FRICTIONLESS DOUBLE CONTACT PROBLEM FOR AN AXISYMMETRIC ELASTIC LAYER BETWEEN AN ELASTIC STAMP AND A FLAT SUPPORT WITH A CIRCULAR HOLE

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FRICTIONLESS DOUBLE CONTACT PROBLEM FOR AN AXISYMMETRIC ELASTIC LAYER BETWEEN AN ELASTIC STAMP AND A FLAT SUPPORT WITH A CIRCULAR HOLE

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ABSTRACT

FRICTIONLESS DOUBLE CONTACT PROBLEM FOR AN AXISYMMETRIC ELASTIC LAYER BETWEEN AN ELASTIC STAMP AND A FLAT SUPPORT WITH A CIRCULAR HOLE

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This study considers the elastostatic contact problem of a semi-infinite cylinder. The cylinder is compressed against a layer lying on a rigid foundation. There is a sharp-edged circular hole in the middle of the foundation. It is assumed that all the contacting surfaces are frictionless and only compressive normal tractions can be transmitted through the interfaces. The contact along interfaces of the elastic layer and the rigid foundation forms a circular area of which outer diameter is unknown. The problem is converted into the singular integral equations of the second kind by means of Hankel and Fourier integral transform techniques. The singular integral equations are then reduced to a system of linear algebraic equations by using Gauss-Lobatto and Gauss-Jacobi integration formulas. This system is then solved numerically. In this study, firstly, the extent of the contact area between the layer and foundation are evaluated. Secondly, contact pressure between the cylinder and layer and contact pressure between the layer and foundation are calculated for various material pairs. Finally, stress intensity factor on the edge of the cylinder and in the end of the sharp-edged hole are calculated.

Keywords: Axisymmetric, Semi-infinite Cylinder, Singular Integral Equations, Contact Problem, Stress Intensity Factor.

ÖZ

BİR ELASTİK SİLİNDİR İLE YUVARLAK BİR DELİĞİ BULUNAN DÜZ BİR DESTEK ARASINDAKİ BİR ELASTİK TABAKA İÇİN SÜRTÜNMESİZ EKSENEL SİMETRİK TEMAS PROBLEMİ

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Bu çalışma, yarı sonsuz bir silindirin elastostatik temas problemini incelemektedir. Silindir, rijit bir temel üzerindeki bir tabakaya karşı sıkıştırılmıştır. Temelin ortasında keskin köşeli dairesel bir delik bulunmaktadır. Tüm temas yüzeylerinin sürtünmesiz olduğu ve arayüzler boyunca sadece basınç normal gerilmelerinin iletildiği varsayılmaktadır. Elastik tabaka ve rijit temel arayüzeyi boyunca temas, dış yarıçapı bilinmeyen dairesel bir alan oluşturmaktadır. Problem, Hankel ve Fourier integral dönüşüm teknikleri kullanılarak ikinci çeşit tekil integral denklemler haline dönüştürülmektedir. Tekil integral denklemler daha sonra Gauss-Lobatto ve Gauss-Jacobi integrasyon formülleri kullanılarak lineer cebirsel denklemlere indirgenmektedir. Bu sistem daha sonra sayısal olarak çözülmektedir. Bu çalışmada ilk olarak, tabaka ve temel arasındaki temas alanının genişliği değerlendirilmektedir. İkinci olarak, silindir ve tabaka arasındaki temas basıncı ve tabaka ile temel arasındaki temas basıncı çeşitli malzeme çiftleri için hesaplanmaktadır. Son olarak, silindirin köşesindeki gerilme şiddeti katsayısı ve keskin köşeli deliğin ucundaki gerilme şiddeti katsayısı hesaplanmaktadır.
Anahtar Sözcükler: Eksenel Simetri, Yarı Sonsuz Silindir, Tekil İntegral Denklemler, Temas Problemi, Gerilme Şiddeti Katsayısı.

To My Family,

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NOMENCLATURE

$2a$	Diameter of semi-infinite cylinder
$2b$	Diameter of hole
$2c$	Outer diameter of contact area
h	Thickness of elastic layer
p_0	Uniform axial compression
u , w	Displacement components in r- and z-directions
σ , τ	Normal and shearing stresses
κ	$(3-4\nu)$
μ	Shear modulus of elasticity
ν	Poisson's ratio
$P_n^{(\alpha,\beta)}$	Jacobi polynomials
J_0 , J_1	Bessel functions of the 1st kind of order zero and one
I_0 , I_1	Modified Bessel functions of the 1st kind of order zero and one
K_0 , K_1	Modified Bessel functions of the 2nd kind of order zero and one
K , E	Complete elliptic integrals of the 1st and the 2nd kinds
U , W	Hankel transforms of u, w
W_i	Weighting constants of the Jacobi polynomials

Γ	Gamma function
D_i	Weighting constants of the Gauss-Lobatto polynomials
h_1 , h_2	Kernels of the integral equations
t	Integration variable
$\bar{k}_{1a}$	Normalized Mode I stress intensity factors at the edge of the semi-infinite cylinder
$\bar{k}_{1b}$	Normalized Mode I stress intensity factors at the edge of the hole
r , z	Cylindrical coordinates
$W_s(r,\alpha)$	Fourier sine transform of $w(r,z)$
α	Fourier transform variable
$U_c(r,\alpha)$	Fourier cosine transform of $u(r,z)$
ρ	Hankel transform variable
c_i	Arbitrary integration constants
x, y, ξ, η	Dimensionless variables
ODE	Ordinary Differential Equation

CHAPTER 1

INTRODUCTION

Elasticity is used widely to obtain a solution for engineering problems. Especially contact problems as a part of the elasticity have a practical importance in the literature. In addition, there are many areas of application for the contact problems like composite materials, roadways, grillages and airfield pavements.

Contact problems are also called Hertz Contact Problems because the first investigations on contact problems were made by Hertz in the second half of the 19th century. There are some rules about the theory of Hertz: bodies are full elastic; surfaces are frictionless; deformations are small; each surface can be treated as an elastic half space. Later, Hertz's theory was developed for axisymmetric contact problems by Bousinessq (1885). Many extensive investigations on contact problems appeared after the publications of Sneddon's studies (1951) on integral transforms in the elasticity theory, and development of complex variable methods by Mushelishvili (1953). The technique of integral transforms is used for solution of different type of contact problems. For example, Mellin transform, Laplace transform, Fourier transform and Hankel transform are applied for polar coordinates problems, vibration problems, cartesian coordinates problems, and cylindrical problems, respectively.

Integral equation methods are one of the first methods to obtain general solutions for contact problems. There are also various other methods for the solution of the contact problems like methods of finite element, finite difference and boundary element. Highly complex problems may be solved using the above methods by means of today's computer technology.

1.1 Literature Review

Lebedev and Ufliand (1958) considered an elastic layer overlying a rigid foundation subjected to pressing a circular cross-sectional stamp. The pressing was occurred via an axial force. The method developed in the study allowed to express the required displacements and stresses with regard to an auxiliary function, which standed for the solution of Fredholm integral equation. Some numerical results were reported to obtain a sample of a stamp by using plane base.

Keer (1964) dealt with the problem of contact stress for an elastic die notching a layer starting from Vorovich and Ustinov notation. The problem was transformed to solution of dual integral equations by the method of Hankel transform. Here, equations were firstly formulated for a rigid die. Then, the problem was solved for the elastic die by using boundary conditions and convenient equations of elasticity. Finally, the influences of the elastic constants were expressed in terms of contact applied load, loading and radius of contact.

Wu, Pao and Chiu (1971) investigated the plane strain problem involving an elastic layer, a half space foundation and cylindrical indenter. Consequences of the contact condition for the interface of half space foundation and thin elastic layer subjected to the contact stresses of indented layer were discussed in detail. In addition, the formulation of integral equations covering the contact stress distribution of the indented layer was expressed for the basic problem of elastic foundation or elastic indenter. After analyzing the problems involving elastic indenter or elastic foundation, the researchers pointed out one similarity and one difference. The similarity between these two was that both had the same integral equations by reformulating the Fredholm integral equations. Yet, these equations were different in their kernel functions. In order to solve these problems, a numerical method based on finite difference approximation was developed.

Parlas and Michalopoulos (1972) analyzed a rigid punch in the shape of a bolt which was pressed into half space. The half space was homogeneous, isotropic and elastic. It also contained a transverse annular cylindrical hole. A set of dual integral equations was obtained from mixed boundary value problem. The first and second kind Bessel functions included in this problem were simplified into the first kind singular Fredholm integral equations. Next, numerical solution for this equation was obtained. The results of displacements and stresses for the half space were presented graphically.

Civelek and Erdogan (1974) investigated axially symmetric situation of double contact problem including three different materials, which are layer, stamp, and half space. The problem was simplified into singular integral equations. Unknown functions in these equations were contact pressures. Comprehensive numerical results for three stamp geometries (flat-ended rigid cylindrical stamp, elastic and rigid spherical stamps) were obtained.

Ufliand and Zlatin (1976) considered an axially symmetric contact problem involving an elastic cylinder and an elastic layer overlying a fixed foundation. When the transform of Hankel for the layer and orthogonality of eigenfunctions for the cylinder were used, the problem was converted into the linear algebraic equations. Consequently, these equations made effective solutions possible by the method of truncation.

Gecit (1980) studied a plane contact problem covering an elastic layer and foundation. Vertical body force, vertical line load and pressure were uniformly applied on the layer. Critical line load was firstly found just after the solution of continuous contact problem was performed. Second, formulation of these continuous contact problems was carried out according to singular integral equations. Distributions of contact stress region of separation and critical line load were numerically found. He also studied with an axisymmetric contact problem consisting of an elastic layer overlying a semi-infinite base in 1981. The elastic layer was pressed towards the base. A line load was vertically applied to the layer. The solution of the formulated problem was obtained for both tensile and compressive line loads. Numerical solutions were obtained for the distributions of the contact stress in terms of combinations of various materials.

Kumar and Hiremath (1984) examined an axially symmetric Boussinesq problem to determine the stress distributions of an isotropic semi-infinite elastic solid. After a rigid annular punch was exposed to heat, it was pressed on the free surface of the solid. At the end of the process, the distributions of temperature on this surface were found to be uneven. In essence, this was a three part mixed boundary value problem and this problem was reduced to triple integral equations' solution. Then, these equations were converted into the solution of simultaneous equations which were infinite provided that $\sigma_{zz}(r,0)$ had in the punch region. Consequently, the variations of $\sigma_{zz}(r,0)$ and total load were presented in graphics.

Gecit (1986) analyzed an elastostatic contact problem including elastic layer and semi-infinite cylinder. The cylinder was pressed towards the layer overlying a rigid base. Frictionless contacting surfaces were assumed. It was thought that tensile tractions were not transmitted along the interfaces. Numerical solutions were performed by means of the integral equations. As a result, contact pressures and stress intensity factors were calculated for some material pairs.

Hara and Suzuki (1988) studied an axially symmetric contact problem including an elastic half space and a rigid foundation with parabolic ended pit and protrusion. The space was pressed against the foundation with a parabolic ended pit or protrusion. The results of the problem were separately discussed for both of these, respectively. Papkovich-Neuber equations were used to solve the problem in oblate spheroidal coordinates. Contact stress and surface displacement distributions were displayed. Then, solutions were found to be in agreement with solutions for flat-ended pit or protrusion.

Li and Dempsey (1990) investigated an axially symmetric contact problem involving a rigid sphere, a rigid flat cylinder, an elastic sphere or a circular plate overlying an elastic layer. In this study, these contact problems were reduced to integral equations. By applying approximation of exponential series, an infinite integral consisting of two Bessel functions was converted to a finite summation. This finite summation was actually achieved by the extraction of a singular term. Numerical values were compared with existing analytical solutions and it was concluded that highly accurate numbers and easiness were achieved through the use of this method. Thus, contact pressure distributions, displacements and contact radii were stated.

Selvadurai (1994) examined an axially symmetric elastostatic problem including two different elastic half spaces and a rigid disc inclusion. The inclusion was embedded between the spaces. The contact problem was reduced to Fredholm type integral equation of second kind. Numerical results showed that precompression stress had an important effect on the radius of the region of separation.

Jaffar (1997) considered an axially symmetric frictional contact problem. It was thought that a thin elastic layer lying on a rigid foundation was indented by a punch. The problem was solved by perturbation theory. Numerical solutions were obtained for three different shapes which are spherical, flat-ended cylindrical and conical. In addition, the effect of friction on results was considered. Then, solutions were compared to the ones in the literature.

Cakiroglu F, Cakiroglu M and Erdol (2001) analyzed the problem of continuous and discontinuous contacts, respectively. The geometry of the problem was defined as two elastic layers overlying an elastic semi-infinite plane. In addition, a frictionless contact between surfaces and a uniform pressure on the top layer were assumed. Singular integral equations were formed according to these continuous contact positions, and then the method of Gauss- Chebyshev integration was used to solve the problem numerically. Finally, graphical forms were obtained for separations and stress distributions.

Chaudhuri and Ray (2003) studied the behaviour of rigid punch on an elastic half space. Basic equations were expressed by Hankel transform. The problem was solved for both flat-ended cylindirical and paraboloidal punches. After that, solutions were obtained from Fredholm integral equation. Calculations were made for different values of the nonhomogenity parameter. Finally, the effect of stress on these values was shown graphically.

Avci, Bulu and Yapici (2006) investigated an axially symmetric contact problem. They used a cylinder which was elastic, thick-walled, hollow and isotropic which was pressed by an inelastic external ring. Fourier transform helped to solve equations of the elastic theory for the problem cylindrical coordinates. Then, basic expressions for the displacements were acquired. A singular integral equation was obtained by simplification of the formulation employing boundary conditions. Next, Gaussian quadrature was used to solve the singular integral equation. Lastly, graphical forms for numerical results were obtained for normalized pressure distributions and distance presenting the contact zone.

Ozsahin, Kahya, Birinci and Cakiroglu (2007) considered a contact problem of a layered composite having different bi-material constants. Navier equations were used to solve two dimensional contact problems. Next, displacement expressions were written by using Fourier transforms for both layers. The problem was converted into algebraic equations by the integral formula of Gauss-Chebyshev. From the solution of these equations, graphs for normalized contact pressure and the axial stress distribution were obtained.

Kahya, Birinci and Erdol (2007) studied a contact problem with no friction between two orthotropic elastic layers. Gravity force was included into the equations. Singular integral equations were obtained from Fourier transforms and the elasticity theory. Graphs were presented for initial separation point, crucial separation load and normalized contact stress, respectively.

Liu, Wang and Zhang (2008) considered the problem of an axially symmetric contact for the half space. The half space was functionally graded and coated. The problem was simplified into a Cauchy singular integral equation by exploiting the Hankel integral transform and transfer matrix method. The equations were numerically solved to calculate indentation, contact zone and pressure.

Rhimi, El-Borgi, Ben Said and Ben Jemaa (2009) studied the problem of receding contact including half space and layer. The layer was elastic functionally graded, and the half space was homogeneous. Two bodies were compressed together, and there was no friction between them. The problem was thought separately for these bodies. Singular integral equation was analytically obtained from the axial symmetric elasticity equations employing Hankel transform. The receding contact radius and the contact pressure were unknown parameters in a singular integral equation. The solution of the equation was obtained numerically with Chebyshev polynomials. In addition, receding contact length was calculated with an iterative method.

1.2 Scope of the Problem and Solution Method

This thesis is involved with the problem of axially symmetric double contact containing an elastic layer, a circular hole and semi-infinite cylinder. The layer overlies a rigid base. There are unlimited contacts between the foundation and layer. In addition, there is a uniform compression applied to the layer by the cylinder. The layer is restricted between z=0 and z=-h planes. There are two assumptions about the problem. First, it is assumed that there is no friction in contact surfaces. Second, only compressive tractions can be transmitted along the interfaces. When pressure stress loses its effect, separation occurs. The separation has infinite length because there is no gravity effect for the problem. Stress and displacement relations are written with Fourier and Hankel transforms for both the layer and cylinder. Then, boundary conditions applied for these expressions make it possible to obtain singular integral equations. All linear algebraic equations are obtained from the singular integral equations after using some integration formulas like Gauss-Jacobi and Gauss-Lobatto. Contact pressures and stress intensity factors are calculated numerically for various material pairs.

CHAPTER 2

DOUBLE CONTACT PROBLEM

2.1 Problem Definition

The axisymmetric double contact problem analyzed is that an infinite layer overlying a rigid base is compressed by a semi-infinite cylinder subjected to a uniform compression p_0 as seen in Figure 2.1. It is assumed that all bodies except rigid foundation have elastic and isotropic properties. Radius of cylinder and layer of thickness are assumed as a and h, respectively. The frictionless elastic layer is resting on a horizontal rigid foundation containing a circular hole of diameter $2b$ at the center with 90^0 sharp corner. The body forces are assumed to be zero. In this case, the contact between the foundation and layer is lost along the outer domain $(c < r < \infty)$ where c is yet unknown. For linearly elastic, isotropic and axisymmetric elasticity problems, the field equations can be listed as follows:

Stress-displacement relations are (Gecit and Erdogan 1978; Gecit 1986) :

$$\sigma_r = \frac{\mu}{\kappa - 1}\left[(\kappa+1)\frac{\partial u}{\partial r} + (3-\kappa)\left(\frac{u}{r} + \frac{\partial w}{\partial z}\right)\right],$$

$$\sigma_z = \frac{\mu}{\kappa - 1}\left[(3-\kappa)\left(\frac{\partial u}{\partial r} + \frac{u}{r}\right) + (\kappa+1)\frac{\partial w}{\partial z}\right],$$

$$\tau_{rz} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right), \qquad \text{(2.1a–c)}$$

where σ and τ denote normal and shearing stresses, μ is named as shear modulus, $\kappa=3-4\,\nu$, ν is defined as Poisson's ratio.

Navier equations (equilibrium equations in point of the displacements) are

$$(\kappa+1)\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}-\frac{u}{r^2}\right)+(\kappa-1)\frac{\partial^2 u}{\partial z^2}+2\frac{\partial^2 w}{\partial r\partial z}=0,$$

$$2\left(\frac{\partial^2 u}{\partial r\partial z}+\frac{1}{r}\frac{\partial u}{\partial z}\right)+(\kappa-1)\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}\right)+(\kappa+1)\frac{\partial^2 w}{\partial z^2}=0. \qquad (2.2a, b)$$

where u and w are displacement components in r-and z-directions, respectively.

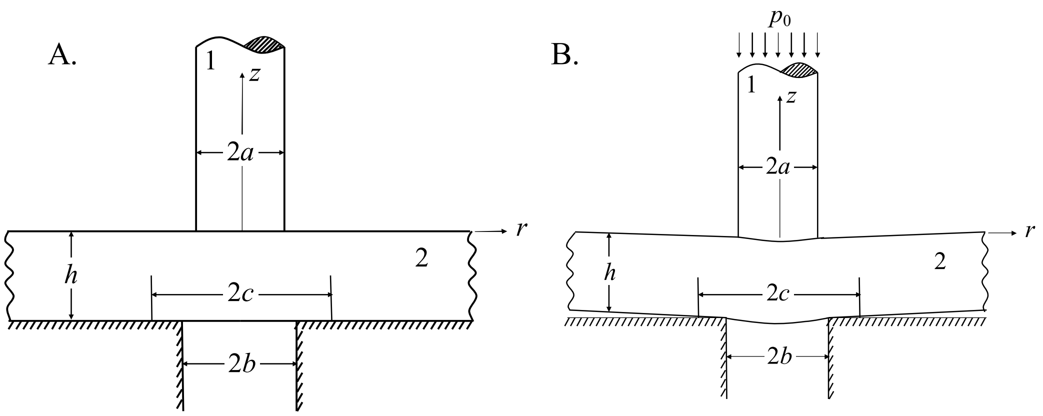

Figure 2.1 Undeformed shape (A) and deformed shape (B) of elastostatic contact problem.

No friction is supposed for the contact among cylinder, layer and foundation, and transmittable tractions across the interfaces are only compressive normal ones. Solution of equations (2.2a, b) are obtained by making use of the following boundary conditions.

$\sigma_{r1}(a,z)=0$,

$\tau_{rz1}(a,z)=0$,

$\sigma_{z1}(r,\infty)=-p_0$,

$\tau_{rz1}(r,0)=\tau_{rz2}(r,0)=0$,

$\sigma_{z1}(r,0)=\sigma_{z2}(r,0)$, $\qquad (0\le r<a)$,

$w_1(r,0)=w_2(r,0)$, $\qquad (0\le r<a)$,

$$\sigma_{z2}(r,-h)=0, \qquad (0 \le r < b,\ c < r < \infty),$$

$$\tau_{rz2}(r,-h)=0,$$

$$w_2(r,-h)=0, \qquad (b < r \le c). \qquad \text{(2.3a–i)}$$

Here, the cylinder and layer are implied by the subscripts 1 and 2, respectively. Note that Eqs. (2.3f, i) are equivalent to

$$\frac{d}{dr}w_1(r,0)=\frac{d}{dr}w_2(r,0), \qquad (0 \le r < a),$$

$$\frac{d}{dr}w_2(r,-h)=0, \qquad (b < r \le c). \qquad \text{(2.4a, b)}$$

Solutions for both the cylinder and layer will be obtained after solving Navier equations. By using also stress displacements, matches through the interface will be performed from these solutions.

Defining $U(\rho,z)$, H_1 Hankel transform of $u(r,z)$ and $W(\rho,z)$, H_0 Hankel transform of $w(r,z)$ in *r*-direction,

$$U(\rho,z)=\int_0^\infty u(r,z)r\,J_1(\rho r)dr,$$

$$W(\rho,z)=\int_0^\infty w(r,z)r\,J_0(\rho r)dr, \qquad \text{(2.5a, b)}$$

Here, J_0 is the Bessel functions of the 1st kind of order zero, and J_1 is the Bessel functions of the 1st kind of order one. Applying H_1 Hankel transform to Equation (2.2a) and H_0 Hankel transform to Equation (2.2b) in *r*-direction

$$(\kappa+1)\,H_1\!\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}-\frac{u}{r^2}\right)+(\kappa-1)\frac{d^2U_1}{dz^2}+2\frac{\partial}{\partial z}H_1\!\left(\frac{\partial w}{\partial r}\right)=0,$$

$$2\frac{\partial}{\partial z}H_0\!\left(\frac{\partial u}{\partial r}+\frac{u}{r}\right)+(\kappa-1)\,H_0\!\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}\right)+(\kappa+1)\frac{d^2W_0}{dz^2}=0,$$

$$(\kappa+1)\left(-\rho^2 U_1\right)+(\kappa-1)\frac{d^2U_1}{dz^2}+2\frac{d}{dz}\left(-\rho W_0\right)=0,$$

$$2\rho\frac{dU_1}{dz}+(\kappa-1)\left(-\rho^2 W_0\right)+(\kappa+1)\frac{d^2W_0}{dz^2}=0. \qquad (2.6a, d)$$

By taking derivative of Eq. (2.6d),

$$2\rho\frac{d^2U_1}{dz^2}-(\kappa-1)\,\rho^2\frac{dW_0}{dz}+(\kappa+1)\frac{d^3W_0}{dz^3}=0, \qquad (2.7)$$

Eq. (2.6c) can be rewritten as

$$\frac{dW_0}{dz}=-\frac{(\kappa+1)\rho}{2}U_1+\frac{(\kappa-1)}{2\rho}\frac{d^2U_1}{dz^2}. \qquad (2.8)$$

When the second and third derivatives of Eq. (2.8) are taken with respect to variable z, following expressions are obtained

$$\frac{d^2W_0}{dz^2}=-\frac{(\kappa+1)\rho}{2}\frac{dU_1}{dz}+\frac{(\kappa-1)}{2\rho}\frac{d^3U_1}{dz^3},$$

$$\frac{d^3W_0}{dz^3}=-\frac{(\kappa+1)\rho}{2}\frac{d^2U_1}{dz^2}+\frac{(\kappa-1)}{2\rho}\frac{d^4U_1}{dz^4}. \qquad (2.9a, b)$$

After substituting Eqs. (2.9b) and (2.8) into Eq. (2.7) and rearranging it, following can be obtained

$$\frac{d^4U}{dz^4}-2\rho^2\frac{d^2U}{dz^2}+\rho^4U=0, \qquad (2.10)$$

where ρ denotes the transform variable of Hankel.

The general solution of Eq. (2.10) is

$$U(\rho,z)=(c_1+c_2z)e^{-\rho z}+(c_3+c_4z)e^{\rho z}, \qquad (2.11)$$

where c_1, c_2, c_3 and c_4 are arbitrary constants with respect to the variable z. By back substitution in the transformed ODEs, one can obtain

$$W(\rho,z)=\left[c_1+\left(z+\frac{\kappa}{\rho}\right)c_2\right]e^{-\rho z}+\left[-c_3-\left(z-\frac{\kappa}{\rho}\right)c_4\right]e^{\rho z}. \qquad (2.12)$$

When the inverse transforms of Equations (2.11) and (2.12) are taken, displacement components are found as follows

$$u(\rho,z)=\int_0^{\infty}\left[(c_1+c_2 z)e^{-\rho z}+(c_3+c_4 z)e^{\rho z}\right]\rho J_1(\rho r)d\rho,$$

$$w(\rho,z)=\int_0^{\infty}\left\{\left[c_1+\left(z+\frac{\kappa}{\rho}\right)c_2\right]e^{-\rho z}+\left[-c_3-\left(z-\frac{\kappa}{\rho}\right)c_4\right]e^{\rho z}\right\}\rho J_0(\rho r)d\rho.$$

(2.13a, b)

The following expressions are acquired for the stress components by substituting Eqs. (2.13) in Eqs. (2.1):

$$\sigma_r(r,z)=\mu\int_0^{\infty}\left\{\left[-2(c_1+c_2 z)\frac{1}{r}J_1(\rho r)\right]+\left[2\rho(c_1+c_2 z)+(\kappa-3)c_2\right]J_0(\rho r)\right\}e^{-\rho z}\rho\, d\rho$$

$$+\mu\int_0^{\infty}\left\{\left[-2(c_3+c_4 z)\frac{1}{r}J_1(\rho r)\right]+\left[2\rho(c_3+c_4 z)-(\kappa-3)c_4\right]J_0(\rho r)\right\}e^{\rho z}\rho\, d\rho,$$

$$\sigma_z(r,z)=\mu\int_0^{\infty}\left\{\left[-2\rho(c_1+c_2 z)-(\kappa+1)c_2\right]e^{-\rho z}\right.$$

$$\left.+\left[-2\rho(c_3+c_4 z)+(\kappa+1)c_4\right]e^{\rho z}\right\}\rho J_0(\rho r)d\rho,$$

$$\tau_{rz}(r,z)=\mu\int_0^{\infty}\left\{\left[-2\rho(c_1+c_2 z)-(\kappa-1)c_2\right]e^{-\rho z}\right.$$

$$\left.+\left[2\rho(c_3+c_4 z)-(\kappa-1)c_4\right]e^{\rho z}\right\}\rho J_1(\rho r)d\rho. \qquad (2.14a–c)$$

Unknown constants may be calculated by employing boundary conditions presented in Eqs. (2.3).

The problem can be investigated as the combination of three basic states for the semi-infinite cylinder. That is,

a) An infinite cylinder exposed to an axial uniform compression p_0,
b) An infinite cylinder symmetric in direction of z=0 plane and exposed to arbitrary axial symmetric loads,
c) An axial symmetric half-space $(z \geq 0)$ exposed to loads on the straight boundary z=0.

The half space solution is anticipated to provide nonzero normal displacement w over z=0. Solution of the problem mentioned in item c can be achieved by employing the sine and cosine transforms of Fourier for Eqs. (2.2) in *z*-direction. Following statements can be obtained when the cosine and sine transforms are applied to Eq. (2.2a) and (2.2b) in *z*- direction, respectively.

$$(\kappa+1)\left(F_c\left(\frac{\partial^2 u}{\partial r^2}\right)+F_c\left(\frac{1}{r}\frac{\partial u}{\partial r}\right)-F_c\left(\frac{u}{r^2}\right)\right)+(\kappa-1)\,F_c\left(\frac{\partial^2 u}{\partial z^2}\right)+2F_c\left(\frac{\partial^2 w}{\partial r\partial z}\right)=0,$$

$$2\,F_s\left(\frac{\partial^2 u}{\partial r\partial z}\right)+\frac{2}{r}F_s\left(\frac{\partial u}{\partial z}\right)+(\kappa-1)\,F_s\left(\frac{\partial^2 w}{\partial r^2}\right)+(\kappa-1)\,F_s\left(\frac{1}{r}\frac{\partial w}{\partial r}\right)$$
$$+(\kappa+1)\,F_s\left(\frac{\partial^2 w}{\partial z^2}\right)=0,$$

$$(\kappa+1)\left[\frac{d^2U_c}{dr^2}+\frac{1}{r}\frac{dU_c}{dr}-\frac{U_c}{r^2}\right]-(\kappa-1)\,\alpha^2U_c+2\alpha\frac{dW_s}{dr}=0,$$

$$-2\alpha\left[\frac{dU_c}{dr}+\frac{1}{r}U_c\right]+(\kappa-1)\left[\frac{d^2W_s}{dr^2}+\frac{1}{r}\frac{dW_s}{dr}\right]-(\kappa+1)\alpha^2W_s=0. \qquad (2.15\text{a–d})$$

Eq. (2.15c) can be rewritten as

$$\frac{dW_s}{dr}=-\frac{(\kappa+1)}{2\alpha}\left[\frac{d^2U_c}{dr^2}+\frac{1}{r}\frac{dU_c}{dr}-\frac{U_c}{r^2}\right]+\frac{(\kappa-1)\,\alpha}{2}U_c\,. \qquad (2.16)$$

by taking second and third derivatives of Eq. (2.16),

$$\frac{d^2W_s}{dr^2}=-\frac{(\kappa+1)}{2\alpha}\left[\frac{d^3U_c}{dr^3}+\frac{1}{r}\frac{d^2U_c}{dr^2}-\frac{2}{r^2}\frac{dU_c}{dr}+\frac{2}{r^3}U_c\right]+\frac{(\kappa-1)\,\alpha}{2}\frac{dU_c}{dr},$$

$$\frac{d^3W_s}{dr^3}=-\frac{(\kappa+1)}{2\alpha}\left[\frac{d^4U_c}{dr^4}+\frac{1}{r}\frac{d^3U_c}{dr^3}-\frac{3}{r^2}\frac{d^2U_c}{dr^2}+\frac{6}{r^3}\frac{dU_c}{dr}-\frac{6}{r^4}U_c\right]$$
$$+\frac{(\kappa-1)\,\alpha}{2}\frac{d^2U_c}{dr^2}. \qquad (2.17\text{a–b})$$

by taking derivative of Eq. (2.15d),

$$-2\alpha\left[\frac{d^2U_c}{dr^2}+\frac{1}{r}\frac{dU_c}{dr}-\frac{1}{r^2}U_c\right]+(\kappa-1)\left[\frac{d^3W_s}{dr^3}+\frac{1}{r}\frac{d^2W_s}{dr^2}-\frac{1}{r^2}\frac{dW_s}{dr}\right]$$
$$-(\kappa+1)\alpha^2\frac{dW_s}{dr}=0\,. \qquad (2.18)$$

After substituting Eqs. (2.16) and (2.17) into Eq. (2.18) and rearranging it, following can be obtained

$$r^4 \frac{d^4U_c}{dr^4} + 2r^3 \frac{d^3U_c}{dr^3} - \left[2\alpha^2 r^2 + 3\right] r^2 \frac{d^2U_c}{dr^2} - \left[2\alpha^2 r^2 - 3\right] r \frac{dU_c}{dr}$$

$$+ \left[\alpha^4 r^4 + 2\alpha^2 r^2 - 3\right] U_c = 0 . \tag{2.19}$$

4^{th} order homogeneous ordinary differential equation is obtained as indicated below

$$\xi^4 \frac{d^4U_c}{d\xi^4} + 2\xi \frac{d^3U_c}{d\xi^3} - \left(2\xi^4 + 3\xi^2\right)\frac{d^2U_c}{d\xi^2} - \left(2\xi^3 - 3\xi\right)\frac{dU_c}{d\xi} + \left(\xi^4 + 2\xi^2 - 3\right)U_c = 0, \tag{2.20}$$

the Fourier cosine transform of $u(r,z)$ is defined as seen in Eq. (2.21)

$$U_c(r,\alpha) = \int_0^\infty u(r,z)\cos(\alpha z)\,dz, \tag{2.21}$$

$\xi = \alpha r$, α is the Fourier transform variable,

Eq.(2.20) can be expressed as follows, Durucan (2010),

$$\Delta_1(\Delta_2 U_c) + \Delta_3(\Delta_4 U_c) = 0, \tag{2.22}$$

where Δ_1, Δ_2, Δ_3 and Δ_4 are 2^{nd} order linear ordinary differential operators with variable coefficients in ξ:

$$\Delta_1 = \xi^2 \frac{d^2}{d\xi^2} - 3\xi \frac{d}{d\xi} - \xi^2 + 3,$$

$$\Delta_2 = \xi^2 \frac{d^2}{d\xi^2} + \xi \frac{d}{d\xi} - \xi^2 - 1,$$

$$\Delta_3 = \xi^3 \frac{d^2}{d\xi^2} + \xi^2 \frac{d}{d\xi} - \xi^3 - 4\xi,$$

$$\Delta_4 = \xi \frac{d^2}{d\xi^2} - \frac{d}{d\xi} - \xi + \frac{1}{\xi}, \tag{2.23a–d}$$

solution of Eq. (2.20) can be obtained from 2^{nd} order ODEs

$$\Delta_2 U_c = 0,\ \Delta_4 U_c = 0, \tag{2.24}$$

in the form of

$$U_c(r,\alpha) = c_5 I_1(\alpha r) + c_6 \alpha r I_0(\alpha r) + c_7 K_1(\alpha r) + c_8 \alpha r K_0(\alpha r), \tag{2.25}$$

in Eq. (2.25), arbitrary constants can be given as c_5, c_6, c_7, c_8 and the modified Bessel functions 1st and 2nd kinds of order zero and one can be shown as I_0, K_0, I_1, K_1, respectively.

c_7 and c_8 arbitrary constants must be zero due to providing the regularity condition at $r = 0$. Thus,

$$U_c(r,\alpha) = c_5 I_1(\alpha r) + c_6 \alpha r I_0(\alpha r). \tag{2.26}$$

Similarly,

$$W_s(r,\alpha) = -c_5 I_0(\alpha r) - c_6 \left[(\kappa+1) I_0(\alpha r) + \alpha r I_1(\alpha r)\right] \tag{2.27}$$

the Fourier sine transform of $w(r,z)$ is defined as seen in Eq. (2.28)

$$W_s(r,\alpha) = \int_0^\infty w(r,z) \sin(\alpha z)\, dz\,. \tag{2.28}$$

Inverse Fourier transforms of Eqs. (2.26) and (2.27) give

$$u(r,z) = \frac{2}{\pi}\int_0^\infty \left[c_5 I_1(\alpha r) + c_6 \alpha r I_0(\alpha r)\right]\cos(\alpha z)\, d\alpha\,,$$

$$w(r,z) = -\frac{2}{\pi}\int_0^\infty \left\{c_5 I_0(\alpha r) + c_6\left[(\kappa+1) I_0(\alpha r) + \alpha r I_1(\alpha r)\right]\right\}\sin(\alpha z)\, d\alpha\,. \tag{2.29a, b}$$

Expressions of stress component are acquired by the substitution of Eqs. (2.29) in Eqs. (2.1):

$$\sigma_r(r,z) = \frac{2\mu}{\pi}\int_0^\infty \left\{2c_5\left[I_0(\alpha r) - \frac{1}{\alpha r} I_1(\alpha r)\right] + c_6\left[(\kappa-1) I_0(\alpha r) + 2\alpha r I_1(\alpha r)\right]\right\}\alpha\cos(\alpha z)\, d\alpha,$$

$$\sigma_z(r,z) = -\frac{2\mu}{\pi}\int_0^\infty \left\{2c_5 I_0(\alpha r) - c_6\left[(\kappa+5) I_0(\alpha r) + 2\alpha r I_1(\alpha r)\right]\right\}\alpha\cos(\alpha z)\, d\alpha,$$

$$\tau_{rz}(r,z) = -\frac{2\mu}{\pi}\int_0^\infty \left\{2c_5 I_1(\alpha r) + c_6\left[(\kappa+1) I_1(\alpha r) + 2\alpha r I_0(\alpha r)\right]\right\}\alpha\sin(\alpha z)\, d\alpha\,. \tag{2.30a–c}$$

2.2 Stress and Displacement Expressions for the Semi-Infinite Cylinder

By renaming the unknown quantities c_1, c_2, c_5 and c_6 in Eqs.(2.13, 14, 29, 30) as follows

$$\rho c_1 = \frac{\kappa - 1}{2} g(\rho),$$

$$c_2 = -g(\rho),$$

$$c_5 = -\frac{\alpha^2}{2\mu} A_1 ,$$

$$c_6 = -\frac{\alpha}{2\mu} B_1 . \qquad (2.31\text{a–d})$$

stress and displacement expressions for semi-infinite cylinder $(0 \le z < \infty)$ are described as (Agarwal 1978; Gecit 1986; Gupta 1974)

$$\pi\mu_1 u_1(r,z) = -\int_0^\infty \left[A_1(\alpha) I_1(\alpha r) + B_1(\alpha)\, r I_0(\alpha r)\right] \alpha^2 \cos(\alpha z)\, d\alpha$$
$$+ \int_0^\infty \left(\frac{\kappa_1 - 1}{2} - \rho z\right) g(\rho) e^{-\rho z} J_1(\rho r)\, d\rho + \frac{\pi p_0}{2} \frac{3-\kappa}{7-\kappa} r ,$$

$$\pi\mu_1 w_1(r,z) = \int_0^\infty \left\{A_1(\alpha) I_0(\alpha r) + B_1(\alpha)\left[\frac{\kappa_1 + 1}{\alpha} I_0(\alpha r) + r I_1(\alpha r)\right]\right\} \alpha^2 \sin(\alpha z)\, d\alpha$$
$$- \int_0^\infty \left(\frac{\kappa_1 + 1}{2} + \rho z\right) g(\rho) e^{-\rho z} J_0(\rho r) d\rho - \frac{\pi p_0}{2} \frac{4z}{7-\kappa} , \qquad (2.32\text{a, b})$$

$$\sigma_{r1}(r,z) = -\frac{2}{\pi}\int_0^\infty \left\{A_1(\alpha)\alpha\left[I_0(\alpha r) - \frac{1}{\alpha r} I_1(\alpha r)\right] + B_1(\alpha)\left[\frac{\kappa_1 - 1}{2} I_0(\alpha r) + \alpha r I_1(\alpha r)\right]\right\}$$
$$\alpha^2 \cos(\alpha z) d\alpha + \frac{2}{\pi}\int_0^\infty \left[(1-\rho z) J_0(\rho r) - \frac{1}{\rho r}\left(\frac{\kappa_1 - 1}{2} - \rho z\right) J_1(\rho r)\right] g(\rho) e^{-\rho z} \rho\, d\rho ,$$

$$\sigma_{z1}(r,z) = \frac{2}{\pi}\int_0^\infty \left\{A_1(\alpha)\, \alpha\, I_0(\alpha r) + B_1(\alpha)\left[\frac{\kappa_1 + 5}{2} I_0(\alpha r) + \alpha r I_1(\alpha r)\right]\right\} \alpha^2 \cos(\alpha z)\, d\alpha$$
$$+ \frac{2}{\pi}\int_0^\infty (1+\rho z) g(\rho) e^{-\rho z} \rho J_0(\rho r) d\rho - p_0 ,$$

$$\tau_{rz1}(r,z)=\frac{2}{\pi}\int_0^\infty\left\{A_1(\alpha)\,\alpha\,I_1(\alpha r)+B_1(\alpha)\left[\frac{\kappa_1+1}{2}I_1(\alpha r)+\alpha r I_0(\alpha r)\right]\right\}\alpha^2\sin(\alpha z)\,d\alpha$$

$$+\frac{2}{\pi}\int_0^\infty \rho\, z\, g(\rho)\, e^{-\rho z}\rho\, J_1(\rho r) d\rho\,. \qquad \text{(2.33a–c)}$$

Note that the boundary condition in Eq. (2.3d) is satisfied with this selection of c_1 and c_2 as in Eqs. (2.31a, b).

If the expressions in Eqs. (2.33a, c) are evaluated at $r=a$

$$\sigma_{r1}(a,z)=-\frac{2}{\pi}\int_0^\infty\left\{A_1\alpha\left[I_0(\alpha a)-\frac{1}{\alpha a}I_1(\alpha a)\right]+B_1\left[\frac{\kappa_1-1}{2}I_0(\alpha a)+\alpha a I_1(\alpha a)\right]\right\}\alpha^2\cos(\alpha z)\,d\alpha$$

$$+\frac{2}{\pi}\int_0^\infty\left[(1-\rho z)J_0(\rho a)-\frac{1}{\rho a}\left(\frac{\kappa_1-1}{2}-\rho z\right)J_1(\rho a)\right]g(\rho)e^{-\rho z}\rho\, d\rho\,,$$

$$\tau_{rz1}(a,\,z)=\frac{2}{\pi}\int_0^\infty\left\{A_1\alpha\, I_1(\alpha a)+B_1\left[\frac{\kappa_1+1}{2}I_1(\alpha a)+\alpha a I_0(\alpha a)\right]\right\}\alpha^2\sin(\alpha z)\,d\alpha$$

$$+\frac{2}{\pi}\int_0^\infty \rho z\, g(\rho)\, e^{-\rho z}\rho\, J_1(\rho a)\, d\rho\,, \qquad \text{(2.34a, b)}$$

are obtained. Conditions in Eqs. (2.3a, b) give then

$$\alpha\left[I_0(\alpha a)-\frac{1}{\alpha a}I_1(\alpha a)\right]A_1+\left[\frac{\kappa_1-1}{2}I_0(\alpha a)+\alpha a I_1(\alpha a)\right]B_1$$

$$=\frac{2}{\pi\alpha^2}\int_0^\infty\int_0^\infty\left[(1-\rho z)J_0(\rho a)-\frac{1}{\rho a}\left(\frac{\kappa_1-1}{2}-\rho z\right)J_1(\rho a)\right]g(\rho)e^{-\rho z}\rho\, d\rho\cos(\alpha z)\,dz\,,$$

$$\alpha I_1(\alpha a)\,A_1+\left[\frac{\kappa_1+1}{2}I_1(\alpha a)+\alpha a I_0(\alpha a)\right]B_1$$

$$=-\frac{2}{\pi\alpha^2}\int_0^\infty\int_0^\infty \rho z\, g(\rho)\, e^{-\rho z}\rho\, J_1(\rho a) d\rho\sin(\alpha z) dz\,, \qquad \text{(2.35a, b)}$$

or, after using certain integral formulas,

$$\alpha\left[I_0(\alpha a)-\frac{1}{\alpha a}I_1(\alpha a)\right]A_1+\left[\frac{\kappa_1-1}{2}I_0(\alpha a)+\alpha a I_1(\alpha a)\right]B_1$$

$$=\frac{2}{\pi\alpha^2}\int_0^\infty\left[\frac{2\alpha^2\rho^2}{\left(\alpha^2+\rho^2\right)^2}J_0(\rho a)-\frac{\frac{\kappa_1-3}{2}\rho^2+\frac{\kappa_1+1}{2}\alpha^2}{\left(\alpha^2+\rho^2\right)^2}\frac{\rho}{a}J_1(\rho a)\right]g(\rho)d\rho\,,$$

$$\alpha I_1(\alpha a)\,A_1+\left[\frac{\kappa_1+1}{2}I_1(\alpha a)+\alpha a I_0(\alpha a)\right]B_1=-\frac{2}{\pi\alpha^2}\int_0^\infty\frac{2\alpha\rho^3}{\left(\alpha^2+\rho^2\right)^2}J_1(\rho a)g(\rho)d\rho$$

are obtained. (2.36a, b)

Noting that the unknown $g(\rho)$ is the H_1 Hankel transform of $G(r)$,

$$g(\rho)=\int_0^\infty G(r)r\,J_1(\rho r)dr\,, \tag{2.37}$$

after lengthy manipulations, following equations may be obtained from Eqs. (2.36)

$$\alpha\left[I_0(\alpha a)-\frac{1}{\alpha a}I_1(\alpha a)\right]A_1+\left[\frac{\kappa_1-1}{2}I_0(\alpha a)+\alpha a I_1(\alpha a)\right]B_1$$

$$=\left[-\frac{\kappa_1+1+2\alpha^2a^2}{2\alpha^2 a}K_1(\alpha a)-\frac{1}{\alpha}K_0(\alpha a)\right]C_1+\left[K_0(\alpha a)+\frac{1}{\alpha a}K_1(\alpha a)\right]C_2\,,$$

$$\alpha I_1(\alpha a)\,A_1+\left[\frac{\kappa_1+1}{2}I_1(\alpha a)+\alpha a I_0(\alpha a)\right]B_1=aK_0(\alpha a)C_1-K_1(\alpha a)C_2\,,$$

(2.38a, b)

where K_0 and K_1 are the modified Bessel functions of the 2^{nd} kind of orders zero and one, respectively, and

$$C_1=\frac{2}{\pi}\int_0^a G(t)I_1(\alpha t)t\,dt\,,$$

$$C_2=\frac{2}{\pi}\int_0^a G(t)I_0(\alpha t)t^2\,dt\,. \tag{2.39a, b}$$

Then, solution of Eqs. (2.38a, b) gives

$$A_1(\alpha)=\left\{\frac{\kappa_1+1}{2\alpha^2}\left[Q_1(\alpha)+1+\frac{2\alpha^2a^2}{\kappa_1+1}\right]C_1-\frac{1}{\alpha}\left[Q_1(\alpha)+\frac{\kappa_1+1}{2}\right]C_2\right\}\Big/Q_2(\alpha)\,,$$

$$B_1(\alpha)=\left[-\frac{1}{\alpha}Q_1(\alpha)C_1+C_2\right]\Big/Q_2(\alpha)\,, \tag{2.40a, b}$$

where

$$Q_1(\alpha)=\alpha^2a^2I_0(\alpha a)K_0(\alpha a)+\left(\frac{\kappa_1+1}{2}+\alpha^2a^2\right)I_1(\alpha a)K_1(\alpha a),$$

$$Q_2(\alpha)=-\alpha^2a^2I_0^{\,2}(\alpha a)+\left(\frac{\kappa_1+1}{2}+\alpha^2a^2\right)I_1^{\,2}(\alpha a). \qquad (2.41\text{a, b})$$

By evaluating the expression in Eq. (2.33b) at $z=0$, one can obtain

$$\sigma_{z1}(r,0)=-p_0+\frac{2}{\pi}\int_0^\infty\left\{\alpha\,I_0(\alpha r)A_1+\left[\frac{\kappa_1+5}{2}I_0(\alpha r)+\alpha rI_1(\alpha r)\right]B_1\right\}\alpha^2d\alpha$$

$$+\frac{2}{\pi}\int_0^\infty g(\rho)\rho\,J_0(\rho r)d\rho\,. \qquad (2.42)$$

Now, substituting Eqs. (2.40a, b) in Eq. (2.42),

$$\sigma_{z1}(r,0)=-p_0+\frac{2}{\pi}\int_0^\infty\left\{\left\{\left[-2Q_1(\alpha)+\frac{\kappa_1+1}{2}+\alpha^2a^2\right]C_1+\left[-Q_1(\alpha)+2\right]\alpha C_2\right\}I_0(\alpha r)\right.$$

$$\left.+\left[-Q_1(\alpha)C_1+\alpha C_2\right]\alpha r\,I_1(\alpha r)\right\}\alpha d\alpha/Q_2(\alpha)$$

$$+\frac{2}{\pi}\int_0^\infty\left[\int_0^a G(t)tJ_1(\rho t)dt\right]\rho\,J_0(\rho r)d\rho\,, \qquad (2.43)$$

is obtained. A similar procedure gives

$$\frac{d}{dr}w_1(r,0)=\frac{\kappa_1+1}{2\pi\mu_1}\int_0^\infty g(\rho)\rho\,J_1(\rho r)d\rho\,. \qquad (2.44)$$

By using Hankel inversion theorem in Eq. (2.37),

$$\frac{d}{dr}w_1(r,0)=\frac{\kappa_1+1}{2\pi\mu_1}G(r) \qquad (2.45)$$

is obtained. Further manipulations on Eq. (2.43) reduces the expression to

$$\sigma_{z1}(r,0)=\frac{4}{\pi^2}\int_0^a\left[h_1(r,t)+tH_1(r,t)\right]G(t)dt-p_0\,, \qquad (2.46)$$

in which

$$h_1(r,t)=\begin{cases}\dfrac{1}{r}K(t/r)+\dfrac{r}{t^2-r^2}E(t/r) & (t<r),\\[2mm] \dfrac{t}{t^2-r^2}E(r/t) & (t>r),\end{cases}$$

$$H_1(r,t)=\int_0^\infty L_1(r,t,\alpha)\,d\alpha,$$

$$L_1(r,t,\alpha)=\{[\alpha t I_0(\alpha t)+Q_1(\alpha)I_1(\alpha t)\,]\,\alpha r\, I_1(\alpha r)+([\,2-Q_1(\alpha)\,]\alpha t\, I_0(\alpha t)$$

$$+\left[\frac{\kappa_1+1}{2}+\alpha^2 a^2-2Q_1(\alpha)\right]I_1(\alpha t)\Big)I_0(\alpha r)\Big\}\alpha/Q_2(\alpha). \qquad (2.47a–c)$$

In above equations, while K is denoted as the complete elliptic integrals of the 1st kind, E is expressed as the complete elliptic integrals of the 2nd kind.

2.3 Stress and Displacement Expressions for the Elastic Layer

Renaming the quantities c_1-c_4 appearing in the general solutions given in Eqs. (2.13, 14) as follows

$$c_1=C_2,$$

$$c_2=\rho D_2,$$

$$c_3=A_2,$$

$$c_4=\rho B_2, \qquad (2.48a–d)$$

one can write these solutions for the elastic layer in the form given by Gecit (1981), Civelek and Erdoğan (1974), Gecit and Erdogan (1978)

$$u_2(r,z)=\int_0^\infty \{[\,A_2(\rho)+\rho z B_2(\rho)\,]e^{\rho z}+[\,C_2(\rho)+\rho z D_2(\rho)\,]e^{-\rho z}\}\,\rho\, J_1(\rho r)\,d\rho,$$

$$w_2(r,z)=\int_0^\infty \{[-A_2(\rho)+(\kappa_2-\rho z)\,B_2(\rho)\,]e^{\rho z}$$

$$+[\,C_2(\rho)+(\kappa_2+\rho z)\,D_2(\rho)\,]e^{-\rho z}\}\,\rho\, J_0(\rho r)\,d\rho\,. \qquad (2.49a, b)$$

$$\sigma_{z2}(r,z)=-2\mu_2\int_0^\infty \left\{\left[\,A_2(\rho)+\left(\rho z-\frac{\kappa_2+1}{2}\right)B_2(\rho)\,\right]e^{\rho z}\right.$$

$$\left.+\left[\,C_2(\rho)+\left(\rho z+\frac{\kappa_2+1}{2}\right)D_2(\rho)\right]e^{-\rho z}\right\}\rho^2\, J_0(\rho r)\,d\rho,$$

$$\tau_{rz2}(r,z) = 2\mu_2 \int_0^\infty \left\{ \left[A_2(\rho) + \left(\rho z + \frac{1-\kappa_2}{2} \right) B_2(\rho) \right] e^{\rho z} \right.$$

$$\left. - \left[C_2(\rho) + \left(\rho z + \frac{\kappa_2 - 1}{2} \right) D_2(\rho) \right] e^{-\rho z} \right\} \rho^2 J_1(\rho r)\, d\rho \,. \qquad (2.50a, b)$$

unknown functions above are defined as A_2, B_2, C_2 and D_2 . Boundary conditions at $(z=0)$ and $(z=-h)$ surfaces makes their calculations possible. For this purpose, introduce unknown functions $p_i(r)$ (i=1, 2) described by

$$\sigma_{z2}(r,0) = p_1(r)\,,$$

$$\sigma_{z2}(r,-h) = p_2(r)\,. \qquad (2.51a, b)$$

Now, evaluate σ_{z2} and τ_{rz2} at $z=0$ and $-h$:

$$\sigma_{z2}(r,0) = -2\mu_2 \int_0^\infty \left\{ \left[A_2(\rho) + \left(-\frac{1+\kappa_2}{2} \right) B_2(\rho) \right] \right.$$

$$\left. + \left[C_2(\rho) + \left(\frac{1+\kappa_2}{2} \right) D_2(\rho) \right] \right\} \rho^2 J_0(\rho r)\, d\rho \,,$$

$$\tau_{rz2}(r,0) = 2\mu_2 \int_0^\infty \left[A_2(\rho) + \left(\frac{1-\kappa_2}{2} \right) B_2(\rho) - C_2(\rho) + \left(\frac{1-\kappa_2}{2} \right) D_2(\rho) \right] \rho^2 J_1(\rho r) d\rho \,,$$

$$\sigma_{z2}(r,-h) = -2\mu_2 \int_0^\infty \left\{ \left[A_2(\rho) - \left(\frac{1+\kappa_2}{2} + \rho h \right) B_2(\rho) \right] e^{-\rho h} \right.$$

$$\left. + \left[C_2(\rho) + \left(\frac{1+\kappa_2}{2} - \rho h \right) D_2(\rho) \right] e^{\rho h} \right\} \rho^2 J_0(\rho r)\, d\rho \,,$$

$$\tau_{rz2}(r,-h) = 2\mu_2 \int_0^\infty \left\{ \left[A_2(\rho) + \left(\frac{1-\kappa_2}{2} - \rho h \right) B_2(\rho) \right] e^{-\rho h} \right.$$

$$\left. + \left[-C_2(\rho) + \left(\frac{1-\kappa_2}{2} + \rho h \right) D_2(\rho) \right] e^{\rho h} \right\} \rho^2 J_1(\rho r)\, d\rho \,. \qquad (2.52a–d)$$

Substitution of Eqs. (2.52a–d) in the conditions given by Eqs. (2.3d) and (2.42a, b) gives

$$-2\mu_2\int_0^\infty\left[\ A_2(\rho)-\frac{1+\kappa_2}{2}\,B_2(\rho)+C_2(\rho)+\frac{1+\kappa_2}{2}\,D_2(\rho)\right]\rho^2 J_0(\rho r)d\rho=p_1(r),$$

$$2\mu_2\int_0^\infty\left[\ A_2(\rho)+\frac{1-\kappa_2}{2}\,B_2(\rho)-C_2(\rho)+\frac{1-\kappa_2}{2}\,D_2(\rho)\right]\rho^2 J_1(\rho r)d\rho=0\,,$$

$$-2\mu_2\int_0^\infty\left\{\left[A_2(\rho)-\left(\frac{1+\kappa_2}{2}+\rho h\right)B_2(\rho)\right]e^{-\rho h}+\left[C_2(\rho)+\left(\frac{1+\kappa_2}{2}-\rho h\right)D_2(\rho)\right]e^{\rho h}\right\}$$

$$\rho^2 J_0(\rho r)d\rho=p_2(r),$$

$$2\mu_2\int_0^\infty\left\{\left[A_2(\rho)+\left(\frac{1-\kappa_2}{2}-\rho h\right)B_2(\rho)\right]e^{-\rho h}+\left[-C_2(\rho)+\left(\frac{1-\kappa_2}{2}+\rho h\right)D_2(\rho)\right]e^{\rho h}\right\}$$

$$\rho^2 J_1(\rho r)d\rho=0\,, \qquad \text{(2.53a–d)}$$

from which one can take the H_0 Hankel transforms of Eqs. (2.53a, c) and H_1 Hankel transforms of Eqs. (2.53b, d) to obtain

$$A_2-\frac{1+\kappa_2}{2}B_2+C_2+\frac{1+\kappa_2}{2}D_2=-\frac{1}{2\mu_2\rho}P_1(\rho),$$

$$A_2+\frac{1-\kappa_2}{2}B_2-C_2+\frac{1-\kappa_2}{2}D_2=0\,,$$

$$A_2-\left(\frac{1+\kappa_2}{2}+\rho h\right)B_2+e^{2\rho h}C_2+\left(\frac{1+\kappa_2}{2}-\rho h\right)e^{2\rho h}D_2=-\frac{e^{\rho h}}{2\mu_2\rho}P_2(\rho),$$

$$A_2+\left(\frac{1-\kappa_2}{2}-\rho h\right)B_2-e^{2\rho h}C_2+\left(\frac{1-\kappa_2}{2}+\rho h\right)e^{2\rho h}D_2=0\,. \qquad \text{(2.54a–d)}$$

In Eqs. (2.54), $P_i(\rho)$ $(i=1,2)$ are the H_0 Hankel transforms of the new unknown functions $p_i(r)$ $(i=1,2)$:

$$P_i(\rho)=\int_0^\infty p_i(r)rJ_0(\rho r)dr\,,\quad (i=1,2)\,. \qquad \text{(2.55a, b)}$$

Note that

$$p_1(r)=0\,,\qquad (a<r<\infty)$$

$$p_2(r)=0\,,\qquad (0<r<b,\ c<r<\infty)\,. \qquad \text{(2.56a, b)}$$

Therefore, Eqs. (2.55) can be stated as

$$P_1(\rho)=\int_0^a p_1(r)rJ_0(\rho r)dr\,,$$

$$P_2(\rho)=\int_b^c p_2(r)rJ_0(\rho r)dr\,. \tag{2.57a, b}$$

Solution of the system in Eqs. (2.54) gives

$$\rho A_2=\frac{1}{2\mu_2}\left\{-\left[\frac{1-\kappa_2}{2}S_1(\rho)-2\rho^2h^2e^{-2\rho h}\right]P_1(\rho)\right.$$
$$\left.+\left[\frac{1-\kappa_2}{2}S_1(\rho)-\kappa_2\rho h\left(1-e^{-2\rho h}\right)\right]e^{-\rho h}P_2(\rho)\right\}\Big/S_3(\rho),$$

$$\rho B_2=\frac{1}{2\mu_2}\left[S_1(\rho)P_1(\rho)-S_2(\rho)e^{-\rho h}P_2(\rho)\right]\Big/S_3(\rho),$$

$$\rho C_2=\frac{1}{2\mu_2}\left\{\left[\frac{1-\kappa_2}{2}S_2(\rho)+2\rho^2h^2\right]e^{-2\rho h}P_1(\rho)\right.$$
$$\left.-\left[\frac{1-\kappa_2}{2}S_1(\rho)+\rho h\left(1-e^{-2\rho h}\right)\right]P_2(\rho)\right\}e^{-\rho h}\Big/S_3(\rho),$$

$$\rho D_2=\frac{1}{2\mu_2}\left[S_2(\rho)e^{-2\rho h}P_1(\rho)-S_1(\rho)P_2(\rho)\right]e^{-\rho h}/S_3(\rho), \tag{2.58a–d}$$

where

$$S_1(\rho)=1-(1-2\rho h)e^{-2\rho h},$$
$$S_2(\rho)=1+2\rho h-e^{-2\rho h},$$
$$S_3(\rho)=\left(1-e^{-2\rho h}\right)^2-4\rho^2h^2e^{-2\rho h}. \tag{2.59a–c}$$

Therefore, expressions of the displacements and stresses of the layer are stated from in the point of the unknown functions $p_i\,(i=1,2)$. For instance,

$$w_2(r,0)=\frac{\kappa_2+1}{4\mu_2}\int_0^\infty\frac{1+4\rho he^{-2\rho h}-e^{-4\rho h}}{\left(1-e^{-2\rho h}\right)^2-4\rho^2h^2e^{-2\rho h}}\int_0^a p_1(t)tJ_0(\rho t)dtJ_0(\rho r)d\rho$$
$$-\frac{\kappa_2+1}{2\mu_2}\int_0^\infty\frac{1+\rho h-(1-\rho h)e^{-2\rho h}}{\left(1-e^{-2\rho h}\right)^2-4\rho^2h^2e^{-2\rho h}}e^{-\rho h}\int_b^c p_2(t)tJ_0(\rho t)dtJ_0(\rho r)d\rho\,, \tag{2.60}$$

and

$$\frac{d}{dr}w_2(r,0) = -\frac{\kappa_2+1}{4\mu_2}\int_0^a p_1(t)t\int_0^\infty \frac{1+4\rho h e^{-2\rho h} - e^{-4\rho h}}{\left(1-e^{-2\rho h}\right)^2 - 4\rho^2 h^2 e^{-2\rho h}} J_1(\rho r) J_0(\rho t)\rho d\rho dt$$

$$+\frac{\kappa_2+1}{2\mu_2}\int_b^c p_2(t)t\int_0^\infty \frac{1+\rho h - (1-\rho h)e^{-2\rho h}}{\left(1-e^{-2\rho h}\right)^2 - 4\rho^2 h^2 e^{-2\rho h}} e^{-\rho h} J_1(\rho r) J_0(\rho t)\rho d\rho dt\,,$$

(2.61)

or, more precisely,

$$\frac{d}{dr}w_2(r,0) = \frac{\kappa_2+1}{2\mu_2}\left\{\int_0^a \left[\frac{1}{\pi}h_2(r,t) - tH_2(r,t)\right] p_1(t)\,dt + \int_b^c tH_3(r,t)\,p_2(t)\,dt\right\}.$$

(2.62)

Similarly,

$$w_2(r,-h) = \frac{\kappa_2+1}{2\mu_2}\int_0^a p_1(t)t\int_0^\infty \frac{\left[1+\rho h - (1-\rho h)e^{-2\rho h}\right]e^{-\rho h}}{\left(1-e^{-2\rho h}\right)^2 - 4\rho^2 h^2 e^{-2\rho h}} J_0(\rho r) J_0(\rho t) d\rho dt$$

$$-\frac{\kappa_2+1}{4\mu_2}\int_b^c p_2(t)t\int_0^\infty \frac{1+4\rho h e^{-2\rho h} - e^{-4\rho h}}{\left(1-e^{-2\rho h}\right)^2 - 4\rho^2 h^2 e^{-2\rho h}} J_0(\rho r) J_0(\rho t) d\rho dt\,,$$

$$\frac{d}{dr}w_2(r,-h) = \frac{\kappa_2+1}{2\mu_2}\left\{\int_0^a -tH_3(r,t)\,p_1(t)\,dt + \int_b^c \left[-\frac{1}{\pi}h_2(r,t) + tH_2(r,t)\right] p_2(t)\,dt\right\},$$

(2.63a, b)

where

$$h_2(r,t) = \begin{cases} \dfrac{t}{t^2-r^2}E(t/r), & (t<r), \\ \dfrac{t^2/r}{t^2-r^2}E(r/t) - \dfrac{1}{r}K(r/t), & (t>r), \end{cases}$$

$$H_i(r,t) = \int_0^\infty L_i(r,t,\rho)d\rho \quad (i=2,3),$$

$$L_2(r,t,\rho) = \left[S_2(\rho) + 2\rho^2 h^2\right]\rho e^{-2\rho h} J_0(\rho t) J_1(\rho r)/S_3(\rho)\,,$$

$$L_3(r,t,\rho) = \left[1+\rho h - (1-\rho h)e^{-2\rho h}\right]\rho e^{-\rho h} J_0(\rho t) J_1(\rho r)/S_3(\rho)\,. \qquad (2.64a–e)$$

CHAPTER 3

INTEGRAL EQUATIONS

3.1 Derivation of Integral Equations

Unknown functions of $G(r)$, $p_1(r)$ and $p_2(r)$ will be calculated from the following boundary conditions:

$$\frac{\mathrm{d}}{\mathrm{d}r} w_1(r,0) - \frac{\mathrm{d}}{\mathrm{d}r} w_2(r,0) = 0 \qquad (0 \le r < a),$$

$$\sigma_{z1}(r,0) - \sigma_{z2}(r,0) = 0 \qquad (0 \le r < a),$$

$$\frac{\mathrm{d}}{\mathrm{d}r} w_2(r,-h) = 0 \qquad (b < r \le c). \qquad (3.1a\text{–}c)$$

If Eqs. (2.35), (2.36), (2.41a), (2.52) and (2.53b) are substituted in Eqs. (3.1a–c), the 2nd kind singular integral equations are presented as follows:

$$\frac{\kappa_1+1}{2\pi\mu_1} G(r) - \frac{\kappa_2+1}{2\mu_2}\left\{\int_0^a \left[\frac{1}{\pi} h_2(r,t) - tH_2(r,t)\right] p_1(t)dt + \int_b^c tH_3(r,t)p_2(t)dt\right\} = 0, \qquad (0 \le r < a)$$

$$\frac{4}{\pi^2}\int_0^a \left[h_1(r,t) + tH_1(r,t)\right] G(t)\, dt - p_1(r) = p_0\,, \qquad (0 \le r < a)$$

$$-\frac{\kappa_2+1}{2\mu_2}\left\{\int_0^a tH_3(r,t)p_1(t)dt + \int_b^c \left[\frac{1}{\pi} h_2(r,t) - tH_2(r,t)\right] p_2(t)dt\right\} = 0\,. \qquad (b < r \le c)$$

$$(3.2a\text{–}c)$$

The kernels of Eq. (3.2) are stated for $r > 0$ and $t > 0$. It is noted that the kernels h_1 and h_2 possess simple Cauchy type singularity at $t = r$ (Muskhelishvili 1953), $1/(t-r)$ becomes unbounded when $t = r$. In addition, it is noteworthy that $K(1) = \infty$ and $E(1) = 1$ (Abramowitz and Stegun

1965). Because of axial symmetry of the problem; it is noted that $p_i(r)\,(i=1,2)$ are even and $G(r)$ is odd:

$$G(-r)=-G(r)\ ,$$

$$p_i(-r)=p_i(r)\ \ ,\ (i=1,2) \qquad\qquad (3.3a–c)$$

Then, the integrals from 0 to a in equations (3.2a–c) can be converted into integrals from $-a$ to a and Eqs. (3.2) can be stated as follows:

$$\frac{2}{\pi^2}\int_{-a}^{a}\left[\frac{1}{t-r}+k_1(r,t)+|t|H_1(r,t)\right]G(t)dt-p_1(r)=p_0\ , \qquad (-a<r<a)$$

$$2\lambda\,G(r)-\int_{-a}^{a}\left[\frac{1}{t-r}+k_2(r,t)-\pi\,|t|H_2(r,t)\right]p_1(t)\,dt-2\pi\int_{b}^{c}t\,H_3(r,t)p_2(t)\,dt=0$$

$$(-a<r<a)$$

$$\int_{-a}^{a}|t|H_3(r,t)\,p_1(t)dt-\frac{2}{\pi}\int_{b}^{c}\left[h_2(r,t)-\pi t H_2(r,t)\right]p_2(t)dt=0 \qquad (b<r<c)$$

(3.4a–c)

where

$$k_i(r,t)=\frac{m_i(r,t)-1}{t-r} \qquad (i=1,2),$$

$$m_1(r,t)=\begin{cases}\dfrac{t^2-r^2}{|rt|}K(|t/r|)+\left|\dfrac{r}{t}\right|E(|t/r|) & (|t|<|r|)\\ E(|r/t|) & (|t|>|r|),\end{cases}$$

$$m_2(r,t)=\begin{cases}\left|\dfrac{t}{r}\right|E(|t/r|) & (|t|<|r|)\\ \dfrac{t^2}{r^2}E(|r/t|)-\dfrac{t^2-r^2}{r^2}K(|r/t|) & (|t|>|r|),\end{cases}$$

$$\lambda=\frac{\mu_2(\kappa_1+1)}{\mu_1(\kappa_2+1)}. \qquad\qquad (3.5a–e)$$

The singular integral equations given in (3.4a–c) are subjected to following equilibrium conditions and symmetry:

$$\int_{-a}^{a} G(t)dt = 0,$$

$$\int_{-a}^{a} |t|\, p_1(t)\, dt = -a^2 p_0,$$

$$\int_{b}^{c} t\, p_2(t)\, dt = -\frac{1}{2}a^2 p_0. \qquad (3.6a–c)$$

The kernels k_1 and k_2 possess logarithmic singularity. Probable singular behavior of H_1 is due to behavior of L_1 at $\alpha = \infty$. Therefore, defining

$$L_{1\infty}(r,t,\alpha) = \lim_{\alpha\to\infty} L_1(r,t,\alpha), \qquad (3.7)$$

the probable singular part of the kernel $H_1(r,t)$ is calculated from

$$H_{1s}(r,t) = \int_{0}^{\infty} L_{1\infty}(r,t,s)\, ds, \qquad (3.8)$$

and its bounded part is calculated from

$$H_{1b}(r,t) = \int_{0}^{\infty} [\, L_1(r,t,\alpha) - L_{1\infty}(r,t,\alpha)\,]\, d\alpha. \qquad (3.9)$$

Here, the subscripts b and s denote the bounded and the singular parts. Therefore, the kernel H_1 is

$$H_1(r,t) = H_{1s}(r,t) + H_{1b}(r,t). \qquad (3.10)$$

By making use of the asymptotic expressions for the modified Bessel functions for $r > 0$ and $t > 0$, (Abramowitz and Stegun, 1965)

$$L_{1\infty}(r,t,\alpha) = -e^{\alpha(2a-r-t)}\left\{(a-r)(a-t)\alpha^2 - \left[2(a-t) + \frac{1}{2}(t-r)\right]\alpha + 1\right\}\Big/\sqrt{rt}. \qquad (3.11)$$

The corresponding singular part, $H_{1s}(r,t)$, may be evaluated for $(-a \le r,t \le a)$ as

$$H_{1s}(r,t) = \frac{1}{\sqrt{rt}}\left\{\left[-\frac{1}{2} + 3(a-r)\frac{d}{dr} - (a-r)^2\frac{d^2}{dr^2}\right]\left(\frac{1}{t+r-2a} + \frac{1}{t-r+2a}\right)\right.$$

$$+\left[-\frac{1}{2}-3(a+r)\frac{d}{dr}-(a+r)^2\frac{d^2}{dr^2}\right]\left(\frac{1}{t-r-2a}+\frac{1}{t+r+2a}\right)\Bigg\} \qquad (3.12)$$

It is noted that $H_1(r,t)$ is singular while $r,t \to \pm a$.

The singular behavior of $p_i(r)\,(i=1,2)$ and $G(r)$ can be determined by writing

$$G(r)=\frac{g_1^*(r)}{\left(a^2-r^2\right)^{\gamma}}, \qquad (0<\mathrm{Re}(\gamma)<1)$$

$$p_1(r)=\frac{g_2^*(r)}{\left(a^2-r^2\right)^{\gamma}}, \qquad (0<\mathrm{Re}(\gamma)<1)$$

$$p_2(r)=\frac{g_3^*(r)(c-r)^{\theta}}{(r-b)^{\theta}} \qquad (0<\mathrm{Re}(\theta)<1) \quad (3.13a–c)$$

where γ and θ are unknown constants, $g_i^*(r)(i=1,2)$ are Hölder-continuous functions in $[-a,a]$ and $g_3^*(r)$ is Hölder-continuous in $[b,c]$. Singularity powers γ and θ are determined by analyzing the singular integral equations, Eqs. (3.4a, b), near $r \to \pm a$ and Eq. (3.4c) close to $r=b$. The technique of complex function provided by Muskhelishvili' work (1953) gives the following characteristic equations

$$\lambda \sin^2 \pi\gamma + \left(1-4\gamma+2\gamma^2\right)\cos\pi\gamma - \cos^2 \pi\gamma = 0,$$

$$\cot\pi\theta = 0. \qquad (3.14a, b)$$

Eqs. (3.14) are consistent with the results of previous studies, Adam and Bogy (1976), Gecit (1986) and Dundurs and Lee (1972).

3.2 Solution of Integral Equations

The singular integral equations are converted into a convenient form in terms of dimensionless quantities to make easier the solution of numerical analysis. For this purpose, dimensionless variables x, y, ξ and η can be introduced by

$$(r,t)=a(x,y) \qquad (-a<(r,t)<a),\ (-1<(x,y)<1)$$

$$(r,t)=\frac{c-b}{2}(\xi,\eta)+\frac{c+b}{2} \qquad (b<(r,t)<c),\ (-1<(\xi,\eta)<1)$$

(3.15a–d)

and Hölder-continuous functions $g_i(i=1-3)$ by

$$G(r)=G(ax)=\frac{\pi}{2}p_0\frac{g_1(x)}{(1-x^2)^\gamma},$$

$$p_1(r)=p_1(ax)=p_0\frac{g_2(x)}{(1-x^2)^\gamma},$$

$$p_2(r)=p_2\left(\frac{c-b}{2}\xi+\frac{c+b}{2}\right)=p_0g_3(\xi)\left(\frac{1-\xi}{1+\xi}\right)^\theta. \qquad (3.16a–c)$$

Then, equation (3.4) and equation (3.6) are rewritten in the form

$$\frac{1}{\pi}\int_{-1}^{1}\left[\frac{1}{y-x}+\widetilde{k}_1(x,y)+|y|\widetilde{H}_1(x,y)\right]\frac{g_1(y)}{(1-y^2)^\gamma}dy-\frac{g_2(x)}{(1-x^2)^\gamma}=1, \qquad (-1<x<1)$$

$$\lambda\pi\frac{g_1(x)}{(1-x^2)^\gamma}-\int_{-1}^{1}\left[\frac{1}{y-x}+\widetilde{k}_2(x,y)-\pi|y|\widetilde{H}_2(x,y)\right]\frac{g_2(y)}{(1-y^2)^\gamma}dy$$

$$-2\pi\int_{-1}^{1}\left(\eta+\frac{c+b}{c-b}\right)\widetilde{H}_3(x,\eta)g_3(\eta)\left(\frac{1-\eta}{1+\eta}\right)^\theta d\eta=0, \qquad (-1<x<1)$$

$$-\int_{-1}^{1}|y|\,\widetilde{H}_4(\xi,y)\frac{g_2(y)}{(1-y^2)^\gamma}dy-\frac{2}{\pi}\int_{-1}^{1}\left[\widetilde{h}_2(\xi,\eta)-\pi\left(\eta+\frac{c+b}{c-b}\right)\widetilde{H}_5(\xi,\eta)\right]g_3(\eta)\left(\frac{1-\eta}{1+\eta}\right)^\theta d\eta=0,$$

$$(-1<\xi<1)$$

(3.17a–c)

and

$$\int_{-1}^{1}\frac{g_1(y)}{(1-y^2)^\gamma}dy=0,$$

$$\int_{-1}^{1}|y|\frac{g_2(y)}{(1-y^2)^\gamma}dy=-1,$$

$$\int_{-1}^{1}\left(\eta+\frac{c+b}{c-b}\right)g_3(\eta)\left(\frac{1-\eta}{1+\eta}\right)^\theta d\eta=\frac{-1/2}{\left(\frac{c-b}{2a}\right)^2}, \qquad (3.18a–c)$$

where

$$\tilde{k}_i(x,y) = ak_i(ax,ay), \qquad (i=1,2)$$

$$\tilde{H}_i(x,y) = a^2 H_i(ax,ay), \qquad (i=1,2)$$

$$\tilde{H}_3(x,\eta) = \left(\frac{c-b}{2}\right)^2 H_3\left(ax, \frac{c-b}{2}\eta + \frac{c+b}{2}\right),$$

$$\tilde{H}_4(\xi,y) = a^2 H_3\left(\frac{c-b}{2}\xi + \frac{c+b}{2}, ay\right),$$

$$\tilde{H}_5(\xi,\eta) = \left(\frac{c-b}{2}\right)^2 H_2\left(\frac{c-b}{2}\xi + \frac{c+b}{2}, \frac{c-b}{2}\eta + \frac{c+b}{2}\right). \qquad \text{(3.19a–g)}$$

By using Gauss-Jacobi integration formula (Erdogan, Gupta and Cook 1973) for integrals of $g_i(i=1,2)$ and Gauss Lobatto integration formula (Krenk, 1978) for integrals of g_3, one may obtain the following linear algebraic equations

$$\sum_{i=1}^{n} W_i \left\{ \frac{1}{\pi}\left[\frac{1}{y_i - x_j} + \tilde{k}_1(x_j,y_i) + |y_i|\tilde{H}_1(x_j,y_i)\right] g_1(y_i)\right\} - \frac{C_{ij}}{\left(1-x_j^{\,2}\right)^{\gamma}} g_2(y_i) = 1,$$

$$(j=1,\ldots,n-1)$$

$$\sum_{i=1}^{n} W_i \left\{ \frac{\lambda C_{ij}}{\left(1-x_j^{\,2}\right)^{\gamma}} g_1(y_i) - \frac{1}{\pi}\left[\frac{1}{y_i - x_j} + \tilde{k}_2(x_j,y_i) - \pi\,|y_i|\tilde{H}_2(x_j,y_i)\right] g_2(y_i)\right\}$$

$$\frac{-4\pi}{2n+1} D_i (1-\eta_i)\left(\eta_i + \frac{c+b}{c-b}\right)\tilde{H}_3(x_j,\eta_i) g_3(\eta_i) = 0, \qquad (j=1,\ldots,n-1)$$

$$-\sum_{i=1}^{n} W_i\,|y_i|\,\tilde{H}_4(\xi_j,y_i) g_2(y_i) - 4\sum_{i=1}^{n} D_i \frac{1-\eta_i}{2n+1}\left[\tilde{h}_2(\xi_j,\eta_i) - \pi\left(\eta_i + \frac{c+b}{c-b}\right)\tilde{H}_5(\xi_j,\eta_i)\right] g_3(\eta_i) = 0.$$

$$(j=1,\ldots,n)$$

(3.20a–c)

from Eqs.(3.17) and

$$\sum_{i=1}^{n} |y_i| W_i\, g_2(y_i) = -1,$$

$$\sum_{i=1}^{n} W_i g_1(y_i) = 0\,,$$

$$\sum_{i=1}^{n} 2\pi D_i \frac{(1-\eta_i)}{(2n+1)}\left(\eta_i + \frac{c+b}{c-b}\right) g_3(\eta_i) = \frac{-1/2}{\left(\frac{c-b}{2a}\right)^2}\,, \qquad (3.21a–c)$$

from Eqs. (3.18) where

$$D_1 = 1/2 = D_n\,;\, D_i = 1\,(i = 2,...,n-1)$$

$$\eta_i = \cos\left(\frac{2i\pi}{2n+1}\right), \qquad (i = 1,\ldots,n)$$

$$\xi_j = \cos\left(\frac{2j-1}{2n+1}\pi\right), \qquad (j = 1,\ldots,n) \qquad (3.22a–c)$$

W_i $(i = 1,\ldots,n)$ are named as the weights of Jacobi polynomial $P_n^{(-\gamma,-\gamma)}$, y_i and x_j are the roots of

$$P_n^{(-\gamma,-\gamma)}(y_i) = 0\,, \qquad (i = 1,\ldots,n)$$

$$P_n^{(1-\gamma,1-\gamma)}(x_j) = 0\,. \qquad (j = 1,\ldots,n-1) \qquad (3.23a, b)$$

The interpolation constants (Gecit, 1986) are defined by

$$C_{ij} = \sum_{l=0}^{n-1} \frac{2l-2\gamma+1}{2^{1-2\gamma}} \frac{l!\,\Gamma(l-2\gamma+1)}{\left[\Gamma(l-\gamma+1)^2\right]} P_l^{(-\gamma,-\gamma)}(x_j) P_l^{(-\gamma,-\gamma)}(y_i)$$

$$(i = 1,\ldots,n\,; j = 1,...,n-1) \qquad (3.24)$$

The system of algebraic equations, Eqs. (3.20) and (3.21), contain $(3n+1)$ equation for $(3n+1)$ unknowns, $g_k(y_i)\,(k = 1,2;\, i = 1,\ldots,n)$, $g_3(\eta_i)$ $(i = 1,\ldots,n)$ and c. The system of algebraic equations given by equation (3.20) and equation (3.21) may be simplified and reduced to the following system:

$$\sum_{i=1}^{n/2} W_i \left\{ \frac{\lambda}{\left(1-x_j^{\,2}\right)^{\gamma}} C_1(x_j, y_i) g_1(y_i) - \left[\frac{1}{\pi}\left(\frac{1}{y_i - x_j} - \frac{1}{y_i + x_j}\right) h_3(x_j, y_i)\right.\right.$$

$$-2\pi\ y_i\widetilde{H}_2\left(x_j,y_i\right)\Big]g_2\left(y_i\right)\Big\}-\sum_{i=1}^{n}\frac{4\pi}{2n+1}D_i\left(1-\eta_i\right)\left(\eta_i+\frac{c+b}{c-b}\right)\widetilde{H}_3\left(x_j,\eta_i\right)g_3\left(\eta_i\right)=0\,,$$

$$\left(j=1,\ldots,\frac{n}{2}-1\right)$$

$$\sum_{i=1}^{n/2}W_iC_1\left(0,y_i\right)g_1\left(y_i\right)=0\,,$$

$$\sum_{i=1}^{n/2}W_i\left\{\frac{1}{\pi}\left[\left(\frac{1}{y_i-x_j}+\frac{1}{y_i+x_j}\right)h_4\left(x_j,y_i\right)+2y_i\widetilde{H}_1\left(x_j,y_i\right)\right]g_1\left(y_i\right)\right\}$$

$$-\frac{W_i}{\left(1-x_j^{\ 2}\right)^{\gamma}}C_2\left(x_j,y_i\right)g_2\left(y_i\right)=1\,,\qquad\left(j=1,\ldots,\frac{n}{2}-1\right)$$

$$\sum_{i=1}^{n/2}W_i\left\{\frac{1}{\pi}\left[\left(\frac{1}{y_i}+\frac{2}{\pi}y_i\widetilde{H}_1\left(0,y_i\right)\right)\right]\right\}g_1\left(y_i\right)-W_iC_2\left(0,y_i\right)g_2\left(y_i\right)=1\,,$$

$$\sum_{i=1}^{n/2}-2W_iy_i\widetilde{H}_4\left(\xi_j,y_i\right)g_2\left(y_i\right)--4\sum_{i=1}^{n}D_i\frac{1-\eta_i}{2n+1}\left[\widetilde{h}_2\left(\xi_j,\eta_i\right)-\pi\left(\eta_i+\frac{c+b}{c-b}\right)\widetilde{H}_5\left(\xi_j,\eta_i\right)\right]g_3\left(\eta_i\right)=0\,,$$

$$\left(j=1,\ldots,n\right)$$

$$\sum_{i=1}^{n/2}W_i\,y_i\,g_2\left(y_i\right)=-\frac{1}{2}\,,$$

$$\sum_{i=1}^{n}2\pi\,D_i\frac{\left(1-\eta_i\right)}{\left(2n+1\right)}\left(\eta_i+\frac{c+b}{c-b}\right)g_3\left(\eta_i\right)=\frac{-1/2}{\left(\frac{c-b}{2a}\right)^2}\,,\qquad(3.25\text{a–g})$$

where

$$h_3\left(x_j,y_i\right)=\begin{cases}\left|\dfrac{y_i}{x_j}\right|E\left(\left|\dfrac{y_i}{x_j}\right|\right) & \left(\left|y_i\right|<\left|x_j\right|\right)\\[2ex] \dfrac{y_i^{\ 2}}{x_j^{\ 2}}E\left(\left|\dfrac{x_j}{y_i}\right|\right)-\dfrac{y_i^{\ 2}-x_j^{\ 2}}{x_j^{\ 2}}K\left(\left|\dfrac{x_j}{y_i}\right|\right) & \left(\left|y_i\right|>\left|x_j\right|\right),\end{cases}$$

$$h_4\left(x_j,y_i\right)=\begin{cases}\dfrac{y_i^{\ 2}-x_j^{\ 2}}{\left|x_jy_i\right|}K\left(\left|\dfrac{y_i}{x_j}\right|\right)+\left|\dfrac{x_j}{y_i}\right|E\left(\left|\dfrac{y_i}{x_j}\right|\right) & \left(\left|y_i\right|<\left|x_j\right|\right)\\[2ex] E\left(\left|\dfrac{x_j}{y_i}\right|\right) & \left(\left|y_i\right|>\left|x_j\right|\right),\end{cases}$$

$$h_5(x_j, y_i) = \frac{2x_j}{y_i^{\,2} - x_j^{\,2}} h_3(x_j, y_i),$$

$$h_6(x_j, y_i) = \frac{2y_i}{y_i^{\,2} - x_j^{\,2}} h_4(x_j, y_i). \qquad (3.26a–d)$$

and

$$C_1(x_j, y_i) = C_{ij}(x, y) - C_{ij}(x, -y),$$

$$C_2(x_j, y_i) = C_{ij}(x, y) + C_{ij}(x, -y). \qquad (3.27a, b)$$

Eqs. (3.25) contain $(2n+2)$equation for $(2n+1)$unknowns. If Eq. (3,25b) is analyzed closely, it is understood that the equation is satisfied automatically at $\xi_j = 0\ (j = n/2)$. Therefore, there would be exactly $(2n+1)$ equations for the $(2n+1)$unknowns.

3.3 Stress Intensity Factors

The stress intensity factors are indicated by the stress state around the edge of the semi-infinite cylinder and hole in the current section. Intensity factors for Mode-I stress, k_a and $k_b$may be defined as

$$k_a = \lim_{r \to a} \sqrt{2}(a-r)^{\gamma} \sigma_z(r,0),$$

$$k_b = \lim_{r \to b} \sqrt{2}(r-b)^{\theta} \sigma_z(r,-h). \qquad (3.28a, b)$$

$\sigma_z(r,0)$ and $\sigma_z(r,-h)$ can be calculated from Eqs.(2.38) and (3.16):

$$\sigma_z(r,0) = p_1(r) = p_0 \frac{g_2(x)}{(1-x^2)^{\gamma}},$$

$$\sigma_z(r,-h) = p_2(r) = p_0 g_3(\xi)\left(\frac{1-\xi}{1+\xi}\right)^{\theta}. \qquad (3.29a, b)$$

Therefore stress intensity factors k_a and $k_b$may be calculated to be

$$k_a = 2^{\frac{1}{2}-\gamma} p_0 a^{\gamma} g_2(1),$$

$$k_b = \sqrt{2(c-b)}\,p_0 g_3(-1)\,, \tag{3.30a, b}$$

since Eq. (3.14b) gives $\theta = 1/2$.

It is convenient to normalize the stress intensity factors:

$$\bar{k}_a = \frac{k_a}{p_0 a^{\gamma}} = 2^{\frac{1}{2}-\gamma}\, g_2(1),$$

$$\bar{k}_b = \frac{k_b}{p_0\sqrt{\frac{c-b}{2}}} = 2g_3(-1)\,. \tag{3.31a, b}$$

CHAPTER 4

RESULTS AND CONCLUSIONS

4.1 Verification of the Solution

In the study of Gecit (1986), a similar problem is considered without a circular hole in the foundation. When the results of the present study are compared with those of Gecit (1986), the pressure distrubitions and stress intensity factor curves show a similar trend. Contact pressure distrubution between the cylinder and layer are shown for a/h=0.25 and ν=0.3 as below. The pressure is dependent on λ. It moves to infinity around the edge of the cylinder in both studies. The pressure increases with increasing of r/a. Similar comparisions can also be made for stress intensity factors.

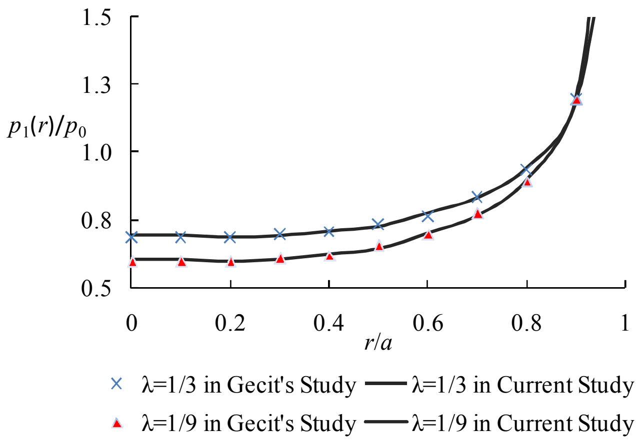

Figure 4.1 Variation of contact pressure vs. r/a for ν=0.3, a/h=0.25

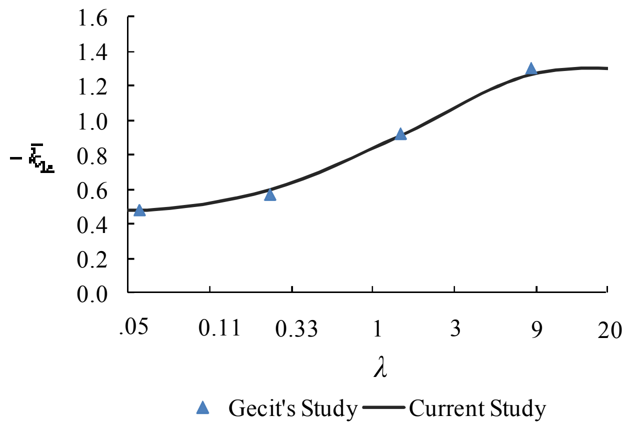

Figure 4.2 Variation of the normalized stress intensity factor $-\bar{k}_{1a}$ vs. λ

The variation of the normalized stress intensity factors $-\bar{k}_{1a}$ at the edges of the semi-infinite cylinder vs. λ for $\nu = 0.3$ and $a/h = 1$ is plotted in Figure (4.1b) for both Gecit and current study. A sharp increase in the values $-\bar{k}_{1a}$ between $\lambda = 1/3$ and 3 is observed for both gecit and current study. Also, $-\bar{k}_{1a}$ values increase more rapidly for higher $\lambda = 3$ in Gecit Study.

The following conclusions may be reached:

1. It is clear that $-\bar{k}_{1a}$ is always greater than $-\bar{k}_{1b}$.
2. It is observed that Poisson's ratio has an effect on contact pressure and stress intensity factors.
3. Contact pressure distributions and variations of stress intensity factor are heavily dependent on the bi-material constant.
4. The magnitudes of $-\bar{k}_{1a}$ and $-\bar{k}_{1b}$ increase in proportional to the width of the cylinder and hole.

4.2 Numerical Results

Normalized contact interval (-1, 1) is considered instead of nominal intervals (0, *a*) and (*b*, *c*) for numerical results. The radius of the semi-infinite cylinder *a*, the radius of circular hole *b*, the thickness of layer *h* and the half length of the contact between rigid foundation and layer *c* are described by geometric parameters. The bi-material constant λ varies from 1/9 to ∞. The case of λ=1 may represent a semi-infinite cylinder and an elastic layer of identical materials. λ is related to the constant μ and κ. Poisson's ratios, ν_1 and ν_2, and shear modulus, μ_1 and μ_2, are defined for the cylinder and layer, respectively. A uniform compression p_0 applied to the cylinder makes it possible to normalize $p_1(r)$, $p_2(r)$ and $G(r)$ unknowns with using dimensionless variables. The cases of $0 < a/h \leq 1.0$ and $0 < b/h \leq 1.0$ are considered. The value of *N* equals to 40 in all computations. It is observed that there are no significant changes in the results when the value of *N* is increased. All graphs are also drawn for $\kappa_1 = 1.8\,(\nu_1 = 0.3)$ as shown in graphical figures 4.3- 4.35.

Contact pressure is applied for both the cylinder-layer and layer-rigid support in contact regions. Therefore, the contact pressure distributions are calculated for various material pairs. In addition, normalized stress intensity factors for the semi-infinite cylinder and hole are very different from each other. Normalized stress intensity factors vs. λ are plotted for different magnitudes of a/h and b/h. It is understood that the bi-material constant λ has an important effect on both contact pressure distributions and stress intensity factors. The case of λ=1 may show the identical materials for the cylinder and layer.

Figures (4.3) to (4.14) show the graphs of normalized contact pressure between the elastic layer and cylinder versus r/a for the various values of a/h and b/h when $\nu = 0.3$. It is noted that the values of λ have a very important role in the distribution of the pressure. Similar figures in figures 4.3 to 4.14 are grouped into separate sections. Each section is explained in detail;

i. Figures (4.3) to (4.8) display the variation of the contact pressure between the cylinder and layer vs. r/a values for b/h = 0.1, 0.5, 1.0 when λ=1/9, 1/3, 1, 3, ∞, ν = 0.3 and a/h = 0.1, 0.25. For figure 4.3, numerical values of $p_1(r)/p_0$ do not considerably change with increasing values of r/a up to r/a = 0.5. Then, $p_1(r)/p_0$ values increase with increasing r/a values between r/a= 0.5 and 1.0. Especially, when r/a approaches 1.0, $p_1(r)/p_0$ moves to infinity (Figure 4.1). Similar explanations may be performed for Figure 4.4 to 4.8, as well.

ii. The variation of the contact pressure between the cylinder and layer vs. r/a values for $b/h = 0.1, 0.5, 1.0$ for $\lambda = 1/9, 1/3, 1, 3, \infty$, $\nu = 0.3$ and $a/h = 0.5$ is shown in figures (4.9) to (4.11). The figures 4.9 and 4.11 are compared with each other in terms of initial contact pressure values. When $p_1(r)/p_0$ value is equal to 0.49 at r/a = 0 for λ = 1/9 in figure 4.11, $p_1(r)/p_0$ value corresponds to 0.63 at r/a = 0 for λ = 1/9 in figure 4.9. In other words, initial contact pressure for λ = 1/9 almost increases by 20 % when b/h changes from 1.0 to 0.1. Similar behavior for the change in values of initial contact pressure is observed for other λ values, as well.

iii. The variation of the contact pressure between the cylinder and layer vs. r/a values for $b/h = 0.1, 0.5, 1.0$ for $\lambda = 1/9, 1/3, 1, 3, \infty$, $\nu = 0.3$ and $a/h = 1.0$ is indicated in figures (4.12) to (4.14). For figure 4.12, all curves drawn for different λ values are very close to each other, and contact pressure values are in the range of 0.75 and 1.0. On the other hand, all curves drawn for different λ values in figure 4.14 are remote from each other. Their values are in the broad range of 0.2 and 1.0. In addition, for figure 4.12, numerical values of $p_1(r)/p_0$ do not significantly alter with increasing values of r/a up to r/a = 0.7. Then, $p_1(r)/p_0$ values increase with increasing r/a values between 0.7 and 1.0. On the other hand, for figure 4.14, numerical values of $p_1(r)/p_0$ do not considerably change with increasing values of r/a up to r/a = 0.2.

However, $p_1(r)/p_0$ values increase with increasing r/a values between 0.2 and 1.0. This means that magnitudes of r/a after a point have a substantial impact on contact pressure. Especially for figure 4.12 and 4.14, while r/a approaches 1.0, $p_1(r)/p_0$ moves to infinity. Similar explanations may be made for Figure 4.13, as well.

Figures (4.15) and (4.18) present the variation of contact pressure vs. r/a for $b/h = 0.1, 0.5, 1.0$ when $\nu = 0.3$, $\lambda = 1/9, 9$ and $a/h = 0.1, 1.0$. Pressure distributions in figure 4.15 and 4.17 overlap for all values of b/h and a/h = 0.1 when the bi-material constant λ is assumed to be 1/9 and 9. Conversely, values of $p_1(r)/p_0$ contact pressures for each r/a values in figure 4.17 are much higher than the values of $p_1(r)/p_0$ contact pressures for each of the r/a values for figure 4.15. While initial contact pressure values are in the range of 0.2 and 0.75 for figure 4.16, the same values in figure 4.16 are in the range of 0.92 and 0.98. That is, curves as a group in figure 4.16 are much closer to each other than the curves in figure 4.16.

Figures (4.19) to (4.22) present the curves of contact pressure between the elastic layer and cylinder vs. r/a for $a/h = 0.1, 0.5, 1.0$ when $\nu = 0.3$, $\lambda = 1/9, 9$ and $b/h = 0.1, 1.0$. There are no significant increments for the values of $p_1(r)/p_0$ up to r/a = 0.5 for all a/h variations in the graph of figure 4.19. A similar trend is observed for lower r/a values in figure 4.20. A sharp increase in $p_1(r)/p_0$ values is observed at r/a = 0.8 in both figure 4.19 and 4.20. The effect of different a/h values on $p_1(r)/p_0$ becomes negligible along all r/a values in figure 4.21. Figure 4.22 shows that the increase in $p_1(r)/p_0$ for the curve for a/h=0.1 is higher than that of the curves for a/h = 0.5 and 1.0 along r/a values. The same figure indicates that $p_1(r)/p_0$ values overlap between r/a=0.65 and r/a =0.75 for $a/h = 0.1, 0.5, 1.0$.

The variation of the normalized stress intensity factors $-\bar{k}_{1a}$ at the edges of the semi-infinite cylinder vs. λ for $b/h = 0.1, 0.5, 1.0$ when $\nu = 0.3$ and $a/h = 0.1, 0.5, 1.0$ is plotted in figures (4.23) to (4.27). The normalized stress intensity factor at the edge of semi-infinite cylinder increases with increasing values of λ for different values of *a/h* and *b/h.* While curves in figure 4.23 overlap for *a*/*h* = 0.1 and *b*/*h* = 0.1, 1.0, the curves in figure 4.24 are clearly separated from each other for *a*/*h* = 1.0 and *b*/*h* = 0.1, 1.0. In other words, different magnitudes of *b*/*h* values do not create a significantly difference in $-\bar{k}_{1a}$ in figure 4.23 while various values of *b*/*h* yield different $-\bar{k}_{1a}$ values in figure 4.24. It is observed that higher *a*/*h* values correspond to lower values of $-\bar{k}_{1a}$ for each bi-material constant in figure 4.25 while higher *a*/*h* values correspond to higher values of $-\bar{k}_{1a}$ for each bi-material constant in figure 4.27.

Figures (4.28) to (4.30) present the variation of contact pressure $p_2(r)/p_0$ along the layer-rigid support vs. *r*/*h* for $b/h = 0.1, 0.5, 0.75, 1.0$ when $\nu = 0.3$, $a/h = 0.1, 0.5, 1.0$ and $\lambda = 1/9, 1.0$. When *r*/*h* approaches 0.1, 0.5, 0.75, 1.0 for *b*/*h* = 0.1, 0.5, 0.75, 1.0, magnitude of $p_2(r)/p_0$ incredibly increases in all curves.

Figure (4.31) indicates the variation of contact pressure $p_2(r)/p_0$ vs. *r*/*h* along the layer-rigid support for $\lambda = 9, 1, 1/9$ when $\nu = 0.3$, $a/h = 0.5$ and $b/h = 0.5$. It is seen in the figure that the values of contact pressure decrease with increasing values of *r*/*h*. It is also understood that the contact pressure has a minimum value at around *r*/*h* = 1.2. No significant effect for different values of λ on curves is observed.

Figure (4.32) and (4.33) show the variation of contact pressure $p_2(r)/p_0$ vs. *r*/*h* along the layer-rigid support for $a/h = 0.1, 0.5, 1.0$ when $\nu = 0.3$, $b/h = 0.5$

and $\lambda = 1/9, 9$. When *r*/*h* approaches 0.5 for *a*/*h*=0.1, 0.5, 1.0, the magnitude of $p_2(r)/p_0$ considerably increases in all curves.

Normalized stress intensity factors $-\bar{k}_{1b}$ versus λ at the edges of the hole for $a/h = 0.1, 1.0$, $\nu = 0.3$ and $b/h = 0.1, 0.5, 1.0$ are plotted in Figures (4.34) and (4.35). It appears that the magnitudes of $-\bar{k}_{1b}$ do not significantly increase with increasing values of λ for figure 4.34. Yet, values of $-\bar{k}_{1b}$ gradually increase with increasing values of λ for figure 4.35.

4.3 Conclusions

In this thesis, the problem is put forward as the frictionless double contact for an axisymmetric elastic layer pressed by an elastic semi-infinite circular cylinder. Therefore, stress–displacement expressions are given by Navier equations for the solution of the contact problem. The general stress-displacement expressions are acquired by employing boundary conditions both the layer and cylinder after applying Fourier and Hankel transforms for Navier Equations. Therefore, a system of three singular integral equations is obtained. This system is converted into linear algebraic equations by means of Gauss-Jacobi and Gauss Lobatto. The values of unknown functions $g_1(y_i)$, $g_2(y_i)$ and $g_3(\eta_i)$ $(i = 1,...,n)$ are calculated from the solution in this system via Fortran program. In addition, *a*/*h*, *b*/*h* and *c*/*h* are expressed as independent variables in the program. Estimated values of *a*/*h* and *b*/*h* are given for the solution of the problem. Therefore, *c*/*h* can be obtained by iterative procedures.

As an alternative method, this problem may also be solved using finite element methods or some package programs such as MARC and ANYS.

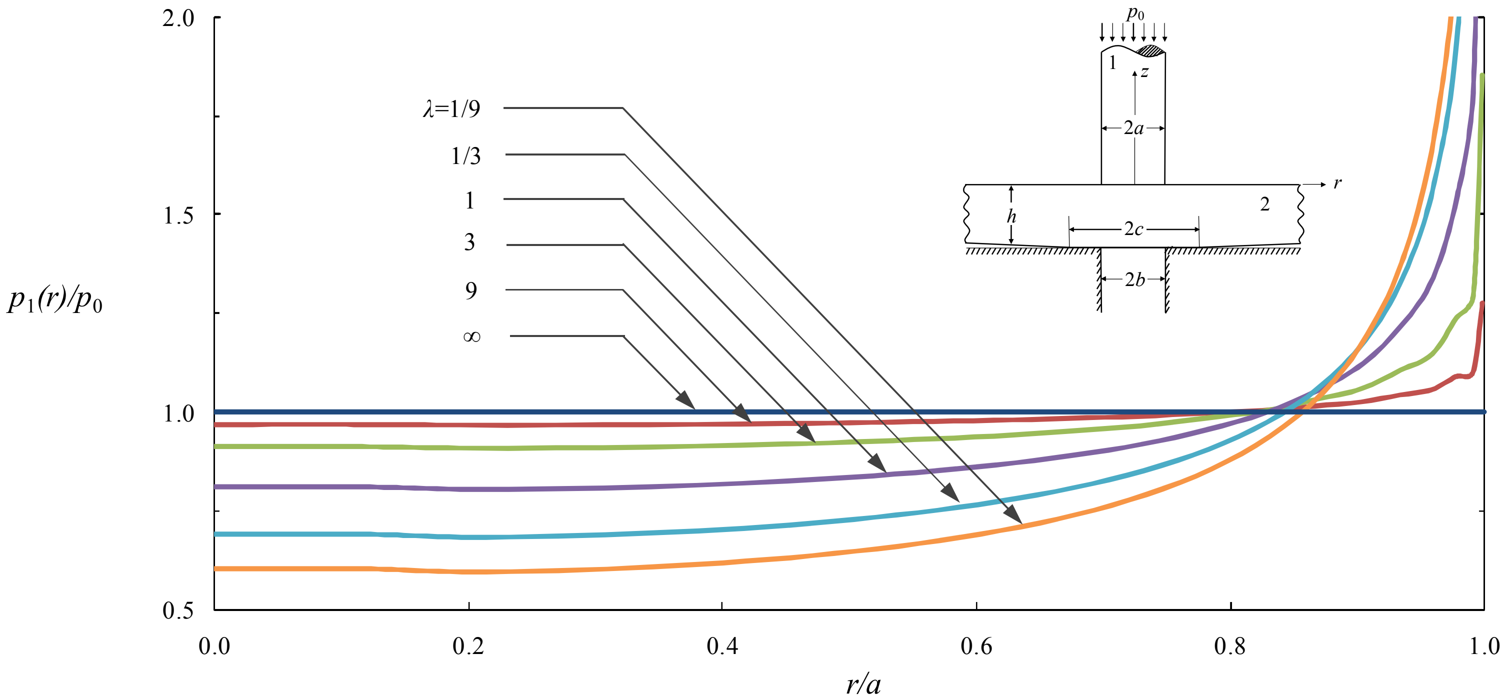

Figure 4.3 Contact pressure between the elastic layer and cylinder for ν=0.3, a/h=0.1 and b/h=0.1.

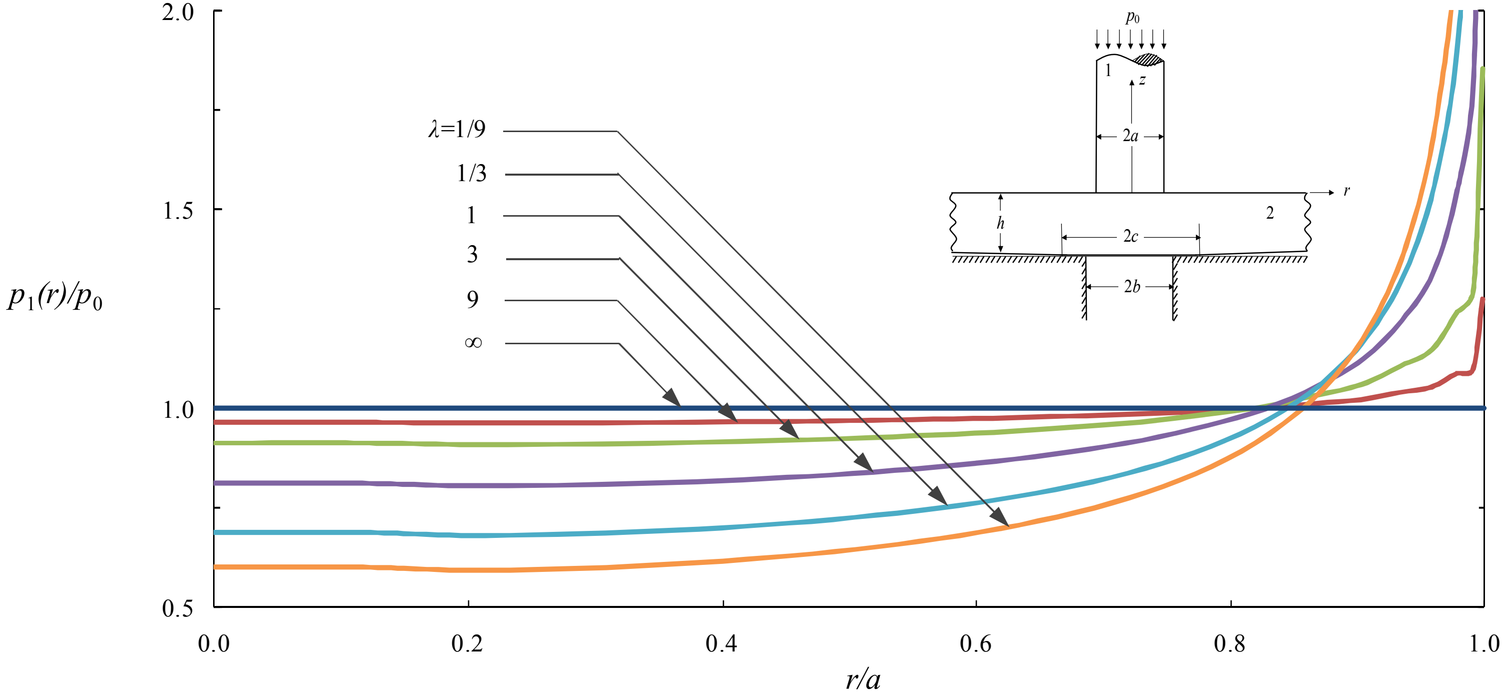

Figure 4.4 Contact pressure between the elastic layer and cylinder for v=0.3, a/h=0.1 and b/h=0.5.

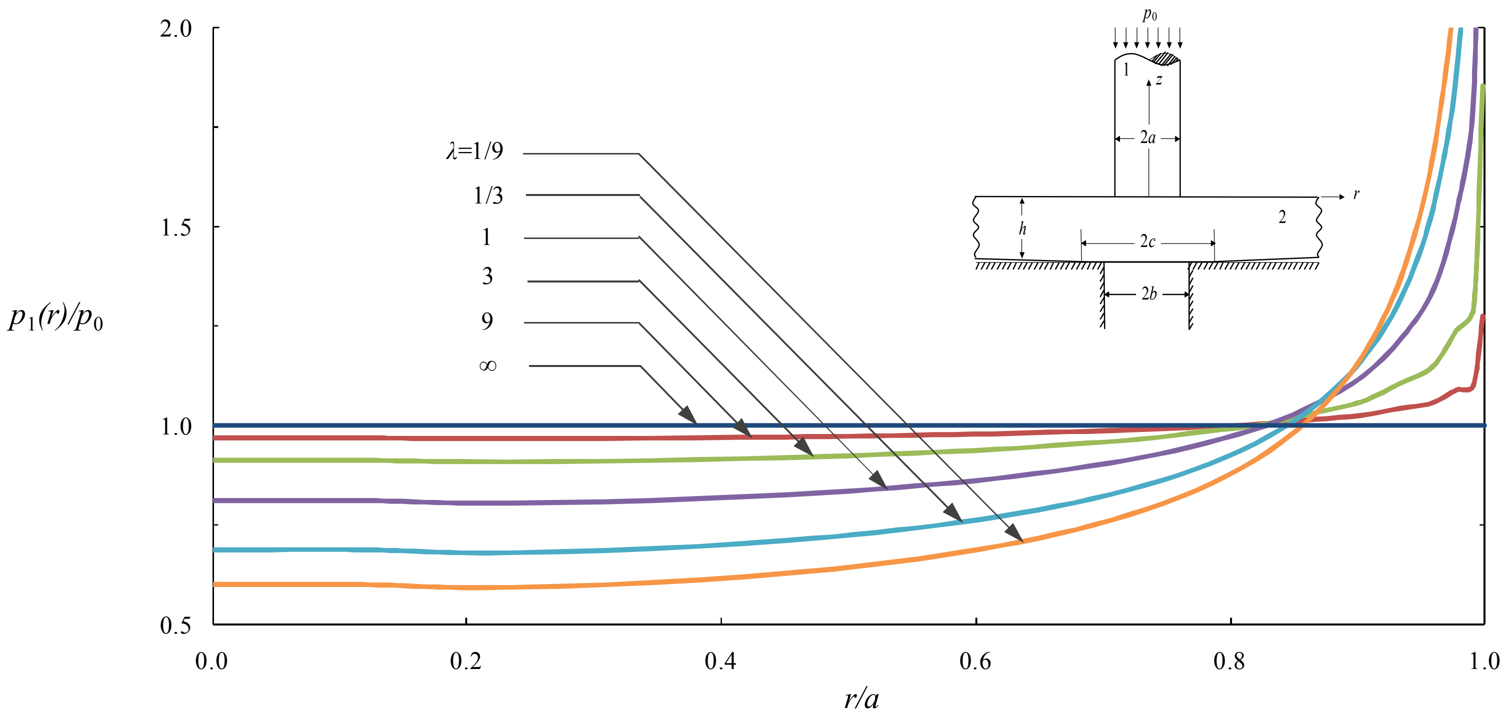

Figure 4.5 Contact pressure between the elastic layer and cylinder for ν=0.3, a/h=0.1and b/h=1.0.

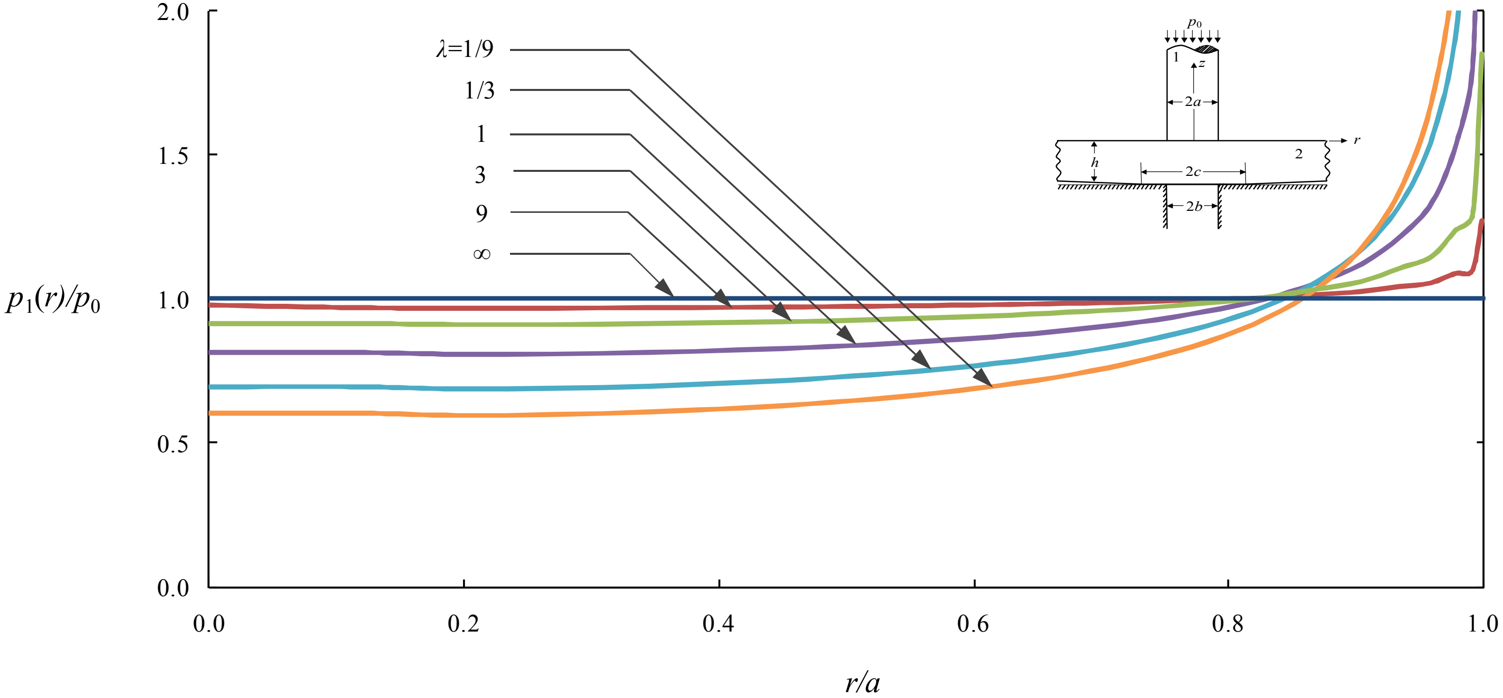

Figure 4.6 Contact pressure between the elastic layer and cylinder for ν=0.3, a/h=0.25 and b/h=0.1.

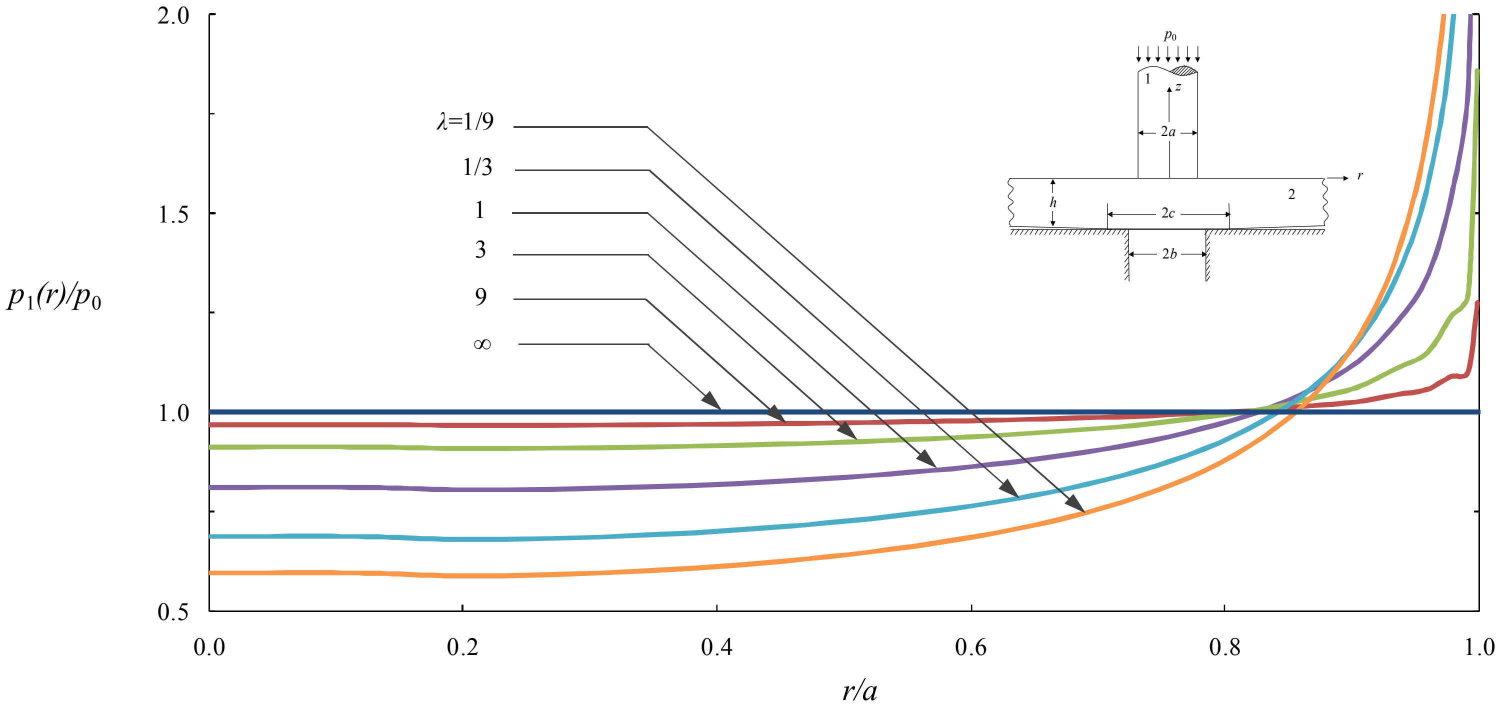

Figure 4.7 Contact pressure between the elastic layer and cylinder for v=0.3, a/h=0.25 and b/h=0.5.

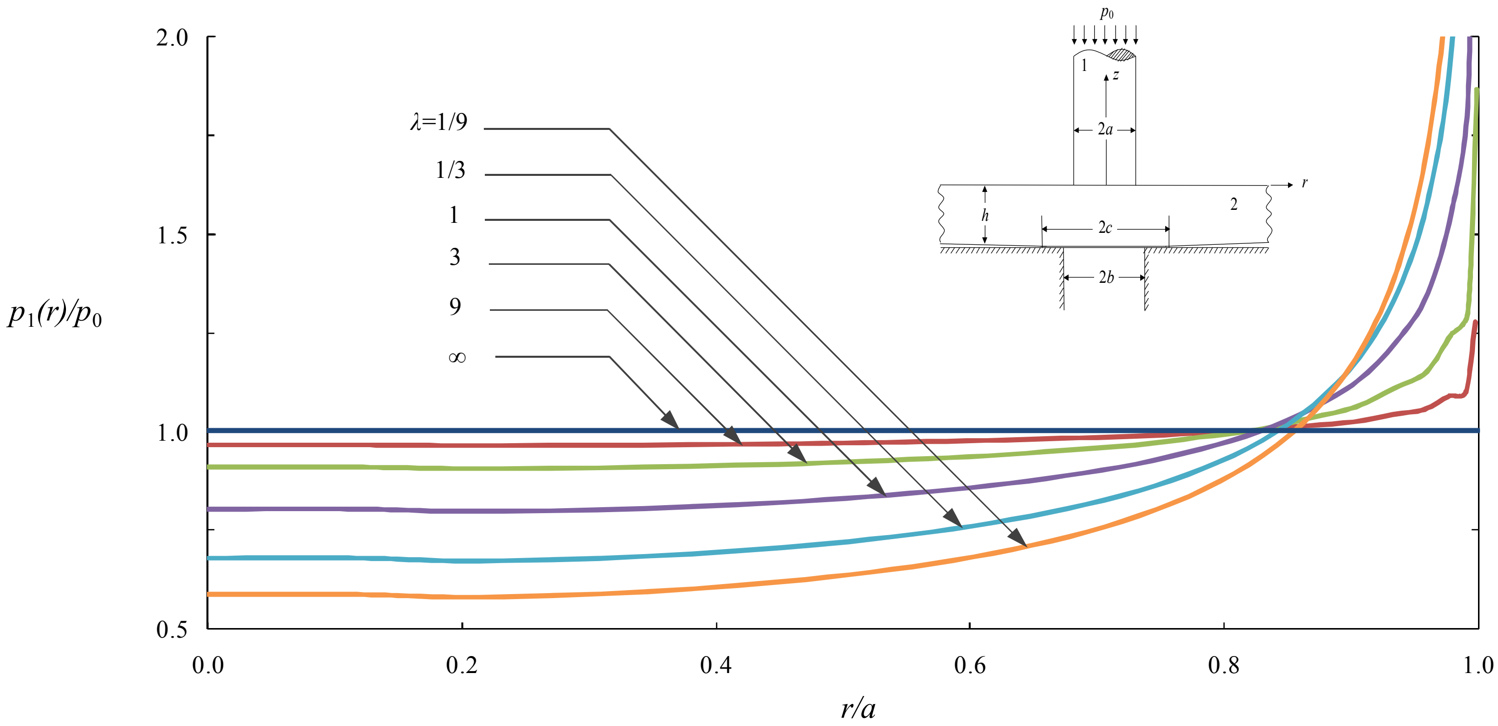

Figure 4.8 Contact pressure between the elastic layer and cylinder for v=0.3, a/h=0.25 and b/h=0.75.

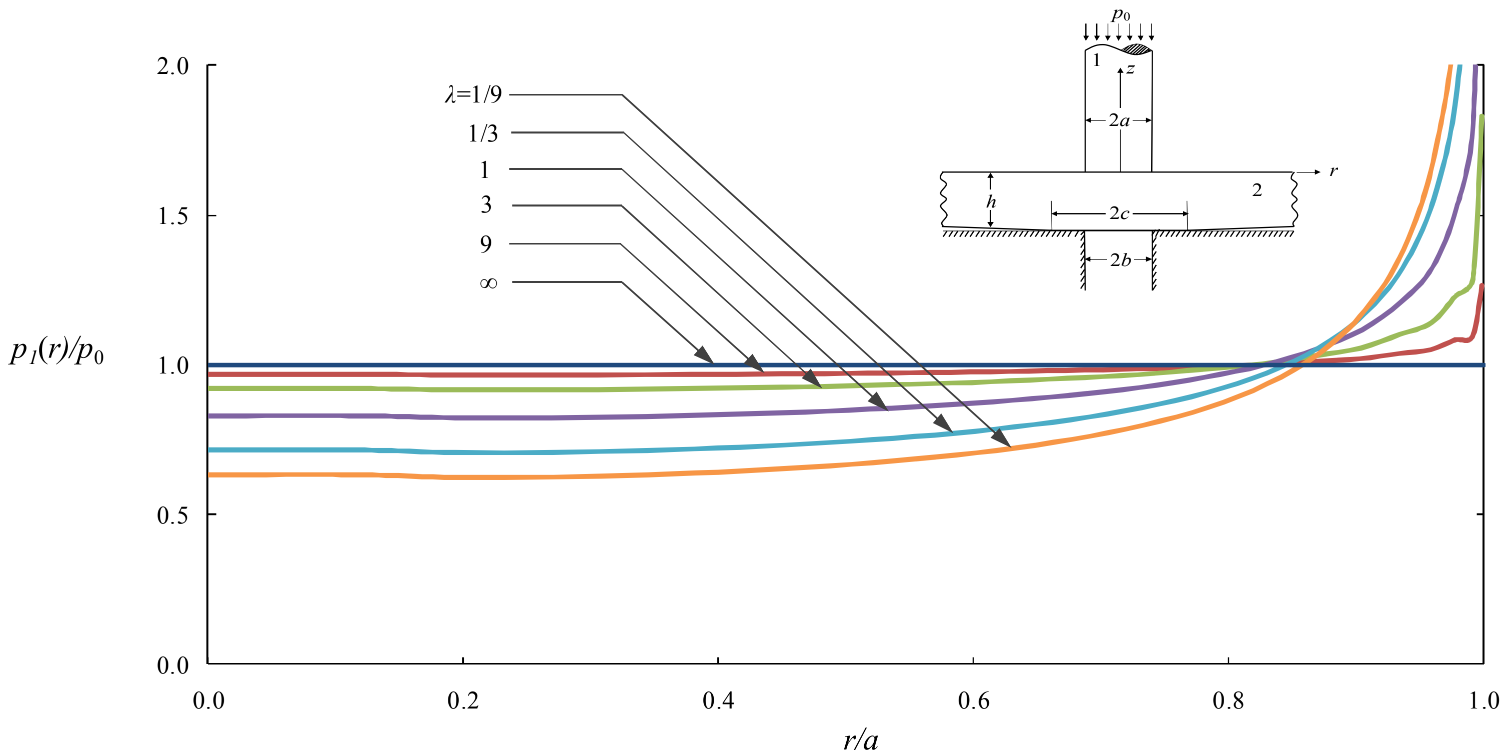

Figure 4.9 Contact pressure between the elastic layer and cylinder for ν=0.3, a/h=0.5 and b/h=0.1.

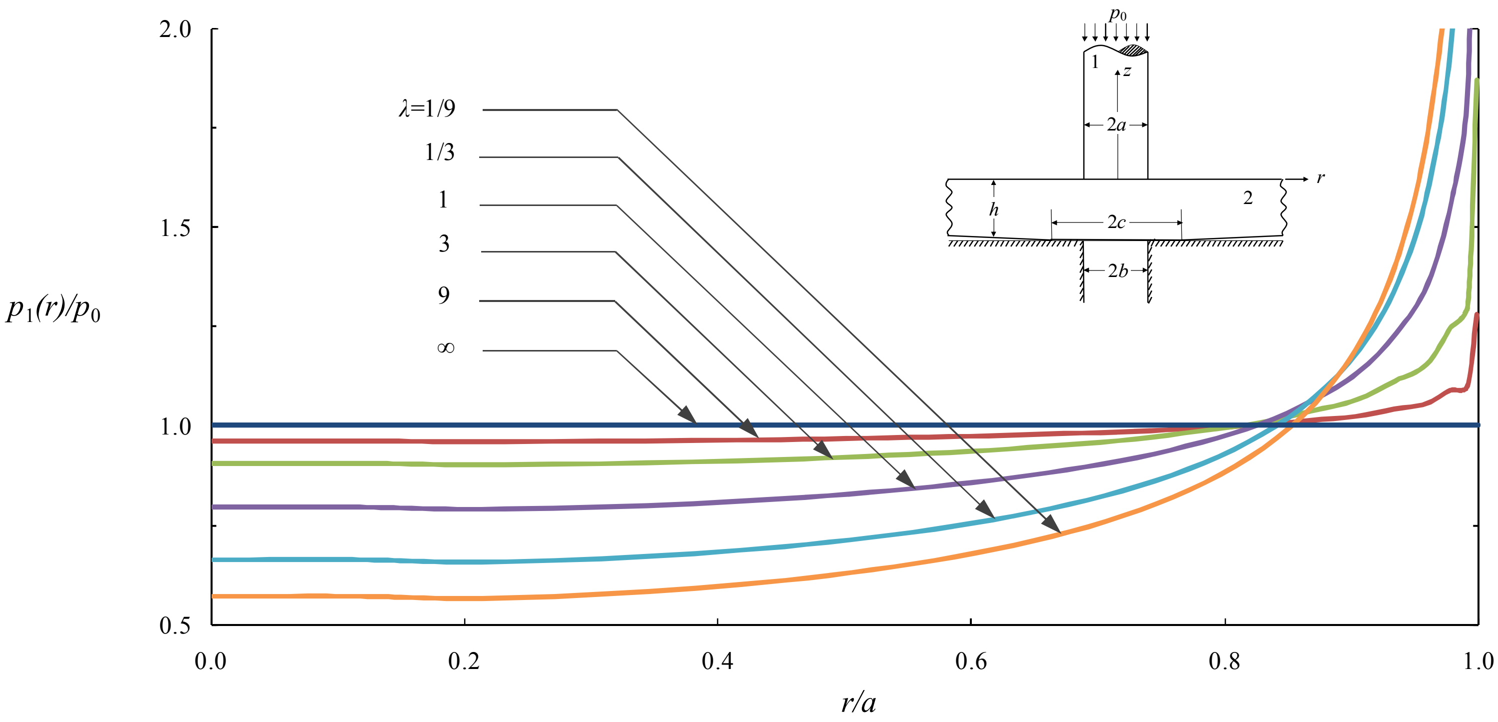

Figure 4.10 Contact pressure between the elastic layer and cylinder for v=0.3, a/h=0.5 and b/h=0.5.

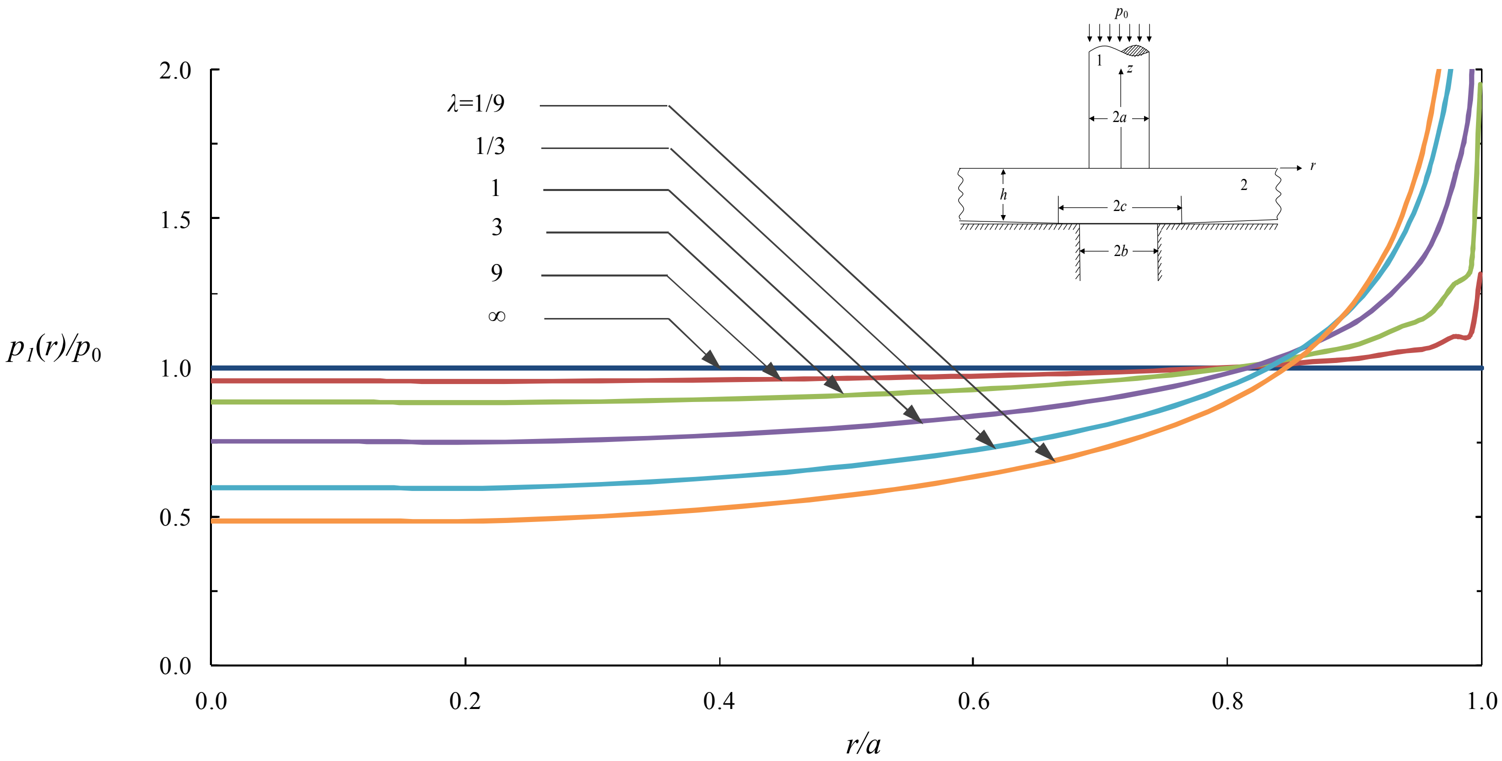

Figure 4.11 Contact pressure between the elastic layer and cylinder for ν=0.3, a/h=0.5 and b/h=1.0.

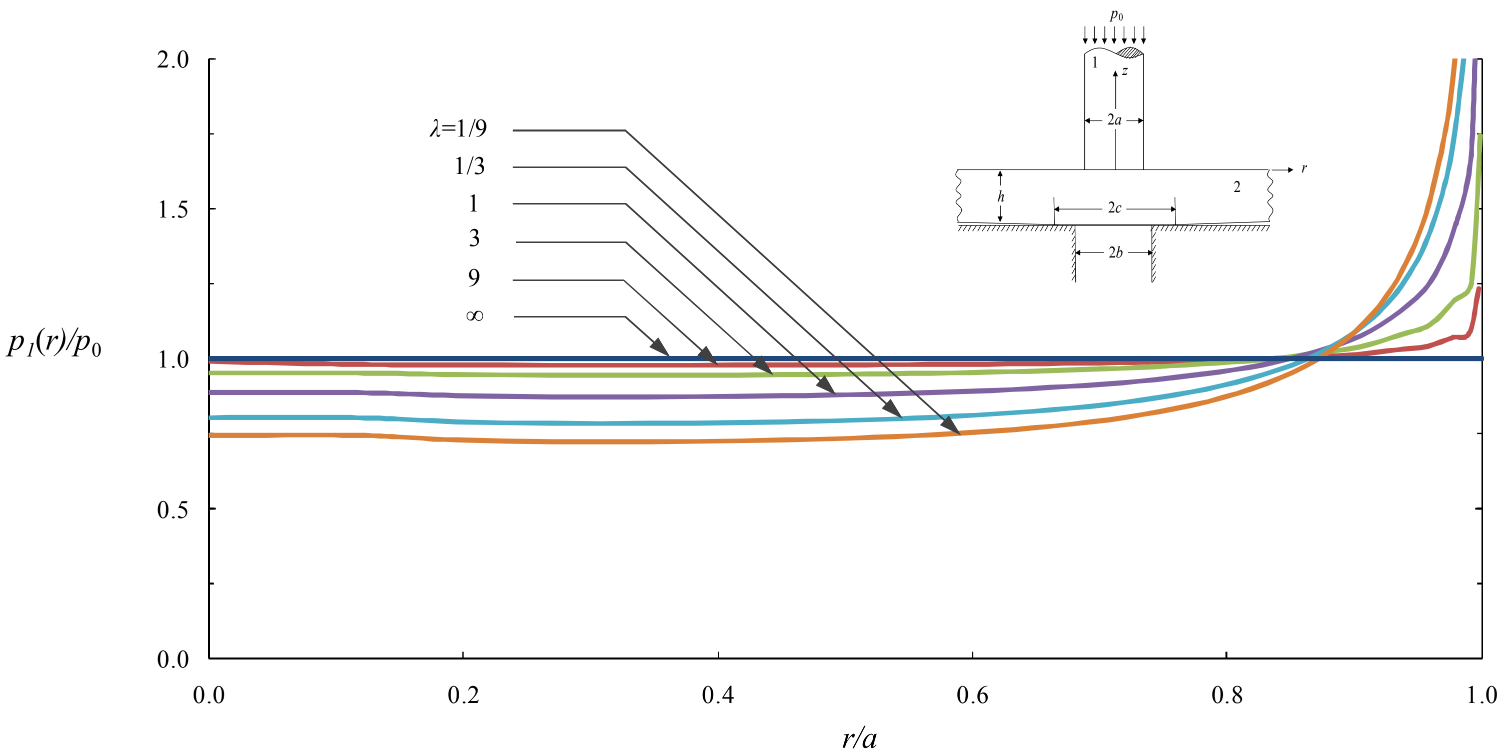

Figure 4.12 Contact pressure between the elastic layer and cylinder for v=0.3, a/h=1.0 and b/h=0.1.

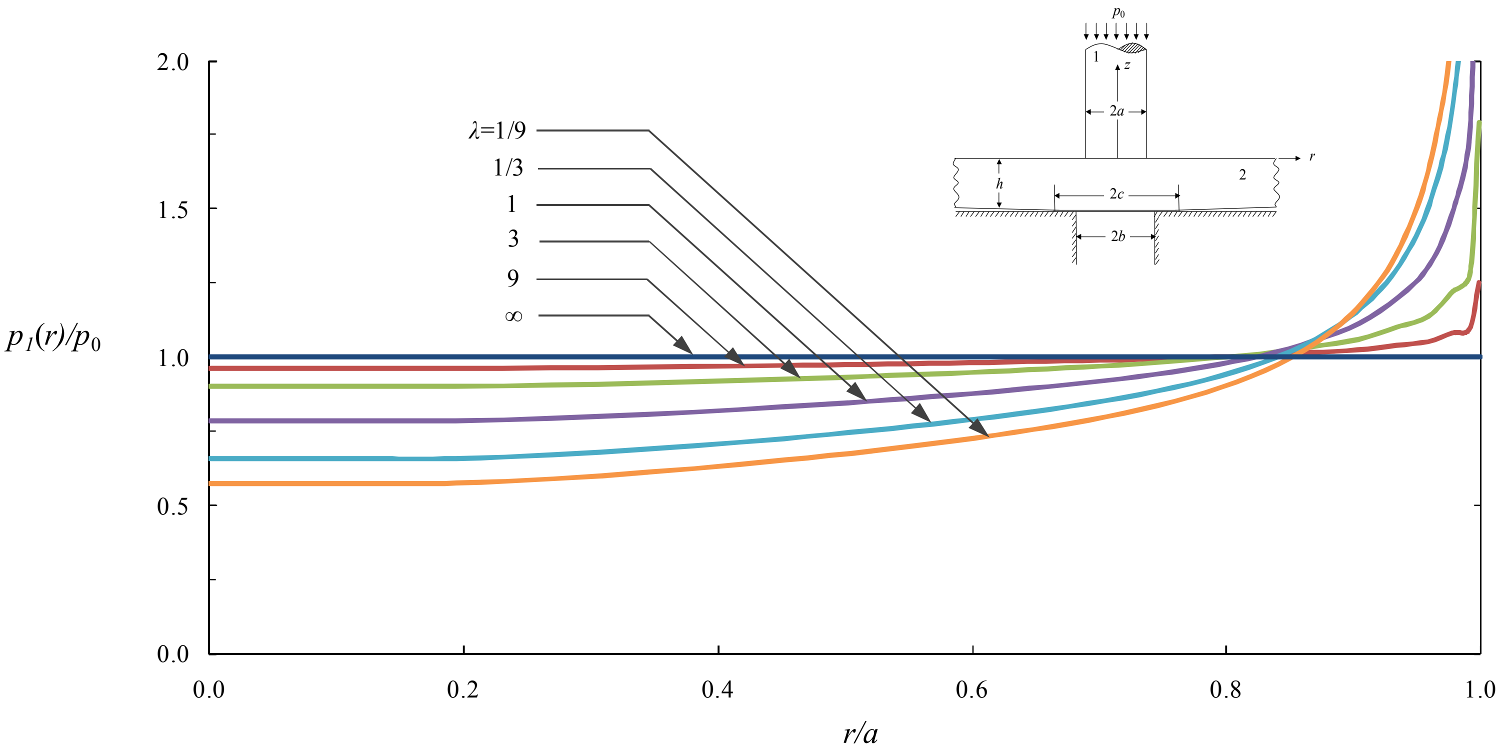

Figure 4.13 Contact pressure between the elastic layer and cylinder for v=0.3, a/h=1.0 and b/h=0.5.

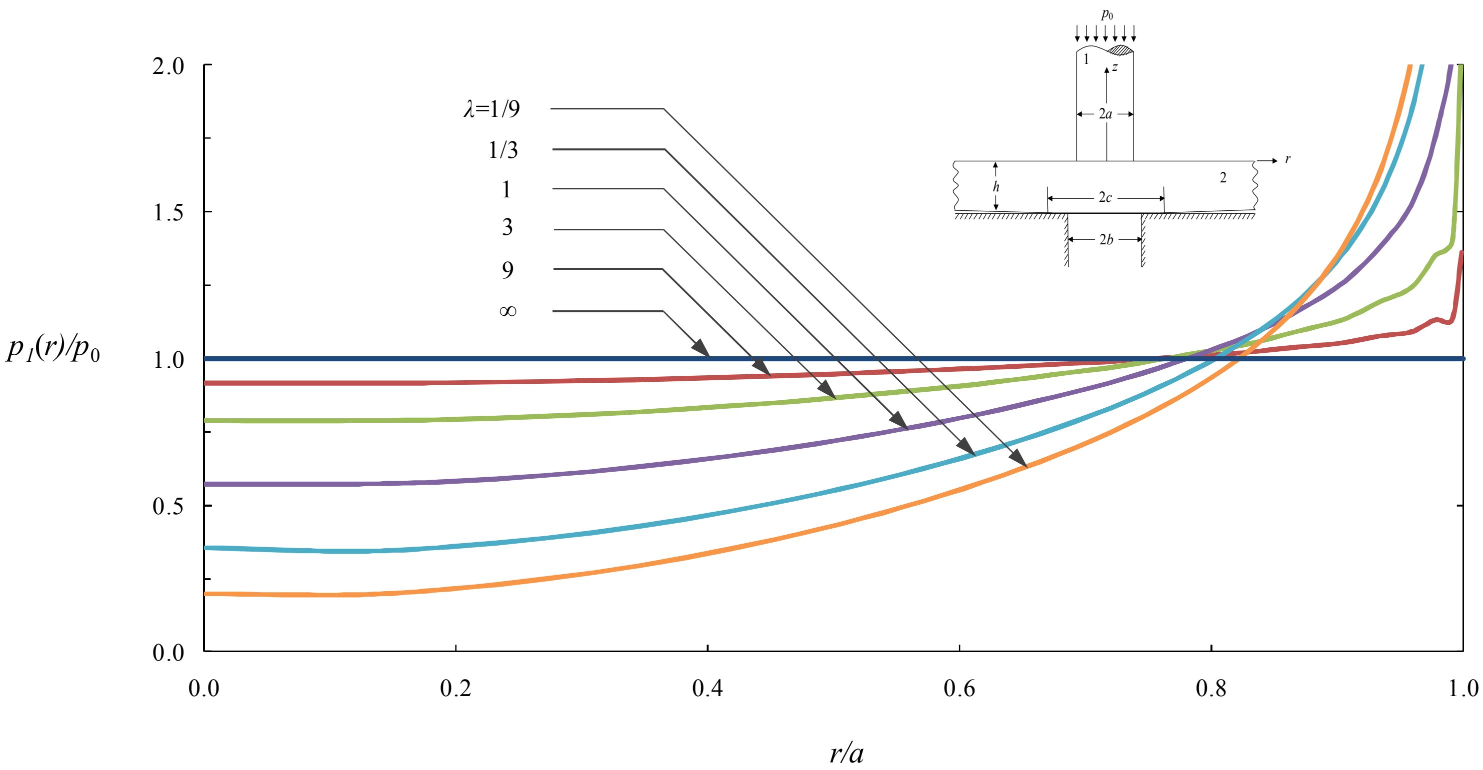

Figure 4.14 Contact pressure between the elastic layer and cylinder for v=0.3, a/h=1.0 and b/h=1.0.

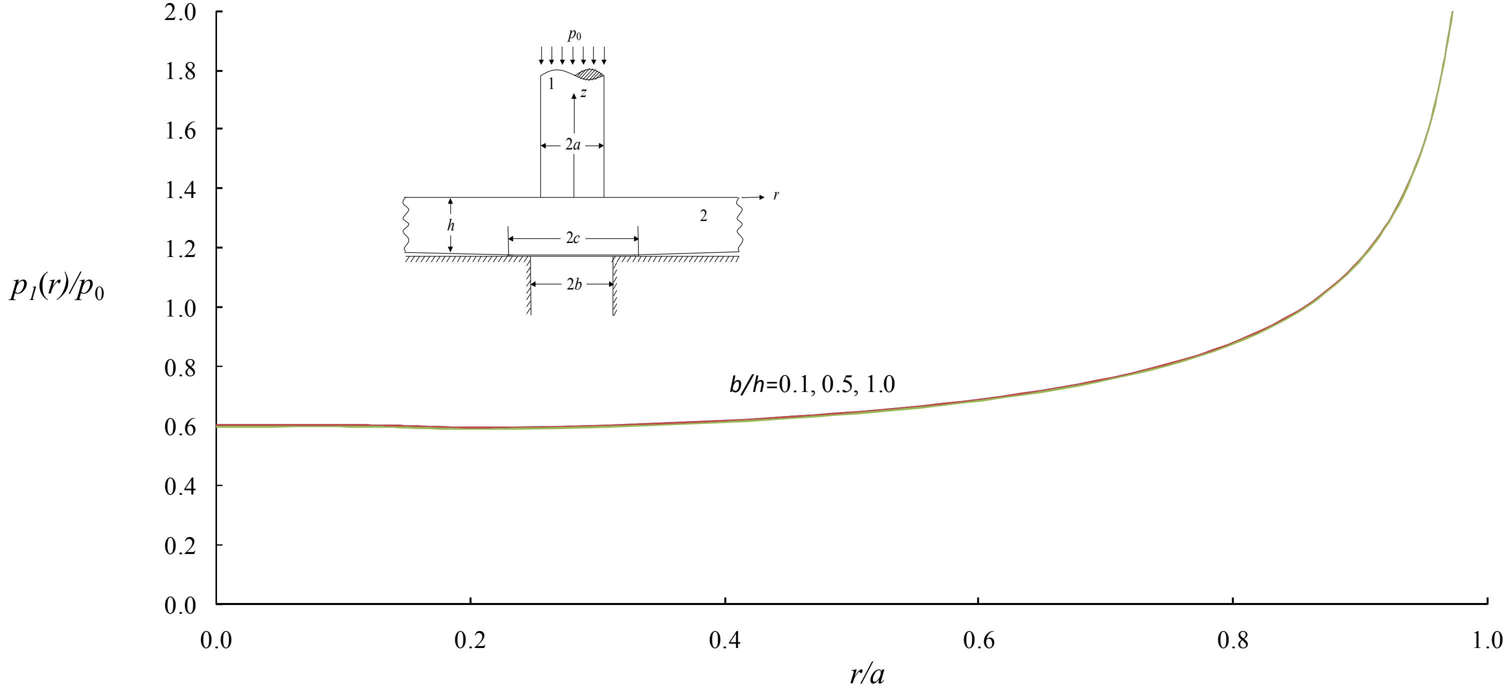

Figure 4.15 Contact pressure between the elastic layer and cylinder for ν=0.3, λ=1/9 and a/h=0.1.

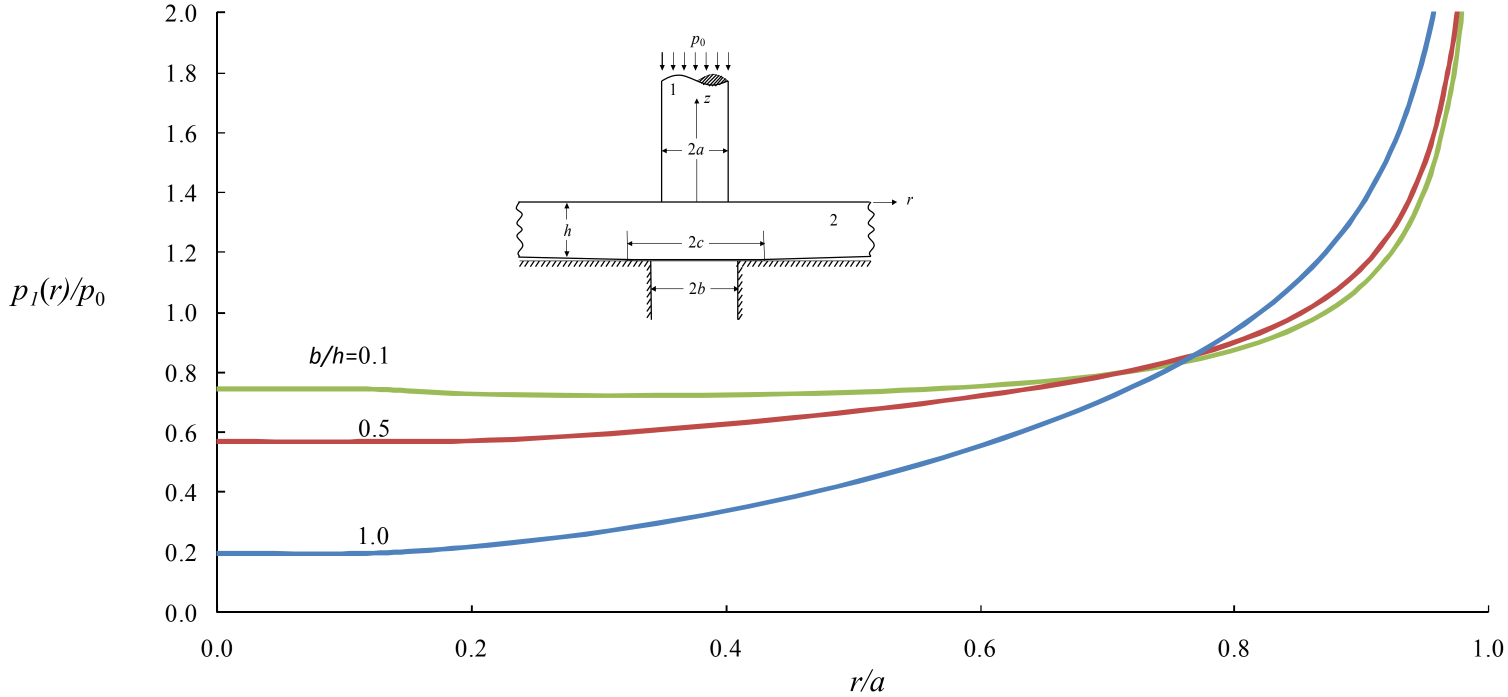

Figure 4.16 Contact pressure between the elastic layer and cylinder for ν=0.3, λ=1/9 and a/h=1.0

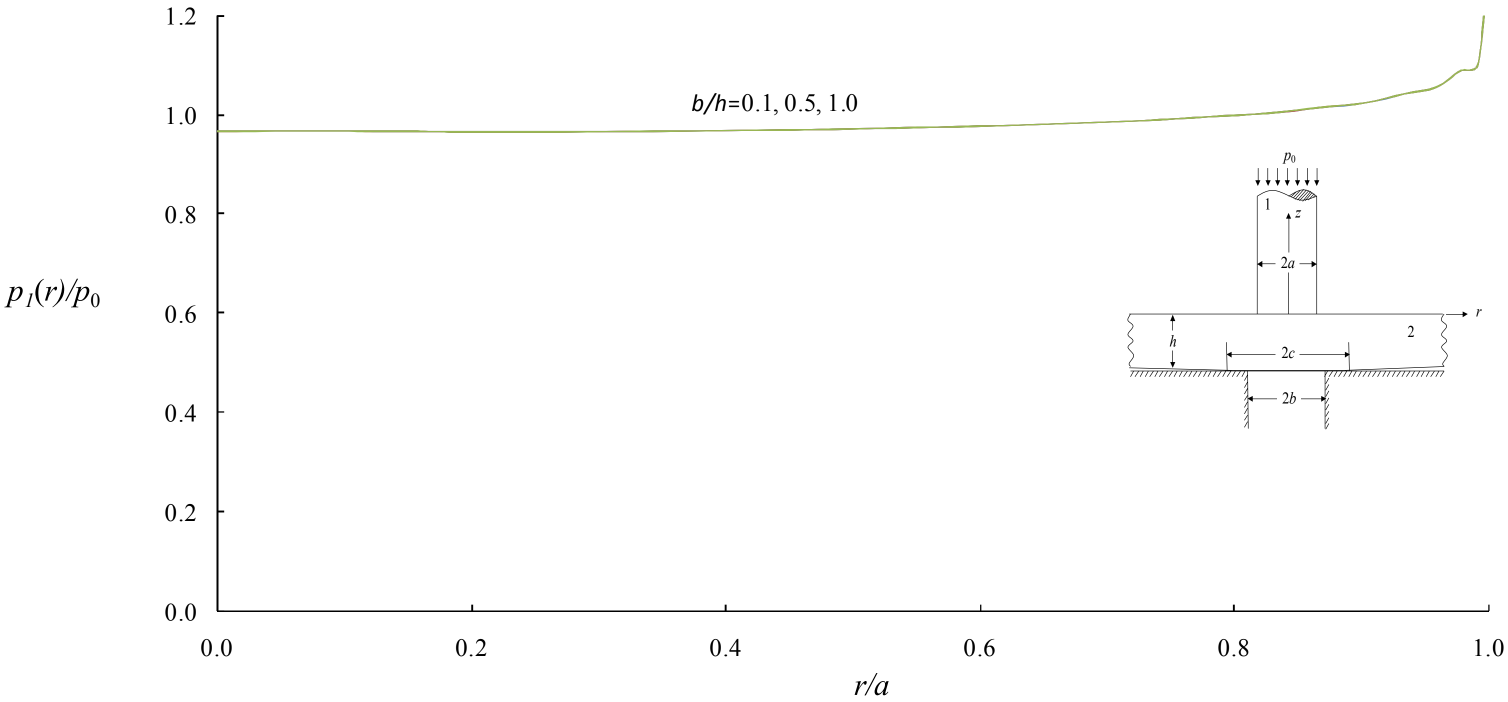

Figure 4.17 Contact pressure between the elastic layer and cylinder for ν=0.3, λ=9 and a/h=0.1.

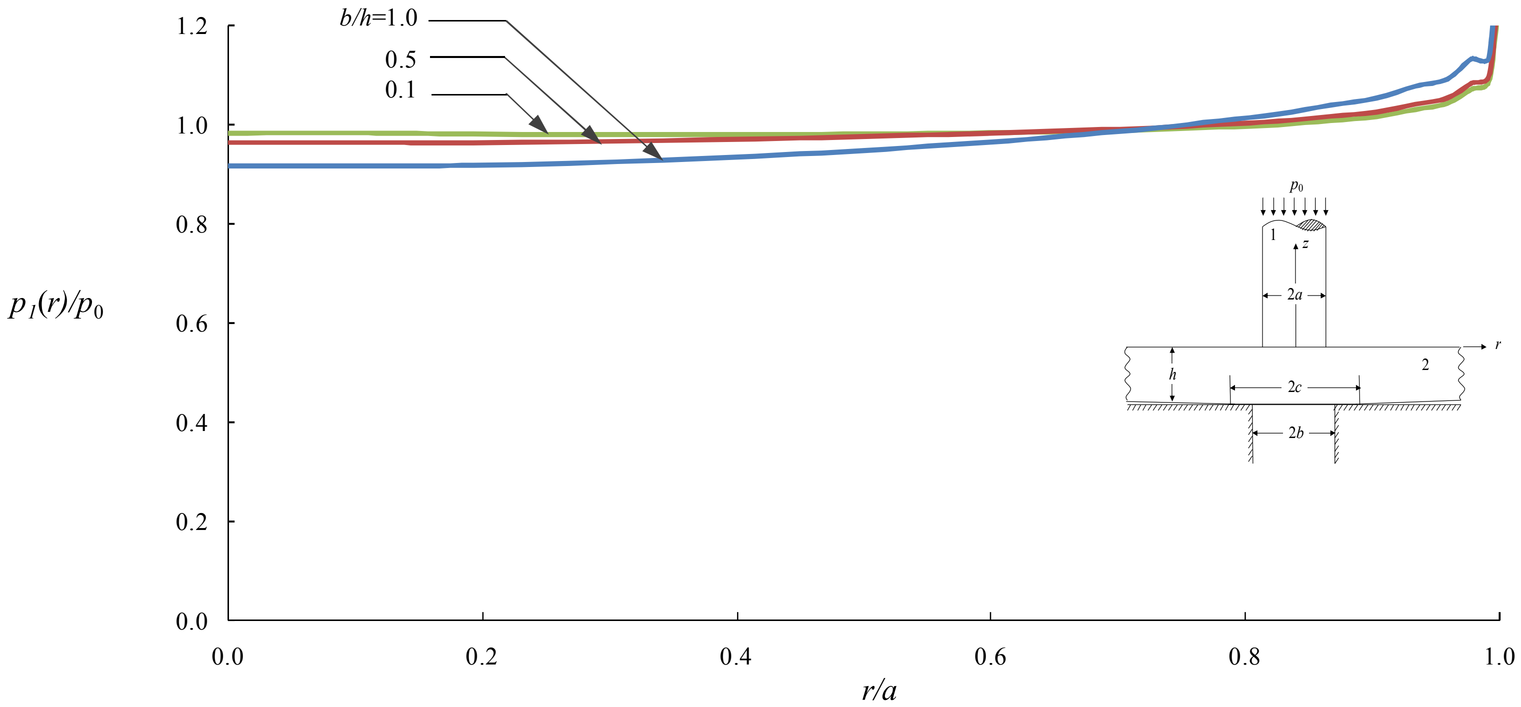

Figure 4.18 Contact pressure between the elastic layer and cylinder for ν=0.3, λ=9 and a/h=1.0.

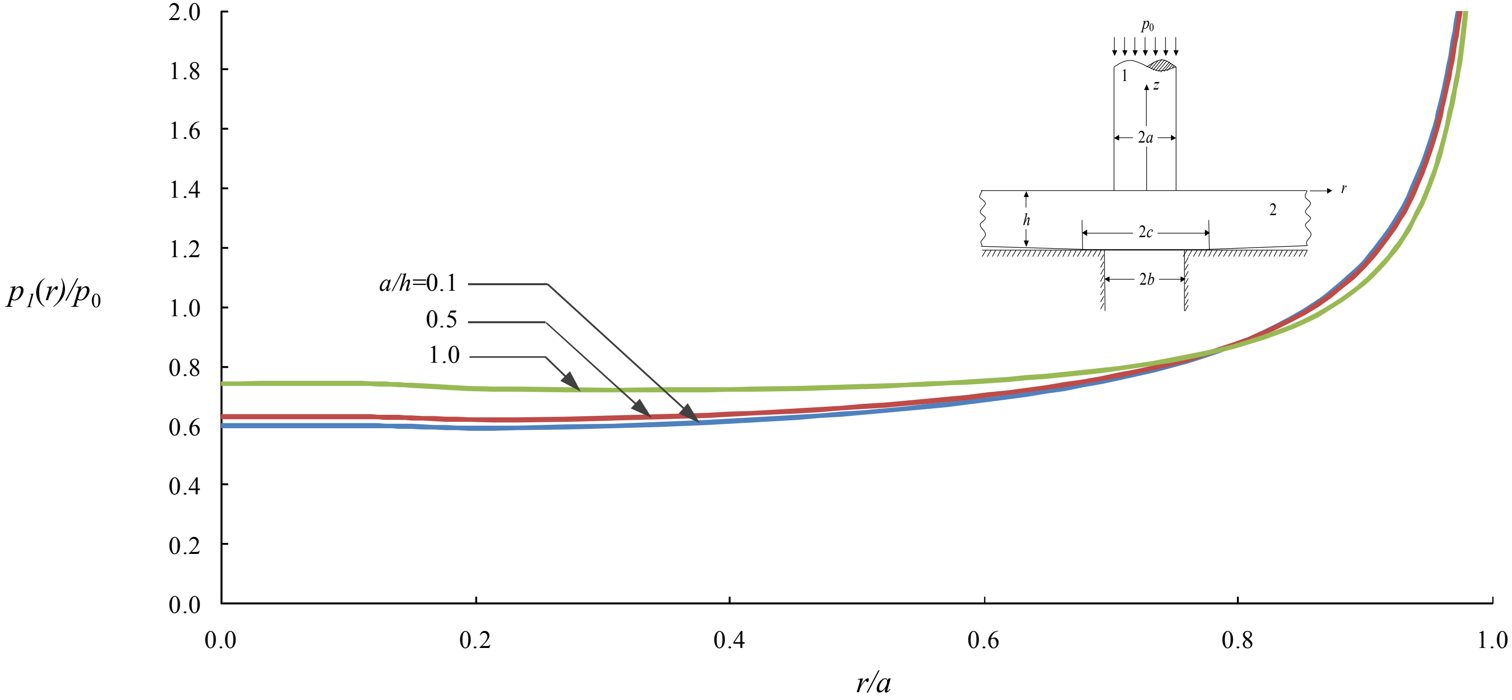

Figure 4.19 Contact pressure between the elastic layer and cylinder for ν=0.3, λ=1/9 and b/h=0.1.

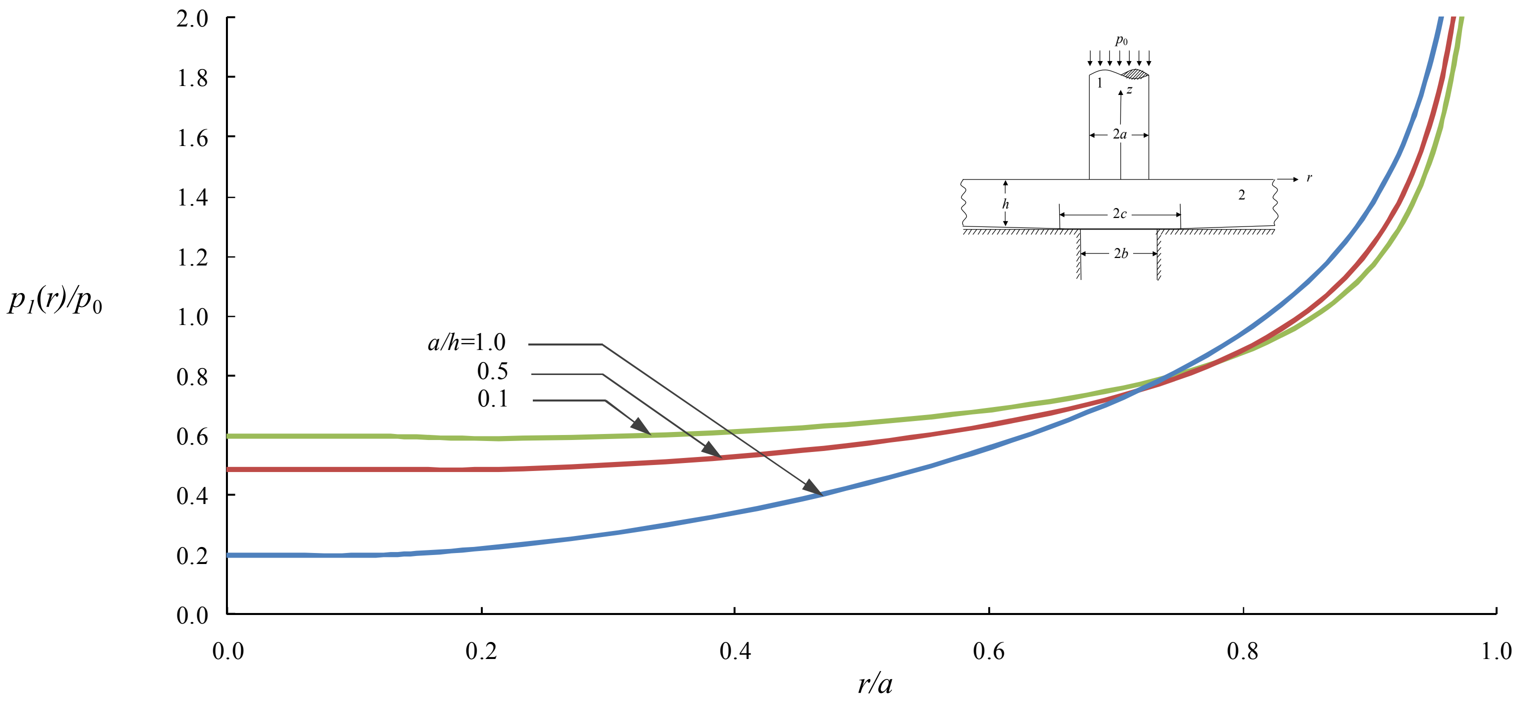

Figure 4.20 Contact pressure between the elastic layer and cylinder for ν=0.3, λ=1/9 and b/h=1.0.

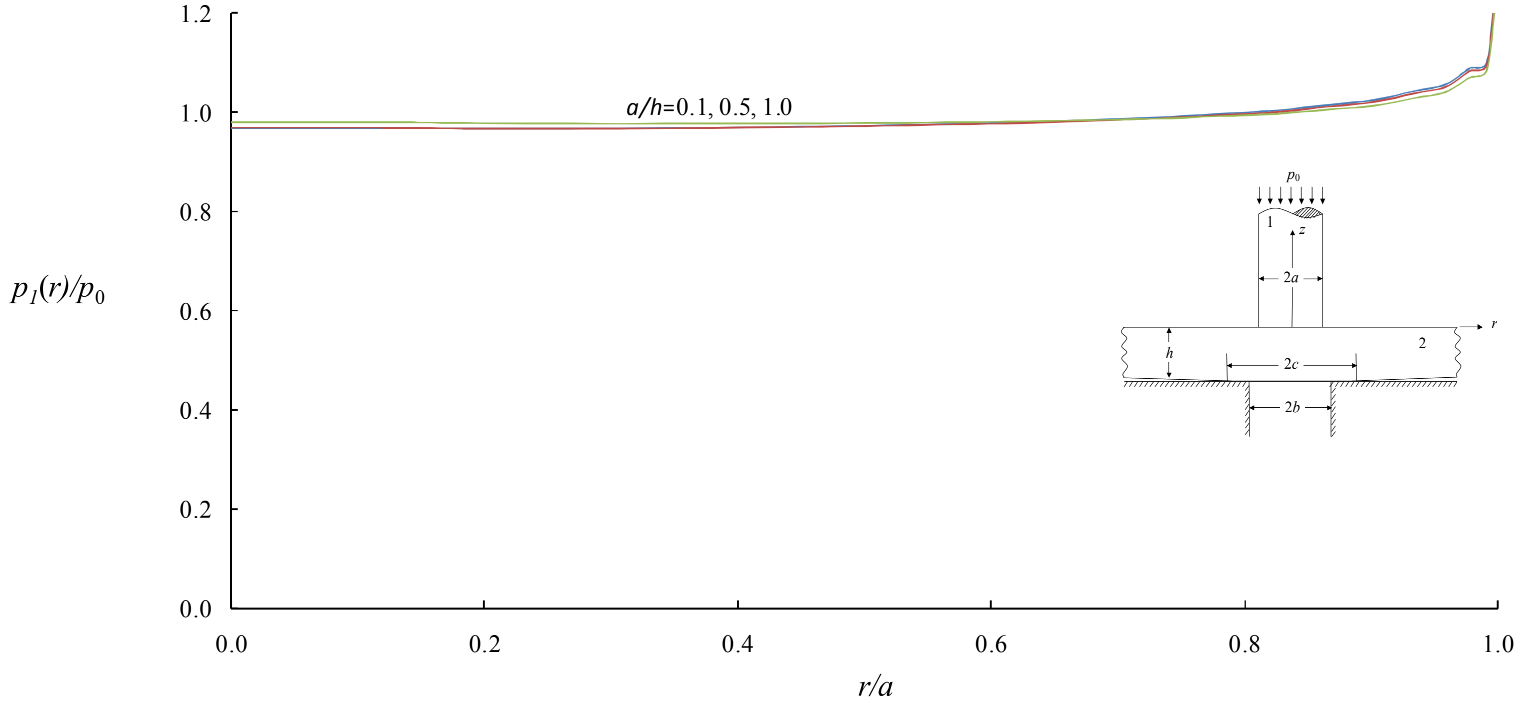

Figure 4.21 Contact pressure between the elastic layer and cylinder for ν=0.3, λ=9 and b/h=0.1.

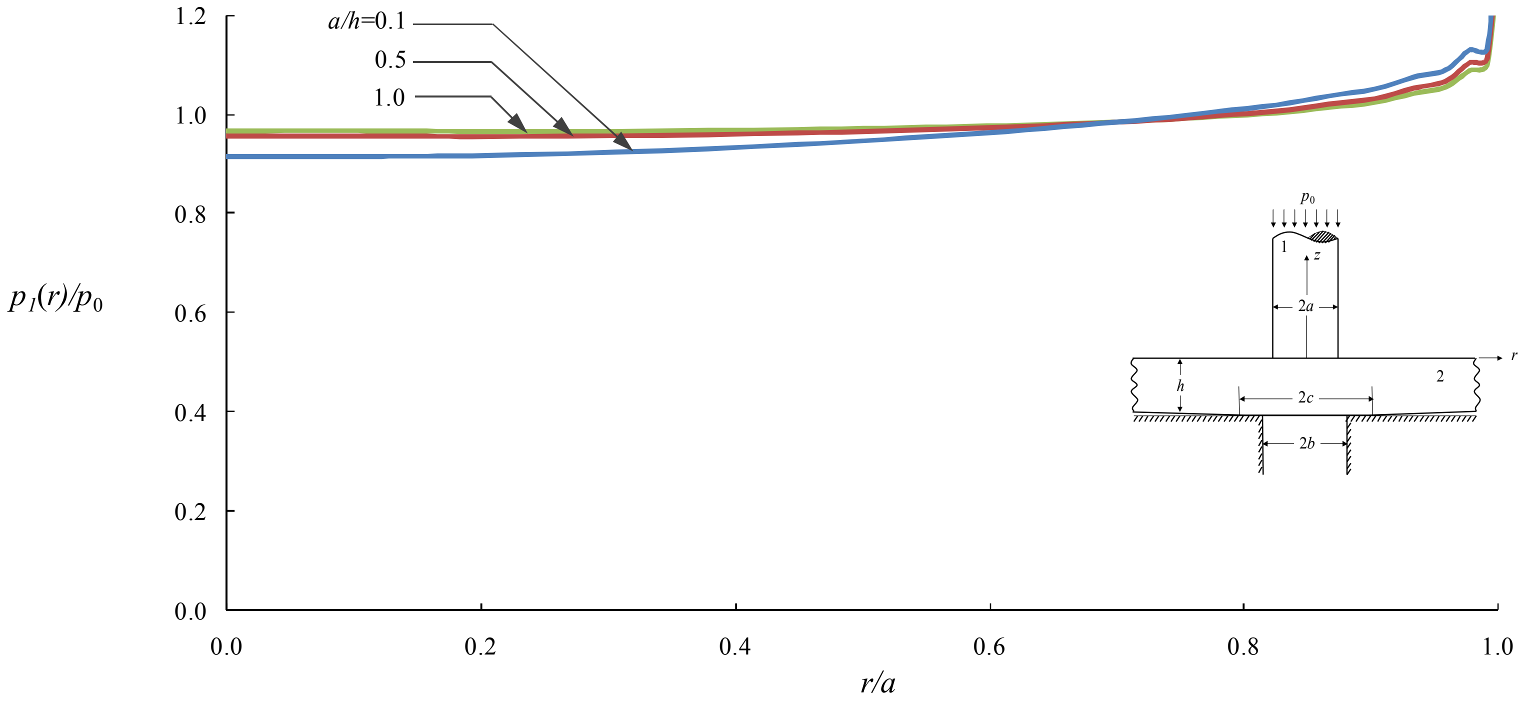

Figure 4.22 Contact pressure between the elastic layer and cylinder for ν=0.3, λ=9 and b/h=1.0.

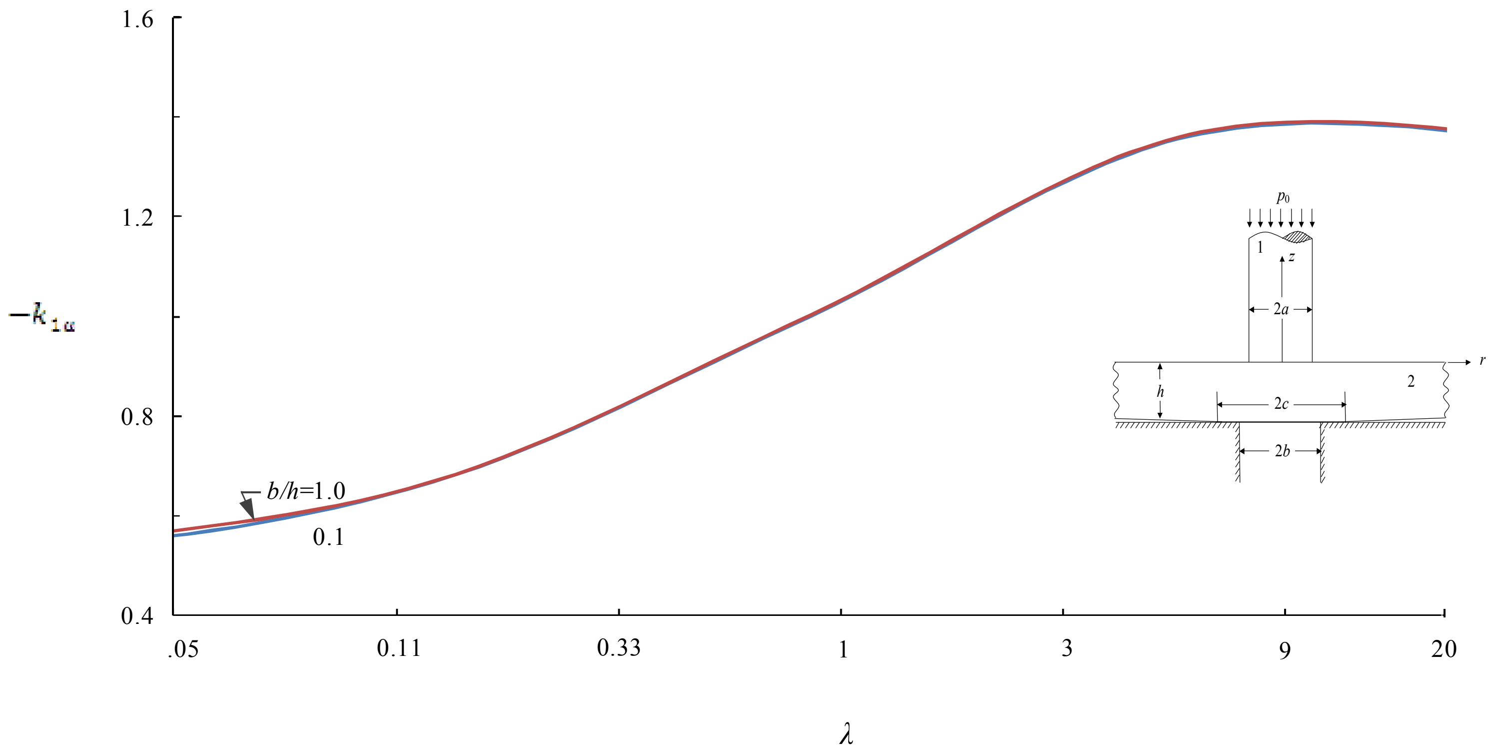

Figure 4.23 Variation of $-\bar{k}_{1a}$ with λ for ν=0.3 and a/h=0.1.

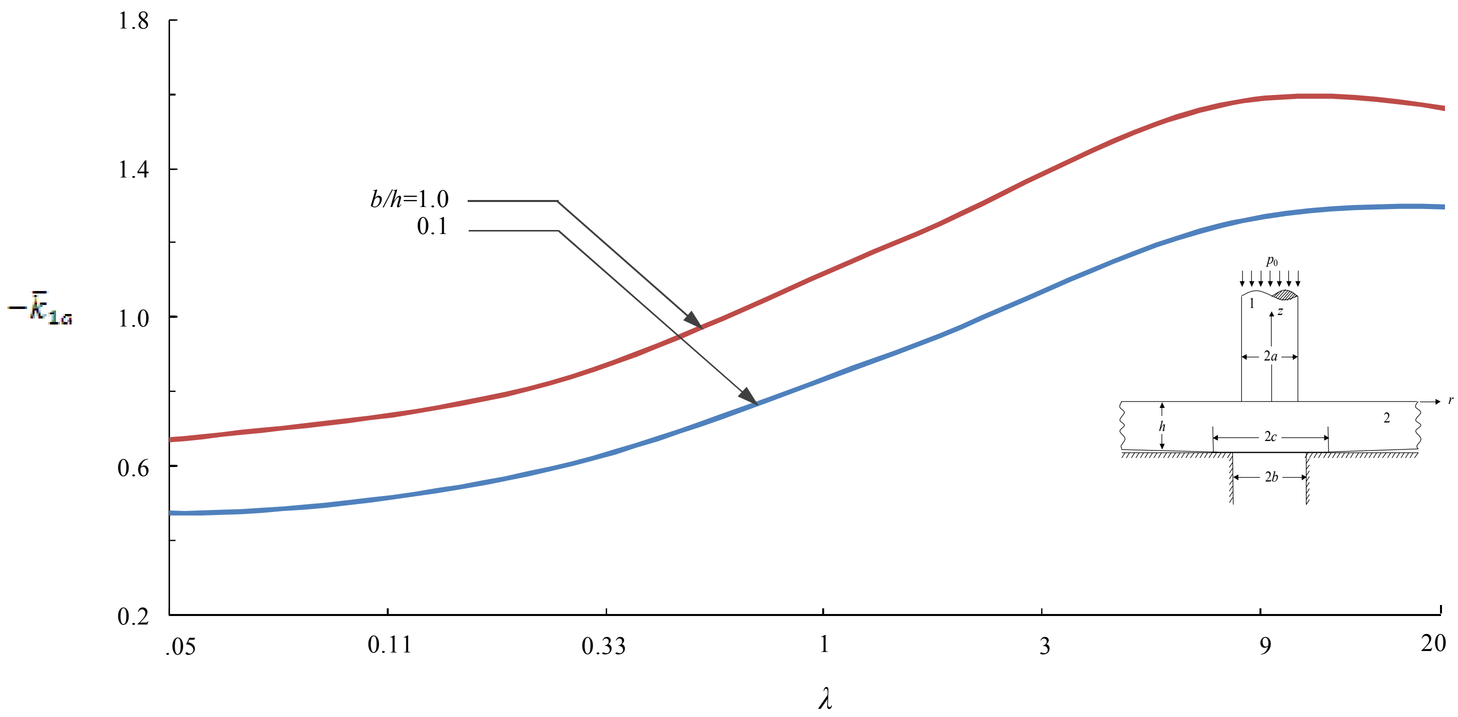

Figure 4.24 Variation of $-\bar{k}_{1a}$ with λ for v=0.3 and a/h=1.0.

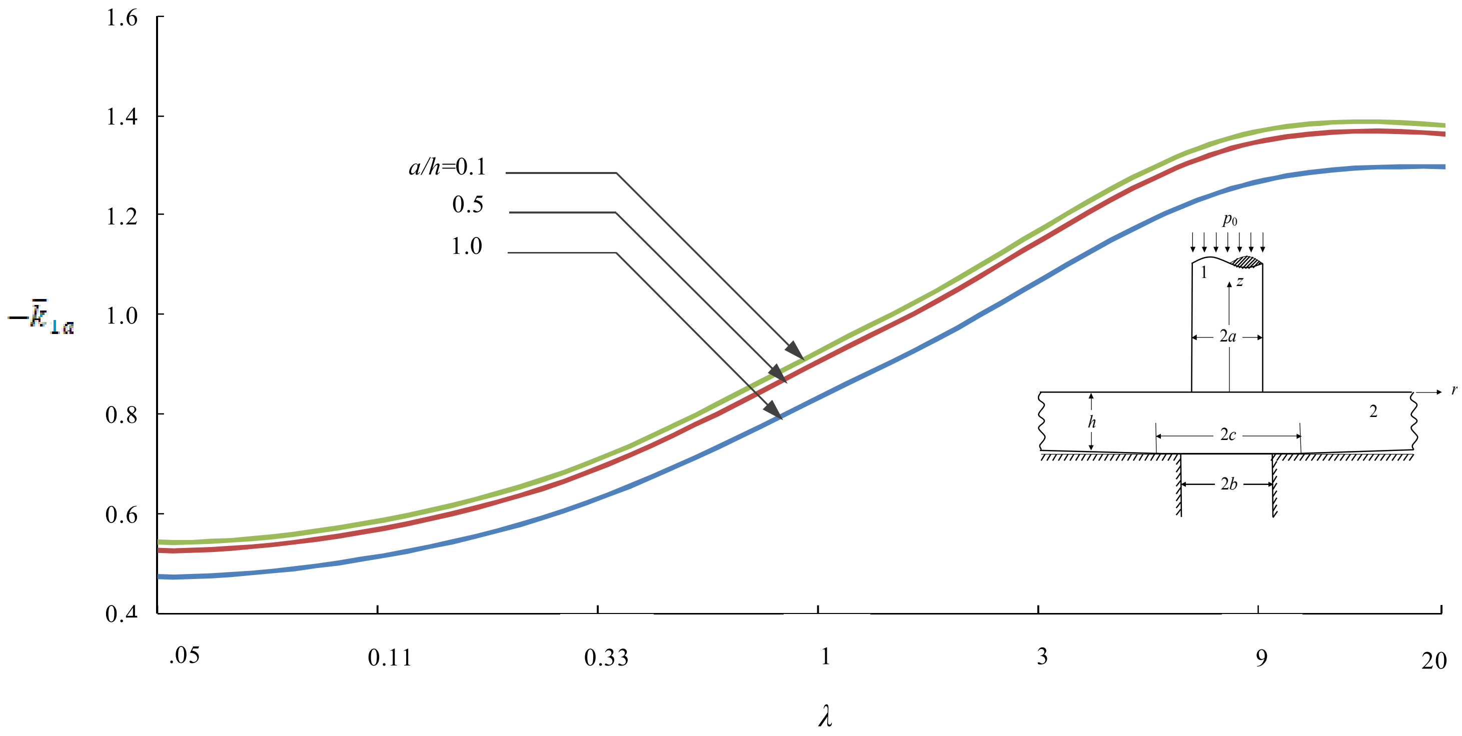

Figure 4.25 Variation of $-\bar{k}_{1a}$ with λ for v=0.3 and b/h=0.1.

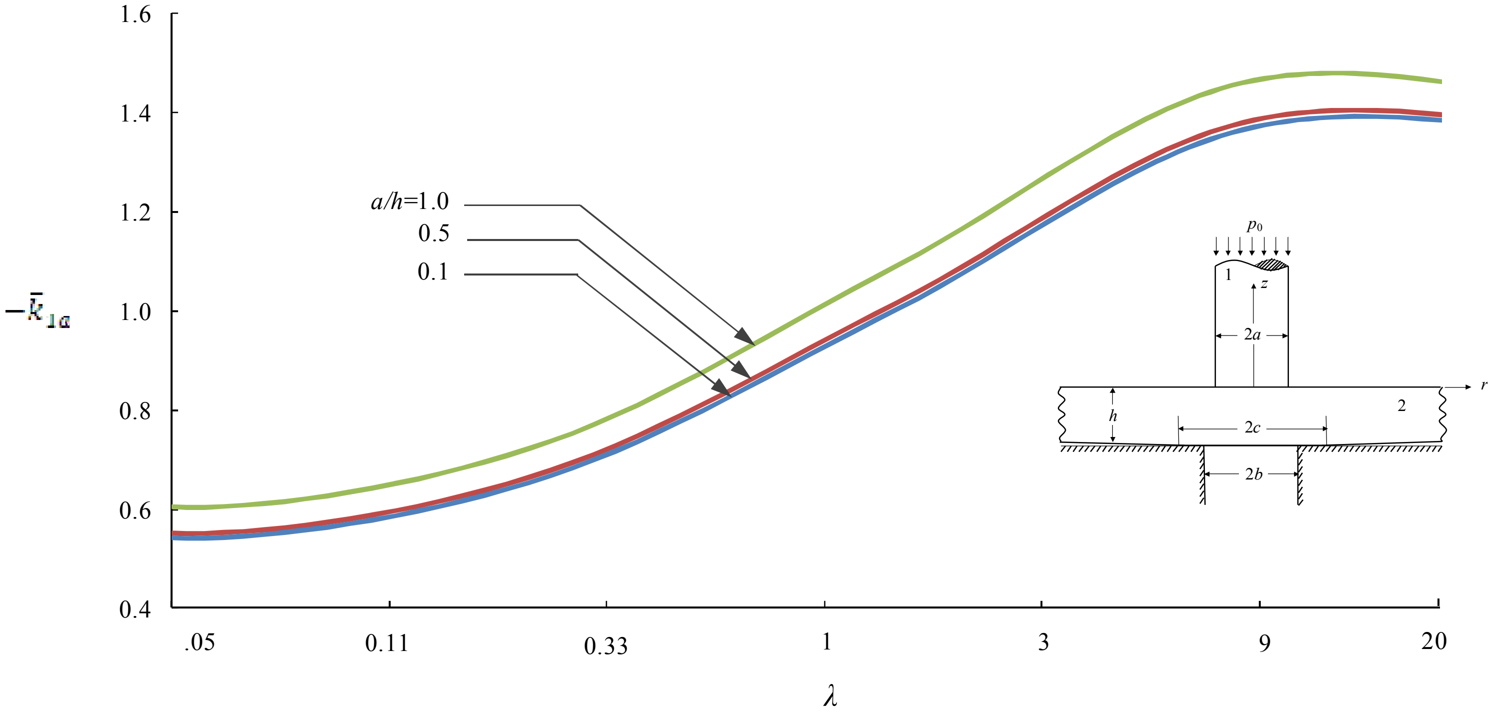

Figure 4.26 Variation of $-\bar{k}_{1a}$ with λ for v=0.3 and b/h=0.5.

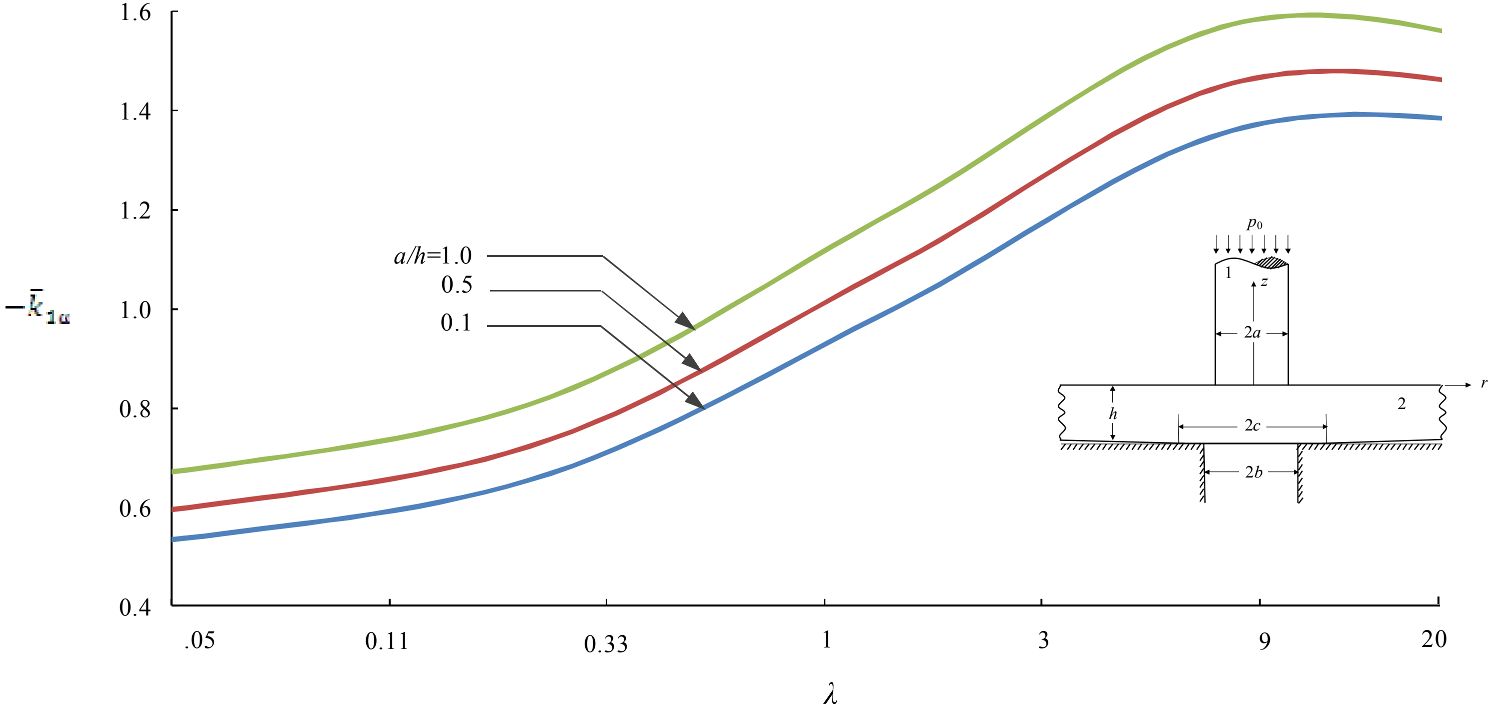

Figure 4.27 Variation of $-\bar{k}_{1a}$ with λ for ν=0.3 and b/h=1.0.

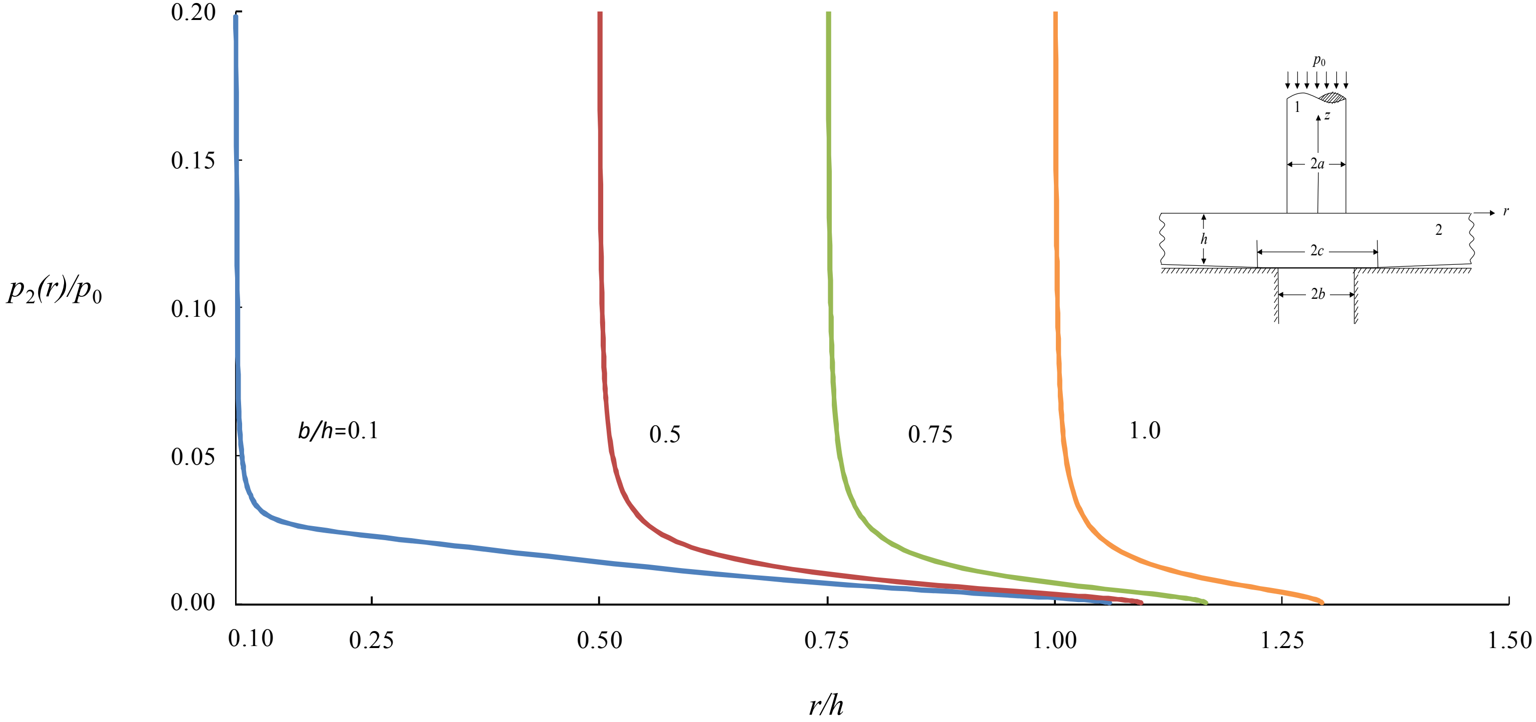

Figure 4.28 Contact pressure between the layer and rigid support for ν=0.3, λ=1/9 and a/h=0.1.

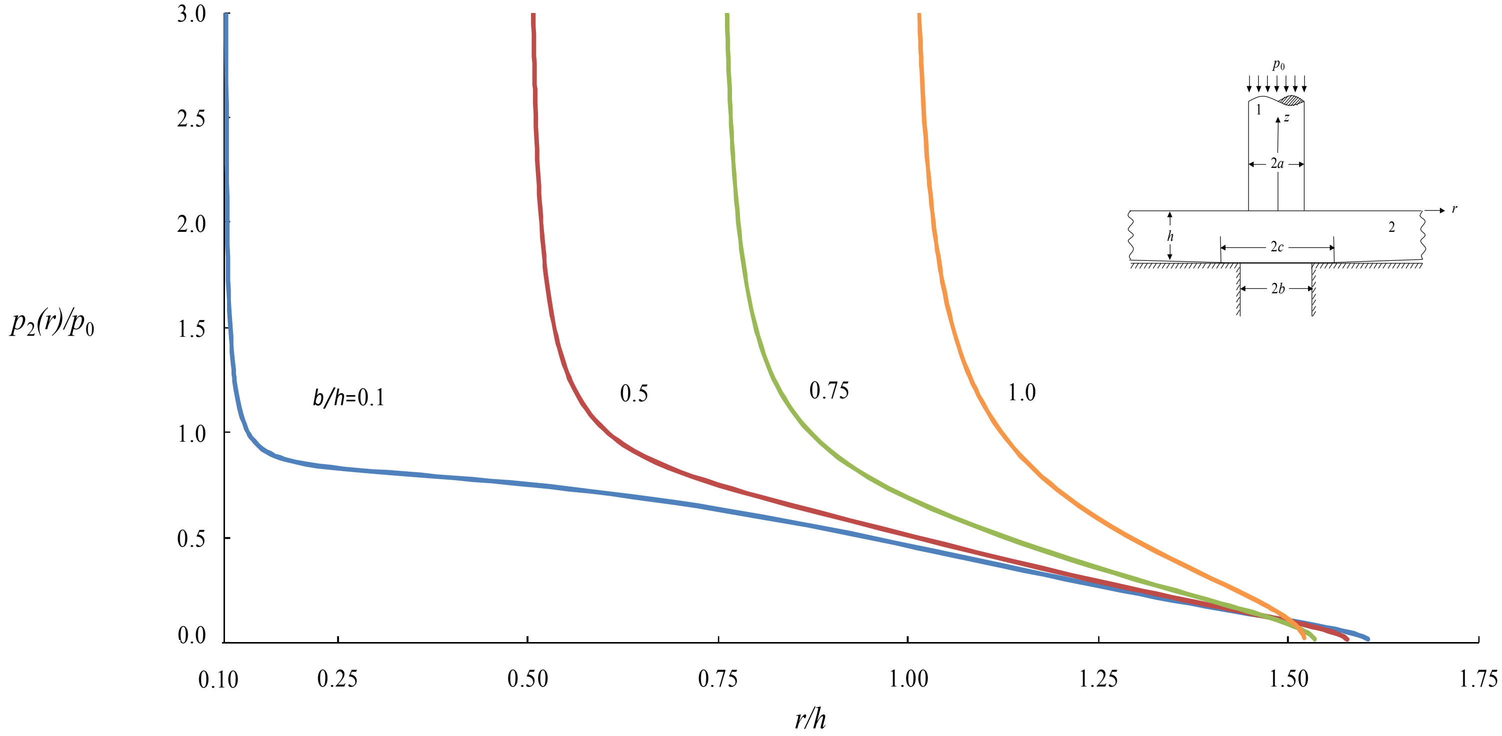

Figure 4.29 Contact pressure between the layer and rigid support for ν=0.3, λ=1/9 and a/h=1.0.

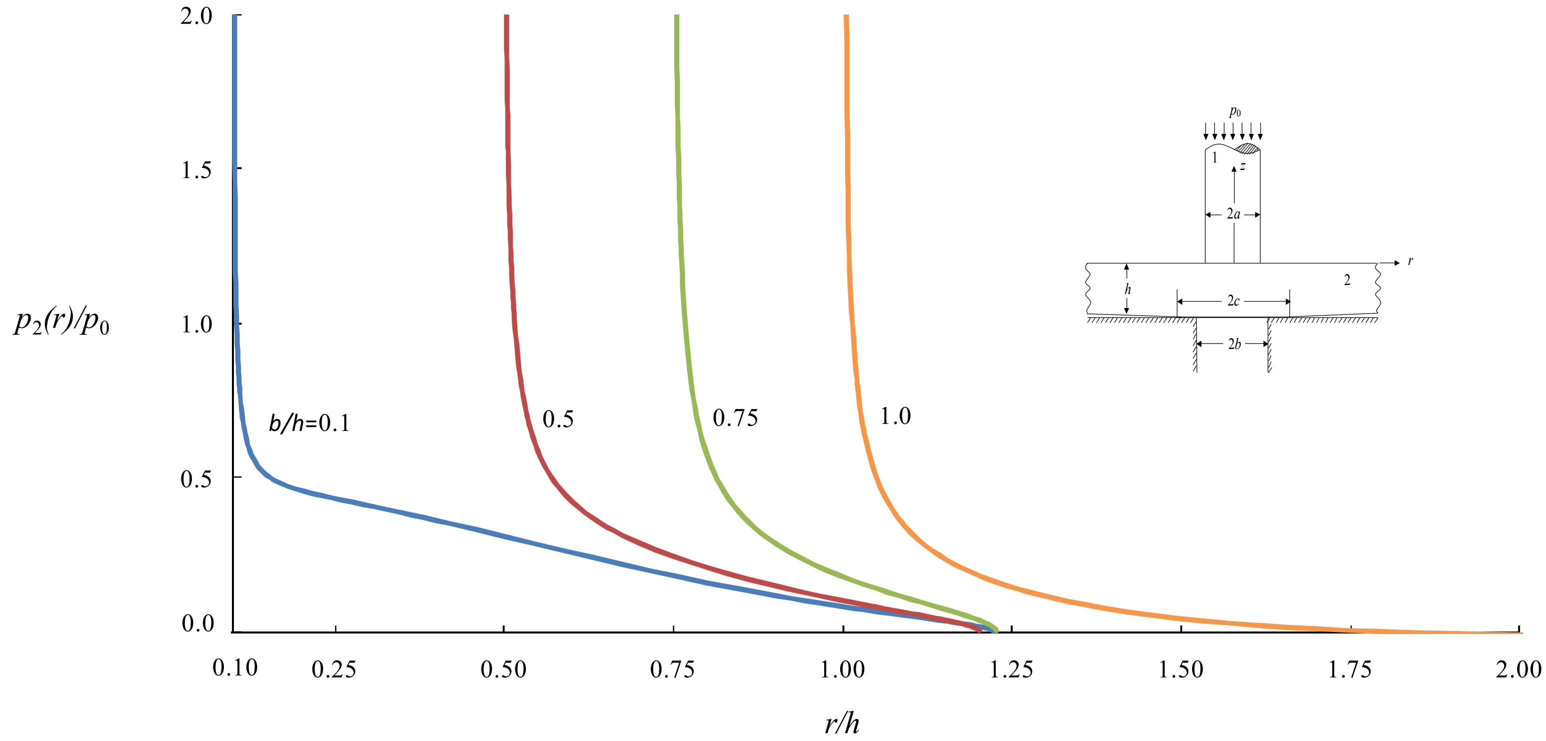

Figure 4.30 Contact pressure between the layer and rigid support for ν=0.3, λ=1and a/h=0.5.

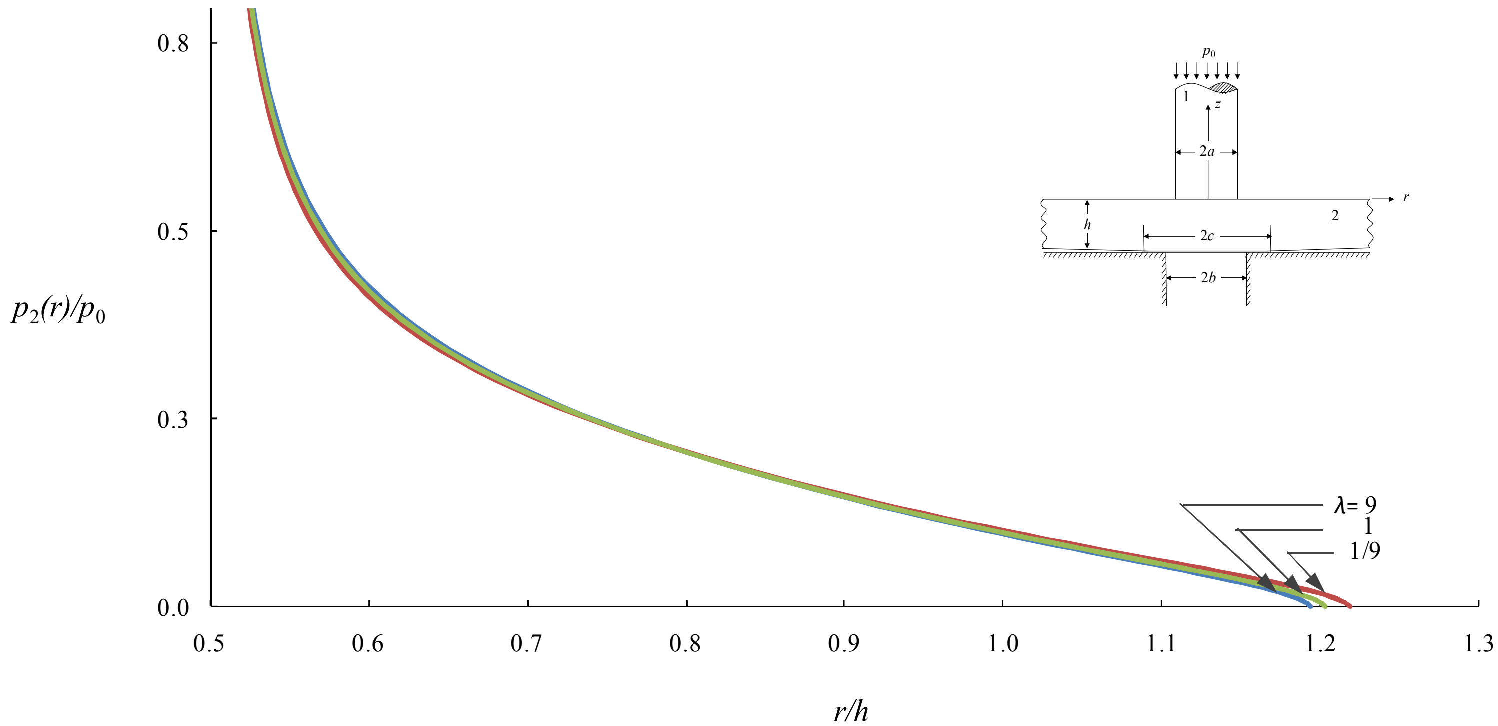

Figure 4.31 Contact pressure between the layer and rigid support for ν=0.3, a/h=0.5 and b/h=0.5.

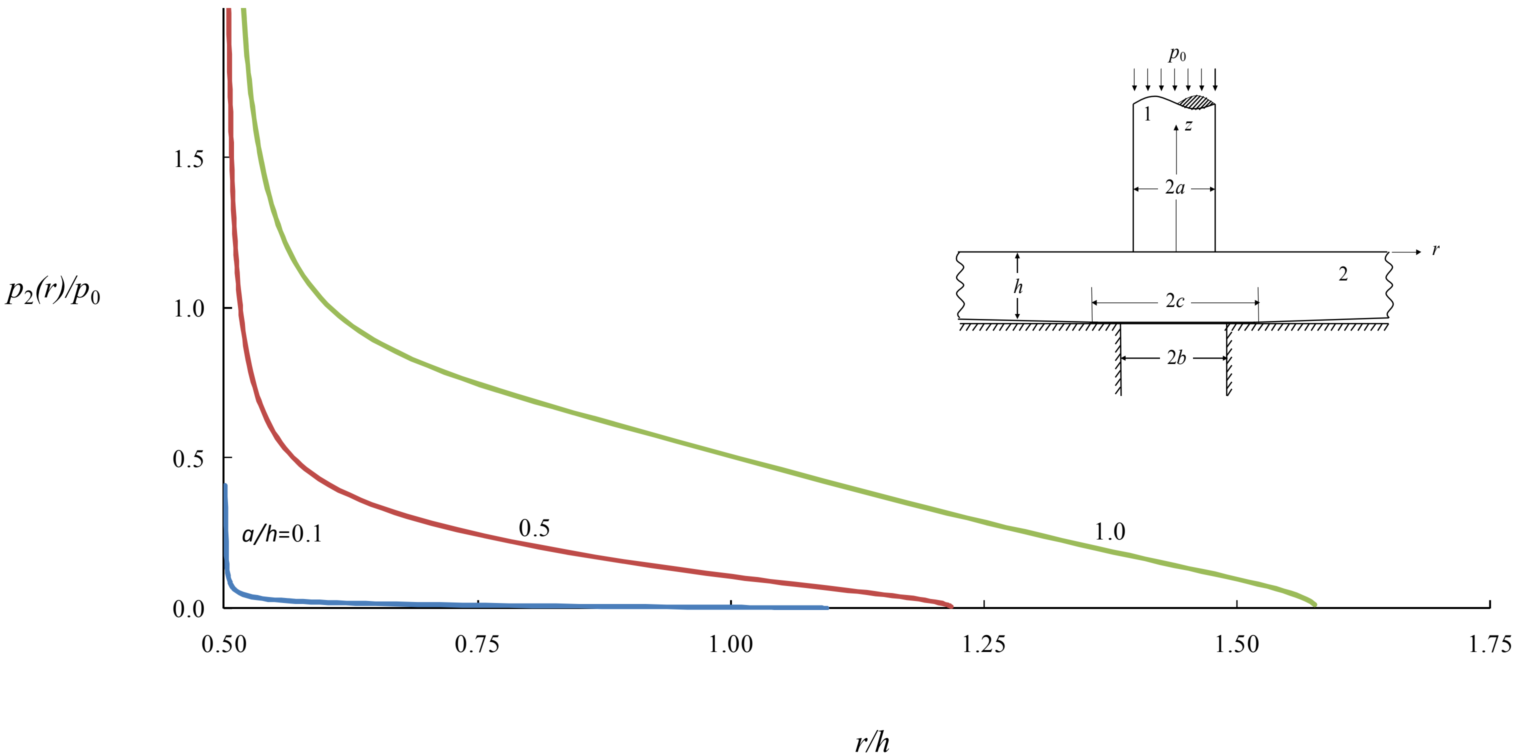

Figure 4.32 Contact pressure between the layer and rigid support for ν=0.3, λ=1/9 and b/h=0.5.

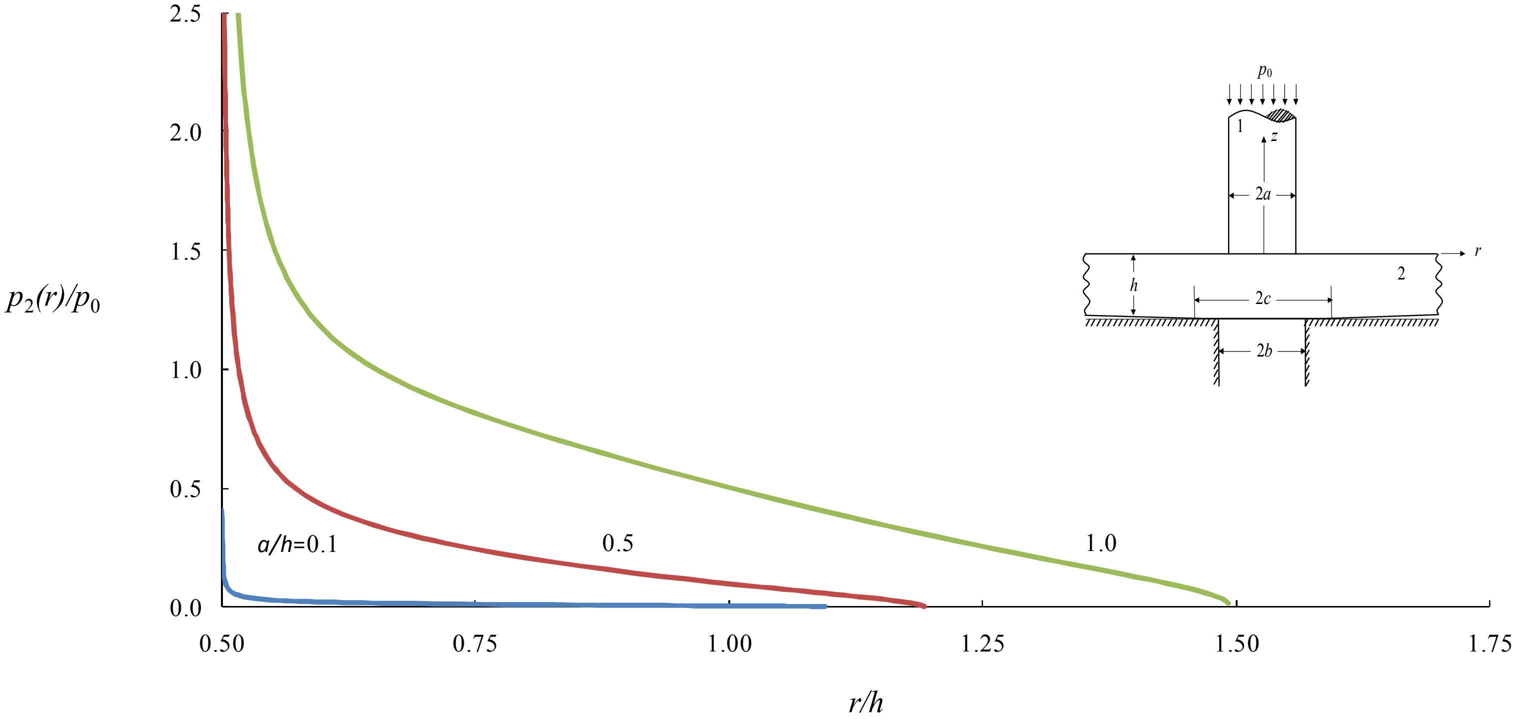

Figure 4.33 Contact pressure between the layer and rigid support for ν=0.3, λ=9 and b/h=0.5.

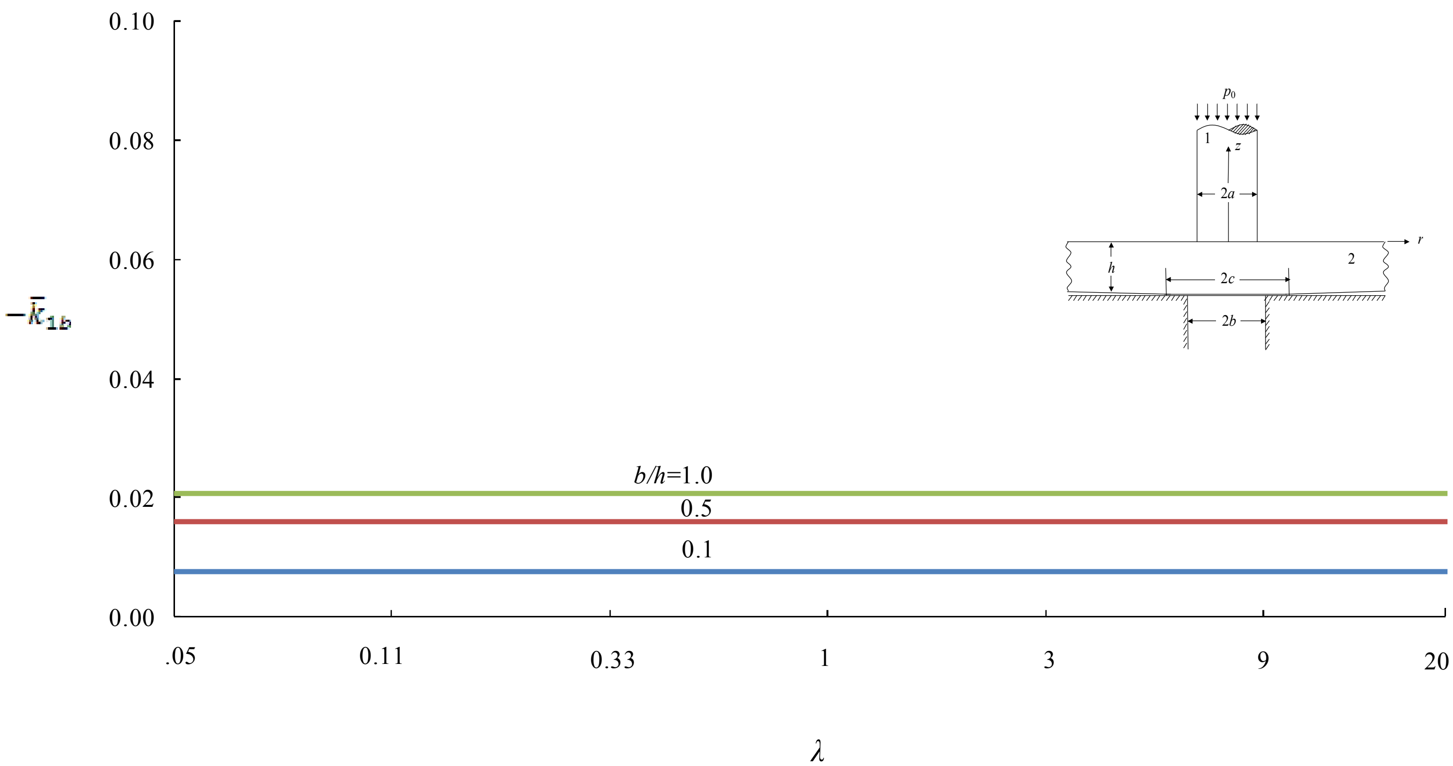

Figure 4.34 Variation of $-\bar{k}_{1b}$ with λ for ν=0.3 and a/h=0.1.

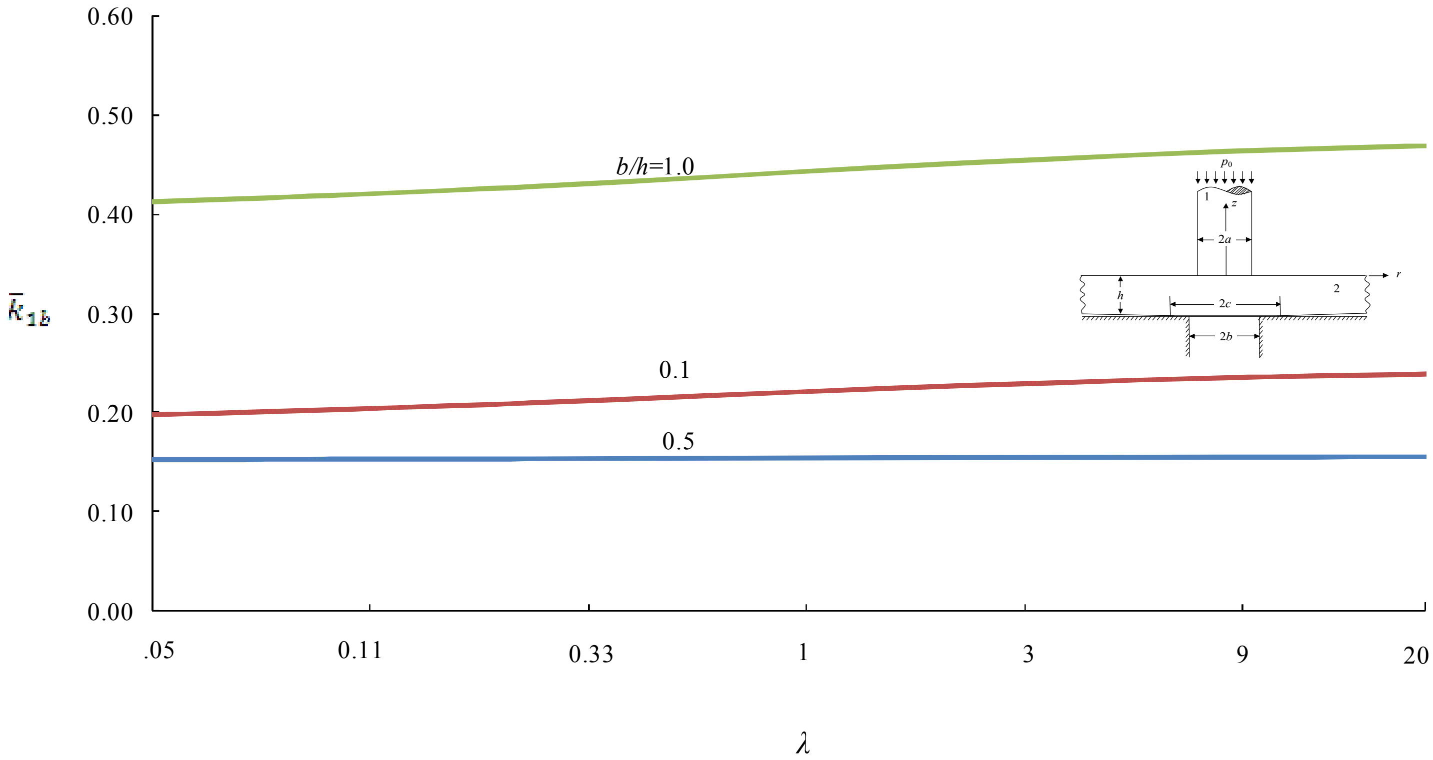

Figure 4.35 Variation of $-\bar{k}_{1b}$ with λ for v=0.3 and a/h=1.0.

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APPENDIX A

DEFINITE INTEGRAL FORMULAS USED IN EQS. (2.35)

Evaluation of some definite integrals from Erdelyi and Magnus (1953):

$$\int_0^\infty e^{-\rho z} \cos(\alpha z)\, dz = \frac{\rho}{\alpha^2 + \rho^2}, \tag{A.1}$$

$$\int_0^\infty e^{-\rho z} \sin(\alpha z)\, dz = \frac{\alpha}{\alpha^2 + \rho^2}, \tag{A.2}$$

$$\int_0^\infty z\, e^{-\rho z} \cos(\alpha z)\, dz = \frac{\rho^2 - \alpha^2}{\left(\alpha^2 + \rho^2\right)^2}, \tag{A.3}$$

$$\int_0^\infty z\, e^{-\rho z} \sin(\alpha z)\, dz = \frac{2\alpha\rho}{\left(\alpha^2 + \rho^2\right)^2}. \tag{A.4}$$

APPENDIX B

INTEGRALS APPEARING IN EQS. (2.36)

$$\int_0^\infty \frac{\rho}{\left(\alpha^2+\rho^2\right)^2} J_1(\rho A) J_1(\rho t)\, d\rho$$

$$= \frac{A}{2\alpha} K_0(\alpha A) I_1(\alpha t) + \frac{1}{\alpha^2} K_1(\alpha A) I_1(\alpha t)$$

$$-\frac{t}{2\alpha} K_1(\alpha A) I_0(\alpha t), \qquad (A \geq t) \qquad \text{(B.1)}$$

$$\int_0^\infty \frac{\rho}{\alpha^2+\rho^2} J_1(\rho A) J_1(\rho t)\, d\rho = K_1(\alpha A) I_1(\alpha t), \qquad (A \geq t) \qquad \text{(B.2)}$$

$$\int_0^\infty \frac{\rho^2}{\alpha^2+\rho^2} J_0(\rho A) J_1(\rho t)\, d\rho = -\alpha K_0(\alpha A) I_1(\alpha t), \qquad (A \geq t) \qquad \text{(B.3)}$$

$$\int_0^\infty \frac{\rho^2}{\left(\alpha^2+\rho^2\right)^2} J_0(\rho A) J_1(\rho t)\, d\rho$$

$$= -\frac{A}{2} K_1(\alpha A) I_1(\alpha t) + \frac{t}{2} K_0(\alpha A) I_0(\alpha t), \qquad (A \geq t) \qquad \text{(B.4)}$$

$$\int_0^\infty \frac{\rho^3}{\left(\alpha^2+\rho^2\right)^2} J_1(\rho A) J_1(\rho t)\, d\rho$$

$$= -\frac{\alpha A}{2\alpha} K_0(\alpha A) I_1(\alpha t) + \frac{\alpha t}{2} K_1(\alpha A) I_0(\alpha t). \qquad (A \geq t) \qquad \text{(B.5)}$$

APPENDIX C

SOME GENERAL EXPANSIONS

C.1 COMPLETE ELLIPTIC INTEGRALS OF THE 1st AND THE 2nd KINDS

$$E(x) = \frac{\pi}{2}\left\{1 - \frac{1}{2^2}x^2 - \frac{1^2 3^2}{2^2 4^2}\frac{x^4}{3} - \cdots - \frac{1^2 3^2 \cdots (2n-1)^2}{2^{2n}(n!)^2}\frac{x^{2n}}{2n-1} - \cdots\right\}$$

$$K(x) = \frac{\pi}{2}\left\{1 + \left(\frac{1}{2}\right)^2 x^2 + \left(\frac{1.3}{2.4}\right)^2 x^4 + \cdots + \left[\frac{1.3\cdots(2n-1)}{2^{2n} n!}\right]^2 x^{2n} + \cdots\right\}$$

$$E(0) = K(0) = \frac{\pi}{2}$$

$$E(1) = 1;\ K(1) = \infty$$

C.2 HANKEL TRANSFORM

$$F_\nu(\rho) \equiv H_\nu\{f(r)\} \equiv \int_0^\infty r\, f(r) J_\nu(\rho\, r)\, dr \qquad r \geq 0$$

$$f(r) \equiv H_\nu^{-1}\{F_\nu(\rho)\} \equiv \int_0^\infty \rho\, F_\nu(\rho) J_\nu(\rho\, r)\, d\rho \quad \nu > -1/2$$

C.3 BESSEL FUNCTIONS OF THE 1st KIND OF ORDER N

$$I_n(x) = \sum_{k=0}^{\infty} \frac{1}{k!(\mathrm{k}+\mathrm{n})!}\left(\frac{x}{2}\right)^{n+2k}$$

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(\mathrm{k}+\mathrm{n})!} \left(\frac{x}{2}\right)^{n+2k}$$

APPENDIX D

GAUSS–LOBATTO INTEGRATION FORMULA USED IN EQS. (3.20b, c) AND EQ. (3.21c) [ERDOGAN (1973)]

$$\frac{1}{\pi}\int_{-1}^{1} k(x_k,t)\left(\frac{1-t}{1+t}\right)^{\frac{1}{2}} g(t)dt = \sum_{i=1}^{n}\frac{2(1-t_i)}{2n+1}k(x_k,t_i)g(t_i) \qquad (k=1,\ldots,n), \qquad \text{(D.1)}$$

$$P_n^{\left(\frac{1}{2},-\frac{1}{2}\right)}(t_i)=0, \quad t_i=\cos\left(\frac{2i\pi}{2n+1}\right), \quad (i=1,\ldots,n), \qquad \text{(D.2)}$$

$$P_n^{\left(-\frac{1}{2},\frac{1}{2}\right)}(x_k)=0, \quad x_k=\cos\left(\frac{2k-1}{2n+1}\pi\right), \quad (k=1,\ldots,n), \qquad \text{(D.3)}$$

GAUSS–JACOBI INTEGRATION FORMULA USED IN EQS. (3.20a–c) AND EQS. (3.21a, b) [ERDOGAN (1973)]

$$\int_{-1}^{1} k(x_j,t)g(t)(1-t)^{\alpha}(1+t)^{\beta}\,dt = \sum_{i=1}^{n} W_i k(x_j,t_i)g(t_i) \qquad (j=1,\ldots,n-1), \qquad \text{(D.4)}$$

where x_j, t_i are the roots of the Jacobi polynomial of degree n.

$$w(t)=(1-t)^{\alpha}(1+t)^{\beta} \qquad (-1<\alpha,\beta<0, \quad -1<t<1) \qquad \text{(D.5)}$$

$$P_n^{(\alpha,\beta)}(t_i)=0 \qquad (i=1,\ldots,n), \qquad \text{(D.6)}$$

$$P_{n-1}^{(1+\alpha,1+\beta)}(x_j)=0 \qquad (j=1,\ldots,n-1), \qquad \text{(D.7)}$$

The weights W_i are given by the formula,

$$W_i = -\frac{2n+\alpha+\beta+2}{(n+1)!(n+\alpha+\beta+1)} \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(n+\alpha+\beta+1)} \frac{2^{\alpha+\beta}}{P_n^{(\alpha,\beta)}(t_i)P_{n+1}^{(\alpha,\beta)}(t_i)}$$

$$(i = 1,...,n), \qquad \text{(D.8)}$$

Special values for Jacobi polynomials [Abramowitz and Stegun (1965)]:

$$f_n = P_n^{(\alpha,\beta)}(t) \qquad \text{(D.9)}$$

$$f_0(t) = 0 \qquad \text{(D.10)}$$

$$f_1(t) = \frac{1}{2}[\alpha - \beta + (\alpha+\beta+2)t] \qquad \text{(D.11)}$$

Recurrence relations,

$$a_{1n} f_{n+1}(t) = (a_{2n} + a_{3n}t) f_n(t) - a_{4n} f_{n-1}(t) \qquad \text{(D.12)}$$

$$a_{1n} = 2(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta) \qquad \text{(D.13)}$$

$$a_{2n} = (2n+\alpha+\beta+1)(\alpha^2 - \beta^2) \qquad \text{(D.14)}$$

$$a_{3n} = (2n+\alpha+\beta) \qquad \text{(D.15)}$$

$$a_{4n} = 2(n+\alpha)(n+\beta) \qquad \text{(D.16)}$$