STOCHASTIC VOLATILITY AND STOCHASTIC INTEREST RATE MODEL WITH JUMP AND ITS APPLICATION ON GENERAL ELECTRIC DATA

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ABSTRACT

STOCHASTIC VOLATILITY AND STOCHASTIC INTEREST RATE MODEL
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In this thesis, we present two different approaches for the stochastic volatility and stochastic interest rate model with jump and analyze the performance of four alternative models. In the first approach, suggested by Scott, the closed form solution for prices on European call stock options are developed by deriving characteristic functions with the help of martingale methods. Here, we study the asset price process and give in detail the derivation of the European call option price process. The second approach, suggested by Bashki-Cao-Chen, describes the closed form solution of European call option by deriving the partial integro-differential equation. In this one we give the derivations of both asset price dynamics and the European call option price process. Finally, in the application part of the thesis, we examine the performance of four alternative models using General Electric Stock Option Data. These models are constructed by using the theoretical results of the second approach.

Keywords: option pricing, stochastic volatility stochastic interest rate model with jump, partial integro-differential equation, martingale methods, direct algorithm

ÖZ

SIÇRAMALI STOKASTİK VOLATİLİTE VE STOKASTİK FAİZ ORANI MODELİ
VE MODELİN GENERAL ELECTRIC VERİSİ ÜZERİNE UYGULAMASI

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Bu çalışmada, stokastik volatilite ve stokastik faiz oranına sahip sıçramalı opsiyon fiyatlama modeli, iki farklı yaklaşımla incelenmiş ve 4 alternatif model üzerinde analizler yapılmıştır. Scott tarafından önerilen ilk yaklaşımda, opsiyon fiyatı karakteristik fonksiyonun bazı martingale yöntemleriyle bulunması sonucu oluşturulmuştur. Sonrasında yaklaşım ayrıntılı bir şekilde incelenip, opsiyon fiyatının çıkarımı orjinal çalışmada bulunmayan ispatlarla birlikte sunulmuştur. Bashki-Cao-Chen tarafından oluşturulan ikinci yaklaşımda ise kısmi integro-diferansiyel denklemleri kullanılarak opsiyon fiyatına ulaşılmıştır. Bu çalışmadaki fiyat süreci dinamikleri ayrıntılı biçimde incelenerek, opsiyon fiyat formülasyonu gerekli görülen ispatlarla beraber oluşturulmuştur. Son olarak ikinci yaklaşımın yardımıyla 4 farklı opsiyon fiyatlama modelinin analitik çözümleri çıkarılmış ve modellerin deneysel performansları General Electric Hisse Senedi Verisi üzerinde ölçülmüştür.

Anahtar Kelimeler: opsiyon fiyatlama, sıçramalı stokastik volatile stokastik faiz oranı modeli, kısmi integro-diferansiyel denklemi, martingale yöntemleri, direct algoritması

To my grandfather Rıza Yıldırım.

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CHAPTER 1

INTRODUCTION

The Black Scholes Model which was articulated by Fischer Black and Myron Scholes in their 1973 paper has an important role in theory and application of financial studies. Obviously, the model is a pioneering work in the area. However it is also know that the assumptions to get the closed form solutions cause some empirical biases such as volatility smile. For this reason in the last two decades, option pricing has witnessed an explosion of new models that each relax some of the restrictive Black Scholes assumptions. Among these assumptions, constant volatility, constant interest rate and no rapid price movements resembling jumps have been the most studied ones. Over the years, there have been many alternative models offered to solve this and other drawbacks of the Black Scholes model. Examples include the stochastic-interest-rate option models of Merton (1973), the jump-diffusion and pure jump models of Bates (1991), Madan and Chang (1996), and Merton (1976), the stochastic-volatility models of Heston (1993), Hull and White(1987), Stein and Stein (1991), the stochastic volatility and stochastic interest rates models of Bakshi and Chen (1997a,b), and Scott (1997) and the stochastic-volatility jump-diffusion models of Bates (1996). In this work, we concentrate on the most generalized model that is stochastic volatility and stochastic interest rate model with jumps with two different angles conducted by Scott [33] and Bashki-Cao-Chen [1].

In the former approach, the closed-form solution for the price of the European call option is obtained by using some martingale properties and mathematical tricks. In addition, by assuming that the volatility and the underlying price have a non-zero correlation and both the interest rate and volatility follow a Cox-Ingersoll-Ross Model [11] captured many properties of the financial data such as non-negative interest rate and volatility. Although the approach has some theoretical advantages, in application, parameter estimation and determination procedure is formidable.

In the latter one studied by Bashki, Cao and Chen [1], the closed-form solution for the price of the European call option is derived by using the partial integro-differential equation. By reducing the difficulty of this complex equation to ordinary differential equations, the call price is obtained. Moreover, the assumption of lognormally distributed jumps provided model improvement and convenience with the market data. Although this approach is theoretically complex and has implementational costs compared to Scott's study, it is more applicable, since it is given only as a function of identifiable variables such that all parameters can be estimated.

The aim of this study is to review these two approaches of the same model in detail and use the results of the second approach in application. The second chapter presents the derivation of the option pricing formula suggested in the study of Scott [33] step-by-step. The way of how the martingale method is used for derivation of characteristic functions is analyzed. In the third chapter, by following the study of Bashki, Cao and Chen [1], the European call option price is obtained. Firstly, the complicated partial integro-differential equation is reduced to simple forms. After that, the solutions of these simple ordinary differential equations are found. Moreover, in the last section of the chapter the results are recorded to use them in application part of the study. In Chapter 4, the results from the previous chapter are calibrated for the General Electric Stock Data. With the help of the calibration procedure, the performance of four alternative models are compared. These alternative models are Black Scholes Model, Stochastic Volatility Model, Stochastic Volatility Jump Model and Stochastic Volatility Stochastic Interest Rate Model. In fact, in the fourth chapter the aim is to answer the following question mainly: ”What do we gain from each generalized feature rather than using simple Black Scholes?” Finally, the conclusion follows.

CHAPTER 2

SCOTT'S APPROACH

Various econometric and numerical studies show that asset prices do not exhibit the assumptions of Black Scholes Model which are constant volatility, constant interest rate and no rapid price movements resembling jumps. Therefore, new extensions of models that capture qualitative feature of the financial data is needed. The model that presented in this chapter is an example of a model that relaxed the assumptions mentioned above. In this chapter, firstly, we introduced the model in Louis Scott's paper [33] and then we studied step-by-step the construction of closed form solution of European call option prices. We only concentrated on European options because the closed form solution with this technique can be reached only for this type options. For path dependent options some numerical methods should be used. The technique to derive a closed form solution to the call option pricing problem under the assumption of stochastic volatility, stochastic interest and jump-diffusion is diverse. The method used in this chapter is effectual from the point of view of getting closed form solution however impractical in application due to the extreme number of parameters.

2.1 Model Construction

Let (Ω, F, P) be our probability space. The underlying risky asset price volatility and interest rate is assumed to follow a standard Cox-Ingersoll-Ross (CIR) process given as follows:

$$dy_j(t) = K_j[\Theta_j - y_j(t)]dt + \sigma_j \sqrt{y_j(t)}dZ_j(t) \qquad (j = 1, 2), \tag{2.1.1}$$

where K_j, Θ_j, and σ_j are respectively the speed of adjustment, long-run mean and volatility coefficient of the process $y_j(t)$. Z_1 and Z_2 being constructed as independent Brownian Motions. It means that their correlation is 0. Moreover, independent Brownian Motions have zero covariation. Therefore, $dZ_1 dZ_2 = 0$. The continuous part of the price process, defined as $S^c(t)$ is assumed to satisfy the following stochastic differential equation:

$$dS^c(t) = r(t)S^c(t)dt + \sigma\sqrt{y_1(t)}S^c(t)dW(t), \tag{2.1.2}$$

where W is a Brownian Motion that is independent of Z_2 but correlated with Z_1 with size ρ . It means that the covariations are $dWdZ_2 = 0$ and $dW(t)dZ_1(t) = \rho d(t)$.

The instantaneous interest rate, r, is assumed to be a linear combination of y_1 and y_2; the reason behind this choice is given in [11] in detail:

$$r(t) = y_1(t) + y_2(t). \tag{2.1.3}$$

The jumps in the log-price process are constructed as a sequence (M_k) of independent normally distributed random variables with parameters μ_j and σ_j^2 as $N(\mu_j, \sigma_j^2)$ and jump counter is modeled as an independent Poisson Process with intensity λ. Consequently, $X(t)$ is assumed to be the sum of all the jumps which occur up to and including time t :

$$\sum_{k=1}^{N(t)} M_k = X(t).$$

Finally, the jump process is formed as $\frac{e^{X(t)}}{E(e^{X(t)})}$ which is a martingale. To examine the jump process, we calculated the expectation of exp(X(t)) by using some characteristics of expectation.

Let B_j's be the events in the sigma algebra F in the probability space (Ω, F, P) such that $\cup B_j = F$. Then $E(X) = \sum_{j=0}^{\infty} E(X|B_j) \cdot P(B_j)$. For more details the reader is referred to the book given in [23]. By using above property, the expectation can be computed as below:

$$
\begin{aligned}
E(e^{uX(t)}) &= E(e^{u\sum_{k=1}^{N(t)} M_k}) \\
&= \sum_{j=0}^{\infty}[E(e^{u\sum_{k=1}^{N(t)} M_k})|N(t) = j] \cdot P(N(t) = j) \\
&= \sum_{j=0}^{\infty}[E(e^{u\sum_{k=1}^{j} M_k})].P(N(t) = j),
\end{aligned} \tag{2.1.4}
$$

The above summation can be written for $j = 0$ and for all $j \geq 1$, separately. For $j = 0$, $\sum_{k=1}^{0} M_k = 0$. Therefore the expectation is 1 ($E(1) = 1$). Consequently,

$$
E[e^{uX(t)}] = (P(N(t) = 0)) + \sum_{j=1}^{\infty}[E(e^{u\sum_{k=1}^{j} M_k})] \cdot P(N(t) = j),
$$

where $N(t)$ is a Poisson Process with intensity λ. Because of this reason, $P(N(t) = k) = \frac{e^{-\lambda t} \cdot (\lambda t)^k}{k!}$ for $k = 0, 1, ...,$ and, hence,

$$
\begin{aligned}
E[e^{uX(t)}] &= e^{-\lambda t} + \sum_{j=1}^{\infty}[E(e^{u\sum_{k=1}^{j} M_k})] \cdot \frac{e^{-\lambda t} \cdot (\lambda t)^j}{j!} \\
&= e^{-\lambda t} + \sum_{j=1}^{\infty}[E(e^{u(M_1+M_2\cdots M_j)})] \cdot \frac{e^{-\lambda t} \cdot (\lambda t)^j}{j!} \\
&= e^{-\lambda t} + \sum_{j=1}^{\infty} E(e^{u(M_1)}) \cdot E(e^{u(M_2)}) \cdots E(e^{u(M_j)}) \cdot \frac{e^{-\lambda t} \cdot (\lambda t)^j}{j!},
\end{aligned} \tag{2.1.5}
$$

Note that $M_j's$ are iid. In addition, $\rho_j(u) = E[e^{u(M_j)}]$ is the moment generating function of normal distribution:

$$
\begin{aligned}
E[e^{uX(t)}] &= e^{-\lambda t} + e^{-\lambda t} \sum_{j=1}^{\infty} \frac{[\rho_j(u) \cdot \lambda t]^j}{j!} \\
&= e^{-\lambda t} \cdot \left[1 + \sum_{j=1}^{\infty} \frac{[\rho_j(u) \cdot \lambda t]^j}{j!}\right] \\
&= e^{-\lambda t} \cdot \left[\sum_{j=0}^{\infty} \frac{[\rho_j(u) \cdot \lambda t]^j}{j!}\right] \\
&= e^{-\lambda t} \cdot e^{\rho_j(u)\lambda t} \\
&= e^{\lambda t[\rho_j(u)-1]}.
\end{aligned} \tag{2.1.6}
$$

Since $\rho_j(u) = e^{\mu_j u + (1/2)\sigma_j^2 u^2}$ is the mgf of $N(\mu_j, \sigma_j^2)$,

$$E[e^{uX(t)}] = \exp(\lambda t(e^{\mu_j u + (1/2)\sigma_j^2 u^2} - 1)). \tag{2.1.7}$$

The jump process is constructed as follows:

$$J(t) = \frac{e^{X(t)}}{E(e^{X(t)})}. \tag{2.1.8}$$

Finally, we established the price process at time t as follows:

$$S(t) = J(t)S^c(t). \tag{2.1.9}$$

2.2 Call Option Pricing Process

In the model, it is assumed that all of the expectations are under a risk-neutral measure, Q under which the discounted asset prices are martingales. This assumption causes many advantages in the upcoming steps. Now, the call option prices can be defined as

$$C[S(0), y(0), T] = E^Q[e^{-R_T}(S(T) - K)^+ | y_1(0) = y_1, y_2(0) = y_2],$$

where $R_T = \int_0^T r(u)du$ and y is a vector such that, $y = (y_1, y_2)$. The logarithmic function is a monotone one. Therefore, instead of $S(T) \geq K$, ln S(T)≥ ln K can be used:

$$C[S(0), y(0), T] = E^Q[e^{-R_T}S(T); lnS(T) \geq lnK] - KE^Q[e^{-R_T}; lnS(T) \geq lnK], \tag{2.2.1}$$

i.e.,

$$C[S(0), y(0), T] = E^Q[e^{-R_T}S(T)\mathbf{1}_{\{\ln S(T) \geq \ln K\}}] - KE^Q[e^{-R_T}\mathbf{1}_{\{\ln S(T) \geq \ln K\}}]. \tag{2.2.2}$$

We separated the equation (2.2.2) into two parts. The first part is, $E^Q[e^{-R_T}S(T)\mathbf{1}_{\{\ln S(T) \geq \ln K\}}]$ and the second part is, $-KE^Q[e^{-R_T}\mathbf{1}_{\{\ln S(T) \geq \ln K\}}]$. To make the calculations easier, in the first expectation *change of numéraire* is computed. A *numéraire* is an asset that is used as a price unit. As an example, if (S_t) is the price process of an asset, the discounted price (S_t/S_0) can be viewed as the price of the asset, when the riskless asset is taken as a *numéraire* [24]. For the expectation given in (2.2.2), the stock process itself is taken as a *numéraire*. Consequently, the new probability measure named as Q_1:

$$\frac{dQ_1}{dQ} = \frac{e^{-R_T}S(T)}{S(0)} \quad \text{and} \quad E^{Q_1}[X] = E^Q\left[X \cdot \frac{dQ_1}{dQ}\right].$$

Note that the above Radon Nikodyn Derivative must be a martingale [34]. Therefore, the martingale property of $\frac{dQ_1}{dQ} = \frac{e^{-R_T}S(T)}{S(0)}$ should be checked. But since the discounted asset prices are martingale under the risk neutral probability measure Q, $\frac{e^{-R_T}S(T)}{S(0)}$ is a martingale automatically.

The expectation in the first part of (2.2.2) takes the form:

$$S(0) \cdot E^Q\left[\frac{e^{-R_T}S(T)}{S(0)} \cdot \mathbf{1}_{\{\ln S(T) \geq \ln K\}}\right] = S(0) \cdot E^{Q_1}[\mathbf{1}_{\{\ln S(T) \geq \ln K\}}]. \tag{2.2.3}$$

On the other hand, in the second part of (2.2.2) we take the T-forward measure as a *numéraire*. T-forward measure is a probability measure P^T defined by

$$\frac{dP^T}{dP} = \frac{e^{-\int_0^T r(s)ds}}{P(0,T)},$$

where $P(0,T)$ is a price process [24].

In our case, bond pricing function $[B(y(0),T)]$ is taken as a T-forward measure. After that, by entitling the new probability measure as Q_2, the Radon Nikodyn derivative becomes

$$\frac{dQ_2}{dQ} = \frac{e^{-R_T}B(T,T)}{B(y(0),T)}, \quad \text{where} \quad B(T,T) = 1.$$

In this case the expectation that we are concerned is as follows:

$$-K \cdot E^Q\left[\frac{e^{-R_T}}{B(y(0),T)} \cdot \mathbf{1}_{\{\ln S(T) \geq \ln K\}}\right] \cdot B(y(0),T) = -K \cdot B(y(0),T) \cdot E^{Q_2}[\mathbf{1}_{\{\ln S(T) \geq \ln K\}}]. \tag{2.2.4}$$

Finally, the equation (2.2.2) occurs as

$$C[S(0), y(0), T] = S(0) \cdot E^{Q_1}[\mathbf{1}_{\{\ln S(T) \geq \ln K\}}] - K \cdot B(y(0), T) \cdot E^{Q_2}[\mathbf{1}_{\{\ln S(T) \geq \ln K\}}]. \qquad (2.2.5)$$

We denoted the density function of $\ln S(T)$ under Q_1, Q_2 as $f^{Q_1}(x)$ and $f^{Q_2}(x)$, respectively, and distribution function as $F^{Q_i}(x)$ for $i = 1, 2,$:

$$\begin{aligned} E^{Q_i}[\mathbf{1}_{\{\ln S(T) \geq \ln K\}}] &= \int_{-\infty}^{\infty} f^{Q_i}(x) dx \quad \text{for} \quad \ln S(T) \geq \ln K. \\ &= F^{Q_i}(\ln S(T) \geq \ln K). \end{aligned} \qquad (2.2.6)$$

As a result the call price process (2.2.5) takes the form

$$\begin{aligned} C[S(0), y(0), T] &= S(0) \cdot F^{Q_1}(\ln S(T) \geq \ln K) - K \cdot B(y(0), T) \cdot F^{Q_2}(\ln S(T) \geq \ln K). \\ &= S(0) \cdot \int_{lnK}^{\infty} F_1(dx) - K \cdot B(y(0), T) \cdot \int_{lnK}^{\infty} F_2(dx), \end{aligned} \qquad (2.2.7)$$

where F_i is the distribution function of $\ln S(T)$ under Q_i for $i = 1, 2$.

2.3 Characteristic Function Using Martingale Methods

To invent the distribution functions F_1 and F_2 under Q_1, Q_2, respectively, firstly we should clarify the form of the characteristic functions $\phi_1(u)$ and $\phi_2(u)$ of the process $\ln S(T)$. Subsequently, by turning into account the inversion formula we can obtain the distribution functions. The function $\phi_1(u)$ is as follows:

$$
\begin{aligned}
\phi_1(u) \quad &= E^{Q_1}[e^{iu\ln S(T)}] \\
&= E^{Q}\left[e^{iu\ln S(T)} \cdot \frac{dQ_1}{dQ}\right] \\
&= E^{Q}\left[e^{iu\ln S(T)-R_T} \cdot \frac{S(T)}{S(0)}\right] \\
&= E^{Q}\left[\frac{e^{[(1+iu)\ln S(T)-R_T]}}{S(0)}\right],
\end{aligned} \tag{2.3.1}
$$

The characteristic function $\phi_2(u)$ can be expressed as follows:

$$
\begin{aligned}
\phi_2(u) \quad &= E^{Q_2}[e^{iu\ln S(T)}] \\
&= E^{Q}\left[e^{iu\ln S(T)} \cdot \frac{dQ_2}{dQ}\right] \\
&= E^{Q}\left[e^{iu\ln S(T)} \cdot \frac{e^{-R_T}}{B(y(0),T)}\right] \\
&= E^{Q}\left[\frac{e^{iu\ln S(T)-R_T}}{B(y(0),T)}\right].
\end{aligned} \tag{2.3.2}
$$

Define $\psi(a) = E^{Q}[\exp(-R_T + a\ln S(T))]$ for real values of a. Instead of the real number a, we wrote a complex number in the following steps. We are allowed to do this analytic extension because the exponential function in $\psi(a)$ is an entire function. The following definition can clarify the fundamental definition in complex analysis, given in [15]:

Definition 2.3.1 *A function f having a derivative at a point $z_0 \in A$ (A open) is said to be differentiable at z_0. If f has a derivative at z_0, as well as at every point of some neighborhood of z_0, it is said to be analytic. If f is analytic at every point in A, then f is analytic in A. A function that is analytic in whole complex plane is said to be entire.*

Afterwards, the equation (2.3.1) can be written as

$$
\phi_1(u) = \frac{\psi(1+iu)}{\psi(1)} \quad \text{since} \quad S(0) = E^{Q}[e^{-R_T} \cdot S(T)] = \psi(1).
$$

Furthermore, the equation (2.3.2) becomes

$$\phi_2(u) = \frac{\psi(iu)}{\psi(0)} \quad \text{since} \quad B(y(0),T) = E^Q[e^{-R_T} \cdot B(T,T)] = E^Q[e^{-R_T}] = \psi(0).$$

Note that $S(T) = J(T)S^c(T)$ implies $\ln S(T) = \ln J(T) + \ln S^c(T)$. To invent the characteristic functions above, the process $\psi(z)$ needs to be calculated but it infers that the log price process should be analyzed. Consequently,

$$\begin{aligned}
\psi(z) &= E^Q[exp(-R_T + z\ln S(T))], \\
&= E^Q[exp(-R_T + z\ln S^c(T) + z\ln J(T))], \\
&= E^Q[e^{z\ln J(T)}] \cdot E^Q[e^{-R_T + z\ln S^c(T)}], \quad \text{since} \quad J(T) \quad \text{and} \quad S^c(T) \quad \text{are independent.}
\end{aligned} \tag{2.3.3}$$

In the above equation, the term with jumps can be obtained easily. However, the part with log-price process should be examined deeply. Therefore, the following lemma in [34] will be one of the supporting steps.

Lemma 2.3.2 (Decomposition of Correlated into Independent Brownian Motions) *Suppose $B_2(t)$ and $B_1(t)$ are Brownian Motions and*

$$dB_1(t)dB_2(t) = \rho(t)d(t),$$

where ρ is a stochastic process taking values strictly between -1 and 1. Define processes $W_1(t)$ and $W_2(t)$ such that

$$\begin{aligned}
B_1(t) &= W_1(t) \\
B_2(t) &= \int_0^t \rho(s)dW_1(s) + \int_0^t \sqrt{1-\rho^2(s)}dW_2(s).
\end{aligned} \tag{2.3.4}$$

then $W_1(t)$ and $W_2(t)$ are independent Wiener Processes.

Proof. We are claiming that $d[W_1, W_2](t) = 0$. From the hypothesis,

$$dB_1(t)dB_2(t) = d[B_1, B_2](t) = \rho(t)d(t).$$

Note that $dB_1(t) = dW_1(t)$ and $dB_2(t) = \rho(t)dW_1(t) + \sqrt{1-\rho^2(t)}dW_2(t)$. Therefore,

$$\begin{aligned}\rho(t)d(t) = d[B_1, B_2](t) &= dW_1(t)\cdot[\rho(t)dW_1(t) + \sqrt{1-\rho^2(t)}dW_2(t)]\\ &= \rho(t)d(t) + \sqrt{1-\rho^2(t)}dW_1(t)dW_2(t).\end{aligned} \tag{2.3.5}$$

We are sure that $\rho(t)$ is strictly in between -1 and 1. Subsequently, equation (2.3.5) is feasible if and only if $dW_1(t)dW_2(t) = 0$. ■

At this step with the help of Lemma 2.3.2 we wrote the following coequal instead of $W(t)$ in our process:

$$W(t) = \rho dZ_1(t) + \sqrt{1-\rho^2}W'(t),$$

with $[W, Z_1] = \rho$ and $[W', Z_1] = 0$.

Hereby, by using the Ito Formula, the continuous part of stock price process ($f(S^c(t)) = log(S^c(t))$), can be expressed as follows:

$$\ln(S^c(t)) = \ln(S^c(0)) + \int_0^t \frac{dS^c(s)}{S^c(s)} - \frac{1}{2}\cdot\int_0^t \frac{d[S^c, S^c]_s}{(S^c(s))^2}.$$

Assume with out loss of generality $S^c(0) = 1$,

$$\begin{aligned}\ln(S^c(t)) \;\; = \;\; & \int_0^t r(s)ds + \sigma\sqrt{1-\rho^2}\int_0^t \sqrt{y_1(s)}dW'(s) + \sigma\rho\int_0^t \sqrt{y_1(s)}dZ_1(s)\\ - \;\; & \frac{1}{2}\cdot\int_0^t \sigma^2\rho^2 y_1(s)ds - \frac{1}{2}\cdot\int_0^t \sigma^2(1-\rho^2)y_1(s)ds.\end{aligned} \tag{2.3.6}$$

We entitle the terms of the above equality as follows:

$$R_t = \int_0^t r(s)ds,$$

$$\xi_t = \sigma\sqrt{1-\rho^2}\int_0^t \sqrt{y_1(s)}dW'(s) - \frac{1}{2}\cdot\sigma^2(1-\rho^2)\int_0^t y_1(s)ds, \tag{2.3.7}$$

$$\eta_t = \sigma\rho\int_0^t \sqrt{y_1(s)}dZ_1(s) - \frac{1}{2}\cdot\int_0^t \sigma^2\rho^2 y_1(s)ds.$$

Therefore, the continuous part of the expectation given in (2.3.3) turn into the following form:

$$\begin{aligned} E^Q[e^{-R_T+\mathrm{z}\ln S^c(T)}] &= E^Q[e^{-R_T+\mathrm{z}R_t+\mathrm{z}\xi_t+z\eta_t}] \\ &= E^Q[\exp((\mathrm{z}-1)R_t+\mathrm{z}\xi_t+z\eta_t)]. \end{aligned} \tag{2.3.8}$$

To clarify the terms given in the above equation, first we defined the sigma algebra generated by Z_1 and Z_2 as $\sigma[Z_1, Z_2]$. Second, by using the iterated expectation property, (2.3.8) is written as

$$E^Q[E^Q[\exp((z-1)R_t+z\xi_t+z\eta_t)|\ \sigma[Z_1,Z_2]]].$$

Note that under the condition, that is sigma algebra generated by Z_1 and Z_2, the terms of expectation are independent, because if Z_1 and Z_2 are known then y_1 and y_2 are also certain. In addition, the correlation between W' and Z_1 is zero. Therefore, (2.3.8) can be expressed as

$$E^Q\Big[E^Q[e^{(z-1)R_t}|\ \sigma[Z_1,Z_2]]\cdot E^Q[e^{z\xi_t}|\ \sigma[Z_1,Z_2]]\cdot E^Q[e^{z\eta_t}|\ \sigma[Z_1,Z_2]]\Big]. \tag{2.3.9}$$

In the above expectation, the terms $e^{(z-1)R_t}$ and $e^{z\eta_t}$ are $\sigma[Z_1, Z_2]$-measurable. For this reason, the following equations obtained:

$$E^Q[e^{(z-1)R_t}|\ \sigma[Z_1,Z_2]] = e^{(z-1)R_t} \quad \text{and} \quad E^Q[e^{z\eta_t}|\ \sigma[Z_1,Z_2]] = e^{z\eta_t}. \tag{2.3.10}$$

Because of the above results, only the term $E^Q[e^{z\xi_t}|\ \sigma[Z_1, Z_2]]$ in equation (2.3.9) is uncertain. To clarify it first we entitled $Y_j(t) = \int_0^t y_j(u)du$ for $j = 1, 2$. Second, note that since $\xi_t|Z_1, Z_2$ is a integral of Brownian Motion with drift given unconditionally in equation (2.3.7), it has a normal distribution with expectation $\frac{-1}{2}\sigma^2(1-\rho^2)Y_1(t)$ and variance $Var(\xi_t|Z_1, Z_2) = \sigma^2(1-\rho^2)Y_1(t)$. Therefore, $E^Q[e^{z\xi_t}|\ \sigma[Z_1, Z_2]]$ is actually the moment generating function of normal distribution with mean $\frac{-1}{2}\sigma^2(1-\rho^2)Y_1(t)$ and variance $\sigma^2(1-\rho^2)Y_1(t)$. Finally, the following equation is obtained:

$$
\begin{aligned}
E^Q[e^{z\xi_T}|\ \sigma[Z_1, Z_2]] &= exp\left(\frac{-z}{2}\sigma^2(1-\rho^2)Y_1(T) + \frac{\sigma^2(1-\rho^2)Y_1(T)z^2}{2}\right) \\
&= exp\left(z(z-1)\left[\frac{\sigma^2(1-\rho^2)Y_1(T)}{2}\right]\right).
\end{aligned}
\tag{2.3.11}
$$

By combining (2.3.10) and (2.3.11) equation (2.3.8) takes the form

$$
\begin{aligned}
E^Q[exp((z-1)R_T + z\xi_T + z\eta_T)] &= E^Q[e^{(z-1)R_T} \cdot e^{z(z-1)(\frac{\sigma^2(1-\rho^2)Y_1(T)}{2})} \cdot e^{z\eta_T}] \\
&= E^Q[e^{(z-1)Y_2(T)} \cdot e^{(z-1)Y_1(T)} \cdot e^{z(z-1)(\frac{\sigma^2(1-\rho^2)Y_1(T)}{2})} \cdot e^{z\eta_T}].
\end{aligned}
\tag{2.3.12}
$$

The above expectation consists of terms with Y_j and a term with η_T. At this step, we wrote η_T in terms of Y_j's. We know the following equation:

$$
\eta_t = \sigma\rho\int_0^t \sqrt{y_1(s)}dZ_1(s) - \frac{1}{2}\cdot\int_0^t \sigma^2\rho^2 y_1(s)ds. \tag{2.3.13}
$$

In addition, we constructed the model as

$$
dy_1(t) = K_1[\Theta_1 - y_1(t)]dt + \sigma_1\sqrt{y_1(t)}dZ_1(t).
$$

The integral form of above equation is as follows:

$$
y_1(t) - y_1(0) = K_1\Theta_1 t - K_1\int_0^t y_1(s)ds + \sigma_1\int_0^t \sqrt{y_1(s)}dZ_1(s).
$$

Therefore,

$$\int_0^t \sqrt{y_1(s)}dZ_1(s) = \frac{y_1(t) - y_1(0) - K_1\Theta_1 t - K_1 Y_1(t)}{\sigma_1}.$$

As a result, (2.3.13) takes the form

$$\eta_t = \frac{\sigma\rho}{\sigma_1} \cdot [y_1(t) - y_1(0) - K_1\Theta_1 t - K_1 Y_1(t)] - \frac{1}{2} \cdot \sigma^2\rho^2 Y_1(t).$$

Then, equation (2.3.12) can be expressed as follows:

$$\begin{aligned} &E^Q[\exp((z-1)R_T + z\xi_T + z\eta_T)] = E^Q\Big[e^{(z-1)Y_2(T)+(z-1)Y_1(T)+z(z-1)(\frac{\sigma^2(1-\rho^2)Y_1(T)}{2})} \\ &\times e^{z\frac{\sigma\rho}{\sigma_1}\cdot(y_1(T)-y_1(0)-K_1\Theta_1 T+K_1 Y_1(T))-\frac{\sigma^2\rho^2 Y_1(T)z}{2}}\Big] \\ &= E^Q[e^{(z-1)Y_2(T)} \cdot e^{wY_1(T)+z\frac{\sigma\rho}{\sigma_1}\cdot(y_1(T)-y_1(0)-K_1\Theta_1 T)}], \end{aligned} \tag{2.3.14}$$

with $w = (z-1) + z(z-1)\frac{1}{2}\sigma^2(1-\rho^2) + z\frac{\sigma\rho}{\sigma_1} \cdot K_1 - \frac{1}{2} \cdot \sigma^2\rho^2 z$.

Since y_1 and y_2 are independent, the right hand side of (2.3.14) can be written as follows:

$$\begin{aligned} E^Q[e^{(z-1)Y_2(T)} \cdot e^{wY_1(T)+z\frac{\sigma\rho}{\sigma_1}\cdot(y_1(T)-y_1(0)-K_1\Theta_1 T)}] &= E^Q[e^{(z-1)Y_2(T)}] \\ &\times E^Q[e^{wY_1(T)}] \cdot e^{z\frac{\sigma\rho}{\sigma_1}\cdot(y_1(T)-y_1(0)-K_1\Theta_1 T)}. \end{aligned} \tag{2.3.15}$$

Eventually, we reached our main purpose with the above equation. To write $\psi(z)$, we are going back to the jump process to calculate $E^Q[e^{z\ln J(T)}]$ where

$$J(T) = e^{X(T)-\lambda T(e^{\mu_j+(1/2)\sigma_j^2}-1)}.$$

Therefore the log-jump process is as follows:

$$\ln J(T) = X(T) - \lambda T(e^{\mu_j + (1/2)\sigma_j^2} - 1).$$

Presently, we are ready to construct the following equation:

$$\begin{aligned} E[e^{z\ln J(T)}] &= E^Q[e^{zX(T) - z\lambda T(e^{\mu_j+(1/2)\sigma_j^2}-1)}] \\ &= e^{-z\lambda T(e^{\mu_j+(1/2)\sigma_j^2}-1)} \cdot E^Q[e^{zX(T)}] \\ &= e^{-z\lambda T(e^{\mu_j+(1/2)\sigma_j^2}-1)+\lambda T(e^{\mu_j+(1/2)\sigma_j^2}-1)}. \end{aligned} \tag{2.3.16}$$

Finally, by combining (2.3.14) and (2.3.16), the equation (2.3.3) is expressed as

$$\begin{aligned} \psi(z) &= e^{-z\lambda T(e^{\mu_j+(1/2)\sigma_j^2}-1)+\lambda T(e^{\mu_j+(1/2)\sigma_j^2}-1)} \\ &\times e^{z\frac{\sigma\rho}{\sigma_1}\cdot(y_1(T)-y_1(0)-K_1\Theta_1 T)} \cdot E^Q[e^{(z-1)Y_2(T)}] \cdot E^Q[e^{wY_1(T)}]. \end{aligned} \tag{2.3.17}$$

Derivation of $\psi(z)$ is the main goal because the characteristic functions can be written subsequently. However, $\psi(z)$ contains unknown expectations. Note that $Y_1(T)$ and $Y_2(T)$ are following CIR processes. Consequently, we will try to get the expectations using the properties of CIR models by using the following lemma in [34] and theorem in [24].

Lemma 2.3.3 (Feyman-Kac) *Consider the stochastic differential equation*

$$\frac{\partial f}{\partial t} + \mu(x,t)\frac{\partial f}{\partial x} + \frac{1}{2}\cdot\sigma^2(x,t)\frac{\partial^2 f}{\partial x^2} = R(x,t)f, \tag{2.3.18}$$

defined or all real x and t in the interval $[0,T]$, subject to the terminal condition

$$f(x,T) = \psi(x),$$

where μ, σ, ψ, R are known functions, T is a parameter, $\tau = T - t$ and f is the unknown. Then the solution can be written as an expectation:

$$f(x,t) = E[e^{-\int_t^T R(X_\tau)d\tau} \cdot \psi(X_T)|X_t = x], \tag{2.3.19}$$

where X is an Ito-process driven by the equation,

$$dX = \mu(X,t)dt + \sigma(X,t)dW,$$

with $W(t)$ is a Brownian Motion and the initial condition for $X(t)$ is $X(0) = x$.

The proof of Feyman-Kac formula can be found in [34] in Chapter 6.

Theorem 2.3.4 *For a process X_t^x starting at x $[X_0 = x]$ and following a Cox-Ingersoll-Ross Model, that means*

$$dX_t = (a - bX_t)dt + \sigma\sqrt{X_t}dW_t \quad on \quad [0,\infty),$$

and for any non-negative λ and μ, we have

$$E[e^{-\lambda X_t^x - \mu\int_0^t X_s^x ds}] = exp(-a\varsigma_{\lambda,\mu}(t))exp(-x\Omega_{\lambda,\mu}(t)), \tag{2.3.20}$$

where the functions $\varsigma_{\lambda,\mu}(t)$ and $\Omega_{\lambda,\mu}(t)$ are given by

$$\varsigma_{\lambda,\mu}(t) = -\frac{2}{\sigma^2}ln\left(\frac{2\gamma e^{\frac{t(\gamma+b)}{2}}}{\sigma^2\lambda(e^{\gamma t}-1)+\gamma-b+e^{\gamma t}(\gamma+b)}\right)$$

and

$$\Omega_{\lambda,\mu}(t) = \frac{\lambda(\gamma+b+e^{\gamma t}(\gamma-b))+2\mu(e^{\gamma t}-1)}{\sigma^2\lambda(e^{\gamma t}-1)+\gamma-b+e^{\gamma t}(\gamma+b)}$$

with $\gamma = \sqrt{b^2+2\sigma^2\mu}$.

Proof.

For the proof, we follow the original proof given in [24] in section 6.2.2. For λ and μ fixed, consider the function $F(t,x)$ defined by

$$F(t,x) = E[e^{-\lambda X_t^x - \mu \int_0^t X_s^x ds}]. \tag{2.3.21}$$

Note that for the initial condition x we can use Feyman-Kac Formula given in Lemma 2.3.3. Therefore, we look for F as a solution of the problem,

$$\frac{\partial F}{\partial t} = \frac{\sigma^2}{2} x \frac{\partial^2 F}{\partial x^2} + (a - bx)\frac{\partial F}{\partial x} - \mu x F \quad \text{and} \quad F(0,x) = e^{-\lambda x}. \tag{2.3.22}$$

Assume that F can be written as $F(t,x) = e^{-a\varsigma(t) - x\Omega(t)}$. The reason behind is the additive property of X_t^x relative to the parameter a and the initial value x [18]. At this step, write the candidate of the solution to the given differential equation (2.3.22):

$$F(t,x)[-a\varsigma'(t) - x\Omega'(t)] = \frac{\sigma^2}{2} x[F(t,x)\Omega^2(t)] - (a - bx)F(t,x)\Omega(t) - \mu x F(t,x).$$

Canceling $F(t,x)$ from both sides of the above equation gives

$$\begin{aligned} -a\varsigma'(t) - x\Omega'(t) &= \frac{\sigma^2}{2} x\Omega^2(t) - (a - bx)\Omega(t) - \mu x \\ &= x\left[\frac{\sigma^2}{2}\Omega^2(t) + b\Omega(t) - \mu\right] - a[\Omega(t)]. \end{aligned} \tag{2.3.23}$$

In equation (2.3.23) the coefficients of a and x should be equal. Therefore, the following two equations are obtained:

$$-\Omega'(t) = \frac{\sigma^2}{2}\Omega^2(t) + b\Omega(t) - \mu \quad \textit{and} \quad \varsigma'(t) = \Omega(t) \tag{2.3.24}$$

By writing the candidate solution into the initial condition given as $F(0,x) = e^{-\lambda x}$, we get $F(0,x) = e^{-a\varsigma(0) - x\Omega(0)}$. Then,

$$e^{-\lambda x} = e^{-a\varsigma(0) - x\Omega(0)}.$$

This means that the initial conditions for (2.3.24) are $\varsigma(0) = 0$ and $\Omega(0) = \lambda$. We need to solve the following differential equations to find out $\Omega(t)$ and $\varsigma(t)$ under the initial conditions:

$$\frac{-d\Omega}{dt} = \frac{\sigma^2}{2}\Omega^2 + b\Omega - \mu.$$

That means,

$$\frac{-d\Omega}{\frac{\sigma^2}{2}\Omega^2 + b\Omega - \mu} = dt.$$

Taking the integral of both sides in the above equation,

$$-\int_0^t \frac{d\Omega}{\frac{\sigma^2}{2}\Omega^2 + b\Omega - \mu} = t + C.$$

Now, by using partial fraction technique of integral, the denominator is separated into two parts as follows:

$$\frac{1}{\gamma}\int_0^t \frac{1}{\Omega + \left(\frac{b+\gamma}{\sigma^2}\right)} d\Omega - \frac{1}{\gamma}\int_0^t \frac{1}{\Omega + \left(\frac{b-\gamma}{\sigma^2}\right)} d\Omega = t + C.$$

By taking the integrals,

$$\frac{1}{\gamma}\ln\left|\frac{\sigma^2\Omega + b + \gamma}{\sigma^2\Omega + b - \gamma}\right| = t + C. \tag{2.3.25}$$

From the initial condition of Ω we know that $\Omega(0) = \lambda$. Therefore,

$$C = \frac{1}{\gamma}\ln\left|\frac{\sigma^2\lambda + b + \gamma}{\sigma^2\lambda + b - \gamma}\right|.$$

This means that the equation (2.3.25) is as follows:

$$\ln\left|\frac{\sigma^2\Omega + b + \gamma}{\sigma^2\Omega + b - \gamma}\right| = \gamma t + \ln\left|\frac{\sigma^2\lambda + b + \gamma}{\sigma^2\lambda + b - \gamma}\right|.$$

We take the exponential of both sides and get

$$\frac{\sigma^2\Omega + b + \gamma}{\sigma^2\Omega + b - \gamma} = e^{\gamma t} \cdot \frac{\sigma^2\lambda + b + \gamma}{\sigma^2\lambda + b - \gamma}. \tag{2.3.26}$$

The above term is denoted as

$$F = e^{\gamma t} \cdot \frac{\sigma^2\lambda + b + \gamma}{\sigma^2\lambda + b - \gamma}.$$

Then equation (2.3.26) comes into the form

$$\sigma^2\Omega + b + \gamma = F\sigma^2\Omega + Fb - F\gamma.$$

Consequently,

$$\Omega = \frac{b + \gamma - Fb + F\gamma}{F\sigma^2 - \sigma^2}.$$

After writing F into the equation and making some arrangements, Ω takes its value as given in the theorem:

$$\Omega(t) = \frac{\lambda(\gamma + b + e^{\gamma t}(\gamma - b)) + 2\mu(e^{\gamma t} - 1)}{\sigma^2\lambda(e^{\gamma t} - 1) + \gamma - b + e^{\gamma t}(\gamma + b)}. \tag{2.3.27}$$

Presently, to clarify $\varsigma(t)$, we can take the given value from the theorem and see, by taking the derivative that it is really $\Omega(t)$. This is because of the equation (2.3.24). In the theorem,

$$\varsigma(t) = -\frac{2}{\sigma^2} log\left(\frac{2\gamma e^{\frac{t(\gamma+b)}{2}}}{\sigma^2\lambda(e^{\gamma t} - 1) + \gamma - b + e^{\gamma t}(\gamma + b)}\right).$$

We entitled the terms as follows:

$$S(t) = 2\gamma e^{\frac{t(\gamma+b)}{2}} \quad \text{and} \quad K(t) = \sigma^2\lambda(e^{\gamma t} - 1) + \gamma - b + e^{\gamma t}(\gamma + b).$$

By taking the derivative of $\varsigma(t)$ we get

$$\varsigma'(t) = -\frac{2}{\sigma^2}\left(\frac{S'(t)K(t) - K'(t)S(t)}{S(t)K(t)}\right).$$

Then,

$$\varsigma'(t) = -\frac{2}{\sigma^2}\left(\frac{S(t)\frac{\gamma+b}{2}K(t) - S(t)[\sigma^2\lambda e^{\gamma t}\gamma + (\gamma+b)e^{\gamma t}\gamma]}{S(t)K(t)}\right).$$

After arranging the terms,

$$\varsigma'(t) = \frac{\lambda(\gamma + b + e^{\gamma t}(\gamma - b)) + 2\mu(e^{\gamma t} - 1)}{\sigma^2\lambda(e^{\gamma t} - 1) + \gamma - b + e^{\gamma t}(\gamma + b)}. \tag{2.3.28}$$

The right hand side of equation (2.3.28) is $\Omega(t)$, which is the main goal of the theorem. ■

We will go back to our fundemental problem that is derivation of $\psi(z)$ from equation (2.3.17). At this step we can write $E^Q[e^{(z-1)Y_2(T)}]$ and $E^Q[e^{wY_1(T)}]$ by using Theorem 2.3.4, since both $Y_2(T)$ and $Y_1(T)$ follow a CIR process. We know that $Y_j(t) = \int_0^t y_j(u)du$ and $y_j(0) = y_j$ for $j = 1, 2$. In addition,

$$dy_j(t) = [K_j\Theta_j - K_jy_j(t)]dt + \sigma_j\sqrt{y_j(t)}dZ_j(t) \qquad (j = 1, 2).$$

Consequently, by using the theorem,

$$E^Q[e^{(z-1)Y_2(T)}] = \exp(-K_2\Theta_2 \cdot \varsigma(T))\exp(-y_2 \cdot \Omega(T)) \tag{2.3.29}$$

where

$$\varsigma(T) = -\frac{2}{\sigma_2^2}\ln\left(\frac{2\gamma e^{\frac{T(\gamma+K_2)}{2}}}{\gamma - K_2 + e^{\gamma T}(\gamma + K_2)}\right)$$

and

$$\Omega(T) = \frac{2(1-z)(e^{\gamma T} - 1)}{\gamma - K_2 + e^{\gamma T}(\gamma + K_2)}$$

with

$$\gamma = \sqrt{K_2^2 + 2\sigma_2^2(1-z)}.$$

The expectation with respect to $Y_1(T)$ is as follows:

$$E^Q[e^{wY_1(T)}] = \exp(-K_1\Theta_1 \cdot \varsigma(T))exp(-y_1 \cdot \Omega(T)) \tag{2.3.30}$$

where

$$\varsigma(T) = -\frac{2}{\sigma_1^2} log\left(\frac{2\gamma e^{\frac{T(\gamma+K_1)}{2}}}{\gamma - K_1 + e^{\gamma T}(\gamma + K_1)}\right)$$

and

$$\Omega(T) = \frac{2(-w)(e^{\gamma T} - 1)}{\gamma - K_1 + e^{\gamma T}(\gamma + K_1)}$$

with

$$\gamma = \sqrt{K_1^2 - 2\sigma_1^2 w}.$$

Finally, we reached to our main concern. Firstly, to find the distribution functions F_1 and F_2, we try to get characteristic functions ϕ_1 and ϕ_2. After that, because of the following relations between characteristic function and $\psi(z)$, the problem turns into finding $\psi(z)$:

$$\phi_1(u) = \frac{\psi(1+iu)}{\psi(1)} \quad \text{and} \quad \phi_2(u) = \frac{\psi(iu)}{\psi(0)}.$$

Now, we are able to calculate each term of $\psi(z)$ by using (2.3.29) and (2.3.30). Remember that,

$$
\begin{aligned}
\psi(z) &= e^{-z\lambda T(e^{\mu_j+(1/2)\sigma_j^2}-1)+\lambda T(e^{\mu_j+(1/2)\sigma_j^2}-1)} \\
&\times \quad e^{z\frac{\sigma\rho}{\sigma_1}\cdot(y_1(T)-y_1(0)-K_1\Theta_1 T)}\cdot E^Q[e^{(z-1)Y_2(T)}]\cdot E^Q[e^{wY_1(T)}].
\end{aligned}
\tag{2.3.31}
$$

Consequently, ϕ_1 and ϕ_2 are available. By using Fourier inversion formula in [33], the distributions F_1 and F_2 can be expressed as follows:

$$
F_j(x) = \frac{1}{2} + \frac{1}{2\pi}\cdot\int_0^\infty \frac{\phi_j(-u)\cdot e^{iux} - \phi_j(u)\cdot e^{-iux}}{iu}\cdot du, \tag{2.3.32}
$$

$$
1 - F_j(x) = \frac{1}{2} - \frac{1}{2\pi}\cdot\int_0^\infty \frac{\phi_j(-u)\cdot e^{iux} - \phi_j(u)\cdot e^{-iux}}{iu}\cdot du \quad \text{for} \quad (j = 1, 2).
$$

After all, the closed form solution for the call option pricing function using (2.2.7) is as follows:

$$
\begin{aligned}
C[S(t), y(t), T-t] &= S(t)\left[\frac{1}{2} - \frac{1}{2\pi}\cdot\int_0^\infty \frac{\phi_1(-u)\cdot e^{iu\ln K} - \phi_1(u)\cdot e^{-iu\ln K}}{iu}\cdot du\right] \\
&- \quad B(y(t), T-t) \\
&\times \quad K\left[\frac{1}{2} - \frac{1}{2\pi}\cdot\int_0^\infty \frac{\phi_2(-u)\cdot e^{iu\ln K} - \phi_2(u)\cdot e^{-iu\ln K}}{iu}\cdot du\right].
\end{aligned}
\tag{2.3.33}
$$

CHAPTER 3

PARTIAL INTEGRO-DIFFERENTIAL EQUATION APPROACH

In this chapter, the closed form solution of European call options under stochastic volatility and stochastic interest rate jump-diffusion model is handled in a different point of view. The method develops a partial integro-differential equation (PIDE) whose solution is the European Call Price. The solution technique suggested in [1] is based on the form of the solution of Black Scholes Model. Beginning with the similar form of Black-Scholes Call Price, the problem to solve the partial integro-differential equation is reduced to deal with some ordinary differential equations, which are easier and less complex. As in the previous chapter, we only concentrated on European call options because the closed form option prices can be reached for only this type options. This method has advantages in applications because of the estimation of few parameters when compared to Scott's approach in the previous chapter. For this reason, this model is used in the next chapter to measure the empirical performance of submodels which can be produced by letting some terms to be zero .

3.1 Asset Price Dynamics

Let (Ω, F, P) be our probability space. The underlying risky asset price at time t is assumed to follow the stochastic differential equation as

$$dS(t) = (R(t) - \lambda\mu_j)S(t)dt + \sqrt{V(t)}S(t)dW_s(t) + J(t)dN(t)S(t^-). \tag{3.1.1}$$

Moreover, $V(t)$ is defined as the diffusion component of return variance conditional on no jump occurring and assumed to follow a standard Cox-Ingersoll-Ross (CIR) process given as follows:

$$dV(t) = [\theta_v - K_v V(t)]dt + \sigma_v \sqrt{V(t)} dW_v(t), \tag{3.1.2}$$

where K_v, θ_v/K_v, and σ_v are respectively the speed of adjustment, long-run mean and variation coefficient of the diffusion volatility $V(t)$.

The underlying risky asset price instantaneous spot interest rate $R(t)$ is defined as follows:

$$dR(t) = [\theta_R - K_R R(t)]dt + \sigma_R \sqrt{R(t)} dW_R(t), \tag{3.1.3}$$

where K_R, θ_R/K_R, and σ_R are respectively the speed of adjustment, long-run mean and volatility coefficient of the process $R(t)$.

In the above equations, W_s, W_v and W_R are standard Brownian Motions. The correlation between W_s and W_v is ρ. That means $d[W_s(t), W_v(t)] = \rho dt$. The rest of the Brownian Motions is assumed to be independent from each other.

In equation (3.1.1), λ is the frequency of jumps per year, $J(t)$ is the percentage jump size conditional on a jump occurring, $N(t)$ is a Poisson Jump Counter with intensity λ, where $N(t)$ and $J(t)$ are uncorrelated. The jump size is assumed to be as follows:

$$\ln[1 + J(t)] \sim N(\ln(1 + \mu_j) - \frac{1}{2}\sigma_j^2, \sigma_j^2).$$

That means under the condition $J(t) > -1$, $1 + J(t)$ is distributed as LogNormal$(\ln(1 + \mu_j) - 1/2\sigma_j^2, \sigma_j^2)$. By using the expectation and variance formula, the following values can be found:

$$E[(1 + J(t))] = e^{ln(1+\mu_j) - 1/2\sigma_j^2 + 1/2\sigma_j^2} = 1 + \mu_j,$$

$$Var[(1 + J(t))] = e^{(\sigma_j^2 - 1)} e^{2[ln(1+\mu_j) - 1/2\sigma_j^2] + \sigma_j^2} = e^{(\sigma_j^2 - 1)} \cdot (1 + \mu_j)^2.$$

Firstly, consider a zero-coupon bond that pays 1 dollar in τ periods from time t, and let $B(t, \tau)$ be its current price. Then,

$$B(t,\tau) = E^Q\left[\exp\left(-\int_t^{t+\tau} R(u)du\right)\right],$$

where R follows a CIR Process. Note that in Theorem 2.3.4, we showed how to find such an expectation for a CIR Process. For this reason, the current value of the zero coupon bond under the risk neutral measure Q can be expressed as

$$B(t,\tau) = E^Q\left[\exp\left(-\int_t^{t+\tau} R(u)du\right)\right] = \exp[-\theta_R \cdot \varsigma(\tau) - R(t) \cdot \Omega(\tau)] \tag{3.1.4}$$

where the functions $\varsigma(\tau)$ and $\Omega(\tau)$ are given by

$$\varsigma(\tau) = -\frac{2}{\sigma_R^2}\ln\left(\frac{2\gamma e^{\frac{\tau(\gamma+K_R)}{2}}}{\gamma - K_R + e^{\gamma\tau}(\gamma + K_R)}\right)$$

and

$$\Omega(\tau) = \frac{2(e^{\gamma\tau} - 1)}{\gamma - K_R + e^{\gamma\tau}(\gamma + K_R)},$$

with $\gamma = \sqrt{K_R^2 + 2\sigma_R^2}$.

3.2 Construction of Partial Integro-Differential Equation

The processes, that are given in the previous section, are defined under the risk neutral probability measure, Q. Therefore discounted prices are martingale under this measure. By using Ito-Doeblin Formula, a partial integro-differential equation, that the discounted call prices satisfy, can be obtained. The discounted call price is formed as follows:

$$f(t,\tau; S, R, V) = e^{-\int_t^{t+\tau} R(s)ds} C(t,\tau; S, R, V),$$

where $C(t, S, R, V)$ is the time t price of the call option.

We denoted the continuous part of the stock process at time t as $S^c(t)$. From equation (3.1.1), the differential equation can be expressed as follows:

$$dS^c(t) = (R(t) - \lambda\mu_j)S(t)dt + \sqrt{V(t)}S(t)dW_s(t). \tag{3.2.1}$$

The quadratic variation of $S^c(t)$ is $d[S^c, S^c]_t = V(t)S^2(t)dt$. Because of the construction of volatility process given in (3.1.2) the quadratic variation is $d[V, V]_t = \sigma_v^2 V(t)dt$ and, similarly, from the equation (3.1.3), $d[R, R]_t = \sigma_R^2 R(t)dt$.

In the Ito Doeblin formula, we also need to know the cross variation of processes. Note that since the Brownian Motions of interest rate and stock process are independent, their cross variation is 0. By the same reasoning, the cross variation between volatility and interest rate process is 0. Furthermore, the jumps in the model is independent from all other stochastic processes. Only volatility and stock price process are correlated with each other as in the following equation,

$$d[S_c, V]_t = d[\sqrt{V(t)}S(t)W_s(t), \sigma_v\sqrt{V(t)}W_v(t)] = \sigma_v V(t)S(t)\rho dt,$$

where ρ is the correlation between two Brownian Motions.

Using above relations, the Ito-Doeblin Formula can be written as follows:

$$\begin{aligned}
&e^{-\int_0^t R(s)ds} C(t, S, R, V) - C(0, S, R, V) = \int_0^t e^{-\int_0^u R(s)ds} \cdot \Big[- R(u)C(u, S, R, V)du \\
&+C_t(u, S, R, V)du + C_S dS^c(u) + C_V dV(u) + C_R dR(u) + \tfrac{1}{2}VS^2 C_{SS}du + \tfrac{1}{2}V\sigma_v^2 C_{VV}du \\
&+\tfrac{1}{2}R\sigma_R^2 C_{RR}du + \tfrac{1}{2}\cdot 2VS\sigma_v\rho C_{SV}du\Big] \\
&+ \sum_{0<u\le t} e^{-\int_0^u R(s)ds}[C(u, S, R, V) - C(u, S^-, R, V)],
\end{aligned} \tag{3.2.2}$$

where $S^- = S(s^-)$.

Note that if s is a jump time, from the stock price process, the following expression is meaningful:

$$\Delta S(s) = J(s)dN(s)S(s^-) \quad \text{where} \quad dN(s) = 1.$$

Therefore, $S(s) - S(s^-) = J(s)S(s^-)$. As a result, $S(s) = (1 + J)S(s^-)$.

Now by using (3.1.2), (3.1.3), (3.2.1) and the above result, the Ito Doeblin Formula becomes arranged as follows:

$$\begin{aligned}
&e^{-\int_0^t R(s)ds} C(t, S, R, V) - C(0, S, R, V) = \int_0^t e^{-\int_0^u R(s)ds} \cdot \Big[- R(u)C(u, S, R, V) + C_t(u, S, R, V) \\
&+[R(t) - \lambda\mu_j]SC_S + \tfrac{1}{2}VS^2C_{SS} + [\theta_v - K_vV]C_V + \tfrac{1}{2}V\sigma_v^2C_{VV} + [\theta_R - K_RR]C_R + \tfrac{1}{2}R\sigma_R^2C_{RR} \\
&+VS\sigma_v\rho C_{SV}\Big]du + \int_0^t e^{-\int_0^u R(s)ds}\Big[\sqrt{V}SC_S dW_s(u) + \sigma_v\sqrt{V}C_V dW_v(u) + \sigma_R\sqrt{R}C_R dW_R(u)\Big] \\
&+ \sum_{0<u\leq t} e^{-\int_0^u R(s)ds}\Big[C(u, S^-(1 + J), R, V) - C(u, S^-, R, V)\Big] \qquad (3.2.3)
\end{aligned}$$

Our aim is to separate (3.2.3) into two parts. The first one will be the martingale part and the other will be the non-martingale part. As the discounted call price is a martingale under the risk neutral measure, the du terms should not exist in the equation. In fact, the partial integro-differential equation will be the the coefficient of du. Since integrals with respect to the Brownian Motions are martingale, they can be added to the martingale part. However, the summation part of the equation should be arranged so that we can separate it into two parts.

Note that the summation above represents the jump component of the call price process. We can write the summation as integral. The variable of integration is a jump measure. We used the notation given in [10] for the jump measure. In intuitive terms, for any measurable set $A \subset \mathbb{R}^d$, $M_X([0, t] \times A)$ is the number of jumps of X occurring between 0 and t, whose amplitude belongs to A. By using this notation we get the the following equation:

$$\begin{aligned}
&\sum_{0<u\leq t} e^{-\int_0^u R(s)ds}[C(u, S^-(1 + J), R, V) - C(u, S^-, R, V)] \\
&= \int_{[0,t]\times\{J>-1\}} e^{-\int_0^u R(s)ds}[C(u, S^-(1 + J), R, V) - C(u, S^-, R, V)]M(du \times dJ).
\end{aligned} \qquad (3.2.4)$$

We compensated the jump part so that it has a martingale and non-martingale part in it. As a notation, the compensated jump measure is $\widetilde{M}(du \times dJ) = M(du \times dJ) - E[M(du \times dJ)]$, where E is the expectation under the risk neutral measure. Note that the compensated jump measure is a martingale. The expectation is $E[M(du \times dJ)] = \lambda F(dJ)du$ where F is the jump size distribution. Therefore, the jump term given in (3.2.4) can be written as follows:

$$\int_{[0,t]\times\{J>-1\}} e^{-\int_0^u R(s)ds} [C(u, S^-(1+J), R, V) - C(u, S^-, R, V)][\widetilde{M}(du \times dJ) + \lambda F(dJ)du].$$

Finally we are ready to write (3.2.3) into two parts. The first part will be equipped with du terms and the second part will be martingale:

$$\begin{aligned}
&e^{-\int_0^t R(s)ds} C(t, S, R, V) - C(0, S, R, V) = \int_0^t e^{-\int_0^u R(s)ds} \cdot \Bigg[-R(u)C(u, S, R, V) + C_t(u, S, R, V) \\
&+[R(u) - \lambda\mu_j]SC_S + \tfrac{1}{2}VS^2C_{SS} + [\theta_v - K_vV]C_V + \tfrac{1}{2}V\sigma_v^2C_{VV} + [\theta_R - K_RR]C_R + \tfrac{1}{2}R\sigma_R^2C_{RR} \\
&+VS\sigma_v\rho C_{SV} + \int_{\{J>-1\}} e^{-\int_0^u R(s)ds}\Big[C(u, S^-(1+J), R, V) - C(u, S^-, R, V)\Big]\lambda F(dJ)\Bigg]du \\
&+\int_0^t e^{-\int_0^u R(s)ds}\Big[\sqrt{V}SC_S dW_s(u) + \sigma_v\sqrt{V}C_V dW_v(u) + \sigma_R\sqrt{R}C_R dW_R(u)\Big] \\
&+\int_{[0,t]\times\{J>-1\}} e^{-\int_0^u R(s)ds}\Big[C(u, S^-(1+J), R, V) - C(u, S^-, R, V)\Big]\widetilde{M}(du \times dJ). \qquad (3.2.5)
\end{aligned}$$

The differential form of above equation looks as follows:

$$\begin{aligned}
&d\Big(e^{-\int_0^t R(s)ds} C(t, S, R, V)\Big) = e^{-\int_0^t R(s)ds}\Bigg[\tfrac{1}{2}VS^2C_{SS} + [R - \lambda\mu_j]SC_S + VS\sigma_v\rho C_{SV} \\
&+\tfrac{1}{2}V\sigma_v^2C_{VV} + [\theta_v - K_vV]C_V + \tfrac{1}{2}R\sigma_R^2C_{RR} + [\theta_R - K_RR]C_R + C_t - RC \\
&+\lambda E[C(t, S^-(1+J), R, V) - C(t, S^-, R, V)]\Bigg]dt \\
&+e^{-\int_0^t R(s)ds}\Bigg[\sqrt{V}SC_S dW_s(t) + \sigma_v\sqrt{V}C_V dW_v(t) + \sigma_R\sqrt{R}C_R dW_R(t) \\
&+\int_{\{J>-1\}}[C(t, S^-(1+J), R, V) - C(t, S^-, R, V)]\widetilde{M}(dt \times dJ)\Bigg]. \qquad (3.2.6)
\end{aligned}$$

We know that the discounted call price above is a martingale. Therefore, in our equation the terms that are not martingale should not exist. We equate these parts to zero. As a result, the following Partial Integro-Differential Equation is obtained:

$$
\begin{aligned}
&\tfrac{1}{2}VS^2C_{SS} + [R-\lambda\mu_j]SC_S + VS\sigma_v\rho C_{SV} + \tfrac{1}{2}V\sigma_v^2 C_{VV} + [\theta_v - K_v V]C_V \\
&+\tfrac{1}{2}R\sigma_R^2 C_{RR} + [\theta_R - K_R R]C_R + C_t - RC \\
&+\lambda E[C(t,S^-(1+J),R,V) - C(t,S^-,R,V)] = 0.
\end{aligned}
$$

Note that the derivative of the call price with respect to t, C_t is equal to minus the derivative of the call price with respect to τ, $-C_\tau$. The reason behind is $t+\tau = T$, where T is a constant. Therefore $C_t + C_\tau = 0$. To construct the same notation with the reference, we will use the following equation in the rest of the study:

$$
\begin{aligned}
&\tfrac{1}{2}VS^2C_{SS} + [R-\lambda\mu_j]SC_S + VS\sigma_v\rho C_{SV} + \tfrac{1}{2}V\sigma_v^2 C_{VV} + [\theta_v - K_v V]C_V \\
&+\tfrac{1}{2}R\sigma_R^2 C_{RR} + [\theta_R - K_R R]C_R - C_\tau - RC \\
&+\lambda E[C(t,S^-(1+J),R,V) - C(t,S^-,R,V)] = 0,
\end{aligned}
\tag{3.2.7}
$$

subject to $C(T,0) = \max[S(T) - K, 0]$.

3.3 Solution of Partial Integro-Differential Equation

In fact the call price that we are trying to find is the solution of equation (3.2.7). The general method to find out the solution of such a partial integro-differential equation is inspired from the call price of Black Scholes Model [7]. The call price calculated in Black Scholes Model is of the following form:

$$C(S,t) = N(d_1)S - N(d_2)Ke^{-r(T-t)},$$

where $N(\cdot)$ is the cumulative distribution function of standard normal distribution,
S is the underlying asset price,
K is the strike price of the European Call Option,

$T - t$ is the time to maturity,
r is the risk free rate.

Inspired from Black Scholes Model European Call Option Price, the European Call Option Price of Stochastic Volatility and Stochastic Interest Rate Jump-Diffusion Model Price can be constructed as follows. Assume that the solution of the partial integro-differential equation (3.2.7) is of the following form:

$$C(t,\tau) = \pi_1(t,\tau,S,R,V)S(t) - KB(t,\tau)\pi_2(t,\tau,S,R,V), \tag{3.3.1}$$

where π_1 and π_2 are the probability functions that are calculated by inverting the respective characteristic functions,
S is the underlying asset price,
K is the strike price of the European Call Option,
τ is the time to maturity,
$B(t,\tau)$ is the zero-coupon bond that pays 1 dollar in τ periods from time t.

As it can be seen above, the form of the solution is similar to the form of Black Scholes. The first difference in our solution is that the discount factor is not in exponential form. Instead, it is the bond price, because the interest rate is not constant in the model. Another difference is that our probabilities are not normal as in the case of Black Scholes.

Now we will apply the transformation $L(t) = \ln(S(t))$ to the terms of equation (3.2.7). Firstly under this transformation the difference of call price $C(t,S^-(1+J),R,V) - C(t,S^-,R,V)$ takes the form $C(t,L+\ln(1+J),R,V) - C(t,L,R,V)$. To get the idea behind the jump size should be converted with logarithmic function. The stock price process itself jumps with an amount $\Delta S = S^-J$ and it changes from S^- to $S^- + S^-J$. Therefore the logarithm of the stock price process will change from $\ln(S^-)$ to $\ln(S^-+S^-J)$. Consequently, $\Delta L = \ln(S^-+S^-J) - \ln(S^-) = \ln(1+J)$.

The conjectured solution given in (3.3.1) will be the starting point of derivatives and differences of call price as follows:

$$C_S = \pi_1 + \frac{\partial \pi_1}{\partial L} - \frac{KB}{S}\frac{\partial \pi_2}{\partial L}, \tag{3.3.2}$$

$$C_{SS} = \frac{\partial \pi_1}{\partial L}\frac{1}{S} + \frac{\partial^2 \pi_1}{\partial L^2}\frac{1}{S} - KB\left[-\frac{1}{S^2}\frac{\partial \pi_2}{\partial L} + \frac{1}{S^2}\frac{\partial^2 \pi_2}{\partial L^2}\right], \tag{3.3.3}$$

$$C_V = S\frac{\partial \pi_1}{\partial V} - KB\frac{\partial \pi_2}{\partial V}, \tag{3.3.4}$$

$$C_{VV} = S\frac{\partial^2 \pi_1}{\partial V^2} - KB\frac{\partial^2 \pi_2}{\partial V^2}, \tag{3.3.5}$$

$$C_{SV} = \frac{\partial \pi_1}{\partial V} + \frac{\partial^2 \pi_1}{\partial L \partial V} - KB\frac{\partial^2 \pi_2}{\partial L \partial V}\frac{1}{S}, \tag{3.3.6}$$

$$C_R = S\frac{\partial \pi_1}{\partial R} - KB\frac{\partial \pi_2}{\partial R} - K\pi_2\frac{\partial B}{\partial R}, \tag{3.3.7}$$

$$C_{RR} = S\frac{\partial^2 \pi_1}{\partial R^2} - KB\frac{\partial^2 \pi_2}{\partial R^2} - K\frac{\partial \pi_2}{\partial R}\frac{\partial B}{\partial R} - K\left[\frac{\partial \pi_2}{\partial R}\frac{\partial B}{\partial R} + \pi_2\frac{\partial^2 B}{\partial R^2}\right], \tag{3.3.8}$$

$$C_t = S\frac{\partial \pi_1}{\partial t} - KB\frac{\partial \pi_2}{\partial t} - K\pi_2\frac{\partial B}{\partial t},$$

$$C_\tau = -S\frac{\partial \pi_1}{\partial \tau} + KB\frac{\partial \pi_2}{\partial \tau} + K\pi_2\frac{\partial B}{\partial \tau}. \tag{3.3.9}$$

The difference of call option prices can be constructed as follows:

$$\begin{aligned} C(t, L + \ln(1+J), R, V) - C(t, L, R, V) &= S^-(1+J)\pi_1(t, L + \ln(1+J), R, V) \\ &- KB\pi_2(t, L + \ln(1+J), R, V) - S^-\pi_1(t, L, R, V) \\ &+ KB\pi_2(t, L, R, V). \end{aligned} \tag{3.3.10}$$

Substituting all of the above results into the partial integro-differential equation (3.2.7), we obtain the following equations with respect to probabilities π_1, π_2 and a differential equation of bond price:

$$
\begin{aligned}
&S^{-}\Bigg[\frac{1}{2}V\frac{\partial^2\pi_1}{\partial L^2}+[R-\lambda\mu_j+\frac{1}{2}V]\frac{\partial\pi_1}{\partial L}+[\theta_v-(K_v-\sigma_v\rho)V]\frac{\partial\pi_1}{\partial V}\\
&+\sigma_v\rho V\frac{\partial^2\pi_1}{\partial L\partial V}+\frac{1}{2}R\sigma_R^2\frac{\partial^2\pi_1}{\partial R^2}+[\theta_R-K_RR]\frac{\partial\pi_1}{\partial R}\\
&+[R-\lambda\mu_j]\pi_1-R\pi_1+\frac{1}{2}V\sigma_v^2\frac{\partial^2\pi_1}{\partial V^2}-\frac{\partial\pi_1}{\partial\tau}+\lambda E[(1+J)\pi_1(t,L+\ln(1+J),R,V)\\
&-\pi_1(t,L,R,V)]\Bigg]=0,
\end{aligned} \tag{3.3.11}
$$

$$
\begin{aligned}
&-KB\Bigg[\frac{1}{2}V\frac{\partial^2\pi_2}{\partial L^2}+[R-\lambda\mu_j-\frac{1}{2}V]\frac{\partial\pi_2}{\partial L}+\sigma_v\rho V\frac{\partial^2\pi_2}{\partial L\partial V}\\
&+\frac{1}{2}V\sigma_v^2\frac{\partial^2\pi_2}{\partial V^2}+[\theta_v-K_vV]\frac{\partial\pi_2}{\partial V}+\frac{1}{2}R\sigma_R^2\frac{\partial^2\pi_2}{\partial R^2}\\
&+\Bigg[\theta_R-(K_R-\frac{\sigma_R^2}{B}\frac{\partial B}{\partial R})R\Bigg]\frac{\partial\pi_2}{\partial R}-\frac{\partial\pi_2}{\partial\tau}+\lambda E[\pi_2(t,L+\ln(1+J),R,V)-\pi_2(t,L,R,V)]\Bigg]=0,
\end{aligned} \tag{3.3.12}
$$

$$
-K\pi_2\Bigg[\frac{1}{2}R\sigma_R^2\frac{\partial^2B}{\partial R^2}+[\theta_R-K_RR]\frac{\partial B}{\partial R}+\frac{\partial B}{\partial t}-RB\Bigg]=0. \tag{3.3.13}
$$

Equations (3.3.11) and (3.3.12) are equal to 0 because these two equations have independent derivatives in them and the partial integro-differential equation itself is equated with zero. The reason of equality to zero of equation (3.3.13) is because of the Ito Formula and martingale property. We can prove (3.3.13) as follows:

$$
d\Bigg(e^{-\int_0^t R(s)ds}B(0,t,R)\Bigg)=e^{-\int_0^t R(s)ds}\Bigg[-R(t)B(0,t,R)dt+dB(0,t,R)\Bigg]. \tag{3.3.14}
$$

This equation should not have term dt since discounted prices are martingale under the risk neutral measure. To find $dB(0,t,R)$ we applied the Ito Doeblin Formula and used the interest rate structure given as:

$$
dR(t)=[\theta_R-K_RR(t)]dt+\sigma_R\sqrt{R(t)}dW_R(t).
$$

By Ito Doeblin Formula,

$$
\begin{aligned}
dB(0,t,R) \quad &= \frac{\partial B}{\partial t}dt + \frac{\partial B}{\partial R}dR(t) + \frac{1}{2}\frac{\partial^2 B}{\partial R^2}R\sigma_R^2 dt \\
&= \frac{\partial B}{\partial t}dt + +\frac{\partial B}{\partial R}\left[[\theta_R - K_R R(t)]dt + \sigma_R\sqrt{R(t)}dW_R(t)\right] \\
&+\frac{1}{2}\frac{\partial^2 B}{\partial R^2}R\sigma_R^2 dt. \qquad (3.3.15)
\end{aligned}
$$

By writing the equation above into (3.3.14) and arranging the terms we get the main goal, that is equation (3.3.13).

Finally the differential equations (3.3.11) and (3.3.12) for probabilities π_1 and π_2, respectively, can be written as follows, since S^- and KB are nonzero:

$$
\begin{aligned}
&\frac{1}{2}V\frac{\partial^2\pi_1}{\partial L^2} + [R - \lambda\mu_j + \frac{1}{2}V]\frac{\partial\pi_1}{\partial L} + [\theta_v - (K_v - \sigma_v\rho)V]\frac{\partial\pi_1}{\partial V} + \sigma_v\rho V\frac{\partial^2\pi_1}{\partial L\partial V} \\
&+\frac{1}{2}R\sigma_R^2\frac{\partial^2\pi_1}{\partial R^2} + [\theta_R - K_R R]\frac{\partial\pi_1}{\partial R} - \lambda\mu_j\pi_1 \\
&+\frac{1}{2}V\sigma_v^2\frac{\partial^2\pi_1}{\partial V^2} - \frac{\partial\pi_1}{\partial\tau} + \lambda E[(1+J)\pi_1(t, L + ln(1+J), R, V) - \pi_1(t, L, R, V)] = 0,
\end{aligned}
\tag{3.3.16}
$$

$$
\begin{aligned}
&\frac{1}{2}V\frac{\partial^2\pi_2}{\partial L^2} + [R - \lambda\mu_j - \frac{1}{2}V]\frac{\partial\pi_2}{\partial L} + \sigma_v\rho V\frac{\partial^2\pi_2}{\partial L\partial V} + \frac{1}{2}V\sigma_v^2\frac{\partial^2\pi_2}{\partial V^2} + [\theta_v - K_v V]\frac{\partial\pi_2}{\partial V} \\
&+\frac{1}{2}R\sigma_R^2\frac{\partial^2\pi_2}{\partial R^2} + \left[\theta_R - (K_R - \frac{\sigma_R^2}{B}\frac{\partial B}{\partial R})R\right]\frac{\partial\pi_2}{\partial R} \\
&-\frac{\partial\pi_2}{\partial\tau} + \lambda E[\pi_2(t, L + ln(1+J), R, V) - \pi_2(t, L, R, V)] = 0.
\end{aligned}
\tag{3.3.17}
$$

The above equations are called *Fokker Planck forward equations for probability functions.* These partial differential equations must be solved separately subject to the terminal conditions:

$$
\pi_i(T, 0) = \mathbf{1}_{\{L(T)\geq K\}} \quad \text{for} \quad i = 1, 2.
$$

In fact, the Fokker-Planck forward equations (also known as the Kolmogorov forward equations) provide a relation between the differential equation of probability functions and the stochastic process itself. The details of these forward equations about continuous processes can be examined in [6] and the processes with jumps in [13]. In fact, using the corresponding characteristic functions instead of probabilities is easier because the closed form of probabilities may not be available but the characteristic functions always exist. The Fokker-Planck forward equations are very important at this step. Firstly, we constructed the corresponding stochastic processes by looking at the Fokker-Planck forward equations above. Then, by using these processes, we showed that the probabilities and characteristic functions satisfy the same PIDE. After that instead of the probabilities we dealt with the characteristic functions. The details of this methodology for the stochastic volatility model are pointed out in the original paper of Heston (1993), and the detailed version can be investigated in [12]. Note that the construction of the process for the probability π_2 is standard, but for π_1 it is slightly trickier. The following property of normal distribution is necessary in that sense. The property is as follows:

$$E(e^z f(z)) = e^{\bar{z}+\frac{1}{2}\sigma_z^2} E(z^*),$$

where $Z \sim N(\bar{z}, \sigma_z^2)$ and $Z^* \sim N(\bar{z}+\sigma_z^2, \sigma_z^2)$.

The proof of the above equation is a consequence of writing the pdf of normal distributions and calculating the expectation as

$$\begin{aligned}
E(e^z f(z)) &= \int_{-\infty}^{\infty} e^z f(z) e^{-\frac{(z-\bar{z})^2}{2\sigma_z^2}} \frac{1}{\sqrt{2\pi\sigma_z^2}} dz \\
&= \int_{-\infty}^{\infty} f(z) e^{-\frac{(z-(\bar{z}+\sigma_z^2))^2}{2\sigma_z^2}} e^{-\frac{\sigma_z^4+2\sigma_z^2\bar{z}}{2\sigma_z^2}} \frac{1}{\sqrt{2\pi\sigma_z^2}} dz \\
&= e^{\bar{z}+\frac{1}{2}\sigma_z^2} \int_{-\infty}^{\infty} f(z) e^{-\frac{(z-(\bar{z}+\sigma_z^2))^2}{2\sigma_z^2}} \frac{1}{\sqrt{2\pi\sigma_z^2}} dz \\
&= e^{\bar{z}+\frac{1}{2}\sigma_z^2} E(z^*), \qquad (3.3.18)
\end{aligned}$$

with $Z \sim N(\bar{z}, \sigma_z^2)$ and $Z^* \sim N(\bar{z}+\sigma_z^2, \sigma_z^2)$.

By using the above result we arranged the terms with jumps in the integro-differential equation of π_1 as follows:

$$\begin{aligned}
-\lambda\mu_j\pi_1 + \lambda \quad E \quad & [(1+J)\pi_1(t, L + \ln(1+J), R, V) - \pi_1(t, L, R, V)] = -\lambda E(J)\pi_1 \\
+ \quad & \lambda E[(1+J)\pi_1(t, L + \ln(1+J), R, V) - \pi_1(t, L, R, V)] \\
= \quad & \lambda E[(1+J)[\pi_1(t, L + \ln(1+J), R, V) - \pi_1(t, L, R, V)]] \\
= \quad & \lambda(1+\mu_j)E[\pi_1(t, L + \ln^*(1+J), R, V) - \pi_1(t, L, R, V)], \qquad (3.3.19)
\end{aligned}$$

with $\ln[1 + J(t)] \sim \ N(\ln(1+\mu_j) - \frac{1}{2}\sigma_j^2, \sigma_j^2)$,

and $\ln^*[1 + J(t)] \sim \ N(\ln(1+\mu_j) + \frac{1}{2}\sigma_j^2, \sigma_j^2)$.

Using the results given in above equations and dividing each term with $(1+\mu_j)$, the integro-differential equation of π_1 can be written as

$$\begin{aligned}
& \frac{1}{2}\frac{V}{(1+\mu_j)}\frac{\partial^2\pi_1}{\partial L^2} + \frac{[R - \lambda\mu_j + \frac{1}{2}V]}{(1+\mu_j)}\frac{\partial\pi_1}{\partial L} + \frac{[\theta_v - (K_v - \sigma_v\rho)V]}{(1+\mu_j)}\frac{\partial\pi_1}{\partial V} + \frac{\sigma_v\rho V}{(1+\mu_j)}\frac{\partial^2\pi_1}{\partial L\partial V} \\
& + \frac{1}{2}\frac{R\sigma_R^2}{(1+\mu_j)}\frac{\partial^2\pi_1}{\partial R^2} + \frac{[\theta_R - K_R R]}{(1+\mu_j)}\frac{\partial\pi_1}{\partial R} \\
& + \frac{1}{2}\frac{V\sigma_v^2}{(1+\mu_j)}\frac{\partial^2\pi_1}{\partial V^2} + \frac{1}{(1+\mu_j)}\frac{\partial\pi_1}{\partial t} + \lambda(E[\pi_1(t, L + \ln^*(1+J), R, V) - \pi_1(t, L, R, V)] = 0.
\end{aligned}$$

Define the following time change in the above equation, we get

$$\frac{\partial\pi_1}{\partial t}\frac{1}{(1+\mu_j)} = \frac{\partial\pi_1}{\partial t^*}, \quad \text{where} \quad t^* = \frac{t}{(1+\mu_j)}.$$

Therefore, the PIDE corresponding to π_1 is as follows:

$$\frac{1}{2}\frac{V}{(1+\mu_j)}\frac{\partial^2\pi_1}{\partial L^2}+\frac{[R-\lambda\mu_j+\frac{1}{2}V]}{(1+\mu_j)}\frac{\partial\pi_1}{\partial L}+\frac{[\theta_v-(K_v-\sigma_v\rho)V]}{(1+\mu_j)}\frac{\partial\pi_1}{\partial V}+\frac{\sigma_v\rho V}{(1+\mu_j)}\frac{\partial^2\pi_1}{\partial L\partial V}$$
$$+\frac{1}{2}\frac{R\sigma_R^2}{(1+\mu_j)}\frac{\partial^2\pi_1}{\partial R^2}+\frac{[\theta_R-K_RR]}{(1+\mu_j)}\frac{\partial\pi_1}{\partial R}$$
$$+\frac{1}{2}\frac{V\sigma_v^2}{(1+\mu_j)}\frac{\partial^2\pi_1}{\partial V^2}+\frac{\partial\pi_1}{\partial t^*}+\lambda(E[\pi_1(t^*,L+\ln^*(1+J),R,V)-\pi_1(t^*,L,R,V)]=0.$$

We wrote the following PIDE for π_2 as a reminder:

$$\frac{1}{2}V\frac{\partial^2\pi_2}{\partial L^2}+[R-\lambda\mu_j-\frac{1}{2}V]\frac{\partial\pi_2}{\partial L}+\sigma_v\rho V\frac{\partial^2\pi_2}{\partial L\partial V}+\frac{1}{2}V\sigma_v^2\frac{\partial^2\pi_2}{\partial V^2}+[\theta_v-K_vV]\frac{\partial\pi_2}{\partial V}$$
$$+\frac{1}{2}R\sigma_R^2\frac{\partial^2\pi_2}{\partial R^2}+\left[\theta_R-(K_R-\frac{\sigma_R^2}{B}\frac{\partial B}{\partial R})R\right]\frac{\partial\pi_2}{\partial R}$$
$$+\frac{\partial\pi_2}{\partial t}+\lambda E[\pi_2(t,L+\ln(1+J),R,V)-\pi_2(t,L,R,V)]=0.$$

The PIDE for π_1 and π_2 can be expressed in a general form as follows:

$$\frac{1}{2}\frac{V}{a_i}\frac{\partial^2P}{\partial L^2}+\frac{[R-\lambda\mu_j+b_iV]}{a_i}\frac{\partial P}{\partial L}+\frac{\sigma_v\rho V}{a_i}\frac{\partial^2P}{\partial L\partial V}+\frac{1}{2}\frac{V\sigma_v^2}{a_i}\frac{\partial^2P}{\partial V^2}+\frac{[\theta_v-c_iV]}{a_i}\frac{\partial P}{\partial V}$$
$$+\frac{1}{2}\frac{R\sigma_R^2}{a_i}\frac{\partial^2P}{\partial R^2}+[\theta_R-(K_R-d_iR)]\frac{\partial P}{\partial R}$$
$$+\frac{\partial P}{\partial t_i}+\lambda E[P(t_i,L+\ln_i(1+J),R,V)-P(t_i,L,R,V)]=0 \tag{3.3.20}$$

for $i=1,2,$ where $a_1=(1+\mu_j)$, $a_2=1$, $b_1=\frac{1}{2}$, $b_2=-\frac{1}{2}$, $c_1=(K_v-\sigma_v\rho)$, $c_2=K_v$,
$d_1=K_R$, $d_2=\frac{\sigma_R^2}{B}\frac{\partial B}{\partial R}$, $t_1=t^*$, $t_2=t$, $\ln_1(1+J)=\ln^*(1+J)$,
$\ln_2(1+J)=\ln(1+J)$.

We can write the corresponding stochastic differential equation by looking at the above probabilities. The spot price process, stochastic volatility and stochastic interest rate dynamics are as follows:

$$
\begin{aligned}
dL_{t_i} &= \frac{[R - \lambda\mu_j + b_i V]}{a_i} dt_i + \sqrt{\frac{V}{a_i}} dW_S + ln_i(1+J)dN, \\
dV_{t_i} &= \frac{[\theta_v - c_i V]}{a_i} dt_i + \sigma_v \sqrt{\frac{V}{a_i}} dW_L, \\
dR_{t_i} &= [\theta_R - (K_R - d_i R)] dt_i + \sigma_R \sqrt{\frac{R}{a_i}} dW_R.
\end{aligned} \tag{3.3.21}
$$

Subsequently, define any twice differentiable function f as a conditional expectation of L, V, R, where $L(t_i)$, $V(t_i)$, $R(t_i)$ follow the risk neutral processes given by (3.3.21). This function is constructed as follows:

$$
f(L, V, R, t_i) := E[g(L(T), V(T), R(T)|L(t_i) = L, V(t_i) = V, R(t_i) = R)]
$$

subject to the terminal condition

$$
f(L, V, R, T) = g(L, V, R).
$$

In fact, this function is a martingale under the risk neutral probability measure. In the thesis given in [12], one of the three conditions of a martingale, that is the tower property, is proved.

We can establish by using Ito-Doeblin Formula the PIDE that f satisfies as follows:

$$
\begin{aligned}
df &= \frac{\partial f}{\partial t_i} dt_i + \frac{\partial f}{\partial L} dL + \frac{\partial f}{\partial V} dV + \frac{\partial f}{\partial R} dR + \frac{1}{2}\frac{\partial^2 f}{\partial L^2} d < L, L > + + \frac{1}{2}\frac{\partial^2 f}{\partial V^2} d < V, V > + \\
&\frac{1}{2}\frac{\partial^2 f}{\partial R^2} d < R, R > + \frac{\partial^2 f}{\partial L \partial V} d < L, V > + \frac{\partial^2 f}{\partial L \partial R} d < L, R > + \frac{\partial^2 f}{\partial V \partial R} d < V, R > + \\
&\int_{[0,t_i]\times\{J>-1\}} [f(u, L + \ln_i(1+J), R, V) - f(u, L, R, V)] M(du \times dJ)
\end{aligned} \tag{3.3.22}
$$

and, after using equations in (3.3.21),

$$
\begin{aligned}
df = &\left[\frac{\partial f}{\partial t_i} + \frac{\partial f}{\partial L}\frac{[R - \lambda\mu_j + b_i V]}{a_i} + \frac{\partial f}{\partial V}\frac{[\theta_v - c_i V]}{a_i} + \frac{\partial f}{\partial R}[\theta_R - (K_R - d_i R)] + \frac{1}{2}\frac{\partial^2 f}{\partial L^2}\frac{V}{a_i}\right. \\
&\frac{1}{2}\frac{\partial^2 f}{\partial V^2}\frac{V}{a_i} + \frac{1}{2}\frac{\partial^2 f}{\partial R^2}\frac{R}{a_i} + \frac{\partial^2 f}{\partial L \partial V}\frac{\sigma_v \rho V}{a_i} + \int_{\{J>-1\}} [f(u, L + \ln_i(1 + J), R, V) - \\
&\left. f(u, L, R, V)]\lambda F(dJ)\right] dt_i + \int_{\substack{[0,t_i]\times \\ \{J>-1\}}} [f(u, L + \ln_i(1 + J), R, V) - f(u, L, R, V)]\widetilde{M}(du \times dJ) + \\
&\frac{\partial f}{\partial L}\sqrt{\frac{V}{a_i}} dW_S + \frac{\partial f}{\partial V}\sigma_v \sqrt{\frac{V}{a_i}} dW_L + \frac{\partial f}{\partial R}\sigma_R \sqrt{\frac{R}{a_i}} dW_R.
\end{aligned} \tag{3.3.23}
$$

Consequently, since f is a martingale, in the above equations the non-martingale part becomes zero as in the following PIDE,

$$
\begin{aligned}
&\frac{1}{2}\frac{V}{a_i}\frac{\partial^2 f}{\partial L^2} + \frac{[R - \lambda\mu_j + b_i V]}{a_i}\frac{\partial f}{\partial L} + \frac{\sigma_v \rho V}{a_i}\frac{\partial^2 f}{\partial L \partial V} + \frac{1}{2}\frac{V\sigma_v^2}{a_i}\frac{\partial^2 f}{\partial V^2} + \frac{[\theta_v - c_i V]}{a_i}\frac{\partial f}{\partial V} \\
&+\frac{1}{2}\frac{R\sigma_R^2}{a_i}\frac{\partial^2 f}{\partial R^2} + [\theta_R - (K_R - d_i R)]\frac{\partial f}{\partial R} + \frac{\partial f}{\partial t_i} \\
&+\lambda E[f(t_i, L + \ln_i(1 + J), R, V) - f(t_i, L, R, V)] = 0
\end{aligned} \tag{3.3.24}
$$

subject to the terminal condition

$$
f(L, V, R, T) = g(L, V, R). \tag{3.3.25}
$$

Note that the above equation has the same form with equation (3.3.20). With the proper choice of g we can get the characteristic functions corresponding to π_1 and π_2. Choose g as

$$
g(L, V, R) = e^{i\phi L}.
$$

Therefore, the function f is in this case as follows:

$$
f(L, V, R, t_i) := E[e^{i\phi L} | L(t_i) = L, V(t_i) = V, R(t_i) = R]. \tag{3.3.26}
$$

It is clear that the solution of equation (3.3.24) gives the characteristic functions of π_1 and π_2.

Finally we reached the desired solution. We can say that the probability functions and their corresponding characteristic functions will satisfy the same PIDEs. By using this result, we employ the PIDEs of characteristic functions instead of probability functions. Then, by using the inversion formula, we can get the probability functions. This transition is essential because, one can view the characteristic function as a contingent claim to be solved using the standard contingent claims' partial integro-differential equation under relatively easy boundary conditions [4].

We entitled the characteristic functions corresponding to π_1 and π_2 as f_1 and f_2, respectively. The following equations can be written for the characteristic functions by going back to equations (3.3.16) and (3.3.17):

$$\begin{aligned}
&\frac{1}{2}V\frac{\partial^2 f_1}{\partial L^2} + [R - \lambda\mu_j + \frac{1}{2}V]\frac{\partial f_1}{\partial L} + [\theta_v - (K_v - \sigma_v\rho)V]\frac{\partial f_1}{\partial V} + \sigma_v\rho V\frac{\partial^2 f_1}{\partial L\partial V} \\
&+\frac{1}{2}R\sigma_R^2\frac{\partial^2 f_1}{\partial R^2} + [\theta_R - K_R R]\frac{\partial f_1}{\partial R} - \lambda\mu_j f_1 \\
&+\frac{1}{2}V\sigma_v^2\frac{\partial^2 f_1}{\partial V^2} - \frac{\partial f_1}{\partial\tau} + \lambda E[(1+J)f_1(t, L + \ln(1+J), R, V) - f_1(t, L, R, V)] = 0
\end{aligned} \tag{3.3.27}$$

and

$$\begin{aligned}
&\frac{1}{2}V\frac{\partial^2 f_2}{\partial L^2} + [R - \lambda\mu_j - \frac{1}{2}V]\frac{\partial f_2}{\partial L} + \sigma_v\rho V\frac{\partial^2 f_2}{\partial L\partial V} + \frac{1}{2}V\sigma_v^2\frac{\partial^2 f_2}{\partial V^2} + [\theta_v - K_v V]\frac{\partial f_2}{\partial V} \\
&+\frac{1}{2}R\sigma_R^2\frac{\partial^2 f_2}{\partial R^2} + \left[\theta_R - (K_R - \frac{\sigma_R^2}{B}\frac{\partial B}{\partial R})R\right]\frac{\partial f_2}{\partial R} \\
&-\frac{\partial f_2}{\partial\tau} + \lambda E[f_2(t, L + \ln(1+J), R, V) - f_2(t, L, R, V)] = 0
\end{aligned} \tag{3.3.28}$$

with the boundary conditions

$$f_i(T, 0, \phi) = e^{i\phi L(T)} \quad \text{for} \quad i = 1, 2.$$

The method to find the solution of the PDEs is very common in applied mathematics. We start with a conjecture solution and insert the solution into the PDE. Finally, we reach the terms of the solution. For this reason assume that the solutions of (3.3.27) and (3.3.28) are in the following form

$$f_1(t, \tau, S, R, V, \phi) = \exp(u(\tau) + x_r(\tau)R(t) + x_v(\tau)V(t) + i\phi\ln[S(t)]). \tag{3.3.29}$$

$$f_2(t,\tau,S,R,V,\phi) = \exp(z(\tau) + y_r(\tau)R(t) + y_v(\tau)V(t) + i\phi\ln[S(t)] - \ln[B(t,\tau)]), \qquad (3.3.30)$$

with $u(0) = x_r(0) = x_v(0) = 0$ and $z(0) = y_r(0) = y_v(0) = 0$ with $B(T,0) = 1$.

Finally, we solved the PDEs by writing the conjecture solutions given above, into the equations of f_1 and f_2. Starting with f_1 we can get the following differentials by taking derivative of conjecture solution,

$$\frac{\partial f_1}{\partial L} = f_1 \cdot (i\phi), \qquad (3.3.31)$$

$$\frac{\partial^2 f_1}{\partial L^2} = f_1 \cdot (i\phi)^2 = -f_1 \cdot \phi^2, \qquad (3.3.32)$$

$$\frac{\partial f_1}{\partial V} = f_1 \cdot x_v(\tau), \qquad (3.3.33)$$

$$\frac{\partial^2 f_1}{\partial V^2} = f_1 \cdot x_v(\tau)^2, \qquad (3.3.34)$$

$$\frac{\partial^2 f_1}{\partial L \partial V} = f_1 \cdot x_v(\tau) \cdot (i\phi), \qquad (3.3.35)$$

$$\frac{\partial f_1}{\partial R} = f_1 \cdot x_r(\tau), \qquad (3.3.36)$$

$$\frac{\partial^2 f_1}{\partial R^2} = f_1 \cdot x_r(\tau)^2, \qquad (3.3.37)$$

$$\frac{\partial f_1}{\partial \tau} = f_1 \cdot (u'(\tau) + x_r'(\tau)R(t) + x_v'(\tau)V(t)), \qquad (3.3.38)$$

and the terms in jump part can be written as

$$f_1(t,\tau,L+\ln(1+J),R,V) = \exp(u(\tau)+x_r(\tau)R(t)+x_v(\tau)V(t)+i\phi L+i\phi\ln(1+J)) = f_1\cdot e^{i\phi\ln(1+J)}, \tag{3.3.39}$$

where $f_1 = f_1(t,\tau,L,R,V)$.

By writing all of the above solutions into the equation (3.3.27), we get

$$\Bigg[-\frac{1}{2}V\phi^2+[R-\lambda\mu_j+\frac{1}{2}V]i\phi+\sigma_v\rho Vx_v(\tau)\cdot(i\phi)+\frac{1}{2}V\sigma_v^2x_v(\tau)^2+[\theta_v-(K_v-\sigma_v\rho)V]x_v(\tau)$$
$$+\frac{1}{2}R\sigma_R^2x_r(\tau)^2+[\theta_R-K_RR]x_r(\tau)-(u'(\tau)+x_r'(\tau)R+x_v'(\tau)V)-\lambda\mu_j$$
$$+\lambda E[(1+J)e^{i\phi ln(1+J)}-1]\Bigg]\times f_1 = 0. \tag{3.3.40}$$

Equation (3.3.40) has terms with the volatility process V, interest rate process R, and terms without V and R which are independent from each other. Therefore each of these terms should be zero. As a result, we can construct the following equations:

$$\Bigg[-\frac{1}{2}\phi^2+\frac{1}{2}i\phi+\sigma_v\rho x_v(\tau)\cdot(i\phi)+\frac{1}{2}\sigma_v^2x_v(\tau)^2-(K_v-\sigma_v\rho)x_v(\tau)-x_v'(\tau)\Bigg]\times V = 0.$$

The above equation can be written now as

$$x_v'(\tau) = \frac{1}{2}\sigma_v^2x_v(\tau)^2+(i\phi\sigma_v\rho-K_v+\sigma_v\rho)x_v(\tau)-\frac{\phi^2-i\phi}{2} = 0. \tag{3.3.41}$$

The terms with respect to R are,

$$\Bigg[i\phi+\frac{1}{2}\sigma_R^2x_r(\tau)^2-K_Rx_r(\tau)-x_r'(\tau)\Bigg]\times R = 0.$$

Arranging, the terms we get

$$x_r'(\tau) = \frac{1}{2}\sigma_R^2x_r(\tau)^2-K_Rx_r(\tau)-\frac{-2i\phi}{2} = 0. \tag{3.3.42}$$

The terms that are not a multiple of V and R are,

$$-\lambda\mu_j i\phi + \theta_v x_v(\tau) + \theta_R x_r(\tau) - u'(\tau) - \lambda\mu_j + \lambda E[(1+J)e^{i\phi\ln(1+J)} - 1] = 0.$$

As a result, the following equation can be formed:

$$u'(\tau) = \theta_v x_v(\tau) + \theta_R x_r(\tau) - \lambda\mu_j[i\phi + 1] + \lambda E[(1+J)e^{i\phi\ln(1+J)} - 1]. \tag{3.3.43}$$

Actually equations (3.3.41) and (3.3.42) are special cases of so called Riccati equations. These equations are first-order ordinary differential equations and in general form, they are given as:

$$\frac{dy}{dx} + a(x)y^2 + b(x)y + c(x) = 0,$$

where $a(x)$, $b(x)$, $c(x)$ are known functions.

If the coefficients a, b, c are constants in the Ricatti equation as in our case, then it allows a separation of the variables and the solution y can be obtained by the general integral

$$C_1 - x = \int \frac{dy}{ay^2 + by + c}.$$

Detailed information about Ricatti equations can be found in [22].

Lemma 3.3.1 *The differential equation $y'(s) = \frac{A}{2}y^2 + By - \frac{C}{2}$ with constants A, B, C and with initial condition $y(0) = 0$ has the following solution:*

$$y(s) = \frac{-C}{\gamma coth(\frac{\gamma s}{2}) - B}, \quad where \quad \gamma = \sqrt{AC + B^2}. \tag{3.3.44}$$

Proof. To show that the given ordinary first order differential equation has the above solution we followed the general method that is writing the conjecture solution into the given differential equation then showing that, really the equation is satisfied.

Firstly, we take the derivative of $y(s)$,

$$y'(s) = \frac{C\gamma(1 - \coth^2(\frac{\gamma s}{2})) \cdot \frac{\gamma}{2}}{(\gamma\coth(\frac{\gamma s}{2}) - B)^2}. \tag{3.3.45}$$

Secondly, we write the terms of the differential equation separately:

$$\frac{A}{2}y^2 = \frac{\frac{AC^2}{2}}{(\gamma\coth(\frac{\gamma s}{2}) - B)^2}, \tag{3.3.46}$$

$$By = \frac{-BC(\gamma\coth(\frac{\gamma s}{2}) - B)}{(\gamma\coth(\frac{\gamma s}{2}) - B)^2}, \tag{3.3.47}$$

$$\frac{-C}{2} = \frac{\frac{-C}{2}(\gamma\coth(\frac{\gamma s}{2}) - B)^2}{(\gamma\coth(\frac{\gamma s}{2}) - B)^2}, \tag{3.3.48}$$

Subsequently, adding (3.3.46), (3.3.47) and (3.3.48), we get

$$\frac{\frac{C}{2}(AC + B^2) - \frac{C}{2}\gamma^2\coth^2(\frac{\gamma s}{2})}{(\gamma\coth(\frac{\gamma s}{2}) - B)^2} = \frac{C\gamma(1 - \coth^2(\frac{\gamma s}{2})) \cdot \frac{\gamma}{2}}{(\gamma\coth(\frac{\gamma s}{2}) - B)^2}.$$

This equation is exactly the same as (3.3.45) which is $y'(s)$.

$\blacksquare$

At this step another proof from differential equations is needed to solve our problem.

Lemma 3.3.2 *Let* $y(s) = \frac{-C}{\gamma coth(\frac{\gamma s}{2}) - B}$, *where* $\gamma = \sqrt{AC + B^2}$, *with the derivative* $y'(s) = \frac{A}{2}y^2 + By - \frac{C}{2}$ *and with initial condition* $y(0) = 0$, *then the integral of this function is as follows:*

$$\int_0^s y(t)dt = \frac{-Bs}{A} - \frac{2ln(cosh\frac{\gamma s}{2} - B\frac{sinh(\frac{\gamma s}{2})}{\gamma})}{A}. \tag{3.3.49}$$

Proof. To prove the lemma, we try to show the derivative of the right-hand side of the equation

(3.3.49) is exactly the same as $y(s)$ given above:

$$\begin{aligned}
\frac{d}{ds}\left(\frac{-Bs}{A} - \frac{2\ln(\cosh\frac{\gamma s}{2} - B\frac{\sinh(\frac{\gamma s}{2})}{\gamma})}{A}\right) &= \frac{-B}{A} - \frac{2}{A}\left[\frac{\sinh(\frac{\gamma s}{2})\frac{\gamma}{2} - \frac{B}{\gamma}\cosh(\frac{\gamma s}{2})\frac{\gamma}{2}}{\cosh(\frac{\gamma s}{2}) - \frac{B}{\gamma}\sinh(\frac{\gamma s}{2})}\right] \\
&= \frac{-B}{A} - \frac{\gamma}{A}\left[\frac{1 - \frac{B}{\gamma}\coth(\frac{\gamma s}{2})}{\coth(\frac{\gamma s}{2}) - \frac{B}{\gamma}}\right] \\
&= \frac{B^2 - AC - B^2}{A(\gamma\coth(\frac{\gamma s}{2}) - B)} \\
&= \frac{-C}{(\gamma\coth(\frac{\gamma s}{2}) - B)} \\
&= y(s).
\end{aligned}$$

Note that the integral may be differ with a constant. However in the statement of the lemma the initial condition is given to be zero. Therefore the integral is exactly the same as (3.3.49).
■

Finally we reached to our main goal that is finding the solutions of (3.3.41), (3.3.42), (3.3.43) of by using the two lemmas above. Note that (3.3.41), (3.3.42) are in the form of the differential equation given in Lemma 3.3.1. Therefore using the lemma we get,

$$x_v(\tau) = \frac{-(\phi^2 - i\phi)}{\gamma\coth(\frac{\gamma\tau}{2}) - (i\phi\sigma_v\rho - K_v + \sigma_v\rho)}, \tag{3.3.50}$$

where $\gamma = \sqrt{\sigma_v^2(\phi^2 - i\phi) + (i\phi\sigma_v\rho - K_v + \sigma_v\rho)^2}$,

and

$$x_r(\tau) = -\frac{-2i\phi}{\eta\coth(\frac{\eta\tau}{2}) + K_R} \tag{3.3.51}$$

with $\eta = \sqrt{\sigma_R^2(-2i\phi) + K_R^2}$.

Using Lemma 3.3.2, equation (3.3.43) can be solved as,

$$
\begin{aligned}
u(\tau) &= \theta_v \int x_v(\tau) + \theta_R \int x_r(\tau) - \tau\lambda\mu_j[i\phi + 1] + \tau\lambda E[(1+J)e^{i\phi\ln(1+J)} - 1] \\
&= \theta_v \Bigg[\frac{(K_v - i\phi\sigma_v\rho - \sigma_v\rho)\tau}{\sigma_v^2} \\
&- \frac{2\ln\Big(\cosh(\frac{\gamma\tau}{2}) + (K_v - i\phi\sigma_v\rho - \sigma_v\rho)\frac{\sinh(\frac{\gamma\tau}{2})}{\gamma}\Big)}{\sigma_v^2} \Bigg] \\
&+ \theta_R \Bigg[\frac{K_R}{\sigma_R^2}\tau - \frac{2\ln\Big(\cosh(\frac{\eta\tau}{2}) + K_R\frac{\sinh(\frac{\eta\tau}{2})}{\eta}\Big)}{\sigma_R^2} \Bigg] - \lambda\mu_j[i\phi + 1]\tau \\
&+ \tau\lambda E[(1+J)e^{i\phi\ln(1+J)} - 1]. \qquad (3.3.52)
\end{aligned}
$$

The expectation $E[(1+J)e^{i\phi\ln(1+J)}]$ can be easily calculated due to the fact that $\ln(1+J)$ is distributed as normal with mean $\ln(1+\mu_j) - \frac{1}{2}\sigma_j^2$ and variance σ_j^2. The general form of the characteristic function of normal distribution $U \sim\ N(\mu, \sigma^2)$ is

$$
E(e^{i\phi u}) = \exp(i\mu\phi - \tfrac{1}{2}\sigma^2\phi^2).
$$

Therefore, by writing the mean and variance of ln(1+J) and arranging the terms as follows, we can get

$$
\begin{aligned}
E[(1+J)e^{i\phi\ln(1+J)}] &= E[e^{\ln(1+J)}e^{i\phi\ln(1+J)}] \\
&= E[e^{(1+i\phi)\ln(1+J)}] \\
&= E[e^{i(\frac{1}{i}+\phi)\ln(1+J)}] \\
&= \exp(i(\ln(1+\mu_j) - \tfrac{1}{2}\sigma_j^2)(\tfrac{1}{i} + \phi) - \tfrac{1}{2}\sigma_j^2(\tfrac{1}{i} + \phi)^2) \\
&= \exp((1+i\phi)\ln(1+\mu_j) + \sigma_j^2\frac{i\phi}{2}(1+i\phi)) \\
&= (1+\mu_j)^{(1+i\phi)}\exp\Big(\sigma_j^2\frac{i\phi}{2}(1+i\phi)\Big). \qquad (3.3.53)
\end{aligned}
$$

Note that in the characteristic function definition ϕ is a real number. However, in equation (3.3.53) we wrote instead of ϕ a complex number. We are allowed to this because the normal distribution has an entire characteristic function. That means instead of a real number a complex number can be put. More detailed results about analytical characteristic functions can be examined in [32].

At this step, by using the lemmas and the conjecture solution of f_2, we can construct the derivatives as follows:

$$\frac{\partial f_2}{\partial L} = f_2 \cdot (i\phi), \tag{3.3.54}$$

$$\frac{\partial^2 f_2}{\partial L^2} = f_2 \cdot (i\phi)^2 = -f_2 \cdot \phi^2, \tag{3.3.55}$$

$$\frac{\partial f_2}{\partial V} = f_2 \cdot y_v(\tau), \tag{3.3.56}$$

$$\frac{\partial^2 f_2}{\partial V^2} = f_2 \cdot y_v(\tau)^2, \tag{3.3.57}$$

$$\frac{\partial^2 f_2}{\partial L \partial V} = f_2 \cdot y_v(\tau) \cdot (i\phi), \tag{3.3.58}$$

$$\frac{\partial f_2}{\partial R} = f_2 \cdot y_r(\tau), \tag{3.3.59}$$

$$\frac{\partial^2 f_2}{\partial R^2} = f_2 \cdot y_r(\tau)^2, \tag{3.3.60}$$

$$\frac{\partial f_2}{\partial \tau} = f_2 \cdot (z'(\tau) + y_r'(\tau)R(t) + y_v'(\tau)V(t)), \tag{3.3.61}$$

and the terms in jump part can be written as

$$\begin{aligned} f_2(t, \tau, L + \ln(1+J), R, V) &= \exp(u(\tau) + x_r(\tau)R(t) + x_v(\tau)V(t) + i\phi L + i\phi \ln(1+J) - \ln[B(t,\tau)] \\ &= f_2 \cdot e^{i\phi \ln(1+J)}, \end{aligned} \tag{3.3.62}$$

where $f_2 = f_2(t, \tau, L, R, V)$.

By writing the above derivatives into equation (3.3.28) we get

$$\begin{aligned}
&\Bigg[-\frac{1}{2}V\phi^2 + [R - \lambda\mu_j - \frac{1}{2}V]i\phi + \sigma_v\rho V y_v(\tau)\cdot(i\phi) + \frac{1}{2}V\sigma_v^2 y_v(\tau)^2 + [\theta_v - K_v V]y_v(\tau) \\
&+\frac{1}{2}R\sigma_R^2 y_r(\tau)^2 + \left[\theta_R - \left(K_R - \frac{\sigma_R^2}{B}\frac{\partial B}{\partial R}\right)R\right]y_r(\tau) - (z'(\tau) + y_r'(\tau)R + y_v'(\tau)V) \\
&+\lambda E[e^{i\phi\ln(1+J)} - 1]\Bigg] \times f_2 = 0. \qquad (3.3.63)
\end{aligned}$$

From a similar argument done for f_1, equation (3.3.63) should have zero multiples of terms with the volatility process V, interest rate process R and terms without V and R which are independent from each other. As a result, we can construct the following equations:

$$\left[-\frac{1}{2}\phi^2 - \frac{1}{2}i\phi + \sigma_v\rho y_v(\tau)\cdot(i\phi) + \frac{1}{2}\sigma_v^2 y_v(\tau)^2 - K_v y_v(\tau) - y_v'(\tau)\right] \times V = 0.$$

The above equation can be written now as,

$$y_v'(\tau) = \frac{1}{2}\sigma_v^2 y_v(\tau)^2 + (i\phi\sigma_v\rho - K_v)y_v(\tau) - \frac{\phi^2 + i\phi}{2} = 0. \qquad (3.3.64)$$

The terms with respect to R are,

$$\left[i\phi + \frac{1}{2}\sigma_R^2 y_r(\tau)^2 - \left(K_R - \frac{\sigma_R^2}{B}\frac{\partial B}{\partial R}\right)y_r(\tau) - y_r'(\tau)\right] \times R = 0,$$

Arranging the terms we get,

$$y_r'(\tau) = \frac{1}{2}\sigma_R^2 y_r(\tau)^2 - (K_R - \frac{\sigma_R^2}{B}\frac{\partial B}{\partial R})y_r(\tau) - \frac{-2i\phi}{2} = 0. \qquad (3.3.65)$$

The terms that are not a multiple of V and R are,

$$-\lambda\mu_j i\phi + \theta_v y_v(\tau) + \theta_R y_r(\tau) - z'(\tau) + \lambda E[e^{i\phi\ln(1+J)} - 1] = 0.$$

As a result the following equation can be formed,

$$z'(\tau) = \theta_v y_v(\tau) + \theta_R y_r(\tau) - \lambda\mu_j i\phi + \lambda E[e^{i\phi \ln(1+J)} - 1]. \tag{3.3.66}$$

Using Lemma 3.3.1, equations (3.3.64) and (3.3.65) can be solved as

$$y_v(\tau) = \frac{-(\phi^2 + i\phi)}{\gamma \coth(\frac{\gamma\tau}{2}) - (i\phi\sigma_v\rho - K_v)}, \tag{3.3.67}$$

where $\gamma = \sqrt{\sigma_v^2(\phi^2 + i\phi) + (i\phi\sigma_v\rho - K_v)^2}$,

and

$$y_r(\tau) = -\frac{-2i\phi}{\eta \coth(\frac{\eta\tau}{2}) + (K_R - \frac{\sigma_R^2}{B}\frac{\partial B}{\partial R})} \tag{3.3.68}$$

with $\eta = \sqrt{\sigma_R^2(-2i\phi) + (K_R - \frac{\sigma_R^2}{B}\frac{\partial B}{\partial R})^2}$.

Using Lemma 3.3.2, equation (3.3.66) can be solved as

$$\begin{aligned}
z(\tau) &= \theta_v \int y_v(\tau) + \theta_R \int y_r(\tau) - \tau\lambda\mu_j i\phi + \tau\lambda E[e^{i\phi\ln(1+J)} - 1] \\
&= \theta_v \Bigg[\frac{(K_v - i\phi\sigma_v\rho)\tau}{\sigma_v^2} \\
&- \frac{2\ln\Bigg(\cosh(\frac{\gamma\tau}{2}) + (K_v - i\phi\sigma_v\rho)\frac{\sinh(\frac{\gamma\tau}{2})}{\gamma}\Bigg)}{\sigma_v^2} \Bigg] + \theta_R \Bigg[\frac{(K_R - \frac{\sigma_R^2}{B}\frac{\partial B}{\partial R})}{\sigma_R^2}\tau \\
&- \frac{2\ln\Bigg(\cosh(\frac{\eta\tau}{2}) + (K_R - \frac{\sigma_R^2}{B}\frac{\partial B}{\partial R})\frac{\sinh(\frac{\eta\tau}{2})}{\eta}\Bigg)}{\sigma_R^2} \Bigg] - \lambda\mu_j i\phi\tau \\
&+ \tau\lambda E[e^{i\phi\ln(1+J)} - 1],
\end{aligned} \tag{3.3.69}$$

where the expectation is the characteristic function of $\ln(1 + J)$ which is normally distributed. The form of the characteristic function of normal distribution is given in the previous part when the differential equations of f_1 constructed. Therefore,

$$E[e^{i\phi\ln(1+J)}] = (1 + \mu_j)^{i\phi}\exp(\sigma_j^2\frac{i\phi}{2}(i\phi - 1)).$$

Eventually, the closed form solution of European Call Option Price is obtained. One can found the results and summary in the next section.

3.4 Final Forms of the European Call Option Prices

In this chapter, the option pricing formula is formed under the asset price process Stochastic Volatility Stochastic Interest Rate Jump Diffusion Model (SVSI-J). The European Call Option Price occurred in a Partial Integro-Differential Equation. We solved this equation to get the solution. To sum up, the call price is found as

$$C(t,\tau) = \pi_1(t,\tau,S,R,V)S(t) - KB(t,\tau)\pi_2(t,\tau,S,R,V), \tag{3.4.1}$$

where

$$\begin{aligned}\pi_j(t,\tau,S,R,V) &= \\ &\frac{1}{2} + \frac{1}{\pi}\int_0^\infty Re\left(\frac{e^{-i\phi\ln[K]f_j(t,\tau,S,R,V)}}{i\phi}\right)d\phi,\end{aligned} \tag{3.4.2}$$

and the characteristic functions $f_j(t,\tau,S,R,V)$ are

$$\begin{aligned}f_1(t,\tau,S,R,V,\phi) &= \exp\Bigg(\theta_v\Bigg[\frac{(K_v - i\phi\sigma_v\rho - \sigma_v\rho)\tau}{\sigma_v^2} \\ &- \frac{2\ln\left(\cosh(\frac{\gamma\tau}{2}) + (K_v - i\phi\sigma_v\rho - \sigma_v\rho)\frac{\sinh(\frac{\gamma\tau}{2})}{\gamma}\right)}{\sigma_v^2}\Bigg] + \theta_R\Bigg[\frac{K_R}{\sigma_R^2}\tau \\ &- \frac{2\ln\left(\cosh(\frac{\eta\tau}{2}) + \frac{K_R\sinh(\frac{\eta\tau}{2})}{\eta}\right)}{\sigma_R^2}\Bigg] - \lambda\mu_j[i\phi + 1]\tau \\ &+ \tau\lambda[(1+\mu_j)^{(1+i\phi)}\exp(\sigma_j^2\frac{i\phi}{2}(1+i\phi)) - 1] \\ &+ \left[-\frac{-2i\phi}{\eta\coth(\frac{\eta\tau}{2}) + K_R}\right]R(t) \\ &+ \left[\frac{-(\phi^2 - i\phi)}{\gamma\coth(\frac{\gamma\tau}{2}) - (i\phi\sigma_v\rho - K_v + \sigma_v\rho)}\right]V(t) + i\phi\ln[S(t)]\Bigg)\end{aligned} \tag{3.4.3}$$

with $\gamma = \sqrt{\sigma_v^2(\phi^2 - i\phi) + (i\phi\sigma_v\rho - K_v + \sigma_v\rho)^2}$,

$\eta = \sqrt{\sigma_R^2(-2i\phi) + K_R^2}$

and

$$\begin{aligned} f_2(t,\tau,S,R,V,\phi) &= \exp\Bigg(\theta_v\Bigg[\frac{(K_v - i\phi\sigma_v\rho)\tau}{\sigma_v^2} \\ &- \frac{2\ln\Big(\cosh(\frac{\gamma\tau}{2}) + (K_v - i\phi\sigma_v\rho)\frac{\sinh(\frac{\gamma\tau}{2})}{\gamma}\Big)}{\sigma_v^2}\Bigg] + \theta_R\Bigg[\frac{(K_R - \frac{\sigma_R^2}{B}\frac{\partial B}{\partial R})}{\sigma_R^2}\tau \\ &- \frac{2\ln\Big(\cosh(\frac{\eta\tau}{2}) + (\frac{K_R - \sigma_R^2}{B}\frac{\partial B}{\partial R})\frac{\sinh(\frac{\eta\tau}{2})}{\eta}\Big)}{\sigma_R^2}\Bigg] - \lambda\mu_j i\phi\tau \\ &+ \tau\lambda[(1+\mu_j)^{i\phi}\exp(\sigma_j^2\frac{i\phi}{2}(i\phi - 1)) - 1] \\ &+ \Bigg[-\frac{-2i\phi}{\eta\coth(\frac{\eta\tau}{2}) + (K_R - \frac{\sigma_R^2}{B}\frac{\partial B}{\partial R})}\Bigg]R(t) \\ &+ \Bigg[\frac{-(\phi^2 + i\phi)}{\gamma\coth(\frac{\gamma\tau}{2}) - (i\phi\sigma_v\rho - K_v)}\Bigg]V(t) \\ &+ i\phi\ln[S(t)] - \ln[B(t,\tau)]\Bigg) \end{aligned} \tag{3.4.4}$$

with $\gamma = \sqrt{\sigma_v^2(\phi^2 + i\phi) + (i\phi\sigma_v\rho - K_v)^2}$ and

$\eta = \sqrt{\sigma_R^2(-2i\phi) + (K_R - \frac{\sigma_R^2}{B}\frac{\partial B}{\partial R})^2}$.

The results are the same as the ones in the paper of Bakshi, Cao and Chen. The only difference is they wrote instead of hyperbolic functions their equals in exponential forms.

This pricing formula is much more applicable when compared to the formula driven by Scott in the second chapter. The reason is that the number of parameters are much more less, especially, it is given only as a function of identifiable variables such that all parameters can be estimated. In the next chapter, the performance of alternative option pricing models is compared. 4 models are chosen to identify the best one from them. These models are Black Scholes Model (BS), Stochastic Volatility Model (SV), Stochastic Volatility Jump Model (SVJ) and Stochastic Volatility Stochastic Interest Rate Model (SVSI). The general form Stochastic Volatility Stochastic Interest Rate Jump Diffusion (SVSI-J) is not included to the empirical study. The reason behind is that in the literature this model performance is evaluated and results were abundantly poor. Therefore many empirical studies are concentrated on the 4 models we already mentioned (see Bakshi, Cao and Chen for details).

From the general formula mentioned above the call prices of sub models BS, SV, SVJ, SVSI can be obtained. Since we need to write the call prices in closed form in the parameter estimation in the next chapter, the sub models prices are established as follows:

If λ is zero in (3.4.3) and (3.4.4), then we get the characteristic functions of SVSI as

$$\begin{aligned}
\widehat{f_1}(t,\tau,S,R,V,\phi) &= \exp\Bigg(\theta_v\Bigg[\frac{(K_v - i\phi\sigma_v\rho - \sigma_v\rho)\tau}{\sigma_v^2} \\
&- \frac{2\ln\Big(\cosh(\frac{\gamma\tau}{2}) + (K_v - i\phi\sigma_v\rho - \sigma_v\rho)\frac{\sinh(\frac{\gamma\tau}{2})}{\gamma}\Big)}{\sigma_v^2}\Bigg] \\
&+ \theta_R\Bigg[\frac{K_R}{\sigma_R^2}\tau - \frac{2\ln\Big(\cosh(\frac{\eta\tau}{2}) + K_R\frac{\sinh(\frac{\eta\tau}{2})}{\eta}\Big)}{\sigma_R^2}\Bigg] \\
&+ \Bigg[-\frac{-2i\phi}{\eta\coth(\frac{\eta\tau}{2}) + K_R}\Bigg]R(t) \\
&+ \Bigg[\frac{-(\phi^2 - i\phi)}{\gamma\coth(\frac{\gamma\tau}{2}) - (i\phi\sigma_v\rho - K_v + \sigma_v\rho)}\Bigg]V(t) + i\phi\ln[S(t)]\Bigg). \qquad (3.4.5)
\end{aligned}$$

$$\begin{aligned}
\widehat{f_2}(t,\tau,S,R,V,\phi) &= \exp\Bigg(\theta_v\Bigg[\frac{(K_v - i\phi\sigma_v\rho)\tau}{\sigma_v^2} \\
&- \frac{2\ln\Big(\cosh(\frac{\gamma\tau}{2}) + (K_v - i\phi\sigma_v\rho)\frac{\sinh(\frac{\gamma\tau}{2})}{\gamma}\Big)}{\sigma_v^2}\Bigg] \\
&+ \theta_R\Bigg[\frac{(K_R - \frac{\sigma_R^2}{B}\frac{\partial B}{\partial R})}{\sigma_R^2}\tau - \frac{2\ln\Big(\cosh(\frac{\eta\tau}{2}) + (K_R - \frac{\sigma_R^2}{B}\frac{\partial B}{\partial R})\frac{\sinh(\frac{\eta\tau}{2})}{\eta}\Big)}{\sigma_R^2}\Bigg] \\
&+ \Bigg[-\frac{-2i\phi}{\eta\coth(\frac{\eta\tau}{2}) + (K_R - \frac{\sigma_R^2}{B}\frac{\partial B}{\partial R})}\Bigg]R(t) \\
&+ \Bigg[\frac{-(\phi^2 + i\phi)}{\gamma\coth(\frac{\gamma\tau}{2}) - (i\phi\sigma_v\rho - K_v)}\Bigg]V(t) \\
&+ i\phi\ln[S(t)] - \ln[B(t,\tau)]\Bigg). \qquad (3.4.6)
\end{aligned}$$

The SVJ model can be obtained by taking $R(t) = R$ (constant). Therefore, the partial derivatives with respect to R will vanish and the bond price will be $B(t,\tau) = e^{-R\tau}$. Hence, the solution of the partial differential equation (3.3.27) becomes

$$\widetilde{f_1}(t, \tau, S, R, V, \phi) = \exp(u(\tau) + x_r(\tau)R + x_v(\tau)V(t) + i\phi\ln[S(t)]).$$

The differential equations of $x_v(\tau)$ given in (3.3.41) does not differ in this case. However, (3.3.42) and (3.3.43) become

$$x_r'(\tau) = -\frac{-2i\phi}{2} = 0$$

and

$$u'(\tau) = \theta_v x_v(\tau) - \lambda\mu_j[i\phi + 1] + \lambda E[(1 + J)e^{i\phi\ln(1+J)} - 1].$$

For this reason, $x_r(\tau) = i\phi\tau$. Finally we can write $x_r(\tau)R = i\phi\tau R = -i\phi\ln(B(t, \tau))$. The coefficients of $u(\tau)$ is the same, only it does not include any terms with R. When the solution of the differential equation in (3.3.28) is considered,

$$\widetilde{f_2}(t, \tau, S, R, V, \phi) = \exp(z(\tau) + y_r(\tau)R + y_v(\tau)V(t) + i\phi\ln[S(t)].$$

The different part of this equation is that there is no coefficient with $\ln(B(t, \tau))$ because we do not have any derivatives with respect to R also. From a similar argument the equation (3.3.64) is the same. However, (3.3.65) and (3.3.66) become

$$y_r'(\tau) = -\frac{-2i\phi}{2} = 0$$

and

$$u'(\tau) = \theta_v y_v(\tau) - \lambda\mu_j i\phi + \lambda E[e^{i\phi\ln(1+J)} - 1].$$

Finally, the characteristic functions corresponding to SVJ are:

$$
\begin{aligned}
\widetilde{f}_1(t,\tau,S,R,V,\phi) &= \exp\Bigg(\theta_v\Bigg[\frac{(K_v - i\phi\sigma_v\rho - \sigma_v\rho)\tau}{\sigma_v^2} \\
&- \frac{2\ln\Big(\cosh(\frac{\gamma\tau}{2}) + (K_v - i\phi\sigma_v\rho - \sigma_v\rho)\frac{\sinh(\frac{\gamma\tau}{2})}{\gamma}\Big)}{\sigma_v^2}\Bigg] \\
&+ \tau\lambda[(1+\mu_j)^{(1+i\phi)}\exp(\sigma_j^2\frac{i\phi}{2}(1+i\phi)) - 1] - \lambda\mu_j[i\phi+1]\tau \\
&+ \Bigg[\frac{-(\phi^2 - i\phi)}{\gamma\coth(\frac{\gamma\tau}{2}) - (i\phi\sigma_v\rho - K_v + \sigma_v\rho)}\Bigg]V(t) + i\phi\ln[S(t)] \\
&- i\phi\ln(B(t,\tau))\Bigg), \qquad (3.4.7)
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{f}_2(t,\tau,S,R,V,\phi) &= \exp\Bigg(\theta_v\Bigg[\frac{(K_v - i\phi\sigma_v\rho)\tau}{\sigma_v^2} \\
&- \frac{2\ln\Big(\cosh(\frac{\gamma\tau}{2}) + (K_v - i\phi\sigma_v\rho)\frac{\sinh(\frac{\gamma\tau}{2})}{\gamma}\Big)}{\sigma_v^2}\Bigg] \\
&+ \tau\lambda[(1+\mu_j)^{i\phi}\exp(\sigma_j^2\frac{i\phi}{2}(i\phi - 1)) - 1] - \lambda\mu_j i\phi\tau \\
&+ \Bigg[\frac{-(\phi^2 + i\phi)}{\gamma\coth(\frac{\gamma\tau}{2}) - (i\phi\sigma_v\rho - K_v)}\Bigg]V(t) \\
&+ i\phi\ln[S(t)] - i\phi\ln[B(t,\tau)]\Bigg). \qquad (3.4.8)
\end{aligned}
$$

The SV model differs from SVJ only with the coefficient λ. By equating λ to zero in the characteristic functions of SVJ, the characteristic functions of SV can be formed as

$$
\begin{aligned}
\check{f}_1(t,\tau,S,R,V,\phi) &= \exp\Bigg(\theta_v\Bigg[\frac{(K_v - i\phi\sigma_v\rho - \sigma_v\rho)\tau}{\sigma_v^2} \\
&- \frac{2\ln\Big(\cosh(\frac{\gamma\tau}{2}) + (K_v - i\phi\sigma_v\rho - \sigma_v\rho)\frac{\sinh(\frac{\gamma\tau}{2})}{\gamma}\Big)}{\sigma_v^2}\Bigg] \\
&+ \Bigg[\frac{-(\phi^2 - i\phi)}{\gamma\coth(\frac{\gamma\tau}{2}) - (i\phi\sigma_v\rho - K_v + \sigma_v\rho)}\Bigg]V(t) + i\phi\ln[S(t)] \\
&- i\phi\ln(B(t,\tau))\Bigg), \qquad (3.4.9)
\end{aligned}
$$

$$
\begin{aligned}
\check{f}_2(t,\tau,S,R,V,\phi) &= \exp\Bigg(\theta_v\Bigg[\frac{(K_v - i\phi\sigma_v\rho)\tau}{\sigma_v^2} \\
&- \frac{2\ln\Big(\cosh(\frac{\gamma\tau}{2}) + (K_v - i\phi\sigma_v\rho)\frac{\sinh(\frac{\gamma\tau}{2})}{\gamma}\Big)}{\sigma_v^2}\Bigg] \\
&+ \Bigg[\frac{-(\phi^2 + i\phi)}{\gamma\coth(\frac{\gamma\tau}{2}) - (i\phi\sigma_v\rho - K_v)}\Bigg]V(t) \\
&+ i\phi\ln[S(t)] - i\phi\ln[B(t,\tau)]\Bigg).
\end{aligned}
\tag{3.4.10}
$$

CHAPTER 4

THE PERFORMANCE OF ALTERNATIVE OPTION PRICING MODELS ON GENERAL ELECTRIC STOCK DATA

There are many studies to construct mathematical description of financial markets and derivative investment instruments. The classical model for stock price fluctuations is the Black-Scholes model. The model assumes that the implicit volatility is constant for all strike prices for options on the same underlying asset with the same remaining time to maturity. However, the actual functional relationship between implicit volatility and strike price is typically shaped like a lopsided smile. This functional relationship is therefore often referred to as the ”volatility smile”. The curve of the volatility smile indicates the deviation of the market’s probability density function from the log-normal distribution assumed in the Black Scholes Model. Volatility smile is the most significant problem of this model. The other deviations of Black Scholes Model from the market behavior are the empirical distributions of log-returns are ’fat-tailed’ and ’sharp-peaked’ compared with normal distributions. Also price jumps of 4 or more standard deviations occur regularly in stock markets, but should be very rare events if log-returns are normal distributed.

The purpose of the empirical analysis in this chapter of this study is to measure the improvements of the model generalization like stochastic volatility, stochastic volatility with jump and stochastic volatility, stochastic interest rate rather than using Black Scholes model. We are trying to answer the questions: Is it really worthy to use models complicated then Black Scholes? Is the gain, if any, worth the additional complexity or implementational costs?

4.1 Data Description

We used the Options on the Stock of General Electric in our empirical performance study. General Electric Company was founded in 1892 and operates as a technology, media and financial services company worldwide. Its Capital Finance segment offers commercial lending and leasing products to manufacturers, distributors, and end-users of equipment of capital assets; consumer financial services to consumers and retailers; capital and investment solutions for real estate; commercial finance to energy and water industries; and commercial aircraft leasing and finance, and fleet and financing solutions. The stock and its options are traded actively in Chicago Board Options Exchange. Observe that, many empirical performance studies are constructed on S&P 500 index. However we chose the stock data on General Electric on purpose. Because Bakshi, Cao, and Chen (1996) suggest that their index option results may not hold for single stock options. The data set is collected on 20 January 2011 at 14:05. At that time the General Electric stock price was 18.43 dollars. The data contains options with 7 different maturities. The days to expiration of these options are changing from 20 days up to 515 days. Time to expiration of the options is founded by dividing days to expiration to 360. For the risk free interest rate the 3 month T-bill Discount rate 0.15 % is used. For dividend payments Forward Annual Dividend Yield of General Electric Stocks 2.8 % is used.

Some exclusion filters are applied to construct the option prices' data set. Firstly, options with unrealistic implied volatilities are ignored so that accurate results can be found. Second, as options with less than six days to expiration may induce liquidity-related biases, they are excluded from the sample. After this filtration, the number of data set is reduced to 75 observations over all. The data set can be found in Appendix A.

The option data is divided into several categories according to either moneyness or term to expiration. $S(t) - K$ is called the time-t intrinsic value of a call. A call option said to be at the money (ATM) if S/K changes in between 0.97 and 1.03; out of the money (OTM) if S/K is less than 0.97; and in the money (ITM) if S/K is greater than 1.03. We applied a finer partition which resulted in 6 moneyness categories. By the term to expiration, we classified the option contract into 3 categories. The first one is short term options with days to expiration less than 60 days. The second type is medium term options that has days to expiration between 60 and 180 days. Finally, there are long term options with days to expiration greater than 180 days. For each of these categories, the following table can be constructed to show the sample properties of the data. Under each moneyness and maturity category the average of option call prices are taken. S denotes the general electric stock price when the data is collected. K is the strike price of options contract. OTM denotes out of the money options, ITM corresponds to in the money options and ATM refers to at the money options.The numbers in parenthesis in Table 4.1 are the number of observations in each category.

Table 4.1: The Sample Properties of General Electric Stock Options

Moneyness(S/K)	**Days to Expiration**		
	< **60**	**60 – 180**	≥ **180**
< **0.94**	\$0.06 (8)	\$0.25 (12)	\$0.68 (9)
0.94 – 0.97	-	-	-
0.97 – 1.00	\$0.37 (2)	\$0.87 (3)	- (-)
1.00 – 1.03	\$0.85 (2)	\$1.373 (3)	- (-)
1.03 – 1.06	- (-)	- (-)	\$2.63 (2)
≥ **1.06**	\$3.19 (8)	\$4.572 (18)	\$7.54 (8)

According to Table 4.1, the average call price ranges from \$0.06 to \$7.54. ITM and ATM options respectively taking up 48 percent and 13.3 percent of the total sample.

4.2 Structural Parameter Estimation

For the empirical work to follow, we concentrate on the four models: the BS, the SV, the SVSI, and the SVJ. The analysis is intended to present what each generalization of the benchmark BS model can really buy in terms of performance improvement and whether each generalization produces a worthy trade off between benefits and costs. To get a sense of what we should look for in any desirable alternative to the BS model, the implied volatility in Table 4.2 is obtained by inverting the Black-Scholes model separately for each call option contract by using the MATLAB comment 'blsimpv'. The implied volatilities of individual calls are then averaged within each moneyness-maturity category to produce an average implied volatility. S denotes the spot General Electric stock level, and K is the exercise price.

Table 4.2: Implied Volatility from the Black-Scholes Model

Moneyness(S/K)	**Days to Expiration**		
	$<$ **60**	**60 – 180**	$\geq$ **180**
$<$ **0.94**	%36.58	%28.96	%30.85
0.94 – 0.97	-	-	-
0.97 – 1.00	%29.35	%29.48	-
1.00 – 1.03	%30.26	%31.05	-
1.03 – 1.06	-	-	%33.6
$\geq$ **1.06**	%58.66	%49.17	%56.21

The findings are consistent with those in the existing literature (e.g., Bates (1996), Bashki-Cao-Chen (1997a)). Clearly, regardless term to expiration, the BS implied volatility exhibits a strong U-shaped pattern (smile) as the call option goes from deep ITM to ATM and then to deep OTM with the deepest ITM call-implied volatilities taking the highest values. Furthermore, the volatility smiles are the strongest for short-term options, indicating that short-term options are the most severely mispriced by the BS model and present perhaps the greatest challenge to any alternative option pricing model. The maturity-related-biases can be seen clearly in Figure 4.1. In the figure, the short-term options (days to expiration less than 60 days) shows the strongest smile if we compare it with medium-term (days to expiration between 60 and 180 days) and long-term options (days to expiration greater or equal to 180 days). The MATLAB m-file regarding to Figure 4.1 can be found in the Appendix.

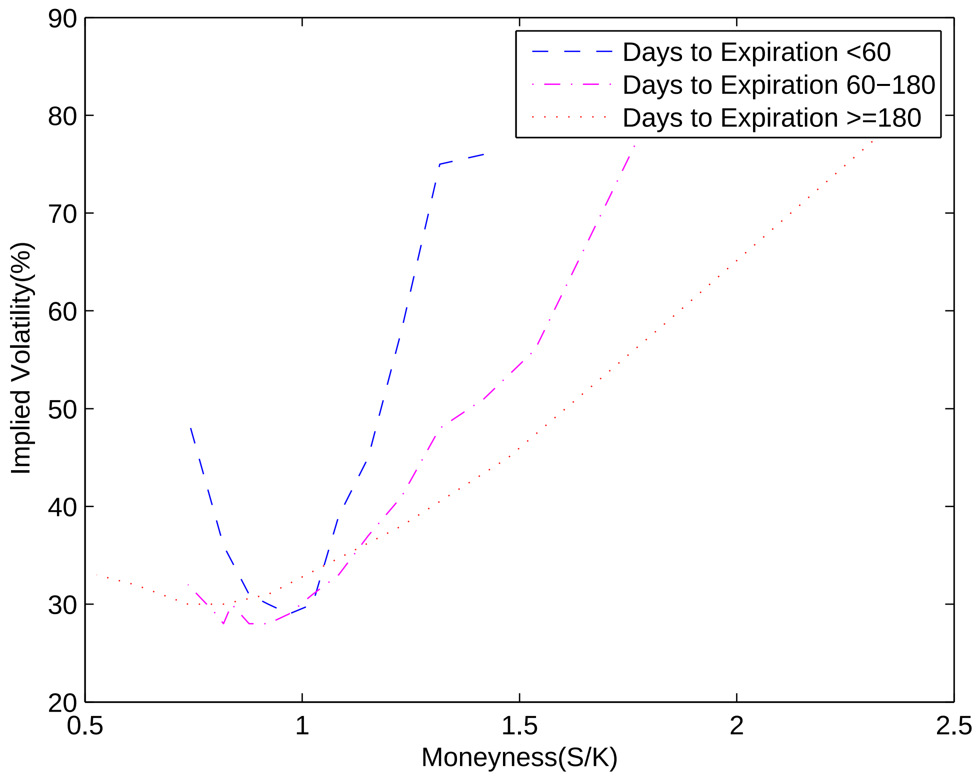

Figure 4.1: Implied Volatility Graph of Black-Scholes Model

Any acceptable alternative to the BS model must show an ability to properly price non-ATM options, especially short-term OTM calls. As the smile evidence is indicative of negatively-skewed implicit return distributions with excess kurtosis, a better model must be based on a distributional assumption that allows for negative skewness and excess kurtosis [1].

4.2.1 Parameter Estimation Procedure

In applying option pricing models, the option implied prices are under risk-neutral distributions, while those estimated from observed time series data are for the true distributions. Thus, before estimating the parameters, we need to change the measure or use some results from the literature. To solve this problem, we rely on the general-equilibrium models of Bakshi and Chen (1997b) and Bates (1996) in which the factor risk premiums are proportional to the respective factors and, consequently, the processes for $V(t)$, $R(t)$, $q(t)$ and $J(t)$ under the true probability measure share the same stochastic structure as their counterparts under the risk-neutral measure. Specifically, $\theta_v, \sigma_v, \rho, \theta_R, \sigma_R, \sigma_j$ are the same under either probability, only K_R, K_v, λ, μ_j will change when the probability measure changes from the risk- neutral to its true counterpart. Let these parameters under the true probability measure be respectively denoted by $\overline{K_R}, \overline{K_v}, \overline{\lambda}, \overline{\mu_j}$. However, according to the study of Bates (1991), when the risk aversion coefficient of the representative agent is bounded within a reasonable range, the parameters of the true distributions will not differ significantly from their risk-neutral counterparts [1]. For this reason, we will use the exact theoretical prices under risk neutral measure in parameter estimation procedure instead of true measure.

Generally, we faced with the difficulty that the spot volatility and the structural parameters are unobservable in option pricing models. Consider the SVSI for instance. The strike price and the term to expiration are specified in the contract, while the spot stock price, the spot interest rate, and the matching T-period bond price can be taken from published market data. But, the spot volatility (conditional on no jump), its related structural parameters $K_R, \theta_R, \sigma_R, K_v, \theta v, \sigma_v, \rho$ need to be estimated. Although the volatility risk premium is internalized in parameter estimates in this study, in some other works like [31] the risk premium is estimated explicitly. However, in general, an explicit estimate of volatility risk premium is not required to implement the models with stochastic volatility. The type of arguments similar in this study are therefore more preferable and the examples can be seen in many other works like Bates (1996), Longstaff (1995), etc.

There are many methods to estimate parameters in the literature. Examples include, maximum likelihood, generalized method of moments, etc. Each method holds its on advantages and disadvantages in it. Generally, the improvement of alternative models and practices on these models end up with the parameter estimation methods that are less depended on historical data. This approach reduced the data requirement and the performance of the model is significantly improved. In the estimation procedure of this study, we focused on the minimization of sum of square dollar pricing error. This error can be constructed in the following steps.

Step 1: Collect N option prices on the same stock taking from the same day, for any N greater than or equal to one plus the number of parameters to be estimated.

Step 2: Let T_i and K_i be the time to expiration and the strike price of the i-th option. Entitle the observed call price obtained from the market as $\widetilde{C}_i(t, T_i, K_i)$ and $C_i(t, T_i, K_i)$ its model price as determined from the formulas of the previous chapter. For example, for SVSI model price the characteristic function equation given in (3.4.5) and (3.4.6) should be used with $S(t)$ and $R(t)$ taken from the market.

Step 3: Define the difference as $H_i[V(t), \Phi] = \widetilde{C}_i(t, T_i, K_i) - C_i(t, T_i, K_i)$ where the parameters in general are $\Phi = (K_R, \theta_R, \sigma_R, K_v, \theta v, \sigma_v, \rho, \lambda, \mu_j, \sigma_j)$ and $V(t)$ is the spot volatility. For instance, for SVSI the only parameters are $\Phi = (K_R, \theta_R, \sigma_R, K_v, \theta v, \sigma_v, \rho)$ and spot volatility $V(t)$.

Step 4: Find the spot volatility $V(t)$ and parameters Φ that minimizes:

$$\sum_{n=1}^{N} |H_n[V(t), \Phi]^2|. \tag{4.2.1}$$

The objective function in (4.2.1) is defined as the Sum of Square Error of Dollar Pricing (SSE). The disadvantage of this objective function is that it may force the estimation to assign more weight to relatively expensive options like ITM and long term options. An alternative could be to minimize the sum of squared percentage pricing errors of all options, but that would lead to a more favorable treatment of cheaper options like OTM. As it can be seen, each estimation procedure has its own advantages and disadvantages. Based on this consideration, we choose to adopt the objective function in (4.2.1). In addition, applying such an implied parameter procedure gives equal chance to each model besides BS model. Note that there are many methods and technical programing languages to minimize the sum of square error. A summary of these techniques are examined in the next chapter. In the literature many studies which measures the performance of these techniques exist. One of this studies can be found in [30]. In our study, we used DIRECT algorithm together with MATLAB function, lsqnonlin.

4.2.2 Some Algorithms and Methods to Minimize SSE

4.2.2.1 MATLAB lsqonlin

The least-squares, non-linear optimizer of MATLAB is the function lsqnonlin(fun,x0,lb,ub). It minimizes the vector-valued function, fun, using the vector of initial parameter values, x0, where the lower and upper bounds of the parameters are the vectors lb and ub, respectively. This method requires the user-defined function which is a vector rather than computing the value of sum of squares. In our problem the vector 'fun' is

$$\begin{bmatrix} H_1[V(t),\Phi] \\ H_2[V(t),\Phi] \\ \vdots \\ H_N[V(t),\Phi] \end{bmatrix}.$$

The result produced by lsqnonlin is dependent on the choice of x0, the initial estimate. This is, therefore, not a global optimizer, but rather, a local one. More detailed information about lsqnonlin can be found in [8] and [9].

4.2.2.2 Excel Solver

The standard Solver supplied with Excel contains an optimizer that can be used for optimization procedure. It uses a Generalized Reduced Gradient (GRG) method and, hence, is a local optimizer. The calibration results are therefore sensitive to the initial estimates of the parameters. The details can be provided from [27].

4.2.2.3 Simulated Annealing (SA) and Adaptive Simulated Annealing (ASA)

Simulated Annealing is a probability-based, non-linear, optimizer inspired by the physical process of annealing. This method statistically guarantee finding an optimal solution. It were Kirkpatrick and Vecchi (1983) who realized the algorithms application to optimization in general.

Adaptive Simulated Annealing (ASA) was developed by the theoretical physicist Lester Ingber. ASA is similar to SA except that it uses statistical measures of the current performance of the algorithm to modify its control parameters i.e. the annealing scheme. For a detailed discussion on ASA the reader is referred to (Ingber 1995).

4.2.2.4 Direct Algorithm

DIRECT is an algorithm developed by Donald R. Jones et al. [20] for finding the global minimum of a multi-variate function subject to simple bounds, using no derivative information. Instead, the algorithm samples points in the domain, and uses the information it has obtained to decide where to search next. In fact, this algorithm is a modification of the standard Lipschitzian approach that eliminates the need to specify a Lipschitz constant. The details about the Lipschitz constant and the relation with the algorithm can be provided from [14]. This algorithm solves the type of optimization problem given as

$$\min_{x} f(x) \quad \text{such that} \quad x_L \leq x \leq x_U$$

with $f(x) \in \mathbb{R}$ and $x, x_L, x_U \in \mathbb{R}^{\mathbb{N}}$.

The first step in the DIRECT algorithm is to transform the search space to be the unit hypercube. In geometry, a hypercube is a n-dimensional analogue of a square (n=2) and a cube (n=3). The function is then sampled at the center-point of this cube. The hypercube is then divided into smaller hyperrectangles whose centerpoints are also sampled. However, when no Lipschitz constant is used, we can not understand the definition of convergence except when the optimal value of the function is known. This problem is solved by a user defined iteration number. The algorithm typically terminates when a user-supplied budget of function evaluations is exhausted. There are many prepared packages to use the DIRECT algorithm in the analysis of studies. Some examples are in the paper of Finkel (2003) and Bjorkman, Holmstrom (1999). In our study, we used the package given in [5]. In this paper, there is a function called gblSolve which applies the DIRECT algorithm into your function in MATLAB. The function that we want to minimize is defined as a real valued function under the name 'fun'. Then the lower bound for the parameter set x_L and upper bound x_U is defined in the vector form. After that the global number of iteration is specified in the name GLOBAL. Finally the printing level of the results in the screen of MATLAB is chosen. Then the user can run the algorithm. The types of printing levels defined under the name PriLev in the algorithm are:

$PriLev \geq 0$ Warnings,

$PriLev > 0$ Small info,

$PriLev > 1$ Each iteration info.

The output is specified with the name Result with $Result = gblSolve(fun, x_L, x_U, GLOBAL, PriLev)$. If the command $f_{opt} = Result.f_k$ is typed in MATLAB, the minimum value of the function that you want to minimize can be reached. In addition, with the command $x_{opt} = Result.x_k$ the estimated parameters can be found. To understand the idea behind Direct algorithm and this specific MATLAB code, the following example from [5] can be examined.

Create a m.file in MATLAB under the name 'funct1' as:

$function f = funct1(x);$

$f = (x(2) - 5 * x(1)^2/(4 * pi^2) + 5 * x(1)/pi - 6)^2 + 10 * (1 - 1/(8 * pi)) * cos(x(1)) + 10;$

Then write the following commands on the screen of MATLAB:

$fun =' funct1';$

$x_L = [-5\ 0]';$

$x_U = [10\ 15]';$

$GLOBAL.iterations = 20;$

$PriLev = 2;$

After that call the gblSolve function, which can be provided from the appendix of the paper [5], we get

$Result = gblSolve(fun, x_L, x_U, GLOBAL, PriLev);$

When 20 iterations are finished we type the following commands:

$f_{opt} = Result.f_k$

$x_{opt} = Result.x_k$

If you want to see a scatter plot of all sampled points in the search space, do:

$C = Result.GLOBAL.C;$

$plot(C(1,:), C(2,:),'.');$

The sampled points in the algorithm can be seen in Figure 4.2 for this example. In the algorithm, firstly, the domain of the function is transformed to a unit rectangle. Then at the center of this rectangle, the function is sampled. After that, the rectangle is divided into smaller ones and the sampling continued. The stopping time of this procedure is determined by the user defined function ”GLOBAL.iterations”. In this example the number of iterations is defined as 20. If you run the commands in MATLAB, you can see that the total number of sampled points is 319.

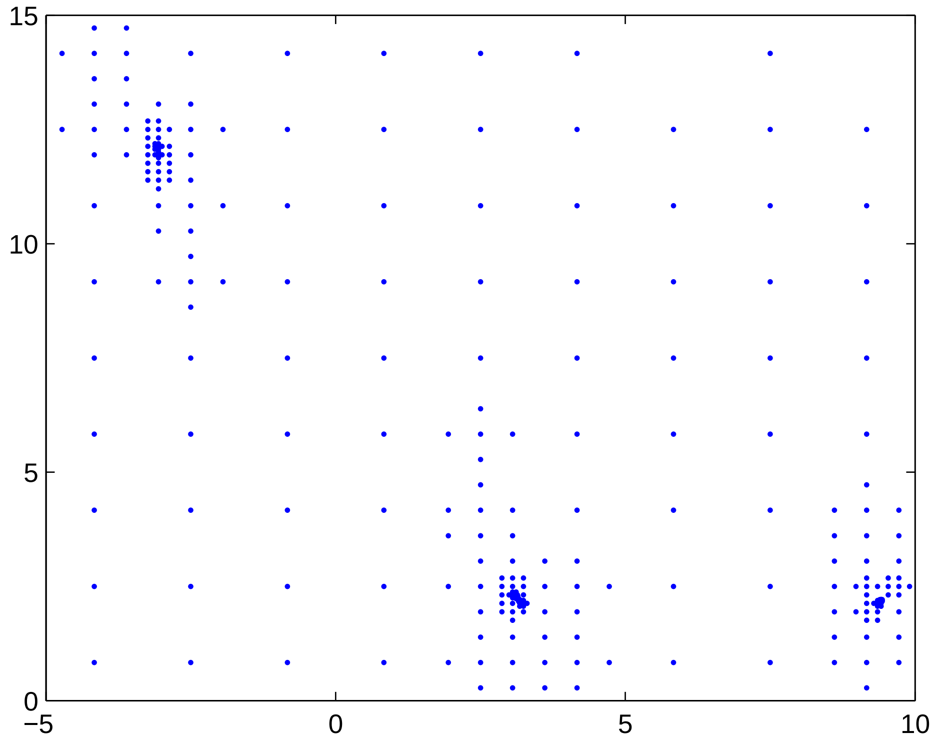

Figure 4.2: Sampled Points with Direct Algorithm

The results are as follows:

$f_{opt} = 0.3979,$

$x_{opt} = [3.1417\ 2.2500].$

In our study, we firstly applied the DIRECT algorithm and then used the resulted parameters in lsqnonlin as initial values. The reason behind this way of parameter estimation is that lsqnonlin function is sensitive to the initial value. Therefore, we used this function when we are sure that the initial estimates are quite close to the optimal parameter set. In addition, note that after finding the characteristic functions formula for each model, we have to invert them by using Fourier Inversion Formulas given in (3.4.2). That means, we encounter with the difficulty to calculate the integral of a complex function in the analysis. We exceed this problem by using the numerical integration function ”quadl” in MATLAB. The function quadl(@fun,a,b) implements an adaptive Gauss Lobatto quadrature rule on the function ‘fun’ over the interval [a,b] where a and b are finite real numbers. A problem arises because quadl evaluates only proper integrals. However, in our problem the integral boundaries were 0 and ∞. For sufficiently large b, the integral can be evaluated with the required accuracy. Therefore we take the integral only from 0 to 200. To save space only SVSI parameter estimation procedure for short-term options is given in the Appendix as m-files. In A.3, the objective function that is used in DIRECT Algorithm can be found with the m.file name ”objvolintshort”. Note that the total number of iterations is determined as 20 in the user defined function. According to this number of iterations, the number of sampled points is 825. The DIRECT Algorithm itself can be provided from A.4 which calls the gblSolve. We used the commands as an m.file but these commands can be written on the screen on MATLAB. In A.5, the ”lsqnonlin” function is called back. Again these commands can be used directly. The objective function of lsqnonlin function can be examined in A.6 with m.file name ”lsqvolintshort”. Finally, the formulation of call option model price of SVSI is in A.7 with the name ”callstockvolint”. Note that this model price is derived theoretically in the previous chapter.

4.3 Implied Parameters and In Sample Performance

In implementing the above procedure the parameters in the groups under ”All Options”, ”Short-Term Options”, and ”At-the-Money Options” are obtained by respectively using all the available options, only short-term options (days to expiration < 60), and only ATM options ($0.97 < S/K < 1.03$) in the day as input into the estimation. For each group the SSE is noted. The structural parameter for each submodel is recorded. The structural parameters K_v, θ_v/K_v and σ_v are respectively the speed of adjustment, the long-run mean, and the variation

Table 4.3: Implied Parameters of All Options

Parameters	**All Options**			
	BS	**SV**	**SVSI**	**SVJ**
K_v		0.8531	0.85	0.86
θ_v		0.5	0.6493	0.4648
σ_v		3.4141	4.4803	3.41
ρ		-0.5594	-0.6164	-0.55
θ_R			0.0139	
K_R			5	
σ_R			0.162	
μ_j				0.0864
σ_j				0
λ				1
$V(t)$	32.76%	13.08%	14.5%	12.17%
SSE	3.7306	0.9084	0.9871	0.8633

coefficient of the diffusion volatility $V(t)$. Similarly, K_R, θ_R/K_R and σ_R are respectively the speed of adjustment, the long-run mean, and the variation coefficient of the spot interest rate $R(t)$. The parameter μ_j represents the mean jump size, λ the frequency of the jumps per year, and σ_j the standard deviation of the logarithm of one plus the percentage jump size. BS, SV, SVSI, and SVJ, respectively, stand for the Black Scholes, the stochastic volatility model, the stochastic volatility and stochastic interest rate model, and the stochastic volatility model with random jumps.

These reported statistics are quite informative about the internal working of the models in three tables. As such, several observations are in order. Firstly, the implied spot volatility among the SV, and the SVSI models are close to each other. Only the implied spot volatility of BS is higher from the others in every category. In fact, this closeness in implied volatility is somewhat surprising. It should, however, be recognized that even small differences in volatility can lead to significantly different pricing and hedging results.

Table 4.4: Implied Parameters of Short-Term Options

Parameters	**Short-Term Options**			
	BS	**SV**	**SVSI**	**SVJ**
K_v		3.1379	3.1380	3.8262
θ_v		0.778	0.6118	0.3716
σ_v		4.9989	5.2716	4.99
ρ		-0.4383	-0.5001	-0.43
θ_R			0.1645	
K_R			5	
σ_R			0.1	
μ_j				0.0843
σ_j				0.0001
λ				0.9997
$V(t)$	31.48%	14.39%	15.81%	15.89%
SSE	0.1618	0.05	0.0546	0.0475

Second, to understand the difference of estimated structural parameters for the SV, the SVSI, and the SVJ models (each assuming stochastic volatility) recall that in the SV model the skewness and kurtosis levels of stock returns are controlled mostly by correlation ρ and volatility variation coefficient σ_v, respectively. The SVSI model relies on the same flexibility, with the additional caveat of having stochastic interest rates to ensure more proper discounting of future payoffs. In addition to inheriting all features of the SV, the SVJ model also allows price jumps to occur, which can internalize more negative skewness and higher kurtosis without making other parameters unreasonable. Therefore, note from the Tables 4.3, 4.4 and 4.5 that

1) The implied speed-of-volatility-adjustment K_v is the highest for the SVJ.

2) The variation coefficient σ_v and the magnitude of ρ are the lowest for the SVJ, then SV and after that SVSI.

These estimates together present the picture of the pricing structure of the calls. Firstly, the

Table 4.5: Implied Parameters of At-The-Money Options

Parameters	**At-the-Money Options**			
	BS	**SV**	**SVSI**	**SVJ**
K_v		0.0002	0.0031	1.7657
θ_v		0.1536	0.4198	0.1667
σ_v		0.8561	2.8547	0.8028
ρ		-0.4776	-0.48	-0.4439
θ_R			0.0462	
K_R			2.2275	
σ_R			0.0002	
μ_j				0.0185
σ_j				0.5
λ				0.1667
$V(t)$	30.32%	8.74%	11.59%	8.33%
SSE	0.0129	0.0026	0.0059	0.0032

SVJ model's demand on the $V(t)$ process is the least strict as it requires both the lowest σ_v and the lowest ρ (in magnitude). However, the SVSI requires σ_v and ρ to be respectively as high as 4.4803 and -0.6164 for all options. The SVJ model attributes part of the implicit negative skewness and excess kurtosis to the possibility of a jump occurring with an average frequency of 1 times per year and an average jump size of 8.64 percent. Secondly, one would expect that adding three extra parameters (related to the interest rate process) will advance the performance of the model. This poor performance by the SVSI can be examined in other measures as well. These results show that adding more parameters to the model does not necessarily better the performance. Finally, by looking at the dollar pricing errors (SSE) one can also see the similar performance of SVSI when compared to SV. From the ”All Options” panel of Table 4.3, the SSE is 3.7306 for the BS and 0.8633 for the SVJ, while it is 0.9084 and 0.9871 for the SV and the SVSI, respectively. Allowing jumps with the model SVJ improves in-sample fit further. The cliff between the SSE of BS model and the other models is generally the case. In our study the number of observations is only 75. If you have a larger data set as in the paper of Bashki, Cao and Chen (1996), this huge cliff becomes more clear. Thirdly, if we compare short-term and all options implied parameters, the volatility coefficient σ_v is higher for each model than before, meaning that for the short-term options to be priced properly the volatility process needs to be more volatile than for all options of any maturity to be priced. Moreover, to price the observed ATM option prices properly, all the three models with stochastic volatility would require volatility shocks to be less negatively correlated with underlying price changes. Finally, as expected, the respective in-sample fits of the four models of short-term options and at-the-money options are better than when the same one set of parameters is applied to all options. This is reflected by the significant reduction in the SSE of each model.

The above results however show something that we do not want to get. That is, if each candidate option pricing model were correctly specified, the six sets of option prices, formed across either moneyness or maturity, should not have resulted in different implied parameter/volatility values. Tables 4.3, 4.4 and 4.5 indicate that every candidate model is misspecified.

4.4 Out-of-Sample Performance of Alternative Models

We have shown that the in sample fit of option prices is increasingly better as we extend from the BS to the SV and then to the SVJ model. However, going from the SV to the SVSI does not necessarily improve the fit much further. In fact, this increasingly better fit might simply due to having an increasingly larger number of structural parameters. To lower the impact of this connection, we turn to examining each model's out-of-sample cross-sectional pricing performance. The presence of more parameters may actually cause over fitting, whereas the structural fitting does not get better. In the following tables, we measured this impact.

Firstly, we calculated model prices of each call option using the parameters estimated for all-options. Then we subtracted the model price from its observed counterpart which is named as the pricing error. To calculate the absolute pricing error, we took the absolute value of each pricing error. In addition, the percentage pricing error is calculated by dividing the pricing error (model price-market price) to market price. This procedure is repeated for every call in the sample, then the average of these errors are recorded. These steps are separately followed for the BS, the SV, the SVSI, and the SVJ models. Table 4.6 reports the absolute pricing errors whereas in Table 4.7 the percentage pricing errors are examined. The minus sign in percentage pricing error implies that in that category the model systematically overprice the options. The plus sign on the contrary means that the options are underpriced. Note that pricing errors are obtained ”All Option” based on the implied parameter/ volatility values. In our main reference Bashki, Cao and Chen (1996) these results are reported in three different categories which are ”Maturity-Based”, ”Moneyness-Based”, ”All-Option-Based”. For our study, one category is enough to measure the out of sample performance of alternative models.

Table 4.6: Absolute Pricing Errors

Moneyness(S/K)	**Model**	**Days to Expiration**		
		< 60	**$60 - 180$**	**≥ 180**
< 0.94	BS	\$0.019	\$0.105	\$0.107
	SV	0.010	0.029	0.059
	SVSI	0.015	0.052	0.091
	SVJ	0.006	0.028	0.060
$0.94 - 0.97$		-	-	-
$0.97 - 1.00$	BS	0.071	0.127	-
	SV	0.033	0.0961	-
	SVSI	0.051	0.126	-
	SVJ	0.02	0.063	-
$1.00 - 1.03$	BS	0.052	0.062	-
	SV	0.054	0.040	-
	SVSI	0.044	0.064	-
	SVJ	0.040	0.06	-
$1.03 - 1.06$	BS	-	-	0.068
	SV	-	-	0.023
	SVSI	-	-	0.028
	SVJ	-	-	0.042
≥ 1.06	BS	0.11	0.169	0.508
	SV	0.064	0.117	0.168
	SVSI	0.066	0.117	0.166
	SVJ	0.062	0.107	0.158

By starting with the absolute and the percentage pricing errors, respectively given in Tables 4.6 and 4.7 we reached the following results. First, both pricing error measures rank the SVJ model first, the SV second, the SVSI next, and the BS last, except that for a few categories either the SV or the SVSI performs slightly better than the others. According to the percentage pricing errors, the SVSI does slightly better than the SVJ in pricing the long-term deepest ITM calls (days to expiration > 180). This is not surprising since the long-term deep ITM calls to be the most sensitive to interest rates. Moreover, according to both measures, the SVJ surpass the SVSI in pricing deep OTM calls (Moneyness < 0.94) which is also expected. In contrast to this result, in [1] the performance of SVSI was better than SVJ while pricing deep OTM calls.

Table 4.7: Percentage Pricing Errors

Moneyness(S/K)	**Model**	**Days to Expiration**		
		< 60	**60 – 180**	**≥ 180**
< 0.94	BS	%22.96	%-45.34	%-18.58
	SV	32.33	-27.52	-20.22
	SVSI	-88.24	0.052	-35.54
	SVJ	-35.47	0.028	-23.69
0.94 – 0.97		-	-	-
0.97 – 1.00	BS	-18.51	-15.04	-
	SV	9.11	11.38	-
	SVSI	13.96	15.04	-
	SVJ	5.71	7.48	-
1.00 – 1.03	BS	-6.06	-4.83	-
	SV	-6.45	2.44	-
	SVSI	-5.39	4.27	-
	SVJ	-4.80	4.98	-
1.03 – 1.06	BS	-	-	22.99
	SV	-	-	0.96
	SVSI	-	-	1.17
	SVJ	-	-	1.65
≥ 1.06	BS	3.71	3.22	6.82
	SV	-0.21	-1.39	0.30
	SVSI	-0.68	-1.51	-0.23
	SVJ	0.19	-0.76	0.94

Second, regardless of option moneyness or maturity, adding stochastic volatility produces a significant improvement over the BS model. This improvement reduces the absolute pricing errors up to 68 percent in the striking cases like deep ITM calls. To see this important progress take a OTM call with moneyness less than 0.94 and with days to expiration between 60 to 180. From Table 4.1 the average price for such a call is \$0.25. When the BS is applied to value this call, the resulting absolute pricing error is, on average, \$0.105 as shown in Table 4.6, but when the SV is applied, the average error goes down to \$0.02. Thus, Table 4.6 suggests that once stochastic volatility is added to the model adding other features lead o a second order development. Thirdly, the absolute error becomes greater when we began from short term followed by medium and finally came to long term options. This result is valid for a given moneyness category and regardless of the pricing model. Looking at the percentage pricing error measure, the BS exhibits clear moneyness and maturity related biases. In addition, from Table 4.7 it can be seen that there is relatively large mispricing of short term options as well as OTM options. The reason behind this result may be the objective function in equation (4.2.1) is biased in favor of more expensive calls (i.e., long-term and ITM calls). As we try to estimate each parameter by minimizing the sum of squared dollar pricing errors, this factor can enlarge the extent of poor fit for short term and OTM options in each pricing model. However, this possible enlargement should not affect the overall conclusion regarding the pricing structure of short-term and OTM options relative to others. The reason is that in Figure 4.1, even when the BS implied volatility is estimated for each option individually (no weighted average), the volatility smile is clearly the sharpest for short-term options. Finally, the magnitude of mispricing varies dramatically across the models, with the BS producing the highest and the SVJ the lowest errors. Among the four models, the SVJ shows the best ability in improving the pricing of short-term options. The same conclusions can be reached regarding the models even according to the percentage pricing errors.

Note that in judging the alternative models, some other yardsticks can be employed. The first one may be showing the consistency of implied structural parameters with those implicit in the relevant times-series data. For more details, [1] and [4] can be examined. Secondly, the hedging errors can be used to measure how well a model captures the dynamic properties of option and underlying security prices. In fact, our theoretical model and closed form solution lead to useful analytical hedge ratios, and contains many known option formulas as special cases.

CHAPTER 5

CONCLUSION

In this study, we have presented two totally different approaches of stochastic volatility and stochastic interest rate model with jumps to the derivation of closed-form European call option problem and used one of these approaches in the empirical analysis. In the first approach, suggested by Scott [33], the characteristic function is obtained via martingale methods. Consequently, by using the inversion formula, the distributions and call option prices are calculated. The other approach proposed by Bashki, Cao and Chen [1] give the option pricing formula and the spot asset price dynamics by constructing partial integro-differential equations. In the empirical analysis, we established 4 alternative models with the help of the closed-form solutions obtained in the second approach. The performance of these models are measured on General Electric Stock Option Data from two perspectives: (1) in-sample fit and (2) out-of sample performance.

In the theoretical part of the study, we have derived the closed-form European call option prices in both approaches step-by-step with the proofs. On the other hand, in the application part, the performances of Black Scholes Model, Stochastic Volatility Model, Stochastic Interest Rate Model, Stochastic Volatility and Stochastic Interest Rate Model are compared. According to our results, incorporating stochastic volatility and jumps improves models on stock options significantly. However, adding stochastic interest rate did not enhance the performance as one would expect. In-sample fit performance is highest for SVJ model then for SV and SVSI and least for BS Model. Out-of sample performance ranks the SVJ model first, the SV second, the SVSI next and the BS last. Only for long-term deep in-the money options SVSI shows better performance then the other models. The short-term options priced best by SVJ model which shows the greatest challenge to any alternative option pricing model.

Moreover, adding volatility into the model improves the out-of sample performance significantly. Therefore, once stochastic volatility is added, adding other features lead a second order improvement.

In this thesis, the author looked for understandable arrangements which could serve the interested reader for further use of the original works. Furthermore, the application part is constructed to guide the readers how much each generalization of alternative models improves option pricing.

Because of the time constraint of this study, we have left the hedging of the models and measuring their performance in this perspective for future works. In fact, the results for index options suggested by Bashki, Cao and Chen [1] has not been applied on a single stock option yet. Therefore, the analysis on single stock options in this study can be improved with some other performance measures. In addition to these, the techniques to price path dependent options can be an extension of the methods given in the thesis.

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APPENDIX A

Table A.1: The General Electric Option Price Data

Strike/Maturity	**18.02.11**	**18.03.11**	**15.04.11**	**17.06.11**	**16.09.11**	**20.01.12**	**18.01.13**
7.5	-	-	-	-	-	11.1	11.13
10	-	-	8.45	8.6	8.57	8.55	8.6
12	-	-	-	6.45	-	-	-
12.5	-	-	-	-	-	6.15	6.3
13	-	5.4	-	5.65	5.45	-	-
14	-	4.65	4.45	4.85	4.55	-	-
15	3.55	3.5	3.65	3.65	3.55	4	4.5
16	2.52	2.56	2.66	2.76	2.96	-	-
17	1.65	1.66	1.81	2.04	2.19	-	-
17.5	-	-	-	-	-	2.27	2.99
18	0.77	0.92	1.1	1.33	1.69	-	-
19	0.30	0.44	0.61	0.85	1.15	-	-
20	0.11	0.18	0.28	0.5	0.76	1.13	1.84
21	0.03	0.09	0.15	0.27	0.47	-	-
22	-	-	0.08	-	-	-	-
22.5	0.01	0.03		0.12	0.25	0.51	1.1
23	-	-	0.03	-	-	-	-
25	0.01	0.02	0.01	0.05	-	0.23	0.73
30	-	-	-	-	-	0.08	0.29
35	-	-	-	-	-	-	0.18

The m.file used to get the Figure 4.1 is as follows:

Table A.2: The Implied Volatility Graph’s M-File

```
l=[1.4177;1.3164;1.2287;1.1519;1.0841;1.0239;0.97;0.9215;0.8776;0.8191;0.7372];
m=[76;75;58;45;39;30;29;30;31;36;49];
plot(l,m,char('–'))
hold on
z=[1.843;1.5358;1.4177;1.3164;1.2287;1.1519;1.0841;1.0239;0.97;0.9215;0.8776;0.8377;
0.8191;0.8013;0.7372];
k=[84;56;51;48;41;37;33;31;29;28;28;30;28;29;32];
plot(z,k,[char('m'),char('-.')])
hold on
x=[2.4573;1.843;1.4744;1.2287;1.0531;0.9215;0.8191;0.7372;0.6143;0.5266];
y=[83;59;45;38;34;31;30;30;32;33];
plot(x,y,[char('r'),char(':')])
xlabel('Moneyness(S/K)')
ylabel('Implied Volatility(%)')
legend('Days to Expiration < 60','Days to Expiration 60 − 180',
'Days to Expiration >= 180')
```

Table A.3: The M-File ”Objvolintshort”

```
function [objective]=objvolintshort(parm)
q=0.028;%dividend yield
r=0.0015;%risk free interest rate
S=18.43;%spot stock price
Cobserved1=[3.55;2.52;1.65;0.77;0.3;0.11;0.03;0.01;0.01];
Strike1=[15;16;17;18;19;20;21;22.5;25];
Strike2=[13;14;15;16;17;18;19;20;21;22.5;25];
Cobserved2=[5.4;4.65;3.5;2.56;1.66;0.92;0.44;0.18;0.09;0.03;0.02];
Time1=0.06;%Time to Maturities
Time2=0.11;
sum1=0;
for j=1:size(Cobserved1,1)
sum1=sum1+(Cobserved1(j)-callstockvolint(parm(1),parm(2),parm(3),parm(4),parm(5),parm(6),
parm(7),parm(8),q,r,Time1,S,Strike1(j)))^2;
end
sum2=0;
for j=1:size(Cobserved2,1)
sum2=sum2+(Cobserved2(j)-callstockvolint(parm(1),parm(2),parm(3),parm(4),parm(5),parm(6),
parm(7),parm(8),q,r,Time2,S,Strike2(j)))^2;
end
objective=sum1+sum2;
end
```

Table A.4: The M-File of Direct Algorithm

```
fun = 'objvolintshort';
%Thetav,Kv,Sigmav,rho,V,Thetar,Kr,Sigmar
x_L = [0 0 5 −1 0 0 0 0.1]';
x_U = [1 3.138 10 −0.439 0.5 1 5 5]';
GLOBAL.iterations = 20;
PriLev = 2;
Result = gblSolve(fun,x_L,x_U,GLOBAL,PriLev);
f_opt = Result.f_k;
x_opt = Result.x_k;
```

Table A.5: The M-File of lsqnonlin

```
%parameters Thetav,Kv,sigmav,rho,V,Thetar,Kr,Sigmar
lb = [0 0 5 −1 0 0 0 0.1];
ub = [1 3.138 10 −0.439 0.5 1 5 5];
x0 = [0.1296;3.0799;8.4259;-0.5325;0.25;0.0185;4.9074;0.1907];
options = optimset('MaxIter',1000,'MaxFunEvals',20000,'TolFun',1e-30);
tic;
[x,resnorm] = lsqnonlin(@lsqvolintshort,x0,lb,ub,options);
tElapsed=toc;
```

Table A.6: The M-File "lsqvolintshort"

```
function [objective]=lsqvolintshort(parm)
q=0.028;%dividend yield
r=0.0015;%risk free interest rate
Time1=0.06;%time to maturities
Time2=0.11;
S=18.43;%spot stock price
Cobserved1=[3.55;2.52;1.65;0.77;0.3;0.11;0.03;0.01;0.01];
K1=[15;16;17;18;19;20;21;22.5;25];
Cobserved2=[5.4;4.65;3.5;2.56;1.66;0.92;0.44;0.18;0.09;0.03;0.02];
K2=[13;14;15;16;17;18;19;20;21;22.5;25];
for j=1:size(Cobserved1,1)
diff1(j)=Cobserved1(j)-callstockvolint(parm(1),parm(2),parm(3),parm(4),parm(5),parm(6),
parm(7),parm(8),q,r,Time1,S,K1(j));
end
for j=1:size(Cobserved2,1)
diff2(j)=Cobserved2(j)-callstockvolint(parm(1),parm(2),parm(3),parm(4),parm(5),parm(6),
parm(7),parm(8),q,r,Time2,S,K2(j));
end
objective=[diff1';diff2'];
end
```

Table A.7: The M-File ”callstockvolint”

```
function CallSVSI= callstockvolint(Thetav,Kv,Sigmav,rho,V,Thetar,Kr,Sigmar,q,r,tau,S,K)
warning off;
y=@(phi)real(exp(-i*log(K)*phi).*f1(phi,Thetav,Kv,Sigmav,rho,V,Thetar,
Kr,Sigmar,q,r,tau,S)./(i * phi ));
p1=0.5 + 1/pi*quadl(y,0,200);
z=@(phi)real(exp(-i*log(K)*phi).*f2(phi,Thetav,Kv,Sigmav,rho,V,Thetar,
Kr,Sigmar,q,r,tau,S)./(i * phi ));
p2=0.5 + 1/pi*quadl(z,0,200);
CallSVSI= exp(-q*tau)*S*p1-Bnd(Thetar,Kr,Sigmar,r,tau)*K*p2;
end
function cf1 = f1(phi,Thetav,Kv,Sigmav,rho,V,Thetar,Kr,Sigmar,q,r,tau,S)
Epsr=sqrt(Kr^2-(2*(Sigmar^2)*i*phi));
Epsv=sqrt([Kv-[(1+(i*phi))*rho*Sigmav]].^2-[(i*phi).*(1+(i*phi))*(Sigmav^2)]);
cf11=[V*(i*phi).*((i*phi)+1).*(1-exp(-Epsv*tau))]./[(2*Epsv)
-[(Epsv-Kv+[(1+(i*phi))*rho*Sigmav]).*(1-exp(-Epsv*tau))]];
cf12=[-Thetav*(Epsv-Kv+[(1+(i*phi))*rho*Sigmav])*tau]./Sigmav^2;
cf13=1i*phi*log(S);
cf14=[-2*Thetav*log(1-((([Epsv-Kv+[(1+(i*phi))*rho*Sigmav]].*
(1-exp(-Epsv*tau))))./(2*Epsv)))]./Sigmav^2;
cf15=[-2*Thetar*log(1-(((Epsr-Kr).*(1-exp(-Epsr*tau)))./(2*Epsr)))]./Sigmar^2;
cf16=(-Thetar*(Epsr-Kr)*tau)./Sigmar^2;
cf17=[2*(r-q)*i*phi.*(1-exp(-Epsr*tau))]
./[(2*Epsr)-[(Epsr-Kr).*(1-exp(-Epsr*tau))]];
cf1=exp(cf11+cf12+cf13+cf14+cf15+cf16+cf17);
end
function cf2=f2(phi,Thetav,Kv,Sigmav,rho,V,Thetar,Kr,Sigmar,q,r,tau,S)
Epsrstar=sqrt(Kr^2-(2*(Sigmar^2)*(i*phi-1)));
Epsvstar=sqrt([Kv-[(i*phi)*rho*Sigmav]].^2-[(i*phi).*((i*phi)-1)*(Sigmav^2)]);
cf21=[V*(i*phi).*((i*phi)-1).*(1-exp(-tau*Epsvstar))]
./[(2*Epsvstar)-[(Epsvstar-Kv+[(i*phi)*rho*Sigmav]).*(1-exp(-tau*Epsvstar))]];
cf22=(1i*phi*log(S))-log(Bnd(Thetar,Kr,Sigmar,r,tau));
cf23=[-Thetav*(Epsvstar-Kv+[(i*phi)*rho*Sigmav])*tau]./Sigmav^2;
cf24=[-2*Thetav*log(1-([[Epsvstar-Kv+[(i*phi)*rho*Sigmav]]
.*(1-exp(-tau*Epsvstar))]./(2*Epsvstar)))]./Sigmav^2;
cf25=[-2*Thetar*log(1-(((Epsrstar-Kr).*(1-exp(-Epsrstar*tau)))
./(2*Epsrstar)))]./Sigmar^2;
cf26=(-Thetar*(Epsrstar-Kr)*tau)./Sigmar^2;
cf27=[2*(r-q)*(i*phi-1).*(1-exp(-Epsrstar*tau))]
./[(2*Epsrstar)-[(Epsrstar-Kr).*(1-exp(-Epsrstar*tau))]];
cf2=exp(cf21+cf22+cf23+cf24+cf25+cf26+cf27);
end
function[bond]=Bnd(Thetar,Kr,Sigmar,r,tau)
bondstar=sqrt(Kr^2+(2*Sigmar^2));
bond1=(-Thetar*(((bondstar-Kr)*tau)+
(2*log(1-(((1-(exp(-bondstar*tau)))*(bondstar-Kr))
/(2*bondstar))))))/Sigmar^2;
bond2=(-2*(1-(exp(-bondstar*tau))))/
(2*bondstar-((bondstar-Kr)*(1-(exp(-bondstar*tau)))));
bond=exp(bond1+r*bond2);
end
```