and it is uniform along the width of the beam. The cracked beam rests on an elastic foundation. The beam is modeled by two different beam theories, which are Euler-Bernoulli beam theory and Timoshenko beam theory. The effect of the crack is considered by representing the crack by rotational springs. The compliance of the spring that represents the crack is obtained by using fracture mechanics theories. Different foundation models are discussed; these models are Winkler Foundation, Pasternak Foundation, and generalized foundation. The equations of motion are derived by applying Newton's 2^{nd} law on an infinitesimal beam element. Non-dimensional parameters are introduced into equations of motion. The beam is separated into pieces at the crack location. By applying the compatibility conditions at the crack location and boundary conditions, characteristic equation whose roots give the non-dimensional natural frequencies is obtained. Numerical solutions are done for a beam with square cross sectional area. The effects of crack ratio, crack location and foundation parameters on transverse vibration natural frequencies are presented. It is observed that existence of crack reduces the natural frequencies. Also the elastic foundation increases the stiffness of the system thus the natural frequencies. The natural frequencies are also affected by the location of the crack.

Keywords: Transverse Vibration, Euler-Bernoulli Beam, Timoshenko Beam, Winkler Foundation, Pasternak Foundation, Generalized Foundation, Edge Crack

ÖZ

ELASTİK MESNET ÜZERİNDE DURAN VE ÇATLAK İÇEREN ÇUBUKLARIN TIMOSHENKO TEORİSİ İLE TİTREŞİM ANALİZİ

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Bu çalışmada elastik mesnet üzerinde duran ve çatlak içeren bir çubuğun yanal titreşimleri incelenmiştir. Elastik mesnetin ve çatlak parametrelerinin çubuğun yanal titreşim doğal frekansları üzerindeki etkileri çalışılmıştır. Tez çalışmasında türetilen formulasyon, dikdörtgensel kesit alanına sahip çubuklar için geçerlidir. Çatlak, sürekli açık kalacak ve çubuğun kenarında olacak şekilde modellenmiştir. Ayrıca, çatlak derinliği çubuğun genişliği boyunca sabit kalacak şekildedir ve çatlak içeren çubuk elastik mesnet üzerinde uzanmaktadır. Çubuk modelleri hem Euler-Bernoulli teorisi hem de Timoshenko teorisi ile hazırlanmıştır. Çatlak modellenirken, burulma yayı olarak temsil edilmiştir. Temsili yayın direngenliği kırılma mekaniği teorileri ile hesaplanmıştır. Değişik elastik mesnet modelleri kullanılmıştır. Bunlar Winkler mesneti, Pasternak mesneti ve Genel mesnet modelleridir. Hareket denklemleri, sonsuz küçüklükteki çubuk elemanı üzerinde Newton'un 2. kanunu uygulanarak hesaplanmıştır. Sonrasında, haraket denklemlerindeki parametreler birimsiz hale getirilmiştir. Çubuk, çatlak noktalarında parçalara bölünerek, her bir parça çatlağı temsil eden burulma yayı ile birbirine bağlanmıştır. Çatlak noktasındaki uyum koşulları ve çubuğun sınır koşulları uygulanrak karakteristik denklem elde edilmiştir. Karakteristik denklemin kökleri hesaplanarak birimsiz doğal frekanslar elde edilmiştir. Takip edilen metod ile, çatlak derinliğinin, çatlak konumunun ve elastik mesnet özelliklerinin çubuğun yanal titreşimleri üzerindeki etkileri hesaplanmıştır. Hesaplamalarda kare kesit alanına sahip çubuk kullanılmıştır. Sonuç olarak, çatlağın doğal frekansları azalttığı, elastik mesnetin ise sistem direngenliğini arttırarak doğal frekansları arttırdığı gözlemlenmiştir. Bunun yanı sıra çatlak konumu da sınır koşullara bağlı olarak doğal frekansları etkilemektedir.

Anahtar Kelimeler: Yanal Titreşim, Euler-Bernoulli, Timoshenko, Winkler, Pasternak, Genellenmiş mesnet, Kenar Çatlağı

To My Family

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A:	Cross sectional area
$A(\omega)$:	Solution parameter
a:	Crack depth
b:	Thickness
$B(\omega)$:	Solution parameter
C:	Compliance of the crack
C_{TR}:	Translational compliance of the crack in Timoshenko Beam
C_R:	Rotational compliance of the crack in Timoshenko Beam
CF:	Clamped-free boundary conditions
D:	Differential operator
E:	Young's modulus
EB:	Euler-Bernoulli beam
F:	Force
$F(\alpha)$:	Shape factor of the crack
$f(\chi)$:	Non-dimensional rotational displacement
G:	Shear modulus. "G" is also used for energy release rate in the derivation steps of compliance.
G_0:	Pasternak foundation constant
I:	Second moment of area
I_G:	Rotational inertia
k:	Timoshenko shear correction coefficient
K_I:	Mode I stress intensity factor
K_{TR}:	Translational compliance of the crack in Euler-Bernoulli beam
K_R:	Rotational compliance of the crack in Euler-Bernoulli beam
k_w:	Winkler foundation constant

k_Φ:	Generalized foundation model 1 constant
k_v:	Generalized foundation model 2 constant
L_c:	Crack position
L_1:	Non-dimensional crack position
M:	Bending moment
m:	Mass per unit length
NC:	No crack
NF:	No foundation
P(x,t):	Force per unit length exerted by spring
q(x):	Reaction of Pasternak foundation per unit length of the beam
S(x,t):	Shear force
SS:	Simply supported boundary conditions
t:	Time
T, T_{EB}:	Time parameter
TB:	Timoshenko beam
U(x,t):	Longitudinal displacement
$U_t(x,t)$:	Total longitudinal displacement
W(x,t):	Transverse displacement
$w(\chi)$:	Non-dimensional transverse displacement
x:	Axial direction
z:	Transverse direction
α:	Crack ratio(crack depth/height of the beam)
$\beta(x,t)$:	Shear deformation
ε_{xx}:	Normal strain in "x" direction
ε_{xz}:	Shear strain
$\Phi(x,t)$:	Rotational displacement
σ_{xx}:	Normal stress in "x" direction
σ_{xz}:	Shear stress
κ , ζ:	Non-dimensional coefficients

χ:	Non-dimensional axial direction
λ_1, λ_2:	Non-dimensional parameter
η:	Non-dimensional parameter
τ:	Non-dimensional time parameter
ρ:	Density of beam element
ν:	Poisson's ratio
ω:	Non-dimensional natural frequency

CHAPTER 1

INTRODUCTION

1.1 Literature Review

Beams are widely used in many engineering applications. Cracks exist in structures due to many reasons affecting the mechanical behavior of the system. Determination of the presence of a crack, its location and its size are important issues since such discontinuities may cause catastrophic failures due to vibratory motion. One of the ways of determining characteristics of cracks in a beam is investigating the vibratory properties of the beam. To know about size, location and geometry of a crack by vibration methods, first of all, behavior of structures that include cracks should be investigated. This led the researchers to study vibration characteristics of beams with cracks. Natural frequencies of continuous beams can be figured out by many models including engineering beam models such as Euler-Bernoulli beam or Timoshenko beam and by using some analytical or approximate methods such as finite element method. Then beams are modeled with a crack in it and natural frequencies of this cracked model are also calculated. Comparison of the results of cracked and uncracked models reveals the effect of crack. For determination of parameters such as crack depth, crack location and crack number, beams with no cracks are compared with those discontinuous beam models respectively. Using the know-how obtained from such data, by reversing the procedure, cracks can be detected and characteristics of cracks can be obtained.

Also in many engineering applications beams are placed on elastic foundations such as plates or beams on elastic medium that constitute parts of any machinery for isolation purposes, concrete structures on soil in civil engineering applications or as the cases in railway applications. Due to such applications, vibration characteristics of beams on elastic foundations have also been subject of research in many studies. Effect of elastic foundation on natural frequencies of cracked and uncracked beams has been investigated in many researches. In literature there exist studies generally done by using Euler-Bernoulli beam theory and Timoshenko beam theory on different types of elastic foundation models such as Winkler type and Pasternak Type.

In literature, Euler-Bernoulli beam theory is widely used for determination of frequencies of flexural vibration. In this theory rotational inertia and shear deformation is neglected, thus for better approximation, Timoshenko beam theory is applied which includes both rotational inertia and shear deformation.

A corrected approximation for dynamic analysis of beams was published by Timoshenko in 1921. Since that time many studies were conducted to apply the Timoshenko beam theory in a more systematic manner. For example, such a study was carried out quite recently, for the systematic solution of flexural vibration problem of beams using Timoshenko beam theory by Van Rensburg et al. (2006). They presented a systematic approach to the solution of eigenvalue problem based on a continuous beam model with Timoshenko's theory. Mode shapes and natural frequencies for various boundary conditions were derived.

In literature there exists hundreds of paper that covers vibration of cracked structures. Out of these studies, the ones including vibration of cracked beams and vibration of beams on elastic foundations are reviewed in this part of the thesis. A comprehensive review article covering the literature on vibration of cracked structures was published by Dimarogonas (1996) covering the studies which had been done until that time. Hence, for the earlier studies one may refer to this review article. In this thesis, most of the papers which are reviewed are the studies that have been conducted since then, mostly in the last decade.

Before starting the review of more recent studies, it is worthwhile to briefly summarize the issues addressed in the review paper of Dimarogonas (1996). In that study, researches about linear and non-linear vibration cases were mentioned. Also, studies concerning open crack, breathing crack, non-linearities due to breathing crack and effect on vibration harmonics were reviewed. Methods developed for local flexibility of cracked regions and vibrations of stationary cracked beams were mentioned in the review. Besides vibration of cracked beams, review of vibration of cracked plates was also done in the aforementioned paper. Methods developed such as continuous cracked beam theory that exist in literature were also reviewed. Torsional vibrations of shafts, vibration of cracked rotors were other subjects that existed in the review study. In addition to the review of theoretical work, studies concerning crack identification were also covered in which identification of cracks in turbine rotors and turbine blades had been investigated. Moreover, studies about hollow structures such as vibration of cracked pipes and shells were included.

For determination of transverse vibration characteristics of a beam, a general approach that exists in literature is the separation of beams into pieces which are connected by rotational springs as shown in Figure 1.1.

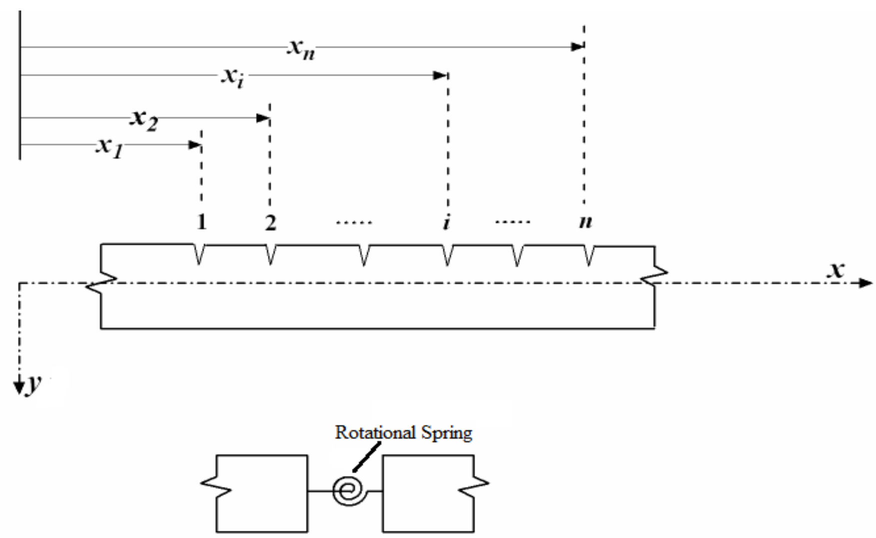

Figure 1.1 Euler-Bernoulli Beam with n cracks, model of a massless rotational spring representing a crack by Aydin (2008)

The rotational spring accounts for the crack and continuity is satisfied by compatibility conditions. Vibration frequencies and mode shape functions of beams containing arbitrary number of cracks was studied by using Euler-Bernoulli beam theory by Aydın (2008). In the study effect of axial load was also investigated. Rotational spring model was used for introducing the compliance caused by the crack and the cracks were assumed to be open for all cases. Natural frequencies and mode shapes for different boundary conditions and arbitrary number of cracks were obtained. The boundary conditions used in the study were: pinned-pinned, clamped-pinned, clamped-free, clamped-clamped and spring-spring with concentrated masses. Comparison of different axial load levels, for different crack depth ratios for each boundary condition showed that tensile load caused an increase in the frequencies of the cracked beam where as a compressive load caused a decrease. Natural frequencies came out to be effected significantly by cracks and axial loading. In the comparisons for different parameters it was shown that cantilever beam case was the one which is most effected by cracks. A crack depth ratio of 0.5 and an axial load which is 30% of the critical load caused almost 50 % reduction in the fundamental frequency. It was also observed that, if a crack coincides with a nodal point, the crack depth has no effect on the natural frequency.

A similar study was done by using a numerical approach by Khiem and Lien (2001). In this study, a new method for determination of natural frequencies of a beam with cracks was developed. In this research the number of the cracks was also arbitrary. Euler-Bernoulli beam theory was used and the cracks were modeled by rotational springs in a similar manner. Through the use of transfer matrix, computation time was significantly reduced. The authors noticed that there existed a set of positions for the cracks for which the existence of cracks do not affect the natural frequencies of the beam. These positions were called as critical points. Also the authors mention that, number of the cracks, position and depth of the crack had significant effects on natural frequencies.

Although the crack is mostly replaced by rotational springs, there also exist studies in which different mathematical models are developed for crack. For determination of lateral vibration of cracked beams, a continuous theory was developed by Chondros et al. (1998). In the theory Euler-Bernoulli beam model was used with single edge or double edge open cracks. Significant point in the study was that the cracked beam was considered as a one dimensional continuum by distributing the local flexibility due to crack along the whole beam. In the paper, a solution in which the crack is replaced by rotational flexibility was also given. Then a comparison was made between the distributed and local flexibility models. The two theoretical models were also compared by experimental results. The approach that was presented in this research agreed well both with the previous methods and the experimental results.

In most of the studies found in the literature the crack is assumed to be open. The alternative approach takes opening and closing of the crack into account. Chondros, Dimarogonas and Yao (2001) presented a study which deals with flexural vibrations of a beam with such a breathing crack. If the crack is breathing, during vibration the crack successively opens and closes. In this study investigating breathing cracks, Euler-Bernoulli beam theory was used by considering the beam as a one dimensional continuum.

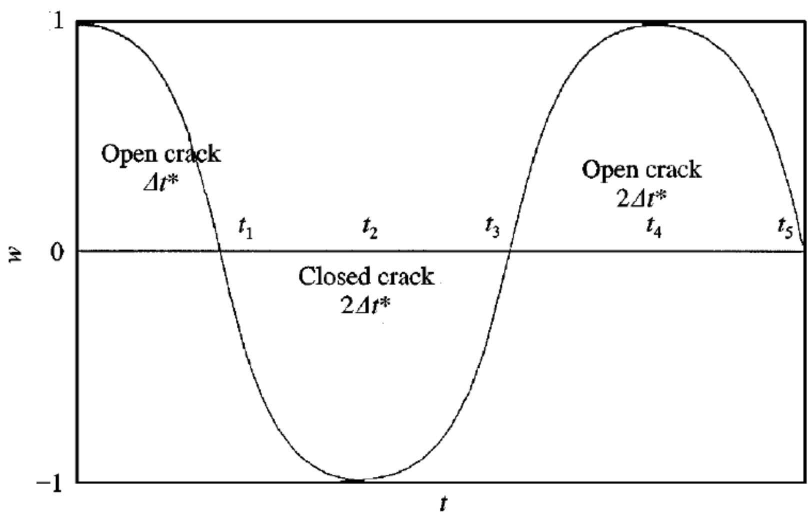

Figure 1.2 Transverse motion of a simply supported prismatic beam with a single-edge crack at mid-span initially bent to its 1^{st} mode, Chondros et al. (2001)

The system characteristics was modeled as bi-linear, in other words the beam was modeled as combination of two linear systems; one linear system for the open crack period and another linear system for the closed crack period. Also, in the study it was assumed that at the transition time when the mode of the crack is passing from open mode to closed mode, the beam was at the undeformed position. From Figure 1.2 it is observed that, at times t_1, t_3 and t_5 , the beam is at undeformed position. The authors concluded that open cracked beam models show a decrease in natural frequencies, but in the case of a breathing crack model, decrease in the natural frequencies are smaller. Also it was mentioned that fatigue cracks behave as breathing cracks if the preload is not sufficient.

An energy based method was developed in order to examine the transverse vibration of non uniform cracked Euler-Bernoulli beams by Mazanoğlu et al. (2009). Rayleigh-Ritz method was used for the solution of natural frequencies. The cracks were taken as open. The open crack assumption was based on the fact that the effect of closing cracks is negligible when the amplitude is small. Effect of multiple cracks was another subject that was studied in this research. Another observation was that if two cracks had come very close to each other, they acted as a single crack. The theoretical results were compared with those obtained by using finite element software. The results seemed to agree well with each other.

There are also studies of cracked beams with mass attachments. In such studies it was found that the attached mass made the effect of the crack more explicit. In one of the studies, transverse vibration of a cracked beam with mass attachment was investigated by Mermertaş et al. (2001). Theoretical model of the beam was built up by using Euler-Bernoulli beam theory. Besides the effect of crack, also how mass attachment affected natural frequencies was also studied. Just like the case in many other researches, the crack was replaced by a rotational spring. The beam was separated into three parts; first part from one end to the mass attachment, second part from the mass attachment to the crack and the third part from the crack to the other end of the beam as shown in Figure 1.3. In the figure W_1,W_2 and W_3 correspond to the flexural displacement of each segment and x_M

and x_C are the corresponding distances of discontinuities caused by mass and crack. Equation of motion for the continuous beam was solved for each part by also including the compatibility equations for the discontinuities and boundary conditions. Using this model authors studied the effect of location of discontinuities and crack depth on transverse vibration, also aiming to provide results for helping with detection of cracks in a beam. In another research addressing the transverse vibration of beams with an edge crack, the cracked beam was modeled with a point mass on an arbitrary place in beam’s span similar to Figure 1.3, by Zhong et al. (2008). Euler-Bernoulli beam theory was used and the calculations were done by extending the Rayleigh method. In other words a polynomial function representing the crack was added to the polynomial function of the simple beam without a crack. The mass attachment was chosen to vary from 0% to 50% of the mass of the beam. The results of the theoretical model were compared with both finite element method results and experimental results. The aim of the study was to investigate effects of crack depth and location of mass attachment on the natural frequency of the beam.

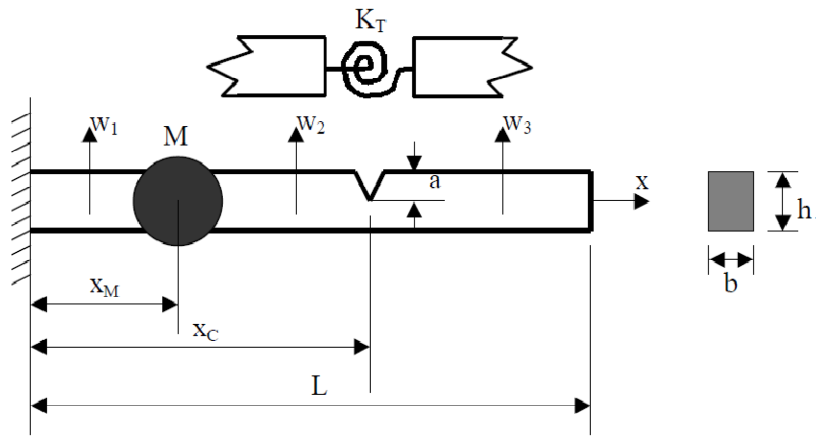

Figure 1.3 Cracked cantilever beam with a point mass, Mermertaş et al. (2001)

The research concluded that the frequencies decreased due to the increase in flexibility of the beam because of the crack. Another important result was that if the mass attachment was in the vicinity of the crack region, the decrease in natural frequency was more significant. A similar study was conducted on a beam containing a crack and with an attached mass whose rotational inertia was neglected by Al-Said et al. (2008). The mathematical model was built by assumed mode methods. Subtracting the frequency of the intact beam with attached mass from the frequency of cracked beam with attached mass, the reduction in the frequency due to crack was investigated. The change in frequency was observed to vary depending on the mass that was attached and its location.

There also exist a number of studies concerning cracked Timoshenko beams. As mentioned previously Timoshenko beam theory makes a difference from Euler-Bernoulli beam theory by including both rotational inertia and shear deformation. Lele et al. (2002) made a study concerning transverse vibration of short Timoshenko beam. In the study, derived method was also used for determination of cracks. The aim of the study was to consider the effects of rotational inertia and shear deformation in short beams. Also, methods were developed for obtaining natural frequencies knowing the beam and crack parameters. In addition, an inverse method was developed through which the location of the crack could be obtained from the knowledge of change in frequency. The crack was modeled by single rotational spring and compliance due to crack was obtained by fracture mechanics. By this way the beam was split into two segments and each segment was connected by the rotational spring that accounts for the crack. After deriving the differential equations that governs the motion and solving them, unknown constants in the solution were determined by applying boundary conditions and compatibility equations due to crack. By using direct methods, natural frequency of the cracked beam was obtained, whereas by using indirect method crack location was obtained, moreover a method was also developed to obtain crack extension.

In one of the researches, a beam with crack was studied by using Timoshenko beam theory by Loya, et al. (2006). For modeling the crack, the beam was split into two segments and then each segment was connected by an extensional and a rotational massless spring as shown in Figure 1.4. In this way discontinuities in both vertical and rotational displacements were implemented. Vertical displacement is proportional to shear force and rotation is proportional to bending moment in the crack section which is replaced by extensional and rotational springs.

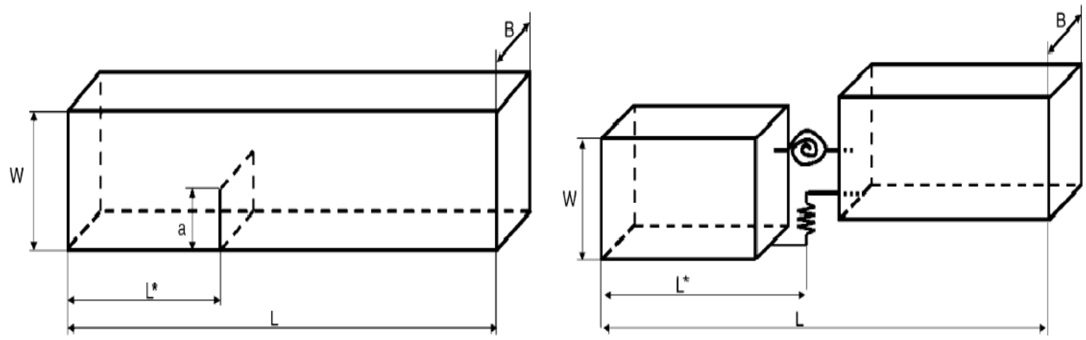

Figure 1.4 Beam with a transverse edge crack, dimensions, and crack model Loya et al. (2006)

The differential equations for transverse vibration is derived and solved for each beam segment. Then boundary conditions for beam ends and compatibility equations for the crack region were applied. Through these steps natural frequency of transverse vibration for cracked Timoshenko beam was obtained. Also perturbation method was applied for the solution of the same problem. It was observed that, perturbation method provided simple expressions for obtaining the natural frequencies. The results for simply supported beam revealed that for shallow cracks; where crack ratio less than 0.3, perturbation method shows agreement with direct solution.

Effect of arbitrary number of cracks on the vibratory characteristics of a beam was also studied by using Timoshenko beam theory, by Li (2003). The beam was modeled by Timoshenko beam theory. Rotational springs were used for modeling the cracks as shown in Figure 1.5. C_1 to C_n correspond to the compliances due to crack region. An analytical approach was proposed by establishing a second order determinant.

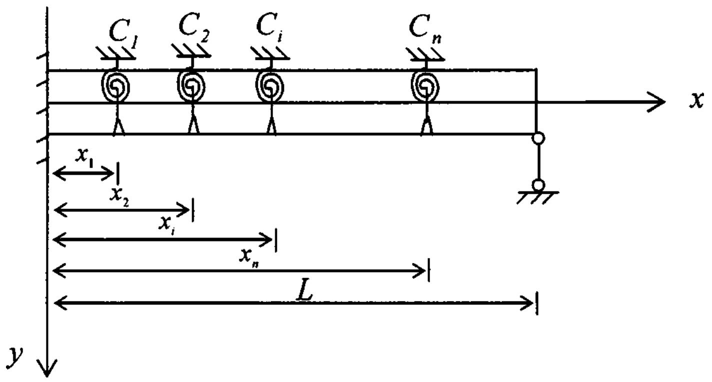

Figure 1.5 Uniform Timoshenko beam with n cracks, Li(2003)

Rotational spring assumption is a linear method and this assumption requires that the crack remains open during vibratory motion. In this study the author mentions that by the analytical approach applied, a second order determinant had been obtained providing a convenient solution by reducing the computation time. Also the numerical example provided by the author shows that the number, depth and the location of the cracks affected the natural frequencies of a Timoshenko beam.

An analysis was presented concerning transverse vibrations of non uniform cross section beams including a crack by Takahashi (1999). The model was developed by using Timoshenko beam theory using a cylindrical beam. In this study, the beam was resting on intermediate supports which were modeled by two springs along the beam span. Local flexibility was carried out by fracture mechanics theories as usual. In the solution procedure, the differential elements were reorganized in terms of first order differential equations providing the transfer matrix. Then by separating the beam into segments between the discontinuities, effect of a crack on a non uniform beam resting on intermediate supports was examined.

Geometrically non linear free vibration of a clamped-clamped beam was studied by El Bikri et al. (2006). The beam was modeled with an open edge crack in its medium. A semi analytic approach was carried out using an extended version of Rayleigh-Ritz method. In the study, admissible functions that satisfy the geometric and natural boundary conditions in addition to the inner boundary conditions were used in the mathematical model. Inner boundary conditions were determined by the crack compatibility equations. These admissible functions were obtained from linear solution of Timoshenko beam with edge crack. As is the usual case in most studies, effect of edge crack on natural fundamental frequency and mode shape was investigated also concerning the non-linear dynamic response near the fundamental resonance.

Chati et al. (1997) studied, modal analysis of a beam containing a transverse edge crack by using finite element method. In this study, opening and closing of the crack was considered which induces a non-linearity in the beam. Nonlinearity caused by opening and closing of the crack was overcome by defining a piecewise linear system which was named as bilinear frequency in the paper. Through the use of finite element method natural frequencies of each linear piece were computed and through these natural frequencies of linear pieces bilinear frequency was obtained according to the method given in the study. In the paper, natural frequencies were also obtained by perturbation method. Comparison of the results showed that piecewise linear system was a good approximation.

There are also studies about beams resting on elastic foundations. In general elastic foundations continuously support the beams along their span and they provide reaction forces or moments which are proportional to the displacements or rotations. Foundation models in literature are mostly one parameter or two parameter models. One parameter foundation model is also referred to as Winkler elastic foundation in which force is directly proportional with flexural displacement of the beam. In two parameter foundation models another parameter is taken into consideration in addition to Winkler parameter. There are different two parameter foundation models in different studies. Different models of two parameter elastic foundation are to be studied in the forthcoming parts of the thesis study. Besides these foundation models a three parameter foundation model was also used in a recent study by Morfidis (2010).

Static analysis of an infinite beam on a two parameter Pasternak foundation was carried out by Ma et al. (2009). The beam was subjected to transverse loads including self weight. The beam was separated into parts depending on the region which was in contact with the foundation and which was not. The results were compared with one parameter Winkler foundation.

A study concerning dynamic behavior of a Timoshenko beam on a single parameter Winkler foundation was conducted by Lee et al. (1992) in order to investigate the effects of foundation modulus, slenderness ratio and elastically restrained boundary conditions. The beam was subjected to a force at both ends. Elastic boundary conditions refer to rotational and extensional springs attached at both ends of the beam. The study concluded that there existed a critical flutter load for a cantilever Timoshenko beam and that flutter load first decreased as the foundation modulus was increased. After a critical foundation modulus, the critical flutter load increased with the foundation modulus.

Some of the studies investigate dynamic characteristics of beams on two parameter foundations. De Rosa (1994) made a study concerning Timoshenko beam resting on a two parameter elastic foundation and two different foundation models were compared. In one of the two parameter foundation models, the second parameter represented the proportionality of the reaction to the bending rotation of the beam whereas in the other model the second parameter represented the proportionality of the reaction to the total rotation of the beam. In a different study a different model for elastic foundation was used for the analysis of a Timoshenko beam on elastic foundation by Wang et al. (1977). The foundation was still modeled as a two parameter foundation where the second parameter is related to the shear layer of the foundation. This model is named as Pasternak foundation model. Effect of elastic foundation parameters on natural frequencies were investigated by using Timoshenko beam theory with different end conditions.

By applying a constant line load on a beam resting on an elastic foundation, the effect of two parameter foundation model was studied by using finite element method by Razaqpur et al. (1991). In this study Euler-Bernoulli beam theory was used. Different types of foundation models were considered such as Filonenko-

Borodich foundation, Pasternak foundation, and Vlasov foundation. Effect of foundation parameters on magnitude of deflection and bending moment was studied.

To see the effect of elastic foundation on flexural vibration, a beam was investigated by using a Timoshenko beam column in a study conducted by Arboleda-Monsalve et al. (2008). The beam was modeled as it was resting on a two parameter elastic foundation. Generalized end conditions were applied for the Timoshenko beam as shown in Figure 1.6.

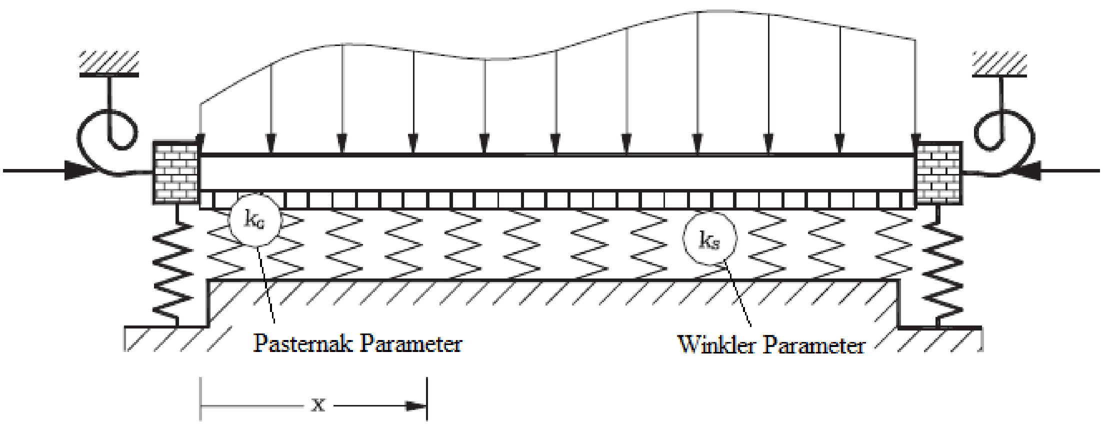

Figure 1.6 Structural Model, Arboleda-Monsalve et al. (2008)

These generalized end conditions are translational and rotational masses, translational and rotational springs and axial forces at both ends of the beam. Also response to the applied distributed shear force on the beam was studied. The model with generalized end conditions resting on an elastic foundation provides solution to the static, dynamic and stability analysis of elastic framed structures. The author points out that the framed structures made of beam columns are highly sensitive to coupling effects such as bending and shear deformation, translational and rotational lumped masses at both ends, translational and rotational masses distributed along the beam span, axial load and shear force due to applied load in terms of static, dynamic and stability behavior.

In another case, the response of Timoshenko beam to a harmonic moving load was studied by Kargarnovin et al. (2004). In the study the beam is considered to be of infinite length having a uniform cross section. The effect of elastic foundation was also another subject of research. The elastic foundation was modeled by a viscoelastic two parameters foundation model or viscoelastic Pasternak model. By this way damping effect due to viscoelastic foundation was also studied. The aim of the study was to simulate the response of the railway and investigate the parameters that affect its vibration characteristics.

For easier computation there are also finite element methods developed for determination of vibration characteristics of beams on elastic foundations. By using finite element method non-linear and forced vibration of a Timoshenko beam resting on two parameter foundation was studied by Zhu et al. (2009). The beam was applied an initial axial load. Also rotary inertia and shear effects were taken into account. The cross section of the beam was non-uniform and axial strain was modeled to be non-linear which results in a non-linear vibration case. Effects of elastic foundation parameters on the natural frequencies were investigated and it was mentioned that due to non linear vibration tensile stress in the beam resting on two parameters elastic foundation came out to be significantly large.

Using the Laplace transform method natural frequencies and mode shapes of a Timoshenko beam with attachments were determined by Magrab (2007). The beam with attachments resting on a Winkler type elastic foundation was modeled as shown in Figure 1.7. The attachments were translational springs, rotational springs and masses without damping. The author used the Laplace transform method, since a solution independent of the number and type of attachment could be obtained by this method. The author concluded that the Laplace transform method had agreed excellently with the other methods and also most of the limitations caused by other methods had been eliminated.

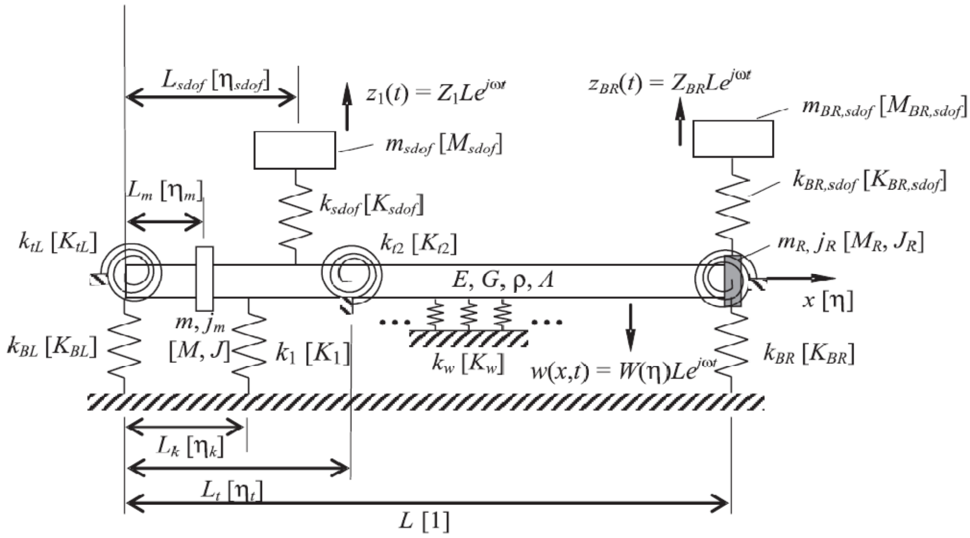

Figure 1.7 Timoshenko beam with attachments, Magrab (2007)

In another study of addressing the problem of transverse vibrations of a Timoshenko beam mounted on an elastic foundation, perturbation method was developed for the solution of the problem by El-Mously (1999). The foundation model was chosen as Pasternak two parameters type as shown in Figure 1.8.

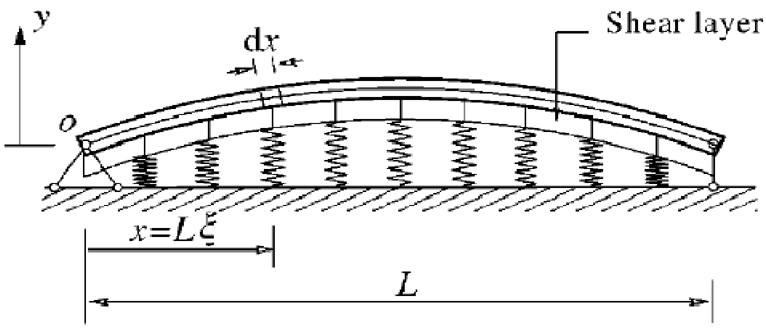

Figure 1.8 Timoshenko beam on Pasternak foundation, El-Mously (1999)

Pasternak foundation considers both extensional stiffness as well as shear stiffness of the elastic foundation. By using perturbation method closed form solutions giving the natural frequencies were obtained. Obtained results were applicable for finite beams mounted on continuous foundations as well as finite beams mounted on finite foundations. However, the author mentioned that the obtained result was not appropriate for beams with free end boundary conditions. In that case, free end boundary conditions cause extra deformation on the elastic foundation axially away from the end of the beam which increases the strain energy.

As pointed out earlier, in literature foundation models are mostly based on one parameter Winkler type foundation or two parameters Pasternak type foundation. In one study concerning the flexural vibration analysis of a beam a three parameter foundation model was used by Morfidis (2010). Vibration characteristics of a beam on three parameter elastic foundation were studied. Three parameters elastic foundation is also known as Kerr elastic foundation. Equations of motions using Timoshenko beam theory on Kerr elastic foundation were derived and the derivations were applied in two numerical examples. The author compared the results of the numerical example by 2D finite element solid models and concluded that use of three parameter foundation model yielded close results to the ones obtained by finite element method.

So far in the literature review, studies including cracked beams and studies including beams on elastic foundations have been considered. There are also studies concerning cracked beams on elastic foundations as depicted in Figure 1.9. In such a study, eigenvalue problem of beams with an edge crack which rests on an elastic foundation was considered by Hsu (2005). The beam was modeled by Euler-Bernoulli beam theory and the foundation model was linear, such that spring force is directly proportional to the transverse displacement of the beam. This is a Winkler type foundation model. For the crack model a compliance value was obtained by fracture mechanics methods. Both opening and closing crack modes were studied. Differential quadrature method was used for modeling the cracked Euler-Bernoulli beam on one parameter elastic foundation. By doing so, the equation of motion for an edge cracked beam is transformed to a discrete form. The author concluded that the first natural frequency increased significantly as the foundation stiffness increased. Also effect of crack depth and location affected the natural frequencies.

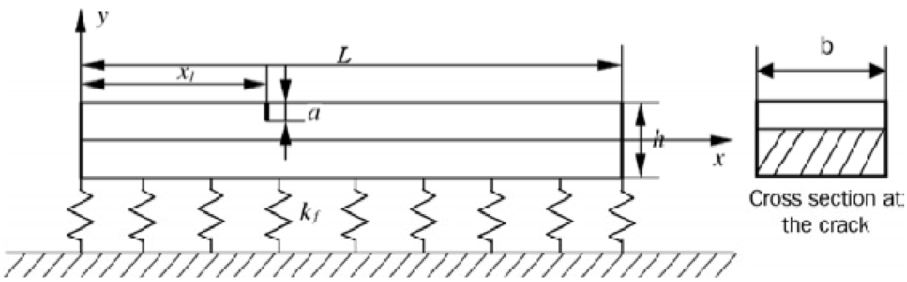

Figure 1.9 Structural system of study, Shin et. al (2006)

In another study of cracked beams on elastic foundations, comparison of different foundation models was carried out by Shin et al. (2006). There were a finite number of transverse edge cracks in the beam. In this research cracks were assumed to be open. For the beam model, Euler-Bernoulli beam theory was used. Two types of foundation models were used; Winkler foundation model and Pasternak foundation model. In the study, the cracks were replaced with massless springs. Spring constant was calculated by using crack compliance value which depends on the crack length. Effect of foundation spring constants, crack length, location of the crack and number of cracks were investigated. Also comparison between two foundation models; Winkler and Pasternak foundation models were done. The study concluded that the natural frequencies of a beam which rests on Pasternak foundation came out to be higher than those of a beam on Winkler foundation except for the case of fixed-fixed boundary conditions. Moreover, a significant effect of crack location on natural frequency was observed. Another observation was the significant decrease of natural frequency as the crack depth increased.

1.2. Objective and Scope of the Thesis

In the thesis study it is aimed to analyze the effect of foundation and crack parameters on natural frequencies of transversely vibrating beams. The beam itself is modeled by both Euler-Bernoulli and Timoshenko beam theories. The equations of motion for transverse vibration, Newton's 2^{nd} law is to be applied on an infinitesimal beam element.

The crack is assumed to be an open edge crack and depth of the crack is constant along the thickness of the beam. Also, the crack is not propagating during vibration, thus there is no crack growth. The representation of the crack will be done by rotational and extensional springs. There exists also an elastic foundation underneath the cracked beam. Depending on the literature review, Winkler, Pasternak and Generalized foundation models are to be used.

In Chapter 2, review of beam theories will be done. Then, equations of motion for Euler-Bernoulli and Timoshenko beam theories are to be derived. The foundation models from literature will presented in detail in Chapter 3 and equations of motion including the elastic foundation will be derived in that chapter. In Chapter 4, a specific review of cracks and crack models from literature are to be carried out. Later, the compatibility equations due to open edge crack will be derived. In Chapters 3 and 4, equations of motion are based on Timoshenko beam theory. In Chapter 5, equations of motion of an edge cracked beam resting on elastic foundation will be derived by Euler-Bernoulli beam theory. Euler-Bernoulli beam solutions of equations of motion are also done in the corresponding chapter. In the solutions simple boundary conditions were applied such as; simply supported, cantilevered, clamped-free. The solution procedure, results of the solutions and the discussion based on the results are all done in Chapter 6.

In the literature reviewed so far, analytical solution of cracked Timoshenko beam on a two parameter elastic foundation has not been considered. In this thesis study, equations of motion of a transversely vibrating beam are derived by considering Timoshenko beam theory, elastic foundation and edge crack at the same time. By this approach natural frequencies are obtained, and effect of crack and foundation parameters on natural frequencies are observed. Timoshenko beam theory solutions are intended to provide benchmark results and facilitate comparison with earlier studies.

CHAPTER 2

BEAM THEORIES

2.1 Beam Theories in Literature

In a study made by Han, Benaroya et al. (1999) dynamics of transversely vibrating beams was investigated by using four different engineering beam theories. One may refer to that study for a brief history of engineering beam theories. According to Han et al. an exact formulation of a vibrating solid cylinder was first investigated in terms of general elasticity equations by Pochhammer (1876) and Chree (1889). However, approximate solutions were needed since Pocchammer's and Chree's studies give more information than necessary. Bending moment, rotation of infinitesimal beam elements, translational and rotational inertias, shear stress and strain are the basic parameters that are used for modeling beams in transverse vibration. Euler-Bernoulli beam, Rayleigh beam, shear bending beam, Timoshenko beam and shear beam are the theories that are developed based on these parameters.

Euler-Bernoulli beam theory is simple and provides reasonable approximations due to the assumptions on which the theory is based. Because of the assumptions of the Euler-Bernoulli beam theory natural frequencies come out to be overestimated. Later, new theories were introduced by eliminating some of the assumptions which were done by Daniel Bernoulli. Later, John William Strutt also known as Lord Rayleigh introduced the rotary inertia to Euler-Bernoulli beam theory and presented Rayleigh beam. Including rotary inertia was a correction of Euler-Bernoulli beam theory but neglecting shear distortion was still a remaining assumption in Rayleigh beam and this gives the results with some overestimation. Another beam theory known as shear bending beam considers shear effect but in this case rotary inertia is neglected. Later it was observed that effect of shear deformation is more important than rotary inertia. The importance of shear deformation is mentioned by Timoshenko, Young and Weaver (1974). Another beam theory was developed which considers only shear deformation, known as shear beam. Shear beam theory can give good results only at high frequency vibrations. In his paper Kausel (2002) mentions that shear beam violates the principles of conservation of momentum for pinned-free and free-free boundary conditions.

Among all these theories Timoshenko beam theory is more complicated and applicable for beams with various thicknesses. In Timoshenko beam theory, rotary inertia and shear distortion are both included in the model whereas in Euler-Bernoulli beam theory, rotary inertia and shear distortion are neglected so that a simpler model is provided. Euler-Bernoulli beam gives closed form expressions for various end conditions and gives good results for beams with high slenderness ratio. Briefly, Timoshenko beam theory is a good correction of Euler-Bernoulli beam theory by including both shear distortion and rotary inertia, also the theory is applicable for beams with various slenderness ratios.

Since Timoshenko beam theory is more comprehensive and accurate compared to the other beam theories, the thesis study is based on this theory. In addition, for comparison purposes also Euler-Bernoulli beam theory is to be studied. The assumptions that both Euler-Bernoulli beam theory and Timoshenko beam theory are based on, derivations of equation of motions with respect to each one of Euler-Bernoulli and Timoshenko beam theories are to be mentioned in detail in the following parts.

2.2 Derivation of Equations of Motion of A Beam in Transverse Vibration by Euler-Bernoulli and Timoshenko Beam Theories

Equation of motion can be derived by either using Newton's approach or Lagrange's method. Newton's method is applied by writing the net force and moment on a differential beam element where as Lagrange's method is an energy method. In both methods, an appropriate mathematical model of the beam must be considered. Euler-Bernoulli beam and Timoshenko beam theories are to be used for modeling the beams due to validity of the assumptions on which the beam theories are based on. In this part, basic knowledge about both of the beam theories are given and the equations of motion are derived by using Newton's 2nd Law. Before focusing on derivation of equation of motions, it is better to mention about the common relations that are to be used in both Euler-Bernoulli and Timoshenko beam theories.

In derivation of the beam models, one dimensional analysis is considered, which means longitudinal dimension of the beam is larger than depth and width of the beam. In the undeformed position all cross sections are placed on a common straight line passing through the center of the cross sections. This straight line is known as the centroidal axis. Centroidal axis and neutral axis are coincident because the cross sectional area is symmetric with respect to z and y axes. In the derivation of the equations, distance along the centroidal axis is denoted as x. Figure 2.1 is a differential beam element on which displacement in longitudinal direction is given by U(x,t) and displacement in transverse direction is given by W(x,t). Rotational motion of a cross section AB is denoted by $\Phi(x,t)$. The dash lines show the deformation of the beam element. As a result of U(x,t), line AB moves axially, thus causing an axial elongation. Transverse displacement W(x,t) causes a rotation of an angle $\Phi(x,t)$ about midpoint of line AB causing point B' to move backward and point A' forward.

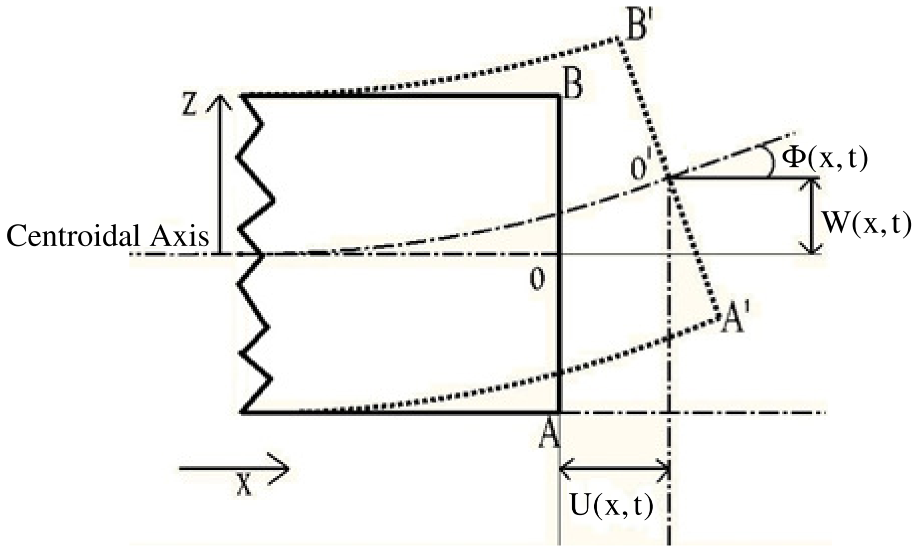

Figure 2.1 Beam Element

In both Euler-Bernoulli beam theory and Timoshenko beam theory deformations are small and angle can be assumed to be small. By this small deformation assumption a linearized model can be derived. So the total displacement on line AB changes linearly by the distance z from the centroidal axis.

$$U_t(x,t) = U(x,t) - z\Phi(x,t) \quad , \tag{2.1}$$

Axial strain in x direction is:

$$\varepsilon_{xx} = \frac{\partial U_t(x,t)}{\partial x} = \frac{\partial U(x,t)}{\partial x} - z\frac{\partial \Phi(x,t)}{\partial x} \quad , \tag{2.2}$$

Another assumption is related to the material of the beams. Linear elastic isotropic homogeneous material model is used in both beam theories. The only non zero normal stress in both theories is the axial one. Furthermore, axial strain is known, so the relation for the normal stress acting on the cross section of the beam element can be found by Hook's Law.

$$\sigma_{xx} = E\varepsilon_{xx} = E\left(\frac{\partial U(x,t)}{\partial x} - z\frac{\partial \Phi(x,t)}{\partial x}\right) \quad , \tag{2.3}$$

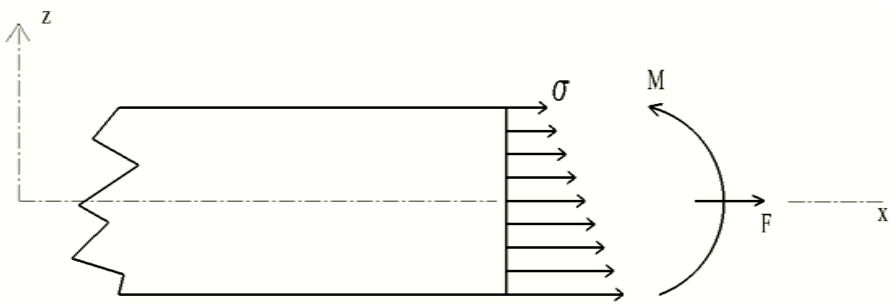

Figure 2.2 Bending moment, axial force and stress distribution

Force acting on an area element on the cross section dA is given by $\sigma_{xx}\,dA$, so the resultant force on the cross section is:

$$F = \int_A \sigma_{xx}\, dA = \int_A E\left(\frac{\partial U(x,t)}{\partial x} - z\frac{\partial \Phi(x,t)}{\partial x}\right) dA \ , \tag{2.4}$$

This force causes a moment about y axis which is perpendicular to centroidal and z axes. Sign convention for the positive bending moment is given in Figure 2.2.

Positive bending moment bends the beam concave up, for example in Figure 2.2 in positive z direction. So depending on this sign convention the differential moment acting on an area element and the resultant moment acting on the cross section are given as:

$$dM = -z\,dF = -z(\sigma_{xx}\, dA) \ , \tag{2.5}$$

$$M = -\int_A z\, E\left(\frac{\partial U(x,t)}{\partial x} - z\frac{\partial \Phi(x,t)}{\partial x}\right) dA \ , \tag{2.6}$$

respectively.

‘A’ is the cross sectional area of the beam. Assuming E to be constant, and U(x,t) and $\Phi(x,t)$ do not depend on y and z, force and moment relations become:

$$F = E\frac{\partial U(x,t)}{\partial x}\int_A dA - \frac{\partial \Phi(x,t)}{\partial x}\int_A z\, dA \,, \tag{2.7}$$

$$M = -E\frac{\partial U(x,t)}{\partial x}\int_A z\,dA + E\frac{\partial \Phi(x,t)}{\partial x}\int_A z^2\,dA\,, \tag{2.8}$$

Where $\int_A z\,dA$ and $\int_A z^2\,dA$ are first and second moments of area respectively. These moments are about y axis which is perpendicular to xz plane. Since z=0 is the centerline of the beam and the origin of the coordinate system is located at the centroid, first moment of area becomes zero. Second moment of area gives I, which is known as area moment of inertia.

$$\int_A z\,dA = 0 \qquad\qquad \int_A z^2\,dA = I\,, \tag{2.9}$$

$$F = EA\frac{\partial U(x,t)}{\partial x}\;, \tag{2.10}$$

$$M = EI\frac{\partial \Phi(x,t)}{\partial x}\;, \tag{2.11}$$

Force and moment relations along the beam axis are derived for a case in which axial and lateral displacements are considered. In some books or papers moment relation is given by a minus sign (etc. -EI…). This minus is due to slope convention, if angle $\Phi(x,t)$ is calculated in the opposite direction of Figure 2.1, minus sign occurs in the moment relation. According to the convention in this work, moment relation is given as in Equation (2.11). As it is seen from Equations (2.10) and (2.11) force is proportional to the first derivative of the axial deformation whereas moment is proportional to the first derivative of rotation of beam element. In this thesis work, axial deformation is not considered so effect of $U(x,t)$ is neglected and only transverse displacement $W(x,t)$ is considered. What $\Phi(x,t)$ depends on shows variations related to the beam theory that is used. Relation of $\Phi(x,t)$ with transverse displacement according to Euler-Bernoulli beam theory and Timoshenko beam theory will be shown respectively.

2.2.1 Derivation of Equation of Motion of a Beam in Transverse Vibration by Euler Bernoulli Beam Theory

In the 18th century Daniel Bernoulli derived the equation of motion of a prismatic bar which is deformed in transverse direction. Derived equations were solved by Leonhard Euler. The theory is also known as classical beam theory or thin beam theory.

Euler-Bernoulli Beam Theory is based on some assumptions. Fundamental assumption for the theory is that planar cross sections remain planar and perpendicular to the centroidal axis when the bar undergoes transverse deformation. So displacement of any point on the cross section depends on the displacement of the centroidal axis. To satisfy the linearity of the theory the curvature of the beam must also be small. By this way a linearized approximation suggests the rotation of the cross section to be equal to the slope of the centroidal axis which is given by $\partial W / \partial x$. So in Euler-Bernoulli beam theory rotation angle $\Phi(x,t)$ is related to W(x,t) by Equation (2.12).

$$\Phi(x,t) = \frac{\partial W(x,t)}{\partial x}, \tag{2.12}$$

In order to apply the theory shear distortion and rotary inertia are neglected. Neglecting rotary inertia and shear distortion is a valid assumption for beams with high slenderness ratios. Since the theory gives accurate results for high slenderness ratios, it is also called as thin beam theory. Stresses are assumed to remain within the elastic limit. One last limitation is that variation of cross sectional area must be small in x direction.

Only displacement in transverse direction is to be considered. The positive sign convention used is as in Figure 2.3.

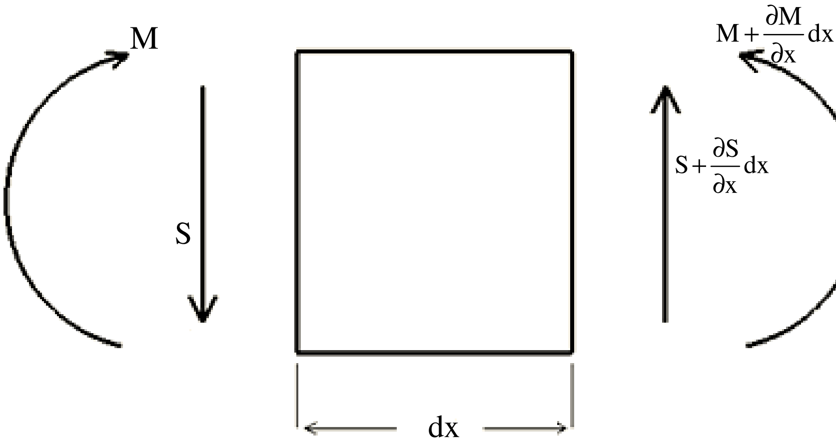

Figure 2.3 Bending moment and shear force

Positive bending moment causes a concave up curvature and a compressive stress on the upper surface of the beam element. Shear stress is positive if it is oriented towards positive z direction on the cross section whose normal is towards the positive x direction.

In Euler-Bernoulli Beam Theory, rotary inertia is neglected. Equation of motion is obtained by applying Newton's 2nd law on the beam element for both translational motion in transverse direction and rotational motion.

Resultant force in transverse direction is proportional to acceleration in the same direction.

$$\sum F_Z = m\ddot{z}, \tag{2.13}$$

If Equation (2.13) is applied for the differential beam element:

$$S+\frac{\partial S}{\partial x}dx - S = dm\,\ddot{W}(x,t) , \tag{2.14}$$

where $dm = \rho A\,dx$ and $\ddot{W}(x,t) = \dfrac{\partial^2 W(x,t)}{\partial t^2}$;

$$S+\frac{\partial S}{\partial x}dx - S = \rho A\,dx\,\frac{\partial^2 W(x,t)}{\partial t^2} , \tag{2.15}$$

$$\frac{\partial S}{\partial x} = \rho A \frac{\partial^2 W(x,t)}{\partial t^2} \quad , \tag{2.16}$$

Total moment on the beam element is proportional to its rotational acceleration.

$$\sum M = I_G \ddot{\Phi}(x,t) \quad , \tag{2.17}$$

Since rotary inertia is neglected $I_G = 0$. So, $\sum M = 0$. Writing the moment balance gives:

$$M + \frac{\partial M}{\partial x}dx - M + (S + \frac{\partial S}{\partial x}dx)\frac{dx}{2} + S\frac{dx}{2} = 0 \quad , \tag{2.18}$$

Higher order derivative terms are neglected. Then equation becomes:

$$S = -\frac{\partial M}{\partial x} \quad , \tag{2.19}$$

Inserting Equation (2.19) into Equation (2.16) gives:

$$-\frac{\partial^2 M}{\partial x^2} = \rho A \frac{\partial^2 W(x,t)}{\partial t^2} \quad , \tag{2.20}$$

Moment is related to W(x,t) through Equation (2.11) and Equation (2.12) , so, after inserting these relations into Equation (2.20), equation of motion for transverse direction according to Euler-Bernoulli Beam theory is obtained.

$$\frac{\partial^2}{\partial x^2}(EI\frac{\partial^2 W(x,t)}{\partial x^2}) + \rho A \frac{\partial^2 W(x,t)}{\partial t^2} = 0 \quad , \tag{2.21}$$

For constant beam properties, Elastic Modulus and second moment of area can be taken out of the derivative parenthesis.

$$EI\frac{\partial^4 W(x,t)}{\partial x^4} + \rho A \frac{\partial^2 W(x,t)}{\partial t^2} = 0 \quad , \tag{2.22}$$

Equation (2.22) is the equation of motion in transverse direction for constant beam properties, derived according to Euler-Bernoulli Beam Theory.

2.2.2 Derivation of Equation of Motion of A Beam in Transverse Vibration by Timoshenko Beam Theory

In the classical beam theory which is also known as Euler-Bernoulli Beam Theory, although there exists a transverse shear stress it is not accompanied by a shear strain. Transverse shear strain is zero as if though modulus of rigidity is infinite. According to Timoshenko beam theory transverse shear stress is related to the shear strain which is taken as constant at a given cross section, whereas in Euler-Bernoulli beam theory transverse shear stress is required to satisfy equilibrium equations. By inspection it is possible to see that for the actual transverse loading, the shear stress and shear strain must vary over the cross section, since shear strain and shear stress are zero at the upper and lower surfaces of the bar. Thus, Timoshenko approximated the effect of the shear as an average over the cross section. As mentioned previously, Euler-Bernoulli beam theory yields good results for beams with high slenderness ratios and at relatively low frequencies, but as the thickness increases or at high frequency vibrations shear distortion becomes important as well as rotary inertia. Timoshenko included effect of rotary inertia which was neglected in Euler-Bernoulli beam theory. Timoshenko beam theory is also known as thick beam theory because the theory is applicable for beams with any slenderness ratio.

In Timoshenko Beam Theory, deformations due to moment and shear stress are imposed by superposition. Distortion due to shear and moment are considered separately. Then total distortion from the undeformed position is obtained by adding the effects of shear strain and flexural deformation. Distortions due to shear and bending moment in Timoshenko Beam are modeled as in Figure 2.4.

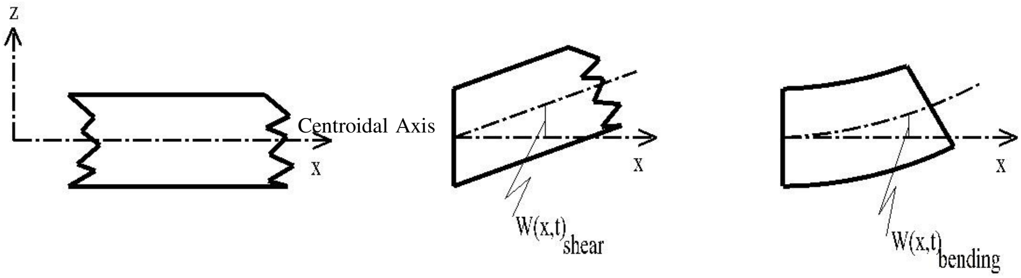

Figure 2.4 Shear distortion and bending distortion

W(x,t) is displacement in transverse direction. This flexural displacement is the same as the one in Figure 2.1. $W(x,t)_{shear}$ is deformation due to only shear strain whereas $W(x,t)_{bending}$ is the deformation due to only bending moment. Both $W(x,t)_{shear}$ and $W(x,t)_{bending}$ are measured vertically from the centroidal axis which is denoted by x.

Shear strain is the change in the angle of an infinitesimal rectangular beam element as shown in Figure 2.5. In Figure 2.5, infinitesimal rectangular beam element has undergone a shear deformation by angle β. Shear deformation β is known as the Engineering Shear Strain. There exists also Tensorial Shear Strain definition which is half of the change in the angle of the rectangular beam element.

$$\varepsilon_{xz} = \frac{1}{2}\beta , \qquad (2.23)$$

From Hooke's Law shear stress and shear strain relation can be obtained.

$$\sigma_{xz} = 2G\varepsilon_{xz} = G\beta \ , \qquad (2.24)$$

G is the shear modulus of the material.

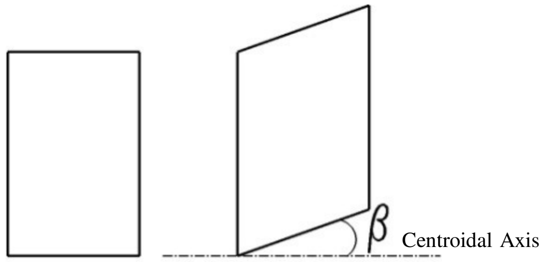

Figure 2.5 Shear strain

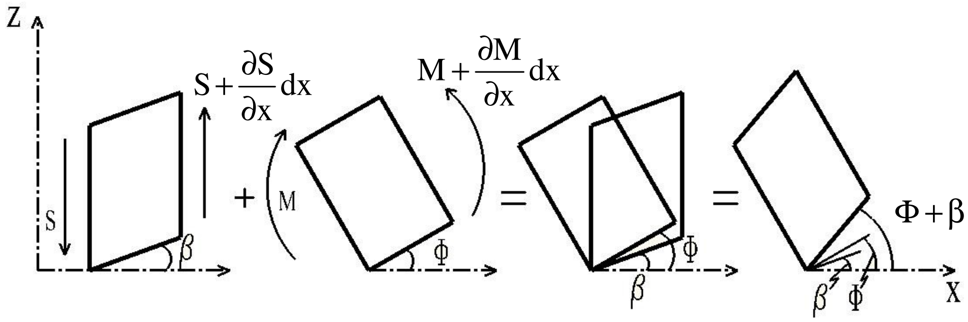

Figure 2.6 Superposition of deformations due to shear and bending moment

In Timoshenko Beam Theory, deformation due to shear and bending are superposed as in Figure 2.6.

Same sign convention which is used for Euler-Beam Theory is also used in this part. Positive shear is towards positive z direction, on the cross section that is oriented towards positive x direction and positive moment is as shown in the figure. Due to shear effect, centroidal axis rotates as $\beta(x,t)$. Also bending moment causes a rotation of the cross section as $\Phi(x,t)$ shown in Figure 2.1. Since small deflections are considered total rotation of the centroidal axis can be given by $\partial W(x,t)/\partial x$. So in Timoshenko beam theory rotation of the infinitesimal beam element and shear distortion add up to give the total slope of the centroidal axis.

$$\Phi(x,t)+\beta(x,t)=\frac{\partial W(x,t)}{\partial x}\ , \tag{2.25}$$

Deformation due to bending causes axial displacement. Consider a point at a distance z from the centroidal axis as shown in Figure 2.1. Points above the centroidal axis move backward while points below the centroidal axis move forward.

Using the stress strain relations given by Hooke's Law, shear force and bending moment can be found in terms of transverse and rotational displacements respectively. Moment relation is given by Equation (2.11). From the derivations it is seen that $\Phi(x,t)$ is not directly equal to $\frac{\partial W(x,t)}{\partial x}$ as in the case of Euler-Bernoulli beam theory.

Shear force S is given by:

$$S = \int_A \sigma_{xz}\, dA = \int_A G\, \beta(x,t)\, dA \quad , \tag{2.26}$$

From Equation (2.25) shear strain relation can be obtained in terms of flexural and rotational displacements.

$$\beta(x,t) = \frac{\partial W(x,t)}{\partial x} - \Phi(x,t) \, , \tag{2.27}$$

Inserting Equation (2.27) into Equation (2.26) and taking the integral shear force is found.

$$S = G\, A\, \left(\frac{\partial W(x,t)}{\partial x} - \Phi(x,t)\right) \quad , \tag{2.28}$$

Timoshenko assumed an average shear force over the cross sectional area. Above expression is not a function of z, therefore constant over the area but actually shear is not constant over the area, it is zero at the top and bottom surfaces of the beam. To take consideration of this variation Timoshenko derived an average shear force; so S is multiplied by a shear coefficient denoted by k. Shear coefficient or the area reduction factor is a function of Poisson's ratio, shape of the cross section and frequency as mentioned in the study made by Han et al. (1999). Literature review of shear correction coefficient is given in detail in Appendix A. Introducing the shear correction equation (2.28) is rewritten as;

$$S = k\,G\,A\left(\frac{\partial W(x,t)}{\partial x} - \Phi(x,t)\right), \tag{2.29}$$

Since the bending moment and shear force equations are known, by applying Newton's Second Law on a differential beam element equation of motion can be obtained for Timoshenko Beam Theory. As a differential beam element Figure 2.3 is to be considered, following the same procedure as in the Euler-Bernoulli Beam case, resultant force in vertical direction gives the relation below.

$$\frac{\partial S}{\partial x} = \rho A \frac{\partial^2 W(x,t)}{\partial t^2} \quad , \tag{2.30}$$

Relation for shear force is already derived; inserting equation (2.29) into equation (2.30) gives:

$$\frac{\partial}{\partial x}\left(k\,G\,A\left(\frac{\partial W(x,t)}{\partial x} - \Phi(x,t)\right)\right) = \rho A \frac{\partial^2 W(x,t)}{\partial t^2} \quad , \tag{2.31}$$

Total moment on the beam element in Figure 2.3 gives:

$$\sum M_y = I_G\,\ddot{\Phi}(x,t) \quad , \tag{2.32}$$

$$M + \frac{\partial M}{\partial x}dx - M + \left(S + \frac{\partial S}{\partial x}dx\right)\frac{dx}{2} + S\frac{dx}{2} = dI_G \frac{\partial^2 \Phi(x,t)}{\partial t^2} \quad , \tag{2.33}$$

For bodies with constant thickness, mass moment of inertia is related to second moment of the cross sectional area of the body.

$$I_G = \int z^2\,dm, \tag{2.34}$$

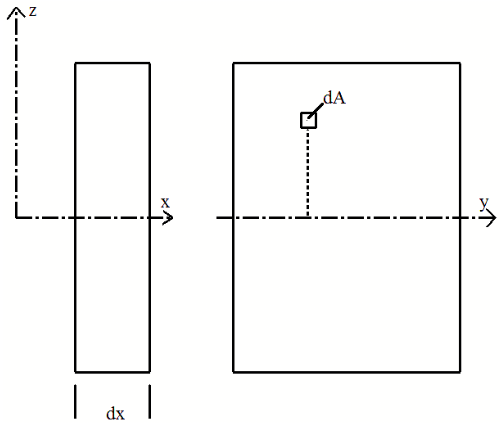

Figure 2.7 Rotational inertia for differential beam element

For a body with constant thickness infinitesimal mass becomes:

$$dm = \Delta x \, \rho \, dA, \tag{2.35}$$

Replacing equation (2.35) into equation (2.34) one obtains:

$$I_G = \int z^2 \, \Delta x \, \rho \, dA = \Delta x \, \rho I, \tag{2.36}$$

Equation (2.36) gives mass moment of inertia for the whole thickness. For the infinitesimal beam element shown in Figure (2.7), thickness is dx so mass moment of inertia for rotational motion of infinitesimal mass element is:

$$dI_G = \rho I \, dx \quad , \tag{2.37}$$

After doing cancellations in Equation (2.33) and replacing rotary inertia the following relation is derived.

$$\frac{\partial M}{\partial x} dx + S \, dx = \rho I \frac{\partial^2 \Phi(x,t)}{\partial t^2} dx, \tag{2.38}$$

Inserting the bending moment equation (2.11) and shear force relations (2.29), which are derived previously, into equation (2.38) gives the second differential equation of Timoshenko Beam Theory.

$$\frac{\partial}{\partial x}(EI\frac{\partial \Phi(x,t)}{\partial x})+kGA(\frac{\partial W(x,t)}{\partial x}-\Phi(x,t))=\rho I\frac{\partial^2\Phi(x,t)}{\partial t^2} \quad , \tag{2.39}$$

Equation of motion is a set of differential equation for Timoshenko Beam. Solution of these differential equations gives the rotational and transverse deformations.

$$\frac{\partial}{\partial x}(kGA(\frac{\partial W(x,t)}{\partial x}-\Phi(x,t)))=\rho A\frac{\partial^2 W(x,t)}{\partial t^2} \quad , \tag{2.40}$$

$$\frac{\partial}{\partial x}(EI\frac{\partial \Phi(x,t)}{\partial x})+kGA(\frac{\partial W(x,t)}{\partial x}-\Phi(x,t))=\rho I\frac{\partial^2\Phi(x,t)}{\partial t^2} \quad , \tag{2.41}$$

If the beam parameters are constant, equations of motion take the form below:

$$kGA(\frac{\partial^2 W(x,t)}{\partial x^2}-\frac{\partial \Phi(x,t)}{\partial x})=\rho A\frac{\partial^2 W(x,t)}{\partial t^2} \quad , \tag{2.42}$$

$$EI\frac{\partial^2 \Phi(x,t)}{\partial x^2}+kGA(\frac{\partial W(x,t)}{\partial x}-\Phi(x,t))=\rho I\frac{\partial^2\Phi(x,t)}{\partial t^2} \quad , \tag{2.43}$$

CHAPTER 3

TRANSVERSE VIBRATION OF TIMOSHENKO AND EULER-BERNOULLI BEAMS ON TWO PARAMETER ELASTIC FOUNDATIONS

3.1Elastic Foundation Models in Literature

A comprehensive article that contains a study of a number of foundation models and development of some models was published by Kerr (1964). In the study of foundation models, the attitude was based on considering the response of the foundation surface to applied loads. Thus, stresses caused within the foundation are not considered. As it was previously stated, in Winkler elastic foundation, force exerted by the foundation is proportional to transverse displacement of the beam. In addition, displacement of the foundation outside the loaded region is zero as shown in Figure 3.1. W(x,t) being the transverse displacement of the beam, the governing equation for Winkler elastic foundation is given by:

$$P(x,t) = k_w W(x,t), \tag{3.1}$$

where P(x,t) is the force per unit length exerted by the foundation and k_w is Winkler foundation constant which has the dimension of force per unit length per unit displacement. Note that Winkler foundation behaves like an array of independent springs as shown in Figure 3.1

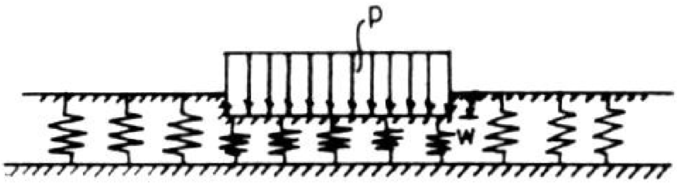

Figure 3.1 Elastic Winkler foundation, Kerr (1964)

Winkler elastic foundation has a single parameter and as it can be seen from equation (3.1) the only foundation parameter is k_w. In literature equation (3.1) was used in the studies made by Shin et al. (2006), Hsu (2005), De Rosa (1994) and Lee et al. (1990).

In two parameter foundation models, first parameter of the foundation is still Winkler elastic foundation parameter. In literature there are different mathematical models for the second elastic foundation parameter. In most of the studies in literature, shearing stiffness of the elastic foundation which is discarded in Winkler elastic foundation is taken into consideration as the second parameter of the elastic foundation. Wang and Stephans (1977) made a study including the vibration characteristics of Timoshenko beam on elastic foundation. In this study shear modulus of the elastic foundation was also included.

Also, Razaqpur and Shah (1991) gave comprehensive definitions for the second parameter of the elastic foundation. Razaqpur and Shah used the following mathematical model for the elastic foundation:

$$P(x) = kW(x) - k_1 \frac{d^2W(x)}{dx^2}, \qquad (3.2)$$

where P(x) is foundation reaction, k is Winkler foundation parameter, and k_1 is second elastic foundation parameter which has the dimension of force. Razaqpur and Shah (1991) and Kerr (1964) mentioned about four different models for k_1 depending on the foundation type. These are Filonenko-Borodich foundation, Pasternak Foundation, Generalized foundation and Vlasov foundation. In two parameter foundation models, the second parameter defines the interaction between the springs of the foundation. In Filonenko-Borodich foundation, k_1 is a constant tension in an elastic membrane which connects the top ends of the springs.

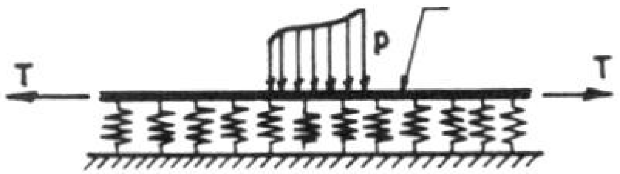

Figure 3.2 Filonenko-Borodich Elastic Foundation, Kerr (1964)

The definition of the two parameter foundation model with shear modulus or transverse modulus is known as Pasternak foundation model as mentioned by Razaqpur and Shah (1991) and Kerr (1964). For Pasternak foundation, it is assumed that there exists a shear interaction between the springs, so k_1 is a parameter of shear layer. For the generalized foundation model k_1 becomes a reaction moment per unit length per unit rotation. According to generalized foundation model, between the beam and foundation besides pressure there is also moment induced by the foundation and this moment is proportional with the angle of rotation of the beam element. In Vlasov foundation, k_1 is obtained in terms of elastic constants and the dimensions of beam and foundation. In this model foundation is assumed to be a semi-infinite medium and k_1 is given by:

$$k_1 = \frac{E_s}{4(1+\nu_s)} \frac{B}{\mu}, \tag{3.3}$$

where, E_s is Young's modulus of foundation, ν_s is Poisson's ratio of foundation, B is width of the beam and μ is a rate at which vertical deformation of foundation decays with. Besides these foundation models, Kerr (1964) mentioned about two different foundation models which are Hetenyi foundation and Reissner foundation. In Hetenyi foundation interaction between spring elements is accomplished by imbedding an elastic beam for two dimensional cases, whereas for three dimensional cases a plate is imbedded to accomplish the interaction between springs of the elastic foundation. Reissner foundation is derived from the equations of continuum and in-plane stresses throughout the foundation layer are very small.

El-Mously (1999) used two parameters elastic foundation model in which the second parameter was shear modulus of the foundation. Shin et al. (2006) also defined the second parameter as shear modulus of the foundation. Same model was also used by Zhu and Leung (2009). Arboleda-Monsalve et al. (2008) used the same model giving a different name to the second parameter of the foundation as transverse modulus. Ma et al. (2009) mentioned that, in Pasternak foundation model, curvature of the elastic foundation was also considered as well as the displacement of the foundation. The governing equation of Pasternak elastic foundation was expressed through equation (3.4) by Ma et al.

$$P(x,y) = k\,W - k_G\,\nabla^2 W\,, \tag{3.4}$$

In the thesis study, variations in y direction are not considered so equation (3.4) becomes for the dynamic case:

$$P(x,t) = kW(x,t) - k_G\,\frac{\partial^2 W(x,t)}{\partial x^2}\;, \tag{3.5}$$

Equation (3.5) is also given in Wang and Stephens (1977).

Kargarnovin and Younesian (2004) presented a generalized Pasternak type viscoelastic foundation in a study investigating vibration of Timoshenko beam under moving load. But in that study model of the second parameter was different. Since viscoleastic foundation model was used damping effect was also included. The governing equations for Pasternak type viscoelastic foundation were given as:

$$P(x,t) = -k\,W(x,t) - c\frac{\partial W(x,t)}{\partial t} + \mu\frac{\partial^3 W(x,t)}{\partial t\,\partial x^2}\,, \tag{3.6}$$

$$M(x,t) = -k_\Phi \Phi(x,t) - c_\Phi\,\frac{\partial \Phi(x,t)}{\partial t}\;, \tag{3.7}$$

where M(x,t) and P(x,t) are moment and pressure induced by the viscoelastic foundation. k and k_Φ are foundation parameters for transverse loading and rotational loading respectively. "c" with subscript and without subscript are damping coefficients and μ is shear viscosity coefficient of the foundation. If viscosity and damping coefficients in equations (3.6) and (3.7) are ignored, two parameter foundation model is defined by only pressure and moment relations. Such a model was used by De Rosa (1995) which is simply the generalized elastic foundation model. In the study of De Rosa, two different second elastic foundation parameters were defined for the generalized foundation. First one is given by the following equation:

$$M(x,t) = k_\Phi \Phi(x,t), \tag{3.8}$$

where M(x,t) is moment stimulated by the elastic foundation and k_Φ is rotational stiffness of the foundation. It is obvious that equation (3.8) is undamped model of equation (3.7). In equation (3.8), $\Phi(x,t)$ is rotation of the beam due to bending moment. De Rosa also presented another model for the second foundation parameter in which, total rotation of the beam is considered instead of rotation due to bending moment. This model has the governing equation below:

$$M(x,t) = k_v \frac{\partial W(x,t)}{\partial x}, \tag{3.9}$$

where the derivative term $\partial W(x,t)/\partial x$ gives the total rotation of the beam and k_v is second parameter of the elastic foundation and refers to the rotational stiffness. As mentioned previously the models used by De Rosa are known as generalized foundation models in literature according to Razaqpur and Shah (1991) and Kerr (1964).

In Morfidis (2010), Kerr foundation, which is a three parameter foundation, was used for modeling Timoshenko beam resting on soil as shown in Figure 3.3. It is mentioned that Kerr foundation was a generalization of two parameter elastic foundation since it also takes into account the effect of soil on either side of the beam ends which means effect of deformation of Region I and Regıon III are also considered. In addition, at the boundaries between the loaded and unloaded region, discontinuous soil surface deformation is also considered by Kerr foundation. In the figure, C, G and K refer to the parameters of the foundation model.

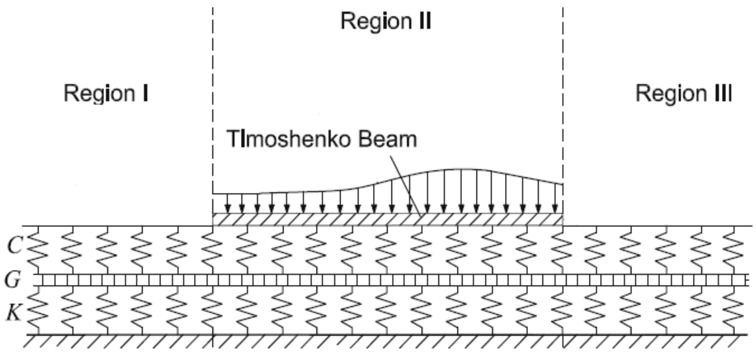

Figure 3.3 Three parameter Kerr elastic foundation, Morfidis (2010)

Depending on the characteristics of the material, different models for elastic foundations may be developed. Kerr (1964) determined that, two parameters Pasternak foundation was the most natural extension of the Winkler model among other homogenous foundation models that exist in literature. This determination is also supported by the fact that most of the researchers chose two parameter Pasternak elastic foundation model in many studies. Also, it was mentioned that two parameter Pasternak foundation, where the second parameter is shear modulus, can be simply applied with satisfactory results in transverse vibration studies.

In the thesis study, elastic foundation on which cracked Timoshenko beam rests is assumed to be homogenous and isotropic. In addition, there is neither separation of the beam from the elastic foundation nor any discontinuity in the boundary between the elastic foundation and Timoshenko beam. So, regarding information from literature, due to validity and simplicity two parameter elastic foundation models are to be used in the thesis. In the forthcoming sections, mathematical models of transversely vibrating Timoshenko and Euler-Bernoulli beams attached to two parameter elastic foundations (Pasternak and generalized) are constructed.

3.2 Derivation of Equations of Motion of A Timoshenko Beam In Transverse Vibration on Elastic Foundations

In this section, equations of motion for Timoshenko beam on elastic foundations will be derived. The foundation models that will be used are two parameter Pasternak foundation and two different models for generalized foundation models. The reason of using these models is verification of their validity by many authors in literature as mentioned previously. After the equations of motion are derived, the equations will be rewritten as a system of differential equations at the end of each sub section, to provide convenience.

3.2.1 Transverse Vibration of A Timoshenko Beam On Two Parameter Pasternak Foundation

As mentioned previously, two parameters Pasternak foundation includes both Winkler elastic foundation parameter and another parameter related to the shear interaction of the foundation. As shown in Figure 3.4, foundation is composed of infinitely many springs and in Pasternak foundation there is shear interaction between these springs which is expressed by shear modulus G_o . In the figure the shear interaction is modeled by an incompressible shear layer connected to the Winkler foundation from top.

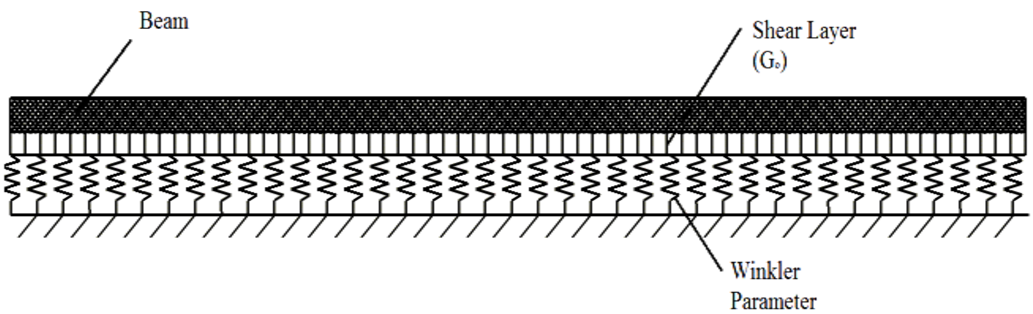

Figure 3.4 Structural model of beam on Pasternak foundation

The reaction of Pasternak foundation per unit length due to transverse displacement W(x,t) of beam is given by:

$$q(x,t) = k_w W(x,t) - G_0 \frac{d^2W(x)}{dx^2}, \tag{3.10}$$

By applying Newton's 2nd law on a differential beam element, equations of motion of Timoshenko beam on two parameters Pasternak foundation can be derived. For the derivation of equations of motion, dynamic force equilibrium in 'z' direction and dynamic moment equilibrium in rotational direction are written. Then using the governing equations for the corresponding beam theory and foundation model, equations defining the motion are obtained. As mentioned in equation (3.10), reaction of the Pasternak foundation is given by per unit length, so force applied by the elastic foundation on Timoshenko beam is:

$$F_{foundation} = q(x,t)\,dx = k_w W(x,t)dx - G_0 \frac{d^2W(x)}{dx^2}dx, \tag{3.11}$$

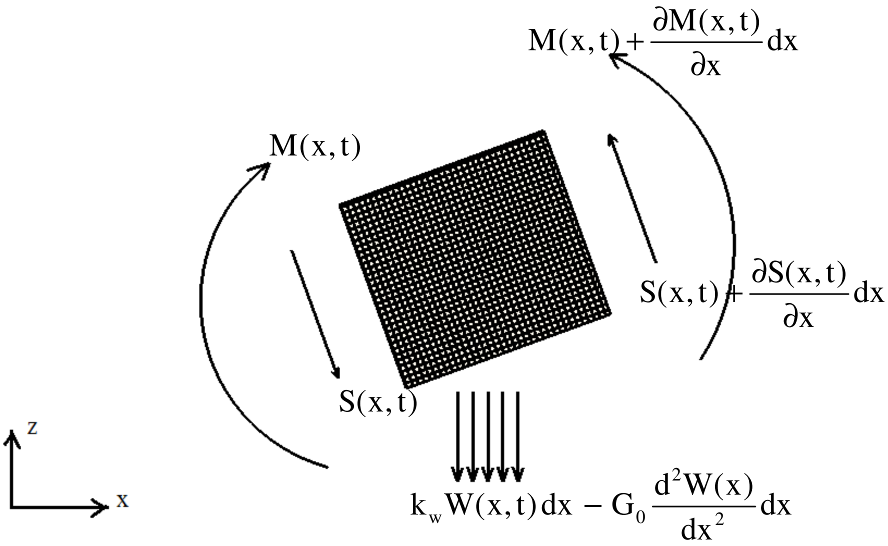

Figure 3.5 Moments and forces for beam on Pasternak foundation

Newton's law in z direction states:

$$\sum F_Z = m\frac{\partial W^2(x,t)}{\partial t^2}, \tag{3.12}$$

$$-S(x,t)+S(x,t)+\frac{\partial S(x,t)}{\partial x}dx-k_w W(x,t)dx+G_0\frac{\partial^2 W(x,t)}{\partial x^2}dx=\rho A dx\frac{\partial^2 W(x,t)}{\partial t^2}, \tag{3.13}$$

Equation (3.13) can be simplified by doing the necessary cancellations:

$$\frac{\partial S(x,t)}{\partial x}-k_w W(x,t)+G_0\frac{\partial^2 W(x,t)}{\partial x^2}=\rho A\frac{\partial^2 W(x,t)}{\partial t^2}\ , \tag{3.14}$$

Shear force S(x,t) for Timoshenko beam was previously mentioned, replacing equation of shear force results in one of the equations of motion.

$$\frac{\partial}{\partial x}\left[kGA(\frac{\partial W(x,t)}{\partial x}-\Phi(x,t))\right]-k_w W(x,t)+G_0\frac{\partial^2 W(x,t)}{\partial x^2}=\rho A\frac{\partial^2 W(x,t)}{\partial t^2}, \tag{3.15}$$

In thesis work, beam properties and cross sectional area is assumed to be homogenous and uniform respectively, so equation (3.15) simplifies to:

$$kGA(\frac{\partial^2 W(x,t)}{\partial x^2}-\frac{\partial \Phi(x,t)}{\partial x})-k_w W(x,t)+G_0\frac{\partial^2 W(x,t)}{\partial x^2}-\rho A\frac{\partial^2 W(x,t)}{\partial t^2}=0, \tag{3.16}$$

Second equation defining the motion of Timoshenko beam on elastic Pasternak foundation is to be derived by using dynamic moment equilibrium.

$$\sum M = I_G\frac{\partial \Phi^2(x,t)}{\partial t^2}, \tag{3.17}$$

$$\begin{aligned}-M(x,t)+M(x,t)+\frac{\partial M(x,t)}{\partial x}dx+S(x,t)\frac{dx}{2}+S(x,t)\frac{dx}{2}\\ +\frac{\partial S(x,t)}{\partial x}dx\frac{dx}{2}=\rho I dx\frac{\partial^2 \Phi(x,t)}{\partial t^2}\end{aligned}, \tag{3.18}$$

By doing necessary cancellations and neglecting higher order derivatives, equation (3.18) is simplified to equation (3.19):

$$\frac{\partial M(x,t)}{\partial x}+S(x,t)=\rho I dx\frac{\partial^2 \Phi(x,t)}{\partial t^2}, \tag{3.19}$$

Inserting moment (2.11) and shear force relations for Timoshenko beam (2.29) into equation (3.19) gives the second equation that defines Timoshenko beam on elastic Pasternak foundation.

$$\frac{\partial}{\partial x}(EI\frac{\partial \Phi(x,t)}{\partial x})+kGA(\frac{\partial W(x,t)}{\partial x}-\Phi(x,t))=\rho I\frac{\partial^2 \Phi(x,t)}{\partial t^2}, \tag{3.20}$$

Since uniform cross section and homogenous material properties are assumed equation (3.20) becomes:

$$EI\frac{\partial^2 \Phi(x,t)}{\partial x^2}+kGA(\frac{\partial W(x,t)}{\partial x}-\Phi(x,t))-\rho I\frac{\partial^2 \Phi(x,t)}{\partial t^2}=0, \tag{3.21}$$

As a result, equations of motion for Timoshenko beam on two parameter elastic Pasternak foundation is defined by the following system of differential equation which includes equations (3.16) and (3.21):

$$kGA(\frac{\partial^2 W(x,t)}{\partial x^2}-\frac{\partial \Phi(x,t)}{\partial x})-k_w W(x,t)+G_0\frac{\partial^2 W(x,t)}{\partial x^2}-\rho A\frac{\partial^2 W(x,t)}{\partial t^2}=0$$

$$EI\frac{\partial^2 \Phi(x,t)}{\partial x^2}+kGA(\frac{\partial W(x,t)}{\partial x}-\Phi(x,t))-\rho I\frac{\partial^2 \Phi(x,t)}{\partial t^2}=0$$

3.2.2 Transverse Vibration of A Timoshenko Beam On Two Parameter Generalized Elastic Foundation

Generalized elastic foundation model was previously mentioned by Kerr (1964) and De Rosa (1995). Briefly generalized elastic foundation includes two parameters. First parameter is simply Winkler foundation parameter. If the elastic foundation is considered as an array of infinitely many springs, the second parameter of generalized foundation defines a moment interaction between these springs. This moment reaction is directly proportional to the rotation of beam elements.

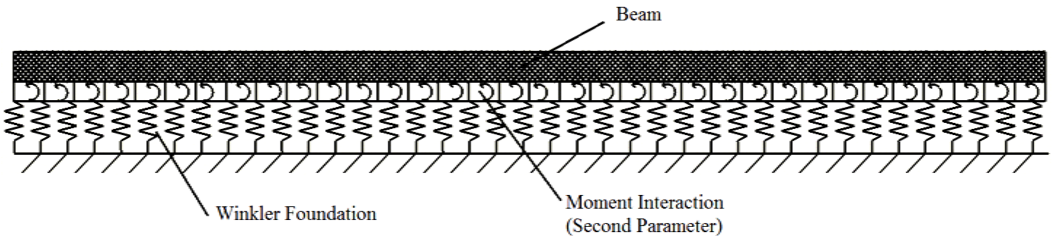

Figure 3.6 Structural model of beam on generalized foundation

Figure 3.6 symbolically shows the reactions of generalized foundation. The layer of moment reaction is incompressible and only deforms in sense of bending.

De Rosa (1995) expressed two different models for the second parameter of generalized elastic foundation. In the thesis study, these two different models are to be denoted by "Generalized Foundation Model 1" and "Generalized Foundation Model 2".

3.2.2.1 Transverse Vibration of A Timoshenko Beam on Generalized Foundation Model 1

It was mentioned previously that the moment reaction of generalized foundation was directly proportional to rotation of the beam elements. In generalized foundation model 1, the moment reaction of the foundation is considered to be proportional to rotation of the beam due to bending only. So, governing equation for the second foundation parameter is defined by equation (3.8).

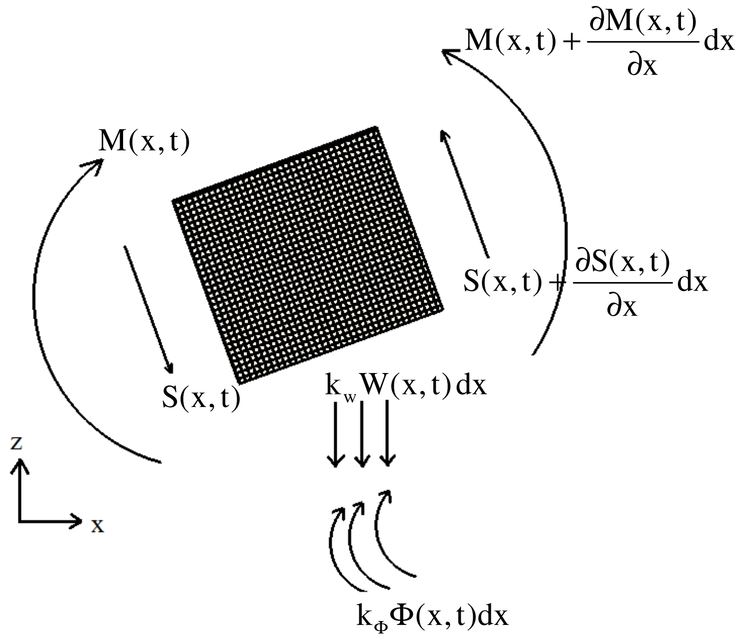

Figure 3.7 Moments and forces for beam on generalized foundation model 1

Equations of motion are to be obtained by applying Newton's 2nd law on a differential beam element as shown in Figure 3.7.

Firstly dynamic force equilibrium in z direction is to be written just regarding Newton's law $\sum F_z = m\,\partial^2 W(x,t)/\partial t^2$.

$$-S(x,t)+S(x,t)+\frac{\partial S(x,t)}{\partial x}dx-k_w W(x,t)dx=\rho A dx\frac{\partial^2 W(x,t)}{\partial t^2}, \tag{3.22}$$

Equation (3.22) is simplified by doing the necessary cancellations.

$$\frac{\partial S(x,t)}{\partial x}-k_w W(x,t)=\rho A\frac{\partial^2 W(x,t)}{\partial t^2}, \tag{3.23}$$

By replacing definition of shear force for Timoshenko beam (2.29) into equation (3.23), all differential terms can be written in terms of transverse displacement and rotation due to bending.

$$\frac{\partial}{\partial x}\left[kGA(\frac{\partial W(x,t)}{\partial x}-\Phi(x,t))\right]-k_w W(x,t)=\rho A\frac{\partial^2 W(x,t)}{\partial t^2}, \tag{3.24}$$

Since uniform cross sectional area and homogenous material properties are assumed, equation (3.24) can be rewritten as below:

$$kGA(\frac{\partial^2 W(x,t)}{\partial x^2}-\frac{\partial\Phi(x,t)}{\partial x})-k_w W(x,t)-\rho A\frac{\partial^2 W(x,t)}{\partial t^2}=0, \tag{3.25}$$

By writing the dynamic moment equilibrium, second equation of motion that is defined for Timoshenko beam on generalized foundation model 1 can be obtained. Moment equilibrium suggests $\sum M(x,t) = I_G\ \partial^2 W(x,t)/\partial x^2$.

$$\begin{aligned}-M(x,t)+M(x,t)+\frac{\partial M(x,t)}{\partial x}dx+S(x,t)\frac{dx}{2}+S(x,t)\frac{dx}{2}\\ +\frac{\partial S(x,t)}{\partial x}dx\frac{dx}{2}-k_\Phi\Phi(x,t)=\rho I dx\frac{\partial^2\Phi(x,t)}{\partial t^2}\end{aligned}, \tag{3.26}$$

After doing necessary cancellations and neglecting higher order derivative terms in equation (3.26) a simplified relation is obtained.

$$\frac{\partial M(x,t)}{\partial x}+S(x,t)-k_\Phi\Phi(x,t)=\rho I\frac{\partial^2\Phi(x,t)}{\partial t^2}, \tag{3.27}$$

Inserting definitions of moment (2.11) and shear force (2.29) with respect to Timoshenko beam into equation (3.27), it can be written in terms of transverse and rotational displacements.

$$\frac{\partial}{\partial x}(EI\frac{\partial\Phi(x,t)}{\partial x})+kGA(\frac{\partial W(x,t)}{\partial x}-\Phi(x,t))-k_\Phi\Phi(x,t)=\rho I\frac{\partial^2\Phi(x,t)}{\partial t^2}, \tag{3.28}$$

Finally, assumption of uniform beam properties simplifies equation (3.28):

$$EI\frac{\partial^2\Phi(x,t)}{\partial x^2}+kGA(\frac{\partial W(x,t)}{\partial x}-\Phi(x,t))-k_\Phi\Phi(x,t)-\rho I\frac{\partial^2\Phi(x,t)}{\partial t^2}=0, \tag{3.29}$$

Briefly equations of motion for Timoshenko beam on two parameter generalized foundation model 1 are equations (3.25) and (3.29) which are given below for convenience:

$$kGA(\frac{\partial^2 W(x,t)}{\partial x^2}-\frac{\partial \Phi(x,t)}{\partial x})-k_w W(x,t)-\rho A\frac{\partial^2 W(x,t)}{\partial t^2}=0$$

$$EI\frac{\partial^2 \Phi(x,t)}{\partial x^2}+kGA(\frac{\partial W(x,t)}{\partial x}-\Phi(x,t))-k_\Phi \Phi(x,t)-\rho I\frac{\partial^2 \Phi(x,t)}{\partial t^2}=0$$

3.2.2.2 Transverse Vibration of A Timoshenko Beam on Generalized Foundation Model 2

Generalized elastic foundation was assumed to be composed of infinitely many springs previously. Also it was mentioned that, the interaction between spring elements had been internal moment reaction between those virtual springs. The only difference of generalized foundation model 2 from the first one is that, the interaction moment is not proportional to bending rotation but it is proportional to total rotational deflection of the beam. In Timoshenko beam it was stated that total rotational deflection of the beam had been sum of shear deformation and bending rotation which is also given by equation (2.25). So, governing equation for generalized foundation model 2 is given by equation (3.9) which defines bending moment per unit length of the beam:

$$M(x,t)=k_v\frac{\partial W(x,t)}{\partial x}$$

There will be a slight change in Figure 3.7 due to moment reaction equation of model 2.

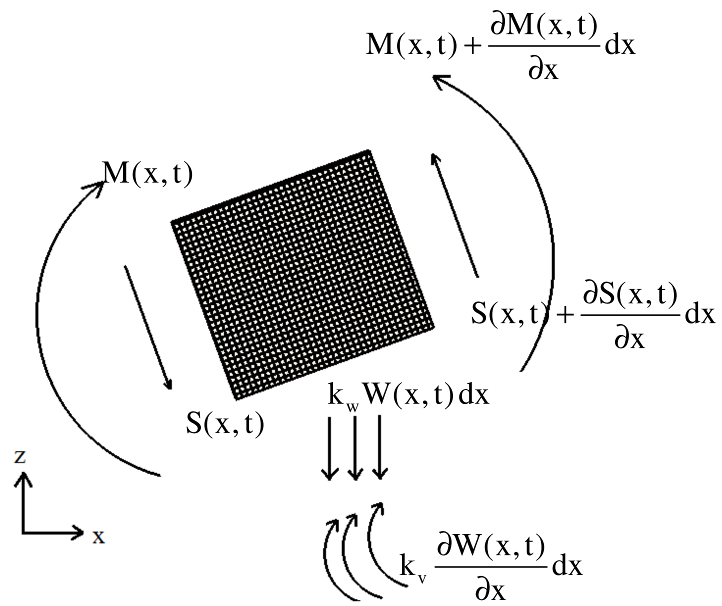

Figure 3.8 Moments and forces for beam on generalized foundation model 2

So following the same procedure as made in derivation of equations motion for model 1, the equations of motion for model 2 can also be derived.

There is no change in dynamic equilibrium in z direction, so one of the equations of motion will remain as same as equation 3.25. But due to change in the governing equation of the bending moment reaction of the foundation dynamic rotational equilibrium changes slightly. So $\sum M = I_G \frac{\partial \Phi^2(x,t)}{\partial t^2}$ for model 2 gives;

$$\begin{aligned}-M(x,t)+M(x,t)+\frac{\partial M(x,t)}{\partial x}dx+S(x,t)\frac{dx}{2}+S(x,t)\frac{dx}{2}\\ +\frac{\partial S(x,t)}{\partial x}dx\frac{dx}{2}-k_v\frac{\partial W(x,t)}{\partial x}=\rho I dx\frac{\partial^2\Phi(x,t)}{\partial t^2},\end{aligned} \tag{3.30}$$

Equation (6) is simplified by doing necessary cancellations and ignoring higher order derivatives.

$$\frac{\partial M(x,t)}{\partial x}+S(x,t)-k_v\frac{\partial W(x,t)}{\partial x}=\rho I\frac{\partial^2\Phi(x,t)}{\partial t^2}, \tag{3.31}$$

Inserting moment and shear force definitions for Timoshenko beam into equation (3.31), equation in terms of transverse displacement and bending rotation is obtained.

$$\frac{\partial}{\partial x}(EI\frac{\partial \Phi(x,t)}{\partial x})+kGA(\frac{\partial W(x,t)}{\partial x}-\Phi(x,t))-k_v\frac{\partial W(x,t)}{\partial x}=\rho I\frac{\partial^2\Phi(x,t)}{\partial t^2}, \quad (3.32)$$

Considering uniform beam properties:

$$EI\frac{\partial^2\Phi(x,t)}{\partial x^2}+kGA(\frac{\partial W(x,t)}{\partial x}-\Phi(x,t))-k_v\frac{\partial W(x,t)}{\partial x}-\rho I\frac{\partial^2\Phi(x,t)}{\partial t^2}=0, \quad (3.33)$$

If equation (3.29) and equation (3.33) are compared, it is observed that the only difference is in the terms containing the second foundation parameters k_Φ and k_v respectively. So equations of motion for model 2 are equations (3.25) and equation (3.33):

$$kGA(\frac{\partial^2 W(x,t)}{\partial x^2}-\frac{\partial \Phi(x,t)}{\partial x})-k_w W(x,t)-\rho A\frac{\partial^2 W(x,t)}{\partial t^2}=0$$

$$EI\frac{\partial^2\Phi(x,t)}{\partial x^2}+kGA(\frac{\partial W(x,t)}{\partial x}-\Phi(x,t))-k_v\frac{\partial W(x,t)}{\partial x}-\rho I\frac{\partial^2\Phi(x,t)}{\partial t^2}=0$$

3.3 Derivation of Equations of Motion of Euler-Bernoulli Beam in Transverse Vibration on Elastic Foundations

Although the thesis study is based on Timoshenko beam theory, for comparison purposes previously defined beam on elastic foundation models are to be also studied by using Euler-Bernoulli beam theory instead of Timoshenko beam theory. In literature it is accepted that Euler-Bernoulli beam theory was easier to apply with the disadvantage of overestimated natural frequencies. In literature survey part, it was also stated that, Timoshenko beam was a good correction of Euler-Bernoulli beam theory. Through obtaining results also by Euler-Bernoulli beam theory, differences between the results due to use of different beam theories is to be investigated. In addition, whether this difference is affected or not by beam properties, elastic foundation and crack parameters will be studied. To be able to achieve these aims, equations of motion of Euler-Bernoulli beam on elastic foundation are to be derived. Elastic foundation models will be the same as the ones used for Timoshenko beam in the previous sections. Briefly Euler-Bernoulli beam will be modeled as resting on two parameters Pasternak foundation, generalized foundation model 1 and generalized foundation model 2.

3.3.1 Transverse Vibration of A Euler Bernoulli Beam Resting On Two Parameter Pasternak Foundation

Figure 3.6 is a general representation of beam on two parameter elastic Pasternak foundation. Subsequently, Figure 3.7 showing the forces on a differential beam element is also appropriate for Euler-Bernoulli beam on Pasternak foundation. After applying Newton's law of motion in z direction, by following the steps of equations (3.12) and (3.13), the same result as equation (3.14) is obtained. Equation (3.14) states that:

$$\frac{\partial S(x,t)}{\partial x} - k_w W(x,t) + G_0 \frac{\partial^2 W(x,t)}{\partial x^2} = \rho A \frac{\partial^2 W(x,t)}{\partial t^2}, \tag{3.34}$$

It is important to recall that, in Euler-Bernoulli beam theory there is no shear deformation and shear force is due to bending moment. Expression of shear force is defined in terms of bending moment which is given by equation (2.19). Bending rotation in Euler-Bernoulli beam is related to W(x,t) through equations (2.11) and (2.12) so bending moment relation can be directly expressed as:

$$M(x,t) = EI \frac{\partial^2 W(x,t)}{\partial x^2}, \tag{3.35}$$

Inserting equation (3.35) into equation (2.19), expression of shear force in terms of transverse deflection can be obtained.

$$S(x,t) = -EI\frac{\partial^3 W(x,t)}{\partial x^3}, \tag{3.36}$$

Then shear force expression is inserted into equation (3.34) which results in the following differential equation in terms of transverse displacement and its derivatives.

$$\frac{\partial}{\partial x}(-EI\frac{\partial^3 W(x,t)}{\partial x^3}) - k_w W(x,t) + G_0\frac{\partial^2 W(x,t)}{\partial x^2} = \rho A\frac{\partial^2 W(x,t)}{\partial t^2}, \tag{3.37}$$

Assumption of uniform beam properties is also valid for Euler-Bernoulli beam.

$$EI\frac{\partial^4 W(x,t)}{\partial x^4} + k_w W(x,t) - G_0\frac{\partial^2 W(x,t)}{\partial x^2} + \rho A\frac{\partial^2 W(x,t)}{\partial t^2} = 0, \tag{3.38}$$

Equation (3.38) is a fourth order partial differential equation that defines motion of Euler-Bernoulli beam resting on two parameter elastic Pasternak foundation.

3.3.2 Transverse Vibration of A Euler Bernoulli Beam Resting On Two Parameters Generalized Elastic Foundation

In the previous sections, two different models were introduced from literature for generalized elastic foundation. Briefly, first parameter of generalized foundation model is simply Winkler parameter, whereas the second parameter is a constant expressing the proportionality between rotation of the beam and bending moment induced in the beam. Depending on the second parameter two different models were developed in literature. The difference between these two models is due to the definition of rotation of beam element. In the first model, rotation due to bending moment is considered, whereas in the second model total rotation of the beam is taken into consideration. For Timoshenko beam it is applicable to build two different models since total rotation and rotation due to bending are different from each other. In case of Euler-Bernoulli beam theory, total rotation of the beam element is equal to rotation due to bending moment in the beam since shear distortion is neglected. As a result, generalized elastic foundation model one and two will yield the same governing equation for Euler-Bernoulli beam resting on generalized elastic foundation.

Governing equations for generalized foundation model one and two are $M(x,t) = k_\Phi \Phi(x,t)$ and $M(x,t) = k_v \, \partial W(x,t)/\partial x$ respectively. For Euler-Bernoulli beam theory, $\Phi(x,t) = \partial W(x,t)/\partial x$, consequently governing equations of the generalized foundation gives the same expression which is $M(x,t) = k_v \partial W(x,t)/\partial x$.

In generalized foundation, first parameter relates transverse displacement of the beam to the force exerted by the elastic foundation through Winkler parameter and the governing equation is $q(x,t)=k_W W(x,t)$ where $q(x,t)$ is force in z direction per unit length of the beam.

For the derivation of equation of motion, dynamic force and moment equilibrium on a differential beam element as shown in Figure 3.8 should be written. Applying dynamic force equilibrium in z direction by following the same steps just like the case for Timoshenko beam generalized foundation model 1, equation (3.23) is obtained.

By writing the dynamic moment equilibrium on differential beam element, expression of $S(x,t)$ in terms of $W(x,t)$ can be obtained for Euler-Bernoulli beam on generalized foundation model 1. Considering no rotational inertia $\sum M(x,t) = 0$ gives:

$$-M(x,t) + M(x,t) + \frac{\partial M(x,t)}{\partial x} dx + S(x,t)\frac{dx}{2} + (S(x,t) + \frac{\partial S(x,t)}{\partial x})\frac{dx}{2} - k_\Phi \Phi(x,t) dx = 0, \quad (3.39)$$

In Euler-Bernoulli beam theory bending rotation $\Phi(x,t)$ is expressed by $W(x,t)$; $\Phi(x,t) = \dfrac{\partial W(x,t)}{\partial x}$, inserting this relation into equation (3.39) and doing the necessary cancellations gives:

$$\frac{\partial M(x,t)}{\partial x}+S(x,t)-k_{\Phi}\frac{\partial W(x,t)}{\partial x}=0\,, \tag{3.40}$$

Inserting expression (3.35) for moment relation for Euler-Bernoulli beam into equation (3.40):

$$S(x,t)=-EI\frac{\partial^3 W(x,t)}{\partial x^3}+k_{\Phi}\frac{\partial W(x,t)}{\partial x}\,, \tag{3.41}$$

results in shear relation in terms of beam, foundation parameters and transverse deflection W(x,t).

Equation (3.23) states that:

$$\frac{\partial S(x,t)}{\partial x}-k_w W(x,t)=\rho A\frac{\partial^2 W(x,t)}{\partial t^2}$$

Where derivative of S(x,t) is:

$$\frac{\partial S(x,t)}{\partial x}=-EI\frac{\partial^4 W(x,t)}{\partial x^4}+k_{\Phi}\frac{\partial^2 W(x,t)}{\partial^2 x}\,, \tag{3.42}$$

Substituting equation (3.42) into equation (3.23) gives:

$$EI\frac{\partial^4 W(x,t)}{\partial x^4}+k_w W(x,t)-k_{\Phi}\frac{\partial^2 W(x,t)}{\partial x^2}+\rho A\frac{\partial^2 W(x,t)}{\partial t^2}=0\,, \tag{3.43}$$

Equation of motion of Euler-Bernoulli beam with uniform cross section area and homogenous properties on generalized elastic foundation is given by equation (3.43).

The equations of motion of Timoshenko beam and Euler-Bernoulli beam on elastic foundation have been obtained for different foundation models. In the next chapter, these equations of motion are to be rewritten in non-dimensional forms. Then, stiffness and mass operators for each set of non-dimensional equations of motion are given in Appendix B. Also in Appendix B it is shown that the stiffness and mass operators are self adjoint and positive definite.

CHAPTER 4

MODELLING THE CRACK AND SOLUTION OF EQUATIONS OF MOTION OF CRACKED TIMOSHENKO BEAM RESTING ON ELASTIC FOUNDATION

4.1 Open Edge Crack Model

Before introducing the open edge crack model into the beam resting on elastic foundation, models that have already been used in literature will be reviewed. After that, the compliance due to open edge crack will be derived by using fracture mechanics relations.

4.1.1 Open Edge Crack Models from Literature

In this part of the thesis, some of the papers from literature that have already been reviewed in Chapter 1 will be further adressed in a specific manner based on the crack models used by the authors.

In literature, in most of the studies the open edge crack was modeled by replacing it by a representative rotational spring. In this way, total beam is separated into pieces that are connected by the representative springs. Generally, only rotational spring was used to represent an open edge crack, but also there exist crack models including both rotational and extensional springs just like the case in the study of Loya et al.(2006). The compliance of the springs was obtained from fracture mechanics theories or from total strain energy of the beam. There are also studies in which effect of crack on stiffness of the beam was distributed along the whole beam which was also mentioned in literature review section of the thesis study.

Dimarogonas (1996), in his literature review paper, mentions about many methods for modeling cracks in a beam. According to this paper, using stress intensity factor, strain energy release rate and Castigliano's theorem cracked region local flexibility for transverse open surface crack for plane strain was obtained. Using the data obtained from experiments following relation was derived, where "c" is rotational compliance.

$$c = M / \Delta\Phi = (6\pi h / bEI)F_1(s)\,, \tag{4.1}$$

In this equation "h" is height and "b" is width of the rectangular cross sectional beam and EI is flexural rigidity. Ratio of crack depth to height of the beam is denoted by "s" where $F_1(s)$ is given by:

$$\begin{aligned} F_1(s) = {} & 1.86s^2 - 3.95s^3 + 16.37s^4 + 37.22s^5 \\ & + 76.81s^6 + 126.9s^7 + 172.5s^8 - 144s^9 + 66.6s^{10} \end{aligned}\,, \tag{4.2}$$

Chondros et al. (1998) distributed the flexibility due to an edge crack along the whole beam by using Euler-Bernoulli beam theory. They also used local flexibility method and compared both methods by experimental results. In the paper it was mentioned that continuous method which is distribution of the flexibility due to crack, gave satisfactory results. In the local flexibility method, the crack was replaced by a rotational spring using the compliance relations given by equations (4.3) and (4.4):

$$\begin{aligned} F(\alpha) = {} & 0.6272\alpha^2 - 1.04533\alpha^3 + 4.5948\alpha^4 - 9.9736\alpha^5 + 20.2948\alpha^6 \\ & - 33.0351\alpha^7 + 47.1063\alpha^8 - 40.7556\alpha^9 + 19.6\alpha^{10} \end{aligned}\,, \tag{4.3}$$

$$c = 6\pi(1 - \nu^2)h\,F(\alpha) / EI\,, \tag{4.4}$$

In equation (4.4) "ν" is Poisson's ratio and "α" is ratio of crack depth to height of the beam. Chondros et al. also mentioned that equations (4.3) and (4.4) gave more accurate results for crack ratio less than 0.6. Later, a study was made by Khiem and Lien (2001) concerning an Euler-Bernoulli beam including multiple open edge cracks. In that study, each crack was represented by a rotational spring and the flexibility of the rotational spring was calculated by using equations (4.3) and (4.4). Same procedure was also followed by Mermertaş et al. (2001).

In the research of Takahashi (1999), Timoshenko beam theory was used for a cylindrical cracked beam. For the surface crack of depth "α" and width "2b", loaded by bending moment and shear force, the local flexibility was introduced by the following relation:

$$C_{ij} = \frac{\partial u_{ij}}{\partial P_j} = \frac{\partial^2}{\partial P_i P_j}\left[\int_{-b}^{b}\int_{0}^{\alpha} J(\alpha)\,d\alpha\,dz\right], \tag{4.5}$$

where $J(\alpha)$ is the strain energy density function, P_j is the load in the same direction as the displacement, u_{ij} strain energy and C_{ij} is crack compliance. Note that, i,j do not imply indicial notation. Using equation (4.5) and corresponding stress intensity factors for bending and shear modes, non-dimensional compliance values were obtained.

In a paper, for determination of natural frequencies of a short beam with open edge cracks, a model was developed by using Timoshenko beam theory by Lele et al. (2002). In the study, the crack was represented by a rotational spring as usual. The author proposed that if the beam had been under pure bending vibration, then it would be enough to represent the crack only by a rotational spring. "W" being the width of the beam and "a" being the crack depth, the stiffness of the rotational spring that represents the edge crack was given by the following relations:

$$K_t = \frac{EBW^4}{72\pi\int_0^a a(f(a/W))^2\,da}, \tag{4.6}$$

where f(a/W) is:

$$\left[f(\frac{a}{W})\right]^2 = a(1.2769 - 3.105\frac{a}{W} + 14.878\frac{a^2}{W^2} - 25.8\frac{a^3}{W^3} + 45.32\frac{a^4}{W^4} - 51.33\frac{a^5}{W^5} + 64.39\frac{a^6}{W^6} - 62.96\frac{a^7}{W^7} + 200.9\frac{a^8}{W^8} - 243.2\frac{a^9}{W^9} + 83.16\frac{a^{10}}{W^{10}} + 225.6\frac{a^{12}}{W^{12}}, \tag{4.7}$$

In a study made by Li (2003), based on vibration characteristics of a Timoshenko beam with arbitrary number of cracks, each crack was represented by a rotational spring. The cracks were modeled as one sided edge cracks and flexibility due to the corresponding crack was given by the following relation:

$$C_i = 5.346\, h\, f(\alpha_i), \tag{4.8}$$

where the flexibility function is given by:

$$\begin{aligned} f(\zeta_i) = {} & 1.862\alpha_i^2 - 3.95\alpha_i^3 + 16.375\alpha_i^4 - 37.226\alpha_i^5 + 76.81\alpha_i^6 \\ & - 126\alpha_i^7 + 172\alpha_i^8 - 143.97\alpha_i^9 + 66.56\alpha_i^{10} \end{aligned}, \tag{4.9}$$

Subscript "i" denotes the i^{th} crack and α_i denotes the crack depth ratio to height of the beam for the corresponding crack. In the paper equation (4.9) was referenced from the paper of Dimarogonas (1996), which is given by equation (4.2), it should be noted that in both equations coefficients show little differences. Equations (4.8) and (4.9) were also used by Shin et al. (2006) and by Aydin (2008) to obtain the stiffness of the rotational spring that represents the open edge crack in an Euler-Bernoulli beam. At this point it should be emphasized that, in the study of Shin et al. the beam was modeled as it was resting on an elastic foundation, but in the calculation of rotational compliance, the effect of elastic foundation was neglected.

Lin (2004) made a study, concerning vibration of cracked Timoshenko beams. Also in this study, the open edge crack was represented by a rotational spring. For the open edge crack non-dimensional crack sectional flexibility was defined as:

$$\theta = 6\pi\gamma^2 f_J(\gamma)\left(\frac{H}{L}\right), \tag{4.10}$$

where "γ" is the ratio of crack length to height "H" of the beam and "L" is the length of the beam. The flexibility function for one sided open edge crack exposed to bending moment was defined by:

$$f_J(\gamma) = 0.6384 - 1.035\gamma + 3.7201\gamma^2 - 5.1773\gamma^3 + 7.553\gamma^4 - 7.332\gamma^5 + 2.4909\gamma^6, \tag{4.11}$$

Hsu (2005) where vibration of cracked Euler-Bernoulli beams are considered, the compliance of the rotational spring which accounts for the crack was calculated by using mode I stress intensity factor. Mode I stress intensity factor is defined for opening mode of the crack. Crack was modeled to be at the surface of the beam. In addition, the beam was resting on a Winkler elastic foundation. The author used the following relations to obtain the stiffness of the rotational spring that represents the crack:

$$\frac{(1-\mu^2)K_I^{\,2}}{E} = \frac{(P_b)^2}{2w}\frac{dC}{da}, \tag{4.12}$$

where "μ" is Poisson's ration, K_I is mode I stress intensity factor, P_b bending moment on the crack section, "w" is width of the beam, "a" is crack depth and "C" is flexibility of the beam. In the study, the magnitude of the stress intensity factor was calculated by the following expression:

$$K_I = \frac{6P_b}{wh^2}\sqrt{\pi r h}\,F_I(r), \tag{4.13}$$

where "h" is height of the beam and "r" is the ratio of the crack depth "a" to beam height "h". $F_I(r)$ in equation (4.13) was given by the following relation:

$$F_I(r) = \sqrt{\frac{2}{\pi r}\tan(\frac{\pi r}{2})}\,\frac{0.923+0.199[1-\sin(\pi r/2)]^4}{\cos(\pi r/2)}, \tag{4.14}$$

Although the beam was resting on Winkler elastic foundation, Hsu neglected the effect of foundation on stress intensity factor while deriving the rotational compliance due to edge crack. Then, by inserting equation (4.14) into equation (4.13), the author obtained flexibility.

$$G = \frac{6(1-\mu^2)h}{EI}Q(r), \tag{4.15}$$

where Q(r) is:

$$Q(r) = \int_0^r \pi r F_I(r)^2 dr, \tag{4.16}$$

Since stiffness is equal to 1/G , finally bending stiffness at the crack section was expressed as shown in equation (4.17) by Hsu.

$$k_T = \frac{EI}{6(1-\mu^2)\,h\,Q(r)}, \tag{4.17}$$

To obtain the rotational flexibility of the spring that represents the open edge crack, El-Bikri el al.(2006) followed a method similar to that of Hsu(2005). In that study El-Bikri used the strain energy at the crack section and Castigliano's theorem to obtain the flexibility of the representative rotational spring. The strain energy at the crack section for a uniform rectangular cross sectional beam of width "b" was given by:

$$V_c = \int_{-b/2}^{b/2}\int_{0}^{a} G(\alpha)\,d\alpha\,dy, \tag{4.18}$$

where "a" is crack depth and energy release rate G(α) is related to mode I stress intensity factor through the following expression.

$$G(\alpha) = \frac{K_I^{\,2}(\alpha)}{E'}, \tag{4.19}$$

Castigliano's theorem states that $\theta=\partial V_c/\partial M_f$ and also considering definition of compliance as $C=\partial\theta/\partial M_f$ and using equations (4.13) and (4.14), El-Bikri obtained the following expression for the flexibility of the representative rotational spring.

$$C = b\frac{\partial^2}{\partial M_f^{\,2}}\int_0^a \frac{K_I^{\,2}}{E'}\,d\alpha, \tag{4.20}$$

In equation (4.19) K_I is Mode I stress intensity factor. The author also mentioned that, for plane stress E' was equal to Young's modulus and for plane strain it was equal to $E/(1-v^2)$. A similar procedure was also used by Mazanoğlu et al. (2009), but in that study $F_I(r)$ was defined differently. The author expressed F(r) as:

$$F(r) = 1.12 - 1.4r + 7.33r^2 - 13.8r^3 + 14r^4 \quad \text{, for } r=a/h<0.6, \tag{4.21}$$

So far, the crack was represented by only rotational spring in the literature. As mentioned in introduction chapter, Loya et al. (2006) represented the edge crack on a Timoshenko beam, by both extensional and rotational springs as shown in Figure 1.4. According to the authors, besides rotation due to bending, also transverse deflection due to shear should also be considered. Consequently, in addition to the previous studies, stiffness of an extensional spring was also calculated in this study. In the study of Loya et al., C_q and C_m denote the extensional and rotational flexibility constants of the representative springs due to open edge crack. These constants are given by:

$$C_q = \frac{W}{EA} q(\alpha), \tag{4.22}$$

$$C_m = \frac{W}{EI} \Theta(\alpha), \tag{4.23}$$

where W is width of the beam and α is the ratio of crack depth to width of the beam. q(α) and Θ(α) are given by the following relations:

$$q(\alpha) = \left(\frac{\alpha}{1-\alpha}\right)^2 (-0.22 + 3.82\alpha + 1.54\alpha^2 - 14.64\alpha^3 + 9.60\alpha^4), \tag{4.24}$$

$$\Theta(\alpha) = 2\left(\frac{\alpha}{1-\alpha}\right)^2 (5.93 - 19.69\alpha + 37.14\alpha^2 - 35.84\alpha^3 + 13.12\alpha^4), \tag{4.25}$$

Equation (4.24) was also used by Zhong and Oyadiji (2008) to obtain the flexibility of a rotational spring which represents an open edge crack on an Euler-Bernoulli beam.

A beam including cracks at both upper and lower surfaces has also been considered in literature. In one of the studies, double sided cracks were considered by Al-Said et al. (2008) in which the crack region was replaced by a rotational spring. Equivalent spring flexibility for double sided open edge crack was given by the following relation:

$$\frac{1}{K_{cr}} = \frac{9\pi D^2}{BH^2 E}(0.5033 - 0.9022 + 3.412D^2 - 3.181D^3 + 5.793D^4), \tag{4.26}$$

where "D" is ratio of crack depth to height of the beam, B is width of the beam and beam height was given by "2H".

As far as the literature based on crack models is concerned, the cracks were represented by only rotational springs if Euler-Bernoulli beam theory was used. This makes sense since shear distortion in Euler-Bernoulli beam theory is neglected. Representation of the crack by single rotational spring was also used in the studies in which Timoshenko beam theory was used. In Timoshenko beam theory deformation due to both bending moment and shear distortion are considered. This fact led Loya et al. (2006) to represent the crack by both rotational and extensional springs. The study of Loya et al. is more recent with respect to the other studies in which the beam was modeled according to Timoshenko beam theory.

Another issue that should be considered is about derivation of compliance of edge cracks when the beam is resting on elastic foundation. As mentioned previously, Shin (2006) and Hsu(2005) studied transverse vibration of cracked beam on elastic foundations. In these studies, if derivation of compliance is investigated, it can be seen that, the effect of elastic foundation on Mode I stress intensity factor is not considered. The compliance expressions were derived as if the edge cracked beam is not resting on elastic foundation.

Consequently, in the thesis study a procedure similar to the information gathered from literature is to be applied. For the cases in which Timoshenko beam is resting on elastic foundation, the open edge crack will be represented by single rotational springs. The extensional compliance was neglected by almost all authors since the extensional compliance is very small with respect to rotational compliance. Also, the effect of elastic foundation on crack compliance was neglected in the literature. To see the effect of foundation on crack compliance, a comparison will be presented between compliances that is calculated by considering the effect of Winkler foundation and that is calculated by neglecting the effect of foundation.

4.1.2 Derivation of Rotational Compliance Due to Open Edge Crack

For calculation of compliance expressions published by Tada et al. are to be used. Energy release rate for an edge crack is given by the following expression.

$$G = \frac{1}{2} M^2 \frac{dC}{dA}, \qquad (4.27)$$

where M is the bending load, C is the crack compliance, A is the crack area.

The stress intensity factor for an edge crack under bending load is given by;

$$K_I = \frac{6M}{bh^2} \sqrt{\pi a} F(a/h), \qquad (4.28)$$

Energy release rate is related to stress intensity factor.

$$G = \frac{K^2_I}{E'}, \qquad (4.29)$$

Where E'=E for plane stress and $E'=E/(1-\nu^2)$ for plane strain. Considering the plane strain, by equating the two energy release rate expressions to each other, compliance and stress intensity factor relation can be derived.

$$\frac{1}{2} M^2 \frac{dC}{dA} = \frac{K^2_I}{E'}, \qquad (4.30)$$

Then inserting the expressions of Mode I stress intensity factor and E'

$$\frac{1}{2} M^2 \frac{dC}{dA} = \frac{(1-\nu^2) 6^2 M^2 \pi a}{Eb^2 h^4} F(a/h)^2, \qquad (4.31)$$

where dA=bda, "a" being the crack depth and "b" width of the beam. After doing the necessary cancellations and taking the integral, following relation is obtained.

$$C = \frac{(1-\nu^2) 6\pi}{EI} \int_0^a \frac{a}{h} F(a/h)^2 da, \qquad (4.32)$$

The integrand "da" can be changed into "α" where "α=a/h".

$$d\alpha = da/h, \tag{4.33}$$

$$C = \frac{(1-\nu^2)6\pi h}{EI}\int_0^{\alpha} \alpha F(\alpha)^2 d\alpha\,, \tag{4.34}$$

Tada et al. presented two different expressions for F(α) in their book. These are:

$$F(\alpha) = 1.122 - 1.40\alpha + 7.33\alpha^2 - 13.08\alpha^3 + 14.0\alpha^4\,, \tag{4.35}$$

According to the information given in the reference book this expression was derived by Brown, by least squares fitting and gives the results with 0.2% error for α<0.6.

Also Tada himself developed an expression for F(α), with an error at most 0.5% for any "α" ratio. Expression of Tada is:

$$F(\alpha) = \sqrt{\frac{2}{\pi\alpha}\tan(\frac{\pi\alpha}{2})}\,\frac{0.923 + 0.199(1-\sin(\pi\alpha/2))^4}{\cos(\pi\alpha/2)}\,, \tag{4.36}$$

By using these expressions rotational compliance due to open edge crack is to be calculated in the thesis study.

4.2 Solution of Equations of Motion of Cracked Timoshenko Beam Resting on Elastic Foundation in Transverse Vibration

So far in the thesis, equations of motion of a beam in transverse vibration, according to Euler-Bernoulli and Timoshenko beam theories have been obtained. Afterwards, parametric effect of elastic foundations were added to these equations of motions leading the equations of motions of Euler-Bernoulli beams and Timoshenko beams on different two parameter elastic foundation models. These foundation models are defined as Pasternak foundation model, generalized foundation model 1 and generalized foundation model 2. In the previous part, using the information from literature, the crack model that is to be used in the thesis have been determined and from here on, the equations of motion of cracked beams on elastic foundation are to be solved for simple boundary conditions.

As mentioned previously, the beam is to be separated into segments which are connected by rotational and in some cases both rotational and extensional springs. The discontinuity due to crack or representative springs will be taken care of by compatibility equations. Before starting the solution, the equations of motion will be written in non-dimensional forms.

4.2.1 Solution of Equations of Motion of Cracked Timoshenko Beam on Pasternak Elastic Foundation in Transverse Vibration

Equations of motion of Timoshenko beam on Pasternak elastic foundation were derived previously as given by equations (3.16) and (3.21). Before starting the solution, the parameters in these equations will be converted to non-dimensional forms. Non-dimensional axial coordinate is obtained by dividing the x coordinate of a point on the beam by the length of beam.

$$\chi = \frac{x}{L}, \tag{4.37}$$

In a similar way, non-dimensional flexural displacement can be defined as:

$$W(\chi, \tau) = \frac{W(x,t)}{L}, \tag{4.38}$$

Slope of the beam which is given by $\Phi(x,t)$ is already non-dimensional since it is given in terms of radians. In addition to the above parameters, non-dimensional time parameter is defined as:

$$\tau = \frac{t}{T}, \tag{4.39}$$

where "T" is to be determined in the following steps.

After defining non-dimensional parameters, equations from (4.37) - (4.39) are to be inserted in to equations (3.16) and (3.21) to obtain non-dimensional equations of motion.

$$\begin{aligned} & kGA\left(\frac{\partial^2 W(\chi,\tau)L}{\partial(\chi L)^2} - \frac{\partial\Phi(\chi,\tau)}{\partial(\chi L)}\right) - k_w W(\chi,\tau)L \\ & \qquad + G_0\frac{\partial^2 W(\chi,\tau)L}{\partial(\chi L)^2} - \rho A\frac{\partial^2 W(\chi,\tau)L}{\partial(\tau T)^2} = 0 \end{aligned} , \tag{4.40}$$

$$EI\frac{\partial^2\Phi(\chi,\tau)}{\partial(\chi L)^2} + kGA\left(\frac{\partial W(\chi,\tau)L}{\partial(\chi L)} - \Phi(\chi,\tau)\right) - \rho I\frac{\partial^2\Phi(\chi,\tau)}{\partial(\tau T)^2} = 0 \ , \tag{4.41}$$

Equations (4.40) and (4.41) can be reduced by doing the necessary cancellations.

$$\begin{aligned} & \frac{kGA}{L}\left(\frac{\partial^2 W(\chi,\tau)}{\partial\chi^2} - \frac{\partial\Phi(\chi,\tau)}{\partial\chi}\right) - k_w L W(\chi,\tau) \\ & \qquad + \frac{G_0}{L}\frac{\partial^2 W(\chi,\tau)}{\partial\chi^2} - \frac{\rho A L}{T^2}\frac{\partial^2 W(\chi,\tau)}{\partial\tau^2} = 0 \end{aligned} , \tag{4.42}$$

$$\frac{EI}{L^2}\frac{\partial^2\Phi(\chi,\tau)}{\partial\chi^2} + kGA\left(\frac{\partial W(\chi,\tau)}{\partial\chi} - \Phi(\chi,\tau)\right) - \frac{\rho I}{T^2}\frac{\partial^2\Phi(\chi,\tau)}{\partial\tau^2} = 0 , \tag{4.43}$$

In equations (4.42) and (4.43) the coefficients still have units. To obtain non-dimensional coefficients necessary cancellations should be done. For instance, dividing equation (4.42) by kGA/L and dividing equation (4.43) by kGA will result in differential set of equations with non-dimensional coefficients.

$$\begin{aligned} & \frac{\partial^2 W(\chi,\tau)}{\partial\chi^2} - \frac{\partial\Phi(\chi,\tau)}{\partial\chi} - \frac{k_w L^2}{kGA}W(\chi,\tau) \\ & \qquad + \frac{G_0}{kGA}\frac{\partial^2 W(\chi,\tau)}{\partial\chi^2} - \frac{\rho L^2}{kGT^2}\frac{\partial^2 W(\chi,\tau)}{\partial\tau^2} = 0 \end{aligned} , \tag{4.44}$$

$$\frac{EI}{kGAL^2}\frac{\partial^2\Phi(\chi,\tau)}{\partial\chi^2} + \frac{\partial W(\chi,\tau)}{\partial\chi} - \Phi(\chi,\tau) - \frac{\rho I}{kGAT^2}\frac{\partial^2\Phi(\chi,\tau)}{\partial\tau^2} = 0 \ , \tag{4.45}$$

If the coefficients are checked in equations (4.44) and (4.45), it can be shown that units cancel each other thus leading non-dimensional coefficients. Also "T" can be found from equation (4.44) as follows.

$$T = L\sqrt{\frac{\rho}{kG}}, \tag{4.46}$$

In order to write the set of differential equations in a compact form, non-dimensional coefficients can be defined by the following relations.

$$\eta_1 = \frac{k_w L^2}{kGA} \qquad \eta_2 = \frac{G_0}{kGA} \qquad \eta_3 = \frac{EI}{kGAL^2} \qquad \eta_4 = \frac{I}{AL^2} \qquad , \tag{4.47}$$

Inserting relations given by equation (4.47), non-dimensional equations of motion are obtained.

$$\frac{\partial^2 W(\chi,\tau)}{\partial \chi^2} - \frac{\partial \Phi(\chi,\tau)}{\partial \chi} - \eta_1 W(\chi,\tau) + \eta_2 \frac{\partial^2 W(\chi,\tau)}{\partial \chi^2} - \frac{\partial^2 W(\chi,\tau)}{\partial \tau^2} = 0, \tag{4.48}$$

$$\eta_3 \frac{\partial^2 \Phi(\chi,\tau)}{\partial \chi^2} + \frac{\partial W(\chi,\tau)}{\partial \chi} - \Phi(\chi,\tau) - \eta_4 \frac{\partial^2 \Phi(\chi,\tau)}{\partial \tau^2} = 0 \ , \tag{4.49}$$

$W(\chi,\tau)$ and $\Phi(\chi,\tau)$ are assumed to be separable functions as the general case in the studies concerning continuous vibration.

$$W(\chi,\tau) = w(\chi)F(\tau), \tag{4.50}$$

$$\Phi(\chi,\tau) = f(\chi)F(\tau), \tag{4.51}$$

Inserting equations (4.50) and (4.51) into the system of non-dimensional differential equations gives the following relations.

$$\frac{d^2 w(\chi)}{d\chi^2}F(\tau) - \frac{df(\chi)}{d\chi}F(\tau) - \eta_1 w(\chi)F(\tau) + \eta_2 \frac{d^2 w(\chi)}{d\chi^2}F(\tau) - w(\chi)\frac{d^2F(\tau)}{d\tau^2} = 0, \tag{4.52}$$

$$\eta_3 \frac{d^2 f(\chi)}{d\chi^2}F(\tau) + \frac{dw(\chi)}{d\chi}F(\tau) - f(\chi)F(\tau) - \eta_4 f(\chi)\frac{d^2F(\tau)}{d\tau^2} = 0, \tag{4.53}$$

Dividing equation (4.52) by $w(\chi)F(\tau)$ and dividing equation (4.53) by $\eta_4 f(\chi)F(\tau)$:

$$\frac{1}{w(\chi)}\left(\frac{d^2w(\chi)}{d\chi^2}-\frac{df(\chi)}{d\chi}-\eta_1 w(\chi)+\eta_2\frac{d^2w(\chi)}{d\chi^2}\right)=\frac{1}{F(\tau)}\frac{d^2F(\tau)}{d\tau^2}, \tag{4.54}$$

$$\frac{1}{\eta_4 f(\chi)}\left(\eta_3\frac{d^2f(\chi)}{d\chi^2}+\frac{dw(\chi)}{d\chi}-f(\chi)\right)=\frac{1}{F(\tau)}\frac{d^2F(\tau)}{d\tau^2}, \tag{4.55}$$

In equations (4.54) and (4.55) it is seen that, left hand side of the equality is dependent on χ whereas right hand side is dependent on τ. This situation can be possible only if both sides of the equality are constant.

$$\frac{1}{w(\chi)}\left(\frac{d^2w(\chi)}{d\chi^2}-\frac{df(\chi)}{d\chi}-\eta_1 w(\chi)+\eta_2\frac{d^2w(\chi)}{d\chi^2}\right)=\frac{1}{F(\tau)}\frac{d^2F(\tau)}{d\tau^2}=-\omega^2, \tag{4.56}$$

$$\frac{1}{\eta_4 f(\chi)}\left(\eta_3\frac{d^2f(\chi)}{d\chi^2}+\frac{dw(\chi)}{d\chi}-f(\chi)\right)=\frac{1}{F(\tau)}\frac{d^2F(\tau)}{d\tau^2}=-\omega^2, \tag{4.57}$$

Equations (4.56) and (4.57) should be equal to a negative constant which is given by "$-\omega^2$". Otherwise, solution of $F(\tau)$ results in two solutions given by "$e^{\omega\tau}$" and "$e^{-\omega\tau}$". These solutions go to plus and minus infinity as "τ" goes to infinity. Since the equations of motion define physical system, these solutions are not appropriate. So to avoid infinity "$-\omega^2$" is used as constant term. To obtain the solutions of $w(\chi)$ and $f(\chi)$, equations (4.56) and (4.57) are rearranged.

$$\frac{d^2w(\chi)}{d\chi^2}-\frac{df(\chi)}{d\chi}-\eta_1 w(\chi)+\eta_2\frac{d^2w(\chi)}{d\chi^2}+\omega^2 w(\chi)=0, \tag{4.58}$$

$$\eta_3\frac{d^2f(\chi)}{d\chi^2}+\frac{dw(\chi)}{d\chi}-f(\chi)+\omega^2\eta_4 f(\chi)=0, \tag{4.59}$$

Equations (4.58) and (4.59) are second order constant coefficient ordinary differential equations where "ω" is non-dimensional natural frequency or eigenvalue that is to be determined. Before continuing with the solution steps, these equations are to be written in operator form as follows:

$$D^2w(\chi)-Df(\chi)-\eta_1 w(\chi)+\eta_2 D^2w(\chi)+\omega^2 w(\chi)=0, \tag{4.60}$$

$$\eta_3 D^2f(\chi)+Dw(\chi)-f(\chi)+\omega^2\eta_4 f(\chi)=0, \tag{4.61}$$

where D means $d/d\chi$. Equations (4.60) and (4.61) can be simplified by writing each equation as multiples of $w(\chi)$ and $f(\chi)$.

$$(D^2 - \eta_1 + \eta_2 D^2 + \omega^2) w(\chi) - Df(\chi) = 0, \tag{4.62}$$

$$Dw(\chi) + (\eta_3 D^2 - 1 + \omega^2 \eta_4) f(\chi) = 0, \tag{4.63}$$

After obtaining the equations in operator form, $f(\chi)$ is eliminated by applying the appropriate operator to each one of the equations (4.62) and (4.63), and then subtracting the equations side by side as follows. Elimination of $f(\chi)$ results in a fourth order constant coefficient differential equation in terms of $w(\chi)$.

$$\begin{aligned}&(\eta_3 D^2 - 1 + \omega^2 \eta_4)\{(D^2 - \eta_1 + \eta_2 D^2 + \omega^2) w(\chi) - Df(\chi)\} = 0\\ &D\{Dw(\chi) + (\eta_3 D^2 - 1 + \omega^2 \eta_4) f(\chi)\} = 0\end{aligned}$$

$$\begin{aligned}\eta_3(\eta_2+1)D^4 w(\chi) + [(\eta_2+1)(\eta_4\omega^2-1) - \eta_3(\eta_1-\omega^2)+1]D^2 w(\chi)&\\ +(\omega^2-\eta_1)(\eta_4\omega^2-1)w(\chi) = 0&\end{aligned}, \tag{4.64}$$

Equation (4.64) can be simplified by dividing the whole equation by "$\eta_3(\eta_2+1)$" and defining new coefficients as functions of non-dimensional natural frequency "ω" , beam properties and foundation parameters.

$$\kappa(\omega) = \frac{\omega^2(\eta_2\eta_4 + \eta_4 + \eta_3) - \eta_2 - \eta_3\eta_1}{\eta_3(\eta_2+1)}, \tag{4.65}$$

$$\zeta(\omega) = \frac{(\omega^2-\eta_1)(\eta_4\omega^2-1)}{\eta_3(\eta_2+1)}, \tag{4.66}$$

Inserting new coefficients given by equations (4.65) and (4.66) into equations (4.64) and writing the operator in open form results in the following constant coefficient fourth order ordinary differential equation.

$$\frac{d^4 w(\chi)}{d\chi^4} + \kappa \frac{d^2 w(\chi)}{d\chi} + \zeta w(\chi) = 0, \tag{4.67}$$

For solving equation (4.67) $w(\chi)$ is assumed to be in the form of an exponential function.

$$w(\chi) = e^{m\chi}, \tag{4.68}$$

where, "m" is a constant to be determined. Inserting equation (4.68) into equation (4.67) gives:

$$m^4 e^{m\chi} + \kappa m^2 e^{m\chi} + \zeta e^{m\chi} = 0, \tag{4.69}$$

Equation (4.69) can be reduced by cancelling the exponential terms.

$$m^4 + \kappa m^2 + \zeta = 0, \tag{4.70}$$

Value of "m" can be found by using the quadratic formula.

$$m^2 = \frac{-\kappa \pm \sqrt{\kappa^2 - 4\zeta}}{2}, \tag{4.71}$$

It is seen that "m" will take four different values. Now let A^2 and B^2 be given by the following expressions:

$$A^2 = \frac{-\kappa + \sqrt{\kappa^2 - 4\zeta}}{2}, \tag{4.72}$$

$$B^2 = -\frac{\kappa + \sqrt{\kappa^2 - 4\zeta}}{2}, \tag{4.73}$$

Note that A=A(ω) and B=B(ω).

Using the definitions given by equations (4.72) , (4.73) and (4.71), values of "m" can be given in terms of "A" and "B".

$$m_1 = A, \qquad m_2 = -A \qquad m_3 = Bi \qquad m_4 = -Bi \tag{4.74}$$

Thus, solution equation (4.67) is a linear combination of four independent solutions.

$$w(\chi) = K_1 e^{A\chi} + K_2 e^{-A\chi} + K_3 e^{Bi\chi} + K_4 e^{-Bi\chi}, \tag{4.75}$$

Let;

$$K_1 = \frac{C_1 + C_2}{2}, \tag{4.76}$$

$$K_2 = \frac{C_1 - C_2}{2}, \tag{4.77}$$

Inserting (4.76) into (4.77) and also using the Euler's formula, the solution can be obtained in terms of trigonometric and hyperbolic functions.

$$\begin{aligned} w(\chi) = \frac{C_1 + C_2}{2} e^{A\chi} + \frac{C_1 - C_2}{2} e^{-A\chi} &+ K_3(\cos(B\chi) + i\sin(B\chi)) \\ &+ K_4(\cos(B\chi) - i\sin(B\chi)) \end{aligned}, \tag{4.78}$$

Hyperbolic cosine and hyperbolic sine are defined as:

$$\cosh(\chi) = \frac{e^{\chi} + e^{-\chi}}{2}, \tag{4.79}$$

$$\sinh(\chi) = \frac{e^{\chi} - e^{-\chi}}{2}, \tag{4.80}$$

By inserting hyperbolic expressions into equation (4.78) and redefining coefficients of $\cos(B\chi)$ and $\sin(B\chi)$ following expression for solution $w(\chi)$ can be obtained.

$$w(\chi) = C_1 \cosh(A\chi) + C_2 \sinh(A\chi) + C_3 \cos(B\chi) + C_4 \sin(B\chi), \tag{4.81}$$

In equation (4.64), $f(\chi)$ was eliminated to find solution of $w(\chi)$. Similarly, if $w(\chi)$ was eliminated a fourth order differential equation of $f(\chi)$ is to be obtained which would be the same as equation (4.81), with similar coefficients. So, $f(\chi)$ is equal to:

$$f(\chi) = C'_1 \cosh(A\chi) + C'_2 \sinh(A\chi) + C'_3 \cos(B\chi) + C'_4 \sin(B\chi), \tag{4.82}$$

The coefficients of equations (4.62) and (4.63) are not multiple of each other, so relation between coefficients of equations (4.81) and (4.82) can be found by inserting these equations into equations (4.58) or (4.59). Before doing this, first and second derivatives of $w(\chi)$ and $f(\chi)$ are necessary and should be calculated.

$$\frac{dw(\chi)}{d\chi} = C_1 A\sinh(A\chi) + C_2 A\cosh(A\chi) - C_3 B\sin(B\chi) + C_4 B\cos(B\chi)\,, \tag{4.83}$$

$$\frac{d^2w(\chi)}{d\chi^2} = C_1 A^2\cosh(A\chi) + C_2 A^2\sinh(A\chi) - C_3 B^2\cos(B\chi) - C_4 B^2\sin(B\chi)\,, \tag{4.84}$$

$$\frac{df(\chi)}{d\chi} = C'_1 A\sinh(A\chi) + C'_2 A\cosh(A\chi) - C'_3 B\sin(B\chi) + C'_4 B\cos(B\chi)\,, \tag{4.85}$$

$$\frac{d^2f(\chi)}{d\chi^2} = C'_1 A^2\cosh(A\chi) + C'_2 A^2\sinh(A\chi) - C'_3 B^2\cos(B\chi) - C'_4 B^2\sin(B\chi)\,, \tag{4.86}$$

In the next step equations (4.83) to (4.86) are to be inserted into equation (4.58).

$$\begin{aligned}
&C_1A^2\cosh(A\chi) + C_2A^2\sinh(A\chi) - C_3B^2\cos(B\chi) - C_4B^2\sin(B\chi)\\
&-C'_1 A\sinh(A\chi) - C'_2 A\cosh(A\chi) + C'_3 B\sin(B\chi) - C'_4 B\cos(B\chi)\\
&-\eta_1(C_1\cosh(A\chi) + C_2\sinh(A\chi) + C_3\cos(B\chi) + C_4\sin(B\chi))\\
&+\eta_2(C_1A^2\cosh(A\chi) + C_2A^2\sinh(A\chi) - C_3B^2\cos(B\chi) - C_4B^2\sin(B\chi))\\
&+\omega^2(C_1\cosh(A\chi) + C_2\sinh(A\chi) + C_3\cos(B\chi) + C_4\sin(B\chi)) = 0
\end{aligned}\quad , \tag{4.87}$$

Then, equation (4.87) is rearranged by collecting the terms under $\cosh(A\chi)$, $\sinh(A\chi)$, $\cos(B\chi)$ and $\sin(B\chi)$ parenthesis.

$$\begin{aligned}
&\cosh(A\chi)\left(C_1\omega^2 - AC'_2 - C_1\eta_1 + A^2C_1 + A^2C_1\eta_2\right)\\
&+\sinh(A\chi)\left(C_2\omega^2 - AC'_1 - C_2\eta_1 + A^2C_2 + A^2C_2\eta_2\right)\\
&\cos(B\chi)\left(C_3\omega^2 - BC'_4 - C_3\eta_1 - B^2C_3 - B^2C_3\eta_2\right)\\
&+\sin(B\chi)\left(C_4\omega^2 + BC'_3 - C_4\eta_1 - B^2C_4 - B^2C_4\eta_2\right) = 0
\end{aligned}\quad , \tag{4.88}$$

Since $\cosh(A\chi)$, $\sinh(A\chi)$, $\cos(B\chi)$ and $\sin(B\chi)$ are independent of each other, to satisfy equation (4.88) the coefficients of trigonometric and hyperbolic functions should be equal to zero. Through this fact expressions that show the relation between the coefficients of $w(\chi)$ and $f(\chi)$ can be derived.

$$C'_1 = C_2 \frac{\omega^2 - \eta_1 + A^2(1+\eta_2)}{A}, \tag{4.89}$$

$$C'_2 = C_1 \frac{\omega^2 - \eta_1 + A^2(1+\eta_2)}{A}, \tag{4.90}$$

$$C'_3 = C_4 \frac{-\omega^2 + \eta_1 + B^2(1+\eta_2)}{B}, \tag{4.91}$$

$$C'_4 = C_3 \frac{\omega^2 - \eta_1 - B^2(1+\eta_2)}{B}, \tag{4.92}$$

Equation (4.82) can be written in terms of coefficients C_1, C_2, C_3 and C_4 by replacing equations from (4.89) to (4.92).

$$\begin{aligned} f(\chi) = {} & C_1 \frac{\omega^2 - \eta_1 + A^2(1+\eta_2)}{A} \sinh(A\chi) + C_2 \frac{\omega^2 - \eta_1 + A^2(1+\eta_2)}{A} \cosh(A\chi) \\ & + C_3 \frac{\omega^2 - \eta_1 - B^2(1+\eta_2)}{B} \sin(B\chi) + C_4 \frac{-\omega^2 + \eta_1 + B^2(1+\eta_2)}{B} \cos(B\chi) \end{aligned}, \tag{4.93}$$

Equations (4.81) and (4.93) are solutions for the set of differential equations for Timoshenko beam resting on two parameter elastic Pasternak foundation. To include the effect of crack, two pieces of Timoshenko beam are to be connected by rotational and translational springs. These springs represent the flexibility due to compliance of the crack. Thus, the span of the beam is to be divided into two.

$$0 \le x_1 \le L_c, \tag{4.94}$$

$$L_c \le x_2 \le L, \tag{4.95}$$

"L_c" denotes the length of the beam section until the crack location. So, the solutions $w(\chi)$ and $f(\chi)$ will be denoted by $w_1(\chi)$ and $f_1(\chi)$ for the portion of the beam on the left of the crack and by $w_2(\chi)$ and $f_2(\chi)$ for the portion of the beam on the right side of the crack. It should be noted that, equations (4.94) and (4.95) are not non-dimensional; to write them in non-dimensional form each of them should be divided by "L" length of the beam. Let the non-dimensional crack position to be $L_1=L_c/L$:

$$0 \le \chi_1 \le L_1, \qquad L_1 \le \chi_2 \le 1 \tag{4.96}$$

Thus, equations (4.81) and (4.93) should be rearranged by considering the effect of crack.

For the portion of the beam on the left side of the crack:

$$w_1(\chi_1) = C_1 \cosh(A\chi_1) + C_2 \sinh(A\chi_1) + C_3 \cos(B\chi_1) + C_4 \sin(B\chi_1), \tag{4.97}$$

$$\begin{aligned} f_1(\chi_1) &= C_1 \frac{\omega^2 - \eta_1 + A^2(1+\eta_2)}{A} \sinh(A\chi_1) + C_2 \frac{\omega^2 - \eta_1 + A^2(1+\eta_2)}{A} \cosh(A\chi_1) \\ &+ C_3 \frac{\omega^2 - \eta_1 - B^2(1+\eta_2)}{B} \sin(B\chi_1) + C_4 \frac{-\omega^2 + \eta_1 + B^2(1+\eta_2)}{B} \cos(B\chi_1) \end{aligned}, \tag{4.98}$$

For the portion of the beam on the right side of the crack:

$$w_2(\chi_2) = C_5 \cosh(A\chi_2) + C_6 \sinh(A\chi_2) + C_7 \cos(B\chi_2) + C_8 \sin(B\chi_2), \tag{4.99}$$

$$\begin{aligned} f_2(\chi_2) &= C_5 \frac{\omega^2 - \eta_1 + A^2(1+\eta_2)}{A} \sinh(A\chi_2) + C_6 \frac{\omega^2 - \eta_1 + A^2(1+\eta_2)}{A} \cosh(A\chi_2) \\ &+ C_7 \frac{\omega^2 - \eta_1 - B^2(1+\eta_2)}{B} \sin(B\chi_2) + C_8 \frac{-\omega^2 + \eta_1 + B^2(1+\eta_2)}{B} \cos(B\chi_2) \end{aligned}, \tag{4.100}$$

The continuity of the beam at crack location is satisfied by the compatibility conditions. These conditions are based on the change of transverse deflection due to translational compliance, change of angular deflection due to rotational compliance, continuity of shear force and continuity of bending moment.

Change of lateral deflection at the crack location is directly proportional to shear force through translational compliance.

$$\Delta W(x,t) = C_{TR} S(x,t), \tag{4.101}$$

After replacing non-dimensional parameters and applying separation of variables, non-dimensional compatibility equation related to shear and transverse displacement is obtained.

$$w_2(L_1) - w_1(L_1) = c_{tr}\left(\frac{dw_1(\chi_1)}{d\chi_1} - f_1(\chi_1)\right)_{L_1}, \tag{4.102}$$

where $c_{tr} = \dfrac{C_{TR}\,kGA}{L}$.

Change of slope at the crack location is directly proportional to moment through rotational compliance of the crack.

$$\Delta\Phi(x,t) = C_R M(x,t) = C_R\, EI\frac{\partial\Phi(x,t)}{\partial x}, \tag{4.103}$$

Non-dimensional parameters and separation of variables should also be used in slope and moment compatibility relation.

$$f_2(L_1) - f_1(L_1) = c_r\left(\frac{df_1(\chi_1)}{d\chi_1}\right)_{L_1}, \tag{4.104}$$

where $c_r = \dfrac{C_R\,EI}{L}$.

Continuity of shear force states that at the crack location, $S_1(L_c,t)=S_2(L_c,t)$. This condition is satisfied by the following compatibility relation.

$$\left(\frac{dw_1(\chi_1)}{d\chi_1} - f_1(\chi_1)\right)_{L_1} = \left(\frac{dw_2(\chi_2)}{d\chi_2} - f_2(\chi_2)\right)_{L_1}, \tag{4.105}$$

The remaining compatibility condition is related to the continuity of the bending moment at the crack location which is expressed by $M_1(L_c,t)=M_2(L_c,t)$.

$$\left(\frac{df_1(\chi_1)}{d\chi_1}\right)_{L_1} = \left(\frac{df_2(\chi_2)}{d\chi_2}\right)_{L_1}, \tag{4.106}$$

By applying the compatibility conditions using the equations from (4.97) to (4.100) four linearly independent equations are obtained.

By applying the first compatibility condition given by equation (4.102):

$$\begin{aligned}&C_1H_{11}(\omega)+C_2H_{12}(\omega)+C_3H_{13}(\omega)+C_4H_{14}(\omega)\\&+C_5H_{15}(\omega)+C_6H_{16}(\omega)+C_7H_{17}(\omega)+C_8H_{18}(\omega)=0\end{aligned}, \tag{4.107}$$

By applying the second compatibility condition given by equation (4.104):

$$\begin{aligned}&C_1H_{21}(\omega)+C_2H_{22}(\omega)+C_3H_{23}(\omega)+C_4H_{24}(\omega)\\&+C_5H_{25}(\omega)+C_6H_{26}(\omega)+C_7H_{27}(\omega)+C_8H_{28}(\omega)=0\end{aligned}, \tag{4.108}$$

By applying the third compatibility condition given by equation (4.105):

$$\begin{aligned}&C_1H_{31}(\omega)+C_2H_{32}(\omega)+C_3H_{33}(\omega)+C_4H_{34}(\omega)\\&+C_5H_{35}(\omega)+C_6H_{36}(\omega)+C_7H_{37}(\omega)+C_8H_{38}(\omega)=0\end{aligned}, \tag{4.109}$$

By applying the fourth compatibility condition given by equation (4.106):

$$\begin{aligned}&C_1H_{41}(\omega)+C_2H_{42}(\omega)+C_3H_{43}(\omega)+C_4H_{44}(\omega)\\&+C_5H_{45}(\omega)+C_6H_{46}(\omega)+C_7H_{47}(\omega)+C_8H_{48}(\omega)=0\end{aligned}, \tag{4.110}$$

The functions of "ω" from H_{11} to H_{48} are given in Appendix C.

The compatibility equations of Timoshenko beam resting on elastic Pasternak Foundation have been derived. These compatibility equations due to crack are independent of the boundary conditions. There are eight unknown coefficients from C_1 to C_8, and yet there are four equations; so four more equations are necessary. The next four equations will be obtained by applying the boundary conditions of Timoshenko beam. The equations that will be obtained by applying the boundary conditions will differ, depending on how the beam is supported.

4.2.1.1 Simply Supported Boundary Conditions

For a simply supported Timoshenko beam, both end surfaces of the beam are free to rotate, but restricted in transverse motion. Thus, moment at both end of the beam become zero as well as the transverse deflection.

$$M(x,t) = EI \left. \frac{\partial \Phi(x,t)}{\partial x} \right|_{x=0,L} = 0, \tag{4.111}$$

$$\left. W(x,t) \right|_{x=0,L} = 0, \tag{4.112}$$

After the equations of motion had been derived, they were introduced non-dimensional parameters. Thus, non-dimensional parameters should also be inserted into the boundary conditions and then the common multiplicative factors can be cancelled. In addition, since the beam consists of two segments, before and after an edge crack, while considering the boundary conditions, the solutions that belong to corresponding part of the beam section should be used.

$$\left. \frac{df_1(\chi_1)}{d\chi_1} \right|_{\chi_1=0} = 0, \tag{4.113}$$

$$\left. \frac{df_2(\chi_2)}{d\chi_2} \right|_{\chi_2=1} = 0, \tag{4.114}$$

$$w_1(0) = 0, \tag{4.115}$$

$$w_2(1) = 0, \tag{4.116}$$

After applying the boundary conditions four additional linearly independent equations are obtained.

$$C_1\left[(\eta_2+1)A^2+\omega^2-\eta_1\right]-C_3\left[(\eta_2+1)B^2-\omega^2+\eta_1\right]=0, \tag{4.117}$$

$$\begin{aligned}&C_5\cosh(A)\left[(\eta_2+1)A^2+\omega^2-\eta_1\right]+C_6\sinh(A)\left[(\eta_2+1)A^2+\omega^2-\eta_1\right]\\&-C_7\cos(B)\left[(\eta_2+1)B^2-\omega^2+\eta_1\right]-C_8\sin(B)\left[(\eta_2+1)B^2-\omega^2+\eta_1\right]=0\end{aligned}, \tag{4.118}$$

$$C_1+C_3=0, \tag{4.119}$$

$$C_5\cosh(A)+C_6\sinh(A)+C_7\cos(B)+C_8\sin(B)=0, \tag{4.120}$$

Previously, by applying the edge crack compatibility conditions functions of "ω" from $H_{11}(\omega)$ to $H_{48}(\omega)$ were obtained. The remaining functions from $H_{51}(\omega)$ to $H_{88}(\omega)$ depend on the boundaries and are to be obtained by application of corresponding boundary conditions. For simply supported boundary conditions the multiples of constants (C_1 to C_8) which are given in equations from (4.117) to (4.120) can be written as functions of "ω" and results in functions from $H_{51}(\omega)$ to $H_{88}(\omega)$. These functions are listed in Appendix D.

After applying all compatibility and boundary conditions, eight equations and eight unknowns are obtained. These linearly independent equations can be written in matrix form.

$$\begin{bmatrix} H_{11}(\omega) & H_{12}(\omega) & H_{13}(\omega) & H_{14}(\omega) & H_{15}(\omega) & H_{16}(\omega) & H_{17}(\omega) & H_{18}(\omega) \\ H_{21}(\omega) & H_{22}(\omega) & H_{23}(\omega) & H_{24}(\omega) & H_{25}(\omega) & H_{26}(\omega) & H_{27}(\omega) & H_{28}(\omega) \\ H_{31}(\omega) & H_{32}(\omega) & H_{33}(\omega) & H_{34}(\omega) & H_{35}(\omega) & H_{36}(\omega) & H_{37}(\omega) & H_{38}(\omega) \\ H_{41}(\omega) & H_{42}(\omega) & H_{43}(\omega) & H_{44}(\omega) & H_{45}(\omega) & H_{46}(\omega) & H_{47}(\omega) & H_{48}(\omega) \\ H_{51}(\omega) & H_{52}(\omega) & H_{53}(\omega) & H_{54}(\omega) & H_{55}(\omega) & H_{56}(\omega) & H_{57}(\omega) & H_{58}(\omega) \\ H_{61}(\omega) & H_{62}(\omega) & H_{63}(\omega) & H_{64}(\omega) & H_{65}(\omega) & H_{66}(\omega) & H_{67}(\omega) & H_{68}(\omega) \\ H_{71}(\omega) & H_{72}(\omega) & H_{73}(\omega) & H_{74}(\omega) & H_{75}(\omega) & H_{76}(\omega) & H_{77}(\omega) & H_{78}(\omega) \\ H_{81}(\omega) & H_{82}(\omega) & H_{83}(\omega) & H_{84}(\omega) & H_{85}(\omega) & H_{86}(\omega) & H_{87}(\omega) & H_{88}(\omega) \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \\ C_7 \\ C_8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{4.121}$$

As given in equation 4.21, the elements of the matrix are functions of non-dimensional frequency "ω" from $H_{11}$to H_{88}. The nontrivial solution of this homogenous system can be obtained if and only if determinant of the coefficient matrix is equal to zero.

$$\begin{vmatrix} H_{11}(\omega) & H_{12}(\omega) & H_{13}(\omega) & H_{14}(\omega) & H_{15}(\omega) & H_{16}(\omega) & H_{17}(\omega) & H_{18}(\omega) \\ H_{21}(\omega) & H_{22}(\omega) & H_{23}(\omega) & H_{24}(\omega) & H_{25}(\omega) & H_{26}(\omega) & H_{27}(\omega) & H_{28}(\omega) \\ H_{31}(\omega) & H_{32}(\omega) & H_{33}(\omega) & H_{34}(\omega) & H_{35}(\omega) & H_{36}(\omega) & H_{37}(\omega) & H_{38}(\omega) \\ H_{41}(\omega) & H_{42}(\omega) & H_{43}(\omega) & H_{44}(\omega) & H_{45}(\omega) & H_{46}(\omega) & H_{47}(\omega) & H_{48}(\omega) \\ H_{51}(\omega) & H_{52}(\omega) & H_{53}(\omega) & H_{54}(\omega) & H_{55}(\omega) & H_{56}(\omega) & H_{57}(\omega) & H_{58}(\omega) \\ H_{61}(\omega) & H_{62}(\omega) & H_{63}(\omega) & H_{64}(\omega) & H_{65}(\omega) & H_{66}(\omega) & H_{67}(\omega) & H_{68}(\omega) \\ H_{71}(\omega) & H_{72}(\omega) & H_{73}(\omega) & H_{74}(\omega) & H_{75}(\omega) & H_{76}(\omega) & H_{77}(\omega) & H_{78}(\omega) \\ H_{81}(\omega) & H_{82}(\omega) & H_{83}(\omega) & H_{84}(\omega) & H_{85}(\omega) & H_{86}(\omega) & H_{87}(\omega) & H_{88}(\omega) \end{vmatrix} = 0$$

(4.122)

The "ω" values that make the determinant zero are eigenvalues of the set of differential equations which denote the equations of motion. These eigenvalues are also the non-dimensional natural frequencies of the Timoshenko beam on Pasternak elastic foundation for the specified boundary conditions.

For different boundary conditions, there will be some changes in the characteristic equation of simply supported boundary conditions. However the elements from $H_{11}(\omega)$ to $H_{48}(\omega)$ will remain the same for different boundary conditions, since these elements are obtained by the application of compatibility conditions. The change of expressions of elements is to be mentioned for the corresponding boundary condition in the following parts of the thesis study.

4.2.1.2 Fixed-Fixed Boundary Conditions

In fixed-fixed boundary conditions, displacement in transverse direction and rotation due to bending moment at both ends of the beam are zero. Also in pinned-pinned boundary condition which was the previous case, the transverse displacement was zero. So in the characteristic equation, the elements obtained due to moment condition of pinned-pinned boundaries are to be replaced by the elements obtained due to zero rotation condition of fixed-fixed boundaries. Since the compatibility equations are independent of boundary conditions, the elements from H_{11} to H_{48} will remain same. In addition, elements of the matrix from H_{71} to H_{88} which were obtained by applying zero transverse displacement at both ends of the beam will remain same. Consequently only the elements from H_{51} to H_{68} are to be replaced by the new expressions that will be obtained by applying zero rotation condition of fixed-fixed boundary conditions.

The rotation due to bending moment at both fixed ends can be expressed by;

$$\Phi(x,t)\big|_{x=0,L} = 0, \tag{4.123}$$

Non dimensional variables should be introduced and separation of variables should be applied.

$$f_1(\chi_1)\big|_{\chi_1=0} = 0, \tag{4.124}$$

$$f_2(\chi_2)\big|_{\chi_2=1} = 0, \tag{4.125}$$

New expressions can be obtained by inserting definition of $f_1(\chi_1)$ and $f_2(\chi_2)$.

$$C_2 \frac{\left[(\eta_2+1)A^2+\omega^2-\eta_1\right]}{A} + C_4 \frac{\left[(\eta_2+1)B^2-\omega^2+\eta_1\right]}{B} = 0, \tag{4.126}$$

$$\begin{aligned} &C_5 \frac{\sinh(A)\left[(\eta_2+1)A^2+\omega^2-\eta_1\right]}{A} + C_6 \frac{\cosh(A)\left[(\eta_2+1)A^2+\omega^2-\eta_1\right]}{A} \\ &-C_7 \frac{\sin(B)\left[(\eta_2+1)B^2-\omega^2+\eta_1\right]}{B} + C_8 \frac{\cos(B)\left[(\eta_2+1)B^2-\omega^2+\eta_1\right]}{B} = 0 \end{aligned}, \tag{4.127}$$

So the elements from $H_{51}(\omega)$ to $H_{68}(\omega)$ are listed in Appendix E.

By replacing these new expressions which are obtained by application of fixed-fixed boundary conditions characteristic equation for the corresponding boundary conditions is obtained. The "ω" values that make the determinant zero are non dimensional natural frequencies of cracked Timoshenko beam on two parameters elastic Pasternak foundation with fixed-fixed boundary conditions.

4.2.1.3 Cantilevered (Fixed-Free) Boundary Conditions

Again, the equations obtained by applying the compatibility conditions will remain the same. Four new linearly independent equations are to be obtained by applying fixed-free boundary conditions.

In fixed free boundary conditions, transverse displacement at the fixed end is zero as well as the rotation due to bending moment. At the free end, no interaction with the elastic foundation is assumed. Thus, bending moment and shear force at the free end are zero.

$$W(x,t)\big|_{x=0} = 0\,, \tag{4.128}$$

$$\Phi(x,t)\big|_{x=0} = 0\,, \tag{4.129}$$

$$EI\frac{\partial \Phi(x,t)}{\partial x}\bigg|_{x=L} = 0\,, \tag{4.130}$$

$$kGA\left(\frac{\partial W(x,t)}{\partial x} - \Phi(x,t)\right)\bigg|_{x=L} = 0\,, \tag{4.131}$$

The fixed-free boundary condition equations should be introduced non-dimensional variables and then separation of variables should be applied.

$$w_1(\chi_1)\big|_{\chi_1=0} = 0\,, \tag{4.132}$$

$$f_1(\chi_1)\big|_{\chi_1=0} = 0\,, \tag{4.133}$$

$$\frac{df_2(\chi_2)}{d\chi_2}\bigg|_{\chi_2=1} = 0\,, \tag{4.134}$$

$$\left(\frac{dw_2(\chi_2)}{d\chi_2} - f_2(\chi_2)\right)\bigg|_{\chi_2=1} = 0\,, \tag{4.135}$$

After applying the boundary conditions given by equations from (4.132) to (4.135), four linearly independent equations of fixed-free boundary conditions are obtained.

$$C_1 + C_3 = 0\,, \tag{4.136}$$

$$C_2 \frac{\left[\left(\eta_2+1\right)A^2+\omega^2-\eta_1\right]}{A} + C_4 \frac{\left[\left(\eta_2+1\right)B^2-\omega^2+\eta_1\right]}{B} = 0\,, \tag{4.137}$$

$$\begin{gathered} C_5\cosh(A)\left[\left(\eta_2+1\right)A^2+\omega^2-\eta_1\right]+C_6\sinh(A)\left[\left(\eta_2+1\right)A^2+\omega^2-\eta_1\right] \\ -C_7\cos(B)\left[\left(\eta_2+1\right)B^2-\omega^2+\eta_1\right]-C_8\sin(B)\left[\left(\eta_2+1\right)B^2-\omega^2+\eta_1\right]=0 \end{gathered}, \tag{4.138}$$

$$\begin{gathered} C_5\left[A\sinh(A)-\frac{\sinh(A)\left[\left(\eta_2+1\right)A^2+\omega^2-\eta_1\right]}{A}\right] \\ +C_6\left[A\cosh(A)-\frac{\cosh(A)\left[\left(\eta_2+1\right)A^2+\omega^2-\eta_1\right]}{A}\right] \\ +C_7\left[\frac{\sin(B)\left[\left(\eta_2+1\right)B^2-\omega^2+\eta_1\right]}{B}-B\sin(B)\right] \\ +C_8\left[B\cos(B)-\frac{\cos(B)\left[\left(\eta_2+1\right)B^2-\omega^2+\eta_1\right]}{B}\right]=0 \end{gathered}, \tag{4.139}$$

Previously an eight by eight matrix was obtained for simply supported boundary conditions. Elements from H_{11} to H_{48} of this matrix were obtained by applying compatibility conditions at the crack position and the rest of the elements were obtained by application of simply supported boundary conditions. So, the elements which are due to simply supported boundary conditions should be replaced by the new elements that are obtained for fixed free boundary conditions to obtain the characteristic equation of fixed free boundary conditions. The "ω" values that make the determinant of obtained matrix zero are non dimensional natural frequencies of cracked Timoshenko beam resting on two parameters Pasternak foundation with fixed free boundary conditions. The elements of the matrix due to cantilevered boundary conditions are given in Appendix F.

The solution of edge cracked Timoshenko beam on elastic Pasternak foundation for simple boundary conditions have been obtained. In the following parts, the solutions for different foundation models are to be derived. These foundation models are generalized foundation model 1 and 2. The change of foundation models will affect only expression of "κ" and "ζ". Thus, depending on the different foundation model the corresponding expressions of "κ" and "ζ" are to be derived and the rest of the solution procedure will remain as same as the solution of edge cracked Timoshenko beam on Pasternak elastic foundation.

4.2.2 Solution of Equations of Motion of Cracked Timoshenko Beam on Generalized Elastic Foundation Model 1 in Transverse Vibration

Equations of motion of Timoshenko beam on generalized foundation model 1 are given by equations (3.25) and (3.29). Before starting the solution, these equations should be transformed into non dimensional forms by replacing the non dimensional variables.

$$kGA\left(\frac{\partial^2 W(\chi,\tau)L}{\partial(\chi L)^2}-\frac{\partial\Phi(\chi,\tau)}{\partial(\chi L)}\right)-k_w LW(\chi,\tau)-\rho A\frac{\partial^2 W(\chi,\tau)L}{\partial(\tau T)^2}=0, \quad (4.140)$$

$$EI\frac{\partial^2\Phi(\chi,\tau)}{\partial(\chi L)^2}+kGA\left(\frac{\partial W(\chi,\tau)L}{\partial(\chi L)}-\Phi(\chi,\tau)\right)-k_\Phi\Phi(\chi,\tau)-\rho I\frac{\partial^2\Phi(\chi,\tau)}{\partial(\tau T)^2}=0, \quad (4.141)$$

After doing necessary cancellations, equations of motions can be rearranged as below:

$$\frac{kGA}{L}\left(\frac{\partial^2 W(\chi,\tau)}{\partial\chi^2}-\frac{\partial\Phi(\chi,\tau)}{\partial\chi}\right)-k_w LW(\chi,\tau)-\frac{\rho AL}{T^2}\frac{\partial^2 W(\chi,\tau)}{\partial\tau^2}=0, \quad (4.142)$$

$$\frac{EI}{L^2}\frac{\partial^2\Phi(\chi,\tau)}{\partial\chi^2}+kGA\left(\frac{\partial W(\chi,\tau)}{\partial\chi}-\Phi(\chi,\tau)\right)-k_\Phi\Phi(\chi,\tau)-\frac{\rho I}{T^2}\frac{\partial^2\Phi(\chi,\tau)}{\partial\tau^2}=0, \quad (4.143)$$

In the next step, divide the first differential equation by "kGA/L" and divide the second differential equation by "kGA".

$$\frac{\partial^2 W(\chi,\tau)}{\partial\chi^2}-\frac{\partial\Phi(\chi,\tau)}{\partial\chi}-\frac{k_w L^2}{kGA}W(\chi,\tau)-\frac{\rho L^2}{kGT^2}\frac{\partial^2 W(\chi,\tau)}{\partial\tau^2}=0\,, \tag{4.144}$$

$$\frac{EI}{kGAL^2}\frac{\partial^2\Phi(\chi,\tau)}{\partial\chi^2}+\frac{\partial W(\chi,\tau)}{\partial\chi}-\Phi(\chi,\tau)-\frac{k_\Phi}{kGA}\Phi(\chi,\tau)-\frac{\rho I}{kGAT^2}\frac{\partial^2\Phi(\chi,\tau)}{\partial\tau^2}=0\,, \tag{4.145}$$

Non dimensional parameters given by equation (4.47) can be replaced into equations of motion except "η_2", since it is non dimensional Pasternak foundation parameter. Besides these non dimensional parameters a new parameter should be defined for the second parameter of generalized foundation model 1.

$$\frac{\partial^2 W(\chi,\tau)}{\partial\chi^2}-\frac{\partial\Phi(\chi,\tau)}{\partial\chi}-\eta_1 W(\chi,\tau)-\frac{\partial^2 W(\chi,\tau)}{\partial\tau^2}=0\,, \tag{4.146}$$

$$\eta_3\frac{\partial^2\Phi(\chi,\tau)}{\partial\chi^2}+\frac{\partial W(\chi,\tau)}{\partial\chi}-\Phi(\chi,\tau)-\eta_5\Phi(\chi,\tau)-\eta_4\frac{\partial^2\Phi(\chi,\tau)}{\partial\tau^2}=0\,, \tag{4.147}$$

where; $\eta_5=\dfrac{k_\Phi}{kGA}$.

After obtaining the non dimensional equations of motion, separation of variables should be applied by replacing $W(\chi,\tau)$ and $\Phi(\chi,\tau)$ with equations (4.50) and (4.51).

$$\frac{d^2 w(\chi)}{d\chi^2}F(\tau)-\frac{df(\chi)}{d\chi}F(\tau)-\eta_1 w(\chi)F(\tau)-w(\chi)\frac{d^2F(\tau)}{d\tau^2}=0\,, \tag{4.148}$$

$$\eta_3\frac{d^2 f(\chi)}{d\chi^2}F(\tau)+\frac{dW(\chi)}{d\chi}F(\tau)-f(\chi)F(\tau)-\eta_5 f(\chi)F(\tau)-\eta_4 f(\chi)\frac{d^2F(\tau)}{d\tau^2}=0\,, \tag{4.149}$$

It is seen that, the functions of "χ" and the functions of "τ" can be gathered on either side of the equality just like the case given by equations (4.56) and (4.57).

$$\frac{1}{w(\chi)}\left(\frac{d^2 w(\chi)}{d\chi^2}-\frac{df(\chi)}{d\chi}-\eta_1 w(\chi)\right)=\frac{1}{F(\tau)}\frac{d^2F(\tau)}{d\tau^2}=-\omega^2\,, \tag{4.150}$$

$$\frac{1}{\eta_4 f(\chi)}\left(\eta_3\frac{d^2 f(\chi)}{d\chi^2}+\frac{dw(\chi)}{d\chi}-f(\chi)-\eta_5 f(\chi)\right)=\frac{1}{F(\tau)}\frac{d^2F(\tau)}{d\tau^2}=-\omega^2\,, \tag{4.151}$$

Then a set of differential equations can be obtained in terms of functions $w(\chi)$ and $f(\chi)$.

$$\frac{d^2w(\chi)}{d\chi^2}-\frac{df(\chi)}{d\chi}-\eta_1 w(\chi)+\omega^2 w(\chi)=0\,, \tag{4.152}$$

$$\eta_3\frac{d^2f(\chi)}{d\chi^2}+\frac{dw(\chi)}{d\chi}-f(\chi)-\eta_5 f(\chi)+\omega^2\eta_4 f(\chi)=0\,, \tag{4.153}$$

These differential equations should be written in operator form.

$$(D^2-\eta_1+\omega^2)w(\chi)-Df(\chi)=0\,, \tag{4.154}$$

$$Dw(\chi)+(\eta_3 D^2-1-\eta_5+\omega^2\eta_4)f(\chi)=0\,, \tag{4.155}$$

By eliminating f(χ), a fourth order differential equation in terms of w(χ) is obtained.

$$\eta_3\frac{d^4w(\chi)}{d\chi^4}+\left(\omega^2\eta_4-\eta_5-\eta_3(\eta_1-\omega^2)\right)\frac{d^2w(\chi)}{d\chi^2}+(\eta_1-\omega^2)(\eta_5-\eta_4\omega^2+1)w(\chi)=0\,, \tag{4.156}$$

Divide the fourth order differential equation by "η_3".

$$\frac{d^4w(\chi)}{d\chi^4}+\frac{\left(\omega^2(\eta_4+\eta_3)-\eta_5-\eta_3\eta_1\right)}{\eta_3}\frac{d^2w(\chi)}{d\chi^2}+\frac{(\omega^2-\eta_1)(\eta_4\omega^2-\eta_5-1)}{\eta_3}w(\chi)=0\,, \tag{4.157}$$

"κ" and "ζ" values should be redefined for Timoshenko beam on generalized elastic foundation model 1.

$$\kappa=\frac{\omega^2(\eta_4+\eta_3)-\eta_5-\eta_3\eta_1}{\eta_3}\,, \tag{4.158}$$

$$\zeta=\frac{(\omega^2-\eta_1)(\eta_4\omega^2-\eta_5-1)}{\eta_3}\,, \tag{4.159}$$

Then fourth order differential equation of w(χ) can be written in a more simple form by using the redefined "κ" and "ζ" values.

$$\frac{d^4w(\chi)}{d\chi^4}+\kappa\frac{d^2w(\chi)}{d\chi}+\zeta w(\chi)=0\,, \tag{4.160}$$

This differential equation is just as same as the one given by equation (4.67). Thus the remaining part of the solution will be as same as the case of Timoshenko beam on Pasternak foundation. It should be noted that, definitions of "κ" and "ζ" values are different for generalized foundation model 1. Expressions of "$w(\chi)$" and "$f(\chi)$" will be as same as equations (4.81) and (4.82), but the coefficients of "$f(\chi)$" in terms of C_1 to C_4 should be redefined. The coefficients of "$f(\chi)$" can be obtained by inserting both "$w(\chi)$" and "$f(\chi)$" into equation (4.152). At this point, it should be noted that the difference between equations (4.152) and (4.58) is "η_2".In equation (4.152) there is no "η_2" term, so the expression of "$f(\chi)$" for generalized foundation model 1 can be obtained by giving zero to "η_2" parameter in equation (4.98). After that, by applying the crack compatibility and the boundary conditions non-dimensional natural frequencies can be obtained by following the same procedure that was explained in detail for transverse vibration of cracked Timoshenko beam on Pasternak foundation. The elements of matrix whose determinant defines the characteristic equation of transversely vibrating cracked Timoshenko beam on generalized foundation 1 can be obtained by using newly defined κ(4.158), ζ(4.159) and giving zero value to "η_2" in the elements of the matrix of Pasternak foundation case.

4.2.3 Solution of Equations of Motion of Cracked Timoshenko Beam on Generalized Elastic Foundation Model 2 in Transverse Vibration

Previously equations of motion for Timoshenko beam on two parameters elastic Pasternak foundation were solved. Then an edge crack was introduced and characteristic equation that results in non dimensional natural frequencies for different simple boundary conditions were given. After that, Timoshenko beam on generalized elastic foundation model 1 was studied, and it was shown that for different foundation models the fourth order simplified differential equation was in the same form which is given by equation (4.67). Only the coefficients "κ" and "ζ" were expressed in slightly different manner due to changes in the elastic foundation. The rest of the solution steps that results in the natural frequencies are as same as the solution steps of Timoshenko beam on Pasternak foundation.

Similar procedure is to be followed for Timoshenko beam on generalized foundation model 2. The expressions "κ" and "ζ" regarding generalized foundation model 2 are to be derived. Then, by following the same solution steps as the ones for the previous foundation models, the natural frequencies of transverse vibration of edge cracked Timoshenko beam on generalized elastic foundation model 2 are to be calculated.

The equations of motion of transversely vibrating Timoshenko beam on generalized elastic foundation model 2 were derived previously which are expressed by equations (3.25) and (3.33). These differential equations should be written in non dimensional form first. The differential equation including the Winkler parameter is common for both generalized model 1 and generalized model 2. But the differential equation including the second parameter is different for model 1 and 2. Non dimensional form of the differential equation with Winkler foundation has already been derived. So, in this part the differential equation with the second parameter is to be rearranged in non dimensional form.

$$EI\frac{\partial^2\Phi(\chi,\tau)}{\partial(\chi L)^2}+kGA(\frac{\partial W(\chi,\tau)L}{\partial(\chi L)}-\Phi(\chi,\tau))-k_v\frac{\partial W(\chi,\tau)L}{\partial\chi L}-\rho I\frac{\partial^2\Phi(\chi,\tau)}{\partial(\tau T)^2}=0, \quad (4.161)$$

Necessary cancellations should be done;

$$\frac{EI}{L^2}\frac{\partial^2\Phi(\chi,\tau)}{\partial\chi^2}+kGA(\frac{\partial W(\chi,\tau)}{\partial\chi}-\Phi(\chi,\tau))-k_v\frac{\partial W(\chi,\tau)}{\partial\chi}-\frac{\rho I}{T^2}\frac{\partial^2\Phi(\chi,\tau)}{\partial\tau^2}=0, \quad (4.162)$$

Then divide whole differential equation by "kGA".

$$\frac{EI}{kGAL^2}\frac{\partial^2\Phi(\chi,\tau)}{\partial\chi^2}+\frac{\partial W(\chi,\tau)}{\partial\chi}-\Phi(\chi,\tau)-\frac{k_v}{kGA}\frac{\partial W(\chi,\tau)}{\partial\chi}-\frac{\rho I}{kGAT^2}\frac{\partial^2\Phi(\chi,\tau)}{\partial\tau^2}=0, \quad (4.163)$$

The differential equation can be written in a simple form by replacing the non dimensional coefficients.

$$\eta_3 \frac{\partial^2 \Phi(\chi,\tau)}{\partial \chi^2} + \frac{\partial W(\chi,\tau)}{\partial \chi} - \Phi(\chi,\tau) - \eta_6 \frac{\partial W(\chi,\tau)}{\partial \chi} - \eta_4 \frac{\partial^2 \Phi(\chi,\tau)}{\partial \tau^2} = 0\,, \tag{4.164}$$

where $\eta_6 = \dfrac{k_v}{kGA}$.

In the next step separation of variables should be applied for $W(\chi,\tau)$ and $\Phi(\chi,\tau)$.

$$\eta_3 \frac{d^2 f(\chi)}{d\chi^2} F(\tau) + \frac{dw(\chi)}{d\chi} F(\tau) - f(\chi)F(\tau) - \eta_6 \frac{dw(\chi)}{d\chi} F(\tau) - \eta_4 \frac{d^2 F(\tau)}{d\tau^2} f(\chi) = 0, \tag{4.165}$$

Differential equation should be arranged by collecting the "χ" terms on the left hand side and "τ" terms on the right hand side and each side of the equality should be equal to a constant.

$$\frac{1}{\eta_4 f(\chi)} \left(\eta_3 \frac{d^2 f(\chi)}{d\chi^2} + \frac{dw(\chi)}{d\chi} - f(\chi) - \eta_6 \frac{dw(\chi)}{d\chi} \right) = \frac{1}{F(\tau)} \frac{d^2 F(\tau)}{d\tau^2} = -\omega^2\,, \tag{4.166}$$

Then the following differential equation is obtained.

$$\eta_3 \frac{d^2 f(\chi)}{d\chi^2} + \frac{dw(\chi)}{d\chi} - f(\chi) - \eta_6 \frac{dw(\chi)}{d\chi} + \omega^2 \eta_4 f(\chi) = 0\,, \tag{4.167}$$

Consequently non dimensional equations of motion of transversely vibration Timoshenko beam on generalized elastic foundation model 2 are:

$$\frac{d^2 w(\chi)}{d\chi^2} - \frac{df(\chi)}{d\chi} - \eta_1 w(\chi) + \omega^2 w(\chi) = 0\,, \tag{4.168}$$

$$\eta_3 \frac{d^2 f(\chi)}{d\chi^2} + \frac{dw(\chi)}{d\chi} - f(\chi) - \eta_6 \frac{dw(\chi)}{d\chi} + \omega^2 \eta_4 f(\chi) = 0\,, \tag{4.169}$$

"$f(\chi)$" should be eliminated. In order to do this elimination, differential equations are to be written in operator forms. Then a fourth order differential equation is to be obtained given in terms of "$w(\chi)$".

$$(D^2 - \eta_1 + \omega^2) w(\chi) - D f(\chi) = 0\,, \tag{4.170}$$

$$D(1-\eta_6) w(\chi) + (\eta_3 D^2 - 1 + \omega^2 \eta_4) f(\chi) = 0\,, \tag{4.171}$$

After eliminating "$f(\chi)$" and doing the necessary cancellations, "κ" and "ζ" values for generalized foundation model 2 can be obtained.

$$\frac{d^4w(\chi)}{d\chi^4}+\frac{\left(\omega^2(\eta_4+\eta_3)-\eta_6-\eta_3\eta_1\right)}{\eta_3}\frac{d^2w(\chi)}{d\chi^2}+\frac{(\omega^2-\eta_1)(\eta_4\omega^2-1)}{\eta_3}w(\chi)=0, \tag{4.172}$$

$$\kappa=\frac{\omega^2(\eta_4+\eta_3)-\eta_6-\eta_3\eta_1}{\eta_3}, \tag{4.173}$$

$$\zeta=\frac{(\omega^2-\eta_1)(\eta_4\omega^2-1)}{\eta_3}, \tag{4.174}$$

Since equation (4.62) is used in the elimination of "C_i", η_3, η_4, η_5, η_6 do not appear explicitly in H_{ij}, but they enter the equations implicitly through κ and ζ. In the rest of the solution a procedure similar to transverse vibration of cracked Timoshenko beam on generalized elastic foundation model 1 is to be followed. Actually the characteristic equation due to generalized model 1 and 2 are the same. The only difference is expressions of "κ" and "ζ". The solution due to generalized model 1 can also be used for generalized model 2 as long as definitions of "κ" and "ζ" due to model 2 are replaced in the derivations.

CHAPTER 5

SOLUTION OF EQUATIONS OF MOTION OF CRACKED EULER-BERNOULLI BEAM RESTING ON ELASTIC FOUNDATION

5.1 Solution of Equations of Motion of Cracked Euler-Bernoulli Beam on Pasternak Elastic Foundation in Transverse Vibration

The equation of motion for cracked Euler-Bernoulli beam on elastic Pasternak foundation in transverse vibration is given by equation (3.38). Before starting the solution, the non-dimensional parameters are to be introduced into equation (3.38).

$$\frac{EI}{L^3}\frac{\partial^4 W(\chi,\tau)}{\partial\chi^4}+k_w W(\chi,\tau)\,L-\frac{G_0}{L}\frac{\partial^2 W(\chi,\tau)}{\partial\chi^2}+\frac{\rho AL}{T_{EB}{}^2}\frac{\partial^2 W(\chi,\tau)}{\partial\tau^2}=0\,, \tag{5.1}$$

where T_{EB} is non-dimensional time parameter for Euler-Bernoulli beam. After dividing Equation (5.1) by EI/L^3 , equation of motion in terms of non-dimensional coefficients is obtained.

$$\frac{\partial^4 W(\chi,\tau)}{\partial\chi^4}+\frac{k_w L^4}{EI}W(\chi,\tau)-\frac{G_0 L^2}{EI}\frac{\partial^2 W(\chi,\tau)}{\partial\chi^2}+\frac{\rho AL^4}{EIT_{EB}{}^2}\frac{\partial^2 W(\chi,\tau)}{\partial\tau^2}=0\,, \tag{5.2}$$

Since all coefficients of equation (5.2) are non-dimensional, then non-dimensional time parameter is equal to:

$$T_{EB}=L^2\sqrt{\frac{\rho A}{EI}}\,, \tag{5.3}$$

The non-dimensional coefficients of equation (5.2) can be introduced as:

$$\lambda_1 = \frac{k_w L^4}{EI}, \tag{5.4}$$

$$\lambda_2 = \frac{G_0 L^2}{EI}, \tag{5.5}$$

By inserting equations form (5.3) to (5.5) into equation (5.2) results in non-dimensional equation of motion of Euler-Bernoulli beam on two parameters Pasternak elastic foundation.

$$\frac{\partial^4 W(\chi,\tau)}{\partial \chi^4} + \lambda_1 W(\chi,\tau) - \lambda_2 \frac{\partial^2 W(\chi,\tau)}{\partial \chi^2} + \frac{\partial^2 W(\chi,\tau)}{\partial \tau^2} = 0, \tag{5.6}$$

$W(\chi,\tau)$ is assumed to be separable which is given by equation (4.50). The following relation is obtained by inserting equation (4.50) into equation (5.6).

$$\frac{d^4 w(\chi)}{d\chi^4} F(\tau) + \lambda_1 w(\chi) F(\tau) - \lambda_2 \frac{d^2 w(\chi)}{d\chi^2} F(\tau) + \frac{d^2 F(\tau)}{d\tau^2} w(\chi) = 0, \tag{5.7}$$

Then, divide equation (5.7) by "$w(\chi)F(\tau)$".

$$\frac{1}{w(\chi)}\left(\frac{d^4 w(\chi)}{d\chi^4} - \lambda_2 \frac{d^2 w(\chi)}{d\chi^2}\right) + \lambda_1 = -\frac{1}{F(\tau)}\frac{d^2 F(\tau)}{d\tau^2},$$

(5.8)

In equation (5.8), left hand side of the equality is given in terms of $w(\chi)$ and right hand side is given in terms of $F(\tau)$. This is possible only if both sides of the equation are equal to a constant value.

$$\frac{1}{w(\chi)}\left(\frac{d^4 w(\chi)}{d\chi^4} - \lambda_2 \frac{d^2 w(\chi)}{d\chi^2}\right) + \lambda_1 = -\frac{1}{F(\tau)}\frac{d^2 F(\tau)}{d\tau^2} = \omega^2, \tag{5.9}$$

$$\frac{d^4 w(\chi)}{d\chi^4} - \lambda_2 \frac{d^2 w(\chi)}{d\chi^2} - (\omega^2 - \lambda_1) w(\chi) = 0, \tag{5.10}$$

In equation (5.10) "ω" is an eigenvalue to be determined. This eigenvalue is also non-dimensional natural frequency of Euler-Bernoulli beam on elastic Pasternak foundation. For the solution equation (5.10) w(χ) is assumed to be:

$$w(\chi) = e^{n\chi}, \tag{5.11}$$

Equation (5.11) is replaced in equation (5.10), then a fourth order equation is obtained.

$$n^4 - \lambda_2 n^2 - (\omega^2 - \lambda_1) = 0, \tag{5.12}$$

Then;

$$n^2 = \frac{\lambda_2 \pm \sqrt{\lambda_2{}^2 + 4(\omega^2 - \lambda_1)}}{2}, \tag{5.13}$$

$$n_1 = A, \qquad n_2 = -A, \qquad n_3 = Bi, \qquad n_4 = -Bi, \tag{5.14}$$

where;

$$A = \sqrt{\frac{\lambda_2 + \sqrt{\lambda_2{}^2 + 4(\omega^2 - \lambda_1)}}{2}}, \tag{5.15}$$

$$B = \sqrt{\frac{\sqrt{\lambda_2{}^2 + 4(\omega^2 - \lambda_1)} - \lambda_2}{2}}, \tag{5.16}$$

So solution of equation (5.10) becomes;

$$w(\chi) = K_1 e^{A\chi} + K_2 e^{-A\chi} + K_3 e^{iB\chi} + K_4 e^{-iB\chi}, \tag{5.17}$$

Using Euler's formula and definitions for hyperbolic cosine and sine, equation (5.17) can be expressed in terms of trigonometric functions.

$$w(\chi) = C_1 \cosh(A\chi) + C_2 \sinh(A\chi) + C_3 \cos(B\chi) + C_4 \sin(B\chi), \tag{5.18}$$

Since a cracked Euler-Bernoulli beam is considered, the whole beam is modeled by two beams connected by a spring, where the spring represents the compliance due to edge crack. In the studies, concerning transverse vibration of cracked Euler-Bernoulli beam, the edge crack was represented only by rotational springs. This subject was mentioned in detail in Chapter 4 literature survey of crack models part. However, in the thesis study, the edge crack in Euler-Bernoulli beam is to be modeled by both rotational and translational springs just similar to the case of Timoshenko beam with edge crack, equations of which were derived in Chapter 4.

Thus the solution (5.18) will depend on different variables and coefficients on either side of the crack.

For the left side of the crack:

$$w_1(\chi_1) = C_1 \cosh(A\chi_1) + C_2 \sinh(A\chi_1) + C_3 \cos(B\chi_1) + C_4 \sin(B\chi_1)\,, \tag{5.19}$$

For the right side of the crack:

$$w_2(\chi_2) = C_5 \cosh(A\chi_2) + C_6 \sinh(A\chi_2) + C_7 \cos(B\chi_2) + C_8 \sin(B\chi_2)\,, \tag{5.20}$$

The compatibility equations are due to jump in transverse displacement, jump in rotation of beam element at crack location, and continuity of shear force and moment at the point of crack. For non-dimensional crack location, relation given by equation (4.96) is to be used.

The compatibility relation due to jump in transverse deflection was expressed by equation (4.101). This equation is rewritten for Euler-Bernoulli case by denoting the translational compliance by K_{TR}.

$$\Delta W(x,t) = K_{TR} S(x,t)\,, \tag{5.21}$$

$$W_2(x,t) - W_1(x,t) = K_{TR} S(x,t)\,, \tag{5.22}$$

S(x,t) for Euler-Bernoulli beam is given by equation (3.36). Expression of S(x,t) for Euler-Bernoulli beam theory and the non-dimensional terms should be replaced in equation (5.22).

$$W_2(\chi,\tau)\,L - W_1(\chi,\tau)\,L = K_{TR}\left(-\frac{EI}{L^2}\frac{\partial^3 W_1(\chi_1,\tau)}{\partial \chi_1{}^3}\right)_{\chi_1=L_1}, \tag{5.23}$$

Then, the necessary cancellations should be done and separation of variables should be applied to equation (5.23).

$$w_2(L_1) - w_1(L_1) = -k_{tr}\left.\frac{d^3 w_1(\chi_1)}{d\chi_1{}^3}\right|_{\chi_1=L_1}, \tag{5.24}$$

where $k_{tr} = \dfrac{K_{TR}EI}{L^3}$.

Another compatibility relation is due to change in slope. The rotational compliance of the edge crack relates moment to change in slope. Similar to equation (4.103) linear relation of slope and moment for Euler-Bernoulli beam can be given by the following relation.

$$\Delta\Phi(x,t) = K_R M(x,t)\,, \tag{5.25}$$

In Euler-Bernoulli beam theory moment is expressed in terms of W(x,t) as mentioned by equation (3.35) and slope is given by;

$$\Phi(x,t) = \frac{\partial W(x,t)}{\partial x}, \tag{5.26}$$

Next, equations (3.35) and (5.26) should be inserted in to equation (5.26) and then non-dimensional variables should be introduced. After that by applying separation of variables the compatibility relation due to rotational compliance of the representative spring can be obtained.

$$\left.\frac{dw_2(\chi_2)}{d\chi_2}\right|_{\chi_2=L_1} - \left.\frac{dw_1(\chi_1)}{d\chi_1}\right|_{\chi_1=L_1} = k_r\left.\frac{d^2 w_1(\chi_1)}{dx_1{}^2}\right|_{\chi_1=L_1}, \tag{5.27}$$

where $k_r = \dfrac{K_R EI}{L}$.

There are two more compatibility relations which are continuity of shear force and bending moment.

Shear force in Euler-Bernoulli beam theory is due to bending moment induced in the beam and it is given by equation (3.36). At the crack location, shear force is continuous thus it is equal for each beam piece. This compatibility relation can be written as:

$$EI\frac{\partial^3 W_1(x_1,t)}{\partial x_1^{\ 3}}\bigg|_{x_1=L_c} = EI\frac{\partial^3 W_2(x_2,t)}{\partial x_2^{\ 3}}\bigg|_{x_2=L_c}, \tag{5.28}$$

Non-dimensional terms should be introduced into equation (5.28) and separation of variables should be applied.

So, the compatibility relation due to continuity of shear force is expressed by the following relation.

$$\frac{d^3 w_1(\chi_1)}{d\chi_1^{\ 3}}\bigg|_{\chi_1=L_1} = \frac{d^3 w_2(\chi_2)}{d\chi_2^{\ 3}}\bigg|_{\chi_2=L_1}, \tag{5.29}$$

Similar to shear force, also the bending moment at the crack location is equal on each side of the crack surface. Thus, bending moments at crack location for each beam piece should be equated to each other. Bending moment for Euler-Bernoulli beam theory is given by equation (3.35).

$$EI\frac{\partial^2 W_1(x_1,t)}{\partial x_1^{\ 2}}\bigg|_{x_1=L_c} = EI\frac{\partial^2 W_2(x_2,t)}{\partial x_2^{\ 2}}\bigg|_{x_1=L_c}, \tag{5.30}$$

Then by applying the similar procedure which is introducing non-dimensional parameters and application of separation of variables, the remaining compatibility relation can be obtained.

$$\frac{d^2 w_1(\chi_1)}{d\chi_1^{\ 2}}\bigg|_{\chi_1=L_1} = \frac{d^2 w_2(\chi_2)}{d\chi_2^{\ 2}}\bigg|_{\chi_1=L_1}, \tag{5.31}$$

These four compatibility equations are independent of the boundary conditions and same for all cases for Euler-Bernoulli beam on two parameters Pasternak foundation with single edge crack. Inserting equations (5.19) and (5.20) into compatibility conditions four equations are obtained.

For compatibility condition given by equation (5.24):

$$\begin{aligned}&C_1U_{11}(\omega)+C_2U_{12}(\omega)+C_3U_{13}(\omega)+C_4U_{14}(\omega)\\&+C_5U_{15}(\omega)+C_6U_{16}(\omega)+C_7U_{17}(\omega)+C_8U_{18}(\omega)=0,\end{aligned} \tag{5.32}$$

For compatibility condition given by equation (5.27):

$$\begin{aligned}&C_1U_{21}(\omega)+C_2U_{22}(\omega)+C_3U_{23}(\omega)+C_4U_{24}(\omega)\\&+C_5U_{25}(\omega)+C_6U_{26}(\omega)+C_7U_{27}(\omega)+C_8U_{28}(\omega)=0,\end{aligned} \tag{5.33}$$

For compatibility condition given by equation (5.29):

$$\begin{aligned}&C_1U_{31}(\omega)+C_2U_{32}(\omega)+C_3U_{33}(\omega)+C_4U_{34}(\omega)\\&+C_5U_{35}(\omega)+C_6U_{36}(\omega)+C_7U_{37}(\omega)+C_8U_{38}(\omega)=0,\end{aligned} \tag{5.34}$$

For compatibility condition given by equation (5.31):

$$\begin{aligned}&C_1U_{41}(\omega)+C_2U_{42}(\omega)+C_3U_{43}(\omega)+C_4U_{44}(\omega)\\&+C_5U_{45}(\omega)+C_6U_{46}(\omega)+C_7U_{47}(\omega)+C_8U_{48}(\omega)=0,\end{aligned} \tag{5.35}$$

The functions of "ω" from U_{11} to U_{48} are listed in Appendix G.

Four more equations are necessary and, they will be obtained by applying the boundary conditions. The equations from (5.32) to (5.35) remain same for different boundary conditions.

5.1.1 Simply Supported Boundary Conditions

As mentioned previously in the case of Timoshenko beam theory, moment reaction at both ends of the beam is zero, as well as vertical displacements. Thus;

$$\left. EI\frac{\partial^2 W(x,t)}{\partial x^2}\right|_{x=0,L} = 0, \tag{5.36}$$

$$\left. W(x,t)\right|_{x=0,L} = 0, \tag{5.37}$$

Non-dimensional parameters should be introduced into the boundary condition relations. Then applying separation of variables, four additional equations can be obtained.

$$\left. \frac{d^2 w_1(\chi_1)}{d\chi_1^{\ 2}}\right|_{\chi_1=0} = 0, \tag{5.38}$$

$$\left. \frac{d^2 w_2(\chi_2)}{d\chi_2^{\ 2}}\right|_{\chi_2=1} = 0, \tag{5.39}$$

$$w_1(0) = 0, \tag{5.40}$$

$$w_2(1) = 0, \tag{5.41}$$

By inserting $w_1(\chi_1)$ and $w_2(\chi_2)$ into boundary condition equations, four linearly independent equations are obtained.

$$C_1 A^2 - C_3 B^2 = 0, \tag{5.42}$$

$$C_5 A^2 \cosh(A) + C_6 A^2 \sinh(A) - C_7 B^2 \cos(B) - C_8 B^2 \sin(B) = 0, \tag{5.43}$$

$$C_1 + C_3 = 0, \tag{5.44}$$

$$C_5 \cosh(A) + C_6 \sinh(A) + C_7 \cos(B) + C_8 \sin(B) = 0, \tag{5.45}$$

The multiples of the coefficients can be given as functions "ω" also for the boundary condition equations. The functions of "ω" from U_{51} to U_{88} are listed in Appendix H for simply supported boundary conditions.

There are eight unknown coefficients and eight equations. These equations can be written in matrix form.

$$\begin{bmatrix} U_{11}(\omega) & U_{12}(\omega) & U_{13}(\omega) & U_{14}(\omega) & U_{15}(\omega) & U_{16}(\omega) & U_{17}(\omega) & U_{18}(\omega) \\ U_{21}(\omega) & U_{22}(\omega) & U_{23}(\omega) & U_{24}(\omega) & U_{25}(\omega) & U_{26}(\omega) & U_{27}(\omega) & U_{28}(\omega) \\ U_{31}(\omega) & U_{32}(\omega) & U_{33}(\omega) & U_{34}(\omega) & U_{35}(\omega) & U_{36}(\omega) & U_{37}(\omega) & U_{38}(\omega) \\ U_{41}(\omega) & U_{42}(\omega) & U_{43}(\omega) & U_{44}(\omega) & U_{45}(\omega) & U_{46}(\omega) & U_{47}(\omega) & U_{48}(\omega) \\ U_{51}(\omega) & U_{52}(\omega) & U_{53}(\omega) & U_{54}(\omega) & U_{55}(\omega) & U_{56}(\omega) & U_{57}(\omega) & U_{58}(\omega) \\ U_{61}(\omega) & U_{62}(\omega) & U_{63}(\omega) & U_{64}(\omega) & U_{65}(\omega) & U_{66}(\omega) & U_{67}(\omega) & U_{68}(\omega) \\ U_{71}(\omega) & U_{72}(\omega) & U_{73}(\omega) & U_{74}(\omega) & U_{75}(\omega) & U_{76}(\omega) & U_{77}(\omega) & U_{78}(\omega) \\ U_{81}(\omega) & U_{82}(\omega) & U_{83}(\omega) & U_{84}(\omega) & U_{85}(\omega) & U_{86}(\omega) & U_{87}(\omega) & U_{88}(\omega) \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \\ C_7 \\ C_8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{5.46}$$

The non-trivial solution of this homogenous system is obtained if and only if determinant of eight by eight matrix is zero.

$$\begin{vmatrix} U_{11}(\omega) & U_{12}(\omega) & U_{13}(\omega) & U_{14}(\omega) & U_{15}(\omega) & U_{16}(\omega) & U_{17}(\omega) & U_{18}(\omega) \\ U_{21}(\omega) & U_{22}(\omega) & U_{23}(\omega) & U_{24}(\omega) & U_{25}(\omega) & U_{26}(\omega) & U_{27}(\omega) & U_{28}(\omega) \\ U_{31}(\omega) & U_{32}(\omega) & U_{33}(\omega) & U_{34}(\omega) & U_{35}(\omega) & U_{36}(\omega) & U_{37}(\omega) & U_{38}(\omega) \\ U_{41}(\omega) & U_{42}(\omega) & U_{43}(\omega) & U_{44}(\omega) & U_{45}(\omega) & U_{46}(\omega) & U_{47}(\omega) & U_{48}(\omega) \\ U_{51}(\omega) & U_{52}(\omega) & U_{53}(\omega) & U_{54}(\omega) & U_{55}(\omega) & U_{56}(\omega) & U_{57}(\omega) & U_{58}(\omega) \\ U_{61}(\omega) & U_{62}(\omega) & U_{63}(\omega) & U_{64}(\omega) & U_{65}(\omega) & U_{66}(\omega) & U_{67}(\omega) & U_{68}(\omega) \\ U_{71}(\omega) & U_{72}(\omega) & U_{73}(\omega) & U_{74}(\omega) & U_{75}(\omega) & U_{76}(\omega) & U_{77}(\omega) & U_{78}(\omega) \\ U_{81}(\omega) & U_{82}(\omega) & U_{83}(\omega) & U_{84}(\omega) & U_{85}(\omega) & U_{86}(\omega) & U_{87}(\omega) & U_{88}(\omega) \end{vmatrix} = 0\,, \tag{5.47}$$

The values of "ω" that make this determinant zero are eigenvalues or non-dimensional natural frequencies of Euler-Bernoulli beam with single edge crack resting on two parameters elastic Pasternak foundation for simply supported boundary conditions.

5.1.2 Fixed-Fixed Boundary Conditions

As mentioned previously compatibility conditions due to crack are independent of the boundary conditions. So, functions of "ω" from U_{11} to U_{48} will remain the same, whereas the functions of "ω" from U_{51} to U_{88} will differ according to the fixed-fixed boundary conditions.

In fixed-fixed boundary conditions, transverse displacements at both ends of the beam are zero as well as the slopes at both ends. Fixed-fixed boundary condition is similar to simply supported boundary condition with respect to transverse displacement condition at the ends of beams. So equations (5.40) and (5.41) are also valid for fixed-fixed boundary condition which means transverse displacement at the end points of the beam is zero. Consequently, most of the elements of eight by eight matrix given by equation (5.46) will remain the same except the elements that depend on moment boundary condition of simply supported end points. In equation (5.46) the elements from U_{51} to U_{68} are due to moment relation of simply supported boundary condition. But, in fixed-fixed boundary condition instead of moment relation, there is slope relation at both ends. Thus, in equation (5.46) only the elements from U_{51} to U_{68} will differ. The slopes at both ends of beam which has fixed boundary conditions at both ends can be given by:

$$\left.\frac{\partial W(x,t)}{\partial x}\right|_{x=0} = \left.\frac{\partial W(x,t)}{\partial x}\right|_{x=L} = 0\,, \tag{5.48}$$

Non-dimensional parameters should be inserted into equation (5.48) and then separation of variables should be applied. Consequently, boundary conditions with respect to slope at both ends of the beam can be obtained.

$$\left.\frac{dw_1(\chi_1)}{d\chi_1}\right|_{\chi_1=0} = 0\,, \tag{5.49}$$

$$\left.\frac{dw_2(\chi_2)}{d\chi_2}\right|_{\chi_2=1} = 0\,, \tag{5.50}$$

The expressions of "$w_1(\chi_1)$" and "$w_2(\chi_2)$" should be inserted into equations (5.49) and (5.50) in order to obtain new expressions which are functions of "ω" from U_{51} to U_{68}.

$$C_2A + C_4B = 0\,, \tag{5.51}$$

$$C_5A\sinh(A) + C_6A\cosh(A) - C_7B\sin(B) + C_8B\cos(B) = 0\,,$$

Expressions of functions from U_{51} to U_{68} for fixed-fixed boundary conditions are given in Appendix I.

Equation (5.47) should be renewed by replacing the new expressions from U_{51} to U_{68} into the determinant and keeping the rest of the elements. The values of "ω" that make the determinant zero will be the non-dimensional natural frequencies of Euler-Bernoulli beam on two parameter Pasternak foundation with fixed-fixed boundary conditions.

5.1.3 Cantilever (Fixed-Free) Boundary Condition

In fixed-free boundary conditions, one end of the beam is rigidly fixed and the other end is free. In the fixed end, transverse deflection is constrained and the slope is zero. In the free end, it is assumed that there is no interaction between the elastic foundation and free surface of the beam end. Thus, bending moment and shear force are zero at the free end. The compatibility conditions due to crack remain the same and only boundary condition equations should be renewed with respect to fixed-free condition. The relations at the fixed boundary can be given in terms of transverse deflection and slope.

$$W(x,t)\big|_{x=0} = 0\,, \tag{5.52}$$

$$\left.\frac{\partial W(x,t)}{\partial x}\right|_{x=0} = 0, \tag{5.53}$$

The relations at the free end can be given in terms of bending moment and shear force relations.

$$\left. EI\frac{\partial^2 W(x,t)}{\partial x^2}\right|_{x=L} = 0, \tag{5.54}$$

$$\left. -EI\frac{\partial^3 W(x,t)}{\partial x^3}\right|_{x=L} = 0, \tag{5.55}$$

Non-dimensional parameters should be introduced into equations (5.52)-(5.55), and then separation of variables should be applied.

$$\left. w_1(\chi_1)\right|_{\chi_1=0} = 0, \tag{5.56}$$

$$\left.\frac{dw_1(\chi_1)}{d\chi_1}\right|_{\chi_1=0} = 0, \tag{5.57}$$

$$\left.\frac{d^2 w_2(\chi_2)}{d{\chi_2}^2}\right|_{\chi_2=1} = 0, \tag{5.58}$$

$$\left.\frac{d^3 w_2(\chi_2)}{d{\chi_2}^3}\right|_{\chi_2=1} = 0, \tag{5.59}$$

Inserting the expressions of $w_1(\chi_1)$ and $w_2(\chi_2)$ into equations (5.56) to (5.59) new set of linearly independent equations of fixed-free boundary conditions are obtained.

$$C_1 + C_3 = 0, \tag{5.60}$$

$$C_2 A + C_4 B = 0, \tag{5.61}$$

$$C_5 A^2 \cosh(A) + C_6 A^2 \sinh(A) - C_7 B^2 \cos(B) - C_8 B^2 \sin(B) = 0, \tag{5.62}$$

$$C_5 A^3 \sinh(A) + C_6 A^3 \cosh(A) + C_7 B^3 \sin(B) - C_8 B^3 \cos(B) = 0, \tag{5.63}$$

The elements of matrix given by equation (5.46) will change according to equations from (5.60) to (5.63) in order to obtain the characteristic equation of fixed-free boundary conditions. In the matrix, the elements due to crack compliance will remain same which are functions of "ω" from U_{11} to U_{48}. However, the functions of "ω" from U_{51} to U_{88} should be replaced by the new expressions obtained by applying the fixed-free boundary conditions. Consequently the functions of "ω" from U_{51} to U_{88} are listed in Appendix J.

After replacing the new expressions for fixed-free boundary conditions into the matrix given by equation (5.46) and keeping the elements obtained by application of crack compatibility conditions unchanged, an eight by eight matrix for fixed-free boundary condition is obtained. The "ω" values that make the determinant of this matrix zero are the non-dimensional natural frequencies of Euler-Bernoulli beam on Pasternak foundation with fixed-free boundary conditions.

5.2 Solution of Equations of Motion of Cracked Euler-Bernoulli Beam on Generalized Elastic Foundation in Transverse Vibration

The equation of motion of Euler-Bernoulli beam on generalized elastic foundation under transverse vibration is given by equation (3.43). If equation (3.38) and equation (3.43) are compared, it can be seen that both equations are in the same form and the only difference is due to second foundation parameter. Equations (3.38) and (3.43) state that;

$$EI\frac{\partial^4 W(x,t)}{\partial x^4}+k_w W(x,t)-G_0\frac{\partial^2 W(x,t)}{\partial x^2}+\rho A\frac{\partial^2 W(x,t)}{\partial t^2}=0$$

$$EI\frac{\partial^4 W(x,t)}{\partial x^4}+k_w W(x,t)-k_\Phi\frac{\partial^2 W(x,t)}{\partial x^2}+\rho A\frac{\partial^2 W(x,t)}{\partial t^2}=0$$

Consequently, for obtaining the solutions of edge cracked Euler-Bernoulli beam on generalized elastic foundation model, "k_Φ" should be used instead of "G_0" in the solution of Euler-Bernoulli beam on Pasternak foundation.

CHAPTER 6

NUMERICAL RESULTS AND DISCUSSION

6.1 Solution Procedure

After obtaining the characteristic equations, numerical examples are to be solved to see the effect of beam, foundation and crack parameters. The characteristic equations are eight by eight matrices and root of the determinant of these matrices will give the transverse vibration natural frequencies of the system.

The roots are to be found by using software "MATLAB R2009b". The characteristic equations are continuous functions of "ω". In a range of "ω" the points at which the characteristic equation changes sign are to be found by incremental search. Then, around these points the roots of the characteristic equations are calculated by using the available function of MATLAB which is "fzero". Then, the obtained root is back substituted into the characteristic equation and using the error criteria less than 10^{-4} roots of the characteristic equation are determined. As mentioned previously, these roots are non-dimensional natural frequencies of transversely vibrating beam on elastic foundation. These roots are transformed into dimensional form by dividing by "T" which was used to make the time term non-dimensional.

If "ζ=0" in equation (4.71), the "m^2" values become:

$m^2 = 0$, and $m^2 = -\kappa$. These values results in rigid body modes for free-free boundary conditions.

For a Timoshenko beam without elastic foundation "κ" and "ζ" values become:

$$\kappa(\omega)=\frac{\omega^2(\eta_4+\eta_3)}{\eta_3}, \tag{6.1}$$

$$\zeta(\omega)=\frac{(\omega^2)(\eta_4\omega^2-1)}{\eta_3}, \tag{6.2}$$

If "ω" is zero "ζ" value also becomes zero and rigid body mode solution is obtained for free-free boundary conditions.

For Timoshenko beam resting on elastic foundation "ζ" expressions are given by equations (4.66), (4.159) and (4.174) depending on the foundation model used.

$$\zeta=\frac{(\omega^2-\eta_1)(\eta_4\omega^2-1)}{\eta_3(\eta_2+1)}, \qquad \text{Equation (4.66)}$$

$$\zeta=\frac{(\omega^2-\eta_1)(\eta_4\omega^2-\eta_5-1)}{\eta_3}, \text{ Equation (4.159)}$$

$$\zeta=\frac{(\omega^2-\eta_1)(\eta_4\omega^2-1)}{\eta_3}, \qquad \text{Equation (4.174)}$$

If equations (4.66) , (4.159) and (4.174) are compared with equation (6.2), it can be seen that the "ω" that makes "ζ" zero is shifted as much as $\sqrt{\eta_1}$. Consequently, for Timoshenko beam on elastic foundation searching for the root of characteristic equation can be started from $\omega=\sqrt{\eta_1}$ value.

A similar case exists also for Euler-Bernoulli beam on elastic foundation.

$$n^2=\frac{\lambda_2\pm\sqrt{\lambda_2{}^2+4(\omega^2-\lambda_1)}}{2}, \text{ Equation (5.13)}$$

If Euler-Bernoulli beam without elastic foundation is considered equation (5.13) becomes:

$$n^2=\pm\sqrt{\omega^2}, \tag{6.3}$$

If "$\omega=0$" , then rigid body mode solution is obtained for free-free boundary conditions. However, for an Euler-Bernoulli beam on elastic foundation the "ω" value that gives rigid body mode is shifted by $\sqrt{\lambda_1}$. As a result, searching for the roots of characteristic equation can be started from $\omega=\sqrt{\lambda_1}$ value.

6.2 Numerical Results and Discussion

The results are obtained by using simple boundary conditions. Comparisons are made between Euler-Bernoulli beam theory and Timoshenko beam theory. The effects of crack and foundation parameters are investigated for different values. The material properties for the beam are chosen as $E=206.10^9$ Pa, $\rho=7850$ kg/m^3, $\nu=0.3$ and the length of the beam is taken to be 1m. Foundation parameters are chosen as $k_w=12.10^6$ N/m^2, $G_o=k_\Phi=k_v=5.4.10^6$ N. The numerical results that are original to thesis study are for a beam with square cross sectional area with height and width which are equal to 0.05m. Although material properties, foundation parameters and beam dimensions are given, it should be noted that, the solutions are based on non-dimensional parameters and there may be several combinations of material properties, foundation parameters and dimensions that result in the same non-dimensional parameters. Using the given parameters following non-dimensional values are obtained:

$\eta_1=0.0699$, $\eta_2=\eta_5=\eta_6=0.0315$, $\eta_3=6.25.10^{-4}$, $\eta_4=2.0833.10^{-4}$, $\lambda_1=111.8447$, $\lambda_2=50.3301$

Throughout the numerical results same material properties and dimensions, thus the same non-dimensional parameters are used unless otherwise mentioned. Only for the comparisons with the literature are made by using different parameters and these parameters are mentioned at the corresponding tables and figures.

For some special cases, the numerical results obtained in the thesis study are compared with the values obtained from literature. The results show good agreement. For example using the parameters given in De Rosa (1995) comparison of the results obtained by the thesis formulation are listed for fixed-free boundary conditions. De Rosa (1995) used the following non-dimensional parameters: (It should be noted that, following parameters are not original to the thesis study and they must not be confused with non-dimensional parameters used in the thesis. The following parameters are originally taken from De Rosa (1995).)

$$\alpha = \frac{k_w L^4}{EI} \qquad S_\Phi = \frac{k_\Phi L^2}{\pi^2 EI} \qquad S_v = \frac{k_v L^2}{\pi^2 EI} \qquad c = \frac{kG}{E}$$

$$r = \sqrt{\frac{AL^2}{I}} \qquad \chi = cr^2 \qquad \mu_i = \sqrt{\sqrt{\frac{\rho A \omega^2 L^4}{EI}}}$$

where μ_i is i^{th} non-dimensional natural frequency. For χ=40, α=10, r=12.5 and S_v=S_Φ=2.5 De Rosa calculated the first three non-dimensional natural frequencies of Timoshenko beam without crack resting on GM1 and GM2. The values used by De Rosa are also used in the thesis study and the results are tabulated in Table 6.1.

Table 6.1 Comparison of the results with De Rosa (1995), TB without crack

	Timoshenko Beam on Generalized 1		Timoshenko Beam on Generalized 2	
μ_i	Thesis	DeRosa(1995)	Thesis	DeRosa(1995)
1^{st}	2.5673	2.5672	2.6585	2.6584
2^{nd}	4.4081	4.4079	4.5509	4.5507
3^{rd}	6.0840	6.0838	6.2231	6.2228

Shin et al. (2006) studied, cracked EB beam on Pasternak foundation. The authors defined the following non-dimensional parameters: $K = \frac{k_f L^4}{EI}$ and $S^2 = \frac{G_o L^2}{EI}$.

For K=10 and S=5, first natural frequencies are obtained for different crack ratio and crack position. Again "K" and "S" are taken originally from Shin et al. and these parameters are not used for the formulation original to the thesis study. The comparison of the results are given in Table 6.2

Table 6.2 Comparison with the results of Shin et al.(2006) EB with crack on Pasternak foundation, SS, first natural frequencies (rad/s)

	L_1=1/8		L_1=1/4	
α	Thesis	Shin et al.	Thesis	Shin et al.
0.2	69.4799	68.8996	69.2561	67.045
0.4	69.1452	66.0965	68.1395	59.2578

Table 6.3 Comparison with the results of Shin et al.(2006) EB with crack on Pasternak foundation, Fixed-Fixed, first natural frequencies (rad/s), L_1=1/8

	Winkler		Pasternak	
α	Shin et al.	Thesis	Shin el al.	Thesis
0.02	83.6699	83.5328	105.669	105.474
0.04	83.5904	83.5239	105.641	105.471
0.2	81.7961	83.2673	105.001	105.38
0.4	78.7395	82.4208	103.889	105.078
0.5	77.4239	81.6737	103.403	104.81

Table 6.4 Comparison with the results of Shin et al.(2006) EB with crack on Pasternak foundation, SS, first natural frequencies (rad/s), L_1=1/8

	Winkler		Pasternak	
α	Shin et al.	Thesis	Shin el al.	Thesis
0.02	38.3844	38.3145	69.6986	69.5721
0.04	38.3705	38.3129	69.6714	69.5691
0.2	37.979	38.2671	68.8996	69.4799
0.4	36.6236	38.0966	66.0965	69.1452
0.5	35.2968	37.9151	64.6815	68.7854

Table 6.5 Comparison with the results of Shin et al.(2006) EB with crack on Pasternak foundation, Fixed-Fixed, first natural frequencies (rad/s), L_1=1/8, α=0.4

	Winkler		Pasternak	
(rad/s)	Shin et al.	Thesis	Shin el al.	Thesis
1^{st}	78.7395	82.4208	103.889	105.678
2^{nd}	228.564	228.262	261.024	260.379
3^{rd}	436.04	444.662	466.031	479.804

The differences between Shin et al.(2006) and the thesis are completely due to crack compliance calculation. If the crack compliance given in Shin et al. is used, similar results are obtained.

Lele and Maiti (2002) studied transverse vibration of cracked Timoshenko beam in the absence of elastic foundation and obtained first three natural frequencies of a short beam with following properties:

α=0.2, L=0.075 h=0.025m, b=0.0125m, E=210 GPa, ρ=7860 kg/m^3 and ν=0.3.

The results obtained by using these properties in the thesis are compared by the ones obtained by Lele and Maiti in Table 6.6.

Table 6.6 Comparison with Lele and Maiti (2002), TB beam with crack, no foundation,CF

	α=0.2		α=0.35		α=0.5	
(rad/s)	Lele&Maiti	Thesis	Lele&Maiti	Thesis	Lele&Maiti	Thesis
1st	3330.6	3345.8	3132	3161.9	2782.5	2834
2nd	14399.9	14601	12643.6	12900	10852.7	11100
3rd	33827.8	34220	33372.4	33797	32938	33375

Table 6.7 Comparison with Lele and Maiti (2002), EB beam with crack, no foundation,CF

	α=0.2		α=0.35		α=0.5	
(rad/s)	Lele&Maiti	Thesis	Lele&Maiti	Thesis	Lele&Maiti	Thesis
1st	3597.7	3495.2	3365.4	3155	2976.3	2653.8
2nd	20669.2	21401	17437.7	1921	14817	17188
3rd	65114	61834	65107.3	58974	64794	56910

A different function is used in Lele and Maiti (2002) that results in different natural frequencies with respect to the thesis study.

In Tables 6.8 and 6.9 first four natural frequencies are obtained by EB and TB for different cross sectional areas. The results demonstrate that, as thickness increases the error between EB and TB also increases. Also for higher frequencies the difference is larger compared to the lower frequencies. As frequency increases effect of rotational inertia increases, thus difference between EB and TB also increases. Percent difference between EB and TB for different cross sectional areas and natural frequencies are given in Table 6.10.

Table 6.8 First Four Transverse Vibration Natural Frequencies (rad/s) of Timoshenko Beam For Different Cross Sectional Areas, SS

(rad/s)	b=h=0.01m	b=h=0.02m	b=h=0.04m	b=h=0.08m	b=h=0.16m
1^{st}	145.927	291.711	582.276	1155.555	2243.960
2^{nd}	583.422	1164.552	2311.110	4487.920	8127.820
3^{rd}	1311.622	2611.736	5135.028	9658.276	16145.869
4^{th}	2329.105	4622.220	8975.841	16255.640	25235.133

Table 6.9 First Four Transverse Vibration Natural Frequencies of Euler-Bernoulli Beam For Different Cross Sectional Areas, SS

(rad/s)	b=h=0.01m	b=h=0.02m	b=h=0.04m	b=h=0.08m	b=h=0.16m
1^{st}	145.951	291.903	583.805	1167.610	2335.221
2^{nd}	583.805	1167.610	2335.221	4670.441	9340.882
3^{rd}	1313.562	2627.123	5254.246	10508.492	21016.984
4^{th}	2335.221	4670.441	9340.882	18681.764	37363.527

Table 6.10 Percent Difference Between Timoshenko Beam Theory And Euler-Bernoulli Beam Theory For Different Cross Sectional Areas

(rad/s)	b=h=0.01m	b=h=0.02m	b=h=0.04m	b=h=0.08m	b=h=0.16m
1^{st}	0.02%	0.07%	0.26%	1.03%	3.91%
2^{nd}	0.07%	0.26%	1.03%	3.91%	12.99%
3^{rd}	0.15%	0.59%	2.27%	8.09%	23.18%
4^{th}	0.26%	1.03%	3.91%	12.99%	32.46%

In literature, for the cracked beam resting on elastic foundation, the effect of foundation on Mode I stress intensity factor, in all the studies reviewed was neglected. In other words the effect of foundation on crack compliance is ignored, and crack compliance is obtained as if the boundaries of the beam are stress free. On the other hand it is known that in the presence of an elastic foundation stress intensity factor and thus crack compliance decreases. In Table 6.11 the percent difference between the natural frequencies obtained by neglecting foundation effect and natural frequencies obtained by considering the foundation effect are given. The shape factor in which effect of elastic foundation is considered is given in Appendix K.

Table 6.11 Percent difference due to effect of foundation on crack compliance, comparison of first four natural frequencies of cracked Timoshenko beam (k_w=6.7318.10^4 N/m^2, L_1=0.4)

		1st	2nd	3rd	4th
CRACK RATIO (α)	0	0.0000%	0.0000%	0.0000%	0.0000%
	0.05	0.0002%	0.0001%	0.0001%	0.0002%
	0.10	0.0011%	0.0004%	0.0004%	0.0010%
	0.15	0.0014%	0.0005%	0.0005%	0.0013%
	0.20	0.0025%	0.0009%	0.0009%	0.0021%
	0.25	0.0151%	0.0054%	0.0052%	0.0120%
	0.30	0.0404%	0.0139%	0.0135%	0.0298%
	0.35	0.0838%	0.0279%	0.0268%	0.0566%
	0.40	0.1558%	0.0495%	0.0474%	0.0945%
	0.45	0.2758%	0.0828%	0.0787%	0.1468%
	0.50	0.4779%	0.1339%	0.1258%	0.2177%
	0.55	0.8130%	0.2089%	0.1930%	0.3073%
	0.60	1.3406%	0.3088%	0.2794%	0.4060%

In Table 6.11 it is seen that, as crack ratio increases the error also increases, but still not much. In Table 6.12, the comparison done in Table 6.11 is extended to the first four natural frequencies of an Euler-Bernoulli beam. So, depending on these data, it is concluded that effect of foundation on crack compliance can be neglected.

Table 6.12 Percent difference due to effect of foundation on crack compliance, comparison of first four natural frequencies of cracked Euler-Bernoulli beam (k_w=6.7318.10^4 N/m^2, L_1=0.4)

		1st	2nd	3rd	4th
CRACK RATIO (α)	0	0%	0%	0%	0%
	0.05	0.0002%	0.0001%	0.0001%	0.0002%
	0.1	0.0011%	0.0004%	0.0004%	0.0010%
	0.15	0.0014%	0.0005%	0.0005%	0.0014%
	0.2	0.0025%	0.0009%	0.0009%	0.0023%
	0.25	0.0152%	0.0055%	0.0055%	0.0130%
	0.3	0.0406%	0.0142%	0.0141%	0.0321%
	0.35	0.0843%	0.0285%	0.0281%	0.0608%
	0.4	0.1567%	0.0505%	0.0496%	0.1011%
	0.45	0.2772%	0.0845%	0.0821%	0.1564%
	0.5	0.4801%	0.1365%	0.1309%	0.2306%
	0.55	0.8164%	0.2126%	0.2004%	0.3237%
	0.6	1.3457%	0.3139%	0.2892%	0.4253%

Loya et al. (2006) used two crack compliances, one of which is rotational compliance as usual in most of the studies, and the other compliance was due to Mode II stress intensity factor of the open edge crack. The compliance of Mode II stress intensity factor was modeled by a translational spring. First natural frequencies of Timoshenko beam with open edge crack were calculated by two different models. In the first model only rotational compliance was used for modeling the crack whereas in the second model both translational and rotational compliances were used as mentioned by Loya et al. The compliances were calculated by using the formulation given in the study of Loya et al., and the differences between both models are listed in Table 6.13 for various crack positions and crack depths.

Table 6.13 Percent difference between first natural frequencies calculated by neglecting Mode II compliance and calculated by considering Mode II compliance (Timoshenko beam theory, Loya et al.2006)

a/h=α	L_1=0.1	L_1=0.2	L_1=0.3	L_1=0.4	L_1=0.5
0	0%	0%	0%	0%	0%
0.05	6.86E-07%	4.96E-07%	2.62E-07%	7.22E-08%	0%
0.10	1.86E-05%	1.34E-05%	7.05E-06%	1.94E-06%	0%
0.15	9.83E-05%	7.07E-05%	3.69E-05%	1.01E-05%	0%
0.20	2.89E-04%	2.07E-04%	1.07E-04%	2.92E-05%	0%
0.25	6.53E-04%	4.65E-04%	2.38E-04%	6.42E-05%	0%
0.30	1.26E-03%	8.88E-04%	4.50E-04%	1.20E-04%	0%
0.35	2.18E-03%	1.52E-03%	7.60E-04%	2.00E-04%	0%
0.40	3.50E-03%	2.42E-03%	1.18E-03%	3.05E-04%	0%
0.45	5.31E-03%	3.59E-03%	1.71E-03%	4.34E-04%	0%
0.50	7.66E-03%	5.06E-03%	2.33E-03%	5.77E-04%	0%
0.55	1.06E-02%	6.80E-03%	3.00E-03%	7.24E-04%	0%
0.60	1.42E-02%	8.70E-03%	3.66E-03%	8.53E-04%	0%

As mentioned in Table 6.13, the difference is so little that it can be neglected, which was the case in most of the studies in literature. In the calculations of the thesis study, depending on this fact the compliance due to Mode II stress intensity factor is neglected, thus only rotational compliance of the crack was used.

The effect of Mode II stress intensity factor is also considered for higher frequencies for a crack position L_1=0.4. The results show that even for higher frequencies the effect of compliance due Mode II stress intensity factor is also negligible. The percent differences are presented in Table 6.14.

Table 6.14 Percent difference between first four natural frequencies calculated by neglecting modeII compliance and calculated by considering modeII compliance (L_1=0.4, Timoshenko beam theory, Loya et al.)

(rad/s)	α=0.1	α=0.2	α=0.3	α=0.4	α=0.5
1st	0.000002%	0.000029%	0.000120%	0.000305%	0.000577%
2nd	0.000052%	0.000805%	0.003457%	0.009460%	0.020200%
3rd	0.000112%	0.001739%	0.007528%	0.020830%	0.045169%
4th	0.000028%	0.000442%	0.002013%	0.006005%	0.014487%

The equations of motion for different cases were introduced non-dimensional parameters. Different set of beam, foundation and crack properties may result in the same non-dimensional properties. For calculation of the numerical results following non-dimensional parameters are used:

Sample parameters for beam properties and beam dimensions that satisfy these non-dimensional parameters were mentioned previously. In addition sample values for foundation parameters that satisfy the non-dimensional parameters are given in the titles of figures and tables.

In Table 6.15, the effect of crack depth and crack position on first natural frequency is given. It is seen that, as the crack depth increased natural frequency decreases. Since simply supported boundaries are applied, the natural frequencies are equal to each other if the crack is at symmetric position with respect to midpoint of the beam.

Table 6.15 First Natural Frequencies (rad/s), TB, SS, No foundation

		CRACK POSITION ($L_c/L=L_1$)				
		0.1	0.3	0.5	0.7	0.9
CRACK RATIO ($a/h=\alpha$)	0	726.778	726.778	726.778	726.778	726.778
	0.05	726.691	726.183	725.870	726.183	726.691
	0.1	726.446	724.507	723.318	724.507	726.446
	0.15	726.046	721.801	719.222	721.801	726.046
	0.2	725.481	718.018	713.543	718.018	725.481
	0.25	724.728	713.040	706.153	713.040	724.728
	0.3	723.747	706.690	696.855	706.690	723.747
	0.35	722.485	698.714	685.376	698.714	722.485
	0.4	720.859	688.757	671.341	688.757	720.859
	0.45	718.743	676.312	654.232	676.312	718.743
	0.5	715.944	660.665	633.352	660.665	715.944
	0.55	712.154	640.856	607.836	640.856	712.154
	0.6	706.897	615.691	576.733	615.691	706.897

Table 6.16 Comparison of First Natural Frequencies (rad/s), TB and EB, SS, No foundation, For Different Crack Depth And Position

	CRACK POSITION (L_c/L=L1)					
	0.1		0.2		0.3	
a/h=α	TB	EB	TB	EB	TB	EB
0	726.778	729.756	726.778	729.756	726.778	729.756
0.05	726.691	729.669	726.464	729.439	726.183	729.155
0.1	726.446	729.420	725.576	728.542	724.507	727.462
0.15	726.046	729.017	724.138	727.089	721.801	724.728
0.2	725.481	728.446	722.113	725.043	718.018	720.907
0.25	724.727	727.685	719.424	722.328	713.040	715.880
0.3	723.747	726.694	715.954	718.823	706.690	709.467
0.35	722.485	725.419	711.532	714.357	698.714	701.415
0.4	720.859	723.777	705.907	708.678	688.757	691.366
0.45	718.743	721.641	698.710	701.413	676.312	678.808
0.5	715.944	718.813	689.392	692.009	660.665	663.025
0.55	712.154	714.988	677.147	679.655	640.856	643.053
0.6	706.897	709.681	660.836	663.206	615.691	617.696

Table 6.17 Percent Difference between First Natural Frequencies, TB and EB, SS, No Foundation

	CRACK POSITION				
a/h=α	L_1=0.1	L_1=0.2	L_1=0.3	L_1=0.4	L_1=0.5
0	0.408%	0.408%	0.408%	0.408%	0.408%
0.05	0.408%	0.408%	0.408%	0.407%	0.407%
0.1	0.408%	0.407%	0.406%	0.405%	0.405%
0.15	0.407%	0.406%	0.404%	0.402%	0.402%
0.2	0.407%	0.404%	0.401%	0.398%	0.397%
0.25	0.406%	0.402%	0.397%	0.393%	0.391%
0.3	0.406%	0.399%	0.392%	0.386%	0.384%
0.35	0.405%	0.396%	0.385%	0.377%	0.374%
0.4	0.403%	0.391%	0.377%	0.367%	0.364%
0.45	0.401%	0.385%	0.368%	0.355%	0.351%
0.5	0.399%	0.378%	0.356%	0.341%	0.336%
0.55	0.396%	0.369%	0.342%	0.324%	0.318%
0.6	0.392%	0.357%	0.325%	0.304%	0.298%

The effect of crack depth and crack position on natural frequencies of a beam resting on different foundation models is tabulated. Also plots of natural frequencies versus crack depth and crack position are presented. These results are obtained for different boundary conditions. In all cases, the crack has a decreasing effect on natural frequency of beam, whereas the foundation has an increasing effect.

Table 6.18 Comparison natural frequencies of EB and TB, SS, no foundation and crack

N.Freq.(rad/s)	TB	EB
1st	726.778	729.756
2nd	2872.4	2919.0
3rd	6340.3	6567.8
4th	10990	11676

Table 6.19 Comparison of natural frequencies of EB and TB, SS, Winkler and Pasternak foundation, no crack

	Winkler		Pasternak	
N.Freq.(rad/s)	TB	EB	TB	EB
1st	1067.0	1069.6	1961.8	1964.6
2nd	2976.2	3021.9	4431.2	4471.6
3rd	6387.5	6614.2	8052.4	8257.7
4th	11017	11702	12793	13431

Table 6.20 Comparison of natural frequencies of EB and TB, SS, GM1 and GM2, no crack

	GM1		GM2	
N.Freq.(rad/s)	TB	EB	TB	EB
1st	1932.6	1964.6	1957.6	1964.6
2nd	4338.9	4471.6	4402.4	4471.6
3rd	7862.1	8257.7	7975.6	8257.7
4th	12477	13431	12649	13431

Table 6.21 Comparison natural frequencies of cracked EB and TB, SS, no foundation, L_1=0.4

	α=0.2		α=0.4		α=0.6	
rad/s	TB	EB	TB	EB	TB	EB
1st	714.768	717.625	675.977	678.468	586.526	588.316
2nd	2854.9	2900.8	2801.7	2845.5	2697.8	2738.1
3rd	6302.6	6526.8	6189.3	6403.7	5975.1	6173.4
4th	10831	11492	10412	11013	9809.1	10340

Table 6.22 Comparison of natural frequencies of cracked EB and TB, SS, Winkler foundation for, L_1=0.4

	α=0.2		α=0.4		α=0.6	
rad/s	TB	EB	TB	EB	TB	EB
1st	1058.8	1061.3	1033.0	1035.3	976.826	978.560
2nd	2959.2	3004.3	2907.9	2951.0	2807.9	2847.6
3rd	6350.2	6573.4	6237.7	6451.3	6025.2	6222.7
4th	10858	11519	10440	11041	9839.4	10369

Table 6.23 Comparison of natural frequencies of cracked EB and TB, SS, Pasternak foundation, L_1,=0.4

	α=0.2		α=0.4		α=0.6	
rad/s	TB	EB	TB	EB	TB	EB
1st	1934.5	1937.1	1845.9	1847.9	1642.5	1643.6
2nd	4405.0	4444.5	4325.7	4362.7	4173.8	4206.8
3rd	8004.9	8206.4	7860.6	8051.4	7587.5	7760.9
4th	12609	13220	12124	12674	11444	11924

Table 6.24 Comparison of natural frequencies of cracked EB and TB, SS, GM1, L_1=0.4

	α=0.2		α=0.4		α=0.6	
rad/s	TB	EB	TB	EB	TB	EB
1st	1928.5	1937.1	1917.4	1847.9	1899.8	1643.6
2nd	4328.3	4444.5	4299.3	4362.7	4253.8	4206.8
3rd	7834.1	8206.4	7757.1	8051.4	7634.7	7760.9
4th	12346	13220	12019	12674	11586	11924

Table 6.25 Comparison of natural frequencies of cracked EB and TB, SS, GM2, L_1,=0.4

	α=0.2		α=0.4		α=0.6	
rad/s	TB	EB	TB	EB	TB	EB
1^{st}	1953.5	1937.1	1942.2	1847.9	1924.3	1643.6
2^{nd}	4391.6	4444.5	4362.3	4362.7	4316.1	4206.8
3^{rd}	7947.2	8206.4	7869.4	8051.4	7745.3	7760.9
4^{th}	12517	13220	12187	12674	11748	11924

Table 6.26 Comparison of natural frequencies of cracked EB and TB, SS, no foundation, α=0.2

	L_1=0.1		L_1=0.2		L_1=0.4	
rad/s	TB	EB	TB	EB	TB	EB
1^{st}	725.5	728.4	722.1	725.0	714.8	717.6
2^{nd}	2854.3	2900.1	2826.0	2870.8	2854.9	2900.8
3^{rd}	6267.0	6488.0	6244.5	6463.5	6302.6	6526.8
4^{th}	10825	11485	10930	11607	10831	11492

Table 6.27 Comparison of natural frequencies of cracked EB and TB, SS, Winkler foundation, α=0.2

	L_1=0.1		L_1=0.2		L_1=0.4	
rad/s	TB	EB	TB	EB	TB	EB
1^{st}	1066.1	1068.7	1063.8	1066.4	1058.8	1061.3
2^{nd}	2958.6	3003.7	2931.4	2975.4	2959.2	3004.3
3^{rd}	6314.8	6535.0	6292.5	6510.7	6350.2	6573.5
4^{th}	10853	11512	10958	11633	10858	11519

Table 6.28 Comparison of natural frequencies of cracked EB and TB, SS, Pasternak foundation, α=0.2

	L_1=0.1		L_1=0.2		L_1=0.4	
rad/s	TB	EB	TB	EB	TB	EB
1^{st}	1958.8	1961.6	1951.1	1953.9	1934.5	1937.1
2^{nd}	4403.9	4443.3	4361.6	4399.6	4405.0	4444.5
3^{rd}	7959.7	8157.6	7932.2	8128.1	8004.9	8206.4
4^{th}	12601	13212	12724	13352	12609	13220

Table 6.29 Comparison of natural frequencies of cracked EB and TB, SS, GM1, $\alpha=0.2$

	$L_1=0.1$		$L_1=0.2$		$L_1=0.4$	
rad/s	TB	EB	TB	EB	TB	EB
1^{st}	1932.1	1961.6	1931.0	1953.9	1928.5	1937.1
2^{nd}	4328.0	4443.3	4310.9	4399.6	4328.3	4444.5
3^{rd}	7808.0	8157.6	7790.5	8128.1	7834.1	8206.4
4^{th}	12342	13212	12427	13352	12346	13220

Table 6.30 Comparison of natural frequencies of cracked EB and TB, SS, GM2, $\alpha=0.2$

	$L_1=0.1$		$L_1=0.2$		$L_1=0.4$	
rad/s	TB	EB	TB	EB	TB	EB
1^{st}	1957.2	1961.6	1956.0	1953.9	1953.5	1937.1
2^{nd}	4391.3	4443.3	4374.1	4399.6	4391.6	4444.5
3^{rd}	7920.8	8157.6	7903.1	8128.1	7947.2	8206.4
4^{th}	12513	13212	12600	13352	12517	13220

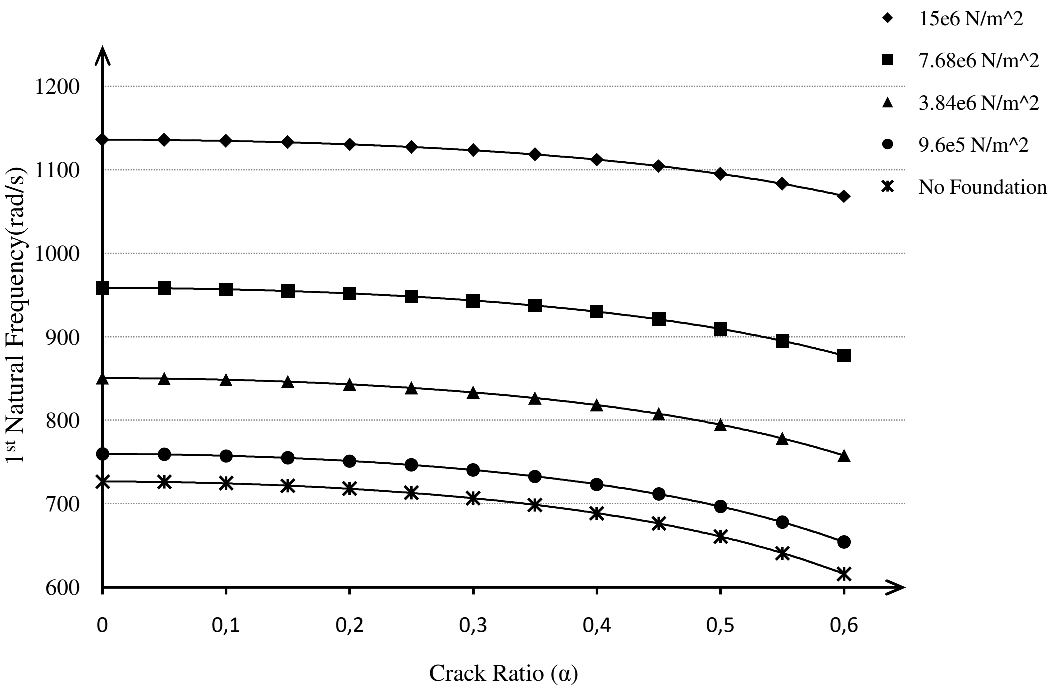

Figure 6.1 1^{st} Nat. Freq. vs. Crack Ratio, TB, SS, Winkler, $L_1=0.3$

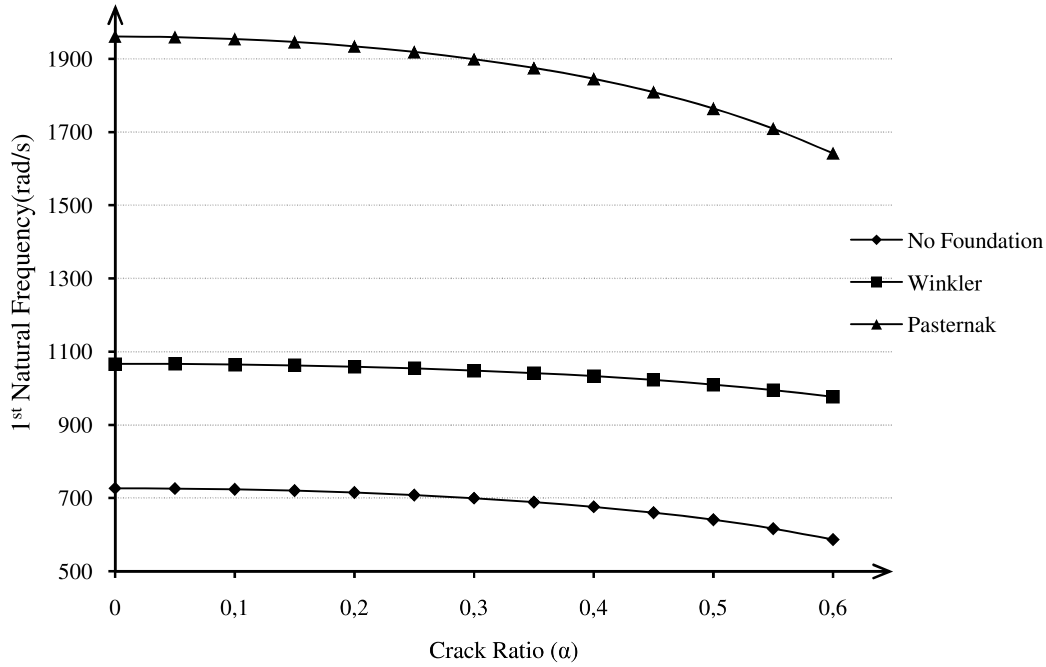

Figure 6.2 1^{st} Nat. Freq. vs. Crack Ratio, TB, SS, Winkler and Pasternak, L_1=0.4

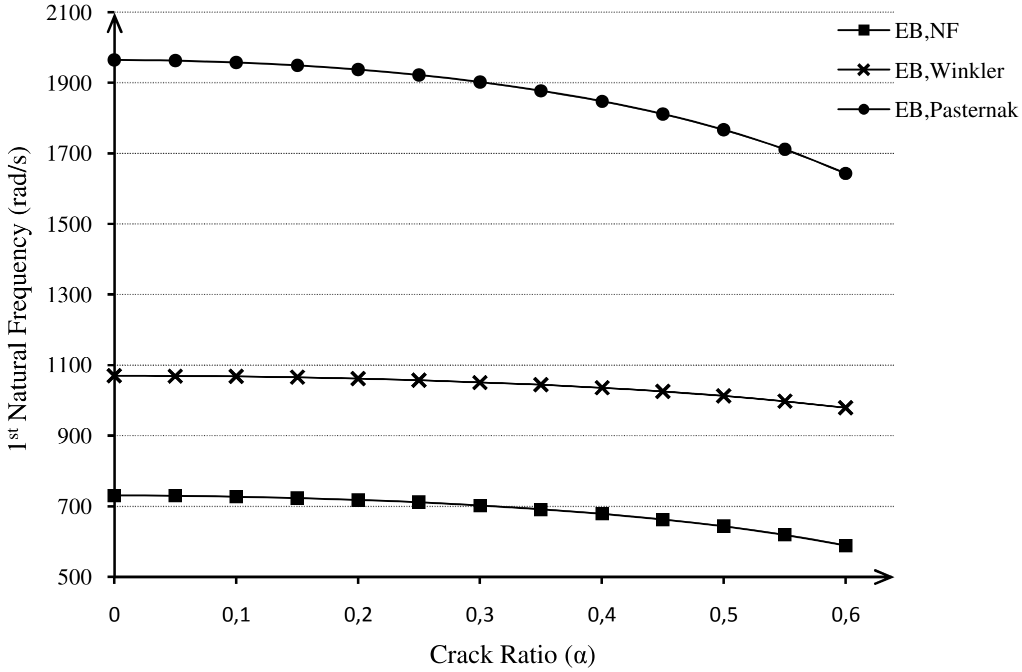

Figure 6.3 1^{st} Nat. Freq. vs. Crack Ratio, EB, SS, Winkler and Pasternak, L_1=0.4

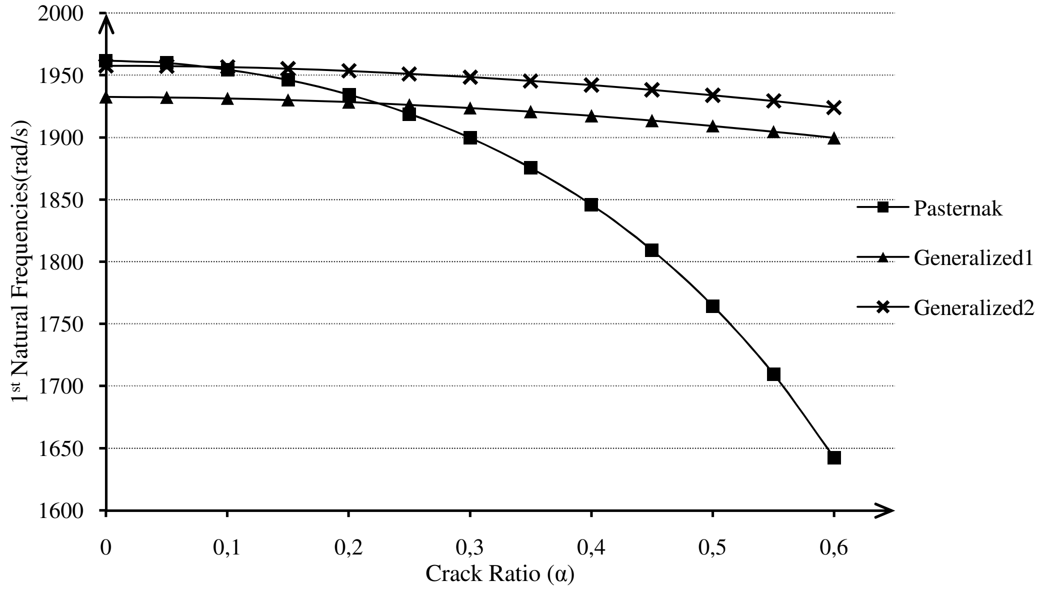

Figure 6.4 1st Nat. Freq. vs. Crack Ratio, TB, SS, Pasternak, GM1, GM2, L_1=0.4

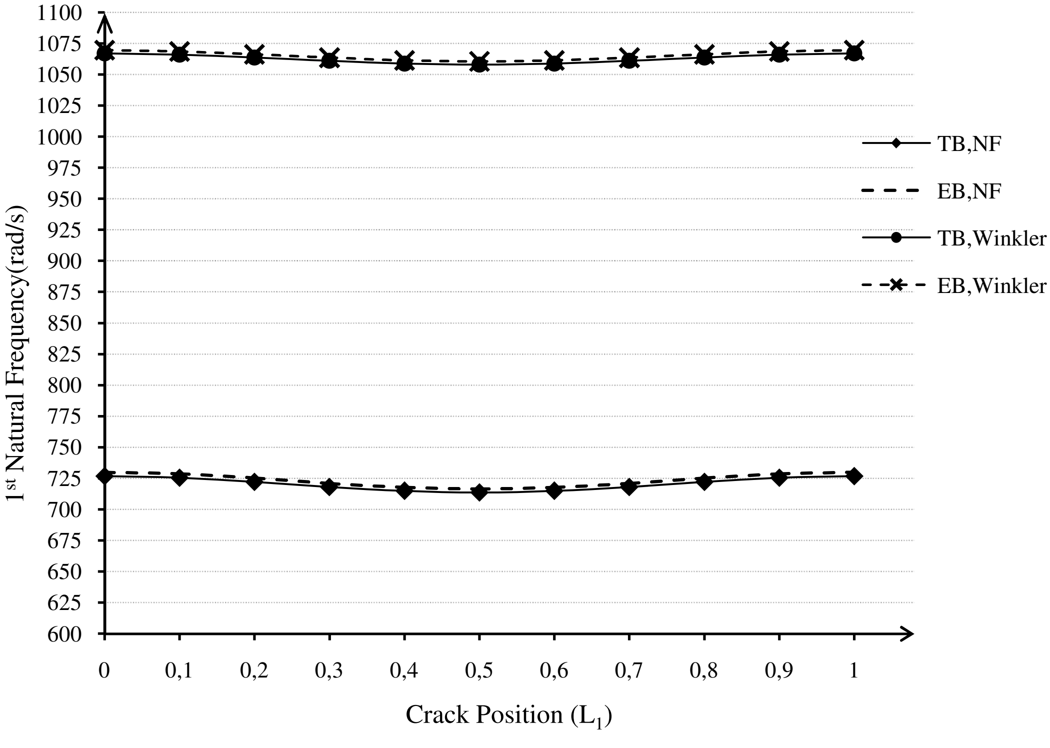

Figure 6.5 1st Nat. Freq. vs. Crack Position, TB and EB, SS, NF, Winkler, α=0.2

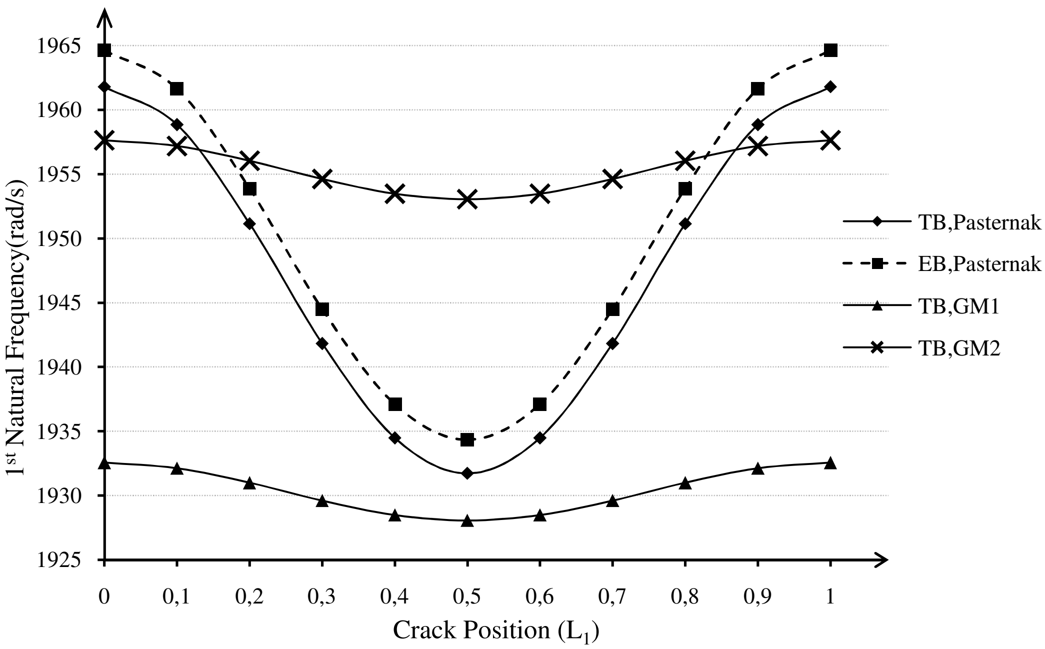

Figure 6.6 1st Nat. Freq. vs. Crack position, TB, EB, SS, Pasternak, GM1, GM2, α=0.2

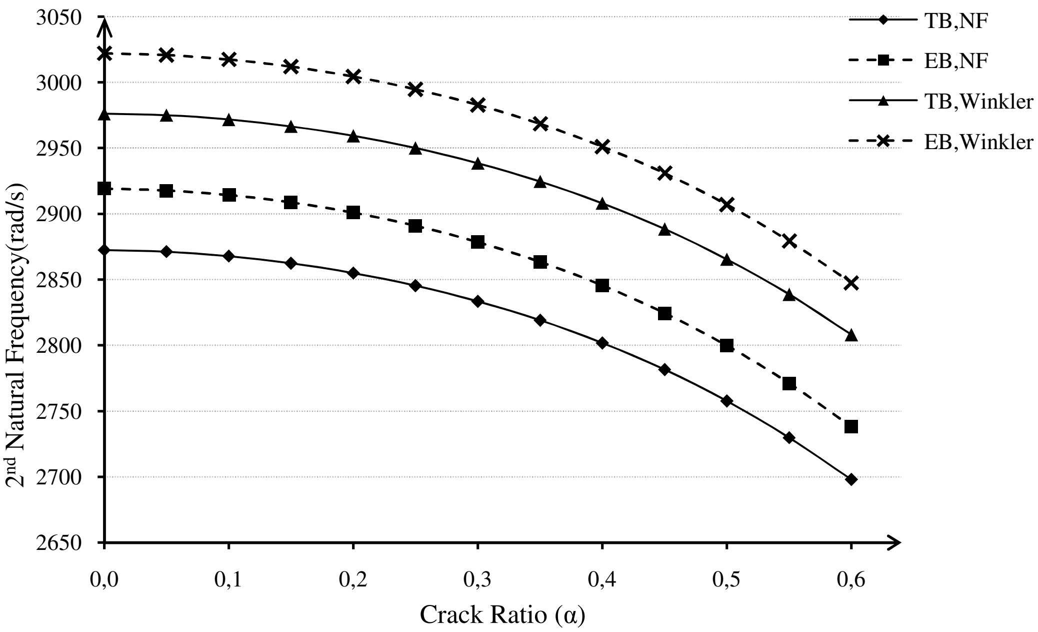

Figure 6.7 2nd Nat. Freq. vs. Crack Ratio, TB, EB, SS, NF, Winkler, L_1=0.4

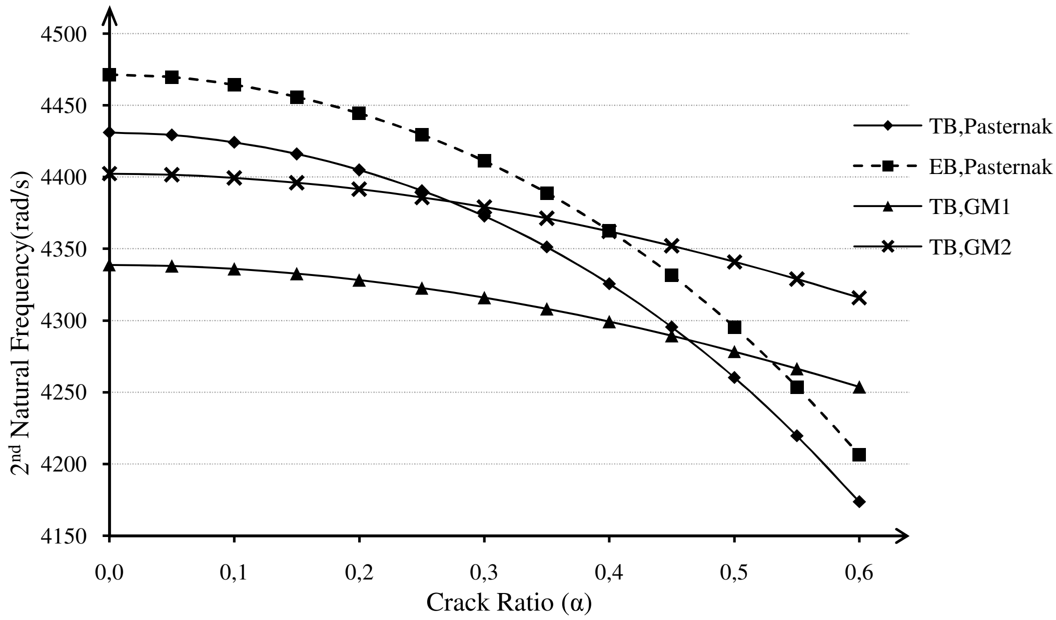

Figure 6.8 2^{nd} Nat. Freq. vs. Crack Ratio, TB, SS, Pasternak, GM1, GM2, L_1=0.4

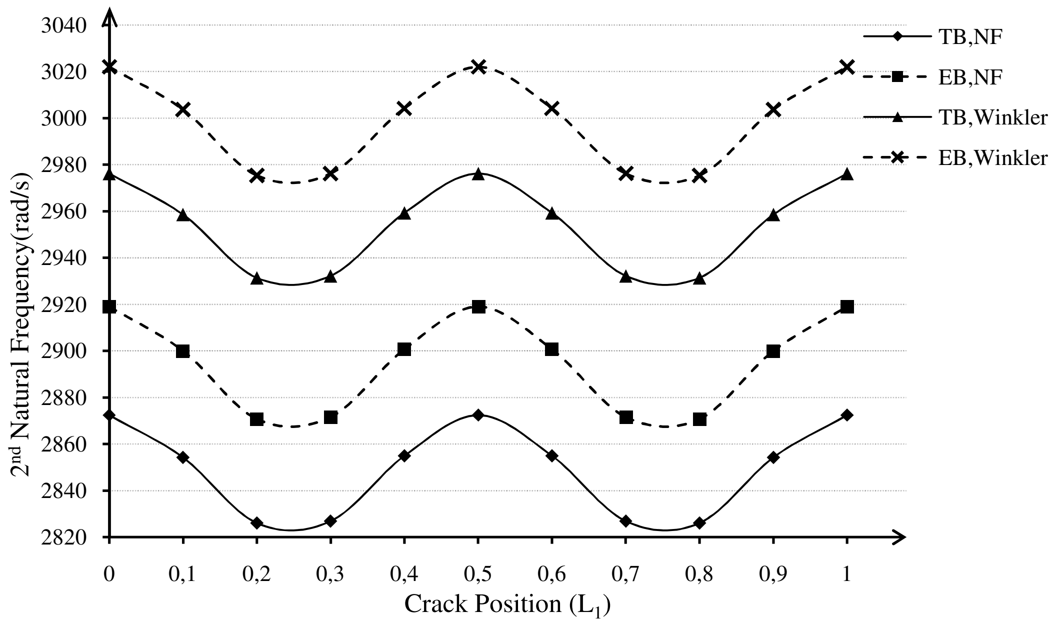

Figure 6.9 2^{nd} Nat. Freq. vs. Crack Position, TB and EB, SS, NF, Winkler , α=0.2

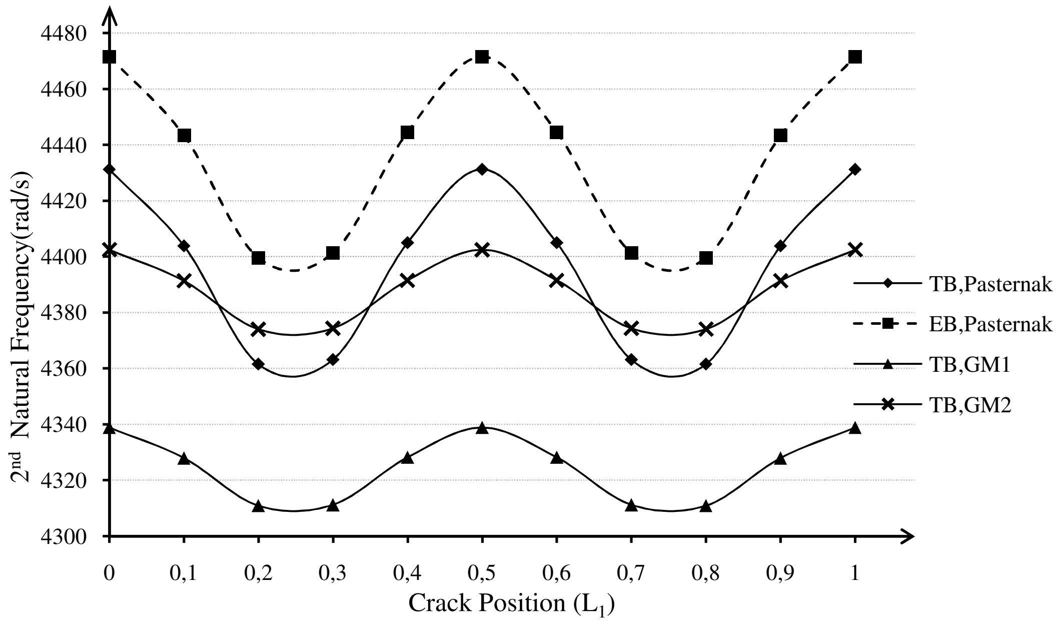

Figure 6.10 2nd Nat. Freq. vs. Crack position, TB, EB, SS, Pasternak, GM1, GM2, α=0.2

The natural frequencies of beam on elastic foundation are higher than the beam without foundation. If Pasternak parameter is added to the system, the frequencies also increase due to increased stiffness. For simply supported boundary conditions, the natural frequencies are the same when the crack is at the symmetric points with respect to the beam center. From the figures and tables, it can be seen that, if the beam is resting on Pasternak foundation the natural frequencies are more affected by change of crack ratio and crack position.

Similar results are also obtained for fixed-fixed boundary conditions. When the boundaries are fixed, the natural frequencies came out to be higher as expected. This is due to the additional restriction of rotation of the beam at the boundaries.

Table 6.31 Comparison natural frequencies of EB and TB, Fixed-Fixed, no foundation and crack

N.Freq.(rad/s)	TB	EB
1^{st}	1627.3	1654.276
2^{nd}	4393.4	4560.1
3^{rd}	8387.9	8939.6
4^{th}	13436.7	14778

Table 6.32 Comparison of natural frequencies of EB and TB, Fixed-Fixed, Winkler and Pasternak foundation, no crack

	Winkler		Pasternak	
N.Freq.(rad/s)	TB	EB	TB	EB
1^{st}	1805.0	1829.8	2550.3	2578.8
2^{nd}	4461.9	4626.6	5666.8	5827.1
3^{rd}	8423.7	8973.7	9849.4	10376.8
4^{th}	13459	14798	15026	16314

Table 6.33 Comparison of natural frequencies of EB and TB, Fixed-Fixed, GM1 and GM2, no crack

	GM1		GM2	
N.Freq.(rad/s)	TB	EB	TB	EB
1^{st}	2499.6	2578.8	2533.5	2578.8
2^{nd}	5534.7	5827.1	5613.3	5827.1
3^{rd}	9607.3	10376.8	9738.8	10376.8
4^{th}	14652	16314	14842	16314

Table 6.34 Comparison natural frequencies of cracked EB and TB, Fixed-Fixed, no foundation, L_1=0.4

	α=0.2		α=0.4		α=0.6	
rad/s	TB	EB	TB	EB	TB	EB
1^{st}	1611.7	1638.0	1563.1	1587.1	1461.1	1481.0
2^{nd}	4358.9	4521.9	4257.9	4410.9	4075.4	4213.0
3^{rd}	8356.6	8905.0	8263.5	8802.9	8092.5	8618.0
4^{th}	13235	14526	12711	13886	11982.1	13023

Table 6.35 Comparison natural frequencies of cracked EB and TB, Fixed-Fixed, Winkler foundation, L_1=0.4

	α=0.2		α=0.4		α=0.6	
rad/s	TB	EB	TB	EB	TB	EB
1^{st}	1791.0	1815.0	1747.4	1769.3	1656.7	1674.7
2^{nd}	4427.9	4589.0	4328.5	4479.6	4149.0	4285.0
3^{rd}	8392.5	8939.2	8299.8	8837.5	8129.6	8653.4
4^{th}	13257	14547	12735	13908	12006.8	13047

Table 6.36 Comparison natural frequencies of cracked EB and TB, Fixed-Fixed, Pasternak foundation, L_1=0.4

	α=0.2		α=0.4		α=0.6	
rad/s	TB	EB	TB	EB	TB	EB
1^{st}	2513.1	2540.3	2394.3	2417.3	2131.6	2146.5
2^{nd}	5621.5	5777.4	5489.0	5633.1	5253.8	5380.7
3^{rd}	9809.9	10333.8	9691.5	10205.9	9472.8	9973.2
4^{th}	14799	16035	14210	15326	13405	14385

Table 6.37 Comparison natural frequencies of cracked EB and TB, Fixed-Fixed, GM1 , L_1=0.4

	α=0.2		α=0.4		α=0.6	
rad/s	TB	EB	TB	EB	TB	EB
1^{st}	2492.1	2540.3	2471.6	2417.3	2439.3	2146.5
2^{nd}	5513.1	5777.4	5455.1	5633.1	5366.5	5380.7
3^{rd}	9581.6	10333.8	9511.3	10205.9	9399.8	9973.2
4^{th}	14484	16035	14067	15326	13525	14385

Table 6.38 Comparison natural frequencies of cracked EB and TB, Fixed-Fixed, GM2, L_1=0.4

	α=0.2		α=0.4		α=0.6	
rad/s	TB	EB	TB	EB	TB	EB
1^{st}	2525.9	2540.3	2505.1	2417.3	2472.3	2146.5
2^{nd}	5591.4	5777.4	5532.8	5633.1	5443.1	5380.7
3^{rd}	9712.8	10333.8	9641.6	10205.9	9528.8	9973.2
4^{th}	14672	16035	14251	15326	13704	14385

Table 6.39 Comparison of natural frequencies of cracked EB and TB, Fixed-Fixed, no foundation, α=0.2

	L_1=0.1		L_1=0.2		L_1=0.4	
rad/s	TB	EB	TB	EB	TB	EB
1^{st}	1611.3	1637.5	1626.8	1653.7	1611.7	1638.0
2^{nd}	4386.4	4551.9	4369.1	4533.9	4358.9	4521.9
3^{rd}	8386.7	8938.7	8276.1	8808.3	8356.6	8905.0
4^{th}	13395	14732	13302	14597	13235	14526

Table 6.40 Comparison of natural frequencies of cracked EB and TB, Fixed-Fixed, Winkler foundation, α=0.2

	L_1=0.1		L_1=0.2		L_1=0.4	
rad/s	TB	EB	TB	EB	TB	EB
1^{st}	1790.6	1814.6	1804.5	1829.3	1791.0	1815.0
2^{nd}	4455.0	4618.5	4437.9	4600.8	4427.9	4589.0
3^{rd}	8422.5	8972.9	8312.3	8843.0	8392.5	8939.2
4^{th}	13417	14753	13324	14617	13257	14547

Table 6.41 Comparison of natural frequencies of cracked EB and TB, Fixed-Fixed, Pasternak foundation, α=0.2

	L_1=0.1		L_1=0.2		L_1=0.4	
rad/s	TB	EB	TB	EB	TB	EB
1^{st}	2546.6	2574.5	2550.5	2578.9	2513.1	2540.3
2^{nd}	5667.8	5827.4	5617.9	5775.5	5621.5	5777.4
3^{rd}	9842.3	10370.9	9704.1	10209.1	9809.9	10333.8
4^{th}	14961	16245	14876	16115	14799	16035

Table 6.42 Comparison natural frequencies of cracked EB and TB, Fixed-Fixed, GM1 , α=0.2

	L_1=0.1		L_1=0.2		L_1=0.4	
rad/s	TB	EB	TB	EB	TB	EB
1^{st}	2492.0	2574.5	2499.6	2578.9	2492.1	2540.3
2^{nd}	5531.6	5827.4	5514.6	5775.5	5513.1	5777.4
3^{rd}	9604.9	10370.9	9517.5	10209.1	9581.6	10333.8
4^{th}	14611	16245	14542	16115	14484	16035

Table 6.43 Comparison natural frequencies of cracked EB and TB, Fixed-Fixed, GM2, α=0.2

	L_1=0.1		L_1=0.2		L_1=0.4	
rad/s	TB	EB	TB	EB	TB	EB
1^{st}	2525.8	2574.5	2533.5	2578.9	2525.9	2540.3
2^{nd}	5610.1	5827.4	5593.0	5775.5	5591.4	5777.4
3^{rd}	9736.4	10370.9	9648.2	10209.1	9712.8	10333.8
4^{th}	14801	16245	14731	16115	14672	16035

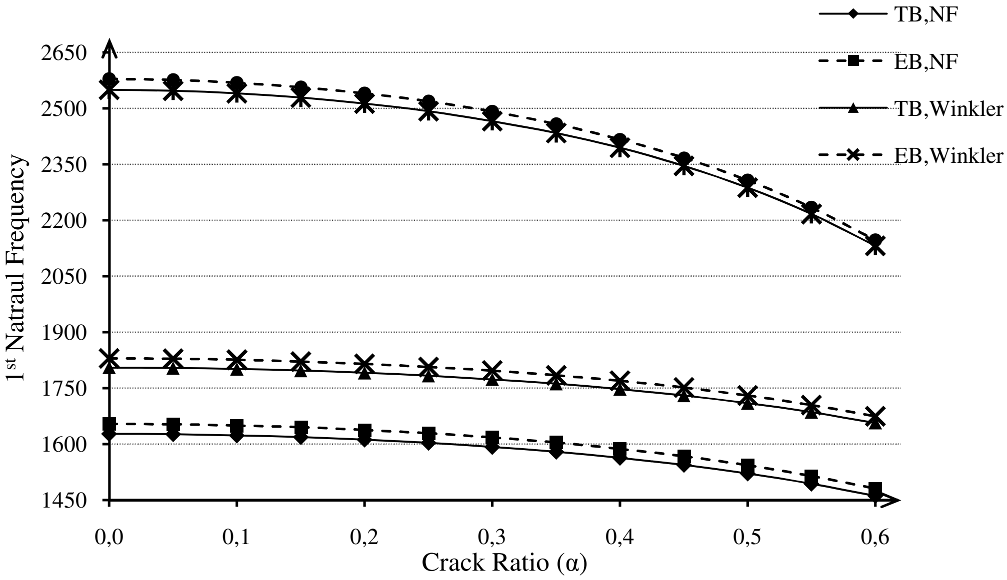

Figure 6.11 1^{st} Nat. Freq. vs. Crack Ratio, TB and EB, Fixed-Fixed, NF, Winkler and Pasternak, L_1=0.4

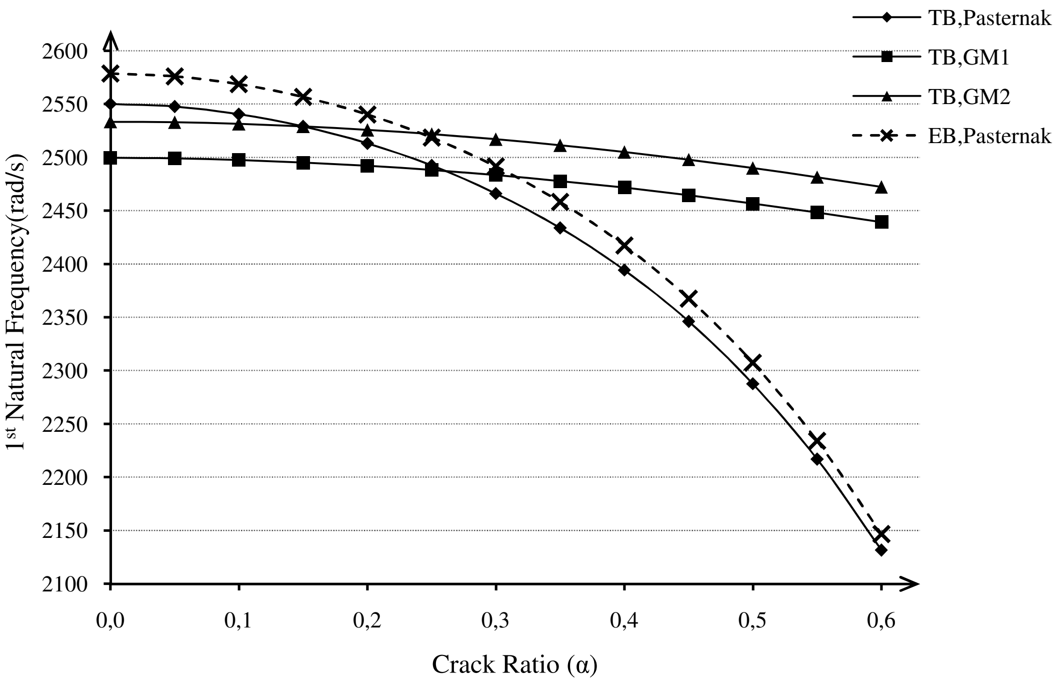

Figure 6.12 1st Nat. Freq. vs. Crack Ratio, TB and EB, Fixed-Fixed, NF, Pasternak, GM1, GM2, L_1=0.4

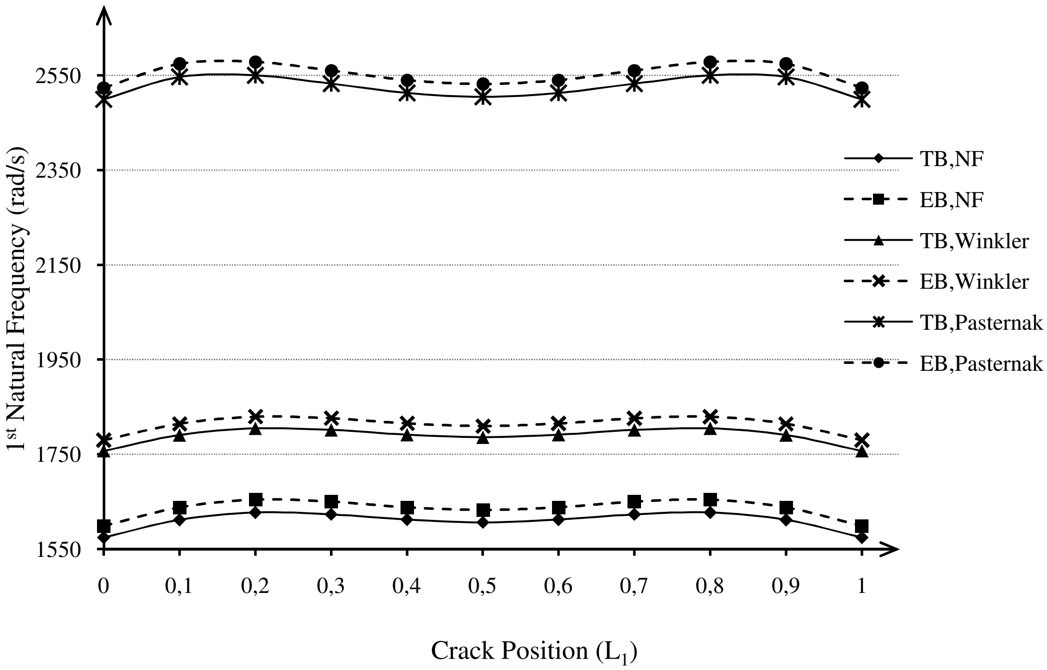

Figure 6.13 1st Nat. Freq. vs. Crack Position, TB, EB, Fixed-Fixed, NF, Winkler, Pasternak, α=0.2

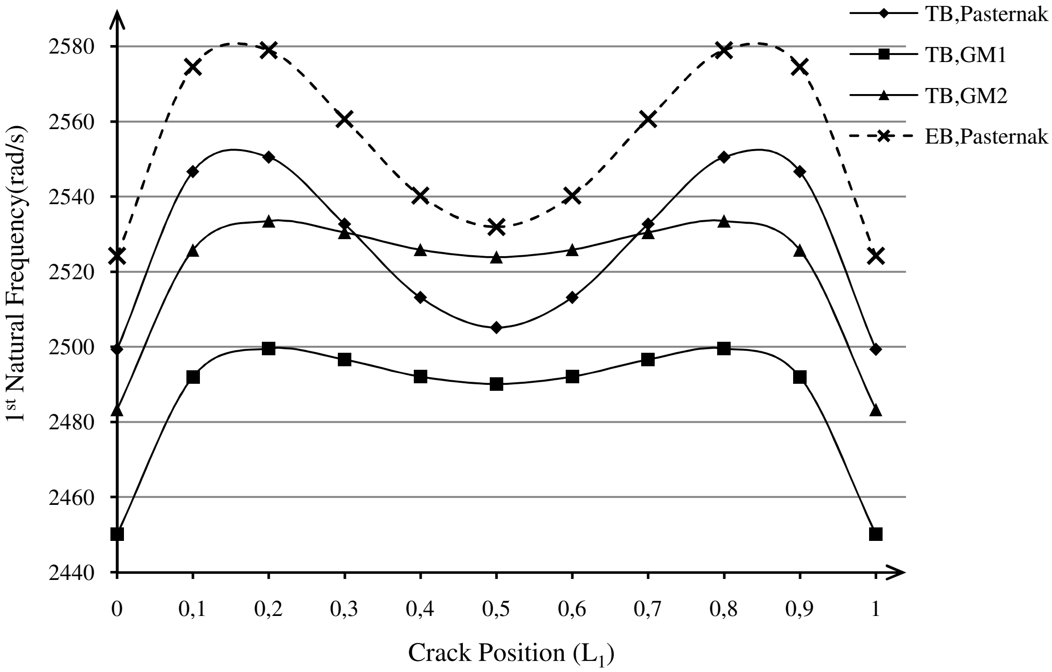

Figure 6.14 1st Nat. Freq. vs. Crack Position, TB, EB, Fixed-Fixed, Pasternak, GM1, GM2, α=0.2

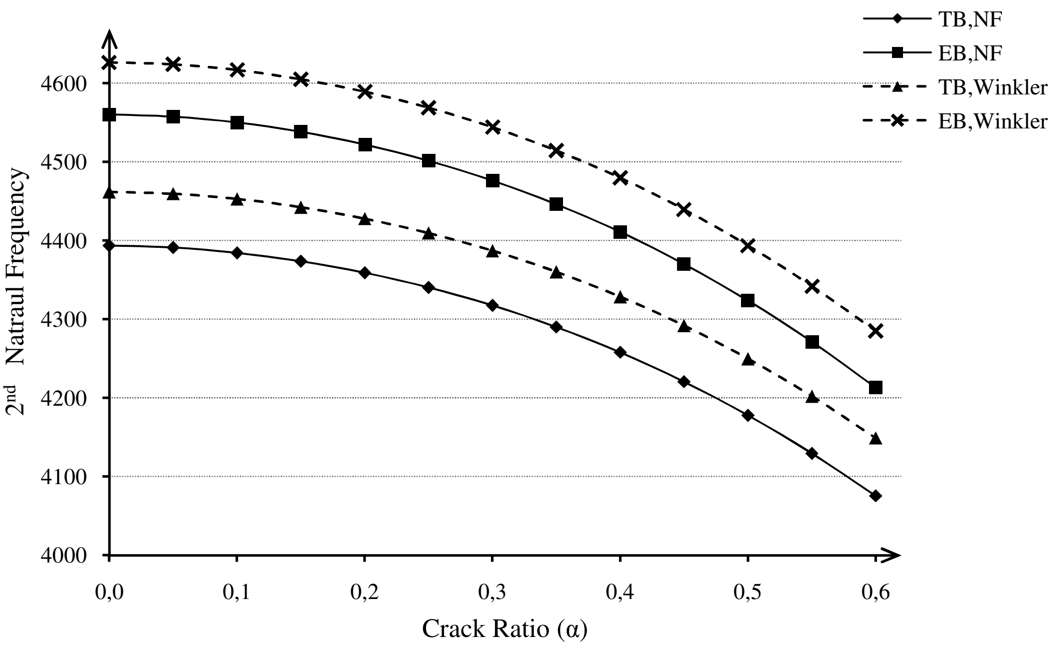

Figure 6.15 2nd Nat. Freq. vs. Crack Ratio, TB and EB, Fixed-Fixed, NF, Winkler, L_1=0.4

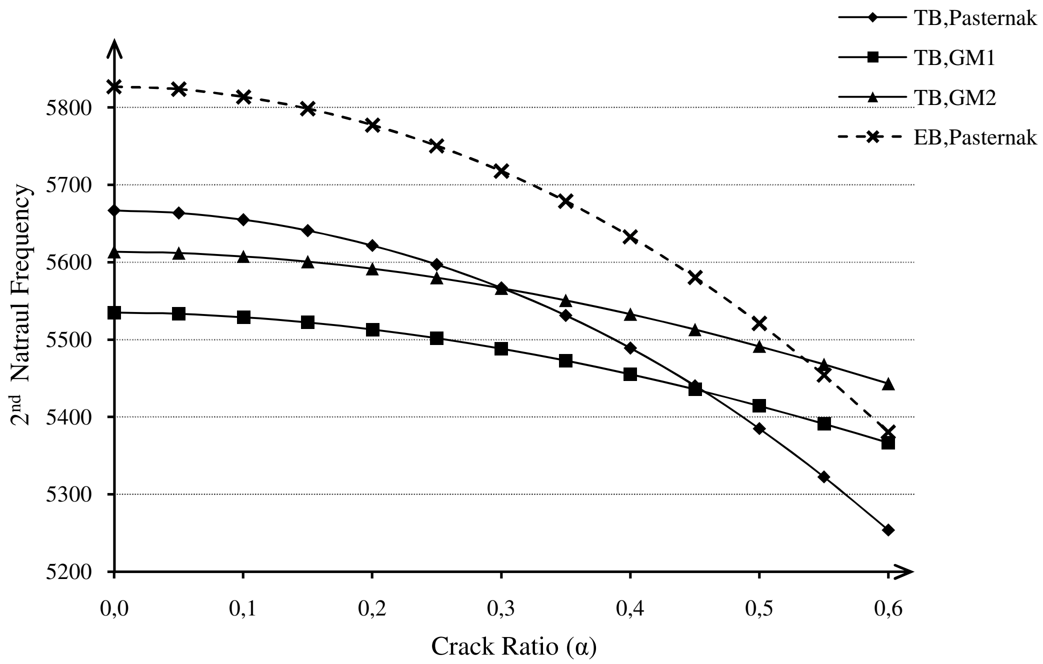

Figure 6.16 2nd Nat. Freq. vs. Crack Ratio, TB and EB, Fixed-Fixed, Pasternak, GM1, GM2, L_1=0.4

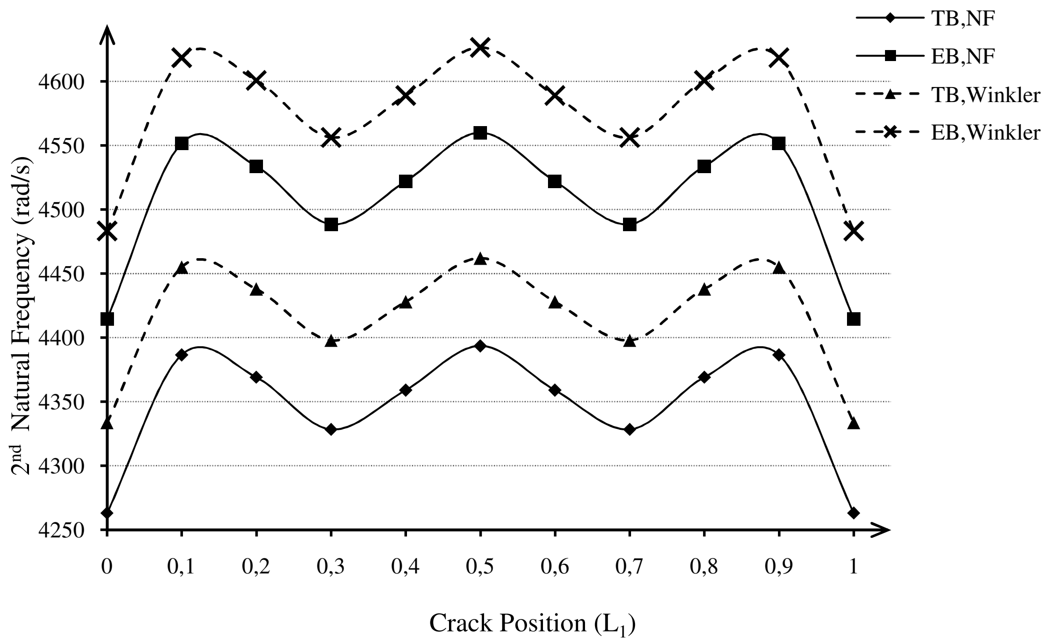

Figure 6.17 2nd Nat. Freq. vs. Crack Position, TB and EB, Fixed-Fixed, NF, Winkler, L_1=0.4

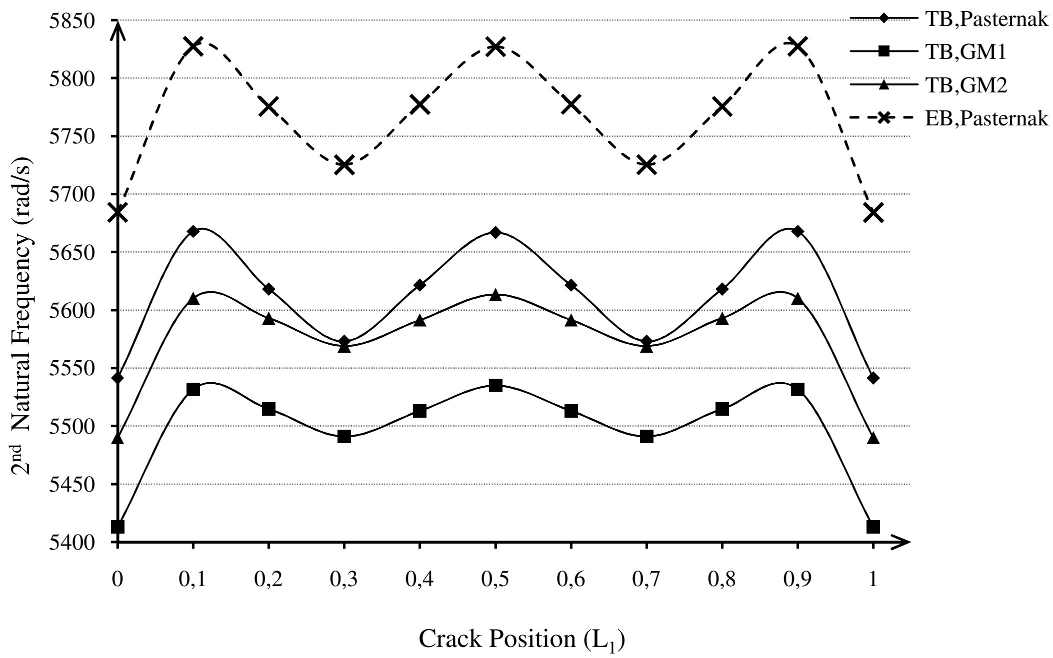

Figure 6.18 2nd Nat. Freq. vs. Crack Position, TB and EB, Fixed-Fixed, Pasternak, GM1, GM2, L_1=0.4

If fixed-free boundary conditions are considered, the natural frequencies tend to decrease with respect to fixed-fixed and simply supported boundaries because of the free end. Restrictions due to vertical displacement and rotation at the free end is removed thus causing a decrease in frequency. The effect of crack and foundation parameters is similar to the cases of boundary conditions that have already been covered. Fixed-free boundary conditions are not symmetric like fixed-fixed and simply supported boundary conditions. So, natural frequency is expected to reach maximum as the crack position reaches to the free end.

Table 6.44 Comparison natural frequencies of EB and TB, Fixed-Free boundaries, no foundation and crack

N.Freq.(rad/s)	TB	EB
1^{st}	259.472	259.973
2^{nd}	1607.7	1629.2
3^{rd}	4422.6	4561.9
4^{th}	8456.8	8939.5

Table 6.45 Comparison of natural frequencies of EB and TB, Fixed-Free, Winkler and Pasternak foundation, no crack

	Winkler		Pasternak	
N.Freq.(rad/s)	TB	EB	TB	EB
1^{st}	823.530	824.046	784.312	784.842
2^{nd}	1786.6	1807.2	2559.3	2577.6
3^{rd}	4490.2	4628.4	5702.5	5827.2
4^{th}	8491.9	8973.6	9930.0	10377

Table 6.46 Comparison of natural frequencies of EB and TB, Fixed-Free, GM1 and GM2, no crack

	GM1		GM2	
N.Freq.(rad/s)	TB	EB	TB	EB
1^{st}	1231.6	784.842	1243.0	784.842
2^{nd}	3247.5	2577.6	3293.9	2577.6
3^{rd}	6204.0	5827.2	6293.6	5827.2
4^{th}	10206	10377	10348	10377

Table 6.47 Comparison natural frequencies of cracked EB and TB, Fixed-Free, no foundation, L_1=0.4

	α=0.2		α=0.4		α=0.6	
rad/s	TB	EB	TB	EB	TB	EB
1^{st}	257.423	257.915	250.435	250.897	231.899	232.287
2^{nd}	1587.6	1608.6	1524.9	1544.3	1393.7	1410.1
3^{rd}	4389.9	4525.2	4294.5	4418.8	4123.7	4231.0
4^{th}	8423.1	8905.1	8323.1	8803.9	8140.3	8620.3

Table 6.48 Comparison natural frequencies of cracked EB and TB, Fixed-Free, Winkler foundation, L_1=0.4

	α=0.2		α=0.4		α=0.6	
rad/s	TB	EB	TB	EB	TB	EB
1^{st}	822.884	823.399	820.714	821.227	815.219	815.734
2^{nd}	1768.6	1788.6	1712.6	1731.0	1596.9	1612.4
3^{rd}	4458.0	4592.3	4364.1	4487.5	4196.0	4302.6
4^{th}	8458.4	8939.4	8358.8	8838.5	8176.9	8655.7

Table 6.49 Comparison natural frequencies of cracked EB and TB, Fixed-Free, Pasternak foundation, L_1=0.4

	α=0.2		α=0.4		α=0.6	
rad/s	TB	EB	TB	EB	TB	EB
1^{st}	784.293	784.828	784.229	784.778	784.045	784.635
2^{nd}	2520.6	2538.1	2396.9	2412.0	2124.3	2134.5
3^{rd}	5658.3	5777.9	5529.0	5635.0	5299.7	5385.2
4^{th}	9886.8	10334	9758.0	10206	9521.9	9974.0

Table 6.50 Comparison natural frequencies of cracked EB and TB, Fixed-Free, GM1 , L_1=0.4

	α=0.2		α=0.4		α=0.6	
rad/s	TB	EB	TB	EB	TB	EB
1^{st}	1231.6	784.828	1231.4	784.778	1231.2	784.635
2^{nd}	3232.1	2538.1	3190.5	2412.0	3125.7	2134.5
3^{rd}	6191.5	5777.9	6158.1	5635.0	6106.7	5385.2
4^{th}	10162	10334	10045	10206	9860.3	9974.0

Table 6.51 Comparison natural frequencies of cracked EB and TB, Fixed-Free, GM2, L_1=0.4

	α=0.2		α=0.4		α=0.6	
rad/s	TB	EB	TB	EB	TB	EB
1^{st}	1242.9	784.828	1242.8	784.778	1242.6	784.635
2^{nd}	3278.3	2538.1	3236.1	2412.0	3170.4	2134.5
3^{rd}	6281.1	5777.9	6247.3	5635.0	6195.4	5385.2
4^{th}	10304	10334	10185	10206	9998	9974.0

Table 6.52 Comparison of natural frequencies of cracked EB and TB, Fixed-Free, no foundation, α=0.2

	L_1=0.1		L_1=0.2		L_1=0.4	
rad/s	TB	EB	TB	EB	TB	EB
1st	252.514	252.982	254.492	254.970	257.423	257.915
2nd	1592.7	1613.5	1607.4	1628.9	1587.6	1608.6
3rd	4415.6	4553.6	4398.0	4536.0	4389.9	4525.2
4th	8455.4	8938.6	8342.6	8808.1	8423.1	8905.1

Table 6.53 Comparison of natural frequencies of cracked EB and TB, Fixed-Free, Winkler foundation, α=0.2

	L_1=0.1		L_1=0.2		L_1=0.4	
rad/s	TB	EB	TB	EB	TB	EB
1st	821.370	821.867	821.975	822.481	822.884	823.399
2nd	1773.2	1793.0	1786.4	1806.9	1768.6	1788.6
3rd	4483.3	4620.2	4465.9	4602.9	4458.0	4592.3
4th	8490.5	8972.7	8378.3	8842.8	8458.4	8939.4

Table 6.54 Comparison of natural frequencies of cracked EB and TB, Fixed-Free, Pasternak foundation, α=0.2

	L_1=0.1		L_1=0.2		L_1=0.4	
rad/s	TB	EB	TB	EB	TB	EB
1st	784.111	784.656	784.218	784.760	784.293	784.828
2nd	2555.7	2573.4	2559.4	2577.7	2520.6	2538.1
3rd	5703.6	5827.6	5652.6	5775.7	5658.3	5777.9
4th	9922.3	10370.8	9782.1	10209.1	9886.8	10333.9

Table 6.55 Comparison natural frequencies of cracked EB and TB, Fixed-Free, GM1, α=0.2

	L_1=0.1		L_1=0.2		L_1=0.4	
rad/s	TB	EB	TB	EB	TB	EB
1st	1228.5	784.656	1231.1	784.760	1231.6	784.828
2nd	3240.9	2573.4	3246.7	2577.7	3232.1	2538.1
3rd	6202.4	5827.6	6176.0	5775.7	6191.5	5777.9
4th	10201	10370.8	10111	10209.1	10162	10333.9

Table 6.56 Comparison natural frequencies of cracked EB and TB, Fixed-Free, GM2, α=0.2

	L_1=0.1		L_1=0.2		L_1=0.4	
rad/s	TB	EB	TB	EB	TB	EB
1^{st}	1239.8	784.656	1242.4	784.760	1242.9	784.828
2^{nd}	3287.2	2573.4	3293.1	2577.7	3278.3	2538.1
3^{rd}	6292.0	5827.6	6265.2	5775.7	6281.1	5777.9
4^{th}	10344	10370.8	10252	10209.1	10304	10333.9

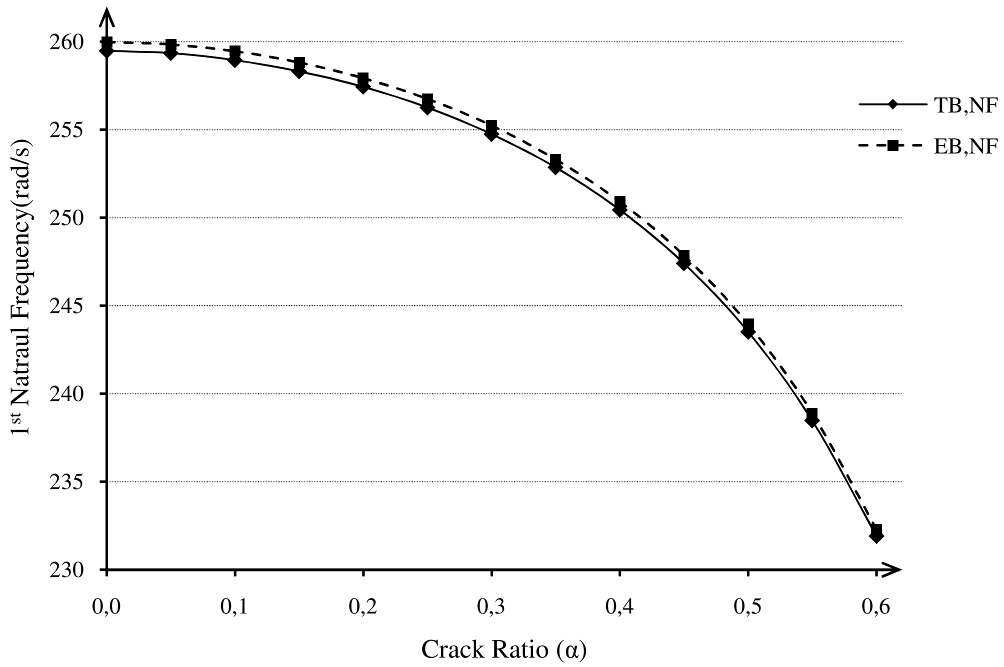

Figure 6.19 1^{st} Nat. Freq. vs. Crack Ratio, TB and EB, Fixed-Free, NF, L_1=0.4

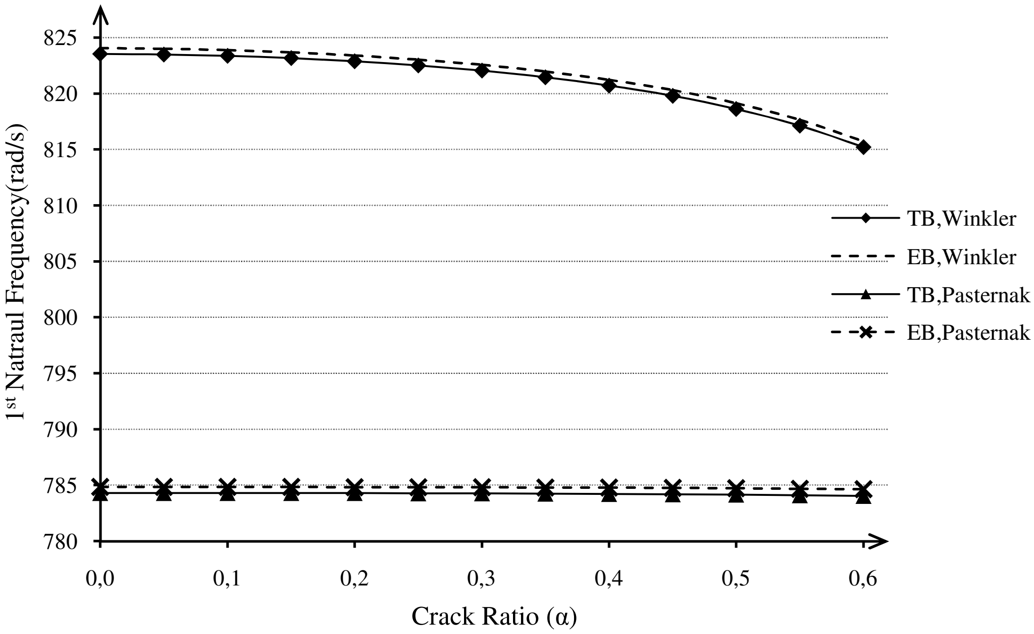

Figure 6.20 1st Nat. Freq. vs. Crack Ratio, TB and EB, Fixed-Free, Winkler and Pasternak, L_1=0.4

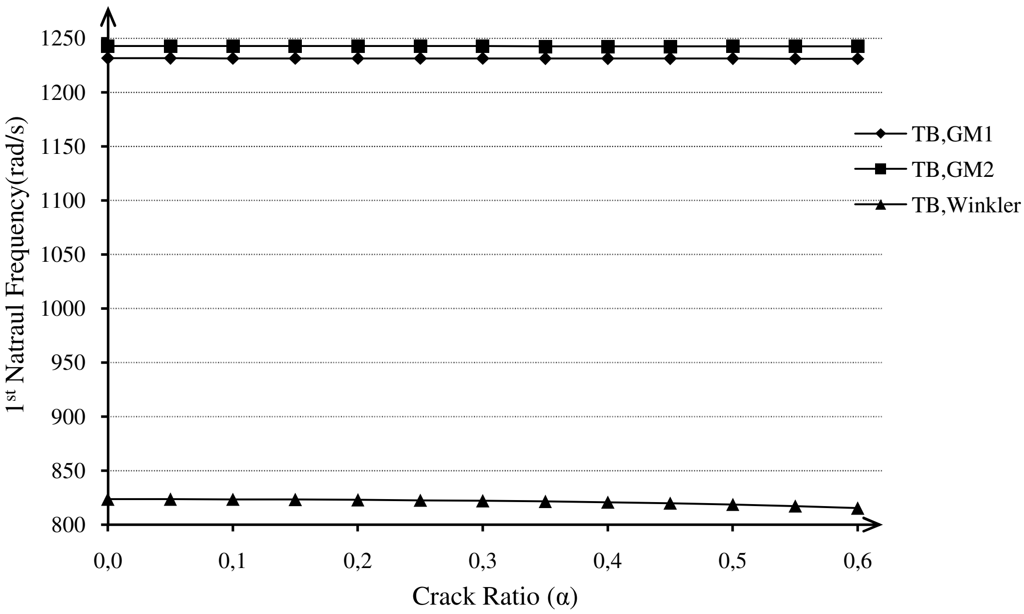

Figure 6.21 1st Nat. Freq. vs. Crack Ratio, TB and EB, Fixed-Fixed, GM1, GM2, L_1=0.4

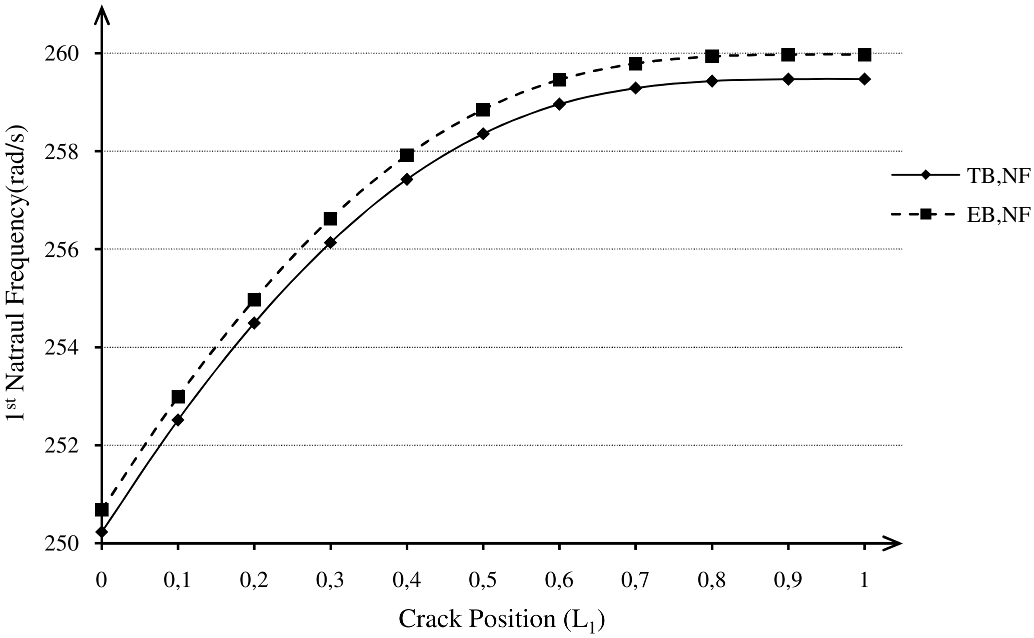

Figure 6.22 1st Nat. Freq. vs. Crack Position, TB and EB, Fixed-Free, NF, α=0.2

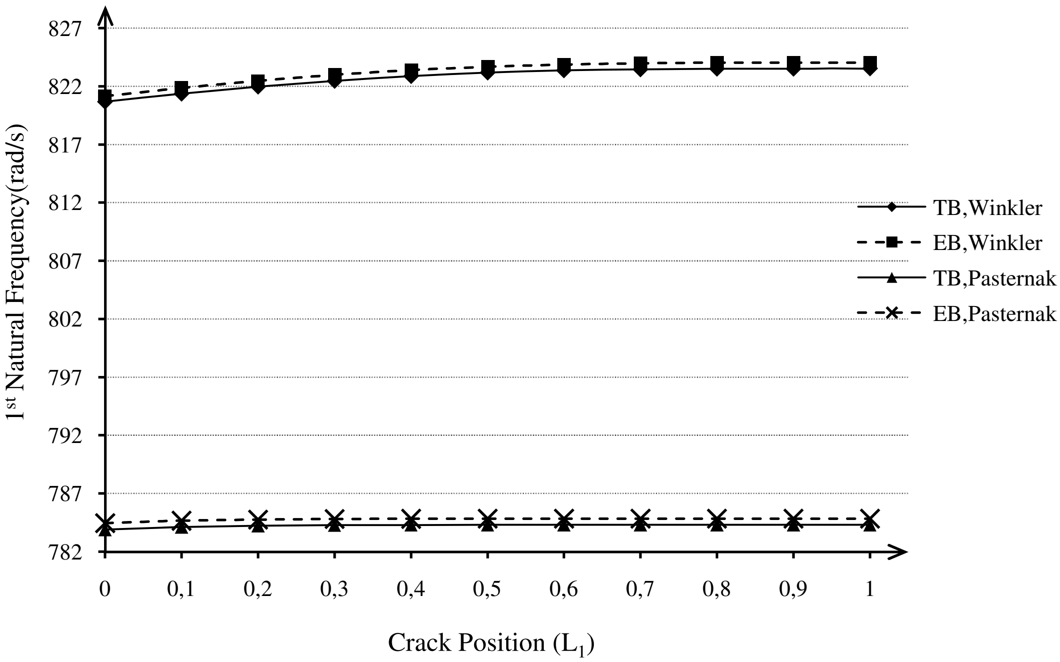

Figure 6.23 1st Nat. Freq. vs. Crack Position, TB and EB, Fixed-Free, Winkler and Pasternak, α=0.2

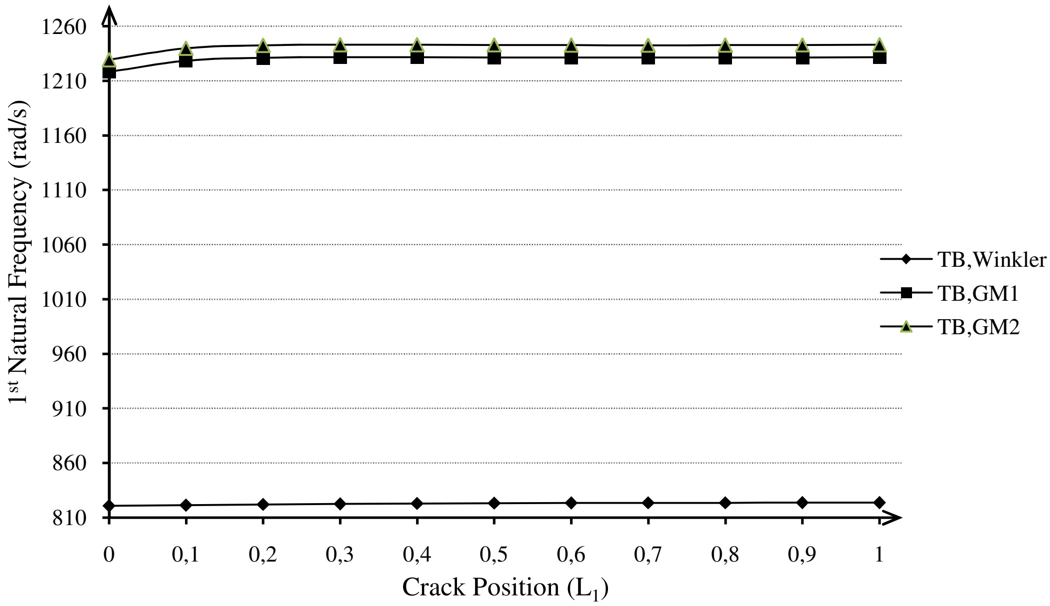

Figure 6.24 1st Nat. Freq. vs. Crack Position, TB, Fixed-Free, Winkler, GM1 and GM2, α=0.2

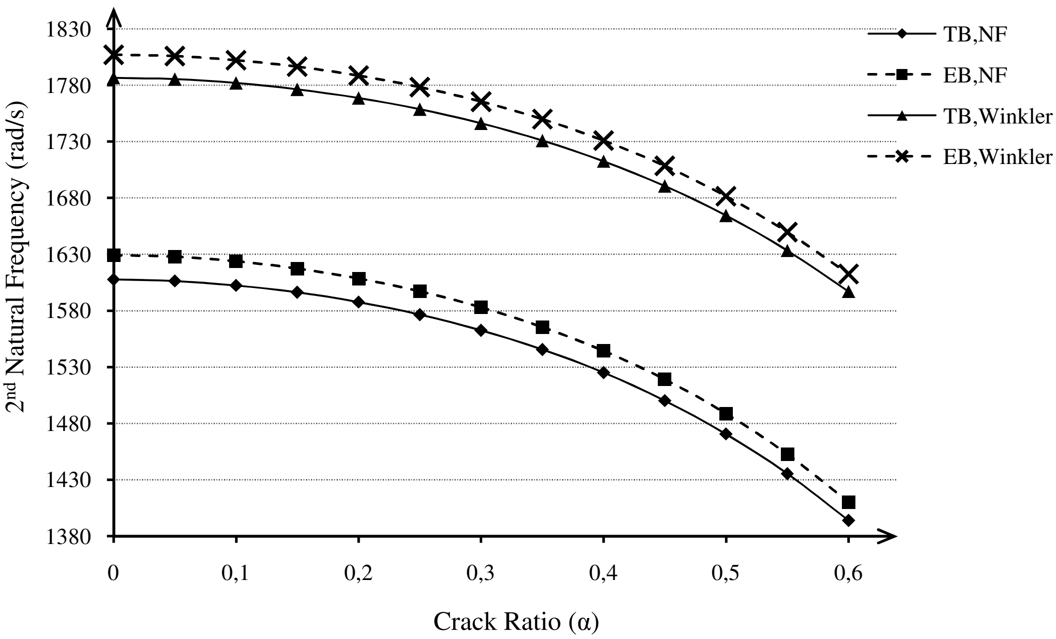

Figure 6.25 2nd Nat. Freq. vs. Crack Ratio, TB and EB, Fixed-Free, NF, Winkler, L_1=0.4

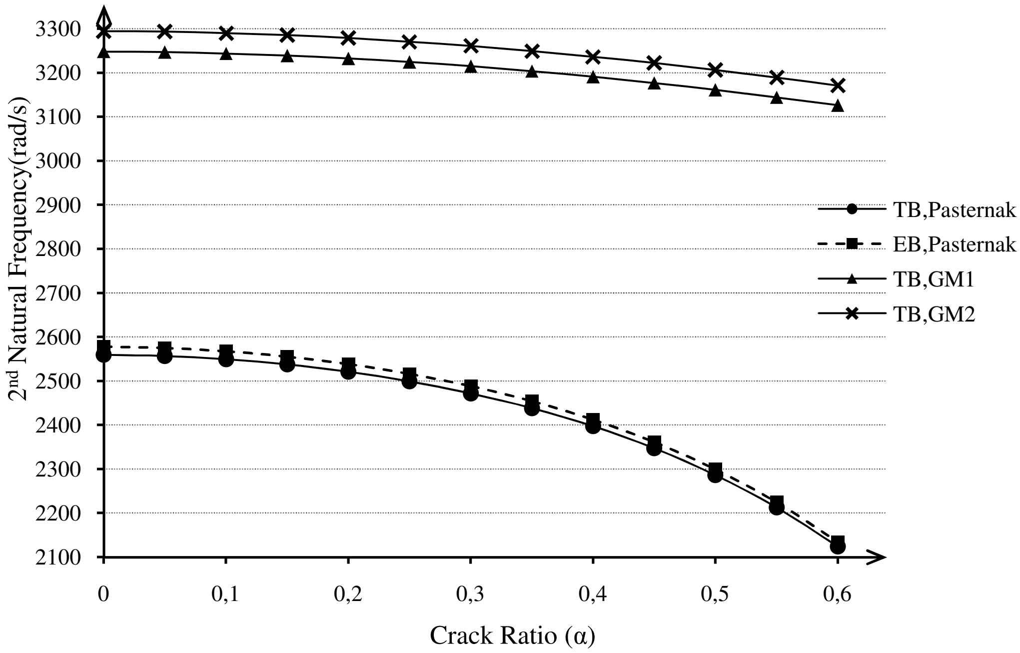

Figure 6.26 2^{nd} Nat. Freq. vs. Crack Ratio, TB and EB, Fixed-Free, Pasternak, GM1, GM2, L_1=0.4

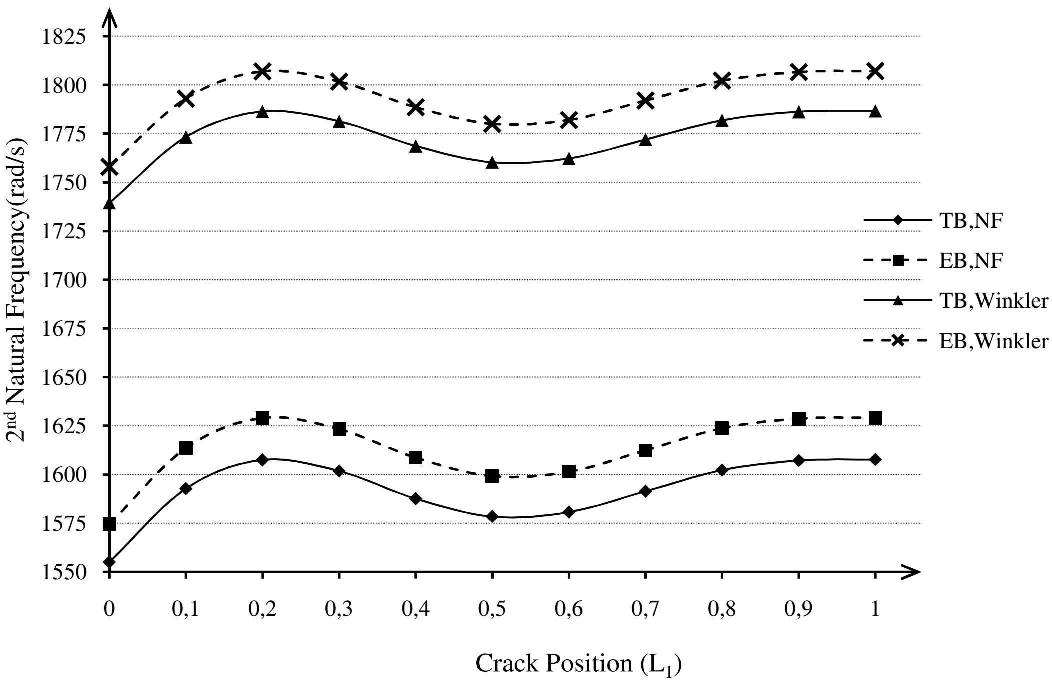

Figure 6.27 2^{nd} Nat. Freq. vs. Crack Position, TB and EB, Fixed-Free, NF, Winkler, L_1=0.4

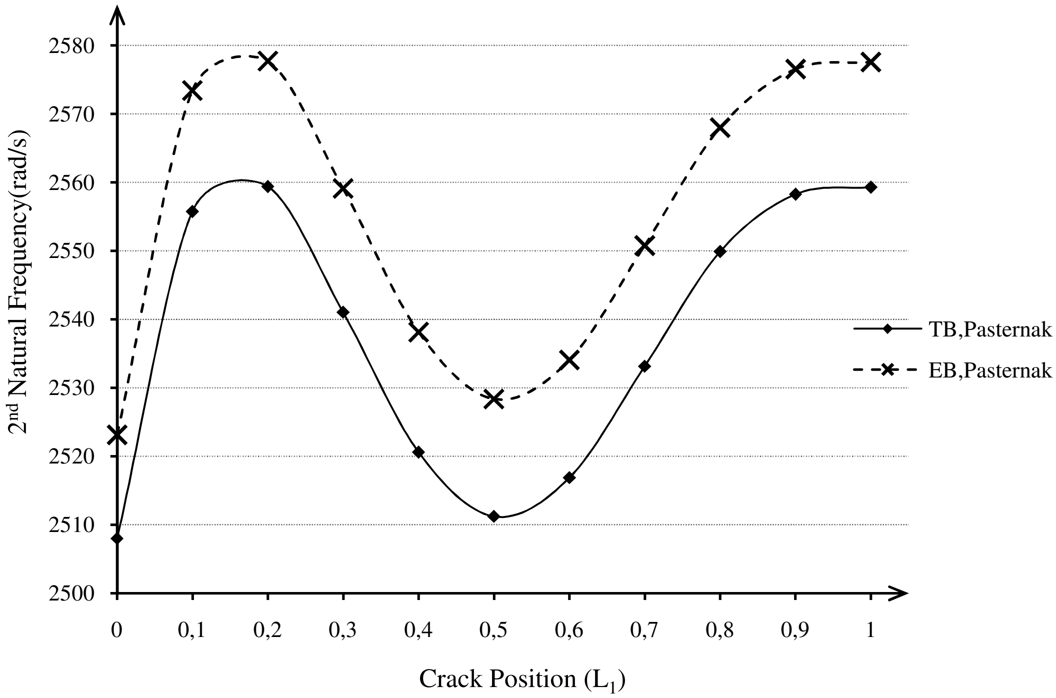

Figure 6.28 2nd Nat. Freq. vs. Crack Position, TB and EB, Fixed-Free, Pasternak, α=0.2

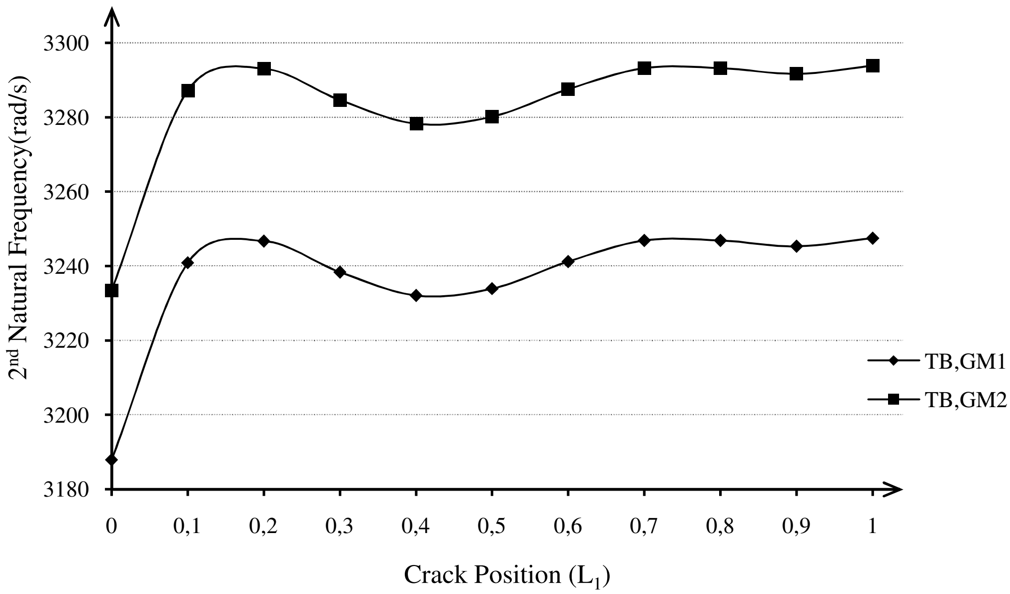

Figure 6.29 2nd Nat. Freq. vs. Crack Position, TB, Fixed-Free, GM1, GM2, α=0.2

In cantilever boundary conditions, addition of elastic foundation caused increase in natural frequencies similar to the boundary conditions mentioned previously. Addition of Winkler foundation increased the natural frequencies as expected. But addition of Pasternak parameter, to Winkler foundation resulted in decrease in the first natural frequency in both Euler-Bernoulli and Timoshenko beam theories. This is an unexpected situation and it is only seen when the boundaries are fixed-free and only for the first natural frequency when the beam is resting on Pasternak foundation. For generalized foundation models or for higher natural frequencies addition of the second parameter to Winkler foundation increases the frequencies.

In a study made by El-Mously (1999), transverse vibration of a beam on Pasternak foundation was solved by perturbation method. The author mentioned in his study that, the results had not been appropriate for free end conditions. This is a similar situation given in Figures 6.20.

In Figure 30, first three mode shapes of a Timoshenko beam without crack on Pasternak foundation is given for simply supported boundaries. The points at which the plot crosses the x axis are nodal points. For the second mode L_1=0.5 is a nodal point. For the third mode, L_1=0.333 and L_1=0.667 are nodal points. If the crack coincides with these nodal points the corresponding natural frequency does not change as expected. In Figure 30, if the crack is at L_1=0.5, the second natural frequency does not affected by the crack. Similar case occurs for the third natural frequency if the crack is at L_1=0.333 and L_1=0.667.

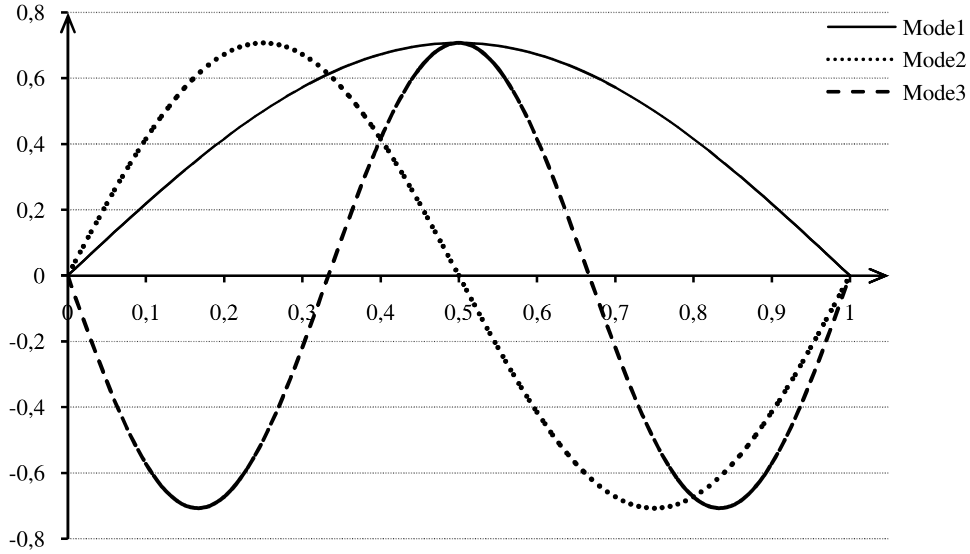

Figure 30 First three mode shapes of TB, SS, and Pasternak Foundation, α=0.6

Table 6.57 Effect of crack when it coincides nodal points

		No Crack	L_1=0.5	L_1=0.333	L_1=0.667
rad/s	1st	1961.8	1623.5	1677.5	1677.5
	2nd	4431.2	4431.2	3919.2	3919.2
	3rd	8052.4	6886.9	8052.4	8052.4

The results show that, transverse vibration natural frequencies decrease due to an edge crack in beam axis. This is an expected result, since the crack reduces the stiffness of the beam. Comparisons show that, effect of compliance due to Mode II stress intensity factor can be negligible. So, the cracks can be modeled by only by rotational springs for transverse vibration. In addition, effect of elastic foundation on crack compliance is also negligible. Increasing the stiffness of the transversely vibrating beam resulted in increase in the natural frequencies. Three different foundation models have been used, and each of them gave different results. The location of the edge crack also affects the frequencies depending on the boundary conditions as well. If the edge crack coincides with the nodal points, the corresponding natural frequency is not affected by the crack as expected.

CHAPTER 7

CONCLUSION

Results obtained in the thesis study give idea about how the transverse vibration natural frequencies are affected by crack depth, crack position and foundation parameters. These results give information about the possible changes in the vibration characteristics of a structure that contains defect in its medium. By the given analytical model in the thesis study, the vibration trend of a cracked structure on elastic foundation may be estimated.

In the thesis study, open crack model was used; a new model may be developed considering the closing effect of the crack. This case may add non-linearity in the solution procedure. But, the model considering also the closing effect of the crack would be more realistic. Also, simple boundary conditions were applied in the thesis study. The boundaries may be generalized by adding masses and springs at the end points of the beam. By this way, by changing the mass and spring parameters at the end points, effect of various boundary conditions can be analyzed. The thesis study is restricted to beams including single edge crack, to see the effect of multiple cracks a new and more general procedure can be developed. Moreover, the change in stiffness due to crack may be distributed along the whole beam.

In the thesis only transverse vibration was considered, degrees of freedom of the beam can be increased and torsional and extensional modes can also be considered at the same time with transverse vibration.

An experimental procedure may be developed to analyze the effect of crack and foundation on transverse vibration natural frequencies. To see the effect of crack and foundation by experimental procedure, the natural frequencies of cracked and uncracked beams can be compared with each other. These comparisons can also be extended to cases with and without elastic foundation. By this way results obtained by the experimental procedure can be compared with the results obtained by the analytical solution. In addition, by inverting the procedure given in the thesis, crack location, crack depth and foundation parameters may be estimated.

Pasternak and generalized foundation models were used in the thesis. Pasternak considers the shear interaction of the foundation, whereas generalized model considers the moment interaction of the foundation. A new foundation model may be developed which considers both shear and moment interaction of the foundation. This foundation model would be combination of both Pasternak and generalized foundation models.

In the thesis study, the crack model was assumed to be at the edge of the beam and uniform through the width and open all the time. In the absence of these assumptions, it would be challenging to see the effect of cracks. In that case, the cracks can be made visible by adding masses on the system. Knowing this fact, a method can be developed to make the cracks visible in real life applications.

Moreover, in the thesis study, no damping in the system was assumed. But in real life applications, structural damping of the beam should be considered as well as damping effect of foundation. So, structural damping of the beam and foundation may also be modeled and the effects may be analyzed.

In the thesis, unexpected results were obtained for cracked beam resting on elastic foundation when the beam is fixed at one end and free at other end. Such a situation was also mentioned by El-Mously (1999). The reason of the unexpected result may be studied for the corresponding boundary condition.

Consequently, in the thesis study, it was aimed to provide a basis for developing a procedure that would be used to detect damage in structures by vibration techniques. Analytical solution gives idea about the vibration trend of the damaged structure. In real life applications, comparison of the vibration data obtained from a damaged structure with the data obtained from undamaged structure may also give information about the location and size of the structural deficiency.

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APPENDIX A

SHEAR CORRECTION FACTOR

Stephen (1978) mentioned that, originally, shear coefficient had been defined by Timoshenko in 1921 as the ratio of average shear stress on a cross section to shear stress at the centroid of the cross section using Saint-Venant flexure theory. However this definition gives unsatisfactory results according to Stephen (1980). According to the paper of Hutchinson (2001), in the first paper published by Timoshenko, shear coefficient was used as 2/3 for rectangular cross sections and 3/4 for circular cross sections. Timoshenko made a comparison of the frequency equation that is based on two dimensional analyses with that is based on one dimensional analysis in connection with Timoshenko beam theory and derived the expression for k for rectangular and circular cross sections as mentioned by Kaneko (1975). The relations derived by Timoshenko are given by Equation (A.1) for rectangular cross section and by Equation (A.2) for circular cross section.

$$k = \frac{5+5\nu}{6+5\nu} , \tag{A.1}$$

$$k = \frac{6+12\nu+6\nu^2}{7+12\nu+4\nu^2} , \tag{A.2}$$

Later Olsson (1934) derived an expression of k for rectangular cross sections.

$$k = \frac{20+20\nu}{24+15\nu} , \tag{A.3}$$

Many researchers including Goens, Roark, Pickett, Mindlin, Goodman and Sutherland, Higuchi, Hearmon, Spinner, Tefft, Zemanek and Rudnick, Nederveen and Schwarzl, Schneider, Cowper, Hardie and Parkins, Hart, Spence and Seldin, Tanji, Ritchie, derived expressions for shear coefficients. Among these expressions Equation (A.1) and (A.2) were implicitly used in papers. Kaneko (1975) made a review of shear coefficients that exist in literature, and after a comparison of theoretical derivations with experimental ones the best expressions for shear coefficient were mentioned. In the review paper, Kaneko mentions that for circular cross sections Equation (A.1) was the best expression. Kaneko determined that Spence and Seldin (1970) provided the best experimental data for comparison with theoretical shear coefficient derivations. This comparison revealed that, Timoshenko, Mindlin, Goodman and Sutherland, Cowper had provided better expressions for shear coefficient for rectangular cross sections among other theories. The relation suggested by Timoshenko is given in Equation (A.1); the other relations are as follows as obtained from the paper of Kaneko (1975):

$$16(1-\nu k)(1-k)=(2-k)^4 \quad \text{(Mindlin, Goodman and Sutherland 1951),} \qquad (A.4)$$

$$k=\frac{10+10\nu}{12+11\nu}\,, \quad \text{(Cowper, 1966),} \qquad (A.5)$$

Finally, after the comparisons of theoretical shear coefficients with experimental data obtained by Spence and Seldin (1970), Kaneko concluded that the shear coefficient obtained by Timoshenko's derivations which are Equations (A.1) and (A.2) gave the best approximation.

Dependence of shear coefficient on frequency of vibration was also investigated by Stephan (1978). It was concluded that, for circular cross sections the coefficient is independent of frequency for frequencies less than that of the thickness shear mode for Poison's ratio $\nu \geq 0.2$. For other cases the shear coefficient slightly changes with frequency.

Hutchinson (2001) made a study on Timoshenko's shear coefficient. In Hutchinson's paper, frequency obtained by the use of displacement field and stress field was compared with Timoshenko's beam theory and a new shear coefficient approximation was developed. Shear coefficients for various cross sections were formulated. For circular section the derivation was the same as

Timoshenko's shear coefficient which is Equation (A.2). For hollow circular cross section the relation developed by Hutchinson (2001) is:

$$k = \frac{6(a^2+b^2)^2(1+\nu)^2}{7a^4+34a^2b^2+7b^4+\nu(12a^4+48a^2b^2+12b^4)+\nu^2(4a^4+16a^2b^2+4b^4)} \quad , \text{(A.6)}$$

where b is the outer radius and a is the inner radius.

For elliptical cross sections following relation was derived by Hutchinson (2001):

$$k = \frac{6a^2(3a^2+b^2)(1+\nu)^2}{20a^4+8a^2b^2+\nu(37a^4+10a^2b^2+b^4)+\nu^2(17a^4+2a^2b^2-3b^4)} \quad , \qquad \text{(A.7)}$$

In Equation (A.7) 'a' and 'b' are semi-minor and semi-major axes of the ellipse that is bounding the cross section defined by the curve $y2/a^2 + z^2/b^2 = 1$.

For rectangular cross sections Hutchinson (2001), derived the shear coefficient as a function of aspect ratio as shown in Equation (A.8).

$$k = -\frac{2(1+\nu)}{[\frac{9}{4a^5b}C_4+\nu(1-\frac{b^2}{a^2})]} \quad , \qquad \text{(A.8)}$$

where C_4 is :

$$C_4 = \frac{4}{45}a^3b(-12a^2-15\nu a^2+5\nu b^2)+\sum_{n=1}^{\infty}\frac{16\nu^2b^5(n\pi a-b\tanh(\frac{n\pi a}{b}))}{(n\pi)^5(1+\nu)}$$

In Equation (A.8) depth of the beam is 2a and width of the beam is 2b.

One last shear coefficient definition is derived for thin walled structures considering thin walled cylinder in Hutchinson's paper (2001). The relation is:

$$k = \frac{1+\nu}{2+\nu} \quad , \qquad \text{(A.9)}$$

Hutchinson (2001) concluded that the shear coefficient for circular cross section was in complete agreement with the values from three dimensional elasticity theory and the plane stress theory. In case of hollow circular cross section, the shear coefficient given by Equation (A.6) was shown to be correct by comparison with three dimensional theory. The shear coefficient for rectangular cross sections given by Equation (A.8) was said to be probably correct after a comparison with three dimensional elasticity, also the experimental proof was inconclusive for the rectangular cross section case.

APPENDIX B

MASS AND STIFFNESS OPERATORS

Timoshenko beam on elastic Pasternak Foundation:

$$[K]=\begin{bmatrix} -\dfrac{\partial^2}{\partial\chi^2}+\eta_1-\eta_2\dfrac{\partial^2}{\partial\chi^2} & \dfrac{\partial}{\partial\chi} \\ -\dfrac{\partial}{\partial\chi} & -\eta_3\dfrac{\partial^2}{\partial\chi^2}+1 \end{bmatrix}, \tag{B.1}$$

$$[M]=\begin{bmatrix} 1 & 0 \\ 0 & \eta_4 \end{bmatrix}, \tag{B.2}$$

M will remain same for different foundation models.

Timoshenko beam on Generalized Foundation Model 1:

$$[K]=\begin{bmatrix} -\dfrac{\partial^2}{\partial\chi^2}+\eta_1 & \dfrac{\partial}{\partial\chi} \\ -\dfrac{\partial}{\partial\chi} & -\eta_3\dfrac{\partial^2}{\partial\chi^2}+1+\eta_5 \end{bmatrix}, \tag{B.3}$$

Timoshenko beam on Generalized Foundation Model 2:

$$[K]=\begin{bmatrix} -\dfrac{\partial^2}{\partial\chi^2}+\eta_1 & \dfrac{\partial}{\partial\chi} \\ -\dfrac{\partial}{\partial\chi}+\eta_6\dfrac{\partial}{\partial\chi} & -\eta_3\dfrac{\partial^2}{\partial\chi^2}+1 \end{bmatrix}, \tag{B.4}$$

Euler-Bernoulli beam on elastic Pasternak and Generalized Foundation:

$$[K]=\frac{\partial^4}{\partial\chi^4}+\lambda_1-\lambda_2\frac{\partial^2}{\partial\chi^2},$$

[M]=1,

Inner product is given by equation (B.5) as mentioned by Kelly.

$$<f,g>=\int_0^1 g^T f\,dx=\int_0^1[f_1(\chi)g_1(\chi)+f_2(\chi)g_2(\chi)]d\chi, \tag{B.5}$$

For the given mass and stiffness operators;

$<Mf,g>=<f,Mg>$

so [M] is self adjoint for defined inner product. Also, $<Mf,f>\ >0$,so [M] is positive semi-definite.

For the stiffness matrix $<Kf,g>=<f,Kg>$ if the boundary conditions are pinned-pinned, cantilevered and fixed-fixed boundaries. Also for the same boundary conditions $<Kf,f>\ >0$, so [K] is positive definite.

In the thesis study, simple boundary conditions are used, so mass and stiffness operators given are positive definite. As a result, rigid body motion is not a solution of the given equations of motion.

APPENDIX C

COMPATIBILITY CONDITIONS FOR TIMOSHENKO BEAM

Functions of "ω" from $H_{11}(\omega)$ to $H_{48}(\omega)$ obtained by application of crack compatibility conditions for Timoshenko beam.

$$H_{11}(\omega) = -\cosh(AL_1) - c_{tr}\left(A\sinh(AL_1) - \frac{\sinh(AL_1)[(\eta_2+1)A^2+\omega^2-\eta_1]}{A}\right),$$

$$H_{12}(\omega) = -\sinh(AL_1) - c_{tr}\left(A\cosh(AL_1) - \frac{\cosh(AL_1)[(\eta_2+1)A^2+\omega^2-\eta_1]}{A}\right),$$

$$H_{13}(\omega) = c_{tr}\left(B\sin(BL_1) - \frac{\sin(BL_1)[(\eta_2+1)B^2-\omega^2+\eta_1]}{B}\right) - \cos(BL_1),$$

$$H_{14}(\omega) = -\sin(BL_1) - c_{tr}\left(B\cos(BL_1) - \frac{\cos(BL_1)[(\eta_2+1)B^2-\omega^2+\eta_1]}{B}\right),$$

$$H_{15}(\omega) = \cosh(AL_1), \qquad H_{16}(\omega) = \sinh(AL_1),$$

$$H_{17}(\omega) = \cos(BL_1), \qquad H_{18}(\omega) = \sin(BL_1),$$

$$H_{21}(\omega) = -\frac{\left(\sinh(AL_1) + A\,c_r\cosh(AL_1)\right)\left(A^2\eta_2 - \eta_1 + A^2 + \omega^2\right)}{A},$$

$$H_{22}(\omega) = -\frac{\left(\cosh(AL_1) + A\,c_r\sinh(AL_1)\right)\left(A^2\eta_2 - \eta_1 + A^2 + \omega^2\right)}{A},$$

$$H_{23}(\omega) = \frac{\left(\sin(BL_1) + B\,c_r\cos(BL_1)\right)\left(B^2\eta_2 + \eta_1 + B^2 - \omega^2\right)}{B},$$

$$H_{24}(\omega) = \frac{\left(B\,c_r\sin(BL_1) - \cos(BL_1)\right)\left(B^2\eta_2 + \eta_1 + B^2 - \omega^2\right)}{B},$$

$$H_{25}(\omega)=\frac{\sinh(AL_1)\left((\eta_2+1)A^2+\omega^2-\eta_1\right)}{A},$$

$$H_{26}(\omega)=\frac{\cosh(AL_1)\left((\eta_2+1)A^2+\omega^2-\eta_1\right)}{A},$$

$$H_{27}(\omega)=-\frac{\sin(BL_1)\left((\eta_2+1)B^2-\omega^2+\eta_1\right)}{B},$$

$$H_{28}(\omega)=\frac{\cos(BL_1)\left((\eta_2+1)B^2-\omega^2+\eta_1\right)}{B},$$

$$H_{31}(\omega)=A\sinh(AL_1)-\frac{\sinh(AL_1)\left((\eta_2+1)A^2+\omega^2-\eta_1\right)}{A},$$

$$H_{32}(\omega)=A\cosh(AL_1)-\frac{\cosh(AL_1)\left((\eta_2+1)A^2+\omega^2-\eta_1\right)}{A},$$

$$H_{33}(\omega)=\frac{\sin(BL_1)\left((\eta_2+1)B^2-\omega^2+\eta_1\right)}{B}-B\sin(BL_1),$$

$$H_{34}(\omega)=B\cos(BL_1)-\frac{\cos(BL_1)\left((\eta_2+1)B^2-\omega^2+\eta_1\right)}{B},$$

$$H_{35}(\omega)=\frac{\sinh(AL_1)\left((\eta_2+1)A^2+\omega^2-\eta_1\right)}{A}-A\sinh(AL_1),$$

$$H_{36}(\omega)=\frac{\cosh(AL_1)\left((\eta_2+1)A^2+\omega^2-\eta_1\right)}{A}-A\cosh(AL_1),$$

$$H_{37}(\omega)=B\sin(BL_1)-\frac{\sin(BL_1)\left((\eta_2+1)B^2-\omega^2+\eta_1\right)}{B},$$

$$H_{38}(\omega)=\frac{\cos(BL_1)\left((\eta_2+1)B^2-\omega^2+\eta_1\right)}{B}-B\cos(BL_1),$$

$$H_{41}(\omega)=\cosh(AL_1)\left((\eta_2+1)A^2+\omega^2-\eta_1\right),$$

$$H_{42}(\omega)=\sinh(AL_1)\left((\eta_2+1)A^2+\omega^2-\eta_1\right),$$

$$H_{43}(\omega)=-\cos(BL_1)\left((\eta_2+1)B^2-\omega^2+\eta_1\right),$$

$$H_{44}(\omega) = -\sin(BL_1)\left((\eta_2 + 1)B^2 - \omega^2 + \eta_1\right),$$

$$H_{45}(\omega) = -\cosh(AL_1)\left((\eta_2 + 1)A^2 + \omega^2 - \eta_1\right),$$

$$H_{46}(\omega) = -\sinh(AL_1)\left((\eta_2 + 1)A^2 + \omega^2 - \eta_1\right),$$

$$H_{47}(\omega) = \cos(BL_1)\left((\eta_2 + 1)B^2 - \omega^2 + \eta_1\right),$$

$$H_{48}(\omega) = \sin(BL_1)\left((\eta_2 + 1)B^2 - \omega^2 + \eta_1\right),$$

APPENDIX D

SIMPLY SUPPORTED BC FOR TIMOSHENKO BEAM

Equations of "ω" from $H_{51}(\omega)$ to $H_{88}(\omega)$ obtained by applying simply supported boundary conditions to Timoshenko beam.

$$H_{51}(\omega) = (\eta_2 + 1)A^2 + \omega^2 - \eta_1 ,$$

$$H_{52}(\omega) = H_{54}(\omega) = H_{55}(\omega) = H_{56}(\omega) = H_{57}(\omega) = H_{58}(\omega) = 0 ,$$

$$H_{53}(\omega) = -\left[(\eta_2 + 1)B^2 - \omega^2 + \eta_1\right],$$

$$H_{61}(\omega) = H_{62}(\omega) = H_{63}(\omega) = H_{64}(\omega) = 0 ,$$

$$H_{65}(\omega) = \cosh(A)\left[(\eta_2 + 1)A^2 + \omega^2 - \eta_1\right],$$

$$H_{66}(\omega) = \sinh(A)\left[(\eta_2 + 1)A^2 + \omega^2 - \eta_1\right],$$

$$H_{67}(\omega) = -\cos(B)\left[(\eta_2 + 1)B^2 - \omega^2 + \eta_1\right],$$

$$H_{68}(\omega) = -\sin(B)\left[(\eta_2 + 1)B^2 - \omega^2 + \eta_1\right],$$

$$H_{71}(\omega) = H_{73}(\omega) = 1 ,$$

$$H_{72}(\omega) = H_{74}(\omega) = H_{75}(\omega) = H_{76}(\omega) = H_{77}(\omega) = H_{78}(\omega) = 0 ,$$

$$H_{81}(\omega) = H_{82}(\omega) = H_{83}(\omega) = H_{84}(\omega) = 0 ,$$

$$H_{85}(\omega) = \cosh(A) , \qquad H_{86}(\omega) = \sinh(A) ,$$

$$H_{87}(\omega) = \cos(B) , \qquad H_{88}(\omega) = \sin(B) ,$$

APPENDIX E

FIXED BOUNDARIES FOR TIMOSHENKO BEAM

Equations of "ω" from $H_{51}(\omega)$ to $H_{68}(\omega)$ obtained by applying fixed-fixed boundary conditions to Timoshenko beam.

$$H_{51}(\omega) = H_{53}(\omega) = H_{55}(\omega) = H_{56}(\omega) = H_{57}(\omega) = H_{58}(\omega) = 0,$$

$$H_{52}(\omega) = \frac{\left[\left(\eta_2 + 1\right) A^2 + \omega^2 - \eta_1\right]}{A},$$

$$H_{54}(\omega) = \frac{\left[\left(\eta_2 + 1\right) B^2 - \omega^2 + \eta_1\right]}{B},$$

$$H_{61}(\omega) = H_{62}(\omega) = H_{63}(\omega) = H_{64}(\omega) = H_{65}(\omega) = H_{67}(\omega) = 0,$$

$$H_{66}(\omega) = \frac{\left[\left(\eta_2 + 1\right) A^2 + \omega^2 - \eta_1\right]}{A},$$

$$H_{68}(\omega) = \frac{\left[\left(\eta_2 + 1\right) B^2 - \omega^2 + \eta_1\right]}{B},$$

APPENDIX F

CANTILEVERED BC FOR TIMOSHENKO BEAM

Equations of "ω" from $H_{51}(\omega)$ to $H_{88}(\omega)$ obtained by applying fixed-free boundary conditions to Timoshenko beam.

$H_{51}(\omega) = 1$, $\quad H_{53}(\omega) = 1$,

$$H_{52}(\omega) = H_{54}(\omega) = H_{55}(\omega) = H_{56}(\omega) = H_{57}(\omega) = H_{58}(\omega) = 0,$$

$$H_{62}(\omega) = \frac{\left[(\eta_2 + 1)\,A^2 + \omega^2 - \eta_1\right]}{A},$$

$$H_{64}(\omega) = \frac{\left[(\eta_2 + 1)\,B^2 - \omega^2 + \eta_1\right]}{B},$$

$$H_{61}(\omega) = H_{63}(\omega) = H_{65}(\omega) = H_{66}(\omega) = H_{67}(\omega) = H_{68}(\omega) = 0,$$

$$H_{71}(\omega) = H_{72}(\omega) = H_{73}(\omega) = H_{74}(\omega) = 0,$$

$$H_{75}(\omega) = \cosh(A)\left[(\eta_2 + 1)\,A^2 + \omega^2 - \eta_1\right],$$

$$H_{76}(\omega) = \sinh(A)\left[(\eta_2 + 1)\,A^2 + \omega^2 - \eta_1\right],$$

$$H_{77}(\omega) = -\cos(B)\left[(\eta_2 + 1)\,B^2 - \omega^2 + \eta_1\right],$$

$$H_{78}(\omega) = -\sin(B)\left[(\eta_2 + 1)\,B^2 - \omega^2 + \eta_1\right],$$

$$H_{81}(\omega) = H_{82}(\omega) = H_{83}(\omega) = H_{84}(\omega) = 0,$$

$$H_{85}(\omega) = A\sinh(A) - \frac{\sinh(A)\left[(\eta_2 + 1)\,A^2 + \omega^2 - \eta_1\right]}{A},$$

$$H_{86}(\omega) = A\cosh(A) - \frac{\cosh(A)\left[\left(\eta_2 + 1\right)A^2 + \omega^2 - \eta_1\right]}{A},$$

$$H_{87}(\omega) = \frac{\sin(B)\left[\left(\eta_2 + 1\right)B^2 - \omega^2 + \eta_1\right]}{B} - B\sin(B),$$

$$H_{88}(\omega) = B\cos(B) - \frac{\cos(B)\left[\left(\eta_2 + 1\right)B^2 - \omega^2 + \eta_1\right]}{B},$$

APPENDIX G

COMPATIBILITY CONDITIONS FOR EULER-BERNOULLI BEAM

Functions of "ω" from $U_{11}(\omega)$ to $U_{48}(\omega)$ obtained by application of crack compatibility conditions for Euler-Bernoulli beam.

$$U_{11} = A^3 k_{tr} \sinh(AL_1) - \cosh(AL_1),$$

$$U_{12} = A^3 k_{tr} \cosh(AL_1) - \sinh(AL_1),$$

$$U_{13} = B^3 k_{tr} \sin(BL_1) - \cos(BL_1),$$

$$U_{14} = -\sin(BL_1) - B^3 k_{tr} \cos(BL_1),$$

$$U_{15} = \cosh(AL_1), \qquad U_{16} = \sinh(AL_1),$$

$$U_{17} = \cos(BL_1), \qquad U_{18} = \sin(BL_1),$$

$$U_{21} = -A \sinh(AL_1) - A^2 k_r \cosh(AL_1),$$

$$U_{22} = -A \cosh(AL_1) - A^2 k_r \sinh(AL_1),$$

$$U_{23} = B \sin(BL_1) + B^2 k_r \cos(BL_1),$$

$$U_{24} = B^2 k_r \sin(BL_1) - B \cos(BL_1),$$

$$U_{25} = A \sinh(AL_1), \qquad U_{26} = A \cosh(AL_1),$$

$$U_{27} = -B \sin(BL_1), \qquad U_{28} = B \cos(BL_1),$$

$U_{31} = -A^3 \sinh(AL_1)$, $U_{32} = -A^3 \cosh(AL_1)$,

$U_{33} = -B^3 \sin(BL_1)$, $U_{34} = B^3 \cos(BL_1)$,

$U_{35} = A^3 \sinh(AL_1)$, $U_{36} = A^3 \cosh(AL_1)$,

$U_{37} = B^3 \sin(BL_1)$, $U_{38} = -B^3 \cos(BL_1)$,

$U_{41} = -A^2 \cosh(AL_1)$, $U_{42} = -A^2 \sinh(AL_1)$,

$U_{43} = B^2 \cos(BL_1)$, $U_{44} = B^2 \sin(BL_1)$,

$U_{45} = A^2 \cosh(AL_1)$, $U_{46} = A^2 \sinh(AL_1)$,

$U_{47} = -B^2 \cos(BL_1)$, $U_{48} = -B^2 \sin(BL_1)$,

APPENDIX H

SIMPLY SUPPORTED BC FOR EULER-BERNOULLI BEAM

Equations of "ω" from $U_{51}(\omega)$ to $U_{88}(\omega)$ obtained by applying simply supported boundary conditions to Euler-Bernoulli beam.

$U_{51}(\omega) = A^2$, $\quad U_{53}(\omega) = -B^2$,

$U_{52}(\omega) = U_{54}(\omega) = U_{55}(\omega) = U_{56}(\omega) = U_{57}(\omega) = U_{58}(\omega) = 0$,

$U_{65}(\omega) = A^2 \cosh(A)$, $\quad U_{66}(\omega) = A^2 \sinh(A)$,

$U_{67}(\omega) = -B^2 \cos(B)$, $\quad U_{68}(\omega) = -B^2 \sin(B)$,

$U_{61}(\omega) = U_{62}(\omega) = U_{63}(\omega) = U_{64}(\omega) = 0$,

$U_{71}(\omega) = U_{73}(\omega) = 1$,

$U_{72}(\omega) = U_{74}(\omega) = U_{75}(\omega) = U_{76}(\omega) = U_{77}(\omega) = U_{78}(\omega) = 0$,

$U_{81}(\omega) = U_{82}(\omega) = U_{83}(\omega) = U_{84}(\omega) = 0$,

$U_{85}(\omega) = \cosh(A)$, $\quad U_{86}(\omega) = \sinh(A)$,

$U_{87}(\omega) = \cos(B)$, $\quad U_{88}(\omega) = \sin(B)$,

APPENDIX I

Fixed Boundaries for Euler-Bernoulli Beam

Equations of "ω" from $U_{51}(\omega)$ to $U_{68}(\omega)$ obtained by applying fixed-fixed boundary conditions to Euler-Bernoulli beam.

$U_{52} = A$, $U_{54} = B$,

$U_{51} = U_{53} = U_{55} = U_{56} = U_{57} = U_{58} = 0$,

$U_{61} = U_{62} = U_{63} = U_{64} = 0$,

$U_{65} = A\sinh(A)$, $U_{66} = A\cosh(A)$,

$U_{67} = -B\sin(B)$, $U_{68} = B\cos(B)$,

APPENDIX J

Cantilevered BC for Euler-Bernoulli Beam

Equations of "ω" from $U_{51}(\omega)$ to $U_{88}(\omega)$ obtained by applying fixed-free boundary conditions to Euler-Bernoulli beam.

$U_{51}(\omega) = U_{53}(\omega) = 1$,

$U_{52}(\omega) = U_{54}(\omega) = U_{55}(\omega) = U_{56}(\omega) = U_{57}(\omega) = U_{58}(\omega) = 0$,

$U_{62}(\omega) = A$, $\quad U_{64}(\omega) = B$,

$U_{61}(\omega) = U_{63}(\omega) = U_{65}(\omega) = U_{66}(\omega) = U_{67}(\omega) = U_{68}(\omega) = 0$,

$U_{71}(\omega) = U_{72}(\omega) = U_{73}(\omega) = U_{74}(\omega) = 0$,

$U_{75}(\omega) = A^2 \cosh(A)$, $\quad U_{76}(\omega) = A^2 \sinh(A)$,

$U_{77}(\omega) = -B^2 \cos(B)$, $\quad U_{78}(\omega) = -B^2 \sin(B)$,

$U_{81}(\omega) = U_{82}(\omega) = U_{83}(\omega) = U_{84}(\omega) = 0$,

$U_{85}(\omega) = A^3 \sinh(A)$, $\quad U_{86}(\omega) = A^3 \cosh(A)$,

$U_{87}(\omega) = B^3 \sin(B)$, $\quad U_{88}(\omega) = -B^3 \cos(B)$,

APPENDIX K

Effect of foundation on K_I

α	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50	0.55	0.60
$F(\alpha)$	1.122	1.069	1.039	1.026	1.025	1.033	1.051	1.076	1.109	1.149	1.197	1.250	1.307

Shape factor for a cracked beam resting on a Winkler foundation of k_w=6.73.10^4 N/m^2.