ALGEBRAIC CURVES, HERMITIAN LATTICES AND HYPERGEOMETRIC FUNCTIONS

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ABSTRACT

ALGEBRAIC CURVES, HERMITIAN LATTICES AND HYPERGEOMETRIC FUNCTIONS

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The aim of this work is to study the interaction between two classical objects of mathematics: the modular group, and the absolute Galois group. The latter acts on the category of finite index subgroups of the modular group. However, it is a task out of reach do understand this action in this generality. We propose a lattice which parametrizes a certain system of "geometric" elements in this category. This system is setwise invariant under the Galois action, and there is a hope that one can explicitly understand the pointwise action on the elements of this system. These elements admit moreover a combinatorial description as quadrangulations of the sphere, satisfying a natural nonnegative curvature condition. Furthermore, their connections with hypergeometric functions allow us to realize these quadrangulations as points in the moduli space of rational curves with 8 punctures. These points are conjecturally defined over a number field and our ultimate wish is to compare the Galois action on the lattice elements in the category and the corresponding points in the moduli space.

Keywords: modular group, absolute Galois group, hypergeometric functions, children's draw-

ings

ÖZ

CEBİRSEL EĞRİLER, HERMİSYEN KAFESLER VE HİPERGEOMETRİK FONKSİYONLAR

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Bu çalışmanın amacı matematiğin iki klasik objesi, modüler grup ve mutlak Galois grubu, arasındaki ilişkiyi irdelemektir. Bu ilişki temel olarak şu şekilde özetlenebilir: mutlak Galois grubu modüler grubun sonlu indeks alt-grupları üzerine etki eder. Ancak, bu etkinin en genel halinde anlaşılmasının günümüz teknikleri ile mümkün olmadığı bir çok çalısmada ortaya komulmuştur. Bu çalışma, temel olarak, modüler grubun sonlu indeks alt-grupları kategorisinin belirli özellikleri sağlayan elemanlarını bir kafes ile temsil edilebileceğini, ve bahsedilen etkinin bu elemanlar üzerinde daha rahat anlaşılabileceğini göstermektedir. Bu elemanlar Galois etkisi altında küme bazında değişmezdirler ve etkinin elemanlar bazında açıkça yazılabileceği umudedilmektedir. Tüm bunlara ek olarak bu elemanlar küre karelemeleri vasıtası ile kombinatoriksel olarak da tarif edilebilinir. Öte yandan hipergeometrik fonksiyonlar bu kafesin elemanlarını 8 delikli rasyonel eğrilerin modüler uzayının elemanları olarak görülmesine imkan verir ki bu noktalar, tahminsel olarak, bir sayı cismi üzerinden tarif edilebilinir. Nihayi hedef kafes noktaları uzerindeki ve modüler uzay üzerindeki Galois etkisini karşılaştırmaktır.

Anahtar Kelimeler: modüler grup, mutlak Galois grubu, hipergeometrik fonksiyonlar, çocuk resimleri

MEÇHULLER

...

Mugaletât-ı riyaziyye eyliyor isbat
Ki hendeseyle hesabın da gayeti zandır.
Çıkarsa vâdi-i zanna reh-i bedihiyyât
Nasıl denir faraziyyât-ı sırfaya "fendir".

...

Cenab Şahabettin, Evrâk-ı Leyâl

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Besides my advisor, this work has benefited a lot from many people. I must thank sincerely to Muhammed Uludağ for sharing his ideas, for his enthusiasm for this thesis. His support and direction was indispensable to this thesis. He has put a lot of effort, both in weekdays and weekends, to put this thesis up to the point where it is.

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CHAPTER 1

INTRODUCTION

One method to understand a group is to define its action on a rather well understood object, and investigate the properties of this action; for example, the stabilizers, the orbits, the structure of the orbits, etc. The group in question is one of the most important and mysterious objects of mathematics: the absolute Galois group, $\mathrm{Gal}(\mathbf{Q})$. $\mathrm{Gal}(\mathbf{Q})$ acts on many different objects, and is related to many different subjects: class field theory, Iwasawa theory, theory of motives, K-theory, moonshine, Teichmüller theory, ... Below, we mention, very briefly, some these relationships; in order to motivate our results.

Besides its importance on its own, the group $\mathrm{PSL}_2(\mathbf{R})$ is one of many different objects related to $\mathrm{Gal}(\mathbf{Q})$. Our aim in this paragraph is to explain the so called *moonshine*. Let $\mathbb{M}$ denote the monster group, which is the largest sporadic finite simple group, $\mathbb{H}$ denote the upper half plane and finally let Γ be a discrete subgroup of $\mathrm{PSL}_2(\mathbf{R})$ of genus zero, i.e. the quotient $\Gamma\backslash\mathbb{H} \cong \mathbf{P}^1_{\mathbf{C}}$. A function $j_\Gamma\colon \mathbb{H} \longrightarrow \mathbf{C}$ is called a *Hauptmodul* for Γ if every Γ invariant function is a rational function of j_Γ. Then there is an infinite dimensional representation, say $V = \bigoplus_{n\in\mathbf{Z}} V_n$ of $\mathbb{M}$ so that for each $g \in \mathbb{M}$ the function:

$$T_g(z) = \sum_{n\in\mathbf{Z}} H_n(g)q^n$$

is a Hauptmodul for some genus zero subgroup $\Gamma \leq \mathrm{PSL}_2(\mathbf{R})$, where H_n stands for the character of V_n and $q = \exp 2\pi\sqrt{-1}z$. In particular, for the simplest case, $g = id$, Γ turns out to be the modular group, $\mathrm{PSL}_2(\mathbf{Z})$. Moreover $H_n(id)$s are related to degrees of the smallest irreducible representations of $\mathbb{M}$. One has to mention the monstrous proposal which predicts that there is a moduli space whose fundamental group is $\mathbb{M}$, as this theory is very much similar to the theory of configuration space of 12 or 8 points in $\mathbf{P}^1$.

One other interesting aspect of the theory is related to mapping class groups, or equivalently, Teichmüller spaces. To every ribbon graph(or sometimes referred to as fat graphs), i.e. graph with prescribed orientation at each vertex, one associates a Riemann surface in which the graph can naturally be embedded. The edges of this graph can be "moved" within its homotopy class so that it becomes a geodesic, and hence for each graph one obtains a map from the Teichmüller space to $\mathbf{R}^{|\{\text{edges}\}|}$. Remark that negative numbers are possible because of orientation. This map is a homeomorphism onto the subset of $\mathbf{R}^{|\{\text{edges}\}|}$ satisfying the relation saying that sum of the lengths of left hand turn loops are equal to 0. The mapping class groupoid of all ribbon graphs are generated by *Whitehead moves*, up to involution, commutativity and pentagon relations. In addition, Whitehead moves may be used to represent elements of the mapping class group of the surface associated to the graph.

We would like to explain a class of zeta functions, graph zeta functions. To every metrized graph, i.e. a graph whose every edge has an associated length, G one may associate the following:

$$\prod_{p \text{ prime in } G} (1 - N(p))^{-1}$$

where by a prime in G we mean paths which does not have any superfluous edges, and N denotes the length of the prime path p. Besides their importance in tropical algebraic geometry, there is a nice Galois theory associated to each metrized graph.

The group $\mathrm{Gal}(\mathbf{Q})$ acts naturally on the set of projective algebraic curves defined over a number field. Indeed, for every projective algebraic curve, X, defined over a number field, K, one can find finitely many homogeneous polynomials in $K[x_0, \cdots, x_n]$, whose zero set is isomorphic to X. Any $\sigma \in \mathrm{Gal}(\mathbf{Q})$ acts on the coefficients of the defining homogeneous polynomials. Even though the description of the action is quite neat, it is almost impossible to work with. At this point, the celebrated theorem of Belyĭ tells us that a projective algebraic curve is defined over a number field, which will be refereed to as an arithmetic curve, exactly when it admits a Belyĭ morphism, i.e. there is a β in the function field of X which is ramified at most over 3 points. One deduces that $\mathrm{Gal}(\mathbf{Q})$ acts on the set of Belyĭ pairs, (X,β).

There is a group theory companion to the above description. Let us recall that the holomorphic automorphisms of upper half plane is $\mathrm{PSL}_2(\mathbf{R})$ and acts 3-transitively on upper half plane. Thus we may assume, without loss of generality, that 3 ramification points of a Belyĭ morphism are $\{0, 1, \infty\}$(In fact, one must pay attention to the field of definition of these 3 ramification points!). It is well known that the congruence subgroup of level two, which we denote by $\Delta_{\infty,\infty,\infty}$, which consists of 2×2 matrices with integer entries which are congruent to 2×2 identity matrix modulo 2, is the surface group of $\mathbf{P}^1 \setminus \{0, 1\infty\}$. So to every X defined over the field of algebraic numbers the map β induces a subgroup, say Γ_X, of $\Delta_{\infty,\infty,\infty}$.

Now, consider the real line segments $[\infty, 0]$, $[0, 1]$, $[1, \infty]$ in $\mathbf{P}^1$ as three edges of a triangle with vertices 0, 1 and ∞. Let us mark the points on X which are inverse images of 0, respectively 1, with a white, respectively black, vertex. Then the inverse image of the real interval $[0, 1]$ is an embedded *bipartite graph* on X. On the other hand the morphism β induces a triangulation of X. Observe that if we color the triangle corresponding to upper half plane by white and lower half plane by black, then the triangulation of X obtained is *bipartite*, i.e. the triangles is the triangulation can be colored black and white so that no two white triangles have a common edge as well as the black triangles. So to every arithmetic curve one may associate a bipartite triangulation. It is clear that not every triangulation is bipartite. However, to every triangulation one may associate a bipartite graph via barycentric subdivision.

The combinatorial arguments explained above may be rephrased in the algebraic category. For this, let X be a non-empty, path connected, locally simply connected topological space(so that it admits a universal covering space, $\widetilde{X}$). For $x_o \in X$ the group of homotopy classes of paths based at x_o is defined to be the fundamental group, $\pi_1(X, x_o)$, of X. There is another interpretation of the fundamental group in term of finite covering maps. However, there is no algebraic analogue of a path. We need another interpretation of the fundamental group.

For this, let $\mathbf{FCov}(X)$ denote the category, in fact it is a so called Galois category, of finite coverings of X whose objects are finite topological coverings of X which having only finitely many connected components and morphisms between two coverings, say $\gamma\colon Y \longrightarrow Y'$, Y and Y' are the covering maps, $\pi\colon Y \longrightarrow X$ and $\pi'\colon Y' \longrightarrow X$ such that $\pi' \circ \gamma = \pi$. If we let $\mathrm{Aut}(\widetilde{X})$ denote the group of all covering maps $\widetilde{\pi}\colon \widetilde{X} \longrightarrow X$, then we obtain an isomorphism from $\mathrm{Aut}(\widetilde{X})$ to $\pi_1(X, x_o)$ described as follows: for any path based at x_o, and any fixed point $\widetilde{x_o} \in \widetilde{\pi}^{-1}(x_o)$, an element, $\gamma \in \mathrm{Aut}(\widetilde{X})$, different from identity, sends $\widetilde{x_o}$ to another element of the set $\widetilde{\pi}^{-1}(x_o)$, hence a path between $\widetilde{x_o}$ and $\gamma(\widetilde{x_o})$ by composing with $\widetilde{\pi}$ gives an element of $\pi_1(X, x_o)$. In fact, this gives us a functor, F, from $\mathbf{FCov}(X)$ to the category of sets sending each covering, $\pi\colon Y \longrightarrow X$ to the set $\pi^{-1}(x_o)$, functorial in Y. Moreover, this functor is representable, i.e. $F(Y) = \mathrm{Hom}_X(\widetilde{X}, Y)$; and we have a natural action of $\mathrm{Aut}(\widetilde{X})$ on $F(Y)$

whose orbits are finite. Thus the target category of the functor are sets with $\mathrm{Aut}(\widetilde{X})$ action with finite orbits. F is, in fact an equivalence between the two categories.

The algebraic counterpart of a covering map is the concept of a finite étale map. We let X be an algebraic variety over a field k and $\mathcal{E}t(X)$ denote the category of finite étale coverings, $\pi\colon Y \longrightarrow X$, of X, whose objects are finite étale coverings of X and morphisms are X-morphisms. In this case, one has to careful in choosing a base point. When X is over an algebraically closed field, k, then a base point may be chosen as an element of $X(k)$. Otherwise, one has to choose a geometric point $x_o \hookrightarrow X$, i.e. a point whose coordinates lie in a separably algebraically closed field. This amounts to choosing a point $x \in X$ and a separably algebraically closed field containing the residue field. Now, we can define a functor $F\colon \mathcal{E}t(X) \longrightarrow \mathrm{Sets}$, sending an étale covering $\pi\colon Y \longrightarrow X$ to x_o-valued points of Y which lie in $\pi^{-1}(x)$. In particular, when k is algebraically closed and $x_o \in X(k)$, then $F(Y) = \pi^{-1}(x_o)$. As in the case of topological coverings, we would like to define a universal object, say $\widetilde{X}$, so that $F(Y) = \mathrm{Hom}_X(\widetilde{X}, Y)$. This is clearly not the case for a very simple reason that the exponential function is not algebraic. Nevertheless there is a projective system $(X_i)_{i\in I}$ of finite étale coverings of X, with I being a directed set so that $F(Y) = \mathrm{Hom}_X(\widetilde{X}, Y) := \varinjlim \mathrm{Hom}_X(X_i, Y)$ functorial in Y. In a similar fashion, one may define then the fundamental group $\pi_1(X) = \mathrm{Aut}(\widetilde{X}) := \varprojlim \mathrm{Aut}(X_i)$. For instance, in the case of the universal covering $\exp\colon \mathbf{C} \longrightarrow \mathbf{C}^\times$, the algebraic fundamental group is the profinite completion of the the topological fundamental group of $\mathbf{C}^\times$, i.e. $\pi_1^{alg}(\mathbf{C}^\times) = \widehat{\mathbf{Z}}$, a fact which is true in general. Now, let $k = \mathbf{Q}$ and $X = \mathbf{P}^1_{\mathbf{Q}} \setminus \{0, 1, \infty\}$. Then, we have the following exact sequence

$$1 \longrightarrow \pi_1^{alg}(X \otimes_{\mathbf{Q}} \overline{\mathbf{Q}}) \longrightarrow \pi_1^{alg}(X) \longrightarrow \mathrm{Gal}(\mathbf{Q}) \longrightarrow 1 \tag{1.1}$$

where as a result of the Riemann Existence Theorem, $\pi_1^{alg}(X \otimes_{\mathbf{Q}} \overline{\mathbf{Q}})$ is the profinite completion of $\Delta_{\infty,\infty,\infty}$. The sequence (1.1) may be used to define an action of $\mathrm{Gal}(\mathbf{Q})$ on $\pi_1^{alg}(X)$, which we have already mentioned above using embedded graphs. One has to admit that studying this extension is much more difficult than studying the action.

An origami is a finite set of euclidean squares which are glued nicely, see Definition 3.2.8 for the precise definition. The set of complex structures on an origami, which is $\mathrm{SL}_2(\mathbf{R})/\mathrm{SO}_2(\mathbf{R}) \cong \mathbb{H}$, can be embedded into the corresponding Teichmüller space which is known to be an isometry with respect to the Poincaré metric on $\mathbb{H}$ and Teichmüller metric on the Teichmüller space. Hence one obtains a so called *Teichmüller disc*. It is, at least intuitively, clear that this construction is closely related to flat surfaces.

It turns out that one should not aim at understanding the action in this full generality. One is led to concentrate on sub-families, sub-categories, etc., which has been the case for several authors, see works of, for example, Wojtkowiak, Deligne, Goncharov on the subject. Our aim in this work is to suggest other sub-systems, which have its origins in geometry. We are going to construct two lattices, Λ and Λ', inside complex Lorentz spaces $\mathbf{C}^{1,9}$ and $\mathbf{C}^{1,5}$, respectively, whose positive cones parametrize non-negatively curved triangulation and quadrangulation, respectively, of S^2. To every triangulation and quadrangulation one may associate their dual graph which is nothing but an embedded(or fat, or ribbon) graph on the sphere. Hence points of Λ and Λ' parametrize in particular Belyĭ pairs (X, β), and our results both in Section 4.2.4 and in [53] provides us evidence that these families are easier to *understand*. As explained above, the lattices Λ and Λ' parametrize certain subgroups of $\Delta_{\infty,\infty,\infty} \leq \mathrm{PSL}_2(\mathbf{R})$, the fundamental group of $\mathbf{P}^1_{\mathbf{C}} \setminus \{0, 1, \infty\}$ or certain covers of $\mathbf{P}^1_{\mathbf{C}} \setminus \{0, 1, \infty\}$; and, similarly, certain subgroups of the profinite completion of free group on 2 generators, the algebraic fundamental group of $\mathbf{P}^1_{\mathbf{Q}}$ or certain connected étale covers of $\mathbf{P}^1_{\mathbf{Q}} \setminus \{0, 1, \infty\}$. Moreover, the euclidean fundamental region, see Definition 2.4.2, associated to an arbitrary element of Λ' can be considered as an origami. Thus, the lattice Λ' parametrizes at the same time Teichmüller discs in certain Teichmüller spaces. To uncover the relationship with moonshine, it is enough to note that as a result of the construction, the surface groups of the Belyĭ pairs appearing as points of Λ and Λ' are of genus 0. Hence they parametrize certain elements of the *monster*. As a final point we want to emphasize that the embedded graphs dual to a triangulation in Λ or a quadrangulation in Λ' of S^2 is also, in a natural fashion, a metric graph, as a result of lying in euclidean triangles or squares. In other words, these objects have *nice* graph zeta functions.

CHAPTER 2

CONE MANIFOLDS OF DIMENSION TWO AND THEIR MODULI

Any Riemann surface $X_{g,N}$ of genus g with N punctures, satisfying $\chi(X_{g,N}) = 2-2g+N < 0$, is uniformized by upper half plane, as a consequence of the uniformization theorem. Hence $X_{g,N}$ admit a canonical hyperbolic metric compatible with the conformal structure. There are, on the other hand many other natural metrics that may be defined on $X_{g,N}$, possibly with singular set which are at the same time compatible with the conformal structure. Our main concern will be metrics with special singular behavior, to which we will refer as *cone metrics*, on surfaces. We are going to recall two representations associated to a cone metric, and show that one factors through the other. Surfaces together with a cone metric will be called cone surfaces. We will see that the set of cone metrics have a nice structure, which will be obtained in the last section via realizing them as the positive cone in certain vector spaces and interpreting this vector space as the cohomology of punctured Riemann sphere with coefficients in a locally constant sheaf of rank one.

2.1 Basics

Our aim is to define what a cone manifold of dimension 2 is and collect some of their properties. Our main reference for this section is [46]. We begin with defining our *singular* neighbourhood:

$$V_\theta := \{(r,t) \,|\, r \in \mathbf{R}_{\geq 0},\ t \in \mathbf{R}/\theta\mathbf{Z}\}.$$

We equip V_θ with the metric $ds_\theta^2 = dr^2 + r^2 dt^2$. Note that (V_θ, ds_θ^2) is nothing but a cone of cone angle θ in $\mathbf{R}^3$ whenever the angle parameter, θ, is a real number between 0 and 2π.

Definition 2.1.1 *A **metric cone** of cone angle θ is defined to be an open neighbourhood of the origin, $(0,0)$, in (V_θ, ds_θ^2).* ■

Now let X be an orientable topological surface, and S be a finite set of points of X.

Definition 2.1.2 *We will say a metric, c, on X is a **Euclidean cone metric**, whenever every element $x \in X - S$ has an open neighbourhood, U_x, isometric to $\mathbb{E}^2 := (\mathbf{C}, ds^2 = |dz|^2)$, and every element $p \in S$ has a neighbourhood, U_p such that there is an isometry, φ_p, between U_p and V_{θ_p} with $\varphi_p(p) = (0,0)$.* ■

We will write cone metric for short instead of Euclidean cone metric. Elements of the set $S = S_c$ will be called *singular points* and the elements of $X - S_c$ will be called *regular points*. We will call the pair (X, c) a Euclidean cone manifold (of dimension 2), or cone manifold, in short. A few remarks are in order:

Remark 2.1.3 *The pair (V_θ, ds_θ^2) is isometric to $(\mathbf{C}, ds^2 = |z|^{2\beta}|dz|^2)$; where $\beta = \frac{\theta}{2\pi} - 1$, referred to as the **residue**. In particular, when $\theta = 2\pi$ then our local model is nothing but $\mathbf{C}$ with its flat metric. We will call $\kappa = 2\pi - \theta$ the **concentrated curvature** at $(0,0)$.*

Remark 2.1.4 *Let (X, c) be a cone manifold. If a point p on X is a regular point then identifying $\mathbb{E}^2$ with $\mathbf{C}$ gives a local analytic chart around p, and if $p \in S_c$ is a singular point then previous remark provides the local analytic map around p. Thus, cone metric induces a complex structure on the surface X. That is, the surface X together with c becomes, in fact, a Riemann surface. However, the singular points, as one can immediately see from the local coordinates, may not be distinguished merely by looking at the complex structure. In fact, any given Riemannian metric on $X_{g,N}$ induces a conformal structure, and the conformal structure induced by the metric is the same as the conformal structure induced by its arbitrary analytic function multiples.*

Let $S_c = \{p_1, \ldots, p_N\}$ be the singular set corresponding to a cone metric c on a surface X, and let $\kappa_i = 2\pi - \theta_i$ the the concentrated curvature at the point p_i, for $i \in \{1, 2, \ldots, N\}$, respectively.

Assume that the surface X is *compact*. In this case, for the pair (X, c) we have the following *singular* version of the well-known Gauß-Bonnet theorem:

Theorem 2.1.5 (Singular Gauß-Bonnet, [46, Proposition 3]) *Let X be a cone surface where the points $p_1, \ldots, p_N \in X$ are singular with concentrated curvatures $\kappa_1, \ldots, \kappa_N$, respectively. Then:*

$$\sum_{i=1}^{N} \kappa_i = 2\pi\chi(X);$$

where $\chi(X)$ is the topological Euler characteristic of X.

And, in fact, this is the only restriction, known as the *Gauss - Bonnet restriction*. In other words, we have the following:

Theorem 2.1.6 ([46, §5, Théorème]) *Let X be as above, $p_1, \cdots, p_N$ are points in X and $\kappa_1, \ldots, \kappa_N$ are rational numbers such that*

$$\sum_{i=1}^{N} \kappa_i = 2\pi\chi(X)$$

Then, X admits a cone metric, c, with concentrated curvature $\kappa_i = 2\pi - \theta_i$, at the point p_i, $i = 1, \ldots, N$. Moreover, this metric is unique up to normalization.

2.2 From Cone Manifolds to Triangulations

As we have seen in previous section, a cone manifold can be obtained simply by choosing finitely many points, $\{p_1, \ldots, p_N\}$ on X and then choosing $\kappa_i \in \mathbf{R}$ so that $\sum_{i=1}^{N} \kappa_i = 2\pi\chi(X)$. There is one other way which will help us to construct and understand the set of all cone metrics on X with pre-determined concentrated curvatures. Even though most of the results that we state are valid for closed surfaces of higher genera, we will restrict ourselves to the sphere, S^2. We further assume that the number of singular points, i.e. $|S_c|$, is at least 3.

Definition 2.2.1 *A (finite)* ***triangulation*** *$\mathcal{T}$ of X is a (finite) set of pairs (f_i, U_i), where $f_i(U_i) = \Delta_i$ is a non-degenerate triangle in $\mathbf{R}^2$, and $f_i : U_i \longrightarrow \Delta_i$ is homeomorphism such that* *i.* $\bigcup_i U_i = X$, *and*

ii. *whenever* $f_i(U_i) \cap f_j(U_j) \neq \emptyset$, *for* $i \neq j$, *then the intersection is a subset (of size* ≤ 3*) of either the set of edges,* $e(\mathcal{T})$, *of* $\mathcal{T}$ *or the set of vertices,* $v(\mathcal{T})$ *of* $\mathcal{T}$*;*

where we define the ***set of vertices, edges*** *of* $\mathcal{T}$*, to be the set of inverse images of all vertices, edges, of the triangles* Δ_i*. We define the* ***set of faces****,* $f(\mathcal{T})$*, to be the set* $\{U_i\}$*.* ■

If, furthermore, a triangulation satisfies the following two properties, then it will be called a *metric triangulation* or a *euclidean triangulation* of X.

iii. Δ_is are subspaces of $\mathbb{E}^2$ with f_is being isometries and

iv. for every pair of distinct triangles $f_i(U_i)$ and $f_j(U_j)$ which intersect in an edge, $e \in e(\mathcal{T})$, there exists an element, $\gamma_{i,j}$ in the group of isometries of the Euclidean plane, $\mathrm{Isom}(\mathbb{E}^2)$, such that $\gamma_{ij}(f_i(e)) = f_j(e)$.

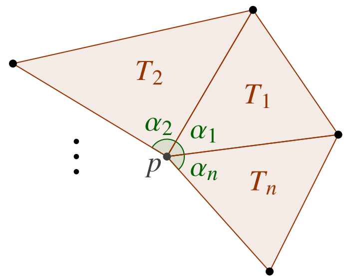


Figure 2.1: Property iv. allows us to glue euclidean triangles.

Let $\mathcal{T}$ be a finite euclidean triangulation on X. Let $p \in v(\mathcal{T})$ be a vertex at which $T_1 = U_{i_1}, \ldots, T_n = U_{i_n}$ meet. In this case we may define the concentrated curvature at p to be:

$$2\pi - \sum_{i=1}^{n} \alpha_i;$$

where α_i is the angle at the point p inside the triangle Δ_{i_n};where $i \in \{1, \ldots, n\}$ see Figure 2.1. Then, Theorem 2.1.6 provides us a cone metric, say $c_{\mathcal{T}}$, associated to $\mathcal{T}$.

Remark 2.2.2 *Whenever such a structure is prescribed on* X*, then* X *becomes a length space. Indeed, let* $\mathcal{T}$ *be a euclidean triangulation on* X *and let* p, q *be two points on* X*. Let* γ :

$[0,1] \longrightarrow X$ *be a rectifiable path between* p *and* q*, and* $0 = t_0 < t_1 < \ldots t_{n-1} < 1 = t_n$ *so that* $\gamma([t_j, t_{j+1}]) \subset U_{i_j}$*; where* $j \in \{0, \ldots, n-1\}$*. Then we define the length of* γ *to be:*

$$l(\gamma) = \sum_{j=0}^{n-1} l_{euc}\Big(f_{i_j}(\gamma([t_j, t_{j+1}]))\Big).$$

And we define the distance between two points $p, q \in X$ *to be the smallest of* $l(\gamma)$*; where* γ *is a rectifiable path from* p *to* q*.*

Thus, if a geodesic triangle, T, is formed on (X, c), then T determines, *unique* up to similarity, a triangle in $\mathbb{E}^2$; where we call a triangle *geodesic* if the edges of the triangle are geodesics and do not contain any element of S_c except possibly at endpoints.

Let c be a cone metric. Fix any singular point $p_1 \in S_c$, and order the elements of S_c with respect to their distance to p_1. By re-indexing if necessary, we may assume that the distance of p_1 to p_i is less than or equal to p_j if and only if $i \leq j$. Let T be the geodesic triangle with vertices at p_1, p_2, p_3; $e_{i,j}$ be the geodesic between p_i and p_j; α_j denote the angle between $e_{i,j}$ and $e_{j,k}$; where $i, j \in \{1, 2, 3\}$.

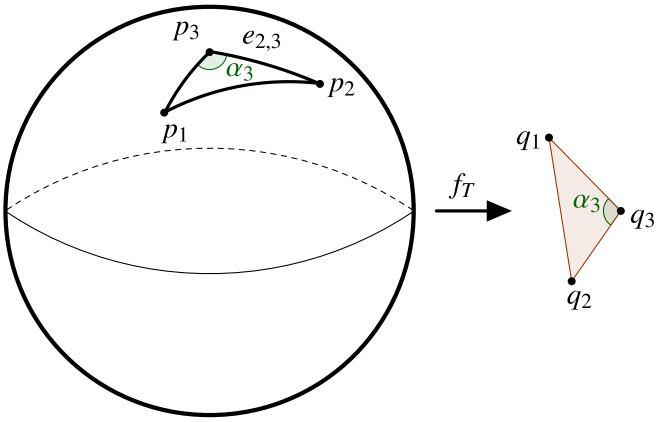


Figure 2.2: A euclidean triangle in (S^2, c) determines a euclidean triangle in $\mathbb{E}^2$.

Observe that edges, $e_{i,j}$, have to have empty intersection with the singular set $S_c \setminus \{p_1, p_2, p_3\}$ because otherwise we could have found a closer singular point. In that case, the triangle T determines, in $\mathbb{E}^2$, a geodesic triangle with the property that the angle at $q_i = f_T(p_i)$ is exactly α_i, and the length of the edge $f_{i,j} = f_T(e_{i,j})$ is equal to that of $e_{i,j}$; where $f_T : T \longrightarrow \mathbb{E}^2$ is the induced isometry in between, see Figure 2.2.

More generally, fix an element $p_1 \in S_c$ and enumerate the remaining singular points so that there is a continuous path, $\gamma : [0,1] \longrightarrow S^2$, joining p_1 to p_N with the following properties:

- there is a sequence of numbers $t_1 = 0 < t_2 < \ldots < 1 = t_N$ satisfying $\gamma(t_i) = p_i$, $i = 1, 2, \ldots, N$,

- $\gamma|_{[t_i, t_{i+1}]}$ is a geodesic with respect to c, for $i = 1, \ldots, N-1$,

- γ is one-to-one.

Proposition 2.2.3 *Let c be a cone metric on X. Then c induces a geodesic triangulation, denoted by $\mathcal{T}_c$, on X with the property that the set of vertices of $\mathcal{T}_c$ is exactly the set of singular points, S_c, of c.*

Proof. Cut X open along γ. where γ is as above. Since all the singular points of c are along $\gamma([0,1])$, one can write a multi-valued map, ψ, from X to a polygon, P in $\mathbb{E}^2$ with *holes* if $X \neq S^2$, so that ψ is an isometry when restricted to $X \setminus \gamma$. Moreover, ψ maps any geodesic $\gamma|_{[t_i, t_{i+1}]}$ to an edge of P, for all $i = 1, \ldots, n-1$, and in fact twice. Remark also that the polygon P is uniquely determined, up to $\mathrm{Isom}(\mathbb{E}^2)$. In order to obtain a euclidean triangulation on X, it is enough to draw the necessary diagonals of P. As every diagonal is a geodesic in $\mathbb{E}^2$, we obtain a euclidean triangulation. ■

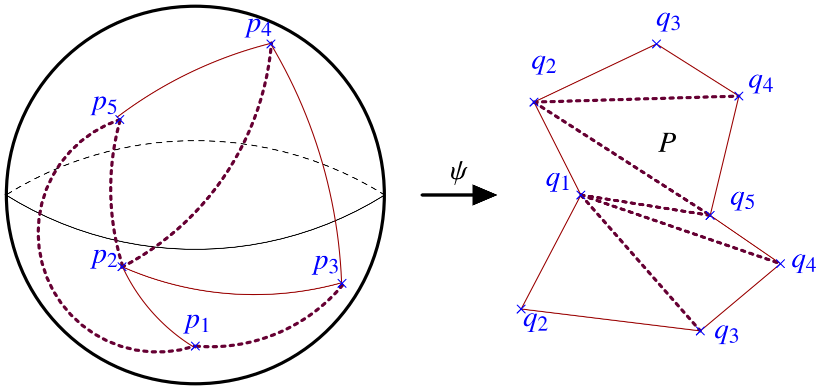


Figure 2.3: From a cone metric to a geodesic triangulation.

We end this section with a remark:

Remark 2.2.4 *The geodesic triangulation associated to a cone metric is* ***not*** *unique. However, there are finitely many such choices. On the other hand, if one allows non-singular vertices to appear inside the triangles of the triangulation then there are infinitely many choices. Therefore, one might call triangulations arising from Proposition 2.2.3* ***minimal****.*

2.3 Two Representations of $\pi_1(\mathbf{P}^1 \setminus S_c)$

In this section, we introduce two representations of $\pi_1(S^2 \setminus S_c)$ associated to a given cone metric c on $X = S^2$. Before proving that two representations, holonomy and monodromy representations, of c are closely related, we will prove that the geometric realization of the universal branched cover is a flat upper half plane, which is obtained by adding cusps of certain subgroups of $\mathrm{PSL}_2(\mathbf{R})$ to $\mathbb{H}$ equipped naturally with a locally euclidean metric. Let us first introduce some notation: by $C(\kappa) = C(\kappa_1, \kappa_2, \ldots, \kappa_N)$ we will denote the set of all cone metrics having N singular points with concentrated curvatures $\kappa_i \neq 0$, $i = 1, 2, \ldots, N$; $N \geq 3$, up to *orientation preserving similarity*; where by an orientation preserving similarity we mean any composition of a rotation a translation and a stretching. And, finally by $\mathbb{H}$, we denote the upper half plane, i.e. the set of complex numbers whose imaginary parts are strictly positive. For technical reasons we assume that $\theta_i = 2\pi - \kappa_i$ are elements of the intersection $\pi\mathbf{Q} \cap (0, 2\pi)$.

2.3.1 Holonomy Representation of a Cone Metric

Let $c \in C(\kappa)$ be a cone metric, with an induced triangulation $\mathcal{T}_c$ on S^2, see Proposition 2.2.3.

Definition 2.3.1 *Let $\gamma : [0,1] \longrightarrow S^2$ be a piecewise smooth path in S^2, such that $\gamma((0,1)) \subset S^2 \setminus S_c$. We will say that γ is **admissible** if $\gamma([0,1])$ intersects the edges, $e(\mathcal{T}_c)$, of $\mathcal{T}_c$ finitely many times. We will call a homotopy $\gamma_t(s)$, $t, s \in [0,1]$, of piecewise smooth paths γ, γ' with $\gamma_0 = \gamma$, and $\gamma_1 = \gamma'$ **admissible** if γ_t is an admissible path for every $t \in [0,1]$.* ■

As in the classical case of fundamental groups, we define two admissible curves to be homotopically equivalent if and only if there is an admissible homotopy from one to another. It is then standard to show that this relation is an equivalence relation. In a similar way to the construction of the fundamental group, if we fix a point in $S^2 \setminus S_c$, the set of homotopy classes of admissible paths form a group, which is isomorphic to the fundamental group of $S^2 \setminus S_c$, as a result of the following lemma:

Lemma 2.3.2 *Let $\gamma\colon [0,1] \longrightarrow S^2$ be a continuous, piecewise differentiable path, with $\gamma((0,1)) \subset S^2 \setminus S_c$. Assume further that γ is not admissible, i.e. there is an edge $e \in e(\mathcal{T}_c)$*

which intersects γ infinitely many times. Then the homotopy class, $[\gamma]$ of γ contains an admissible path.

Proof. Suppose that γ intersects $e_o \in e(\mathcal{T}_c)$ infinitely many times. We exclude the case when γ, or a part of γ follows a portion of e, as in that case we may perturb γ so that it intersects e in only two points. There is, then, an increasing sequence, r_n, of elements of $(0,1)$, not necessarily non-constant, with the property that $e_o \cap \gamma(0,1) = \{\gamma(r_n) \mid n \in \mathbf{N}\}$. One can find a sufficiently large $M \in \mathbf{N}$ such that for every $n > M$ the restriction of the path γ to the closed interval $[r_n, r_{n+1}]$ is homotopic to the path that follows e with initial point $\gamma(r_n) \in e$ and terminal point $\gamma(r_{n+1}) \in e$. So, γ is homotopic to the path that follows e from γ_{r_M} to γ_{r_∞}; where γ_∞ denotes the limit of the sequence $(\gamma(r_k))_{k\in\mathbf{N}}$. As noted above, this last path is homotopic to a path which intersects e in only two points. ■

Corollary 2.3.3 *The group of homotopy classes of admissible paths is independent of the chosen triangulation.*

Now, regard $\mathcal{T}_c$ as a simplicial complex on S^2 and fix a base point $p_I \in S^2 \setminus (S_c \cup e(\mathcal{T}_c))$. By $\widehat{\mathcal{T}_c}$ denote the set of all pairs $(\sigma, [\gamma])$; where σ is a $0, 1$ or 2-simplex of $\mathcal{T}_c$, and $[\gamma]$ is *the* admissible homotopy class of an admissible curve $\gamma : [0,1] \longrightarrow \mathbf{P}^1$ which connects p_I to a point, call p_F, in σ. Note that $\widehat{\mathcal{T}_c}$ is by definition a simplicial complex. Let $\widehat{X}$ denote the geometric realization of $\widehat{\mathcal{T}_c}$, i.e. there is a bijection, B between the set of vertices of $\widehat{\mathcal{T}_c}$ and the set of vertices of $\widehat{X}$ such that $x \in \widehat{X}$ if and only if the convex hull of $B(x) \in \widehat{\mathcal{T}_c}$. Note that $\widehat{X}$ comes together with a projection map:

$$\begin{aligned} \widehat{\pi} : \qquad \widehat{X} &\longrightarrow S^2 \\ (\sigma, [\gamma]) &\mapsto p_F = \gamma(1) \end{aligned}$$

In order to understand $\widehat{X}$ better, we look at the complex structure on S^2 induced by c, and from now on we will write $\mathbf{P}^1$, instead of S^2, and by $\mathbf{P}_c$ we will denote the set $S^2 \setminus S_c$. As $N \geq 3$, by uniformization theorem $\mathbb{H}$ is the universal covering space of $\mathbf{P}_c$. So there is a torsion-free subgroup $\Gamma_c \leq \mathrm{PSL}_2(\mathbf{R})$. Note that, Γ_c will have cusps. By adding the set of cusps of Γ_c to $\mathbb{H}$, see [43, Chapter 1], we obtain a map $\widetilde{\pi} : \mathbb{H}_\flat = \mathbb{H} \cup \{\text{cusps of } \Gamma_c\} \longrightarrow \mathbf{P}^1$. More precisely, let $z \in \mathbf{R} \cup \{\infty\}$ be a cusp. Let $\Gamma_{c,z}$ denote the stabilizer of z in Γ_c. Then there is a neighbourhood, $U_z \subset \mathbb{H}$ containing z, so that $\gamma(U_z \cap U_z) = \emptyset$ for every $\gamma \in \Gamma_{c,z}$. Let θ_z denote the cone angle at

$\widetilde{\pi}^{-1}(z)$. Let $\rho \in \mathrm{PSL}_2(\mathbf{R})$ be an element sending z to ∞, so that $\rho\Gamma_{c,z}\rho^{-1} = \langle z \mapsto z+t \rangle$, for some positive t. Then the complex structure around any element of $\widetilde{\pi}^{-1}(z)$ can be obtained from the natural map $pr\colon \Gamma_{c,z}\backslash U_z \longrightarrow \mathbf{C}$ such that $pr \circ \widehat{\pi}(x) = \exp 2\pi\sqrt{-1}\rho(x)/t$; where $t = \theta/2\pi$. Now we pull back the triangulation $\mathcal{T}_c$ by $\widetilde{\pi}$, to obtain a triangulation on $\mathbb{H}_\flat$, denoted by $\widetilde{\mathcal{T}_c}$.

Proposition 2.3.4 *The geometric realization $\widehat{X}$ of $\widehat{\mathcal{T}_c}$ is nothing but $\mathbb{H}_\flat$.*

Proof. Call the triangle, in $\mathcal{T}_c$ which contains p_I, the *base triangle* and denote it by T_I. Choose a point, $\widetilde{p_I}$, in the set $\pi^{-1}(p_I)$. Consider the map from $\widehat{\mathcal{T}_c}$ to $\widetilde{\mathcal{T}_c}$ described as follows: Take an element $(\sigma, [\gamma]) \in \widehat{X}$, let $p_F \in \sigma$ denote the endpoint of γ. We may lift the path γ to a path in $\mathbb{H} \subset \mathbb{H}_\flat$ in a unique way as we chose already an initial point, $\widetilde{p_I}$. Therefore the final point $\widetilde{p_F}$ is already determined, which we define to be the image of the pair $(\sigma, [\gamma])$.

$$\begin{tikzcd}[column sep=small]
(\mathbb{H}_\flat, \widetilde{\mathcal{T}_c}) \arrow[rr, leftrightarrow] \arrow[dr, "\widetilde{\pi}"'] & & (\widehat{X}, \widehat{\mathcal{T}_c}) \arrow[dl, "\widehat{\pi}"] \\
& (\mathbf{P}^1, \mathcal{T}_c) &
\end{tikzcd}$$

For the inverse map, take any point $\widetilde{x}$ in $\mathbb{H}_\flat$, and any piecewise smooth path, $\gamma_{\widetilde{p_I},\widetilde{x}}$ from $\widetilde{p_I}$ to $\widetilde{x}$ so that it has empty intersection with $v(\widetilde{\mathcal{T}_c})$ for every $t \in (0, 1)$. Then the path $\gamma_{p_I,x} = \pi(\gamma_{\widetilde{p_I},\widetilde{x}})$ is a path in $\mathbf{P}_c$, which does not pass through the vertices of $\mathcal{T}_c$ except possibly at endpoints. Then there is a 0, 1 or 2-simplex, σ, of $\mathcal{T}_c$ to which $x = \pi(\widetilde{x})$ belongs. Map this element to the pair $(\sigma, \gamma_{p_I,x})$.

In order to see that this map is well-defined, suppose that we have chosen another path, $\gamma'_{\widetilde{p_I},\widetilde{x}}$. Then the concatenation of two paths $\gamma'_{\widetilde{p_I},\widetilde{x}} \cdot \gamma^{-1}_{\widetilde{p_I},\widetilde{x}}$ is homotopic to the identity(there can only be cusps) and hence stay identity when pushed down by π. So that

$$(\sigma, \gamma_{p_I,x}) = (\sigma, \gamma'_{p_I,x} \cdot \gamma^{-1}_{p_I,x} \cdot \gamma_{p_I,x}) = (\sigma, \gamma'_{p_I,x})$$

From the construction, we deduce that the composition is identity.

■

Definition 2.3.5 *The pair $(\mathbb{H}_\flat, \widetilde{\mathcal{T}_c})$, together with the locally flat metric obtained by lifting c is called the* ***universal branched cover*** *of $(\mathbf{P}^1, c)$.* ■

By a locally flat metric we mean that every point has an open neighbourhood which is isometric to an open set in the Euclidean space and should not be confused with the term locally flat connection on a principal fiber bundle.

We would like to admit at this point that the term branched cover is somewhat misleading as the branching index at cusps are not finite, yet it has been accepted in the literature.

As in the proof of Proposition 2.2.3 we fix a base triangle $T_I \in f(\mathcal{T}_c)$; where $c \in C(\kappa)$ and we fix a face $\widetilde{T_I} \in f(\widetilde{\mathcal{T}_c})$ in the set $\widetilde{\pi}^{-1}(T_I)$. The triangulation $\widetilde{\mathcal{T}_c}$ is a euclidean triangulation so we have an isometry $\widetilde{\varphi_T}$ sending every face $\widetilde{T}$ of $\widetilde{\mathcal{T}_c}$ to a euclidean triangle. Now, consider another face $\widetilde{T'} \in f(\widetilde{\mathcal{T}_c})$ with isometry $\widetilde{\varphi_{T'}}$, which has one edge, $\widetilde{e}$ in common with $\widetilde{T}$. By definition there is an element, $\widetilde{g_{0,1}}$ in Isom$(\mathbb{E}^2)$ so that $\widetilde{g_{0,1}}(\widetilde{\varphi_T}(e)) = \widetilde{\varphi_{T'}}(e)$. Thus, we may define an isometry from the union:

$$\widetilde{\varphi_{T,T'}} : \widetilde{T} \cup \widetilde{T'} \longrightarrow \mathbb{E}^2.$$

Proceeding as above, we obtain a local isometry $\widetilde{\varphi} : \mathbb{H}_\flat \longrightarrow \mathbb{E}^2$.

Definition 2.3.6 *The map $\widetilde{\varphi} = \widetilde{\varphi}_c$ is called the **developing map** associated to c.* ■

Furthermore, we have:

Proposition 2.3.7 ([48, Proposition 2.8]) *Under the above settings, i.e. when a base triangle and an isometry of the base triangle into $\mathbb{E}^2$ are fixed, the developing map $\widetilde{\varphi}$ is unique.*

Now, take a path, γ_0, and consider the self-map, φ_{γ_0}, of the universal branched cover, $\mathbb{H}_\flat$, sending every pair $(\sigma, [\gamma])$ to $(\sigma, [\gamma \cdot \gamma_0])$. It is clear that φ_{γ_0} is independent of the representative considered in the homotopy class $[\gamma_0]$.

Lemma 2.3.8 *Suppose in particular that $\gamma_0 \in \pi_1(\mathbf{P}_c, p_I)$, for some fixed base point $p_I \in \mathbf{P}_c$. Fix also a base triangle $\widetilde{T} \in \mathbb{H}_\flat$ containing an element, $\widetilde{p_I}$, in $\widetilde{\varphi}^{-1}(p_I)$. Then φ_{γ_0} induces an element $h_{[\gamma_0]} \in Isom(\mathbb{E}^2)$ making the following diagram commutative.*

Furthermore, $h_{[\gamma_0]}$ is independent of the choice of the representative.

$$\begin{tikzcd}
(\mathbb{H}_\flat, \widetilde{\mathcal{T}_c}) \arrow[r, "\varphi_{\gamma_0}"] \arrow[d, "\widetilde{\varphi}"] & (\mathbb{H}_\flat, \widetilde{\mathcal{T}_c}) \arrow[d, "\widetilde{\varphi}"] \\
\mathbb{E}^2 \arrow[r, "h_{[\gamma_0]}"] & \mathbb{E}^2
\end{tikzcd}$$

Proof. As both p_I and $\widetilde{T} \in \mathbb{H}_\flat$ is fixed the associated developing map is uniquely determined. Consider the element, $t \in \mathrm{Isom}(\mathbb{E}^2)$, sending every $z = (x, y) \in \mathbb{E}^2$ to $z + v$; where $v = \left(\widetilde{\varphi} \circ \varphi_{\gamma_0}\right)(\widetilde{p_I}) - \widetilde{\varphi}(\widetilde{p_I})$. Moreover there is a rotation $r \in \mathrm{Isom}(\mathbb{H})$ with the property that

$$r\left(t\left(\widetilde{\varphi}(\widetilde{T})\right)\right) = \widetilde{\varphi}\left(\varphi_{\gamma_0}\right)(\widetilde{T}).$$

It is enough to set h_{γ_0} as the composition $r \circ t$. Suppose γ_0' is another representative. Since the final points agree, the corresponding translations, t and t' must agree. To see $r = r'$ note the vertices of the triangle $\widetilde{T}$ is mapped to the exact same point under both φ_{γ_0} and $\varphi_{\gamma_0'}$. The commutativity of the diagram can be proven as follows: take any point $(\sigma, [\gamma]) \in \mathbb{H}_\flat$. We have $\varphi_{\gamma_0((\sigma,[\gamma]))} = (\sigma, [\gamma \cdot \gamma_0])$. Then we obtain the image of $(\sigma, [\gamma \ldots \gamma_0])$ by "continuing" the isometry conformally along the concatenation of γ and γ_0. On the other hand the image of the pair (σ, γ) is obtained by continuation of the fixed isometry of $\widetilde{T}$. Composing with $r \circ t$ means that one passes exactly through the triangles that appear along the path $\widetilde{\varphi}\left(\widetilde{\pi}^{-1}\right)(\gamma)$. ■

For any other homotopy class, $[\gamma_1]$, in $\pi_1(\mathbf{P}_c, p_I)$, if we repeat the above arguments we arrive at the following diagram which is also commutative:

$$\begin{tikzcd}
(\mathbb{H}_\flat, \widetilde{\mathcal{T}_c}) \arrow[r, "\varphi_{\gamma_0}"] \arrow[d, "\widetilde{\varphi}"] & (\mathbb{H}_\flat, \widetilde{\mathcal{T}_c}) \arrow[r, "\varphi_{\gamma_1}"] \arrow[d, "\widetilde{\varphi}"] & (\mathbb{H}_\flat, \widetilde{\mathcal{T}_c}) \arrow[d, "\widetilde{\varphi}"] \\
\mathbb{E}^2 \arrow[r, "h_{[\gamma_0]}"] & \mathbb{E}^2 \arrow[r, "h_{[\gamma_1]}"] & \mathbb{E}^2
\end{tikzcd}$$

Hence we have a homomorphism $hol : \pi_1(\mathbf{P}^1 \setminus S_c, p_I) \longrightarrow \mathrm{Isom}(\mathbb{E}^2)$, called the *holonomy representation* associated to the cone metric $c \in C(\kappa)$.

2.3.2 Local Systems and the Monodromy Representation

In this section we will introduce the monodromy representation. We will define a local system of rank one, or equivalently a flat line bundle on $\mathbf{P}^1 \setminus S_c$, associated to any given cone metric $c \in C(\kappa)$ and hence obtain another representation of $\pi_1(\mathbf{P}^1 \setminus \{S_c\})$. Let us recall basic definitions first.

Definition 2.3.9 *Let $\mathcal{F}$ be a sheaf on a ringed space (X, O_X), which admits a universal cover. $\mathcal{F}$ is called* ***locally constant*** *if every point $x \in X$ has a neighbourhood U_x such that for every element $y \in U_x$ the natural map*

$$\mathcal{F}(U_x) \longrightarrow \mathcal{F}_y \tag{2.1}$$

is an isomorphism. ■

When X is connected, the dimension of the stalks of $\mathcal{F}$ are same, which is called the *rank* of $\mathcal{F}$. $\mathcal{F}$ is called a *local system* whenever $\mathcal{F}$ is a locally constant sheaf of finite rank. As is well-known, [18, Exercise 5.18], the terms locally free sheaf and vector bundles may be used interchangeably as there is an equivalence in between.

To describe the monodromy representation associated to a local system $\mathcal{F}$ on X, fix a base point $x_0 \in X$, and take any path $\gamma : [0, 1] \longrightarrow X$ in X based at x_0 representing a homotopy class in $\pi_1(X, x_0)$. Cover γ with open sets for which the map in Equation 2.1 is an isomorphism. From compactness of γ reduce to finitely many of them, say $U_1, U_2, \ldots, U_n$. By renumbering, if necessary, we may, without loss of generality, assume that $x_0 \in U_1 \cap U_n$, and for each $i \in \{1, 2, \ldots, n-1\}$, the intersection $U_i \cap U_{i+1} \neq \emptyset$, so that we have an element x_i in each intersection; where we assume

$$0 = \gamma^{-1}(x_0) < \gamma^{-1}(x_1) < \cdots < \gamma^{-1}(x_{n-1}) < 1 = \gamma^{-1}(x_0)$$

We have the following sequence of isomorphisms:

$$\mathcal{F}_{x_0} \xrightarrow{\simeq} \mathcal{F}(U_1) \xrightarrow{\simeq} \mathcal{F}_{x_2} \xrightarrow{\simeq} \cdots \xrightarrow{\simeq} \mathcal{F}_{x_{n-1}} \xrightarrow{\simeq} \mathcal{F}(U_n) \xrightarrow{\simeq} \mathcal{F}_{x_0}$$

giving an automorphism of the stalk $\mathcal{F}_{x_0}$, φ_γ. One may check that the map $\gamma \mapsto \varphi_\gamma$ induces a homomorphism $\pi_1(X) \longrightarrow \mathrm{Aut}(\mathcal{F}_{x_0})$, i.e. a representation of $\pi_1(X)$, see Figure 2.4.

Conversely, let X be a topological space admitting a universal cover, and let a representation $\varphi : \pi_1(X) \longrightarrow \mathrm{Aut}(\mathcal{F}_{x_0})$ be given, where x_0 is an element of X. Let $\widetilde{X}$ denote the universal cover of X so that $\pi_1(X) = \pi_1(X, x_0)$ acts on $\widetilde{X}$ and $\pi_1(X) \setminus \widetilde{X} \cong X$. Consider

$$\widetilde{X} \times_{\pi_1(X)} \mathbf{C}^n := \widetilde{X} \times \mathbf{C}^n / \sim \tag{2.2}$$

where we say that $(x, v) \sim (x', v')$ whenever there is an element $\gamma \in \pi_1(X)$ with the property that $x = \gamma x'$ and $v = \varphi(\gamma)v'$. Then fibers of the natural projection $\pi : \widetilde{X} \times_{\pi_1(X)} \mathbf{C}^n \longrightarrow X$

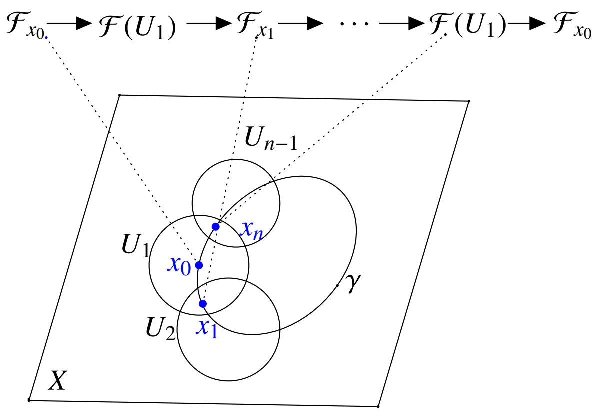


Figure 2.4: Monodromy Representation.

sending each equivalence class $[x, v]$ to the point $x \in X$ are isomorphic to $\mathbf{C}^n$. Thus we obtain a locally constant vector bundle.

Theorem 2.3.10 ([8, Corollaire 1.4]) *When X has universal cover, the functor we described, called the fiber functor at x_0, is an equivalence between the category of local systems of rank r over $\mathbf{C}$ on X and the category of r-dimensional complex representations of $\pi_1(X, x_0)$.*

Example 2.3.11 *Set $\mathbf{P}_c = \mathbf{P}^1 \backslash S_c$, where $c \in C(\kappa)$, as usual. Let γ_i denote a positively oriented curve in $\mathbf{P}_c$ based at p_I that rotates once, positively around p_i for $i \in \{1, \ldots, N\}$. There exists a rank one complex local system $\mathcal{F}_\kappa$ on $\mathbf{P}_c$ which gives multiplication by $e^{\theta_i \sqrt{-1}}$, defined as in Equation 2.2. We will refer to this representation as the* ***monodromy representation associated to*** *c.*

Remark 2.3.12 *When the image of the monodromy representation is an abelian group, for instance when the local system is of rank one, as in Example 2.3.11, the representation we have written factors through the homology, i.e.*

$$\pi_1(X, x_0)^{ab} = \pi_1(X, x_0)/[\pi_1(X, x_0), \pi_1(X, x_0)] = H_1(X, \mathbf{Z}).$$

2.3.3 The Relation between Two Representations

In this part, we will show that the monodromy representation and the holonomy representation are somewhat related. For this, as before we fix a cone metric $c \in C(\kappa)$. It is well-known that there are three types of elements in $\mathrm{Isom}(\mathbb{E}^2)$, namely translations, rotations and reflections. Within these elements the translations form a normal subgroup, say $Tr \triangleleft \mathrm{Isom}(\mathbb{E}^2)$, and the quotient $\mathrm{Isom}(\mathbb{E}^2)/Tr$ is called the orthogonal group, $O(2)$. We are now ready to define:

Definition 2.3.13 *We will call the image of $hol\,(\pi_1(\mathbf{P}_c, p_I))$ in $O(2)$ under the natural projection the **orthogonal part** of the holonomy representation and denote by hol_o the composition.*

■

Take a vertex $p \in v(\mathcal{T}_c)$, i.e. a singular point $p \in S_c$. Let U_p be an open neighbourhood of p so that U_p contains no other singular point of c. Suppose that there are l triangles having p as a vertex with angles at p being $\alpha_1, \ldots, \alpha_l$, see Figure 2.5. If by θ_p we denote the cone angle at p, then we have

$$\theta_p = \sum_{i=1}^{l} \alpha_i.$$

Recall that the generating set for the fundamental group $\pi_1(\mathbf{P}_c, p_I)$ may be chosen as positively oriented simple closed curves that rotates once around every element of S_c. Let γ_p denote the positively oriented simple closed curve which rotates *once* around p. Without loss of generality we may assume that $p_I \in U_p$. It follows that, γ_p is a generator of the local fundamental group $\pi_1(U_p, p_I) \cong \mathbf{Z}$. Then, the pair (T_1, id) is send to $(T_1, \gamma_p) \in \mathbb{H}_\flat$. And hence, the element induced by γ_p, $h_{[\gamma_p]}$ is then nothing but rotation by an angle of θ_p, r_{θ_p}.

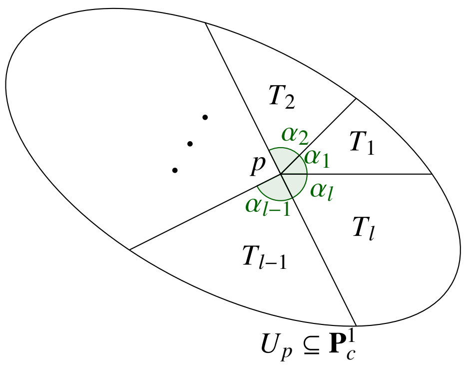


Figure 2.5: Neighbourhood of p.

So, we proved:

Proposition 2.3.14 *The image of $hol_o : \pi_1(\mathbf{P}_c, p_I) \longrightarrow O(2)$ is generated by rotations of angle*

θ_p, *for* $p \in S_c$.

The following proposition implies in particular that the holonomy representation factor through the monodromy representation.

Proposition 2.3.15 *The orthogonal part of the holonomy representation and the monodromy representation associated to c are isomorphic.*

Proof. Consider $f : hol_o(\pi_1(\mathbf{P}_c, p_I)) \longrightarrow \mathrm{GL}_1(\mathbf{C}) \cong \mathbf{C}^\times$ defined as $f(r_{\theta_p}) = e^{\theta_p\sqrt{-1}}$, for every $p \in S_c$. As generators are mapped to generators taking into account the orders, f is an isomorphism. ■

2.4 Combinatorics and Cohomology

In this section, we will begin with introducing a vector space, which is closely related to the space of cone metrics. After recalling the options one has to understand the cohomology of a locally constant sheaf, we shall interpret this vector space as the cohomology of a particular local system. Throughout we fix the curvature parameters $\kappa = (\kappa_1, \dots, \kappa_N) \in \pi \cdot \mathbf{Q}^N \cap (0, 2\pi)$ and assume that they satisfy the Gauss-Bonnet condition, see Theorem 2.1.5.

2.4.1 Cone Metrics as Cocycles

Take any cone metric $c \in C(\kappa)$, $N \geq 3$. If we identify $\mathbb{E}^2$ with $\mathbf{C}$ then one can use the developing map $\widetilde{\varphi}$, see Definition 2.3.6, to associate two complex numbers to each edge, namely the difference between the endpoints, $+$ or $-$ according to orientation. Denote this association by $Z_c : e(\mathcal{T}_c) \times \mathbf{Z}/2\mathbf{Z} \longrightarrow \mathbf{C}$, where the group $\mathbf{Z}/2\mathbf{Z}$ is used for keeping track of the orientation. Observe the following two properties of Z_c:

i. $Z_c(e, +) + Z_c(e, -) = 0$, for every edge e of $\mathcal{T}_c$

ii. if $(e_1, +), (e_2, +), (e_3, +)$ denote the oriented boundary of some triangle in $\mathcal{T}_c$, then $\sum_i Z_c(e_i, +) = 0$.

These properties encourage us to call Z_c a *cocycle*. Remark that such *cocycles*, that is, maps $Z : e(\mathcal{T}_c) \longrightarrow \mathbf{C}$ satisfying i. and ii. above, form a $\mathbf{C}$ vector space, say H_κ, depending on κ. Define the following *hermitian form* on H_κ:

$$A(c) := \frac{1}{4} \sum_{\text{triangles} \in \mathcal{T}_c} Z_c(e_1)\overline{Z_c(e_2)} - \overline{Z_c(e_1)}Z_c(e_2); \tag{2.3}$$

where e_is denote the positively oriented edges of each triangle. Note that the form A defined above is nothing but a measure of the area of a given cone metric on the sphere. We also have:

Proposition 2.4.1 ([45, Propositions 3.2, 3.3]) *The hermitian form A on the vector space H_κ has signature $(1, N-3)$, where N is the number of singular points of c. In particular, the* complex *dimension of the vector space of cocycles is $N-2$.*

Before the proof, we would like to make:

Definition 2.4.2 *For a cone metric $c \in C(\kappa)$, a subset $F \subseteq \mathbb{E}^2$ will be called a* ***euclidean fundamental region****, whenever the followings hold:*

- *F is connected,*
- *the developing map $\widetilde{\varphi_c}$ has a well defined inverse when restricted to F,*
- $\widetilde{\varphi_c}(\mathbf{P}^1_c) = \overline{F}$.

■

Remark 2.4.3 *Using proper identifications of the boundary of F we can easily recover $\mathbf{P}^1$ and c. If singular vertices appear on the boundary of F then the identified boundary components have to have equal length with respect to the metric c.*

Proof.[Proposition 2.4.1]. For any given c we may choose the euclidean fundamental region so that its euclidean boundary consists of edges of the triangulation $\mathcal{T}_c$. Hence F is a polygon, in fact $2(N-1)$-gon, in $\mathbb{E}^2$, not necessarily convex. And Remark 2.4.3 tells us that the lengths of partner edges must agree. So, it is enough to consider the dimension of all such polygons.

The degree of freedom we have is then $N-2$, as whenever we are given $N-2$ edges, we know the lengths of their partner edges as well as the angles in between. The remaining 2 edges are used to satisfy the curvature condition, or Gauß-Bonnet restriction. Hence H has dimension $N-2$. To prove that the signature of the Hermitian form is $(1, N-3)$, first set $N = 4$. Then the value of A, see Equation 2.3, is either positive or negative. Thus, the form is of signature $(1, 1)$. Suppose now that c has $k > 4$ singular vertices. In this case there is always pair of singular vertices, say p and q in S_c so that the sum of cone angles, θ_p and θ_q, at p and q is not equal to 2π. We connect these two edges by a geodesic, e. Let us denote the length of e by $l(e)$. We cut $(\mathbf{P}^1, c)$ along e and replace the vertices p and q by a single vertex, denoted by $p+q$, so that the cone angle θ_{p+q} at the point $p+q$ is $\theta_p + \theta_q$. Call the resulting new pair $(\mathbf{P}^1, c')$. Figure 2.6 shows the effect of the above operation on the euclidean fundamental region. One can then see that the area of $(\mathbf{P}^1, c')$ is equal to the area of $(\mathbf{P}^1, c)$ plus a constant times $l(e)^2$. ■

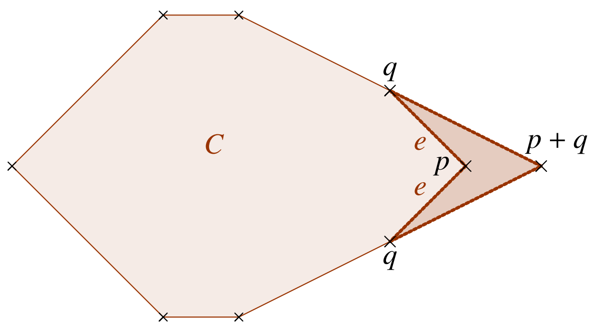


Figure 2.6: Obtaining $(\mathbf{P}^1, c')$ from $(\mathbf{P}^1, c)$.

Remark 2.4.4 *Inside the cocycle space, H_κ, there are elements whose norm is negative, which cannot come from a cone metric, as elements of $C(\kappa)$ automatically have positive area. It is known that, see Section 3.2.1, the projectivization of the set of elements in H_κ which are of positive norm with respect to a hermitian form of signature $(1, N-3)$ form a complex ball inside H_κ of dimension $N-3$, which can be regarded as a model for the complex hyperbolic space, $\mathbf{C}\mathbb{H}^{N-3}$, together with a negatively curved hermitian metric induced by the form A, for details see Section 3.2.1.*

Remark 2.4.5 *One can also compute, with a little bit more work, the signature of the pair (H_κ, A) as follows: Combining Theorem 2.4.12 and Equation 2.6, we obtain the following isomorphism:*

$$H_\kappa \cong H^1(X_c, \mathbf{C})^\chi \tag{2.4}$$

where χ denotes the tautological character the Galois group of an abelian cover of $\mathbf{P}_c$ ramified only over the singular points with compatible *orders. In that case [7] tells us that it is of signature* $(1, N-3)$*. In Section 2.4.3 we will explain how one constructs such covers and computes the signature of the associated Hermitian form from another perspective.*

2.4.2 An Hypercohomology Approach to $H^1(\mathbf{P}_c, \mathcal{F}_\kappa)$

We may define the vector space $H^1(\mathbf{P}_c, \mathcal{F}_\kappa)$ using the usual theory of derived functors, see [18, §§1-4]. One may also take the path to use hypercohomology of the log-complex which we will describe shortly. We will follow [49] to introduce the log-complex, and hypercohomology. Further treatments of cohomology of rank one local systems can be found in [10, §2].

Let us start with a ringed space (X, O_X), together with a sheaf, $\mathcal{F}$, of O_X-modules on X. By $\mathcal{M}\mathrm{od}(X)$ (resp. $\mathcal{A}\mathrm{b}(X)$), denote the category of sheaf of O_X modules(resp. the category of sheaves of abelian groups) on X. As a consequence of [18, Chapter 3, Proposition 2.2], we obtain $\mathcal{M}\mathrm{od}(X) = \mathcal{A}\mathrm{b}(X)$, in particular both categories have enough injectives, i.e. every object of $\mathcal{A}\mathrm{b}(X)$ can be embedded, isomorphically as a sub-object, into an injective object of $\mathcal{A}\mathrm{b}(X)$; where an object I is called *injective* if the functor $\mathrm{Hom}(\cdot, I)$ is exact. An injective resolution of any object A in a category, $\mathcal{A}$, is a complex, $I^\cdot$; $i \geq 0$, with a morphism $\varepsilon : A \longrightarrow I^0$ so that the sequence

$$0 \longrightarrow A \xrightarrow{\varepsilon} I^0 \xrightarrow{d_0} I^1 \xrightarrow{d_1} \dots$$

is exact. When a category has enough injectives then every object in that category has an injective resolution. The i^{th} cohomology object, h^i, is in this case defined as $\ker / \mathrm{im} \subseteq I^i$. It can be shown that, the cohomology objects are independent of the chosen injective resolution. Now, for every object of $\mathcal{A}$ fix an injective resolution and define the i^{th} right derived functor, R^i, of a functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ to associate each object the i^{th} cohomology object of the image of its injective resolution, $R^i F(A) = h^i(F(I^\cdot))$. In particular, the cohomology functors, $H^i(X, \cdot)$, are defined to be the right derived functors of the global section functor $\Gamma(X. \cdot)$ from $\mathcal{M}\mathrm{od}(X)$ to the category of abelian groups, $\mathcal{A}\mathrm{b}$.

Recall that a category, C, is called abelian if the following properties are satisfied by C:

- for any two objects, A, B, of C the morphisms, $\mathrm{Hom}(A, B)$, can be given the structure of an abelian group so that composition is bilinear,

- there is an object which is both initial and terminal,
- finite direct sums exists,
- every morphism has a kernel and a cokernel,
- every monomorphism is the kernel of its cokernel and every epimorphism is the cokernel of its kernel
- every morphism can be factored into an epimorphism composed with a monomorphism.

For instance the category, $\mathscr{A}$b, of abelian groups is an abelian category. Let, now, $\mathscr{A}$ and $\mathscr{B}$ be two abelian categories and suppose that $\mathscr{A}$ has enough injectives. From now on, we assume that for every complex, $\mathbb{A}^{\cdot}$, in a category we have $A^i = 0$ for $i < 0$. Say $(A^{\cdot}, d_A^{\cdot})$ is a complex of objects of $\mathscr{A}$. By assumption, $A^i = 0$ for every $i < 0$. Then [49, Proposition 8.4] assures the existence of a complex $(I^{\cdot}, d_I^{\cdot})$ of injective objects of $\mathscr{A}$ and a morphism of complexes $\phi^{\cdot} : A^{\cdot} \longrightarrow I^{\cdot}$ so that

i. for every i, ϕ^i is injective,

ii. ϕ is a quasi-isomorphism.

Recall that a morphism of complexes is called a quasi-isomorphism if

$$\ker(d_A^i)/\mathrm{im}(d_A^{i-1}) = h^i(A^{\cdot}) \cong h^i(I^{\cdot}) = \ker(d_I^i)/\mathrm{im}(d_I^{i-1}).$$

Now, let $F : \mathscr{A} \longrightarrow \mathscr{B}$ be a left-exact functor. For a complex $A^{\cdot}$ of $\mathscr{A}$ we consider its injective resolution $I^{\cdot}$, together with ϕ as above. Then we define the i^{th} derived object $R^i(F)(A^{\cdot}) = H^i(F(I^{\cdot}))$, which is independent of the resolution, see [49, Proposition 8.6]. Once again, if one chooses $\mathscr{A} = \mathscr{A}\mathrm{b}(X)$, and $F(\cdot) = \Gamma(X, \cdot)$, then the group $R^i(\Gamma)(\mathcal{F}^{\cdot})$, denoted by $\mathbb{H}(X, \mathcal{F}^{\cdot})$; where $\mathcal{F}^{\cdot}$ is a complex of sheaves of abelian groups, is called the *hypercohomology* of the complex $\mathcal{F}^{\cdot}$. In fact, one has a little more freedom as the following proposition suggests:

Proposition 2.4.6 ([16, Lemma, pp.447]) *Let $\phi^{\cdot} : A^{\cdot} \longrightarrow B^{\cdot}$ be a quasi-isomorphism of complexes. Then:*

$$\mathbb{H}(X, A^{\cdot}) \cong \mathbb{H}(X, B^{\cdot}).$$

In what follows we would like to demonstrate the above machinery through some examples.

Example 2.4.7 (The (Holomorphic) deRham Complex) *By* **R** *let us denote the constant sheaf, whose fiber is* **R**, *on X(by abuse of notation) where X is a smooth manifold and let $\mathcal{P}^i_X$ denote the sheaf of smooth i-forms on X. Then the following sequence is a resolution of*

$$0 \longrightarrow \mathbf{R} \longrightarrow \mathcal{P}^0_X \xrightarrow{d} \mathcal{P}^1_X \xrightarrow{d} \dots$$

the constant sheaf **R**. *As a result of [49, Corollary 8.14], we get*

$$H^i(X, \mathbf{R}) \cong \mathbb{H}^i(X, \mathcal{P}^{\cdot}_X).$$

The same argument applies in the case when X is a complex manifold and when we consider the sheaf of holomorphic i-forms, Ω^i_X, as a resolution of the constant sheaf **C**. *Once again we obtain,*

$$H^i(X, \mathbf{C}) \cong \mathbb{H}^i(X, \Omega^{\cdot}_X).$$

Example 2.4.8 (The log-complex) *Let X be an n dimensional complex manifold and D be a* normal crossings *divisor, i.e. every point $p \in D$ has a neighbourhood, $U_p \subseteq X$ and local coordinate system $\{z_1 \dots z_n\}$, so that in U_p the divisor D is given locally by $\Pi_{i=1}^k z_i = 0$, $k < n$. Set $U = X \setminus D$. Let $\Omega^i_X(*D)$ denote the sheaf of i-forms on X which have poles of* finite *order along D. Define a sub-sheaf, $\Omega^i_X(\log D)$, of $\Omega^i_X(*D)$ to be those forms, α, satisfying the following properties:*

i. α has at most first order pole along D,

ii. $d\alpha$ has at most first order pole along D.

In other words, sections of $\Omega^i_X(\log D)(U)$; where $U \subseteq X$ is an open subset, are given locally by

$$\sum_{|I|+|J|=i} f_{I,J} \frac{dz_I}{z_I} \wedge dz_J;$$

where we utilize the multi-index notation and $I \subseteq \{1, \ldots, k\}$, $J \subseteq \{k+1, \ldots, n\}$ *and* $f_{I,J} \in \mathcal{O}_X(U)$. *We thus conclude that the sheaves* $\Omega^i_X(\log D)$ *are, in fact, free* $\mathcal{O}_X$*-modules. We, moreover, have the following:*

$$\Omega^{\cdot}_X(\log D) \hookrightarrow j_*\Omega^{\cdot}_U \hookrightarrow j_*\mathcal{P}^{\cdot}_U.$$

where $j : U \hookrightarrow X$ *denotes the inclusion. From [9, §3.1], one concludes that the morphism* $\Omega^{\cdot}_X(\log D) \longrightarrow j_*\mathcal{P}^{\cdot}_U$ *is a quasi-isomorphism. And Proposition 2.4.6, gives us*

$$\mathbb{H}^{\cdot}(X, \Omega_X(\log D)) \cong \mathbb{H}^{\cdot}(X, j_*\mathcal{P}^{\cdot}_U).$$

We further have $\mathbb{H}^{\cdot}(X, \Omega_X(\log D)) \cong H^{\cdot}(U, \mathbf{C})$, *see [49, Corollary 8.19].*

We end our examples with local systems:

Example 2.4.9 (Local Systems) *Let* $\mathcal{F}$ *be a local system on X. It is well-known that* $\mathcal{F}$ *is a flat vector bundle together with a connection* ∇. *We form the following sequence of sheaves:*

$$0 \longrightarrow \mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{F} \xrightarrow{\nabla} \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{F} \xrightarrow{\nabla} \Omega^2_X \otimes_{\mathcal{O}_X} \mathcal{F} \xrightarrow{\nabla} \ldots$$

where we define our differential operator as $\nabla(\alpha \otimes e) = d\alpha \otimes e + (-1)^{\deg e}\alpha \otimes \nabla(e)$. *We have:*

$$\begin{aligned}
\nabla(\nabla(\alpha \otimes e)) &= \nabla\left(d\alpha \otimes e + (-1)^{\deg e}\alpha \otimes \nabla(e)\right) \\
&= \nabla(d\alpha \otimes e) + (-1)^{\deg e}\nabla(\alpha \otimes \nabla(e)) \\
&= \underbrace{d^2\alpha \otimes e}_{} + \underbrace{(-1)^{\deg e} d\alpha \otimes \nabla(e) + (-1)^{\deg \nabla(e)} d\alpha \otimes \nabla(e)}_{} + \underbrace{(-1)^{\deg e + \deg \nabla(e)}\alpha \otimes \nabla^2(e)}_{} \\
&= \quad 0 \quad + \quad\quad 0 \quad\quad + \quad 0 \\
&= 0.
\end{aligned}$$

So we, indeed, have a complex of sheaves. In fact, the complex $(\Omega^{\cdot}_X \otimes_{\mathcal{O}_X} \mathcal{F}, \nabla)$ *is a resolution of* $\mathcal{F}$, *and hence we get:*

$$\mathbb{H}^i(X, \Omega^{\cdot}_X \otimes_{\mathcal{O}_X} \mathcal{F}) \cong H^i(X, \mathcal{F}).$$

Note here that the same process is still valid if we replace the sheaf of holomorphic differentials, $\Omega^{\cdot}_X$ *with the sheaf of smooth differentials,* $\mathcal{P}^{\cdot}_X$.

Remark 2.4.10 *As far as we are concerned, the pair* $(X, D) = (\mathbf{P}^1, S_c)$ *for* $c \in C(\kappa)$ *satisfies the above condition automatically.*

2.4.3 $H^1(\mathbf{P}_c, \mathcal{F}_\kappa)$ via Abelian Covers

Let $c \in C(\kappa)$ be arbitrary with singular set $S_c = \{p_1, \ldots, p_N\}$. Suppose further that the concentrated curvature at p_i is κ_i, $i = 1, 2, \ldots, N$ with $\sum_{i=1}^N \kappa_i = 4\pi$. As each $\kappa_i \in 2\pi \cdot \mathbf{Q}$, we may write $\frac{\kappa_i}{2\pi} = \frac{\delta_i}{\eta}$, where $\delta_1, \ldots, \delta_N, \eta \in \mathbf{N}$ with the property that the greatest common divisor of $\delta_1, \ldots, \delta_N$ and η is 1. Without loss of generality assume that $p_N = \infty$ and to c we associate the normalization of the plane algebraic curve, X_c, given by the equation:

$$y^\eta = \prod_{i=1}^{N-1} (x - p_i)^{\delta_i}. \tag{2.5}$$

The projection map $(x, y) \mapsto x$ from X_c to $\mathbf{P}^1$ is then ramified precisely over the singular set S_c. And, we get the following commutative diagram:

$$\begin{array}{ccc} X_c \times \mathbf{C} & \longrightarrow & (pr_x)_* \mathbf{C} \\ \downarrow & & \downarrow \\ X_c & \xrightarrow{pr_x} & \mathbf{P}^1 \end{array}$$

Observe that, the sheaf $\mathcal{F}_\kappa$ is nothing but the sheaf of sections of the line bundle $(pr_x)_*\mathbf{C}$ on $\mathbf{P}_c$. Let $\mathrm{Gal}(pr_x)$ denote the Galois group of the covering $X_c \longrightarrow \mathbf{P}^1$, where we define a cover to be *Galois* whenever the corresponding function field extension is Galois. In this case we have a Kummer extension, it is cyclic of order η. Let $\chi\colon \mathrm{Gal}(pr_x) \longrightarrow \mathbf{C}^\times$ denote the tautological character. In this setup χ acts on the space, Ω_{X_c}, of holomorphic one forms on X_c and we have

$$H^1(X_c, \mathbf{C})^\chi \cong H^1(\mathbf{P}_c, \mathcal{F}_\kappa); \tag{2.6}$$

where $H^1(X_c, \mathbf{C})^\chi$ denotes the set of cohomology classes of 1-forms on X_c which are invariant under χ. Moreover, we have the usual cup product on $H^1(X_c, \mathbf{C})$, given by:

$$\alpha \cup \beta = \frac{\sqrt{-1}}{2} \int_{X_c} \alpha \wedge \overline{\beta},$$

which is known to have signature (g, g); where g is the genus of X_c. The multivalued form

$$\omega_c := \prod_{i=1}^{N-1} (x - p_i)^{\delta_i/\eta} dx \tag{2.7}$$

on $\mathbf{P}^1$ pulls back to $pr_x^*(\omega_c) = \frac{dx}{y}$, a single-valued holomorphic form on X_c. For $a \in \mathrm{Gal}(pr_x)$ let us denote the map induced by $\chi(a)$ on Ω_{X_c} by χ_a. Then

$$\chi_a^*(\frac{dx}{y}) = \overline{\chi(a)}\frac{dx}{y}.$$

We conclude that $pr_x^*(\omega_c)$ is an eigenform for $\chi_{a^{-1}}$, i.e. $pr_x^*(\omega_c) \in \Omega_{X_c}^{\chi}$. On the other hand, χ acts on the space of anti-holomorphic one forms, $\overline{\Omega_{X_c}}$, on X_c, and we have a canonical identification

$$\overline{\Omega_{X_c}}^{\chi} \cong \Omega_{X_c}^{\overline{\chi}} = \Omega_{X_c}^{\chi^{\eta-1}}$$

via complex conjugation. To compute the dimension of $\Omega_{X_c}^{\chi^{\eta-1}}$ it is enough to note that only 1-forms of type $f(x)\frac{dx}{y}$ can be an eigenform; where $f(x) \in \mathbf{C}[x]$ of degree less than $N-2$.

Lemma 2.4.11 $\dim_{\mathbf{C}} \Omega_{X_c}^{\chi^{\eta-1}} = N-3$.

Proof. We will prove our claim only for the case where $\delta_i = 1$ for each $i = 1, \dots N-1$. The general case follows the same line of arguments, only somewhat more complicated. To prove, let us show that the set $\mathcal{B} := \{\frac{dx}{y}, x\frac{dx}{y}, \dots, x^{N-2}\frac{dx}{y}\}$ is a basis. $\mathbf{C}$-linear independence of $\mathcal{B}$ is clear. Let $(x)_0 = D$, $(x)_\infty = D' = \sum_{j=1}^{\eta} q_j$ denote the zero and pole divisor of x, respectively. At any ramification point p_i of the projection pr_x, the function $x - p_i$ is locally of order η, hence dx is of order $\eta - 1$. On the other hand at each pole, q_j, of x, the function $x - q_j$ is locally of the form $\frac{1}{z_j}e^h$; where h is a holomorphic function. Thus $(dx)_\infty = 2D'$. The zero divisor of the function y is nothing but the ramification divisor of pr_x, i.e. $(y)_0 = R = \sum_{i=1}^{N-1} p_i$. As $\deg y = N-1$ we must have $(y)_\infty = \frac{N-1}{\eta}D'$. So:

$$\begin{aligned}\left(x^k \frac{dx}{y^{\eta-1}}\right) &= kD - kD' + (\eta-1)R - 2D' - (\eta-1)(R - \frac{N-1}{\eta}D') \\ &= kD - \left((\eta-1)\,\frac{N-1}{\eta} - k - 2\right)D'.\end{aligned}$$

So, we must have $k \geq 0$ and $k < N-3$. Hence the claim follows. ■

2.4.4 *H* and $H^1(\mathbf{P}_c, \mathcal{F}_\kappa)$

After introducing the vector space of cocycles and reviewing cohomology of rank one local systems and various approaches to it, let us state the following:

Theorem 2.4.12 *The vector space H of cocycles and $H^1(\mathbf{P}_c, \mathcal{F}_\kappa)$ are isomorphic.*

Remark 2.4.13 *Theorem 2.4.12 can be considered not only as an explanation of the comment "This turns out to be closely related to work of Picard and Mostow and Deligne." made in [45], but also as a combinatorial description of some cohomology groups.*

Before proving Theorem 2.4.12, we have:

Lemma 2.4.14 ([10, Proposition 2.3.1]) $\dim_{\mathbf{C}}(H^1(\mathbf{P}_c, \mathcal{F}_\kappa) = N - 2$.

Proof. We first replace $\mathbf{P}_c$ by a compact object, Y_c, obtained via removing disks, D_{p_i}, around singular points, $S_c = \{p_1, \ldots, p_N\}$ so that for every distinct $i, j \in \{1, \ldots, N\}$ we have $\overline{D_{p_i}} \cap \overline{D_{p_j}} = \emptyset$. Remark that this process does not change the cohomology groups. As Y_c is compact, we may choose a finite triangulation, $\mathcal{T}_c'$ of Y_c and consider the restriction of the sheaf $\mathcal{F}_\kappa$ to Y_c which will be denoted same by abuse of notation. And we have the isomorphism $H^{\cdot}(\mathbf{P}_c, \mathcal{F}_\kappa) \cong H^{\cdot}(Y_c, \mathcal{F}_\kappa)$. We will see during the proof of Theorem 2.4.12 that the cohomology of Y_c can be described as the cohomology of the complex of $\mathcal{F}_\kappa$-valued cochains of $\mathcal{T}_c'$. This tells us that the rank of the i^{th} cohomology group is independent of the chosen locally constant sheaf. As $\mathbf{P}_c$ is locally contractible, $H^1_{sing}(\mathbf{P}_c, \mathbf{C}) \cong H^1(\mathbf{P}_c, \mathbf{C})$; where H^i_{sing} denotes the singular cohomology and $\mathbf{C}$ is used also to denote the constant sheaf with fiber $\mathbf{C}$ on $\mathbf{P}_c$. As the fundamental group of $\mathbf{P}_c$ is isomorphic to the free group on $N - 2$ letters $\dim_{\mathbf{C}}(H^1_{sing}(\mathbf{P}_c, \mathbf{C})) = N - 2$. ■

Proof.[Theorem 2.4.12] Combining Lemma 2.4.14, Proposition 2.4.1, and Remark 2.4.5 we conclude the result. Nevertheless, in what follows we describe the map explicitly. Let $c \in C(\kappa)$, and consider the associated triangulation $\mathcal{T}_c$ as a CW-complex on $\mathbf{P}^1$. In § 2.4.2 we have seen that we may define $H^1(\mathbf{P}_c, \mathcal{F}_\kappa)$ as the hypercohomology of the the complex of $\mathcal{F}_\kappa$-valued smooth differentials, $\mathcal{P}^{\cdot}_{\mathbf{P}_c} \otimes_{O_X} \mathcal{F}_\kappa$. For an $\mathcal{F}_\kappa$-valued 1-form, and σ an edge of $\mathcal{T}_c$ we define the map:

$$\omega \mapsto Z_\omega(\sigma) := \int_\sigma \omega \tag{2.8}$$

from the complex $\mathcal{P}^{\cdot}_{\mathbf{P}_c} \otimes_{O_X} \mathcal{F}_\kappa$, to the $\mathcal{F}_\kappa$-valued cochains of $\mathcal{T}_c$. It is easy to see that this map is a quasi-isomorphism, i.e. it induces an isomorphism on the cohomology. On the other hand, any cocycle $Z \in H$ is a function from the free abelian group of 1-cells to $\mathbf{C}$. Now, let $p \in S_c$

be any singular point with concentrated curvature κ_p, γ_p be a positively oriented loop around p with winding number 1 and let e_p be an edge of $\mathcal{T}_c$ incident to p. Without loss of generality assume that $Z(p) = 0$. We have:

$$Z(\gamma(e), \pm) = e^{-2\pi\sqrt{-1}\kappa_p} Z(e, \pm) \tag{2.9}$$

so that induces an element $w_Z \in H^1(\mathbf{P}_c, \mathcal{F}_\kappa)$, see § 2.4.1. Say Z and Z' in H induces w_Z and $w_{Z'}$ in $H^1(\mathbf{P}_c, \mathcal{F}_\kappa)$. Then for any edge $e \in e(\mathcal{T}_c)$ we have $w_Z - w_{Z'}(e) = 0$, hence it is a coboundary. The result can be deduced from Lemma 2.4.14. ■

CHAPTER 3

QUADRANGULATIONS OF THE SPHERE AS A LATTICE

In this section, our aim is to obtain a classification of a family of quadrangulations of the sphere satisfying certain curvature conditions. That is: we will obtain a lattice in a specific space of cone metrics whose points parametrize quadrangulations of non-negative curvature. The result we obtain is an analogue of [45, Theorem 0.1]. As before, we assume that our curvature parameters $\kappa_1, \ldots, \kappa_N$ are rational multiples of π lying in the open interval $(0, 2\pi)$. The result of Thurston and ours may be regarded as classification of certain subgroups of the modular group and of the group $\mathbf{Z}/2\mathbf{Z} * \mathbf{Z}/4\mathbf{Z}$, with $*$ denoting the free product, hence the points of the lattices correspond to algebraic curves defined over a number field. One further aspect of the lattice we found in that it also parametrizes certain flat surfaces, called origamis, which corresponds to Teichmüller discs.

3.1 Quadrangulations...

3.1.1 Basic Definitions

Definition 3.1.1 *A (finite) metric (or euclidean) quadrangulation, Q, of the sphere is a family of pairs $(f_i, \Box_i)$, for $i = 1, 2, \cdots, n$, where each $\Box_i$ is a non-degenerate quadrangle in $\mathbb{E}^2$, and each $f_i : \Box_i \longrightarrow S^2$ is an isometry such that:*

i. $\bigcup_{i=1}^{n} f_i(\Box_i) = S^2$,

ii. for $i \neq j$, whenever we have a non-trivial intersection $f_i(\Box_i) \cap f_j(\Box_j) \neq \varnothing$, then this intersection is a subset of the set of edges, $e(Q)$, of Q or a subset of the set of vertices, $v(Q)$, of Q; where we define the edges and vertices of Q in the usual manner, *iii. if an edge $e \in f_i(\Box_i) \cap f_j(\Box_j) \subset e(Q)$, then there exists some element $\gamma_{i,j} \in Isom(\mathbb{E}^2)$ such that $f_i(e) = \gamma_{i,j} \cdot f_j(e)$. If there is more than one edge in the intersection, then the isometry is expected to bring all edges together.*

■

If we have a set of pairs $(f_i, \Box_i)$ which satisfies only the first two properties, i. and ii., then we will refer to this collection as a *quadrangulation*.

Just as in the case of euclidean triangulation, there exists a flat metric on a given euclidean quadrangulation, which in turn induces a complex structure on the sphere, S^2. Hence, we are allowed to consider the sphere with a euclidean quadrangulation as the projective line $\mathbf{P}^1$.

Example 3.1.2 *The first example is, of course, the cube. Suppose each side of the cube is of unit length, so that it gives a euclidean quadrangulation on the sphere. We have 8 vertices, and at each vertex the cone angle is $3\pi/2$, hence at these points the curvature is $2\pi - 3\pi/2 = \pi/2$. Observe here that the sum of curvatures, $8 \cdot \pi/2$, is nothing but $2\pi \cdot \chi(S^2)$, see Figure 3.1. One*

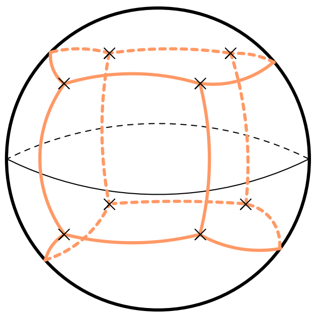


Figure 3.1: The cube as a quadrangulation of S^2.

can extend the examples of this type by an easy iteration progress. Namely, put an extra mid-vertex to the midpoint of each edge of the cube, and connect the new mid-points of opposite edges with a straight line, and introduce a new vertex at the intersection point of the two newly introduced lines. In each of the 6 faces we get now four squares instead of one, each square is of half-unit length. In the next step, we divide each edge into three equal pieces instead of two, and connect the edges which are opposite to each other. The reader is encouraged *to carry on this process. We would like to draw attention of one point here. This family of quadrangulations does not bring anything new in the sense that, all newly introduced vertices are not singular as there are 4 new squares meet at each new vertex appearing in the middle of the faces. Hence the curvature is* 0 *at the these vertices. So, the metric, and hence the complex structure, is only multiplied by an integer constant.*

Motivated by the previous example, we make

Definition 3.1.3 *We will say that a quadrangulation, Q, is non-negatively curved whenever Q has no vertex at which more than four quadrangles meet. In other words for all vertices v of Q, there may meet at most four faces of Q.* ■

In particular, each of the quadrangulations appeared in Example 3.1.2 is of non-negative combinatorial curvature, whereas the quadrangulation in Figure 3.2 contains both positive, zero and negative curvature.

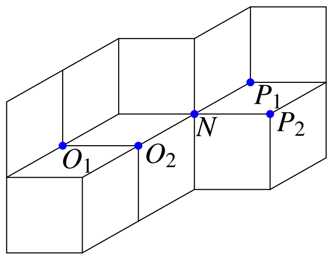


Figure 3.2: An example of a stepped surface, as a quadrangulation.

For future reference we state the following:

Lemma 3.1.4 *Let c be a cone metric on the sphere. Then, there is an associated metric quadrangulation of the sphere.*

Proof. Suppose we are given an element $c \in C(\kappa)$. Let F_c denote the *euclidean fundamental region* corresponding to c, see Definition 2.4.2. Without loss of generality, we assume that the singular vertices, $S_c = \{p_1, \ldots, p_N\}$, appear on ∂F and the boundary segments connecting singular vertices are geodesics with respect to the cone metric c, hence they are, possibly broken, straight lines. We will use induction on the cardinality of S_c. If $N = 3$ then the euclidean fundamental region is itself a quadrangle. For the general case, take 4 consecutive singular points, call p_1, p_2, p_3 and p_4 so that there are no other elements of the singular set on the path from p_1 to p_4 along the boundary of F, which is a $2(N-1)$-gon, see Figure 3.3. Note that possible identifications of the chosen vertices do not pose any problems, for we are only interested in the existence of a quadrangulation. We now connect p_1 to p_4 with a straight line to obtain the first quadrangle. The remaining is now a $2(N-2)$-gon, which, by induction assumption, can be divided into quadrangles finishing the proof. ■

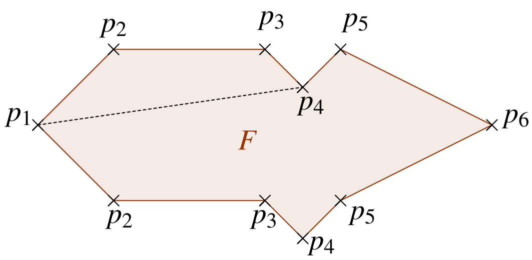


Figure 3.3: Induction step for the case $N = 6$.

3.1.2 Shapes of Quadrangulations in $\mathbb{E}^2$

Let $\mathbf{Z}[\sqrt{-1}]$ be the ring of Gaußian integers considered as as subset of $\mathbb{E}^2$, or equivalently $\mathbf{C}$. In this section, we will analyze Gaußian lattice quadrangles whose sides are parallel to the sides of a standard qaudrangle and whose vertices are at Gaußian integers, to which we will refer simply as a lattice quadrangle, see Figure 3.4 for an example.

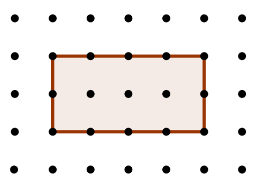


Figure 3.4: A Lattice Quadrangle.

Such an object is given by two parameters, the number of quadrangles in the vertical direction to which we will refer as n_1, and number of quadrangles in the horizontal direction to which we will refer as n_2. Moreover, a quadrangle having n_1 many vertical and n_2 many horizontal has $A(n_1, n_2) = n_1 n_2$ many quadrangles. In this coordinates, however, our area form is not diagonal with respect to this basis. There is a geometric way of achieving this, see Figure 3.5. Given any n_1 and n_2 we consider the following area form:

$$A(n_1, n_2) := \frac{1}{4}\left((n_1 + n_2)^2 - (n_1 - n_2)^2\right).$$

which measures the area of a lattice quadrangulation in terms of number of quadrangles. Note that the area form is of signature $(1, 1)$. One may extend our definition to the case where n_1 are n_2 are real numbers. In that case, of course, the real parameters do not lead to a lattice quadrangulation. So one obtains an $\mathbf{R}$-vector space with a form of signature $(1, 1)$. As $n_1, n_2 \geq 0$, forms a cone, say C, the possible *shapes* of lattice quadrangulations are elements of the projective image of C.

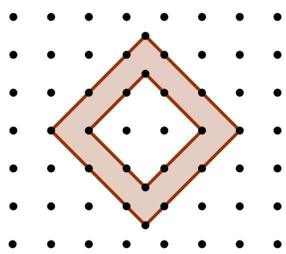


Figure 3.5: Diagonalizing the area form, for $n_1 = 1, n_2 = 4$.

Remark 3.1.5 *Possible shapes of lattice hexagons, i.e. hexagons whose vertices are at the Eisenstein integers,* $\mathbf{Z}[e^{2\pi\sqrt{-1}/3}]$*, sides are parallel to the sides of a standard hexagon are analyzed in [45, §1]. In the case when* $\mathbf{R}$ *is replaced by* $\mathbf{C}$*, the space that one obtains is a hermitian form on* $\mathbf{C}^{1,1}$.

3.2 ...as a Lattice

In this section we will generalize the results of Section 3.1.2 to shapes of quadrangulations of the sphere. We are going to prove that quadrangulations of the sphere are given by a lattice inside a complex Lorentzian vector space. In order to do so, we will recall basics of complex hyperbolic geometry.

3.2.1 Complex Hyperbolic Geometry

We are going to describe the ball model of the n dimensional complex hyperbolic space. For this let $\mathbf{C}^{1,n}$ be the vector space of dimension $n+1$ over $\mathbf{C}$, which consists of $n+1$-tuples $Z = (Z_1, \ldots, Z_n, Z_{n+1})$ together with the following Hermitian form

$$\langle Z, W \rangle := Z_{n+1}\overline{W_{n+1}} - \sum_{i=1}^{n} Z_i \overline{W_i}. \tag{3.1}$$

We will call a vector, Z *negative*, *null*, *positive* whenever $\langle Z, Z\rangle < 0$, $\langle Z, Z\rangle = 0$, $\langle Z, Z\rangle > 0$, respectively. Observe that if a vector, Z, is negative, null, positive then for any $\lambda \in \mathbf{C} \setminus \{0\}$, λZ is also negative, null, positive, respectively. So the following is well defined:

Definition 3.2.1 *The complex hyperbolic space $\mathbb{H}^n_{\mathbf{C}}$ is defined to be the set of all positive* lines *in the projectivization, $\mathbb{P}(\mathbf{C}^{1,n})$, of the vector space $\mathbf{C}^{1,n}$.* ■

One can give a complex manifold structure to $\mathbb{P}(\mathbf{C}^{1,n})$ if one considers the usual quotient map $\mathbf{C}^{1,n} \setminus \{\overrightarrow{0}\} \longrightarrow \mathbb{P}(\mathbf{C}^{1,n})$. Thus we may regard $\mathbb{H}^n_{\mathbf{C}}$ as a complex manifold. Consider $\mathbf{C}^n$ with its usual Hermitian form:

$$\langle\langle z, w\rangle\rangle := \sum_{i=1}^{n} z_i \overline{w_i}.$$

By $\mathbb{B}^n$ denote the set of all elements, $z \in \mathbf{C}^n$, with $\langle\langle z, z\rangle\rangle < 1$. Define the map $\Xi\colon \mathbf{C}^n \longrightarrow \mathbb{H}^n_{\mathbf{C}}$ as:

$$z = (z_1, \ldots, z_n) \mapsto [z_1 : \ldots : z_n : 1].$$

The map Ξ embeds $\mathbf{C}^n$ onto the subset of $\mathbb{P}(\mathbf{C}^{1,n})$ defined by $Z_{n+1} \neq 0$. Moreover, for any element $z \in \mathbb{B}^n$ we have

$$\begin{aligned}\langle \Xi(z), \Xi(z)\rangle &= \langle [z_1 : \ldots : z_n : 1], [z_1 : \ldots : z_n : 1]\rangle \\ &= 1 - \sum_{i=1}^{n} z_i \overline{z_i} \\ &> 0.\end{aligned}$$

So $\Xi(\mathbb{B}^n) = \mathbb{H}^n_{\mathbf{C}}$, and as Ξ is a holomorphic embedding $\mathbb{B}^n$ and $\mathbb{H}^n_{\mathbf{C}}$ are complex analytically isomorphic.

Example 3.2.2 *This situation has already appeared in Section 3.1.2 where it is proved that possible shapes of quadrangulations of lattice quadrangles in $\mathbb{E}^2$ are parametrized by a cone C inside the projectivization of $\mathbf{R}^{1,1}$. It is, in fact, an example of the above machinery except the base field was $\mathbf{R}$ instead of $\mathbf{C}$.*

Remark 3.2.3 *We have proven in Proposition 2.4.1 that we have a complex vector space, H, of cocycles with a Hermitian form, A of signature $(1, N-3)$, where N is the number of singular points. Recalling that the form A is a measure of the area associated to a cocycle,* *to get an honest cone metric we have to consider only those cocycles, $Z \in H$, for which we have $A(Z) > 0$. In addition, the multiplicative group $\mathbf{C}^\times$ acts on the space of cocycles, whose orbits include elements which differ by a rotation followed by a dilation. The quotient is then nothing but the image of the complex ball inside the projectivization of* complex Lorentzian *vector space $\mathbf{C}^{1,N-3}$. This, in particular, implies that the space of cone metrics for given curvature parameters, $C(\kappa)$, is a complex hyperbolic manifold.*

3.2.2 Non-negatively Curved Quadrangulations of the Sphere

A lattice, Λ, in a vector space V is a free $\mathbf{Z}$-module together with a symmetric bilinear form, $\langle\cdot,\cdot\rangle$. More generally, an Eisenstein(respectively Gaußian) lattice is a free $\mathbf{Z}[e^{2\pi\sqrt{-1}/3}]$-module(respectively $\mathbf{Z}[\sqrt{-1}]$-module) with a Hermitian form. Λ is called integral whenever the Hermitian form takes values in $\mathbf{Z}[e^{2\pi\sqrt{-1}/3}]$(respectively in $\mathbf{Z}[\sqrt{-1}]$). Every lattice comes with a group, namely the group of its symmetries. The *automorphism group*, $\mathrm{Aut}(\Lambda)$, of Λ is defined to be the set of isometries of the vector space V fixing $\overrightarrow{0} \in V$, and send Λ to Λ. We would like to note here two equivalences. Given an Eisenstein lattice multiplication by the element $e^{2\pi\sqrt{-1}/3}$ is an order 3 automorphism of the lattice Λ fixing only $0 \in \Lambda$. Hence the associated $\mathbf{Z}$-lattice, $\Lambda_{\mathbf{Z}}$ has an automorphism of order 3 with a single fixed point. Conversely, if we start with a $\mathbf{Z}$-lattice, $\Lambda_{\mathbf{Z}}$ with an automorphism, ϑ, of order 3 fixing only $0 \in \Lambda_{\mathbf{Z}}$, then we may define a Hermitian, $h(Z, W)$, form on Λ as:

$$h(Z_{\mathbf{Z}}, W_{\mathbf{Z}}) := \frac{3}{2}\left(\langle Z_{\mathbf{Z}}, W_{\mathbf{Z}}\rangle + \frac{\sqrt{-1}}{\sqrt{3}}\langle Z_{\mathbf{Z}}, \vartheta(W_{\mathbf{Z}}) - \vartheta^2(W_{\mathbf{Z}})\rangle\right)$$

where $\frac{1}{\sqrt{3}}\left(\vartheta\left(W_{\mathbf{Z}}\right) - \vartheta^2\left(W_{\mathbf{Z}}\right)\right)$ is nothing but the complex structure on $\Lambda_{\mathbf{Z}} \otimes_{\mathbf{Z}} \mathbf{R}$. Similarly, a Gaußian lattice possesses an automorphism of order 4 fixing only 0: multiplication by $\sqrt{-1}$. Repeating the above arguments we conclude that a $\mathbf{Z}$-lattice with an automorphism of oder 4 fixing the origin is equivalent to a Gaußian lattice.

We state now the following:

Theorem 3.2.4 ([45, Theorem 0.1]) *There is an integral* ***Eisenstein*** *lattice Λ in $\mathbf{C}^{1,9}$ and a subgroup, Γ, of $Aut(\Lambda)$ such that Λ_+/Γ parametrizes non-negatively curved triangulations of the sphere which have 5 triangles meeting at 12 marked vertices; where Λ_+ is the set of lattice points with positive square-norm, denoting the number of triangles in the triangulation. The quotient of $\mathbb{H}^9_{\mathbf{C}}$ by the action of Γ has finite volume.*

Analogously, we have:

Theorem 3.2.5 *There is an integral **Gaußian** lattice, Λ' in $\mathbf{C}^{1,5}$ a subgroup, Γ', of $Aut(\Lambda)$ such that Λ'_{+}/Γ' parametrizes non-negatively curved quadrangulations of the sphere having 3 quadrangles that meet at 8 marked vertices; where Λ'_{+} is the set of lattice points with positive square-norm, which is the number of quadrangles in the quadrangulation. The quotient of $\mathbb{H}^5_{\mathbf{C}}$ by the action of Γ' has finite volume.*

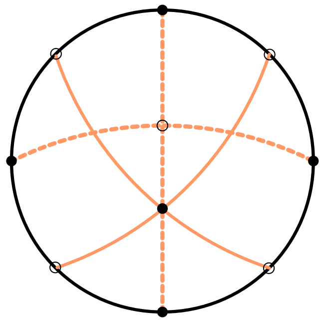


Figure 3.6: A sample element of Λ'.

Given a cone metric c, Lemma 3.1.4 provides us with a metric quadrangulation. And given a metric quadrangulation, Q, of the sphere, for every quadrangle in Q we draw one of the two diagonals so as to obtain a triangulation, $\mathcal{T}_Q$. There are $2^{|f(Q)|}$ distinct choices for $\mathcal{T}_Q$; where $f(Q)$ denotes the set of faces of a quadrangulation. We, however, have:

Lemma 3.2.6 *$A(\mathcal{T}_Q)$ is independent of the choice of $\mathcal{T}_Q$; where A is the hermitian form defined on the vector space of cocycles, see Equation 2.3 for the definition of A, and by abuse of notation we write $A(\mathcal{T}_Q)$ to denote the* area *of the cocycle associated to the metric triangulation $\mathcal{T}_Q$.*

Proof. It is enough to concentrate on one quadrangle. Let $\omega_1, \omega_2, \omega_3, \omega_4$ denote the edges and d_1, d_2 denote the two possible diagonals of a single quadrangle $q \in f(Q)$, see Figure 3.7. Let us denote by A_i the value of the hermitian form obtained by subdividing q using d_i, $i = 1, 2$

and write:

$$\begin{aligned}
A_1 - A_2 = {} & [\omega_1(\overline{-\omega_2}) - \overline{\omega_1}(-\omega_2) + \omega_4(\overline{-\omega_3}) - \overline{\omega_4}(-\omega_3)] - \\
& [\omega_3(\overline{-\omega_1}) - \overline{\omega_3}(-\omega_1) + \omega_2(\overline{-\omega_4}) - \overline{\omega_2}(-\omega_4)] \\
= {} & -\omega_1\overline{\omega_2} + \overline{\omega_1}\omega_2 - \omega_4\overline{\omega_3} + \overline{\omega_4}\omega_3 + \\
& \omega_3\overline{\omega_1} - \overline{\omega_3}\omega_1 + \omega_2\overline{\omega_4} - \overline{\omega_2}\omega_4 \\
= {} & -\omega_1(\overline{\omega_2} + \overline{\omega_3}) + \overline{\omega_4}(\omega_2 + \omega_3) + \overline{\omega_1}(\omega_2 + \omega_3) - \omega_4(\overline{\omega_2} + \overline{\omega_3}) \\
= {} & (\omega_2 + \omega_3)(\overline{\omega_1} + \overline{\omega_4}) - (\overline{\omega_2} + \overline{\omega_3})(\omega_1 + \omega_4) \\
= {} & -(\omega_1 + \omega_4)(\overline{\omega_1 + \omega_4}) + (\overline{\omega_1 + \omega_4})(\omega_1 + \omega_4), \text{ as } \sum_{i=1}^{4} \omega_i = 0. \\
= {} & 0.
\end{aligned}$$

■

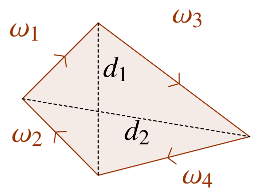


Figure 3.7: A quadrangle, q, may be divided into two triangles using both d_1 and d_2.

Proof.[Theorem 3.2.5] Let us choose $\kappa_i = \pi/2$ for $i \in \{1, 2, \ldots, 8\}$ as curvatures, see Theorem 2.1.6. The vector space of cocycles, H, associated to chosen curvature parameters has signature $(1, 8-3)$ by Proposition 2.4.1. We would like to note at this point that as a consequence of Lemma 3.2.6, we may and will write $A(Q)$ for the value of the hermitian form on a euclidean quadrangulation, instead of a euclidean triangulation. Now, let Q be a non-negatively curved quadrangulation of the sphere having 8 marked vertices at which exactly 3 quadrangles meet. To Q we associate the cone metric, c_Q, on S^2 obtained by declaring that every quadrangle of Q is a unit square. The cocycle, Z_{c_Q}, associated to c_Q is by its very definition an element of H. Moreover, as every $q \in f(Q)$ is a unit square, the difference between the endpoints of the edges under the developing map, see Definition 2.3.6, are naturally elements of $\mathbf{Z}[\sqrt{-1}]$. Let Λ' denote the set of all cocycles. Multiplying and $c \in \Lambda'$ by an element of $\mathbf{Z}[\sqrt{-1}]$ produces an element of Λ'. Finally, any two elements of Λ', say c_1 and c_2 gives us the following sum:

$$A(c_1, c_2) = \sum_i Z_{c_1}(e_i)\overline{Z_{c_2}(e_i)} - \overline{Z_{c_1}(e_i)}Z_{c_2}(e_i)$$

each of whose elements are in $\mathbf{Z}[\sqrt{-1}]$, hence the sum is an element of $\mathbf{Z}[\sqrt{-1}]$. ■

Proof.[Theorem 3.2.4, Sketch] Following the same lines of the proof of Theorem 3.2.5, we choose $\kappa_i = \pi/3$, $i = 1, \ldots, 12$ as curvature parameters and consider the vector space of cocycles, H, associated to these parameters. For any given triangulation, $\mathcal{T}$, we declare that each triangle is equilateral of unit side length in order to obtain a euclidean triangulation. We then consider the associated cone metric, $c_{\mathcal{T}}$, which is by construction an element of H. The *Eisenstein* lattice, Λ, is comprised of all such triangulations inside H which is of signature $(1, 12-3)$. ■

Let us now concentrate on the lattice Λ. One has the following inclusion relations:

$$\begin{array}{ccc} \Lambda_+ & \subseteq & \mathbf{C}^{1,9} \\ \downarrow & & \downarrow \\ \mathrm{Proj}(\Lambda_+) & \subseteq & \mathrm{Proj}(\mathbf{C}^{1,9}) \end{array}$$

Let now $\overline{Z}$ be a cocycle in $\mathrm{Proj}(\Lambda_+)$. Then the elements above $\overline{Z}$ may be obtained by subdivision, see Figure 3.8 for an example.

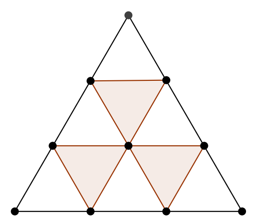


Figure 3.8: Sub-dividing edges of a triangle.

Remark 3.2.7 *As in the case of quadrangulations of lattice quadrangles, the possible* shapes *of quadrangulations of the sphere is obtained via considering the action of* $\mathbf{C}^\times$ *on* H*, after considering the cocycles with positive norm.*

We end this section with two aspects of Theorem 3.2.4, and Theorem 3.2.5, both of which are related to the absolute Galois group, Gal($\mathbf{Q}$). The first one is that, by dualizing the triangulation, of quadrangulation, one obtains a bipartite graph on S^2. This way, each point of

$\mathrm{Proj}(\Lambda_+)$ and $\mathrm{Proj}(\Lambda'_+)$ may be considered as an arithmetic curve, or a genus zero subgroup of $\mathrm{PSL}_2(\mathbf{R})$.

To demonstrate another aspect we make a little pause, and introduce origamis and Veech groups, see [39] or [24] for further details:

Definition 3.2.8 *An origami is defined to be a finite set of Euclidean squares of side length one that are glued according to following set of rules:*

i. every left edge is identified with a right edge(by a translation),

ii. every upper edge is identified with a lower one(by a translation),

iii. the closed surface obtained after the identifications is oriented and connected.

■

The simplest origami, which we call E^*, is obtained by considering only one unit square. The above rules leaves us no choice but to glue the upper edge with the lower one, and left edge with the right one. Hence, if we mark a vertex of the square, then every other vertex of the square has the same marking, and we get a punctured, or marked, torus, see Figure 3.9.

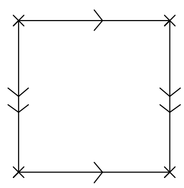


Figure 3.9: The simplest origami, E^*.

For an arbitrary origami, if one marks the vertices considering the identifications then, one gets a ramified covering of E^*, which is unramified away from the vertices. A surface together with a complex atlas whose every transition function is a translation is called a *translation surface*. If one identifies $\mathbb{E}^2$ with $\mathbf{C}$ then to every origami, one associated a translation surface, which becomes a Riemann surface under $\mathbb{E}^2 \cong \mathbf{C}$. For a translation surface, call X, we define the associated *affine group* as:

$$Aff(X) := \{\sigma\colon X \longrightarrow X \,|\, \sigma \text{ is an affine diffeomorphism preserving orientation}\}. \tag{3.2}$$

In other words, σ can be locally written as $Az + t$, for some $A \in \mathrm{GL}_2(\mathbf{R})$ and $t \in \mathbf{C}$. When X is of finite volume, the matrix $A \in \mathrm{SL}_2(\mathbf{R})$. Also, for any matrix $B \in \mathrm{SL}_2(\mathbf{R})$ one gets another Riemann surface structure, which is essentially the same structure whenever $B \in \mathrm{SO}_2(\mathbf{R})$. Hence the embedding $\mathrm{SL}_2(\mathbf{R}) \hookrightarrow \mathcal{T}_{g,N}$ factors through the quotient $\mathbb{H} \cong \mathrm{SL}_2(\mathbf{R})/\mathrm{SO}_2(\mathbf{R}) \hookrightarrow \mathcal{T}_{g,N}$; where X is a surface of genus g with N punctures and $\mathcal{T}_{g,N}$ stands for the Teichmüller space of genus g surfaces with N punctures. The embedding is an isometry with respect to the Poincaré metric on $\mathbb{H}$ and Teichmüller metric on $\mathcal{T}_{g,N}$, and the image is called a *Teichmüller disc*, which is geodesic, see [14]. In view of the above constructions, every origami is represented by a point Λ'_+; hence we conclude that Λ'_+ parametrizes Teichmüller discs in $\mathcal{T}_{g,N}$ for any g and N corresponding to curves having exactly 8 points at which meets 3 squares instead of 4.

CHAPTER 4

TWO APPLICATIONS

This section is devoted to demonstrate two arithmetical applications of the theory we have developed so far. After a brief overview of machinery to be used, we demonstrate an idea to find $\overline{\mathbf{Q}}$-rational points on moduli of pointed rational curves after considering a particular case. We will, then consider another application concerning the action of the absolute Galois group, Gal($\mathbf{Q}$), on dessins d'enfants. Namely, we will employ a technique used already in [53], and we further introduce a family of rational functions, what we call Gausß- Chebyshev functions, which are at the same time Belyĭ morphisms.

4.1 Rational Points on Moduli of Pointed Rational Curves

4.1.1 Configuration Spaces and Braid Groups

Let us recall basic definitions for sake of fixing notation. Let M denote a (real or complex) manifold of dimension dimension at least 2. The n-dimensional *fat diagonal* of M is defined to be

$$\Delta_M^n := \{(x_1, \dots, x_n) \in M^n : \ x_i = x_j \text{ for some } i \neq j\}.$$

The configuration space, $C_n M$, of n points of M may then be defined as:

$$\begin{aligned} C_n M &:= \left(M^n \setminus \Delta_M^n\right) / \sim \\ &= \{(x_1, \dots, x_n) \in M^n : \ x_i \neq x_j \text{ for every } i \neq j\} / \sim; \end{aligned}$$

where we call two points in $M^n \setminus \Delta^n_M$ equivalent whenever one may be obtained as a permutation of coordinates of the other. Namely, the natural projection from $M^n \setminus \Delta^n_M$ to $C_n(M)$ is a Galois covering whose Galois group is Σ_n, symmetric group on n letters. Fundamental groups of configuration spaces play a central role as we will see:

Example 4.1.1 *Let us choose $M = \mathbf{C}$. The fundamental group of the configuration space of n points on $\mathbf{C}$(or $\mathbb{E}^2$), $C_n\mathbf{C}$, is called the* ***Artin (or full) braid group****. The natural covering $M^n \setminus \Delta^n_M \longrightarrow C_nM$ is, as noted above, regular covering, in other words $\pi_1(C_nM)$ is an Σ_n extension of $\pi_1(M_n \setminus \Delta^n_M)$. Precisely, $\pi_1(C_n(M))\backslash\pi_1(M^n \setminus \Delta^n_M) \cong \Sigma_n$. The group $\pi_1(M_n \setminus \Delta^n_M)$ is called the* ***pure braid group****.*

A presentation of the full braid group is very well-known:

Theorem 4.1.2 ([3]) *The full braid group on n strings is generated by σ_i, see Figure 4.1 for a geometric description, subject to the following relations*

i. $\sigma_i\sigma_j = \sigma_j\sigma_i$ *for* $|i - j| \geq 2$, $i, j \in \{1, \ldots, n-1\}$,

ii. $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$, *for* $i \in \{1, \ldots, n-2\}$.

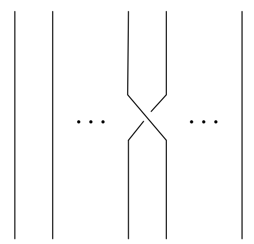


Figure 4.1: Geometric description of σ_i.

Example 4.1.3 *Let now M be the sphere. For any given pair of points in S^2, say z and w, and any path, $\gamma(t) = (z(t), w(t))$, based at $(z(0), w(0)) = (z, w)$, the following argument tells us that $\pi_1((S^2)^2 \setminus \Delta^n_{S^2})$, the pure braid group of the sphere on two strings, is trivial. Indeed, $z \neq w$ allows us to write continuous functions $z_s(t)$, $w_s(t)\colon [0,1] \times [0,1] \longrightarrow (S^2)^2 \setminus \Delta^n_{S^2}$ so that*

i. for fixed $s_o \in [0,1]$, $\gamma_{s_o} = (z_{so}, w_{s_o})$ is a path in $(S^2)^2 \setminus \Delta^n_{S^2}$, *ii.* $z_1(t) = z(0)$*, and* $w_1(t) = w(0)$ *for every* $t \in [0,1]$*, i.e.* $\gamma_1(t)$ *is the trivial path.*

More explicitly, for every $z(t)$*, respectively* $w(t)$*, define the functions* $z_s(t) = sz(t) + (1-s)z(0))$*, respectively* $w_s(t) = sw(t) + (1-s)w(0))$*. Then the homotopy* $\gamma_s(t) = (z_s(t), w_s(t))$ *is a path in* $(S^2)^2 \setminus \Delta^2_{S^2}$*. Regularity of the covering* $C_2S^2 \longrightarrow \left(\left(S^2\right)^2 \setminus \Delta^2_{\mathbf{C}}\right)$*, see Example 4.1.1, tells us that the full braid group of the sphere,* $\pi_1(C_2S^2)$*, is isomorphic to* $\Sigma_2 = \mathbf{Z}/2\mathbf{Z}$*.*

More generally, we have

Theorem 4.1.4 ([11]) $\pi_1(C_nS^2)$ *may be generated by* $n-1$ *elements,* $\delta_1, \ldots \delta_{n-1}$*, subject to:*

i. $\delta_i\delta_j = \delta_j\delta_i$ *for* $|i-j| \geq 2$*,* $i, j \in \{1, \ldots, n-1\}$*,*

ii. $\delta_i\delta_{i+1}\delta_i = \delta_{i+1}\delta_i\delta_{i+1}$*, for* $i \in \{1, \ldots, n-2\}$*,*

iii. $\delta_1 \ldots \delta_{n-1}\delta_{n-1} \ldots \delta_1 = 1$*.*

4.1.2 Mapping Class Groups and Teichmüller Spaces

Let $S_{g,N}$ denote a surface of genus $g \in \mathbf{Z}_{\geq 0}$ with $N \in \mathbf{Z}_{\geq 0}$ marked points(or punctures), $\{p_1, \ldots, p_N\}$. We list now some groups of self diffeomorphisms of $S_{g,N}$:

1. $Diff^+(S_{g,N})$:=orientation preserving self diffeomorphisms of $S_{g,N}$ which maps the N marked points to itself set-wise,
2. $Diff_0^+(S_{g,N})$:=elements of $Diff^+(S_{g,N})$ which are isotopic to $id\colon S_{g,N} \longrightarrow S_{g,N}$ on $S_{g,N} \setminus \{p_1, \ldots, p_N\}$.

There are infinitely many inequivalent complex structures on $S_{g.N}$ provided $\chi(S_{g,N}) \geq 1$ except for the case when $S_{g,N} = \mathbf{P}^1 \setminus \{0, 1, \infty\}$.

Definition 4.1.5 *The* ***mapping class group*** *or* ***(Teichmüller) modular group*** *of the surface* $S_{g.N}$*, denoted* $\Gamma_{g,N}$*, is defined to be the quotient* $Diff^+(S_{g,N})/Diff_0^+(S_{g,N})$*.* ■

We may now state:

Theorem 4.1.6 ([6, Theorem 4.5]) *For $N \geq 3$ the mapping class group of the sphere minus N points may be generated by $\rho_1, \ldots \rho_{n-1}$ satisfying the following set of relations:*

i. $\rho_i\rho_j = \rho_j\rho_i$ *for* $|i - j| \geq 2$, $i, j \in \{1, \ldots, n-1\}$,

ii. $\rho_i\rho_{i+1}\rho_i = \rho_{i+1}\rho_i\rho_{i+1}$, *for* $i \in \{1, \ldots, n-2\}$,

iii. $\rho_1 \ldots \rho_{n-1}\rho_{n-1} \ldots \rho_1 = 1$,

iv. $(\rho_1 \ldots \rho_{n-1})^n = 1$.

Two remarks are in order:

Remark 4.1.7 *In fact, the generators, as in the case of Artin braid group, admit a geometric description. Let $\mathbb{D}_i$ denote a disc containing no marked points other than p_i and p_{i+1}. There exists a self diffeomorphism, call f_i, of S^2 which interchanges p_i with p_{i+1} leaving $S^2 \setminus \mathbb{D}_i$ fixed. The maps f_i may be identified with ρ_i. Moreover, the functions f_i may be regarded as Dehn twists along a curve containing p_i and p_{i+1}.*

Remark 4.1.8 *It is known that, [6, Lemma 4.2.3], the center of the full braid group of S^2 on N strings, $\pi_1(C_N S^2)$, is generated by the element*

$$(\delta_1 \ldots \delta_{N-1})^N$$

which is of order 2.

4.1.3 Appell - Lauricella Functions

Let $p_1, \ldots, p_N$ be distinct points on the projective line, $\mathbf{P}^1_{\mathbf{C}}$, together with positive rational numbers $\mu_1 \ldots, \mu_N$, to which we will refer as weights, with the property that $\sum_{i=1}^N \mu_i = 1$. For technical reasons we assume that μ_is are not integers. Consider the following differential

$$\omega = \prod_{i=1}^{N-1} (x - p_i)^{-\mu_i}\, dx. \tag{4.1}$$

Remark 4.1.9 *One can choose a linear fractional transformation to reduce the above differential form in the following form:*

$$\omega_{norm} = x^{-\mu_1}(x-1)^{-\mu_2}\prod_{i=3}^{N-1}(x-p_i)^{-\mu_i}\,dx, \tag{4.2}$$

as $PGL_2(\mathbf{C})$ *acts* 3*-transitively on* $\mathbf{P}^1$.

Definition 4.1.10 *The* ***Appell - Lauricella*** *function in variables* $p_3, \ldots, p_{N-1}$*, see Remark 4.1.9, is defined to be the integral:*

$$\int \omega.$$

■

Remark 4.1.11 *The above integral is multi-valued. One has to choose a branch, which can be taken to be the set of all* x *so that argument of* x *lies in* $(-\pi/\sum_{i=1}^{N-1}\mu_i, \pi/\sum_{i=1}^{N-1}\mu_i)$.

Choose the *least possible* positive integer η with the property that $\eta\mu_i \in \mathbf{N}$ for each $i \in \{1, \ldots, N\}$. Then the pull-back of ω on the normalization of the curve with affine equation $y^\eta = \prod_{i=1}^{N-1}(x-p_i)^{\eta\mu_i}$ is a differential of the first kind.

Remark 4.1.12 *The above theory is closely related to hypergeometric differential equations. More precisely, suitably chosen Pochhammer cycles, [37], in x-coordinate around singular points* $\{0, 1, p_3, \ldots, p_{N-1}, \infty\}$*, say* $\gamma_1, \ldots, \gamma_{N-2}$*, produces the solutions of the hypergeometric differential equation corresponding to the above system. Namely, the functions* $\int_{\gamma_i}\omega$ *forms a basis for the solution space for such a system. This implies, in particular that the map* $\mathfrak{Sch}$ *defined as:*

$$p = (p_3, \ldots, p_{N-1}) \mapsto \left[\int_{\gamma_1}\omega_{norm} : \ldots : \int_{\gamma_{N-2}}\omega_{norm}\right] \in \mathbf{P}^{N-3}$$

is well defined and is called the ***Schwarz map****. Remark also that the above integrals are periods of the curve* $y^\eta = \prod_{i=1}^{N-1}(x-p_i)^{\eta\mu_i}$*. Further, the image, as we have noted in Remark 3.2.3,* $N-3$ *dimensional complex ball inside* $\mathbf{P}^{N-3}$ *on which acts the monodromy group, denoted by* $\Delta(\kappa_1, \ldots, \kappa_N)$.

Schwarz map is multivalued. Depending on the corresponding monodromy group the arithmeticity of the values of $\mathfrak{Sch}$ may be determined in view of the following theorem which is a corollary to [42, Main Theorem].

Theorem 4.1.13 *If the complex numbers $p_3, \ldots, p_{N-1} \in \overline{\mathbf{Q}}$ and the monodromy group is arithmetic then the Prym variety associated to $y^\eta = \prod_{i=1}^{N-1}(x-p_i)^{\eta\mu_i}$, which we explain below, has complex multiplication if and only if all periods*

$$\int_{\gamma_1} \omega_{norm}, \ldots, \int_{\gamma_{N-2}} \omega_{norm}$$

are $\overline{\mathbf{Q}}$ multiples of each other.

Jacobian and Prym Varieties. Let X be a smooth projective algebraic curve of genus g. Every element $\gamma \in H_1(X, \mathbf{Z})$ can be realized as an element of the dual of $H^0(X, \Omega_X)$, sending every ω to $\int_\gamma \omega$. The Jacobian of X, denoted by $\mathrm{Jac}(X)$ is the quotient $H^0(X, \Omega_X)^*/H_1(X, \mathbf{Z})$. Let $\mathrm{Pic}^0(X)$ denote divisors on X which are linearly equivalent to 0 modulo linear equivalence. Any divisor, $\sum_i p_i - \sum_i q_i$, in $\mathrm{Pic}^0(X)$ can be mapped into $\mathrm{Jac}(X)$ via $(\sum_i \int_{q_i}^{p_i} \omega_1, \ldots, \sum_i \int_{q_i}^{p_i} \omega_g)$. which is an isomorphism(Abel - Jacobi Theorem). We may thus identify $\mathrm{Jac}(X)$ with $\mathrm{Pic}^0(X)$. Suppose now we are given a surjective morphism of smooth projective curves $f\colon X \longrightarrow Y$. The norm map, N_f, from $\mathrm{Jac}(X)$ to $\mathrm{Jac}(Y)$ is defined as:

$$O_X\left(\sum_i n_i p_i\right) \mapsto O_Y\left(\sum_i n_i f(p_i)\right).$$

The **Prym variety** associated to the map f is the connected component of 0 in $\ker N_f$. Let us now restrict ourselves to the case where X is the curve given by the equation $y^\eta = \prod_{i=1}^{N-1}(x-p_i)^{\eta\mu_i}$ and Y_d is given by $y^d = \prod_{i=1}^{N-1}(x-p_i)^{\eta\mu_i}$; where d runs over all *proper* divisors of η. Then, we define the **Prym variety associated to** X as the connected component of 0 in $\bigcap_d \ker N_{f_d}$ where:

$$\begin{aligned} f_d\colon \qquad X &\longrightarrow Y_d \\ (x,y) &\mapsto (x, y^{\eta/d}) \end{aligned}$$

with d being a *proper* divisor of η.

4.1.4 From Lattices to $\overline{\mathbf{Q}}$-Rational Points

Recall that we denoted the space of all Euclidean cone manifolds having N singular points with concentrated curvatures $\kappa = (\kappa_1, \ldots, \kappa_N)$ up to orientation preserving similarity by $C(\kappa)$. If one labels the singular vertices, then one obtains a finite covering of $C(\kappa)$, call $P(\kappa) = P(\kappa_1, \ldots, \kappa_N)$, whose fundamental group is the pure braid group of the sphere on N strings. The fundamental group of $C(\kappa)$ depends solely on the curvature parameters, κ_i. If, for instance, $\kappa_i \neq \kappa_j$ for every $i \neq j$ then $\pi_1(C(\kappa))$ is nothing but the pure braid group of the sphere. In the other extreme case, i.e. when $\kappa_i = \kappa_j$ for each i, j, the fundamental group of $C(\kappa)$ becomes the full braid group of the sphere on N strings, which is possible only for finitely many cases as opposed to the first extreme case, in which there are infinitely many possibilities.

Given a cone metric $c \in C(\kappa)$, we saw that c defines a complex structure on S^2. Hence if we forget about the metric c and consider only the complex structure, then we obtain a mapping from $C(\kappa)$ to the moduli space of smooth N-pointed rational curves. However, the target space cannot be $\mathcal{M}_{0,N}$, for the induced homomorphism from the fundamental group of $C(\kappa)$ to the mapping class group is not always an isomorphism. Nevertheless the target is a finite cover of $\mathcal{M}_{0,N}$. The following relates $C(\kappa)$ to moduli of pointed rational curves:

Theorem 4.1.14 ([45, Theorem 8.1]) *The map from $C(\kappa)$ to $\mathcal{M}_{0,N}$, denoted by $\mathfrak{S}$, described above is a homeomorphism. In particular, when $\kappa_i = \kappa_j$ for each i, $j \in \{1, \ldots, N\}$ we have an isomorphism.*

There is an inverse to the map $\mathfrak{S}$, denoted by $\mathfrak{S}^{-1}$, explained in the proof of [45, Theorem 8.1]]. We, on the other hand, already know an inverse to $\mathfrak{S}$. Any element of $\mathcal{M}_{0,N}$ comes with a distinguished set of points which forms the singular set. The metric, which is unique up to normalization, is the provided by Theorem 2.1.6. For further details see [47].

Similarly, we have a map $\mathfrak{T}\colon C(\kappa) \longrightarrow X(\kappa) = X(\kappa_1, \ldots, \kappa_N)$; where $X(\kappa)$ denotes a finite covering of $C_N\mathbf{P}^1$. The map sends every euclidean cone metric c to its singular set S_c. And as in the case of $\mathfrak{S}$, the degree of the covering depends on the curvature parameters, see Commutative Diagram 4.2 for an overview of the described maps.

Example 4.1.15 *A classical case of the above phenomenon occurs when one considers the configuration space of* 4 *points on* $\mathbf{P}^1$, *in other words when one chooses the parameters as* $\frac{2\pi}{2}, \frac{2\pi}{2}, \frac{2\pi}{2}, \frac{2\pi}{2}$. *One obtains the following diagram:*

$$\begin{CD}
\mathbf{P}^1 \setminus \{0,1,\infty\} @>>> \Delta_{\infty,\infty,\infty}\backslash\mathbb{H} \\
@VVV @VVV \\
C_4\mathbf{P}^1 @>>> \mathrm{PSL}_2(\mathbf{Z})\backslash\mathbb{H} = \mathcal{M}_1
\end{CD}$$

Commutative Diagram 4.1: The particular case: 4 points on $\mathbf{P}^1$.

On the other hand, one has a natural quadrangulation of each such configuration consisting of two quadrangles. As we did in the proof of Theorem 3.2.5, let us set each quadrangle to be a unit square. Then the curve in $\mathcal{M}_1$ one gets has the affine equation $y^2 = x^3 - x$, which is defined over $\mathbf{Q}$. As the map between $\mathcal{M}_1$ and $\mathcal{M}_{0,4}$ is algebraic the corresponding pointed rational curve is also defined over $\mathbf{Q}$. Another way to see this, which is somewhat more appropriate for generalization is the following: for the curve corresponding to the quadrangulation consisting of two unit squares glued from their boundary, one may take the fourth ramification point to be defined over $\mathbf{Z}[\sqrt{-1}]$, hence the Jacobian has complex multiplication, [30, Theorem 12.8] or Theorem 4.1.13, thus we get an algebraic point.

$$\begin{tikzcd}
S\,(C\,(\kappa)) \arrow[d] & & C(\kappa) \arrow[ll, "s"'] \arrow[rr, bend left, "\mathfrak{S}"] & & \mathfrak{S}\,(C\,(\kappa)) \arrow[ll, bend left, "\mathfrak{S}^{-1}"] \arrow[d] \\
C_N\mathbf{P}^1/\mathrm{PGL}_2(\mathbf{C}) \arrow[drr, "\mathfrak{S}ch"] & & & & \mathcal{M}_{0,N} \arrow[dll, "\cong"'] \\
& & \mathbb{B}^{N-3}/\Delta(\kappa_1,\ldots,\kappa_N) & &
\end{tikzcd}$$

Commutative Diagram 4.2: Configuration spaces, moduli of cone metrics and pointed rational curves.

Remark 4.1.16 *We are interested in the cases where we obtained the lattices Λ and Λ', namely the following cases*

$$\kappa = \underbrace{\left(\frac{1}{6},\ldots,\frac{1}{6}\right)}_{12\ \textit{times}} \ \textit{and}\ \kappa' = \underbrace{\left(\frac{1}{4},\ldots,\frac{1}{4}\right)}_{8\ \textit{times}},$$

respectively. The vertical arrows in Commutative Diagram 4.2 reduces to identity and we get the following simpler diagram:

$$\begin{tikzcd} C(\kappa) \arrow[rr, bend left, "\mathfrak{S}"] \arrow[d, "S"] & & \mathcal{M}_{0,N} \arrow[ll, bend left, "\mathfrak{S}^{-1}"] \arrow[d, "\cong"] \\ C_N\mathbf{P}^1/\mathrm{PGL}_2(\mathbf{C}) \arrow[rr, "\mathfrak{S}ch"] & & \mathbb{B}^{N-3}/\Delta(\kappa_1, \ldots, \kappa_N) \end{tikzcd}$$

Commutative Diagram 4.3: The cases κ and κ'.

Thus, one expects that the points of Proj(Λ_+)*, respectively Proj*(Λ'_+)*, corresponds to* $\overline{\mathbf{Q}}$ *rational points of* $\mathcal{M}_{0,12}$*, respectively* $\mathcal{M}_{0,8}$.

4.2 Graphs on Surfaces

In this section we will begin with defining the algebraic fundamental group, then concentrate on the concept of embedded graphs, which are called *dessins d'enfants* by Grothendieck in [17]. Our main motivation for this introductory section is [35].

4.2.1 Analogy Between Galois and Fundamental Groups

We will try to point out the analogy between Galois groups and fundamental groups and then try to introduce the concept of arithmetic fundamental groups which unifies both approaches.

Galois Groups. To every given extension, K/k, of fields one may associate a group known as the Galois group, $\mathrm{Gal}(K/k)$ defined as:

$$\mathrm{Gal}(K/k) = \{\sigma : K \longrightarrow K \in \mathrm{Aut}(K) \,|\, \sigma|_k = id_k\}\,.$$

Now fix an extension K/k. To every subfield L of K which is also an extension of k, one may associate the corresponding Galois group, $\mathrm{Gal}(L/k)$ which is clearly a subgroup of $\mathrm{Gal}(K/k)$. And conversely, to every subgroup of $H \leq \mathrm{Gal}(K/k)$ we may associate the *fixed field of* H:

$$K^H := \{a \in K \,|\, h(a) = a, \ \forall\, h \in H\}.$$

In this setup the fundamental theorem of Galois theory may be stated as:

Theorem 4.2.1 (Fundamental Theorem of Galois Theory) *If the extension K/k is finite and Galois, then one has a one to one correspondence between the subfields L/k of K/k and subgroups of $Gal(K/k)$.*

Pictorially we have:

$$\begin{tikzcd}[row sep=small]
\mathrm{Gal}(K/K) = \{id\} \arrow[d, dash, dashed] & \leq & \mathrm{Gal}(K/L) \arrow[d, dash, dashed] & \leq & \mathrm{Gal}(K/k) \arrow[d, dash, dashed] \\
K^{\mathrm{Gal}(K/K)} = K & \geq & K^{\mathrm{Gal}(K/L)} & \geq & K^{\mathrm{Gal}(K/k)} = k
\end{tikzcd}$$

Fundamental Groups. Let X be a path connected space. One may then speak about the homotopy class of paths between any two given points in X. If, further, one fixes a base point $x \in X$, then the set of paths whose both initial and terminal point is x is a group called the fundamental group of X, denoted by $\pi_1(X, x)$. Now, suppose furthermore that X is locally path connected and locally simply connected, so that X admits a universal cover, $\widetilde{X}$. Then $\pi_1(X, x)$ acts on $\widetilde{X}$ and to every connected covering Y of X, one has a corresponding subgroup, Γ_Y, of $\pi_1(X, x)$. We conclude:

$$\begin{tikzcd}[row sep=small]
\pi_1(X) \arrow[d, dash, dashed] & \geq & \Gamma_Y \arrow[d, dash, dashed] & \geq & \pi_1(\widetilde{X}) = \{id\} \arrow[d, dash, dashed] \\
X & & Y \arrow[ll, two heads] & & \widetilde{X} \arrow[ll, two heads]
\end{tikzcd}$$

4.2.2 Arithmetic Fundamental Groups

One may unify the above two parallel theories in a quite general setting. Our notation is that of [33]. For this, we start with a definition:

Definition 4.2.2 *Let C be a category which satisfies the following properties:*

- *there is an initial and a terminal object, I and T respectively, in C,*
- *for $A, B, S \in Ob(C)$ with $A \longrightarrow S$, $B \longrightarrow S$, $A \times_S B \in Ob(C)$,*
- *$A, B \in Ob(C)$ implies that the disjoint union $A \coprod B \in Ob(C)$,*

- *any morphism in C can be written as a composition of an effective epimorphism and a monomorphism,*

- *if $A \in Ob(C)$ and G is a finite group of automorphisms of A then the quotient $A/G \in Ob(C)$ and the morphism $A \longrightarrow A/G$ is an effective epimorphism in C.*

Assume further that C is equipped with a functor $\mathcal{F} : C \longrightarrow$ F-Set; where F-Set denotes the category of finite sets, so that:

- *$\mathcal{F}(A) = \emptyset$ if and only if $A = \emptyset$,*

- *the set $\mathcal{F}(T)$ has cardinality one, i.e. it is* the *terminal object, and $\mathcal{F}(A \times_S B) = \mathcal{F}(A) \times_{\mathcal{F}(S)} \mathcal{F}(B)$ for any $A, B, S \in Ob(C)$,*

- *$\mathcal{F}(A \coprod B) = \mathcal{F}(A) \coprod \mathcal{F}(B)$,*

- *any effective epimorphism is mapped to an onto map,*

- *for $A \in Ob(C)$ with a finite group of automorphisms G, the induced map from $\mathcal{F}(A)/G$ to $\mathcal{F}(A/G)$ is a bijection,*

- *whenever $f : A \longrightarrow B$ is a morphism in C whose image $\mathcal{F}(f) : \mathcal{F}(A) \longrightarrow \mathcal{F}(B)$ is a bijection then f itself is a bijection;*

where by F-Set we mean the category of finite sets. Then we refer to C as a ***Galois category*** *and the functor $\mathcal{F}$ is called* ***fundamental functor***. ■

Example 4.2.3 (Finite Sets with identity functor) *The category F-Set with the functor ID sending every set as well as every morphism to itself is a* Galois category. *Indeed, set with one element is* the *terminal object, recall here that terminal object is unique up to isomorphism. Empty set is the initial object. Whenever we are given $\alpha : A \longrightarrow S$ and $\beta : B \longrightarrow S$, then the set*

$$A \times_S B = \{(a, b) \in A \times B \,|\, \alpha(a) = \beta(b)\}$$

is in $Ob(C)$. Disjoint union of two finite sets is again finite, hence in F-Set. And the quotient of a set by a finite group of automorphisms is again a finite set together with the natural projection. Further, it is quite clear that the identity functor is a fundamental functor.

Example 4.2.4 (Category of étale coverings) *Let S be a locally noetherian, connected scheme and by $\mathcal{E}t(S)$ denote the category of étale coverings of S. The empty set is, as in Example 4.2.3, the initial object, and S itself is the terminal object. [33, 3.3.3, (4)] implies that given two arrows $\alpha : A \longrightarrow X$ and $\beta : B \longrightarrow X$ in $\mathcal{E}t(S)$, the fiber product $\alpha \times_X \beta : A \times_X B$ is in $\mathcal{E}t(S)$. The property concerning disjoint unions is straightforward. One may obtain the decomposition of an arrow, $\alpha : A \longrightarrow X$ in $\mathcal{E}t(S)$ simply by choosing $B = \alpha(X)$ and let $X = B \coprod B_1$ to write:*

$$\begin{tikzcd} A \arrow[rr, "\alpha"] \arrow[dr] & & X \\ & B \arrow[ur, hook] & \end{tikzcd}$$

Now, we fix a point $s \in S$ and an algebraically closed field K containing $k(s)$. Then we have: Now, we define the functor $\mathcal{F} : \mathcal{E}t(S) \longrightarrow F\text{-}Set$ to send an element $X \in Ob(\mathcal{E}t(S))$ to the

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}(k(s)) \arrow[r] & S \end{tikzcd}$$

set of all S-morphisms for which the above diagram commutes. In other words every X is sent to the set of K-points of X over s. It is clear that $\mathcal{F}(A) = \emptyset \Leftrightarrow A = \emptyset$. $\mathcal{F}(S)$ has one element and $\mathcal{F}(A \times_X B) = \mathcal{F}(A) \times_{\mathcal{F}(X)} \mathcal{F}(B)$ and $\mathcal{F}(A \coprod B) = \mathcal{F}(A) \coprod \mathcal{F}(B)$ for any $A, B, X \in Ob(\mathcal{E}t(S))$. Say we are given an effective epimorphism $\alpha : A \longrightarrow X$. Take any element $x \in X$ lying above s, or equivalently, take a $k(s)$ monomorphism of $k(y)$ into K. Then we may extend this monomorphism to a $k(s)$ monomorphism of $k(a)$ into K; where $\alpha(a) = x$. Suppose that G is a finite group of S automorphisms of A. Then we have a natural surjective morphism $pr : A \longrightarrow A/G$, together with the following commutative diagram:

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] & A \arrow[rr, "g \in G"] \arrow[dr] & & A \\ & & A/G \arrow[ur] & \end{tikzcd}$$

So we obtain a surjective morphism $\mathcal{F}(A)/G \longrightarrow \mathcal{F}(A/G)$ which is, as a consequence of [33, Lemma 4.2.1], in fact a bijection. Finally, for a given $f : A \longrightarrow B$ with $\mathcal{F}(f) : \mathcal{F}(A) \xrightarrow{\cong} \mathcal{F}(B)$, we have that the degree(rank) of f is one, hence it is an isomorphism. In particular, let $S = Spec(k)$ for some field k. Then we know that objects of $\mathcal{E}t(k)$ are *finite separable field extensions of k, and we have $\mathcal{F}(X) = Hom_k(L, k^{sep})$; where $X = Spec(L)$, k^{sep} is the separable closure of k in K which is a fixed algebraically closed field containing k, as above.*

Example 4.2.5 (Category of topological coverings) *Let X be a connected, locally path connected, locally simply connected topological space, and $\mathcal{T}op(X)$ be the category of connected topological coverings of X with morphisms being covering maps. Fix a point $x \in X$ and for any topological cover $\pi : Y \longrightarrow X$ define $\mathcal{F}(Y) = \pi^{-1}(x)$.*

Theorem 4.2.6 ([33, 4.4.1]) *Given a Galois category C with fundamental functor $\mathcal{F}$, there exists a pro-finite group Π with the property that C and the category of Π-sets, i.e. sets on which Π acts continuously are equivalent.*

Definition 4.2.7 *The group Π is called the fundamental group of the category C.* ■

In Example 4.2.4 we have seen that the category of étale coverings of a scheme X is a Galois category with the functor $\mathcal{F}$. The corresponding pro-finite group is called the **étale** fundamental group of X. When we fix our base field as $\mathbf{Q}$ then we refer to this group also as the *arithmetic fundamental group*. On the other hand, the fundamental group Π we obtain from Example 4.2.5 is the usual topological fundamental group, $\pi_1(X, x)$. In fact, every topological cover of X, i.e. every element in $\mathrm{Ob}(\mathcal{T}op(X))$ determines a subgroup in $\pi_1(X, x)$, and conversely.

4.2.3 Embedded Graphs

We will begin this section by recalling the celebrated result due to Belyĭ:

Theorem 4.2.8 ([4]) *An algebraic curve X may be defined over the field of algebraic numbers, $\overline{\mathbf{Q}}$, if and only if X admits a meromorphic function (or a Belyĭ morphism), $f : X \longrightarrow \overline{\mathbb{C}}$, ramified at most over 3 points which may be chosen to be 0, 1 and ∞.*

Combining Theorem 4.2.6 and the well-known Riemann existence theorem we obtain the following:

Theorem 4.2.9 *The arithmetic fundamental group of* $\mathbf{P}^1_{\overline{\mathbf{Q}}} \setminus \{0, 1, \infty\}$ *is isomorphic to the profinite completion of the fundamental group of* $\mathbf{P}^1_{\mathbf{C}} \setminus \{0, 1, \infty\}$. *Furthermore, the following categories are equivalent:*

- *finite topological covers of* $\mathbf{P}^1_{\mathbf{C}} \setminus \{0, 1, \infty\}$,
- *finite étale covers of* $\mathbf{P}^1_{\overline{\mathbf{Q}}} \setminus \{0, 1, \infty\}$*(or* $\mathbf{P}^1_{\mathbf{C}} \setminus \{0, 1, \infty\}$*),*
- *finite sets with the action of the algebraic fundamental group* $\pi_1^{alg}(\mathbf{P}^1_{\overline{\mathbf{Q}}} \setminus \{0, 1, \infty\})$,
- *finite sets with the action of the fundamental group* $\pi_1(\mathbf{P}^1_{\mathbf{C}} \setminus \{0, 1, \infty\})$,
- *subgroups of* $\pi_1(\mathbf{P}^1_{\mathbf{C}} \setminus \{0, 1, \infty\}, x)$ *up to conjugation.*

There is one more category which is equivalent to the ones listed above. Namely, the category of *embedded graphs*:

Definition 4.2.10 *An* ***embedded graph or a map*** *is a graph,* Γ*, embedded into a topological surface, X, i.e. a* ***closed***, ***oriented***, *two dimensional topological manifold so that*

- *edges intersect only at vertices,*
- *each connected component of* $X \setminus \{\text{image of } \Gamma\}$ *is homeomorphic to a disc.*

The embedding of the graph into X is denoted by ι. ■

It is common to regard graphs as cell complex comprised only of 0 and 1 cells, and hence embedded graphs as an *injection* $\iota : \Gamma \longrightarrow X$ satisfying $X \setminus \iota(\Gamma)$ is a union of open sets each of which is homeomorphic to a disc.

Definition 4.2.11 *Each connected component of* $X \setminus \iota(\Gamma)$ *is called a* ***face*** *of* Γ. ■

Observe that since X is oriented, around each vertex of Γ there is a canonical orientation of the edges of Γ coming out of this vertex. Keeping in mind these observations we define two embedded graphs to be equivalent if there is a map between vertices and edges respecting orientation.

Then, to every curve admitting a meromorphic function ramified over at most three point, we associate an embedded graph, which is by construction *bipartite*. In fact, the graph is nothing but the inverse image of the closed interval [0, 1]. Conversely, every bipartite embedded graph defines a complex structure, hence a Riemann surface(or equivalently an algebraic curve), [50]. In the light of Theorem 4.2.8, we conclude the equivalence we mentioned.

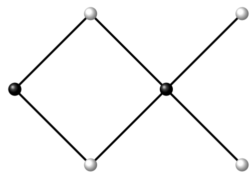


Figure 4.2: A Graph Embedded in the Riemann Sphere.

4.2.4 A Computation

Hypergeometric Differential Equation. The equation:

$$\frac{d^2\omega}{dx^2} + p(x)\frac{d\omega}{dx} + q(x)\omega = 0 \tag{4.3}$$

where $p(x)$ and $q(x)$ are functions of the complex variable x, is referred to as a hypergeometric differential equation. A point $x = x_s$ is called a *singular point* of (Equation 4.3) if p or q, or both, have a pole at x_s. x_s is, further, called a *regular singular* point of (Equation 4.3), when p has at most a pole of order 1, and q has at most a pole of order 2 at x_s; and the equation is called *Fuchsian*, exactly when all the singular point are regular. The first *non-trivial* Fuchsian differential equation occurs when it possesses 3 regular singular points, i.e. when we have a *hypergeometric differential equation*. In suitable coordinates hypergeometric differential equation can be put into the form:

$$\frac{d^2\omega}{dx^2} + \left[\frac{1-\lambda}{x} + \frac{1-\mu}{x-1}\right]\frac{d\omega}{dx} + \left[\frac{(1-\lambda-\mu)^2 - \nu^2}{4x(x-1)}\right]\omega = 0 \tag{4.4}$$

where, the numbers λ, μ, ν are referred to as the *exponent differences*. The solutions of Equation 4.4 exist, and are linearly independent, by the fundamental theorem of Cauchy([51, Section 2.2]), and are given by hypergeometric series. Throughout we will name these solutions η_1 and η_2.

It was observed by Schwarz that the behavior of the quotient $y(x) = \frac{\eta_1(x)}{\eta_2(x)}$ is very special. More precisely, one can show

Proposition 4.2.12 ([13]) *Any branch of the function $y(x)$ maps $\mathbf{C}$ to two neighboring triangles in*

i. $\mathbb{H}$ *(considered as equipped with its usual hyperbolic structure, when $\lambda + \mu + \nu < 1$)*

ii. $\mathbf{C}$ *(considered with its euclidean structure, when $\lambda + \mu + \nu = 1$)*

iii. $\mathbf{P}^1$ *(considered with its spherical metric, when $\lambda + \mu + \nu > 1$)*

with angles $\lambda\pi$, $\mu\pi$ and $\nu\pi$.

Triangle Groups and Dessins d'Enfants. Recall first that a triangle group of signature (k, l, m), $k, l, m \in \mathbf{Z}_{>0}$ has the following presentation

$$\Delta_{k,l,m} := \langle \sigma,\, \tau \,|\, \sigma^k = \tau^l = (\sigma \cdot \tau)^m = 1 \rangle.$$

Suppose we are given an embedded graph, Γ, on an oriented surface, X. The embedding ι, see Definition 4.2.10, gives us an orientation around every vertex. Hence if we number the edges of Γ, or in short a marking of Γ, then we obtain a subgroup, called the *cartographic group*, denoted by $C(\Gamma)$, of the symmetric group on $|e(\Gamma)|$ letters whose generators are rotations around black vertices, call σ, and rotations around white vertices, call τ. Let k, l, m be the least common multiple of valencies of black vertices, white vertices, faces of Γ. Thus we obtain an epimorphism:

$$\begin{aligned} \varphi\colon \Delta_{k,l,m} := &\longrightarrow C(\Gamma) \\ \widetilde{\sigma} &\mapsto \sigma \\ \widetilde{\tau} &\mapsto \tau, \end{aligned}$$

whose kernel is isomorphic to the surface group of the curve X; where by a *surface group* we mean a torsion free subgroup of $\mathrm{PSL}_2(\mathbf{R})$, say, with the property that $\ker\varphi \backslash \mathbb{H} \cong X$.

Computing Belyĭ Morphisms. There is a natural family of curves, say Y_n, each of which is defined over a number field by Theorem 4.2.8, whose n^{th} element can be constructed as follows:

1. Take a unit Euclidean square,
2. Divide the edges of the square into n equal parts,
3. Connect the possible edges by *new* lines parallel to edges of the square, call the resulting square Q_n,
4. Mark the midpoints of the squares with a black vertex, and connect the black vertices lying in neighboring squares,
5. Put a white vertex at every point where lines connecting black vertices and *new* lines intersects,
6. Identify the top edges with bottom and left edge with right to get a torus, see Figure 4.3.
7. Use the inclusion relation between $\mathbf{Z}[\sqrt{-1}]$ and $\Delta_{2,4,4}$ to project down to $\mathbf{P}^1$, see Figure 4.4 for a geometric description, and obtain Y_n.

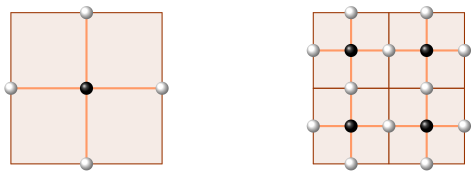
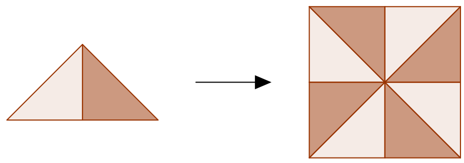


Figure 4.3: First two tori with embedded graphs.

Remark 4.2.13 *The steps 4., 5., 6., are referred to taking barycentric subdivision in literature.*

The embedded graph defining the curve Y_n will be referred to as Γ_n. Figure 4.5 displays the curve Y_3 together with Γ_3.

Remark 4.2.14 *To every Γ_n, one may associate a quadrangulation of the sphere by connecting the midpoint of each face by the white vertices lying on the boundary of the face. Observe*



Fundamental region for $\Delta_{2,4,4}$ | Fundamental region for $\mathbf{Z}[\sqrt{-1}]$

Figure 4.4: Geometric description of the natural projection between $\Delta_{2,4,4}$ and $\mathbf{Z}[\sqrt{-1}]$.

that such a quadrangulation is an element of the compactification of the space in which the lattice Λ' found in Theorem 3.2.5 lies.

The computation uses the following commutative diagram:

$$\begin{tikzcd}
\mathbf{Z}[\sqrt{-1}]\backslash\mathbf{C} \arrow[r, "m_n"] \arrow[d] & \mathbf{Z}[\sqrt{-1}]\backslash\mathbf{C} \arrow[d] & \\
H_n\backslash\mathbf{C} \arrow[r] & \Delta(2,4,4)\backslash\mathbf{C} \arrow[r, "\frac{\eta_1}{\eta_2}"] & \mathbf{P}^1
\end{tikzcd}$$

where the functions η_1 and η_2 refers to the solutions of the hypergeometric differential equation for $\Delta_{2,4,4}$, H_i corresponds to the subgroup of $\mathrm{PSL}_2(\mathbf{R})$ making the square commutative, m_i refers to the multiplication by i self-morphism of the elliptic curve $\mathbf{Z}\sqrt{-1}\backslash\mathbf{C}$, which has Weierstraß form $y^2 = 4x^3 - x$.

The corresponding Belyĭ morphisms in this case are composition of the arrows on the bottom. However, we know the ramification points are the i-division values of a particular elliptic function, where by an i-division value we mean the value of an elliptic function at points $x \in \mathbf{Z}\sqrt{-1}\backslash\mathbf{C}$ so that $i \cdot x \in \mathbf{Z}[\sqrt{-1}]$. Our aim is thus to find the ramification points of the Belyĭ morphism. For our purposes it is enough to consider the elliptic function, $w = \varepsilon(z)$,

$$\begin{aligned}
z \mapsto w &= \frac{1}{(\wp(\omega_3) - \wp(\omega_1))(\wp(\omega_3) - \wp(\omega_1))}(\wp(z) - \wp(\omega_1))(\wp(z) - \wp(\omega_2)) \\
&= -4\wp^2(z) + 1,
\end{aligned}$$

where ω_1 is the real and ω_2 is the purely imaginary period of $y^2 = 4x^3 - x$, and $\omega_3 = \frac{1}{2}(\omega_1 + \omega_2)$. The last equality is a result of the fact that $\wp(\omega_1) = \frac{1}{2} = -\wp(\omega_2)$, which implies $\wp(\omega_3) = 0$.

Then the Belyĭ morphism corresponding to Y_n, call g_n, up to a constant, has the following general form:

$$g_n(w) := c_n \frac{\prod_{z \in \text{ white vertices}} (w - \varepsilon(z))^{\mathrm{ord}(z)}}{\prod_{z \in \text{ poles}} (w - \varepsilon(z))^{\mathrm{ord}(z)}}$$

where by poles we mean midpoints of faces, and by order the valency of corresponding vertex or face, and c_n is a constant which will be described in Example 4.2.16.

Definition 4.2.15 *The corresponding functions g_n are referred to as the **Gauss-Chebyshev** functions.* ■

Example 4.2.16 *We would like to demonstrate the case $n = 3$, whose dessin can be found in Figure 4.5. The list of ramification points may be found in Table 4.2. Thus, g_3 is equal to:*

$$c_3 \frac{\left[\prod_{p \in P_3} (w - \varepsilon(p)) \prod_{q \in Q_3} (w - \varepsilon(q))^2\right]}{(w - \varepsilon(0))(w - \varepsilon(\frac{2\omega_1}{3}))^2 (w - \varepsilon(\frac{2\omega_3}{3}))^2 (w - \varepsilon(\frac{4\omega_1}{3}))^2}$$

where c_3 is the normalization constant and $Q_3 := \{\frac{2\omega_1}{3}) + \frac{\omega_2}{3}, \omega_1 + \frac{2\omega_2}{3}, \frac{4\omega_1}{3}) + \frac{\omega_2}{3}\}$, $P_3 = \{\frac{\omega_1}{3}, \omega_1, \frac{5\omega_1}{3}\}$. As 1 is a ramification value, we choose $c_3 = \frac{1}{g_3(\omega_3)/3}$, and in general, $c_n = \frac{1}{g_n(\omega_3/n)}$.

Remark 4.2.17 *The well-known formula*

$$\wp(z + z') = \frac{1}{4}\left[\frac{\wp'(z) - \wp'(z')}{\wp(z) - \wp(z')}\right] - \wp(z) - \wp(z')$$

together with the fact that $\wp(\omega_1), \wp(\omega_2) \in \mathbf{Q}$ implies that for every n the values of ε are algebraic. However as n assumes larger values the degree of the algebraic number gets larger, too. Nevertheless, as a result of dessin being symmetric with respect to $\mathbf{R} \cup \{\infty\} \subseteq \mathbf{P}^1$ the field of definition of the Belyĭ morphism is a subfield of a totally real field.

Numerical data for the ramification points of g_3 may be found in Table 4.2.

Remark 4.2.18 *A similar family for the lattice Λ appeared in Theorem 3.2.4 may be defined. The description of the family and corresponding calculations of Belyĭ morphisms as well as an application to curves of higher genera may be found in [53].*

white vertices (inverse image of 0)	black vertices (inverse image of 1)	poles (inverse image of ∞)
$\frac{1}{3}\omega_1$	$\frac{1}{3}\omega_3$	0
$\frac{2}{3}\omega_1 + \frac{1}{3}\omega_2$	$\omega_1 + \frac{1}{3}\omega_2$	$\frac{2}{3}\omega_1$
ω_1	ω_3	$\frac{2}{3}\omega_3$
$\omega_1 + \frac{2}{3}\omega_2$	$\frac{5}{3}\omega_1 + \frac{1}{3}\omega_2$	$\frac{4}{3}\omega_1$
$\frac{4}{3}\omega_1 + \frac{1}{3}\omega_2$		$\frac{4}{3}\omega_1 + \frac{2}{3}\omega_2$
$\frac{5}{3}\omega_1$		$2\omega_1 = 0 \bmod \mathbf{Z}[\sqrt{-1}]$

Table 4.1: Points on E whose values give ramification data of g_3.

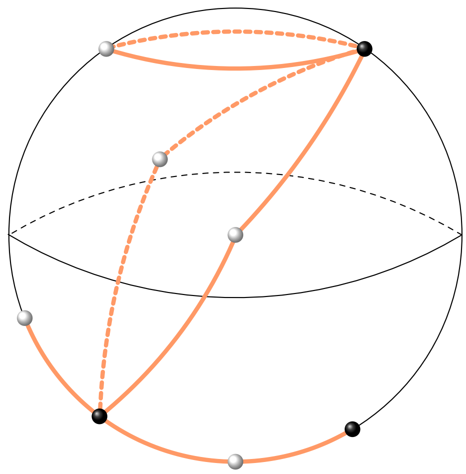


Figure 4.5: The curve Y_3.

zeros of g_3
$\varepsilon(\frac{1}{3}\omega_1) = -26.8204616940335$
$\varepsilon(\frac{2}{3}\omega_1 + \frac{1}{3}\omega_2) = 0.9282032302755 + 0.9974192818755\sqrt{-1}$
$\varepsilon(\omega_1) = 0$
$\varepsilon(\omega_1 + \frac{2}{3}\omega_2) = 0.9640552334825$
$\varepsilon(\frac{4}{3}\omega_1 + \frac{1}{3}\omega_2) = 0.9282032302755 - 0.9974192818755\sqrt{-1}$
$\varepsilon(\frac{5}{3}\omega_1) = -26.8204616940335$

poles of g_3
$\varepsilon(0) = \infty$
$\varepsilon(\frac{2}{3}\omega_1) = -1.15470053837925$
$\varepsilon(\frac{2}{3}\omega_3) = 1.15470053837925$
$\varepsilon(\frac{4}{3}\omega_1) = -1.15470053837925$
$\varepsilon(\frac{4}{3}\omega_1 + \frac{2}{3}\omega_2) = +1.15470053837925$

Table 4.2: Zeros and poles of g_3.

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